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Robust Global Trajectory Tracking for a Class of Underactuated Vehicles ⋆

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Abstract

In this paper, we tackle the problem of trajectory tracking for a particular class of underactuated vehicles with full torque actuation and a single force direction (thrust), which is fixed relative to a body attached frame. Additionally, we consider that thrust reversal is not available. Under some given assumptions, the control law that we propose is able to track a smooth reference position trajectory while minimizing the angular distance to a desired orientation. This objective is achieved robustly, with respect to bounded state disturbances, and globally, in the sense that it is achieved regardless of the initial state of the vehicle. The proposed controller is tested in an experimental setup, using a small scale quadrotor vehicle and a VICON motion capture system.

Key words: Guidance Navigation and Control of Vehicles, Application of Nonlinear Analysis and Design, Robust Control of Nonlinear Systems, Hybrid Control Systems

1 Introduction

In recent years, the advent of miniaturized electronics allowed for the development of small-scale aerial vehicles that are able to perform efficiently a number of different tasks, such as surveillance, targeting, structure inspection, among others (c.f. Herrick (2000), Kinsey et al. (2006)). In order to take full advantage of the capabilities of these vehicles, several controllers have been proposed that make use of different parametrizations of the attitude of the system, such as: Euler-angles, quaternions, rotation matrix and angle-axis parametrization, just to name a few. Euler-angles arise when linearization is used in the controller design and even though they are very intuitive in nature and might be used effectively for local stabilization around a given set-point, they are not singularity-free, i.e., there exist points in the attitude space that cannot be represented with a given set of Euler-angles, so they cannot be used for the purpose of global stabilization. The rotation matrix provides a singularity-free injective representation of the orientation of the vehicle, and it can be used for controller design as suggested by Koditschek (1989). The unit-quaternions and the angle-axis are representations of attitude that provide a double cover of the attitude space, meaning that for every orientation there exist

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two quaternions (or two different angle-axis combinations) that represent that rotation. So, even though they are singularity-free, they might lead to *inconsistent* behaviour, namely the unwinding phenomenon (c.f. Mayhew et al. (2011*a*)). In order to avoid such problems, one is required to select the sign of the unit-quaternion so that the kinematic equations of motion are satisfied. In practice, a memory state is required to keep track of past values, as suggested by Mayhew et al. (2013). For a more in-depth discussion on attitude representation, the reader is referred to the work of Shuster (1993).

Every attitude representation has its advantages and drawbacks and, depending on the particular application at hand, some might prove more useful that others. In particular, unit-quaternion representations have been applied to the control of spacecraft by Joshi et al. (1995), Li et al. (2010), Kristiansen et al. (2009), Wisniewski & Kulczycki (2003), unmanned aerial vehicles by Tayebi & McGilvray (2006) and underwater vehicles by Fjellstad & Fossen (1994). A good overview of these different control techniques can be found in the work by Chaturvedi et al. (2011).

Despite their relative success, the aforementioned papers provide a control solution for fully actuated vehicles, ruling out very common vehicles such as helicopters and underwater vehicles. In order to address this issue, other control solutions have been presented by Goodarzi et al. (2013), Lee et al. (2011) and Aguiar & Hespanha (2007). However, such strategies rely on continuous controllers and it has been shown by Bhat & Bernstein (2000) that global asymptotic stabilization of a given setpoint is not possible by means of continuous feedback. In order to solve this problem, discontinuous control laws have been proposed (see e.g. Fjellstad & Fossen (1994)) but these are not robust to small measurement noise, as shown by Mayhew & Teel (2011*a*). Recent advances in hybrid control theory have shown that hybrid systems satisfying the so-called *hybrid basic conditions* are inherently robust to small measurement noise (c.f. Goebel et al. (2012)), making hybrid control techniques a suitable candidate for the problem at hand. In fact, hybrid control strategies using both quaternion feedback and rotation matrix feedback have been proposed by Mayhew et al. (2011*a*) and Mayhew & Teel (2011*b*), respectively.

In this paper, we will make use of the hybrid quaternion feedback strategy that is presented by Mayhew et al. (2011*a*), in order to design a controller for a class of underactuated vehicles that have a single force direction, known as thrust, and full torque actuation. Resorting to the backstepping of hybrid feedback laws given by Mayhew et al. (2011*b*), we design a controller that is able to globally asymptotically stabilize a given smooth reference position trajectory while minimizing the rotation angle to a given attitude configuration. The proposed strategy is, in part, similar to that of Zhao et al. (2013), however, the controller we propose includes an integral term that makes it robust to static acceleration perturbations, we use a robust hybrid system in order to extract the desired unit-quaternion and our solution is evaluated in an experimental setup using an optical motion capture system. A preliminary version of this article that does not consider the additive disturbance and without the experimental results was presented at the 2013 American Control Conference (c.f. Casau et al. (2013)). Another preliminary version of the present work with a system model that does not include the attitude dynamics was presented at the 2014 International Conference on Robotics and Automation (c.f. Casau et al. (2014)).

The remainder of the paper is organized as follows. In Section 2, we present some of the notation and basic concepts that are used throughout the paper. In Section 3, we rigorously define the problem at hand and presents some of the assumptions that render the proposed controller a feasible solution to the given problem. In Section 4, we devise a controller for the position subsystem, considering the orientation and the thrust as inputs, while in the sections 5 and 6 we follow the backstepping procedures in order to devise a controller in terms of the torque and the thrust. In Section 7, we present some experimental results. Finally, in Section 8 provides some concluding remarks to this work.

2 Preliminaries

In this paper, we make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space equipped with the inner product $\langle x, y\rangle := x^\top y$ for each $x, y \in \mathbb{R}^n$ which induces the norm $|x| := \sqrt{\langle x, x\rangle}$; $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ matrices; $\mathrm{vec} : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ is given by $\mathrm{vec}\,(A) := \begin{bmatrix} e_1^\top A^\top & \dots & e_n^\top A^\top \end{bmatrix}^\top$ for each $A \in \mathbb{R}^{m\times n}$, where $e_i \in \mathbb{R}^n$ is a vector of zeros except for the i-th entry which is 1; $|v|_\infty := \max_{i\in\{1,\dots,n\}} v_i$ for each $v \in \mathbb{R}^n$; $|A|_2$ denotes the maximum singular value of a matrix $A \in \mathbb{R}^{m\times n}$; given $M > 0$, we have that $M\mathbb{B} := \{x \in \mathbb{R}^n : |x| \le M\}$; given a set valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the range of M is the set $\mathrm{rge}\, M = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{ such that } y \in M(x)\}$.

We follow the same notation of Magnus & Neudecker (1985) to represent the derivatives of differentiable functions. Let $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ be a differentiable function, then

$$\mathcal{D}_X\,(F) := \frac{\partial \mathrm{vec}\,(F)}{\partial \mathrm{vec}\,(X)^\top}. \tag{1}$$

We also define the saturation function:

Definition 1 *A* K-saturation function *is a smooth strictly increasing function* $\sigma_K : \mathbb{R} \to \mathbb{R}$ *that satisfies the following properties:*

(1) $\sigma_K(0) = 0$,
(2) $s\sigma_K(s) > 0$ *for all* $s \neq 0$,
(3) $\lim_{s\to\pm\infty} \sigma_K(s) = \pm K$, *for some* $K > 0$.

