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Perturbative S-matrix for massive scalar fields in global de Sitter space

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Abstract

We construct a perturbative S-matrix for interacting massive scalar fields in global de Sitter space. Our S-matrix is formulated in terms of asymptotic particle states in the far past and future, taking appropriate care for light fields whose wavefunctions decay only very slowly near the de Sitter conformal boundaries. An alternative formulation expresses this S-matrix in terms of residues of poles in analytically-continued Euclidean correlators (computed in perturbation theory), making it clear that the standard Minkowski-space result is obtained in the flat-space limit. Our S-matrix transforms properly under CPT, is invariant under the de Sitter isometries and perturbative field redefinitions, and is unitary. This unitarity implies a de Sitter version of the optical theorem. We explicitly verify these properties to second order in the coupling for a general cubic interaction, including both tree- and loop-level contributions. Contrary to other statements in the literature, we find that a particle of any positive mass may decay at tree level to any number of particles, each of arbitrary positive masses. In particular, even very light fields (in the complementary series of de Sitter representations) are not protected from tree-level decays.

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Contents

1	Introduction	3
2	Preliminaries	7
2.1	de Sitter geometry	7
2.2	Scalar fields on dS	8
2.3	Interacting fields	12
3	Asymptotic states	16
3.1	General formalism	16
3.2	Interacting theories and IR divergences	18
3.3	The R_σ projector	19
4	The S-matrix	21
5	Lorentz-signature perturbation theory	22
5.1	Schwinger-Keldysh correlation functions	23
5.2	Scattering amplitudes	25
6	Example with heavy fields	27
6.1	$\mathcal{O}(g)$ amplitudes	27
6.2	$\mathcal{O}(g^2)$ state corrections	31
6.3	The Optical Theorem	33
7	Light fields	37
8	Discussion	41
A	KG mode linearization	43
B	The explicit form of good operators	45
	Bibliography	48

1 Introduction

The S-matrix is an invaluable tool for studying quantum field theory (QFT) on Minkowski space. Even leaving aside the all-important connection to experiments, its gauge invariance and invariance under field redefinitions make the S-matrix a powerful way to organize our understanding of quantum fields. The Coleman-Mandula [1], Haag-Lopuszanski-Sohnius [2], and Weinberg-Witten [3] theorems are prime examples of the utility of an S-matrix approach to field theory, and the above properties allow the S-matrix to be well-defined even in perturbative (and perhaps also nonperturbative) string theory.

At a more mundane level, the fact that all correct calculations of an S-matrix element must agree allows a clean comparison of different approaches, choices of gauge, etc., that helps to resolve potential controversies and hastens the advance of knowledge. In contrast, the lack of an S-matrix-like object has been sorely felt in years of controversy regarding quantum fields in de Sitter space. Arguments continue over the interpretation of calculations in both simple self-interacting scalar theories on a fixed de Sitter background (compare e.g. [4, 5, 6, 7, 8, 9, 10]) and also in the more complicated case of gravitational theories (compare e.g. [11, 12, 13, 14, 15, 16, 17]).

In this work, we introduce an S-matrix for weakly-coupled quantum field theories in de Sitter space that can be computed order-by-order in perturbation theory. The associated spaces of asymptotic states are defined using the interacting Hartle-Hawking state $|\Omega\rangle$ as the vacuum. For theories of massive ($M^2 > 0$) scalar fields, we show our S-matrix to be unitary, de Sitter invariant, and invariant under perturbative field redefinitions. It also transforms properly under CPT and reduces to the usual S-matrix in the flat-space limit. Our analysis is strictly perturbative and we consider in detail only theories of interacting scalars. Our final discussion (section 8) will comment briefly on extensions to gauge fields and why gauge-invariance is to be expected. Perhaps our construction will also be of use in perturbative string theory on (likely meta-stable) de Sitter backgrounds.

To be specific, in this work we use the term “de Sitter (dS)” to refer to *global* de Sitter space, including both the contracting and expanding cosmological regions; see figure 1. Thus we formulate a global de Sitter S-matrix below, though we believe that an analogous S-matrix can be defined for both the Poincaré patch (also known as the $k = 0$ cosmological patch), and the hyperbolic ($k = -1$) cosmological patch of dS. Furthermore, these S-matrices should all be closely related through appropriate analytic continuations¹.

One sometimes hears the claim that there can be no S-matrix on de Sitter space. We are aware of the following concerns regarding potential S-matrices:

- i) The Minkowski S-matrix is defined using in/out perturbation theory, but it is well-known that in/out perturbation theory in dS suffers from infrared (IR) divergences. So this definition does not work in de Sitter space.
- ii) There is no positive-definite energy-like conserved quantity in de Sitter space. As a

¹While there should also be an S-matrix for the static patch, it would naturally be defined using asymptotic particle states built on the static vacuum in contrast to the Hartle-Hawking state used in our construction.

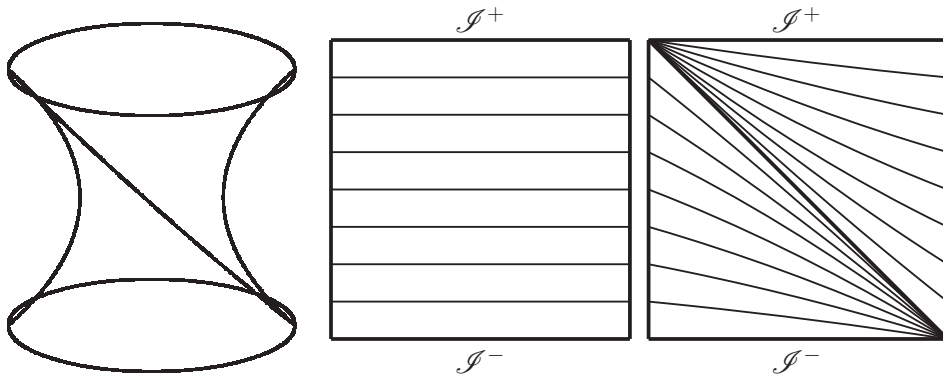


Figure 1: **Left:** A finite piece of global D -dimensional de Sitter space represented as a timelike hyperboloid in $D + 1$ Minkowski space. The diagonal line is a cosmological horizon. **Center:** A conformal (Carter-Penrose) diagram for global de Sitter marked with horizontal cross-sections, each representing an S^{D-1} . The left and right edges are the poles. **Right:** The dS conformal diagram showing a cosmological horizon \mathcal{H} . Each point on the diagram represents an S^{D-2} which contracts to zero size at the left and right edges.

result, one-particle states can decay and all particles are unstable; thus there are no viable asymptotic states. This is directly related to the concerns of e.g. [18, 19, 20, 21].

- iii) The contracting phase of global de Sitter space tends to blueshift particles to high energies. In a theory with dynamical gravity, many states which are weakly-coupled near past infinity induce large gravitational back-reaction near the minimal-radius sphere (the de Sitter “neck,” $\eta = 0$ in the coordinates of (2.2) below). Semi-classically, this should result in gravitational collapse to a cosmological singularity. There is then no reason to expect late-time behavior described by weakly-coupled asymptotic states near the future de Sitter boundary.
- iv) At least in string theory, all known de Sitter vacua are at best meta-stable. So one expects that mere particle excitations of a de Sitter background cannot provide a complete set of outgoing states.
- v) The causal structure of global de Sitter space, and in particular the fact that its past/future boundary is spacelike, prevents any one observer from interacting with a complete set of ingoing/outgoing states. This means that the S-matrix is not experimentally accessible to a single observer and hence need not necessarily be a well-defined object in a fundamental theory.

The reader will note that comments (ii)-(v) are not directly relevant at the level we wish to work. This is particularly manifest for issues (iii), (iv), and (v), which concern dynamical gravity, string theory, or other supposed fundamental theories. Issues (iii) and (iv) (and arguably also (v)) are also intrinsically non-perturbative, and so do not obstruct the more modest goal of formulating a perturbative S-matrix even for gravity or string theory. While

issue (v) implies that a direct connection to experiment is unlikely, a de Sitter S-matrix should nevertheless provide a useful theoretical tool along the lines noted above.

Let us briefly elaborate on the irrelevance of issue (ii), the expected decay of all particles in de Sitter space. First recall the usual situation in flat space: at the perturbative level, each free field is associated with an appropriate set of asymptotic particle states, whether or not these particles turn out to be stable. In the unstable case there is a nonzero 1-to-2 (or more) amplitude that can be computed order by order, and this amplitude is related by the optical theorem to the self-energy correction for that particle. While this implies certain IR divergences in the computation of the strict order-by-order S-matrix,² the physics of these divergences is well-understood. We will show that all of this works analogously in de Sitter space with our definition of the perturbative S-matrix.

It thus remains only to address the more technical concern (i). Dealing with the associated potential IR divergences (in this case, those not associated with self-energy corrections) is in fact the main focus of our work below. But the basic idea is simply that while most textbooks do use in/out perturbation theory to construct the Minkowski S-matrix, this choice is far from unique. The LSZ formalism allows one to extract the S-matrix from vacuum correlators, no matter how they have been computed [22]. In particular, one may define the Minkowski S-matrix by applying LSZ to time-ordered correlators computed using in/in perturbation theory via the closed-time-path formalism [23, 24]; see e.g. [25, 26] for reviews. To do so, one considers a path integral that begins at past infinity, runs up to future infinity, and then back to past infinity. One then inserts operators near past infinity (say, at the beginning of the closed-time path) and also at future infinity (in the middle of the closed-time path). Applying the LSZ formalism yields an S-matrix. The result is manifestly equal to the usual S-matrix, except that the out-vacuum has been replaced by the in-vacuum. But the positivity of the Minkowski conserved energy means that the in- and out-vacua agree, so that this closed-time-path definition does indeed reproduce the usual in/out S-matrix. Since in/in perturbation theory is well-defined in dS and leads to good asymptotic behavior (e.g., an analogue of cluster decomposition) [9, 8, 27, 10], the de Sitter analogue can lead to a good S-matrix.

One complication is that in/in perturbation theory in de Sitter space is IR finite only when the closed time path begins and ends at a finite time. In particular, it is best behaved when the path begins and ends on a cosmological horizon so that it constructs correlators in the interacting Hartle-Hawking state (called $|\Omega\rangle$ below); see [28] for a proof valid for scalar theories with $M^2 > 0$. Thus the natural de Sitter analogue of the flat-space construction just outlined would be an S-matrix for the part of dS to the future of a cosmological horizon. This region is called the Poincaré patch of dS by some authors and the expanding $k = 0$ cosmological patch by others. While we believe that this object does in fact exist, we save its construction and analysis for future work. Instead, we construct an S-matrix for global de Sitter space below – roughly speaking by gluing together two copies of the Poincaré patch construction just described, one for the expanding patch and one for the corresponding contracting patch to the past of the horizon – along the de Sitter horizon. More precisely, we

²That is, without resummation of the 1PI self-energy corrections to the propagator.

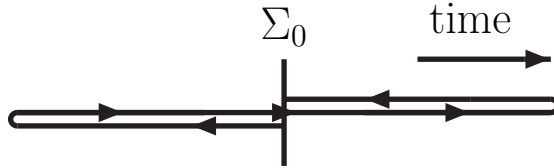


Figure 2: The time contour used to construct our global de Sitter S-matrix. The surface Σ_0 represents a cosmological horizon. See section 5 for details.

use an out/out-in/in closed time path which begins on a de Sitter horizon, travels backward to past infinity, returns to the de Sitter horizon and keeps on going to future infinity, and then finally returns to the de Sitter horizon once again; see figure 2. Operators are then inserted near past and future infinity and a de Sitter version of LSZ is used to extract an S-matrix.

This final LSZ-like step requires some care, as one must sort out further IR issues to find an appropriate dS analogue of the flat-space LSZ formalism. Some of these issues are already documented in the literature (see e.g. [29, 30, 31, 32]). The point is again that several procedures which are equivalent in flat space do not agree in dS, and that some seemingly natural dS generalizations lead to IR divergences for light (so-called complementary series) fields with masses $0 < M^2 \ell^2 < (D - 1)^2/4$; here ℓ is the de Sitter length scale and D is the spacetime dimension. Nevertheless, we show in section 3.3 that a particular definition (based on extracting an S-matrix from poles in correlation functions) is free of unphysical IR divergences for generic³ $M^2 > 0$. An alternate more explicit (but ultimately equivalent) procedure is described in appendix B. While we find it enlightening to understand this explicit structure, it quickly becomes cumbersome at higher orders in perturbation theory.

Before proceeding, we should warn the reader of two further technical issues. The first is that our choice of time contour and the use of the associated Schwinger-Keldysh-like formalism means that perturbation theory involves several distinct types of vertices and propagators. This makes explicit computations more complicated than in flat space. It may well turn out that other computational techniques, such as analytic continuation from the Euclidean sphere or Euclidean AdS, will prove more efficient for computing scattering amplitudes.

The second technical issue is that the lack of a positive-definite energy-like conserved quantity makes representation theory of the de Sitter group somewhat less powerful than its flat-space analogue. In particular, de Sitter representation theory alone does not guarantee the orthogonality of what we call distinct multi-particle asymptotic states. While such states are orthogonal for free theories, they cease to be orthogonal for general interacting theories. Even states with different particle numbers develop non-zero inner products. This, however, is not an obstacle to defining a useful S-matrix. We view it as essentially an accounting issue, though admittedly one that provides a sense in which our asymptotic states are not

³Our formulation fails for a measure zero subset of masses where self-energy corrections to the locations of poles are always large, even at small coupling g ; see section 2.3.

‘free.’ A formal construction of orthonormal asymptotic states (e.g., by the Gram-Schmidt procedure) is always possible. However, such a construction obscures the physics, and so we will not emphasize it below.

The remainder of the paper is organized as follows. We begin by reviewing basic properties of de Sitter space and de Sitter QFTs in §2. We discuss the construction of asymptotic states in §3 and show that the resulting S-matrix has the desired properties in §4. The tools necessary to actually calculate S-matrix elements in Lorentz signature are provided in §5. Using these tools we compute the S-matrix of a model cubic theory of heavy fields to second order in §6. We then revisit our model theory in §7 allowing for fields of arbitrary positive mass. One interesting result is that, consistent with [33, 34] but in contrast to the claims of [35, 36, 37], we find that complementary series particles generically decay. In all cases, we explicitly verify unitarity by showing that our S-matrix satisfies the de Sitter version of the optical theorem. We conclude with a summary and discussion of open issues in §8.

2 Preliminaries

We begin by briefly reviewing de Sitter space and some relevant aspects of de Sitter quantum field theory.

2.1 de Sitter geometry

The D -dimensional de Sitter manifold dS_D may be defined as the single-sheet hyperboloid in an ambient $(D + 1)$ -dimensional Minkowski space:

$$dS_D = \{X \in \mathbb{R}^{D,1} \mid X \cdot X = \ell^2\}. \quad (2.1)$$

The line element of de Sitter space may be written in the following convenient coordinates:

$$\frac{ds^2}{\ell^2} = \left[-\frac{1}{1 + \eta^2} d\eta^2 + (1 + \eta^2) d\Omega_{D-1}^2 \right], \quad \eta \in \mathbb{R}. \quad (2.2)$$

Here ℓ is the de Sitter radius and $d\Omega_{D-1}^2$ denotes the line element on unit S^{D-1} . The time coordinate η is related to the more familiar global de Sitter time coordinate t with $g_{tt} = -1$ via $\eta = \sinh(t/\ell)$. In these coordinates the volume element is $\sqrt{-g(x)} d^D x = \ell^D (1 + \eta^2)^{(D-2)/2} d\eta d\Omega_{D-1}(\vec{x})$. The metric (2.2) describes spatial sections which are spheres S^{D-1} of radius $\ell^2(1 + \eta^2)$. The conformal boundary of the chart consists of two disjoint spheres at $\eta \rightarrow \pm\infty$ denoted \mathcal{I}^\pm .

At times it will be useful to use a second coordinate chart with line element

$$\frac{ds^2}{\ell^2} = \tau^2 \left[-\frac{1}{\tau^4} d\tau^2 + d\vec{x}^2 \right], \quad \tau \in \mathbb{R}, \quad (2.3)$$

which also covers the entire manifold. Here $d\vec{x}^2$ denotes the line element on \mathbb{R}^{D-1} . This chart has a coordinate singularity at $\tau = 0$ corresponding to a preferred cosmological horizon \mathcal{H} . In

the regions $\tau < 0$ ($\tau > 0$) one may use instead the time coordinate $\lambda = \pm\tau^{-1}$ to recover the more familiar Poincaré coordinate chart (a.k.a. the expanding cosmological chart). Penrose diagrams which depict the above charts are shown at center (τ -chart) and right (η -chart) of Fig. 1.

While dS is maximally symmetric, it is important to remember that it has no globally timelike Killing vector field. In particular, neither ∂_η nor ∂_τ are Killing vectors. As a result, global de Sitter does not possess a conserved quantity associated with flow in only timelike directions; in this sense there is no conserved “energy.” For further details of de Sitter spacetime we refer the reader to [38, 39, 40].

2.2 Scalar fields on dS

Free scalar QFT on a fixed de Sitter background is a well-understood subject with many good references (e.g., [39, 41, 27]). Below we focus only on establishing notation and on certain group theoretical aspects which will be useful later. At the end of this section, we comment briefly on interacting fields.

Scalar fields on de Sitter are associated with representations of the de Sitter isometry group $SO(D, 1)$. For instance, the one-particle states of a free scalar field $\phi_\sigma(x)$ described by the canonical Lagrangian

$$\mathcal{L}_0[\phi_\sigma] = -\frac{1}{2}\nabla_\mu\phi_\sigma\nabla^\mu\phi_\sigma(x) - \frac{M^2}{2}\phi_\sigma^2(x), \quad (2.4)$$

form a unitary irreducible representation (UIR) of $SO(D, 1)$. We define the *weight* σ via

$$M^2(\sigma)\ell^2 = -\sigma(\sigma + D - 1). \quad (2.5)$$

The right-hand side is invariant under $\sigma \rightarrow -(\sigma + D - 1)$; we choose σ to be given by

$$\sigma = -\frac{(D-1)}{2} + \left[\frac{(D-1)^2}{4} - M^2\ell^2 \right]^{1/2}. \quad (2.6)$$

The UIRs of the de Sitter group may be classified as follows [42]:

1. principal series:

$$\frac{(D-1)^2}{4} \leq M^2\ell^2, \quad \Rightarrow \quad \sigma = -\frac{(D-1)}{2} + i\rho, \quad \rho \in \mathbb{R}, \quad \rho \geq 0, \quad (2.7)$$

2. complementary series:

$$0 < M^2\ell^2 < \frac{(D-1)^2}{4}, \quad \Rightarrow \quad \sigma \in \left(-\frac{(D-1)}{2}, 0 \right), \quad (2.8)$$

3. for some D , there is also a discrete series:

$$M^2\ell^2 = -n(n + D - 1) \text{ for } n \in \mathbb{N}_0, \quad \Rightarrow \quad \sigma = n. \quad (2.9)$$

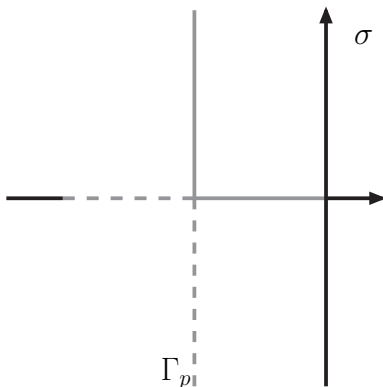


Figure 3: Scalar de Sitter UIRs are depicted by the solid gray line in the complex σ plane. For complementary series representations the weight σ takes values along the negative real axis $\sigma \in (-(D-1)/2, 0)$ while for the principal series σ takes complex values $\sigma = -(D-1)/2 + i\rho$, $\rho \geq 0$. For each series the ‘conjugate weights’ $-(\sigma + D - 1)$ are depicted with a dashed gray line. We denote by Γ_p the $\text{Re } \sigma = -(D-1)/2$ contour. Representations with σ values to the left of Γ_p are reducible; they may be represented as an integral over the principal series UIRs. Representations with $\text{Re } \sigma > -(D-1)/2$ and $\text{Im } \sigma \neq 0$ are not unitary.

