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THE PROPAGATION OF SHOCK WAVES IN NONLINEAR VISCOELASTIC MATERIALS

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Summary

The possibility of a shock wave generated by impact on a long bar of a general nonlinear viscoelastic ("simple") material is discussed. Characteristic relations are derived, as well as the equation governing the decay of the shock amplitude as the shock is propagated into a previously undeformed bar. It is shown how these equations may be used to generate a numerical integration procedure.

1. Introduction

The present study concerns the propagation of an impact wave into a previously undeformed bar of a nonlinear viscoelastic material. The bar is assumed to be straight and prismatic, and extends from $X = 0$ to $X = \infty$, X being a material coordinate corresponding to the undeformed state.

At every point the mechanical state is described by the stress σ (based on the original area), the strain ϵ, and the particle velocity v. Wherever these functions (of X and t, t being the time) are continuously differentiable, they are related by the equations of continuity and motion, namely

$$v_X = \epsilon_t , \tag{1}$$

and

$$\sigma_X = \rho_0 v_t . \tag{2}$$

where ρ_0 denotes the density in the undeformed state, and the subscript notation indicates partial differentiation.

In the case of a velocity impact taking place at $t=0$ (prior to which the bar is undisturbed, i.e., $\sigma = \epsilon = v = 0$), we have the boundary condition

$$v(0,t) = g(t) , \quad t > 0. \tag{3}$$

If the function $g(t)$ is continuous at $t=0$, but has a discontinuous derivative of any order, then the disturbance will be propagated into the bar as a generalized acceleration wave.* One-dimensional acceleration waves in nonlinear viscoelastic materials have been thoroughly studied by Coleman,

*For a discussion of the possible growth of an acceleration wave into a shock wave, see Ref. [2].

Gurtin, and Herrera [1], and Coleman and Gurtin [2]. If $g(t)$ is discontinuous at $t=0$, then the disturbance may be propagated as a shock wave. The authors [8] have considered shock waves in semilinear viscoelastic materials (i.e., materials with linear instantaneous response) in another paper, and here we shall discuss shock waves in general nonlinear viscoelastic materials.

2. Definitions and Wave Properties

We shall now review some definitions and establish certain results pertaining to waves.

A wave front Σ is a surface dividing the material into an undisturbed region ahead of the wave, and a disturbed region behind. We designate the value of a quantity $\alpha(x,t)$ just ahead of Σ by α^+ and just behind by α^-. If $\alpha^+ = \alpha^-$ the quantity α is continuous across Σ; if $\alpha^+ \neq \alpha^-$ we define the "jump" $[\alpha]$ in α across Σ by

$$[\alpha] = \alpha^- - \alpha^+. \tag{4}$$

At any time t, Σ occupies a unique position in space, given by the Lagrangian coordinate $x_\Sigma(t)$ of Σ. We define the wave speed by

$$c^*(t) = \frac{dx_\Sigma}{dt}. \tag{5}$$

We shall denote the inverse functional relation to $x_\Sigma(t)$ by $t_0(x)$. Clearly any quantity related to the wave front may be regarded as only a function of x, or only a function of t. The relationship between x_Σ and t_0 is given by

$$t_0 = \int_0^{x_\Sigma} \frac{d\xi}{c(\xi)} , \tag{6}$$

where $c(x)$ is the wave speed as a function of x, or by

$$x_\Sigma = \int_0^{t_0} c^*(\eta)\, d\eta . \tag{7}$$

When we do not show any explicit dependence of a wave front quantity on x or t, we may choose that dependence for convenience.

If Σ is a shock wave, then the functions σ, ϵ and v are discontinuous there. The discontinuities are related by the kinematic compatibility equation

$$-c[\epsilon] = [v] , \tag{8}$$

and by the dynamic compatibility equation

$$[\sigma] = -\rho_0 c [v] . \tag{9}$$

From (8) and (9) we find (Cf. [1])

$$c^2 = \frac{[\sigma]}{\rho_0 [\epsilon]} . \tag{10}$$

3. The Constitutive Law and Its Consequences

We shall consider the constitutive equation

$$\sigma(t) = \mathcal{F}_{s=0}^{\infty} \{ \epsilon(t-s) \} , \tag{11}$$

where $\mathcal{F}$ is a functional. (The dependence of σ and ϵ on X will not always be specified, but is to be understood).