Moreover, for each $x \in \mathbb{R}^n$ *we define*

$$\Sigma_K(x) := \left[\sigma_K(x_1)\ \ldots\ \sigma_K(x_n)\right]^\top. \qquad \square$$

The attitude of a rigid-body can be described by an element R of $SO(3)$ given by $SO(3) := \{R \in \mathbb{R}^{3\times3} : R^\top R = I_3, \det(R) = 1\}$. Flows in $SO(3)$ satisfy $\dot{R} = RS(\omega)$, where $\omega \in \mathbb{R}^3$ denotes the angular velocity and

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$

(c.f. (Bullo & Lewis 2005, Section 4.1.5)). Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ denote the n-dimensional sphere, defined by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$. The attitude of a rigid body may also be represented by unit quaternions $q := [\eta\ \epsilon^\top]^\top$, where η and ϵ denote the scalar and vector components of q. The mapping $R : \mathbb{S}^3 \to SO(3)$, given by

$$R(q) := I_3 + 2\eta S(\epsilon) + 2S(\epsilon)^2, \qquad (2)$$

maps a given unit-quaternion to a rotation matrix (c.f. (Wertz 1978, Eq. (12-47))). This map is a double cover of $SO(3)$, since $R(q) = R(-q)$. It is important to note that for any continuous path $\mathcal{R} : [0,1] \mapsto SO(3)$ and for any $q(0) \in \mathbb{S}^3$ such that $R(q(0)) = \mathcal{R}(0)$, there exists a unique continuous path $q(t) : [0,1] \mapsto \mathbb{S}^3$ such that $R(q(t)) = \mathcal{R}(t)$ for all $t \in [0,1]$ (c.f. Bhat & Bernstein (2000)). This is known as the *path lifting property* and, in particular, it means that the solution $R(t)$ to $\dot{R} = RS(\omega)$ can be uniquely lifted to a path $q(t)$ in $\mathbb{S}^3$ that satisfies

$$\dot{q} = \frac{1}{2}\begin{bmatrix} -\epsilon^\top \\ \eta I_3 + S(\epsilon) \end{bmatrix}\omega := \frac{1}{2}\Pi(q)\omega.$$

We make use of recent developments on hybrid systems theory which are described by Goebel et al. (2012). Under this framework, a hybrid system $\mathcal{H}$ is defined as

$$\mathcal{H} = \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases}, \qquad (3)$$

where the data (F, C, G, D) is given as follows: the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *flow map* and governs the continuous dynamics (also known as flows) of the hybrid system; the set $C \subset \mathbb{R}^n$ is the *flow set* and defines the set of points where the system is allowed to flow; the set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *jump map* and defines the behavior of the system during jumps; the set $D \subset \mathbb{R}^n$ is the *jump set* and defines the set of points where the system is allowed to jump. A solution x to $\mathcal{H}$ is parametrized by (t, j), where t denotes ordinary time and j denotes the jump time, and its domain $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain: for each $(T, J) \in \operatorname{dom} x$, $\operatorname{dom} x \cap ([0,T] \times \{0, 1, \ldots J\})$ can be written in the form $\cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$, where $I_j := [t_j, t_{j+1}]$ and the t_j's define the jump times. For a definition of asymptotic stability for hybrid systems see (Goebel et al. 2012, Definition 7.1).

3 Problem Formulation

In this paper, we consider the problem of designing a controller for a class of rigid bodies with a single thrust direction and full torque actuation. This includes, for example, different types of helicopter vehicles. For controller design purposes, we consider that the dynamics of such vehicles can be described by the following set of differential equations:

$$\dot{p} = v, \qquad (4a)$$
$$\dot{v} = -Re_3\frac{T}{m} + ge_3 + L(p, v)b, \qquad (4b)$$
$$\dot{R} = RS(\omega), \qquad (4c)$$
$$\dot{\omega} = -J^{-1}S(\omega)J\omega + J^{-1}M, \qquad (4d)$$

where $p \in \mathbb{R}^3$ denotes the position of the rigid body in the inertial reference frame, $v \in \mathbb{R}^3$ represents its linear velocity, expressed in inertial coordinates, $R \in SO(3)$ represents the orientation of the body fixed frame with respect to the inertial reference frame, $\omega \in \mathbb{R}^3$ denotes the angular velocity, expressed in the body attached frame, $g \in \mathbb{R}$ denotes the acceleration of gravity, $(p, v) \mapsto L(p, v) \in \mathbb{R}^{3\times\ell}$ is smooth function that represents state dependent disturbances that scale linearly with an unknown constant parameter $b \in \mathbb{R}^\ell$ for some $\ell \in \mathbb{N}$, $T \in \mathbb{R}$ is the thrust magnitude, $M \in \mathbb{R}^3$ is the torque, $m \in \mathbb{R}$ denotes the mass of the rigid body and $J \in \mathbb{R}^{3\times3}$ denotes its tensor of inertia. This model is similar to those that were used by Frazzoli et al. (2000) and Hua et al. (2009). For more details, the reader is referred to Padfield (2007), Betty (1986) and references therein.

Suppose that we are given a function $t \mapsto (p_d^{(4)}(t), \dot{\omega}_d(t)) \in M_p\mathbb{B} \times M_\omega\mathbb{B}$ for some $M_p, M_\omega > 0$ and for each $t \geq 0$. Then, the position and attitude reference trajectories, denoted by p_d and R_d, respectively, are obtained by integration of this function, given a set of suitable initial conditions. In particular, the attitude trajectory $t \mapsto R_d(t)$ is obtained by integration of the differential equation $\dot{R}_d = R_dS(\omega_d)$, hence guaranteeing that $R_d(t)$ belongs to $SO(3)$ for each $t \geq 0$.

This procedure gives rise to a map $t \mapsto r(t) \in \mathbb{R}^{12} \times SO(3) \times \mathbb{R}^3$ defined for each $t \geq 0$, where

$$r(t) := (p_d(t), p_d^{(1)}(t), p_d^{(2)}(t), p_d^{(3)}(t), R_d(t), \omega_d(t)) \quad (5)$$

collects not only the position and attitude trajectories, but their derivatives up to a certain order. In the sequel, we restrict our attention to bounded reference trajectories and disturbances satisfying the following assumption.

Assumption 1 *Given $M_p, M_\omega > 0$, a* reference trajectory *is a solution r to*

$$\dot{r} \in F_d(r) := (p_d^{(1)}, p_d^{(2)}, p_d^{(3)}, M_p\mathbb{B}, R_d S(\omega_d), M_\omega\mathbb{B}), \quad (6)$$

such that $\operatorname{rge} r \in \Omega_r$ *for some compact set $\Omega_r \subset \mathbb{R}^{12} \times SO(3) \times \mathbb{R}^3$, satisfying $e_3^\top R_d(t) e_3 \geq 0$ for each $t \geq 0$. Moreover, given disturbances $(p, v, b) \mapsto L(p, v)b$ for (4), the following holds*

$$\sup_{r \in \Omega_r} \left| p_d^{(2)} \right| + (\sqrt{3} + \sqrt{\ell} \sup_{(p,v) \in \mathbb{R}^6} |L(p, v)|_2) |b|_\infty < g. \quad (7)$$

□

In general, it is not possible for an underactuated vehicle to track an arbitrary reference trajectory (c.f. Levine & Mllhaupt (2011)). Therefore, given a reference trajectory r satisfying Assumption 1, the controller proposed in this paper is able to track the attitude trajectory $R_0(r, X)$ obtained by solving the optimization problem

$$\begin{array}{rl} \text{minimize} & \frac{1}{2}\text{trace}\left(I_3 - R R_d^\top\right) \\ \text{subject to} & R \in X, \end{array} . \quad (8)$$

where $X \subset SO(3)$ is such that $(p_d, R_0(r, X))$ is a feasible trajectory for the system (4). Under these considerations we may now state the objective of the controller proposed in this paper.