“Heavy” fields belong to the principal series while extremely light fields with masses of order ℓ^{-2} belong to the complementary series. These will suffice for our purposes as we limit our analysis below to fields with mass $M^2 > 0$. In contrast, UIRs of the discrete series are associated with $M^2 = 0$ and certain tachyonic masses. It is useful to visualize the de Sitter representations in the complex σ plane – see Fig. 3.

For a given mass $M^2 \geq 0$ solutions to the Klein-Gordon (KG) equation $(\square - M^2)\phi_\sigma(x) = 0$ are endowed with a positive-definite “Klein-Gordon norm.” Working in the global chart, there is a natural basis of Klein-Gordon modes $u_{\sigma\vec{L}}(x)$ which are distinguished by their angular momenta \vec{L} on the spatial S^{D-1} and are orthonormal with respect to the Klein-Gordon norm

$$\begin{aligned}
(u_{\sigma\vec{L}_1}, u_{\sigma\vec{L}_2})_{\text{KG}} &:= -i \int d\Sigma^\nu(x) \left[u_{\sigma\vec{L}_1}(x) \overleftrightarrow{\nabla}_\nu u_{\sigma\vec{L}_2}^*(x) \right] \Big|_{\eta=\text{const.}} \\
&= -i \ell^{D-2} (1 + \eta^2)^{D/2} \int d\Omega_{D-1}(\vec{x}) \left[u_{\sigma\vec{L}_1}(x) \overleftrightarrow{\nabla}_\eta u_{\sigma\vec{L}_2}^*(x) \right] \Big|_{\eta=\text{const.}} \\
&= \delta_{\vec{L}_1\vec{L}_2}.
\end{aligned} \tag{2.10}$$

The Klein-Gordon modes may be written explicitly as

$$u_{\sigma\vec{L}}(x) = \ell^{(2-D)/2} f_{\sigma L}(\eta) Y_{\vec{L}}(\vec{x}), \tag{2.11}$$

where $Y_{\vec{L}}(\vec{x})$ are spherical harmonics on S^{D-1} parametrized by angular momenta \vec{L} with

total angular momentum L and the functions $f_{\sigma L}(\eta)$ may be written, e.g., as

$$f_{\sigma L}(\eta) = N_{\sigma L}(1 + \eta^2)^{L/2} {}_2F_1 \left[L - \sigma, L + \sigma + D - 1; L + \frac{D}{2}; \frac{1 - i\eta}{2} \right], \quad (2.12)$$

$$N_{\sigma L} = \frac{2^{-L-(D-1)/2}}{\Gamma(L + \frac{D}{2})} [\Gamma(L - \sigma)\Gamma(L + \sigma + D - 1)]^{1/2}, \quad (2.13)$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function [43]. The behavior of the mode functions in the asymptotic regions $|\eta| \rightarrow \infty$ is given by [10]

$$f_{\sigma L}(\eta) = (1 + \eta^2)^{L/2} [K_{\sigma L}(2i\eta)^{\sigma-L} + K_{-(\sigma+D-1)L}(2i\eta)^{-(\sigma+D-1)-L}] [1 + \mathcal{O}(\eta^{-2})],$$

$$|\eta| \gg (L - \sigma), \quad (2.14)$$

with $K_{\sigma L}$ a coefficient whose value is

$$K_{\sigma L} = \frac{2^{\sigma+(D-3)/2}}{\sqrt{\pi}} \Gamma\left(\sigma + \frac{D-1}{2}\right) \left[\frac{\Gamma(L - \sigma)}{\Gamma(L + \sigma + D - 1)} \right]^{1/2}. \quad (2.15)$$

It is important to note, however, that the Klein-Gordon current does not generally lead to a useful inner product between functions with support on multiple de Sitter representations. Integrals of the form

$$-i \int d\Sigma^\nu(x) \left[u_{\sigma \vec{L}_1}(x) \overleftrightarrow{\nabla}_\nu F(x) \right] \Big|_{\eta=\eta_0} \quad (2.16)$$

for general $F(x)$ are complex, depend upon η_0 , and may diverge as $|\eta_0| \rightarrow \infty$. This includes the case where $F(x) = u_{\sigma_2 \vec{L}_2}^*(x)$ is a Klein-Gordon mode corresponding to a mass $\sigma_2 \neq \sigma$. Indeed, the asymptotic behavior (2.14) shows that the expression

$$\lim_{\eta_0 \rightarrow \pm\infty} \left[-i \ell^{D-2} (1 + \eta^2)^{D/2} \int d\Omega_{D-1}(\vec{x}) \left[u_{\sigma_1 \vec{L}_1}(x) \overleftrightarrow{\nabla}_\eta u_{\sigma_2 \vec{L}_2}^*(x) \right] \Big|_{\eta=\eta_0} \right] \quad (2.17)$$

generally diverges for $\sigma_1 \neq \sigma_2$, though it is finite for $\sigma_1 = \sigma_2$ and is oscillatory for $\sigma_1 \neq \sigma_2$ both in the principal series (in which case it converges to zero as a distribution).

Free massive scalar fields in de Sitter have a unique normalizable maximally symmetric Hadamard state $|0\rangle$ known variously as the Hartle-Hawking (HH) state, the Euclidean vacuum, and Bunch-Davies state [41, 44]. The multiplicity of names can be attributed to the multiplicity of ways to construct the state. The first two names come from the fact that the state, defined by its set of correlation functions, may be constructed by analytic continuation from the Euclidean section S^D . Alternatively, one may construct this state in either global de Sitter or the Poincaré chart by imposing adiabatic or ‘‘Bunch-Davies’’ vacuum conditions on the cosmological horizon. Within the static chart this state has yet another interpretation as the thermal state at the de Sitter temperature (it is the unique thermal state regular on the cosmological horizons).

For a scalar field $\phi_\sigma(x)$ we denote the associated Wightman and time-ordered 2-point functions by

$$W_\sigma(x_1, x_2) := \langle 0 | \phi_\sigma(x_1) \phi_\sigma(x_2) | 0 \rangle, \quad (2.18)$$

$$G_\sigma(x_1, x_2) := \langle 0 | T \phi_\sigma(x_1) \phi_\sigma(x_2) | 0 \rangle. \quad (2.19)$$

They may of course be written in terms of the Klein-Gordon modes (2.11) as

$$W_\sigma(x_1, x_2) = \sum_{\vec{L}} u_{\sigma\vec{L}}(x_1) u_{\sigma\vec{L}}^*(x_2), \quad (2.20)$$

$$G_\sigma(x_1, x_2) = W_\sigma(x_1, x_2) \theta(\eta_1 - \eta_2) + W_\sigma(x_2, x_1) \theta(\eta_2 - \eta_1). \quad (2.21)$$

In the latter expression $\theta(\eta)$ is the Heaviside step function taking values in $\{0, 1\}$.

For interacting theories, perturbative corrections to such correlators often involve products of several free 2-point functions W_σ or G_σ . Such computations can be simplified by making use of so-called linearization formulae, which express these products as weighted integrals of W_μ or G_μ over some contour in the complex μ -plane; see e.g. [8]. We will make use below of the slightly more complicated linearization formula

$$\begin{aligned} u_{\sigma_1\vec{L}_1}(x) u_{\sigma_2\vec{L}_2}(x) &= \sum_{\vec{K}} \text{CGC}(\vec{L}_1, \vec{L}_2; \vec{K}) \int_{\mu} (2\mu + D - 1) \rho_{\sigma_1\sigma_2 L_1 L_2}(\mu, K) u_{\mu\vec{K}}(x) \\ &=: \sum_{\vec{K}} \int_{\mu} (2\mu + D - 1) \rho_{12}(\mu, \vec{K}) u_{\mu\vec{K}}(x), \end{aligned} \quad (2.22)$$

for the Klein-Gordon mode functions themselves (which are just de Sitter harmonics). To explain this formula, first recall that spherical harmonics obey their own linearization formula

$$Y_{\vec{L}_1}(\vec{x}) Y_{\vec{L}_2}(\vec{x}) = \sum_{\vec{K}} \text{CGC}(\vec{L}_1, \vec{L}_2; \vec{K}) Y_{\vec{K}}(\vec{x}), \quad (2.23)$$

where $\text{CGC}(\dots; \dots)$ denote the generalized Clebsch-Gordon coefficients of $SO(D)$ [45]. Equation (2.22) then follows by using (2.11) and the analogous formula

$$f_{\sigma_1 L_1}(\eta) f_{\sigma_2 L_2}(\eta) = \int_{\mu} (2\mu + D - 1) \rho_{\sigma_1\sigma_2 L_1 L_2}(\mu, K) f_{\mu K}(\eta). \quad (2.24)$$

for the time-dependent parts of the modes $u_{\sigma\vec{L}}$. The kernel $\rho_{\sigma_1\sigma_2 L_1 L_2}(\mu, K)$ may be computed the methods of appendix A of [8]. Indeed, up to normalization the $\rho_{\sigma_1\sigma_2}(\mu)$ found there is our $\rho_{\sigma_1\sigma_2 L_1 L_2}(\mu, K)$ evaluated at $L_1 = L_2 = K = 0$. A useful Mellin-Barnes representation (A.5) of $\rho_{\sigma_1\sigma_2 L_1 L_2}(\mu, K)$ is given in our own appendix A, though we will not need the details here.

In (2.24) the symbol $\int_{\mu} \dots$ denotes a contour integral in the complex μ plane with measure $d\mu/(2\pi i)$. The contour is traversed from $\mu = -i\infty$ to $\mu = +i\infty$ within the strip $\text{Re}(\sigma_1 + \sigma_2) < \text{Re} \mu < 0$. The representation (A.5) can be used to show that $\rho_{\sigma_1\sigma_2 L_1 L_2}(\mu, K)$

is a meromorphic function of μ within this strip and that it decays sufficiently rapidly as $|\operatorname{Im} \mu| \rightarrow \infty$ for the above integral to converge absolutely. At generic arguments σ_1, σ_2 the function $\rho_{\sigma_1 \sigma_2 L_1 L_2}(\mu, K)$ in this strip has only simple poles at

$$\begin{aligned} \mu &= \sigma_1 + \sigma_2 - 2n, & \mu &= \sigma_1 - \sigma_2 - D + 1 - 2n, \\ \mu &= -\sigma_1 + \sigma_2 - D + 1 - 2n, & \mu &= -\sigma_1 - \sigma_2 - 2D + 2 - 2n, \end{aligned} \quad \text{for } n \in \mathbb{N}_0; \quad (2.25)$$

higher order poles arise at arguments σ_1, σ_2 where two or more of the above poles coalesce. These poles are required in order for the left-hand side of (2.24) to have the same asymptotic behavior in the regime $|\eta| \gg L_1 - \sigma_1$, $|\eta| \gg L_2 - \sigma_2$ as the right-hand side.⁴ It also useful to know that for σ_1, σ_2 corresponding to positive mass-squared $\rho_{\sigma_1 \sigma_2 \vec{L}_1 \vec{L}_2}(\mu, K)$ satisfies

$$[\rho_{\sigma_1 \sigma_2 L_1 L_2}(\mu, K)]^* = \rho_{\sigma_1 \sigma_2 L_1 L_2}(\mu^*, K). \quad (2.26)$$

so in particular $\rho_{\sigma_1 \sigma_2 L_1 L_2}(\mu, K) \in \mathbb{R}$ for μ in the complementary series and generally complex for μ in the principal series. See appendix A for further comments.

2.3 Interacting fields

Finally, we review some relevant features of interacting massive scalar de Sitter QFTs. When the potential is bounded below, such theories admit a maximally symmetric state $|\Omega\rangle$ which is calculable at the level of perturbation theory, and for which perturbative correlators decay near past and future infinity. As a result, general states in the Hilbert space have the property that local correlators approach those of $|\Omega\rangle$ in the distant past and future. We say that $|\Omega\rangle$ is an attractor state for local operators [9, 27]. Since correlation functions in $|\Omega\rangle$ may be defined by Wick rotation from S^D , we refer to $|\Omega\rangle$ as the Hartle-Hawking state. However, these correlators may also be computed using the Lorentz-signature Schwinger-Keldysh formalism (see e.g. [28]). We delay a detailed discussion of these techniques until §5.1 when such technical details become necessary.

The structure of our S-matrix will be intimately tied to the de Sitter Källén-Lehmann representation of the 2-point function (and to corresponding generalizations for higher correlators). Using the results of [8, 9, 27], it was argued in [46] that the 2-point function of any (perhaps composite) scalar operator $\Phi(x)$ in a general interacting theory may be written

$$\langle \Phi(x_1) \Phi(x_2) \rangle := \langle \Omega | \Phi(x_1) \Phi(x_2) | \Omega \rangle \quad (2.27)$$

$$= \int_{\mu} \rho(\mu) W_{\mu}(x_1, x_2), \quad (2.28)$$

where $\rho(\mu)$ is the Källén-Lehmann weight of Φ and $\int_{\mu} \dots$ denotes a contour integral in the complex μ plane (corresponding to complex operator weight μ) with measure $d\mu/(2\pi i)$. The integration contour is traversed from $-i\infty$ to $+i\infty$ to the left of the imaginary axis and to

⁴ This can be verified by inserting the asymptotic expansion (2.14) for $f_{\mu K}(\eta)$ into (2.22), closing the μ integration contour appropriately and using the Cauchy integral formula to equate the contour integrals with the sum of residues due to the poles (2.25).

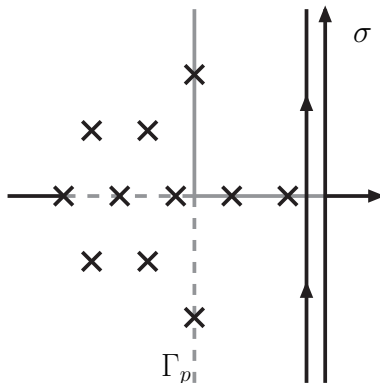


Figure 4: An example contour of integration for the Lehmann-Källén 2-point function. The contour is traversed from $-i\infty$ to $+i\infty$ in the strip to the left of the imaginary axis and to the right of any singularities (x's for our example) in the Källén-Lehmann weight with $\text{Re } \sigma < 0$. The particular example involves the theory discussed in section 6, which involves three species of scalar field. We have shown the poles in $\langle \phi_1(x)\phi_1(y) \rangle$ at $\mathcal{O}(g^2)$ for a case where ϕ_1, ϕ_2 lie in the principle series and ϕ_3 lies in the complementary series.

the right of the singularities in $\rho(\mu)$ – see Fig. 4. See also [47, 48, 49, 36] for earlier related results.

For example, in a free theory (2.28) is simply

$$\langle 0 | \phi_\sigma(x_1)\phi_\sigma(x_2) | 0 \rangle = \int_\mu \frac{(2\mu + D - 1)}{(\mu - \sigma)(\mu + \sigma + D - 1)} W_\mu(x_1, x_2) = W_\sigma(x_1, x_2). \quad (2.29)$$

By deforming the contour of integration to lie along the UIRs one may convert the contour integral in (2.28) into an integral over positive mass-squared $M^2 > 0$ (see Appendix A.2 of [46]) but the form (2.28) is more useful for our purposes. Other 2-point functions (time-ordered, (anti-)symmetric, retarded/advanced) may be obtained by exchanging $W_\mu(x_1, x_2)$ in the integrand of (2.29) with the appropriate free 2-point function.

The singularities in the Källén-Lehmann representation are particularly important. As emphasized in [8, 9, 27] they determine the large- η behavior of the two-point function just as the behavior of flat-space correlators at large spacelike separations is governed by their singularities in the complex m^2 . Simple poles at UIRs contribute power law terms that appear in the corresponding free fields while higher poles contribute additional logarithms. More complicated singularities do not arise; in particular, there are no branch cuts⁵. This fact was shown in [9, 27], which analyzed the Mellin transforms of correlation functions in such theories in the perturbative expansion⁶. The point may be argued without going into technical details as follows. The Mellin transform for any correlation function of a

⁵The familiar branch cuts of flat-space field theory arise in the limit $\ell \rightarrow \infty$ through the coalescence of an infinite family of poles.

⁶The Mellin transform of the 2-point function is closely related to the Källén-Lehmann weight – see Appendix A.2 of [46].

free field $\phi_\sigma(x)$ contains only simple poles. The Mellin transform of correlation functions of composite operators built from $\phi_\sigma(x)$ in the free theory are obtained by repeated use of the linearization formula (2.22), and from this one easily sees that these transforms also contain only poles. Now consider an interacting theory constructed to first order in the coupling g . The correlation functions in this theory obey Schwinger-Dyson equations of the form

$$(\square_x - M^2(\sigma)) \langle \phi_\sigma(x) \dots \rangle^{(1)} = -g \left\langle \frac{\delta \Gamma_{\text{int}}}{\delta \phi_\sigma(x)} \dots \right\rangle^{(0)}, \quad (2.30)$$

where $g\Gamma_{\text{int}}$ is the interacting part of the effective quantum effective action for $\phi_\sigma(x)$. The right-hand side is just a correlation function of the linear theory, so its Mellin transform contains only poles. We obtain the Mellin transform of the left-hand side by inverting the Klein-Gordon operator on the right-hand side; this adds pole singularities to the MB transform. By repeating this procedure for all correlators we deduce that the Mellin transforms of all $\mathcal{O}(g)$ correlators of $\phi_\sigma(x)$ contain only poles. The transforms of correlators of composite operators at $\mathcal{O}(g)$ likewise contain only poles. Thus the argument can be repeated at higher orders and extends to all orders in perturbation theory.

In perturbation theory about a free theory with weights σ_i , the following additional properties of $\rho(\mu)$ also hold. Property (ii) follows directly from causality. The remaining properties follow from the argument above, though they can alternatively be read off from the Mellin-Barnes analysis of [9, 27]. These results will play key roles in constructing our S-matrix, showing that it is finite, and demonstrating the desired properties.

- i) **Mass gap:** All operators are “massive,” by which we mean that their Källén-Lehmann weight $\rho(\mu)$ is analytic on a strip in the complex μ plane $\epsilon < \text{Re } \mu < 0$ for some $\epsilon < 0$. As a result, there always exists an allowed path for the contour of integration.
- ii) **Causality:** The 2-point function of real scalar operators is real at spacelike separations. This requires that

$$[\rho(\mu)]^* = \rho(\mu^*). \quad (2.31)$$

Any singularity in the lower half-plane has an image in the upper half-plane.

- iii) **Representation theory:** The singularities of $\rho(\mu)$ are contained in the region

$$R_U = \left\{ \text{Re } \mu \leq -\frac{(D-1)}{2} \cup \mu \in (-(D-1)/2, 0) \right\}. \quad (2.32)$$

This is the largest region consistent with scalar fields forming unitary representations of the de Sitter group. This is a direct corollary of property (iv) below, though we state it separately to highlight its importance.

- iv) **Lattice of poles:** For any scalar operator $\Phi(x)$, all poles in $\rho(\mu)$ lie on the lattice built by taking linear combinations (with non-negative coefficients) of $\sigma_i - 2n$, $-[(D-1) + \sigma_i] - 2n$ for non-negative integers n ($n \in \mathbb{N}_0$). Here σ_i are the weights σ_i of the free fields about which we perturb.

