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Games with filters I

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This paper has two parts. The first is concerned with a variant of a family of games introduced by Holy and Schlicht, that we call *Welch games*. Player II having a winning strategy in the Welch game of length ω on κ is equivalent to weak compactness. Winning the game of length 2^κ is equivalent to κ being measurable. We show that for games of intermediate length γ , II winning implies the existence of precipitous ideals with γ -closed, γ -dense trees.

The second part shows the first is not vacuous. For each γ between ω and κ^+ , it gives a model where II wins the games of length γ , but not γ^+ . The technique also gives models where for all $\omega_1 < \gamma \leq \kappa$ there are κ -complete, normal, κ^+ -distributive ideals having dense sets that are γ -closed, but not γ^+ -closed.

Keywords: Games played with filters; weakly compact cardinals; measurable cardinals; precipitous ideals.

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1. Introduction

Motivated by ideas of generalizing properties of the first inaccessible cardinal ω , Tarski [22] came up with the idea of considering uncountable cardinals κ such that $\mathcal{L}_{\kappa\kappa}$ -compactness holds for languages of size κ . This became the definition of a *weakly compact* cardinal. Hanf [12], showed that weakly compact cardinals

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are Mahlo. Work of Keisler [16] and Keisler and Tarski [17] showed the following theorem:

Theorem. *Let κ be an uncountable inaccessible cardinal. Then the following are equivalent to weak compactness:*

- (1) *Whenever $R \subseteq V_\kappa$ there is a transitive set X and $S \subseteq X$ such that*

$$\langle V_\kappa, \in, R \rangle \prec \langle X, \in, S \rangle.$$

- (2) *If $\mathcal{B} \subseteq \mathcal{P}(\kappa)$ is a κ -complete Boolean subalgebra with $|\mathcal{B}| = \kappa$ and F is a κ -complete filter on \mathcal{B} , then F can be extended to a κ -complete ultrafilter on \mathcal{B} .*

Items (1) and (2) are clearly implied by their analogues for measurable cardinals:

- (1') There is an elementary embedding of V into a transitive class M that has critical point κ .
 (2') There is a non-atomic, κ -complete ultrafilter on $\mathcal{P}(\kappa)$.

Holy–Schlicht Games. This paper concerns several of a genre of games originating in the paper [13] of Holy and Schlicht, which were modified and further explored by Nielsen and Welch [19]. The following small variant of the Holy–Schlicht–Nielsen–Welch games was suggested to us by Welch.

Players I and II alternate moves:

$$\begin{array}{c|c|c|c|c|c|c} \text{I} & \mathcal{A}_0 & \mathcal{A}_1 & \dots & \mathcal{A}_\alpha & \mathcal{A}_{\alpha+1} & \dots \\ \hline \text{II} & U_0 & U_1 & \dots & U_\alpha & U_{\alpha+1} & \dots \end{array}$$

The game proceeds for some length $\ell \leq \gamma$ determined by the play. The sequence $\langle \mathcal{A}_\delta : 0 \leq \delta < \ell \leq \gamma \rangle$ is an increasing sequence of κ -complete subalgebras of $\mathcal{P}(\kappa)$ of cardinality κ and $\langle U_\delta : 0 \leq \delta < \ell \rangle$ is sequence of uniform κ -complete filters, each U_α is a uniform ultrafilter on \mathcal{A}_α and $\alpha < \alpha'$ implies that $U_\alpha \subseteq U_{\alpha'}$. We assume without loss of generality that \mathcal{A}_0 contains all singletons. Player I goes first at limit stages. The game continues until either Player II can't play or the play has length γ . If Player II can't play, the game ends and ℓ is the length of the sequence already played.^a We denote this game by \mathcal{G}_γ^W .

The winning condition. Player II wins if the game continues through all stages below γ .

There are two extreme cases: $\gamma \leq \omega$ and $\gamma = 2^\kappa$. Using item (2) of the characterization of weakly compact cardinals, one sees easily that if κ is weakly compact then II wins the game of length ω .

The situation with the converse is slightly complicated. If κ is inaccessible and Player II can win the Welch game of length 2, then κ is weakly compact. If κ is

^aWe could omit “uniform” and simply require \mathcal{A}_0 to include the $\text{co-}<\kappa$ subsets of κ and U_0 to extend the $\text{co-}<\kappa$ -filter. As noted in Sec. 2, if κ is inaccessible, then Player I always has a legal play in the Welch game.

not inaccessible, then either Player I does not have an opening move, or Player II loses. This follows from work in [1], though stated in a different way there. For completeness, it is proved in Sec. 2.

At the other extreme if κ is measurable one can fix in advance a κ -complete uniform ultrafilter \mathcal{U} on $\mathcal{P}(\kappa)$ and at stage α play $U_\alpha = U \cap \mathcal{A}_\alpha$. The converse is also immediate: if the second player has a winning strategy in the game of length 2^κ , and the first player plays a sequence of algebras with $\bigcup_{\alpha < 2^\kappa} \mathcal{A}_\alpha = \mathcal{P}(\kappa)$, then the union of the U_α 's in Player II's responses gives a κ -complete ultrafilter on κ .

In [19], Nielsen–Welch proved that Player II having a winning strategy in the game of length $\omega+1$ implies that there is an inner model with a measurable cardinal. This motivated the following:

Welch's Question. Welch asked whether Player II having a winning strategy in the game of length ω_1 implies the existence of a non-principal precipitous ideal.

For the readers' convenience we recall the definition of *precipitousness*. An ideal \mathcal{I} on a set X is *precipitous* if for all generic $G \subseteq \mathcal{P}(X)/\mathcal{I}$ the generic ultrapower V^X/G is well-founded. See [14] or [8] for details of the definition.

The main result of this paper is as follows:

Theorem. *If κ is inaccessible, $2^\kappa = \kappa^+$ and Player II can win the game of length $\omega + 1$ then there is a uniform normal precipitous ideal on κ .*

In Sec. 2, we show that even the Welch game of length one is not meaningful if κ is not inaccessible.

We note here that for γ a limit, there is an intermediate property between “Player II wins the game of length γ ” and “Player II wins the game of length $\gamma + 1$ ”. It is the game \mathcal{G}_γ^* of length γ that is played the same way as the original Welch game \mathcal{G}_γ^W , but with a different winning condition: For Player II to win, there must be an extension of $\bigcup_{\alpha < \gamma} U_\alpha$ to a uniform κ -complete ultrafilter on the κ -complete subalgebra of $\mathcal{P}(\kappa)$ generated by $\bigcup_{\alpha < \gamma} \mathcal{A}_\alpha$.

Precipitous ideals. We are fortunate that Welch's question leads to a number of more refined results about the structure of the quotients of the Boolean algebras $\mathcal{P}(\kappa)/\mathcal{I}$. We begin by discussing a strong hypothesis:

A κ -complete, uniform ideal \mathcal{I} on κ such that the Boolean algebra $\mathcal{P}(\kappa)/\mathcal{I}$ has the κ^+ -chain condition is called a *saturated ideal*.

It follows from results of Solovay in [20] that if \mathcal{I} is a saturated ideal on κ then \mathcal{I} is precipitous. Thus to show that a property P implies that there is a non-principal precipitous ideal on κ it suffices to consider only the case where κ does not carry a saturated ideal.

The most direct answer to Welch's question is given by the following theorem:

Theorem 1.1. *Assume that $2^\kappa = \kappa^+$ and that κ does not carry a saturated ideal. If Player II has a winning strategy in the game \mathcal{G}_ω^* , then there is a uniform normal precipitous ideal on κ .*

We recall that a normal uniform ideal on κ is κ -complete. As a corollary we obtain the following corollary:

Corollary. *Under the assumptions of Theorem 1.1, if Player II has a winning strategy in either \mathcal{G}_ω^* or \mathcal{G}_γ^W for any $\gamma \geq \omega + 1$, then there is a uniform normal precipitous ideal on κ .*

While this is the result with the simplest statement, its proof gives a lot of structural information about the quotient algebra $\mathcal{P}(\kappa)/\mathcal{I}$. We prove the following theorem in Sec. 5:

Theorem 1.2. *Assume that $2^\kappa = \kappa^+$ and that κ does not carry a saturated ideal. Let $\gamma > \omega$ be a regular cardinal less than κ^+ . If Player II has a winning strategy in the Welch game of length γ , then there is a uniform normal ideal \mathcal{I} on κ and a set $D \subseteq \mathcal{I}^+$ such that*

- (1) $(D, \subseteq_{\mathcal{I}})$ is a downward growing tree of height γ ,
- (2) D is closed under $\subseteq_{\mathcal{I}}$ -decreasing sequences of length less than γ ,
- (3) D is dense in $\mathcal{P}(\kappa)/\mathcal{I}$.

In fact, it is possible to construct such a dense set D where (1) and (2) above hold with the almost containment \subseteq^ in place of $\subseteq_{\mathcal{I}}$.*

Definition 1.3. Let \mathcal{I} be a κ -complete ideal on $\mathcal{P}(\kappa)$ and $\gamma > \omega$ be a regular cardinal. Then \mathcal{I} is γ -densely treed if there is a set $D \subseteq \mathcal{I}^+$ such that

- (1) $(D, \subseteq_{\mathcal{I}})$ is a downward growing tree,
- (2) D is closed under $\subseteq_{\mathcal{I}}$ -decreasing sequences of length less than γ ,
- (3) D is dense in $\mathcal{P}(\kappa)/\mathcal{I}$.

Note that this is weaker than the conclusions of Theorem 1.2.

We will abuse notation slightly and say “ D is dense in \mathcal{I}^+ ” to mean that D is a dense subset of $\mathcal{P}(\kappa)/\mathcal{I}$.

We will say that an ideal \mathcal{I} is (κ, ∞) -distributive if $\mathcal{P}(\kappa)/\mathcal{I}$ is a (κ, ∞) -distributive Boolean Algebra.

In this language, Theorem 1.2 can be restated as saying that Player II having a winning strategy in the Welch game implies the existence of a normal γ -densely treed ideal and the tree has height γ .

We have a partial converse to Theorem 1.2:

Theorem 1.4. *Let $\gamma \leq \kappa$ be uncountable regular cardinals and \mathcal{J} be a uniform κ -complete ideal over κ which is (κ^+, ∞) -distributive and has a dense γ -closed subset. Then Player II has a winning strategy in the game \mathcal{G}_γ^W which is constructed in a natural way from the ideal \mathcal{J} , and which we denote by $\mathcal{S}_\gamma(\mathcal{J})$.*

A proof of Theorem 1.4 is at the end of Sec. 5. We note that if κ carries a uniform, κ -complete ideal which is (κ^+, ∞) -distributive, then κ must be inaccessible.

How does precipitousness arise? In [11], Galvin *et al.* introduced the following game of length ω . Fix an ideal \mathcal{I} . Players I and II alternate playing

$$\begin{array}{c|c|c|c|c|c|c} \text{I} & A_0 & A_1 & \dots & A_n & A_{n+1} & \dots \\ \hline \text{II} & B_0 & B_1 & \dots & B_n & B_{n+1} & \dots \end{array}$$

With $A_n \supseteq B_n \supseteq A_{n+1}$ and each $A_n, B_n \in \mathcal{I}^+$. Player II wins the game if $\bigcap_n B_n \neq \emptyset$. We will call this game the *Ideal Game* for \mathcal{I} . They proved the following theorem.

Theorem ([11]). *Let \mathcal{I} be a countably complete ideal on a set X . Then \mathcal{I} is precipitous if and only if Player I does not have a winning strategy in the ideal game for \mathcal{I} .*

In the proof of Theorem 1.1, we construct an ideal \mathcal{I} and show that Player II has a winning strategy in the ideal game for \mathcal{I} . In Theorem 1.2, the existence of a dense set D closed under descending ω -sequences immediately gives that Player II has a winning strategy in the ideal game. (See [7] for some information about the relationship between games and dense closed subsets of Boolean Algebras.) The proofs of both Theorems 1.1 and 1.2 are in Sec. 5.

Is this vacuous? So far we haven't addressed the question of the existence of strategies in the Welch games if κ is *not* measurable. We answer this with the following theorem. We use the terminology regarding closure and distributivity properties of forcing partial orderings from [4].

Theorem 1.5. *Assume κ is measurable and $\mathbf{V} = \mathbf{L}[E]$ is a fine structural extender model. Then there is a generic extension in which κ is inaccessible, carries no saturated ideals (in particular, κ is non-measurable) and for all regular γ with $\omega < \gamma \leq \kappa$ there is a uniform, normal γ -densely treed ideal \mathcal{J}_γ on κ that is (κ^+, ∞) -distributive. The Boolean algebra $\mathcal{P}(\kappa)/\mathcal{J}_\gamma$ does not contain a dense γ^+ -closed subset.*

Corollary 1.6. *It follows from Theorems 1.4 and 1.5 that in the forcing extension of Theorem 1.5*

- (a) *Player II has a winning strategy $\mathcal{S}_\gamma \stackrel{\text{def}}{=} \mathcal{S}(\mathcal{J}_\gamma)$ in \mathcal{G}_γ^W .*
- (b) *There is an ideal \mathcal{I}_γ as in Theorem 1.2.*

It will follow from the proof of Theorem 1.5 that the winning strategies \mathcal{S}_γ in (a) are incompatible with winning strategies $\mathcal{S}_{\gamma'}$ for Player II in $\mathcal{G}_{\gamma'}^W$ for $\gamma' \neq \gamma$ in the following sense: If $\gamma, \gamma' \leq \kappa$ are regular and $\gamma \neq \gamma'$ then it is possible for Player I to play the first round \mathcal{A}_0 in such a way that the responses of \mathcal{S}_γ and $\mathcal{S}_{\gamma'}$ to $\langle \mathcal{A}_0 \rangle$ are distinct.

We give a proof of Theorem 1.5 in Sec. 6. The existence of winning strategies \mathcal{S}_γ as in (a) for Player II in \mathcal{G}_γ^W is a direct consequence of Theorem 1.4. A proof of the incompatibility of strategies \mathcal{S}_γ , as formulated at the end of Corollary 1.6, is at the end of Sec. 6.

Strengthenings of Theorem 1.5. We have two variants of Theorem 1.5 that are proved in Part II of this paper. The first deals with a single regular uncountable $\gamma < \kappa$, and shows that it is consistent that γ is the only cardinal such that there is a normal γ -densely treed ideal on κ . The second shows that it is consistent that for all such γ there is a normal γ -densely treed ideal \mathcal{J}_γ on κ but that they are all incompatible under inclusion.

Similar statements about the relevant strategies in the Welch games are also included. Explicitly

Theorem 1.7. *Assume κ is a measurable cardinal, $\gamma < \kappa$ is regular uncountable and $\mathbf{V} = \mathbf{L}[E]$ is a fine structural extender model. Then there is a generic extension in which κ is inaccessible, carries no saturated ideals (in particular, κ is non-measurable) and there is a uniform, normal γ -densely treed ideal \mathcal{J}_γ on κ that is (κ^+, ∞) -distributive. Moreover, in the generic extension:*

- (a*) *There does not exist a uniform ideal \mathcal{J}' over κ such that $\mathcal{P}(\kappa)/\mathcal{J}'$ has a dense γ' -closed subset for any $\gamma' > \gamma$.*
- (b*) *Player II does not have any winning strategy in $\mathcal{G}_{\gamma'}^W$ where $\gamma' > \gamma$.*

In particular, it is a consequence of (a) that*

- (c*) *For all regular $\gamma' > \gamma$ there is no uniform normal γ' -densely treed ideal on κ .*

Another modification of the proof of Theorem 1.5 which is based on Theorem 1.9 below yields the following variant of Theorem 1.5.

Theorem 1.8. *Assume κ is a measurable cardinal, and $\mathbf{V} = \mathbf{L}[E]$ is a fine structural extender model. Then there is a generic extension in which κ is inaccessible, carries no saturated ideals (in particular, κ is non-measurable) and for all regular γ with $\omega < \gamma \leq \kappa$ there is a uniform, normal γ -densely treed ideal \mathcal{J}_γ that is (κ^+, ∞) -distributive. The relationship between the ideals and strategies for different γ 's is as follows:*

- (a*) *There does not exist a uniform normal ideal $\mathcal{J}' \subseteq \mathcal{J}_\gamma$ over κ such that $\mathcal{P}(\kappa)/\mathcal{J}'$ has a dense γ' -closed subset for any $\gamma' > \gamma$.*
- (b*) *The strategy $\mathcal{S}_\gamma \stackrel{\text{def}}{=} \mathcal{S}(\mathcal{J}_\gamma)$ is not included in any winning strategy for Player II in $\mathcal{G}_{\gamma'}^W$ where $\gamma' > \gamma$.*
- (c*) *Letting \mathcal{I}_γ be the ideal arising from the strategy \mathcal{S}_γ , there does not exist an ideal $\mathcal{I} \subseteq \mathcal{I}_\gamma$ which is γ' -densely treed as witnessed by a tree $D \subseteq \mathcal{I}^+$ of height γ' , for any $\gamma' > \gamma$.^b*

^bThere are two general techniques used in this paper for building ideals. One is the conventional method of starting with a large cardinal embedding and extending it generically. We use the notation \mathcal{J}_γ for these. The second is the new technique of *hopeless ideals*, built in Theorems 1.1 and 1.2 from the strategies \mathcal{S}_γ . These will be denoted by \mathcal{I}_γ or very similar notation.

In other words, the ideals \mathcal{I} in (c^*) in Theorem 1.8 are like ideals \mathcal{I} in Theorem 1.2, with γ' in place of γ .

The models constructed in Theorems 1.7 and 1.8 require more sophisticated techniques than those used in the proof of Theorem 1.5. They involve the relationship between the fine structure in the base model and the forcing extension.

The most substantial difference is that the model in Theorem 1.5 is built by iteratively shooting clubs through the complements of non-reflecting stationary sets which have been added generically, however the proofs of Theorems 1.7 and 1.8 shoot club sets through non-reflecting stationary sets built from canonical square sequences constructed in the fine structural extender model. Unlike the partial orderings used in the construction of a model in the proof of Theorem 1.5, those partial orderings will have low closure properties, but high degree of distributivity. It is the proof of distributivity of the iterations of club shooting partial orderings which uses the significant fine structural properties of the extender model. Here is the result allowing the desired iteration.

Theorem 1.9. *Assume $\mathbf{V} = \mathbf{L}[E]$ is a fine structural extender model and κ is a measurable cardinal as witnessed by an extender on the extender sequence E . Assume further that*

- (i) $(c_\xi \mid \xi < \alpha^+)$ is a canonical square sequence,^c
- (ii) $S_\alpha \subseteq \alpha^+ \cap \text{cof}(< \alpha)$,
- (iii) $S_\alpha \cap c_\xi = \emptyset$ for all ξ

whenever α is a cardinal.

Let \mathbb{P}^δ be the Easton support iteration of length κ of club shooting partial orderings with initial segments where each active stage α is an inaccessible $\geq \delta$ and the club subset of α^+ generically added at stage α is disjoint from S_α . Then there is an ordinal $\varrho < \kappa$ such that for every inaccessible δ such that $\varrho < \delta < \kappa$ the following holds.

- (a) \mathbb{P}^δ is δ^+ -distributive.
- (b) If G is generic for \mathbb{P}^ϱ over \mathbf{V} and $j : \mathbf{V} \rightarrow M$ is an elementary embedding in some generic extension \mathbf{V}' of \mathbf{V} which preserves κ^+ then $j(\mathbb{P}^\varrho)/G$ is κ^+ -distributive in \mathbf{V}' .

Although Theorem 1.9 is formulated for Easton support iterations with inaccessible active stages, variations which involve iterations with supports which are not necessarily Easton, but still sufficiently large, and with active stages that are not necessarily inaccessible can also be proved.

^cBy a canonical square sequence we mean a square sequence obtained by a slight variation of Jensen's fine structural construction, generalized to extender models. This is made precise in Part II of this paper.

As the proof of Theorem 1.9 is of considerable length and (we believe) has broader applicability and is of interest on its own, we will postpone the proof to Part II of this paper.

Basic definitions and notation. We now present terminology and notation we use throughout the paper. We will use the phrases “ideal on κ ” and “ideal on $\mathcal{P}(\kappa)$ ” interchangeably. Perhaps ideals should be viewed as subsets of Boolean algebras, but the former phrase is the more common colloquialism.

Fix a regular cardinal κ and \mathcal{I} a κ -complete ideal on κ . We say that $A \subseteq_{\mathcal{I}} B$ if $A \setminus B \in \mathcal{I}$, and $\supseteq_{\mathcal{I}}$ is the converse relation. The notations \subseteq^* , \supseteq^* are these notions when \mathcal{I} is the ideal of bounded subsets of κ . The notation $A \not\subseteq_{\mathcal{I}} B$ abbreviates the conjunction of $A \subseteq_{\mathcal{I}} B$ and $A \Delta B \notin \mathcal{I}$, where Δ means symmetric difference.

The ideal \mathcal{I} induces an equivalence relation on $\mathcal{P}(\kappa)$ by $[A] = [B]$ if and only if $A \Delta B \in \mathcal{I}$. The notion $\subseteq_{\mathcal{I}}$ induces a partial ordering on $\mathcal{P}(\kappa)/\mathcal{I}$, we will sometimes call this $\leq_{\mathcal{I}}$ and refer to the set of \mathcal{I} equivalence classes of $\mathcal{P}(\kappa)$ that don't contain the empty set as \mathcal{I}^+ . We will force with $(\mathcal{P}(\kappa)/\mathcal{I}, \subseteq_{\mathcal{I}}^*)$ viewed either as a Boolean algebra, or removing the equivalence class of the empty set as a partial ordering. These are equivalent forcing notions. Occasionally, we will abuse language by saying “forcing with \mathcal{I}^+ ” when we mean this forcing.

Definition 1.10. κ -complete sub-Boolean algebras of $\mathcal{P}(\kappa)$ that have cardinality κ are called κ -algebras.

If σ and τ are sequences we will use $\sigma \frown \tau$ to mean the concatenation of σ and τ . We will abuse this slightly when τ has length one. For example, given $\sigma = \langle \alpha_i : i < \beta \rangle$ and δ we will write $\langle \alpha_i : i < \beta \rangle \frown \delta$ for the sequence of length $\beta + 1$ whose first β elements coincide with σ and whose last element is δ .

Usually our trees grow downwards, with longer branches extending shorter branches. A tree \mathcal{T} is γ -closed if when b is a branch through \mathcal{T} whose length has cofinality less than γ there is a node $\sigma \in \mathcal{T}$ such that σ is below each element of b . Occasionally, we will say $<\gamma$ -closed to mean γ -closed.

2. Weak Compactness

In this section, we clarify the relationship between these games and weak compactness and discuss the role of inaccessibility in the work of Keisler and Tarski. It has been pointed out to us that these results appear in work of Abramson *et al.* [1] stated slightly differently and with different proofs. We include them here for completeness and because these techniques are relevant to the topics in this paper.

If κ is inaccessible and \mathcal{A} is a κ -algebra and $B \subseteq [\kappa]^\kappa$ then $\mathcal{A} \cup B$ generates a κ -complete subalgebra of $\mathcal{P}(\kappa)$ that has cardinality κ (i.e. another κ -algebra). The situation where κ is not inaccessible is quite different.

Proposition 2.1. *Suppose that κ is an infinite cardinal and either*

- *a singular strong limit cardinal^d or*
- *for some $\gamma < \kappa$, $2^\gamma > \kappa$ but for all $\gamma' < \gamma$, $2^{\gamma'} < \kappa$.*

Then there is no Boolean subalgebra $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{A}| = \kappa$, \mathcal{A} is κ -complete.