Problem 1 *Design a hybrid controller*

$$\dot{x}_c \in F_c(x),\ x \in C_c, \quad (9a)$$
$$x_c^+ \in G_c(x),\ x \in D_c, \quad (9b)$$

with output $(T(x), M(x))$, where $x := (r, p, v, R, \omega, x_c)$ belongs to $\mathcal{X} := \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \times \mathcal{X}_c$, for some $\mathcal{X}_c$, such that the set

$$\mathcal{A} := \left\{ x \in \mathcal{X} : p = p_d,\ v = p_d^{(1)},\ R = R_0(r, X) \right\}, \quad (10)$$

is globally asymptotically stable *for the interconnection between (4) and the controller (9) and there exists $\overline{T} > 0$ such that $0 < T(x) \leq \overline{T}$ for each solution x to the closed-loop system.* □

To solve this problem, we separate it into three simpler tasks. In Section 4, we design a controller for the position subsystem and then, in Sections 5 and 6, we design a control law for the whole system using backstepping techniques.

4 Robust Position Tracking by Saturated Feedback

In this section, let us consider $R \in SO(3)$ as a virtual input. Then, given a reference trajectory satisfying Assumption 1, the position and velocity tracking errors are given by

$$p_0 := p - p_d, \qquad v_0 := v - p_d^{(1)}, \quad (11)$$

respectively. Then, using (4a) and (4b), we find that the dynamics of the tracking errors are given by

$$\dot{p}_0 = v_0, \quad (12a)$$
$$\dot{v}_0 = -Re_3\frac{T}{m} + ge_3 + L(p, v)b - p_d^{(2)}. \quad (12b)$$

Since we are considering both $R \in SO(3)$ and $T \in \mathbb{R}$ as inputs, the term $-Re_3T/m$ can be set to an arbitrary vector $\mu \in \mathbb{R}^3$ using the thrust input

$$T(\mu) := m|\mu|, \quad (13)$$

and the attitude input as the solution to the optimization problem (8) with

$$X_\mu := \{R \in SO(3) : Re_3 = -\mu/|\mu|\}, \quad (14)$$

which is given by

$$R_0(r, X_\mu) = \left(I_3 + S(\gamma) + \frac{1}{1 - e_3^\top R_d^\top \frac{\mu}{|\mu|}} S(\gamma)^2 \right) R_d, \quad (15)$$

where

$$\gamma := -S(R_d e_3) \frac{\mu}{|\mu|}, \quad (16)$$

for each $\mu \in \mathbb{R}^3$ (c.f. Frazzoli et al. (2000)). Then, let us define the feedback law

$$\begin{aligned} \mu(r, p_0, v_0, z) := & -\Sigma_K(k_p p_0 + k_v v_0) \\ & - L(p, v)\Sigma_K(z) \\ & - ge_3 + p_d^{(2)}, \end{aligned} \quad (17)$$

where $\Sigma_K : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a K-saturation function with the properties given in Definition 1, $k_p, k_v > 0$ and $z \in \mathbb{R}^\ell$ is a integral state satisfying

$$\dot{z} := k_z L(p, v)^\top \mathcal{D}_{v_0}\left(\overline{V}_0(p_0, v_0)\right)^\top,$$

for each $(r, p_0, v_0) \in \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3$, where

$$\overline{V}_0 := \sum_{i=1}^{3} \frac{1}{2} \left[\sigma_K\left(\widetilde{r}_i\right)\ e_i^\top v_0\right] P \begin{bmatrix} \sigma_K\left(\widetilde{r}_i\right) \\ e_i^\top v_0 \end{bmatrix} + \int_0^{r_i} \sigma_K\left(\xi\right) \mathrm{d}\xi, \tag{18}$$

with $\widetilde{r} := k_p p_0 + k_v v_0$, σ_K given in Definition 1 and

$$P := \begin{bmatrix} \frac{k_v}{k_p}\beta & -\beta \\ -\beta & k_p \end{bmatrix},$$

for some $\beta \in (0, k_v)$.

Clearly, $R_0(r, X_\mu)$ is not defined when $e_3^\top R_d \mu = |\mu|$. However, if $e_3^\top R_d(t) e_3 \geq 0$ for each $t \geq 0$ and (7) is satisfied then this situation does not happen for any solution to the closed-loop system. Additionally, (7) is required to guarantee that the thrust input satisfies the lower saturation bound $T(\mu) > 0$ as described in Problem 1. Notice that the model disturbances $L(p, v)b$ must be bounded. If the disturbance term does not satisfy this requirement, then the results presented in this paper do not hold globally, but rather on a subset of the state space where (7) is satisfied.

Replacing (17) into (12) using the inputs $(T, R) = (T(\mu), R_0(r, X_\mu))$, given by (15) and (13), we obtain the closed-loop system

$$\begin{aligned} \dot{p}_0 &= v_0, \\ \dot{v}_0 &= -\Sigma_K(k_p p_0 + k_v v_0) + L\left(p, v\right)\left(b - \Sigma_K(z)\right), \\ \dot{z} &= k_z L(p, v)^\top \mathcal{D}_{v_0}\left(\overline{V}_0(p_0, v_0)\right)^\top. \end{aligned} \tag{19}$$

with the important stability properties that are given in the following lemma.

Lemma 1 *Let Assumption 1 hold. Then, for each $k_p, k_v, k_z > 0$, there exists $K > |b|_\infty$ such that the set*

$$\mathcal{A}_0 := \{(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = v_0 = 0\}, \tag{20}$$

is globally asymptotically stable for the system (19) and its solutions are bounded. Moreover, there exists $\overline{T} > 0$ such that $0 < T(\mu(r, p_0, v_0, z)) < \overline{T}$ for each $r \in \Omega_r$ and for each solution (p_0, v_0, z) to (12).

PROOF. See Appendix 9.

5 Global Asymptotic Stabilization of the Attitude Kinematics by Hybrid Quaternion Feedback

In this section, we develop a controller that solves Problem 1 when the ω is taken as a virtual input. To do so, let us define the rotation error as

$$R_1 := R R_0^\top, \tag{21}$$

where R_0 is given by (15). Since $R_0(t) \equiv R_0(r(t), X_{\mu(t)})$ belongs to $SO(3)$ for each $t \geq 0$, then its derivative satisfies

$$\begin{aligned} \mathcal{D}_t\left(R_0(t)\right) &= \operatorname{vec}\left(R_0(t) S\left(\omega_0(t)\right)\right) \\ &= -\Gamma(R_0(t)) R_0(t) \omega_0(t), \end{aligned} \tag{22}$$

for each $t \geq 0$, with

$$\Gamma(R) := -\left[S\left(Re_1\right)\ S\left(Re_2\right)\ S\left(Re_3\right)\right]^\top.$$

Moreover, solving (22) for ω_0, we obtain

$$\omega_0 = -\frac{1}{2} R_0^\top \Gamma(R_0)^\top \mathcal{D}_t\left(R_0\right), \tag{23}$$

and, from (4c) and (21), we conclude that

$$\dot{R}_1 = R_1 S\left(R_0(\omega - \omega_0)\right).$$

The design of a controller such that $R = R_0$ is globally asymptotically stable is equivalent to the design of a controller that stabilizes $R_1 = I_3$. Although strategies for the global stabilization of an attitude reference by matrix feedback exist (c.f. Mayhew & Teel (2011*b*)), it is not clear how they can be extended to the stabilization of the class of underactuated vehicles presented in this paper. Instead, we resort to attitude stabilization by hybrid quaternion feedback introduced in Mayhew et al. (2011*a*).