We will call a theory *exceptional* if some weight σ_i , or some ‘conjugate weight’ $\tilde{\sigma}_i = -[(D-1) + \sigma_i]$ can be expressed as a non-trivial combination of the above generators with non-negative integer coefficients. By ‘non-trivial,’ we mean that the sum of the coefficients should be greater than 1; i.e., that they should not just be $(1, 0, 0, \dots)$ or some permutation thereof. Our definition of the de Sitter S-matrix will fail precisely for this (measure zero) set of exceptional theories. We will call a theory *generic* if it is not exceptional in the above sense.

Note that for generic theories the bare weights σ_i may be recovered given only the lattice of poles; i.e., given only the correlators computed to some perturbative order.

- v) **Smooth self-energy:** For generic theories, the one-particle irreducible (1PI) diagrams for two-point functions of composite operators have no poles at the bare weights σ_i or the bare conjugate weights $\tilde{\sigma}_i$. This may be shown by the same methods used in [9] to show that all diagrams (even if one-particle reducible) decay at large arguments at least as the two-point function of the lightest field involved. As a result, the corresponding self-energy $\Pi(\mu)$ (see below) is smooth at $\mu = \sigma_i, \tilde{\sigma}_i$.

Finally, when $\Phi(x) = \phi_\sigma(x)$ is a massive operator with bare mass $M^2(\sigma)$ it can be convenient to write the Källén-Lehmann form in terms of the self-energy $\Pi(\mu)$ defined as usual by the sum of 1PI $1 \rightarrow 1$ graphs:

$$\langle \Phi(x_1)\Phi(x_2) \rangle = \int_\mu \frac{(2\mu + D - 1)}{(\mu - \sigma)(\mu + \sigma + D - 1) - \Pi(\mu)} W_\mu(x_1, x_2). \quad (2.33)$$

In a strict perturbative expansion of the 2-point function about small couplings the denominator of (2.33) must be expanded. Suppose the coupling constant is g ; then the perturbative expansion of $\Pi(\mu)$ is

$$\Pi(\mu) = \sum_{n=1}^{\infty} \Pi^{(n)}(\mu), \quad (2.34)$$

where $\Pi^{(n)}(\mu)$ is $\mathcal{O}(g^n)$, and the expansion of the 2-point function is

$$\langle \Phi(x_1)\Phi(x_2) \rangle = \int_\mu \frac{(2\mu + D - 1)}{(\mu - \sigma)(\mu + \sigma + D - 1)} W_\mu(x_1, x_2) \left[1 + \frac{\Pi^{(1)}(\mu)}{(\mu - \sigma)(\mu + \sigma + D - 1)} + \frac{\Pi^{(2)}(\mu)}{(\mu - \sigma)(\mu + \sigma + D - 1)} - \left(\frac{\Pi^{(1)}(\mu)}{(\mu - \sigma)(\mu + \sigma + D - 1)} \right)^2 + \mathcal{O}(g^3) \right]. \quad (2.35)$$

Clearly then, order by order, each term in (2.34) inherits the five characteristics of $\rho(\mu)$ mentioned above.

One may also consider the 2-point function obtained by truncating the sum (2.34) to some finite order N and inserting the result into (2.33). While this object includes contributions from an infinite number of graphs at infinite orders in the strict perturbative expansion, its study is well-motivated when, say, the physics of interest is governed by the nature of the

singularities (poles) in the Källén-Lehmann weight. As in flat space, the self-energy can be used to compute perturbative shifts of these poles at each order. In generic theories, the fact that $\Pi(\mu)$ is smooth at both the bare weights and the bare conjugate weights (see point (v) above) implies that these shifts are finite at each order.

Correlation functions of three or more fields are then computed order-by-order via a diagrammatic expansion with the same general structure as various flat space constructions; see section 5.1 below. In particular, all higher-point correlation functions have a perturbative expansion analogous to (2.35) [27, 9]. Thus, for generic theories, the order of any pole at $\sigma_i, \tilde{\sigma}_i$ in any n -point correlation function is controlled by the perturbative expansion in precisely the same manner as in flat space.

3 Asymptotic states

The quantum states involved in traditional scattering experiments may be described in terms of multiple widely-separated, non-interacting particles in the far past/future. While the construction of such particle states is trivial in free theories, we do not expect to be able to construct such states in interesting interacting field theories at finite separations. The best we can do is construct *asymptotic particle states* which enjoy a particle-state interpretation in the limits $\eta \rightarrow \pm\infty$. In the Heisenberg picture, inner products between past- and future-asymptotic states define the S-matrix (a more precise definition will be provided in §4).

The construction of asymptotic particle states will present the main technical challenge below. We therefore devote this section to addressing this issue in detail. Our definition of asymptotic particle states is a straightforward generalization of definitions that appear in standard QFT textbooks [50, 22, 51]. However, we warn the reader that one must use caution in applying the term ‘non-interacting’ to our asymptotic particle states. While our states transform in a simple way under the de Sitter group (and thus may be said to evolve freely in time), for interacting theories the inner product on our space of asymptotic multi-particle states will differ from that associated with any free theory.

3.1 General formalism

The starting point for our construction is a (bosonic) local quantum field $\Phi(x)$ and the corresponding de Sitter Källén-Lehmann spectral representation reviewed in section 2.3. As in flat space, we take the masses of the asymptotic particle states to be associated with values of σ at which the integrand is singular. In particular, for interacting fields we take them to be given by the bare weights σ_i defined by the lattice of poles; see point iv in section 2.3. Precisely free fields have a Källén-Lehmann representation supported only on a single dS UIR which defines the associated particles.

Our goal is to use $\Phi(x)$ to construct asymptotic particle states labeled by the quantum numbers of a de Sitter representation $n = (\sigma, \vec{L})$. We will give particular prescriptions for doing so below. For now, let us simply list the desired properties of such initial and final asymptotic states $\{|\psi\rangle_{i/f}\}$:

1. The states are normalizable: ${}_{i/f}\langle\psi|\psi\rangle_{i/f} < \infty$.
2. The Hilbert-space is spanned by multi-particle states, i.e. any initial (final) state is a sum of states labeled by strings of quantum⁷ numbers. So the states $|\psi\rangle_{i/f} = |n_1, n_2, \dots, n_k\rangle_{i/f}$ form a basis.
3. Such basis states are symmetric with respect to their particle labels, i.e., $|n_1, n_2, \dots\rangle_{i/f} = |n_2, n_1, \dots\rangle_{i/f}$.
4. In the asymptotic past (future) the initial (final) asymptotic states transform as symmetric tensor products under the action of the (diagonal) de Sitter group. This requirement is most easily understood in the Heisenberg picture. If $U(g)$ is a de Sitter transformation with parameter g then a k -particle asymptotic state transforms as

$$U(g) |n_1, n_2, \dots, n_k\rangle_{i/f} = |gn_1, gn_2, \dots, gn_k\rangle_{i/f}, \quad gn_i = (\sigma, g\vec{L}_i). \quad (3.1)$$

The notation $g\vec{L}$ denotes an appropriate linear combination of angular momenta. The weight σ is unchanged by de Sitter transformations because it denotes the eigenvalue of the quadratic Casimir of $SO(D, 1)$ which commutes with $U(g)$. We say that states satisfying (3.1) transform covariantly under the the de Sitter group. We also require that (3.1) be unitary.

Each set of multi-particle states should include a vacuum state $|v\rangle_{i/f}$. From criteria 1 and 4 we conclude that this state must be both normalizable and invariant under the action of the de Sitter group: $U(g)|v\rangle_{i/f} = |v\rangle_{i/f}$. For free fields these requirements limit the possible choices of $|v\rangle_{i/f}$ to the so-called Mottola-Allen vacua [44, 41]. If we require that the short-distance structure be Hadamard, the free Hartle-Hawking state becomes the unique allowed choice. It is thus natural to take both $|v\rangle_{i/f}$ to be the Hartle-Hawking state $|\Omega\rangle$ in the interacting case as well. The fact that $|\Omega\rangle$ is an attractor state for local operators [8, 9, 27, 10] means that it is also a natural choice from the perspective of local physics in the asymptotic regions⁸.

In the free case, the full space of states is readily constructed by acting with ladder operators

$$|n_1, n_1, \dots, n_k\rangle_{i/f} := a_{n_1}^\dagger a_{n_2}^\dagger \dots a_{n_k}^\dagger |0\rangle_{i/f}, \quad (3.2)$$

where

$$\begin{aligned} a_{\sigma\vec{L}}^\dagger &:= -i \int d\Sigma^\nu(\vec{x}) \left[u_{\sigma\vec{L}}(x) \overleftrightarrow{\nabla}_\nu \phi_\sigma(x) \right] \Big|_{\eta=\text{const}}, \\ a_{\sigma\vec{L}} &:= -i \int d\Sigma^\nu(\vec{x}) \left[\phi_\sigma(x) \overleftrightarrow{\nabla}_\nu u_{\sigma\vec{L}}^*(x) \right] \Big|_{\eta=\text{const}}, \end{aligned} \quad (3.3)$$

⁷ Depending on the details of the theory more quantum numbers may be necessary to fully specify the state, but $n = (\sigma, \vec{L})$ will be sufficient for our purposes.

⁸ The authors [52, 53, 54, 55] have considered asymptotic states constructed out of the non-Hadamard Mottola-Allen vacua. Though they typically have divergent stress-tensors and do not reduce to the Minkowski asymptotic particle states in the flat-space limit [56], it has been argued that these asymptotic states may nevertheless play an important role in a dS/CFT correspondence.

and as usual we also have

$$\phi_\sigma(x) = \sum_{\vec{L}} \left[u_{\sigma\vec{L}}(x) a_{\sigma\vec{L}} + u_{\sigma\vec{L}}^*(x) a_{\sigma\vec{L}}^\dagger \right], \quad [a_{\sigma\vec{L}}, a_{\sigma\vec{K}}] = 0 = [a_{\sigma\vec{L}}^\dagger, a_{\sigma\vec{K}}^\dagger], \quad [a_{\sigma\vec{L}}, a_{\sigma\vec{K}}^\dagger] = \delta_{\vec{L}\vec{K}}. \quad (3.4)$$

From the commutations relations (3.4) we readily verify that these particle states are normalizable and symmetric, and from (3.3) we see that the ladder operators transform covariantly under the de Sitter group: $U(g)a_{\sigma\vec{L}}^\dagger = a_{\sigma g\vec{L}}^\dagger$ where $g\vec{L}$ denotes the covariantly transformed angular momenta. Unitarity of $U(g)$ follows from the manifest de Sitter symmetry. Thus the states (3.2) satisfy our criteria for asymptotic particle states in both past and future. As an added bonus, this set of states also forms an orthonormal basis. Since $|0\rangle_i = |0\rangle_f = |0\rangle$, the S-matrix is trivial.

We will describe extensions of this construction to interacting fields in sections 3.2 and 3.3 below. For now, we mention that the initial (final) states define two multi-particle Hilbert spaces $\mathcal{H}_{i/f}$ and that for perturbation theory in small couplings we expect both $\mathcal{H}_{i/f}$ to be isomorphic to the entire Hilbert space \mathcal{H} . So each set of multi-particle asymptotic states should provide a complete basis for \mathcal{H} . We simply take this as an assumption, though our explicit verification of the optical theorem at low orders of perturbation theory in sections 6 and 7 provides supporting evidence.

We noted above that the states (3.2) are orthonormal for free fields. However, we have imposed no requirement on the inner product ${}_f\langle a|b\rangle_f$ between two final states (or ${}_i\langle a|b\rangle_i$ between two initial states) for more general theories. In particular, because requirement (4) involves only the diagonal action of the de Sitter group (with all g 's in (3.1) being the same), it is much weaker than requiring our asymptotic multi-particle states to be tensor products of asymptotic single-particle states, and this in principle allows the inner products ${}_f\langle a|b\rangle_f$ to depend on the interactions. In Minkowski space, the continuous spectrum of the energy-momentum operator means that the associated conservation laws enforce orthogonality of distinct asymptotic particle states even in the interacting theory. But de Sitter space does not enjoy such simple conservation laws: there is no positive-definite conserved energy, and conservation of the spatial momenta is governed by the addition of angular momenta (i.e., the Clebsch-Gordon coefficients of $SO(D)$) which is less stringent than the addition of linear momenta. So long as $|L_1 - L_2| \leq L_3 \leq L_1 + L_2$, dS group theory alone does not require the vanishing of overlaps like ${}_i\langle n_1 n_2 | n_3 \rangle_i$. Instead, we will be led to compute such inner products order by order in perturbation theory using appropriate Feynman-like diagrams.

3.2 Interacting theories and IR divergences

We now turn to the construction of asymptotic states for (massive) interacting fields; i.e., we seek a generalization of the LSZ formalism to de Sitter space. Unfortunately, the generalization of LSZ to de Sitter space is not unique. We describe one generalization below that is both conceptually and computationally straightforward. It turns out to suffice for explicit computations of heavy (principal series) fields but becomes IR divergent in the presence of light (complementary series) fields. Closely related IR divergences were noted previously in

[29, 30, 31, 32]. After identifying the relevant issues below, we present a modified procedure in section 3.3 that is free of unphysical divergences. The procedure of §3.3 is also required to give a field-redefinition-invariant definition of the S-matrix even for heavy fields.

Our first attempt at an LSZ-like prescription involves defining (now time-dependent) ladder operators in analogy with those of the free theory (3.3):

$$a_{\sigma\bar{L}}^\dagger(\eta) := -i \int d\Sigma^\nu(\vec{x}) \left[u_{\sigma\bar{L}}(x) \overleftrightarrow{\nabla}_\nu \phi_\sigma(x) \right] \Big|_\eta, \quad a_{\sigma\bar{L}}(\eta) := -i \int d\Sigma^\nu(\vec{x}) \left[\phi_\sigma(x) \overleftrightarrow{\nabla}_\nu u_{\sigma\bar{L}}^*(x) \right] \Big|_\eta. \quad (3.5)$$

We hope to construct initial (final) states via

$$|n_1, n_2, \dots, n_k\rangle_{i/f} = \lim_{\eta \rightarrow \mp\infty} \left[\prod_{j=1}^k a_{n_j}^\dagger(\eta) \right] |\Omega\rangle. \quad (3.6)$$

As mentioned above, asymptotic particle states will not generically remain orthonormal in the presence of interactions. But so long as the inner products are finite this need not be seen as an obstruction. Indeed, they may be computed order by order in perturbation theory as we discuss in sections 6 and 7 below. If desired, one may then choose to construct orthonormal bases via e.g. the Gram-Schmidt procedure. The resulting orthonormal asymptotic states then enjoy all of the properties discussed in §3.1 except that they do not have well-defined particle numbers with respect to the field operator $\phi_\sigma(x)$.

However, it turns out the inner products of these states can diverge even at lowest order in a coupling (i.e., at tree level) when complementary series fields are involved. In such cases the Klein-Gordon inner products used to define the ladder operators (3.5) may fail to have well-defined limits as $\eta \rightarrow \pm\infty$. Nothing is wrong with the theory per se: the $\phi_\sigma(x)$ correlation functions are well-defined and for non-coincident configurations enjoy power-law decay as the operators are taken to large times $|\eta| \gg 1$. The problem is simply that the Klein-Gordon inner products contain a competing power-law growth $d\Sigma u_{\sigma\bar{L}} \sim (1 + \eta^2)^{D/2} \eta^\sigma$. The single potentially-dangerous term cancels precisely for free fields, but not with interactions.

3.3 The R_σ projector

The IR divergences found above appear to result from technical rather than physical issues. In Minkowski space, treating the KG inner product in a natural distributional sense usefully extracts the residue of the operator at a simple pole and allows us to define asymptotic states. When correlators of ϕ_σ can be written as integrals over only principle series UIRs, the same is true in dS. But this procedure fails in dS when correlators receive contributions from complementary series UIRs. As already mentioned in §2.2, in de Sitter space the KG norm fails to converge on generic functions in the asymptotic regions, even when these functions decay exponentially. So it seems that the KG norm is simply not the right tool for extracting the complementary-series residues and thus the desired particle content from the correlation functions.

We will resolve this technical issue by modifying (3.5) so as to remove contributions from any support away from the designated value of σ . This may be accomplished by inserting

the operator

$$R_\sigma := \lim_{b \rightarrow \infty} \exp \left[-\frac{b^2}{(\square_x - M^2(\sigma))^2} \right] \quad (3.7)$$

into the definition of the ladder operators:

$$\begin{aligned} a_{\sigma\bar{L}}^\dagger(\eta) &:= -i \int d\Sigma^\nu(\vec{x}) \left[u_{\sigma\bar{L}}(x) \overleftrightarrow{\nabla}_\nu R_\sigma \Phi(x) \right] \Big|_\eta, \\ a_{\sigma\bar{L}}(\eta) &:= -i \int d\Sigma^\nu(\vec{x}) \left[R_\sigma \Phi(x) \overleftrightarrow{\nabla}_\nu u_{\sigma\bar{L}}^*(x) \right] \Big|_\eta. \end{aligned} \quad (3.8)$$

Note that R_σ annihilates smooth functions of σ as well as poles elsewhere in the complex plane, though for any finite value of b it leaves the residues at σ unchanged. Since all derivatives of R_σ with respect to σ vanish at the pole, the higher order coefficients in the Laurent expansion about σ are unchanged as well. Thus the multi-particle states defined by acting on $|\Omega\rangle$ with (3.8) are normalizable when the pole at σ is simple, and these states agree with those built from (3.5) when all fields lie in the principal series.

When the pole at σ is not simple, it is important to point out that acting on $|\Omega\rangle$ with (3.8) still generally yields non-normalizable states. However, the remaining divergences are only logarithmic in η , as opposed to the power-law divergences that arose for light fields in section 3.2. Similar logarithmic divergences are encountered in Minkowski space, where they are associated with perturbative self-energy corrections that shift the original (bare) simple poles to new locations in the complex plane. The general structure of the 1PI expansion (2.35) shows that for generic theories (in which the self-energy $\Pi(\mu)$ is smooth at the poles) the same is true in dS; i.e., the series (2.35) can be summed to yield (2.33) which has only simple poles near the original bare poles $\sigma, -[\sigma + D - 1]$. While we will not work with the resummed series below, it is clear that logarithmic divergences of this type are physically meaningful. We therefore take them to be acceptable in an order-by-order computation of the perturbative S-matrix that does not use resummed propagators. We then say that, despite these divergences, our S-matrix “exists” for generic theories. We will comment further on the exceptional theories (in which $\Pi(\mu)$ is not smooth at the poles) in section 8.

An operator Φ for which the naive LSZ expressions (3.5) converge up to logarithms will be called a “good operator”. As we have discussed, examples include free fields of all masses, as well as general local operators in theories containing only principal series fields. The operator $\Phi(x) := R_\sigma \phi_\sigma(x)$ is a good operator in any theory. Some might object that this $\Phi(x)$ is a rather formal object, as the application of R_σ involves taking an infinite number of derivatives. But it is useful to note that there is an entire class of operators that are in some sense equivalent to $\Phi(x)$. Let us write $\tilde{\Phi}(x) \cong \Phi(x)$ when the operators differ only by terms which decay faster than $\mathcal{O}(\eta^{-(\sigma+D-1)})$, i.e. only up to terms which do not contribute to amplitudes of asymptotic states. Clearly any such $\tilde{\Phi}(x)$ is also a good operator. At a fixed order in perturbation theory it is straightforward (if tedious) to construct operators $\tilde{\Phi}(x)$ equivalent, in this sense, to $R_\sigma \phi_\sigma(x)$ whose explicit expressions involve only a finite number of terms and contain only a finite number of derivatives of ϕ_σ . This procedure is illustrated in appendix B.