Proof. In the first case, since κ is singular, if \mathcal{A} is κ -complete, it is κ^+ -complete. For $\delta < \kappa$, let $a_\delta = \bigcap \{A \in \mathcal{A} : \delta \in A\}$. Then a_δ is an atom of \mathcal{A} and each non-empty $A \in \mathcal{A}$ contains some a_δ . Moreover distinct a_δ 's are disjoint. Thus, the $\{a_\delta : \delta \in \kappa\}$ generate \mathcal{A} as a κ -algebra. If there are κ many distinct a_δ then $|\mathcal{A}| = 2^\kappa$. Otherwise, since κ is a strong limit, $|\mathcal{A}| < \kappa$.

Assume now that $\gamma < \kappa$, $2^\gamma > \kappa$ and for all $\gamma' < \gamma$, $2^{\gamma'} < \kappa$. Since $|\mathcal{A}| = \kappa$, \mathcal{A} must have fewer than γ atoms. If $\langle a_\delta : \delta < \gamma' \rangle$ is the collection of these atoms, the κ -algebra \mathcal{B} generated by the atoms of \mathcal{A} has cardinality at most $2^{\gamma'}$. Let $B = \bigcup_{\delta < \gamma'} a_\delta$ and $C = \kappa \setminus B$. Since $2^{\gamma'} < \kappa$ and \mathcal{A} has cardinality κ , there is an $a \in \mathcal{A}$ that does not belong to \mathcal{B} . Since a is not in \mathcal{B} the set $a' = a \setminus \bigcup_{\delta < \gamma'} a_\delta \neq \emptyset$. Hence, C is non-empty. Replacing \mathcal{A} with $\{A \setminus B : A \in \mathcal{A}\}$ we get an atomless, κ -complete algebra on the set C .

Since no element of \mathcal{A} is an atom we can write each $A \in \mathcal{A}$ as a disjoint union of non-empty elements A_0, A_1 of \mathcal{A} . Build a binary splitting tree \mathcal{T} of elements of \mathcal{A} of height γ by induction on $\delta < \gamma$ as follows:^e

- $(\mathcal{T})_0$ is the \subseteq -maximal element C of \mathcal{A} .
- Suppose $(\mathcal{T})_\delta$ is built. For each $A \in \mathcal{T}_\delta$, write A as the disjoint union $A_0 \cup A_1$ such that $A_i \in \mathcal{A}$, and $A_i \neq \emptyset$ and let $(\mathcal{T})_{\delta+1} = \{A_0, A_1 : A \in (\mathcal{T})_\delta\}$.
- Suppose that δ is a limit ordinal. Let $(\mathcal{T})_\delta = \{\bigcap b : b \text{ is a branch through } (\mathcal{T})_{<\delta} \text{ and } \bigcap b \neq \emptyset\}$.

Note that for all $\delta < \gamma$, each element $c \in C$ determines a unique path through $(\mathcal{T})_\delta$ of length δ . Hence $\bigcup (\mathcal{T})_\delta = C$.

Fix a $c \in C$ and let $b = \langle A_\delta : \delta < \gamma \rangle$ be the branch through \mathcal{T} determined by c . For $\delta < \gamma$, the tree splits $A_\delta = A_{\delta+1} \cup A'_{\delta+1}$ with $A_{\delta+1}, A'_{\delta+1} \in (\mathcal{T})_{\delta+1}$. The sets $\langle A'_{\delta+1} : \delta < \gamma \rangle$ each belong to \mathcal{A} and form a collection of disjoint subsets of C of size γ . By taking unions of these sets we see that $|\mathcal{A}| \geq 2^\gamma > \kappa$. \square

In contrast to Proposition 2.1, we have

Proposition 2.2. *Let κ be infinite and $2^\gamma = \kappa$. Then $\mathcal{P}(\gamma) \subseteq \mathcal{P}(\kappa)$ and $\mathcal{P}(\gamma)$ is κ -complete.*

Proof. Immediate. \square

^dWe would like to thank James Cummings for giving significant help in understanding this case.

^eWe use the notation $(\mathcal{T})_\alpha$ for level α of \mathcal{T} .

The upshot of Propositions 2.1 and 2.2 is the following theorem and corollary, which show that the Welch game is only interesting when κ is inaccessible.

Theorem 2.3. *Suppose that κ is an accessible infinite cardinal. Then either*

- (1) *There is no κ -algebra $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{A}| = \kappa$ or*
- (2) *There is a κ -algebra $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{A}| = \kappa$ but every κ -complete ultrafilter U on \mathcal{A} is principal*

Corollary 2.4. *Consider the Welch game of length 1. Suppose that there is a κ -algebra \mathcal{A}_0 that is a legal move for Player I and that Player II has a winning strategy in \mathcal{G}_1^W . Then κ is inaccessible.*

If κ is inaccessible we have the following result, which can also be deduced directly from the results of Abramson *et al.* [1]:

Theorem 2.5. *Suppose that κ is inaccessible and Player II wins the Welch game of length 1. Then κ is weakly compact.*

Proof. To show κ is weakly compact, it suffices to show it has the tree property. Let \mathcal{T} be a κ -tree. For $\alpha < \kappa$, let A_α be the set of $\beta < \kappa$ such that $\alpha \leq_{\mathcal{T}} \beta$. Let \mathcal{A} be the κ -algebra generated by $\{A_\alpha : \alpha < \kappa\}$ and \mathcal{U} be a uniform κ -complete ultrafilter on \mathcal{A} .

For each $\gamma < \kappa$, $\kappa = \bigcup_{\alpha \in (\mathcal{T})_\gamma} A_\alpha \cup R$ where $|R| < \kappa$. It follows that for each γ there is an $\alpha \in (\mathcal{T})_\gamma$ such that $A_\alpha \in \mathcal{U}$. But then $\{\alpha : A_\alpha \in \mathcal{U}\}$ is a κ -branch through \mathcal{T} . \square

3. Hopeless Ideals

In this section, we define the notion of a *hopeless ideal* in a general context, and toward the end of the section we will narrow our focus to the context of games. Fix an inaccessible cardinal κ . Assume F is a function with domain R such that for every $r \in R$ the value $F(r)$ is a sequence of length ξ_r of the form

$$F(r) = \langle \mathcal{A}_i^r, U_i^r \mid i < \xi_r \rangle \tag{1}$$

where for every $i < \xi_r$,

- (i) $\mathcal{A}_i^r \subseteq \mathcal{P}(\kappa)$ and
- (ii) U_i^r is a κ -complete ultrafilter on the κ -algebra of subsets of κ generated by $\bigcup_{j \leq i} \mathcal{A}_j^r$.
- (iii) For all $r \in R$, the sequence $\langle U_i^r \mid i < \xi_r \rangle$ is monotonic with respect to the inclusion.
- (iv) (Density) For every $r \in R$, $j < \xi_r$ and $\mathcal{B} \in [\mathcal{P}(\kappa)]^{\leq \kappa}$ there is $s \in R$ such that $F(r) \upharpoonright j = F(s) \upharpoonright j$ and $\mathcal{B} \subseteq \mathcal{A}_j^s$.

We will call functions F with the properties (i)–(iv) *assignments*.

One can also formulate a variant with normal ultrafilters U_i^r . Denote the maximo-lexicographical ordering of $\kappa \times \kappa$ by $<_{\text{mlex}}$. Let $h : (\kappa \times \kappa, <_{\text{mlex}}) \rightarrow (\kappa, \in)$ be the natural isomorphism. For a set $A \subseteq \kappa$, let $A_i = (h^{-1}[A])_i$ be the i th section of $h^{-1}(A)$. The sequence $\langle A_i \mid i < \kappa \rangle$ is *associated to* A . We will say that a κ -algebra \mathcal{A} of subsets of κ is *normal* if for all $A \in \mathcal{A}$, each A_i belongs to \mathcal{A} and the diagonal intersection $\Delta_{i < \kappa} A_i$ also belongs to \mathcal{A} . We will say that a sequence $\langle A_i \mid i < \kappa \rangle$ belongs to \mathcal{A} if it is associated to an element of \mathcal{A} . Finally we say that an ultrafilter U on a normal κ -algebra \mathcal{A} is *normal* if and only if for every sequence $\langle A_i \mid i < \kappa \rangle \in \mathcal{A}$,

$$(\forall i < \kappa)(A_i \in U) \Rightarrow \Delta_{i < \kappa} A_i \in U. \quad (2)$$

A variant of this definition is an *assignment with normal ultrafilters* where we require, instead of (ii) above, that

(ii)' U_i^r is a κ -complete normal ultrafilter on the normal κ -algebra of subsets of κ generated by $\bigcup_{j \leq i} \mathcal{A}_j^r$.

If (ii)' is satisfied we say that F is *normal*. Note that there is no need to modify clause (iv), as normal κ -algebras are able to decode κ -sequences of subsets of κ from other subsets of κ via the pairing function h introduced above. However, instead of families $\mathcal{B} \in [\mathcal{P}(\kappa)]^{\leq \kappa}$, it is convenient in (iv) to consider sets $B \in \mathcal{P}(\kappa)$ that code \mathcal{B} .

Definition 3.1. Given an *assignment* F , we define the ideal $\mathcal{I}(F)$ as follows.

$$\mathcal{I}(F) = \text{the set of all } A \subseteq \kappa \text{ such that } A \notin U_i^r \text{ for any } i < \xi_r \text{ and any } r \in R. \quad (3)$$

The ideal $\mathcal{I}(F)$ is called the *hopeless ideal on* $\mathcal{P}(\kappa)$ *induced by* F .

Although in the above definition we say we are defining an ideal, an argument is needed to see that $\mathcal{I}(F)$ is indeed an ideal. It follows immediately that $\emptyset \in \mathcal{I}(F)$ and $\mathcal{I}(F)$ is downward closed under inclusion. The rest is given by the following proposition.

Proposition 3.2. *Given an assignment F , the ideal $\mathcal{I}(F)$ is κ -complete. If all ultrafilters U_i^r are uniform then $\mathcal{I}(F)$ is uniform. If additionally F is normal then $\mathcal{I}(F)$ is normal. If F' is an assignment on $R' \supseteq R$ and $F' \upharpoonright R = F$, then $\mathcal{I}(F') \subseteq \mathcal{I}(F)$.*

Proof. We first verify κ -completeness of $\mathcal{I}(F)$. We noted above that $\emptyset \in \mathcal{I}(F)$ and $\mathcal{I}(F)$ is downward closed under inclusion; hence it suffices to check that $\mathcal{I}(F)$ is closed under unions of cardinality $< \kappa$. If $\langle A_\eta \mid \eta < \xi \rangle$ is such that $\xi < \kappa$ and $A = \bigcup_{\eta < \xi} A_\eta \notin \mathcal{I}(F)$, then there is some $r \in R$ and some $i < \xi_r$ such that $A \in U_i^r$. By the density condition, there is some $s \in R$ such that $\mathcal{A}_j^s = \mathcal{A}_j^r$ and $U_j^s = U_j^r$ for all $j \leq i$, and $\{A_\eta \mid \eta < \xi\} \subseteq \mathcal{A}_{i+1}^s$. In particular, $A \in U_i^s \subseteq U_{i+1}^s$ and all sets A_η , $\eta \leq \xi$ are in the κ -algebra generated by $\bigcup_{j \leq i+1} \mathcal{A}_j$. By κ -completeness of U_{i+1}^s then $A_\eta \in U_{i+1}^s$ for some $\eta < \xi$, hence $A_\eta \notin \mathcal{I}(F)$.

The proof of normality of $\mathcal{I}(F)$ for normal F is the same, with $\nabla_{\eta < \kappa} A_\eta$ in place of $\bigcup_{\eta < \xi} A_\eta$. The conclusion on uniformity of $\mathcal{I}(F)$ follows by a straightforward argument from the definition of $\mathcal{I}(F)$.

Finally, if R', F' are as in the statement of the proposition, then any $A \in \mathcal{I}(F')$ trivially avoids all ultrafilters U_i^r where $r \in R$ and $i < \xi_r$, so $A \in \mathcal{I}(F)$. \square

Now assume \mathcal{G} is a two player game of perfect information, and \mathcal{S} is a strategy for Player II in \mathcal{G} . Denote the set of all runs of \mathcal{G} according to \mathcal{S} by $R_{\mathcal{S}}$ (by a run we mean a complete play). Assume every $r \in R_{\mathcal{S}}$ is associated with a sequence of fragments $\mathcal{A}_i^r \subseteq \mathcal{P}(\kappa)$ and ultrafilters U_i^r , in a way that makes the function

$$F_{\mathcal{S}} : r \mapsto \langle \mathcal{A}_i^r, U_i^r \mid i < \text{lh}(r) \rangle \quad (4)$$

an assignment/normal assignment with domain $R_{\mathcal{S}}$. Here of course $\xi_r = \text{lh}(r)$ when compared with (1). In all concrete situations we will consider, the rules of the game \mathcal{G} will guarantee that the function $F_{\mathcal{S}}$ is really an assignment. As the strategy \mathcal{S} makes it clear which game is played, we suppress writing \mathcal{G} explicitly in our notation.

Here are some examples. If \mathcal{G} is the Welch game \mathcal{G}_γ^W then $F_{\mathcal{S}}$ is the identity function. In the following section, we introduce games $\mathcal{G}_1^-, \mathcal{G}_1$ and \mathcal{G}_2 . These games are defined relative to a sequence of models $\langle N_\alpha \mid \alpha < \kappa^+ \rangle$ increasing with respect to the inclusion, and Player I plays ordinals $\alpha < \kappa^+$ which refer to these models. In the games \mathcal{G}_1^- and \mathcal{G}_1 Player II plays uniform κ -complete ultrafilters on $\mathcal{P}(\kappa) \cap N_\alpha$; in \mathcal{G}_1 these ultrafilters are required to be normal. Thus, if r is a run in one of these games according to \mathcal{S} , say $r = \langle \alpha_i^r, U_i^r \mid i < \text{lh}(r) \rangle$ then

$$F_{\mathcal{S}}(r) = \langle \mathcal{P}(\kappa) \cap N_{\alpha_i^r}, U_i^r \mid i < \text{lh}(r) \rangle.$$

In the game \mathcal{G}_2 Player II plays sets $Y \subseteq \kappa$ which determine uniform normal κ -complete ultrafilters U on $\mathcal{P}(\kappa) \cap N_\alpha$ defined by $U = \{X \in \mathcal{P}(\kappa) \cap N_\alpha \mid Y \subseteq^* X\}$. Thus, if r is a run in the game \mathcal{G}_2 according to \mathcal{S} , say $r = \langle \alpha_i^r, Y_i^r \mid i < \text{lh}(r) \rangle$ then

$$F_{\mathcal{S}}(r) = \langle \mathcal{P}(\kappa) \cap N_{\alpha_i^r}, \{X \in \mathcal{P}(\kappa) \cap N_{\alpha_i^r} \mid Y_i^r \subseteq^* X\} \mid i < \text{lh}(r) \rangle.$$

If P is a position in \mathcal{G} played according to \mathcal{S} we let

$$R_{\mathcal{S}, P} = \text{the set of all } r \in R_{\mathcal{S}} \text{ extending } P \quad (5)$$

and

$$F_{\mathcal{S}, P} = F_{\mathcal{S}} \upharpoonright R_{\mathcal{S}, P}. \quad (6)$$

We are now ready to define the central object of our interest.

Definition 3.3. Assume \mathcal{G} is a game of perfect information played by two players, \mathcal{S} is a strategy for Player II in \mathcal{G} , and $F_{\mathcal{S}}$ is an assignment with domain $R_{\mathcal{S}}$ as in (4). Consider a position P in \mathcal{G} according to \mathcal{S} . We define

$$\mathcal{I}(\mathcal{S}, P) = \mathcal{I}(F_{\mathcal{S}, P})$$

to be the *hopeless ideal with respect to \mathcal{S} conditioned on P* . Here we suppress mentioning the assignment $F_{\mathcal{S}}$ in the notation, as in all situations we will consider

it will be given by the strategy \mathcal{S} in a natural way. The ideal $\mathcal{I}(\mathcal{S}, \emptyset)$ is called the unconditional hopeless ideal with respect to \mathcal{S} . We will write $\mathcal{I}(\mathcal{S})$ for $\mathcal{I}(\mathcal{S}, \emptyset)$.

When the strategy \mathcal{S} is clear from the context we suppress referring to it, and will talk briefly about the “hopeless ideal conditioned on P ” and the “unconditional hopeless ideal”. By Proposition 3.2, we have the following as an immediate consequence.

Proposition 3.4. *Given a game \mathcal{G} of limit length, a strategy \mathcal{S} for Player II in \mathcal{G} and a position P as in Definition 3.3, the ideal $\mathcal{I}(\mathcal{S}, P)$ is κ -complete. If all ultrafilters U_i^r associated with $F_{\mathcal{S}, P}$ are uniform then $\mathcal{I}(\mathcal{S}, P)$ is uniform. If moreover $F_{\mathcal{S}, P}$ is normal, then $\mathcal{I}(\mathcal{S}, P)$ is normal as well.*

4. Games We Play

In this section, we introduce a sequence of games \mathcal{G}_k closely related to Welch’s game \mathcal{G}_γ^W . The last one will be \mathcal{G}_2 , and we will be able to show that if \mathcal{S} is a winning strategy for II in \mathcal{G}_2 of sufficient length then we can construct a winning strategy \mathcal{S}^* for Player II in \mathcal{G}_2 such that $\mathcal{I}(\mathcal{S}^*)$ is precipitous and more, depending on the length of the game and the payoff set.

To unify the notation, we let \mathcal{G}_0 of length γ be the Welch game \mathcal{G}_γ^W . Thus, a run of the game continues until either Player II cannot play or else until γ rounds are played. The set of all runs of \mathcal{G}_0 of length γ is denoted by R_γ . As usual with these kinds of games, a set $B \subseteq R_\gamma$ is called a payoff set. We say that Player II wins a run R of the game \mathcal{G}_0 of length γ with payoff set B if R has γ rounds and the resulting run is an element of B . We call this game $\mathcal{G}_0(B)$. Thus, if $B = R_\gamma$ then $\mathcal{G}_0(B)$ is just the game \mathcal{G}_0 . With this notation, the game \mathcal{G}_γ^* is just the game $\mathcal{G}_0(Q_\gamma)$ of length γ where

$$\begin{aligned} Q_\gamma = & \text{The set of all runs } \langle \mathcal{A}_i, U_i \mid i < \gamma \rangle \in R_\gamma \text{ such that there is} \\ & \text{a } \kappa\text{-complete ultrafilter on the } \kappa\text{-algebra generated by } \bigcup_{i < \gamma} \mathcal{A}_i \\ & \text{extending all } U_i, i < \gamma. \end{aligned} \tag{7}$$

As already discussed in the introduction, the existence of a winning strategy for Player II in the game $\mathcal{G}_0(Q_\gamma)$ of length γ is a strengthening of the requirement that Player II has a winning strategy in \mathcal{G}_0 of length γ . This strengthening is among the weakest ones which increase the consistency strength in the case $\gamma = \omega$. From the point of view of increasing the consistency strength, the case $\gamma = \omega$ is of primary interest, as follows from (TO1) combined with Corollary 1.6. Here are some trivial observations.

(TO1) $\mathcal{G}_0(Q_\gamma)$ is the same game as \mathcal{G}_0 whenever γ is a successor ordinal, so a winning strategy for Player II in $\mathcal{G}_0(Q_\gamma)$ gives us something new only when γ is a limit.

- (TO2) A winning strategy for Player II in $\mathcal{G}_0(Q_\gamma)$ is a winning strategy for Player II in \mathcal{G}_0 , but the converse may not be true in general.
- (TO3) If \mathcal{S} is a winning strategy for Player II in \mathcal{G}_0 of length $> \gamma$ then the restriction of \mathcal{S} to positions of length $< \gamma$ is a winning strategy for Player II in $\mathcal{G}_0(Q_\gamma)$ of length γ .
- (TO4) Given $\xi < \kappa$ and sequences $\langle \mathcal{A}_i \mid i < \xi \rangle$ and $\langle U_i \mid i < \xi \rangle$ where $\mathcal{A}_i \subseteq \mathcal{P}(\kappa)$ and U_i is a κ -complete ultrafilter on the κ -algebra (respectively, normal κ -algebra) of subsets of κ generated by $\bigcup_{j \leq i} \mathcal{A}_j$ such that $U_i \subseteq U_j$ whenever $i \leq j$, there is at most one κ -complete (respectively, normal) ultrafilter U on the (normal) κ -algebra \mathcal{B} of subsets of κ generated by $\bigcup_{i < \xi} \mathcal{A}_i$ which extends all U_i . Thus, if we changed the rules of \mathcal{G}_0 to require that Player II goes first at limit stages then Player II has a winning strategy in this modified \mathcal{G}_0 if and only if Player II has a winning strategy in the original game \mathcal{G}_0 .
- (TO5) Let \mathcal{S} be a winning strategy for Player II in \mathcal{G}_0 or $\mathcal{G}_0(Q_\gamma)$ and

$$\begin{array}{c} \text{I} \\ \hline \mathcal{A}_0 \mid \mathcal{A}_1 \mid \dots \mid \mathcal{A}_\alpha \mid \mathcal{A}_{\alpha+1} \mid \dots \\ \hline \text{II} \\ \hline U_0 \mid U_1 \mid \dots \mid U_\alpha \mid U_{\alpha+1} \mid \dots \end{array}$$

be a play of the game \mathcal{G}_0 or $\mathcal{G}_0(Q_\gamma)$ according to \mathcal{S} . Let $\mathcal{B}_i \subseteq \mathcal{A}_i$ be another sequence of κ -complete algebras. Then, the play:

$$\begin{array}{c} \text{I} \\ \hline \mathcal{B}_0 \mid \mathcal{B}_1 \mid \dots \mid \mathcal{B}_\alpha \mid \mathcal{B}_{\alpha+1} \mid \dots \\ \hline \text{II} \\ \hline U_0 \upharpoonright \mathcal{B}_0 \mid U_1 \upharpoonright \mathcal{B}_1 \mid \dots \mid U_\alpha \upharpoonright \mathcal{B}_\alpha \mid U_{\alpha+1} \upharpoonright \mathcal{B}_{\alpha+1} \mid \dots \end{array}$$

is a run of the game where Player II wins.

In what follows, we will consider θ a regular cardinal much larger than κ , and fix a well-ordering of H_θ which we denote by $<_\theta$. We augment our language of set theory by a binary relation symbol denoting this well-ordering, and work in this language when taking elementary hulls of H_θ . We will thus work with the structure $(H_\theta, \in, <_\theta)$, but will frequently suppress the symbols denoting \in and $<_\theta$ in our notation.

The common background setting for the games we are going to describe is an internally approachable sequence $\langle N_\alpha \mid \alpha < \kappa^+ \rangle$ of elementary substructures of H_θ . That is: a continuous sequence such that for all $\alpha < \kappa^+$ the following hold.

- (a) $\kappa + 1 \subseteq N_\alpha$ and $\text{card}(N_\alpha) = \kappa$,
- (b) ${}^{<\kappa}N_{\alpha+1} \subseteq N_{\alpha+1}$,
- (c) $\langle N_\xi \mid \xi \leq \alpha' \rangle \in N_\alpha$ whenever $\alpha' < \alpha$.

The following are standard remarks:

- If we are playing any of the games $\mathcal{G}_0, \mathcal{G}_1^-, \mathcal{G}_1, \mathcal{G}_2$ then the game has length $\gamma \leq \kappa$. Since $\kappa + 1 \subseteq N_0, \gamma \subseteq N_\alpha$ for all α .
- If $\langle N_\alpha : \alpha < \kappa \rangle$ is an internally approachable sequence then there is a closed unbounded set $C \subseteq \kappa^+$ such that for $\alpha \in C, N_\alpha \cap \kappa = \alpha$.

