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Properties of the Multivariate Cauchy Estimator

Author
Bai, Yu

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Abstract of the Dissertation

Properties of the Multivariate Cauchy Estimator

by

Yu Bai
Doctor of Philosophy in Mechanical Engineering
University of California, Los Angeles, 2016
Professor Jason L. Speyer, Chair

In this dissertation, the fundamental structure of a multivariate discrete-time state estimator with Cauchy distributed process noise and measurement noise is discussed in depth. The characteristic function (CF) of the unnormalized conditional probability density function (ucpdf) is found to be a sum of elements that increases at each update of the current measurement. Each term in this sum is composed of a coefficient term which contains the measurement history operating on an exponential term composed of a sum of absolute values whose argument is the inner product of a direction vector with the spectral variable. The objective is to understand the structure of the CF so as to simplify this sum. We shows that directions in the terms of the CF-s are co-aligned only along certain directions which are functions of a unique fundamental basis. Based on the knowledge of combining co-aligned directions, an indexing scheme, called "S" matrix, is developed to indicate which exponential terms can be combined without the necessity of numerical comparison. The S matrix is invariant for systems of the same dimension regardless of the system parameters. The coefficient terms are also restructured and simplified by eliminating all the redundant zero elements. For two-state systems, we show that there are no more than three non-zero elements in each layer of any new coefficient term. Furthermore, with these newly uncovered properties the Cauchy estimator is implemented efficiently using a pre-computational technique. The simulations of three-state and four-state systems illustrate the performance of Cauchy estimator compared with the Kalman Filter.

The dissertation of Yu Bai is approved.

Tetsuya Iwasaki

James S. Gibson

Lieven Vandenberghe

Jason L. Speyer, Committee Chair

University of California, Los Angeles

2016

To my family.

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Preface

I still remember one day in a warm early afternoon, Prof. Speyer, my thesis adviser, and I were brainstorming in his office on the third floor of the Engineering IV building as usual. When I sit beside Prof. Speyer and randomly wrote down the form of the child directions, I noticed that their parent directions must have some special structure, which we did not look at before. I was excited. In another three days, this finding was extended to become Theorem 2.1.2 (Co-Alignment Condition), which plays a fundamental role to establish the understanding of estimator properties in this dissertation. And it seems so simple.

Before that afternoon, I was stuck when trying to better uncover the estimator structure for three-state system. The integral in the update formula just produces so many terms. Most properties we have already had for the two-state system seemed not to work for higher-order cases in a understandable way. For more than a year, I spent day and night with these equations, sometimes even made notes after I "derived" some equations in my dream. I tried all possible approaches that I could think of, and then ended up not really going very far at that time.

Prof. Speyer has always given me the greatest patience since the very beginning. His extensive expertise in this area gradually trained me how to conduct scientific research. In the meanwhile his diligence and positive attitude influenced me time after time. No matter I have progresses or not, Prof. Speyer sit down with me at least twice a week to discuss the current problems. After that afternoon, my thesis research started to progress smoothly; I could write up working reports for my new findings every one to two weeks. Without Prof. Speyer's effort, I would never get to where I am.

Research is about uncovering unknowns. And we study uncertainties. I used to have a lot of anxieties about my research progress, about all those uncertain future. After this experience at UCLA, I realized that we have to accept uncertainties. We have to accept that we may or may not find the solution today, and hard working oftentimes may not guarantee

the desired outcome. But that should not stop our hope. That reminds me a famous quote in *Gone with the Wind* which we sometimes chat about: After all, tomorrow is another day.

I owe my most sincere appreciation to my adviser, Prof. Jason L. Speyer. Over the past four years, Prof. Speyer has devoted tremendous amount of efforts to guide, educate and support me.

I would also like to thank Prof. Moshe Idan in Technion, Israel.

At UCLA, many professors and stuffs have provided me with great support whenever I need them. I would like to say thank you to all of them.

I would like to thank my friends too. The understanding and support from my peers raised me up during those difficult days.

Finally, I am deeply thankful to my family for their unconditional love and support which makes me stronger. Without them, this thesis would never have been written. In addition, I dedicate this thesis to the memory of my grandparents. Because of studying abroad for these years, I did not seize my last chance to see them ever again.

Yu Bai

September 2016, Los Angeles

Vita

2010	Bachelor in Automotive Engineering at Tsinghua University, China
2012	MS in Mechanical Engineering at Columbia University in the City of New York
2012–2016	Graduate Research Assistant at the University of California, Los Angeles
2013–2016	Teaching Assistant at the University of California, Los Angeles

Publications

Yu Bai, Jason Speyer, Moshe Idan, Scott Ploen, David Bayard, *The Cauchy Estimator for Europa Flyby Attitude Estimation*, SIAM Conference on Control and Its Applications, 2015.

Yu Bai, Jason Speyer, Moshe Idan, *Efficient Cauchy Estimation via a Pre-Computational Technique*, 55th IEEE Conference on Decision and Control, 2016.

CHAPTER 1

Introduction

1.1 Overview

A multivariate state estimator for linear systems with additive Cauchy noise is studied in this work. Many uncertainties in nature, engineering systems, finance and other fields have been found to have heavy-tailed characteristics [1] [2]. Unlike the light-tailed distributions, including the most common Gaussian distributions, heavy-tailed uncertainties are very impulsive, hence can hardly be described by any light-tailed probability density function (pdf). Under many practical circumstances, the traditional Kalman Filter cannot obtain desired performance in the presence of heavy-tailed noises. A state estimator developed for heavy-tailed uncertainties has significant research potential.

Cauchy distribution has heavy tail which upper bounds many distribution densities. It has closed form probability density function (pdf) and characteristic function (CF), which enables us to develop the estimation problem under Cauchy distributed uncertainties. Unlike Gaussian distribution, Cauchy has undefined mean and infinite second moment. However, we show that the conditional mean given the measurement history is well-defined, and the conditional variance given the measurements is finite.

In [3], [4], [5] and [6], scalar, multivariate, and two-state Cauchy estimators with additive Cauchy noise was developed in a recursive, closed form that in the Cauchy environment produces the conditional mean and finite conditional variance. This Cauchy estimator is the only algorithm besides the Gaussian to have a recursive, closed-form. It was shown to have almost as good performance as the Kalman Filter when the process noise and measurement noise are Gaussian distributed. In addition, in the presence of Cauchy distributed uncertainties, the Cauchy estimator outperforms the Kalman Filter. Therefore, although physically there does not exist Cauchy noise, we hypotheses that the Cauchy estimator is robust in many noise environments. The major challenge of the multivariate Cauchy estimator is the computational efficiency and memory requirement, due to the complexity of the estimator structure.

The goal of this dissertation is to uncover several important fundamental properties of the multivariate Cauchy estimator structure, and provide an offline - online technique which allows the estimator to process a large sequence of measurement data. Aligned with [4] [5] and [6], the CF approach is used. In other words, the CF of the conditional pdf of the state given the measurement history at each step is propagated and updated, from which the conditional mean and conditional variance is evaluated. These newly found fundamental properties as well as the pre-computational technique presented in this dissertation is demonstrated to have enhanced the computational efficiency of the estimator significantly.

1.2 Cauchy Distribution

The Cauchy distribution is included in the symmetric α-stable (SαS) distributions, which is a class of heavy-tailed distributions [7]. It is defined by its characteristic function (CF) as,

$$\phi(\nu) = \exp\left(j\mu\nu - \gamma^{\alpha}|\nu|^{\alpha}\right) \tag{1.1}$$

where $\alpha \in (0, 2]$ is the stability parameter, $\gamma \in (0, \infty)$ is the scale parameter, and $\mu \in (-\infty, \infty)$ is the location parameter, and the median. When $\alpha > 1$, μ is the mean. When $\alpha \leq 1$, the mean is undefined. In addition, when $\alpha = 2$, equation (1.1) describes a Gaussian distribution, with the variance well-defined to be $2\gamma^2$. When $\alpha < 2$, the random variable has infinite second moment.

The Cauchy distribution can be described by equation (1.1) when $\alpha = 1$. Cauchy distribution has undefined mean and infinite second moment. The probability density function (pdf) of a random variable $X \sim Cauchy(\mu, \gamma)$ is expressed as,

$$f(x) = \frac{\gamma/\pi}{(x-\mu)^2 + \gamma^2}, \quad -\infty < x < \infty \tag{1.2}$$

The probability density functions of SαS distribution for different cases of Cauchy distributions compared with Gaussian distribution is shown in Figure 1.1.

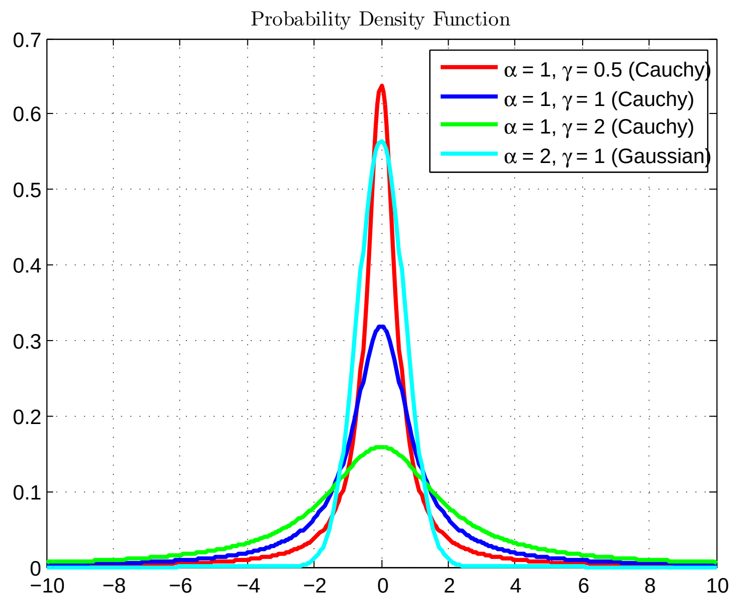

Figure 1.1: SαS probability density functions

1.3 Problem Statement

Consider the linear state space system,

$$x_{k+1} = \Phi x_k + \Gamma w_k \tag{1.3}$$

$$z_k = H x_k + v_k \tag{1.4}$$

The state vector is $x_k \in \mathbb{R}^n$ and z_k is the scalar measurement. The state transition matrix $\Phi \in \mathbb{R}^{n\times n}$, the process noise matrix $\Gamma \in \mathbb{R}^{n\times m}$, and the measurement matrix $H \in \mathbb{R}^{1\times n}$ are known. The process noise w_k and the measurement noise v_k are assumed to be independent Cauchy distributed random variables. In the following of this work, we study the vector case particularly, based on which the solutions and results are presented. Although not necessary, for convenience we let $m = 1$ for the rest of this dissertation. And let w_k and v_k have zero medians and a scale parameter β and γ, respectively, i.e. the characteristic function of w_k and v_k is $\phi_W(\bar{\nu}) = \exp(-\beta|\bar{\nu}|)$ and $\phi_V(\bar{\nu}) = \exp(-\gamma|\bar{\nu}|)$. $\bar{\nu}$ is a scalar spectral variable. k is the stage time. The elements of the initial state vector x_1 is assumed to be independent, Cauchy distributed, with a zero median and scale parameters α_i, $i = 1, 2, ..., n$. The characteristic function of the initial state vector is $\phi_{X_1}(\nu) = \exp\left(-\sum_{i=1}^{n} \alpha_i \left|e_i \nu\right| + j{u_1^1}^T \nu\right)$, where the spectral vector $\nu \in \mathbb{R}^n$, the unit directions $e_1 = [1 \quad 0 \quad ... \quad 0]$, ..., $e_n = [0 \quad 0 \quad ... \quad 1]$, and the median of the distribution of the initial state x_1 is $u_1^1 \in \mathbb{R}^n$.

1.4 Estimator Structure

State estimators with additive Cauchy noise was developed in [4] [5] and [6] which is recursive and in closed form. The estimator structure was established on characteristic functions (CF) of the conditional probability density functions (cpdf), instead of updating the pdf directly.

1.4.1 Propagation and Update

The multivariate Cauchy estimator in [5] is formulated by propagating and updating the characteristic function (CF) of the conditional probability density function of the state x_k given the measurement history $y_k = \{z_1, \cdots, z_k\}$ at step k. At the first measurement update, the characteristic function is given by,

$$\phi_{X_1|Z_1}(\nu) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1|Z_1}(x_1|z_1) e^{j\nu^T x_1} dx_1 \tag{1.5}$$

where $\nu \in \mathbb{R}^n$. Because of Bayes's theorem, the conditional pdf can be written as,

$$f_{X_1|Z_1}(x_1|z_1) = \frac{f_V(z_1 - Hx_1) f_{X_1}(x_1)}{f_{Z_1}(z_1)} \tag{1.6}$$

Then the CF of the cpdf becomes,

$$\phi_{X_1|Z_1}(\nu) = \frac{1}{f_{Z_1}(z_1)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_V(z_1 - Hx_1) f_{X_1}(x_1) e^{j\nu^T x_1} dx_1 \tag{1.7}$$

and the CF of the unnormalized conditional probability density function (ucpdf) at step $k = 1$ was defined in [5] as,

$$\bar{\phi}_{X_1|Z_1}(\nu) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_V(z_1 - Hx_1) f_{X_1}(x_1) e^{j\nu^T x_1} dx_1 \tag{1.8}$$

This integral, shown in Appendix A in [5], can be expressed by utilizing the property of Fourier transforms, as

$$\bar{\phi}_{X_1|Z_1}(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{X_1}(\nu - H^T\eta) \phi_V(-\eta) e^{jz_1\eta} d\eta \tag{1.9}$$

This integral is solved using Appendix B in [5]. Note that the solution of Appendix B in [5] is summarized and presented in Appendix A for the readers' reference.

The CF of the ucpdf was propagated and updated at every step in a recursive manner. If starting from the CF of the ucpdf $\bar{\phi}_{X_k|Y_k}(\nu)$ at step k where Y_k is the random variable of the measurement history, then after time propagation according to the equation $x_{k+1} = \Phi x_k + \Gamma w_k$ from step k to step $k+1$, the CF was derived in Appendix C in [5] as,

$$\bar{\phi}_{X_{k+1}|Y_k}(\nu) = \bar{\phi}_{X_k|Y_k}(\Phi^T\nu) \phi_W(\Gamma^T\nu) \tag{1.10}$$

The CF of the ucpdf at the $(k+1)^{th}$ measurement update according to the equation $z_{k+1} = Hx_{k+1} + v_k$ is updated as

$$\bar{\phi}_{X_{k+1}|Y_{k+1}}(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\phi}_{X_{k+1}|Y_k}(\nu - H^T\eta) \phi_V(-\eta) e^{jz_{k+1}\eta} d\eta \tag{1.11}$$

This integral is solved using Appendix B in [5].

The conditional mean and second moment were then evaluated by taking the first and second derivatives of the CF of the ucpdf around the origin $\{0\}_n$ at each measurement step. A constant direction $\hat{\nu}$ was picked *a priori*. Let the spectral variable ν approach the origin $\{0\}_n$ along the direction of $\hat{\nu}$, i.e. $\nu = \epsilon\hat{\nu}$ while letting $\epsilon \to 0$. As shown in [5], the conditional mean of the state x_k given the measurement sequence y_k was given by,

$$\hat{x}_k = \mathbf{E}\left[x_k|y_k\right] = \frac{1}{j f_{Y_k}(y_k)} \left(\frac{\partial \bar{\phi}_{X_k|Y_k}(\epsilon\hat{\nu})}{\partial(\nu)} \right)^T \Bigg|_{\epsilon=0} \tag{1.12}$$

and the second conditional moment of x_k given y_k was given by,

$$\mathbf{E}\left[x_k x_k^T | y_k\right] = \frac{1}{j^2 f_{Y_k}(y_k)} \left(\frac{\partial^2 \bar{\phi}_{X_k|Y_k}(\epsilon\hat{\nu})}{\partial(\nu)\partial(\nu)^T}\right)^T \Bigg|_{\epsilon=0} \tag{1.13}$$

where $f_{Y_k}(y_k)$ was determined by the value of the CF at the origin, i.e. $f_{Y_k}(y_k) = \bar{\phi}_{X_k|Y_k}(\epsilon\hat{\nu})\big|_{\epsilon=0}$.

1.4.2 Structure of the CF

As presented in Appendix B.1 and B.2 in [5], at each measurement update, the CF of the ucpdf, $\bar{\phi}_{X_k|Y_k}(\nu)$, (the solution of (1.9) and (1.11)) was expressed as a sum of products of an exponential term $\mathcal{E}_i^{k|k}(\nu)$ and a coefficient term $G_i^{k|k}(\nu)$, i.e.

$$\bar{\phi}_{X_k|Y_k}(\nu) = \sum_{i=1}^{N_t^{k|k}} G_i^{k|k}(\nu)\mathcal{E}_i^{k|k}(\nu) \tag{1.14}$$

where Y_k is the random variable of the measurement history and y_k is a realization of the measurement history.

The exponential part can be expressed as,

$$\mathcal{E}_i^{k|k}(\nu) = \exp\left(-\sum_{l=1}^{N_{ei}^{k|k}} P_{i,l}^{k|k}\left|B_{i,l}^{k|k}\nu\right| + j\zeta_i^{k|k}\nu\right) \tag{1.15}$$

where the notations $P_{i,l}^{k|k} \in \mathbb{R}$, $B_{i,l}^{k|k} \in \mathbb{R}^{1\times n}$, $\zeta_i^{k|k} \in \mathbb{R}^{1\times n}$ and $N_{ei}^{k|k} \in \mathbb{R}$ are corresponding to the original notation $p_{il}^{k|k}$, ${a_{i,l}^{k|k}}^T$, ${b_i^{k|k}}^T$, and $n_{ei}^{k|k}$, respectively, in equation (4.3) in [5]. $P_{i,l}^{k|k}$ are scalars. The "directions" $B_{i,l}^{k|k}$ are row vectors. The imaginary part of the argument of the exponential, $\zeta_i^{k|k}$, is also a row vector that contains the measurements. At step k, the number of directions in the i^{th} term is denoted as $N_{ei}^{k|k}$, and there are a total of $N_t^{k|k}$ terms in the characteristic function.

The coefficient part $G_i^{k|k}(\nu)$ were described as layers of fractional forms in equation (4.2) in [5].

$$\begin{aligned} G_i^{k|k}(\nu) &= g_i^{k|k}\left(y_{gi}^{k|k}(\nu)\right) \\ &= \frac{1}{2\pi}\left[\frac{g_{r_i^{k|k}}^{k-1|k-1}\left(y_{gi1}^{k|k}(\nu) + h_i^{k|k}\right)}{jc_i^{k|k} + d_i^{k|k} + y_{gi2}^{k|k}(\nu)} - \frac{g_{r_i^{k|k}}^{k-1|k-1}\left(y_{gi1}^{k|k}(\nu) - h_i^{k|k}\right)}{jc_i^{k|k} - d_i^{k|k} + y_{gi2}^{k|k}(\nu)}\right] \end{aligned} \tag{1.16}$$

where

$$y_{gi}^{k|k}(\nu) = \sum_{l=1}^{n_{ei}^{k|k}} q_{il}^{k|k} sgn\left(B_{i,l}^{k|k}\nu\right) \in \mathbb{R}^k \tag{1.17}$$

The coefficient functions $g_i^{k|k}(\cdot)$ was determined by the parameters including $c_i^{k|k}$, $d_i^{k|k}$, the offsets $h_i^{k|k}$ and the index $r_i^{k|k}$ of the parent term. $y_{gi}^{k|k}(\cdot)$ and $q_{il}^{k|k}$ are k-dimensional vectors. $y_{gi}^{k|k}(\cdot)$ breaks down to two components: the first $k-1$ components of $y_{gi}^{k|k}(\cdot)$ constructs $y_{gi1}^{k|k}(\cdot)$, while the last component of $y_{gi}^{k|k}(\cdot)$ comprises the scalar $y_{gi2}^{k|k}(\cdot)$. The derivations for these parameters were given in Appendix B1 - B2 in [5]. These parameters were all obtained recursively from the last measurement update.

In [5], for the first time a multivariate state estimator with Cauchy distributed uncertainties was expressed in a recursive, closed form. The form of the CF of the ucpdf was described by a sum of terms, composed of a product of exponential and coefficient elements, shown in equation (1.14). For systems of a given dimension, the number of terms in this sum, $N_t^{k|k}$, grows as k gets large. Furthermore, there were significantly more terms in the CF of a higher-order system than a lower-order system. Due to the complexity of the estimator structure, processing speed and storage requirement becomes large when time step k or dimension n increases. This is the essential difficulty that needs to be overcome to make the Cauchy estimator a practical algorithm.

[6] developed an efficient recursive estimator structure for two-state systems. This structure reduced substantially the number of terms, i.e. $N_t^{k|k}$, that were needed to propagate and update the CF. Moreover, the estimator was computationally simplified by truncating the measurement sequence by a "sliding window" approximation allowing an unlimited number of measurements to be processed while maintaining good performance. This structure was derived based on some special properties that are only valid for two-state systems. We try to generalize the special structure of the two-state estimator to higher-order systems, but only partial results have been obtained.

1.5 Objectives

The objective of this research is to break down the fundamental structure of the multivariate Cauchy estimator, and seek appropriate techniques so that the computational efficiency of the estimator can be largely enhanced. In this dissertation, several interesting properties behind this estimator structure are newly uncovered.

Firstly, not all of the directions, i.e. $B_{i,l}^{k|k}$, are distinct: many of them are co-aligned. These co-aligned directions have to be combined to avoid singularity [5]. The reduction of the total number of directions due to this combination can also simplify the estimator structure. Let us call the directions in earlier steps as "parent" directions, and the directions in later steps which are produced from those parent directions as "child" directions. Analytically, it is proven in this dissertation that parent directions of a certain form can produce co-aligned child directions. All co-aligned child directions are functions of a unique fundamental basis. This is presented in Chapter 2.

Secondly, Chapter 3 shows that many of exponential terms in the sum of the CF have the same argument analytically and can be combined. For two-state systems, this property is summarized by two rules for term combination. It is also shown that there exists only such two rules which indicate how the argument of the exponential terms can be combined. For three-state systems, four such rules are found. These term combination rules start to reveal the fundamental properties of analytically combining exponential terms for more general higher-order cases.

Thirdly, the way that repeated exponential terms are *a priori* combined is represented by an indexing matrix, called "S" matrix, discussed in Chapter 4. It determines which terms to be combined without the need of online comparison. For two-state case, particularly, a recursive structure of the S matrix is derived analytically, given that the term combination rules for two-state systems are fully uncovered. For higher-order cases, the S matrix can always be obtained numerically.

Fourthly, Chapter 5 proposes a new structure of the coefficient terms that reduces the memory requirement by eliminating all the redundant zeros, and makes the offline - online separation implementable. The coefficient terms have multiple layers of offsets and sign functions. However, many of these offsets are zero, because the integral solved in Appendix B in [5] artificially introduces zero elements for consistency. This results in many zero elements in many layers of the G terms. An alternative structure for the G terms is proposed in order to avoid these artificial zeros. A comprehensive study on two-state system shows that for new terms, there are at most three non-zero elements in each layer of the G terms.

Finally, in Chapter 6 we develop a pre-computational technique to separate the component of the estimator structure that is independent of the measurements (offline), with the component that is dependent upon the measurements (online). By storing the results of the offline stage as *a priori* data, the actual online processing efficiency is significantly enhanced. The implementations for a three and four state system in both Cauchy and Gaussian simulations compared to the Kalman Filter are discussed in Chapter 7. Conclusions are presented in Chapter 8.

CHAPTER 2

Co-Alignment of Directions

As mentioned earlier in Chapter 1, many directions $B_{i,l}^{k|k}$ are co-aligned with each other and hence can be combined. This fact was observed in [5] where it was found that it was necessary to make these combinations. In this chapter, analytically we explain when and how such direction co-alignment occurs. Firstly, three distinct parent directions being linearly dependent is proven to be the necessary and sufficient condition for their two child directions to be co-aligned. Next, it is shown that these co-aligned child directions can be determined by functions of a fundamental basis matrix. The analytic form of the fundamental basis is found so far to be unique for systems up to five-state. By induction, it can be inferred that co-aligned directions in general case are all in forms of fundamental basis, and such a fundamental basis is unique. This property helps to simplify the estimator structure and hence contribute to the computational efficiency.

2.1 A Necessary and Sufficient Condition for Co-Alignment

In order to investigate how the directions are co-aligned, we need to explicitly write down the form of the child directions in terms of its parent directions in a recursive manner. At step k, suppose an exponential term is expressed as,

$$\mathcal{E}^{k|k}(\nu) = \exp\left(-\sum_{i=1}^{N_{ei}^{k|k}} P_i^{k|k}\left|B_i^{k|k}\nu\right| + j\zeta^{k|k}\nu\right) \tag{2.1}$$

Then at step $k+1$, the l^{th} child term is given by,

$$\mathcal{E}_l^{k+1|k+1}(\nu) = \exp\left(- \sum_{i=1, i\neq l}^{N_{ei}^{k|k}} P_i^{k|k} \left| B_i^{k|k} \Phi^T H^T \right| \left| \frac{B_i^{k|k}\Phi^T \nu}{B_i^{k|k}\Phi^T H^T} - \frac{B_l^{k|k}\Phi^T \nu}{B_l^{k|k}\Phi^T H^T} \right| + j\zeta_l^{k+1|k+1}\nu \right) \tag{2.2}$$

This derivation is presented by Appendix A, referring to Appendix B.1 and B.2 in [5]. Then the child direction $B_{i,l}^{k+1|k+1}$ at step $k+1$ in terms of the parent directions $B_i^{k|k}$ and $B_l^{k|k}$ at step k can be rewritten as follow.

$$\begin{aligned} B_{i,l}^{k+1|k+1} &= \frac{B_i^{k|k}\Phi^T}{B_i^{k|k}\Phi^T H^T} - \frac{B_l^{k|k}\Phi^T}{B_l^{k|k}\Phi^T H^T} \\ &= \frac{H\Phi\left(B_l^{k|k^T} B_i^{k|k} - B_i^{k|k^T} B_l^{k|k} \right)\Phi^T}{(B_i^{k|k}\Phi^T H^T)(B_l^{k|k}\Phi^T H^T)} \end{aligned} \tag{2.3}$$

One can immediately notice that the child direction $B_{i,l}^{k+1|k+1}$ is in the form of HC, where C is a rank two skew-symmetric matrix, i.e. $C^T = -C$. This can be stated as the following lemma.

Lemma 2.1.1. *Any two non-zero parent directions at step k will produce a child direction in the form of HC at step $k+1$, where C is a rank two skew-symmetric matrix.*

Proof. Recall equation (2.3). The matrix C can be written as $C = \frac{\Phi\left(B_l^{k|k^T} B_i^{k|k} - B_i^{k|k^T} B_l^{k|k} \right)\Phi^T}{(B_i^{k|k}\Phi^T H^T)(B_l^{k|k}\Phi^T H^T)}$, then the child direction is $B_{i,l}^{k+1|k+1} = HC$. It is obvious that $C = -C^T$. In addition, $rank(C) = rank(B_l^{k|k^T} B_i^{k|k} - B_i^{k|k^T} B_l^{k|k}) \leq rank(B_l^{k|k^T} B_i^{k|k}) + rank(B_i^{k|k^T} B_l^{k|k}) = 2$, while C cannot be rank 1. Therefore C is skew-symmetric rank two matrix. □

A necessary and sufficient condition on the co-alignment of directions within a term is presented next. With the direction update being explicitly defined in (2.3), we will look at three parent directions at step k, and consider under what conditions the child directions at step $k+1$ are co-aligned.

Theorem 2.1.2 (Co-Alignment Condition)**.** *Suppose at step k, b_1, b_2 and $b_3 \in \mathbb{R}^{1\times n}$ are three directions in one term. b_1, b_2 and b_3 are not co-aligned with each other. And at step $k+1$, the child directions can be written as,*

$$c_{1,2} = H\Phi(b_1^T b_2 - b_2^T b_1)\Phi^T \tag{2.4}$$

and

$$c_{1,3} = H\Phi(b_1^T b_3 - b_3^T b_1)\Phi^T \tag{2.5}$$

Then $c_{1,2}$ and $c_{1,3}$ are co-aligned if and only if the matrix $M = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^{3\times n}$ is not full row rank.

Proof. 1. Necessity

If the matrix M is not full row rank, then there exist a scalar $\alpha \in \mathbb{R}$, such that the linear combination $b_2 + \alpha b_3$ is co-aligned with b_1. Normalize them using $\Phi^T H^T$, assuming $b_1\Phi^T H^T$ (or $(b_2+\alpha b_3)\Phi^T H^T$) is non-zero. Therefore,

$$\frac{b_2 + \alpha b_3}{(b_2+\alpha b_3)\Phi^T H^T} = \frac{b_1}{b_1 \Phi^T H^T} \tag{2.6}$$

Then

$$(b_1\Phi^T H^T)(b_2+\alpha b_3)\Phi^T = (b_2+\alpha b_3)\Phi^T H^T \cdot b_1 \Phi^T \tag{2.7}$$

Rearrange equation (2.7) to obtain the following equation.

$$H\Phi b_1^T (b_2+\alpha b_3)\Phi^T = H\Phi(b_2+\alpha b_3)^T b_1 \Phi^T \tag{2.8}$$

Expand both side of the equation,

$$H\Phi b_1^T b_2 \Phi^T + \alpha H\Phi b_1^T b_3 \Phi^T = H\Phi b_2^T b_1 \Phi^T + \alpha H \Phi b_3^T b_1 \Phi \tag{2.9}$$

$$H\Phi(b_1^T b_2 - b_2^T b_1)\Phi^T = -\alpha H\Phi(b_1^T b_3 - b_3^T b_1)\Phi^T \tag{2.10}$$

or

$$c_{1,2} = -\alpha \cdot c_{1,3} \tag{2.11}$$

In this case, $c_{1,2}$ and $c_{1,3}$ are co-aligned.

2. Sufficiency

Suppose $c_{1,2}$ and $c_{1,3}$ are co-aligned, then there exists α such that equation (2.11) holds. Reversing the proof for necessity, it can be concluded that b_1 and $(b_2 + \alpha b_3)$ are co-aligned. This implies that the matrix M is not full row rank. □

Theorem 2.1.2 shows the necessary and sufficient condition for directions to be co-aligned in general case. Using this theorem, the rest of this chapter investigates what types of parent directions can produce co-aligned child directions, and what those co-aligned child directions look like. These pieces provide us adequate information to uncover the direction co-alignment process analytically.

2.2 Parent Directions, Child Directions, and Fundamental Basis

According to Lemma 2.1.1, only three linearly dependent parent directions can produce co-aligned child directions. As discussed in [5] in detail, each parent term at step k produces many child terms at step $k+1$. Generating different child terms involves different pairing of those parent directions at step k. For instance, the directions in the j^{th} child term involves the operation between the j^{th} parent direction $b_j\Phi^T$ with all the rest of the parent directions. The last child term is called the "old term". The last parent direction after time propagation is the zero vector $\{0\}_n$, which can be found in [5]. The rest of the child terms are called "new terms". As a result of Lemma 2.1.1, for a new term, most of the directions are in the HC form, and only one (the last) direction is not. Furthermore, for an old term, some directions are of the $HC\Phi^{\theta T}$ form instead, where the positive integer θ represents how old the term is. The form of HC (or $HC\Phi^{\theta T}$) is special, because when b_1, b_2 and b_3 are all in the HC (or $HC\Phi^{\theta T}$) form where C is skew-symmetric matrix, it is possible that the child directions are co-aligned, resulting from Theorem 2.1.2. This is due to the fact that for any skew-symmetric matrix C, $HCH^T = 0$ for any n-dim row vector H. Thus, H is already in the null space of M in Theorem 2.1.2. In addition, the co-aligned directions are always in a form of a unique fundamental basis.

2.2.1 Two-State Case

Many interesting properties, although somewhat specialized, are obtained for this case. Consider the argument of the exponential term of a two-state system. Generally, a new term with p elements in the exponential argument has the directions in the forms of $[b_1, b_2, ..., b_p]$, where $b_i \in \mathbb{R}^{1\times 2}$ for $i = 1, 2, \cdots, p$. Starting from this term, the 1-step old child term has the directions $[b_1\Phi^T, b_2\Phi^T, ..., b_p\Phi^T, \Gamma^T]$. Furthermore, for a general θ-step old term, the directions are in the forms of $[b_1\Phi^{T\theta}, b_2\Phi^{T\theta}, ..., b_p\Phi^{T\theta}, \Gamma^T\Phi^{T(\theta-1)}, ..., \Gamma^T]$. What we now show is how in the next update these p elements collapse.

A necessary and sufficient condition for the child directions to be co-aligned is that the three parent directions are not full row rank, according to Theorem 2.1.2. The two-state systems is special in that the direction co-alignment will always occur from any parent directions, regardless of new parent terms or old parent terms. This is because M in Theorem 2.1.2 becomes a 3 by 2 matrix and can never be full row rank. As a consequence, all child directions that are produced from any two non-zero directions in a parent term will co-align. Then, any new term for a two-state system will have only two directions and will not increase as time goes on. In other words, for any new term, the number of elements in the argument of the exponential is $p = 2$. This is stated in the following theorem,

Theorem 2.2.1. *For two-state systems, any new term has only two directions (or elements) in the argument of the exponential term.*

In fact, repeated child directions in different terms can be co-aligned to common direction that is a function of a unique fundamental basis. In order to show the uniqueness, let $b_1 = [\beta_1 \quad \beta_2]$, $b_2 = [\alpha_1 \quad \alpha_2]$, where β_1, β_2, α_1 and α_2 are arbitrary scalars. Construct a matrix $b = [b_2^T \quad b_1^T] = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Suppose b_1 and b_2 are any two independent directions at step k, then after time propagation and measurement update, at step $k+1$, the child direction that will be produced from b_1 and b_2 is denoted as c with the scaling coefficients neglected, as seen in (2.3).

$$\begin{aligned} c &= H\Phi \left(b_2^T b_1 - b_1^T b_2 \right) \Phi^T && (2.12) \\ &= H\Phi \left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} - \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \right) \Phi^T \\ &= H\Phi \left(\begin{bmatrix} 0 & \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_2\beta_1 - \alpha_1\beta_2 & 0 \end{bmatrix} \right) \Phi^T \\ &= \left(\alpha_1\beta_2 - \alpha_2\beta_1 \right) H\Phi\mathbf{A}\Phi^T && (2.13) \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.14}$$

Because,

$$\begin{aligned} \Phi\mathbf{A}\Phi^T &= \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \\ &= \begin{bmatrix} -\phi_{12} & \phi_{11} \\ -\phi_{22} & \phi_{21} \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \phi_{11}\phi_{22} - \phi_{12}\phi_{21} \\ \phi_{12}\phi_{21} - \phi_{11}\phi_{22} & 0 \end{bmatrix} \\ &= \det(\Phi) \cdot \mathbf{A} && (2.15) \end{aligned}$$

and

$$\left(\alpha_1\beta_2 - \alpha_2\beta_1 \right) = \det \left(\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \right) = \det(b)$$

Therefore, the child direction c that is produced from b_1 and b_2 in equation (2.12) is,

$$c = \det(\Phi) \cdot \det(b) \cdot \mathbf{A} \tag{2.16}$$

b_1 and b_2 here are set to be arbitrary, meaning that they can be either new parent directions or old parent directions. There is no necessity to discuss new parent and old parent separately. The child directions of any two non-zero parent directions in any term will be aligned onto the $H\mathbf{A}$ direction. $\mathbf{A}$ is the fundamental basis for two-state problem. The form of $\mathbf{A}$ is unique, if ignoring the scaling factor. This result can be summarized into the following theorem,

Theorem 2.2.2. *For two-state systems, all the co-aligned directions can only be combined along the direction of* $H\boldsymbol{A}$*, where* $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *is the fundamental basis.*

2.2.2 Three-State Case

Consider three-state systems. A parent term at the k^{th} measurement update can produce several child terms at the $(k+1)^{th}$ measurement update. Certain directions (or elements in the argument of the exponential part) in some of the child terms are co-aligned. The new parent terms and the old parent terms are discussed separately.

2.2.2.1 From New Parent Terms

First, let's look at the directions in a new parent term at step k. Suppose there are in total $N_e = p+1$ directions in this term. Derived from Lemma 2.1.1, the directions can be written as $[HC_1, HC_2, ..., HC_p, b]$, where C_i's are skew-symmetric matrices, and b is a single direction that cannot be expressed in HC form.

From Theorem 2.1.2 in section 2.1, we understand that whenever two child-directions are co-aligned, their three parent-directions have to be linearly dependent. For a new term, this will happen when the three parent direction all have the form of HC, where $C \in \mathbb{R}^{3\times 3}$

is an anti-symmetric matrix. This is because of one property of anti-symmetric matrices $C \in \mathbb{R}^{n\times n}$, $HCH^T = 0$ holds for any $H \in \mathbb{R}^{1\times n}$. Then $MH^T = \{0\}_3$. Hence M is not full row rank.

In fact, this is also the only scenario that can produce co-aligned child directions, since the parent directions other than HC_i's is merely b. During time propagation, the directions of this term will become $HC_i\Phi^T$, $b\Phi^T$ and Γ^T. It is assumed that $b\Phi^T$ and Γ^T are generically linearly independent with each other and with all the HC_i directions. In order for the matrix M in the statement to be not full row rank, each row of M has to be in the form of HC before time propagation.

This can be stated in the following theorem,

Theorem 2.2.3. *For new parent terms of the three-state systems, only the parent directions in the forms of H multiplied by a skew-symmetric matrix can produce co-aligned child directions.*

Furthermore, if the two parent directions in a new term, b_1 and b_2, at step k are,

$$b_1 = H(\alpha_1 A_{12} + \alpha_2 A_{13} + \alpha_3 A_{23}) \tag{2.17}$$

and

$$b_2 = H(\beta_1 A_{12} + \beta_2 A_{13} + \beta_3 A_{23}) \tag{2.18}$$

where A_{ij} is defined to be $A_{ij} = e_i^T e_j - e_j^T e_i$, for $i, j = 1, 2, 3$, and $i \neq j$. Thus b_1 and b_2 are two arbitrary directions that are in the form of HC where $C = -C^T$.

Then at step $k+1$, using equation (2.3), the child direction is given by,

$$c = \frac{b_1\Phi^T}{b_1\Phi^T H^T} - \frac{b_2\Phi^T}{b_2\Phi^T H^T} = \frac{H\Phi(b_2^T b_1 - b_1^T b_2)\Phi^T}{(b_1\Phi^T H^T)(b_2\Phi^T H^T)} \tag{2.19}$$

Ignoring the denominator $(b_1\Phi^T H^T)$ and $(b_2\Phi^T H^T)$ since they are scalars, and looking at the numerator, the child direction becomes as follows,

$$c = H\Phi[(\alpha_1 A_{21} + \alpha_2 A_{31} + \alpha_3 A_{32})H^T H(\beta_1 A_{12} + \beta_2 A_{13} + \beta_3 A_{23}) \tag{2.20}$$

$$- (\beta_1 A_{21} + \beta_2 A_{31} + \beta_3 A_{32})H^T H(\alpha_1 A_{12} + \alpha_2 A_{13} + \alpha_3 A_{23})]\Phi^T \tag{2.21}$$

$$= H\Phi(\sigma \mathbf{B})\Phi^T \tag{2.22}$$

where

$$\mathbf{B} = He_1^T A_{23} + He_2^T A_{31} + He_3^T A_{12} \tag{2.23}$$

and

$$\sigma = He_1^T[\alpha_1\beta_2 - \alpha_2\beta_1] + He_2^T[\alpha_1\beta_3 - \alpha_3\beta_1] + He_3^T[\alpha_2\beta_3 - \alpha_3\beta_2] \tag{2.24}$$

Therefore, the possible form of co-aligned child directions that are produced from a new parent term can be expressed exactly as $H\Phi\mathbf{B}\Phi^T$. This form is unique. Aligned with Theorem 2.2.2, this important finding of three-state systems can be concluded in Theorem 2.2.4.

Theorem 2.2.4. *For three-state systems, all the co-aligned directions in new terms can only be combined along the direction of* $H\Phi\boldsymbol{B}\Phi^T$*, where* $\boldsymbol{B} = He_1^T A_{23} + He_2^T A_{31} + He_3^T A_{12}$*,* $A_{ij} = e_i^T e_j - e_j^T e_i$*,* $i, j = 1, 2, 3$*,* $i \neq j$*.* $\boldsymbol{B}$ *is the unique fundamental basis.*

2.2.2.2 From Old Parent Terms

Next, consider what the directions look like for an old term. Every old term at step $k+1$ is the last child term of its parent term at step k. If a parent term is "new", meaning that the directions in its parent term can be written as $[HC_1, HC_2, ..., HC_p, b]$, then we can write down the directions of its old child term (the last child term) as,

$$[HC_1\Phi^T, HC_2\Phi^T, ..., HC_p\Phi^T, b\Phi^T, \Gamma^T] \tag{2.25}$$

We call this old term as the "newest old" term, or "1-step old" term.

Let the above term be a parent term. Then the directions of the last child term of the above term at the next measurement update becomes,

$$[HC_1\Phi^{T2}, HC_2\Phi^{T2}, ..., HC_p\Phi^{T2}, b\Phi^{T2}, \Gamma^T\Phi^T, \Gamma^T] \tag{2.26}$$

This is the form of the directions of a "2-step old" term.

More generally, for a "θ-step old" term, the directions can be expressed by induction.

$$[HC_1\Phi^{T\theta}, HC_2\Phi^{T\theta}, ..., HC_p\Phi^{T\theta}, b\Phi^{T\theta}, \Gamma^T\Phi^{T(\theta-1)}, ..., \Gamma^T] \tag{2.27}$$

Look at θ-step backward from equation (2.27), the original parent term is a new parent term with the number of HC_i's as p. And at current measurement update the number of directions is $(p + 1 + \theta)$.

Unlike those in new terms, some of the directions in the old terms are not in the form of HC_i, but in the form of $HC_i\Phi^{T\theta}$, where $C_i \in \mathbb{R}^{3\times3}$ is a skew-symmetric matrix and θ is a positive integer. Generically, the one and the only situation to produce two co-aligned child directions is that the three parent directions are in the form of $HC_i\Phi^{T\theta}$ in order to form a new anti-symmetric matrix at the next measurement update. The matrix M that is defined in Theorem 2.1.2 constructed by $HC_i\Phi^{T\theta}$'s is not full row rank. This is because $HC_i\Phi^{T\theta}(\Phi^{-T\theta}H^T) = 0$, i.e. $\Phi^{-T\theta}H^T$ is in the null space of all $HC_i\Phi^{T\theta}$ directions. The rest of the directions, for example b and $\Gamma^T\Phi^{T\theta}$, are generically linearly independent with each other and the $HC_i\Phi^{T\theta}$'s. In other words,

Theorem 2.2.5. *For old parent terms of three-state systems, only the parent directions in the forms of $HC_i\Phi^{T\theta}$ where C_i is a skew-symmetric matrix and θ a positive integer can produce co-aligned child directions.*

Similarly, suppose two arbitrary parent directions of an old term which have the form of $HC_i\Phi^{T\theta}$ are,

$$b_1 = H(\alpha_1 A_{12} + \alpha_2 A_{13} + \alpha_3 A_{23})\Phi^{T\theta} \tag{2.28}$$

and

$$b_2 = H(\beta_1 A_{12} + \beta_2 A_{13} + \beta_3 A_{23})\Phi^{T\theta} \tag{2.29}$$

At the $(k+1)^{th}$ measurement update, consider the numerator of the child direction,

$$\begin{aligned} c = H\Phi[&\Phi^{\theta}(\alpha_1 A_{21} + \alpha_2 A_{31} + \alpha_3 A_{32})H^T H(\beta_1 A_{12} + \beta_2 A_{13} + \beta_3 A_{23})\Phi^{T\theta} \\ &- \Phi^{\theta}(\beta_1 A_{21} + \beta_2 A_{31} + \beta_3 A_{32})H^T H(\alpha_1 A_{12} + \alpha_2 A_{13} + \alpha_3 A_{23})\Phi^{T\theta}]\Phi^T \\ = H\Phi^{\theta+1}&(\sigma\mathbf{B})\Phi^{T(\theta+1)} \end{aligned} \tag{2.30}$$

and

$$\sigma = He_1^T[\alpha_1\beta_2 - \alpha_2\beta_1] + He_2^T[\alpha_1\beta_3 - \alpha_3\beta_1] + He_3^T[\alpha_2\beta_3 - \alpha_3\beta_2] \tag{2.31}$$

This σ is the same with that of the new terms.

Analog to what has been concluded for new parent terms, Theorem 2.2.6 is stated as follow.

Theorem 2.2.6. *For the three-state systems, all the co-aligned directions in old terms can only be combined along the direction of $H\Phi^{\theta+1}(\sigma\boldsymbol{B})\Phi^{T(\theta+1)}$, where $A_{ij} = e_i^T e_j - e_j^T e_i$, $i, j = 1, 2, 3$, $i \neq j$, θ is a positive integer, and $\boldsymbol{B} = He_1^T A_{23} + He_2^T A_{31} + He_3^T A_{12}$ is the unique fundamental basis.*

Theorem 2.2.3 to Theorem 2.2.6 shows that for three-state systems, the co-aligned directions will only appear in the forms of $H\Phi^{(\theta+1)}\mathbf{B}\Phi^{T(\theta+1)}$, where $\theta = 0, 1, 2, ...$ is non-negative integer. In particular, when the parent term is a new term, the repeated directions of its child terms fall into the case when $\theta = 0$. When the parent term is an old term, then $\theta > 0$, and θ depends on how old the parent term is. That term is called a θ-step old term. In addition, the number of terms with fundamental directions $\theta > 0$ is fairly rare, because next time through in the new terms it will became $H\Phi\mathbf{B}\Phi^T$ again. We summarize this conclusion into the following corollary.

Corollary 2.2.7. *For three-state systems, only the parent directions in the forms of $HC_i\Phi^{T\theta}$ where C_i is a skew-symmetric matrix and θ is a non-negative integer can produce co-aligned* *child directions. All the co-aligned directions can only be combined along the direction of* $H\Phi^{\theta+1}(\sigma\boldsymbol{B})\Phi^{T(\theta+1)}$, *where* $A_{ij} = e_i^T e_j - e_j^T e_i$, *and* $\boldsymbol{B} = He_1^T A_{23} + He_2^T A_{31} + He_3^T A_{12}$ *is the fundamental basis. Furthermore, the structure of* $\boldsymbol{B}$ *is unique.*

Corollary 2.2.7 is important because it analytically provides complete information of direction for co-alignment for three-state systems. With this corollary, one is able to explicitly determine the directions, especially when and how to combine them. Unlike the two-state systems where the fundamental basis $\mathbf{A}$ is fairly straightforward, the one additional dimension for three state systems results in structural complexity. Using Corollary 2.2.7, the directions of three-state systems can be combined analytically without the necessity of numerical comparison during implementation. It saves tremendous amount of computation time. In addition, the studies on other component of the estimator structure, including the term combination and the coefficient terms, are established heavily on the understanding of direction co-alignment. Through these theorems and corollary additional fundamental simplifications of the estimator structure can be deduced.

2.2.3 Higher-Order Cases

Similar properties have been observed for higher-order cases as well. For four and five-state systems, the form of such fundamental basis exist and is unique. Since it has higher dimension, there are more freedom in the child directions. If we still want to reach a stationary child direction like we did for $H\mathbf{A}$ in two-state case and $H\Phi\mathbf{B}\Phi^T$ for three-state case, we need more update. Take the directions in the new child terms for instance. For two-state case, it takes one update to reach $H\mathbf{A}$. For three-state case, it takes two update to reach $H\Phi\mathbf{B}\Phi^T$. It has been demonstrated numerically that for four-state case, it takes three updates to reach $H\Phi^2\mathbf{C}\Phi^{2T}$, where $\mathbf{C}$ is the fundamental basis matrix. And for five-state case, it takes four updates to reach $H\Phi^3\mathbf{D}\Phi^{3T}$, where $\mathbf{D}$ is the fundamental basis.