4 The S-matrix

We have assumed that both the initial and final spaces of asymptotic states form complete bases for our Hilbert space. For generic theories we then define the S-matrix to be the operator which induces the corresponding change of basis. To be explicit, given bases $|A\rangle_f$ and $|B\rangle_i$ for the initial and final spaces of asymptotic states, we define the corresponding S-matrix S_{BA} such that

$$|B\rangle_i = \sum_A S_{BA} |A\rangle_f. \quad (4.1)$$

For orthonormal bases, this gives

$$S_{BA} = {}_f\langle A|B\rangle_i. \quad (4.2)$$

The essential steps in computing the S-matrix are the construction of correlators in the Hartle-Hawking state $|\Omega\rangle$ and the application to those correlators of the LSZ-like operations described in sections 3.2 and 3.3. Perturbatively, n -point functions in $|\Omega\rangle$ can be computed either by analytic continuation from the Euclidean D-sphere S^D or equivalently (see e.g. [28]) by using Schwinger-Keldysh perturbation theory along a time-contour that begins and ends on some cosmological horizon. In order to insert operators near both I^+ and I^- , we take the Schwinger-Keldysh contour to be as shown in figure 2. In the τ -coordinates of section 2, this may be described as the contour which for fixed x^i starts at $\tau = 0$, runs backward to $\tau = -\infty$, then runs forward to $\tau = +\infty$, and finally returns to $\tau = 0$. We will use this scheme for our explicit computations in section 6 and 7. The associated diagrammatic expansion will be reviewed in section 5 below.

However, regardless of the calculational scheme employed, the S-matrix enjoys the following properties:

1. **The vacuum-to-vacuum amplitude is unity:** By construction the initial and final vacuum states are explicitly the same state $|\Omega\rangle$. By definition this state is normalized to unity.
2. **Covariance under the de Sitter group:** This follows from the covariance of the asymptotic states. E.g.,

$${}_f\langle A|B\rangle_i = {}_f\langle A|1|B\rangle_f = {}_f\langle A|U^{-1}(g)U(g)|B\rangle_f = {}_f\langle gA|gB\rangle_i. \quad (4.3)$$

3. **Behavior under CPT:** Under the action of the CPT operator Θ the S-matrix transforms as

$$\Theta S = S^{-1}\Theta. \quad (4.4)$$

We note that for real fields like the scalar operators we consider section 6 and 7 the CPT operation is equivalent to the antipodal map A , so property (4.4) is simply a consequence of (4.3).

4. **Invariance under perturbative field-redefinitions:** For simplicity, consider a theory defined by a single fundamental scalar field ϕ_σ and consider the field

$$\Phi(x) = \phi_\sigma(x) + \mathcal{O}(x) \quad (4.5)$$

for some local operator $\mathcal{O}(x)$. We assume that $\mathcal{O}(x)$ may be expanded in powers of ϕ_σ . Since taking $\mathcal{O}(x)$ linear in the fundamental fields would induce only a trivial field renormalization (which might be matrix-valued if the theory involves multiple fields), we take this expansion to have no linear terms. As noted in section 2.3, the poles in the Källén-Lehmann representation of $\mathcal{O}(x)$ lie on the same lattice as those of ϕ_σ . They are thus associated with the same bare weight σ , and the same bare conjugate weight $\tilde{\sigma}$, and define asymptotic particles of the same mass as ϕ_σ . Since section 2.3 also noted there that for generic theories the operator $\mathcal{O}(x)$ has no poles at $\sigma, \tilde{\sigma}$ themselves. It follows that $R_\sigma \Phi = R_\sigma \phi_\sigma$ and thus that $\Phi(x)$ and ϕ_σ define the same S-matrix.

5. **Unitarity:** The physical content of S-matrix unitarity is implicit in our assumption that the initial and final spaces of asymptotic states each form complete bases for the same Heisenberg-picture Hilbert space, with the same norm. When expressed in terms of orthonormal bases, this assumption implies that the matrix S_{BA} is unitary in the standard sense that $S^\dagger S = 1 = S S^\dagger$. The unitarity of such an S is often phrased in terms of the transition matrix \mathcal{T} where $S = 1 + i\mathcal{T}$. Then the unitarity of S is equivalent to the optical theorem

$$2 \operatorname{Im} \mathcal{T} = \mathcal{T}^\dagger \mathcal{T}. \quad (4.6)$$

While this remains an assumption at present, sections 6 and 7 will provide evidence in its favor by explicitly verifying the optical theorem at low orders of perturbation theory.

6. **Flat-space limit:** Consider the flat-space limit $\ell \rightarrow \infty$ (with all masses and couplings held fixed in physical units). As mentioned above, the correlators used to define our S-matrix may be obtained via analytic continuation from Euclidean signature; i.e., from the path integral on S^D . The latter clearly approaches the usual flat-space result for large S^D . In particular, the generating poles defined by the dS correlators approach the 1-particle poles of the Minkowski correlators. We noted above that the recipe (3.8) relates our S-matrix to the residues of such poles. Since this is analogous to the construction of the Minkowski S-matrix from residues of 1-particle poles, in the flat-space limit they must agree up to possible normalizations. The agreement of such normalizations will be verified by explicitly calculating the S-matrix for a ϕ^3 theory in section 6. We will find there that the de Sitter corrections are exponentially small in $M^2 \ell^2$, and are thus nonperturbative in $1/\ell$.

5 Lorentz-signature perturbation theory

The above sections give a general definition of our de Sitter S-matrix. In order to demonstrate the properties (1-6) from section 4, sections 6 and 7 below will explicitly calculate the S-

matrix for cubic theories through 2nd order. We choose to use Lorentz-signature Schwinger-Keldysh perturbation theory [23, 24] (see also [57, 58, 13, 59]). Since we are unaware of a presentation in the literature of this technique in the global de Sitter chart we give a brief introduction in section 5.1 below⁹. Section 5.2 then states the associated diagrammatic rules for computing our scattering amplitudes. For now, these rules are valid only for sufficiently heavy fields such that there are no IR divergences.

5.1 Schwinger-Keldysh correlation functions

The generating functional for time-ordered Hartle-Hawking correlators in global de Sitter may be constructed as follows. Consider for simplicity a theory of a single scalar field $\phi(x)$. We note that

$$1 = \langle \Omega | \Omega \rangle = \sum_f \sum_p \langle \Omega | f \rangle \langle f | p \rangle \langle p | \Omega \rangle, \quad (5.1)$$

where in the second equality we have inserted two complete sets of states $\{|f\rangle\}$ and $\{|p\rangle\}$ defined at a time in the far future and past, respectively, of an initial surface Σ_0 . We take Σ_0 to be a cosmological horizon so that regions to “future”/“past” correspond to expanding/contracting Poincaré charts. The state $|\Omega\rangle$ may be defined by imposing adiabatic boundary conditions on Σ_0 (see e.g. [28]). The expression (5.1) may then be understood as a path integral with a three-legged time contour of integration: reading (5.1) from right to left, the time contour begins at the horizon, is traversed into the far past, then to the far future, then back to the horizon. In the coordinates of (2.3), Σ_0 is simply the $\tau = 0$ hypersurface, the “future”/“past” are the regions $\tau > 0/\tau < 0$, and the time contour is $\tau = 0 \rightarrow -L \rightarrow +L \rightarrow 0$ for any sufficiently large L ; see Fig. 2.

By inserting classical sources J_f, J_o, J_p into the three legs of the path integral we obtain the generating functional for $\phi(x)$ correlation functions with respect to $|\Omega\rangle$:

$$\begin{aligned} Z[J_f, J_o, J_p] &:= \sum_f \sum_p \langle \Omega | f \rangle_{J_f} \langle f | p \rangle_{J_o} \langle p | \Omega \rangle_{J_p} \\ &= \int [\mathcal{D}\phi_f][\mathcal{D}\phi_o][\mathcal{D}\phi_p] \exp \left[-i \left(S[\phi_f] + \int_x \phi_f(x) J_f(x) \theta_f(x) \right) \right. \\ &\quad \left. + i \left(S[\phi_o] + \int_x \phi_o(x) J_o(x) \right) \right. \\ &\quad \left. - i \left(S[\phi_p] + \int_x \phi_p(x) J_p(x) \theta_p(x) \right) \right]. \end{aligned} \quad (5.2)$$

Here $\theta_f(x), \theta_p(x)$ are step functions with unit support in the future/past of the horizon, e.g.

⁹ The Lorentz signature discussion of [60] involves a similar 3-legged contour but uses the de Sitter “neck” $\eta = 0$ as the initial and final surface. In order to successfully produce the Hartle-Hawking correlators, this approach requires an independent computation of interacting correlators at $\eta = 0$. This latter step is not required in our approach. See [28] for further comments on the use of spacelike initial surfaces.

$\theta(\tau)$, $\theta(-\tau)$. Time-ordered correlation functions of $\phi(x)$ are generated via

$$\langle \Omega | T \phi(x_1) \dots \phi(x_n) | \Omega \rangle = \frac{\delta}{i\delta J_o(x_1)} \dots \frac{\delta}{i\delta J_o(x_n)} Z[J_p, J_o, J_f] \Big|_{J_f=J_o=J_p=0, \phi_f=\phi_o=\phi_p=\phi}. \quad (5.4)$$

To generate diagrammatic rules one may, in the standard fashion, re-write the interacting part of the action $S_{\text{int}}[\phi]$ in terms of functional derivatives, then perform the Gaussian path integral over the fields.¹⁰ The result is

$$Z[J_f, J_o, J_p] = \exp \left(-iS_{\text{int}} \left[-\frac{\delta}{i\delta J_f} \right] + iS_{\text{int}} \left[\frac{\delta}{i\delta J_o} \right] - iS_{\text{int}} \left[-\frac{\delta}{i\delta J_p} \right] \right) Z_0[J_f, J_o, J_p], \quad (5.5)$$

with

$$\begin{aligned} Z_0[J_f, J_p, J_o] := \exp \int_x \int_{\bar{x}} \left[-\frac{1}{2} J_o(x) G(x, \bar{x}) J_o(\bar{x}) - \frac{1}{2} J_f(x) G^*(x, \bar{x}) J_f(\bar{x}) \theta_f(x) \theta_f(\bar{x}) \right. \\ \left. - \frac{1}{2} J_p(x) G^*(x, \bar{x}) J_p(\bar{x}) \theta_p(x) \theta_p(\bar{x}) + J_o(x) W(x, \bar{x}) J_p(\bar{x}) \theta_p(\bar{x}) \right. \\ \left. - J_f(x) \theta_f(x) W(x, \bar{x}) J_p(\bar{x}) \theta_p(\bar{x}) + J_f(x) \theta_f(x) W(x, \bar{x}) J_o(\bar{x}) \right]. \end{aligned} \quad (5.6)$$

For correlation functions with arguments contained in a single Poincaré chart the generating function (5.5) reduces to the more familiar generating functions of “in-in” or “out-out” closed-time-path perturbation theory [57, 58, 61, 62]. Likewise, for arguments contained within a static chart (5.5) reduces to the generating function of thermal field theory [26].

The generating function (5.5) differs from generating functional of “in-out” perturbation theory in that it has two additional path integrals (or “time contours”). These may be interpreted as providing the necessary corrections to $\langle \Omega |$ and $| \Omega \rangle$ respectively. Their origin lies in the fact that we have defined $\langle \Omega |$ and $| \Omega \rangle$ at an intermediate time rather than at the conformal boundaries $\eta = \pm\infty$. In terms of Feynman graphs, these additional contours lead to extra vertices (which we label by f and p) in addition to those (which label o , for ordinary) arising “in-out” perturbation theory; i.e., we label a vertex f, o, p to indicate the term in (5.5) from which it arises.

The resulting graphs can contain legs which are Wightman functions rather than time-ordered Green’s functions. These have the important role of restricting the region of space-time integration involved in the construction of any correlator to the union of i) the past light cones of operators in the upper Poincaré chart, and ii) the future light cones of operators in the lower Poincaré chart. As a result, the perturbative construction of correlation functions via (5.5) is manifestly causal.

¹⁰We assume the Lagrangian has the decomposition $\mathcal{L}_0 + \mathcal{L}_{\text{int}}$ with \mathcal{L}_0 the canonical free field Lagrangian (2.4).

5.2 Scattering amplitudes

The above prescription for computing vacuum correlators leads directly to diagrammatic rules for scattering amplitudes. We work in the asymptotic particle bases and use the ladder operators (3.5), if necessary evaluating the states (3.6) at a finite cutoff $\eta = \pm\eta_0$. This allows us to deal with potential logarithmic divergences. The $\eta_0 \rightarrow \pm\infty$ limits give rise to our S-matrix so long as the fields used in (5.4) are good operators in the sense of section 3.3. In particular, the rules below may be used if all fields involved are sufficiently heavy or if, in the case of complementary series fields, we have performed a field redefinition equivalent in the sense of section 3.3 to $\phi \rightarrow R_\sigma\phi$; see appendix B for an explicit example.

With the above understanding, we may interchange the order of the KG inner products and these spacetime integrals so as to first evaluate the KG inner products between mode functions and the Green's functions connected to external points, then take the $\eta \rightarrow \pm\infty$ limit, and finally evaluate the rest of the Feynman diagram. This is precisely what one does in the traditional Minkowski setting.

Let us focus on the amplitudes of types ${}_i\langle a|b\rangle_i$, ${}_f\langle a|b\rangle_f$, or ${}_f\langle a|b\rangle_i$ for which we may use the time-ordered field correlators. Such amplitudes involve the p , o , and f -type vertices defined above according to the following prescription. First draw every Feynman diagram that contributes to an amplitude, paying no regard to different varieties of vertices. Then for each diagram:

1. For every vertex connected directly to a final bra, include only p and o varieties, and replace the Green's function connecting to a final bra with $u_n^*(y)$, where y is the integration variable.
2. For every vertex connected directly to a final ket, include only the f variety, and replace the Green's function connecting to a final bra with $u_n(y)$. See also rule 5 below.
3. For every vertex connected directly to an initial bra, include only the p variety, and replace the Green's function connecting to a final bra with $u_n^*(y)$.
4. For every vertex connected directly to an initial ket, include only the f and o varieties, and replace the Green's function connecting to a final bra with $u_n(y)$. See also rule 5 below.
5. The diagram vanishes if it contains a vertex connected directly to both a final bra and final ket (in the computation of some ${}_f\langle a|b\rangle_f$) or to both an initial bra and an initial ket (in the computation of some ${}_i\langle a|b\rangle_i$). This removes any conflicts between rules 1 and 2 and between rules 3 and 4.
6. For every vertex that is not connected to an external state, include all varieties o , f , p .
7. Include multiplicative factors of (ig) for every o vertex and $-(ig)$ for every f and p vertex in the diagram.

8. Each f/p vertex includes a theta function $\theta_{f/p}(y)$ where y is the integration variable.
9. The Green's functions for lines between vertices are as follows:

x vertex	y vertex	Green's function
o	o	$G_\sigma(x, y)$
o	f	$W_\sigma(y, x)$
o	p	$W_\sigma(x, y)$
f	f	$G_\sigma^*(x, y)$
f	p	$W_\sigma(x, y)$
p	p	$G_\sigma^*(x, y)$

These rules allow us to write scattering amplitudes as multiple spacetime integrals whose integrands involve products of KG modes, Green's functions, and the theta functions $\theta_{f/p}(y)$. These integrals are not easily computed in closed form but they they do reveal key properties of the amplitudes.

A useful tool for analyzing amplitudes is the antipodal map A which maps a point x in global de Sitter to its antipodal point Ax . For a fixed coordinate chart (2.3), if x is in the past Poincaré chart then Ax is in the future Poincaré chart. A acts upon our basic ingredients as

$$A : Y_{\vec{L}}(\vec{x}) \mapsto (-1)^L Y_{\vec{L}}(\vec{x}), \quad A : f_{\sigma L}(\eta) \mapsto f_{\sigma \vec{L}}(-\eta). \quad (5.7)$$

Complex conjugation acts upon our ingredients as

$$Y_{\vec{L}}^*(\vec{x}) = (-1)^m Y_{\vec{L}}(\vec{x}), \quad f_{\sigma L}^*(\eta) = f_{\sigma L}(-\eta). \quad (5.8)$$

Here m is the quantum number associated with the azimuthal direction in the standard spherical coordinates.¹¹ Combining these results we obtain the relations

$$u_{\sigma \vec{L}}(Ax) = (-1)^{L+m} u_{\sigma \vec{L}}^*(x), \quad (5.9)$$

$$W_\sigma(Ax, Ay) = W_\sigma(y, x) = [W_\sigma(x, y)]^*, \quad (5.10)$$

$$G_\sigma(Ax, Ay) = G_\sigma(x, y), \quad (5.11)$$

$$\theta_{p/f}(Ax) = \theta_{f/p}(x), \quad (5.12)$$

$$\sqrt{-g(Ax)} = \sqrt{-g(x)}. \quad (5.13)$$

The final equality follows from the fact that A is a de Sitter isometry.

¹¹Our conventions for spherical harmonics are those of [63]. All we really need for our analysis is that $Y_{\vec{L}}(\vec{x}) \propto e^{im\varphi}$ where φ is the azimuthal coordinate.

6 Example with heavy fields

We now provide an explicit example of the construction of section 4. Consider a model theory of three massive scalars with a cubic interaction described by the Lagrangian

$$\mathcal{L}[\vec{\phi}] = \sum_{i=1}^3 \mathcal{L}_0[\phi_i] + \mathcal{L}_{\text{int}}[\vec{\phi}] + \mathcal{L}_{\text{c.t.}}[\vec{\phi}], \quad (6.1)$$

$$\mathcal{L}_{\text{int}}[\vec{\phi}] = g\phi_3\phi_2\phi_1(x), \quad (6.2)$$

$$\mathcal{L}_{\text{c.t.}}[\vec{\phi}] = \sum_{i=1}^3 \left[-\frac{(Z_{\phi_i} - 1)}{2} \nabla_\mu \phi_i \nabla^\mu \phi_i(x) - \frac{(Z_{M_i} - 1)M_i^2}{2} \phi_i^2(x) \right] + \mathcal{O}(g^3). \quad (6.3)$$

We work in perturbation theory with $g \ll 1$, and we will examine S-matrix elements through $\mathcal{O}(g^2)$. The dS Hartle-Hawking correlators of this theory were studied previously [8]. The field and mass renormalization counterterms may be chosen to be $Z_{\phi_i} = 1 + \mathcal{O}(g^2)$ and $Z_{M_i} = 1 + \mathcal{O}(g^2)$; no other counterterms are necessary until $\mathcal{O}(g^3)$.

In this section we restrict the bare weights to the regime

$$\text{Re}(\sigma_1 + \sigma_2 + \sigma_3) < -(D - 1). \quad (6.4)$$

This condition is automatically satisfied by principal series fields (for which $\text{Re } \sigma = -(D - 1)/2$), but it can also be satisfied when some or all of the fields are sufficiently heavy members of the complementary series. To $\mathcal{O}(g^2)$ in the above theory, the condition (6.4) will suffice to qualify the fields as “heavy.” In particular, the naive LSZ prescription (3.5) gives finite results and the diagrammatic rules of section 5.2 may be applied directly. We will revisit this theory in §7 in order to lift the restriction (6.4).

Below, we use subscripts 1, 2, 3 to indicate which of the 3 flavors of scalar field in (6.1) are described by a given object. E.g. u_1, u'_1 denote wavefunctions of ϕ_1 -particles, n_2, n'_2 denote occupation numbers for ϕ_2 -particles, etc. More primes will be added as needed.

6.1 $\mathcal{O}(g)$ amplitudes

Contributions to the S-matrix at $\mathcal{O}(g)$ arise from the 3-point function $\langle \phi_3(x_3)\phi_2(x_2)\phi_1(x_1) \rangle^{(1)}$. Here and below we use parenthesized superscripts (e.g., $^{(1)}$) to denote the order in g to which a quantity is calculated. We begin with the amplitude

$${}_f \langle n_3 n_2 | n_1 \rangle_i^{(1)} = o \text{ term} = ig \int_y u_3^* u_2^* u_1(y), \quad (6.5)$$

where we abbreviate $u_1 u_2 \dots u_k^*(y) = u_{n_1}(y) u_{n_2}(y) \dots u_{n_k}^*(y)$. In the first equality we note, following the rules of §5.2, that only the ‘o’ vertex contributes. The final integral expression is simply the de Sitter analogue of the familiar Minkowski result.