- If $2^\kappa = \kappa^+$, then there is a well ordering of $\mathcal{P}(\kappa)$ of order type κ^+ in H_θ . Hence, if $\langle N_\alpha : \alpha < \kappa^+ \rangle$ is an internally approachable sequence then $\mathcal{P}(\kappa) = \bigcup_{\alpha < \kappa^+} (\mathcal{P}(\kappa) \cap N_\alpha)$. Clearly, $\mathcal{P}(\kappa) \subseteq \mathcal{P}(\kappa) \cap \bigcup_{\alpha < \kappa^+} N_\alpha$ implies that $2^\kappa = \kappa^+$, which we stated as an assumption in Theorems 1.1 and 1.2.

Definition 4.1 (The Game \mathcal{G}_1^-). The rules of the game \mathcal{G}_1^- are as follows. Fix an ordinal $\gamma \leq \kappa^+$.

- Player I plays an increasing sequence of ordinals $\alpha_i < \kappa^+$.
- Player II plays an increasing sequence of uniform κ -complete ultrafilters U_i on $\mathcal{P}(\kappa)^M$ where $M = N_{\alpha_i+1}$.
- Player I plays first at limit stages.

A run of \mathcal{G}_1^- continues until Player II cannot play or until it reaches length γ . Player II wins a run in \mathcal{G}_1^- if and only if the length of the run is γ .

Payoff sets R_γ and Q_γ for \mathcal{G}_1^- are defined analogously to the definition for the game \mathcal{G}_0 . So R_γ consists of all runs of \mathcal{G}_1^- of length γ , and Q_γ consists of all runs $\langle \alpha_i, U_i \mid i < \gamma \rangle \in R_\gamma$ such that there is a κ -complete ultrafilter on the κ -algebra generated by $\mathcal{P}(\kappa) \cap N_\alpha$, where $\alpha = \sup_{i < \gamma} \alpha_i$, extending all U_i , $i < \gamma$.

The symbols R_γ and Q_γ have a double usage: They were also defined in connection with the game \mathcal{G}_0 and were different, but analogous to that in Definition 4.1. Thus, to determine the exact meaning of R_γ and Q_γ one always needs to take into account which game is being considered.

In the case where $\gamma = \kappa^+$, if Player II has a winning strategy in any of the games then κ is measurable. So for the purposes of this paper we can assume that $\gamma \leq \kappa$, in particular $\gamma \in N_\alpha$ for every α .

Remark. Let $\langle \alpha_i, U_i \mid i < \xi \rangle$ be a full or partial play of the game \mathcal{G}_1^- and $\alpha = \sup_{i < \xi} \alpha_i$.

- (1) If ξ has cofinality κ then $\mathcal{P}(\kappa) \cap N_\alpha$ is a κ -algebra.
- (2) If $\xi = \zeta + 1$, then $\alpha = \alpha_\zeta$, and again $\mathcal{P}(\kappa) \cap N_{\alpha_\zeta+1}$ is a κ -algebra.
- (3) if ξ is a limit ordinal of cofinality less than κ , then the κ -algebra of sets generated by $\bigcup_{i < \xi} N_{\alpha_i}$ is not a κ -algebra, the κ -algebra it generates is strictly larger.

Finally, let us stress that remarks analogous to the remarks (TO1)–(TO5) that stated below formula (7) for games \mathcal{G}_0 and $\mathcal{G}_0(Q_\gamma)$ also hold for \mathcal{G}_1^- and $\mathcal{G}_1^-(Q_\gamma)$.

Proposition 4.2. *Assuming $2^\kappa = \kappa^+$ and $\gamma \leq \kappa^+$ is an infinite regular cardinal, the following hold.*

- (a) *Player II has a winning strategy in \mathcal{G}_1^- of length γ if and only if Player II has a winning strategy in \mathcal{G}_0 of length γ .*
- (b) *Player II has a winning strategy in $\mathcal{G}_1^-(Q_\gamma)$ of length γ if and only if Player II has a winning strategy in $\mathcal{G}_0(Q_\gamma)$ of length γ .*

Moreover, the analogues of the above equivalences (a) and (b) also hold for winning strategies for Player I in the respective games.

Although the last statement in the above proposition concerning winning strategies for Player I is not strictly relevant for this paper, we include it for the sake of completeness.

Proof. This is an easy application of auxiliary games. Regarding (a), if \mathcal{S}_0 is a winning strategy for Player II in \mathcal{G}_0 then \mathcal{S}_0 induces a winning strategy \mathcal{S}_1^- for Player II in \mathcal{G}_1^- the output of which at step i is the same as the output of \mathcal{S} at step i in the auxiliary game \mathcal{G}_0 where Player I plays $\mathcal{P}(\kappa) \cap N_{\alpha_{i+1}}$ at step i (where α_i is the move of Player I in \mathcal{G}_1^- at step i). For the converse, we proceed similarly. This time a winning strategy \mathcal{S}_1^- for Player II in \mathcal{G}_1^- induces a strategy \mathcal{S}_0 for Player II in \mathcal{G}_0 as follows. If Player I plays \mathcal{A}_i at step i in \mathcal{G}_0 then Player I plays

$$\alpha_i = \text{the least } \alpha > \alpha_{i'} \text{ for all } i' < i \text{ such that } \mathcal{A}_i \subseteq N_{\alpha+1}$$

in the auxiliary game \mathcal{G}_1^- . Letting U_i^- be the output of \mathcal{S}_1^- at step i , we let the output of \mathcal{S}_0 at step i to be $U_i^- \cap \mathcal{A}_i$. That \mathcal{S}_0 is a winning strategy for Player II in \mathcal{G}_0 is immediate.

It is straightforward to verify that this choice of strategies also works in the case of games with payoff sets Q_γ in (b).

Because we will not study winning strategies for Player I in the games we consider, we leave the proof of the last statement in the proposition concerning these strategies to the reader. The proof is based on the same ideas as the proof of (a), (b) above. \square

We will use the following lemma:

Lemma 4.3. *Suppose that \mathcal{S}_0 is the $<_\theta$ least winning strategy for Player II in \mathcal{G}_0 and \mathcal{S}_1^- be the strategy defined from \mathcal{S}_0 as in Proposition 4.2. Suppose that $\beta < \gamma$ and $\langle \alpha_i : i < \beta \rangle$ is a sequence of ordinals such that for all $i, \alpha_{i+1} < \alpha$. Then Player II's response to $\langle \alpha_i : i < \beta \rangle$ in \mathcal{G}_1^- belongs to $N_{\alpha+1}$.*

Proof. Because the sequence $\langle N_\alpha : \alpha < \kappa^+ \rangle$ is internally approachable and $\alpha_i < \alpha$, $\alpha_i + 1 < \alpha$. Since we are taking $\gamma \leq \kappa$ and $N_{\alpha+1}$ is closed under $< \kappa$ -sequences, the sequence of κ -algebras $\langle P(\kappa) \cap N_{\alpha_{i+1}} : i < \beta \rangle$ belongs to N_α . Since \mathcal{S}_0 is $<_\theta$ -least, the sequence of responses by Player II to $\langle P(\kappa) \cap N_{\alpha_{i+1}} : i < \beta \rangle$ in \mathcal{G}_0 belongs to $N_{\alpha+1}$, and hence the sequence of responses by Player II according to \mathcal{S}_1^- as defined in Proposition 4.2 belongs to $N_{\alpha+1}$. \square

Definition 4.4 (The Game \mathcal{G}_1). The rules of \mathcal{G}_1 are exactly the same as those of \mathcal{G}_1^- with the only difference that the ultrafilters U_i played by Player II are required to be normal with respect to $N_{\alpha_{i+1}}$.

As before, the payoff set R_γ is defined for \mathcal{G}_1 the same way as it was for \mathcal{G}_0 and \mathcal{G}_1^- , that is, R_γ consists of all runs of \mathcal{G}_1 of length γ . For \mathcal{G}_1 we define a payoff set

W_γ as follows:

$W_\gamma =$ the set of all $\langle \alpha_i, U_i \mid i < \gamma \rangle \in R_\gamma$ such that if $\langle X_i \mid i < \gamma \rangle$ is a sequence satisfying $X_i \in U_i$ for all $i < \gamma$ then $\bigcap_{i < \gamma} X_i \neq \emptyset$.

Note that $W_\gamma = \emptyset$ whenever $\gamma \geq \kappa$, so the game $\mathcal{G}_1(W_\gamma)$ is of interest only for $\gamma < \kappa$. The existence of a winning strategy for Player II in $\mathcal{G}_1(W_\omega)$ of length ω seems to be exactly what is needed to run the proof of precipitousness of the hopeless ideal $\mathcal{I}(\mathcal{S}^*)$ in Sec. 5; see Proposition 5.7. As we will see shortly, the existence of such a winning strategy follows from the existence of a winning strategy for Player II in $\mathcal{G}_1^-(Q_\omega)$ of length ω .

In the case of \mathcal{G}_1 we will not make use of a payoff set for \mathcal{G}_1 that would be an analogue of what was Q_γ for \mathcal{G}_0 and \mathcal{G}_1^- , so we will not introduce it formally. We note that Q_γ is a subset of W_γ , so the winning condition for Player II is weaker using W_γ .

Let us also note that the somewhat abstract notion of normality of an ultrafilter U_i on $\mathcal{A}_i = \mathcal{P}(\kappa) \cap N_{\alpha_i+1}$ introduced in Sec. 3 is identical with the usual notion of normality with respect to the model N_{α_i+1} where it is required that U_i is closed under diagonal intersections of sequences $\langle A_\xi \mid \xi < \kappa \rangle \in N_{\alpha_i+1}$ such that $A_\xi \in U_i$ for all $\xi < \kappa$.

Remark 4.5. If we have a strategy \mathcal{S} defined for either \mathcal{G}_1^- or \mathcal{G}_1 , then a play of the game according to this strategy is determined by Player I's moves. Thus, if \mathcal{S} is clear from context we can save notation by referring to plays as sequences of ordinals $\langle \alpha_i : i < \beta \rangle$. Similarly if \mathcal{S} is a partial strategy defined on plays of length at most β we can index these plays according to \mathcal{S} by $\langle \alpha_i : i < \beta^* \rangle$, where $\beta^* \leq \beta$. This allows strategies to be defined by induction on the lengths of the plays.

Proposition 4.6 (Passing to normal measures). *The following correspondences between the existence of winning strategies for \mathcal{G}_1^- and \mathcal{G}_1 hold.*

- (a) *Let $\gamma \leq \kappa^+$ be an infinite regular cardinal. If Player II has a winning strategy in \mathcal{G}_1^- of length γ then Player II has a winning strategy in \mathcal{G}_1 of length γ . (So in fact we have “if and only if” here, as the converse holds trivially.)*
- (b) *If Player II has a winning strategy in $\mathcal{G}_1^-(Q_\gamma)$ of length γ then Player II has a winning strategy in $\mathcal{G}_1(W_\gamma)$ of length γ .*

We do not know whether there is an analogue of Proposition 4.6 with respect to strategies for Player I.

Proof. We begin with some conventions and settings. Let M_α be the transitive collapse of N_α . We will work with models M_α in place of N_α .

Since $\kappa + 1 \subseteq N_\alpha$, we have

$$\mathcal{P}(\kappa)^{N_\alpha} = \mathcal{P}(\kappa) \cap N_\alpha = \mathcal{P}(\kappa) \cap M_\alpha = \mathcal{P}(\kappa)^{M_\alpha},$$

so the games \mathcal{G}_1^- and \mathcal{G}_1 can be equivalently defined using structures M_α instead of N_α .

If U is an M -ultrafilter over κ we denote the internal ultrapower of M by U by $\text{Ult}(M, U)$. Then $\text{Ult}(M, U)$ is formed using all functions $f : \kappa \rightarrow M$ which are elements of M . If U is κ -complete then $\text{Ult}(M, U)$ is well-founded, and we will always consider it transitive; moreover the critical point of the ultrapower map $\pi_U : M \rightarrow \text{Ult}(M, U)$ is precisely κ . Recall also that U is normal if and only if $\kappa = [\text{id}]_U$, that is, κ is represented in the ultrapower by the identity map. As $M \models \text{ZFC}^-$ (by ZFC^- we mean ZFC without the power set axiom), the Łoś Theorem holds for all formulae, hence the ultrapower embedding π_U is fully elementary. Finally recall that the M -ultrafilter derived from π_U , which we denote by U^* , is defined by

$$X \in U^* \Leftrightarrow \kappa \in \pi_U(X) \tag{8}$$

and U^* is normal with respect to M .

Assume $\alpha < \alpha'$ and U' is a κ -complete $M_{\alpha'}$ -ultrafilter. Suppose that $U = U' \cap M_\alpha$. We have the following diagram:

$$\begin{array}{ccc} M_{\alpha'} & \xrightarrow{\pi_{U'}} & \text{Ult}(M_{\alpha'}, U') \\ \sigma \uparrow & & \uparrow \sigma' \\ M_\alpha & \xrightarrow{\pi_U} & \text{Ult}(M_\alpha, U) \end{array} \tag{9}$$

Here $\sigma : M_\alpha \rightarrow M_{\alpha'}$ is the natural map arising from collapsing the inclusion map from N_α to $N_{\alpha'}$, and σ' is the natural embedding of the ultrapowers defined by

$$[f]_U \mapsto [\sigma(f)]_{U'}. \tag{10}$$

Note that $\text{cr}(\sigma) = (\kappa^+)^{M_\alpha}$. Using the Łoś theorem, it is easy to check that the diagram is commutative, σ' is fully elementary, and $\sigma' \upharpoonright \kappa = \text{id} \upharpoonright \kappa$. It follows that

$$\sigma'(\kappa) \geq \kappa. \tag{11}$$

Given a set $X \in \mathcal{P}(\kappa) \cap M_\alpha$,

$$X \in U^* \Leftrightarrow \kappa \in \pi_U(X) \Leftrightarrow \sigma'(\kappa) \in \sigma'(\pi_U(X)) = \pi_{U'}(\sigma(X)) = \pi_{U'}(X). \tag{12}$$

Thus, using (11) combined with (12),

$$U^* \not\subseteq (U')^* \Rightarrow \sigma'(\kappa) > \kappa. \tag{13}$$

The property that $\sigma'(\kappa) > \kappa$ can be restated as saying that if f represents κ in $\text{Ult}(M_\alpha, U)$ and g represents κ in $\text{Ult}(M_{\alpha'}, U')$, then $\{\delta : f(\delta) > g(\delta)\} \in U'$. Equation (13) can also be rephrased in this way.

Note that since $U^*, (U')^*$ are ultrafilters on the respective models, the statement $U^* \subseteq (U')^*$ can be equivalently expressed as $U^* = (U')^* \cap M_\alpha$.

Before we define the winning strategies for Player II in \mathcal{G}_1 , we prove two useful facts about the normalization process. The first says there can't be an infinite sequence of ultrafilters that disagree on their normalizations.

Lemma 4.7. *Let $\langle \alpha_n : n \in \mathbb{N} \rangle$ be an increasing sequence of ordinals between κ and κ^+ . Then there is no sequence of ultrafilters $\langle U_n : n \in \mathbb{N} \rangle$ such that*

- U_n is a κ -complete ultrafilter on M_{α_n+1}
- $(U_n)^* \not\subseteq (U_{n+1})^*$
- there is a countably complete ultrafilter V on $\bigcup_n (\mathcal{P}(\kappa) \cap M_{\alpha_n+1})$ with $V \supseteq U_n$ for all n .

Proof. For each n let $f_n \in M_{\alpha_n+1}$ represent κ in $\text{Ult}(M_{\alpha_n+1}, U_n)$, and let σ'_n be the map from $\text{Ult}(M_{\alpha_n+1}, U_n)$ to $\text{Ult}(M_{\alpha_{n+1}+1}, U_{n+1})$ defined as in Eq. (10). Then there is a set $X_{n+1} \in U_{n+1}$ such that for all $\delta \in X_{n+1}$, $f_n(\delta) > f_{n+1}(\delta)$. The X_n 's all belong to V and intersecting them we get a $\delta \in \kappa$ such that for all n , $f_n(\delta) > f_{n+1}(\delta)$, a contradiction. \square

We note that Lemma 4.7 implies that in a play $\langle \alpha_i, U_i : i < \gamma \rangle$ there is no infinite increasing sequence $\langle i_n : n \in \mathbb{N} \rangle$ such that $(U_{\alpha_{i_n}})^* \not\subseteq (U_{\alpha_{i_{n+1}}})^*$.

Let $\langle \alpha_i, U_i : i < \beta \rangle$ be a partial or complete play of the game \mathcal{G}_1^- of limit length β . Suppose that $N_\infty = \bigcup_{i < \beta} N_{\alpha_i+1}$. Then the transitive collapse of N_∞ is the direct limit of $\langle M_{\alpha_i+1} : i < \beta \rangle$ along the canonical functions $\sigma_{j,j'} : M_{\alpha_j+1} \rightarrow M_{\alpha_{j'}+1}$ in diagram (9). Denote the transitive collapse of N_∞ by M_∞ . Let U_∞ be a κ -complete ultrafilter defined on the κ -algebra generated by $\mathcal{P}(\kappa) \cap M_\infty$ that extends $\bigcup_{i < \beta} U_i$. The following lemma implies that if for some $i < \beta$, $(U_i)^* \not\subseteq (U_\infty)^*$ then for some j with $i < j < \beta$, $(U_i)^* \not\subseteq (U_j)^*$.

Lemma 4.8. *Let $M_\alpha \prec M_\beta \prec M_\gamma$ with α, β, γ members of the $\alpha_i + 1$'s. Let $U_\alpha \subseteq U_\beta \subseteq U_\gamma$ be κ -complete ultrafilters on the respective $\mathcal{P}(\kappa)$'s of $M_\alpha, M_\beta, M_\gamma$. Suppose that $(U_\alpha)^* \not\subseteq (U_\gamma)^*$. Let $X \in M_\alpha \cap \mathcal{P}(\kappa)$ be such that $X \notin (U_\alpha)^*$, $X \in (U_\gamma)^*$.*

Then we can choose $f^\alpha, g^\alpha, f^\gamma, g^\gamma$ such that

$$\begin{aligned} f^\alpha, g^\alpha \in M_\alpha, \quad [f^\alpha]_{U_\alpha} = \kappa, \quad [g^\alpha]_{U_\alpha} = X, \\ f^\gamma, g^\gamma \in M_\gamma, \quad [f^\gamma]_{U_\gamma} = \kappa, \quad [g^\gamma]_{U_\gamma} = X. \end{aligned}$$

Suppose that $f^\gamma, g^\gamma \in M_\beta$. Then $(U_\alpha)^ \not\subseteq (U_\beta)^*$.*

Proof. If any of $U_\alpha, U_\beta, U_\gamma$ are principal the hypothesis clearly fails. It follows that each of the ultrafilters is uniform.

The point of the proof is showing that if $f^\alpha, g^\alpha, f^\gamma, g^\gamma$ belong to M_β , then $X \in (U_\beta)^*$. Since $X \notin (U_\alpha)^*$ but does belong to M_α , it follows that $(\kappa \setminus X) \in U_\alpha^*$. So $\kappa \setminus X$ witnesses the conclusion of the lemma.

Using the notation of diagram (9), since

$$\sigma' : \text{Ult}(M_\beta, U_\beta) \rightarrow \text{Ult}(M_\gamma, U_\gamma)$$

is order preserving and $[f^\gamma]_{U_\gamma} = \kappa$, we must have $[f^\gamma]_{U_\beta} = \kappa$.

Since $X \in (U_\gamma)^*$, $[f^\gamma]_{U_\gamma} = \kappa$ and $[g^\gamma]_\gamma = X$, we must have $\{\delta : f^\gamma(\delta) \in g^\gamma(\delta)\} \in U_\gamma$. Since f^γ and g^γ belong to M_β and $U_\beta \subseteq U_\gamma$ we have $\{\delta : f^\gamma(\delta) \in g^\gamma(\delta)\} \in U_\beta$, and hence $[g^\gamma]_{U_\beta} \in (U_\beta)^*$.

To finish it suffices to show that $[g^\gamma]_{U_\beta} = X$. Since $\{\delta : \text{sup}(g^\gamma(\delta)) = f^\gamma(\delta)\} \in U_\gamma$, we must have $\{\delta : \text{sup}(g^\gamma(\delta)) = f^\gamma(\delta)\} \in U_\beta$. Thus, $\text{sup}([g]_{U_\beta}) = \kappa$.

For $\alpha < \kappa$, let $c_\alpha : \kappa \rightarrow \kappa$ be the constant function α . Then $\{\delta : c_\alpha(\delta) < f^\gamma(\delta)\} \in U_\beta$, by κ -completeness. Using induction and the κ -completeness of U_β , one proves that $[c_\alpha]_{U_\beta} = \alpha$. But then

$$\begin{aligned} \alpha \in [g^\gamma]_{U_\beta} & \text{ if and only if } \{\delta : c_\alpha(\delta) \in g^\gamma(\delta)\} \in U_\beta \\ & \text{ if and only if } \{\delta : c_\alpha(\delta) \in g^\gamma(\delta)\} \in U_\gamma \\ & \text{ if and only if } \alpha \in [g^\gamma]_{U_\gamma}. \end{aligned}$$

Since $[g^\gamma]_{U_\gamma} = X$ we have $[g^\gamma]_{U_\beta} = X$. □

It follows from Lemmas 4.7 and 4.8 that if $\langle \alpha_i, U_i : i < \beta + k \rangle$ is a play of \mathcal{G}_1^- where β is zero or a limit ordinal and $k \in \omega$, then there is a finite set $i_0 = 0 < i_1 < i_2 < \dots < i_n = \beta$ such that for all $1 \leq m < n$

- (A) for all $i < j \in [i_{m-1}, i_m)$, it holds that $(U_i)^* \subseteq (U_j)^*$,
- (B) for all $i \in [i_{m-1}, i_m)$, $(U_i)^* \not\subseteq (U_{i_m})^*$.

We will call the stages i_1, \dots, i_n together with $\{0 \leq j < k-1 : (U_{\beta+j})^* \not\subseteq (U_{\beta+j+1})^*\}$ *drops*. Note that in clause (B), $m < n$ so this does not imply that β is a drop.

A position P of the game \mathcal{G}_1^- has the form

$$P = \langle \alpha_i^P, U_i^P \mid i < \beta^P \rangle \tag{14}$$

where α_i^P are moves of Player I and U_i^P are moves of Player II, and we will not use the superscripts P if there is no danger of confusion. We will take $\beta = 0$ as the length of the empty position. Given an infinite regular cardinal γ and a strategy \mathcal{S} for Player II in the game \mathcal{G}_1^- of length $\geq \gamma$, let \mathcal{Z}_γ be the set of all positions in \mathcal{G}_1^- of length $< \gamma$ according to \mathcal{S} that have successor length, where the last move of Player I is a drop. As stated in Remark 4.5 we can index plays in \mathcal{Z}_γ by increasing sequences of ordinals. On \mathcal{Z}_γ we define a binary relation \blacktriangleleft as follows. Given two positions $P, Q \in \mathcal{Z}_\gamma$, we let

$$P \blacktriangleleft Q \tag{15}$$

if and only if P properly extends Q .

Claim 4.9. *Assume one of the following holds*

- (a) $\gamma > \omega$ is regular and \mathcal{S} is a winning strategy for Player II in \mathcal{G}_1^- of length γ .
- (b) $\gamma = \omega$ and \mathcal{S} is a winning strategy for Player II in $\mathcal{G}_1^-(Q_\gamma)$ of length γ .

Then \blacktriangleleft is a well-founded tree.

Proof. It is immediate that \blacktriangleleft is a tree. The well-foundedness follows from the fact that there can be only finitely many drops along a play of the game. \square

The proof of Claim 4.9 implies that if \mathcal{S} is a winning strategy in any of the variants of \mathcal{G}_1^- of any length γ , then \blacktriangleleft is well-founded. Note for well-foundedness the only relevant γ are limit ordinals. As stated, the Claim handles all of the cases relevant to the theorems we are proving.