We assume that, $\mathcal{F}$ may be represented to any desired degree of accuracy [3] by a multiple-integral expansion of the form

$$\mathcal{F}\left\{ \underset{s=0}{\overset{\infty}{\epsilon(t-s)}} \right\} = \int_0^\infty R^{(1)}(s)\, \epsilon_t(t-s)\, ds + \cdots + \tag{12}$$
$$+ \int_0^\infty \cdots \int_0^\infty R^{(n)}(s_1, \cdots, s_n)\, \epsilon_t(t-s_1) \cdots$$
$$\cdot\, \epsilon_t(t-s_n)\, ds_1 \cdots ds_n\,.$$

If a shock wave propagates along the bar, at any point X the strain ϵ (as well as the stress and the velocity) is discontinuous at $t = t_0(x)$. Let us write

$$\epsilon(x,t) = \epsilon^*(x,t)\, H(t - t_0(x)), \tag{13}$$

where $H(—)$ is the Heaviside step function, and ϵ^* is a continuously differeniable function of X and t . The integrands in (12) then become distributions, specifically

$$\epsilon_t = \epsilon_t^*(x,t)\, H(\tau) + E\, \delta(\tau), \tag{14}$$

where we write τ for $t - t_0(x)$, and E for $\epsilon^*(x, t_0(x)) = [\epsilon]$; $\delta(\tau)$ is the Dirac delta function.

The stress just behind the wave, i.e., the stress discontinuity, is given by

$$[\sigma] = \lim_{\tau \to 0^+} \mathcal{F}\left\{ E\,H(\tau - s) \right\}_{s=0}^{\infty}, \tag{15}$$

and in the case of $\mathcal{F}\{-\}$ as given by (12),

$$[\sigma] = R^{(1)}(0^+)\,E + \cdots + R^{(n)}(0^+,\cdots,0^+)\,E^n . \tag{16}$$

The wave velocity is therefore given by

$$\begin{aligned} \rho_0 c^2 &= \frac{1}{E} \lim_{\tau \to 0^+} \mathcal{F}\left\{ E\,H(\tau - s) \right\}_{s=0}^{\infty} \\ &= R^{(1)}(0^+) + \cdots + R^{(n)}(0^+,\cdots,0^+)\,E^{n-1} . \end{aligned} \tag{17}$$

and is clearly a function of E. (The right-hand side of (17) is the "instantaneous secant modulus" of [1]). Let the positive root of (17) be denoted by c_E; then c_0 represents the limit of c_E as $E \to 0$, and is also the velocity of propagation of an acceleration wave into a previously undisturbed material [1]. The condition for the existence of a shock wave with strain discontinuity E is ([4], Chap. III)

$$c_E \geq c_0 . \tag{18}$$

In the material described by (12) we have

$$\rho_0 c_0^2 = R^{(1)}(0^+) . \tag{19}$$

For a material such that $R^{(n)}(0^+,\cdots,0^+) = 0$ for $n \geq 2$, Eqns. (17) and (19) show that $c_E = c_0$. Such materials, which are designated "semilinear", have been treated in detail by the authors [8].

4. Characteristics and the Characteristic Relations

Discontinuities are propagated from the shock front along the characteristics. In order to determine the characteristics in the xt plane we differentiate (11) with respect to x to yield

$$\sigma_x = \delta \mathcal{F} \left\{ \epsilon(t-\sigma) \Big| \epsilon_x(t-s) \right\}_{\sigma=0}^{\infty}{}_{s=0}^{\infty}, \tag{20}$$

where $\delta\mathcal{F}$ denotes the Fréchet differential (see [5] pp. 22-28). Since σ_x is a linear functional of ϵ_x we can write (20) in the form [6]

$$\sigma_x = \int_0^\infty G(s)\, \epsilon_{xt}(t-s)\, ds, \tag{21}$$

where $G(s)$ is a functional of the history of ϵ and a function of s :

$$G(s) = \mathcal{G}\left\{ \epsilon(t-\sigma) \,;\, s \right\}_{\sigma=0}^{\infty}. \tag{22}$$