In this direction, let us point out that there exists a unique unit-quaternion satisfying $R_1 = R(q_1)$ and the kinematic equations

$$\dot{q}_1 = \frac{1}{2} \Pi(q_1) R_0(\omega - \omega_0). \tag{24}$$

In order to retrieve the unit-quaternion uniquely, we make use of the robust path-lifting strategy that was introduced in Mayhew et al. (2013). We discuss the implementation of this technique at the end of Section 6, but, for now, we assume that q_1 is readily available from the measurements.

In standard backstepping we would add a feedforward term to ω in order to cancel out ω_0. However, due to the presence of an unknown constant $b \in \mathbb{R}^\ell$ in the dynamics of the plant, we cannot determine ω_0. Instead, we use an estimate $\omega_{0,1}$, given by

$$\omega_{0,1} := -\frac{1}{2} R_0^\top \Gamma(R_0)^\top \left.\mathcal{D}_t\left(R_0\right)\right|_{b=b_1},$$

which is in all aspects identical to (23) but where we replace the unknown disturbance b by an estimate b_1.

It is possible to verify that the difference between $\omega_{0,1}$ and ω_0 is given by

$$\omega_{0,1} - \omega_0 = -\frac{1}{2} R_0^\top \Gamma(R_0)^\top \mathcal{D}_{v_0}(R_0) L(p,v) \widetilde{b}_1, \quad (25)$$

where we have used the definition of the estimation error $\widetilde{b}_1 := b_1 - b$.

Let η_1 and ϵ_1 denote the scalar and vector components of q_1, respectively, $H := \{-1, 1\}$,

$$\begin{aligned} Q_\delta^+ &:= \{(q,h) \times \in \mathbb{S}^3 \times H : h\eta \geq -\delta\}, \\ Q_\delta^- &:= \{(q,h) \times \in \mathbb{S}^3 \times H : h\eta \leq -\delta\}, \end{aligned} \quad (26)$$

$x_1 := (r, p_0, v_0, q_1, z, h, \widetilde{b}_1)$ belongs to $\mathcal{X}_1 := \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{R}^\ell \times H \times \mathbb{R}^\ell$ and

$$\dot{b}_1 := \frac{1}{2} k_{b_1} k h L(p,v)^\top \mathcal{D}_{v_0}(R_0)^\top \Gamma(R_0) \epsilon_1, \quad (27)$$

then define the hybrid system $\mathcal{H}_1 := (C_1, F_1, D_1, G_1)$ as follows

$$\dot{x}_1 \in F_1(x_1)$$

$$F_1(x_1) := \begin{pmatrix} F_d(r) \\ v_0 \\ -R(q_1) R_0 e_3 \frac{T(\mu)}{m} + g e_3 - p_d^{(2)} + L(p,v) b \\ \frac{1}{2} \Pi(q_1) R_0 (\omega_1 - \omega_0) \\ k_z L(p,v)^\top \mathcal{D}_{v_0}\left(\overline{V}_0(p_0, v_0)\right)^\top \\ 0 \\ \frac{1}{2} k_{b_1} k h L(p,v)^\top \mathcal{D}_{v_0}(R_0)^\top \Gamma(R_0) \epsilon_1 \end{pmatrix}$$

$$x_1 \in C_1 := \{x_1 \in \mathcal{X}_1 : (q_1, h) \in Q_\delta^+\} \quad (28a)$$

$$x_1^+ \in G_1(x_1) := (r, p_0, v_0, q_1, q_0, z, -h, \widetilde{b}_1)$$

$$x_1 \in D_1 := \{x \in \mathcal{X}_1 : (q_1, h) \in Q_\delta^-\}, \quad (28b)$$

for some $\delta \in (0,1)$, where $T(\mu)$ is given by (13) and ω_1 is given by

$$\omega_1 := \omega_{0,1} + R_0^\top \left(-\omega_1^\star - k_q h \epsilon_1\right), \quad (29)$$

with

$$\omega_1^\star := \frac{2 k_z k_{V_0}}{kh} \left(\eta_1 S(\mu) - S(\mu) S(\epsilon_1)\right) \mathcal{D}_{v_0}(V_0)^\top. \quad (30)$$

and $h \in H$ is a logic variable that enables controller switching and $H := \{-1, 1\}$ is a discrete set endowed with the discrete topology, but it can be regarded as a subset of $\mathbb{R}$ with the subspace topology. In particular, if we consider any function $V : \mathbb{R}^n \times H \to \mathbb{R}$ such that the map $x \mapsto V(x,h)$ is continuous for each $h \in H$, then $V(x,h)$ is continuous on $\mathbb{R} \times H$. This fact is used in the proof of the following theorem.

Theorem 2 *Let Assumptions 1 hold. Then, for each $k_p, k_v, k_{V_0}, k_q, k, k_z, k_{b_1} > 0$, there exists $K > |b|_\infty$ such that the solutions to the hybrid system (28) are bounded and the set*

$$\mathcal{A}_1 := \left\{ x_1 \in \mathcal{X}_1 : p_0 = 0, v_0 = 0, q_1 = \begin{bmatrix} h & 0^\top \end{bmatrix}^\top \right\}$$

is globally asymptotically stable.

PROOF. First of all, we prove that the hybrid system (28) meets the hybrid basic conditions (as given in Goebel et al. (2012)): (1) since $\phi(x_1) := h\eta_1$ is continuous, the pre-image of closed sets under ϕ is also closed, thus both C_1 and D_1 are closed; (2) since $F_1(x_1)$ given in (28a), is a single valued function and continuous, it is locally bounded, convex and outer semicontinuous; (3) by (Goebel et al. 2012, Lemma 5.10), the jump map $G_1(x_1)$ is outer semicontinuous if and only if $D_1 \times G_1(D_1)$ is closed. Notice that the jump map changes the logic variable but not the states, therefore $G_1(D_1)$ is closed and $G_1(x_1)$ is locally bounded for each $x_1 \in D_1$. Since D_1 is closed, we conclude that the jump map is outer-semicontinuous.

Next, let us prove that every maximal solution to $\mathcal{H}_1$ is precompact, i.e. complete and bounded. Consider the following definition

$$\begin{aligned} V_1(x_1) := & k_{V_0} V_0(p_0, v_0, z) + 2k(1 - h\eta_1) \\ & + \frac{1}{2k_{b_1}} \widetilde{b}_1^\top \widetilde{b}_1. \end{aligned} \quad (31)$$

From the properties of V_0 and knowing that both H and $\mathbb{S}^3$ are compact we have that for any $c > 0$, $V_1^{-1}(c)$ is compact. From Assumption 1 we have that the reference trajectory r belongs to a compact set Ω_r and, since q_0 belongs to the compact set $\mathbb{S}^3$, then for any initial condition $(r, p_0, v_0, q_1, , z, h, \widetilde{b}_1)(0,0)$ we have that the set

$$U_1 := \{x_1 \in \mathcal{X}_1 : V_1(x_1) \leq V_1(x_1(0,0))\}, \quad (32)$$

is compact. Using (64), (2) and (28a) we have that the time derivative of V_1 is given by

$$\begin{aligned} \left\langle \mathcal{D}_{x_1}(V_1)^\top, F_1(x_1) \right\rangle = & -k_{V_0} k_z W_0(p_0, v_0) \\ & + k_{V_0} \mathcal{D}_{v_0}(V_0)(-2\eta_1 S(\epsilon_1) - 2S(\epsilon_1)^2)\mu \\ & + kh\epsilon_1^\top R_0(\omega_1 - \omega_0) + \frac{1}{k_b} \widetilde{b}^\top \dot{b}_1. \end{aligned} \quad (33)$$