Now assume \mathcal{S} is as in (a) or (b) in Claim 4.9. For $P \in \mathcal{Z}_\gamma$, let i_P be the largest drop in P if P does have a drop, and $i_P = 0$ otherwise. Fix a \blacktriangleleft -minimal $P \in \mathcal{Z}_\gamma$. By the minimality of P , if P' extends P then $i_{P'} = i_P$; in other words, P' has no drops above i_P , hence $(U_i^{P'})^* \subseteq (U_{i'}^{P'})^*$ whenever $i_P \leq i < i'$.

Let $\alpha^* = \alpha_{i_P}$ and $V^* = (U_{\alpha^*})^*$.

We define a winning strategy \mathcal{S}_P for Player II in \mathcal{G}_1 of length γ . Viewing \mathcal{S} as defined on sequences of ordinals $\langle \alpha_i : i < \beta \rangle$, we define \mathcal{S}_P on such sequences $\langle \alpha_i : i < \beta \rangle$ by induction on their length β .

For ordinals $\alpha_i < \alpha^*$ played by the first player we assume inductively that the normal ultrafilter V_i played by the second player is $(V)^* \cap N_{\alpha_i+1}$.

Suppose we have defined \mathcal{S}_P on sequences of length less than β , where $\beta = 0$ corresponds to the empty position. Formally, to $\langle \alpha_i : i < \beta \rangle$ we inductively associate the play $\langle (\alpha_i, V_i) : i < \beta \rangle$ where V_i is the response by Player II according to \mathcal{S}_P . We need to define \mathcal{S}_P on $\langle \alpha_i : i < \beta \rangle \frown \alpha_\beta$.

Case 1. $\alpha_\beta \leq \alpha^*$. In this case

$$\mathcal{S}_P(\langle \alpha_i : i < \beta \rangle \frown \alpha_\beta) = (V)^* \cap N_{\alpha_\beta+1}.$$

Case 2. $\alpha_\beta > \alpha^*$.

Let j be least such that $\alpha_j > \alpha^*$. Let

$$\mathcal{S}_P(\langle \alpha_i : i < \beta \rangle \frown \alpha_\beta) = (\mathcal{S}(P \frown \langle \alpha_i : j \leq i \leq \beta \rangle))^*.$$

Note that in Case 1, it is trivial that Player II's move is a legal move. In Case 2, all of the filters played in response to ordinals less than α^* are sub-filters of V^* and hence are legal plays and sub-filters of $\mathcal{S}(P)^*$. Going beyond P , the plays of \mathcal{S}_P are extensions of plays according to \mathcal{S} that have initial segment P . Since P is \blacktriangleleft minimal there are no drops for those plays — in other words, there is inclusion of the normalized responses according to \mathcal{S} .

From this we conclude that Player II wins the game of length γ in part (a) of Claim 4.9.

We only prove (b) for $\gamma = \omega$ because that is the most relevant case for this paper. A straightforward generalization of this argument gives the result for general γ . The strategy \mathcal{S}_P is defined using a winning play by \mathcal{S} in the game $\mathcal{G}_1^-(Q_\omega)$. Since \mathcal{S} is a winning strategy in that game, if $\langle (\alpha_n, U_n) : n \in \mathbb{N} \rangle$ is that play according to \mathcal{S} , there is a κ -complete ultrafilter $U_\infty \supseteq \bigcup U_n$ defined on the κ -algebra generated by $\bigcup_n M_{\alpha_n}$. By Lemmas 4.7 and 4.8 and the remarks preceding them, $(U_\infty)^*$ extends V_n for all n . Part (b) follows.

Remark 4.10. Arguing exactly as in Lemma 4.3, if $\langle \alpha_i : i < \beta \rangle \in N_\alpha$ is a sequence of ordinals and a \blacktriangleleft -minimal position P in the game \mathcal{G}_1^- belongs to N_α then the sequence of responses by Player II to $\langle \alpha_i : i < \beta \rangle$ using \mathcal{S}_P belongs to N_α . In particular, if β is a successor ordinal $j + 1$ then \mathcal{S}_P 's responses belong to N_{α_j+2} .

Definition 4.11 (The Game \mathcal{G}_2). The rules of the game \mathcal{G}_2 are as follows.

- Player I plays an increasing sequence of ordinals $\alpha_i < \kappa^+$ as before.
- Player II plays distinct sets $Y_i \subseteq \kappa$ such that the following are satisfied.
 - (i) $Y_j \subseteq^* Y_i$ whenever $i < j$, and
 - (ii) Letting $U_i = \{X \in \mathcal{P}(\kappa) \cap N_{\alpha_i+1} \mid Y_i \subseteq^* X\}$, the family U_i is a uniform normal ultrafilter on $\mathcal{P}(\kappa) \cap N_{\alpha_i+1}$.
- Player I goes first at limit stages.

A run of \mathcal{G}_2 of length $\gamma \leq \kappa^+$ continues until Player II cannot play or else until it reaches length γ .

Payoff sets R_γ and W_γ for the game \mathcal{G}_2 are defined analogously to those for \mathcal{G}_1 . So R_γ consists of all runs in \mathcal{G}_2 of length γ and W_γ consists of all those runs $\langle \alpha_i, Y_i \mid i < \gamma \rangle \in R_\gamma$ such that if $\langle X_i \mid i < \gamma \rangle$ is a sequence satisfying $X_i \in N_{\alpha_i+1}$ and $Y_i \subseteq^* X_i$ for all $i < \gamma$ then $\bigcap_{i < \gamma} X_i \neq \emptyset$.

Note that $Y_i \notin N_{\alpha_i+1}$ in (ii). Note also that since the ultrafilters U_i are required to be uniform, the sets Y_i are unbounded in κ . As with \mathcal{G}_1 , we will not make any use of what would be an analogue of payoff set Q_γ .

Proposition 4.12. *Assume $\gamma \leq \kappa^+$ is an infinite regular cardinal.*

- (a) *Player II has a winning strategy in \mathcal{G}_1 of length γ if and only if Player II has a winning strategy in \mathcal{G}_2 of length γ .*
- (b) *Player II has a winning strategy in $\mathcal{G}_1(W_\gamma)$ of length γ if and only if Player II has a winning strategy in $\mathcal{G}_2(W_\gamma)$ of length γ .*

Proof. For (a), it is immediate that a winning strategy for Player II in \mathcal{G}_2 gives a winning strategy in \mathcal{G}_1 : if Player II plays Y_i at turn i , then Y_i generates a normal ultrafilter on N_{α_i+1} which is Player II's move in \mathcal{G}_1 .

For the non-trivial direction, assume Player II has a winning strategy \mathcal{S} in \mathcal{G}_1 of length γ . As noted before Definition 4.11, such a strategy exists in N_0 . We build a winning strategy \mathcal{S}' for Player II in \mathcal{G}_2 of length γ by induction.

Induction Hypothesis. Suppose that Player I plays $\langle \alpha_i : i < \beta \rangle$ in the game \mathcal{G}_2 , and $\langle U_i : i < \beta \rangle$ is the play by Player II using \mathcal{S} in the game \mathcal{G}_1 . Then Player II plays $\langle Y_i : i < \beta \rangle$ where Y_i is a definable diagonal intersection of the members of U_i .

For each i , let $\langle X_\xi^i \mid \xi < \kappa \rangle$ be the $<_\theta$ -least enumeration of U_i of length κ (recall that $<_\theta$ is the well-ordering of H_θ fixed at the beginning of this section; see the paragraphs immediately above Definition 4.1). The induction hypothesis is that for all $i < \beta$, Player II's responses according to the strategy \mathcal{S}' to the sequence $\langle \alpha_i : i \leq \delta \rangle$ are $\langle Y_i : i \leq \delta \rangle$ where

$$Y_i = \Delta_{\xi < \kappa} X_\xi^i.$$

This induction hypothesis is automatically preserved at limit stages. Suppose that it holds up to β and Player I plays α_β . Then Player II plays an ultrafilter U_β on $\mathcal{P}(\kappa) \cap N_{\alpha_\beta+1}$ in the game \mathcal{G}_1 using the strategy defined in Proposition 4.6. Then, as in Remark 4.10, $N_{\alpha_\beta+2}$ contains the information that U_β is Player II's response as well as the $<_\theta$ -least enumeration $\langle X_\xi^\beta : \xi < \kappa \rangle$ of U_β . Let $Y_\beta = \Delta_{\xi < \kappa} X_\xi^\beta$ and let Y_β be Player II's response in \mathcal{G}_2 using \mathcal{S}' .

Suppose now that $\langle \alpha_i : i < \gamma \rangle$ is a run of the game \mathcal{G}_2 according to \mathcal{S}' . Then, since U_{i+1} is normal each Y_i belongs to U_{i+1} . Since, $Y_j \subseteq^* X$ for all $X \in U_j$, for $i < j$, $Y_j \subseteq^* Y_i$. Moreover, since Y_i is a diagonal intersection of the ultrafilter U_i , clause (ii) in Definition 4.11 is immediate.

Since the relevant ultrafilters are the same, whether II is playing by \mathcal{S} in \mathcal{G}_1 or \mathcal{S}' in \mathcal{G}_2 , clause (b) in Proposition 4.12 is immediate. \square

Remark 4.13. The definition of the winning strategy \mathcal{S}' for Player II in the previous proof depends on the position P in \mathcal{G}_1^- , beyond which there are no drops. Suppose that Player I plays $\langle \alpha_i : i < \beta \rangle$ in the game \mathcal{G}_2 and Player II responds with $\langle Y_i : i < \beta \rangle$ using the winning strategy \mathcal{S}' . Then for all $j < \beta$ with $P \in N_j$,

- $Y_j \notin N_{\alpha_j+1}$ because it induces an ultrafilter on N_{α_j+1} ,
- $Y_j \in N_{\alpha_j+2}$ because $\langle N_{\alpha_i} : i \leq j \rangle \in N_{\alpha_j+2}$ and Player II's response to $\langle \alpha_i : i \leq j \rangle$ according to \mathcal{S}' is definable from Player II's response to $\langle \alpha_i : i \leq j \rangle$ according to the strategy \mathcal{S} for \mathcal{G}_1 , which in turn is definable from P and Player II's response according to her strategy in \mathcal{G}_1^- and thus from the original strategy in \mathcal{G}_0 .

It follows that for all $i < j$, $Y_j \subseteq^* Y_i$ and $|Y_i \setminus Y_j| = \kappa$. (Restating this $Y_j \subsetneq^* Y_i$.)

We complete this section with a corollary which will be used in studying properties of the strategies constructed in Sec. 6.

Corollary 4.14. *Assume \mathcal{S}_1 is a winning strategy for Player II in the game \mathcal{G}_1 of length γ and \mathcal{S}_2 is the winning strategy for Player II in the game \mathcal{G}_2 of length γ*

obtained as in Proposition 4.12. Then for every $A \in \mathcal{P}(\kappa)$,

$$A \in \mathcal{I}(\mathcal{S}_1) \Leftrightarrow A \in \mathcal{I}(\mathcal{S}_2).$$

5. Strategies \mathcal{S}^* and \mathcal{S}_γ

Consider a winning strategy \mathcal{S}_0 for Player II in \mathcal{G}_0 of length γ and a position P in \mathcal{G}_0 according to \mathcal{S}_0 . Given a set $X \in \mathcal{I}(\mathcal{S}_0, P)^+$, there may exist different runs of \mathcal{G}_0 extending P which witness that X is $\mathcal{I}(\mathcal{S}_0, P)$ -positive. This causes difficulties in proving that $\mathcal{I}(\mathcal{S}_0, P)$ has strong properties like precipitousness or the existence of a dense subset with a high degree of closure. To address this issue, we construct a winning strategy \mathcal{S}^* for Player II in \mathcal{G}_2 of length γ such that for each position Q in \mathcal{G}_2 according to \mathcal{S}^* and each $X \notin \mathcal{I}(\mathcal{S}^*, Q)$ there is a unique run witnessing that X is $\mathcal{I}(\mathcal{S}^*, Q)$ -positive, and show that using \mathcal{S}^* one can prove the precipitousness of $\mathcal{I}(\mathcal{S}^*, \emptyset)$ and the existence of a dense subset with a high degree of closure, thus proving Theorems 1.1 and 1.2.

Recall from the introduction that when we talk about saturated ideals over κ , we always mean uniform κ -complete and κ^+ -saturated ideals over κ . The results in this section are formulated under the assumption of the non-existence of a normal saturated ideal over κ , as this allows to fit the results together smoothly. That the results actually constitute a proof of Theorem 1.2, which is stated under a seemingly stronger requirement on the non-existence of a saturated ideal over κ , is a consequence of the following standard proposition.

Proposition 5.1. *Given a regular cardinal $\kappa > \omega$, the following are equivalent.*

- (a) κ carries a saturated ideal.
- (b) κ carries a normal saturated ideal.

Proof. A standard elementary argument shows that any uniform normal ideal over κ is κ -complete, hence (a) follows immediately from (b).

To see that (b) follows from (a), assume \mathcal{I} is a saturated ideal over κ . Let $\mathbb{P}_{\mathcal{I}}$ be the partial ordering $(\mathcal{I}^+, \subseteq_{\mathcal{I}})$ and \dot{U} be a $\mathbb{P}_{\mathcal{I}}$ -term for the normal \mathbf{V} -ultrafilter over κ derived from the generic embedding $j_G : \mathbf{V} \rightarrow M_G$ associated with $\text{Ult}(\mathbf{V}, G)$ where G is $(\mathbb{P}_{\mathcal{I}}, \mathbf{V})$ -generic. Let $\mathcal{I}^* \in \mathbf{V}$ be the ideal over κ defined by

$$a \in \mathcal{I}^* \Leftrightarrow \Vdash_{\mathbb{P}_{\mathcal{I}}}^{\mathbf{V}} \check{a} \notin \dot{U}.$$

Equivalently

$$a \in \mathcal{I}^* \Leftrightarrow \Vdash_{\mathbb{P}_{\mathcal{I}}}^{\mathbf{V}} \check{\kappa} \notin j(\check{a}).$$

A standard argument shows that \mathcal{I}^* is a uniform normal ideal over κ . To see that \mathcal{I}^* is saturated, we construct an incompatibility-preserving map $e : (\mathcal{I}^*)^+ \rightarrow \mathcal{I}^+$. Let $f : \kappa \rightarrow \kappa$ be a function in \mathbf{V} which represents κ in $\text{Ult}(\mathbf{V}, G)$ whenever G

is $(\mathbb{P}_{\mathcal{I}}, \mathbf{V})$ -generic. Since \mathcal{I} is saturated, such a function can be constructed using standard techniques (see [20]). Let

$$e(a) \stackrel{\text{def}}{=} \{\xi < \kappa \mid f(\xi) \in a\}.$$

Note that for every $a \in \mathcal{P}(\kappa)^{\mathbf{V}}$ and every $(\mathbb{P}_{\mathcal{I}}, \mathbf{V})$ -generic G

$$a \in \dot{U}^G \Leftrightarrow \kappa \in j_G(a) \Leftrightarrow [f]_G \in [c_a]_G \Leftrightarrow e(a) \in G.$$

It follows from these equivalences that indeed $e(a) \in \mathcal{I}^+$ whenever $a \in (\mathcal{I}^*)^+$. To see that e is incompatibility preserving, we prove the contrapositive. Assume $e(a), e(b)$ are compatible, so $e(a) \cap e(b) \in \mathcal{I}^+$. Let G be $(\mathbb{P}_{\mathcal{I}}, \mathbf{V})$ -generic such that $e(a) \cap e(b) \in G$. Then $e(a), e(b) \in G$, so $a, b \in \dot{U}^G$ by the above equivalences. But then $a \cap b \in \dot{U}^G$, which tells us that $a \cap b \in (\mathcal{I}^*)^+$. \square

We are now ready to formulate the main technical result of this section.

Proposition 5.2. *Assume $2^\kappa = \kappa^+$ and there is no normal saturated ideal over κ . Let $\gamma \leq \kappa^+$ be an infinite regular cardinal and \mathcal{S} be a winning strategy for Player II in \mathcal{G}_2 of length γ . Then there is a tree $T(\mathcal{S})$ which is a subtree of the partial ordering $(\mathcal{P}(\kappa), \subseteq^*)$ such that the following hold.*

- (a) *The height of $T(\mathcal{S})$ is γ and $T(\mathcal{S})$ is γ -closed.*
- (b) *If $Y, Y' \in T(\mathcal{S})$ are \subseteq^* -incomparable then Y, Y' are almost disjoint.*
- (c) *There is an assignment $Y \mapsto P_Y$ assigning to each $Y \in T(\mathcal{S})$ a position P_Y in \mathcal{G}_2 of successor length according to \mathcal{S} in which the last move by Player II is Y ; we denote the last move of Player I in P_Y by $\alpha(Y)$. The assignment $Y \mapsto P_Y$ has the following property:*

$$Y' \subsetneq^* Y \Rightarrow \alpha(Y) < \alpha(Y') \text{ and } P_{Y'} \text{ is an extension of } P_Y.$$

- (d) *If b is a branch of $T(\mathcal{S})$ of length $< \gamma$, let $P_b = \bigcup_{Y \in b} P_Y$. Then P_b is a position in \mathcal{G}_2 according to \mathcal{S} , and the set of all immediate successors of b in $T(\mathcal{S})$ is of cardinality κ^+ . Moreover the assignment $Y \mapsto \alpha(Y)$ is injective on this set.*

Finally, if $A \in \mathcal{I}(\mathcal{S})^+$ then it is possible to construct the tree $T(\mathcal{S})$ in such a way that

$$A \in T(\mathcal{S}). \tag{16}$$

Clause (d) in the above definition treats both successor and limit cases for γ . The successor case in (d) simply says that if $Y \in T(\mathcal{S})$ then the conclusions in (d) apply to the set of all immediate successors of Y in $T(\mathcal{S})$.

Proof. The tree $T(\mathcal{S})$ is constructed by induction on levels. Limit stages of this construction are trivial: If $\bar{\gamma} < \gamma$ is a limit and we have already constructed initial segments T_{γ^*} of $T(\mathcal{S})$ of height γ^* for all $\gamma^* < \bar{\gamma}$ so that (b)–(d) hold with T_{γ^*} in place of $T(\mathcal{S})$ and $T_{\gamma'}$ end-extends T_{γ^*} whenever $\gamma^* < \gamma' < \bar{\gamma}$ then it is easy to see

that $T_{\bar{\gamma}} = \bigcup_{\gamma^* < \bar{\gamma}} T_{\gamma^*}$ is a tree with tree ordering \supseteq^* end-extending all T_{γ^*} , $\gamma^* < \bar{\gamma}$, and such that (b)–(d) hold with $T_{\bar{\gamma}}$ in place of $T(\mathcal{S})$. We will thus focus on the successor stages of the construction.

Assume $\bar{\gamma} < \gamma$ and $T(\mathcal{S})$ is constructed at all levels up to level $\bar{\gamma}$; our task now is to construct the $\bar{\gamma}$ th level of $T(\mathcal{S})$. Let b be a cofinal branch through this initial segment of $T(\mathcal{S})$, so b is of length $\bar{\gamma}$. We construct the set of immediate successors of b in $T(\mathcal{S})$, along with the assignment $Y \mapsto P_Y$ on this set, as follows. As we are assuming there is no normal saturated ideal over κ , we can pick an antichain \mathcal{A} in $\mathcal{I}(\mathcal{S}, P_b)^+$ of cardinality κ^+ . For each $X \in \mathcal{A}$ there is a position Q_X in \mathcal{G}_2 of successor length $< \gamma$ according to \mathcal{S} extending P_b such that the last move by Player II in Q_X is almost contained in X . For the sake of definability we can let this position to be $<_\theta$ -least, where recall that $<_\theta$ is the fixed well-ordering of H_θ .

Now construct the set $\langle Y_\xi \mid \xi < \kappa^+ \rangle$ of all immediate successors of b in $T(\mathcal{S})$ recursively as follows. Assume $\xi < \kappa^+$ and we have already constructed the set $\langle Y_{\bar{\xi}} \mid \bar{\xi} < \xi \rangle$ along with the assignment $Y_{\bar{\xi}} \mapsto P_{Y_{\bar{\xi}}}$ with the desired properties. Since each model N_β is of cardinality κ , we can pick the $<_\theta$ -least set $X \in \mathcal{A}$ which is not an element of any $N_{\alpha(Y_{\bar{\xi}})+1}$ where $\bar{\xi} < \xi$. Now let Player I extend Q_X by playing the least ordinal α such that

$$\{X\} \cup \{Y_{\bar{\xi}} \mid \bar{\xi} < \xi\} \subseteq N_{\alpha+1}. \tag{17}$$

This is a legal move in \mathcal{G}_2 following Q_X . Let Y be the response of the strategy \mathcal{S} to $Q_X \hat{\ } \langle \alpha \rangle$. We let Y_ξ be this Y and $P_Y = Q_X \hat{\ } \langle \alpha, Y \rangle$. Note that $Y_\xi \subseteq^* X$, as Y_ξ , being played according to \mathcal{S} , is almost contained in the last move by Player II in Q_X .

We show

$$\text{Any two sets } Y \neq Y' \text{ on the } \bar{\gamma}\text{th level are almost disjoint.} \tag{18}$$

If Y, Y' are above two distinct cofinal branches then this follows immediately from the induction hypothesis: Letting Z , respectively, Z' be the immediate successor of $b \cap b'$ in b , respectively, b' , we have $Y \subseteq^* Z$ and $Y' \subseteq^* Z'$, and the induction hypothesis tells us that Z, Z' are almost disjoint.

Now assume Y, Y' are above the same branch b ; without loss of generality we may assume $Y = Y_\xi$ and $Y' = Y_{\xi'}$ in the above enumeration and $\xi' < \xi$. Then we have $X, X', P_Y, P_{Y'}$ as in the construction, with $Y \subseteq^* X$ and $Y' \subseteq^* X'$. Also $\alpha(Y') < \alpha(Y)$.

If $Y \subseteq^* Y'$ then $Y \subseteq^* X \cap X'$, thus witnessing $X \cap X' \in \mathcal{I}(\mathcal{S}, P_b)^+$. This contradicts the fact that \mathcal{A} is an antichain in $\mathcal{I}(\mathcal{S}, P_b)^+$. It follows that $Y \not\subseteq^* Y'$. Now for every $Z \in N_{\alpha(Y)+1}$ the set Y is either almost contained in or almost disjoint from Z . As $Y' \in N_{\alpha(Y)+1}$ by our choice of $\alpha(Y)$ in (17), necessarily Y is almost disjoint from Y' . This proves (18).

To verify that (b)–(d) hold with the tree obtained by adding the immediate successors of a single branch b as described in the previous paragraph in place of $T(\mathcal{S})$, note that (c) and (d) immediately follow from the construction just described,

so all we need to check is clause (b) and the fact that \supseteq^* is still a tree ordering after adding the entire $\bar{\gamma}$ th level. But clause (b) follows from the combination of (18) with the induction hypothesis and the fact that every set on the $\bar{\gamma}$ th level is almost contained in some set on an earlier level. Finally, that adding the $\bar{\gamma}$ th level keeps \supseteq^* a tree ordering follows from clause (b). More generally, any collection $\mathcal{X} \subseteq \mathcal{P}(\kappa)$ which satisfies (b) with \mathcal{X} in place of $T(\mathcal{S})$ has the property that the set of all $Y' \in \mathcal{X}$ which are \supseteq^* -predecessors of a set $Y \in \mathcal{X}$ is linearly ordered under \supseteq^* . What now remains is to see that clause (a) holds, but this is immediate once we have completed all γ steps of the construction.