When $\mathcal{F}$ has a multiple-integral expansion of the form (12), $\mathcal{G}$ has the form

$$\begin{aligned} G(s) = \mathcal{G}\left\{ \epsilon(t-\sigma); s \right\}_{\sigma=0}^{\infty} &= R^{(1)}(s) + \\ &+ 2\int_0^\infty R^{(2)}(s,\sigma_1)\, \epsilon_t(t-\sigma_1)\, d\sigma_1 + \cdots + \\ &+ n\int_0^\infty R^{(n)}(s,\sigma_1,\cdots,\sigma_{n-1})\, \epsilon_t(t-\sigma_1)\cdots \\ &\qquad \cdot \epsilon_t(t-\sigma_{n-1})\, d\sigma_1 \cdots d\sigma_{n-1} . \end{aligned} \tag{23}$$

We integrate (21) by parts, obtaining

$$\sigma_x = G(0^+)\, \epsilon_x(t) + \int_0^\infty G'(s)\, \epsilon_x(t-s)\, ds, \tag{24}$$

where $G'(s) = \frac{\partial}{\partial s} \mathcal{G}\{-\}$. The quantity $G(0^+)$ is, for every x and t, a functional of the history of ϵ; it is the "instantaneous tangent modulus" corresponding to the history [1]. Equations (1), (2), and (24) form a system of quasi-linear partial integro-differential equations for the three functions σ, ϵ and v. This system is hyperbolic if $G(0^+)$ is positive, and by the usual method [4,7] can be shown to have three families of characteristics in the xt plane, whose directions are given by

$$\frac{dx}{dt} = \pm a, 0, \tag{25}$$

where a is given by

$$\rho_0 a^2 = G(0^+). \tag{26}$$

It is possible, however, (and convenient) to eliminate σ from the given system, and to consider the system consisting of (1) and

$$v_t = a^2 \epsilon_x + b, \tag{27}$$

where b is given by

$$\rho_0 b = \int_0^\infty G'(s)\, \epsilon_x(t-s)\, ds. \tag{28}$$

There remain only two families of characteristics, namely, those with the directions

$$\frac{dx}{dt} = \pm a, \tag{29}$$

along which the characteristic relations are

$$dv \mp a\, d\epsilon = b\,dt\, . \tag{30}$$

To study the characteristic directions and relations at points immediately behind the wave front, we consider the function

$$G_E(s) = \lim_{\tau \to 0^+} g\left\{ E H(\tau-\sigma) \Big|_{\sigma=0}^{\infty} ; s \right\} . \tag{31}$$

Just behind Σ the quantity a becomes a_E , a function of E , given by

$$\rho_0 a_E^2 = G_E(0^+) \tag{32}$$

To determine the corresponding quantity b_E , we differentiate (13) with respect to x , obtaining

$$\epsilon_x = \epsilon_x^*(x,t) H(\tau) - \frac{E}{c_E}\delta(\tau) , \tag{33}$$

whence

$$\rho_0 b = \int_0^\tau G'(s)\, \epsilon_x^*(t-s)\, ds - \frac{E}{c_E} G'(\tau) . \tag{34}$$

Taking the limit of (34) as $\tau \to 0^+$, we obtain

$$\rho_0 b_E = -\frac{E}{c_E} G'_E(0^+) . \tag{35}$$

It can be shown, however, that the boundary condition (3), the jump condition (8), and the characteristic relations (30) are not sufficient to determine numerically the values of v and ϵ at all points behind the shock front. Consider the characateristic network shown in Fig. 1, in which the heavy line represents the shock front. At points 0, 1, and 2 we wish to know the six quantities v_i, ϵ_i $(i = 0, 1, 2)$. We may apply (3)