The analytic form of the fundamental basis $\mathbf{C}$ for four-state system is found to be in the following form,

$$\mathbf{C} = \begin{bmatrix} 0 & -\beta_{34} & \beta_{24} & -\beta_{23} \\ \beta_{34} & 0 & -\beta_{14} & \beta_{13} \\ -\beta_{24} & \beta_{14} & 0 & -\beta_{12} \\ \beta_{23} & -\beta_{13} & \beta_{12} & 0 \end{bmatrix} \tag{2.32}$$

where

$$\beta_{il} = \left(He_i^T\right)\left(H\Phi e_l^T\right) - \left(He_l^T\right)\left(H\Phi e_i^T\right), \quad 1 \le i, l \le 4, \quad i \ne l \tag{2.33}$$

For five-state system, the fundamental basis **D** is in the following form,

$$\mathbf{D} = \begin{bmatrix} 0 & \beta_{345} & -\beta_{245} & \beta_{235} & -\beta_{234} \\ -\beta_{345} & 0 & \beta_{145} & -\beta_{135} & \beta_{134} \\ \beta_{245} & -\beta_{145} & 0 & \beta_{125} & -\beta_{124} \\ -\beta_{235} & \beta_{135} & -\beta_{125} & 0 & \beta_{123} \\ \beta_{234} & -\beta_{134} & \beta_{124} & -\beta_{123} & 0 \end{bmatrix} \tag{2.34}$$

where

$$\begin{aligned} \beta_{ijk} = \quad & \left(He_i^T\right)\left[(H\Phi e_j^T)(H\Phi^2 e_k^T) - (H\Phi e_k^T)(H\Phi^2 e_j^T)\right] \\ & + \left(He_j^T\right)\left[(H\Phi e_k^T)(H\Phi^2 e_i^T) - (H\Phi e_i^T)(H\Phi^2 e_k^T)\right] \\ & + \left(He_k^T\right)\left[(H\Phi e_i^T)(H\Phi^2 e_j^T) - (H\Phi e_j^T)(H\Phi^2 e_i^T)\right], \quad 1 \le i, j, k \le 5, \quad i \ne j \ne k \end{aligned} \tag{2.35}$$

For detail derivations of **C** and **D**, refer to Appendix B .

We have proved the existence and uniqueness of the fundamental basis for two and three-state systems analytically, and have found the form of **A** and **B**. For four and five-state case, we have observed the same properties, and also found the form of the fundamental basis **C** and **D**. Using the same technique, by induction one can reach the following conclusion,

Corollary 2.2.8. *For multivariate systems, the co-aligned child direction can be reached within finite update. In particular, for n-state system, this direction co-alignment can be obtained in* $(n-1)$ *updates. Co-aligned directions are along a direction that is a function of a fundamental basis matrix. This fundamental basis is unique.*

2.2.4 Additional Properties of the Fundamental Basis

The fundamental basis has a special structure. In this subsection, some interesting properties of the fundamental basis are raised. Suppose an n-dim state space dynamic system. Its fundamental basis is denoted as $\mathbf{F}$.

Theorem 2.2.9. *The fundamental basis $\boldsymbol{F}$ of an n-dim state space dynamic system is always rank 2.*

Proof. $\mathbf{F}$ is obtained by $v_1^T v_2 - v_2^T v_1$, where v_1 and v_2 are n-dim row vectors. $rank(v_1^T v_2) = 1$. Then $rank(M) \leq rank(v_1^T v_2) + rank(-v_2^T v_1) = 1 + 1 = 2$. But $\mathbf{F}$ cannot be rank 1, since the diagonal elements of $\mathbf{F}$ are all zeros. Therefore the rank of $\mathbf{F}$ is 2. □

Theorem 2.2.10. *The dimension of the null space of $\boldsymbol{F}$ is $n-2$, spanned by a set of vectors associated with the measurement matrix H and the transition matrix Φ, i.e. $N(\boldsymbol{F}) = \boldsymbol{sp}\{H^T, (H\Phi)^T, \cdots, (H\Phi^{n-3})^T\}$.*

In particular, for a two-state case, the fundamental basis $\mathbf{A}$ is full rank, and the null space is empty. For three-state case, it can be shown analytically that $\mathbf{B}$ and H are orthogonal, i.e. $\mathbf{B}H^T = \{0\}_3$. Because the matrix $\mathbf{B}$ is rank 2, the null space $N(\mathbf{B})$ is 1-dim, and $N(\mathbf{B}) = \mathbf{sp}\left\{H^T\right\}$. Similarly, for a four-state case, it can be shown that $\mathbf{C}H^T = \{0\}_4$ and $\mathbf{C}\left(H\Phi\right)^T = \{0\}_4$. We know that the matrix $\mathbf{C}$ is rank 2. Hence the null space $N(\mathbf{C})$ is 2-dim, and $N(\mathbf{C}) = \mathbf{sp}\left\{H^T, (H\Phi)^T\right\}$. For a five-state case, it can be shown that $\mathbf{D}H^T = \{0\}_5$, $\mathbf{D}\left(H\Phi\right)^T = \{0\}_5$ and $\mathbf{D}\left(H\Phi^2\right)^T = \{0\}_5$. The matrix $\mathbf{D}$ is rank 2. Hence, the null space $N(\mathbf{D})$ is 3-dim, and $N(\mathbf{D}) = \mathbf{sp}\left\{H^T, \left(H\Phi\right)^T, \left(H\Phi^2\right)^T\right\}$.

CHAPTER 3

Term Combination Rules

Many exponential terms are identical in the sum of equation (1.14) at each measurement update. For a given state dimension, some exponential terms are functionally the same with respect to any system parameters including Φ, H, Γ, α, β, γ and with respect to any spectral variable ν. Hence they can be combined so that the total number of exponential terms can be largely reduced. In other words, equation (1.14) can be rewritten as,

$$\bar{\phi}_{X_k|Y_k}(\nu) = \sum_{i=1}^{\tilde{N}_t^{k|k}} \left[\left(\sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} G_{i,l}^{k|k}(\nu) \right) \mathcal{E}_i^{k|k}(\nu) \right] \tag{3.1}$$

where $\tilde{N}_t^{k|k}$ is the total number of distinct exponential terms, and $\tilde{N}_{t,i}^{k|k}$ is the number of coefficient terms associated with each distinct exponential. Because certain exponential terms $\mathcal{E}_i^{k|k}(\nu)$ are combined, the number of distinct exponential terms, $\tilde{N}_t^{k|k}$, is much less than the total number of terms originally before the term combination, i.e. $N_t^{k|k}$ in equation (1.14). This reduction of the number of exponential terms enhances the computational efficiency significantly. Numerical simulations are conducted in latter chapters. Take a three-state system for instance, it will be shown later that at step $k = 7$, the offline computation can save about 99% if repeated exponential terms are combined using equation (3.1).

In this chapter, the argument of certain exponential terms are shown to have the same expression analytically. For two-state systems, we find two rules for combining terms by comparing the exponential terms analytically. Using these two rules, total number of distinct exponential terms and the number of new distinct exponential terms are derived in a recursive matrix form. This matrix form exactly matches the empirical results in [6]. Therefore, it is concluded that there are only such two rules for combining terms for two-state systems. In addition, some term combination rules for three-state systems are also uncovered, implying that for higher-order cases in general certain exponential terms can be combined due to their identical expressions.

3.1 Rules for Two-State Case

This section derives two rules for term combination for two-state systems. Starting from the grandparent term $\mathcal{E}_i^{k|k}$ at step k which contains m elements in the argument of the exponential, the combination rules study the grandchild term $\mathcal{E}_{i,l,r}^{k+2|k+2}$ at step $k+2$. Note that the notation $\mathcal{E}_{i,l,r}^{k+2|k+2}$ has three subscripts. It represents the r^{th} grandchild term at step $k+2$ from the l^{th} child term $\mathcal{E}_{i,l}^{k+1|k+1}$ at step $k+1$. The first combination rule shows that the first grandchild terms at step $k+2$ from all new child terms at step $k+1$ will always combine. This also brings up the notion of "invariance" of the exponential terms. The second combination rule states that the grandchild term $\mathcal{E}_{i,l,2}^{k+2|k+2}$ and $\mathcal{E}_{i,m+2,l}^{k+2|k+2}$ for $1 \leq l \leq m+1$ can be combined. Based on these rules, the recursive structure on the number of distinct exponential terms $\tilde{N}_t^{k|k}$ and the number of new distinct exponential terms, denoted as $\tilde{N}_{t,new}^{k|k}$, is found analytically. This recursion matches the empirical results of number of terms shown in [6] exactly, which also indicates that we have exhausted the combination rules for two-state systems.

3.1.1 Combination Rule for the First Grandchild Terms

In this section we show that the exponentials from the first grandchild terms are always combined. In addition, these terms are independent of the initial condition parameters. Suppose at step k, the exponential of any two terms are expressed as,

$$\mathcal{E}_i^{k|k}(\nu) = \exp\left(-P_1\,|b_1\nu| - P_2\,|b_2\nu| - ... - P_m\,|b_m\nu| + j\zeta_i^k\nu\right) \tag{3.2}$$

$$\mathcal{E}_p^{k|k}(\nu) = \exp\left(-Q_1\,|c_1\nu| - Q_2\,|c_2\nu| - ... - Q_n\,|c_n\nu| + j\zeta_p^k\nu\right) \tag{3.3}$$

where b, c and ζ are 2-dim row vectors. Note that ζ in equation (3.2) and (3.3) represents the imaginary part of the argument in the exponential terms. We are going to show the first combination rule by analytically showing that for any $i \leq \tilde{N}_t^{k|k}$ and $p \leq \tilde{N}_t^{k|k}$, and $l = 1, 2, \cdots, m+1$, $q = 1, 2, \cdots, n+1$,

$$\mathcal{E}_{i,l,1}^{k+2|k+2} = \mathcal{E}_{p,q,1}^{k+2|k+2} \tag{3.4}$$

Remark 3.1.1. Here, we are only interested in the real component of the argument, because we have found out numerically that whenever the argument of the exponential of two terms are identical, the imaginary parts always match too. In latter derivations, the imaginary parts, ζ, will be omitted.

At step $k+1$, $\mathcal{E}_i^{k|k}$ has $(m+2)$ child terms while $\mathcal{E}_p^{k|k}$ has $(n+2)$ child terms. The first $(m+1)$ child terms from $\mathcal{E}_i^{k|k}$ and the first $(n+1)$ child terms from $\mathcal{E}_p^{k|k}$ only have 2 elements in the argument of the exponential, according to Theorem 2.2.1. Take $\mathcal{E}_i^{k|k}$ for instance. After time propagation to step $k+1$, the exponential term becomes,

$$\begin{aligned}\mathcal{E}_i^{k+1|k}(\nu) = \exp\big(&-P_1\left|b_1\Phi^T\nu\right| - P_2\left|b_2\Phi^T\nu\right| - ... - P_m\left|b_m\Phi^T\nu\right| \\ &-\beta\left|\Gamma^T\nu\right| + j\zeta_i^{k+1|k}\nu\big)\end{aligned} \tag{3.5}$$

Then, at step $k+1$, the child terms are derived from the solution of the update integral shown as follow,

$$\begin{aligned}\bar{\phi}_{k+1|k+1}(\nu) = \int_{-\infty}^{+\infty}\sum_{i=1}^{\tilde{N}_t^{k|k}}\Bigg\{&\left(\sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} G_{i,l}^{k+1|k}(\nu - H^T\eta)\right)\cdot\exp\Bigg(-P_1\left|b_1\Phi^T H^T\right|\left|\frac{b_1\Phi^T\nu}{b_1\Phi^T H^T} - \eta\right| - ... \\ &-P_m\left|b_m\Phi^T H^T\right|\left|\frac{b_m\Phi^T\nu}{b_m\Phi^T H^T} - \eta\right| - \beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \eta\right| \\ &-\gamma\left|\eta\right| + jz_{k+1}\eta + j\zeta_i^{k+1|k}(\nu - H^T\eta)\Bigg)\Bigg\}\,d\eta\end{aligned} \tag{3.6}$$

where $\tilde{N}_t^{k|k}$ is the number of distinct exponential terms at step k, and $\tilde{N}_{t,i}^{k|k}$ is the number of coefficient terms that are associated with the i^{th} distinct exponential term.

At step $k+1$, the l^{th} child term when $1 \leq l \leq m$ has the following exponential terms, using the form of the solution given in Appendix A of this dissertation. The method was provided in Appendix B in [5]. Note that the notation l here is different from the index l that appears in the inner sum of the coefficient terms $G_{i,l}^{k+1|k}$.

$$\begin{aligned}\mathcal{E}_{i,l}^{k+1|k+1}(\nu) = \exp\Bigg(&-P_1\left|b_1\Phi^T H^T\right|\left|\frac{b_1\Phi^T\nu}{b_1\Phi^T H^T} - \frac{b_l\Phi^T\nu}{b_l\Phi^T H^T}\right| - \ldots \\ &- P_m\left|b_m\Phi^T H^T\right|\left|\frac{b_m\Phi^T\nu}{b_m\Phi^T H^T} - \frac{b_l\Phi^T\nu}{b_l\Phi^T H^T}\right| \\ &- \beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{b_l\Phi^T\nu}{b_l\Phi^T H^T}\right| \\ &-\gamma\left|\frac{b_l\Phi^T\nu}{b_l\Phi^T H^T}\right| + j\zeta_{i,l}^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{3.7}$$

Define $B_{l,i} = \begin{bmatrix} b_l^T & b_i^T \end{bmatrix} \in \mathbb{R}^{2\times 2}$, and $C_l = \begin{bmatrix} \Phi^{-1}\Gamma & b_l^T \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Then,

$$\begin{aligned}\mathcal{E}_{i,l}^{k+1|k+1}(\nu) = \exp\Bigg(&-\frac{P_1}{\left|b_l\Phi^T H^T\right|}\left|\det(\Phi)\cdot\det(B_{l,1})\right|\cdot\left|H\mathbf{A}\nu\right| - \ldots \\ &- \frac{P_m}{\left|b_l\Phi^T H^T\right|}\left|\det(\Phi)\cdot\det(B_{l,m})\right|\cdot\left|H\mathbf{A}\nu\right| \\ &- \frac{\beta}{\left|b_l\Phi^T H^T\right|}\left|\det(\Phi)\cdot\det(C_l)\right|\cdot\left|H\mathbf{A}\nu\right| \\ &-\frac{\gamma}{\left|b_l\Phi^T H^T\right|}\left|b_l\Phi^T\nu\right| + j\zeta_{i,l}^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{3.8}$$

Combine the first m elements, which are all along the direction of $H\mathbf{A}$.

$$\begin{aligned}\mathcal{E}_{i,l}^{k+1|k+1}(\nu) = \exp\Bigg(&-\frac{\left|\det(\Phi)\right|\cdot\left[\left(\sum_{q=1,q\neq l}^{m} P_q\left|\det(B_{l,q})\right|\right) + \beta\left|\det(C_l)\right|\right]}{\left|b_l\Phi^T H^T\right|}\left|H\mathbf{A}\nu\right| \\ &-\frac{\gamma}{\left|b_l\Phi^T H^T\right|}\left|b_l\Phi^T\nu\right| + j\zeta_{i,l}^{k+1|k+1}\nu\Bigg), \quad 1 \le l \le m\end{aligned} \tag{3.9}$$

Similarly, the exponential part of the $(m+1)^{th}$ child term at step $k+1$ can be expressed as,

$$\begin{aligned}\mathcal{E}_{i,m+1}^{k+1|k+1}(\nu) = \exp\Bigg(&-P_1\left|b_1\Phi^T H^T\right|\left|\frac{b_1\Phi^T\nu}{b_1\Phi^T H^T} - \frac{\Gamma^T\nu}{\Gamma^T H^T}\right| - \ldots \\ &-P_m\left|b_m\Phi^T H^T\right|\left|\frac{b_m\Phi^T\nu}{b_m\Phi^T H^T} - \frac{\Gamma^T\nu}{\Gamma^T H^T}\right| - \gamma\left|\frac{\Gamma^T\nu}{\Gamma^T H^T}\right| + j\zeta_{i,m+1}^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{3.10}$$

With C_l defined earlier, we have,

$$\mathcal{E}_{i,m+1}^{k+1|k+1}(\nu) = \exp\left(-\frac{P_1}{|\Gamma^T H^T|}\left|\det(\Phi)\cdot\det(C_1)\right|\cdot|H\mathbf{A}\nu| - ... \right.$$
$$\left. - \frac{P_m}{|\Gamma^T H^T|}\left|\det(\Phi)\cdot\det(C_m)\right|\cdot|H\mathbf{A}\nu| - \frac{\gamma}{|\Gamma^T H^T|}\left|\Gamma^T\nu\right| + j\zeta_{i,m+1}^{k+1|k+1}\nu\right) \quad (3.11)$$

Combine the first m elements.

$$\mathcal{E}_{i,m+1}^{k+1|k+1}(\nu) = \exp\left(-\frac{|\det(\Phi)|\cdot\left(\sum_{q=1}^{m} P_q\left|\det(C_q)\right|\right)}{|\Gamma^T H^T|}|H\mathbf{A}\nu| - \frac{\gamma}{|\Gamma^T H^T|}\left|\Gamma^T\nu\right| \right.$$
$$\left. + j\zeta_{i,m+1}^{k+1|k+1}\nu\right) \quad (3.12)$$

Equation (3.9) gives the general form of the exponentials of the first m child terms from the parent term $\mathcal{E}_i^{k|k}$, and equation (3.12) describes the $(m+1)^{th}$ child term at step $k+1$. They are all the new child terms at step $k+1$ from $\mathcal{E}_i^{k|k}$. Observing equation (3.9) and (3.12), one can see that all new child terms of any parent term only have two elements in the argument of the exponential. This fact aligns with Theorem 2.2.1 in Chapter 2. For convenience of further derivation of the first combination rule, rewrite the exponential part of any new term in the following general form,

$$\tilde{\mathcal{E}}^{k+1|k+1}(\nu) = \exp\left(-P_1\,|H\mathbf{A}\nu| - \frac{\gamma}{|bH^T|}\,|b\nu| + j\zeta\nu\right) \quad (3.13)$$

where the fundamental basis $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and b can be any 2-dim row vector. Note that for convenience, the subscript is neglected, and the new notation $\tilde{\mathcal{E}}^{k+1|k+1}$ is used.

After time propagation to step $k+2$, this exponential term becomes,

$$\tilde{\mathcal{E}}^{k+2|k+1}(\nu) = \exp\left(-P_1\left|H\mathbf{A}\Phi^T\nu\right| - \frac{\gamma}{|bH^T|}\left|b\Phi^T\nu\right| - \beta\left|\Gamma^T\nu\right| + j\zeta\Phi^T\nu\right) \quad (3.14)$$

Deriving from the update integral, the 1^{st} child term at step $k+2$ is

$$\tilde{\mathcal{E}}_1^{k+2|k+2}(\nu) = \exp\left(-\frac{\gamma}{|bH^T|}\left|b\Phi^T H^T\right|\left|\frac{b\Phi^T\nu}{b\Phi^T H^T} - \frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^T H^T}\right| \right.$$
$$- \beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^T H^T}\right|$$
$$\left. -\gamma\left|\frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^T H^T}\right| + j\zeta_1^{k+2|k+2}\nu\right) \quad (3.15)$$

Because,

$$\mathbf{A}^T H^T b - b^T H \mathbf{A} = -(bH^T) \cdot \mathbf{A} \tag{3.16}$$

and

$$\mathbf{A}^T H^T \Gamma^T \Phi^{-T} - \Phi^{-1} \Gamma H \mathbf{A} = -(\Gamma^T \Phi^{-T} H^T) \cdot \mathbf{A} \tag{3.17}$$

Substitute (3.16) and (3.17) back to (3.15).

$$\begin{aligned}\tilde{\mathcal{E}}_1^{k+2|k+2}(\nu) = \exp\Bigg(& -\frac{\gamma \cdot |\det(\Phi)|}{|H\mathbf{A}\Phi^T H^T|} |H\mathbf{A}\nu| - \frac{\beta \cdot \left|\Gamma^T \Phi^{-T} H^T\right| \cdot |\det(\Phi)|}{|H\mathbf{A}\Phi^T H^T|} |H\mathbf{A}\nu| \\ & -\frac{\gamma}{|H\mathbf{A}\Phi^T H^T|} \left|H\mathbf{A}\Phi^T \nu\right| + j\zeta_1^{k+2|k+2}\nu \Bigg)\end{aligned} \tag{3.18}$$

Combine the first 2 elements,

$$\begin{aligned}\tilde{\mathcal{E}}_1^{k+2|k+2}(\nu) = \exp\Bigg(& -\frac{\left(\gamma + \beta \cdot \left|\Gamma^T \Phi^{-T} H^T\right|\right) \cdot |\det(\Phi)|}{|H\mathbf{A}\Phi^T H^T|} |H\mathbf{A}\nu| - \frac{\gamma}{|H\mathbf{A}\Phi^T H^T|} \left|H\mathbf{A}\Phi^T \nu\right| \\ & + j\zeta_1^{k+2|k+2}\nu \Bigg)\end{aligned} \tag{3.19}$$

Remark 3.1.2. It is interesting to notice that the form in (3.19) is not in terms of P_1 anymore; instead, it is only related to the system parameters, including H, Φ, Γ, γ and β. Therefore, the first child exponential term of any 2-element parent exponential term should be in exactly the same form in (3.19), which can be combined.

This leads to the first term combination rule, expressed as,

$$\mathcal{E}_{i,l,1}^{k+2|k+2} = \mathcal{E}_{p,q,1}^{k+2|k+2}, \quad 1 \le l \le m+1, \quad 1 \le q \le n+1, \quad \text{for any } i, p \tag{3.20}$$

Remark 3.1.3. Equation (3.19) also brings out the notion of "invariance" in exponential terms. Since $\tilde{\mathcal{E}}_1^{k+2|k+2}$ in (3.19) is not a function of P_1 nor the initial conditions any more, it indicates that the first child term at step $k+3$ will have the same real component of the argument of the exponential as well. There exists an invariant form for these first child terms as k increases. This will be discussed later in section 3.3.

In the next section, we discuss the second combination rule.

3.1.2 Second Rule for Term Combination

Consider the grandchild terms $\mathcal{E}_{i,l,2}^{k+2|k+2}$ and $\mathcal{E}_{i,m+2,l}^{k+2|k+2}$ from the same grandparent term $\mathcal{E}_{i}^{k|k}$ which has m elements. We show that $\mathcal{E}_{i,l,2}^{k+2|k+2} = \mathcal{E}_{i,m+2,l}^{k+2|k+2}$ for $1 \leq l \leq m+1$. For different value of l, the exponential terms are derived analytically in the following subsections.

3.1.2.1 Case 1: $1 \leq l \leq m$

First, consider the case when $1 \leq l \leq m$. Start with $\mathcal{E}_{i,j}^{k+1|k+1}$ for $1 \leq j \leq m$ expressed in (3.9). After time propagation to step $k+2$, the argument of the exponential becomes,

$$\begin{aligned}\mathcal{E}_{i,l}^{k+2|k+1}(\nu) = \exp\Bigg(& -\frac{|\det(\Phi)| \cdot \left[\left(\sum_{q=1,q\neq l}^{m} P_q \left|\det(B_{l,q})\right|\right) + \beta\left|\det(C_l)\right|\right]}{|b_l \Phi^T H^T|} \left|H\mathbf{A}\Phi^T\nu\right| \\ & -\frac{\gamma}{|b_l\Phi^T H^T|}\left|b_l\Phi^{2T}\nu\right| - \beta\left|\Gamma^T\nu\right| + j\zeta_{i,l}^{k+2|k+1}\nu\Bigg)\end{aligned} \tag{3.21}$$

After measurement update derived from the update integral, the 2^{nd} child term at step $k+2$ is,

$$\begin{aligned}\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu) = \exp\Bigg(& -\frac{|\det(\Phi)| \cdot \left[\left(\sum_{q=1,q\neq l}^{m} P_q \left|\det(B_{l,q})\right|\right) + \beta\left|\det(C_l)\right|\right]}{|b_l \Phi^T H^T|} \left|H\mathbf{A}\Phi^T H^T\right| \\ & \cdot\left|\frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^T H^T} - \frac{b_l\Phi^{2T}\nu}{b_l\Phi^{2T}H^T}\right| - \beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{b_l\Phi^{2T}\nu}{b_l\Phi^{2T}H^T}\right| \\ & -\gamma\left|\frac{b_l\Phi^{2T}\nu}{b_l\Phi^{2T}H^T}\right| + j\zeta_{i,l,2}^{k+2|k+2}\nu\Bigg)\end{aligned} \tag{3.22}$$

From simple algebra, we have,

$$\Phi b_l^T H\mathbf{A} - \mathbf{A}^T H^T b_l \Phi^T = \left(H\Phi b_l^T\right) \cdot H\mathbf{A} \tag{3.23}$$

Also, define

$$D_l = \begin{bmatrix}\Phi^{-2}\Gamma & b_l^T\end{bmatrix} \in \mathbb{R}^{2\times 2} \tag{3.24}$$

Note that the determinant of the product equals to the product of the determinant. Using all these properties and definitions to simplify the argument of the exponential in (3.22).

$$\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu) = \exp\left(-\frac{|\det(\Phi^2)| \cdot \left[\left(\sum_{q=1,q\neq l}^{m} P_q |\det(B_{l,q})|\right) + \beta |\det(C_l)|\right]}{|b_l \Phi^{2T} H^T|} |H\mathbf{A}\nu| \right.$$
$$\left. -\frac{\beta |\det(\Phi^2) \cdot \det(D_l)|}{|b_l \Phi^{2T} H^T|} |H\mathbf{A}\nu| - \frac{\gamma}{|b_l \Phi^{2T} H^T|} \left|b_l \Phi^{2T} \nu\right| + j\zeta_{i,l,2}^{k+2|k+2} \nu\right) \quad (3.25)$$

Combine the first two elements in (3.25):

$$\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu) = \exp\left(-\frac{|\det(\Phi^2)| \cdot \left[\left(\sum_{q=1,q\neq l}^{m} P_q |\det(B_{l,q})|\right) + \beta |\det(C_l)|\right.}{|b_l \Phi^{2T} H^T|}\right.$$
$$\left.\frac{+\beta |\det(D_l)|]}{} \cdot |H\mathbf{A}\nu| - \frac{\gamma}{|b_l \Phi^{2T} H^T|} \left|b_l \Phi^{2T} \nu\right| + j\zeta_{i,l,2}^{k+2|k+2} \nu\right) \quad (3.26)$$

where $1 \leq l \leq m$.

We consider $\mathcal{E}_{i,m+2,l}^{k+2|k+2}$ for $1 \leq l \leq m$ and show that it is identical to equation (3.26). Start with the form of $\mathcal{E}_{i,m+2}^{k+1|k+1}$ as the old child term at step $k+1$,

$$\mathcal{E}_{i,m+2}^{k+1|k+1}(\nu) = \exp\left(-P_1 \left|b_1 \Phi^T \nu\right| - \ldots - P_m \left|b_m \Phi^T \nu\right| - \beta \left|\Gamma^T \nu\right| + j\zeta_{i,m+2}^{k+1|k+1} \nu\right) \quad (3.27)$$

The l^{th} child term for $1 \leq l \leq m$ at step $k+2$ has the argument of the exponential as follows.

$$\mathcal{E}_{i,m+2,l}^{k+2|k+2}(\nu) = \exp\left(-P_1 \left|b_1 \Phi^{2T} H^T\right| \left|\frac{b_1 \Phi^{2T} \nu}{b_1 \Phi^{2T} H^T} - \frac{b_l \Phi^{2T} \nu}{b_l \Phi^{2T} H^T}\right| - \ldots\right.$$
$$- P_m \left|b_m \Phi^{2T} H^T\right| \left|\frac{b_m \Phi^{2T} \nu}{b_m \Phi^{2T} H^T} - \frac{b_l \Phi^{2T} \nu}{b_l \Phi^{2T} H^T}\right|$$
$$- \beta \left|\Gamma^T \Phi^T H^T\right| \left|\frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T} - \frac{b_l \Phi^{2T} \nu}{b_l \Phi^{2T} H^T}\right|$$
$$\left. -\beta \left|\Gamma^T H^T\right| \left|\frac{\Gamma^T \nu}{\Gamma^T H^T} - \frac{b_l \Phi^{2T} \nu}{b_l \Phi^{2T} H^T}\right| - \gamma \left|\frac{b_l \Phi^{2T} \nu}{b_l \Phi^{2T} H^T}\right| + j\zeta_{i,m+2,l}^{k+2|k+2} \nu\right) \quad (3.28)$$

And,

$$H\left(\Phi^2 b_l^T b_q \Phi^{2T} - \Phi^2 b_q^T b_l \Phi^{2T}\right) = \det(B_{l,q}) \cdot \det(\Phi^2) \cdot H\mathbf{A} \quad (3.29)$$

$$H\left(\Phi^2 b_l^T \Gamma^T \Phi^T - \Phi \Gamma b_l \Phi^{2T}\right) = \det(C_l) \cdot \det(\Phi^2) \cdot H\mathbf{A} \quad (3.30)$$

$$H\left(\Phi^2 b_l^T \Gamma^T - \Gamma b_l \Phi^{2T}\right) = \det(D_l) \cdot \det(\Phi^2) \cdot H\mathbf{A} \quad (3.31)$$

where $B_{l,q}$, C_l and D_l have been defined earlier.

Substitute (3.29) - (3.31)into (3.28), and combine the first $(m+1)$ elements. One can obtain the following form,

$$\mathcal{E}_{i,m+2,l}^{k+2|k+2}(\nu) = \exp\left(-\frac{|\det(\Phi^2)| \cdot \left[\left(\sum_{q=1,q\neq l}^{m} P_q \left|\det(B_{l,q})\right|\right) + \beta\left|\det(C_l)\right|}{|b_l \Phi^{2T} H^T|}\right.$$
$$\left.\frac{+\beta\left|\det(D_l)\right|]}{} \cdot |H\mathbf{A}\nu| - \frac{\gamma}{|b_l \Phi^{2T} H^T|}\left|b_l \Phi^{2T}\nu\right| + j\zeta_{i,m+2,l}^{k+2|k+2}\nu\right) \tag{3.32}$$

where $1 \leq l \leq m$.

Again, as what has been mentioned earlier, numerical simulations have shown that whenever the real parts of two exponential terms combine, the imaginary parts match too. Therefore for the case when $1 \leq l \leq m$, the two grandchild terms $\mathcal{E}_{i,l,2}^{k+2|k+2}$ in equation (3.26) and $\mathcal{E}_{i,m+2,l}^{k+2|k+2}$ in equation (3.32) are identical.

3.1.2.2 Case 2: $l = m+1$

Next, consider the case when $l = m+1$. Starting from equation (3.12), the $(m+1)^{th}$ child term at step $k+1$ has the exponential term as,

$$\mathcal{E}_{i,m+1}^{k+1|k+1}(\nu) = \exp\left(-\frac{|\det(\Phi)| \cdot \left(\sum_{q=1}^{m} P_q \left|\det(C_q)\right|\right)}{|\Gamma^T H^T|}|H\mathbf{A}\nu| - \frac{\gamma}{|\Gamma^T H^T|}\left|\Gamma^T \nu\right|\right.$$
$$\left. + j\zeta_{i,m+1}^{k+1|k+1}\nu\right) \tag{3.33}$$

After time propagation to step $k+2$, it becomes,

$$\mathcal{E}_{i,m+1}^{k+2|k+1}(\nu) = \exp\left(-\frac{|\det(\Phi)| \cdot \left(\sum_{q=1}^{m} P_q \left|\det(C_q)\right|\right)}{|\Gamma^T H^T|}\left|H\mathbf{A}\Phi^T\nu\right| - \frac{\gamma}{|\Gamma^T H^T|}\left|\Gamma^T \Phi^T \nu\right|\right.$$
$$\left. -\beta\left|\Gamma^T\nu\right| + j\zeta_{i,m+1}^{k+2|k+1}\nu\right) \tag{3.34}$$

The 2^{nd} child term at step $k+2$ has the argument of the exponential as follows.

$$\mathcal{E}_{i,m+1,2}^{k+2|k+2}(\nu) = \exp\Bigg(-\frac{|\det(\Phi)| \cdot \left(\sum_{q=1}^{m} P_q |\det(C_q)|\right)}{|\Gamma^T H^T|} \left|H\mathbf{A}\Phi^T H^T\right| \left|\frac{H\mathbf{A}\Phi^T \nu}{H\mathbf{A}\Phi^T H^T} - \frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T}\right|$$
$$-\beta \left|\Gamma^T H^T\right| \left|\frac{\Gamma^T \nu}{\Gamma^T H^T} - \frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T}\right| - \frac{\gamma}{|\Gamma^T \Phi^T H^T|} \left|\Gamma^T \Phi^T \nu\right| + j\zeta_{i,m+1,2}^{k+2|k+2} \nu \Bigg) \tag{3.35}$$

For any row vector v, $v^T H\mathbf{A} - \mathbf{A}^T H^T v = \left(Hv^T\right) \cdot H\mathbf{A}$ holds. Let $v = \Gamma^T$. Then,

$$\Gamma H\mathbf{A} - \mathbf{A}^T H^T \Gamma^T = (H\Gamma) \cdot H\mathbf{A} \tag{3.36}$$

Define $E = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} \in \mathbb{R}^{2\times 2}$, then,

$$H\left(\Phi\Gamma\Gamma^T - \Gamma\Gamma^T\Phi^T\right) = -\det(E) \cdot H\mathbf{A} \tag{3.37}$$

Substitute equation (3.36) and (3.37) back into equation (3.35).

$$\mathcal{E}_{i,m+1,2}^{k+2|k+2}(\nu) = \exp\Bigg(-\frac{|\det(\Phi^2)| \cdot \left(\sum_{q=1}^{m} P_q |\det(C_q)|\right)}{|\Gamma^T \Phi^T H^T|} |H\mathbf{A}\nu|$$
$$-\frac{\beta |\det(E)|}{|\Gamma^T \Phi^T H^T|} |H\mathbf{A}\nu| - \frac{\gamma}{|\Gamma^T \Phi^T H^T|} \left|\Gamma^T \Phi^T \nu\right| + j\zeta_{i,m+1,2}^{k+2|k+2} \nu \Bigg) \tag{3.38}$$

Combine the first two elements which are co-aligned onto the $H\mathbf{A}$ direction. Then one can obtain the exponential term of the second child term at step $k+2$ from $\mathcal{E}_{i,m+1}^{k+1|k+1}$,

$$\mathcal{E}_{i,m+1,2}^{k+2|k+2}(\nu) = \exp\Bigg(-\frac{|\det(\Phi^2)| \cdot \left(\sum_{q=1}^{m} P_q |\det(C_q)|\right) + \beta |\det(E)|}{|\Gamma^T \Phi^T H^T|} |H\mathbf{A}\nu|$$
$$-\frac{\gamma}{|\Gamma^T \Phi^T H^T|} \left|\Gamma^T \Phi^T \nu\right| + j\zeta_{i,m+1,2}^{k+2|k+2} \nu \Bigg) \tag{3.39}$$

Next, starting from $\mathcal{E}_{i,m+2}^{k+1|k+1}$ expressed in equation (3.27). Using the same approach, at step $k+2$, the exponential term becomes,

$$\mathcal{E}_{i,m+2,m+1}^{k+2|k+2}(\nu) = \exp\Bigg(-P_1 \left|b_1 \Phi^{2T} H^T\right| \left|\frac{b_1 \Phi^{2T} \nu}{b_1 \Phi^{2T} H^T} - \frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T}\right| - \ldots$$
$$- P_m \left|b_m \Phi^{2T} H^T\right| \left|\frac{b_m \Phi^{2T} \nu}{b_m \Phi^{2T} H^T} - \frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T}\right|$$
$$-\beta \left|\Gamma^T H^T\right| \left|\frac{\Gamma^T \nu}{\Gamma^T H^T} - \frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T}\right| - \gamma \left|\frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T}\right| + j\zeta_{i,m+2,m+1}^{k+2|k+2} \nu \Bigg) \tag{3.40}$$

Because,

$$H\left(\Phi\Gamma b_l\Phi^{2T} - \Phi^2 b_l^T\Gamma^T\Phi^T\right) = \det(C_l)\cdot\det(\Phi^2)\cdot H\mathbf{A} \tag{3.41}$$

$$H\left(\Phi\Gamma\Gamma^T - \Gamma\Gamma^T\Phi^T\right) = -\det(E)\cdot H\mathbf{A} \tag{3.42}$$

Substitute equation (3.41) and (3.42) into (3.40).

$$\begin{aligned}\varepsilon_{m+2,m+1}^{k+2|k+2}(\nu) = \exp\Bigg(&-\frac{-P_1\left|\det(C_1)\cdot\det(\Phi^2)\right|}{|\Gamma^T\Phi^T H^T|}|H\mathbf{A}\nu| - ...\\ &-\frac{-P_m\left|\det(C_m)\cdot\det(\Phi^2)\right|}{|\Gamma^T\Phi^T H^T|}|H\mathbf{A}\nu|\\ &-\frac{\beta\left|\det(E)\right|}{|\Gamma^T\Phi^T H^T|}|H\mathbf{A}\nu| - \frac{\gamma}{|\Gamma^T\Phi^T H^T|}\left|\Gamma^T\Phi^T\nu\right| + j\zeta_{i,m+2,m+1}^{k+2|k+2}\nu\Bigg)\end{aligned} \tag{3.43}$$

Combine the first $(m+1)$ elements to obtain the $(m+1)^{th}$ child term at step $k+2$ from $\mathcal{E}_{i,m+2}^{k+1|k+1}$,

$$\begin{aligned}\mathcal{E}_{i,m+2,m+1}^{k+2|k+2}(\nu) = \exp\Bigg(&-\frac{\left|\det\left(\Phi^2\right)\right|\cdot\left(\sum_{q=1}^{m}P_q\left|\det\left(C_q\right)\right|\right)+\beta\left|\det(E)\right|}{|\Gamma^T\Phi^T H^T|}|H\mathbf{A}\nu|\\ &-\frac{\gamma}{|\Gamma^T\Phi^T H^T|}\left|\Gamma^T\Phi^T\nu\right| + j\zeta_{i,m+2,m+1}^{k+2|k+2}\Bigg)\end{aligned} \tag{3.44}$$

where $E = \begin{bmatrix}\Gamma & \Phi\Gamma\end{bmatrix} \in \mathbb{R}^{2\times 2}$.

Observe equation (3.39) and (3.44), these two types of grandchild terms also have identical form for the case when $l = m+1$.

In sum, the second term combination rule for two-state system can be expressed as follows,

$$\mathcal{E}_{i,l,2}^{k+2|k+2} = \mathcal{E}_{i,m+2,l}^{k+2|k+2}, \quad \text{for } 1 \le l \le m+1 \tag{3.45}$$

Remark 3.1.4. As it turns out later, the two combination rules of equation (3.20) and (3.45) are all the rules and as shown in the next section, the number of terms can be computed.

3.1.3 Number of Terms after Term Combination

In this section, the term combination rules proposed earlier are to utilized to theoretically derive the number of distinct exponential terms after term combination. It turns out that this theoretical model of the total number of distinct exponential terms matches the empirical structure in [6]. The fact that this theoretical model predicts the right number of terms, implies that there exists only two such rules for two-state systems.

Figure 3.1 illustrates how a general term with m elements at step $k-1$ produces grandchild terms at step $k+1$. Let us consider different types of exponential terms at $(k+1)^{th}$ update. All the terms with 2 elements will be the $(j,1)$, $(j,2)$, $(j,3)$, $(m+2,1)$, $(m+2,2)$, ..., $(m+2,m+2)$ terms, where j can be anything between 1 and $m+1$. For different grandparent term at step $k-1$, there can be different grandchild terms. However, all the grandchild terms at step $k+1$ must belong to one of these types.

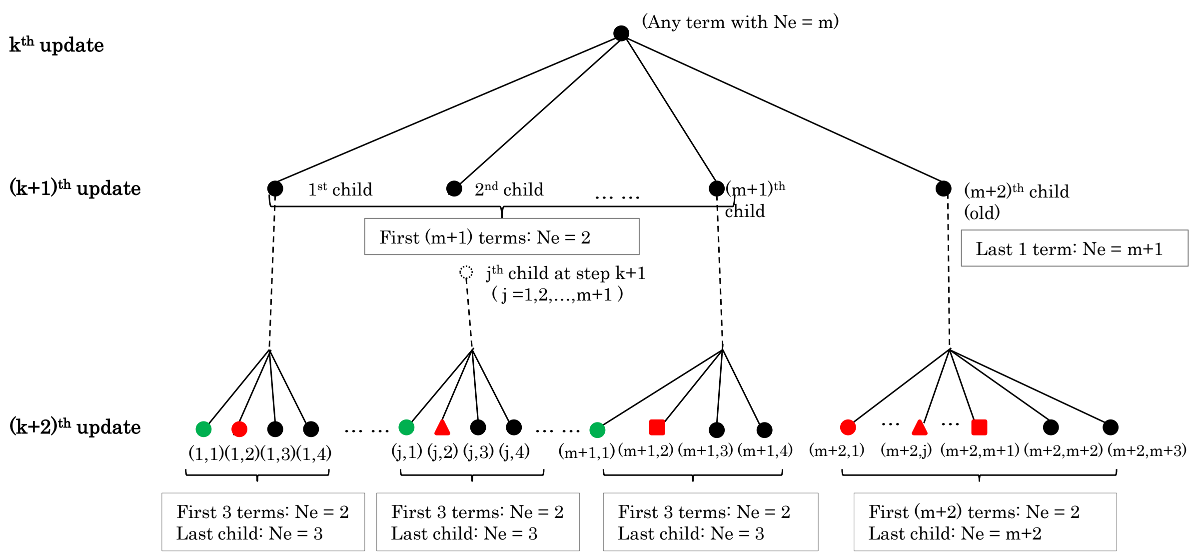

Figure 3.1: Term combination rules for 2-state systems

For two-state systems, exponential terms that contains more than two elements are always old child terms being produced from their parent terms. Any two distinct parent terms will produce different old child terms. Therefore, none of the child terms with more than two elements will combine with each other. All the new terms have two elements; and all the 2-element terms starting from the 2^{nd} measurement update are new terms.

Now, recall the definition of $\tilde{N}_t^{k|k}$ as the number of distinct exponential terms after combination at step k, and $\tilde{N}_{t,new}^{k|k}$ as the number of new distinct exponential terms after combination at step k. Also define $\tilde{N}_{t,old}^{k|k}$ to be the number of old distinct exponential terms after combination at step k. The relation of the three quantities can be written as,

$$\tilde{N}_{t}^{k|k} = \tilde{N}_{t,new}^{k|k} + \tilde{N}_{t,old}^{k|k} \tag{3.46}$$

The first rule for term combination is expressed as $\mathcal{E}_{i,l,1}^{k+2|k+2} = \mathcal{E}_{p,q,1}^{k+2|k+2}$, for $1 \leq l \leq m+1$, $1 \leq q \leq n+1$, and for any i, p. Suppose the number of new distinct exponential terms is $\tilde{N}_{t,new}^{k+1|k+1}$ at step $k+1$, then each of the new parent terms at step $k+1$ produces the first child term of the $(j,1)$ type at step $k+2$, which are colored in green in Figure 3.1. These $(j,1)$ type of grandchild terms at step $k+2$ will combine to one single exponential term. Therefore, **there are at most 1 exponential term of $(j,1)$ type at step $k+2$.**

The second rule for combining terms says that $\mathcal{E}_{i,l,2}^{k+2|k+2} = \mathcal{E}_{i,m+2,l}^{k+2|k+2}$, for $1 \leq l \leq m+1$. These two types of terms are colored in red in fig 3.1. Consider the 2-element child terms at step $k+2$ which come from a new parent term at step $k+1$. **At step $k+2$, there are at most $\tilde{N}_{t,new}^{k+1|k+1}$ terms of $(j,2)$ type; and at most $\tilde{N}_{t,new}^{k+1|k+1}$ terms of $(j,3)$ type,** where $1 \leq j \leq m+1$.

Now consider the 2-element child terms at step $k+2$ which come from an old parent term at step $k+1$. At step $k+2$, according to the second combination rule, the $(m+2, j)$ type of terms will combine with the $(j,2)$ type of terms, where $1 \leq j \leq m+1$. And all the old terms at step $k+1$ can be referred to the "$(m+2)^{th}$ child" at step $k+1$ in Figure 3.1. Therefore, all the $(m+2, j)$, where $1 \leq j \leq m+1$ types of terms will combine with the corresponding $(j,2)$ counterparts which have the same grandparent term with them.

Now the only left type of 2-element terms is the $(m+2, m+2)$ type. If there are $\tilde{N}_{t,old}^{k+1|k+1}$ old terms at step $k+1$, **then there will be in total $\tilde{N}_{t,old}^{k+1|k+1}$ terms of $(m+2, m+2)$ type at step $k+2$.**

In sum, counting all types of 2-element terms together at step $k+2$, one obtains the upper limit of the total number of new distinct exponential terms at step $k+2$ after conducting the two combination rules.

$$\tilde{N}_{t,new}^{k+2|k+2} \leq 1 + \tilde{N}_{t,new}^{k+1|k+1} + \tilde{N}_{t,new}^{k+1|k+1} + \tilde{N}_{t,old}^{k+1|k+1} \tag{3.47}$$

Because,

$$\tilde{N}_{t}^{k+1|k+1} = \tilde{N}_{t,new}^{k+1|k+1} + \tilde{N}_{t,old}^{k+1|k+1} \tag{3.48}$$

Substitute (3.48) back into (3.47),

$$\tilde{N}_{t,new}^{k+2|k+2} \leq 1 + \tilde{N}_{t,new}^{k+1|k+1} + \tilde{N}_{t}^{k+1|k+1} \tag{3.49}$$

Also because the number of old term at step $k+2$ equals to the total number of terms at step $k+1$, i.e.,

$$\tilde{N}_{t,old}^{k+2|k+2} = \tilde{N}_{t}^{k+1|k+1} \tag{3.50}$$

Then one can write down the total number of distinct exponential terms at step $k+2$ as,

$$\begin{aligned} \tilde{N}_{t}^{k+2|k+2} &= \tilde{N}_{t,new}^{k+2|k+2} + \tilde{N}_{t,old}^{k+2|k+2} \\ &\leq \left(1 + \tilde{N}_{t,new}^{k+1|k+1} + \tilde{N}_{t}^{k+1|k+1}\right) + \tilde{N}_{t}^{k+1|k+1} \\ &\leq 1 + \tilde{N}_{t,new}^{k+1|k+1} + 2\tilde{N}_{t}^{k+1|k+1} \end{aligned} \tag{3.51}$$

Rewrite the inequalities (3.49) and (3.51) into matrix form,

$$\begin{bmatrix} \tilde{N}_{t}^{k+2|k+2} \\ \tilde{N}_{t,new}^{k+2|k+2} \end{bmatrix} \leq \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{N}_{t}^{k+1|k+1} \\ \tilde{N}_{t,new}^{k+1|k+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.52}$$

The inequality in equation (3.52) provides an upper limit for the number of distinct exponential terms as well as new terms for two state cases. In [6], an equality with exactly the same matrix structure was verified through empirical simulations. Therefore, the statement of (3.52) can be stronger.