Finally, given a set $A \in \mathcal{I}(\mathcal{S})^+$, to see that we can construct the tree $T(\mathcal{S})$ so that (16) holds, note that we can put A into the first level of $T(\mathcal{S})$ at the first step in the inductive construction. This involves a slight modification of the construction of the first level of $T(\mathcal{S})$, and is left to the reader. \square

The new strategy \mathcal{S}^* for Player II in \mathcal{G}_2 is now obtained by, roughly speaking, playing down the tree $T(\mathcal{S})$. More precisely:

Definition 5.3. Assume $\gamma \leq \kappa^+$ is an infinite regular cardinal, \mathcal{S} is a winning strategy for Player II in \mathcal{G}_2 of length γ , and $T(\mathcal{S})$ is a subtree of the partial ordering $(\mathcal{P}(\kappa), \supseteq^*)$ satisfying (a)–(d) in Proposition 5.2. We define a strategy \mathcal{S}^* for Player II in \mathcal{G}_2 of length γ associated with $T(\mathcal{S})$ recursively as follows.

Assume

$$P = \{(\alpha_i, Y_i) \mid i < j\}$$

is a position in \mathcal{G}_2 of length $j < \gamma$ according to \mathcal{S}^* . Denote the corresponding branch in $T(\mathcal{S})$ by b_P , that is,

$$b_P = \{Y_i \mid i < j\}.$$

If α_j is a legal move of Player I in \mathcal{G}_2 at position P then

$$\begin{aligned} \mathcal{S}^*(P \frown \langle \alpha_j \rangle) &= \text{the unique immediate successor } Y \text{ of } b_P \text{ in } T(\mathcal{S}) \\ &\text{with minimal possible } \alpha(Y) \geq \alpha_j. \end{aligned}$$

Here recall that $\alpha(Y)$ is the last move of Player I in P_Y .

As an immediate consequence of the properties of $T(\mathcal{S})$ we obtain:

Proposition 5.4. Let $\gamma \leq \kappa^+$ be an infinite regular cardinal and assume $T(\mathcal{S})$ is as in Proposition 5.2. Then \mathcal{S}^* is a winning strategy for Player II in \mathcal{G}_2 of length γ .

Moreover, if

$$r^* = \langle \alpha_i, Y_i \mid i < \gamma \rangle$$

is a run of \mathcal{G}_2 of length γ according to \mathcal{S}^* then

$$r = \bigcup_{i < \gamma} P_{Y_i}$$

is a run of \mathcal{G}_2 of length γ according to \mathcal{S} .

Before giving a proof of Theorem 1.1, we record the following obvious fact, which will be useful in Sec. 6 in studying properties of winning strategies for Player II in games \mathcal{G}_i of length γ , and to which we will refer later.

Corollary 5.5. *Under the assumptions of Proposition 5.2, assume $A \in \mathcal{I}(\mathcal{S})^+$ and $T(\mathcal{S})$ is constructed in such a way that (16) holds, that is, $A \in T(\mathcal{S})$. Let \mathcal{S}^* be the winning strategy for Player II constructed as in Definition 5.3 using this $T(\mathcal{S})$. Then $A \in \mathcal{I}(\mathcal{S}^*)^+$.*

One of the main points of passing to \mathcal{S}^* is the following remark.

Remark 5.6. For any position P of a partial run according to \mathcal{S}^* of successor length with Y being the last move by Player II, the conditional hopeless ideal $\mathcal{I}(\mathcal{S}^*, P)$ is equal to the unconditional hopeless ideal restricted to Y

$$\mathcal{I}(\mathcal{S}^*, P) = \mathcal{I}(\mathcal{S}^*) \upharpoonright Y.$$

We now turn to a proof of Theorem 1.1. If there is a normal saturated ideal over κ then there is nothing to prove. Otherwise Player II has a winning strategy in $\mathcal{G}_2(W_\omega)$ of length ω , as follows from Propositions 4.2(b), 4.6(b) and 4.12(b). The conclusion in Theorem 1.1 then follows from a more specific fact we prove, namely, Proposition 5.7 in what follows. In the proof of this proposition, we will make use of the criterion for precipitousness in terms of the ideal game, see Sec. 1.

Proposition 5.7. *Assume there is no normal saturated ideal over κ . Let*

- \mathcal{S} be a winning strategy for Player II in $\mathcal{G}_2(W_\omega)$ of length ω , and
- \mathcal{S}^* be the winning strategy constructed from \mathcal{S} as in Definition 5.3.

Then Player I does not have a winning strategy in the ideal game $\mathcal{G}(\mathcal{I}(\mathcal{S}^))$. Consequently, the ideal $\mathcal{I}(\mathcal{S}^*)$ is precipitous.*

Proof. Assume \mathcal{S}_I is a strategy for Player I in the ideal game $\mathcal{G}(\mathcal{I}(\mathcal{S}^*))$. We construct a run in $\mathcal{G}(\mathcal{I}(\mathcal{S}^*))$ according to \mathcal{S}_I which is winning for Player II. Odd stages in this run will come from positions in \mathcal{G}_2 played according to \mathcal{S}^* ; more precisely, they will be tail-ends of sets on those positions. So suppose

$$Q = \langle X_0, X_1, X_2, X_3 \dots, X_{2n-1} \rangle$$

is the finite run of $\mathcal{G}(\mathcal{I}(\mathcal{S}^*))$ constructed so far, and

$$\beta_0, Z_0, \beta_1, Z_1, \dots, \beta_{n-1}, Z_{n-1}$$

is the associated auxiliary run of \mathcal{G}_2 according to \mathcal{S}^* such that $Z_i \subseteq^* X_{2i}$ and

$$X_{2i+1} = \text{the longest tail-end of } Z_i \text{ that is contained in } X_{2i}$$

for all $i < n$. Let X_{2n} be the response of \mathcal{S}_I to Q in $\mathcal{G}(\mathcal{I}(\mathcal{S}^*))$. As $X_{2n} \in \mathcal{I}(\mathcal{S}^*)^+$, there is a finite position in \mathcal{G}_2 according to \mathcal{S}^* where the last move of Player II is a set almost contained in X_{2n} and, letting Z_n be this set, we also have $X_{2n} \in N_{\alpha(Z_n)+1}$.

As the sets Z_n constitute an \subseteq^* -decreasing chain of nodes in $T(\mathcal{S})$, the positions P_{Z_n} extend P_{Z_m} whenever $m < n$. By Proposition 5.4

$$r = \bigcup_{n \in \omega} P_{Z_n}$$

is a run in \mathcal{G}_2 of length ω according to \mathcal{S} . Let

$$r = \langle \alpha_i, Y_i \mid i \in \omega \rangle$$

be this run. For each $i \in \omega$ let

$$X'_i = X_{2n} \text{ where } n \text{ is such that } \text{lh}(P_{Z_n}) \leq i < \text{lh}(P_{Z_{n+1}}).$$

Then

$$\bigcap_{n \in \omega} X_n = \bigcap_{n \in \omega} X_{2n} = \bigcap_{i \in \omega} X'_i \neq \emptyset.$$

Here the equality on the left comes from the fact that the sets X_n , $n \in \omega$ constitute an \subseteq -descending chain, and the inequality on the right follows from the fact that $X'_i \in N_{\alpha_i+1}$ and $Y_i \subseteq^* X'_i$ for all $i \in \omega$, and that \mathcal{S} is a winning strategy for Player II in $\mathcal{G}_2(W_\omega)$ of length ω ; see the last paragraph in Definition 4.11. \square

We remark that the proof of Proposition 5.7 shows Player II has a winning strategy in the ideal game $\mathcal{I}(\mathcal{S}^*)$.

The following proposition gives a proof of Theorem 1.2. Recall that all background we have developed so far was under the assumption that κ is inaccessible and $2^\kappa = \kappa^+$. Also recall that by trivial observation (TO3) at the beginning of Sec. 4 and results in Sec. 4, if Player II has a winning strategy in \mathcal{G}_0 of length $\gamma > \omega$ then Player II has a winning strategy in $\mathcal{G}_0(Q_\omega)$ of length ω and in $\mathcal{G}_2(W_\omega)$ of length ω , as well as in \mathcal{G}_2 of length γ whenever γ is regular. By a similar argument, if Player II has a winning strategy in \mathcal{G}_2 of length $\gamma > \omega$ then Player II has a winning strategy in $\mathcal{G}_2(W_\omega)$ of length ω . Thus, under the assumptions of Theorem 1.2, the assumptions of Proposition 5.8 below are not vacuous.

Proposition 5.8. *Assume there is no normal saturated ideal over κ and $2^\kappa = \kappa^+$. Let $\gamma \leq \kappa^+$ be an uncountable regular cardinal. Assume further that \mathcal{S} and \mathcal{S}^* are strategies as in Proposition 5.7, with γ in place of ω .*

Then $T(\mathcal{S})$ is a γ -closed dense subset of $\mathcal{I}(\mathcal{S}^)^+$. It follows that Player I does not have a winning strategy in the ideal game $\mathcal{G}(\mathcal{I}(\mathcal{S}^*))$. Consequently, the ideal $\mathcal{I}(\mathcal{S}^*)$ is precipitous.*

Proof. That $T(\mathcal{S})$ is a γ -closed dense subset of $\mathcal{I}(\mathcal{S}^*)^+$ follows immediately from the properties of $T(\mathcal{S})$. If $A \in \mathcal{I}(\mathcal{S}^*)^+$, then there is a play of the game such that A is in the ultrafilter determined by some Y_ξ played by Player II using \mathcal{S}^* . But then $Y_\xi \subseteq^* A$. Since Y_ξ is on $T(\mathcal{S})$, we have shown that for every element of $\mathcal{I}(\mathcal{S}^*)^+$ there is an element of the tree below it. Hence, the tree is dense.

To see that $\mathcal{I}(S^*)$ is precipitous, we use an argument originally due to Laver. It follows the idea of Proposition 5.7 and shows that Player II has a winning strategy in the game $\mathcal{G}(\mathcal{I}(S^*))$. At stage n of the game suppose that Player I plays X_{2n} . Player II chooses an $X'_{2n+1} \in T(S)$ (so $X'_{2n+1} \in \mathcal{I}(S^*)^+$) and $X'_{2n+1} \subseteq_{\mathcal{I}(S^*)}^* X_{2n}$.

Let $A_n \in \mathcal{I}(S^*)$ be such that $X'_{2n+1} \setminus A_n \subset X_{2n}$. Player II's response to X_{2n} in $\mathcal{G}(\mathcal{I}(S^*))$ is $X_{2n+1} \stackrel{\text{def}}{=} X'_{2n+1} \setminus A_n$. Let $A = \bigcup_n A_n$. Since $\mathcal{I}(S^*)$ is countably complete, $A \in \mathcal{I}(S^*)$. Let $X_\infty \in T(S)$, with $X_\infty \subseteq^* X'_{2n+1}$ for all n . Then:

$$\bigcap_n X_n \supseteq \bigcap_n X_n \setminus A \supseteq_{\mathcal{I}(S^*)}^* X_\infty \setminus A.$$

It follows that there is a set $B \in \mathcal{I}(S^*)$ such that $\bigcap X_n \supseteq X_\infty \setminus B$. Since $X_\infty \notin \mathcal{I}(S^*)$, $X_\infty \setminus B$ is not empty. Hence $\bigcap X_n \neq \emptyset$. \square

Proof of Theorem 1.4

Proof. Consider a uniform κ -complete ideal \mathcal{J} over κ such that $\mathcal{P}(\kappa)/\mathcal{J}$ is (κ^+, ∞) -distributive and has a dense γ -closed set. Because of notational convenience we will work with the partial ordering $\mathbb{P}_{\mathcal{J}} = (\mathcal{J}^+, \subseteq_{\mathcal{J}})$. (See also the partial ordering $\mathbb{P}_{\mathcal{I}}$ used in the proof of Proposition 5.1.) Since $a \mapsto [a]_{\mathcal{J}}$ is a dense embedding of $\mathbb{P}_{\mathcal{J}}$ onto $\mathcal{P}(\kappa)/\mathcal{J}$, we can fix a dense γ -closed set $D \subseteq \mathbb{P}_{\mathcal{J}}$. We work inside H_θ for a sufficiently large θ and will use the fixed well-ordering $<_\theta$ introduced in Sec. 4 to define a winning strategy \mathcal{S}_γ for Player II in \mathcal{G}_γ^W . As usual, \mathcal{S}_γ is defined inductively on the length of runs.

So assume

$$\mathcal{A}_0, U_0, \mathcal{A}_1, U_1, \dots, \mathcal{A}_j, U_j, \dots$$

is a run of \mathcal{G}_γ^W according to \mathcal{S}_γ for $j < i$. Along the way, we define auxiliary moves X_j played by Player II; these moves are elements of D , constitute a descending chain in the ordering by $\subseteq_{\mathcal{J}}$, and for each $j < i$,

$$X_j \Vdash_{\mathbb{P}_{\mathcal{J}}} \dot{G} \cap \check{\mathcal{A}}_j = \check{U}_j. \tag{19}$$

At step $i < \gamma$ Player I plays a κ -algebra \mathcal{A}_i on κ of cardinality κ extending all \mathcal{A}_j , $j < i$. As D is γ -closed and $i < \gamma$, there is an element $X \in D$ below all X_j in $\mathbb{P}_{\mathcal{J}}$, $j < i$. If G is a $(\mathbb{P}_{\mathcal{J}}, \mathbf{V})$ -generic filter such that $X \in G$ then by (19), $U_j \subseteq G$ whenever $j < i$. Since $\mathbb{P}_{\mathcal{J}}$ is (κ^+, ∞) -distributive and $\mathcal{A}_i \in \mathbf{V}$ is of cardinality κ , the intersection $G \cap \mathcal{A}_i$ is an element of \mathbf{V} , and is a uniform κ -complete ultrafilter on \mathcal{A}_i extending all U_j where $j < i$. This is then forced by some condition $Y \in G$ such that $Y \subseteq_{\mathcal{J}} X$, hence $Y \subseteq_{\mathcal{J}} X_j$ for all $j < i$. As D is dense in $\mathbb{P}_{\mathcal{J}}$, Y can be chosen to be an element of D . The following is thus not vacuous. We define

$$X_i = \text{the } <_\theta\text{-least element } Y \text{ of } D \text{ such that } Y \subseteq_{\mathcal{J}} X_j \text{ for all } j < i \text{ and there is a } U \in \mathbf{V} \text{ satisfying } Y \Vdash_{\mathbb{P}_{\mathcal{J}}} \dot{G} \cap \check{\mathcal{A}}_i = \check{U}$$

and

$$U_i = \text{the unique } U \in \mathbf{V} \text{ such that } X_i \Vdash_{\mathbb{P}_\mathcal{J}} \dot{G} \cap \check{A}_i = \check{U}.$$

Letting

$$\mathcal{S}_\gamma(\langle \mathcal{A}_j, U_j \mid j < i \rangle \wedge \langle \mathcal{A}_i \rangle) = U_i,$$

it is straightforward to verify that \mathcal{S}_γ is a winning strategy for Player II in \mathcal{G}_γ^W . \square

6. The Model

In this section, we give a construction of a model where the following holds.

$$\kappa \text{ is inaccessible and carries no saturated ideals} \quad (20)$$

and

$$\begin{aligned} &\text{For every regular uncountable } \gamma \leq \kappa \text{ there is an ideal } \mathcal{J}_\gamma \text{ on } \mathcal{P}(\kappa) \\ &\text{as in Theorem 1.5, that is, } \mathcal{J}_\gamma \text{ is uniform, normal, } \gamma\text{-densely treed} \quad (21) \\ &\text{and } (\kappa^+, \infty)\text{-distributive.} \end{aligned}$$

The model is a forcing extension of a universe \mathbf{V} in which the following are satisfied.

- (A) GCH.
- (B) U is a normal measure on κ .
- (C) $\langle T_{\alpha, \xi} \mid \xi < \alpha^+ \rangle$ is a disjoint sequence of stationary subsets of $\alpha^+ \cap \text{cof}(\alpha)$ whenever $\alpha \leq \kappa$ is inaccessible.
- (D) Assume $\mathbf{V}[K]$ is a generic extension via a set-size forcing which preserves κ^+ , and, in $\mathbf{V}[K]$
 - there is a definable class elementary embedding $j' : \mathbf{V} \rightarrow M'$ where M' is transitive, and
 - Letting

$$\begin{aligned} &\langle \langle T'_{\alpha, \xi} \mid \xi < \alpha^+ \rangle \mid \alpha \leq j'(\kappa) \text{ is inaccessible in } M' \rangle \\ &= j'(\langle \langle T_{\alpha, \xi} \mid \xi < \alpha^+ \rangle \mid \alpha \leq \kappa \text{ is inaccessible} \rangle) \end{aligned}$$

$$\mathbf{V}, M' \text{ agree on what } H_{\kappa^+} \text{ is and } T'_{\kappa, \xi} = T_{\kappa, \xi} \text{ whenever } \xi < \kappa^+.$$

We will informally explain the purpose of the sets $T_{\alpha, \xi}$ before we begin with the construction of the model. These sets are not needed for the construction of ideals \mathcal{J}_γ in Theorem 1.5, but only for the proof that κ does not carry a saturated ideal in our model. To understand this proof, it suffices to accept (D) as a black box, that is, it is not necessary to understand how the system of sets $T_{\alpha, \xi}$ is constructed.

Proper class models satisfying (A)–(D) are known to exist, and can be produced via the so-called background certified constructors. The two most used background certified constructions are K^c -constructions and fully background certified constructions. If there is a proper class inner model with a measurable cardinal then any \mathbf{K}^c -construction (see for instance [21] for \mathbf{K}^c -constructions of models with Mitchell-Steel indexing of extenders, and [25] for \mathbf{K}^c -constructions with Jensen's λ -indexing)

performed inside such a model gives rise to a fine structural proper class model satisfying (A)–(D). We will sketch a proof of this fact below in Proposition 6.1. Similar conclusions are true of fully background certified constructions, but one needs to assume that a measurable cardinal exists in \mathbf{V} .

There is some similarity in the argument in Proposition 6.1 of the existence of a sequence of mutually disjoint stationary subsets $T_{\kappa,\xi}$ of κ^+ which behave nicely with respect to the ultrapower by a normal ultrafilter on κ to a similar claim in [10] where it is proved that one can have such sequence of stationary sets in $\mathbf{L}[U]$.

A background certified construction as above gives rise to a model of the form $\mathbf{L}[E]$ where $E = \langle E_\alpha \mid \alpha \in \mathbf{On} \rangle$ is such that each E_α either codes an extender in a way made precise, or $E_\alpha = \emptyset$. Additionally, a model of this kind admits a detailed fine structure theory. There is an entire family of such models, so-called fine structural models; the internal first-order theory of these models is essentially the same, up to the large cardinal axioms. There are $L[E]$ models with the properties needed for the construction in this paper that satisfy the statement:

There is a Woodin cardinal κ that is a limit of Woodin cardinals,

as is shown in [18].

We now list some notation, terminology and general facts which will be used for the proof of (C) and (D). Clauses (A) and (B) follow from the construction of the $L[E]$ model, and their proofs can be found in [21] or [25]. In fact, each proper initial segment of the model is acceptable in the sense of fine structure theory. We omit the technical definition here and merely say that acceptability is a local form of GCH, and is proved along the way the model is constructed.

From now on assume $W = \mathbf{L}[E]$ is a fine structural extender model with indexing of extenders as in [21] or in [25, Chap. 9] (which covers all $\mathbf{L}[E]$ models discussed above). We often write E^W in place of E to emphasize that E is the extender sequence of W .

- FS1 $W \parallel \alpha$ is the initial segment of W of height $\omega\alpha$ with the top predicate, that is, $W \parallel \alpha = (J_\alpha^E, E_{\omega\alpha})$.
- FS2 If α is a cardinal of W then $E_\alpha = \emptyset$. Thus, in this case $W \parallel \alpha = (J_\alpha^E, \emptyset)$ and we identify this structure with J_α^E .
- FS3 If μ is a cardinal of W then the structure $W \parallel \mu$ calculates all cardinals and cofinalities $\leq \mu$ the same way as W . This is a consequence of acceptability.
- FS4 $\beta(\tau)$ is the unique β such that τ is a cardinal in $W \parallel \beta$ but not in $W \parallel (\beta + 1)$.
- FS5 ϱ^1 stands for the first projectum; that $\varrho^1(W \parallel \beta) \leq \alpha$ is equivalent to saying that there is a surjective partial map $f : \alpha \rightarrow J_\beta^E$ which is Σ_1 -definable over $W \parallel \beta$ with parameters.
- FS6 (Coherence.) If $i : W \rightarrow W'$ is a Σ_1 -preserving map in possibly some outer universe of W such that κ is the critical point of i and $\tau = (\kappa^+)^W$ then $E^{W'} \upharpoonright \tau = E^W \upharpoonright \tau$.

FS7 (Cores.) Assume α is a cardinal in W and N is a structure such that $\varrho^1(N) = \alpha$ and there is a Σ_1 -preserving map π of N into a level of W such that $\pi \upharpoonright \alpha = \text{id}$. Let p_N be the $<^*$ -least finite set of ordinals p such that there is a set of ordinals a which is $\Sigma_1(N)$ -definable in the parameter p and satisfies $a \cap \alpha \notin N$. Here $<^*$ is the usual well-ordering of finite sets of ordinals, that is, finite sets of ordinals are viewed as descending sequences and $<^*$ is the lexicographical ordering of these sequences. Let X be the Σ_1 -hull of $\alpha \cup \{p_N\}$ and $\sigma : \bar{N} \rightarrow N$ be the inverse of the collapsing isomorphism. Then $\rho^1(\bar{N}) = \alpha$, the models \bar{N}, N agree on what $\mathcal{P}(\alpha)$ is, and π is Σ_1 -preserving and maps \bar{N} cofinally into N . In this situation, \bar{N} is called the core of N and σ is called the core map.

FS8 (Condensation lemma.) Assume α is a cardinal in W and N, \bar{N}, π and σ are as in FS7. Then \bar{N} is a level of W , that is, $\bar{N} = W \parallel \beta$ for some β .

Proposition 6.1. *There is a formula $\varphi(u, v, w)$ in the language of extender models such that the following holds. If $W = \mathbf{L}[E]$ is a fine structural extender model, α is an inaccessible cardinal of W and $\xi < \alpha^+$, letting*

$$T_{\alpha, \xi} = \{\tau \in \alpha^+ \cap \text{cof}(\alpha) \mid W \parallel (\alpha^+)^W \models \varphi(\tau, \alpha, \xi)\},$$

each $T_{\alpha, \xi}$ is a stationary subset of $\alpha^+ \cap \text{cof}(\alpha)$ in W , and $T_{\alpha, \xi} \cap T_{\alpha, \xi'} = \emptyset$ whenever $\xi \neq \xi'$. Moreover, the sequence $\langle T_{\alpha, \xi} \mid \xi < \alpha^+ \rangle$ is inaccessible in W and satisfies clause (D) above with W in place of \mathbf{V} .

Proof. Since the definition of $\langle T_{\alpha, \xi} \mid \xi < \alpha^+ \rangle$ takes place inside $W \parallel (\alpha^+)^W$, any two extender models W, W' such that $(\alpha^+)^W = (\alpha^+)^{W'}$ and $E^W \upharpoonright \alpha^+ = E^{W'} \upharpoonright \alpha^+$ calculate this sequence the same way (here α^+ stands for the common value of the cardinal successor of α in both models). Now if $\mathbf{V} = W$ and j is as in (D) above then

$$T'_{\alpha, \xi} = \{\tau \in (\alpha^+)^{M'} \cap \text{cof}(\alpha) \mid M' \parallel (\alpha^+)^{M'} \models \varphi(\tau, \alpha, \xi)\},$$

whenever $\alpha \leq j'(\kappa)$ is inaccessible in M' , so to see that $T'_{\kappa, \xi} = T_{\kappa, \xi}$ for all $\xi < \kappa^+$ it suffices to prove that $(\kappa^+)^{M'} = (\kappa^+)^{\mathbf{V}}$ and $E^{\mathbf{V}} \upharpoonright \kappa^+ = E^{M'} \upharpoonright \kappa^+$ (where again κ^+ stands for the common value of the cardinal successor of κ in \mathbf{V} and M'). Regarding the former, the inequality $(\kappa^+)^{\mathbf{V}} \leq (\kappa^+)^{M'}$ is entirely general and follows from the fact that $\mathcal{P}(\kappa^{\mathbf{V}}) \subseteq \mathcal{P}(\kappa)^{M'}$. The reverse inequality follows from the assumption that the generic extension preserves κ^+ , so $(\kappa^+)^{\mathbf{V}}$ remains a cardinal in M' . The latter is then a consequence of the coherence property FS6.