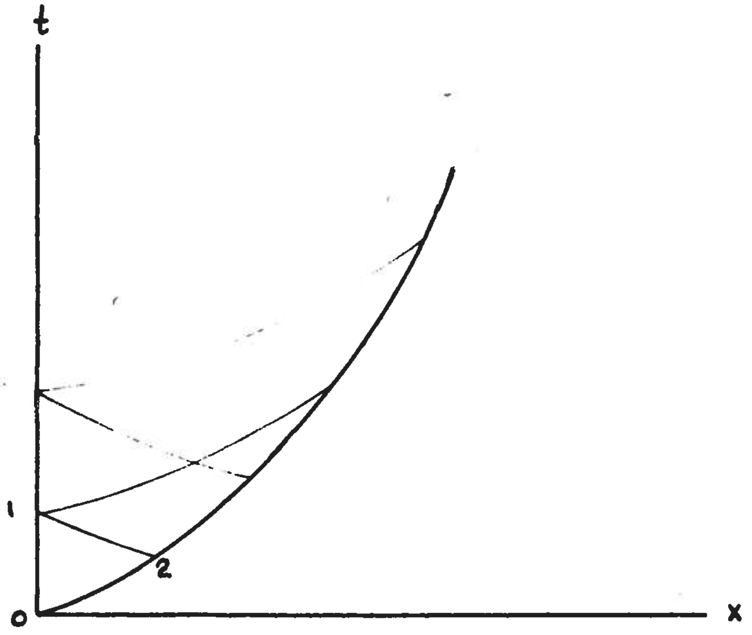

Fig. 1. Characteristic Network for Shock Propagation

at points 0 and 1, and (8) at points 0 and 2, and $(30)_2$ along the characteristic segment joining points 1 and 2 - a total of five equations. We must, consequently, develop an additional equation.

5. The Equation Governing the Shock Amplitude

We shall consider the behavior of E as a function of time. The total time derivative of a function f which is discontinuous across Σ is given by [2, Eqn. 2.4]

$$\frac{d}{dt}[f] = [f_t] + c[f_x] \,. \tag{36}$$

Letting f stand for ϵ and v, respectively, and using (1), (2), and (8), we obtain

$$\frac{1}{\rho_0}[\sigma_x] = -2c_E\frac{d}{dt}[\epsilon] + c_E^2[\epsilon_x] - [\epsilon]\frac{dc_E}{dt}. \tag{37}$$

From (24), (26), and (28) we obtain also

$$\frac{1}{\rho_0}[\sigma_x] = [a^2\epsilon_x] + [b]. \tag{38}$$

Noting that σ, ϵ, v and their derivatives vanish ahead of the wave front, we combine (37) and (38) to obtain the desired equation

$$\left(2 + \frac{E}{c_E}\frac{dc_E}{dE}\right)\frac{dE}{dt} + \left(\frac{a_E^2}{c_E} - c_E\right)\epsilon_x^- + \frac{b_E}{c_E} = 0. \tag{39}$$

With the exception of ϵ_x^-, every quantity appearing in (39) is a function of E: c_E is given by (17), a_E by (32), and b_E by (35). For a material characterized by the expansion (12), (32) and (35) become, respectively,

$$\begin{aligned}\rho_0 a_E^2 &= R^{(1)}(0^+) + 2ER^{(2)}(0^+,0^+) + \cdots + \\ &+ nE^{n-1}R^{(n)}(0^+,\cdots,0^+),\end{aligned} \tag{40}$$

and

$$\begin{aligned}\rho_0 b_E = -\frac{E}{c_E}\Big\{&\frac{d}{ds}R^{(1)}(s) + 2E\frac{\partial}{\partial s}R^{(2)}(s,0^+) + \cdots \\ &+ (n)E^{n-1}\frac{\partial}{\partial s}R^{(n)}(s,0^+,\cdots,0^+)\Big\}\Big|_{s=0^+}.\end{aligned} \tag{41}$$

In a <u>semilinear</u> viscoelastic material, $a_E = c_E = c_0$; hence (39) reduces to

$$2\frac{dE}{dt} + \frac{b_E}{c_o} = 0 \tag{42}$$

a differential equation soluable explicitly in the form

$$t = -2c_o \int_{E_o}^{E} \frac{dE}{b_E}\,, \tag{43}$$

where

$$-c_o E_o = q(0^+)\,. \tag{44}$$

We refer for further information to our study of waves in semilinear materials [8].

In an <u>elastic</u> material b_E vanishes identically. If a bar of such a material is subjected to constant-velocity impact, the governing equations are satisfied by a constant state: the shock is propagated at constant speed and with constant strength.

In the general case, in order to apply (39) to the triangle 0-1-2, we need an estimate for $\bar{\epsilon}_x$ along the segment 0-2. For this purpose it seems reasonable to assume that ϵ varies linearly in x and t throughout the triangle, so that an average value for ϵ_x may be expressed in terms of the ϵ_i $(i=0,1,2)$.

It should be further noted that, because the characteristics are curved, a numerical solution by the method of characteristics would be extremely complicated. It appears simpler to perform a finite-difference solution of the partial differential equations (1) and (27). However, the approach by characteristics must be used near the origin in order to permit a start. Thus, if the data obtained for points 0, 1, and 2 are interpolated on the line $t = t_2$, then this line may be used as a base line for forward integration in the t direction.

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