Replacing (29), (25) and (27) into (33) yields

$$\langle \mathcal{D}_{x_1}(V_1), F_1(x_1)\rangle = -k_{V_0} k_z W_0(p_0, v_0) - k k_q \epsilon_1^\top \epsilon_1. \tag{34}$$

Defining

$$u_{c_1}(x_1) := \begin{cases} -k_{V_0} k_z W_0(p_0, v_0) - k k_q \epsilon_1^\top \epsilon_1 & \text{if } x_1 \in C_1 \\ -\infty & \text{otherwise} \end{cases}, \tag{35}$$

it is straightforward to see that $\langle \mathcal{D}_{x_1}(V_1), F_1(x_1)\rangle = u_{c_1}(x_1) \leq 0$, for all $x_1 \in C_1 \cap U_1$. If $x_1 \in U_1 \cap D_1$ then

$$V_1(G_1(x_1)) - V_1(x_1) = 4kh\eta_1, \tag{36}$$

From (28b), we have than $h\eta_1 \leq -\delta$ thus

$$V_1(G_1(x_1)) - V_1(x_1) \leq -4k\delta. \tag{37}$$

Defining

$$u_{d_1}(x_1) = \begin{cases} -4k\delta & \text{if } x \in D_1 \\ -\infty & \text{otherwise} \end{cases}, \tag{38}$$

we have that $V_1(G_1(x_1)) - V_1(x_1) = u_{d_1}(x_1) < 0$ for all $x_1 \in U_1 \cap D_1$. These results show that any solution $x_1(t,j)$ to $\mathcal{H}_1$ remains in U_1 for all $(t,j) \in \operatorname{dom} x$. This together with the fact that $G_1(D_1) \subset C_1$ implies the completeness (from (Goebel et al. 2012, Proposition 2.10)) and the boundedness of solutions. Additionally, the relation $\overline{\operatorname{rge} x_1} \subset U_1$ is also verified and the growth of V_1 along solutions to $\mathcal{H}_1$ is bounded by u_{c_1}, u_{d_1} on U_1. Then, since $\mathcal{H}_1$ satisfies the hybrid basic conditions and V_1 is continuous, by (Goebel et al. 2012, Theorem 8.2), the precompact solutions $x_1(t,j)$ to $\mathcal{H}_1$ approach the largest weakly invariant set $\Omega(x_1)$ on

$$V_1^{-1}(c) \cap U_1 \cap \left[\overline{u_{c_1}^{-1}(0)} \cup \left(u_{d_1}^{-1}(0) \cap G(u_{d_1}^{-1}(0))\right)\right], \tag{39}$$

for some $c > 0$. Since $u_{d_1}^{-1}(0) = \emptyset$ we have that, in particular, $\Omega(x) \subset \overline{u_{c_1}^{-1}(0)}$, with

$$u_{c_1}^{-1}(0) = \left\{ x_1 \in \mathcal{X}_1 : p_0 = v_0 = 0, q_1 = \begin{bmatrix} h & 0 \end{bmatrix}^\top \right\} = \mathcal{A}_1. \tag{40}$$

Since every solution to $\mathcal{H}_1$ is precompact, then the solutions $x_1(t,j)$ to $\mathcal{H}_1$ converge to $\mathcal{A}_1$. We conclude that $\mathcal{A}_1$ is globally attractive for the closed-loop hybrid system (28). Since V_1 is positive-definite relative to $\mathcal{A}_1$ and non-increasing along solutions to (28), then $\mathcal{A}_1$ is globally stable for the closed-loop hybrid system. Hence, we conclude that it is globally asymptotically stable. □

In the next section, we take advantage of Theorem 2 and backstepping techniques in order to solve Problem 1.

6 Global Asymptotic Stabilization of the Full Dynamic System

In this section, we develop a hybrid feedback law that is obtained from that of the previous section by means of backstepping. As before, due to the unknown disturbance b, the derivative of ω_1 has to be estimated using a second estimator for b, denoted by $b_2 \in \mathbb{R}^3$. Let $\widetilde{\omega} := \omega - \omega_1$, $\widetilde{b}_2 := b_2 - b$ and

$$M = S(\omega)J\omega + J(\dot{\omega}_{1,2} + u), \tag{41}$$

where $k_\omega > 0$, $u \in \mathbb{R}^3$ denotes a new virtual input variable and $\dot{\omega}_{1,2} := \mathcal{D}_t(\omega_1)|_{b=b_2}$ denotes the estimate of $\dot{\omega}_1$ when using the estimate b_2. The difference between $\dot{\omega}_{1,2}$ and $\dot{\omega}_1$ is given by

$$\dot{\omega}_{1,2} - \dot{\omega}_1 = \mathcal{D}_{v_0}(\omega_1) L(p,v)\widetilde{b}_2. \tag{42}$$

Let $x_2 := (x_1, \widetilde{\omega}, \widetilde{b}_2)$ belong to $\mathcal{X}_2 := \mathcal{X} \times \mathbb{R}^3 \times \mathbb{R}^3$. Then, replacing (41) into (4d), we obtain

$$\dot{\omega} = u + \dot{\omega}_{1,2}, \tag{43}$$

allowing us to define the hybrid system $\mathcal{H}_2 := (C_2, F_2, D_2, G_2)$ as follows:

$$\dot{x}_2 \in F_2(x_2) := \begin{pmatrix} F_1(x_1) \\ -k_\omega \widetilde{\omega} + \mathcal{D}_{v_0}(\omega_1) L(p,v)\widetilde{b}_2 - khR_0^\top \epsilon_1 \\ -L(p,v)^\top \mathcal{D}_{v_0}(\omega_1)^\top \widetilde{\omega} \end{pmatrix}$$
$$x_2 \in C_2 = \{x_2 \in \mathcal{X}_2 : (q_1, h) \in Q_\delta^+\} \tag{44a}$$

$$x_2^+ \in G_2(x_2) := (G_1(x_1), \widetilde{\omega}, \widetilde{b}_2)$$
$$x \in D_2 := \{x_2 \in \mathcal{X}_2 : (q_1, h) \in Q_\delta^-\} \tag{44b}$$

where we have used

$$u := -k_\omega \widetilde{\omega} - khR_0^\top \epsilon_1, \tag{45a}$$
$$\dot{b}_2 := -L(p,v)^\top \mathcal{D}_{v_0}(\omega_1)^\top \widetilde{\omega}. \tag{45b}$$

With these definitions, we are able to state the main result of this paper.

Theorem 3 *Let Assumptions 1 hold. Then, for each $k_p, k_v, k_{V_0}, k_q, k, k_z, k_{b_1}, k_{b_2} > 0$, there exists $K > |b|_\infty$ such that the solutions to the hybrid system (44) are bounded and the set*

$$\mathcal{A}_2 := \{x_2 \in \mathcal{X}_2 : x_1 \in \mathcal{A}_1, \widetilde{\omega} = 0\},$$

is globally asymptotically stable.

PROOF. The proof of this theorem follows very closely the proof of Theorem 2. Namely, the proof that $\mathcal{H}_2$ meets the hybrid basic conditions is essentially the same so we dismiss it here.