It remains to come up with a formula φ such that the sets $T_{\alpha, \xi}$ are stationary in W for all α, ξ of interest, and pairwise disjoint. Here we make a more substantial use of the fine structure theory of W . Given an inaccessible α and a $\xi < \alpha^+$, letting

$$T_{\alpha, \xi} \stackrel{\text{def}}{=} \text{the set of all } \tau \in \alpha^+ \cap \text{cof}(\alpha) \text{ such that } \varrho^1(W \parallel \beta(\tau)) = \alpha \quad (22)$$

and $W \parallel \beta(\tau)$ has $\xi + 1$ cardinals above α ,

it is clear that $T_{\alpha,\xi} \cap T_{\alpha,\xi'} = \emptyset$ whenever $\xi \neq \xi'$. Then it suffices to show that

$$T_{\alpha,\xi} \text{ is stationary in } W, \tag{23}$$

as we can then take φ be the defining formula for the system $(T_{\alpha,\xi})_{\alpha,\xi}$.

The first step toward the proof of (23) is the following observation.

$$\begin{aligned} &\text{Assume } \nu > \alpha \text{ is regular in } W, p \in W \parallel \nu \text{ and } X \text{ is the } \Sigma_1\text{-hull of} \\ &\alpha \cup \{p\} \text{ in } W \parallel \nu. \text{ Let } \nu^X = \sup(X \cap \nu). \text{ Then } \text{cof}^W(\nu^X) = \alpha. \end{aligned} \tag{24}$$

Proof. Obviously, $\gamma = \text{cof}^W(\nu^X) \leq \alpha$. Assume for a contradiction that $\gamma < \alpha$. Let $\langle \nu_i \mid i < \gamma \rangle$ be an increasing sequence converging to ν^X such that $\nu_i \in X$ for every $i < \gamma$. For each such i pick a $j_i \in \omega$ and an ordinal $\eta_i < \alpha$ such that $\nu_i = h_{W \parallel \nu}(j_i, \langle \eta_i, p \rangle)$ where $h_{W \parallel \nu}$ is the standard Σ_1 -Skolem function for $W \parallel \nu$. Here $W \parallel \nu$ is of the form $\langle J_{\nu}^E, \emptyset \rangle$ (see FS2), and we identify it with the structure J_{ν}^E . The Skolem function $h_{W \parallel \nu}$ has a Σ_1 -definition of the form $(\exists w)\psi(w, u_0, u_1, v)$ where ψ is a Δ_0 -formula in the language of extender models. (The standard Σ_1 -Skolem function has a uniform Σ_1 -definition, which means that there is a Σ_1 -formula which defines a Σ_1 -Skolem function h_N over every acceptable structure N . However, the argument below does not make use of uniformity of the definition.) Since $\nu > \alpha$ is regular

$$(\exists \bar{v})(J_{\bar{v}}^E \models (\forall i < \gamma)(\exists w)(\exists v)\psi(w, j_i, \langle \eta_i, p \rangle, v)). \tag{25}$$

Since the statement in (25) is Σ_1 , there is some such \bar{v} with $J_{\bar{v}}^E \in X$. To justify this note that the sequences $\langle \eta_i \mid i < \gamma \rangle$ and $\langle j_i \mid j < \gamma \rangle$ are elements of X as $J_{\alpha}^E \subseteq X$, and we can view these sequences as parameters in the formula in (25). Fix such an ordinal \bar{v} . Now consider $i < \gamma$ such that $\nu_i > \omega \bar{v}$. Using (25) pick z and ν^* in $J_{\bar{v}}^E$ such that $J_{\bar{v}}^E \models \psi(z, j_i, \langle \eta_i, p \rangle, \nu^*)$. Since ψ is Δ_0 , we actually have $J_{\nu}^E \models \psi(z, j_i, \langle \eta_i, p \rangle, \nu^*)$, which tells us that $\nu^* = h_{W \parallel \nu}(j_i, \langle \eta_i, p \rangle) = \nu_i$. As $\nu_i > \omega \bar{v}$, this is a contradiction. This completes the proof of (24). \square

Now let C be a club subset of α^+ , X be the Σ_1 -hull of $\alpha \cup \{C, \xi, \alpha^{+\xi+1}\}$ in $W \parallel \alpha^{+\xi+2}$, N be the transitive collapse of X , and $\pi : N \rightarrow W \parallel \alpha^{+\xi+2}$ be the inverse of the collapsing isomorphism. Let further $\tau = X \cap \alpha^+ = \text{cr}(\pi)$. Then $\tau > \xi$ as $\alpha \cup \{\xi\} \subseteq X$. It is a standard fact that $\text{cof}^W(\tau) = \text{cof}^W(\sup(X \cap \mathbf{On}))$ (and can be proved similarly as (24) above). Now $\text{cof}^W(\sup(X \cap \mathbf{On})) = \alpha$ by (24), hence $\text{cof}^W(\tau) = \alpha$. Moreover $\tau \in C$ as C is closed and τ is a limit point of C . Thus, the proof of (22) will be complete once we show that $\rho^1(W \parallel \beta(\tau)) = \alpha$ and $W \parallel \beta(\tau)$ has $\xi + 1$ cardinals above κ . We first look at the set of cardinals in N .

By acceptability, the structures $W \parallel \alpha^{+\xi+1}$ and $W \parallel \alpha^{+\xi+2}$ agree on what is a cardinal below $\alpha^{+\xi+1}$. It follows that in $W \parallel \alpha^{+\xi+2}$, the statement

“The order type of the set of cardinals in the interval $(\alpha, \alpha^{+\xi+1})$ is ξ ”

can be expressed in a Σ_1 -way as

“The order type of the set of cardinals above α in the structure $W \parallel \alpha^{+\xi+1}$ is ξ .” (26)

Since π is Σ_1 -preserving and $\text{cr}(\pi) = \tau$, this Σ_1 -statement can be pulled back to N via π . Also by the Σ_1 -elementarity of π we have $\pi^{-1}(\alpha^{+\xi+1})$ is the largest cardinal in N . Then, using acceptability in N , we conclude:

The order type of the set of cardinals above α in N is $\xi + 1$. (27)

By construction, the Σ_1 -Skolem function of N induces a partial surjection of α onto N . Then $\varrho^1(N) \leq \alpha$ by FS5. Since α is a cardinal in W , we conclude $\varrho^1(N) = \alpha$. Let \bar{N} be the core of N and $\sigma : \bar{N} \rightarrow N$ be the core map. By FS7, $\varrho^1(\bar{N}) = \alpha$ and $\mathcal{P}(\alpha)^{\bar{N}} = \mathcal{P}(\alpha)^N$, so in particular $\tau = (\alpha^+)^N = (\alpha^+)^{\bar{N}}$. By FS8, $\bar{N} = W \parallel \beta$ for some β . Since $\varrho^1(\bar{N}) = \alpha$, FS5 implies $\beta = \beta(\tau)$. To see that $\bar{N} = W \parallel \beta(\tau)$ has $\xi + 1$ cardinals above α , first note that, since by FS7 the map σ is cofinal, the largest cardinal in N must be in the range of σ . This along with (27) provides a Σ_1 -definition of ξ in N from parameters in $\text{rng}(\sigma)$. The point here is that we can reformulate the notion of cardinal in N below $\alpha^{+\xi+1}$ as the cardinal in the sense of the structure $N \parallel \alpha^{+\xi+1}$, similarly as in (26). It follows that $\xi \in \text{rng}(\sigma)$, and since $\xi < (\alpha^+)^N$ we have $\xi < \text{cr}(\sigma)$. Then, using the Σ_1 -reformulation of (27) one more time, we conclude that $\alpha^{+\eta} \in \text{rng}(\sigma)$ for every $\eta \leq \xi$, which means that $W \parallel \beta(\tau) = \bar{N}$ has $\xi + 1$ cardinals above α . This completes the proof of (22) and thereby the proof of Proposition 6.1.

6.1. The tools

Two main tools we will use to construct the forcing used to build our model are club shooting with initial segments, and adding non-reflecting stationary sets with initial segments. We then use variations of standard techniques for building ideals using elementary embeddings. The background information on the first two can be found in [4–6] and on ideal constructions in [9], but we review the relevant facts for the reader’s convenience. When discussing the successor of a regular cardinal λ we will often assume GCH even when it is known that $\lambda^{<\lambda}$ suffices. Since the models we work in satisfy the GCH this is not important for our results.

Recall that if $S \subseteq \lambda^+$ is a stationary set (where λ is a cardinal) then the club shooting partial ordering $\mathbb{CS}(S)$ consists of closed bounded subsets of λ^+ which are contained in S , and is ordered by end-extension. In general, this partial ordering may not have good preservation properties, but if S is sufficiently large then it is known to be highly distributive. The following is standard.

Proposition 6.2 (See [3, 5, 6]). *Assume λ is regular, $\lambda^{<\lambda} = \lambda$ and T is a subset of λ^+ such that $T \cap \alpha$ is non-stationary in α whenever $\alpha < \lambda^+$, and $(\lambda^+ \cap \text{cof}(\lambda)) \setminus T$*

is stationary. Then the following hold.

- (a) $\mathbb{CS}(\lambda^+ \setminus T)$ is (λ^+, ∞) -distributive, that is, it does not add any new function $f : \lambda \rightarrow \mathbf{V}$. In particular, generic extensions of \mathbf{V} via $\mathbb{CS}(\lambda^+ \setminus T)$ agree with \mathbf{V} on all cardinals and cofinalities $\leq \lambda^+$, and on what H_{λ^+} is.
- (b) If $\gamma \leq \lambda$ is regular and $T \subseteq \lambda^+ \cap \text{cof}(\gamma)$ then $\mathbb{CS}(\lambda^+ \setminus T)$ has a dense set which is γ -closed but if T is stationary then it does not have a dense set which is γ^+ -closed.
- (c) If G is $(\mathbb{CS}(\lambda^+ \setminus T), \mathbf{V})$ -generic then $C_G = \bigcup G$ is a closed unbounded subset of λ^+ such that $C_G \subseteq \lambda^+ \setminus T$.

To show that there is no saturated ideal in the model of Theorem 1.5 and Corollary 1.6 we will need to see that the forcing for shooting a closed unbounded set through the complement of a non-reflecting stationary set A preserves stationary sets disjoint from A . This is the content of the following proposition that appears in [3, 5, 6]. We give the proof here for the reader's convenience.

Lemma 6.3. *Assume λ is an uncountable cardinal with $\lambda^{<\lambda} = \lambda$, A_1, A_2 are disjoint stationary subsets of λ^+ and that for all $\delta < \lambda^+$, $A_2 \cap \delta$ is non-stationary. If $G \subseteq \mathbb{CS}(\lambda^+ \setminus A_2)$ is generic, then A_1 remains stationary in $V[G]$.*

Proof. Let $p \in \mathbb{CS}(\lambda^+ \setminus A_2)$ force that \dot{D} is a closed unbounded subset of λ^+ with $\dot{D} \cap A_1 = \emptyset$. Let $\theta > (2^{2^\lambda})$ be a regular cardinal and let $\langle N_\alpha : \alpha < \lambda^+ \rangle$ be an internally approachable sequence of elementary substructures of $\langle H_\theta, \epsilon, <_\theta, \{A_1, A_2, p, \dot{D}\} \rangle$. Then $\langle N_\alpha \cap \lambda^+ : \alpha \text{ is a limit} \rangle$ is a closed unbounded subset of λ^+ and for each such α , $N_\alpha^{<\text{cof}(\alpha)} \subseteq N_\alpha$.

Choose a limit δ such that $N_\delta \cap \lambda^+ \in A_1$. Let $\gamma = \text{cof}(\delta)$ and $C_\delta \subseteq (\delta \setminus A_2)$ be a closed unbounded set of order type γ . Build a decreasing sequence of conditions $\langle p_\alpha : \alpha < \gamma \rangle$ such that

- $p_0 = p$
- for each $\beta < \gamma$, $\langle p_\alpha : \alpha < \beta \rangle \in N_\delta$
- if i is the α th member of C_δ , then for some ordinal $\xi < \delta$, with $i < \xi$

$$p_{\alpha+1} \Vdash \xi \in \dot{D}.$$

- $\text{sup}(p_{\alpha+1}) > i$.

Such a sequence is possible to build, because $N_\delta^{<\gamma} \subseteq N_\delta$.

But then $\text{sup}(\bigcup_{\alpha < \gamma} p_\alpha) = \delta$ and $\delta \notin A_2$, hence

$$q = \bigcup_{\alpha < \gamma} p_\alpha \cup \{\delta\} \in \mathbb{CS}(\lambda^+ \setminus A_2).$$

Moreover $q \Vdash \dot{D} \cap A_1 \neq \emptyset$. This contradiction establishes Lemma 6.3. □

Our application of the following definition and the following lemmas will be with $\mu = \lambda^+$ for a regular λ .

Definition 6.4. Let μ be a regular cardinal.

- (a) The partial ordering $\text{NR}(\mu)$ for adding a non-reflecting stationary subset of μ consists of functions $p : \alpha \rightarrow \{0, 1\}$ for some $\alpha < \mu$ and letting

$$S_p = \{\xi < \alpha \mid p(\xi) = 1\},$$

for every limit $\bar{\alpha} \leq \alpha$ there is a closed unbounded set $C \subseteq \bar{\alpha}$ such that $S_p \cap C = \emptyset$.

- (b) Let $\gamma < \mu$ be regular. The partial ordering $\text{NR}(\mu, \gamma)$ for adding a non-reflecting stationary subset of $\mu \cap \text{cof}(\gamma)$ consists of those conditions $p \in \text{NR}(\mu)$ which concentrate on $\mu \cap \text{cof}(\gamma)$:

$$p(\xi) = 0 \quad \text{whenever } \text{cof}(\xi) \neq \gamma.$$

Let $\gamma < \mu$ be uncountable regular cardinals and define the map

$$\pi_\gamma : \text{NR}(\mu) \rightarrow \text{NR}(\mu, \gamma)$$

by setting

$$\pi_\gamma(p)(\xi) = \begin{cases} p(\xi) & \text{if } \text{cof}(\xi) = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

We will use the following lemma which relates $\text{NR}(\mu)$ with $\text{NR}(\mu, \gamma)$.

Lemma 6.5. *Let $\gamma < \mu$ be uncountable regular cardinals.*

- (a) *If G is generic for $\text{NR}(\mu)$, then*

$$S_G \stackrel{\text{def}}{=} \{\xi < \mu : \text{for some } p \in G, p(\xi) = 1\}$$

is a non-reflecting stationary subset of μ .

- (b) *If H is generic for $\text{NR}(\mu, \gamma)$ then*

$$S_H =_{\text{def}} \{\xi < \mu : \text{for some } p \in H, p(\xi) = 1\}$$

is a non-reflecting stationary subset of $\mu \cap \text{cof}(\gamma)$.

- (c) *If $G \subseteq \text{NR}(\mu)$ is generic over V , and $H = \pi_\gamma G$, then H is generic over V for $\text{NR}(\mu, \gamma)$. (In other words the map π_γ is a projection.)*

Proof. The first two items are immediate. For the third note that for all $p \in \text{NR}(\mu)$ and all $q \in \text{NR}(\mu, \gamma)$ with $q \leq_{\text{NR}(\mu, \gamma)} \pi_\gamma(p)$, there is a $p' \leq_{\text{NR}(\mu)} p$ with $\pi_\gamma(p') \leq_{\text{NR}(\mu, \gamma)} q$. This is the standard criterion for being a projection. \square

It is an easy remark that in (a) $V[G] = V[S_G]$ and in (b) $V[H] = V[S_H]$. For this reason, we will frequently write $V[S]$ for the extension, when it is clear from context whether we are in case (a) or (b).

We will make use of these partial orderings in the special case where μ is of the form λ^+ . For this reason we formulate the next proposition for cardinals of the form λ^+ , although it is true for any regular $\mu > \omega$.

Proposition 6.6 (See [4–6]). *Assume $\gamma \leq \lambda$ where γ is regular and $\lambda^{<\lambda} = \lambda$. Then the following hold.*

- (a) *Both $\text{NR}(\lambda^+)$ and $\text{NR}(\lambda^+, \gamma)$ are strategically λ^+ -closed. In particular, both $\text{NR}(\lambda^+)$ and $\text{NR}(\lambda^+, \gamma)$ preserve stationarity of stationary subsets of λ^+ , are (λ^+, ∞) -distributive, so they do not add any new functions $f : \lambda \rightarrow \mathbf{V}$, and generic extensions of \mathbf{V} via these partial orderings agree with \mathbf{V} on all cardinals and cofinalities $\leq \lambda^+$ and on what H_{λ^+} is.*
- (b) *$\text{NR}(\lambda^+, \gamma)$ is γ -closed but not γ^+ -closed.*
- (c) *If G is $(\text{NR}(\lambda^+, \gamma), \mathbf{V})$ -generic then $S_G = \bigcup\{S_p \mid p \in G\}$ is a non-reflecting stationary subset of $\lambda^+ \cap \text{cof}(\gamma)$.*
- (d) *If G is $(\text{NR}(\lambda^+), \mathbf{V})$ -generic then $S_G = \bigcup\{S_p \mid p \in G\}$ is a non-reflecting stationary subset of λ^+ such that S_G has stationary intersection with each stationary subset of λ^+ that lies in V . In particular, $S_G \cap \lambda^+ \cap \text{cof}(\gamma)$ is stationary for all regular $\gamma < \lambda^+$.*

Proof. Only (d) is not explicitly proved in the earlier literature (though it was known). Let T be a stationary subset of λ^+ in V . Let $G \subseteq \text{NR}(\lambda^+)$ be generic and $S \subseteq \lambda^+$ be the generic stationary set added by G . We claim that S has stationary intersection with T . We assume without loss of generality that every ordinal in T has the same cofinality $\gamma \leq \lambda$.

If the claim fails let $p \in \text{NR}(\lambda^+)$ force over V that $\dot{S} \cap T \cap \dot{D} = \emptyset$ where \dot{D} is a term for a closed unbounded subset of λ^+ in $V[G]$.

Let $\theta > (2^{2^\lambda})$ be a regular cardinal and let $\langle N_\alpha : \alpha < \lambda^+ \rangle$ be an internally approachable sequence of elementary substructures of

$$\langle H_\theta, \epsilon, <_\theta, \{\dot{S}, T, \dot{D}\} \rangle.$$

Then $\langle N_\alpha \cap \lambda^+ : \alpha \text{ is a limit} \rangle$ is a closed unbounded subset of λ^+ and for each such α , $N_\alpha^{<\text{cof}(\alpha)} \subseteq N_\alpha$. Choose a limit ordinal δ such that $N_\delta \cap \lambda^+ \in T$ and $N_\delta \cap \lambda^+ = \delta$. Then δ has cofinality γ . Let $C_\delta \subseteq \delta$ be closed and unbounded in δ with order type γ such that every initial segment of C_δ belongs to N_δ .

By recursion on β build a decreasing sequence of conditions $\langle p_\alpha : \alpha < \delta \rangle$ in $\text{NR}(\lambda^+)$ such that

- $p_0 = p$
- for each $\beta < \gamma$, $\langle p_\alpha : \alpha < \beta \rangle \in N_\delta$
- if i is the α^{th} member of C_δ , then for some ordinal $\zeta < \delta$, with $i < \zeta$

$$p_\alpha \Vdash \zeta \in \dot{D}.$$

- $\text{sup}(\text{dom}(p_\alpha)) > i$.
- If β is a limit ordinal, then $p_\beta = \bigcup_{\bar{\beta} < \beta} p_{\bar{\beta}}$ and if $\delta_\beta = \text{sup}(\bigcup_{\bar{\beta} < \beta} \text{dom}(p_{\bar{\beta}}))$, then $p_{\beta+1}$ forces $\delta_\beta \notin S$.

Let $p^* = \bigcup_{\beta < \delta} p_\beta$. Then $\text{dom}(p^*)$ has supremum δ and forces that

- $\dot{S} \cap \delta$ is non-stationary
- $\dot{D} \cap \delta$ is cofinal in δ

Extending p^* by one point to get a condition q that forces $\delta \in \dot{S}$ gives a condition $q \in \text{NR}(\lambda^+)$ that forces $\delta \in \dot{S} \cap T \cap \dot{D}$. This contradiction shows that in the extension by $\text{NR}(\lambda^+)$, S intersects every stationary T . \square

Although both partial orderings $\text{NR}(\lambda^+, \gamma)$ and $\text{CS}(S)$ have a low degree of closure in general, the iteration $\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S})$ that generically adds a non-reflecting stationary set S followed by adding a closed unbounded subset of the complement of S does have a high degree of closure.

Proposition 6.7. *Assume λ is a cardinal, $\gamma \leq \lambda$ is regular, and \dot{S} is the canonical $\text{NR}(\lambda^+, \gamma)$ -term for the generic non-reflecting stationary subset of $\lambda^+ \cap \text{cof}(\gamma)$. Then the composition*

$$\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S})$$

has a dense λ^+ -closed subset $D \subseteq H_{\lambda^+}$. In particular, this two-step iteration preserves stationarity of stationary subsets of λ^+ .

Proof. Let D be the collection of all $(p, \dot{c}) \in \text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S}) \cap H_{\lambda^+}$ such that

- $\{\xi : p(\xi) = 0\}$ is closed (so has successor order type), and
- $p \Vdash \dot{c} = \check{c}$ for some closed unbounded set $c \subseteq \text{dom}(p)$ with $p(\xi) = 0$ for all $\xi \in c$.

Then D is dense in $\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S})$. For details, see [4, 5] or [6]. \square

Fix a regular cardinal λ . At successor steps in the iteration used to prove Theorem 1.5, we will use an iteration of the form

$$\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{T}) * \text{CS}(\lambda^+ \setminus \dot{S}), \tag{28}$$

where \dot{S} is a term for the generic non-reflecting stationary subset of $\lambda^+ \cap \text{cof}(\gamma)$ given by $\text{NR}(\lambda^+, \gamma)$ and \dot{T} will be a term for a certain subset of $\lambda^+ \cap \text{cof}(\lambda)$. We note in passing that the realization of \dot{T} is a non-reflecting stationary set. Since both \dot{S} and \dot{T} lie in $V^{\text{NR}(\lambda^+, \gamma)}$, the following three forcing notions are equivalent:

Version 1: $\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{T}) * \text{CS}(\lambda^+ \setminus \dot{S})$

Version 2: $\text{NR}(\lambda^+, \gamma) * (\text{CS}(\lambda^+ \setminus \dot{T}) \times \text{CS}(\lambda^+ \setminus \dot{S}))$

Version 3: $\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S}) * \text{CS}(\lambda^+ \setminus \dot{T})$.

Lemma 6.8. *Let $\mathbb{P} = \text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{T}) * \text{CS}(\lambda^+ \setminus \dot{S})$. Then \mathbb{P} has a dense set D such that*

- (i) D has cardinality λ^+ ,
- (ii) $D \subseteq H_{\lambda^+}$,
- (iii) D is λ -closed, and
- (iv) D is (λ^+, ∞) -distributive.