Theorem 3.1.5. *For two-state systems, the number of distinct exponential terms and new distinct exponential terms satisfies the following equality,*

$$\begin{bmatrix} \tilde{N}_{t}^{k+1|k+1} \\ \tilde{N}_{t,new}^{k+1|k+1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{N}_{t}^{k|k} \\ \tilde{N}_{t,new}^{k|k} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \textit{for } k \geq 2 \tag{3.53}$$

Remark 3.1.6. Theorem 3.1.5 were already brought up in [6], discovered by observing the empirical data. In this dissertation, it is better understood from an analytic viewpoint.

In addition, Theorem 3.1.5 implies that no other exponential terms except what has been discussed in this chapter can be combined. We have exhausted all the term combination rules for two-state systems. This conclusion can be summarized in the following theorems.

Theorem 3.1.7. *For two-state systems, suppose at step k, the exponential term $\mathcal{E}_i^{k|k}$ has m elements in the argument of the exponential, and $\mathcal{E}_p^{k|k}$ has n elements in the argument of the exponential, then there exists only 2 combination rules,*

$$(a) \quad \mathcal{E}_{i,l,1}^{k+2|k+2} = \mathcal{E}_{p,q,1}^{k+2|k+2}, \quad 1 \leq l \leq m+1, \quad 1 \leq q \leq n+1, \quad \textit{for any } i, p \tag{3.54}$$

$$(b) \quad \mathcal{E}_{i,l,2}^{k+2|k+2} = \mathcal{E}_{i,m+2,l}^{k+2|k+2}, \quad \textit{for } 1 \leq l \leq m+1 \tag{3.55}$$

3.2 Three-State Case

Similarly, the argument of some exponential terms for three-state systems also has the same functional expression. As known, in two-state problems, there are only two elements in the argument of the exponential for any new term, and only those 2-element exponential terms are involved in term combination rules. However, for three-state case, the minimum number of elements in the argument of the exponential term is four, while other terms have more elements, as time step k increases. We observe that some exponential terms with the same number of elements could be combined, which are not limited to 4-element terms. In this section, we show four combination rules for three-state systems, illustrated in Figure 3.2. Each rule is highlighted by different color and shape in the figure. These colored dots (combined terms) covers all the scenarios of 4-element exponential terms which are produced from a new grandparent term. For those exponential terms with more than four elements, and for those terms that are produced from old grandparent terms, combination rules are still to be discovered. Although these four rules proven in this section are not sufficient to fully describe all exponential terms that can be combined, the three-state study reveals that the argument of some exponential terms are functionally the same and can be combined for general higher-order systems.

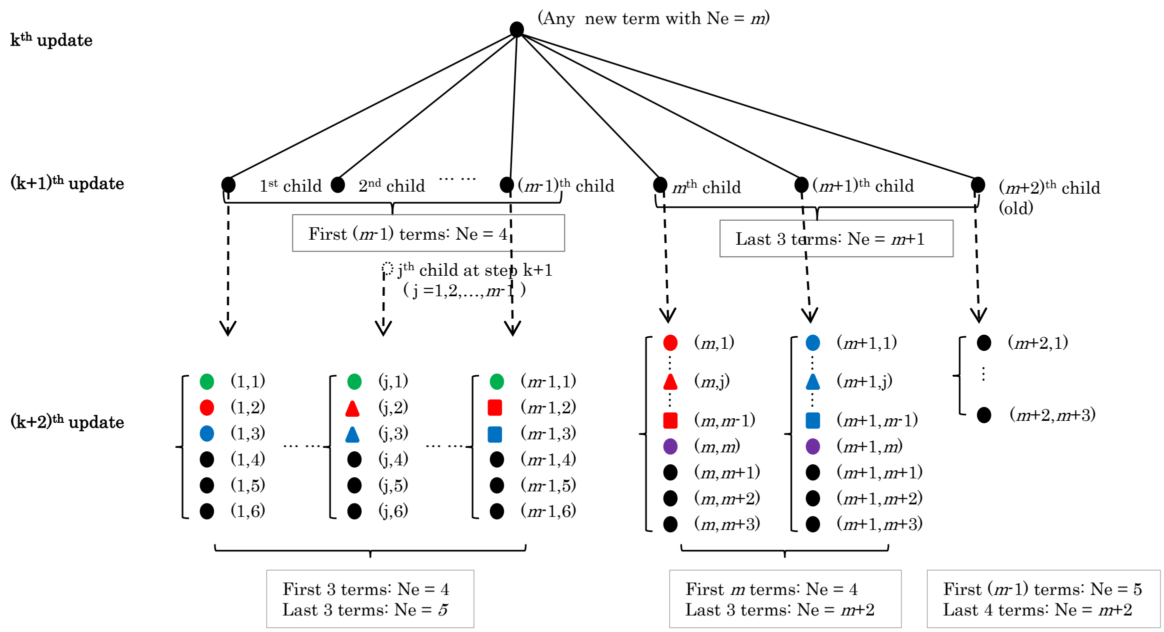

Figure 3.2: Some combination rules for three-state systems: 4-element grandchild terms from a new grandparent term

3.2.1 Derivation of $\mathcal{E}_{i,l,1}^{k+2|k+2} = \mathcal{E}_{p,q,1}^{k+2|k+2}$

Analog to the two-state case, here we show that the first child term of all 4-element parent terms can be combined, which is highlighted by green circles in Figure 3.2.

Consider the first child term of each of the parent term after the 3^{rd} update which has 4 elements, i.e. $N_e^{parent} = 4$. All exponential terms with four elements is a new term, when it is during or after the 3^{rd} measurement update. This is because all terms at the 2^{nd} measurement update have four elements. When they get propagated and updated to the step $k = 3$, 5 elements are produced in the argument of the exponential. But some of the new child terms will produce co-aligned directions and the number of elements N_e will be reduced to 4. On the contrary, an old parent term will produce child terms with at least

5 elements. This is because old child terms does not involve directions co-alignment. Old terms will have one more element than their parent terms. The minimal number of elements in a new term is four, thus there are at least 5 elements in the argument of the exponential for an old term. Starting from the 3^{rd} measurement update, all four-element terms will be one of the first several child terms from a new parent term at the last step.

At step k, the exponential of any new parent term can be written as,

$$\mathcal{E}_i^{k|k}(\nu) = \exp\left(-P_1|HC_1\nu| - P_2|HC_2\nu| - ... - P_{m-1}|HC_{m-1}\nu| - P_m|b_1\nu| + j\zeta_i^{k|k}\nu\right) \quad (3.56)$$

where C_q, $q = 1, 2, ..., m-1$, are skew-symmetric matrices, and b_1 is a 3-dim row vector which cannot be expressed in the HC form.

The number of elements of this term $\mathcal{E}_i^k$ is m. At step $k+1$, the first $(m-1)$ child terms will have 4 elements, i.e. $N_e = 4$. The l^{th} child at step $k+1$ when $1 \leq l \leq m-1$ has the argument of the exponential as follow,

$$\begin{aligned}\mathcal{E}_{i,l}^{k+1|k+1}(\nu) = \exp\Bigg(&-\frac{\left(\sum_{q=1,q\neq l}^{m-1} P_q|HC_qA_{21}C_l^TH^T|\right)}{|HC_l\Phi^TH^T|\cdot|He_3^T|}|H\Phi\mathbf{B}\Phi^T\nu| \\ &- \frac{\frac{\gamma}{|b_1H^T|}}{|HC_l\Phi^TH^T|}|H\Phi D_l\Phi^T\nu| - \frac{\beta}{|HC_l\Phi^TH^T|}|H\Phi E_l\Phi^T\nu| \\ &- \frac{\gamma}{|HC_l\Phi^TH^T|}|HC_l\Phi^T\nu| + j\zeta_{i,l}^{k+1|k+1}\nu\Bigg) \end{aligned} \quad (3.57)$$

where

$$D_l = C_l^TH^Tb_1 - b_1^TH^TC_l, \quad E_l = C_l^TH^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma HC_l \quad (3.58)$$

For detail derivation, see Appendix C.1.1.

Equation (3.57) is regarded as the general form of a parent term with four elements, i.e. $N_e^{parent} = 4$, at step $k+1$ when $k \geq 3$. Starting from parent terms of such form at step $k+1$, the first child terms at step $k+2$ has the exponential term expressed as follows,

$$\begin{aligned}\mathcal{E}_{i,l,1}^{k+2|k+2}(\nu) = \exp\Bigg(&-\frac{\frac{\gamma}{|b_1H^T|}|H\Phi D_l\Phi^TA_{21}\Phi\mathbf{B}^T\Phi^TH^T| + \beta|H\Phi E_l\Phi^TA_{21}\Phi\mathbf{B}^T\Phi^TH^T|}{|HC_l\Phi^TH^T|\cdot|H\Phi B\Phi^{2T}H^T|\cdot|He_3^T|}|H\Phi\mathbf{B}\Phi^T\nu| \\ &- \frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}|H\Phi^2\mathbf{B}\Phi^{2T}\nu| - \frac{\beta}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}|H\left(\Phi^2\mathbf{B}^T\Phi^TH^T\Gamma^T - \Gamma H\Phi\mathbf{B}\Phi^{2T}\right)\nu| \\ &- \frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}|H\Phi\mathbf{B}\Phi^{2T}\nu| + j\zeta_{i,l,1}^{k+2|k+2}\nu\Bigg) \end{aligned} \quad (3.59)$$

As defined earlier, $D_l = C_l^T H^T b_1 - b_1^T H^T C_l$ and $E_l = C_l^T H^T \Gamma^T \Phi^{-T} - \Phi^{-1} \Gamma H C_l$.

In equation (3.59), the argument of the exponential consists of four elements, each of which has a coefficient (the quantity outside the absolute value) and a direction (the quantity inside the absolute value). This grandchild term $\mathcal{E}_{i,l,1}^{k+2|k+2}$ at step $k+2$ is updated from the term $\mathcal{E}_{i,l}^{k+1|k+1}$ at step $k+1$. Besides the system parameters Φ, Γ, H, γ, and β, the value of the term $\mathcal{E}_{i,l}^{k+1|k+1}$ also relies on the value of P_l, C_l and b_1. However, the child term $\mathcal{E}_{i,l,1}^{k+2|k+2}$ expressed in equation (3.59) are independent of the value of P_l, C_l and b_1, but only functions of Φ, Γ H, γ, and β. We show this fact by observation and by numerical checks.

Firstly, one can immediately notice that all of the four directions, $H\Phi\mathbf{B}\Phi^T$, $H\Phi^2\mathbf{B}\Phi^{2T}$, $H\left(\Phi^2\mathbf{B}^T\Phi^T H^T \Gamma^T - \Gamma H \Phi \mathbf{B} \Phi^{2T}\right)$, and $H\Phi\mathbf{B}\Phi^{2T}$ in the child term $\mathcal{E}_{i,l,1}^{k+2|k+2}$ are independent with the value of P_l, C_l and b_1, but only functions of system parameters. Moreover, the second, third, and fourth coefficients in the grandchild term $\mathcal{E}_{i,l,1}^{k+2|k+2}$ are also uniquely determined by only system parameters and not the initial value of P_l, C_l or b_1.

Next, look at the coefficient of the first element in equation (3.59). Although it is expressed in terms of C_l and b_1, the value is actually independent of C_l and b_1. The first coefficient is uniquely determined by system parameters as well. It is not obvious, but it has been verified by simple numerical computations. Later, it will be shown that this first child term will stay invariant as k grows, which analytically proves that the expression of this exponential term is independent of the value of its parent terms. Later in Section 3.3, this invariance property of the argument of the exponential of first child terms is further discussed.

Therefore, the elements including both the coefficients and the directions in the 1^{st} child term $\mathcal{E}_{i,l,1}^{k+2|k+2}$ at step $k+2$ are uniquely determined by system parameters. All of the first child terms at step $k+2$ from all parent terms with $N_e^{parent} = 4$ at $(k+1)^{th}$ update when $k \geq 3$ will combine universally, no matter which particular parents they comes from, i.e.

$$\mathcal{E}_{i,l,1}^{k+2|k+2} = \mathcal{E}_{p,q,1}^{k+2|k+2} \tag{3.60}$$

where $\mathcal{E}_i^{k|k}$ and $\mathcal{E}_p^{k|k}$ are new terms at step k with m and n elements in the argument of the exponential term respectively, $1 \leq l \leq m-1$ and $1 \leq q \leq n-1$.

3.2.2 Derivation of $\mathcal{E}_{i,l,2}^{k+2|k+2} = \mathcal{E}_{i,m,l}^{k+2|k+2}$ from a New Grandparent Term $\mathcal{E}_i^{k|k}$ with $N_e^k = m$ for $1 \leq l \leq m-1$

Consider a new term at step k which has m elements in the argument of the exponential. Then the exponential can be expressed as,

$$\mathcal{E}_i^{k|k} = \exp\left(-P_1|HC_1\nu| - P_2|HC_2\nu| - ... - P_{m-1}|HC_{m-1}\nu| - \frac{\gamma}{|b_1H^T|}|b_1\nu| + j\zeta^{k|k}\nu\right) \tag{3.61}$$

where C_i, $i = 1, 2, ..., m-1$, are skew-symmetric matrices, and b_1 is another vector which cannot be expressed in the HC form.

At the $(k+1)^{th}$ measurement update, the l^{th} child term $\mathcal{E}_{i,l}^{k+1|k+1}$ for $1 \leq l \leq m-1$ has four elements; while the m^{th} child term $\mathcal{E}_{i,m}^{k+1|k+1}$ has $(m+1)$ elements. Then, at the $(k+2)^{th}$ measurement update, the 2^{nd} child term $\mathcal{E}_{i,l,2}^{k+2|k+2}$ from the parent term $\mathcal{E}_{i,l}^{k+1|k+1}$ has four elements. And the l^{th} child term $\mathcal{E}_{i,m,l}^{k+2|k+2}$ from the parent term $\mathcal{E}_{i,m}^{k+1|k+1}$ also has four elements.

In this section, we show that $\mathcal{E}_{i,l,2}^{k+2|k+2} = \mathcal{E}_{i,m,l}^{k+2|k+2}$, for $1 \leq l \leq m-1$. In Figure 3.2, these exponential terms are colored in red.

From Appendix C.1.1, the l^{th} child term $\mathcal{E}_{i,l}^{k+1|k+1}$ for $1 \leq l \leq m-1$ at step $k+1$ is,

$$\begin{aligned}\mathcal{E}_{i,l}^{k+1|k+1}(\nu) = \exp\Bigg(&-\frac{\left(\sum_{q=1,q\neq l}^{m-1} P_q \cdot \left|HC_qA_{21}C_l^TH^T\right|\right)}{|HC_l\Phi^TH^T| \cdot \left|He_3^T\right|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| - \frac{\frac{\gamma}{|b_1H^T|}}{|HC_l\Phi^TH^T|}\left|H\Phi D_l\Phi^T\nu\right| \\ &- \frac{\beta}{|HC_l\Phi^TH^T|}\left|H\Phi E_l\Phi^T\nu\right| - \frac{\gamma}{|HC_l\Phi^TH^T|}\left|HC_l\Phi^T\nu\right| + j\zeta_{i,l}^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{3.62}$$

where

$$D_l = C_l^TH^Tb_1 - b_1^THC_l \quad \text{and} \quad E_l = C_l^TH^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma HC_l \tag{3.63}$$

Next, from Appendix C.2.2, the 2^{nd} child term at step $k+2$ from $\mathcal{E}_{i,l}^{k+1|k+1}$ has the exponential term as follow,

$$
\begin{aligned}
\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu) = & \exp\left(-\rho_1 \left|H\Phi\mathbf{B}\Phi^T\nu\right|\right. \\
& - \frac{\gamma}{|H\Phi D_l\Phi^{2T}H^T|}\left|H\Phi^2 D_l\Phi^{2T}\nu\right| \\
& - \frac{\beta}{|H\Phi D_l\Phi^{2T}H^T|}\left|H\left(\Phi^2 D_l^T\Phi^T H^T\Gamma^T - \Gamma H\Phi D_l\Phi^{2T}\right)\nu\right| \\
& \left. - \frac{\gamma}{|H\Phi D_l\Phi^{2T}H^T|}\left|H\Phi D_l\Phi^{2T}\nu\right| + j\zeta_{i,l,2}^{k+2|k+2}\nu\right)
\end{aligned} \tag{3.64}
$$

where

$$
\begin{aligned}
\rho_1 = & \frac{\left(\sum_{q=1,q\neq l}^{m-1} P_q\cdot\left|HC_qA_{21}C_l^TH^T\right|\right)}{|HC_l\Phi^TH^T|\cdot|He_3^T|}\frac{\left|H\Phi D_l\Phi^TA_{21}\Phi\mathbf{B}^T\Phi^TH^T\right|}{|He_3^T|\cdot|H\Phi D_l\Phi^{2T}H^T|} \\
& + \frac{\beta}{|HC_l\Phi^TH^T|}\frac{\left|H\Phi D_l\Phi^TA_{21}\Phi E_l^T\Phi^TH^T\right|}{|He_3^T|\cdot|H\Phi D_l\Phi^{2T}H^T|}
\end{aligned} \tag{3.65}
$$

and D_l and E_l has been defined earlier.

Again, from Appendix C.1.2, at the $(k+1)^{th}$ update, the m^{th} child term $\mathcal{E}_{i,m}^{k+1|k+1}$ is expressed as,

$$
\begin{aligned}
\mathcal{E}_{i,m}^{k+1|k+1} = & \exp\left(-\frac{P_1}{|b_1\Phi^TH^T|}\left|H\Phi D_1\Phi^T\nu\right| - ... - \frac{P_{m-1}}{|b_1\Phi^TH^T|}\left|H\Phi D_{m-1}\Phi^T\nu\right|\right. \\
& \left. - \frac{\beta}{|b_1\Phi^TH^T|}\left|H\Phi D_{gb}\Phi^T\nu\right| - \frac{\gamma}{|b_1\Phi^TH^T|}\left|b_1\Phi^T\nu\right| + j\zeta_{i,m}^{k+1|k+1}\nu\right)
\end{aligned} \tag{3.66}
$$

where

$$
D_l = -b_1^THC_l + C_l^TH^Tb_1, \quad l = 1, ..., m-1 \tag{3.67}
$$

$$
D_{gb} = b_1^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma b_1 \tag{3.68}
$$

Next, at the $(k+2)^{th}$ measurement update, the l^{th} child term has the exponential term as follows. For detail derivations, please refer to Appendix C.2.4.

$$
\begin{aligned}
\mathcal{E}_{i,m,l}^{k+2|k+2}(\nu) = & \exp\left(-\rho_2 \left|H\Phi\mathbf{B}\Phi^T\nu\right|\right. \\
& - \frac{\gamma}{|H\Phi D_l\Phi^{2T}H^T|}\left|H\Phi^2 D_l\Phi^{2T}\nu\right| \\
& - \frac{\beta}{|H\Phi D_l\Phi^{2T}H^T|}\left|H\left(\Phi^2 D_l^T\Phi^T H^T\Gamma^T - \Gamma H\Phi D_l\Phi^{2T}\right)\nu\right| \\
& \left. - \frac{\gamma}{|H\Phi D_l\Phi^{2T}H^T|}\left|H\Phi D_l\Phi^{2T}\nu\right| + j\zeta_{i,m,l}^{k+2|k+2}\nu\right)
\end{aligned} \tag{3.69}
$$

where

$$\rho_2 = \frac{\left(\sum_{q=1,q\neq l}^{m-1} P_q \cdot \left|H\Phi D_q \Phi^T A_{21} \Phi D_l^T \Phi^T H^T\right|\right)}{|b_1 \Phi^T H^T|\,|H\Phi D_l \Phi^{2T} H^T|\,|He_3^T|} + \frac{\beta \cdot \left|H\Phi D_{gb} \Phi^T A_{21} \Phi D_l^T \Phi^T H^T\right|}{|b_1 \Phi^T H^T|\,|H\Phi D_l \Phi^{2T} H^T|\,|He_3^T|} \quad (3.70)$$

Now compare $\mathcal{E}_{i,l,2}^{k+2|k+2}$ in equation (3.64) with $\mathcal{E}_{i,m,l}^{k+2|k+2}$ in equation (3.69). Both of them have four elements. It is obvious by first glance that the second, third and fourth elements of $\mathcal{E}_{i,l,2}^{k+2|k+2}$ and $\mathcal{E}_{i,m,l}^{k+2|k+2}$ are identical to each other, respectively. Moreover, the first direction of both two terms is $H\Phi B\Phi^T$. Hence, one will just need to consider the first coefficients, i.e., ρ_1 in equation (3.65), and ρ_2 in equation (3.70).

By looking at the structure of ρ_1 and ρ_2, we can tell that the sufficient conditions for $\rho_1 = \rho_2$ are,

$$\frac{\left|HC_q A_{21} C_l^T H^T\right| \cdot \left|H\Phi D_l \Phi^T A_{21} \Phi B^T \Phi^T H^T\right|}{|HC_l \Phi^T H^T| \cdot |He_3^T|} = \frac{\left|H\Phi D_q \Phi^T A_{21} \Phi D_l^T \Phi^T H^T\right|}{|b_1 \Phi^T H^T|} \quad (3.71)$$

and

$$\frac{\left|H\Phi D_l \Phi^T A_{21} \Phi E_l^T \Phi^T H^T\right|}{|HC_l \Phi^T H^T|} = \frac{\left|H\Phi D_{gb} \Phi^T A_{21} \Phi D_l^T \Phi^T H^T\right|}{|b_1 \Phi^T H^T|} \quad (3.72)$$

where

$$D_q = C_q^T H^T b_1 - b_1^T H C_q, \quad q = 1, ..., m-1 \quad (3.73)$$

$$D_{gb} = b_1^T \Gamma^T \Phi^{-T} - \Phi^{-1} \Gamma b_1 \quad (3.74)$$

$$E_l = C_l^T H^T \Gamma^T \Phi^{-T} - \Phi^{-1} \Gamma H C_l \quad (3.75)$$

Numerical results show that for arbitrary skew-symmetric matrices $C_q, C_l \in \mathbb{R}^{3\times 3}$, $l \neq i$ and arbitrary 3-dim vector b_1, equation (3.71) and equation (3.72) always hold. Therefore $\rho_1 = \rho_2$.

Till now, we have shown that the two grandchild terms $\mathcal{E}_{i,l,2}^{k+2|k+2}$ and $\mathcal{E}_{i,m,l}^{k+2|k+2}$ from a new grandparent term $\mathcal{E}^k$ with $N_e^k = m$ for $1 \leq i \leq m-1$ are equal, i.e.,

$$\mathcal{E}_{i,l,2}^{k+2|k+2} = \mathcal{E}_{i,m,l}^{k+2|k+2} \quad (3.76)$$

3.2.3 Derivation of $\mathcal{E}_{i,l,3}^{k+2|k+2} = \mathcal{E}_{i,m+1,l}^{k+2|k+2}$ from a New Grandparent Term $\mathcal{E}_i^{k|k}$ with $N_e^k = m$ for $1 \le l \le m-1$

Again, consider a new term at step k which has m elements in the argument of the exponential,

$$\mathcal{E}_i^{k|k}(\nu) = \exp\left(-P_1|HC_1\nu| - P_2|HC_2\nu| - ... - P_{m-1}|HC_{m-1}\nu| - \frac{\gamma}{|b_1 H^T|}|b_1\nu| + j\zeta_i^{k|k}\nu\right) \tag{3.77}$$

where C_q, $q = 1, 2, ..., m-1$, are skew-symmetric matrices, and b_1 is another vector which cannot be expressed in the HC form. In this section we show that $\mathcal{E}_{i,l,3}^{k+2|k+2} = \mathcal{E}_{i,m+1,l}^{k+2|k+2}$ for $1 \le l \le m-1$. These terms are colored in blue in Figure 3.2.

The number of elements $N_e^k = m$, then at the $(k+1)^{th}$ measurement update, the l^{th} child term $\mathcal{E}_{i,l}^{k+1|k+1}$ for $1 \le l \le m-1$ has four elements; while the $(m+1)^{th}$ child term $\mathcal{E}_{i,m+1}^{k+1|k+1}$ has $(m+1)$ elements. Refer to Appendix C.1.1, the l^{th} child term $\mathcal{E}_{i,l}^{k+1|k+1}$ for $1 \le l \le m-1$ at step $k+1$ is,

$$\begin{aligned}\mathcal{E}_{i,l}^{k+1|k+1}(\nu) = \exp\Bigg(&-\frac{\left(\sum_{q=1,q\neq l}^{m-1} P_q \cdot \left|HC_qA_{21}C_l^TH^T\right|\right)}{|HC_l\Phi^TH^T| \cdot |He_3^T|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| - \frac{\frac{\gamma}{|b_1H^T|}}{|HC_l\Phi^TH^T|}\left|H\Phi D_l\Phi^T\nu\right| \\ &-\frac{\beta}{|HC_l\Phi^TH^T|}\left|H\Phi E_l\Phi^T\nu\right| - \frac{\gamma}{|HC_l\Phi^TH^T|}\left|HC_l\Phi^T\nu\right| j\zeta_{i,l}^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{3.78}$$

where

$$D_l = C_l^TH^Tb_1 - b_1^THC_l \quad \text{and} \quad E_l = C_l^TH^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma HC_l \tag{3.79}$$

According to Appendix C.2.3, at step $k+2$, the 3^{nd} child term from $\mathcal{E}_{i,l}^{k+1|k+1}$ has the exponential term written as,

$$\begin{aligned}\mathcal{E}_{i,l,3}^{k+2|k+2}(\nu) = \exp\Big(&-\rho_1\left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ &- \frac{\gamma}{|H\Phi E_l\Phi^{2T}H^T|}\left|H\Phi^2E_l\Phi^{2T}\nu\right| \\ &- \frac{\beta}{|H\Phi E_l\Phi^{2T}H^T|}\left|H\left(\Phi^2E_l^T\Phi^TH^T\Gamma^T - \Gamma H\Phi E_l\Phi^{2T}\right)\nu\right| \\ &-\frac{\gamma}{|H\Phi E_l\Phi^{2T}H^T|}\left|H\Phi E_l\Phi^{2T}\nu\right| + j\zeta_{i,l,3}^{k+2|k+2}\nu\Big)\end{aligned} \tag{3.80}$$

where

$$\rho_1 = \frac{\left(\sum_{q=1,q\neq l}^{m-1} P_q \cdot \left|HC_qA_{21}C_l^TH^T\right|\right)}{|HC_l\Phi^TH^T| \cdot |He_3^T|} \frac{\left|H\Phi E_l\Phi^TA_{21}\Phi\mathbf{B}^T\Phi^TH^T\right|}{|He_3^T| \cdot |H\Phi E_l\Phi^{2T}H^T|} + \frac{\frac{\gamma}{|b_1H^T|}}{|HC_l\Phi^TH^T|} \frac{\left|H\Phi D_l\Phi^TA_{21}\Phi E_l^T\Phi^TH^T\right|}{|He_3^T| \cdot |H\Phi E_l\Phi^{2T}H^T|} \tag{3.81}$$

and D_l and E_l has been defined earlier.

At the $(k+1)^{th}$ measurement update, refer to Appendix C.1.3, the $(m+1)^{th}$ child term $\mathcal{E}_{i,m+1}^{k+1|k+1}$ is expressed as,

$$\mathcal{E}_{i,m+1}^{k+1|k+1} = \exp\left(-\frac{P_1}{|\Gamma^TH^T|}\left|H\Phi E_1\Phi^T\nu\right| - ... - \frac{P_{m-1}}{|\Gamma^TH^T|}\left|H\Phi E_{m-1}\Phi^T\nu\right| - \frac{\gamma}{|b_1H^T| \cdot |\Gamma^TH^T|}\left|H\Phi D_{gb}\Phi^T\nu\right| - \frac{\gamma}{|\Gamma^TH^T|}\left|\Gamma^T\nu\right| + j\zeta_{i,m+1}^{k+1|k+1}\nu\right) \tag{3.82}$$

where

$$E_q = \Phi^{-1}\Gamma HC_q - C_q^TH^T\Gamma^T\Phi^{-T}, \quad q = 1, ..., m-1 \tag{3.83}$$

and

$$D_{gb} = \Phi^{-1}\Gamma b_1 - b_1^T\Gamma^T\Phi^{-T} \tag{3.84}$$

Next, at the $(k+2)^{th}$ update, the l^{th} child term from $\mathcal{E}_{i,m+1}^{k+1|k+1}$ has the exponential term expressed as follows, refer to Appendix C.2.6.

$$\mathcal{E}_{i,m+1,l}^{k+2|k+2}(\nu) = \exp\Big(-\rho_2\left|H\Phi\mathbf{B}\Phi^T\nu\right| - \frac{\gamma}{|H\Phi E_l\Phi^{2T}H^T|}\left|H\Phi^2E_l\Phi^{2T}\nu\right| - \frac{\beta}{|H\Phi E_l\Phi^{2T}H^T|}\left|H\left(\Phi^2E_l^T\Phi^TH^T\Gamma^T - \Gamma H\Phi E_l\Phi^{2T}\right)\nu\right| - \frac{\gamma}{|H\Phi E_l\Phi^{2T}H^T|}\left|H\Phi E_l\Phi^{2T}\nu\right| + j\zeta_{i,m+1,l}^{k+2|k+2}\nu\Big) \tag{3.85}$$

where

$$\rho_2 = \frac{\left(\sum_{q=1,q\neq l}^{m-1} P_q \cdot \left|H\Phi E_q\Phi^TA_{21}\Phi E_l^T\Phi^TH^T\right|\right)}{|\Gamma^TH^T|\,|H\Phi E_l\Phi^{2T}H^T|\,|He_3^T|} + \frac{\gamma \cdot \left|H\Phi D_{gb}\Phi^TA_{21}\Phi E_l^T\Phi^TH^T\right|}{|b_1H^T| \cdot |\Gamma^TH^T| \cdot |H\Phi E_l\Phi^{2T}H^T| \cdot |He_3^T|} \tag{3.86}$$

Now compare $\mathcal{E}_{i,l,3}^{k+2|k+2}$ in equation (3.80) with $\mathcal{E}_{i,m+1,l}^{k+2|k+2}$ in equation (3.85). Both of them have four elements. The second, third and fourth elements of $\mathcal{E}_{i,l,3}^{k+2|k+2}$ and $\mathcal{E}_{i,m+1,l}^{k+2|k+2}$ are identical to each other, respectively. Moreover, the first direction of both two terms is $H\Phi\mathbf{B}\Phi^T$. Hence, one will just need to consider the first coefficients, i.e., ρ_1 in equation (3.81), and ρ_2 in equation (3.86). The sufficient conditions for $\rho_1 = \rho_2$ are,

$$\frac{\left|HC_q A_{21} C_l^T H^T\right|}{|HC_l \Phi^T H^T|} \frac{\left|H\Phi E_l \Phi^T A_{21} \Phi \mathbf{B}^T \Phi^T H^T\right|}{|He_3^T|} = \frac{\left|H\Phi E_q \Phi^T A_{21} \Phi E_l^T \Phi^T H^T\right|}{|\Gamma^T H^T|} \tag{3.87}$$

and

$$\frac{\left|H\Phi D_l \Phi^T A_{21} \Phi E_l^T \Phi^T H^T\right|}{|HC_l \Phi^T H^T|} = \frac{\left|H\Phi D_{gb} \Phi^T A_{21} \Phi E_l^T \Phi^T H^T\right|}{|\Gamma^T H^T|} \tag{3.88}$$

where

$$D_l = C_l^T H^T b_1 - b_1^T H C_l \tag{3.89}$$

$$E_q = \Phi^{-1}\Gamma H C_q - C_q^T H^T \Gamma^T \Phi^{-T}, \quad q = 1, ..., m-1 \tag{3.90}$$

$$D_{gb} = \Phi^{-1}\Gamma b_1 - b_1^T \Gamma^T \Phi^{-T} \tag{3.91}$$

Numerical results show that for arbitrary skew-symmetric matrices $C_q, C_l \in \mathbb{R}^{3\times3}$, $q \neq l$ and arbitrary 3-dim vector b_1, equation (3.87) and equation (3.88) always hold. Therefore $\rho_1 = \rho_2$.

We just showed that the two grandchild terms $\mathcal{E}_{i,l,3}^{k+2|k+2}$ and $\mathcal{E}_{i,m+1,l}^{k+2|k+2}$ from a new grandparent term $\mathcal{E}^k$ with $N_e^k = m$ for $1 \leq l \leq m-1$ are equal, i.e.,

$$\mathcal{E}_{i,l,3}^{k+2|k+2} = \mathcal{E}_{i,m+1,l}^{k+2|k+2} \tag{3.92}$$

3.2.4 Derivation of $\mathcal{E}_{i,m,m}^{k+2|k+2} = \mathcal{E}_{i,m+1,m}^{k+2|k+2}$ from a New Grandparent Term $\mathcal{E}^{k|k}$ with $N_e^k = m$

This is the last scenario of 4-element terms that are produced from a new grandparent term. A new term at step k which has m elements in the argument of the exponential can be expressed as,

$$\mathcal{E}_i^{k|k}(\nu) = \exp\left(-P_1|HC_1\nu| - P_2|HC_2\nu| - ... - P_{m-1}|HC_{m-1}\nu| - \frac{\gamma}{|b_1 H^T|}|b_1\nu| + j\zeta_i^{k|k}\nu\right) \tag{3.93}$$

where C_q, $q = 1, 2, ..., m-1$, are skew-symmetric matrices, and b_1 is another vector which cannot be expressed in the HC form. Because $N_e^k = m$, at the $(k+1)^{th}$ measurement update, the m^{th} child term $\mathcal{E}_{i,m}^{k+1|k+1}$ and the $(m+1)^{th}$ child term $\mathcal{E}_{i,m+1}^{k+1|k+1}$ have $(m+1)$ elements. Then, at the $(k+2)^{th}$ measurement update, the m^{th} child terms from both $\mathcal{E}_{i,m}^{k+1|k+1}$ and $\mathcal{E}_{i,m+1}^{k+1|k+1}$ must have 4 elements. This section shows that $\mathcal{E}_{i,m,m}^{k+2|k+2} = \mathcal{E}_{i,m+1,m}^{k+2|k+2}$. These terms are colored in purple in Figure 3.2.

From Appendix C.1.2, at the $(k+1)^{th}$ update, the m^{th} child term $\mathcal{E}_{i,m}^{k+1|k+1}$ is expressed as,

$$\begin{aligned}\mathcal{E}_{i,m}^{k+1|k+1}(\nu) = \exp\Bigg(&-\frac{P_1}{|b_1\Phi^T H^T|}\left|H\Phi D_1 \Phi^T \nu\right| - ... - \frac{P_{m-1}}{|b_1\Phi^T H^T|}\left|H\Phi D_{m-1}\Phi^T\nu\right| \\ &-\frac{\beta}{|b_1\Phi^T H^T|}\left|H\Phi D_{gb}\Phi^T\nu\right| - \frac{\gamma}{|b_1\Phi^T H^T|}\left|b_1\Phi^T\nu\right| + j\zeta_{i,m}^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{3.94}$$

where

$$D_q = -b_1^T H C_q + C_q^T H^T b_1, \quad q = 1, ..., m-1 \tag{3.95}$$

$$D_{gb} = b_1^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma b_1 \tag{3.96}$$

Next, refer to Appendix C.2.5, $\mathcal{E}_{i,m,m}^{k+2|k+2}$ can be written as,

$$\begin{aligned}\mathcal{E}_{i,m,m}^{k+2|k+2}(\nu) = \exp\Bigg(&-\frac{\sum_{q=1}^{m-1}\left(P_q\left|H\Phi D_q\Phi^T A_{21}\Phi D_{gb}^T\Phi^T H^T\right|\right)}{|b_1\Phi^T H^T|\cdot|H\Phi D_{gb}\Phi^{2T}H^T|\cdot|He_3^T|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ &-\frac{\gamma}{|H\Phi D_{gb}\Phi^{2T}H^T|}\left|H\Phi^2 D_{gb}\Phi^{2T}\nu\right| \\ &-\frac{\beta}{|H\Phi D_{gb}\Phi^{2T}H^T|}\left|H\left(\Phi^2 D_{gb}^T\Phi^T H^T\Gamma^T - \Gamma H\Phi D_{gb}\Phi^{2T}\right)\nu\right| \\ &-\frac{\gamma}{|H\Phi D_{gb}\Phi^{2T}H^T|}\left|H\Phi D_{gb}\Phi^{2T}\nu\right| + j\zeta_{i,m,m}^{k+2|k+2}\nu\Bigg)\end{aligned} \tag{3.97}$$

where

$$D_q = -b_1^T H C_q + C_q^T H^T b_1, \quad q = 1, ..., m-1 \tag{3.98}$$

$$D_{gb} = b_1^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma b_1 \tag{3.99}$$

At the $(k+1)^{th}$ measurement update, the $(m+1)^{th}$ child term $\mathcal{E}_{i,m+1}^{k+1|k+1}$ is expressed as follow, refer to Appendix C.1.3.

$$\mathcal{E}_{i,m+1}^{k+1}(\nu) = \exp\left(-\frac{P_1}{|\Gamma^T H^T|}\left|H\Phi E_1\Phi^T\nu\right| - ... - \frac{P_{m-1}}{|\Gamma^T H^T|}\left|H\Phi E_{m-1}\Phi^T\nu\right| \right.$$
$$\left. -\frac{\gamma}{|b_1 H^T|\cdot|\Gamma^T H^T|}\left|H\Phi D_{gb}\Phi^T\nu\right| - \frac{\gamma}{|\Gamma^T H^T|}\left|\Gamma^T\nu\right| + j\zeta_{i,m+1}^{k+1|k+1}\nu\right) \tag{3.100}$$

where

$$E_l = \Phi^{-1}\Gamma H C_l - C_l^T H^T \Gamma^T \Phi^{-T}, \quad l = 1, ..., m-1 \tag{3.101}$$

$$D_{gb} = \Phi^{-1}\Gamma b_1 - b_1^T \Gamma^T \Phi^{-T} \tag{3.102}$$

Next, refer to Appendix C.2.7, the m^{th} child term at step $k+2$ can be written as,

$$\mathcal{E}_{i,m+1,m}^{k+2|k+2}(\nu) = \exp\left(-\frac{\sum_{q=1}^{m-1}\left(P_q\left|H\Phi E_q\Phi^T A_{21}\Phi D_{gb}^T\Phi^T H^T\right|\right)}{|\Gamma^T H^T|\cdot|He_3^T|\cdot|H\Phi D_{gb}\Phi^{2T}H^T|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| \right.$$
$$-\frac{\gamma}{|H\Phi D_{gb}\Phi^{2T}H^T|}\left|H\Phi^2 D_{gb}\Phi^{2T}\nu\right|$$
$$-\frac{\beta}{|H\Phi D_{gb}\Phi^{2T}H^T|}\left|H\left(\Phi^2 D_{gb}^T\Phi^T H^T\Gamma^T - \Gamma H\Phi D_{gb}\Phi^{2T}\right)\nu\right|$$
$$\left. -\frac{\gamma}{|H\Phi D_{gb}\Phi^{2T}H^T|}\left|H\Phi D_{gb}\Phi^{2T}\nu\right| + j\zeta_{i,m+1,m}^{k+2|k+2}\nu\right) \tag{3.103}$$

where

$$E_q = \Phi^{-1}\Gamma H C_q - C_q^T H^T \Gamma^T \Phi^{-T}, \quad q = 1, ..., m-1 \tag{3.104}$$

$$D_{gb} = b_1^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma b_1 \tag{3.105}$$

Compare $\mathcal{E}_{i,m,m}^{k+2|k+2}$ in equation (3.97) with $\mathcal{E}_{i,m+1,m}^{k+2|k+2}$ in equation (3.103). Both of them have four elements. The second, third and fourth elements of $\mathcal{E}_{i,m,m}^{k+2|k+2}$ and $\mathcal{E}_{i,m+1,m}^{k+2|k+2}$ are identical to each other, respectively. Moreover, the first direction of both two terms is $H\Phi\mathbf{B}\Phi^T$. Again, we just need to consider the first coefficients.

By comparing the structure of the two terms, we can tell that $\mathcal{E}_{i,m,m}^{k+2|k+2} = \mathcal{E}_{i,m+1,m}^{k+2|k+2}$ if

$$\frac{\left|H\Phi D_q\Phi^T A_{21}\Phi D_{gb}^T\Phi^T H^T\right|}{|b_1\Phi^T H^T|} = \frac{\left|H\Phi E_q\Phi^T A_{21}\Phi D_{gb}^T\Phi^T H^T\right|}{|\Gamma^T H^T|} \tag{3.106}$$

holds, where

$$D_q = C_q^T H^T b_1 - b_1^T H C_q, \quad q = 1, ..., m-1 \tag{3.107}$$

$$E_q = C_q^T H^T \Gamma^T \Phi^{-T} - \Phi^{-1} \Gamma H C_q, \quad q = 1, ..., m-1 \tag{3.108}$$

$$D_{gb} = b_1^T \Gamma^T \Phi^{-T} - \Phi^{-1} \Gamma b_1 \tag{3.109}$$

Numerical checks show that for any skew-symmetric matrices $C_q \in \mathbb{R}^{3\times3}$ and any 3-dim vector b_1, equation (3.106) always holds. Therefore $\mathcal{E}_{i,m,m}^{k+2|k+2} = \mathcal{E}_{i,m+1,m}^{k+2|k+2}$.

So far, the term combination rules for 4-element terms that are produced from new grandparent terms for three-state systems are shown, highlighted in Figure 3.2. Not as special as the two-state cases, three state cases involve more scenarios of term combination. Rules for 4-element terms that are from old grandparent terms are still to be found. Furthermore, terms with more than 4 elements in the argument of the exponential also have the chance to be combined. Referred to Prof. Moshe Idan's numerical results, for a three-state case, at step $k = 5$ for example, 2512 4-element exponential term are combined to be only 97 distinct terms; 1408 5-element exponential terms are combined to be 93 distinct terms; 672 6-element exponential terms are combine to be 80 terms; and 848 7-element exponential terms are combined to be 262 terms. Although more such combination rules are still puzzles, this investigation of three-state systems starts to uncover the properties from a more general perspective that certain exponential terms have the same functional structure. They can be combined in order to reduce the amount of computation required.

3.3 Invariance

The first combination rule for both two-state and three-state systems indicates that the first grandchild term $\mathcal{E}_{i,l,1}^{k+2|k+2}$ from the first several parent term $\mathcal{E}_{i,l}^{k+1|k+1}$ that are produced from any new grandparent term $\mathcal{E}_{i}^{k|k}$ at step k can be combined universally. It can be expressed by a form that only depends on system parameters including H, Φ, Γ, α, β and γ, and is independent with any particular value of $\mathcal{E}_{i}^{k|k}$.

At step $k+1$, the term $\mathcal{E}_{i,l}^{k+1|k+1}$ is also a new term. Suppose this term at step $k+1$ has N_e^{k+1} elements in the argument of the exponential. Then, at step $k+3$, its corresponding grandchild term $\mathcal{E}_{i,l,p,1}^{k+3|k+3}$ where $p \leq N_e^{k+1} - 1$ should also be independent with any particular value of its grandparent term $\mathcal{E}_{i,l}^{k+1|k+1}$. This implies that the exponential term $\mathcal{E}_{i,l,p,1}^{k+3|k+3}$ at step $k+3$ and the exponential term $\mathcal{E}_{i,l,1}^{k+2|k+2}$ at step $k+2$ have the same real component of the argument of the exponential. Note that the imaginary component of the exponential is associated with all measurement data, hence will be different after one more measurement update. Moreover, if we let $p = 1$, then $\mathcal{E}_{i,l,1,1}^{k+3|k+3}$ and $\mathcal{E}_{i,l,1}^{k+2|k+2}$ has the same real component of the argument of the exponential.

Therefore, for two-state and three-state cases, the real component of the argument of the exponential of first child terms will remain an invariant form as k gets larger. In particular, for two-state cases, it is obvious that this invariant exponential term is expressed only with respect to system parameters, as shown in equation (3.19). For three-state systems, the expression of the invariant exponential term, (3.59), still contains the quantity of C_i, although it is independent of its value. Inspired by the invariance property, one is able to eliminate the expression of C_i by deriving its first child term at step $k+3$. Refer to Appendix C.3.1, the exponential term of the first child term $\mathcal{E}_{i,l,1,1}^{k+3|k+3}$ can be written as,

$$\begin{aligned}\mathcal{E}_{i,l,1,1}^{k+3|k+3}(\nu) = \exp\Bigg(& -\frac{\rho_1 \left|H\Phi^2\mathbf{B}\Phi^{2T}A_{21}\Phi\mathbf{B}^T\Phi^T H^T\right| + \rho_2\left|HCA_{21}\Phi\mathbf{B}^T\Phi^T H^T\right|}{\left|H\Phi\mathbf{B}\Phi^{2T}H^T\right| \cdot \left|He_3^T\right|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ & - \rho_3\left|H\Phi^2\mathbf{B}\Phi^{2T}\nu\right| - \frac{\beta}{\left|H\Phi\mathbf{B}\Phi^{2T}H^T\right|}\left|H\left(\Phi^2\mathbf{B}^T\Phi^T H^T\Gamma^T - \Gamma H\Phi\mathbf{B}\Phi^{2T}\right)\nu\right| \\ & - \frac{\gamma}{\left|H\Phi\mathbf{B}\Phi^{2T}H^T\right|}\left|H\Phi\mathbf{B}\Phi^{2T}\nu\right| + j\zeta_{i,l,1,1}^{k+3|k+3}\nu\Bigg)\end{aligned} \tag{3.110}$$

where

$$\rho_1 = \rho_3 = \frac{\gamma}{\left|H\Phi\mathbf{B}\Phi^{2T}H^T\right|}, \quad \rho_2 = \frac{\beta}{\left|H\Phi\mathbf{B}\Phi^{2T}H^T\right|} \tag{3.111}$$

Here we introduce a lemma,

Lemma 3.3.1. *Let $b \in \mathbb{R}^{1\times 3}$ be an arbitrary row vector, $C \in \mathbb{R}^{3\times 3}$ be any skew-symmetric matrix, and $\boldsymbol{B}$ be the fundamental basis of three-state systems. Then,*

$$\left|\frac{H\Phi\left(C^T H^T b - b^T HC\right)\Phi^T A_{21}\Phi\boldsymbol{B}^T\Phi^T H^T}{bH^T \cdot HC\Phi^T H^T}\right| = \left|\frac{H\Phi^2\boldsymbol{B}\Phi^{2T}A_{21}\Phi\boldsymbol{B}^T\Phi^T H^T}{H\Phi\boldsymbol{B}\Phi^{2T}H^T}\right| \tag{3.112}$$

Proof. This algebraic result has been verified numerically. ☐

Then one can obtain the following two equalities,

$$\frac{\frac{\gamma}{|b_1 H^T|}|H\Phi D_l \Phi^T A_{21}\Phi\mathbf{B}^T\Phi^T H^T|}{|HC_l\Phi^T H^T|} = \frac{\gamma|H\Phi^2\mathbf{B}\Phi^{2T}A_{21}\Phi\mathbf{B}^T\Phi^T H^T|}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} \tag{3.113}$$

and

$$\begin{aligned}\frac{\beta|H\Phi E_l\Phi^T A_{21}\Phi\mathbf{B}^T\Phi^T H^T|}{|HC_l\Phi^T H^T|} &= \frac{\beta|H\left(\Phi^2\mathbf{B}^T\Phi^T H^T\Gamma^T - \Gamma H\Phi\mathbf{B}\Phi^{2T}\right)A_{21}\Phi\mathbf{B}^T\Phi^T H^T|}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} \\ &= \frac{\beta|H\Phi^2\mathbf{B}\Phi^{2T}A_{21}\Phi\mathbf{B}^T\Phi^T H^T|}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}\cdot|\Gamma^T\Phi^{-T}H^T|\end{aligned} \tag{3.114}$$

Equation (3.113) and (3.114) show that the real component of the argument of the exponential term $\mathcal{E}_{i,l,1,1}^{k+3|k+3}$ in (3.110) and $\mathcal{E}_{i,l,1}^{k+2|k+2}$ in (3.59) are the same. Furthermore, let us define a coefficient σ as,

$$\sigma = \frac{|H\Phi^2\mathbf{B}\Phi^{2T}A_{21}\Phi\mathbf{B}^T\Phi^T H^T|}{|H\Phi\mathbf{B}\Phi^{2T}H^T|^2\cdot|He_3^T|} \tag{3.115}$$

Then equation (3.110) can be rewritten as,

$$\begin{aligned}\mathcal{E}_{i,l,1,1}^{k+3|k+3}(\nu) = \exp\Bigg[&-\sigma\left(\gamma+\beta|\Gamma^T\Phi^{-T}H^T|\right)|H\Phi\mathbf{B}\Phi^T\nu| - \frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}|H\Phi^2\mathbf{B}\Phi^{2T}\nu| \\ &- \frac{\beta}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}|H\left(\Phi^2\mathbf{B}^T\Phi^T H^T\Gamma^T - \Gamma H\Phi\mathbf{B}\Phi^{2T}\right)\nu| \\ &- \frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}|H\Phi\mathbf{B}\Phi^{2T}\nu| + j\zeta_{i,l,1,1}^{k+3|k+3}\nu\Bigg]\end{aligned} \tag{3.116}$$

Equation (3.116) provides an alternative expression for the first child term from any parent term with four elements during or after the third measurement update.