Let us define the following Lyapunov function candidate

$$\begin{aligned}V_2(x_2) &:= k_{V_0}V_0(p_0, v_0, z) + 2k(1 - h\eta_1)\\ &\quad + \frac{1}{2}\widetilde{\omega}^\top\widetilde{\omega} + \frac{1}{2k_{b_1}}\widetilde{b}_1^\top\widetilde{b}_1 + \frac{1}{2k_{b_2}}\widetilde{b}_2^\top\widetilde{b}_2.\end{aligned} \tag{46}$$

Since V_0 is positive-definite relative to $\{(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = 0, v_0 = 0, z = \Sigma_K^{-1}(b)\}$ and, by Assumption 1, $r \in \Omega_r$ for some compact set Ω_r we conclude that, for each initial condition $x_2(0, 0)$, the set

$$U_2 := \{x_2 \in \mathcal{X}_2 : V_2(x_2) \leq V_2(x_2(0, 0))\}, \tag{47}$$

is compact. Using (33) and (44a) we have that the time derivative of V_2 is given by

$$\begin{aligned}\left\langle \mathcal{D}_{x_2}(V_2)^\top, F_2(x_2)\right\rangle &= -k_{V_0}k_zW_0(p_0, v_0) - kk_q\epsilon_1^\top\epsilon_1\\ &\quad - k_\omega\widetilde{\omega}^\top\widetilde{\omega}.\end{aligned} \tag{48}$$

Defining

$$u_{c_2}(x_2) := \begin{cases}-k_{V_0}k_zW_0 - kk_q\epsilon_1^\top\epsilon_1 - k_\omega\widetilde{\omega}^\top\widetilde{\omega} & \text{if } x_2 \in C_2\\ -\infty & \text{otherwise}\end{cases} \tag{49}$$

it is straightforward to see that $\left\langle \mathcal{D}_{x_2}(V_2)^\top, F_2(x_2)\right\rangle = u_{c_2}(x_2) \leq 0$, for all $x_2 \in C_2 \cap U_2$. From (44b), we have $h\eta_1 \leq -\delta$, thus the following holds

$$V_2(G_2(x_2)) - V_2(x_2) \leq -4k\delta \quad \forall x_2 \in D_2. \tag{50}$$

Defining

$$u_{d_2}(x_2) = \begin{cases}-4k\delta & \text{if } x_2 \in D_2\\ -\infty & \text{otherwise}\end{cases}, \tag{51}$$

we have that $V_2(G_2(x_2)) - V(x_2) = u_{d_2}(x_2) < 0$ for all $x_2 \in U_2 \cap D_2$. These results show that any give solution $x_2(t, j)$ to $\mathcal{H}_2$ remains in U_2 for all $(t, j) \in \operatorname{dom} x_2$. This together with the fact that $G_2(D_2) \subset C_2$ implies the completeness (from (Goebel et al. 2012, Proposition 2.10)) and the boundedness of solutions. Additionally, the relation $\overline{\operatorname{rge} x_2} \subset U_2$ is also verified and the growth of V_2 along solutions to $\mathcal{H}_2$ is bounded by u_{c_2}, u_{d_2} on U_2. Then, since $\mathcal{H}_2$ satisfies the hybrid basic conditions and V_2 is continuous, by (Goebel et al. 2012, Theorem 8.2) or (Sanfelice et al. 2007, Theorem 4.7), the precompact solutions to $x_2(t, j)$ approach the largest weakly invariant set $\Omega(x_2)$ on

$$V_2^{-1}(r) \cap U_2 \cap \left[\overline{u_{c_2}^{-1}(0)} \cup \left(u_{d_2}^{-1}(0) \cap G(u_{d_2}^{-1}(0))\right)\right], \tag{52}$$

for some $r > 0$. Since $u_{d_2}^{-1}(0) = \emptyset$ we have that, in particular, $\Omega(x_2) \subset \overline{u_{c_2}^{-1}(0)}$, with

$$\begin{aligned}u_{c_2}^{-1}(0) &= \left\{x_2 \in \mathcal{X}_2 : p_0 = v_0 = 0, q_1 = \begin{bmatrix}h & 0\end{bmatrix}^\top, \widetilde{\omega} = 0\right\}\\ &= \mathcal{A}_2.\end{aligned} \tag{53}$$

Since every solution to $\mathcal{H}_2$ is precompact, then the solutions $x_2(t, j)$ to $\mathcal{H}_2$ converge to $\mathcal{A}_2$. We conclude that $\mathcal{A}_2$ is globally attractive for the hybrid system $\mathcal{H}_2$. Since V_2 is positive-definite relative to $\mathcal{A}_2$ and non-increasing along solutions to (44), then $\mathcal{A}_2$ is globally stable for the closed-loop hybrid system. Hence, we conclude that $\mathcal{A}_2$ is globally asymptotically stable for (44). It follows from the fact that solutions to the hybrid system remain in U_2 that $\widetilde{b}_2$ is bounded. $\square$

To show that Problem 1 is solved using the proposed control law, let us define the controller variables $x_c := (z, h, b_1, b_2, \widehat{q}_1)$, where $\widehat{q}_1 \in \mathbb{S}^3$ is a memory variable that is part of the robust path-lifting strategy introduced in Mayhew et al. (2013). Then, the state variables of the closed-loop system are $x := (r, p, v, R, \omega, x_c) \in \mathcal{X}$ with $\mathcal{X} := \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \times \mathbb{R}^\ell \times H \times \mathbb{R}^\ell \times \mathbb{R}^\ell \times \mathbb{S}^3$. Furthermore, let us define the hybrid controller

$$\begin{aligned}F_c(x) &:= \begin{pmatrix} k_zL(p, v)^\top\mathcal{D}_{v_0}\left(\overline{V}_0(p_0, v_0)\right)^\top\\ 0\\ \frac{1}{2}k_{b_1}khL(p, v)^\top\mathcal{D}_{v_0}(R_0)^\top\Gamma(R_0)\epsilon_1\\ -L(p, v)^\top\mathcal{D}_{v_0}(\omega_1)^\top\widetilde{\omega}\\ 0\end{pmatrix}\\ x \in C_c &:= \{x \in \mathcal{X} : (\Phi(\widehat{q}_1, RR_0^\top), h) \in Q_\delta^+,\\ &\qquad \operatorname{dist}(\widehat{q}_1, \mathcal{Q}(RR_0^\top)) \leq \alpha\},\\ G_c(x) &:= \begin{cases}(z, -h, b_1, b_2, \widehat{q}_1) & \text{if } x \in D_1,\\ (z, h, b_1, b_2, \Phi(\widehat{q}_1, RR_0^\top)) & \text{if } x \in D_2\end{cases}\\ x \in D_c &:= D_1 \cup D_2,\end{aligned} \tag{54}$$

where $0 < \alpha < 1$, $\mathcal{Q}(R)$ denotes the set of quaternions $\{q, -q\} \subset \mathbb{S}^3$ satisfying $R(q) = R(-q) = R$ for each $R \in SO(3)$,

$$\begin{aligned}D_1 &:= \{x \in \mathcal{X} : (\Phi(\widehat{q}_1, RR_0^\top), h) \in Q_\delta^-\},\\ D_2 &:= \{x \in \mathcal{X} : \operatorname{dist}(\widehat{q}_1, \mathcal{Q}(RR_0^\top)) \geq \alpha\},\end{aligned} \tag{55}$$

$\operatorname{dist}(p, Q) := \inf\{p^\top q : q \in Q\}$ for each $p \in \mathbb{S}^3$ and $Q \subset \mathbb{S}^3$, and

$$\Phi(q, R) := \arg\max_{p \in \mathcal{Q}(R)} q^\top p,$$

is such that $q_1 = \Phi(\widehat{q}_1, RR_0^\top)$. Moreover, the output of the controller is

$$\begin{aligned} T &= m\,|\mu|\,, \\ M &= S(\omega)\,J\omega + J^{-1}(-k_\omega\widetilde{\omega} - khR_0^\top\epsilon_1 + \dot{\omega}_{1,2}). \end{aligned} \tag{56}$$

Backtracking the definitions of the variables in (56), it follows from (Mayhew et al. 2013, Theorem 9) and Theorem 3 that (10) is globally asymptotically stable for the interconnection between (4) and (54).