Proof. Proposition 6.7 shows that $\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S})$ has a dense λ^+ -closed subset. Since \dot{T} consists of ordinals of cofinality λ , $\text{CS}(\lambda^+ \setminus \dot{T})$ is λ -closed and (λ^+, ∞) -distributive. Since \mathbb{P} is isomorphic to $\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S}) * \text{CS}(\lambda^+ \setminus \dot{T})$, items (iii) and (iv) follow. Now (i) is immediate, since $\text{CS}(\lambda^+ \setminus \dot{T})$ has a dense set of size λ^+ after forcing with the first two partial orderings.

To see (ii), use Version 3 of the partial ordering \mathbb{P} . The first step is clearly a subset of H_{λ^+} . By Proposition 6.7 there is a dense subset of the first two steps that lies in H_{λ^+} and is λ^+ -closed. After forcing with $\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S})$ the conditions in $\text{CS}(\lambda^+ \setminus \dot{T})$ belong to H_{λ^+} and can be realized by elements of V using the closure of $\text{NR}(\lambda^+, \gamma) * \text{CS}(\lambda^+ \setminus \dot{S})$. Hence, there is a dense subset of Version 3 consisting of triples (p, c, d) where each coordinate belongs to H_{λ^+} . Rearranging, we get (ii). \square

In the iteration, we will construct \dot{T} as a coding tool. Let $\{T_\xi : \xi < \lambda^+\}$ be a sequence of disjoint stationary subsets of $\lambda^+ \cap \text{cof}(\lambda)$. Let $S \subset \lambda^+$ and define

$$T(S) = \bigcup_{\xi \in S} T_{2\xi} \cup \bigcup_{\xi \notin S} T_{2\xi+1}. \tag{29}$$

We will use $T(S)$ for a set S that is V -generic for $\text{NR}(\lambda^+)$. When forcing with $\text{NR}(\lambda^+, \gamma)$ we will use the following variant:

$$T_\gamma(S) = \bigcup_{\xi \in S \cap \text{cof}(\gamma)} T_{2\xi} \cup \bigcup_{\xi \in \text{cof}(\gamma) \cap (\lambda^+ \setminus S)} T_{2\xi+1}. \tag{30}$$

Given an $\text{NR}(\lambda^+)$ -generic $S \subseteq \lambda$, and a sequence of sets S_γ for each regular uncountable $\gamma \leq \lambda$ with $S_\gamma = S \cap \text{cof}(\gamma)$, the following holds:

$$T(S) = \bigcup_{\gamma} T_\gamma(S_\gamma) \tag{31}$$

In particular, if $\delta \notin T(S)$ then $\delta \notin T_\gamma(S)$.

Proposition 6.9. *Suppose λ is regular and the GCH holds. Let \mathbb{P} be the partial ordering*

$$\text{NR}(\lambda^+) * \text{CS}(\lambda^+ \setminus T(\dot{S})) * \text{CS}(\lambda^+ \setminus \dot{S})$$

where \dot{S} is the canonical $\text{NR}(\lambda^+)$ -term for the generically added non-reflecting stationary set S and $T(\dot{S})$ is the canonical $\text{NR}(\lambda^+)$ -term for the set $T(S)$. If $G \subseteq \mathbb{P}$ is generic then in $V[G]$:

- (a) If $\xi \in \dot{S}^G$, then $T_{2\xi}$ is non-stationary and $T_{2\xi+1}$ is stationary.
- (b) If $\xi \notin \dot{S}^G$, then $T_{2\xi+1}$ is non-stationary, and $T_{2\xi}$ is stationary.

Proof. Force with $\text{NR}(\lambda^+)$ to get a generic stationary set S and let \dot{C} be a term for the closed unbounded set added by $\text{CS}(\lambda^+ \setminus T(S))$. If $H \subseteq \text{CS}(\lambda^+ \setminus T(S))$ is $V[S]$ -generic then $\dot{C}^H \cap T(S)$ is empty which shows the non-stationarity claims in both (a) and (b).

What is left is to show that the appropriate T_η 's stationarity is preserved. The argument in each case is the same, so assume we argue for case (a). Since the partial ordering $\mathbb{NR}(\lambda^+) * \mathbb{CS}(\lambda^+ \setminus \dot{S})$ has a dense $<\lambda^+$ -closed subset, it preserves the stationarity of each T_ξ .

Suppose that $\xi \in \dot{S}^G$. Applying Lemma 6.3 in $V[G]$ with $A_1 = T_{2\xi+1}$ and $A_2 = T(S)$ shows that $T_{2\xi+1}$ is stationary in $V[G][H]$. \square

Essentially, the same proof shows:

Proposition 6.10. *Suppose λ is regular, $\gamma \leq \lambda$ is regular and uncountable, and that the GCH holds. Let \mathbb{P} be the partial ordering*

$$\mathbb{NR}(\lambda^+, \gamma) * \mathbb{CS}(\lambda^+ \setminus T_\gamma(\dot{S})) * \mathbb{CS}(\lambda^+ \setminus \dot{S})$$

where \dot{S} and $T_\gamma(\dot{S})$ are defined as in Proposition 6.9. If $G \subseteq \mathbb{P}$ is generic then in $V[G]$:

- (a) *If $\xi \in \dot{S}^G \cap \text{cof}(\gamma)$, then $T_{2\xi}$ is non-stationary and $T_{2\xi+1}$ is stationary.*
- (b) *If $\xi \in \text{cof}(\gamma) \cap (\lambda^+ \setminus \dot{S}^G)$, then $T_{2\xi+1}$ is non-stationary, and $T_{2\xi}$ is stationary.*
- (c) *If $\xi \notin \text{cof}(\gamma)$, then T_ξ is stationary.*

The point of this coding is that using the forcing in either Proposition 6.9 or 6.10, for ξ of the appropriate cofinality we have

$$\xi \in S \text{ if and only if } T_{2\xi} \text{ is non-stationary and } T_{2\xi+1} \text{ is stationary.}$$

Proposition 6.11. *Under the hypotheses of Proposition 6.9 (or Proposition 6.10), the set S added by $\mathbb{NR}(\lambda^+)$ (respectively, $\mathbb{NR}(\lambda^+, \gamma)$) remains stationary after forcing with $\mathbb{CS}(\lambda^+ \setminus T(\dot{S}))$ (respectively, $\mathbb{CS}(\lambda^+ \setminus T_\gamma(\dot{S}))$).*

Proof. We prove it with the hypotheses of Proposition 6.9, the proof using the hypotheses of Proposition 6.10 is essentially the same.

Let $G \subseteq \mathbb{NR}(\lambda^+)$ be generic and $S \subseteq \lambda^+$ be the generic stationary set constructed by G . By Proposition 6.6 item (d), in $V[G]$, S has stationary intersection with each T_ξ . Choose a ξ_0 such that $T_{\xi_0} \cap T(S) = \emptyset$. Let $A_1 = S \cap T_{\xi_0}$ and $A_2 = T(S)$. The A_1 and A_2 satisfy the hypotheses of Lemma 6.3 for the forcing $\mathbb{CS}(\lambda^+ \setminus T(S))$. Hence, $S \cap T_{\xi_0}$ is stationary in the generic extension of $V[S]$ by $\mathbb{CS}(\lambda^+ \setminus T(S))$, and so S is stationary after the forcing $\mathbb{NR}(\lambda^+) * \mathbb{CS}(\lambda^+ \setminus T(\dot{S}))$. \square

6.2. The construction

Let U be the normal measure as in (B) above and

$$j : \mathbf{V} \rightarrow M \tag{32}$$

be the ultrapower embedding by U where M is transitive. Let κ be the critical point of j .

The forcing will be an iteration of length $\kappa + 2$ with Easton supports. If $\alpha < \kappa$ is inaccessible we will choose a regular uncountable $\gamma \leq \alpha$ and do a three step forcing. First we add a non-reflecting stationary set S . We then force to code the non-reflecting stationary set using the stationary sets $T_{\alpha,\xi}$. The last step is to shoot a club through the complement of the stationary set S created in the first step.

At stage κ we do the analogous forcing except that we only use the first two steps.

Description of the Forcing. We now formally define the partial orderings used in the construction. For an inaccessible cardinal α fix the stationary sets $\langle T_{\alpha,\xi} : \xi < \alpha^+ \rangle$ from Proposition 6.1. Fix a regular uncountable $\gamma \leq \alpha$. For this γ , let \mathbb{Q}_α^γ be the partial ordering

$$\text{NR}(\alpha^+, \gamma) * \text{CS}(\alpha^+ \setminus T_{\alpha,\gamma}(\dot{S}_{\alpha,\gamma})) * \text{CS}(\alpha^+ \setminus \dot{S}_{\alpha,\gamma}), \quad (33)$$

defined as in Proposition 6.10, with α in place of λ , $T_{\alpha,\gamma}$ in place of T_γ , and $\dot{S}_{\alpha,\gamma}$ in place of \dot{S} . (We will often suppress γ in the notation if γ is clear from the context, and write simply \dot{S}_α .)

The final partial ordering \mathbb{P}^* will be an iteration with Easton supports of length $\kappa + 2$. We define the initial segment of length κ , \mathbb{P}_κ , as follows. \mathbb{P}_κ will be the direct limit of the forcing iteration

$$(\mathbb{P}_\alpha \mid \alpha \leq \kappa)$$

satisfying the following.

FI-1 For inaccessible α , conditions in each \mathbb{P}_α are partial functions p with $\text{dom}(p)$ contained in inaccessibles below α such that $\text{dom}(p) \cap \beta$ is bounded in β whenever $\beta \leq \alpha$ is inaccessible.

FI-2 If $p \in \mathbb{P}_\alpha$ and $\bar{\alpha} \in \text{dom}(p)$ then

$$p(\bar{\alpha}) = (\gamma^p(\bar{\alpha}), w^p(\bar{\alpha}))$$

is an ordered pair such that

$$\gamma^p(\bar{\alpha}) \in R_{\bar{\alpha}} = \{\gamma \leq \bar{\alpha} \mid \gamma \text{ is regular uncountable}\},$$

and $w^p(\bar{\alpha}) \in H_{\alpha^+}$ is a $\mathbb{P}_{\bar{\alpha}}$ -term for a condition in the three step forcing $\mathbb{Q}_{\bar{\alpha}}^{\gamma^p(\bar{\alpha})}$ defined in Eq. (33).^f

The ordering on \mathbb{P}_α is defined in the standard way, that is,

FI-3 $p \leq q$ if and only if the following hold:

- (1) $\text{dom}(p) \supseteq \text{dom}(q)$ and

^fWe can view $w^p(\bar{\alpha})$ as a triple $(w^p(\bar{\alpha}), w^p(\bar{\alpha} + 1), w^p(\bar{\alpha} + 2))$ but the notation $w^p(\bar{\alpha})$ is frequently more convenient.

- (2) for every $\bar{\alpha} \in \text{dom}(q)$:
- (a) $\gamma^p(\bar{\alpha}) = \gamma^q(\bar{\alpha})$ and
 - (b) $p \upharpoonright \bar{\alpha} \Vdash_{\mathbb{P}_{\bar{\alpha}}} \text{“}w^p(\bar{\alpha}) \text{ extends } w^q(\bar{\alpha}) \text{ in } \dot{Q}_{\bar{\alpha}}^{\gamma^p(\bar{\alpha})}\text{”}$
 (where, by $p \upharpoonright \bar{\alpha}$ we mean $p \upharpoonright (\text{dom}(p) \cap \bar{\alpha})$.)

From Lemmas 6.6 to 6.8, we conclude that

- (i) For all inaccessible α , $\mathbb{P}_{\alpha} \subseteq V_{\alpha}$
- (ii) For α Mahlo, \mathbb{P}_{α} is α -c.c.
- (iii) If G is $(\mathbb{P}_{\alpha}, \mathbf{V})$ -generic then in $\mathbf{V}[G]$ the partial ordering $(\dot{Q}_{\alpha}^{\gamma^p(\bar{\alpha})})^G$ contains a dense α -closed set and is (α^+, ∞) -distributive.
- (iv) For $\alpha < \kappa$, if $\mathbb{P}_{\kappa} = \mathbb{P}_{\alpha} * \dot{\mathbb{P}}_{\kappa}^{\alpha}$ is the canonical factorization, and G is $(\mathbb{P}_{\alpha}, \mathbf{V})$ -generic, then

$$\mathbf{V}[G] \models \text{“}(\dot{\mathbb{P}}_{\kappa}^{\alpha})^G \text{ has an } \alpha\text{-closed dense subset”}.$$

- (v) For each inaccessible $\alpha < \kappa$, if $p \in \mathbb{P}_{\kappa}$ then $(p(\bar{\alpha}), p(\bar{\alpha} + 1), p(\bar{\alpha} + 3)) \in H_{\bar{\alpha}^+}$.
- (vi) For all cardinals α , $\mathbb{P}_{\alpha+3}$ preserves both α and α^+ .
- (vii) \mathbb{P}_{κ} preserves all cardinals.

Now define a partial ordering \mathbb{P}^* as the $\kappa + 2$ length iteration:

$$\mathbb{P}^* = \mathbb{P}_{\kappa} * \text{NR}(\kappa^+) * \text{CS}(\kappa^+ \setminus T(\dot{S})) \tag{34}$$

where \dot{S} and $T(\dot{S})$ are as in Proposition 6.9, with κ in place of λ .

We claim that any generic extension via \mathbb{P}^* produces a model as in Theorem 1.5. We will first focus on the proof of the following proposition.

Proposition 6.12. *In any generic extension via \mathbb{P}^* all cardinals and cofinalities are preserved, κ remains inaccessible, and for each regular uncountable $\gamma \leq \kappa$ there is a uniform normal (κ^+, ∞) -distributive ideal \mathcal{J}_{γ} such that $\mathcal{P}(\kappa)/\mathcal{J}_{\gamma}$ has a dense γ -closed set, but no dense γ^+ -closed set.*

Proof. Fix a regular uncountable cardinal $\gamma \leq \kappa$.

By GCH in \mathbf{V} , any generic extension via \mathbb{P}_{κ} satisfies $2^{\kappa} = \kappa^+$, so in any such generic extension the partial ordering $\text{NR}(\kappa^+)$ has cardinality κ^+ . Using the strategic closure of $\text{NR}(\kappa^+)$ we conclude that $2^{\kappa} = \kappa^+$ in the generic extension via $\mathbb{P}_{\kappa} * \text{NR}(\kappa^+)$. Let S be the non-reflecting stationary set added by $\text{NR}(\kappa^+)$. Then $\text{CS}(\kappa^+ \setminus T(S))$ has cardinality κ^+ in any such generic extension. All of this combined with the distributivity properties of $\text{NR}(\kappa^+)$ and $\text{CS}(\kappa^+ \setminus T(S))$, shows that $2^{\kappa} = \kappa^+$. Similar arguments show that

$$\mathbb{P}^* \text{ preserves all cardinals and cofinalities and also the GCH.} \tag{35}$$

Now return to the map j from (32). Let G be $(\mathbb{P}_{\kappa}, \mathbf{V})$ -generic. Because $\text{card}(\mathbb{P}_{\kappa}) = \kappa$ and \mathbb{P}_{κ} is κ -c.c., $M[G]$ is closed under κ -sequences in $\mathbf{V}[G]$ and the models $M[G], \mathbf{V}[G]$ agree on what H_{κ^+} is. It follows that the models $M[G], \mathbf{V}[G]$ agree on what $\text{NR}(\kappa^+)$ and $\text{NR}(\kappa^+, \gamma)$ are.

Let $G' = G'_0 * G'_1$ be $(\mathbb{NR}(\kappa^+) * \mathbb{CS}(\kappa^+ \setminus T(\dot{S})), \mathbf{V}[G])$ -generic where \dot{S} is as in Eq. (34). It follows that $S = \dot{S}^{G'_0} = \bigcup G'_0$.

Let $G_{\kappa,0} = \pi_\gamma(G'_0)$ where π_γ is as in Lemma 6.5. Then $G_{\kappa,0}$ is generic for $\mathbb{NR}(\kappa, \gamma)$ over both $M[G]$ and $V[G]$. Let \dot{S} be the term for the non-reflecting stationary set coming from G'_0 . Then $\dot{S}_{\kappa,\gamma} = \dot{S} \cap \text{cof}(\gamma)$. Denote $S_{\kappa,\gamma}$ by S_κ .

Since $\mathbb{NR}(\kappa^+)$ and $\mathbb{NR}(\kappa^+, \gamma)$ are (κ^+, ∞) -distributive in the models where they live

$$M[G, G_{\kappa,0}] \text{ is closed under } \kappa\text{-sequences lying in } \mathbf{V}[G, G'_0]. \quad (36)$$

In particular, $M[G, G_{\kappa,0}]$ and $\mathbf{V}[G, G'_0]$ agree on what H_{κ^+} and $\mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(S_\kappa))$ are.

Let $C \in \mathbf{V}[G, G']$ be the closed unbounded subset of $\kappa^+ \setminus T(S)$ associated with the generic ultrafilter G'_1 for $\mathbb{CS}(\kappa^+ \setminus T(S))$ over $\mathbf{V}[G, G'_0]$.

Note that $T_{\kappa,\gamma}(S_\kappa) \in M[G, G_{\kappa,0}]$ and $C \cap T_{\kappa,\gamma}(S_\kappa) = \emptyset$ because $T_{\kappa,\gamma}(S_\kappa) \subseteq T(S)$. From the point of view of $\mathbf{V}[G, G'_0]$ there are only κ^+ many dense subsets of $\mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(S_\kappa))$ which are in $M[G, G_{\kappa,0}]$.

We can construct a $(\mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(S_\kappa)), M[G, G_{\kappa,0}])$ -generic filter $G_{\kappa,1} \in \mathbf{V}[G, G']$ as follows. In $\mathbf{V}[G, G'_0]$ fix an enumeration $\langle D_\beta \mid \beta < \kappa^+ \rangle$ of dense subsets of $\mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(S_\kappa))$ which belong to $M[G, G_{\kappa,0}]$. Using recursion on $\beta < \kappa^+$ construct a descending chain $\langle c_\beta, c'_\beta \mid \beta < \kappa^+ \rangle$ in $\mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(S_\kappa))$ as follows:

- Let $c'_0 = \emptyset$.
- Given c'_β , pick $c_\beta \in D_\beta$ such that $c_\beta \leq c'_\beta$ in $\mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(S_\kappa))$.
- Given c_β , let $c'_{\beta+1} = c_\beta \cup \{\delta_{\beta+1}\}$ where $\delta_{\beta+1}$ is the least element of C larger than $\max(c_\beta)$.
- If β is a limit let $c'_\beta = \left(\bigcup_{\bar{\beta} < \beta} c'_{\bar{\beta}} \cup \{\delta_\beta\} \right)$ where $\delta_\beta = \sup\{\max(c'_{\bar{\beta}}) \mid \bar{\beta} < \beta\}$.

To see that this works, note that for every $\beta < \kappa^+$ both c_β and c'_β are elements of $M[G, G_{\kappa,0}]$, which is verified inductively on β . The only non-trivial step in the induction is to see that $c'_\beta \in M[G, G_{\kappa,0}]$ for β limit. That the sequence $\langle c'_\beta : \bar{\beta} < \beta \rangle$ belongs to $M[G, G_{\kappa,0}]$ follows from the κ -closure property of $M[G, G_{\kappa,0}]$. For the union to be a condition requires that the supremum δ of c'_β does not belong to $T_{\kappa,\gamma}(S_\kappa)$. However by Eq. (31), since $\delta \notin T(S)$ we know that $\delta \notin T_{\kappa,\gamma}(S_\kappa)$.

Now let $G_{\kappa,1}$ be the filter on $\mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(S_\kappa))$ generated by the sequence $\langle c_\beta \mid \beta < \kappa^+ \rangle$; it is clear that $G_{\kappa,1} \in \mathbf{V}[G, G']$ and is $(\mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(S_\kappa)), M[G, G_{\kappa,0}])$ -generic. Finally, set $G_\kappa = G_{\kappa,0} * G_{\kappa,1}$.

We note here that by Proposition 6.11, S_κ is stationary in $\mathbf{V}[G, G']$. Thus in $\mathbf{V}[G, G']$, S_κ is a non-reflecting stationary set.

Consider a $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic filter H . Then the filter $G_\kappa * H$ is $(\mathbb{NR}(\kappa^+, \gamma) * \mathbb{CS}(\kappa^+ \setminus T_{\kappa,\gamma}(\dot{S}_\kappa)) * \mathbb{CS}(\kappa^+ \setminus \dot{S}_\kappa), M[G])$ -generic. It follows that $G * G_\kappa * H$ is $(j(\mathbb{P}_\kappa) \upharpoonright (\kappa+3), M)$ -generic. By the Factor Lemma applied inside $M[G, G_\kappa, H]$, the quotient $j(\mathbb{P}_\kappa)/G * G_\kappa * H$ is isomorphic to the iteration $\mathbb{P}_{j(\kappa)}^{\kappa+3}$ as calculated in $M[G, G_\kappa, H]$. Let μ be the least inaccessible of M above κ . Using (iv) in the list

of the properties of the iteration stated below FI-3, we conclude that $M[G, G_\kappa, H]$ satisfies the following:

$$j(\mathbb{P}_\kappa)/G * G_\kappa * H \text{ has a dense } \mu\text{-closed subset.} \quad (37)$$

Since $\mathbb{NR}(\kappa^+) * \mathbb{CS}(\kappa^+ \setminus T(\dot{S}_\kappa)) * \mathbb{CS}(\kappa^+ \setminus \dot{S}_\kappa)$ is (κ^+, ∞) -distributive in $\mathbf{V}[G]$,

$$M[G, G_\kappa, H] \text{ is closed under } \kappa\text{-sequences in } \mathbf{V}[G, G', H]. \quad (38)$$

Working in $\mathbf{V}[G, G', H]$: since the cardinality of $\mathbb{P}_{j(\kappa)}^{\kappa+3}$ is κ^+ , we have an enumeration $\langle D_\beta \mid \beta < \kappa^+ \rangle$ of all dense subsets of $j(\mathbb{P}_\kappa)/G * G_\kappa * H$ which are in $M[G, G_\kappa, H]$. Using (iv) in the list of the properties of the iteration stated below FI-3, the sentences labeled (37), (38) above and the fact that $\mu > \kappa^+$, we can construct a descending sequence $\langle p_\beta \mid \beta < \kappa^+ \rangle$ with each proper initial segment being an element of $M[G, G_\kappa, H]$ and such that $p_\beta \in D_\beta$ for all $\beta < \kappa^+$. Let K_1 be the filter on $j(\mathbb{P}_\kappa)/G * G_\kappa * H$ generated by this sequence. Then K_1 is $(j(\mathbb{P}_\kappa)/G * G_\kappa * H, M[G, G_\kappa, H])$ -generic and $K_1 \in \mathbf{V}[G, G', H]$. Let $K = G * G_\kappa * H * K_1$. Then K can be viewed as a $(j(\mathbb{P}_\kappa), M)$ -generic filter, so we can extend j to an elementary embedding

$$j_{H,K} : \mathbf{V}[G] \rightarrow M[K]$$

defined by setting $j_{H,K}(\dot{x}^G) = j(\dot{x})^K$ whenever $\dot{x} \in \mathbf{V}$ is a \mathbb{P}_κ -term. Since K_1 can be constructed inside $\mathbf{V}[G, G', H]$, there is a $\mathbb{CS}(\kappa^+ \setminus S_\kappa)$ -term $\dot{K}_1 \in \mathbf{V}[G, G']$ such that \dot{K}_1^H is $(j(\mathbb{P}_\kappa)/G * G_\kappa * H, M[G, G_\kappa, H])$ -generic whenever H is $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic. In particular, there is a M -generic $K^H \subseteq j(\mathbb{P})$ determined by forcing over $V[G, G_\kappa]$ to get a generic $H \subseteq \mathbb{CS}(\kappa^+ \setminus S_\kappa)$.