CHAPTER 4

S Matrix

It has been discussed in detail that certain exponential terms can be combined analytically in the last chapter. In this chapter, the "S" matrix is constructed as an indexing scheme to keep track of how exponential terms are repeated functionally. Furthermore, for the same system dimension, an S matrix stays invariant regardless of different system parameters. As brought up in chapter 1, all the analytic solutions presented in this dissertation is based on the assumption that none of the directions can be orthogonal to H. If some directions are orthogonal to H, the solution can also be obtained, but will have a slightly different form. The special case of the orthogonality was discussed in [5]. Finally, the explicit recursive structure of the S matrix for the two-state dynamic systems is determined and proved analytically. For higher order systems, it appears that the S matrix can always be computed. The S matrix allows for combination of terms without the need of numerical comparison during the estimation process. This saves a tremendous amount of implementation time.

4.1 General Structure of S Matrix

Suppose at step $k-1$, there are $\tilde{N}_t^{k-1|k-1}$ distinct exponential terms in the CF of the ucpdf described in (3.1). The i^{th} exponential term at step $k-1$ has $N_{ei}^{k-1|k-1}$ elements in the argument of the exponential and $\tilde{N}_{t,i}^{k-1|k-1}$ coefficient terms. Each exponential term at step $k-1$ might have a different number of elements depending on how many directions combine at each measurement update. Hence, different terms at step $k-1$ can produce a different number of child terms at step k. Each parent term at step k can produce at most $(k+n)$ child terms at step $k+1$. Define a matrix S at step k to have $(k+n)$ rows and $\tilde{N}_t^{k-1|k-1}$

columns.

$$S_k = \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,\tilde{N}_t^{k-1|k-1}} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,\tilde{N}_t^{k-1|k-1}} \\ \vdots & & & \\ s_{k+n,1} & s_{k+n,2} & \cdots & s_{k+n,\tilde{N}_t^{k-1|k-1}} \end{bmatrix} \tag{4.1}$$

Each element in matrix S_k, s_{ij}, is an identifier for the i^{th} child term at step k of the j^{th} parent term at step $k-1$, represented by a non-negative integer. All the elements in matrix S provide complete information of the numbering system of all children terms at step k. If $s_{ij} = s_{kl}$, then the argument of the exponential of the i^{th} child term (at step k) of the j^{th} parent term (at step $(k-1)$) and that of the k^{th} child term (at step k) of the l^{th} parent term (at step $k-1$) are functionally the same. They are numbered and stored as the ${s_{ij}}^{th}$ term at step k. The parent terms at step $(k-1)$, that has directions less than $(k+n-2)$, will not produce as many as $(k+n)$ child terms at step k. The elements for their child terms are also aligned on the specific column of S_k from the top. For the last few places in that column where there is a lack of child terms, zero is placed to indicate that there is no child term at that specific place.

To be more explicit, let the capital letter S^* and S^o represents the matrix to be examined, and the lower case $s^*_{i,j}$ and $s^o_{i,j}$ represents the $(i,j)^{th}$ element of S^* and S^o. The $(i,j)^{th}$ exponential term is expressed as the functional form of the particular functions that comprise the estimation process, denoted as $\mathcal{E}_{i,j} = f(\Phi, \Gamma, H, \alpha, \beta, \gamma; \nu)$, where $f(\bullet)$ is a function of all the system parameters as well as the spectral variable, structured in (1.15). If two such expressions $\mathcal{E}_{i,j} = \mathcal{E}_{l,m}$ for all Φ, H, Γ, α, β, γ and ν, then the two exponential terms can be combined. With these notations, let us define viable S matrix as shown in Definition 4.1.1.

Definition 4.1.1. A matrix S^* is a viable S matrix when the following statement holds:

$s^*_{i,j} = s^*_{l,m} \in \mathbb{N}$ implies $\mathcal{E}_{i,j} = \mathcal{E}_{l,m}$ for all system parameters Φ, H, Γ, α, β, γ and for all spectral variable ν.

From the above definition, one can see that viable S matrix is not unique. Furthermore, we define the minimal S matrix to be a viable S matrix that has the minimum number of distinct terms by making the statement a necessary and sufficient condition.

Definition 4.1.2. A matrix S^o is the minimal S matrix when the following statement holds:

$s^o_{i,j} = s^o_{l,m} \in \mathbb{N}$ if and only if $\mathcal{E}_{i,j} = \mathcal{E}_{l,m}$ for all system parameters Φ, H, Γ, α, β, γ and for all spectral variable ν.

The definition of a viable S matrix is important. It provides the minimum requirement for a S matrix to store the terms combination information. By definition it states that if $s_{i,j} = s_{l,m}$, then $\mathcal{E}_{i,j} = \mathcal{E}_{l,m}$ must hold for all parameters and ν. However, when $s_{i,j} \neq s_{l,m}$, it is not necessary that $\mathcal{E}_{i,j} \neq \mathcal{E}_{l,m}$. In fact, during update processing, when the program looks up the S matrix to decide which term to combine, it does not cause any issue when it computes the repeated exponential term twice; however, it will raise problems if two distinct exponential terms were combined by mistake. The notion of the viable S matrix defines all possible constructions of S matrix that will not cause such computation problems. If making the one-side statement stronger by a necessary and sufficient condition, one will get the special viable S that contains the minimum number of distinct exponential terms analytically, as brought up in Definition 4.1.2. In further content, the viable S matrix is also named S matrix for short.

4.2 Invariance of S Matrix

The viable S matrix stays invariant for systems of the same dimension, regardless of different values of the system parameters Φ, H, Γ, α, β and γ. This is because analytically, a viable S matrix is obtained by comparing the functional expressions $\mathcal{E}_{i,j}$ of the exponential terms. Only if $\mathcal{E}_{i,j} = \mathcal{E}_{l,m}$ for all system parameters $p = \{\Phi, H, \Gamma, \alpha, \beta, \gamma\}$ and for all ν, we say that these two exponential terms can be combined analytically. If there exists some values of system parameters $p_1 = \{\Phi_1, H_1, \Gamma_1, \alpha_1, \beta_1, \gamma_1\}$ and $p_2 = \{\Phi_2, H_2, \Gamma_2, \alpha_2, \beta_2, \gamma_2\}$, such that $\mathcal{E}_{i,j}(p_1) = \mathcal{E}_{l,m}(p_1)$ but $\mathcal{E}_{i,j}(p_2) \neq \mathcal{E}_{l,m}(p_2)$, it means that these two exponential can only be combined under special numerical cases. Hence, the terms are not considered to be combined analytically. In this case, $s_{i,j} \neq s_{l,m}$. Now, we have seen that only terms with the same analytic expressions are considered to have the same $s_{i,j}$ in the S matrix. In other words, these repeated exponential terms which are indicated by the same integer element in the S matrix, should be equal with each other with respect to all system parameters $p = \{\Phi, H, \Gamma, \alpha, \beta, \gamma\}$ and for all ν. For systems of same dimension, the S matrix is independent of the particular values of parameters.

4.3 Analytic Recursive Structure of S for Two-State Case

In this section, for two-state system we present an analytic recursive structure of the S matrix. This recursion is derived by examining different part of S in an order such that given S_k, the form of S_{k+1} can be uniquely determined.

The recursive structure is described explicitly as follows. As a proper initialization, there are 3 terms at the first measurement update. They are all distinct. The S matrix at step $k = 1$ is expressed as,

$$S_{k=1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{4.2}$$

At the second measurement update, the S matrix is,

$$S_{k=2} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{4.3}$$

Suppose at step k, where $k = 2, 3, ...$, the S matrix for a two-state dynamics is known as,

$$S_k = \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,\tilde{N}_t^{k-1|k-1}} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,\tilde{N}_t^{k-1|k-1}} \\ \vdots & & & \\ s_{k+2,1} & s_{k+2,2} & \cdots & s_{k+2,\tilde{N}_t^{k-1|k-1}} \end{bmatrix} \in \mathbb{N}^{(k+2)\times \tilde{N}_t^{k-1|k-1}} \tag{4.4}$$

where $\tilde{N}_t^{k-1|k-1}$ stands for the number of distinct exponential terms at step $k-1$. This notation is consistent with equation (3.1). S_k has a "staircase"-shape structure, with its bottom left corner to be all zeros. This property will be elaborated in detail in latter subsections. The staircase part of the array is denoted as $S^*_{k,new}$, shown in equation (4.5). The asterisk in the superscript means that this is not a rectangular matrix in the most common sense.

$$
S^*_{k,new} = \begin{bmatrix}
s^k_{1,1} & \cdots & s^k_{1,\tilde{N}^{k-1|k-1}_{t,new}} & s^k_{1,\tilde{N}^{k-1|k-1}_{t,new}+1} & \cdots & s^k_{1,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}} & s^k_{1,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1} & \cdots & s^k_{1,\tilde{N}^{k-1|k-1}_t} \\
s^k_{2,1} & \cdots & s^k_{2,\tilde{N}^{k-1|k-1}_{t,new}} & s^k_{2,\tilde{N}^{k-1|k-1}_{t,new}+1} & \cdots & s^k_{2,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}} & s^k_{2,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1} & \cdots & s^k_{2,\tilde{N}^{k-1|k-1}_t} \\
s^k_{3,1} & \cdots & s^k_{3,\tilde{N}^{k-1|k-1}_{t,new}} & s^k_{3,\tilde{N}^{k-1|k-1}_{t,new}+1} & \cdots & s^k_{3,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}} & s^k_{3,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1} & \cdots & s^k_{3,\tilde{N}^{k-1|k-1}_t} \\
NaN & \cdots & NaN & s^k_{4,\tilde{N}^{k-1|k-1}_{t,new}+1} & \cdots & s^k_{4,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}} & s^k_{4,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1} & \cdots & s^k_{4,\tilde{N}^{k-1|k-1}_t} \\
NaN & \cdots & NaN & NaN & \cdots & NaN & s^k_{5,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1} & \cdots & s^k_{5,\tilde{N}^{k-1|k-1}_t} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
NaN & \cdots & NaN & NaN & \cdots & NaN & NaN & \cdots & s^k_{k+1,\tilde{N}^{k-1|k-1}_t}
\end{bmatrix} \tag{4.5}
$$

Then, at step $k+1$, $S_{k+1} \in \mathbb{N}^{(k+3)\times \tilde{N}_t^{k|k}}$ is expressed as,

$$
S_{k+1} = \begin{bmatrix}
1 & ... & 1 & & & \\
2 & ... & \tilde{N}_{t,new}^{k|k}+1 & & S_{k,new}^{*}+[1]^{*} & \\
\tilde{N}_{t,new}^{k|k}+2 & ... & 2\tilde{N}_{t,new}^{k|k}+1 & & & \\
\tilde{N}_{t,new}^{k+1|k+1}+1 & ... & \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k} & 2\tilde{N}_{t,new}^{k|k}+2 & ... & ... \\
0 & ... & 0 & \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k}+1 & ... & \\
\vdots & & & \vdots & & \\
0 & ... & & & & \\
0 & ... & ... & & ... & \tilde{N}_{t,new}^{k+1|k+1} \\
0 & ... & ... & & ... & \tilde{N}_{t}^{k+1|k+1}
\end{bmatrix} \tag{4.6}
$$

where $S_{k,new}^{*}+[1]^{*}$ means adding 1 to each of the entry in $S_{k,new}^{*}$, described in equation (4.7).

$$S^*_{k,new} + [1]^* =$$

$$\begin{bmatrix}
s^k_{1,1}+1 & \cdots & s^k_{1,\tilde{N}^{k-1|k-1}_{t,new}}+1 & s^k_{1,\tilde{N}^{k-1|k-1}_{t,new}+1}+1 & \cdots & s^k_{1,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}}+1 & s^k_{1,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1}+1 & \cdots & s^k_{1,\tilde{N}^{k-1|k-1}_t}+1 \\
s^k_{2,1}+1 & \cdots & s^k_{2,\tilde{N}^{k-1|k-1}_{t,new}}+1 & s^k_{2,\tilde{N}^{k-1|k-1}_{t,new}+1}+1 & \cdots & s^k_{2,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}}+1 & s^k_{2,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1}+1 & \cdots & s^k_{2,\tilde{N}^{k-1|k-1}_t}+1 \\
s^k_{3,1}+1 & \cdots & s^k_{3,\tilde{N}^{k-1|k-1}_{t,new}}+1 & s^k_{3,\tilde{N}^{k-1|k-1}_{t,new}+1}+1 & \cdots & s^k_{3,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}}+1 & s^k_{3,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1}+1 & \cdots & s^k_{3,\tilde{N}^{k-1|k-1}_t}+1 \\
NaN & \cdots & NaN & s^k_{4,\tilde{N}^{k-1|k-1}_{t,new}+1}+1 & \cdots & s^k_{4,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}}+1 & s^k_{4,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1}+1 & \cdots & s^k_{4,\tilde{N}^{k-1|k-1}_t}+1 \\
NaN & \cdots & NaN & NaN & \cdots & NaN & s^k_{5,\tilde{N}^{k-1|k-1}_{t,new}+\tilde{N}^{k-2|k-2}_{t,new}+1}+1 & \cdots & s^k_{5,\tilde{N}^{k-1|k-1}_t}+1 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
NaN & \cdots & NaN & NaN & \cdots & NaN & NaN & \cdots & s^k_{k+1,\tilde{N}^{k-1|k-1}_t}+1
\end{bmatrix} \tag{4.7}$$

For the rest of Section 4.3, the recursive formulation of the S matrix for two-state systems is proved. We break down the S matrix into different parts. By showing what each part of S looks like in a sequential manner, the S structure is uniquely determined. This is also verified by numerical experiments, though not presented here. While this structure seems to be a complicated algorithm, in fact, the implementation is very efficient and straightforward. This recursion contributes to the computational efficiency significantly, since it provides a practical implementation that can *a priori* combine terms without the need of numerical comparison.

4.3.1 Part 1: The First Child Terms from New Parent Terms

Starting from this subsection, we prove the recursive structure by dividing the S matrix into seven different parts. First, consider the first child terms form new parent terms. From the first term combination rule for two-state systems described in equation (3.20), the first child terms at step $k+1$ of all new parent terms at step k can be combined. These terms are stated in S_{k+1} as the first row of the first $\tilde{N}_{t,new}^{k|k}$ columns. Accordingly, let us assign these entries to be 1 in S_{k+1}, meaning that they will be combined and stored as the 1^{st} term at step $k+1$ after term combination.

Hence, the S_{k+1} becomes,

$$S_{k+1} = \begin{bmatrix} 1 & 1 & \dots & 1 & s_{1,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{1,\tilde{N}_t^{k|k}} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,\tilde{N}_{t,new}^{k|k}} & s_{2,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{2,\tilde{N}_t^{k|k}} \\ \vdots & & & & & & \vdots \\ s_{k+3,1} & s_{k+3,2} & \cdots & & & & s_{k+3,\tilde{N}_t^{k|k}} \end{bmatrix} \tag{4.8}$$

where $S_{k+1} \in \mathbb{N}^{(k+3)\times \tilde{N}_t^{k|k}}$.

4.3.2 Part 2: The Second Child Terms from New Parent Terms

Here we consider the second row of the first $\tilde{N}_{t,new}^{k|k}$ columns. We have exhausted combination rules for two-state case, where none of the second child terms from new parent terms can be combined. In the matrix S_{k+1}, those terms are represented as the first $\tilde{N}_{t,new}^{k|k}$ entries of the second row. We assign the indices of those terms from 2 to $\tilde{N}_{t,new}^{k|k}+1$. Then rewrite S_{k+1} more specifically as,

$$S_{k+1} = \begin{bmatrix} 1 & 1 & \dots & 1 & s_{1,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{1,\tilde{N}_t^{k|k}} \\ 2 & 3 & \dots & \tilde{N}_{t,new}^{k|k}+1 & s_{2,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{2,\tilde{N}_t^{k|k}} \\ \vdots & & & & & & \vdots \\ s_{k+3,1} & s_{k+3,2} & \cdots & & & & s_{k+3,\tilde{N}_t^{k|k}} \end{bmatrix} \tag{4.9}$$

where $S_{k+1} \in \mathbb{N}^{(k+3)\times \tilde{N}_t^{k|k}}$.

4.3.3 Part 3: The First Child Terms from the First $\tilde{N}_{t,new}^{k-1|k-1}$ Old Parent Terms

At step k, the number of old distinct exponential terms is $\tilde{N}_{t,old}^{k|k} = \tilde{N}_t^{k-1|k-1} = \tilde{N}_{t,new}^{k-1|k-1} + \tilde{N}_{t,old}^{k-1|k-1}$. Consider the first row from the $(\tilde{N}_{t,new}^{k|k}+1)^{th}$ column to the $(\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1})^{th}$ column. These entries represent the first child terms from the first $\tilde{N}_{t,new}^{k-1|k-1}$ old parent terms at step k.

Rewrite S_{k+1} in more detail. The first $\tilde{N}_{t,new}^{k|k}$ columns of S_{k+1} represent all child terms at step $k+1$ from new parent terms at step k, denoted as $S_{k+1}(:,1:\tilde{N}_{t,new}^{k|k})$.

$$S_{k+1}(:,1:\tilde{N}_{t,new}^{k|k}) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 3 & \dots & \tilde{N}_{t,new}^{k|k}+1 \\ \vdots & & & \vdots \\ s_{k+3,1} & s_{k+3,2} & \cdots & s_{k+3,\tilde{N}_{t,new}^{k|k}+1} \end{bmatrix} \in \mathbb{N}^{(k+3)\times \tilde{N}_{t,new}^{k|k}} \tag{4.10}$$

The last $\tilde{N}_{t,old}^{k|k} = \tilde{N}_t^{k-1|k-1}$ columns of S_{k+1} represent all child terms at step $k+1$ from old parent terms at step k, shown as follow.

$$S_{k+1}(:,\tilde{N}_{t,new}^{k|k}+1:\tilde{N}_t^{k|k})$$
$$=\begin{bmatrix} s_{1,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}} & s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \cdots & s_{1,\tilde{N}_t^{k|k}} \\ s_{2,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}} & s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \cdots & s_{2,\tilde{N}_t^{k|k}} \\ \vdots & & & & & \vdots \\ s_{k+3,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{k+3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}} & s_{k+3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \cdots & s_{k+3,\tilde{N}_t^{k|k}} \end{bmatrix} \quad (4.11)$$

where $S_{k+1}(:,\tilde{N}_{t,new}^{k|k}+1:\tilde{N}_t^{k|k})\in\mathbb{N}^{(k+3)\times\tilde{N}_t^{k-1|k-1}}$.

In $S_{k+1}(:,\tilde{N}_{t,new}^{k|k}+1:\tilde{N}_t^{k|k})$, the first $\tilde{N}_{t,new}^{k-1|k-1}$ columns represent the grandchild terms from new grandparent terms at step $k-1$, and the rest columns of $S_{k+1}(:,\tilde{N}_{t,new}^{k|k}+1:\tilde{N}_t^{k|k})$ represents the grandchild terms from old grandparent terms at step $k-1$.

Recall the second combination rules expressed in (3.45),

$$\mathcal{E}_{i,l,2}^{k+1|k+1}=\mathcal{E}_{i,N_{e,i}^{k-1|k-1}+2,l}^{k+1|k+1},\quad \text{for } 1\le l\le N_{e,i}^{k-1|k-1}+1 \quad (4.12)$$

where $N_{e,i}^{k-1|k-1}$ is the number of elements of the i^{th} grandparent term at step $k-1$.

Consider the first child terms at step $k+1$ from the first $\tilde{N}_{t,new}^{k-1|k-1}$ old parent terms at step k. It is obvious that the first $\tilde{N}_{t,new}^{k-1|k-1}$ old parent terms at step k are the old child terms of all the new terms at step $k-1$. In other words, these first $\tilde{N}_{t,new}^{k-1|k-1}$ old terms at step k are "1-step" old.

Therefore, in this case, the grandparent term has only 2 elements, i.e. $N_{e,i}^{k-1|k-1}=2$. Then the combination rule can be written as,

$$\mathcal{E}_{i,l,2}^{k+1|k+1}=\mathcal{E}_{i,4,l}^{k+1|k+1},\quad i=1,2,...,\tilde{N}_{t,new}^{k-1|k-1} \quad (4.13)$$

Let $l=1$. Then

$$\mathcal{E}_{i,1,2}^{k+1|k+1}=\mathcal{E}_{i,4,1}^{k+1|k+1},\quad i=1,2,...,\tilde{N}_{t,new}^{k-1|k-1} \quad (4.14)$$

Look at the left hand side of the equation. Because the grandparent terms at step $k-1$ are all new terms, the parent term at step k, $\mathcal{E}_{i,1}^{k|k}$, where $i=1,2,...,\tilde{N}_{t,new}^{k-1|k-1}$ will always combine according to the first combination rule. Recall equation (4.8), we already assign them to be 1 in S_k. Hence, the term $\mathcal{E}_{i,1,2}^{k+1|k+1}$ where $i = 1, 2, ..., \tilde{N}_{t,new}^{k-1|k-1}$ is represented by $s_{2,1} = 2$ in the matrix S_{k+1}.

Remark 4.3.1. A trick is used here to convert the three-subscript system of $\mathcal{E}_{i,1,2}^{k+1|k+1}$ into the two-subscript system of $s_{2,1}$ in S_{k+1}. The first subscripts system is for exponential terms. The first letter i represents the i^{th} grandparent term at step $k-1$. i can be very large, according to the specific term. The second subscript "1" represents the first child at step k. The third subscript "2" represents the second child at step $k+1$. The second subscripts system is for the element in the S matrix. Therefore, the first two subscripts $(i, 1)$ in $\mathcal{E}_{i,1,2}^{k+1|k+1}$ corresponds to the second subscript "1" of $s_{2,1}$ in S_{k+1}. The third subscript "2" in $\mathcal{E}_{i,1,2}^{k+1|k+1}$ corresponds to the first subscript of $s_{2,1}$ in S_{k+1}. This trick is frequently used in latter subsections.

Now consider the right hand side of equation (4.14). The term $\mathcal{E}_{i,4}^{k|k}$ at step k for $i = 1, 2, ..., \tilde{N}_{t,new}^{k-1|k-1}$ are represented from $s_{4,1} = \tilde{N}_{t,new}^{k|k} + 1$ through $s_{4,\tilde{N}_{t,new}^{k-1|k-1}} = \tilde{N}_{t,new}^{k|k} + \tilde{N}_{t,new}^{k-1|k-1}$ in the matrix S_k. The reason to count these terms from $\tilde{N}_{t,new}^{k|k} + 1$ is trivial – there are $\tilde{N}_{t,new}^{k|k}$ new distinct exponential terms at step k and $s_{4,1}$ represents the first old term. Therefore, the child terms $\mathcal{E}_{i,4,1}^{k+1|k+1}$ where $i = 1, 2, ..., \tilde{N}_{t,new}^{k-1|k-1}$ are represented from $s_{1,\tilde{N}_{t,new}^{k|k}+1}$ to $s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}}$ in the matrix S_{k+1}.

We can conclude at this stage that in the matrix S_{k+1},

$$s_{1,\tilde{N}_{t,new}^{k|k}+1} = \cdots = s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}} = s_{2,1} = 2 \tag{4.15}$$

The last $\tilde{N}_{t,old}^{k|k}$ columns of S_{k+1} is,

$$S_{k+1}(:, \tilde{N}_{t,new}^{k|k} + 1 : \tilde{N}_t^{k|k}) \tag{4.16}$$

$$= \begin{bmatrix} 2 & \ldots & 2 & s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \cdots & s_{1,\tilde{N}_t^{k|k}} \\ s_{2,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}} & s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \cdots & s_{2,\tilde{N}_t^{k|k}} \\ \vdots & & & & & \vdots \\ s_{k+3,\tilde{N}_{t,new}^{k|k}+1} & \cdots & s_{k+3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}} & s_{k+3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \cdots & s_{k+3,\tilde{N}_t^{k|k}} \end{bmatrix}$$

where $S_{k+1}(:, \tilde{N}_{t,new}^{k|k} + 1 : \tilde{N}_t^{k|k}) \in \mathbb{N}^{(k+3)\times\tilde{N}_t^{k-1|k-1}}$

4.3.4 Part 4: The Second Child Terms from the First $\tilde{N}_{t,new}^{k-1|k-1}$ Old Parent Terms

Consider the second row from the $(\tilde{N}_{t,new}^{k|k}+1)^{th}$ column to the $(\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1})^{th}$ column. These entries represent the second child terms from the first $\tilde{N}_{t,new}^{k-1|k-1}$ old parent terms at step k.

Again, look at equation (4.13). For the second child terms at step $k+1$ from the first $\tilde{N}_{t,new}^{k-1|k-1}$ old parent term at step k, let $l=2$. Then,

$$\mathcal{E}_{i,2,2}^{k+1|k+1}=\mathcal{E}_{i,4,2}^{k+1|k+1}, \quad i=1,2,...,\tilde{N}_{t,new}^{k-1|k-1} \tag{4.17}$$

Look at the left hand side of equation (4.17). The term $\mathcal{E}_{i,2}^{k}$ for $i=1,2,...,\tilde{N}_{t,new}^{k-1|k-1}$ at step k are represented from $s_{2,1}=2$ through $s_{2,\tilde{N}_{t,new}^{k-1|k-1}}=\tilde{N}_{t,new}^{k-1|k-1}+1$ in the matrix S_k. Then the term $\mathcal{E}_{i,2,2}^{k+1|k+1}$ at step $k+1$ are represented from $s_{2,2}=3$ through $s_{2,\tilde{N}_{t,new}^{k-1|k-1}+1}=\tilde{N}_{t,new}^{k-1|k-1}+2$ in the matrix S_{k+1}, refer to equation (4.10). Again, the trick described in Remark 4.3.1 is used here to convert the three-subscript system for exponential terms to the two-subscript system for elements in the S matrix.

Now look at the right hand side of equation (4.17). The term $\mathcal{E}_{i,4}^{k}$ at step k for $i=1,2,...,\tilde{N}_{t,new}^{k-1|k-1}$ at step k are represented from $s_{4,1}=\tilde{N}_{t,new}^{k|k}+1$ through $s_{4,\tilde{N}_{t,new}^{k-1|k-1}}=\tilde{N}_{t,new}^{k|k}+N_{t,new}^{k-1|k-1}$ in the matrix S_k. Then the term $\mathcal{E}_{i,4,2}^{k+1|k+1}$ at step $k+1$ are represented from $s_{2,\tilde{N}_{t,new}^{k|k}+1}$ through $s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}}$ in the matrix S_{k+1}. Then,

$$s_{2,\tilde{N}_{t,new}^{k|k}+1}=3 \tag{4.18}$$

$$\cdots$$

$$s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}}=\tilde{N}_{t,new}^{k-1|k-1}+2 \tag{4.19}$$

Or,

$$s_{2,\tilde{N}_{t,new}^{k|k}+l}=2+l, \quad l=1,2,...,\tilde{N}_{t,new}^{k-1|k-1} \tag{4.20}$$

Therefore, equation (4.16) can be rewritten as,

$$S_{k+1}(:, \tilde{N}_{t,new}^{k|k} + 1 : \tilde{N}_t^{k|k})$$

$$= \begin{bmatrix} 2 & \dots & 2 & s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & s_{1,\tilde{N}_t^{k|k}} \\ 3 & \dots & \tilde{N}_{t,new}^{k-1|k-1}+2 & s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & s_{2,\tilde{N}_t^{k|k}} \\ \vdots & & & & & \vdots \\ s_{k+3,\tilde{N}_{t,new}^{k|k}+1} & \dots & s_{k+3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}} & s_{k+3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & s_{k+3,\tilde{N}_t^{k|k}} \end{bmatrix} \tag{4.21}$$

where $S_{k+1}(:, \tilde{N}_{t,new}^{k|k} + 1 : \tilde{N}_t^{k|k}) \in \mathbb{N}^{(k+3)\times\tilde{N}_t^{k-1|k-1}}$.

4.3.5 Part 5: Last Two Child Terms from All Parent Terms

The combination rules show that none of the last two child terms from any two distinct parent terms can be combined. Also, since we already exhausted the combination rules for the two-state case, all the non-zero entries in equation (4.21) except for the last 2 non-zero entries in each column will be the same with some entries in the second row of $S_{k+1}(:, 1 : \tilde{N}_{t,new}^{k|k})$ described in equation (4.10). This is illustrated in Fig. 3.1. In other words, no new integers will show up in the top right corner of the matrix S_{k+1}.

In addition, for the S matrix to have a simple and implementable recursion, one approach is to pile all the old terms to the end of the new terms. Newer terms come first, and older terms come after the newer ones. By giving the old child terms a consecutive sequence, the order of these child terms is kept exactly the way what their several-step old parent terms were.

Since no new integers show up in the top right corner of S_{k+1}, and we need to point the old child terms continuously, the indexing method for this part of S matrix becomes straightforward.

Let us assign the $(3, 1)$ entry in the matrix S_{k+1} to be $\tilde{N}_{t,new}^{k|k} + 2$, and assign the rest of the second last non-zero entry of each column sequentially by adding 1. Finally, assign the last non-zero entry of each column sequentially starting from the first column of S_{k+1} by adding 1 sequentially as well.

After these indexing, equation (4.10) and equation (4.21) can be rewritten in more detail.

$$S_{k+1}(:,1:\tilde{N}_{t,new}^{k|k}) = \begin{bmatrix} 1 & 1 & ... & 1 \\ 2 & 3 & ... & \tilde{N}_{t,new}^{k|k}+1 \\ \tilde{N}_{t,new}^{k|k}+2 & \tilde{N}_{t,new}^{k|k}+3 & ... & 2\tilde{N}_{t,new}^{k|k}+1 \\ \tilde{N}_{t,new}^{k+1|k+1}+1 & \tilde{N}_{t,new}^{k+1|k+1}+2 & ... & \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k} \\ 0 & ... & & 0 \\ \vdots & & & \vdots \\ 0 & ... & & 0 \end{bmatrix} \tag{4.22}$$

where $S_{k+1}(:,1:\tilde{N}_{t,new}^{k|k}) \in \mathbb{N}^{(k+3)\times\tilde{N}_{t,new}^{k|k}}$. Note that only the top four rows of $S_{k+1}(:,1:\tilde{N}_{t,new}^{k|k})$ have non-zero entries. For convenience, write the rest columns of S matrix in two parts, and fill in the last two child terms with ascending integers.

$$\begin{aligned} & S_{k+1}(:,\tilde{N}_{t,new}^{k|k}+1:\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}) \\ = & \begin{bmatrix} 2 & ... & 2 \\ 3 & ... & \tilde{N}_{t,new}^{k-1|k-1}+2 \\ s_{3,\tilde{N}_{t,new}^{k|k}+1} & ... & s_{3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}} \\ 2\tilde{N}_{t,new}^{k|k}+2 & ... & 2\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1 \\ \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k}+1 & ... & \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1} \\ 0 & ... & 0 \\ \vdots & & \vdots \\ 0 & ... & 0 \end{bmatrix} \end{aligned} \tag{4.23}$$

where $S_{k+1}(:, \tilde{N}_{t,new}^{k|k}+1 : \tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}) \in \mathbb{N}^{(k+3)\times \tilde{N}_{t,new}^{k-1|k-1}}$. And

$$S_{k+1}(:, \tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1 : \tilde{N}_t^{k|k})$$
$$= \begin{bmatrix} s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & \dots & s_{1,\tilde{N}_t^{k|k}} \\ s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & \dots & s_{2,\tilde{N}_t^{k|k}} \\ s_{3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & \dots & s_{3,\tilde{N}_t^{k|k}} \\ s_{4,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & \dots & s_{4,\tilde{N}_t^{k|k}} \\ 2\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+2 & \dots & \dots & \dots \\ \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1 & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots \\ \dots & & & \\ 0 & \dots & \dots & \tilde{N}_{t,new}^{k+1|k+1} \\ 0 & \dots & \dots & \tilde{N}_t^{k+1|k+1} \end{bmatrix} \tag{4.24}$$

where $S_{k+1}(:, \tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1 : \tilde{N}_t^{k|k}) \in \mathbb{N}^{(k+3)\times(\tilde{N}_t^{k|k}-\tilde{N}_{t,new}^{k|k}-\tilde{N}_{t,new}^{k-1|k-1})}$.

Look at equation(4.24). Newer old parent terms at step k produce less child terms at step $k+1$, and older old parent terms produce more child terms. These old parent terms are already in an order that older terms always come after newer terms. Hence the non-zero entries form a "staircase" shape. The entries at the bottom left corner are all zeros. Those at the top right corner are positive integers less than or equal to $\tilde{N}_{t,new}^{k|k}+1$. Between the 2 corners are these 2 "staircase"-shape rows, indicating the second last child terms and the old child terms at step $k+1$. Fig. 4.1 put together all parts of S_{k+1} and illustrates the "staircase" shape in the middle of the matrix.

4.3.6 Part 6: The Third Last Child Terms from All Old Parent Terms

In this subsection, consider the third last row of the last $\tilde{N}_{t,old}^{k|k}$ colomuns in S_{k+1}.

For all the $\tilde{N}_t^{k-1|k-1}$ terms at step $k-1$, suppose in the i^{th} term at step $k-1$, there are $N_{e,i}^{k-1|k-1}$ elements in the argument of the exponential.

Recall the second term combination rule for two-state case described in equation (3.45). At step $k+1$,

$$\mathcal{E}_{i,l,2}^{k+1|k+1} = \mathcal{E}_{i,N_{e,i}^{k-1|k-1}+2,l}^{k+1|k+1}, \quad \text{for } 1 \leq l \leq N_{e,i}^{k-1|k-1} + 1 \tag{4.25}$$

Look at the right hand side of the equation (4.25). Its parent term, $\mathcal{E}_{i,N_{e,i}^{k-1|k-1}+2}^{k|k}$, is the i^{th} old term at step k, which has $(N_{e,i}^{k-1|k-1}+1)$ elements. At step $k+1$, it will produce $(N_{e,i}^{k-1|k-1}+3)$ child terms. The third last child terms from all old parent terms at step k are the $\mathcal{E}_{i,N_{e,i}^{k-1|k-1}+2,N_{e,i}^{k-1|k-1}+1}^{k+1|k+1}$. This is the case when l in equation (4.25) is equal to,

$$l = N_{e,i}^{k-1|k-1} + 1 \tag{4.26}$$

When $1 \leq i \leq \tilde{N}_t^{k-1|k-1}$, the term $\mathcal{E}_{i,N_{e,i}^{k-1|k-1}+2}^{k|k}$ is represented as $s_{N_{e,i}^{k-1|k-1}+2,i} = \tilde{N}_{t,new}^{k|k} + i$ in the matrix S_k, in equation (4.22). Then at step $k+1$, the term $\mathcal{E}_{i,N_{e,i}^{k-1|k-1}+2,N_{e,i}^{k-1|k-1}+1}^{k+1|k+1}$ is represented as $s_{N_{e,i}^{k-1|k-1}+1,\tilde{N}_{t,new}^{k|k}+i}$ in the matrix S_{k+1}. These are exactly the third last child terms of all old parent terms in sequence, and occupy the third last non-zero entries of each of the last $\tilde{N}_t^{k-1|k-1}$ columns in S_{k+1}.

Now look at the left hand side of equation (4.25). At step k, the parent term $\mathcal{E}_{i,N_{e,i}^{k-1|k-1}+1}^{k|k}$ is represented as $s_{N_{e,i}^{k-1|k-1}+1,i} = \tilde{N}_{t,new}^{k-1|k-1}+1+i$ in the matrix S_k, from equation (4.22) – (4.24). Then at step $(k+1)$, the term $\mathcal{E}_{i,N_{e,i}^{k-1|k-1}+1,2}^{k+1|k+1}$ is represented as $s_{2,\tilde{N}_{t,new}^{k-1|k-1}+i+1} = \tilde{N}_{t,new}^{k-1|k-1}+i+2$ in the matrix S_{k+1}, from equation (4.22).

Therefore, combine the understanding of both sides of equation (4.25), we can get the third last non-zero entries of the last $\tilde{N}_t^{k-1|k-1}$ columns in the matrix S_{k+1} at step $k+1$.

$$s_{N_{e,i}^{k-1|k-1}+1,\tilde{N}_{t,new}^{k|k}+i} = \tilde{N}_{t,new}^{k-1|k-1} + i + 2, \quad 1 \leq i \leq \tilde{N}_t^{k-1|k-1} \tag{4.27}$$

With this knowledge, S_{k+1} can be rewritten as,

$$\begin{aligned} &S_{k+1}(:, \tilde{N}_{t,new}^{k|k}+1 : \tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}) \\ &= \begin{bmatrix} 2 & \dots & 2 \\ 3 & \dots & \tilde{N}_{t,new}^{k-1|k-1}+2 \\ \tilde{N}_{t,new}^{k-1|k-1}+3 & \dots & 2\tilde{N}_{t,new}^{k-1|k-1}+2 \\ 2\tilde{N}_{t,new}^{k|k}+2 & \dots & 2\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1 \\ \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k}+1 & \dots & \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \\ &\in \mathbb{N}^{(k+3)\times \tilde{N}_{t,new}^{k-1|k-1}} \end{aligned} \tag{4.28}$$

And

$$\begin{aligned} &S_{k+1}(:, \tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1 : \tilde{N}_{t}^{k|k}) \\ &= \begin{bmatrix} s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots\ \dots & s_{1,\tilde{N}_{t}^{k|k}} \\ s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots\ \dots & s_{2,\tilde{N}_{t}^{k|k}} \\ s_{3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots\ \dots & s_{3,\tilde{N}_{t}^{k|k}} \\ 2\tilde{N}_{t,new}^{k-1|k-1}+3 & \dots\ \dots & s_{4,\tilde{N}_{t}^{k|k}} \\ 2\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+2 & \dots\ \dots & \dots \\ \tilde{N}_{t,new}^{k+1|k+1}+\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1 & \dots\ \dots & \dots \\ \dots & & \\ \dots & & \tilde{N}_{t,new}^{k-1|k-1}+\tilde{N}_{t}^{k-1|k-1}+2 \\ 0 & \dots\ \dots & \tilde{N}_{t,new}^{k+1|k+1} \\ 0 & \dots\ \dots & \tilde{N}_{t}^{k+1|k+1} \end{bmatrix} \\ &\in \mathbb{N}^{(k+3)\times(\tilde{N}_{t}^{k|k}-\tilde{N}_{t,new}^{k|k}-\tilde{N}_{t,new}^{k-1|k-1})} \end{aligned} \tag{4.29}$$

Refer to equation (3.49),

$$\tilde{N}_{t,new}^{k|k} = \tilde{N}_{t}^{k-1|k-1} + \tilde{N}_{t,new}^{k-1|k-1} + 1, \quad k \geq 2 \tag{4.30}$$

Hence,

$$\tilde{N}_{t,new}^{k-1|k-1} + \tilde{N}_{t}^{k-1|k-1} + 2 = \tilde{N}_{t,new}^{k|k} + 1 \tag{4.31}$$

Then equation (4.29) is,

$$\begin{aligned}
&S_{k+1}(:, \tilde{N}_{t,new}^{k|k} + \tilde{N}_{t,new}^{k-1|k-1} + 1 : \tilde{N}_{t}^{k|k}) \\
&= \begin{bmatrix}
s_{1,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & \dots & s_{1,\tilde{N}_{t}^{k|k}} \\
s_{2,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & \dots & s_{2,\tilde{N}_{t}^{k|k}} \\
s_{3,\tilde{N}_{t,new}^{k|k}+\tilde{N}_{t,new}^{k-1|k-1}+1} & \dots & \dots & s_{3,\tilde{N}_{t}^{k|k}} \\
2\tilde{N}_{t,new}^{k-1|k-1} + 3 & \dots & \dots & s_{4,\tilde{N}_{t}^{k|k}} \\
2\tilde{N}_{t,new}^{k|k} + \tilde{N}_{t,new}^{k-1|k-1} + 2 & \dots & \dots & \dots \\
\tilde{N}_{t,new}^{k+1|k+1} + \tilde{N}_{t,new}^{k|k} + \tilde{N}_{t,new}^{k-1|k-1} + 1 & \dots & \dots & \dots \\
\dots & & & \\
\dots & & & \tilde{N}_{t,new}^{k|k} + 1 \\
0 & \dots & \dots & \tilde{N}_{t,new}^{k+1|k+1} \\
0 & \dots & \dots & \tilde{N}_{t}^{k+1|k+1}
\end{bmatrix} \\
&\in \mathbb{N}^{(k+3)\times(\tilde{N}_{t}^{k|k} - \tilde{N}_{t,new}^{k|k} - \tilde{N}_{t,new}^{k-1|k-1})}
\end{aligned} \tag{4.32}$$

4.3.7 Part 7: The "Top Right Corner" of S_{k+1}

One should realize that the only thing left to be determined is the top right corner of S_{k+1}, i.e. the upper region of $S_{k+1}(:, \tilde{N}_{t,new}^{k|k} + \tilde{N}_{t,new}^{k-1|k-1} + 1 : \tilde{N}_{t}^{k|k})$ in equation (4.32). As we introduced this notion earlier, the S_{k+1} can be split into different areas, illustrated in Fig. 4.1. In particular, the last two child terms at step $k+1$ of each of the parent terms at step k form a "staircase"-shape, which split the rest of the S matrix into its bottom left corner and top right corner. The bottom left corner should be all zeros. Regarding the top right corner of S_{k+1} at step $k+1$, suppose that the i^{th} term at step $k-1$ have $N_{e,i}^{k-1|k-1}$ elements. Each of them produces an old term at step k, and contain $(N_{e,i}^{k-1|k-1} + 1)$ elements. Then at step $k+1$, it produce $(N_{e,i}^{k-1|k-1} + 3)$ child terms. The top right corner of S_{k+1} represents the first $(N_{e,i}^{k-1|k-1} + 1)$ child terms.

According to the second combination rule, these terms at the last $\tilde{N}_t^{k-1|k-1}$ columns of the top right corner will combine with the terms at the second row of the first $\tilde{N}_{t,new}^{k|k}$ columns in $S_{k+1|k+1}$. Note that the total number of columns of S_{k+1} is $\tilde{N}_t^{k-1|k-1} + \tilde{N}_{t,new}^{k|k} = \tilde{N}_t^{k|k}$. Therefore, all the numbers in the top right corner of S_{k+1} should be greater than or equal to 2, and less than or equal to $\tilde{N}_{t,new}^{k|k} + 1$.

Next, we need to introduce the following lemma.

Lemma 4.3.2. *Consider the exponential part of the i^{th} term, $\mathcal{E}_i^{k-1|k-1}$, at step $k-1$ and the m^{th} term, $\mathcal{E}_m^{k-1|k-1}$ at step $k-1$, then the l^{th} child term at step k of $\mathcal{E}_i^{k-1|k-1}$ and the p^{th} child term at step k of $\mathcal{E}_m^{k-1|k-1}$ can be combined if and only if the l^{th} grandchild term at step $k+1$ of the old child term $\mathcal{E}_{i,old}^{k|k}$ at step k and the p^{th} grandchild term at step $k+1$ of the old child term $\mathcal{E}_{m,old}^{k|k}$ at step k can be combined, i.e.*

$$\mathcal{E}_{i,l}^{k|k} = \mathcal{E}_{m,p}^{k|k} \quad \textit{if and only if} \quad \mathcal{E}_{i,old,l}^{k+1|k+1} = \mathcal{E}_{m,old,p}^{k+1|k+1} \tag{4.33}$$

The proof is straightforward, shown in the Appendix D.

Lemma 4.3.2 indicates that for which ever terms that can be combined at step k, the corresponding child terms at step $k+1$ of its old parent terms combine. If any two numbers in S_k are the same, then in S_{k+1}, the top right corner starting from the $(\tilde{N}_{t,new}^{k|k}+1)^{th}$ column will show the same pattern: the two numbers at the corresponding places in S_{k+1} are also the same.

Observe equation (4.22), (4.28), and (4.32) closely. Only the top right corner in equation (4.32) is unknown. The known integers in the top right corner of S_{k+1} already cover all integers from 2 to $\tilde{N}_{t,new}^{k|k} + 1$. Hence every unknown integer in part 7 of the S matrix must be identical with some integers that has been determined in part 1 to part 6 in earlier subsections. Then, our statement can be stronger. Not only that the two numbers at the corresponding places in S_{k+1} are the same, but also we can uniquely determine the integers at each unknown places in part 7.

Note that the top 3 rows of $S_{k+1}(:, \tilde{N}_{t,new}^{k|k} + 1 : \tilde{N}_{t,new}^{k|k} + \tilde{N}_{t,new}^{k-1|k-1})$ is really copying the pattern of the top 3 rows in $S_k(:, 1 : \tilde{N}_{t,new}^{k-1|k-1})$ by adding 1 on each entry. Let us assume that the last $\tilde{N}_t^{k-2|k-2}$ columns of the top right corner in S_k copy S_{k-1} except its last non-zero entry at each column by adding 1. Then the last $\tilde{N}_t^{k-1|k-1}$ columns of the top right corner in S_{k+1} copy S_k except its last non-zero entry at each column by adding 1 as well. Since these entries are the only entries left to be determined, this assumption is proved immediately by itself in a recursive manner.

Up till now, we have analytically derived the recursive structure for S matrix for two-state systems.