The integral may be decomposed as

$${}_f \langle n_3 n_2 | n_1 \rangle_i^{(1)} = ig \int_{-\infty}^{+\infty} d\eta (1 + \eta^2)^{(D-2)/2} f_{\sigma_3 L_3}^* f_{\sigma_2 L_2}^* f_{\sigma_1 L_1}(\eta) \int_{\vec{x}} Y_{\vec{L}_3}^* Y_{\vec{L}_2}^* Y_{\vec{L}_1}(\vec{x}). \quad (6.6)$$

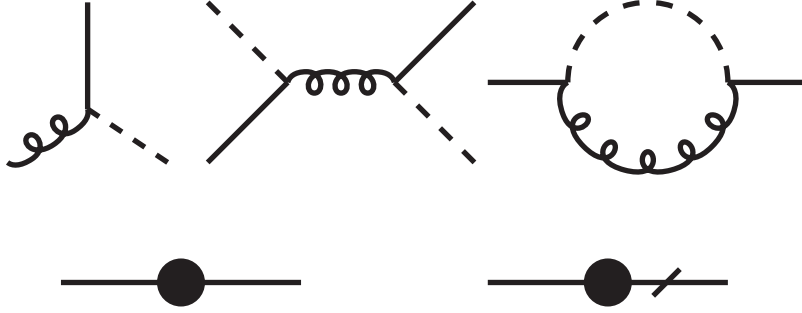


Figure 5: Feynman graphs contributing to connected corrections at $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$ in $g\phi_3\phi_2\phi_1(x)$ theory. We use solid lines for $\phi_1(x)$, dashed lines for $\phi_2(x)$, and curly lines for $\phi_3(x)$. In the top row, from left to right: the corrections $\langle\phi_3(x_3)\phi_2(x_2)\phi_1(x_1)\rangle^{(1)}$, $\langle\phi_2(\bar{x}_2)\phi_1(\bar{x}_1)\phi_2(x_2)\phi_1(x_1)\rangle^{(1)}$, and $\langle\phi_1(\bar{x})\phi_1(x)\rangle^{(2)}$. On the bottom row are the mass and field renormalization counterterms present in $\langle\phi_1(\bar{x})\phi_1(x)\rangle^{(2)}$. There are also graphs of the same topology but with different permutations of the fields.

The integral over \vec{x} provides an $SO(D)$ Clebsch-Gordon coefficient (CGC). These CGCs are always real, and this particular CGS is non-vanishing when the angular momenta satisfy the triangle inequalities [45]:

$$|L_i - L_j| \leq L_k \leq L_i + L_j, \quad \forall\{i, j, k\} = \{1, 2, 3\}. \quad (6.7)$$

The remaining integral over η is also real, as can be seen by changing integration variables $\eta \rightarrow -\eta$ and using (5.8).¹² Thus, the entire amplitude (6.5) is imaginary. Note that this integral converges only for masses satisfying (6.4).

To obtain an explicit expression for (6.5) we will compute the integral

$$I_1 := \int_y u_3 u_2 u_1^*(y), \quad (6.9)$$

where we have taken a complex conjugate relative to (6.5) so that we may more directly employ the linearization formula (2.24) to replace the product $u_3 u_2$ with a weighted integral over a single mode function.

We will also make use of two further tricks. The first involves the cutoff integral

$$\int_{y, \eta_0} u_1 u_2^*(y) := \int_{-\eta_0}^{+\eta_0} d\eta (1 + \eta^2)^{(D-2)/2} \int_{\vec{x}} u_1 u_2^*(y). \quad (6.10)$$

¹²More generally one can use time reversal and parity to show that integrals of the form

$$I := \int_y \prod_{i=1}^k u_i(y) \prod_{j=1}^l u_j^*(y), \quad \text{Re} \left(\sum_{i=1}^k \sigma_i + \sum_{j=1}^l \sigma_j \right) < -(D-1), \quad (6.8)$$

are real and vanish when either $L_{\text{tot}} = \sum_{i=1}^{k+l} L_i$ or $m_{\text{tot}} = \sum_{i=1}^{k+l} m_i$ is odd.

Because $u_{1,2}$ satisfy the Klein-Gordon equation with mass $M_{1,2}^2$ respectively we have

$$\int_{y,\eta_0} u_1 u_2^*(y) = \frac{1}{M_1^2 - M_2^2} \int d\Sigma^\nu(y) \left[[u_1 \overleftrightarrow{\nabla}_\nu u_2^*(y)]_{\eta=\eta_0} - [u_1 \overleftrightarrow{\nabla}_\nu u_2^*(y)]_{\eta=-\eta_0} \right]. \quad (6.11)$$

This formula is valid for all complex values of σ_1 and σ_2 apart from $\sigma_i = L_i + n$ or $\sigma_i = -L_i - (D-1) - n$ where n is a non-negative integer.

The other trick is to note that when $|\eta| > 1$ the functions $f_{\sigma L}(\eta)$ may be split into two functions each of which solves the Klein-Gordon equation individually but contains only one asymptotic behavior.¹³ We label the two behaviors “fast” and “slow”:

$$f_{\sigma L}(|\eta| > 1) = f_{\sigma L}^s(\eta) + f_{\sigma L}^f(\eta), \quad f_{\sigma L}^s(\eta) = \mathcal{O}(\eta^\sigma), \quad f_{\sigma L}^f(\eta) = \mathcal{O}(\eta^{-(\sigma+D-1)}), \quad (6.12)$$

$$f_{-(\sigma+D-1)L}(\eta) = f_{\sigma L}^f(\eta), \quad f_{\sigma L}^f(\eta) = f_{-(\sigma+D-1)L}^s(\eta). \quad (6.13)$$

The explicit form of $f_{\sigma L}^{s/f}(\eta)$ may be found using identities of the Gauss hypergeometric function, but we will only need the leading behaviors given in (2.14). Complex conjugation affects $f_{\sigma L}^{s/f}(\eta)$ differently depending on the series to which σ belongs:

$$\begin{aligned} [f_{\sigma L}^s(\eta)]^* &= f_{\sigma L}^s(-\eta), & [f_{\sigma L}^f(\eta)]^* &= f_{\sigma L}^f(-\eta), & \sigma \in \text{c. series}, \\ [f_{\sigma L}^s(\eta)]^* &= f_{\sigma L}^f(-\eta), & [f_{\sigma L}^f(\eta)]^* &= f_{\sigma L}^s(-\eta), & \sigma \in \text{p. series}. \end{aligned} \quad (6.14)$$

Returning to (6.9), introducing the time cutoff $|\eta| < \eta_0$, and using the linearization formula for $u_2 u_3(y)$ yields:

$$\begin{aligned} I_1(H) &= \sum_{\vec{K}} \int_\mu (2\mu + D - 1) \rho_{23}(\mu, \vec{K}) \int_{y,\eta_0} u_{\mu\vec{K}} u_1^*(y) \\ &= \int_\mu (2\mu + D - 1) \rho_{23}(\mu, \vec{L}_1) \int_{y,\eta_0} u_{\mu\vec{L}_1} u_1^*(y), \end{aligned} \quad (6.15)$$

where the final step is just orthogonality of distinct spherical harmonics. The integration contour may be traversed anywhere in the strip $\text{Re}(\sigma_2 + \sigma_3) \leq \mu < 0$; for concreteness let the contour traverse $\text{Re} \mu = -\epsilon$ for infinitesimal positive ϵ . We then integrate over y using (6.11) to obtain

$$\begin{aligned} I_1(H) &= - \int_\mu \left\{ \frac{(2\mu + D - 1)}{(\mu - \sigma_1)(\mu + \sigma_1 + D - 1)} \rho_{23}(\mu, \vec{L}_1) \right. \\ &\quad \left. \times \int d\Sigma^\nu(y) \left[[u_{\mu\vec{L}_1} \overleftrightarrow{\nabla}_\nu u_1^*(y)]_{\eta=\eta_0} - [u_{\mu\vec{L}_1} \overleftrightarrow{\nabla}_\nu u_1^*(y)]_{\eta=-\eta_0} \right] \right\}. \end{aligned} \quad (6.16)$$

Next we split $f_{\mu K}(\eta)$ into fast and slow parts as in (6.12). The $f_{\mu L_1}^{s*}(\eta)$ contributes terms $\mathcal{O}(H^{\sigma_1+D-1-\epsilon})$ while the $f_{\mu L_1}^{f*}(\eta)$ contributes terms $\mathcal{O}(H^{\sigma_1-\epsilon})$. Thus the latter, “fast-decay”

¹³The function $f_{\sigma L}(\eta)$ may be decomposed this way for $|\eta| > 1$, but the asymptotic behavior does not dominate the behavior until $|\eta| \gg L - \sigma$ [10].

terms vanish in the limit $\eta_0 \rightarrow \infty$ and we need only keep only the contributions of $f_{\sigma L_1}^{s*}(\eta)$. We now deform the integration contour to the left from $\text{Re } \mu = -\epsilon$ to $\text{Re } \mu = \text{Re}(\sigma_2 + \sigma_3) + \epsilon$. From (6.4) we know

$$\text{Re}(\sigma_2 + \sigma_3) < \text{Re} -(\sigma_1 + D - 1) \leq \text{Re } \sigma_1 < 0, \quad (6.17)$$

so the only poles we encounter in moving the contour to $\text{Re } \mu = \text{Re}(\sigma_2 + \sigma_3) + \epsilon$ are the simple poles at $\mu = \sigma_1$ and $\mu = -(\sigma_1 + D - 1)$ (examine the denominator (6.16)). The remaining contour integral behaves like $\mathcal{O}(\eta_0^{\text{Re}(\sigma_2 + \sigma_3 + \sigma_1) + D - 1 + \epsilon})$ and so vanishes as $\eta_0 \rightarrow \infty$. Thus

$$\begin{aligned} I_1(\eta_0) = & - \left\{ \rho_{23}(\sigma_1, \vec{L}_1) \int d\Sigma^\nu(\vec{y}) \left[[u_{\mu\vec{L}}^s \overleftarrow{\nabla}_\nu u_1^*(y)]_{\eta=\eta_0} - [u_{\mu\vec{L}}^s \overleftarrow{\nabla}_\nu u_1^*(y)]_{\eta=-\eta_0} \right] \right. \\ & \left. + \rho_{23}(-(\sigma_1 + D - 1), \vec{L}_1) \int d\Sigma^\nu(\vec{y}) \left[[u_{\mu\vec{L}}^f \overleftarrow{\nabla}_\nu u_1^*(y)]_{\eta=\eta_0} - [u_{\mu\vec{L}}^f \overleftarrow{\nabla}_\nu u_1^*(y)]_{\eta=-\eta_0} \right] \right\}. \end{aligned} \quad (6.18)$$

It is straight-forward to evaluate the remaining KG norms and take the limit $\eta_0 \rightarrow \infty$. We finally obtain

$$I_1 = -\cot \left[\pi \left(\sigma_1 + \frac{D-1}{2} \right) \right] \left[\rho_{23}(\sigma_1, \vec{L}_1) - \rho_{23}(-(\sigma_1 + D - 1), \vec{L}_1) \right]. \quad (6.19)$$

For σ_1 in the complementary series $\cot \left[\pi \left(\sigma + \frac{D-1}{2} \right) \right]$ is real and so too is the second term in brackets, as $\rho_{23}(\mu \in \mathbb{R}, L_1) \in \mathbb{R}$. For σ_1 in the principal series $\cot \left[\pi \left(\sigma + \frac{D-1}{2} \right) \right]$ is imaginary and so too is the term in brackets which is equal to $2i \text{Im } \rho_{23}(\sigma_1, \vec{L}_1)$. So I_1 is real in all cases. Buried within $\rho_{23}(\sigma_1, \vec{L}_1)$ is the $SO(D)$ Clebsch-Gordon coefficient.

The most salient feature of the result (6.19) is that the amplitude ${}_f \langle n_3 n_2 | n_1 \rangle_i^{(1)} = ig I_1$ is generically nonzero whenever allowed by addition of angular momentum. This is true for *any* set of masses satisfying $\text{Re}(\sigma_1 + \sigma_2 + \sigma_3) < (D - 1)$. This includes configurations in which some of the fields belong to the complementary series. The expression (6.19) contains $\rho_{23}(\mu, \vec{L}_1)$ evaluated in the region of the complex μ -plane for which $\rho_{23}(\mu, \vec{L}_1)$ is analytic and nontrivial; thus, it can vanish at most at isolated points. We emphasize this point in order to contrast with the statements of [35, 36] that complementary series particles can decay only in what we have called exceptional theories. Instead, our conclusions support the analysis of [33, 34]; the discrepancy with [35, 36] will be explained in section 8. Due to the detailed form of $\rho_{\sigma_1 \sigma_2 L_1 L_2}(\mu, K)$, the amplitude decays exponentially at large masses; see Appendix A. We have checked these results by direct numerical integration of (6.9).

There is also a non-vanishing connected contribution to the amplitude

$$\langle \Omega | n_3 n_2 n_1 \rangle_i^{(1)} = f + o \text{ terms} = i \int_y \theta_p(y) u_3 u_2 u_1(y). \quad (6.20)$$

This amplitude may be understood as both a transition amplitude for a $3 \rightarrow 0$ process and as a correction to the overlap between two initial states. As with the previous example (6.20)

converges absolutely only for masses in the regime (6.4). The same physics may be accessed different ways: we may compute

$${}_i\langle n_3 n_2 n_1 | \Omega \rangle^{(1)} = p \text{ term} = -ig \int_y \theta_p(y) u_3^* u_2^* u_1^*(y), \quad (6.21)$$

which is clearly the complex conjugate of (6.20), or we could compute the amplitudes

$${}_f\langle n_3 n_2 n_1 | \Omega \rangle^{(1)} = o + p \text{ term} = ig \int_y \theta_f(y) u_3^* u_2^* u_1^*(y), \quad (6.22)$$

$$\langle \Omega | n_3 n_2 n_1 \rangle_f^{(1)} = f \text{ term} = -ig \int_y \theta_f(y) u_3 u_2 u_1(y), \quad (6.23)$$

which are also clearly complex conjugates. Moreover, by letting $y \rightarrow Ay$ in any of these expressions we verify that

$${}_i\langle n_3 n_2 n_1 | \Omega \rangle^{(1)} = \langle \Omega | n_3 n_2 n_1 \rangle_f^{(1)}, \quad \text{and} \quad {}_f\langle n_3 n_2 n_1 | \Omega \rangle^{(1)} = \langle \Omega | n_3 n_2 n_1 \rangle_i^{(1)}, \quad (6.24)$$

as is required by CPT.

Evaluating these amplitudes explicitly is rather awkward because they are expressed as an integral of *global* KG modes over a single Poincaré chart. We will be content to show that the amplitude is generically non-zero when $L_1 + L_2 + L_3$ is even. Consider the imaginary part of the amplitude,

$$2i \text{ Im} \langle \Omega | n_3 n_2 n_1 \rangle_i^{(1)} = \langle \Omega | n_3 n_2 n_1 \rangle_i^{(1)} - \langle \Omega | n_3 n_2 n_1 \rangle_f^{(1)} = ig \int_y u_3 u_2 u_1(y) =: igI_2. \quad (6.25)$$

The integral I_2 may be evaluated by following the same procedure as was used to compute I_1 . The result is

$$I_2 = \frac{e^{-i\pi D} \cos \left[\pi \left(\frac{D-1}{2} \right) \right]}{\sin \left[\pi \left(\sigma + \frac{(D-1)}{2} \right) \right]} \left[\rho_{23}(\sigma_1, \vec{L}_1) - \rho_{23}(-(\sigma_1 + D - 1), \vec{L}_1) \right]. \quad (6.26)$$

Indeed, this expression is generically non-zero. We plot an example in Fig. 6 below.

Note that all other corrections to either the initial or final inner products (e.g. ${}_i\langle n_1 | n_2 n_3 \rangle_i$ or ${}_f\langle n_1 | n_2 n_3 \rangle_f$) vanish by rule 5 of section 5.2.

6.2 $\mathcal{O}(g^2)$ state corrections

We now compute corrections to the initial state inner products at $\mathcal{O}(g^2)$. The corresponding final state corrections are then easily obtained by applying CPT. Since explicit expressions for amplitudes at $\mathcal{O}(g^2)$ are both lengthy and time-consuming to attain, we will compute results only at the level needed to verify the optical theorem in section 6.3. Due to the way that the various terms enter the optical theorem, it will be convenient to delay computation of the actual S-matrix elements until section 6.3.

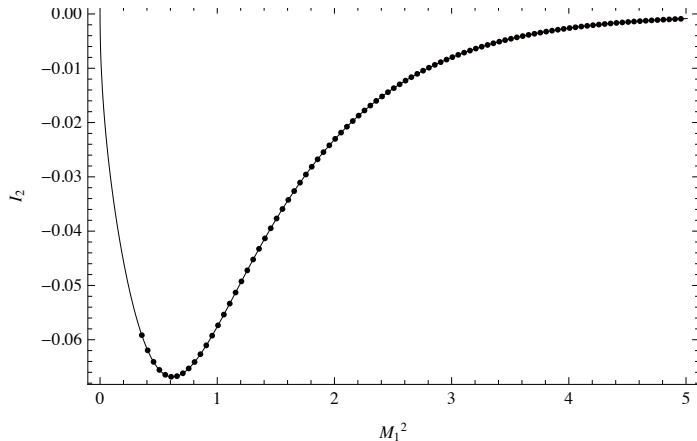


Figure 6: The amplitude I_2 in $D = 3$ as a function of M_1^2 with $\vec{L}_1 = \vec{L}_2 = \vec{L}_3 = 0$, $M_2^2 = 2/\ell^2$, and $M_3^2 = 1.25/\ell^2$. The continuous curve is plotted using (A.5) while the dots represent direct numerical integration of (6.25). The amplitude is small, tends rapidly to zero as M_1^2 increased, and approaches zero for $M_1^2 \rightarrow 0$. Since M_2 and M_3 lie in the principal series, (6.25) converges for all $M_1^2 > 0$ – though the slow convergence near $M_1^2 \approx 0$ makes direct numerical evaluation of (6.25) difficult in that regime.

At $\mathcal{O}(g^2)$ there are two types of connected corrections to vacuum correlators functions: tree-level corrections to 4-point functions and 1-loop corrections to the 2-point functions (recall Fig. 5). Tree-level corrections to 4-point functions contribute to the connected part of the amplitudes

$$\begin{aligned}
& {}_i \langle n_1 n_2 n'_1 n'_2 | \Omega \rangle_C^{(2)} = pp \text{ term} \\
& = (ig)^2 \int_{y_1} \int_{y_2} \theta_p(y_1) \theta_p(y_2) [u_1 u_2(y_1) u'_1 u'_2(y_2) + u_1 u'_2(y_1) u'_1 u_2(y_2)] G_3^*(y_{1,2}), \quad (6.27)
\end{aligned}$$

as well as to amplitudes obtained by permuting the species. As we will see in section 6.3, such vacuum-to-many amplitudes cancel out completely in any check of the optical theorem. There is also a tree-level correction to the connected part of the 2-particle amplitude ${}_i \langle n'_2 n'_1 | n_2 n_1 \rangle_{i,C}^{(2)}$. In principle both s- and u-channel graphs may contribute, though the u-channel contributions vanish by rule 5 of section 5.2. Thus we compute only

$$\begin{aligned}
& {}_i \langle n'_1 n'_2 | n_1 n_2 \rangle_{i,s\text{-channel}}^{(2)} = pf + po \text{ terms} \\
& = g^2 \int_{\bar{y}} \int_y \theta_p(\bar{y}) \theta_p(y) u_1^* u_2^*(\bar{y}) u_1 u_2(y) W_3(y, \bar{y}) \\
& = \sum_{\vec{K}} \left[(ig) \int_{\bar{y}} \theta_p(\bar{y}) u_1^* u_2^* u_{\sigma_3 \vec{K}}(\bar{y}) \right]^* \left[(ig) \int_y \theta_p(y) u_1 u_2 u_{\sigma_3 \vec{K}}(y) \right]. \quad (6.28)
\end{aligned}$$

These amplitudes are UV-finite because the $SO(D)$ CGCs limit the range of the sum over

\vec{K} to

$$\min \{|L'_2 - L'_1|, |L_2 - L_1|\} \leq K \leq \max \{L'_2 + L'_1, L_2 + L_1\}. \quad (6.29)$$

The right-hand side can correct amplitudes both on and off the diagonal $\delta_{\vec{L}'_1 \vec{L}_1} \delta_{\vec{L}'_2 \vec{L}_2}$. On the diagonal the right-hand side is a sum of absolute values of integrals which were shown to be generally non-vanishing above.