In the next section we present some experimental results that show the behaviour of the closed-loop system using the given controller.

7 Experimental Results

In order to experimentally evaluate the performance of the controller described in Section 6, we make use of the following components:(1) *Blade mQX quadrotor* Horizon Hobby Inc. (2012), (2) VICON Bonita motion capture system VICON (2012), (3) MATLAB/Simulink software, and (4) custom made RF interface. The *Blade mQX quadrotor* weighs 80 g and has a radius of approximately 11 cm. This vehicle accepts the thrust and the angular velocity as inputs. It is readily available in the market and allows for easy integration with the other components of the control architecture. However, the development of a custom made RF interface board is required in order to enable the communication between the MATLAB/Simulink software that is running the control algorithms and the vehicle itself. In order to estimate the position and velocity of the quadrotor we make use of the VICON Bonita motion capture system. This is a high performance system that operates with sub-millimetre precision and up to a frequency of 120 Hz. In these experiments we have used a frequency of 50 Hz for improved accuracy. The overall control architecture is depicted in Figure 1.

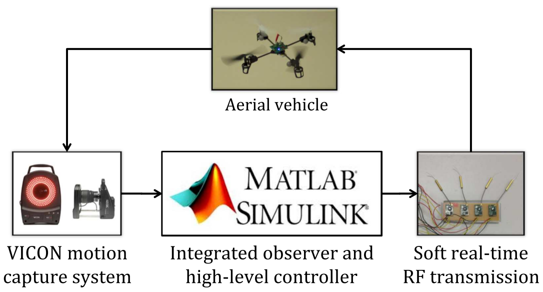

Fig. 1. Quadrotor control architecture.

We performed the identification of the platform by applying different constant inputs over several experiments and applying a linear regression to the experimental results. The maximum allowed thrust is approximately 1.37 N, the maximum allowed angular velocity input in the x-axis and the y-axis is 200 deg/s and 300 deg/s in the z-axis direction. Moreover, changes in the actuation are not instantaneous but can be modelled by a linear first order system with a pole at 1.5 Hz. For more details on the system architecture and identification, the reader is referred to Cabecinhas et al. (2014). Since the VICON motion capture system outputs the rotation matrix of the vehicle we resort to the strategy outlined in Mayhew et al. (2013) so as to obtain consistent quaternion representations of attitude.

In order to assess that the hybrid controller was working as intended, we carried out the following experiment

- Set the desired position trajectory to (57);
- Set the initial yaw of the quadrotor to be approximately 180 degrees away from the desired orientation;
- Run the experiment for $h(0,0) = 1$ and $h(0,0) = -1$.

Let us consider that $L(p,v) = I_3$ and $\sigma_K(s) = \frac{2K}{\pi}\arctan(s)$. Moreover, we have chosen the controller parameters $k_{V_0} = 0.01$, $k_z = 0.3$, $k_p = 3$, $k_v = 6$, $k = 3$, $K = 1$, $k_\omega = 40$, $k_{b_2} = 1$ and $k_{b_1} = 1$. The controller parameters for the position subsystem were obtained using LQR synthesis techniques and, even though, the performance attained by the LQR controller does not translate to this application due to saturation, it works fairly well when the tracking error is small. Also, one must select the attitude controller gains much higher that the position controller gains to ensure that the commanded orientation R_0 is closely tracked. The reference trajectory is given by

$$p_d(t) := \begin{bmatrix} a\sin(2\pi\nu_0 t) \\ a\cos(2\pi\nu_0 t) \\ -h_0 \end{bmatrix}, \quad R_d := I_3, \tag{57}$$

where $a = 1$ m, $f_0 = (2\pi)^{-1}$ Hz and $h_0 = 1$ m. This trajectory satisfies Assumptions 1.

In this experiment, we test specifically the hybrid nature of the proposed controller since, if working as intended, different values of the logic variable produce different outcomes when the quadrotor is near a rotation error of 180 degrees. From the analysis of Figure 2, it is possible to verify that this is indeed the case. For the experiment where $h(0,0) = -1$, the initial yaw angle is approximately 175° (or −185° if we subtract 360°) and it is quickly brought to zero. On the other hand, for the experiment where $h(0,0) = 1$, the initial yaw angle is 174° and, even though the initial yaw angles are only 1° apart, the quadrotor corrects its yaw angle by rotating in opposite directions. It may also be verified that this correction of the yaw angle has nothing but a small effect on the pitch and the roll angles. By introducing a hysteresis gap around rotations of 180°, where the behavior of the vehicle depends on the value of a logic variable, the hybrid controller reduces the possibility of chattering due to noise. This feature of the hybrid quaternion-based feedback is discussed in more detail in Mayhew et al. (2011*a*).

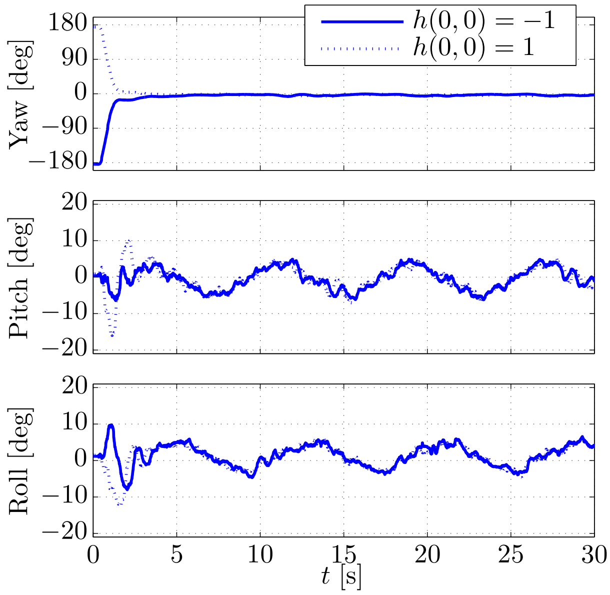

Fig. 2. Euler angles for two experiment runs starting roughly with the same initial.

In Figure 3, we present a comparison between the reference and the actual position of the vehicle for the two experiments. We see that there is little mismatch between the trajectories in both experiments and that the controller is able to converge to the given trajectory within the first 10 seconds of the experiments. The (component-wise) position tracking errors do not exceed 10 cm after the first 10 seconds, which we attribute mostly to delays in the system, since it is possible to see that there is some lag in the tracking of the given trajectory.

From the analysis of Figures 2 and 3 it can be concluded that the controller proposed in this paper yields very good results, despite the very simple model that was considered in its design.

8 Conclusion

In this paper, we designed a quaternion-based hybrid controller that globally asymptotically stabilizes a class of underactuated vehicles to a smooth reference position trajectory. In particular, the proposed controller is robust to bounded state dependent acceleration disturbances and small measurement noise. Moreover, the proposed controller also minimizes the angle to a reference orientation. The proposed controller was tested in an experimental setup using a optical motion capture system.