Remark 4.3.3. This approach to derive the two-state case S matrix recursion is tricky in the sense that each part of the S matrix is constructed sequentially. The proof of the parts of S in earlier subsections turns out to be the premise of the proof presented in its following subsections.

$$S_{k+1} = \begin{bmatrix} & & & & & \textit{Non-Zero Elements} & & & & \\ \tilde{N}^{k|k}_{t,new}+2 & ... & 2\tilde{N}^{k|k}_{t,new}+1 & & & & & & & \\ \tilde{N}^{k+1|k+1}_{t,new}+1 & ... & \tilde{N}^{k+1|k+1}_{t,new}+\tilde{N}^{k|k}_{t,new} & 2\tilde{N}^{k|k}_{t,new}+2 & ... & 2\tilde{N}^{k|k}_{t,new}+\tilde{N}^{k-1|k-1}_{t,new}+1 & & & & \\ 0 & ... & 0 & \tilde{N}^{k+1|k+1}_{t,new}+\tilde{N}^{k|k}_{t,new}+1 & ... & \tilde{N}^{k+1|k+1}_{t,new}+\tilde{N}^{k|k}_{t,new}+\tilde{N}^{k-1|k-1}_{t,new} & 2\tilde{N}^{k|k}_{t,new}+\tilde{N}^{k-1|k-1}_{t,new}+2 & ... & ... & ... \\ 0 & ... & 0 & 0 & ... & 0 & \tilde{N}^{k+1|k+1}_{t,new}+\tilde{N}^{k|k}_{t,new}+\tilde{N}^{k-1|k-1}_{t,new}+1 & ... & ... & ... \\ 0 & ... & 0 & 0 & ... & 0 & 0 & ... & ... & ... \\ \vdots & & & \vdots & \vdots & & \vdots & & & \\ 0 & ... & 0 & 0 & ... & 0 & 0 & ... & ... & \tilde{N}^{k+1|k+1}_{t,new} \\ 0 & ... & 0 & 0 & ... & 0 & 0 & ... & ... & \tilde{N}^{k+1|k+1}_{t} \end{bmatrix}$$

Figure 4.1: The “staircase” shape of S matrix

CHAPTER 5

G Terms

In this chapter, the G terms, i.e. the coefficient terms $G_i^{k|k}(\nu)$ in equation (1.14) are reconstructed. As described in [5], the G terms were found to be in a fractional form, of which each layer contains an imaginary and a real component. The real component is determined by a sum of real scalars, so called "offsets", and some sign functions scaled by coefficients. The real component is independent of the measurements, while the imaginary component is a function of the measurements. The new structure of G terms updates the offsets and the sign functions as well as the imaginary component in a recursive manner as k increases. Many zeros are added artificially into the structure due to the update integral discussed in [5]. By introducing this new structure, we are able to eliminate all the redundant zeros, hence reduce the computing and memory requirement. Next, a comprehensive study of the G terms for two-state case is presented, revealing the interesting property that in each layer of G of a new term, there are at most three non-zero elements in the real component. Furthermore, this approach of breaking down the G terms provides the separation of the part of structure that is independent of the measurement history with the part that is relevant to the measurements. This allows the offline - online implementation, presented later in Chpater 6.

5.1 General Structure of G Terms

In this section, a general structure of the G terms is proposed. Recall the recursive form of the G terms in equation (1.16). The coefficient functions $g_i^{k|k}(\cdot)$ at step k are functions of $g_{r_i^{k|k}}^{k-1|k-1}(\cdot)$ at step $k-1$, where $r_i^{k|k}$ is the index of the parent terms. If we rewrite $g_{r_i^{k|k}}^{k-1|k-1}(\cdot)$

with respect to the coefficient functions at earlier steps and keep expanding the numerator of the functional form of G, eventually equation (1.16) can be expressed as a multi-layer structure, shown as follow,

$$G^{k|k} = \frac{1}{(2\pi)^k} \left\{ \frac{\frac{\frac{\frac{1}{jIm^{(1)}+R^{(1)}_{k,1}} - \frac{1}{jIm^{(1)}+R^{(1)}_{k,2}}}{} - \cdots}{\vdots}}{jIm^{(k-1)}+R^{(k-1)}_{k,1}} - \cdots}{jIm^{(k)} + R^{(k)}_{k,1}} - \frac{\frac{\cdots}{jIm^{(k-1)}+R^{(k-1)}_{k,3}} - \frac{\cdots}{jIm^{(k-1)}+R^{(k-1)}_{k,4}}}{jIm^{(k)} + R^{(k)}_{k,2}} \right\} \tag{5.1}$$

The notation $R^{(m)}_{k,l}$ represents the real part of the corresponding denominator, and $Im^{(m)}$ represents the imaginary part. The superscript (m) represents the m^{th} layer from the top. The subscript k means this term is at step k. The second subscript of $R^{(m)}_{k,l}$ represents the l^{th} element in the sequence. Each scalar $R^{(m)}_{k,l}$ is a linear combination of the offsets and coefficients that multiply sign functions. The real part of the G terms, $R^{(m)}_k$, can be formulated as the product of a vector $\rho^{(m)}_k$ and a matrix $F^{(m)}_k$, i.e.

$$R^{(m)}_k = \left[R^{(m)}_{k,1}, R^{(m)}_{k,2}, \cdots\right] = \rho^{(m)}_k \cdot F^{(m)}_k \tag{5.2}$$

where the dimension varies with specific term.

Separate ρ and F into the offset component ρ_o, F_o and the sign function component ρ_c, F_c. Let ,

$$\rho^{(m)}_k = \left[\rho^{(m)}_{ok} \quad | \quad \rho^{(m)}_{ck}\right], \quad F^{(m)}_k = \begin{bmatrix} F^{(m)}_{ok} \\ --- \\ F^{(m)}_{ck} \end{bmatrix}, \tag{5.3}$$

The offset component, $F^{(m)}_{ok}$, is a matrix that contains only 1 and -1. The sign function component, $F^{(m)}_{ck}$, is in the following,

$$F^{(m)}_{ck} = \begin{bmatrix} s_1 & \cdots & s_1 \\ \vdots & & \vdots \\ s_q & \cdots & s_q \end{bmatrix}, \quad 1 \leq q \leq N^{k|k}_e \tag{5.4}$$

where $s_i = sgn\left(B_i^{k|k}\nu\right)$, $1 \le i \le q$.

The imaginary part of each layer $Im^{(m)}$ will not change from step to step. Once they are produced during a certain step, they stay fixed as functions of the measurement sequence. At each layer for any particular term, all the imaginary parts are the same.

There are three major benefit from restructuring the G terms into this form and, in particular, the split of the offset component and the sign function component in the real part of each G layer. First, the emphasis on the real part R and the imaginary part Im in each layer allows separation of the component of the estimator structure, that is independent of the measurement data, from the component that is dependent on the measurements. This contributes to a pre-computational technique set-up, aimed at improving online computational efficiency, which will be discussed in a later section. Second, the split of the offset and sign function successfully eliminates all the zeros that have been artificially added into the offsets during the update process. These artificial zeros were brought into the coefficient of the exponential so as to simplify the integration formula in [5]. Therefore, many of the offsets are zero, and those zeros cannot be distinguished from non-zero offsets treated via the existing method. Third, the G terms discussed in this section provides a better understanding of the fundamental structure. One interesting property is that for a two-state system, for any new term, there are at most 3 non-zero elements in the sequence $\rho = [\rho_o \quad | \quad \rho_c]$. This fact is uncovered by derivations of recursive update of the G terms using the proposed new structure, presented in the following section which concludes this chapter.

5.2 A Comprehensive Study on Two-State Case

In this section, the recursion of ρ and F for two-state systems is completely analyzed. For two-state systems, there are no more than three non-zero elements in the sequence of ρ in any layer of any new term. To show this property, the sequence ρ and the matrix F are not split apart. However, they will be split into the offset part and the sign function part when implemented in the offline - online structure in latter sections. We first present the update at $k = 1$ in Section 5.2.1 and for $k = 2$ in Section 5.2.2. Then by induction, the general form of G is obtained in Section 5.2.3 for new terms and Section 5.2.4 for old terms. Then a recursion is developed starting in Section 5.2.5 to Section 5.2.7. The interesting property that no more than three non-zero elements in the sequence of ρ in any layer of any new term is presented in Section 5.2.8.

5.2.1 1^{st} Measurement Update

Refering to [5], the CF is obtained from the update integral at step $k = 1$.

$$\begin{aligned}\bar{\phi}_{X_1|Z_1}(\nu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_{X_1}(\nu - H^T\eta)\phi_V(-\eta)e^{jz_1\eta}d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\alpha_1\left|e_1H^T\right|\left|\frac{e_1\nu}{e_1H^T}-\eta\right| - \alpha_2\left|e_2H^T\right|\left|\frac{e_2\nu}{e_2H^T}-\eta\right|\right. \\ &\qquad \left. -\gamma\left|-\eta\right| + jz_1\eta\right)d\eta \\ &= \sum_{i=1}^{3} G_i^{1|1}(\nu)\cdot\mathcal{E}_i^{1|1}(\nu)\end{aligned} \tag{5.5}$$

There are 3 terms at step $k = 1$. The exponential part of these three terms are,

$$\mathcal{E}_1^{1|1} = \exp\left(-\frac{\alpha_2}{|e_1H^T|}\left|H\mathbf{A}\nu\right| - \gamma\left|-\frac{e_1\nu}{e_1H^T}\right| + j\zeta_1^{1|1}\nu\right) \tag{5.6}$$

$$\mathcal{E}_2^{1|1} = \exp\left(-\frac{\alpha_1}{|e_2H^T|}\left|H\mathbf{A}\nu\right| - \gamma\left|-\frac{e_2\nu}{e_2H^T}\right| + j\zeta_2^{1|1}\nu\right) \tag{5.7}$$

$$\mathcal{E}_3^{1|1} = \exp\left(-\alpha_1\left|e_1\nu\right| - \alpha_2\left|e_2\nu\right| + j\zeta_3^{1|1}\nu\right) \tag{5.8}$$

The G part are as follows.

$$
\begin{aligned}
G_1^{1|1} = \frac{1}{2\pi} \Bigg\{ & \frac{1}{jz_1 + \alpha_1 \left|e_1 H^T\right| + \alpha_2 \left|e_2 H^T\right| sgn\left(\frac{e_2\nu}{e_2 H^T} - \frac{e_1\nu}{e_1 H^T}\right) + \gamma sgn\left(0 - \frac{e_1\nu}{e_1 H^T}\right)} \\
& - \frac{1}{jz_1 - \alpha_1 \left|e_1 H^T\right| + \alpha_2 \left|e_2 H^T\right| sgn\left(\frac{e_2\nu}{e_2 H^T} - \frac{e_1\nu}{e_1 H^T}\right) + \gamma sgn\left(0 - \frac{e_1\nu}{e_1 H^T}\right)} \Bigg\} \\
= \frac{1}{2\pi} \Bigg\{ & \frac{1}{jz_1 + \alpha_1 \left|e_1 H^T\right| + \alpha_2 \left|e_2 H^T\right| sgn\left(e_1 H^T \cdot e_2 H^T\right) sgn\left(H\mathbf{A}\nu\right) + \gamma sgn\left(-\frac{e_1\nu}{e_1 H^T}\right)} \\
& - \frac{1}{jz_1 - \alpha_1 \left|e_1 H^T\right| + \alpha_2 \left|e_2 H^T\right| sgn\left(e_1 H^T \cdot e_2 H^T\right) sgn\left(H\mathbf{A}\nu\right) + \gamma sgn\left(-\frac{e_1\nu}{e_1 H^T}\right)} \Bigg\}
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
G_2^{1|1} = \frac{1}{2\pi} \Bigg\{ & \frac{1}{jz_1 + \alpha_2 \left|e_2 H^T\right| + \alpha_1 \left|e_1 H^T\right| sgn\left(\frac{e_1\nu}{e_1 H^T} - \frac{e_2\nu}{e_2 H^T}\right) + \gamma sgn\left(0 - \frac{e_2\nu}{e_2 H^T}\right)} \\
& - \frac{1}{jz_1 - \alpha_2 \left|e_2 H^T\right| + \alpha_1 \left|e_1 H^T\right| sgn\left(\frac{e_1\nu}{e_1 H^T} - \frac{e_2\nu}{e_2 H^T}\right) + \gamma sgn\left(0 - \frac{e_2\nu}{e_2 H^T}\right)} \Bigg\} \\
= \frac{1}{2\pi} \Bigg\{ & \frac{1}{jz_1 + \alpha_2 \left|e_2 H^T\right| + \alpha_1 \left|e_1 H^T\right| sgn\left(-e_1 H^T \cdot e_2 H^T\right) sgn\left(H\mathbf{A}\nu\right) + \gamma sgn\left(-\frac{e_2\nu}{e_2 H^T}\right)} \\
& - \frac{1}{jz_1 - \alpha_2 \left|e_2 H^T\right| + \alpha_1 \left|e_1 H^T\right| sgn\left(-e_1 H^T \cdot e_2 H^T\right) sgn\left(H\mathbf{A}\nu\right) + \gamma sgn\left(-\frac{e_2\nu}{e_2 H^T}\right)} \Bigg\}
\end{aligned} \tag{5.10}
$$

$$G_3^{1|1} = \frac{1}{2\pi}\left\{\frac{1}{jz_1 + \gamma + \alpha_1 \left|e_1 H^T\right| sgn\left(\frac{e_1\nu}{e_1 H^T} - 0\right) + \alpha_2 \left|e_2 H^T\right| sgn\left(\frac{e_2\nu}{e_2 H^T} - 0\right)}\right.$$

$$\left. - \frac{1}{jz_1 - \gamma + \alpha_1 \left|e_1 H^T\right| sgn\left(\frac{e_1\nu}{e_1 H^T} - 0\right) + \alpha_2 \left|e_2 H^T\right| sgn\left(\frac{e_2\nu}{e_2 H^T} - 0\right)}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{jz_1 + \gamma + \alpha_1 \left|e_1 H^T\right| sgn\left(\frac{e_1\nu}{e_1 H^T}\right) + \alpha_2 \left|e_2 H^T\right| sgn\left(\frac{e_2\nu}{e_2 H^T}\right)}\right.$$

$$\left. - \frac{1}{jz_1 - \gamma + \alpha_1 \left|e_1 H^T\right| sgn\left(\frac{e_1\nu}{e_1 H^T}\right) + \alpha_2 \left|e_2 H^T\right| sgn\left(\frac{e_2\nu}{e_2 H^T}\right)}\right\} \tag{5.11}$$

Rewrite the G parts in the following form.

$$G_i^{1|1} = \frac{1}{2\pi}\left\{\frac{1}{jz_1 + R_{1,1}^{(1)}(i)} - \frac{1}{jz_1 + R_{1,2}^{(1)}(i)}\right\} \tag{5.12}$$

where R is the real component of the denominator, structured as,

$$R_1^{(1)}(i) = \left[R_{1,1}^{(1)}(i), \quad R_{1,2}^{(1)}(i)\right] = \rho_1^{(1)}(i) \cdot F_1^{(1)}(i) \tag{5.13}$$

Now look at the three terms at step $k = 1$. When $i = 1$,

$$R_{1,1}^{(1)}(i=1) = \alpha_1 \left|e_1 H^T\right| + \alpha_2 \left|e_2 H^T\right| sgn\left(e_1 H^T \cdot e_2 H^T\right) sgn\left(HA\nu\right) + \gamma sgn\left(-\frac{e_1\nu}{e_1 H^T}\right) \tag{5.14}$$

and

$$R_{1,2}^{(1)}(i=1) = -\alpha_1 \left|e_1 H^T\right| + \alpha_2 \left|e_2 H^T\right| sgn\left(e_1 H^T \cdot e_2 H^T\right) sgn\left(HA\nu\right) + \gamma sgn\left(-\frac{e_1\nu}{e_1 H^T}\right) \tag{5.15}$$

If we let $s_1 = sgn(H\mathbf{A}\nu)$, $s_2 = sgn(-\frac{e_1\nu}{e_1 H^T})$ and let the sequence $\rho_1^{(1)}(i=1)$ and the matrix $F_1^{(1)}(i=1)$ be,

$$\begin{aligned}\rho_1^{(1)}(i=1) &= \left[\alpha_1 \left|e_1 H^T\right|, \quad \alpha_2 \left|e_2 H^T\right| \cdot sgn(e_1 H^T \cdot e_2 H^T), \quad \gamma\right] \\ &= \left[\alpha_1 \left|e_1 H^T\right|, \quad \alpha_2 \left(e_2 H^T\right) \cdot sgn(e_1 H^T), \quad \gamma\right]\end{aligned} \tag{5.16}$$

$$F_1^{(1)}(i=1) = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix}, \tag{5.17}$$

then

$$R_1^{(1)}(i=1) = \left[R_{1,1}^{(1)}(i=1), R_{1,2}^{(1)}(i=1)\right] = \rho_1^{(1)}(i=1) \cdot F_1^{(1)}(i=1) \tag{5.18}$$

Similarly, when $i = 2$, from equation (5.10), one obtains,

$$\begin{aligned}\rho_1^{(1)}(i=2) &= \left[\alpha_2 \left|e_2 H^T\right|, \quad \alpha_1 \left|e_1 H^T\right| \cdot sgn(-e_1 H^T \cdot e_2 H^T), \quad \gamma\right] \\ &= \left[\alpha_2 \left|e_2 H^T\right|, \quad \alpha_1 \left(e_1 H^T\right) \cdot sgn(-e_2 H^T), \quad \gamma\right]\end{aligned} \tag{5.19}$$

$$F_1^{(1)}(i=1) = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{5.20}$$

where $s_1 = sgn(H\mathbf{A}\nu)$, $s_2 = sgn(-\frac{e_2\nu}{e_2 H^T})$.

Finally, when $i = 3$, from equation (5.11), one gets,

$$\rho_1^{(1)}(i=3) = \left[\gamma, \quad \alpha_1 \left(e_1 H^T\right), \quad \alpha_2 \left(e_2 H^T\right)\right] \tag{5.21}$$

$$F_1^{(1)}(i=3) = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{5.22}$$

where $s_1 = sgn(e_1\nu)$, $s_2 = sgn(e_2\nu)$.

5.2.2 2^{nd} Measurement Update

Consider the i^{th} term at the first measurement update, and express it in the following form.

$$\mathcal{E}_i^{1|1} = \exp\left(-P_{i,1}^{1|1} \left|B_{i,1}^{1|1}\nu\right| - P_{i,2}^{1|1} \left|B_{i,2}^{1|1}\nu\right| + j\zeta_i^{1|1}\nu\right) \tag{5.23}$$

$$G_i^{1|1} = \frac{1}{2\pi}\left\{\frac{1}{jz_1 + R_{1,1}^{(1)}(i)} - \frac{1}{jz_1 + R_{1,2}^{(1)}(i)}\right\} \tag{5.24}$$

$$R_1^{(1)}(i) = \left[R_{1,1}^{(1)}(i), \quad R_{1,2}^{(1)}(i)\right] = \rho_1^{(1)}(i) \cdot F_1^{(1)}(i) \tag{5.25}$$

$$\rho_1^{(1)}(i) = \left[\rho_{1,1}^{(1)}(i), \quad \rho_{1,2}^{(1)}(i), \quad \rho_{1,3}^{(1)}(i)\right] \tag{5.26}$$

$$F_1^{(1)}(i) = \begin{bmatrix} 1 & -1 \\ s_1(i) & s_1(i) \\ s_2(i) & s_2(i) \end{bmatrix} \tag{5.27}$$

where $s_1(i) = sgn(B_1\nu)$, $s_2(i) = sgn(B_2\nu)$.

At the 2^{nd} measurement update, each parent term will produce 4 child terms. The l^{th} child term has the G part expressed in the following form.

$$G_{i,l}^{2|2} = \frac{1}{(2\pi)^2}\left\{\frac{\frac{1}{jz_1+R_{2,1}^{(1)}(i,l)} - \frac{1}{jz_1+R_{2,2}^{(1)}(i,l)}}{j(z_2-\zeta_i^{2|1}H^T)+R_{2,1}^{(2)}(i,l)} - \frac{\frac{1}{jz_1+R_{2,3}^{(1)}(i,l)} - \frac{1}{jz_1+R_{2,4}^{(1)}(i,l)}}{j(z_2-\zeta_i^{2|1}H^T)+R_{2,2}^{(2)}(i,l)}\right\} \tag{5.28}$$

5.2.2.1 The First Child Term $i = 1$

Using the first child term as an example, we will show how the sequence of R in the denominators are updated.

At step $k = 1$, the first and the only layer of the i^{th} term has the sequence R as follows. The notation omits the "(i)" for simplicity.

$$R_{1,1}^{(1)} = \rho_{1,1}^{(1)} + \rho_{1,2}^{(1)} \cdot s_1 + \rho_{1,3}^{(1)} \cdot s_2 \tag{5.29}$$

$$R_{1,2}^{(1)} = -\rho_{1,1}^{(1)} + \rho_{1,2}^{(1)} \cdot s_1 + \rho_{1,3}^{(1)} \cdot s_2 \tag{5.30}$$

At step $k = 2$, the fractional form now has two layers as shown in (5.28). The first (top) layer $R_2^{(1)}$ at step $k = 2$ is updated from the layer $R_1^{(1)}$ at step $k = 1$ as

$$R_2^{(1)} = \begin{bmatrix} R_{2,1}^{(1)}, & R_{2,2}^{(1)}, & R_{2,3}^{(1)}, & R_{2,4}^{(1)} \end{bmatrix} = \rho_2^{(1)} \cdot F_2^{(1)} \tag{5.31}$$

and

$$\begin{aligned} R_{2,1}^{(1)} =& \rho_{1,1}^{(1)} + \rho_{1,2}^{(1)} sgn\left(B_1^{1|1}\Phi^T H^T\right) \\ &+ \rho_{1,3}^{(1)} sgn\left(B_2^{1|1}\Phi^T H^T\right) \cdot sgn\left(\frac{B_2^{1|1}\Phi^T\nu}{B_2^{1|1}\Phi^T H^T} - \frac{B_1^{1|1}\Phi^T\nu}{B_1^{1|1}\Phi^T H^T}\right) \end{aligned} \tag{5.32}$$

where $\rho_{1,2}^{(1)} sgn\left(B_1^{1|1}\Phi^T H^T\right)$ in the above equation is the new offset. The reason why there is a sign function $sgn\left(B_1^{1|1}\Phi^T H^T\right)$ is that in the update integral, the variable $(\nu - H^T\eta)$

extracts the factor $\left(B_1^{1|1}\Phi^T H^T\right)$ out of the absolute value, and leaves the direction in the form of $\frac{B_1^{1|1}\Phi^T}{B_1^{1|1}\Phi^T H^T}$ for further operations.

From Chapter 2, the direction $\left(\frac{B_2^{1|1}\Phi^T}{B_2^{1|1}\Phi^T H^T} - \frac{B_1^{1|1}\Phi^T}{B_1^{1|1}\Phi^T H^T}\right)$ will be align with the direction of $H\mathbf{A}$, i.e.,

$$\frac{B_2^{1|1}\Phi^T}{B_2^{1|1}\Phi^T H^T} - \frac{B_1^{1|1}\Phi^T}{B_1^{1|1}\Phi^T H^T} = c \cdot H\mathbf{A} \tag{5.33}$$

where c is a scalar. However, c can be either positive or negative.

$H\mathbf{A}$ serves as the fundamental direction in the two-state case. Whenever we have co-aligned directions, we always want to represent them in the $H\mathbf{A}$ format by scaling that direction appropriately. If we want to "convert" the sign function in equation (5.32) into $sgn(H\mathbf{A}\nu)$, we must deal with the sign of c. One way to represent the sign of c is to compare the first element of direction $\left(\frac{B_2^{1|1}\Phi^T}{B_2^{1|1}\Phi^T H^T} - \frac{B_1^{1|1}\Phi^T}{B_1^{1|1}\Phi^T H^T}\right)$ and the first element of the direction $H\mathbf{A}$, i.e.,

$$sgn(c) = sgn\left(\frac{\left(\frac{B_2^{1|1}\Phi^T}{B_2^{1|1}\Phi^T H^T} - \frac{B_1^{1|1}\Phi^T}{B_1^{1|1}\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right) \tag{5.34}$$

Define $sgn(c)$ to be a scalar t_1, i.e. $t_1 = sgn(c)$. Looking at (5.32),

$$R_{2,1}^{(1)} = \rho_{1,1}^{(1)} + \rho_{1,2}^{(1)} sgn\left(B_1^{1|1}\Phi^T H^T\right) + \rho_{1,3}^{(1)} sgn\left(B_2^{1|1}\Phi^T H^T\right) \cdot t_1 \cdot sgn\left(H\mathbf{A}\nu\right). \tag{5.35}$$

Conducting similar analysis, we can obtain the rest of the R sequence as

$$R_{2,2}^{(1)} = -\rho_{1,1}^{(1)} + \rho_{1,2}^{(1)} sgn\left(B_1^{1|1}\Phi^T H^T\right) + \rho_{1,3}^{(1)} sgn\left(B_2^{1|1}\Phi^T H^T\right) \cdot t_1 \cdot sgn\left(H\mathbf{A}\nu\right), \tag{5.36}$$

$$R_{2,3}^{(1)} = \rho_{1,1}^{(1)} - \rho_{1,2}^{(1)} sgn\left(B_1^{1|1}\Phi^T H^T\right) + \rho_{1,3}^{(1)} sgn\left(B_2^{1|1}\Phi^T H^T\right) \cdot t_1 \cdot sgn\left(H\mathbf{A}\nu\right), \tag{5.37}$$

$$R_{2,4}^{(1)} = -\rho_{1,1}^{(1)} - \rho_{1,2}^{(1)} sgn\left(B_1^{1|1}\Phi^T H^T\right) + \rho_{1,3}^{(1)} sgn\left(B_2^{1|1}\Phi^T H^T\right) \cdot t_1 \cdot sgn\left(H\mathbf{A}\nu\right). \tag{5.38}$$

Organize the R sequences into the form of $\rho \cdot F$. Then,

$$\begin{aligned} \rho_2^{(1)} &= \left[\rho_{2,1}^{(1)}, \quad \rho_{2,2}^{(1)}, \quad \rho_{2,3}^{(1)}\right] \\ &= \left[\rho_{1,1}^{(1)}, \quad \rho_{1,2}^{(1)} \cdot sgn(B_1^{1|1}\Phi^T H^T), \quad \rho_{1,3}^{(1)} \cdot sgn(B_2^{1|1}\Phi^T H^T) \cdot t_1\right] \end{aligned} \tag{5.39}$$

$$F_2^{(1)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \end{bmatrix} \tag{5.40}$$

where $s_1 = sgn(H\mathbf{A}\nu)$.

The second (bottom) layer is,

$$R_2^{(2)} = \left[R_{2,1}^{(2)}, \quad R_{2,2}^{(2)} \right] = \rho_2^{(2)} \cdot F_2^{(2)} \tag{5.41}$$

Again, look at the first element of the $R_2^{(2)}$ sequence.

$$\begin{aligned} R_{2,1}^{(2)} = P_1^{1|1} \left| B_1^{1|1} \Phi^T H^T \right| + P_2^{1|1} \left| B_2^{1|1} \Phi^T H^T \right| sgn \left(\frac{B_2^{1|1} \Phi^T \nu}{B_2^{1|1} \Phi^T H^T} - \frac{B_1^{1|1} \Phi^T \nu}{B_1^{1|1} \Phi^T H^T} \right) \\ + \beta \left| \Gamma^T H^T \right| sgn \left(\frac{\Gamma^T \Phi^T \nu}{\Gamma^T H^T} - \frac{B_1^{1|1} \Phi^T \nu}{B_1^{1|1} \Phi^T H^T} \right) + \gamma sgn \left(0 - \frac{B_1^{1|1} \Phi^T \nu}{B_1^{1|1} \Phi^T H^T} \right) \end{aligned} \tag{5.42}$$

Define,

$$t_2 = sgn \left(\frac{\left(\frac{\Gamma^T}{\Gamma^T H^T} - \frac{B_1^{1|1} \Phi^T}{B_1^{1|1} \Phi^T H^T} \right) e_1^T}{(H\mathbf{A}) e_1^T} \right), \quad s_2 = sgn \left(- \frac{B_1 \Phi^T \nu}{B_1 \Phi^T H^T} \right) \tag{5.43}$$

Then equation (5.42) becomes,

$$\begin{aligned} R_{2,1}^{(2)} &= P_1^{1|1} \left| B_1^{1|1} \Phi^T H^T \right| + P_2^{1|1} \left| B_2^{1|1} \Phi^T H^T \right| \cdot t_1 \cdot sgn \left(HA\nu \right) \\ &\quad + \beta \left| \Gamma^T H^T \right| \cdot t_2 \cdot sgn \left(HA\nu \right) + \gamma sgn \left(- \frac{B_1^{1|1} \Phi^T \nu}{B_1^{1|1} \Phi^T H^T} \right) \\ &= P_1^{1|1} \left| B_1^{1|1} \Phi^T H^T \right| + \left(P_2^{1|1} \left| B_2^{1|1} \Phi^T H^T \right| \cdot t_1 + \beta \left| \Gamma^T H^T \right| \cdot t_2 \right) \cdot sgn \left(HA\nu \right) \\ &\quad + \gamma sgn \left(- \frac{B_1^{1|1} \Phi^T \nu}{B_1^{1|1} \Phi^T H^T} \right) \\ &= P_1^{1|1} \left| B_1^{1|1} \Phi^T H^T \right| + \left(P_2^{1|1} \left| B_2^{1|1} \Phi^T H^T \right| \cdot t_1 + \beta \left| \Gamma^T H^T \right| \cdot t_2 \right) \cdot s_1 + \gamma \cdot s_2 \end{aligned} \tag{5.44}$$

Similarly,

$$R_{2,2}^{(2)} = -P_1^{1|1} \left| B_1^{1|1} \Phi^T H^T \right| + \left(P_2^{1|1} \left| B_2^{1|1} \Phi^T H^T \right| \cdot t_1 + \beta \left| \Gamma^T H^T \right| \cdot t_2 \right) \cdot s_1 + \gamma \cdot s_2 \tag{5.45}$$

Therefore, the second layer of the R sequence can be organized in the following manner.

$$\begin{aligned} \rho_2^{(2)} &= \left[\rho_{2,1}^{(2)}, \quad \rho_{2,2}^{(2)}, \quad \rho_{2,3}^{(2)}\right] \\ &= \left[P_1^{1|1}\left|B_1^{1|1}\Phi^T H^T\right|, \quad P_2^{1|1}\left|B_2^{1|1}\Phi^T H^T\right| \cdot t_1 + \beta\left|\Gamma^T H^T\right| \cdot t_2, \quad \gamma\right] \end{aligned} \tag{5.46}$$

$$F_2^{(2)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{5.47}$$

Till now, we have obtained the G part of the first child term at step $k = 2$ given by equation (5.28). The R sequences of each layer in the fractional form can be recursively updated. In the following sections, we will omit the intermediate process but present the necessary analytic results.

5.2.2.2 The Second Child Term $i = 2$

When $i = 2$, the first layer of the second child term is, same as equation (5.31),

$$R_2^{(1)} = \left[R_{2,1}^{(1)}, \quad R_{2,2}^{(1)}, \quad R_{2,3}^{(1)}, \quad R_{2,4}^{(1)}\right] = \rho_2^{(1)} \cdot F_2^{(1)} \tag{5.48}$$

Unlike the first child term, in this case the element $\rho_{1,3}^{(1)} \cdot sgn(B_2^{1|1}\Phi^T H^T)$ becomes the new offset.

Define

$$t_1 = sgn\left(\frac{\left(\frac{B_1^{1|1}\Phi^T}{B_1^{1|1}\Phi^T H^T} - \frac{B_2^{1|1}\Phi^T}{B_2^{1|1}\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right), \quad t_2 = sgn\left(\frac{\left(\frac{\Gamma^T}{\Gamma^T H^T} - \frac{B_2^{1|1}\Phi^T}{B_2^{1|1}\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right) \tag{5.49}$$

Then,

$$\begin{aligned} \rho_2^{(1)} &= \left[\rho_{2,1}^{(1)}, \quad \rho_{2,2}^{(1)}, \quad \rho_{2,3}^{(1)}\right] \\ &= \left[\rho_{1,1}^{(1)}, \quad \rho_{1,3}^{(1)} \cdot sgn(B_2^{1|1}\Phi^T H^T), \quad \rho_{1,2}^{(1)} \cdot sgn(B_1^{1|1}\Phi^T H^T) \cdot t_1\right] \end{aligned} \tag{5.50}$$

$$F_2^{(1)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \end{bmatrix} \tag{5.51}$$

where $s_1 = sgn(H\mathbf{A}\nu)$.

The second layer is,

$$R_2^{(2)} = \left[R_{2,1}^{(2)}, \quad R_{2,2}^{(2)}\right] = \rho_2^{(2)} \cdot F_2^{(2)} \tag{5.52}$$

$$\begin{aligned} \rho_2^{(2)} &= \left[\rho_{2,1}^{(2)}, \quad \rho_{2,2}^{(2)}, \quad \rho_{2,3}^{(2)}\right] \\ &= \left[P_2^{1|1} \left|B_2^{1|1}\Phi^T H^T\right|, \quad P_1^{1|1} \left|B_1^{1|1}\Phi^T H^T\right| \cdot t_1 + \beta \left|\Gamma^T H^T\right| \cdot t_2, \quad \gamma\right] \end{aligned} \tag{5.53}$$

$$F_2^{(2)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{5.54}$$

where $s_2 = sgn\left(-\frac{B_2\Phi^T\nu}{B_2\Phi^T H^T}\right)$.

5.2.2.3 The Third Child Term $i = 3$

When $i = 3$, define

$$t_1 = sgn\left(\frac{\left(\frac{B_1^{1|1}\Phi^T}{B_1^{1|1}\Phi^T H^T} - \frac{\Gamma^T}{\Gamma^T H^T}\right) e_1^T}{(H\mathbf{A})e_1^T}\right), \quad t_2 = sgn\left(\frac{\left(\frac{B_2^{1|1}\Phi^T}{B_2^{1|1}\Phi^T H^T} - \frac{\Gamma^T}{\Gamma^T H^T}\right) e_1^T}{(H\mathbf{A})e_1^T}\right) \tag{5.55}$$

The first layer of the third child term is,

$$R_2^{(1)} = \left[R_{2,1}^{(1)}, \quad R_{2,2}^{(1)}, \quad R_{2,3}^{(1)}, \quad R_{2,4}^{(1)}\right] = \rho_2^{(1)} \cdot F_2^{(1)} \tag{5.56}$$

$$\begin{aligned} \rho_2^{(1)} &= \left[\rho_{2,1}^{(1)}, \quad \rho_{2,2}^{(1)}, \quad \rho_{2,3}^{(1)}\right] \\ &= \left[\rho_{1,1}^{(1)}, \quad 0, \quad \rho_{1,2}^{(1)} \cdot sgn(B_1^{1|1}\Phi^T H^T) \cdot t_1 + \rho_{1,3}^{(1)} \cdot sgn(B_2^{1|1}\Phi^T H^T) \cdot t_2\right] \end{aligned} \tag{5.57}$$

$$F_2^{(1)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \end{bmatrix} \tag{5.58}$$

where $s_1 = sgn(H\mathbf{A}\nu)$.

Note that a zero offset is added in $\rho_2^{(1)}$ in equation (5.57). This is because at step $k = 1$, the real part of the denominator at the top layer looks like $R_{1,1}^{(1)} = \rho_{1,1}^{(1)} + \rho_{1,2}^{(1)} \cdot s_1 + \rho_{1,3}^{(1)} \cdot s_2$. If we consider the third child, the coefficient of the third sign function will be pulled out as the new offset in $\rho_2^{(1)}$, and that is a zero.

The second layer is,

$$R_2^{(2)} = \left[R_{2,1}^{(2)}, \quad R_{2,2}^{(2)}\right] = \rho_2^{(2)} \cdot F_2^{(2)} \tag{5.59}$$

$$\begin{aligned} \rho_2^{(2)} &= \left[\rho_{2,1}^{(2)}, \quad \rho_{2,2}^{(2)}, \quad \rho_{2,3}^{(2)}\right] \\ &= \left[\beta \left|\Gamma^T H^T\right|, \quad P_1^{1|1} \left|B_1^{1|1} \Phi^T H^T\right| \cdot t_1 + P_2^{1|1} \left|B_2^{1|1} \Phi^T H^T\right| \cdot t_2, \quad \gamma\right] \end{aligned} \tag{5.60}$$

$$F_2^{(2)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{5.61}$$

where $s_1 = sgn(H\mathbf{A}\nu)$, $s_2 = sgn\left(-\frac{\Gamma^T \nu}{\Gamma^T H^T}\right)$.

5.2.2.4 The Fourth Child Term $i = 4$

When $i = 4$, the fourth child term is the old term. In this case, zero offsets are added to the ρ sequence.

The first layer of this old child term is,

$$R_2^{(1)} = \left[R_{2,1}^{(1)}, \quad R_{2,2}^{(1)}, \quad R_{2,3}^{(1)}, \quad R_{2,4}^{(1)}\right] = \rho_2^{(1)} \cdot F_2^{(1)} \tag{5.62}$$

$$\begin{aligned} \rho_2^{(1)} &= \left[\rho_{2,1}^{(1)}, \quad \rho_{2,2}^{(1)}, \quad \rho_{2,3}^{(1)}, \quad \rho_{2,4}^{(1)}\right] \\ &= \left[\rho_{1,1}^{(1)}, \quad 0, \quad \rho_{1,2}^{(1)}, \quad \rho_{1,3}^{(1)}\right] \end{aligned} \tag{5.63}$$

$$F_2^{(1)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \\ s_2 & s_2 & s_2 & s_2 \end{bmatrix} \tag{5.64}$$

where $s_1 = sgn\left(B_1^{1|1}\Phi^T\nu\right)$, and $s_2 = sgn\left(B_2^{1|1}\Phi^T\nu\right)$.

Note that in this case, we also have a new zero offset being added in the sequence of $\rho_2^{(1)}$ in equation (5.63). The reason is similar. The new offset should come from the fourth coefficient of the sign function s_4. However, s_4 is really added as a dummy zero before the update integral. Therefore the new offset in $\rho_2^{(1)}$ is a zero.

The second layer is,

$$R_2^{(2)} = \left[R_{2,1}^{(2)}, \quad R_{2,2}^{(2)}\right] = \rho_2^{(2)} \cdot F_2^{(2)} \tag{5.65}$$

$$\begin{aligned} \rho_2^{(2)} &= \left[\rho_{2,1}^{(2)}, \quad \rho_{2,2}^{(2)}, \quad \rho_{2,3}^{(2)}, \quad \rho_{2,4}^{(2)}\right] \\ &= \left[\gamma, \quad P_1^{1|1}\left(B_1^{1|1}\Phi^T H^T\right), \quad P_2^{1|1}\left(B_2^{1|1}\Phi^T H^T\right), \quad \beta\left(\Gamma^T H^T\right)\right] \end{aligned} \tag{5.66}$$

$$F_2^{(2)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \\ s_3 & s_3 \end{bmatrix} \tag{5.67}$$

where $s_1 = sgn\left(B_1^{1|1}\Phi^T\nu\right)$, $s_2 = sgn\left(B_2^{1|1}\Phi^T\nu\right)$, $s_3 = sgn\left(\Gamma^T\nu\right)$.

From the solutions shown above, it is obvious that, for new terms up to step $k = 2$, there are at most three non-zero elements in the sequence of ρ at each layer. In the rest of this analysis for the G terms of two-state case, it will be shown that the number of non-zero elements will not be more than three as well.

5.2.3 General Forms of G for New Terms at Step k

We already understand that starting from the second measurement update, any new term at step k has the exponential part that can be written as,

$$\mathcal{E}^{k|k} = \exp\left(-P_1 \left|H\mathbf{A}\nu\right| - P_2 \left|B_2\nu\right| + j\zeta^{k|k}\nu\right) \tag{5.68}$$

There are k (number of) layers in the G part in (5.1). Since this is a new term, only the last (bottom) layer will involve 2 different sign functions $s_1 = sgn(H\mathbf{A}\nu)$ and $s_2 = sgn(B_2\nu)$. All the other layers only contain s_1 in the sum of the sign functions. And s_1 is invariant across all the new terms in every step.

To better describe $R_k^{(m)}$ at the m^{th} layer as $R_k^{(m)} = \rho_k^{(m)} \cdot F_k^{(m)}$, when $1 \leq m \leq k-1$,

$$\rho_k^{(m)} = \left[\rho_{k,1}^{(m)}, \quad \rho_{k,2}^{(m)}, \quad \cdots, \quad \rho_{k,k+2-m}^{(m)}\right] \tag{5.69}$$

$$F_k^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix} \in \mathbb{R}^{(k+2-m)\times 2^{(k+1-m)}} \tag{5.70}$$

When $m = k$, the bottom layer is,

$$\rho_k^{(k)} = \left[\rho_{k,1}^{(k)}, \quad \rho_{k,2}^{(k)}, \quad \rho_{k,3}^{(k)}\right], \quad F_k^{(k)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \in \mathbb{R}^{3\times 2} \tag{5.71}$$

where $\rho_k^{(m)}$ for $1 \leq m \leq k$ only contains at most three non-zero elements. This is shown in the rest of this chapter and summarized in Theorem 5.2.1 in Section 5.2.8.

5.2.4 General Forms of G for Old Terms at Step k

There are two types of old terms. Firstly, most of the old terms come from new parent terms if tracked several steps back. They can be expressed in a general form. They are called "type-I" old terms. Secondly, there are three old terms that are originated from the three old parent terms at step $k = 2$. These three old terms are the "oldest" terms, since they never had a new parent. They are called "type-II" old terms. The general forms of G for both types of old terms are presented as follows.

5.2.4.1 Old Terms: Type I

Old terms of type I refer to those terms that are originated from a new parent term at certain step $k > 1$. Let us find the general form for an θ-step old term at step k. Then at step $k-\theta$, notated as p, i.e. $p = k - \theta$, the original parent term is a new term.

The general form of any new term at step p can be found from equation (5.68), (5.1), (5.69) - (5.71).

"1-step old" term ($\theta = 1$) is the old child term at step $(p+1)$ from a new parent term at step p. The exponential part of this 1-step old child term becomes,

$$\mathcal{E}^{p+1|p+1} = \exp\left(-P_1\left|H\mathbf{A}\Phi^T\nu\right| - P_2\left|B_2\Phi^T\nu\right| - \beta\left|\Gamma^T\nu\right| + j\zeta^{p+1|p+1}\nu\right) \tag{5.72}$$

The G part is in the structure in (5.1), with the parameters $s_1 = sgn\left(H\mathbf{A}\Phi^T\nu\right)$, $s_2 = sgn\left(B_2\Phi^T\nu\right)$, $s_3 = sgn\left(\Gamma^T\nu^T\right)$. The real part R at each layer m is the product of the sequence ρ and the matrix F, i.e. $R^{(m)}_{p+1} = \rho^{(m)}_{p+1} \cdot F^{(m)}_{p+1}, \quad 1 \le m \le p+1$.

When $1 \le m \le p-1$

$$\rho^{(1)}_{p+1} = \left[\rho^{(1)}_{p,1}, \quad \rho^{(1)}_{p,2}, \quad \cdots, \quad \cdots, \quad \rho^{(1)}_{p,p}, \quad 0, \quad \rho^{(1)}_{p,p+1}\right] \tag{5.73}$$

$$\rho^{(2)}_{p+1} = \left[\rho^{(2)}_{p,1}, \quad \rho^{(2)}_{p,2}, \quad \cdots, \quad \rho^{(2)}_{p,p-1}, \quad 0, \quad \rho^{(2)}_{p,p}\right] \tag{5.74}$$

$$\vdots$$

$$\rho^{(p-1)}_{p+1} = \left[\rho^{(p-1)}_{p,1}, \quad \rho^{(p-1)}_{p,2}, \quad 0, \quad \rho^{(p-1)}_{p,3}\right] \tag{5.75}$$

or equivalently,

$$\rho^{(m)}_{p+1} = \left[\rho^{(m)}_{p,1}, \quad \rho^{(m)}_{p,2}, \quad \cdots, \quad \rho^{(m)}_{p,p+1-m}, \quad 0, \quad \rho^{(m)}_{p,p+2-m}\right] \tag{5.76}$$

and

$$F^{(m)}_{p+1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix} \in \mathbb{R}^{(p+3-m)\times 2^{(p+2-m)}} \tag{5.77}$$

When $m = p$

$$\rho_{p+1}^{(p)} = \left[\rho_{p,1}^{(p)}, \quad 0, \quad \rho_{p,2}^{(p)}, \quad \rho_{p,3}^{(p)}\right], \quad F_{p+1}^{(p)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \\ s_2 & s_2 & s_2 & s_2 \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{5.78}$$

When $m = p + 1$

$$\rho_{p+1}^{(p+1)} = \left[\gamma, \quad P_1\left(H\mathbf{A}\Phi^T H^T\right), \quad P_2\left(B_2\Phi^T H^T\right), \quad \beta\left(\Gamma^T H^T\right)\right] \tag{5.79}$$

$$F_{p+1}^{(p+1)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \\ s_3 & s_3 \end{bmatrix} \in \mathbb{R}^{4\times 2} \tag{5.80}$$

By looking at how the ρ sequences for each layer are updated, one can see that at step $p+1$, a zero is added to the ρ sequence of the top p layers, being the last offset that occurs before the coefficients of sign functions, as shown in equation (5.76) and (5.78). Therefore, the number of non-zero elements in the ρ sequence of the top p layers does not increase; still at most 3 of these elements are non-zero.

For the bottom layer $m = p+1$, all 4 elements are non-zero. The first element, γ, is the only offset. The rest 3 elements are the coefficients of sign functions s_1, s_2, and s_3.

Next, examine the old child term ($\theta = 2$) at step $p+2$ from the old parent term at step $p+1$. The exponential part becomes,

$$\begin{aligned} \mathcal{E}^{p+2|p+2} = \exp\big(&-P_1\left|H\mathbf{A}\Phi^{2T}\nu\right| - P_2\left|B_2\Phi^{2T}\nu\right| - \beta\left|\Gamma^T\Phi^T\nu\right| \\ &-\beta\left|\Gamma^T\nu\right| + j\zeta^{p+2|p+2}\nu\big) \end{aligned} \tag{5.81}$$

The real part R at each layer m is the product of the sequence ρ and the matrix F. Let $s_1 = sgn\left(H\mathbf{A}\Phi^{2T}\nu\right)$, $s_2 = sgn\left(B_2\Phi^{2T}\nu\right)$, $s_3 = sgn\left(\Gamma^T\Phi^T\nu^T\right)$, $s_4 = sgn\left(\Gamma^T\nu^T\right)$.