Changing notation slightly to emphasize the dependence on H , define j_H be as follows:

$$j_H = j_{H, \dot{K}^H} : \mathbf{V}[G] \rightarrow M[\dot{K}^H]. \quad (39)$$

We also have a $\mathbb{CS}(\kappa^+ \setminus S_\kappa)$ -term $\dot{U} \in \mathbf{V}[G, G']$ such that \dot{U}^H is the normal $\mathbf{V}[G]$ -measure over κ derived from j_H . That is,

$$\dot{U}^H = \{x \in \mathcal{P}(\kappa)^{\mathbf{V}[G]} \mid \kappa \in j_H(x)\} \quad (40)$$

whenever H is a $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic filter. It is a standard fact that

$$M[\dot{K}^H] = \text{Ult}(\mathbf{V}[G], \dot{U}^H) \text{ and } j_H : \mathbf{V}[G] \rightarrow M[\dot{K}^H] \quad (41)$$

is the associated ultrapower map.

Since the composition $\mathbb{NR}(\kappa^+) * \mathbb{CS}(\kappa^+ \setminus T(\dot{S}_\kappa)) * \mathbb{CS}(\kappa^+ \setminus \dot{S}_\kappa)$ is (κ^+, ∞) -distributive in $\mathbf{V}[G]$, the models $\mathbf{V}[G]$ and $\mathbf{V}[G, G']$ agree on what $\mathcal{P}(\kappa)$ is, so \dot{U}^H is also a normal $\mathbf{V}[G, G', H]$ -measure over κ . Since $\dot{U}^H \in \mathbf{V}[G, G', H]$ we record that

$$\kappa \text{ is measurable in } \mathbf{V}[G, G', H]. \quad (42)$$

We now define the ideal \mathcal{J}_γ on $\mathcal{P}(\kappa)$ in $\mathbf{V}[G, G']$. For every $x \in \mathcal{P}(\kappa)^{\mathbf{V}[G, G']}$,

$$x \in \mathcal{J}_\gamma \iff \Vdash_{\mathbb{CS}(\kappa^+ \setminus S_\kappa)}^{\mathbf{V}[G, G']} \check{x} \notin \dot{U}. \quad (43)$$

Note that this definition takes place in $\mathbf{V}[G, G']$ so $\mathcal{J}_\gamma \in \mathbf{V}[G, G']$ and standard arguments show that \mathcal{J}_γ is a uniform normal ideal on $\mathcal{P}(\kappa)$ in $\mathbf{V}[G, G']$.

Recall that $S_\kappa \subseteq \kappa^+ \cap \text{cof}(\gamma)$ where γ was fixed at the in $V[G]$. This is crucial for determining the closure properties of $\mathcal{P}(\kappa)/\mathcal{J}_\gamma$. The main tool for analyzing properties of \mathcal{J}_γ is the duality theory developed in [9]. Rather than simply citing theorems there, we show the following proposition:

Proposition 6.13. *In $\mathbf{V}[G, G']$ there is a dense embedding*

$$e : \mathbb{CS}(\kappa^+ \setminus S_\kappa) \rightarrow \mathcal{P}(\kappa)/\mathcal{J}_\gamma.$$

Proof. In \mathbf{V} , fix an assignment $x \mapsto f_x$ where $x \in M$ and $f_x : \kappa \rightarrow \mathbf{V}$ is such that

$$x = [f_x]_U = j(f_x)(\kappa). \tag{44}$$

The partial ordering $\mathbb{CS}(\kappa^+ \setminus S_\kappa)$ in the generic extension $M[G, G_\kappa]$ can be viewed as the quotient $(j(\mathbb{P}_\kappa) \upharpoonright (\kappa+1))/G * G_\kappa$, so we can consider conditions in $\mathbb{CS}(\kappa^+ \setminus S_\kappa)$ as elements of M that are ordered the same way as conditions in $j(\mathbb{P}_\kappa)$. Hence each such condition p is represented in the ultrapower by U by the function f_p .

Next, recall that at each inaccessible $\alpha < \kappa$, stages $\alpha, \alpha + 1$ and $\alpha + 2$ of \mathbb{P}_κ are a composition of three partial orderings where the last one is $\mathbb{CS}(\alpha^+ \setminus S_\alpha)$. The $\alpha + 1, \alpha + 2, \alpha + 3$ components of the generic filter G are then of the form $G_{\alpha,0} * G_{\alpha,1} * h(\alpha)$ where $h(\alpha)$ is $(\mathbb{CS}(\alpha^+ \setminus S_\alpha), \mathbf{V}[G \upharpoonright \alpha * G_{\alpha,0} * G_{\alpha,1}])$ -generic. The function h is thus an element of $\mathbf{V}[G]$ and represents the filter H in the ultrapower by \dot{U}^H , that is, $H = j_H(h)(\kappa)$; see (41).

Then for any $p \in \mathbb{CS}(\kappa^+ \setminus S_\kappa)$ we have the following:

$$p \in H \Leftrightarrow j_H(f_p)(\kappa) \in j_H(h)(\kappa) \Leftrightarrow a_p \stackrel{\text{def}}{=} \{\alpha < \kappa \mid f_p(\alpha) \in h(\alpha)\} \in \dot{U}^H. \tag{45}$$

We show that in $\mathbf{V}[G, G']$, the map $e : \mathbb{CS}(\kappa^+ \setminus S_\kappa) \rightarrow \mathcal{P}(\kappa)/\mathcal{J}_\gamma$ defined by

$$e(p) = [a_p]_{\mathcal{J}_\gamma} \tag{46}$$

is a dense embedding. The proof is a standard variant of the duality argument, which we include for the reader's convenience. We write briefly $[a]$ for $[a]_{\mathcal{J}_\gamma}$.

To see that e is order-preserving, consider $p \leq q$ in $\mathbb{CS}(\kappa^+ \setminus S_\kappa)$. By the above remarks on the ordering of the quotient, we have $p \leq q$ in $j(\mathbb{P}_\kappa)$, hence $j(f_p)(\kappa) \leq j(f_q)(\kappa)$ in $j(\mathbb{P}_\kappa)$. It follows that

$$b_{p,q} \stackrel{\text{def}}{=} \{\xi < \kappa \mid f_p(\xi) \leq f_q(\xi)\} \in U,$$

and so $b_{p,q} \in \dot{U}^H$ whenever H is a $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic filter. It follows that $\kappa \setminus b_{p,q} \in \mathcal{J}_\gamma$. Since $a_p \setminus a_q \subseteq \kappa \setminus b_{p,q}$, we have $[a_p] \leq_{\mathcal{J}_\gamma} [a_q]$.

To see that the map e is incompatibility preserving, we prove the contrapositive. Assume $p, q \in \mathbb{CS}(\kappa^+ \setminus S_\kappa)$ are such that $a_p \cap a_q \in \mathcal{J}_\gamma^+$. It follows that there is some $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic filter H such that $a_p \cap a_q \in \dot{U}^H$. Then $a_p \in \dot{U}^H$ and $a_q \in \dot{U}^H$. Using (45) we conclude that $p, q \in H$. Hence p, q are compatible.

To see that the range of e is dense, assume that $a \in \mathcal{J}_\gamma^+$. It follows that there is some $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic filter H such that $a \in \dot{U}^H$. So there is some $p \in H$ such that

$$p \Vdash_{\mathbb{CS}(\kappa^+ \setminus S_\kappa)}^{\mathbf{V}[G, G']} \check{a} \in \dot{U}. \tag{47}$$

Now for every $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic filter H we have

$$a_p \in \dot{U}^H \Rightarrow p \in H \Rightarrow a \in \dot{U}^H.$$

Here the first implication follows from (45) and the second implication from (47). We thus conclude that $a_p \setminus a \notin \dot{U}^H$ whenever H is a $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic filter, which means that $a_p \setminus a \in \mathcal{J}_\gamma$, or equivalently, $[a_p] \leq_{\mathcal{J}_\gamma} [a]$. \square

We can now complete the proof of Proposition 6.12 by looking at the properties of the partial ordering $\mathbb{CS}(\kappa^+ \setminus S_\kappa)$ in $\mathbf{V}[G, G']$. By Proposition 6.11, S_κ is stationary in $\mathbf{V}[G, G']$, so $\mathbb{CS}(\kappa^+ \setminus S_\kappa)$ is a standard forcing for killing a non-reflecting stationary subset of κ^+ . The (κ^+, ∞) -distributivity follows from Proposition 6.2(a). The existence of a dense γ -closed set as well as the non-existence of a dense γ^+ -closed set follows from Proposition 6.2(b) and the fact that $S_\kappa \subseteq \kappa^+ \cap \text{cof}(\gamma)$.

The last major step toward the proof of Theorem 1.5 is the following proposition.

Proposition 6.14. *κ does not carry a saturated ideal in a generic extension via \mathbb{P}^* .*

Proof. Assume for a contradiction that κ does carry a saturated ideal in $\mathbf{V}[G, G']$ where G, G' are as above. Denote this ideal by \mathcal{I} , and let L be a $(\mathbb{P}_\mathcal{I}, \mathbf{V}[G, G'])$ -generic filter where $\mathbb{P}_\mathcal{I}$ is the partial ordering $(\mathcal{I}^+, \subseteq)$ and

$$j' : \mathbf{V}[G, G'] \rightarrow N$$

be the generic embedding associated with the ultrapower $\text{Ult}(\mathbf{V}[G, G'], L)$. Letting $M' = j'(\mathbf{V})$ and $(K, K') = j'(G, G')$, we have $N = M'[K, K']$. The partial ordering $\mathbb{P}^* * \mathbb{P}_\mathcal{I}$ preserves κ^+ , which allows us to refer to (D) at the beginning of this section. It follows that the models \mathbf{V}, M' and all transitive extensions of these models which are contained in $\mathbf{V}[G, G', L]$ have a common cardinal successor of κ , which we denote by κ^+ .

Now look at the κ th step of the iteration $j'(\mathbb{P}_\kappa)$. Obviously $j'(\mathbb{P}_\kappa) \upharpoonright \kappa = \mathbb{P}_\kappa$ and $K \cap \mathbb{P}_\kappa = G$. Let $\gamma \in R_\kappa^{M'} = R_\kappa$ be the ordinal chosen by the generic filter K at step κ of the iteration $j'(\mathbb{P}_\kappa)$ (see FI-2). Then steps $\kappa, \kappa + 1$ and $\kappa + 2$ are thus forcing with

$$\text{NR}(\kappa^+, \gamma) * \mathbb{CS}(\kappa^+ \setminus T(\dot{S}_\kappa)) * \mathbb{CS}(\kappa^+ \setminus \dot{S}_\kappa)$$

over $M'[G]$. This composition of partial orderings is computed the same way in $M'[G]$ and $\mathbf{V}[G]$, as by (D) at the beginning of this section, the models \mathbf{V} and M' agree on what H_{κ^+} is, but we don't use this directly. What is relevant is the agreement of the models on what κ^+ is, along with the fact that $T'_{\kappa, \xi} = T_{\kappa, \xi}$ for all $\xi < \kappa^+$ where the sets $T_{\kappa, \xi}$ and $T'_{\kappa, \xi}$ are as in (D) quoted above.

The κ th component K_κ of K has the form $K_{\kappa,0} * K_{\kappa,1} * K_{\kappa,2}$. Let S_κ be the generic non-reflecting stationary subset of $\kappa^+ \cap \text{cof}(\gamma)$ added by $K_{\kappa,0}$ over $M'[G]$. Since $\bigcup K_{\kappa,2} \in M'[K] \subseteq \mathbf{V}[G, G', L]$ is a closed unbounded subset of κ^+ disjoint from S_κ , the set S_κ is non-stationary in $\mathbf{V}[G, G', L]$.

By elementarity, the generic filter $K_{\kappa,1}$ codes the set S_κ inside $M'[K]$ as follows. Given an ordinal $\xi \in \kappa^+ \cap \text{cof}(\gamma)$,

$$\xi \in S_\kappa \Leftrightarrow T'_{\kappa,2\xi+1} \text{ is stationary and } T'_{\kappa,2\xi} \text{ is non-stationary.}$$

By the agreement $T'_{\kappa,\xi} = T_{\kappa,\xi}$ coming from (D) and mentioned above

$$\xi \in S_\kappa \Leftrightarrow T_{\kappa,2\xi+1} \text{ is stationary and } T_{\kappa,2\xi} \text{ is non-stationary}$$

for all such ξ . Recall that S is the subset of κ^+ with characteristic function $\bigcup G'_0$, and the generic filter G'_1 codes S in $\mathbf{V}[G, G']$ the same way as the generic filter $K_{\kappa,1}$ codes the set S_κ inside $M'[K]$, that is,

$$\xi \in S \Leftrightarrow T_{\kappa,2\xi+1} \text{ is stationary and } T_{\kappa,2\xi} \text{ is non-stationary}$$

whenever $\xi < \kappa^+$. It follows that for every $\xi \in \kappa^+ \cap \text{cof}(\gamma)$,

$$\begin{aligned} \xi \in S_\kappa &\Rightarrow T_{\kappa,2\xi} \text{ is non-stationary in } M'[K] \\ &\Rightarrow T_{\kappa,2\xi} \text{ is non-stationary in } \mathbf{V}[G, G', L] \\ &\Rightarrow T_{\kappa,2\xi} \text{ is non-stationary and } T_{\kappa,2\xi+1} \text{ is stationary in } \mathbf{V}[G, G', L] \\ &\Rightarrow \xi \in S \end{aligned}$$

Here the third implication follows from the fact that in $\mathbf{V}[G, G']$, if $\xi < \kappa^+$ then exactly one of $T_{\kappa,2\xi}, T_{\kappa,2\xi+1}$ is stationary. As $\mathbb{P}_{\mathcal{I}}$ is κ^+ -c.c., for each $\xi < \kappa^+$ exactly one of $T_{\kappa,2\xi}, T_{\kappa,2\xi+1}$ is stationary in $\mathbf{V}[G, G', L]$, namely, the one which is stationary in $\mathbf{V}[G, G']$. Similarly we verify the implication $\xi \notin S_\kappa \implies \xi \notin S$ whenever $\xi \in \kappa^+ \cap \text{cof}(\gamma)$. Altogether we then conclude that $S_\kappa = S \cap \text{cof}(\gamma)$. But then, by Proposition 6.11, S_κ is stationary in $\mathbf{V}[G, G']$. Then, again by the κ^+ -c.c. of $\mathbb{P}_{\mathcal{I}}$, S_κ remains stationary in $\mathbf{V}[G, G', L]$, a contradiction. \square

Finally, we give a proof of incompatibility of strategies \mathcal{S}_γ from Corollary 1.6(a), as formulated at the end of Corollary 1.6.

The point here is that in the construction of \mathcal{J}_γ , the ordinal γ at the κ th stage in $j(\mathbb{P}_\kappa)$ is chosen before the generic filter H comes into play. Therefore, the set x_γ is defined by

$$x_\gamma = \{\alpha < \kappa \mid \gamma^p(\alpha) = \gamma \text{ for some/all } p \in G \text{ with } \alpha \in \text{dom}(p)\} \quad \text{if } \gamma < \kappa$$

and

$$x_\gamma = \{\alpha < \kappa \mid \gamma^p(\alpha) = \alpha \text{ for some/all } p \in G \text{ with } \alpha \in \text{dom}(p)\} \quad \text{if } \gamma = \kappa$$

is an element of \dot{U}^H for all $(\mathbb{CS}(\kappa^+ \setminus S_\kappa), \mathbf{V}[G, G'])$ -generic filters H , hence x_γ is in the filter dual to \mathcal{J}_γ . Now if Player I plays \mathcal{A}_0 such that $x_\gamma, x_{\gamma'} \in \mathcal{A}_0$ and Player II responds with U_0 according to \mathcal{S}_γ then $x_\gamma \in U_0$, as $U_0 = W \cap \mathcal{A}_0$ for some

$(\mathbb{P}_{\mathcal{J}_\gamma}, \mathbf{V}[G, G'])$ -generic filter W . Similarly as above, $x_{\gamma'} \in U'_0$ for the response U'_0 of $\mathcal{S}_{\gamma'}$ to $\langle \mathcal{A}_0 \rangle$. Since $x_\gamma \cap x_{\gamma'} = \emptyset$, we have $U_0 \neq U'_0$.

Remark 6.15. We could do the construction without the “lottery” aspect, aiming at a single γ . Indeed that works for that γ , but leaves open the problem of whether ideals exist with dense trees of height γ' for $\gamma' \neq \gamma$ and for which γ' strategies exist in the Welch game. These questions are thorny and are left to the second part of this paper. The solutions there use extensive fine structural arguments.

7. Open Problems

In this section, we raise questions we don't know the answer to. We do not guarantee any of these questions are deep, difficult or even make sense.

Open Problem 1. Removing Hypotheses: Theorem 1.2 requires the GCH and the non-existence of saturated ideals on κ . Are either of these hypotheses necessary? Can some variant of the proof work without those hypotheses?

Open Problem 2. What can be said about correspondence between ideals and strategies? Theorem 1.4 says that starting with a nice ideal \mathcal{J}_γ one can build a winning strategy \mathcal{S}_γ^* for Player II in \mathcal{G}_γ . In turn, \mathcal{S}_γ^* can be used to build the ideal \mathcal{I}_γ with the methods in Theorems 1.1 and 1.2

$$\mathcal{J}_\gamma \Rightarrow \mathcal{S}_\gamma^* \Rightarrow \mathcal{I}_\gamma.$$

Inspection of the proof shows that $\mathcal{J}_\gamma \subseteq \mathcal{I}_\gamma$. Is there anything else one can say? For example, are the two ideals equal?

An Ulam Game. Consider the following variant of the cut-and-choose game of length ω derived from games introduced by Ulam in [23] (see [15]).[§]

I	(A_0^0, A_1^0)	(A_0^1, A_1^1)	...	(A_0^n, A_1^n)	(A_0^{n+1}, A_1^{n+1})	...
II	B_0	B_1	...	B_n	B_{n+1}	...

At stage 0, Player I plays a partition (A_0^0, A_1^0) of κ . At stage $n \geq 0$, Player II lets B_n be either A_0^n or A_1^n , and plays B_n . At stage $n \geq 1$, Player I plays a partition (A_0^{n+1}, A_1^{n+1}) of B_n . The winning condition for Player II is that $|\bigcap_{n \in \omega} B_n| \geq 2$.

These games generalize to lengths $\gamma > \omega$ as follows:

- (1) At successor stages $\alpha + 1$, Player I partitions B_α into two pieces and Player II chooses one of the pieces.
- (2) At limit stages α , let $B_\alpha = \bigcap_{\beta < \alpha} B_\beta$ and then Player I partitions B_α into two pieces, and Player II chooses one of the pieces.
- (3) The winning condition is the same: the intersection of the pieces that player II chooses has to have at least two elements.

[§]Velickovic [24] calls these Mycielski games.

Observation. If Player II has a winning strategy in the game \mathcal{G}_ω^* , then Player II has a winning strategy in the Ulam game.

This is immediate: Player II follows her strategy in an auxiliary play of the game \mathcal{G}_ω^* against the Boolean Algebras \mathcal{A}_n generated by $\{A_0^i, A_1^i : i \leq n\}$. In the game \mathcal{G}_ω^* , she then plays as B_n whichever of A_0^n or A_1^n belongs to U_n . By the winning condition on \mathcal{G}_ω^* , $\bigcap_{n \in \omega} B_n$ belongs to a κ -complete, uniform filter. Hence $|\bigcap_n B_n| = \kappa > 1$.

Silver and Solovay (see [15, p. 249]) showed that if Player II wins the Ulam game, then there is an inner model with a measurable cardinal. This provides an alternate proof that the consistency strength of the statement “Player II has a winning strategy in \mathcal{G}_ω^* ” is that of a measurable cardinal.

What is unclear is the exact relationship between the Ulam Game and the Welch Game. Laver showed that if a measurable cardinal is collapsed to ω_2 by the Lévy collapse and \mathcal{I} is the ideal generated by the original normal measure on κ , then in the extension $\mathcal{P}(\omega_2)/\mathcal{I}$ has a dense countably closed subset ([9]). He showed that it follows from this that Player II has a winning strategy in the Ulam game.

In Sec. 2, it is shown that the Welch games only make sense at regular cardinals κ such that for all $\gamma < \kappa$, $2^\gamma \leq \kappa$. At successor cardinals κ there is a single play by Player I (the algebra in part (2) of Theorem 2.3) that defeats Player II in the game of length 1. Moreover at non-weakly compact inaccessible cardinals κ , the Keisler–Tarski Theorem shows player I has a winning strategy in the game of length 1. But if κ is weakly compact, Player II has a winning strategy in the game of length ω .

The upshot of this discussion is that a comparison between the Ulam games and the Welch games should occur at weakly compact cardinals.

Open Problem 3. Suppose that κ is weakly compact and that Player II has a winning strategy in the Ulam game of length γ (for $\gamma \geq \omega$), does Player II have a winning strategy in \mathcal{G}_γ^* ?

Determinacy of the Welch Games. The discussion in the paragraphs before Problem 3 (based on Sec. 2 of this paper) shows that questions about the determinacy of Welch Games really only make sense at inaccessible cardinals. Moreover at non-weakly-compact inaccessible cardinals Player I wins the game of length 1 and at weakly compact cardinals Player II wins the game of length ω . By work of Nielsen and Welch if II has a winning strategy in the game of length ω_1 , then there is an inner model with a measurable cardinal — so Player II can’t have such a winning strategy in L . (Theorem 1.1 in this paper also gives this result.) Welch showed that for all regular γ , \mathcal{G}_γ^W is determined in L (this also follows immediately from [13, Theorem 5.6]).

However the following seems to be an open problem:

Open Problem 4. Is there a model of ZFC + GCH with a measurable cardinal where the Welch games are determined? With a supercompact cardinal?

Welch Games on Larger cardinals. In this paper, the Welch games are shown to provide intermediary properties between weakly compact cardinals and measurable

cardinals. What is the analogue for cardinals that are at least measurable? Perhaps the most interesting question is the following:

Open Problem 5. Are there $\mathcal{P}_\kappa(\lambda)$ versions of the game?

It is not trivial to even formulate a reasonable analogue of Welch games on supercompact cardinals. The classical ultrafilter extension properties on $\mathcal{P}_\kappa(\lambda)$ that follow from large cardinals suggest one, but it is not clear how to proceed.

Another technical obstacle that would have to be overcome is the following: in the proofs in this paper one passes from a κ -filter U on an N_α to its normal derivative U^* . Normality presents an obstacle for $\mathcal{P}_\kappa(\lambda)$ because this is the crucial difference between supercompact and strongly compact cardinals.

In [2], Buhagiar and Dzamonja found analogies of strongly compact cardinals that Dzamonja suggested might be candidates for this game.

Extender Algebras. Large cardinals whose embeddings are determined by Extender Algebras also form candidates for places games like this can be played. If E is an extender with generators $\lambda^{<\omega}$ one might consider games where Player I plays elements of $\lambda^{<\omega}$ and sequences of κ -algebras in a coherent way, and player II plays ultrafilters on the associated algebras.

In this manner, one might hope to extend these results to $\mathcal{P}^2(\kappa)$ or further.

Games on accessible cardinals:

Open Problem 6. Are there small cardinal versions of these games?

The results in Sec. 2 limit the Welch games to inaccessible cardinals. However one might hope that there is some version of these games that end up creating ideals on cardinals that are not weakly compact. A random suggestion is to require Player II to play ideals with some combinatorial property at each stage (rather than ultrafilters). One target would be to define a game similar to the Welch games that gives ω -closed densely treed ideals on ω_2 (the original Laver ideals).

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Publisher's Note

Due to an error during production, there was a mistake in the online published version of this article: on page 21, line 13, “By the minimality of P , if P_0 extends P then $iP_0 = iP$ ” should be replaced with “By the minimality of P , if P' extends P then $iP' = iP$ ”. This mistake has been corrected as of December 11, 2023.

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