The 1-loop correction to the 2-point function $\langle \phi_1(\vec{x}) \phi_1(x) \rangle^{(2)}$ alters the normalization of the 1-particle states ${}_i \langle n_1 | n_1 \rangle_i^{(2)}$, though conservation of angular momentum prevents it contributing to off-diagonal 1-particle amplitudes. The 1-loop correction is UV-divergent for $D > 3$, so we must include the standard perturbative counterterms that follow from (6.3). Only the field renormalization counterterm contributes to this amplitude:

$$\begin{aligned} {}_i \langle n_1 | n_1 \rangle_i^{(2)} &= p_0 + p_f \text{ terms} + (Z_{\phi_1} - 1) \\ &= -(ig)^2 \int_{\vec{y}} \int_y \theta_p(\vec{y}) \theta_p(y) u_1^*(\vec{y}) u_1(\vec{y}) W_3(y, \vec{y}) W_2(y, \vec{y}) + (Z_{\phi_1} - 1) \\ &= \sum_{\vec{L}_2} \sum_{\vec{L}_3} \left[(ig) \int_{\vec{y}} \theta_p(\vec{y}) u_3 u_2 u_1(\vec{y}) \right]^* \left[(ig) \int_y \theta_p(y) u_3 u_2 u_1(y) \right] + (Z_{\phi_1} - 1). \end{aligned} \quad (6.30)$$

This expression is IR-finite but the sum over momenta does not terminate and the expression contains an ultraviolet divergences for $D \geq 6$. We choose Z_{ϕ_1} to cancel this divergence. All of this is very much analogous to the computation in Minkowski space. The 1-loop correction $\langle \phi_1(\vec{x}) \phi_1(x) \rangle^{(2)}$ also contributes a correction to the 2-to-vacuum amplitude $\langle \Omega | \sigma_i \vec{0}, \sigma_i \vec{0} \rangle_i^{(2)}$, which will again cancel out entirely in any check of the optical theorem.

There are of course analogous formulae for similar amplitudes obtained by permuting the flavors (1, 2, 3) above, as well as those between final states which may be obtained by using CPT symmetry and conjugation to relate final state amplitudes to the results above. For the remaining inner products between initial bras and initial kets which have yet to be mentioned, the $O(g^2)$ contributions vanish either manifestly (there are no relevant diagrams) or by rule 5 of section 5.2. In particular, we have

$${}_i \langle n'_2 n'_1 n_2 | n_1 \rangle_{iC}^{(2)} = 0, \quad {}_i \langle n'_2 n_2 | n'_1 n_1 \rangle_{iC}^{(2)} = 0. \quad (6.31)$$

6.3 The Optical Theorem

We wish to explicitly verify the optical theorem, and thus the unitarity of our S-matrix, to order g^2 for our model theory. In terms of an orthonormal basis, the optical theorem takes the standard form

$$-2 \operatorname{Re} {}_f \widetilde{\langle B | A \rangle}_i = \sum_C {}_i \widetilde{\langle B | C \rangle}_f {}_f \widetilde{\langle C | A \rangle}_i, \quad (6.32)$$

where ${}_f \widetilde{\langle B | A \rangle}_i := {}_f \langle B | A \rangle_i - \delta_{AB}$. We may restrict attention to amplitudes involving at most 4-particles as only these have fully-connected contributions.

To verify (6.32), we first construct orthonormal bases from our multi-particle asymptotic states. At $\mathcal{O}(g)$ the 1- and 2-particle states require no modification; we need only remove

the overlap of 3-particle states with the vacuum, as well as overlaps between higher-number particle states due to disconnected contributions. We therefore define

$$|n_3 n_2 n_1\rangle_i^{ON} = |n_3 n_2 n_1\rangle_i - |\Omega\rangle \langle\Omega|n_3 n_2 n_1\rangle_i^{(1)} + \mathcal{O}(g^2). \quad (6.33)$$

The only non-zero connected contributions to the S-matrix at $\mathcal{O}(g)$ are the amplitudes ${}_f\langle n_3 n_2 | n_1 \rangle_i$ which were computed in section 6.1. Recall that these amplitudes are imaginary, so that (6.32) is satisfied at $\mathcal{O}(g)$.

The vacuum-to-three amplitude in the orthonormal bases vanishes by construction:

$${}_{f}^{ON}\langle n_3 n_2 n_1 | \Omega \rangle_i^{ON(1)} = 0. \quad (6.34)$$

Quite generally, converting in this way from a particle basis to an orthonormal basis removes contributions from any diagram which is attached to only the final bra or initial ket (in the particle basis). It is natural to think of such diagrams as computing corrections to the relevant bra or ket, rather than representing genuine particle scattering processes, so it is not surprising that these diagrams do not ultimately contribute to the S-matrix despite being non-zero.

At $\mathcal{O}(g^2)$, transforming to orthonormal bases once again removes overlaps between particle states and the vacuum; the transition also removes overlaps between non-identical pairs of 2-particle states and corrects normalizations. Below we use the orthonormal states

$$|n_1\rangle_i^{ON} = |n_1\rangle_i \left[1 - \frac{1}{2} {}_i\langle n_1 | n_1 \rangle_i^{(2)} \right] + \mathcal{O}(g^3), \quad (6.35)$$

$$|n'_1 n_1\rangle_i^{ON} = |n'_1 n_1\rangle_i \left[1 - \frac{1}{2} {}_i\langle n'_1 n_1 | n'_1 n_1 \rangle_i^{(2)} \right] + \mathcal{O}(g^3), \quad (6.36)$$

$$|n_2 n_1\rangle_i^{ON} = |n_2 n_1\rangle_i - \frac{1}{2} \sum_{n'_1, n'_2} |n'_1 n'_2\rangle_i {}_i\langle n'_1 n'_2 | n_2 n_1 \rangle_i^{(2)} + \mathcal{O}(g^3). \quad (6.37)$$

Returning to (6.32), we begin with the $\phi_1 \rightarrow \phi_1$ scattering process. Again, angular momentum conservation sets the off-diagonal terms to zero so that the non-trivial part is just

$${}_{f}^{ON}\langle n_1 | n_1 \rangle_i^{ON(2)} = {}_f\langle n_1 | n_1 \rangle_i^{(2)} - \frac{1}{2} \left[{}_i\langle n_1 | n_1 \rangle_i^{(2)} + {}_f\langle n_1 | n_1 \rangle_f^{(2)} \right] = (\text{graph}) + (\text{c.t.}) \quad (6.38)$$

Here (graph) denotes the contribution from the 1-loop diagram while (c.t.) denotes contributions from the mass- and field-renormalization counterterms. Explicitly these are:

$$\begin{aligned} (\text{graph}) &= of + oo + po + pf - \frac{1}{2} [po + pf + of + pf] \text{ terms} \\ &= oo + \frac{1}{2} [of + po] \text{ terms} \\ &= (ig)^2 \int_{\bar{y}} \int_y u_1^*(\bar{y}) u_1(y) \mathcal{M}(\bar{y}, y), \end{aligned} \quad (6.39)$$



Figure 7: Corrections to $\phi_2\phi_1 \rightarrow \phi_2\phi_1$ scattering at $\mathcal{O}(g^2)$. From left to right are the s-channel, the u-channel, and two disconnected 1-loop corrections.

where

$$\mathcal{M}(\bar{y}, y) := G_3(y, \bar{y})G_2(y, \bar{y}) - \frac{1}{2}W_3(y, \bar{y})W_2(y, \bar{y}) (\theta_f(y) + \theta_p(\bar{y})), \quad (6.40)$$

and

$$(\text{c.t.}) = o \text{ term} = i(Z_{M_1} + Z_{\phi_1} - 2) \int_y u_1^* u_1(y). \quad (6.41)$$

Note that the integrand in (6.41) is positive-definite and behaves at large $|\eta| \gg 1$ like $\eta^{2\sigma_1}$ so that the integral is IR-divergent for all values of σ_1 . This divergence turns out to cancel a similar divergence that appears in (graph) under precisely the same conditions that UV counterterms are necessary. There is also a logarithmic IR divergence proportional to $\Pi^{(1)}(\sigma_1)$ as expected from the second term in (2.35), which we therefore regard as physical. Recall that a similar divergence in Minkowski space is associated with a finite decay rate per unit time (“Fermi’s golden rule”) and plays a key role in satisfying the optical theorem.

To verify our dS the optical theorem we compute

$$\begin{aligned} -2 \operatorname{Re} \langle n_1 | n_1 \rangle_i^{ON(2)} &= -(\text{graph}) - (\text{graph})^* \\ &= -(ig)^2 \int_{\bar{y}} \int_y u_1^*(\bar{y}) u_1(y) [\mathcal{M}(\bar{y}, y) + \mathcal{M}^*(A\bar{y}, Ay)]. \end{aligned} \quad (6.42)$$

In the second equality we have used (5.9). From (5.11) and (5.10) it follows that

$$\begin{aligned} \mathcal{M}(\bar{y}, y) + \mathcal{M}^*(A\bar{y}, Ay) &= G_3(y, \bar{y})G_2(y, \bar{y}) + G_3^*(\bar{y}, y)G_2^*(\bar{y}, y) - W_3(y, \bar{y})W_2(y, \bar{y}) \\ &= W_3(\bar{y}, y)W_2(\bar{y}, y). \end{aligned} \quad (6.43)$$

Inserting (6.43) into (6.42) we obtain

$$\begin{aligned} -2 \operatorname{Re} \langle n_1 | n_1 \rangle_i^{ON(2)} &= -(ig)^2 \int_{\bar{y}} \int_y u_1^*(\bar{y}) u_1(y) W_3(\bar{y}, y) W_2(\bar{y}, y) \\ &= \sum_{\vec{L}_2} \sum_{\vec{L}_3} \left| (ig) \int_y u_1 u_2^* u_3^*(y) \right|^2 = \sum_{\vec{L}_2} \sum_{\vec{L}_3} \left| {}_f \langle n_2 n_3 | n_1 \rangle_i^{(1)} \right|^2. \end{aligned} \quad (6.44)$$

The final equality is precisely what is required by the optical theorem.

Next we examine the $\phi_2\phi_1 \rightarrow \phi_2\phi_1$ amplitude

$${}^{ON}_f \langle n'_1 n'_2 | n_1 n_2 \rangle_i^{ON(2)} = {}_f \langle n'_1 n'_2 | n_1 n_2 \rangle_i^{(2)} - \frac{1}{2} \left[{}_f \langle n'_1 n'_2 | n_1 n_2 \rangle_f^{(2)} + {}_i \langle n'_1 n'_2 | n_1 n_2 \rangle_i^{(2)} \right]. \quad (6.45)$$

This expression naturally splits into a connected part ${}_f \langle n'_1 n'_2 | n_1 n_2 \rangle_{iC}^{(2)}$ which receives tree-level corrections and a disconnected part that receives 1-loop corrections; see Fig. 7. The connected part may further be split into s-channel and u-channel contributions with the former including the state corrections $-\frac{1}{2} \left[{}_f \langle n'_1 n'_2 | n_1 n_2 \rangle_f^{(2)} + {}_i \langle n'_1 n'_2 | n_1 n_2 \rangle_i^{(2)} \right]$. Using our diagrammatic rules it is straightforward to compute

$$\begin{aligned} {}^{ON}_f \langle n'_1 n'_2 | n_1 n_2 \rangle_{i\text{s-channel}}^{ON(2)} &= +oo + of + po + pf - \frac{1}{2} [pf + po + pf + of] \\ &= +oo + \frac{1}{2}(of + po) \\ &= (ig)^2 \int_{\bar{y}} \int_y u_1'^* u_2'^*(\bar{y}) u_1 u_2(y) \mathcal{M}(\bar{y}, y), \end{aligned} \quad (6.46)$$

with

$$\mathcal{M}(\bar{y}, y) := G_3(\bar{y}, y) - \frac{1}{2} W_3(y, \bar{y}) (\theta_p(\bar{y}) + \theta_f(y)). \quad (6.47)$$

To verify the optical theorem we compute

$$-2 \operatorname{Re} {}^{ON}_f \langle n'_1 n'_2 | n_1 n_2 \rangle_{i\text{s-channel}}^{ON(2)} = -(ig)^2 \int_{\bar{y}} \int_y u_1'^* u_2'^*(\bar{y}) u_1 u_2(y) [\mathcal{M}(\bar{y}, y) + \mathcal{M}^*(A\bar{y}, y)], \quad (6.48)$$

where in the second term we have once again used (5.9). From (5.11) and (5.10) it follows that

$$\mathcal{M}(\bar{y}, y) + \mathcal{M}^*(A\bar{y}, y) = G_3(y, \bar{y}) + G_3^*(y, \bar{y}) - W_3(y, \bar{y}) = W_3(\bar{y}, y), \quad (6.49)$$

and inserting this into (6.48) we obtain

$$\begin{aligned} -2 \operatorname{Re} {}^{ON}_f \langle n'_1 n'_2 | n_1 n_2 \rangle_{i\text{s-channel}}^{ON(2)} &= -(ig)^2 \int_{\bar{y}} \int_y u_1'^* u_2'^*(\bar{y}) u_1 u_2(y) W_3(\bar{y}, y) \\ &= \sum_{\vec{L}_3} \left[-(ig) \int_{\bar{y}} u_3 u_2'^* u_1'^*(\bar{y}) \right] \left[(ig) \int_y u_3^* u_2 u_1(y) \right] \\ &= \sum_{\vec{L}_3} {}_i \langle n'_2 n'_1 | n_3 \rangle_f^{(1)} {}_f \langle n_3 | n_2 n_1 \rangle_i^{(1)}, \end{aligned} \quad (6.50)$$

where the summand vanishes unless

$$\max \{|L'_2 - L'_1|, |L_2 - L_1|\} \leq L_3 \leq \min \{L'_2 + L'_1, L_2 + L_1\}. \quad (6.51)$$

This is indeed a piece of the necessary result. To complete the desired check of unitarity, we must show that the u-channel amplitude combines with the 1-loop contributions to yield

$$\begin{aligned}
& -2 \operatorname{Re} \langle n'_1 n'_2 | n_1 n_2 \rangle_{i \text{ u-channel} + 1\text{-loop}}^{ON(2)} \\
&= \sum_{\bar{L}'_1} \sum_{\bar{L}'_3} \sum_{\bar{L}'_1} \langle n'_2 n'_1 | n''_1 n''_3 n''_1 \rangle_f^{(1)} \langle n''_1 n''_3 n''_1 | n_2 n_1 \rangle_i^{(1)} \\
&+ \sum_{\bar{L}'_2} \sum_{\bar{L}'_3} \sum_{\bar{L}'_2} \langle n'_2 n'_1 | n''_2 n''_3 n''_2 \rangle_f^{(1)} \langle n''_2 n''_3 n''_2 | n_2 n_1 \rangle_i^{(1)}. \tag{6.52}
\end{aligned}$$

It is easy to repeat the above steps for the u-channel, as there is only one term. Doing so verifies that it contributes the part of the right-hand side of (6.52) arising from a cut tree diagram. Our earlier result for $-2 \operatorname{Re} \langle N_1 | N_1 \rangle_i^{(2)}$ shows that it indeed provides the part of the right-hand side of corresponding to cut loop diagrams. Thus the optical theorem is satisfied.

7 Light fields

Section 6 restricted study of the model theory (6.1)-(6.41) to bare weights satisfying $\sigma_1 + \sigma_2 + \sigma_3 < -(D-1)$. We now extend the treatment of this model to arbitrarily small positive masses using the R_σ projector of section 3.3. As before, our goal is to verify the optical theorem to $\mathcal{O}(g^2)$. A more explicit but ultimately equivalent method of computing the S-matrix for light fields is described in appendix B, which studies the same model theory to $\mathcal{O}(g^2)$.

Let us begin by analyzing amplitudes at $\mathcal{O}(g)$. It is important to note that the actions of $R_\sigma(x)$ and time-ordering do not commute, e.g.:

$$R_1(x)R_1(\bar{x}) \langle T \phi_1(x) \phi_1(\bar{x}) \rangle = \langle T [R_1 \phi_1(x) R_1 \phi_1(\bar{x})] \rangle + (\text{contact}). \tag{7.1}$$

While these contact terms ultimately do not affect scattering amplitudes, they do complicate the computations. We find it simpler to first apply the $R_i(x)$ operators on the Wightman functions of the theory, and then to use the diagrammatic rules of §5.2 to compute amplitudes. It is useful to define $\tilde{R}_i(x)$ through

$$R_\sigma(x) = 1 + \tilde{R}_\sigma(x)(\square_x - M^2(\sigma)), \tag{7.2}$$

so that

$$\begin{aligned}
& R_3(x_3)R_2(x_2)R_1(x_1) \langle \phi_3(x_3)\phi_2(x_2)\phi_1(x_1) \rangle^{(1)} \\
&= \langle \phi_3(x_3)\phi_2(x_2)\phi_1(x_1) \rangle^{(1)} - g\tilde{R}_3(x) \langle \phi_2\phi_1(x_3)\phi_2(x_2)\phi_1(x_1) \rangle^{(0)} \\
&\quad - g\tilde{R}_2(x) \langle \phi_3(x_3)\phi_3\phi_1(x_2)\phi_1(x_1) \rangle^{(0)} - g\tilde{R}_1(x) \langle \phi_3(x_3)\phi_2(x_2)\phi_3\phi_2(x_1) \rangle^{(0)} \\
&= \langle \phi_3(x_3)\phi_2(x_2)\phi_1(x_1) \rangle^{(1)} - g\tilde{R}_3(x)W_2(x_3, x_2)W_1(x_3, x_1) \\
&\quad - g\tilde{R}_2(x)W_3(x_3, x_2)W_1(x_2, x_1) - g\tilde{R}_1(x)W_3(x_3, x_1)W_2(x_2, x_1), \tag{7.3}
\end{aligned}$$

where we abbreviate $R_i := R_{\sigma_i}$.