When $1 \leq m \leq p-1$

$$\begin{aligned}\rho^{(1)}_{p+2} &= \left[\rho^{(1)}_{p+1,1}, \quad \rho^{(1)}_{p+1,2}, \quad \cdots, \quad \cdots, \quad \rho^{(1)}_{p+1,p+1}, \quad 0, \quad \rho^{(1)}_{p+1,p+2}\right] \\ &= \left[\rho^{(1)}_{p,1}, \quad \rho^{(1)}_{p,2}, \quad \cdots, \quad \rho^{(1)}_{p,p}, \quad 0, \quad 0, \quad \rho^{(1)}_{p,p+1}\right] \end{aligned} \tag{5.82}$$

$$\begin{aligned}\rho^{(2)}_{p+2} &= \left[\rho^{(2)}_{p+1,1}, \quad \rho^{(2)}_{p+1,2}, \quad \cdots, \quad \rho^{(2)}_{p+1,p}, \quad 0, \quad \rho^{(2)}_{p+1,p+1}\right] \\ &= \left[\rho^{(2)}_{p,1}, \quad \rho^{(2)}_{p,2}, \quad \cdots, \quad \rho^{(2)}_{p,p-1}, \quad 0, \quad 0, \quad \rho^{(2)}_{p,p}\right] \end{aligned} \tag{5.83}$$

$$\vdots$$

$$\begin{aligned}\rho^{(p-1)}_{p+2} &= \left[\rho^{(p-1)}_{p+1,1}, \quad \rho^{(p-1)}_{p+1,2}, \quad \rho^{(p-1)}_{p+1,3}, \quad 0, \quad \rho^{(p-1)}_{p+1,4}\right] \\ &= \left[\rho^{(p-1)}_{p,1}, \quad \rho^{(p-1)}_{p,2}, \quad 0, \quad 0, \quad \rho^{(p-1)}_{p,3}\right] \end{aligned} \tag{5.84}$$

or equivalently,

$$\rho^{(m)}_{p+2} = \left[\rho^{(m)}_{p,1}, \quad \rho^{(m)}_{p,2}, \quad \cdots, \quad \rho^{(m)}_{p,p+1-m}, \quad 0, \quad 0, \quad \rho^{(m)}_{p,p+2-m}\right] \tag{5.85}$$

and

$$F^{(m)}_{p+2} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix} \in \mathbb{R}^{(p+4-m)\times 2^{(p+3-m)}} \tag{5.86}$$

When $m = p$

$$\begin{aligned}\rho^{(p)}_{p+2} &= \left[\rho^{(p)}_{p+1,1}, \quad \rho^{(p)}_{p+1,2}, \quad 0, \quad \rho^{(p)}_{p+1,3}, \quad \rho^{(p)}_{p+1,4}\right] \\ &= \left[\rho^{(p)}_{p,1}, \quad 0, \quad 0, \quad \rho^{(p)}_{p,2}, \quad \rho^{(p)}_{p,3}\right] \end{aligned} \tag{5.87}$$

$$F_{p+2}^{(p)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 & s_1 & s_1 & s_1 & s_1 \\ s_2 & s_2 & s_2 & s_2 & s_2 & s_2 & s_2 & s_2 \end{bmatrix} \in \mathbb{R}^{5\times 8} \tag{5.88}$$

When $m = p + 1$

$$\begin{aligned} \rho_{p+2}^{(p+1)} &= \left[\rho_{p+1,1}^{(p+1)}, \quad 0, \quad \rho_{p+1,2}^{(p+1)}, \quad \rho_{p+1,3}^{(p+1)}, \quad \rho_{p+1,4}^{(p+1)}\right] \\ &= \left[\gamma, \quad 0, \quad P_1\left(H\mathbf{A}\Phi^T H^T\right), \quad P_2\left(B_2\Phi^T H^T\right), \quad \beta\left(\Gamma^T H^T\right)\right] \end{aligned} \tag{5.89}$$

$$F_{p+2}^{(p+1)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \\ s_2 & s_2 & s_2 & s_2 \\ s_3 & s_3 & s_3 & s_3 \end{bmatrix} \in \mathbb{R}^{5\times 4} \tag{5.90}$$

When $m = p + 2$

$$\rho_{p+2}^{(p+2)} = \left[\gamma, \quad P_1\left(H\mathbf{A}\Phi^{2T} H^T\right), \quad P_2\left(B_2\Phi^{2T} H^T\right), \quad \beta\left(\Gamma^T\Phi^T H^T\right), \quad \beta\left(\Gamma^T H^T\right)\right] \tag{5.91}$$

$$F_{p+2}^{(p+2)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \\ s_3 & s_3 \\ s_4 & s_4 \end{bmatrix} \in \mathbb{R}^{5\times 2} \tag{5.92}$$

Repeat this approach to find the general form of an θ-step old term at step k. There are in total $(\theta + 2)$ elements in the argument of the exponential.

$$\mathcal{E}^{k|k} = \exp\left(-P_1 \left|H\mathbf{A}\Phi^{T\theta}\nu\right| - P_2 \left|B_2\Phi^{T\theta}\nu\right| - \beta \left|\Gamma^T\Phi^{T(\theta-1)}\nu\right| \right.$$
$$\left. - \cdots - \beta \left|\Gamma^T\nu\right| + j\zeta^{k|k}\nu\right) \tag{5.93}$$

$$R_k^{(m)} = \rho_k^{(m)} \cdot F_k^{(m)}, \quad 1 \leq m \leq k \tag{5.94}$$

Let $s_1 = sgn\left(H\mathbf{A}\Phi^{T\theta}\nu\right)$, $s_2 = sgn\left(B_2\Phi^{T\theta}\nu\right)$, $s_3 = sgn\left(\Gamma^T\Phi^{T(\theta-1)}\nu\right)$, $\cdots$, $s_{\theta+2} = sgn\left(\Gamma^T\nu\right)$.

When $1 \leq m \leq p-1$

$$\rho_k^{(m)} = \left[\rho_{p,1}^{(m)}, \quad \rho_{p,2}^{(m)}, \quad \cdots, \quad \rho_{p,p+1-m}^{(m)}, \quad 0_1, \quad 0_2, \quad \cdots, \quad 0_\theta, \quad \rho_{p,p+2-m}^{(m)}\right] \tag{5.95}$$

The total number of zero elements in equation (5.95) is θ. The subscripts of the zeros are for convenience. No matter what the subscripts are, they all represent scalar zeros. As child terms get older (θ becomes larger), these zeros are always inserted as offsets, in front of the last entry of the ρ sequence.

$$F_k^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix} \in \mathbb{R}^{(k+2-m)\times 2^{(k+1-m)}} \tag{5.96}$$

When $m = p$

$$\rho_k^{(p)} = \left[\rho_{p,1}^{(p)}, \quad 0_1, \quad \cdots, \quad 0_\theta, \quad \rho_{p,2}^{(p)}, \quad \rho_{p,3}^{(p)}\right] \tag{5.97}$$

Again, number of zeros in equation (5.97) is θ.

$$F_k^{(p)} = \begin{bmatrix} 1 & -1 & ... \\ 1 & 1 & ... \\ \vdots & & \\ 1 & 1 & ... \\ s_1 & s_1 & ... \\ s_2 & s_2 & ... \end{bmatrix} \in \mathbb{R}^{(\theta+3)\times 2^{(\theta+1)}} \tag{5.98}$$

When $p+1 \leq m \leq k-1$

$$\rho_k^{(m)} = \left[\gamma, 0_1, \cdots, 0_{k-m}, P_1\left(H\mathbf{A}\Phi^{T(m-p)}H^T\right), P_2\left(B_2\Phi^{T(m-p)}H^T\right),\right.$$
$$\left.\beta\left(\Gamma^T\Phi^{T(m-p-1)}H^T\right), \cdots, \beta\left(\Gamma^T H^T\right)\right] \tag{5.99}$$

Unlike the previous cases, the zeros in equation (5.99) are added behind the first entry of the ρ sequence. The number of zeros in equation (5.99) is $(k-m)$. There are $(\theta+3)$ rows in $F_k^{(p)}$. The total number of columns in $F_k^{(p)}$ is $2^{(\theta+3)-(m+2-p)} = 2^{\theta-m+p+1} = 2^{k-m+1}$.

$$F_k^{(p)} = \begin{bmatrix} 1 & -1 & ... \\ 1 & 1 & ... \\ \vdots & & \\ 1 & 1 & ... \\ s_1 & s_1 & ... \\ \vdots & & \\ s_{m+2-p} & s_{m+2-p} & ... \end{bmatrix} \in \mathbb{R}^{(\theta+3)\times 2^{k+1-m}} \tag{5.100}$$

When $m = k$ **(The bottom layer)**

$$\rho_k^{(m)} = \left[\gamma, P_1\left(H\mathbf{A}\Phi^{T\theta}H^T\right), P_2\left(B_2\Phi^{T\theta}H^T\right), \beta\left(\Gamma^T\Phi^{T(\theta-1)}H^T\right), \cdots, \beta\left(\Gamma^T H^T\right)\right] \tag{5.101}$$

There are $(\theta+3)$ rows in $F_k^{(k)}$ as well. The total number of columns in $F_k^{(k)}$ is 2. Sign functions are from s_1 to $s_{\theta+2}$.

$$F_k^{(k)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ \vdots & \\ s_{\theta+2} & s_{\theta+2} \end{bmatrix} \in \mathbb{R}^{(\theta+3)\times 2} \tag{5.102}$$

5.2.4.2 Old Terms: Type II

Old terms of type I cover the majority of old terms at a general measurement step k. However, at each step there are 3 old terms that cannot be described by (5.93). They are the old descendants from the 3 terms at the 1^{th} measurement update in (5.6) - (5.8). We call these three old descendants as "Type II" old terms.

Recall the fourth child terms at step $k=2$. There are two layers in the G part. The real part R of each layer m is a product of a row vector ρ and the matrix F as

$$\rho_2^{(1)} = \left[\rho_{1,1}^{(1)}, \quad 0, \quad \rho_{1,2}^{(1)}, \quad \rho_{1,3}^{(1)}\right] \tag{5.103}$$

$$\rho_2^{(2)} = \left[\gamma, \quad P_1^{1|1}\left(B_1^{1|1}\Phi^T H^T\right), \quad P_2^{1|1}\left(B_2^{1|1}\Phi^T H^T\right), \quad \beta\left(\Gamma^T H^T\right)\right] \tag{5.104}$$

$$F_2^{(1)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \\ s_2 & s_2 & s_2 & s_2 \end{bmatrix}, \quad F_2^{(2)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \\ s_3 & s_3 \end{bmatrix} \tag{5.105}$$

where $s_1 = sgn\left(B_1^{1|1}\Phi^T\nu\right)$, $s_2 = sgn\left(B_2^{1|1}\Phi^T\nu\right)$, $s_3 = sgn\left(\Gamma^T\nu\right)$.

These 3 terms will keep producing old child terms as k becomes large. At each step there will be 3 of them. The form of such "oldest" child terms at step k can be easily derived.

At step k, the argument of the exponential has the form of,

$$\mathcal{E}^{k|k} = \exp\Big(-P_1^{1|1}\left|B_1^{1|1}\Phi^{T(k-1)}\nu\right| - P_2^{1|1}\left|B_2^{1|1}\Phi^{T(k-1)}\nu\right| - \beta\left|\Gamma^T\Phi^{T(k-2)}\nu\right| - \cdots - \beta\left|\Gamma^T\nu\right| + j\zeta^{k|k}\nu\Big) \tag{5.106}$$

When $m = 1$

$$\rho_k^{(1)} = \left[\rho_{1,1}^{(1)}, \quad 0_1, \quad \cdots, \quad 0_{k-1}, \quad \rho_{1,2}^{(1)}, \quad \rho_{1,3}^{(1)}\right] \tag{5.107}$$

When $2 \leq m \leq k-1$

$$\rho_k^{(m)} = \Big[\gamma, \quad 0_1, \quad \cdots, \quad 0_{k-m}, \quad P_1^{1|1}\left(B_1^{1|1}\Phi^{T(m-1)}H^T\right), \quad P_2^{1|1}\left(B_2^{1|1}\Phi^{T(m-1)}H^T\right), \quad \beta\left(\Gamma^T\Phi^{T(m-2)}H^T\right), \quad \cdots, \quad \beta\left(\Gamma^T H^T\right)\Big] \tag{5.108}$$

When $m = k$

$$\rho_k^{(k)} = \Big[\gamma, \quad P_1^{1|1}\left(B_1^{1|1}\Phi^{T(k-1)}H^T\right), \quad P_2^{1|1}\left(B_2^{1|1}\Phi^{T(k-1)}H^T\right), \quad \beta\left(\Gamma^T\Phi^{T(k-2)}H^T\right), \quad \cdots, \quad \beta\left(\Gamma^T H^T\right)\Big] \tag{5.109}$$

Every layer has $(k+2)$ elements such that

$$F_k^{(m)} = \begin{bmatrix} 1 & \cdots \\ \vdots & \\ 1 & \cdots \\ s_1 & \cdots \\ \vdots & \\ s_{m+1} & \cdots \end{bmatrix} \in \mathbb{R}^{(k+2)\times 2^{k+1-m}}, \quad 1 \leq m \leq k \tag{5.110}$$

5.2.5 The Recursion: New Terms at Step k to Child Terms at Step $k+1$

Consider the exponential part of a new term at step k in equation (5.68). During time propagation at $k+1$, the exponential part becomes,

$$\mathcal{E}^{k+1|k} = \exp\left(-P_1\left|H\mathbf{A}\Phi^T\nu\right| - P_2\left|B_2\Phi^T\nu\right| - \beta\left|\Gamma^T\nu\right| + j\zeta^{k|k}\Phi^T\nu\right) \tag{5.111}$$

At step $k+1$, this term will produce four child terms. The first three child terms are new terms, and the last child term is an old child term. The four child terms are examined as follows.

5.2.5.1 The First Child Term at Step $k+1$

Let us look at each layer m of the fractional form separately.

When $1 \leq m \leq k-1$, consider equation (5.69). Each of the first $(k-1)$ layers of the fractional form from the top contains a sequence of offsets and one coefficient of sign function s_1, if any. The first child term extracts that one coefficient at step k and has it as the last offset at step $(k+1)$. In the mean time, because there is only one row of sign function s_1 in $F_k^{(m)}$ and two zero will be added to complete the update integral, it will produce a zero coefficient for s_1 at step $(k+1)$. This means that the top $(k-1)$ layer does not contain sign functions any more. And if they do not have sign functions, any newly added offset in the future will be zero. Therefore they become "invariant" as time proceeds. Only the bottom 2 layers contain sign functions. This property of "invariance" aligns with our earlier findings that the real component of the argument of exponential of the first child terms stay invariant, presented in Chapter 3.3.

Therefore, the real part of the denominators $R_{k+1}^{(m)} = \rho_{k+1}^{(m)} \cdot F_{k+1}^{(m)}$ will be updated as,

$$\rho_{k+1}^{(m)} = \left[\rho_{k,1}^{(m)}, \rho_{k,2}^{(m)}, \cdots, \rho_{k,k+1-m}^{(m)}, \rho_{k,k+2-m}^{(m)} \cdot sgn\left(HA\Phi^T H^T\right), 0\right] \tag{5.112}$$

$$F_{k+1}^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix}$$
$$\in \mathbb{R}^{(k+3-m)\times 2^{(k+2-m)}} \tag{5.113}$$

When $m = k$, consider equation (5.71). At step $k+1$, there will be 2 offsets and 1 coefficient of sign function. The last offset at step $k+1$ will be the coefficient of the first sign function s_1 at step k. The real part of the denominators $R_{k+1}^{(k)} = \rho_{k+1}^{(k)} \cdot F_{k+1}^{(k)}$ will be updated as,

$$\rho_{k+1}^{(k)} = \left[\rho_{k,1}^{(k)}, \quad \rho_{k,2}^{(k)} \cdot sgn\left(H\mathbf{A}\Phi^T H^T\right), \quad \rho_{k,3}^{(k)} \cdot sgn\left(B_2\Phi^T H^T\right) \cdot t_1\right] \tag{5.114}$$

$$F_{k+1}^{(k)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \end{bmatrix} \in \mathbb{R}^{3\times 4} \tag{5.115}$$

where

$$t_1 = sgn\left[\frac{\left(\frac{B_2\Phi^T}{B_2\Phi^T H^T} - \frac{HA\Phi^T}{HA\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})\,e_1^T}\right], \quad s_1 = sgn\left(H\mathbf{A}\nu\right) \tag{5.116}$$

For two-state system, any two non-zero parent directions will produce a child direction that is aligned with $H\mathbf{A}$. The reason we have t_1 in (5.114) is for convenience of direction combination, such that the sign function s_1 in $F_{k+1}^{(k)}$ in (5.115) can be always normalized to $sgn(H\mathbf{A}\nu)$.

When $m = k+1$, this layer is the bottom layer of the fractional form. Its offsets and coefficients at step $k+1$ should directly come from the exponential part at step k. The real part of the denominators $R_{k+1}^{(k+1)} = \rho_{k+1}^{(k+1)} \cdot F_{k+1}^{(k+1)}$ at step $(k+1)$ is,

$$\rho_{k+1}^{(k+1)} = \left[P_1\left|H\mathbf{A}\Phi^T H^T\right|, \quad P_2\left|B_2\Phi^T H^T\right| \cdot t_1 + \beta\left|\Gamma^T H^T\right| \cdot t_2, \quad \gamma\right] \tag{5.117}$$

where t_1 has already been defined earlier and t_2 is defined as,

$$t_2 = sgn\left[\frac{\left(\frac{\Gamma^T}{\Gamma^T H^T} - \frac{H\mathbf{A}\Phi^T}{H\mathbf{A}\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})\,e_1^T}\right] \tag{5.118}$$

where again, t_1 and t_2 here are for the convenience of elements combination. And we also have,

$$F_{k+1}^{(k+1)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \in R^{3\times 2} \tag{5.119}$$

$$s_1 = sgn\left(H\mathbf{A}\nu\right), \quad s_2 = sgn\left(-\frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^T H^T}\right) \tag{5.120}$$

5.2.5.2 The Second Child Term at Step $k+1$

Similarly, consider the second child term at step $k+1$. Again, consider the m^{th} layer of the fractional form separately, where $1 \le m \le k+1$.

When $1 \le m \le k-1$, consider equation (5.70). There is only one row of sign function in $F_k^{(m)}$, i.e. the coefficient of a second sign function s_2 is zero. Hence, the second child term at step $(k+1)$ will have a new zero offset in the real part of the denominators. The process of obtaining zero offsets is similar.

Define

$$t_1 = sgn\left[\frac{\left(\frac{H\mathbf{A}\Phi^T}{HA\Phi^T H^T} - \frac{B_2\Phi^T}{B_2\Phi^T H^T}\right)e_1^T}{\left(H\mathbf{A}\right)e_1^T}\right] \tag{5.121}$$

Then, the real part of the denominators $R_{k+1}^{(m)} = \rho_{k+1}^{(m)} \cdot F_{k+1}^{(m)}$ becomes,

$$\rho_{k+1}^{(m)} = \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad 0, \quad \rho_{k,k+2-m}^{(m)} \cdot sgn\left(H\mathbf{A}\Phi^T H^T\right) \cdot t_1\right] \tag{5.122}$$

$$F_{k+1}^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix}$$

$$\in \mathbb{R}^{(k+3-m)\times 2^{(k+2-m)}} \tag{5.123}$$

When $m = k$, consider equation (5.71). At step $k+1$, there will be 2 offsets and 1 coefficient of sign function. The last offset at step $(k+1)$ will be the second coefficient of the sign function, s_2, at step k.

The real part of the denominators $R_{k+1}^{(k)} = \rho_{k+1}^{(k)} \cdot F_{k+1}^{(k)}$ will be updated as,

$$\rho_{k+1}^{(k)} = \left[\rho_{k,1}^{(k)}, \quad \rho_{k,3}^{(k)} \cdot sgn\left(B_2 \Phi^T H^T\right), \quad \rho_{k,2}^{(k)} \cdot sgn\left(H\mathbf{A}\Phi^T H^T\right) \cdot t_1\right] \tag{5.124}$$

$$F_{k+1}^{(k)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \end{bmatrix} \in \mathbb{R}^{3\times 4} \tag{5.125}$$

where $s_1 = sgn\,(H\mathbf{A}\nu)$.

When $m = k+1$, this layer is the bottom layer. The real part of the denominators $R_{k+1}^{(k+1)} = \rho_{k+1}^{(k+1)} \cdot F_{k+1}^{(k+1)}$ at step $(k+1)$ is,

$$\rho_{k+1}^{(k+1)} = \left[P_2 \left|B_2 \Phi^T H^T\right|, \quad P_1 \left|H\mathbf{A}\Phi^T H^T\right| \cdot t_1 + \beta \left|\Gamma^T H^T\right| \cdot t_2, \quad \gamma\right] \tag{5.126}$$

$$t_2 = sgn\left[\frac{\left(\frac{\Gamma^T}{\Gamma^T H^T} - \frac{B_2 \Phi^T}{B_2 \Phi^T H^T}\right) e_1^T}{(H\mathbf{A})\, e_1^T}\right] \tag{5.127}$$

and,

$$F_{k+1}^{(k+1)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \in R^{3\times 2} \tag{5.128}$$

$$s_1 = sgn\,(H\mathbf{A}\nu)\,, \quad s_2 = sgn\left(-\frac{B_2 \Phi^T \nu}{B_2 \Phi^T H^T}\right) \tag{5.129}$$

5.2.5.3 The Third Child Term at Step $k+1$

Consider the third child term at step $k+1$. Look at the m^{th} layer of the fractional form separately, where $1 \le m \le k+1$. The method is similar with the second child term.

Define

$$t_1 = sgn\left[\frac{\left(\frac{H\mathbf{A}\Phi^T}{H\mathbf{A}\Phi^T H^T} - \frac{\Gamma^T}{\Gamma^T H^T}\right) e_1^T}{(H\mathbf{A})\, e_1^T}\right], \quad t_2 = sgn\left[\frac{\left(\frac{B_2\Phi^T}{B_2\Phi^T H^T} - \frac{\Gamma^T}{\Gamma^T H^T}\right) e_1^T}{(H\mathbf{A})\, e_1^T}\right] \tag{5.130}$$

When $1 \le m \le k-1$, the real part of the denominator is,

$$R_{k+1}^{(m)} = \rho_{k+1}^{(m)} \cdot F_{k+1}^{(m)} \tag{5.131}$$

where

$$\rho_{k+1}^{(m)} = \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad 0, \quad \rho_{k,k+2-m}^{(m)} \cdot sgn\left(H\mathbf{A}\Phi^T H^T\right) \cdot t_1\right] \tag{5.132}$$

$$F_{k+1}^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix}$$
$$\in \mathbb{R}^{(k+3-m)\times 2^{(k+2-m)}} \tag{5.133}$$

When $m = k$, consider equation (5.71). At step $k+1$, there will be 2 offsets and 1 coefficient of sign function. The second offset at step $k+1$ will be the third coefficient of the sign function at step k, which is zero.

The real part of the denominators $R_{k+1}^{(k)} = \rho_{k+1}^{(k)} \cdot F_{k+1}^{(k)}$ will be updated as,

$$\rho_{k+1}^{(k)} = \left[\rho_{k,1}^{(k)}, \quad 0, \quad \rho_{k,2}^{(k)} \cdot sgn\left(H\mathbf{A}\Phi^T H^T\right) \cdot t_1 + \rho_{k,3}^{(k)} \cdot sgn\left(B_2\Phi^T H^T\right) \cdot t_2\right] \tag{5.134}$$

$$F_{k+1}^{(k)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \end{bmatrix} \in \mathbb{R}^{3\times 4} \tag{5.135}$$

where $s_1 = sgn\left(H\mathbf{A}\nu\right)$.

When $m = k+1$, the real part of the denominators $R_{k+1}^{(k+1)} = \rho_{k+1}^{(k+1)} \cdot F_{k+1}^{(k+1)}$ at step $(k+1)$ is,

$$\rho_{k+1}^{(k+1)} = \left[\beta\left|\Gamma^T H^T\right|, \quad P_1\left|H\mathbf{A}\Phi^T H^T\right| \cdot t_1 + P_2\left|B_2\Phi^T H^T\right| \cdot t_2, \quad \gamma\right] \tag{5.136}$$

$$F_{k+1}^{(k+1)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \in R^{3\times 2} \tag{5.137}$$

where $s_1 = sgn\left(H\mathbf{A}\nu\right)$, and $s_2 = sgn\left(-\frac{\Gamma^T\nu}{\Gamma^T H^T}\right)$.

5.2.5.4 The Fourth Child Term at Step $k+1$

The fourth child term is the old term. Zero offsets are added to each layer of the fractional form. Explicitly,

When $1 \leq m \leq k-1$

$$R_{k+1}^{(m)} = \rho_{k+1}^{(m)} \cdot F_{k+1}^{(m)} \tag{5.138}$$

where

$$\rho_{k+1}^{(m)} = \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad 0, \quad \rho_{k,k+2-m}^{(m)} \cdot sgn\left(H\mathbf{A}\Phi^T H^T\right)\right] \tag{5.139}$$

$$F_{k+1}^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix}$$

$$\in \mathbb{R}^{(k+3-m)\times 2^{(k+2-m)}} \tag{5.140}$$

and $s_1 = sgn\left(\frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^T H^T}\right)$.

When $m = k$

$$R_{k+1}^{(k)} = \rho_{k+1}^{(k)} \cdot F_{k+1}^{(k)} \tag{5.141}$$

where

$$\rho_{k+1}^{(k)} = \left[\rho_{k,1}^{(k)}, \quad 0, \quad \rho_{k,2}^{(k)} \cdot sgn\left(HA\Phi^T H^T\right), \quad \rho_{k,3}^{(k)} \cdot sgn\left(B_2\Phi^T H^T\right)\right] \tag{5.142}$$

and

$$F_{k+1}^{(k)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ s_1 & s_1 & s_1 & s_1 \\ s_2 & s_2 & s_2 & s_2 \end{bmatrix} \in R^{4\times 4} \tag{5.143}$$

where

$$s_1 = sgn\left(\frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^T H^T}\right), \quad s_2 = sgn\left(\frac{B_2\Phi^T\nu}{B_2\Phi^T H^T}\right) \tag{5.144}$$

When $m = k+1$, the real part of the denominators $R_{k+1}^{(k+1)} = \rho_{k+1}^{(k+1)} \cdot F_{k+1}^{(k+1)}$ at step $(k+1)$ is,

$$\rho_{k+1}^{(k+1)} = \left[\gamma, \quad P_1\left(HA\Phi^T H^T\right), \quad P_2\left(B_2\Phi^T H^T\right), \quad \beta\left(\Gamma^T H^T\right)\right] \tag{5.145}$$

$$F_{k+1}^{(k+1)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \\ s_3 & s_3 \end{bmatrix} \in \mathbb{R}^{4\times 2} \tag{5.146}$$

where $s_1 = sgn\left(H\mathbf{A}\Phi^T\nu\right)$, $s_2 = sgn\left(B_2\Phi^T\nu\right)$, and $s_3 = sgn\left(\Gamma^T\nu\right)$.

It is interesting to notice that for the first, second and third child term at step $k+1$, the number of non-zero elements in the sequence ρ at each layer does not increase. There are at most three non-zero elements.

5.2.6 The Recursion: From Type-I Old Parent Terms

In this section, the scenarios of old parent terms producing child terms will be evaluated. We start with the general form of an arbitrary θ-step old term at step k. There are $(\theta+2)$ elements in the argument of the exponential. Hence, at step $k+1$, $(\theta+4)$ child terms will be produced, among which the first $(\theta+3)$ of them are new, and the last one is old.

5.2.6.1 The First Child Term at Step $k+1$

At step $k+1$, the child term has $(k+1)$ layers in the G part. Since this is the first child term, the two child directions at step $k+1$ become $H\mathbf{A}$ and $-\frac{H\mathbf{A}\Phi^{T(\theta+1)}}{H\mathbf{A}\Phi^{T(\theta+1)}H^T}$, i.e.

$$s_1 = sgn\left(H\mathbf{A}\nu\right), \quad s_2 = sgn\left(-\frac{H\mathbf{A}\Phi^{T(\theta+1)}}{H\mathbf{A}\Phi^{T(\theta+1)}H^T}\right) \tag{5.147}$$

The F matrix is also in a simple form,

$$F_{k+1}^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix}$$

$$\in \mathbb{R}^{(k+3-m)\times 2^{(k+2-m)}}, \quad 1 \le m \le k \tag{5.148}$$

$$F_{k+1}^{(k+1)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \in \mathbb{R}^{3\times 2} \tag{5.149}$$

Since $R_{k+1}^{(m)} = \rho_{k+1}^{(m)} \times F_{k+1}^{(m)}$, next we are going to find the sequence $\rho_{k+1}^{(m)}$ for each layer m.

When $1 \le m \le p-1$, recall equation (2.27). In the G part of the first child term at step $k+1$, a new offset is added into the ρ sequence, and the only coefficient of sign function s_1 becomes 0, for the case when $1 \le m \le p-1$.

$$\rho_{k+1}^{(1)} = \Big[\rho_{p,1}^{(1)}, \quad \rho_{p,2}^{(1)}, \quad \cdots, \quad \rho_{p,p}^{(1)}, \quad 0_1, \quad \cdots, \quad 0_\theta, \quad \rho_{p,p+1}^{(1)} \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right), \quad 0\Big] \tag{5.150}$$

$$\rho_{k+1}^{(2)} = \Big[\rho_{p,1}^{(2)}, \quad \rho_{p,2}^{(2)}, \quad \cdots, \quad \rho_{p,p-1}^{(2)}, \quad 0_1, \quad \cdots, \quad 0_\theta, \quad \rho_{p,p}^{(2)} \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right), \quad 0\Big] \tag{5.151}$$

$$\vdots$$

$$\rho_{k+1}^{(p-1)} = \Big[\rho_{p,1}^{(p-1)}, \quad \rho_{p,2}^{(p-1)}, \quad 0_1, \quad \cdots, \quad 0_\theta, \quad \rho_{p,3}^{(p-1)} \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right), \quad 0\Big] \tag{5.152}$$

Equivalently,

$$\rho_{k+1}^{(m)} = \Big[\rho_{p,1}^{(m)}, \quad \rho_{p,2}^{(m)}, \quad \cdots, \quad \rho_{p,p+1-m}^{(m)}, \quad 0_1, \quad \cdots, \quad 0_\theta,$$

$$\rho_{p,p+2-m}^{(m)} \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right), \quad 0\Big] \tag{5.153}$$

There are θ zeros in the middle of (5.153), and a zero at the last entry.

When $m = p$. Starting from the p^{th} layer, there are more than one sign function in the F matrix. The number of zero offsets start to vary as well.

$$\rho_{k+1}^{(p)} = \Big[\rho_{p,1}^{(p)}, \quad 0_1, \quad \cdots, \quad 0_\theta, \quad \rho_{p,2}^{(p)} \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right),$$

$$\rho_{p,3}^{(p)} \cdot sgn\left(B_2\Phi^{T(\theta+1)}H^T\right) \cdot t_1\Big] \tag{5.154}$$

where

$$t_1 = sgn\left(\frac{\left(\frac{B_2\Phi^{T(\theta+1)}}{B_2\Phi^{T(\theta+1)}H^T} - \frac{H\mathbf{A}\Phi^{T(\theta+1)}}{HA\Phi^{T(\theta+1)}H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right) \tag{5.155}$$

When $p + 1 \leq m \leq k$, start with the $(p+1)^{th}$ layer,

$$\rho_{k+1}^{(p+1)} = \left[\gamma, 0_1, \cdots, 0_{\theta-1}, P_1\left(H\mathbf{A}\Phi^T H^T\right) \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right), q_1\right] \tag{5.156}$$

where

$$q_1 = P_2\left(B_2\Phi^T H^T\right) \cdot sgn\left(B_2\Phi^{T(\theta+1)}H^T\right) \cdot t_1 + \beta\left(\Gamma^T H^T\right) \cdot sgn\left(\Gamma^T\Phi^{T\theta}H^T\right) \cdot t_2 \tag{5.157}$$

t_1 is defined in equation (5.155), and,

$$t_2 = sgn\left(\frac{\left(\frac{\Gamma^T\Phi^{T\theta}}{\Gamma^T\Phi^{T\theta}H^T} - \frac{H\mathbf{A}\Phi^{T(\theta+1)}}{H\mathbf{A}\Phi^{T(\theta+1)}H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right) \tag{5.158}$$

The $(p+2)^{th}$ layer is,

$$\rho_{k+1}^{(p+2)} = \left[\gamma, 0_1, \cdots, 0_{\theta-2}, P_1\left(H\mathbf{A}\Phi^{2T}H^T\right) \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right), q_2\right] \tag{5.159}$$

where

$$\begin{aligned} q_2 =& P_2\left(B_2\Phi^{2T}H^T\right) \cdot sgn\left(B_2\Phi^{T(\theta+1)}H^T\right) \cdot t_1 + \beta\left(\Gamma^T\Phi^T H^T\right) \cdot sgn\left(\Gamma^T\Phi^{T\theta}H^T\right) \cdot t_2 \\ &+ \beta\left(\Gamma^T H^T\right) \cdot sgn\left(\Gamma^T\Phi^{T(\theta-1)}H^T\right) \cdot t_3 \end{aligned} \tag{5.160}$$

t_1 and t_2 is defined in equation (5.155) and (5.158). t_3 is defined as follow.

$$t_3 = sgn\left(\frac{\left(\frac{\Gamma^T\Phi^{T(\theta-1)}}{\Gamma^T\Phi^{T(\theta-1)}H^T} - \frac{H\mathbf{A}\Phi^{T(\theta+1)}}{H\mathbf{A}\Phi^{T(\theta+1)}H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right) \tag{5.161}$$

Keep doing this until we find the ρ's for the k^{th} layer.

$$\rho_{k+1}^{(k)} = \left[\gamma, \quad P_1\left(H\mathbf{A}\Phi^{T\theta}H^T\right) \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right), \quad q_3\right] \tag{5.162}$$

where

$$\begin{aligned} q_3 =& P_2\left(B_2\Phi^{T\theta}H^T\right) \cdot sgn\left(B_2\Phi^{T(\theta+1)}H^T\right) \cdot t_1 \\ &+ \beta\left(\Gamma^T\Phi^{T(\theta-1)}H^T\right) \cdot sgn\left(\Gamma^T\Phi^{T\theta}H^T\right) \cdot t_2 \\ &+ \beta\left(\Gamma^T\Phi^{T(\theta-2)}H^T\right) \cdot sgn\left(\Gamma^T\Phi^{T(\theta-1)}H^T\right) \cdot t_3 + \cdots \\ &+ \beta\left(\Gamma^T H^T\right) \cdot sgn\left(\Gamma^T\Phi^T H^T\right) \cdot t_{\theta+1} \end{aligned} \tag{5.163}$$

where the scalar t_l when $2 \le l \le \theta+1$ is defined to be,

$$t_l = sgn\left(\frac{\left(\frac{\Gamma^T\Phi^{T(\theta+2-l)}}{\Gamma^T\Phi^{T(\theta+2-l)}H^T} - \frac{H\mathbf{A}\Phi^{T(\theta+1)}}{H\mathbf{A}\Phi^{T(\theta+1)}H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right) \tag{5.164}$$

Note that in (5.156), there are $(\theta-1)$ zeros. In (5.159), there are $(\theta-2)$ zeros. For each layer downward, there will be one less zero in the ρ sequence. At the k^{th} layer, there is no zero at all, see equation (5.162).

In summary, when $p+1 \le m \le k$,

$$\rho_{k+1}^{(m)} = \left[\gamma, 0_1, \cdots, 0_{k-m}, P_1\left(H\mathbf{A}\Phi^{T(m-p)}H^T\right) \cdot sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right), q_4\right] \tag{5.165}$$

Especially, when $m = k$, the subscript of the zero is $k - m = 0$. This means that there are no zeros in $\rho_{k+1}^{(k)}$.

$$\begin{aligned} q_4 =& P_2 \left(B_2 \Phi^{T(m-p)} H^T\right) \cdot sgn \left(B_2 \Phi^{T(\theta+1)} H^T\right) \cdot t_1 \\ & + \beta \left(\Gamma^T \Phi^{T(m-p-1)} H^T\right) \cdot sgn \left(\Gamma^T \Phi^{T\theta} H^T\right) \cdot t_2 \\ & + \beta \left(\Gamma^T \Phi^{T(m-p-2)} H^T\right) \cdot sgn \left(\Gamma^T \Phi^{T(\theta-1)} H^T\right) \cdot t_3 + \cdots \\ & + \beta \left(\Gamma^T H^T\right) \cdot sgn \left(\Gamma^T \Phi^{T(k-m+1)} H^T\right) \cdot t_{m-p+1} \end{aligned} \tag{5.166}$$

or, written as a sum,

$$\begin{aligned} q_4 =& P_2 \left(B_2 \Phi^{T(m-p)} H^T\right) \cdot sgn \left(B_2 \Phi^{T(\theta+1)} H^T\right) \cdot t_1 \\ & + \beta \cdot \sum_{l=1}^{m-p} \left(\Gamma^T \Phi^{T(m-p-l)} H^T\right) sgn \left(\Gamma^T \Phi^{T(\theta+1-l)} H^T\right) t_{l+1} \end{aligned} \tag{5.167}$$

The Bottom Layer where $m = k + 1$ is directly derived from the exponential part in equation (5.93).

$$\rho_{k+1}^{(k+1)} = \left[P_1 \left|H\mathbf{A}\Phi^{T(\theta+1)} H^T\right|, \quad q_5, \quad \gamma\right] \tag{5.168}$$

Like q_4, the quantity q_5 again should be a sum. This is due to the element combination which are co-aligned onto the $H\mathbf{A}$ direction.

$$\begin{aligned} q_5 =& P_2 \left|B_2 \Phi^{T(\theta+1)} H^T\right| \cdot t_1 + \beta \left|\Gamma^T \Phi^{T\theta} H^T\right| \cdot t_2 + \beta \left|\Gamma^T \Phi^{T(\theta-1)} H^T\right| \cdot t_3 + \cdots \\ & + \beta \left|\Gamma^T \Phi^T H^T\right| \cdot t_{\theta+1} \end{aligned} \tag{5.169}$$

or, as a sum,

$$q_5 = P_2 \left|B_2 \Phi^{T(\theta+1)} H^T\right| \cdot t_1 + \beta \cdot \sum_{l=1}^{\theta+1} \left|\Gamma^T \Phi^{T(\theta+2-l)} H^T\right| t_l \tag{5.170}$$

5.2.6.2 The i^{th} Child Term when $2 \le i \le \theta + 3$

Recall the θ-step old term at step k in equation (5.93). For convenience, rewrite the exponential term as,

$$\mathcal{E}^{k|k} = \exp\left(-P_1^{k|k}\left|B_1^{k|k}\nu\right| - P_2^{k|k}\left|B_2^{k|k}\nu\right| - \cdots - P_{\theta+2}^{k|k}\left|B_{\theta+2}^{k|k}\nu\right| + j\zeta^{k|k}\nu\right) \tag{5.171}$$

where,

$$P_l^{k|k} = \begin{cases} P_l & l = 1,2 \\ \beta & 3 \le l \le \theta + 2 \end{cases}, \quad B_l^{k|k} = \begin{cases} H\mathbf{A}\Phi^{T\theta} & l = 1 \\ B_2\Phi^{T\theta} & l = 2 \\ \Gamma^T\Phi^{T(\theta+2-l)} & 3 \le l \le \theta + 2 \end{cases} \tag{5.172}$$

Furthermore, define the sign of the difference between two parent directions.

$$t_l = sgn\left(\frac{\left(\frac{B_l^{k|k}\Phi^T}{B_l^{k|k}\Phi^T H^T} - \frac{B_i^{k|k}\Phi^T}{B_i^{k|k}\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})\,e_1^T}\right), \quad l < i \tag{5.173}$$

$$t_l = sgn\left(\frac{\left(\frac{B_{l+1}^{k|k}\Phi^T}{B_{l+1}^{k|k}\Phi^T H^T} - \frac{B_i^{k|k}\Phi^T}{B_i^{k|k}\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})\,e_1^T}\right), \quad l \ge i \tag{5.174}$$

At step $k+1$, since the i^{th} child term when $2 \le i \le \theta + 3$ is new, the F matrix is simply constructed.

$$F_{k+1}^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix}$$

$$\in \mathbb{R}^{(k+3-m)\times 2^{(k+2-m)}}, \quad 1 \le m \le k \tag{5.175}$$

$$F_{k+1}^{(k+1)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \in \mathbb{R}^{3\times 2} \tag{5.176}$$

where

$$s_1 = sgn\left(H\mathbf{A}\nu\right), \quad s_2 = sgn\left(-\frac{B_i^{k|k}\Phi^T}{B_i^{k|k}\Phi^T H^T}\right) \tag{5.177}$$

Now, consider each layer of the i^{th} child term at step $(k+1)$.

When $1 \leq m \leq p-1$, the top $(p-1)$ layer at step k only has one sign function. Therefore, starting from the second child term, a zero offset will be added to the ρ sequence at step $k+1$.

$$\rho_{k+1}^{(m)} = \left[\rho_{k,1}^{(m)}, \quad \rho_{k,2}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad 0, \quad \rho_{k,k+2-m}^{(m)} sgn\left(B_1^{k|k}\Phi^T H^T\right) \cdot t_1\right] \tag{5.178}$$

In particular, if we substitute in the quantities that we already know in equation (5.95), the ρ sequence is,

$$\begin{aligned}\rho_{k+1}^{(m)} = \Big[\rho_{p,1}^{(m)}, \quad \cdots, \quad \rho_{p,p+1-m}^{(m)}, \quad 0_1, \quad \cdots, \quad 0_\theta, \quad 0_{\theta+1}, \\ \rho_{p,p+2-m}^{(m)} sgn\left(H\mathbf{A}\Phi^{T(\theta+1)}H^T\right) \cdot t_1\Big]\end{aligned} \tag{5.179}$$

Look at the above expression, we see that the number of nonzero entries in $\rho_{k+1}^{(m)}$ does not increase compared to the prior sequence $\rho_p^{(m)}$ from its original parent at step p.

When $p \leq m \leq p-3+i$**, only if** $p \leq p-3+i$. At step k, starting from the p^{th} layer, there are more than one sign functions in the sequence. In fact, the ρ sequence at step k for layers $p \leq m \leq k$ can be summarized as,

$$\rho_k^{(m)} = \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad \rho_{k,k+2-m}^{(m)}, \quad \cdots, \quad \rho_{k,\theta+3}^{(m)}\right], \quad p \leq m \leq k \tag{5.180}$$

The specific value of each ρ has already been defined earlier in equation (5.97) and (5.99). The first $(k+1-m)$ elements in $\rho_k^{(m)}$ are offsets, while the rest of the elements are coefficients associated with the sign functions. Except for the first offset, all the other offsets are zero, i.e. $\rho_{k,2}^{(m)} = \rho_{k,3}^{(m)} = \cdots = \rho_{k,k+1-m}^{(m)} = 0$. As m increases by 1, the number of offsets will decrease by 1 and the number of sign functions will increase by 1. The total number of entries in $\rho_k^{(m)}$ for $p \leq m \leq k$ is always $\theta+3$. Hence, for the i^{th} sign function s_i, the first $(p-3+i)$ layers do not contain s_i, and s_i start to appear from layer $(p-2+i)$. For example, when $i=2$, the first $(p-3+i=p-3+2=p-1)$ layers do not have s_2. The p^{th} layer and lower layers contain s_2. Similarly, the third sign function s_3 does not appear until the $(p+1)^{th}$ layer. This is the reason why to split the case of $p \leq m \leq k$ into two scenarios.

Note that this subsection only apply to the scenario when $p \leq p-3+i$, i.e. $i \geq 3$. When $i=2$, the rest subsections already exhaust the recursion of all the layers.

For the m^{th} layer where $p \leq m \leq p-3+i$, the new offset at step $k+1$ is zero.

$$\begin{aligned}\rho_{k+1}^{(m)} &= \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad 0, \quad q_1\right] \\ &= \left[\rho_{k,1}^{(m)}, \quad 0_1, \quad \cdots, \quad 0_{k-m}, \quad 0_{k-m+1}, \quad q_1\right], \\ &p \leq m \leq p-3+i, \quad i \geq 3 \end{aligned} \tag{5.181}$$

$$\begin{aligned} q_1 &= \rho_{k,k+2-m}^{(m)} sgn\left(B_1^{k|k}\Phi^T H^T\right)\cdot t_1 + \rho_{k,k+3-m}^{(m)} sgn\left(B_2^{k|k}\Phi^T H^T\right)\cdot t_2 + \cdots \\ &\quad + \rho_{k,\theta+3}^{(m)} sgn\left(B_{m+2-p}^{k|k}\Phi^T H^T\right)\cdot t_{m+2-p} \\ &= \sum_{l=1}^{m+2-p} \rho_{k,k+1-m+l}^{(m)} sgn\left(B_l^{k|k}\Phi^T H^T\right)\cdot t_l \end{aligned} \tag{5.182}$$

In the ρ sequence at step $k+1$ in (5.181), there are $(k+3-m)$ entries. The first $(k+2-m)$ entries are offsets. Only $\rho_{k+1,1}^{(m)} = \rho_{k,1}^{(m)}$ is non-zero. All the rest offsets are zero. There are in total two non-zero entries in the sequence of $\rho_{k+1}^{(m)}$ for $p \leq m \leq p-3+i$, one of which is an offset, while the other one is a coefficient of the sign function s_1.

When $p-2+i \leq m \leq k$. From layer $p-2+i$ to layer k, every layer contains the sign function s_i. A non-zero offset will be introduced to the ρ sequence, and there will be a coefficient of the sign function s_1 as well.

$$\begin{aligned}\rho_{k+1}^{(m)} &= \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad \rho_{k,k+1-m+i}^{(m)} sgn\left(B_i^{k|k}\Phi^T H^T\right), \quad q_2\right] \\ &= \left[\rho_{k,1}^{(m)}, \quad 0_1 \quad \cdots, \quad 0_{k-m}, \quad \rho_{k,k+1-m+i}^{(m)} sgn\left(B_i^{k|k}\Phi^T H^T\right), \quad q_2\right], p-2+i \le m \le k\end{aligned} \tag{5.183}$$

where

$$\begin{aligned}q_2 &= \rho_{k,k+2-m}^{(m)} sgn\left(B_1^{k|k}\Phi^T H^T\right)\cdot t_1 + \rho_{k,k+3-m}^{(m)} sgn\left(B_2^{k|k}\Phi^T H^T\right)\cdot t_2 + \cdots \\ &\quad + \rho_{k,k-m+i}^{(m)} sgn\left(B_{i-1}^{k|k}\Phi^T H^T\right)\cdot t_{i-1} + \rho_{k,k-m+2+i}^{(m)} sgn\left(B_{i+1}^{k|k}\Phi^T H^T\right)\cdot t_i \\ &\quad + \cdots + \rho_{k,\theta+3}^{(m)} sgn\left(B_{m+2-p}^{k|k}\Phi^T H^T\right)\cdot t_{m+1-p} \\ &= \sum_{l=1}^{i-1}\rho_{k,k+1-m+l}^{(m)} sgn\left(B_l^{k|k}\Phi^T H^T\right)\cdot t_l + \sum_{l=i}^{m+1-p}\rho_{k,k+2-m+l}^{(m)} sgn\left(B_{l+1}^{k|k}\Phi^T H^T\right)\cdot t_l\end{aligned} \tag{5.184}$$

The sequence in (5.183) has three non-zero entries.

The Bottom Layer where $m = k+1$ has the ρ sequence updated as,

$$\rho_{k+1}^{(k+1)} = \left[P_i^{k|k}\left|B_i^{k|k}\Phi^T H^T\right|, \quad q_3, \quad \gamma\right] \tag{5.185}$$

where

$$q_3 = \sum_{l=1}^{i-1} P_l^{k|k}\left|B_l^{k|k}\Phi^T H^T\right| t_l + \sum_{l=i}^{\theta+1} P_{l+1}^{k|k}\left|B_{l+1}^{k|k}\Phi^T H^T\right| t_l \tag{5.186}$$

Therefore the bottom also contain 3 elements. They are all non-zero.

5.2.6.3 The Last Child Term $i = \theta + 4$ (old)

The last child term is the old one. It preserves the general structure of an old term that was derived in Section 5.2.4.