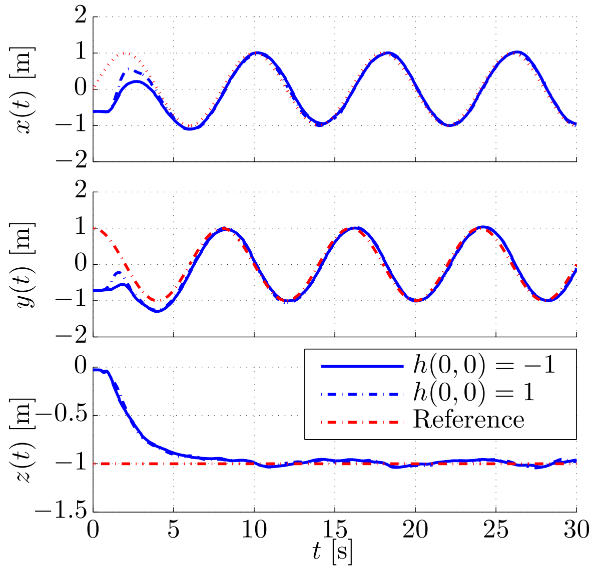

Fig. 3. Comparison between the position of the quadcopter and the reference trajectory for two experimental runs with initial yaw error of approximately 180° and different initial values of $h(0,0)$.

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9 Proof of Lemma 1

In what follows, let

$$\overline{u}_0(p_0, v_0) := -\Sigma_K(k_p p_0 + k_v v_0). \tag{58}$$

Proposition 4 ((Casau et al. 2013, Proposition 1)) *For each $K, k_p, k_v > 0$ there exists a positive definite and symmetric matrix $P \in \mathbb{R}^{2\times 2}$ such that*

$$\left\langle \mathcal{D}_{(p_0,v_0)} \left(\overline{V}_0(p_0, v_0)\right)^\top, [v_0^\top \; \overline{u}_0(p_0, v_0)^\top]^\top \right\rangle < 0,$$

for each $(p_0, v_0) \in \mathbb{R}^6 \backslash \{0\}$ and

$$\left\langle \mathcal{D}_{(p_0,v_0)} \left(\overline{V}_0(p_0, v_0)\right)^\top, [v_0^\top \; \overline{u}_0(p_0, v_0)^\top]^\top \right\rangle = 0,$$

for $(p_0, v_0) = 0$.

Next, we use the following intermediary result to show that the input T satisfies the given bounds.

Lemma 5 *Given a reference trajectory satisfying Assumption 1 and disturbances $(p, v, b) \mapsto L(p, v)b$ for (12), if (7) holds for each $(p, v) \in \mathbb{R}^3$, then there exists $K > |b|_\infty$ such that*

$$0 < |\mu(r, p_0, v_0, z)| \le g + (\sqrt{3} + \sqrt{\ell} \sup_{(p,v)\in\mathbb{R}^6} |L(p, v)|_2)K + \sup_{r\in\Omega_r} \left|p_d^{(2)}\right|, \tag{59}$$

for each $(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell$.

PROOF. It follows from (17) and from the reverse triangle inequality that

$$|\mu(r, p_0, v_0, z)| \ge g - (\sqrt{3} + \sqrt{\ell}\, |L(p, v)|_2)K - \sup_{r\in\Omega_r} \left|p_d^{(2)}\right|.$$

Then, it follows from (7) that it is possible to select $K > |b|_\infty$ such that $|\mu(r, p_0, v_0, z)| > 0$ for each $(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell$. It follows from the triangle inequality that (59) holds. $\square$

Consider the following continuous function

$$\begin{aligned} V_0(p_0, v_0, z) = {} & k_z \overline{V}_0(p_0, v_0) \\ & + \sum_{i=1}^{\ell} \left(\int_0^{|z_i|} \sigma_K(\xi)\, \mathrm{d}\xi + \int_0^{|b_i|} \sigma_K^{-1}(\xi)\, \mathrm{d}\xi \right) \\ & - b^\top z. \end{aligned} \tag{60}$$

From an application of Young's inequality it follows that

$$b_i z_i \le |b_i||z_i| \le \int_0^{|z_i|} \sigma_K(\xi)\, \mathrm{d}\xi + \int_0^{|b_i|} \sigma_K^{-1}(\xi)\, \mathrm{d}\xi, \tag{61}$$

it is possible to conclude that (60) is positive definite relative to the compact set $\{(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = v_0 = 0, z = \Sigma_K^{-1}(b)\}$. To prove that V_0 is radially unbounded, let us consider the following limit

$$\lim_{|z_i|\to+\infty} \int_0^{|z_i|} \sigma_K(\xi)\, \mathrm{d}\xi - b|z_i|, \tag{62}$$

where $b > 0$ and $i \in \{1, 2, 3\}$. Suppose that the limit in (62) is finite, then

$$\begin{aligned} \lim_{|z_i|\to+\infty} & \int_0^{|z_i|} \sigma_K(\xi)\, \mathrm{d}\xi - b|z_i| = \\ & = \lim_{|z_i|\to+\infty} \frac{|z_i| \int_0^{|z_i|} \sigma_K(\xi)\, \mathrm{d}\xi - b|z_i|^2}{|z_i|} \\ & = \lim_{|z_i|\to+\infty} \int_0^{|z_i|} \sigma_K(\xi)\, \mathrm{d}\xi - b|z_i| \\ & + \lim_{|z_i|\to+\infty} |z_i|(\sigma_K(|z_i|) - b). \end{aligned} \tag{63}$$

However, using the properties of the K-saturation function, since $K > |b|_\infty$ we have that $\lim_{|z_i|\to+\infty} |z_i|(\sigma_K(|z_i|) - b)$ does not converge, thus the limit in (62) cannot converge. By (61) we have that

$$\lim_{|z_i|\to+\infty} \int_0^{|z_i|} \sigma_K(\xi)\, \mathrm{d}\xi - b|z_i| = +\infty. \qquad \square$$

Since $\overline{V}_0$ is also radially unbounded, then V_0 is radially unbounded.

Let

$$F = \begin{pmatrix} v_0 \\ -\Sigma_K(k_p p_0 + k_v v_0) + L(p, v)(b - \Sigma_K(z)) \\ k_z L(p, v)^\top \mathcal{D}_{(p_0,v_0)} \left(\overline{V}_0(p_0, v_0)\right)^\top \end{pmatrix},$$

such that $(\dot{p}_0, \dot{v}_0, \dot{z}) = F(r, p_0, v_0, z)$, then the time derivative of (60) is given by

$$\left\langle \mathcal{D}_{(p_0,v_0)} (V_0(p_0, v_0))^\top, F \right\rangle = -k_z W_0(p_0, v_0), \tag{64}$$

thus (64) is negative definite relative to $p_0 = v_0 = 0$. Since (60) is radially unbounded, for any initial condition $(p_0, v_0, z)(0)$ then the sub-level set

$$\begin{aligned} \Omega_0 := \{ & (p_0, v_0, z) \in \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : \\ & V_0(p_0, v_0, z) \le V_0((p_0, v_0, z)(0))\} \end{aligned} \tag{65}$$

is compact. It follows from (Khalil 2002, Theorem 4.8) that the set $\{(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = v_0 = 0, z = \Sigma_K^{-1}(b)\}$ is globally stable for the system (12), and it follows from (Khalil 2002, Theorem 8.4) that $V_0(p_0, v_0, z)$ converges to 0, therefore each solution converges to

$$\Omega_0 \cap \{(p_0, v_0, z) \in \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = v_0 = 0\}, \tag{66}$$

which is a subset of $\mathcal{A}_0$.