Let us compute the amplitude ${}_f\langle n_3 n_2 | n_1 \rangle_i^{(1)}$. We may time order the first term in (7.3) and use our diagrammatic rules. The contributions of the remaining terms in (7.3) are easily evaluated because they are free theory correlators:

$$\begin{aligned} {}_f\langle n_3 n_2 | n_1 \rangle_i^{(1)} &= ig \int_{y, \eta_0} u_3^* u_2^* u_1(y) - g \int d\Sigma^\nu \left[u_1 \overleftrightarrow{\nabla}_\nu (\tilde{R}_1 u_3^* u_2^*) \right]_{-\eta_0} \\ &\quad - \frac{g}{2} \left[\int d\Sigma^\nu \left[(\tilde{R}_3 u_2^* u_1) \overleftrightarrow{\nabla}_\nu u_3^* \right]_{+\eta_0} + (2 \leftrightarrow 3) \right]. \end{aligned} \quad (7.4)$$

Note that we have temporarily implemented a cutoff $|\eta| < \eta_0$, and that we have explicitly symmetrized across the final state operators for convenience. We may clean up this expression by writing the second term as an integral over the regulated space:

$$\begin{aligned} {}_f\langle n_3 n_2 | n_1 \rangle_i^{(1)} &= ig \int_{y, \eta_0} u_3^* u_2^* u_1(y) + ig \int_{y, \eta_0} u_1 (\square - M_1^2) \tilde{R}_1 (u_3^* u_2^*) \\ &\quad - \frac{g}{2} \left[\int d\Sigma^\nu \left[(\tilde{R}_3 u_2^* u_1) \overleftrightarrow{\nabla}_\nu u_3^* \right]_{+\eta_0} + (2 \leftrightarrow 3) \right] + g \int d\Sigma^\nu \left[u_1 \overleftrightarrow{\nabla}_\nu (\tilde{R}_1 u_3^* u_2^*) \right]_{+\eta_0} \\ &= ig \int_{y, \eta_0} (R_1 u_3^* u_2^*) u_1(y) - \frac{g}{2} \left[\int d\Sigma^\nu \left[(\tilde{R}_3 u_2^* u_1) \overleftrightarrow{\nabla}_\nu u_3^* \right]_{+\eta_0} + (2 \leftrightarrow 3) \right] \\ &\quad + g \int d\Sigma^\nu \left[u_1 \overleftrightarrow{\nabla}_\nu (\tilde{R}_1 u_3^* u_2^*) \right]_{+\eta_0}. \end{aligned} \quad (7.5)$$

We need to take a moment to analyze integrals of the form

$$I_3 := \int_y (R_1 u_3^* u_2^*) u_1(y). \quad (7.6)$$

This integral is finite for all positive masses. When the weights satisfy $\text{Re}(\sigma_1 + \sigma_2 + \sigma_3) < -(D-1)$ we may integrate by parts to make $R_1(y)$ act on $u_1(y)$; noting that $R_1(x)u_1(x) = u_1(x)$ it follows that

$$I_3 = \int_y (R_1 u_3^* u_2^*) u_1(y) = \int_y u_3^* u_2^* u_1(y) = I_1, \quad \text{for } \text{Re}(\sigma_1 + \sigma_2 + \sigma_3) < -(D-1). \quad (7.7)$$

Using the linearization formula for $u_3^* u_2^*$ we may compute I_3 in precisely the same way we computed I_1 in §6.1. The only difference is that now the $R_1(x)$ annihilates any contributions from poles in the complex μ plane other than the two at $\mu = \sigma$ and $\mu = -(\sigma + D - 1)$. Thus for all positive masses I_3 is given by

$$I_3 = (\text{RHS of eq. (6.19)}) \quad \forall M_i^2 > 0. \quad (7.8)$$

It follows that in fact

$$I_3 = \int_y (R_1 u_3^* u_2^*) u_1(y) = \int_y (R_2 u_3^* u_1) u_2^*(y) = \int_y (R_3 u_2^* u_1^*) u_3^*(y). \quad (7.9)$$

All three expressions are analytic (up to poles at the exceptional theories). So (7.9) is clear from the fact that the integrals agree for $\text{Re}(\sigma_1 + \sigma_2 + \sigma_3) < -(D - 1)$ when the actions of R_1, R_2, R_3 are trivial.

We now return to the amplitude (7.5). Noting that

$$\begin{aligned} \int_{y, \eta_0} (R_1 u_3^* u_2^*) u_1(y) &= \int_{y, \eta_0} u_3^* u_2^* u_1(y) \\ &+ \int d\Sigma^\nu \left[u_1 \overleftrightarrow{\nabla}_\nu (\tilde{R}_1 u_3^* u_2^*) \right]_{-\eta_0} - \int d\Sigma^\nu \left[u_1 \overleftrightarrow{\nabla}_\nu (\tilde{R}_1 u_3^* u_2^*) \right]_{+\eta_0} \end{aligned} \quad (7.10)$$

and using (7.9), one sees that the surface integrals in (7.5) cancel each other completely so that the final expression for the amplitude is given by ig times (6.19) for all $M_i^2 > 0$. We see that the amplitudes are analytic in the weights σ_i , though with poles at the exceptional theories¹⁴ defined in point iv of section 2.3. As in section 6, the $\mathcal{O}(g)$ S-matrix element ${}_f \langle n_3 n_2 | n_1 \rangle_i^{(1)}$ is imaginary in agreement with the optical theorem.

A similar analysis (with corresponding results) can be performed for the remaining amplitudes at $\mathcal{O}(g)$. In particular, for all positive masses we have

$$2i \text{Im} \langle \Omega | n_3 n_2 n_1 \rangle_i = (ig) (\text{R.H.S. of (6.26)}) \quad \forall M_i^2 > 0, \quad (7.11)$$

see Fig. 6 for an example. The construction of an orthonormal basis also proceeds just as for heavy fields and so again has no effect on the 1- and 2-particle states.

We now proceed to $\mathcal{O}(g^2)$. Once the R_σ operators are in place the calculation is similar to that described in section 6.3 for heavy fields. We will thus proceed rather quickly.

The inner products which vanish for heavy fields continue to do so for light fields as well, e.g. ${}_i \langle n'_2 n'_1 n_2 | n_1 \rangle_{iC}^{(2)} = 0$, so it is natural to again define orthonormal states via the analogues of (6.35). Since the quantities of interest for the optical theorem are the real parts of the $\mathcal{O}(g^2)$ S-matrix amplitudes ${}_f \langle A | B \rangle_i^{(2)}$, we focus only on these objects below.

We begin with the $1 \rightarrow 1$ S-matrix element ${}^N_f \langle n_1 | n_1 \rangle_i^{ON(2)}$. The real part may be expressed as

$$-2 \text{Re} {}^N_f \langle n_1 | n_1 \rangle_i^{ON(2)} = {}_i \langle n_1 | n_1 \rangle_i^{(2)} + {}_f \langle n_1 | n_1 \rangle_f^{(2)} - {}_f \langle n_1 | n_1 \rangle_i^{(2)} - {}_i \langle n_1 | n_1 \rangle_f^{(2)}. \quad (7.12)$$

Integrating by parts and using the fact that the $\phi_1(x)$ 2-point function is de Sitter-invariant, one may recast the right-hand side as

$$-2 \text{Re} {}^N_f \langle n_1 | n_1 \rangle_i^{ON(2)} = \int_y u_1(y) (\square_y - M_1^2) \int_{\bar{y}} u^*(\bar{y}) (\square_{\bar{y}} - M_1^2) R_1(y) R_1(\bar{y}) \langle \phi_1(\bar{y}) \phi_1(y) \rangle^{(2)}. \quad (7.13)$$

We may freely commute the Klein-Gordon operators through the R_1 projectors and let them act on the correlation function. The KG operators reduce the correlator to an $\mathcal{O}(g^0)$

¹⁴These poles precisely match those found in [35, 36]. The coefficients c_k in (B.7) also have poles at these theories.

correlator:

$$\begin{aligned}
-2 \operatorname{Re} \langle n_1 | n_1 \rangle_i^{ON(2)} &= +g^2 \int_y u_1(y) \int_{\bar{y}} u^*(\bar{y}) R_1(y) R_1(\bar{y}) \langle \phi_3 \phi_2(\bar{y}) \phi_3 \phi_2(y) \rangle^{(0)} \\
&= +g^2 \int_y u_1(y) \int_{\bar{y}} u^*(\bar{y}) R_1(y) R_1(\bar{y}) [W_3(\bar{y}, y) W_2(\bar{y}, y)] \\
&= \sum_{\vec{L}_2} \sum_{\vec{L}_3} \left| ig \int_y (R_1 u_3^* u_2^*) u_1(y) \right|^2. \tag{7.14}
\end{aligned}$$

The right-hand side is precisely what is required by the optical theorem.

We conclude this section by verifying the optical theorem for the $2 \rightarrow 2$ S-matrix element $\langle n'_2 n'_1 | n_2 n_1 \rangle_i^{ON(2)}$. We focus on the s-channel contribution as this is the most non-trivial; the u-channel contribution to the optical theorem involves fewer diagrams, and the disconnected contributions follow from (7.14). The real part of $\langle n'_2 n'_1 | n_2 n_1 \rangle_i^{ON(2)}$ may be written

$$\begin{aligned}
-2 \operatorname{Re} \langle n'_2 n'_1 | n_2 n_1 \rangle_i^{ON(2)} &= \frac{1}{4} \int_{\bar{x}_2} \int_{\bar{x}_1} \int_{x_2} \int_{x_1} \left\{ u_1'^*(\bar{x}_1) u_2'^*(\bar{x}_2) u_2(x_2) u_1(x_1) \right. \\
&\quad (\square_{\bar{x}_2} - M_2^2) (\square_{\bar{x}_1} - M_1^2) (\square_{x_2} - M_2^2) (\square_{x_1} - M_1^2) \\
&\quad R_2(\bar{x}_2) R_1(\bar{x}_1) R_2(x_1) R_1(x_1) \\
&\quad \left. \langle \{ \phi_2(\bar{x}_2), \phi_1(\bar{x}_1) \} \{ \phi_2(x_2), \phi_1(x_1) \} \rangle^{(2)} \right\}. \tag{7.15}
\end{aligned}$$

We have explicitly symmetrized the states for convenience. Because the correlator is $\mathcal{O}(g^2)$, only two $R_i(x)$ operators act non-trivially on the correlator. Let us define the ‘‘s-channel part’’ to be the part for which $\phi_2(\bar{x}_2)$ and $\phi_1(\bar{x}_1)$ are connected by a vertex. Then the s-channel contribution to (7.15) is

$$\begin{aligned}
-2 \operatorname{Re} \langle n'_2 n'_1 | n_2 n_1 \rangle_{i \text{ s-channel}}^{ON(2)} &= \frac{1}{4} \int_{\bar{x}_2} \int_{\bar{x}_1} \int_{x_2} \int_{x_1} \left\{ u_1'^*(\bar{x}_1) u_2'^*(\bar{x}_2) u_2(x_2) u_1(x_1) \right. \\
&\quad (\square_{\bar{x}_2} - M_2^2) (\square_{\bar{x}_1} - M_1^2) (\square_{x_2} - M_2^2) (\square_{x_1} - M_1^2) \\
&\quad [R_2(\bar{x}_2) R_2(x_2) + R_2(\bar{x}_2) R_1(x_1) + R_1(\bar{x}_1) R_2(x_2) + R_1(\bar{x}_1) R_1(x_1)] \\
&\quad \left. \langle \{ \phi_2(\bar{x}_2), \phi_1(\bar{x}_1) \} \{ \phi_2(x_2), \phi_1(x_1) \} \rangle_{\text{s-channel}}^{(2)} \right\}. \tag{7.16}
\end{aligned}$$

Once again we may freely commute the Klein-Gordon operators past the $R_i(x)$ operators and let them act on the correlator. Note that

$$\begin{aligned}
&(\square_{\bar{x}_2} - M_2^2) (\square_{\bar{x}_1} - M_1^2) (\square_{x_2} - M_2^2) (\square_{x_1} - M_1^2) \langle \{ \phi_2(\bar{x}_2), \phi_1(\bar{x}_1) \}, \{ \phi_2(x_2) \phi_1(x_1) \} \rangle_{\text{s-channel}}^{(2)} \\
&= +g^2 \delta(\bar{x}_2, \bar{x}_1) \delta(x_2, x_1) [W_3(\bar{x}_2, x_2) + W_3(\bar{x}_2, x_1) + W_3(\bar{x}_1, x_2) + W_3(\bar{x}_1, x_1)] \tag{7.17}
\end{aligned}$$

Upon inserting (7.17) into (7.16) we obtain a number of terms. In each term we may use the delta functions to integrate over one of the \bar{x}_i and one of the x_i . For each term we expand

the Wightman function $W_3(\bar{x}_i, x_j)$ in a sum over harmonics. Cleaning up a bit and using (7.9) yields the desired result:

$$\begin{aligned}
-2 \operatorname{Re} \langle n'_2 n'_1 | n_2 n_1 \rangle_i^{ON(2)} &= g^2 \int_{\bar{y}} (R_3 u'_2 u'_1(\bar{y})) \int_y (R_3 u_2 u_1(y)) W_3(\bar{y}, y) \\
&= \sum_{\vec{L}_3} \left| (ig) \int_y (R_3 u_2 u_1) u_3^*(y) \right|^2, \tag{7.18}
\end{aligned}$$

which explicitly verifies our optical theorem.

8 Discussion

We have introduced a perturbative S-matrix for interacting theories of massive ($M^2 > 0$) scalar fields in global de Sitter space. Our S-matrix reduces in the flat-space limit to the usual Minkowski S-matrix, is invariant under perturbative field redefinitions, and is unitary in the sense stated in section 4. In particular, we have verified the associated optical theorem through order g^2 in a $g\phi^3$ theory by explicit calculation. While there are certain qualitative differences from Minkowski space (lack of branch cuts, non-orthogonality of distinct asymptotic multi-particle states), we expect our S-matrix to be a useful tool for better understanding interacting de Sitter quantum fields.

As expected from the lack of a positive-definite energy-like conserved quantity, we find that all particles generically decay for any $M^2 > 0$; i.e., all particles in dS are unstable. This result agrees with [33, 34] which followed a rather different approach. However, let us pause briefly to explain the discrepancy with [35, 36, 37], which claim that general complementary series particles can decay only in theories that are exceptional in the sense of section 2.3. Although refs. [35, 36, 37] do not define a full S-matrix, they study what they call the inclusive decay rate Γ per unit time at $O(g^2)$. They compute the result in the principle series (where the IR issues are straightforward) and analytically continue to the complementary series. Their Γ is defined by generalizing certain formulae from Minkowski space, and in the principle series it agrees precisely with the right-hand-side of our $O(g^2)$ optical theorem (6.44). In the principle series it is thus $-2 \operatorname{Re} \langle n_1 | n_1 \rangle_i^{ON(2)}$. The technical discrepancy with our work then arises because while the full amplitude $\langle n_1 | n_1 \rangle_i^{ON(2)}$ is analytic in our formalism, the real part alone fails to be analytic outside the principle series. The issue is analogous to attempting to compute $\operatorname{Re} e^{iz} = 1/e$ at $z = i$ by analytically continuing from the real axis the function $\cos(x) = \operatorname{Re} e^{ix}$ and obtaining $\cos(i) = e + e^{-1}$. The forthcoming work [64] will comment further on this issue.

In Minkowski space, unstable particles are ultimately unsuitable as asymptotic states. In particular, this issue can be seen in perturbation theory once self-energy corrections are included in external lines representing the unstable particles. Before resumming 1PI graphs, the correlation functions develop higher order poles that lead to IR divergences in the S-matrix. After resumming 1PI graphs and interpreting the result as a self-energy correction,

the relevant poles move off the real axis and the associated wavefunctions are no longer normalizable.

We find similar behavior for unstable particles in the principal series of de Sitter representations; i.e., with $M^2\ell^2 > (D-1)^2/4$. Thus, a theory with only principal-series particles apparently does not have a well-defined S-matrix outside of perturbation theory. In Minkowski space, the notion of a generalized S-matrix with unstable particles as additional asymptotic states has been sometimes considered (see, e.g., section 4.9 of [65]). In de Sitter space with principle-series fields only, such a formalism could be used to define a generalized S-matrix despite the lack of stable particles.

The behavior of complementary series fields is more subtle. As always, the optical theorem relates the 1-to-2 decay amplitude to the imaginary part of the loop-corrected 1-to-1 amplitude. However, as will be explained in detail in [64], the fact that the wavefunctions do not oscillate near infinity means that the self-energy remains real; i.e., the imaginary part of the 1-to-1 amplitude is not then given by the imaginary part of the self-energy, and the poles remain on the real axis. This raises the possibility that one might be able to define a finite S-matrix using normalizable asymptotic states built out of complementary-series fields, though this seems far from straightforward; see [64] for additional comments.

With the above understanding, our S-matrix is free of unphysical IR divergences for generic masses. However, extra IR divergences do arise when the self-energy Π is singular at some generating pole; i.e., for the exceptional theories described in section 2.3. As a result, the locations of the poles fail to admit a good perturbative expansion. In this sense, such exceptional theories are always strongly coupled even for small g . Our exceptional theories coincide precisely with the special cases previously identified in [35, 36, 37].

There are of course many possible generalizations of our construction above. For example, we worked here in global de Sitter space. We expect that one could use the future half of our Schwinger-Keldysh contour (see figure 2) to define a similar S-matrix in the expanding Poincaré patch, though we have not explored this case in detail. In addition, we have thus far restricted attention to scalars with $M^2 > 0$. Massless minimally coupled ($M^2 = 0$) scalars do not have a good dS Hartle-Hawking state in perturbation theory (see e.g. [66]), but massless higher spin fields are better behaved. In particular, both the free Maxwell field and graviton admit de Sitter-invariant vacuum states with good analytic continuations to the Euclidean sphere (i.e., Hartle-Hawking states). [67, 68, 69] One may therefore hope to extend our S-matrix to such cases. So long as the associated correlation functions decay sufficiently rapidly¹⁵, our use of Klein-Gordon inner products (and the invariance of such inner products under linearized gauge transformations) suggests that the final result will be gauge-invariant. While naively the fact that $\sigma = -1$ for Maxwell fields and $\sigma = 0$ for gravitons would seem to qualify these theories as exceptional, we expect that as in flat space gauge-invariance will prevent the self-energy from acquiring singularities at these values. A discussion of such issues will appear soon [74]. It would be very interesting to understand if our definition could be further extended to perturbative string theory¹⁶.

¹⁵This is expected for the Maxwell field but is a matter of significant controversy for the graviton. See e.g. [70, 71, 72, 73].

¹⁶We thank Eva Silverstein for raising this issue.

As a final comment, one would also like to understand any relation of our S-matrix to quantities of interest in dS/CFT [75, 76, 77]. Since our S-matrix focuses on asymptotic quantities near future and past infinity, one may expect some relation to exist. However, because the Hartle-Hawking state is not itself dual to a particular CFT with fixed sources, it is in some sense a derived object in dS/CFT as opposed to a fundamental one [77, 78]. Our reliance on the Hartle-Hawking state will thus make any connection somewhat indirect.

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A KG mode linearization

This appendix gives some details of the function $\rho_{\sigma_1\sigma_2L_1L_2}(\mu, K)$ appearing in the linearization formula (2.24). As noted in the main text, the results below may be derived using the methods of appendix A of [8], and in particular techniques for manipulating Mellin-Barnes representations of hypergeometric functions as described in [79, 43].

We introduce $\alpha := (D - 1)/2$ as well as the notation

$$\Gamma \left[\begin{matrix} a_1, a_2, \dots, a_k \\ b_1, b_2, \dots, b_j \end{matrix} \right] := \frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_k)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_j)} \quad (\text{A.1})$$

$$\psi_{\sigma L}(\nu) := \Gamma \left[L - \sigma + \nu, L + \sigma + 2\alpha + \nu, \frac{1}{2} - L - \alpha - \nu, -\nu \right], \quad (\text{A.2})$$

$$\mathcal{N}_{\sigma L} := \frac{\cos \pi(\sigma + \alpha)}{2^{L+\alpha}\pi} \{ \Gamma [L - \sigma, L + \sigma + 2\alpha] \}^{-1/2}, \quad (\text{A.3})$$

and finally the terminating, “regulated” hypergeometric series

$${}_3F_2 \left[\begin{matrix} a, b, -p \\ c, d \end{matrix} ; 1 \right] := \sum_{n=0}^p \frac{(a)_n (b)_n (-p)_n}{\Gamma [1 + n, c + n, d + n]}, \quad (\text{A.4})$$

with $(a)_n$ the Pochhammer symbol. This series is an entire function of each parameter a, b, c, d .