5.2.7 The Recursion: From Type-II Old Parent Terms

Recall the general form of type-II old term at step k, described in equation (5.106) - (5.110). At step k, there are $(k+1)$ elements in the argument of the exponential. Therefore, at step $k+1$, there will be $(k+3)$ child terms. The approach is similar. Rewrite equation (5.106) in the following form for convenience.

$$\mathcal{E}^{k|k} = \exp\left(-P_1^{k|k}\left|B_1^{k|k}\nu\right| - P_2^{k|k}\left|B_2^{k|k}\nu\right| - \cdots - P_{k+1}^{k|k}\left|B_{k+1}^{k|k}\nu\right| + j\zeta^{k|k}\nu\right) \tag{5.187}$$

where

$$P_l^{k|k} = \begin{cases} P_l & l = 1,2 \\ \beta & 3 \le l \le k+1 \end{cases}, \quad B_l^{k|k} = \begin{cases} B_i^{1|1}\Phi^{T(k-1)} & l = 1,2 \\ \Gamma^T\Phi^{T(k+1-l)} & 3 \le l \le k+1 \end{cases} \tag{5.188}$$

Let $P_{k+2}^{k|k} = \beta$ and $B_{k+2}^{k|k} = \Gamma^T\Phi^{-T}$. Also define the sign of the difference between two parent directions.

$$t_l = sgn\left(\frac{\left(\frac{B_l^{k|k}\Phi^T}{B_l^{k|k}\Phi^T H^T} - \frac{B_i^{k|k}\Phi^T}{B_i^{k|k}\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})\,e_1^T}\right), \quad l < i \tag{5.189}$$

$$t_l = sgn\left(\frac{\left(\frac{B_{l+1}^{k|k}\Phi^T}{B_{l+1}^{k|k}\Phi^T H^T} - \frac{B_i^{k|k}\Phi^T}{B_i^{k|k}\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})\,e_1^T}\right), \quad l \ge i \tag{5.190}$$

At step $(k+1)$, the i^{th} child terms when $1 \le i \le k+2$ are new child terms. The F matrix for those cases can be constructed as follows.

$$F_{k+1}^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots & \cdots \\ \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & \cdots & & & -1 \\ s_1 & s_1 & & & & \cdots & & & & s_1 \end{bmatrix}$$
$$\in \mathbb{R}^{(k+3-m)\times 2^{(k+2-m)}}, \quad 1 \le m \le k \tag{5.191}$$

$$F_{k+1}^{(k+1)} = \begin{bmatrix} 1 & -1 \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \in \mathbb{R}^{3\times 2} \tag{5.192}$$

where

$$s_1 = sgn\left(H\mathbf{A}\nu\right), \quad s_2 = sgn\left(-\frac{B_i^{k|k}\Phi^T}{B_i^{k|k}\Phi^T H^T}\right) \tag{5.193}$$

Now, consider each layer of the i^{th} child term at step $k+1$.

5.2.7.1 New Child Terms When $1 \leq i \leq k+2$

Under some circumstances, a zero offset will be introduced into ρ sequence while sometimes a non-zero offset is produced. According to the value of m, the layers will be discussed in three different scenarios.

When $1 \leq m \leq i-2$**, only if** $i \geq 3$. The sequence ρ at step k is,

$$\rho_k^{(m)} = \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad \rho_{k,k+2-m}^{(m)}, \quad \cdots, \quad \rho_{k,k+2}^{(m)}\right], \quad 1 \leq m \leq k \tag{5.194}$$

where $\rho_{k,2}^{(m)} = \rho_{k,3}^{(m)} = \cdots = \rho_{k,k+1-m}^{(m)} = 0$.

In the first $(i-2)$ layers, the coefficients associated with s_i are zero. At step $k+1$, ρ becomes,

$$\begin{aligned} \rho_{k+1}^{(m)} &= \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad 0, \quad q_1\right], \quad 1 \leq m \leq i-2, \quad i \geq 3 \\ &= \left[\rho_{k,1}^{(m)}, \quad 0_1, \quad \cdots, \quad 0_{k-m+1}, \quad q_1\right] \end{aligned} \tag{5.195}$$

where

$$q_1 = \sum_{l=1}^{m+1} \rho_{k,k+1-m+l}^{(m)} sgn\left(B_l^{k|k}\Phi^T H^T\right) \cdot t_l \tag{5.196}$$

Thus there are only two nonzero elements (not even three) in the sequence of $\rho_{k+1}^{(m)}$.

When $i-1 \leq m \leq k$, every layer contains a non-zero coefficient of the sign function s_i. Therefore, a non-zero offset will be introduced into the ρ sequence at step $k+1$, and there will be a coefficient of the sign function s_1 as well.

Because $\rho_{k,2}^{(m)} = \rho_{k,3}^{(m)} = \cdots = \rho_{k,k+1-m}^{(m)} = 0$, $\rho_{k+1}^{(m)}$ only has 3 non-zero entries.

$$\begin{aligned} \rho_{k+1}^{(m)} &= \left[\rho_{k,1}^{(m)}, \quad \cdots, \quad \rho_{k,k+1-m}^{(m)}, \quad \rho_{k,k+1-m+i}^{(m)} sgn\left(B_i^{k|k}\Phi^T H^T\right), \quad q_2\right] \\ &= \left[\rho_{k,1}^{(m)}, \quad 0_1, \quad \cdots, \quad 0_{k-m}, \quad \rho_{k,k+1-m+i}^{(m)} sgn\left(B_i^{k|k}\Phi^T H^T\right), \quad q_2\right] \end{aligned} \tag{5.197}$$

where

$$q_2 = \sum_{l=1}^{i-1} \rho_{k,k+1-m+l}^{(m)} sgn\left(B_l^{k|k}\Phi^T H^T\right) \cdot t_l + \sum_{l=i}^{m} \rho_{k,k+2-m+l}^{(m)} sgn\left(B_{l+1}^{k|k}\Phi^T H^T\right) \cdot t_l \tag{5.198}$$

When $m = k+1$, consider the bottom layer. The sequence ρ is updated as,

$$\rho_{k+1}^{(k+1)} = \left[P_i^{k|k}\left|B_i^{k|k}\Phi^T H^T\right|, \quad q_3, \quad \gamma\right] \tag{5.199}$$

where

$$q_3 = \sum_{l=1}^{i-1} P_l^{k|k}\left|B_l^{k|k}\Phi^T H^T\right| t_l + \sum_{l=i}^{k+2} P_{l+1}^{k|k}\left|B_{l+1}^{k|k}\Phi^T H^T\right| t_l \tag{5.200}$$

5.2.7.2 Old Child Term when $i = k+3$

At step $k+1$, the formula of the old child term is consistent with previous discussions in equation (5.93) - (5.102), hence omitted.

5.2.8 Properties

The recursive structure of G layers uncovers an interesting property:

Theorem 5.2.1. *Consider the two-state case. For any new term, there are no more than three non-zero elements in ρ sequence of each layer of G.*

Proof. This theorem is proven by induction.

1. Examine the ρ sequence of each layer of G at step $k = 2$ for new terms. The first three child terms at step $k = 2$ from each of the three parent terms are new terms. The G term of each one has two layers. Check the ρ sequence expressed in equation (5.39) (5.46) (5.50) (5.53) (5.57) and (5.60). Each layer at step $k = 2$ only has three elements in ρ.

2. Assume the theorem statement holds at step k, we show that the number of non-zero elements does not increase at step $k + 1$.

   (a) Consider the new child terms at step $k + 1$ that are produced from new parent terms at step k. In this case, the first three child terms at step $k + 1$ are new.

      i. For the first child term at step $k + 1$, look at equation (5.112) (5.114) and (5.117). Particularly, compare (5.112) with (5.69). One zero element is introduced into $\rho_{k+1}^{(m)}$ for $1 \leq m \leq k - 1$ after propagated and updated from $\rho_k^{(m)}$. Therefore, there should be still no more than three non-zero elements in the ρ sequence at step $k + 1$ for this scenario.

      ii. For the second child term at step $k + 1$, look at equation (5.122) (5.124) and (5.126). Again, compare (5.122) with (5.69). One zero element is introduced into $\rho_{k+1}^{(m)}$ for $1 \leq m \leq k - 1$ after propagated and updated from $\rho_k^{(m)}$. Therefore, there should be still no more than three non-zero elements in the ρ sequence at step $k + 1$ for this scenario.

      iii. For the third child term at step $k + 1$, look at equation (5.132) (5.134) and (5.136). Compare (5.132) with (5.69). One zero element is introduced into $\rho_{k+1}^{(m)}$ for $1 \leq m \leq k - 1$ after propagated and updated from $\rho_k^{(m)}$. Therefore, there should be still no more than three non-zero elements in the ρ sequence at step $k + 1$ for this scenario.

   (b) Consider the new child terms at step $k + 1$ that are produced from type-I old parent terms at step k. In this case, the first $(\theta+3)$ child terms at step $k+1$ are new, where θ indicates how old this term is, formulated earlier in Section 5.2.4.

i. For the first child term at step $k+1$, look at equation (5.153) (5.154) (5.165) and (5.168). Compare (5.153) with (5.69). $(\theta+1)$ zero element is introduced into $\rho_{k+1}^{(m)}$ for $1 \leq m \leq p-1$ after propagated and updated from $\rho_p^{(m)}$ at step p. Therefore, there should be still no more than three non-zero elements in the ρ sequence at step $k+1$ for this scenario.

ii. For the i^{th} child term when $2 \leq i \leq \theta+3$ at step $k+1$, look at equation (5.179) (5.181) (5.183) and (5.185). There should be no more than three non-zero elements in the ρ sequence at step $k+1$ for this scenario.

(c) Consider the new child terms at step $k+1$ that are produced from type-II old parent terms at step k. In this case, the first $(k+2)$ child terms at step $k+1$ are new terms. Consider equation (5.195) (5.197) and (5.199). There are at most three non-zero elements in ρ sequence at step $k+1$.

By exhausting all scenarios of new terms at step $k+1$, we have shown that there are at most three non-zero elements in ρ sequence. □

Remark 5.2.2. For old terms, there are more than three non-zero entries for certain layers. However, among those non-zero entries, only one of them is the offset, while the rests are all associated with the sign functions. When this old term produces new child terms, the number of non-zero entries in the new ρ will again collapse to at most three, due to the direction combination aligned with $H\mathbf{A}$.

Furthermore, this comprehensive analysis on the real component of the G terms for two-state systems also shows the fundamental mechanism of how zeros are introduced into the G structure. In the update integral, two zeros are added artificially in order to complete the integral properly. Now we understand analytically that these added zeros in the integral appear as offsets of each layer in G. Based on this understanding, the structure for a general multi-dimensional system is simplified by splitting the offset ρ_o, F_o and coefficient ρ_c, F_c

apart, as proposed up front in Chapter 5.1. This technique will stop adding zeros into the structure anymore, which potentially enhances the computational efficiency.

CHAPTER 6

A Pre-Computational Technique

In the estimator structure, much of the computation is independent of the measurement data. According to the analysis in the previous chapter, the argument of the exponential part, i.e. $P_{i,j}^{k|k}$ and $B_{i,j}^{k|k}$ in (1.15), are not functions of the measurement sequence. In the G terms, the real component of each layer, $R_{k,l}^{(m)}$, does not depend on the measurements. They can be pre-computed "offline" and stored *a priori*. This makes the "online" process easier because all the directions along with their coefficients in the argument of the exponential term collapse into a single scalar. Furthermore, all the offsets and sign functions in each denominator of the G term also become a scalar by *a priori* picking the spectral variable ν. An offline - online separation of the estimator structure allows a significant amount of computational efficiency. From our analysis of the two and three state systems, and our numerical experiments with four state system, it appears that the S matrix, in general, is not only independent of the measurements, but also independent of system parameters. It has been shown numerically that the offline efficiency is greatly enhanced by utilizing the S matrix to combine exponential terms without comparison at each step.

Based on these observations, a pre-computational implementation for the Cauchy estimator is proposed as follows.

6.1 The Offline Stage

Recall the form of the characteristic function described in (3.1). In order to express the imaginary part of the exponential term $\zeta^{k|k}$ more explicitly, rewrite it as $\zeta^{k|k} = y_k Q_1^{k|k} +$

${u_1^1}^T Q_2^{k|k}$, where u_1^1 is the median of the initial states, $Q_1^{k|k} \in \mathbb{R}^{k\times n}$, and $Q_2^{k|k} \in \mathbb{R}^{n\times n}$. Also, rewrite the imaginary part of each G layer at step k as $Im_k^{(m)} = y_k N_{1k}^{(m)} - {u_1^1}^T N_{2k}^{(m)}$, and when $m = k$, $N_{1k}^{(k)} \in \mathbb{R}^{k\times 1}$, $N_{2k}^{(k)} \in \mathbb{R}^{n\times 1}$. Altogether, during the offline stage, the parameters $P^{k|k}$, $B^{k|k}$, $\rho_{ok}^{(m)}$, $\rho_{ck}^{(m)}$, $F_{ok}^{(m)}$, $F_{ck}^{(m)}$, $Q_1^{k|k}$, $Q_2^{k|k}$, $N_{1k}^{(m)}$ and $N_{2k}^{(m)}$ are computed recursively from step to step. These parameters are independent of the measurements.

6.1.1 Initialization

At the first measurement update, there are $(n+1)$ terms. For the i^{th} term where $1 \le i \le n$, the exponential term can be written as,

$$\mathcal{E}_i^{1|1}(\nu) = \exp\left\{ -\sum_{l=1,l\neq i}^{n} \frac{\alpha_l}{|e_i H^T|} |HA_{il}\nu| - \gamma \left| -\frac{e_i\nu}{e_i H^T} \right| \right.$$
$$\left. +j \left[z_1 \frac{e_i\nu}{e_i H^T} + {u_1^1}^T \left(I - \frac{H^T e_i}{e_i H^T} \right) \right] \nu \right\} \tag{6.1}$$

where A_{il} are defined as $A_{il} = e_i^T e_l - e_l^T e_i$ and e_i are the unit row vectors. This form is obtained by solving the update integral in Appendix B in [5]. Hence, the $P^{1|1}$'s are the corresponding coefficients $\frac{\alpha_l}{|e_i H^T|}$ and γ, and the directions $B^{k|k}$'s are HA_{il} and $-\frac{e_i\nu}{e_i H^T}$. The imaginary part of the exponential is expressed by $Q_1^{1|1}(i) = \frac{e_i\nu}{e_i H^T}$ and $Q_2^{1|1}(i) = I - H^T Q_1^{1|1}(i)$.

Next, consider the G terms of the first n terms at step $k = 1$. Again, by taking the update integral and organizing the form into the proposed structure in (5.1) - (5.4), one can obtain,

$$\rho_{o1}^{(1)}(i) = \left[\alpha_i \left| e_i H^T \right| \right] \tag{6.2}$$

$$\begin{aligned} \rho_{c1}^{(1)}(i) = \big[&\alpha_1 \left(e_1 H^T \right) sgn(e_i H^T) \cdot sgn(HA_{i1}\nu), \\ &\cdots, \alpha_{i-1} \left(e_{i-1} H^T \right) sgn(e_i H^T) \cdot sgn(HA_{i(i-1)}\nu), \\ &\alpha_{i+1} \left(e_{i+1} H^T \right) sgn(e_i H^T) \cdot sgn(HA_{i(i+1)}\nu), \\ &\cdots, \alpha_n \left(e_n H^T \right) sgn(e_i H^T) \cdot sgn(HA_{in}\nu), \big] \end{aligned} \tag{6.3}$$

$$F_{o1}^{(1)}(i) = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c1}^{(1)}(i) = \begin{bmatrix} s_1 & s_1 \\ \vdots & \\ s_n & s_n \end{bmatrix} \tag{6.4}$$

where the sign functions are $s_1 = sgn(HA_{i1}\nu)$, ..., $s_{i-1} = sgn(HA_{i(i-1)}\nu)$, $s_i = sgn(HA_{i(i+1)}\nu)$, ..., $s_{n-1} = sgn(HA_{in}\nu)$, $s_n = sgn(-\frac{e_i\nu}{e_iH^T})$. The imaginary component of the G term consists of $N_1^{(1)}(i) = 1$ and $N_2^{(1)}(i) = H^T$.

For the $(n+1)^{th}$ term, initialize the parameters as below.

$$\mathcal{E}_{n+1}^{1|1}(\nu) = \exp\left(-\sum_{l=1}^{n} \alpha_l \left|e_l\nu\right| + ju_1^{1^T}\nu\right) \tag{6.5}$$

$$\rho_{o1}^{(1)}(i = n+1) = \gamma, \quad \rho_{c1}^{(1)}(i = n+1) = \left[\alpha_1\left(e_1H^T\right), \cdots, \alpha_n\left(e_nH^T\right)\right] \tag{6.6}$$

$$F_{o1}^{(1)}(i = n+1) = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c1}^{(1)}(i = n+1) = \begin{bmatrix} s_1 & s_1 \\ \vdots & \\ s_n & s_n \end{bmatrix} \tag{6.7}$$

where $s_1 = sgn(e_1\nu)$, $s_2 = sgn(e_2\nu)$,..., $s_n = sgn(e_n\nu)$. Let $Q_1^{1|1}(i = n+1) = \{0\}_{1\times n}$, $Q_2^{1|1}(i = n+1) = I_{n\times n}$, $N_1^{(1)}(i = n+1) = 1$, $N_2^{(1)}(i = n+1) = H^T$.

6.1.2 Update

Suppose at the k^{th} measurement update, the parameters to construct a complete characteristic function are known. The goal, then, is to derive the parameters $P^{k+1|k+1}$, $B^{k+1|k+1}$, $\rho_{o(k+1)}^{(m)}$, $\rho_{c(k+1)}^{(m)}$, $F_{o(k+1)}^{(m)}$, $F_{c(k+1)}^{(m)}$, $Q_1^{k+1|k+1}$, $Q_2^{k+1|k+1}$, $N_{1(k+1)}^{(m)}$ and $N_{2(k+1)}^{(m)}$ for $1 \leq m \leq k+1$, as a recursive function of their value at step k.

First, look at the exponential term. Consider (3.1), (1.15) at step k. Define $P_{i,N_e+1}^{k|k} = \beta$ and $B_{i,N_e+1}^{k|k} = \Gamma^T\Phi^{-T}$. At step $k+1$, the exponential part of the r^{th} child term can be expressed as,

$$\mathcal{E}_{i,r}^{k+1|k+1}(\nu) = \exp\left(- \sum_{l=1, l\neq r}^{N_{ei}^{k+1|k+1}} P_{i,l}^{k|k} \left| B_{i,l}^{k|k} \Phi^T H^T \right| \right.$$
$$\left. \cdot \left| \frac{B_{i,l}^{k|k}\Phi^T \nu}{B_{i,l}^{k|k}\Phi^T H^T} - \frac{B_{i,r}^{k|k}\Phi^T \nu}{B_{i,r}^{k|k}\Phi^T H^T} \right| + j\zeta_{i,r}^{k+1|k+1}\nu \right) \tag{6.8}$$

The coefficients $P_{i,l}^{k|k} \left| B_{i,l}^{k|k}\Phi^T H^T \right|$ forms the new P's and the absolute value of the differences of parent directions forms the new B's. Some directions are co-aligned, recall Chapter 2. Note that $P^{k+1|k+1}$ and $B^{k+1|k+1}$ are obtained after directions are combined.

The imaginary part is constructed as $\zeta_{i,r}^{k+1|k+1} = y_{k+1}Q_1^{k+1|k+1} + {u_1^1}^T Q_2^{k+1|k+1}$. From equation (3.43b) in [5] and be consistent with the notation in this dissertation,

$$\zeta_{i,r}^{k+1|k+1} = \left(z_{k+1} - H\Phi{\zeta_i^{k|k}}^T \right) \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} + \zeta_i^{k|k}\Phi^T \tag{6.9}$$

Substitute $\zeta_i^{k|k} = y_k Q_1^{k|k} + {u_1^1}^T Q_2^{k|k}$ where $y_k = \begin{bmatrix} z_1 & z_2 & \cdots & z_k \end{bmatrix}$ into equation (6.9). Then,

$$\begin{aligned}
\zeta_{i,r}^{k+1|k+1} &= z_{k+1}\frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} + \zeta_i^{k|k}\Phi^T \left(I - H^T \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} \right) \\
&= z_{k+1}\frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} + y_k Q_1^{k|k}\Phi^T \left(I - H^T \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} \right) + {u_1^1}^T Q_2^{k|k}\Phi^T \left(I - H^T \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} \right) \\
&= y_{k+1} \begin{bmatrix} Q_1^{k|k}\Phi^T (I - H^T \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T}) \\ \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} \end{bmatrix} + {u_1^1}^T Q_2^{k|k}\Phi^T \left(I - H^T \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} \right) \\
&\stackrel{def}{=} y_{k+1}Q_1^{k+1|k+1} + {u_1^1}^T Q_2^{k+1|k+1}
\end{aligned} \tag{6.10}$$

Therefore,

$$Q_1^{k+1|k+1} = \begin{bmatrix} Q_1^{k|k}\Phi^T (I - H^T \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T}) \\ \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} \end{bmatrix} \tag{6.11}$$

$$Q_2^{k+1|k+1} = Q_2^{k|k}\Phi^T \left(I - H^T \frac{B_i^{k|k}\Phi^T\nu}{B_i^{k|k}\Phi^T H^T} \right) \tag{6.12}$$

Next, consider the G term. Suppose at step k, the G terms are described in equation (5.1)-(5.4). There are three different scenarios depending upon which child term i is examined.

- When $1 \leq i \leq q$

  The offset component of the top k layer can be written as,

$$\rho_{o(k+1)}^{(m)} = \left[\rho_{ok}^{(m)}, \rho_{ck,i}^{(m)} sgn\left(B_i^{k|k}\Phi^T H^T\right)\right] \tag{6.13}$$

$$F_{o(k+1)}^{(m)} = \begin{bmatrix} F_{ok}^{(m)} & | & F_{ok}^{(m)} \\ ---- & + & ---- \\ 1\cdots 1 & | & -1\cdots -1 \end{bmatrix} \tag{6.14}$$

  where $1 \leq m \leq k$.

  The offset component of the bottom layer ($(k+1)^{th}$ layer) can be expressed as follows:

$$\rho_{o(k+1)}^{(k+1)} = \left[P_i^{k|k}\left|B_i^{k|k}\Phi^T H^T\right|\right], \quad F_{o(k+1)}^{(k+1)} = \begin{bmatrix}1 & -1\end{bmatrix} \tag{6.15}$$

  The sign function component of the top k layer can be calculated as,

$$\begin{aligned}\rho_{c(k+1)}^{(m)} = \Big[&\rho_{ck,1}^{(m)} sgn\left(B_1^{k|k}\Phi^T H^T\right), \quad \cdots, \quad \rho_{ck,i-1}^{(m)} sgn\left(B_{i-1}^{k|k}\Phi^T H^T\right), \\ &\rho_{ck,i+1}^{(m)} sgn\left(B_{i+1}^{k|k}\Phi^T H^T\right), \quad \cdots, \quad \rho_{ck,q}^{(m)} sgn\left(B_q^{k|k}\Phi^T H^T\right)\Big]\end{aligned} \tag{6.16}$$

$$F_{c(k+1)}^{(m)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \\ s_2 & s_2 & \cdots & s_2 \\ \vdots & & & \\ s_{q-1} & s_{q-1} & \cdots & s_{q-1} \end{bmatrix} \tag{6.17}$$

  where $1 \leq m \leq k$.

  For the bottom layer, the sign function component is expressed as,

$$\begin{aligned}\rho_{c(k+1)}^{(k+1)} = \Big[&P_1^{k|k}\left|B_1^{k|k}\Phi^T H^T\right|, \cdots, P_{i-1}^{k|k}\left|B_{i-1}^{k|k}\Phi^T H^T\right|, \\ &P_{i+1}^{k|k}\left|B_{i+1}^{k|k}\Phi^T H^T\right|, \cdots, P_{N_e+1}^{k|k}\left|B_{N_e+1}^{k|k}\Phi^T H^T\right|, \gamma\Big]\end{aligned} \tag{6.18}$$

$$F_{c(k+1)}^{(k+1)} = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \\ \vdots & \\ s_{N_e+1} & s_{N_e+1} \end{bmatrix} \tag{6.19}$$

s_1 through s_{N_e+1} in (6.17) and (6.19) are the sign function of the first $N_e + 1$ child directions multiplied by the variable ν. Note that $\rho_{c(k+1)}^{(m)}$ and $F_{c(k+1)}^{(m)}$ in (6.16)-(6.19) at each layer should also be refined by combining the co-aligned directions in the consistent way as the exponential term. If $q = 1$ for the m^{th} layer at step k, the $\rho_{c(k+1)}^{(m)}$ and $F_{c(k+1)}^{(m)}$ is empty.

- When $q + 1 \leq i \leq N_e + 1$

In this case, the offset component of the top k layer is,

$$\rho_{o(k+1)}^{(m)} = \rho_{ok}^{(m)}, \quad F_{o(k+1)}^{(m)} = \left[F_{ok}^{(m)} \quad | \quad F_{ok}^{(m)} \right] \tag{6.20}$$

where $1 \leq m \leq k$.

The offset component of the bottom layer is in the same form as the $1 \leq i \leq q$ case, i.e.

$$\rho_{o(k+1)}^{(k+1)} = P_i^{k|k} \left| B_i^{k|k} \Phi^T H^T \right|, \quad F_{o(k+1)}^{(k+1)} = \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{6.21}$$

Similarly, the sign function component of the top k layer is,

$$\rho_{c(k+1)}^{(m)} = \left[\rho_{ck,1}^{(m)} sgn\left(B_1^{k|k} \Phi^T H^T \right), \quad \cdots, \quad \rho_{ck,q}^{(m)} sgn\left(B_q^{k|k} \Phi^T H^T \right) \right] \tag{6.22}$$

$$F_{c(k+1)}^{(m)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \\ s_2 & s_2 & \cdots & s_2 \\ \vdots & & & \\ s_q & s_q & \cdots & s_q \end{bmatrix} \tag{6.23}$$

where $1 \leq m \leq k$. And the sign function component of the bottom layer is,

$$\rho_{c(k+1)}^{(k+1)} = \Big[P_1^{k|k} \left|B_1^{k|k}\Phi^T H^T\right|, \quad \cdots, \quad P_{i-1}^{k|k} \left|B_{i-1}^{k|k}\Phi^T H^T\right|, \\ P_{i+1}^{k|k} \left|B_{i+1}^{k|k}\Phi^T H^T\right|, \quad \cdots, \quad P_{N_e+1}^{k|k} \left|B_{N_e+1}^{k|k}\Phi^T H^T\right|, \quad \gamma\Big] \tag{6.24}$$

$F_{c(k+1)}^{(k+1)}$ is given in (6.19). Again, s_1 through s_{N_e+1} are the sign function of the first $N_e + 1$ child directions multiplied by the variable ν. Finally, co-aligned directions at step $k + 1$ must be combined to find $\rho_{c(k+1)}^{(m)}$ and $F_{c(k+1)}^{(m)}$ for all m.

- When $i = N_e + 2$

  The offset component of the top k layer is,

$$\rho_{o(k+1)}^{(m)} = \rho_{ok}^{(m)}, \quad F_{o(k+1)}^{(m)} = \left[F_{ok}^{(m)} \quad | \quad F_{ok}^{(m)}\right] \tag{6.25}$$

  The offset component of the bottom layer is,

$$\rho_{o(k+1)}^{(k+1)} = \gamma, \quad F_{o(k+1)}^{(k+1)} = \begin{bmatrix}1 & -1\end{bmatrix} \tag{6.26}$$

  The sign function component of the top k layer is,

$$\rho_{c(k+1)}^{(m)} = \rho_{ck}^{(m)}, \quad F_{c(k+1)}^{(m)} = \left[F_{ck}^{(m)} \quad | \quad F_{ck}^{(m)}\right] \tag{6.27}$$

  And the sign function component of the bottom layer is,

$$\rho_{c(k+1)}^{(k+1)} = \left[P_1^{k|k}\left(B_1^{k|k}\Phi^T H^T\right), \quad \cdots, \quad P_{N_e}^{k|k}\left(B_{N_e}^{k|k}\Phi^T H^T\right), \beta\left(\Gamma^T H^T\right)\right] \tag{6.28}$$

  where $F_{c(k+1)}^{(k+1)}$ is given by (6.19).

  The sign functions are $s_1 = sgn\left(B_1\Phi^T\nu\right)$, $s_2 = sgn\left(B_2\Phi^T\nu\right)$, ..., $s_{N_e} = sgn\left(B_{N_e}\Phi^T\nu\right)$, $s_{N_e+1} = sgn\left(\Gamma^T\nu\right)$. In this case, child directions will not be co-aligned.

Finally, let's construct the imaginary part of each layer in the G terms. Still consider step $k+1$. The update process will not change the imaginary part of the top k layers, i.e. $Im_{k+1}^{(m)} =$

$Im_k^{(m)}$, $1 \leq m \leq k$. For the bottom layer, construct $Im_{k+1}^{(k+1)} = y_{k+1} N_{1,k+1}^{(k+1)} - u_1^{1\,T} N_{2,k+1}^{(k+1)}$. According to Appendix B.1 and B.2 in [5], $Im_{k+1}^{(k+1)} = z_{k+1} - \zeta_i^{k|k} \Phi^T H^T$. Substitute the expression of $\zeta_i^{k|k}$, then one can obtain the recursion of $N_{1,k+1}^{(k+1)}$ and $N_{2,k+1}^{(k+1)}$ in the following form.

$$N_{1,k+1}^{(k+1)} = \begin{bmatrix} -Q_1^{k|k} \Phi^T H^T \\ 1 \end{bmatrix}, \quad N_{2,k+1}^{(k+1)} = Q_2^{k|k} \Phi^T H^T \tag{6.29}$$

Remark 6.1.1. Certain directions are co-aligned and hence need to be combined in order to implement this algorithm. That involves equation (6.16) – (6.19) and (6.22) – (6.24). Generally, this combination process can all be done by numerically comparing the directions. However, for two-state and three-state case, it has been fully uncovered how to combine the directions analytically without the need of numerical comparison. This is developed from the extensive studies of directions co-alignment in Chapter 2.

The explicit, analytic form of updating R in each layer of G terms without numerically comparing directions for two-state and three-state cases are provided in Appendix E and F.

6.1.3 Construction of the Offline Stage

During the offline stage, the parameters P, B, Q_1, Q_2, ρ_o, ρ_c, F_o, F_c, N_1 and N_2 of each term are updated and stored according to the S matrix scheme. They are computed inside the offline loop, providing complete information for estimator update. However, not all of them are passed to the online stage directly. In order to improve the online efficiency, some of the parameters can be combined.

First, pick *a priori* ν as $\hat{\nu}$. $\hat{\nu}$ can be randomly chosen, as long as it is not orthogonal with any directions $B_i^{k|k}$. Define a_i as a n-dim column vector,

$$a_i^{k|k} = -\sum_{l=1}^{N_{ei}^{k|k}} P_i^{k|k} sgn\left(B_i^{k|k} \hat{\nu} \right) B_i^{k|k^T}, \tag{6.30}$$

which is related to the real part of the exponential term in (1.15), and is used in the online stage.

Also, compute the real part of each layer in the G terms. This manipulation simplifies the online computation by reducing the sequences of the parameters, ρ's and F's, into single scalars $R_k^{(m)}$ as

$$R_k^{(m)} = \left[\rho_{ok}^{(m)} \quad | \quad \rho_{ck}^{(m)}\right] \cdot \begin{bmatrix} F_{ok}^{(m)} \\ --- \\ F_{ck}^{(m)} \end{bmatrix} \tag{6.31}$$

As a result, for the exponential part, $a_i^{k|k}$, $Q_1^{k|k}$ and $Q_2^{k|k}$ will be passed onto the online stage in a reduced form. For the G part, $R^{k|k}$, $N_{1,k}^{(m)}$ and $N_{2,k}^{(m)}$ will be passed onto the online stage.

6.2 The Online Stage

During the online stage, the conditional mean and conditional variance given the measurement data is determined by taking derivatives of the characteristic function with respect to a picked spectral variable $\hat{\nu}$ and evaluating at the origin. [5] gives closed form expressions for the derivatives.

To determine the online values of the conditional mean and conditional variance, the following parameters are determined. $a_i^{k|k}$ is received directly from the offline stage. Define the column vector $b_i^{k|k} \in \mathbb{R}^n$ to be,

$$b_i^{k|k} = \zeta^{k|k^T} = Q_1^{k|k^T} \cdot y_k^T + Q_2^{k|k^T} \cdot u_1^1 \tag{6.32}$$

Also define $c_i^{k|k}$ and $d_i^{k|k}$ to be the real and imaginary parts of each G term, $G_i^{k|k}(\hat{\nu})$

$$G_i^{k|k}(\nu) = G_i^{k|k}(\hat{\nu}) \stackrel{\text{def}}{=} c_i^{k|k} + j d_i^{k|k} \tag{6.33}$$

where $c_i^{k|k}$ and $d_i^{k|k}$ are scalars. At each layer, the imaginary component equals $Im_k^{(m)} = y_m N_{1,k}^{(m)} - u_1^{1^T} N_{2,k}^{(m)}$. With the real component $R_k^{(m)}$ and the imaginary component $Im_k^{(m)}$ in each layer of G as deterministic scalars, $c_i^{k|k}$ and $d_i^{k|k}$ can be obtained by algebraically computing the value of all the layers.

Let $M_1 = a_i^{k|k} a_i^{k|k^T} - b_i^{k|k} b_i^{k|k^T}$, $M_2 = a_i^{k|k} b_i^{k|k^T} + b_i^{k|k} a_i^{k|k^T}$ be n by n matrices. Starting with the equation (5.29) and (5.32) in [5] and utilizing our definition of $a_i^{k|k}$, $b_i^{k|k}$, $c_i^{k|k}$ and $d_i^{k|k}$, the conditional mean $\hat{x}_k$ and second moment $E\left[x_k x_k^T | Y_k\right]$ can be rewritten in the following form.

$$\hat{x}_k = \frac{\sum_{i=1}^{N_t^{k|k}} \left(d_i^{k|k} a_i^{k|k} + c_i^{k|k} b_i^{k|k}\right)}{\sum_{i=1}^{n_t^{k|k}} c_i^{k|k}} \tag{6.34}$$

$$E\left[x_k x_k^T | Y_k\right] = \frac{\sum_{i=1}^{N_t^{k|k}} \left[c_i^{k|k} M_1 - d_i^{k|k} M_2\right]}{-\sum_{i=1}^{N_t^{k|k}} c_i{}^{k|k}} \tag{6.35}$$

The conditional error variance is $E\left[e_k e_k^T | Y_k\right] = E\left[x_k x_k^T | Y_k\right] - \hat{x}_k \hat{x}_k^T$.

The pre-computational technique is illustrated in Figure 6.1.

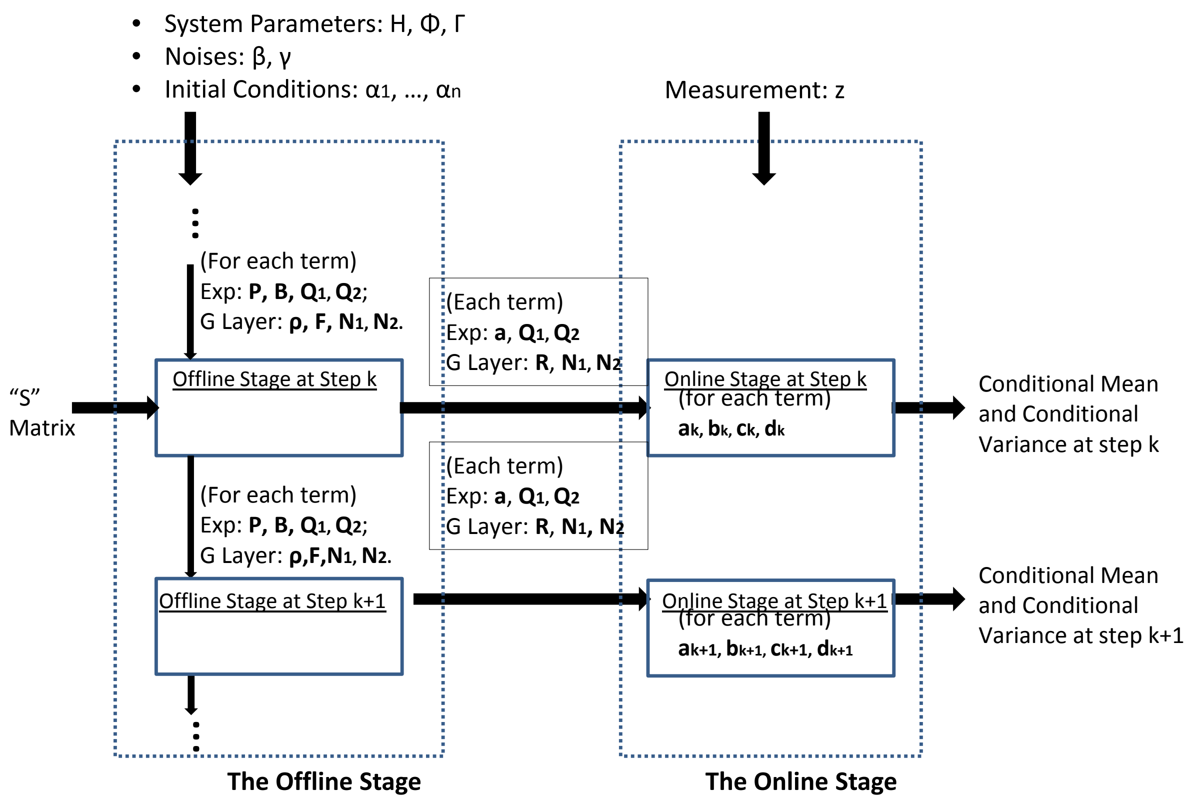

Figure 6.1: The pre-computational technique

6.3 Finite Approximation

The size of the estimator structure, i.e. the number of terms in the characteristic function, grows significantly as more measurement history data is processed. However, not all measurements have the same influence on the estimates. It has been shown that as time goes on, the measurement from the distant past, as well as the uncertainties of the initial state, has less and less influence on the current estimates. We use a "sliding window" method to process a fixed number of recent measurements. This approximation keeps a finite estimator structure.

To implement this finite approximation, first initialized the "sliding window" with appropriate initial mean of the state at step k, defined earlier as u_1^1, which matches the estimated conditional mean $\hat{x}_k$ propagated by the transition matrix Φ, i.e. let $u_1^1 = \Phi\hat{x}_k$. Then, by inputting the initial mean of the window, as well as the measurement sequence $y_{k+L}^* = [z_{k+1} \quad \cdots \quad z_{k+L}]$ with a sliding window size of L into the offline characteristic function structure at step $k = L$, the algorithm produces the approximated estimate at step $k + L$. The performance of the Cauchy estimator with the sliding window approximation is demonstrated numerically by various examples, presented in the next chapter.

CHAPTER 7

Simulation Results

7.1 A Three-State System

For a three-state system, the system dynamics are chosen to be,

$$\Phi = \begin{bmatrix} 1.5374 & -0.9874 & 0.4924 \\ 1.1026 & -0.3642 & 0.5942 \\ 0.3853 & -0.8192 & 1.3268 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.1 \\ 0.3 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0.5 & 0.2 \end{bmatrix} \tag{7.1}$$

The eigenvalues of Φ are 0.90 and $0.80 \pm 0.55j$. Process noise, measurement noise and initial states are all assumed to be Cauchy distributed with $\beta = 0.2$, $\gamma = 0.2$, $\alpha_1 = 0.2$, $\alpha_2 = 0.2$, $\alpha_3 = 0.2$, $u_1^1 = \{0\}_3$. The simulation runs 101 steps with a sliding window size of $L = 7$.

Fig. 7.1 shows the estimates error and standard deviation of the Cauchy estimator under Cauchy-type process noise and measurement noise. The correlation of the estimates is shown in fig. 7.2. "KF" in the figure stands for the Kalman Filter, whose parameters are chosen to least square fit the system's Cauchy pdfs as discussed in [3]. The Cauchy estimator performs well compared to the standard Kalman Filter. The Cauchy state estimate error stays within the 1-σ range generated in the Cauchy estimator implementation, as shown in the figure by dashed lines. When there is an impulsive fluctuation in either the process or measurement noise, the Cauchy estimator yields a smaller estimation error and converges faster than the Kalman Filter.

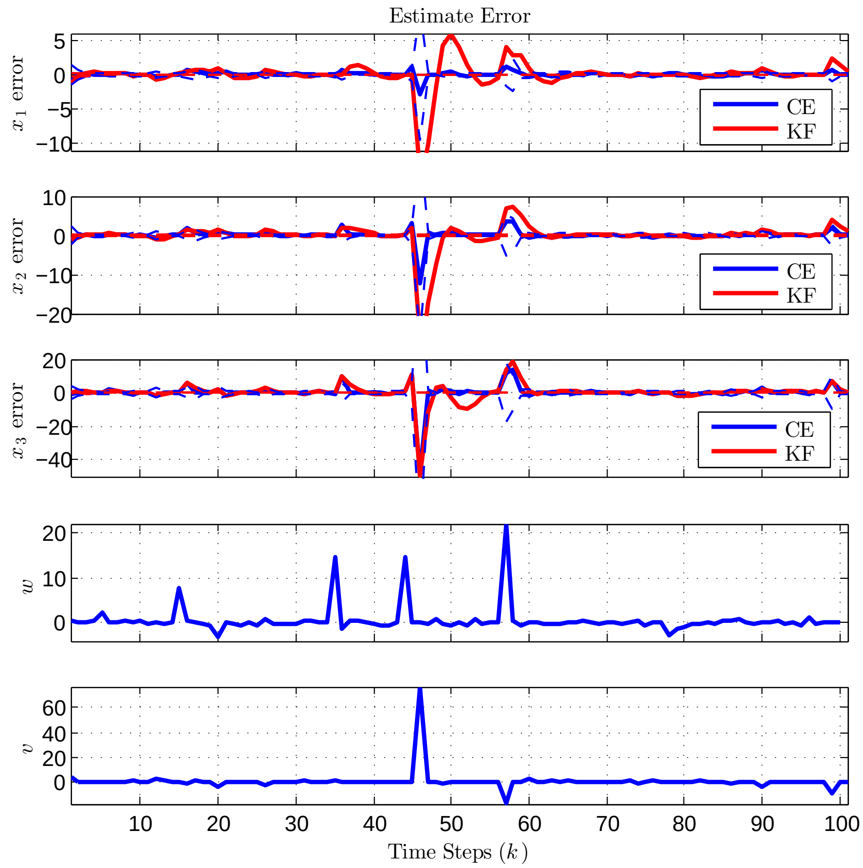

Figure 7.1: Three state estimation error with window size of 7 compared with a standard Kalman Filter, Cauchy noises

In Gaussian noise environment, the Cauchy Estimator also functions well, matching the performance of a standard Kalman Filter. Fig. 7.3 shows the estimation error and the standard deviation of Cauchy Estimator compared with the Kalman Filter, under Gaussian-distributed process noise and measurement noise. The correlation is shown in fig. 7.4. It is interesting to observe that, the multivariate Cauchy estimator, though developed using heavy-tailed noise profile, also has good estimation performance under light-tailed noise circumstances. Since the tail of Cauchy distribution upper bounds many practical uncertainties, the fact that the Cauchy estimator performs well under both Cauchy noise environment and Gaussian noise environment indicates its robustness under various noise environment.

In terms of computational efficiency, on our machine with the CPU at 2.40 GHz and

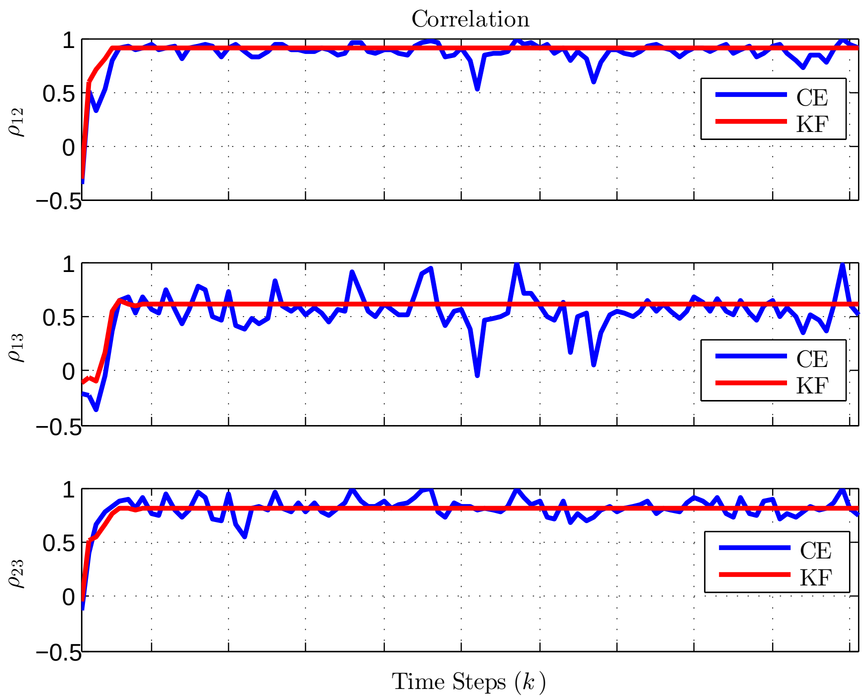

Figure 7.2: Three state estimation correlation with window size of 7 compared with a standard Kalman Filter, Cauchy noises

memory of 8 GB, the offline stage at step $k = 7$ takes around 164 seconds, while the online computation takes 89 seconds per step with a window size of $L = 7$. The online computation roughly save 65% of the total time consumption. In addition, without the use of S matrix, the offline computation itself takes around 14,000 seconds at step $k = 7$. However, it only takes 164 seconds if using the S matrix. The S matrix technique is able to save nearly 99% of the offline computation. The computation time per step is shown in Table 7.1

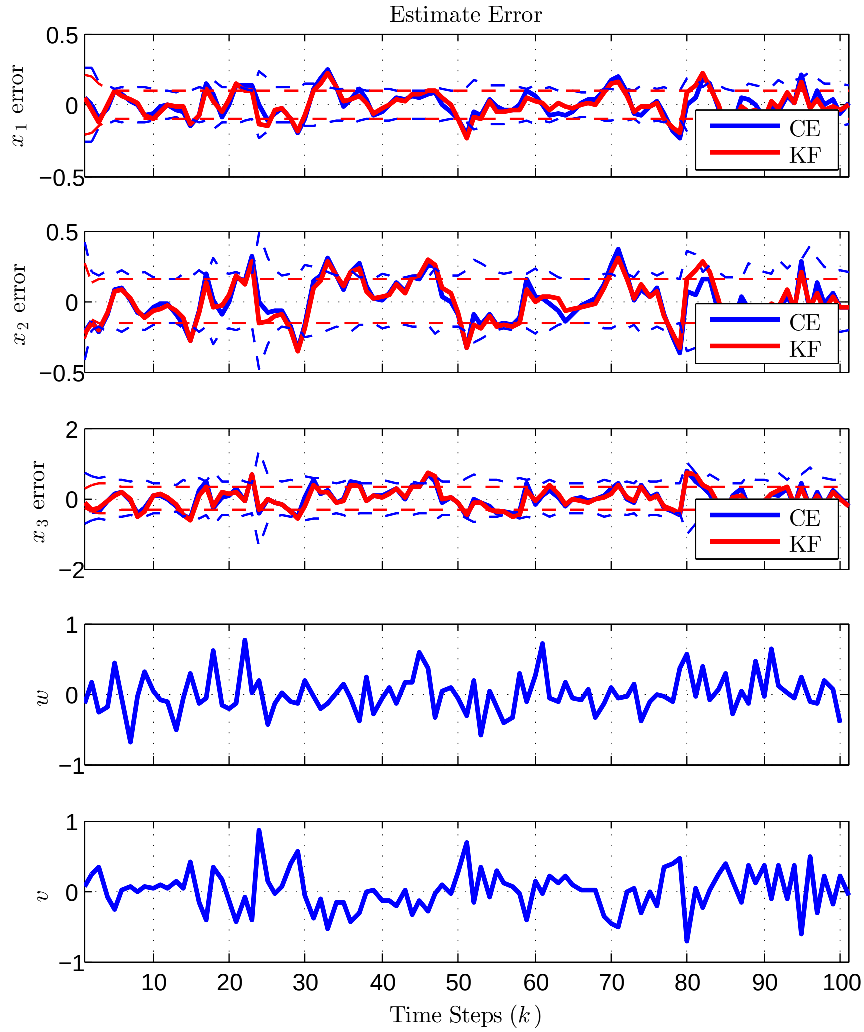

Figure 7.3: Three state estimation error with window size of 7 compared with a standard Kalman Filter, Gaussian noises

7.2 A Four-State System

Considers a four-state system with parameters:

$$\Phi = \begin{bmatrix} 2.0014 & -1.4605 & 0.8927 & -1.2017 \\ 0.3075 & -0.5191 & 1.7812 & -1.1497 \\ 1.1703 & -1.3713 & 1.4875 & -1.0113 \\ 0.6122 & -0.6357 & 0.5183 & 0.3302 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 1 \end{bmatrix}$$

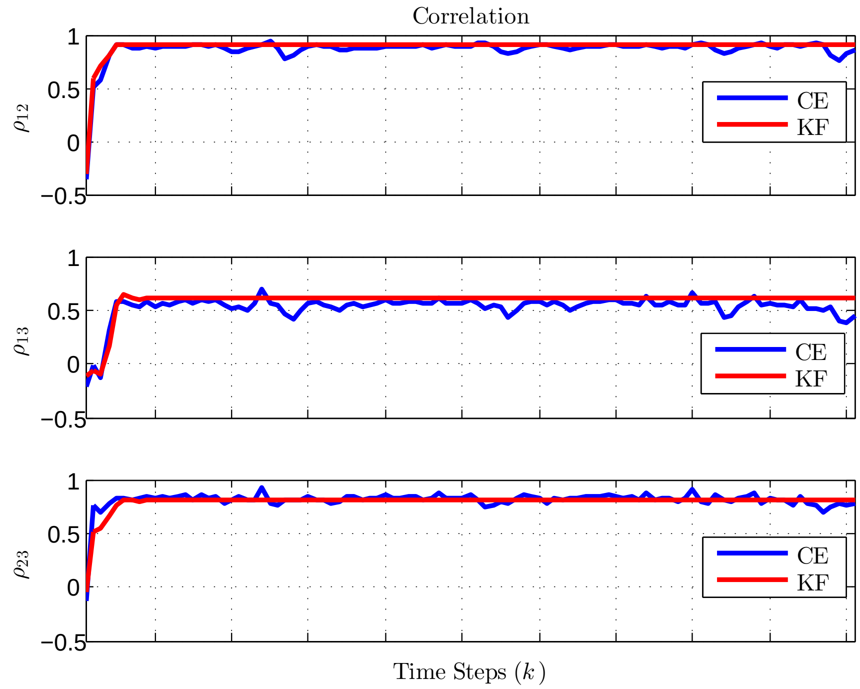

Figure 7.4: Three state estimation correlation with window size of 7 compared with a standard Kalman Filter, Gaussian noises

$$H = \begin{bmatrix} 1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \tag{7.2}$$

The eigenvalues of Φ are $0.75 \pm 0.60j$ and $0.90 \pm 0.30j$. Process noise, measurement noise and initial states are all assumed to be Cauchy distributed as $\beta = 0.2$, $\gamma = 0.2$, $\alpha_1 = 0.2$, $\alpha_2 = 0.2$, $\alpha_3 = 0.2$, $\alpha_4 = 0.2$, $u_1^1 = \{0\}_4$. The simulation runs 101 steps with a window size of $L = 6$.