The linearization kernel may be then expressed as the double Mellin-Barnes integral

$$\begin{aligned}
\rho_{\sigma_1\sigma_2L_1L_2}(\mu, K) &= (-1)^{(L_1+L_2-K)/2} 2^{\alpha+L_1+L_2} \mathcal{N}_{\sigma_1L_1} \mathcal{N}_{\sigma_2L_2} \{\Gamma[K - \mu, K + 2\alpha + \mu]\}^{1/2} \\
&\times \int_{\nu_1} \int_{\nu_2} \left\{ \psi_{\sigma_1L_1}(\nu_1 - L_1) \psi_{\sigma_2L_2}(\nu_2 - L_2) \right. \\
&\times \Gamma \left[\begin{matrix} \mu - \nu_1 - \nu_2, 1 - \mu + \nu_1 + \nu_2, \frac{1+K-L_1-L_2}{2} + \alpha + \nu_1 + \nu_2 \\ \frac{K+L_1+L_2}{2} - \nu_1 - \nu_2 \end{matrix} \right] \\
&\times \left. {}_3F_2 \left[\begin{matrix} \frac{1+K-L_1-L_2}{2} + \alpha + \nu_1 + \nu_2, & 1 - \left(\frac{K+L_1+L_2}{2}\right) + \nu_1 + \nu_2, & \frac{K-L_1-L_2}{2} \\ 1 + \frac{K-L_1-L_2}{2} + 2\alpha + \mu + \nu_1 + \nu_2, & 1 + \frac{K-L_1-L_2}{2} - \mu + \nu_1 + \nu_2 \end{matrix} ; 1 \right] \right\}. \tag{A.5}
\end{aligned}$$

The well-known asymptotic behavior of the Gamma function may be used to show that the Mellin-Barnes integrals are absolutely convergent. Equation (2.26) follows from the identity $[\Gamma(x)]^* = \Gamma(x^*)$.

We are primarily interested locating singular points of $\rho_{\sigma_1\sigma_2L_1L_2}(\mu, K)$ as a function of μ . The singularities of the double MB integral in (A.5) may be determined using standard techniques presented, e.g., in Ch. 4 of [80]. Within the integrand there are potentially poles due to the Gamma functions $\Gamma(\mu - \nu_1 - \nu_2)$ and $\Gamma(1 - \mu + \nu_1 + \nu_2)$. However, the poles in the latter Gamma function are canceled by zeros in each term of the hypergeometric series, so this Gamma function does not in fact yield singularities. Thus the only source of singularities from the double integral is the Gamma function $\Gamma(\mu - \nu_1 - \nu_2)$. The ν_i integrals have poles at $\nu_i = \sigma_i - n$ and $\nu_i = -(\sigma + 2\alpha) - n$, $n \in \mathbb{N}_0$, respectively, so it follows that the double integral contributes poles to μ at poles at precisely the locations (2.25).

The observant reader will note that the first line of (A.5) contains combinations of gamma functions which, when combined with an identical factor in $f_{\mu K}(\eta)$, contributes additional poles in the integrand at

$$\mu = K + n, \quad \mu = -2\alpha - K - n, \quad n \in \mathbb{N}_0. \tag{A.6}$$

We did not mention these poles in section 2.2 for two reasons: first, they lie outside the strip in the complex μ plane in which we utilize $\rho_{\sigma_1\sigma_2L_1L_2}(\mu, K)$; second, these poles do not contribute to the behavior of (2.24) at large $|\eta| \gg 1$, as we now show. Recall the asymptotic form (2.14) of $f_{\sigma L}(\eta)$ which we re-write here as

$$\begin{aligned}
f_{\sigma L}(|\eta| \gg 1) &= \mathcal{N}_{\sigma L} \left[\Gamma \left[L - \sigma, \frac{1}{2} - \alpha - \sigma, 2(\sigma + \alpha) \right] \left(\frac{i\eta}{2} \right)^\sigma + (\sigma \rightarrow -(\sigma + 2\alpha)) \right] \\
&\times [1 + \mathcal{O}(\eta^{-2})], \quad |\eta| \gg |L - \sigma|. \tag{A.7}
\end{aligned}$$

When $|\eta| \gg L_1 + L_2$, inserting (A.7) into (2.24) yields

$$\begin{aligned}
f_{\sigma_1 L_1} f_{\sigma_2 L_2} (|\eta| \gg L_1 + L_2) &= \int_{\mu} 2(\mu + \alpha) \rho_{\sigma_1 \sigma_2 L_1 L_2}(\mu, K) \mathcal{N}_{\mu K} \\
&\times \left\{ \Gamma \left[K - \mu, \frac{1}{2} - \alpha - \mu, 2(\mu + \alpha) \right] \left(\frac{i\eta}{2} \right)^{\mu} \right. \\
&\quad \left. + \Gamma \left[K + \mu + 2\alpha, \frac{1}{2} + \alpha + \mu, -2(\mu + \alpha) \right] \left(\frac{i\eta}{2} \right)^{-(\mu + 2\alpha)} \right\} \\
&\times [1 + \mathcal{O}(\eta^{-2})]. \tag{A.8}
\end{aligned}$$

Notice that the factor $\mathcal{N}_{\mu K}$ contains the precisely the combination of gamma functions needed to cancel the similar factor in $\rho_{\sigma_1 \sigma_2 L_1 L_2}(\mu, K)$. Upon inspection one finds that $\rho_{\sigma_1 \sigma_2 L_1 L_2}(\mu, K) \mathcal{N}_{\mu K}$ contains zeros which cancel the poles due to $\Gamma(\pm 2(\mu + \alpha))$. To obtain the asymptotic form of $f_{\sigma_1 L_1}(\eta) f_{\sigma_2 L_2}(\eta)$ from (A.8) we close the integration contour with an arc at infinity in the left-half plane for the first term, and with an arc at infinity in the right-half plane for the second term, and use Cauchy's integral formula to relate these two contour integrals to the sum of residues obtained. The second term in (A.8) yields zero because the integrand contains no poles in the right-half plane. The first term in (A.8) acquires residues from the poles (2.25), and these poles precisely reproduce the asymptotic expansion of $f_{\sigma_1 L_1}(\eta) f_{\sigma_2 L_2}(\eta)$.

B The explicit form of good operators

This appendix explores a rather brute force but very explicit method of constructing good operators in the context of the model theory (6.1)-(6.41). The basic idea is that, at each order, one simply chooses a perturbative field redefinition that removes the unwanted IR-divergent contributions but which leaves the behavior at the desired pole unchanged. We find such explicit results enlightening, though they become cumbersome at higher orders. As a result, we work here only to $\mathcal{O}(g)$. The results agree completely with those of section 7.

Recall that the difficulty with the naive prescription (3.5) was that the $\eta_0 \rightarrow \infty$ limit of

$$\langle a_{\sigma_3 \vec{L}_3}(\eta_0) \dots \rangle - i \int d\Sigma^\nu \langle \phi_3(x) \dots \rangle \overleftrightarrow{\nabla}_\nu u_{\sigma_3 \vec{L}_3}^*(x) \Big|_{\eta=\eta_0} \tag{B.1}$$

is well-defined only when the correlator satisfies

$$\langle \phi_3(x) \dots \rangle = (\text{homogeneous}) + \mathcal{O}(\eta^{-(\sigma+3+D-1)}), \quad \eta \gg 1. \tag{B.2}$$

Here (homogeneous) denotes solutions to the homogeneous Klein-Gordon equation $(\square - M_3^2)f(x) = 0$. We refer to the rest of $\phi_3(x)$ as the inhomogeneous part. It is straightforward

to perturbatively compute the asymptotics of the inhomogeneous part from the Schwinger-Dyson equations. For example, at $\mathcal{O}(g)$ we have

$$\begin{aligned}
(\square_3 - M_3^2) \langle \phi_3(x_3) \phi_2(x_2) \phi_1(x_1) \rangle^{(1)} &= -g \langle \phi_2 \phi_1(x_3) \phi_2(x_2) \phi_1(x_1) \rangle^{(0)} \\
&= -g W_2(x_3, x_2) W_1(x_3, x_1) \\
&= \left[\mathcal{O}(\eta_3^{\sigma_2 + \sigma_1}) + \mathcal{O}(\eta_3^{\sigma_2 - \sigma_1 - (D-1)}) + \mathcal{O}(\eta_3^{-\sigma_2 + \sigma_1 - (D-1)}) \right. \\
&\quad \left. + \mathcal{O}(\eta_3^{-\sigma_2 - \sigma_1 - 2(D-1)}) \right] [1 + \mathcal{O}(\eta_3^{-2})] \quad \text{for } |\eta_3| \gg 1 \quad (\text{B.3})
\end{aligned}$$

where the final equality summarizes the asymptotic behavior of the Wightman functions. Because the action of $(\square_3 - M_3^2)$ does not change the asymptotic behavior of power laws of η , the right hand side of (B.3) also describes the inhomogeneous part of $\langle \phi_3(x_3) \phi_2(x_2) \phi_1(x_1) \rangle^{(1)}$. This violates the condition (B.2) for $-(D-1) < \text{Re}(\sigma_1 + \sigma_2 + \sigma_3)$, and moreover contributes unwanted information to (B.1).

Acting with the R_σ projector of section 3.3 is thus equivalent (in the sense of section 3.3) to constructing some $\Phi_3(x)$ whose asymptotics contain the same homogeneous part as $\phi_3(x)$ but which lacks the divergent inhomogeneous parts. So for $-(D-1) < \text{Re}(\sigma_1 + \sigma_2 + \sigma_3)$ let $P_3 = \{\sigma_2 + \sigma_1, \dots\}$ be the set of weights belonging to terms in (B.3) violating (B.2) and consider

$$\Phi_3(x) = \phi_3(x) + g \sum_{p \in P_3} \mathcal{O}_p(x) + \mathcal{O}(g^2), \quad (\text{B.4})$$

where $\mathcal{O}_p(x)$ denotes an operator with asymptotic behavior $\mathcal{O}(\eta^p) + \mathcal{O}(\eta^{-(\sigma_3 + D-1)})$ at $\mathcal{O}(g^0)$. We choose the $\mathcal{O}_p(x)$ to be normalized so that

$$\mathcal{O}_p(x) = \frac{1}{M^2(p) - M_3^2} \phi_2 \phi_1(x) \Big|_p + \mathcal{O}(\eta^{-(\sigma_3 + D-1)}), \quad (\text{B.5})$$

where $\phi(x)|_p$ denotes the $\mathcal{O}(\eta^p)$ part of $\phi(x)$ at $\mathcal{O}(g^0)$ and $M^2(p)$ is the squared-mass which gives rise to this fall off at $\mathcal{O}(g^0)$. The choice of operators $\mathcal{O}_p(x)$ is not unique, but a natural choice is

$$\begin{aligned}
\mathcal{O}_p(x) &= \frac{1}{M^2(p) - M_3^2} \left[\prod_{q \in P_3, q \neq p} \frac{(\square - M^2(q))}{(M^2(p) - M^2(q))} \right] \phi_2 \phi_1(x) \\
&= \frac{1}{M^2(p) - M_3^2} \phi_2 \phi_1(x) \Big|_p + \mathcal{O}(\eta^{-(\sigma_3 + D-1)}). \quad (\text{B.6})
\end{aligned}$$

With this choice for the $\mathcal{O}_p(x)$ the new operator $\Phi_3(x)$ may also be written

$$\Phi_3(x) = \phi_3(x) + g \sum_{k=0} c_k \square^k \phi_2 \phi_1(x), \quad (\text{B.7})$$

with appropriate coefficients c_k . A similar procedure should be applied to $\Phi_2(x)$ and $\Phi_1(x)$.

It is sufficient to simply change variables from $\phi_i(x)$ to $\Phi_i(x)$ in the generating functional (the path integral). This is most easily accomplished with $\Phi_i(x)$ defined as in (B.7). The

lagrangian for the $\Phi_i(x)$ contains a quadratic part in which each $\Phi_i(x)$ has a canonically normalized kinetic term and a mass term that agrees with that of the original field $\phi_i(x)$. At order $\mathcal{O}(g)$ the lagrangian contains the cubic interaction $cg\Phi_3\Phi_2\Phi_1(x)$, now with an additional coefficient c , as well as higher-derivative cubic interactions. These higher-derivative interactions, taken on their own, produce correlators whose asymptotics have similar fall-off properties to the simple cubic interaction. But these higher-derivative interactions conspire to cancel the unwanted inhomogeneous terms in (B.3) without altering the homogeneous part. One could say that the new higher-derivative interactions are ‘‘IR counterterms,’’ though we stress that the theory has not been altered nor does the theory need such terms in order for correlation functions to be well-defined.

We can now compute $\mathcal{O}(g)$ amplitudes. To keep the equations tractable let us consider the case where there is only one problematic term at $\mathcal{O}(\eta^{\sigma_1+\sigma_2+\sigma_3})$. In this case the $\Phi_i(x)$ defined in (B.6) are

$$\begin{aligned}\Phi_3(x) &= \phi_3(x) + \frac{g}{M^2(\sigma_2 + \sigma_1) - M_3^2} \phi_2\phi_1(x) \\ &= \phi_3(x) + \frac{g}{(\sigma_3 - \sigma_1 - \sigma_2)(\sigma_1 + \sigma_2 + \sigma_3 + D - 1)} \phi_2\phi_1(x)\end{aligned}\quad (\text{B.8})$$

and likewise for $\Phi_2(x)$, $\Phi_1(x)$. The $\mathcal{O}(g)$ 3-pt. function of the corrected operators is

$$\begin{aligned}\langle \Phi_3(x_3)\Phi_2(x_2)\Phi_1(x_1) \rangle^{(1)} &= \langle \phi_3(x_3)\phi_2(x_2)\phi_1(x_1) \rangle^{(1)} \\ &\quad + \frac{g}{M^2(\sigma_2 + \sigma_1) - M_3^2} \langle \phi_2\phi_1(x_3)\phi_2(x_2)\phi_1(x_1) \rangle^{(0)} \\ &\quad + \frac{g}{M^2(\sigma_3 + \sigma_1) - M_2^2} \langle \phi_3(x_3)\phi_3\phi_1(x_2)\phi_1(x_1) \rangle^{(0)} \\ &\quad + \frac{g}{M^2(\sigma_3 + \sigma_2) - M_1^2} \langle \phi_3(x_3)\phi_2(x_2)\phi_3\phi_2(x_1) \rangle^{(0)}.\end{aligned}\quad (\text{B.9})$$

The $1 \rightarrow 2$ transition amplitude is then

$$\begin{aligned}_f \langle n_3 n_2 | n_1 \rangle_i^{(1)} &= ig \int_{\eta(y) \leq \eta_0} u_3^* u_2^* u_1(y) \\ &\quad - ig \int d\Sigma^\nu \left[\frac{1}{2} \frac{u_2^* u_1}{(M^2(\sigma_2 + \sigma_1) - M_3^2)} \overleftrightarrow{\nabla}_\nu u_3^*(x) + (2 \leftrightarrow 3) \right]_{\eta=+\eta_0} \\ &\quad - ig \int d\Sigma^\nu \left[u_1 \overleftrightarrow{\nabla}_\nu \frac{u_3^* u_2^*(x)}{M^2(\sigma_3 + \sigma_2) - M_1^2} \right]_{\eta=-\eta_0}.\end{aligned}\quad (\text{B.10})$$

Here we have regulated right-hand-side by imposing $|\eta| < \eta_0$ since each term is individually divergent as $|\eta_0| \rightarrow \infty$. We have also explicitly symmetrized the operators of the final state.

The first term is just the regulated version $igI_1(\eta_0)$ of the integral considered in §6.1. Recall that our computation of $I_1(\eta_0)$ for heavy fields used the linearization formula (2.22) on $u_3 u_2(x)$ then deformed the μ integration contour to $\text{Re } \mu < -\text{Re}(\sigma_1 + D - 1)$. In the process we acquired residues due to simple poles at $\mu = \sigma_1$ and $\mu = -(\sigma_1 + D - 1)$. A similar

procedure can be used in the present case but now, since $-\text{Re}(\sigma_1 + D - 1) < \text{Re}(\sigma_2 + \sigma_3)$, deforming the contour to the left picks up an additional residue from the simple pole in the linearization kernel at $\mu = \sigma_2 + \sigma_3$. This new pole now provides the leading behavior of $u_3 u_2(x)$. Thus

$$I_1(\eta_0) = (\text{RHS of eq. 6.19}) \\ - \frac{1}{M^2(\sigma_2 + \sigma_3) - M_1^2} \int d\Sigma^\nu(y) \left\{ \left[u_3^* u_2^* \overleftrightarrow{\nabla}_\nu u_1(y) \right]_{\eta=\eta_0} - \left[u_3^* u_2^* \overleftrightarrow{\nabla}_\nu u_1(y) \right]_{\eta=-\eta_0} \right\} \\ + \dots, \quad (\text{B.11})$$

where the ellipses denote the remaining integral over some contour with $\text{Re} \mu < -\text{Re}(\sigma_1 + D - 1)$. Contributions from this remaining contour integral vanish in the limit $\eta_0 \rightarrow \infty$.

We now examine the terms on the second and third lines of (B.10). Note that only the leading term of each KG mode $u_i(x) = \mathcal{O}(\eta^{\sigma_i})$ will contribute to these surface integrals in the limit $\eta_0 \rightarrow \infty$. Noting that

$$\frac{1}{2} \left[\frac{(\sigma_3 - \sigma_2 - \sigma_1)}{M^2(\sigma_2 + \sigma_1) - M_3^2} + (2 \leftrightarrow 3) \right] = - \frac{(\sigma_3 + \sigma_2 - \sigma_1)}{M^2(\sigma_3 + \sigma_2) - M_1^2}, \quad (\text{B.12})$$

it follows that the contribution of the integrals at $\eta = +H$ may also be written as

$$-ig \int d\Sigma^\nu \left[\frac{1}{2} \frac{u_2^* u_1}{(M^2(\sigma_2 + \sigma_1) - M_3^2)} \overleftrightarrow{\nabla}_\nu u_3^*(x) + (2 \leftrightarrow 3) \right]_{\eta=+\eta_0} \\ = \frac{ig}{M^2(\sigma_3 + \sigma_2) - M_1^2} \int d\Sigma^\nu \left[u_1 \overleftrightarrow{\nabla}_\nu u_3^* u_2^*(y) \right]_{\eta=+\eta_0} + \dots, \quad (\text{B.13})$$

where again ellipses denotes terms that vanish as $\eta_0 \rightarrow \infty$. It follows that for $\eta_0 \rightarrow \infty$ the last two lines of (B.10) precisely cancel the new residue term in $I_1(\eta_0)$. Thus the amplitude continues to be given by ig times (6.19).

It will not surprise the reader that the same result turns out to be valid for all $M_i^2 > 0$. A detailed check shows that similar cancellations occur in the presence of further problematic terms. For every divergent term $\mathcal{O}(\eta^{\sigma_1+\sigma_2+\sigma_3})$, $\mathcal{O}(\eta^{\sigma_1+\sigma_2+\sigma_3-2})$, $\mathcal{O}(\eta^{-\sigma_1+\sigma_2+\sigma_3-(D-1)})$, etc., one acquires a set of surface integrals arising from the residue of a pole in the derivation of $I_1(\eta_0)$. These surface integrals are exactly canceled by the contribution of the appropriate $\mathcal{O}_p(x)$ operators. It is also straightforward (if tedious) to verify similar cancellations of the divergent terms in the remaining $\mathcal{O}(g)$ amplitudes. In agreement with section 7, we see that the results are again given by the corresponding final formulae in section 6.3.

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