Similar to the three-state example, the Cauchy estimator obtains reasonably good performance under Cauchy-distributed process noise and measurement noise, as shown in Fig. 7.5. When the estimation error of the Cauchy estimator becomes larger, the dashed line of standard deviation also opens up, which implies that the Cauchy estimator understands and handles that uncertainty well. This performance cannot be observed from a Kalman Filter. The correlation of the states from the Cauchy estimator somewhat matches that from the Kalman Filter, illustrated in fig. 7.6. When both process noise and measurement noise are

Stage	Step	Without S Matrix	With S Matrix
Offline	$k = 1$	0.0014 sec	0.0025 sec
	$k = 2$	0.0200 sec	0.0325 sec
	$k = 3$	0.0785 sec	0.1127 sec
	$k = 4$	0.6618 sec	0.3880 sec
	$k = 5$	9.2660 sec	2.8333 sec
	$k = 6$	$\sim$ 282 sec	20.7938 sec
	$k = 7$	$\sim$ 14000 sec	164.3458 sec
Online	$LL = 5$	-	1.2 sec
	$LL = 6$	-	10.2 sec
	$LL = 7$	-	88.6 sec

Table 7.1: Computation time per step for three-state case

Gaussian-distributed, the Cauchy Estimator performs well too, see the estimation error in fig. 7.7 and the correlation in fig. 7.8.

Again, implemented on our machine with the CPU at 2.40 GHz and memory of 8 GB, the offline stage at step $k = 6$ takes around 87 seconds, while the online computation takes 37 seconds per step with a window size of $L = 6$. The online computation roughly save 70% of the total time consumption, illustrated in Table 7.2.

The successful implementation of four-state example shows that the pre-computational technique of the Cauchy estimator proposed in this dissertation is to a large extent general for linear systems of different dimension. But the implementation is also tailored to the specific system order, by *a priori* combining directions onto fundamental basis, and combining terms using the S matrix for the particular system dimension. Furthermore, the online computation is simplified by *a priori* picking $\hat{\nu}$, such that the G structure is large reduced. The numerical illustrations in this chapter show the significant computational efficiency enhancement for the three-state system, and for the first time makes the four-state system implementable.

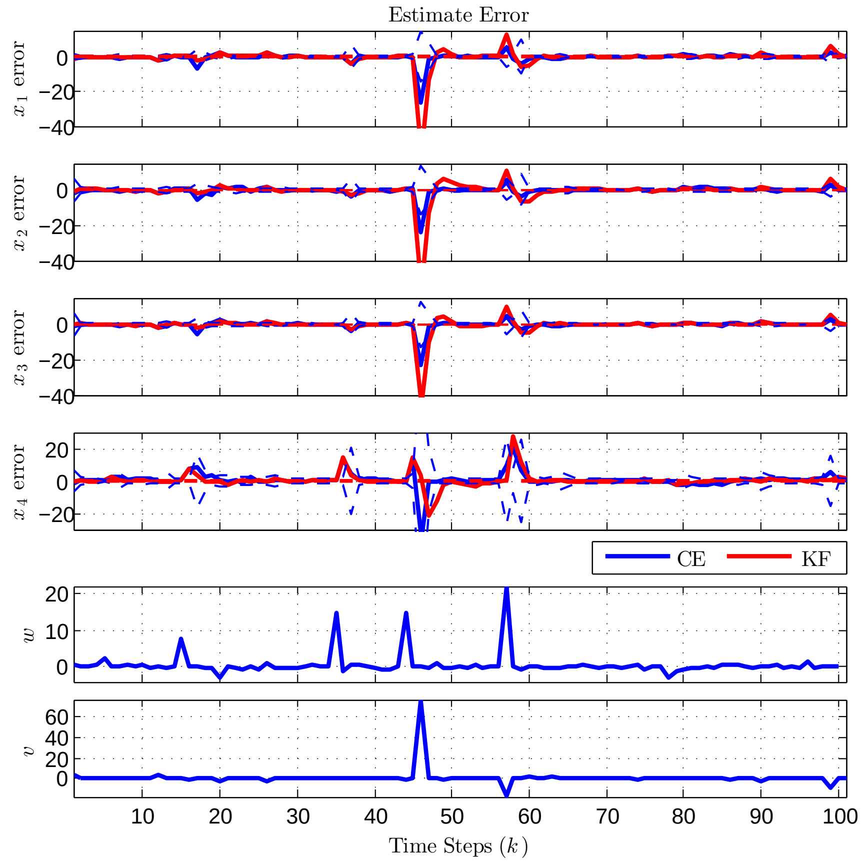

Figure 7.5: Four state estimation error with window size of 6 compared with a standard Kalman Filter, Cauchy noises

Stage	**Step**	**Time per Step**
Offline	$k = 5$	8.7750 sec
	$k = 6$	86.8517 sec
Online	$LL = 5$	3.2 sec
	$LL = 6$	37.4 sec

Table 7.2: Computation time per step for four-state case

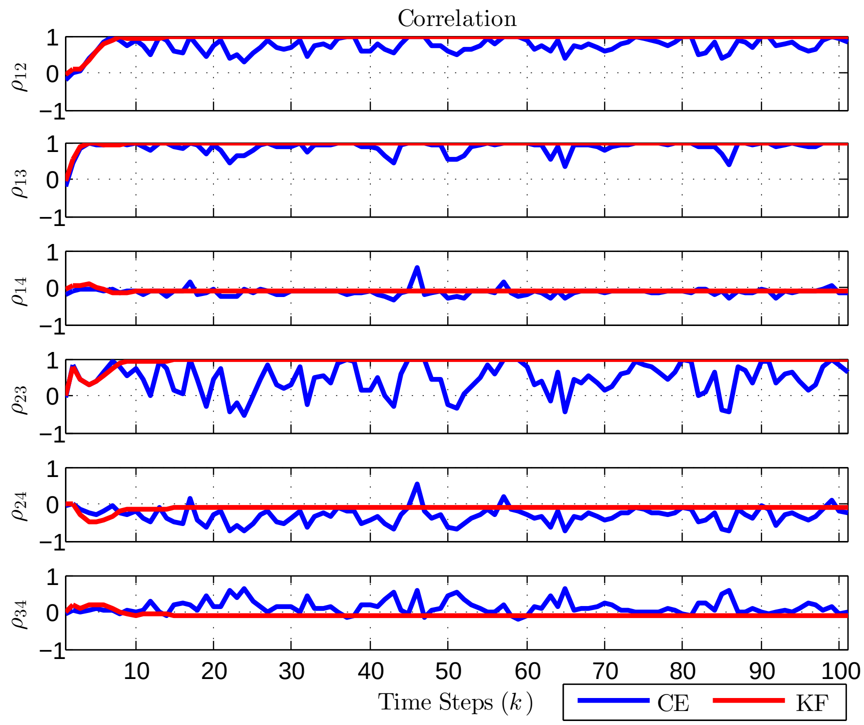

Figure 7.6: Four state correlation with window size of 6 compared with a standard Kalman Filter, Cauchy noises

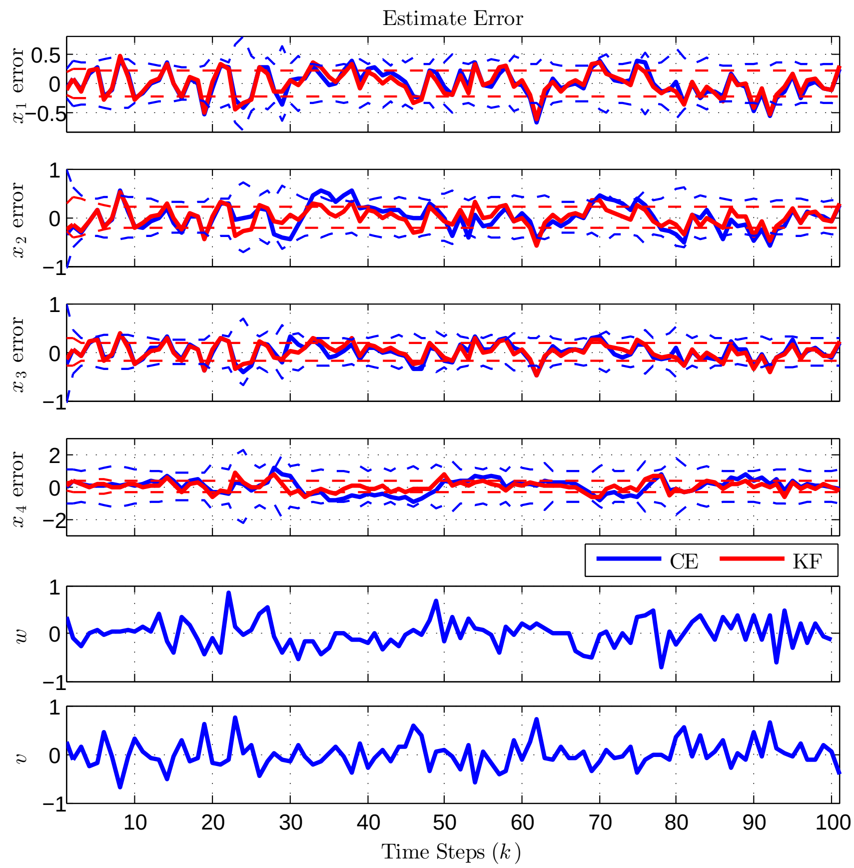

Figure 7.7: Four state estimation error with window size of 6 compared with a standard Kalman Filter, Gaussian noises

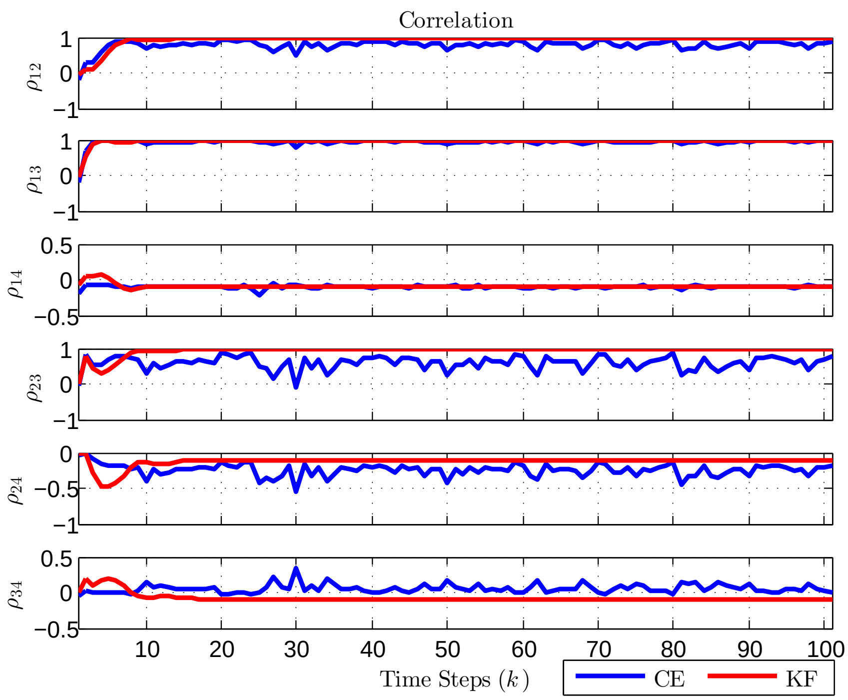

Figure 7.8: Four state correlation with window size of 6 compared with a standard Kalman Filter, Gaussian noises

CHAPTER 8

Conclusions and Future Work

8.1 Conclusions

This dissertation aims at understanding and developing the fundamental structure of the multivariate Cauchy estimator. Established on the newly uncovered properties, it then presents an implementation of the Cauchy estimator that is able to significantly enhance the computational efficiency. In particular, this dissertation has the following contribution.

1. It is uncovered that three parent directions that are linearly dependent can produce co-aligned child directions. These co-aligned child directions can always be expressed as a function of a unique fundamental basis matrix. For two-state case, any two parent directions can produce a co-aligned child direction along $H\mathbf{A}$, where $\mathbf{A}$ is the fundamental basis. For three-state case, any two parent directions that are in the form of $HC\Phi^{T\theta}$ where $C = -C^T$ can produce a co-aligned child direction along $H\Phi^{\theta+1}\mathbf{B}\Phi^{T(\theta+1)}$, where $\mathbf{B}$ is the fundamental basis. The analytic form of the fundamental basis is derived for up to five-state systems.

2. Based on the properties of directions co-alignment, certain exponential terms are shown to have the same functional form and hence can be combined. For two-state case, all identical exponential terms can be combined using only two combination rules. For three-state case, several rules are presented as it starts to reveal the term combination property for multi-dimentional case in general. This combination is analytic, regardless of system parameters.

3. An indexing scheme, S matrix, is constructed to allow for terms combination without numerical comparison during the estimation process. The recursion of the S matrix for two-state case is derived analytically in closed form. For the first time we are able to *a priori* describe which exponential terms to combine. By utilizing S matrix to combine identical exponential terms, the (offline) computation saves roughly 99% of the time consumption if not using S matrix to combine terms.

4. The coefficient terms, i.e. G terms, is reconstructed into a recursive structure. This structure reduce the memory requirement by completely eliminating all the artificial zeros in the formulas. Particularly, this new structure helps to prove that for two-state case, there are no more than three non-zero elements in the real component of each layer of any new term. Furthermore, this new structure provides an approach to separate the intermediate parameters that are independent of the measurement history from those that are relevant to the measurements. It makes the offline - online separation implementable.

5. Finally, the Cauchy estimator is implemented through a pre-computational technique. This technique separates the part of the estimator structure that is independent of the measurements, from the part of the estimator that is dependent upon the measurements. A sliding window method is used to truncate the complexity of the structure and provides a reasonable approximation. For the first time the Cauchy estimator can be implemented efficiently on three and four state systems. Cauchy estimator performs well under both Cauchy and Gaussian environment compared with a standard Kalman Filter. This indicates the robustness of the Cauchy estimator.

8.2 Future Work

There are several potential directions in the future, established on the current understanding of the Cauchy estimator.

1. The combination of G terms.

   Many exponential terms are shown to have the same functional form regardless of the system parameters and hence can be combined. However, the number of G terms are not reduced the same way. Instead, the G terms that are associated with the same exponential term are grouped and stored as multiple individual G terms. One potential research direction is to seek to combine or simplify the group of G terms that are associated with the same exponential term. The G terms have layers of divisions, each of which are constructed by the product of ρ sequence and F matrix, as well as the imaginary component. We observe that the ρ and F among different individual G terms are very similar, varying by only some elements or the sequential order. Once the G terms that are associated with the same exponential term can be combined, the algorithm can be significantly simplified.

2. The analytic form of S matrix for higher order cases.

   For two-state case, the analytic form of S matrix is derived in a recursive manner. That is because we know all the combination rules for exponential terms. However, for higher order cases, the S matrix is obtained numerically. When the dimension gets higher, the computational S matrix may become more sensitive to the tolerance. One direction is to find the analytic recursive structure of S matrix for higher order cases, or a scheme that can better indicate the term combination.

3. Convergence of the sliding window approximation.

   In the numerical experiments, a sliding window approximation is applied to process only a fixed number of the most recent measurements so that the estimator can proceed the implementation for longer time sequence. It has been observed that the initial conditions of the states have less and less influence on the current conditional mean and conditional variance as time goes on. It can be inferred that the estimator needs only some recent measurements to obtain reasonable performance. We check the sensitivity of the current estimates to the initial conditions by various approach numerically to support our infer. In the future, one direction is to show this convergence of the sliding window estimations analytically.

4. Parallel computation

   Another potential research direction is parallel computation. The Cauchy estimator structure is very suitable for distributed computation, because each measurement update contains a large number of terms. These terms are independent of each other. For each distinct exponential term, the multiple G terms are independent of each other as well. One approach of parallel implementation is to distribute the terms at each measurement update to different processors during the online stage, obtain the intermediate parameters $a_i^{k|k}$, $b_i^{k|k}$, $c_i^{k|k}$, $d_i^{k|k}$ of each term in different processors using equation (6.30) (6.32) and (6.33), and collect all these values together to evaluate the conditional mean and conditional second moment using equation (6.34) and (6.35). In Chapter 7, the simulations are timed via regular sequential computing in Matlab. If the algorithm is implemented by parallel computation technique, e.g. GPU, it is very promising that the computational efficiency may be enhanced to nearly real-time.

APPENDIX A

Solutions of Update Integral Formula

This chapter of appendix summarizes the solution of the update integral formula that is proved in Appendix B in [5], for the readers' convenience to refer to when necessary.

A.1 Exponent-only integral

In the first measurement update, the update integral is in the exponent-only form,

$$I = \int_{-\infty}^{\infty} \exp\left[\left(-\sum_{l=1}^{n} \rho_l \left|\xi_l - \eta\right|\right) + jz\eta\right] d\eta \tag{A.1}$$

where z is the measurement, ρ_l-s are positive constants, and the ξ_l-s are variables linear in ν.

The integral was solved by assuming a particular order of ξ_l-s, according to [5]. The solution is in the following form,

$$I = \sum_{i=1}^{n} g_i \left(\sum_{l=1,l\neq i}^{n} \rho_l sgn(\xi_l - \xi_i)\right) \exp\left[\left(-\sum_{l=1,l\neq i}^{n} \rho_l |\xi_l - \xi_i|\right) + jz\xi_i\right], \tag{A.2}$$

where

$$\begin{aligned} & g_i\left(\sum_{l=1,l\neq i}^{n} \rho_l sgn(\xi_l - \xi_i)\right) \\ & = \frac{1}{jz + \rho_i + \sum_{l=1,l\neq i}^{n} \rho_l sgn(\xi_l - \xi_i)} - \frac{1}{jz - \rho_i + \sum_{l=1,l\neq i}^{n} \rho_l sgn(\xi_l - \xi_i)} \end{aligned} \tag{A.3}$$

A.2 Generalized integral

In the second and subsequent measurement update steps, the more general integral form is involved,

$$I = \int_{-\infty}^{\infty} g\left(\sum_{l=1}^{n} \varrho_l sgn(\xi_l - \eta)\right) \exp\left[\left(-\sum_{l=1}^{n} \rho_l \left|\xi_l - \eta\right|\right) + jz\eta\right] d\eta \tag{A.4}$$

The solution of this integral was solved in [5] as,

$$I = \sum_{i=1}^{n} G_i \left(\sum_{l=1,l\neq i}^{n} \varrho_l sgn(\xi_l - \xi_i), \sum_{l=1,l\neq i}^{n} \rho_l sgn(\xi_l - \xi_i)\right) \exp\left[\left(-\sum_{l=1,l\neq i}^{n} \rho_l |\xi_l - \xi_i|\right) + jz\xi_i\right], \tag{A.5}$$

where

$$G_i \left(\sum_{l=1,l\neq i}^{n} \varrho_l sgn(\xi_l - \xi_i), \sum_{l=1,l\neq i}^{n} \rho_l sgn(\xi_l - \xi_i)\right)$$
$$= \frac{g\left(\varrho_i + \sum_{l=1,l\neq i}^{n} \varrho_l sgn(\xi_l - \xi_i)\right)}{jz + \rho_i + \sum_{l=1,l\neq i}^{n} \rho_l sgn(\xi_l - \xi_i)} - \frac{g\left(-\varrho_i + \sum_{l=1,l\neq i}^{n} \varrho_l sgn(\xi_l - \xi_i)\right)}{jz - \rho_i + \sum_{l=1,l\neq i}^{n} \rho_l sgn(\xi_l - \xi_i)} \tag{A.6}$$

APPENDIX B

Fundamental Basis for Higher-Order Systems

For two-state case, the co-aligned direction becomes $H\mathbf{A}$ immediately after one update. For three-state case, repeated directions align with $H\Phi\mathbf{B}\Phi^T$ in two updates. Similarly, it can be deduced that three updates are needed for four-state cases, and four updates are needed for five-state cases, to obtain the fundamental basis. This update process has been verified by numerical simulations. It has been demonstrated numerically that such fundamental basis exists and is unique for four-state and five-state cases. Based on the fact that the form of the fundamental basis is unique, the analytic structure for four-state and five-state system can be derived as follows.

B.1 Four-State Case

Suppose B_1, B_2, and B_3 to be arbitrary 4-by-4 skew-symmetric matrices. Define D_1 and D_2 as,

$$D_1 = B_1^T H^T H B_2 - B_2^T H^T H B_1 \tag{B.1}$$

$$D_2 = B_1^T H^T H B_3 - B_3^T H^T H B_1 \tag{B.2}$$

Consider the parent directions at step k. Then B_1, B_2, and B_3 represent the child directions at step $k+1$, due to the skew-symmetry. The formulation of D_1 and D_2 stands for the child directions at step $k+2$. Next, the differences of the outer product will produce the fundamental basis $\mathbf{C}$ for four-state case at step $k+3$, i.e.

$$\sigma\mathbf{C} = D_2^T \Phi^T H^T H \Phi D_1 - D_1^T \Phi^T H^T H \Phi D_2 \tag{B.3}$$

Substitute (B.1) and (B.2) into equation (B.3),

$$\sigma\mathbf{C} = \left(B_1^T H^T H B_3 - B_3^T H^T H B_1\right)^T \Phi^T H^T H \Phi \left(B_1^T H^T H B_2 - B_2^T H^T H B_1\right) - \left(B_1^T H^T H B_2 - B_2^T H^T H B_1\right)^T \Phi^T H^T H \Phi \left(B_1^T H^T H B_3 - B_3^T H^T H B_1\right) \tag{B.4}$$

Expand (B.4) further,

$$\begin{aligned}\sigma\mathbf{C} =& B_3^T H^T H B_1 \Phi^T H^T H \Phi B_1^T H^T H B_2 - B_3^T H^T H B_1 \Phi^T H^T H \Phi B_2^T H^T H B_1 \\ &- B_1^T H^T H B_3 \Phi^T H^T H \Phi B_1^T H^T H B_2 + B_1^T H^T H B_3 \Phi^T H^T H \Phi B_2^T H^T H B_1 \\ &- B_2^T H^T H B_1 \Phi^T H^T H \Phi B_1^T H^T H B_3 + B_1^T H^T H B_2 \Phi^T H^T H \Phi B_1^T H^T H B_3 \\ &+ B_2^T H^T H B_1 \Phi^T H^T H \Phi B_3^T H^T H B_1 - B_1^T H^T H B_2 \Phi^T H^T H \Phi B_3^T H^T H B_1\end{aligned} \tag{B.5}$$

Then,

$$\begin{aligned}\sigma\mathbf{C} = \left(H B_1 \Phi^T H^T\right) \cdot [&\left(H B_1 \Phi^T H^T\right) \cdot \left(B_3^T H^T H B_2 - B_2^T H^T H B_3\right) \\ &+ \left(H B_2 \Phi^T H^T\right) \cdot \left(B_1^T H^T H B_3 - B_3^T H^T H B_1\right) \\ &+ \left(H B_3 \Phi^T H^T\right) \cdot \left(B_2^T H^T H B_1 - B_1^T H^T H B_2\right)]\end{aligned} \tag{B.6}$$

The fundamental basis $\mathbf{C}$ is *a priori* matrix regardless the value of B_1, B_2, and B_3. Hence, by choosing specific values of B_1, B_2, and B_3, the form of $\mathbf{C}$ can be solved.

Pick

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{B.7}$$

Then,

$$B_3^T H^T H B_2 - B_2^T H^T H B_3 = \begin{bmatrix} 0 & 0 & -h_1 h_4 & h_1 h_3 \\ 0 & 0 & 0 & 0 \\ h_1 h_4 & 0 & 0 & -h_1^2 \\ -h_1 h_3 & 0 & h_1^2 & 0 \end{bmatrix} \tag{B.8}$$

$$B_1^T H^T H B_3 - B_3^T H^T H B_1 = \begin{bmatrix} 0 & h_1 h_4 & 0 & -h_1 h_2 \\ -h_1 h_4 & 0 & 0 & h_1^2 \\ 0 & 0 & 0 & 0 \\ h_1 h_2 & -h_1^2 & 0 & 0 \end{bmatrix} \tag{B.9}$$

$$B_2^T H^T H B_1 - B_1^T H^T H B_2 = \begin{bmatrix} 0 & -h_1 h_3 & h_1 h_2 & 0 \\ h_1 h_3 & 0 & -h_1^2 & 0 \\ -h_1 h_2 & h_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{B.10}$$

where the lower case of h's represent the elements of the H matrix, i.e. $H = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \end{bmatrix}$.

Also,

$$H B_1 \Phi^T H^T = (-h_2)(H\Phi e_1) + h_1 (H\Phi e_2) \tag{B.11}$$

$$H B_2 \Phi^T H^T = (-h_3)(H\Phi e_1) + h_1 (H\Phi e_3) \tag{B.12}$$

$$H B_3 \Phi^T H^T = (-h_4)(H\Phi e_1) + h_1 (H\Phi e_4) \tag{B.13}$$

Now define

$$\beta_{il} = h_i (H\Phi e_l) - h_l (H\Phi e_i), \quad 1 \le i, l \le 4, i \ne l \tag{B.14}$$

Substitute (B.7) - (B.14) back into (B.6). And let $\sigma = \left(H B_1 \Phi^T H^T\right) \cdot h_1^2$. Then the fundamental basis $\mathbf{C}$ is,

$$\mathbf{C} = \begin{bmatrix} 0 & -\beta_{34} & \beta_{24} & -\beta_{23} \\ \beta_{34} & 0 & -\beta_{14} & \beta_{13} \\ -\beta_{24} & \beta_{14} & 0 & -\beta_{12} \\ \beta_{23} & -\beta_{13} & \beta_{12} & 0 \end{bmatrix} \tag{B.15}$$

The form of $\mathbf{C}$ described in (B.15) has been proved numerically.

B.2 Five-State Case

For five-state case, consider the fundamental basis $\mathbf{D}$ to be obtained through four updates at step $k+4$.

$$\sigma\mathbf{D} = E_1^T \Phi^{2T} H^T H \Phi^2 E_2 - E_2^T \Phi^{2T} H^T H \Phi^2 E_1 \tag{B.16}$$

$$E_1 = C_1^T \Phi^T H^T H \Phi C_2 - C_2^T \Phi^T H^T H \Phi C_1 \tag{B.17}$$

$$E_2 = C_1^T \Phi^T H^T H \Phi C_3 - C_3^T \Phi^T H^T H \Phi C_1 \tag{B.18}$$

where E_1 and E_2 are produced at step $k+3$, and C_1, C_2 and C_3 are produced at step $k+2$. Substitute (B.17) (B.18) back to (B.16),

$$\begin{aligned}\sigma\mathbf{D} =&(C_2^T \Phi^T H^T H \Phi C_1 - C_1^T \Phi^T H^T H \Phi C_2) \Phi^{2T} H^T H \Phi^2 \\ &\qquad\qquad \cdot (C_1^T \Phi^T H^T H \Phi C_3 - C_3^T \Phi^T H^T H \Phi C_1) \\ &- (C_3^T \Phi^T H^T H \Phi C_1 - C_1^T \Phi^T H^T H \Phi C_3) \Phi^{2T} H^T H \Phi^2 \\ &\qquad\qquad \cdot (C_1^T \Phi^T H^T H \Phi C_2 - C_2^T \Phi^T H^T H \Phi C_1)\end{aligned} \tag{B.19}$$

Expanding every elements in (B.19) is tedious, not to mention that even the C_i's are produced from matrices B_j's in the directions at step $k+1$. One will need to trace back even further. Hence, let us just write down the first element in equation (B.19).

$$\begin{aligned}&C_2^T \Phi^T H^T H \Phi C_1 \Phi^{2T} H^T H \Phi^2 C_1^T \Phi^T H^T H \Phi C_3 \\ &= \left(H \Phi C_1 \Phi^{2T} H^T\right) \left(H \Phi^2 C_1^T \Phi^T H^T\right) C_2^T \Phi^T H^T H \Phi C_3\end{aligned} \tag{B.20}$$

C_1, C_2, and C_3 are produced at step $k+2$. In particular, define,

$$C_1 = B_1^T H^T H B_2 - B_2^T H^T H B_1 \tag{B.21}$$

And consider the coefficient in equation (B.20),

$$\begin{aligned}H \Phi C_1 \Phi^{2T} H^T &= H \Phi \left(B_1^T H^T H B_2 - B_2^T H^T H B_1\right) \Phi^{2T} H^T \\ &= (H \Phi B_1^T H^T)(H B_2 \Phi^{2T} H^T) - (H \Phi B_2^T H^T)(H B_1 \Phi^{2T} H^T)\end{aligned} \tag{B.22}$$

Now, pick

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{B.23}$$

Then,

$$H\Phi B_1^T H^T = h_1(H\Phi e_2) - h_2(H\Phi e_1) \tag{B.24}$$

$$H\Phi B_2^T H^T = h_1(H\Phi e_3) - h_3(H\Phi e_1) \tag{B.25}$$

$$HB_1\Phi^{2T} H^T = h_1(H\Phi^2 e_2) - h_2(H\Phi^2 e_1) \tag{B.26}$$

$$HB_2\Phi^{2T} H^T = h_1(H\Phi^2 e_3) - h_3(H\Phi^2 e_1) \tag{B.27}$$

where the lower case of h's represent the elements of the H matrix, i.e. $H = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 \end{bmatrix}$. Substitute (B.24) - (B.27) back into (B.22).

$$\begin{aligned} H\Phi C_1\Phi^{2T} H^T =& (h_1(H\Phi e_2) - h_2(H\Phi e_1))(h_1(H\Phi^2 e_3) - h_3(H\Phi^2 e_1)) \\ & - (h_1(H\Phi e_3) - h_3(H\Phi e_1))(h_1(H\Phi^2 e_2) - h_2(H\Phi^2 e_1)) \end{aligned} \tag{B.28}$$

Then,

$$\begin{aligned} H\Phi C_1\Phi^{2T} H^T =& h_1^2 \left[(H\Phi e_2)(H\Phi^2 e_3) - (H\Phi e_3)(H\Phi^2 e_2)\right] \\ & + h_1 h_2 \left[(H\Phi e_3)(H\Phi^2 e_1) - (H\Phi e_1)(H\Phi^2 e_3)\right] \\ & + h_1 h_3 \left[(H\Phi e_1)(H\Phi^2 e_2) - (H\Phi e_2)(H\Phi^2 e_1)\right] \end{aligned} \tag{B.29}$$

Take this form, and define a set of quantities as,

$$\begin{aligned} \beta_{ijk} = \quad & h_i \left[(H\Phi e_j)(H\Phi^2 e_k) - (H\Phi e_k)(H\Phi^2 e_j)\right] \\ & + h_j \left[(H\Phi e_k)(H\Phi^2 e_i) - (H\Phi e_i)(H\Phi^2 e_k)\right] \\ & + h_k \left[(H\Phi e_i)(H\Phi^2 e_j) - (H\Phi e_j)(H\Phi^2 e_i)\right], \quad 1 \le i, j, k \le 5, i \ne j \ne k \end{aligned} \tag{B.30}$$

Finally, the fundamental basis $\mathbf{D}$ can be constructed as follows.

$$\mathbf{D} = \begin{bmatrix} 0 & \beta_{345} & -\beta_{245} & \beta_{235} & -\beta_{234} \\ -\beta_{345} & 0 & \beta_{145} & -\beta_{135} & \beta_{134} \\ \beta_{245} & -\beta_{145} & 0 & \beta_{125} & -\beta_{124} \\ -\beta_{235} & \beta_{135} & -\beta_{125} & 0 & \beta_{123} \\ \beta_{234} & -\beta_{134} & \beta_{124} & -\beta_{123} & 0 \end{bmatrix} \tag{B.31}$$

Numerical results show that the $\mathbf{D}$ matrix described in (B.31) is the fundamental basis for five-state case.

APPENDIX C

Some Derivations of Term Combination Rules for 3-State Cases

Consider a new term at step k. The exponential term is expressed as,

$$\mathcal{E}^{k|k} = \exp\left(-P_1|HC_1\nu| - P_2|HC_2\nu| - ... - P_{m-1}|HC_{m-1}\nu| - P_m|b_1\nu| + j\zeta^{k|k}\nu\right) \tag{C.1}$$

where C_i, $i = 1, 2, ..., m-1$, are skew-symmetric matrices, and b_1 is another row vector which cannot be expressed in the HC form.

C.1 Child Terms at step $k+1$

After time propagation, the exponential term becomes,

$$\begin{aligned}\mathcal{E}^{k+1|k} =& \exp\left(-P_1|HC_1\Phi^T\nu| - P_2|HC_2\Phi^T\nu| - ... - P_{m-1}|HC_{m-1}\Phi^T\nu|\right.\\ &\left.-P_m|b_1\Phi^T\nu| - \beta|\Gamma^T\nu| + j\zeta^{k+1|k}\nu\right)\end{aligned} \tag{C.2}$$

Then at step $k+1$, the CF is obtained from the update integral.

$$\begin{aligned}\bar{\phi}_{k+1}(\nu) =& \int_{-\infty}^{+\infty} G \cdot \exp\Bigg(-P_1\left|HC_1\Phi^T H^T\right|\left|\frac{HC_1\Phi^T\nu}{HC_1\Phi^T H^T} - \eta\right| - cdots\\ &- P_{m-1}\left|HC_{m-1}\Phi^T H^T\right|\left|\frac{HC_{m-1}\Phi^T\nu}{HC_{m-1}\Phi^T H^T} - \eta\right| - P_m\left|b_1\Phi^T H^T\right|\left|\frac{b_1\Phi^T\nu}{b_1\Phi^T H^T} - \eta\right|\\ &-\beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \eta\right| - \gamma\left|\eta\right| + jz_{k+1}\eta + j\zeta^{k+1|k}\left(\nu - H^T\eta\right)\Bigg)\,d\eta\end{aligned} \tag{C.3}$$

C.1.1 $\mathcal{E}_i^{k+1|k+1}$ when $1 \le i \le m-1$

The i^{th} child term when $1 \le i \le m-1$ has the exponential term expressed as,

$$\begin{aligned}\mathcal{E}_i^{k+1|k+1} = \exp\Bigg(& -P_1 \left|HC_1\Phi^T H^T\right| \left|\frac{HC_1\Phi^T\nu}{HC_1\Phi^T H^T} - \frac{HC_i\Phi^T\nu}{HC_i\Phi^T H^T}\right| - \cdots \\ & - P_{m-1}\left|HC_{m-1}\Phi^T H^T\right| \left|\frac{HC_{m-1}\Phi^T\nu}{HC_{m-1}\Phi^T H^T} - \frac{HC_i\Phi^T\nu}{HC_i\Phi^T H^T}\right| \\ & - P_m \left|b_1\Phi^T H^T\right| \left|\frac{b_1\Phi^T\nu}{b_1\Phi^T H^T} - \frac{HC_i\Phi^T\nu}{HC_i\Phi^T H^T}\right| \\ & -\beta\left|\Gamma^T H^T\right| \left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{HC_i\Phi^T\nu}{HC_i\Phi^T H^T}\right| - \gamma\left|\frac{HC_i\Phi^T\nu}{HC_i\Phi^T H^T}\right| + j\zeta_i^{k+1|k+1}\nu\Bigg) \end{aligned} \tag{C.4}$$

For two 3-dim row vectors $b_1 = HC_1$ and $b_2 = HC_2$, $b_2^T b_1 - b_1^T b_2 = \sigma\mathbf{B}$, and $\sigma = \frac{b_1 A_{21} b_2^T}{He_3^T}$. Then,

$$\begin{aligned}\mathcal{E}_i^{k+1|k+1} = \exp\Bigg(& -\frac{P_1 \cdot \left|HC_1 A_{21} C_i^T H^T\right|}{\left|HC_i\Phi^T H^T\right| \cdot \left|He_3^T\right|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| - \cdots \\ & - \frac{P_{m-1} \cdot \left|HC_{m-1} A_{21} C_i^T H^T\right|}{\left|HC_i\Phi^T H^T\right| \cdot \left|He_3^T\right|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ & - \frac{P_m}{\left|HC_i\Phi^T H^T\right|} \left|H\Phi\left(C_i^T H^T b_1 - b_1^T HC_i\right)\Phi^T\nu\right| \\ & - \frac{\beta}{\left|HC_i\Phi^T H^T\right|} \left|H\Phi\left(C_i^T H^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma HC_i\right)\Phi^T\nu\right| \\ & - \frac{\gamma}{\left|HC_i\Phi^T H^T\right|} \left|HC_i\Phi^T\nu\right| + j\zeta_i^{k+1|k+1}\nu\Bigg) \end{aligned} \tag{C.5}$$

Combine the first $(m-2)$ elements. Then,

$$\begin{aligned}\mathcal{E}_i^{k+1|k+1} = \exp\Bigg(& -\frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l \cdot \left|HC_l A_{21} C_i^T H^T\right|\right)}{\left|HC_i\Phi^T H^T\right| \cdot \left|He_3^T\right|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| - \frac{P_m}{\left|HC_i\Phi^T H^T\right|} \left|H\Phi D_i\Phi^T\nu\right| \\ & -\frac{\beta}{\left|HC_i\Phi^T H^T\right|} \left|H\Phi E_i\Phi^T\nu\right| - \frac{\gamma}{\left|HC_i\Phi^T H^T\right|} \left|HC_i\Phi^T\nu\right| + j\zeta_i^{k+1|k+1}\nu\Bigg) \end{aligned} \tag{C.6}$$

where

$$D_i = C_i^T H^T b_1 - b_1^T HC_i \quad \text{and} \quad E_i = C_i^T H^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma HC_i \tag{C.7}$$

C.1.2 $\mathcal{E}_m^{k+1|k+1}$

At step $k+1$, the m^{th} child term has the exponential term expressed as,

$$\begin{aligned}\mathcal{E}_m^{k+1|k+1} = &\exp\Bigg(-P_1\left|HC_1\Phi^TH^T\right|\left|\frac{HC_1\Phi^T\nu}{HC_1\Phi^TH^T}-\frac{b_1\Phi^T\nu}{b_1\Phi^TH^T}\right|-\cdots\\ &-P_{m-1}\left|HC_{m-1}\Phi^TH^T\right|\left|\frac{HC_{m-1}\Phi^T\nu}{HC_{m-1}\Phi^TH^T}-\frac{b_1\Phi^T\nu}{b_1\Phi^TH^T}\right|\\ &-\beta\left|\Gamma^TH^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^TH^T}-\frac{b_1\Phi^T\nu}{b_1\Phi^TH^T}\right|-\gamma\left|\frac{b_1\Phi^T\nu}{b_1\Phi^TH^T}\right|+j\zeta_m^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{C.8}$$

Then,

$$\begin{aligned}\mathcal{E}_m^{k+1|k+1} = &\exp\Bigg(-\frac{P_1}{|b_1\Phi^TH^T|}\left|H\Phi\left(b_1^THC_1-C_1^TH^Tb_1\right)\Phi^T\nu\right|-\cdots\\ &-\frac{P_{m-1}}{|b_1\Phi^TH^T|}\left|H\Phi\left(b_1^THC_{m-1}-C_{m-1}^TH^Tb_1\right)\Phi^T\nu\right|\\ &-\frac{\beta}{|b_1\Phi^TH^T|}\left|H\Phi\left(b_1^T\Gamma^T\Phi^{-T}-\Phi^{-1}\Gamma b_1\right)\Phi^T\nu\right|-\frac{\gamma}{|b_1\Phi^TH^T|}\left|b_1\Phi^T\nu\right|+j\zeta_m^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{C.9}$$

As defined earlier, $D_i = C_i^TH^Tb_1 - b_1^THC_i, \quad i = 1,...,m-1$. Also define $D_{gb} = b_1^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma b_1$. Then $\mathcal{E}_m^{k+1|k+1}$ can be rewrite as follows.

$$\begin{aligned}\mathcal{E}_m^{k+1|k+1} = &\exp\Bigg(-\frac{P_1}{|b_1\Phi^TH^T|}\left|H\Phi D_1\Phi^T\nu\right|-\cdots-\frac{P_{m-1}}{|b_1\Phi^TH^T|}\left|H\Phi D_{m-1}\Phi^T\nu\right|\\ &-\frac{\beta}{|b_1\Phi^TH^T|}\left|H\Phi D_{gb}\Phi^T\nu\right|-\frac{\gamma}{|b_1\Phi^TH^T|}\left|b_1\Phi^T\nu\right|+j\zeta_m^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{C.10}$$

C.1.3 $\mathcal{E}_{m+1}^{k+1|k+1}$

At step $(k+1)$, the $(m+1)^{th}$ child term has the exponential term expressed as,

$$\begin{aligned}\mathcal{E}_{m+1}^{k+1|k+1} = &\exp\Bigg(-P_1\left|HC_1\Phi^TH^T\right|\left|\frac{HC_1\Phi^T\nu}{HC_1\Phi^TH^T}-\frac{\Gamma^T\nu}{\Gamma^TH^T}\right|-\cdots\\ &-P_{m-1}\left|HC_{m-1}\Phi^TH^T\right|\left|\frac{HC_{m-1}\Phi^T\nu}{HC_{m-1}\Phi^TH^T}-\frac{\Gamma^T\nu}{\Gamma^TH^T}\right|\\ &-\frac{\gamma}{|b_1H^T|}\left|b_1\Phi^TH^T\right|\left|\frac{b_1\Phi^T\nu}{b_1\Phi^TH^T}-\frac{\Gamma^T\nu}{\Gamma^TH^T}\right|-\gamma\left|\frac{\Gamma^T\nu}{\Gamma^TH^T}\right|+j\zeta_{m+1}^{k+1|k+1}\nu\Bigg)\end{aligned} \tag{C.11}$$

Then,

$$\begin{aligned}\mathcal{E}_{m+1}^{k+1|k+1} = \exp\Bigg(& -\frac{P_1}{|\Gamma^T H^T|} \left| H\Phi \left(\Phi^{-1}\Gamma H C_1 - C_1^T H^T \Gamma^T \Phi^{-T}\right) \Phi^T \nu \right| - \cdots \\ & - \frac{P_{m-1}}{|\Gamma^T H^T|} \left| H\Phi \left(\Phi^{-1}\Gamma H C_{m-1} - C_{m-1}^T H^T \Gamma^T \Phi^{-T}\right) \Phi^T \nu \right| \\ & - \frac{\gamma}{|b_1 H^T| \cdot |\Gamma^T H^T|} \left| H\Phi \left(\Phi^{-1}\Gamma b_1 - b_1^T \Gamma^T \Phi^{-T}\right) \Phi^T \nu \right| \\ & - \frac{\gamma}{|\Gamma^T H^T|} \left|\Gamma^T \nu\right| + j\zeta_{m+1}^{k+1|k+1}\nu \Bigg) \end{aligned} \tag{C.12}$$

As defined earlier, $E_l = C_l^T H^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma H C_l$, $l = 1, ..., m-1$ and $D_{gb} = b_1^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma b_1$. Then $\mathcal{E}_{m+1}^{k+1|k+1}$ can be rewrite as follows.

$$\begin{aligned}\mathcal{E}_{m+1}^{k+1|k+1} = \exp\Bigg(& -\frac{P_1}{|\Gamma^T H^T|} \left| H\Phi E_1 \Phi^T \nu \right| - ... - \frac{P_{m-1}}{|\Gamma^T H^T|} \left| H\Phi E_{m-1} \Phi^T \nu \right| \\ & - \frac{\gamma}{|b_1 H^T| \cdot |\Gamma^T H^T|} \left| H\Phi D_{gb} \Phi^T \nu \right| - \frac{\gamma}{|\Gamma^T H^T|} \left|\Gamma^T \nu\right| + j\zeta_{m+1}^{k+1|k+1}\nu \Bigg) \end{aligned} \tag{C.13}$$

Furthermore, look at the exponential terms at step $k+1$ in equation (C.6), (C.10) and (C.13), one can find the value of P_m in equation (C.1), as $P_m = \frac{\gamma}{b_1 H^T}$.

C.2 Grandchild Terms at step $k+2$

C.2.1 From $\mathcal{E}_i^{k+1|k+1}$, $1 \le i \le m-1$ to $\mathcal{E}_{i,1}^{k+2|k+2}$

Consider equation (C.6) at step $k+1$. Then, after time propagation, the exponential term becomes,

$$\begin{aligned}\mathcal{E}_i^{k+2|k+1} = \exp\Bigg(& -\frac{\left(\sum_{l=1, l\neq i}^{m-1} P_l \cdot \left| H C_l A_{21} C_i^T H^T \right|\right)}{|H C_i \Phi^T H^T| \cdot |H e_3^T|} \left| H\Phi \mathbf{B} \Phi^{2T} \nu \right| - \frac{\frac{\gamma}{b_1 H^T}}{|H C_i \Phi^T H^T|} \left| H\Phi D_i \Phi^{2T} \nu \right| \\ & - \frac{\beta}{|H C_i \Phi^T H^T|} \left| H\Phi E_i \Phi^{2T} \nu \right| - \frac{\gamma}{|H C_i \Phi^T H^T|} \left| H C_i \Phi^{2T} \nu \right| - \beta \left|\Gamma^T \nu\right| + j\zeta_i^{k+2|k+1}\nu \Bigg) \end{aligned} \tag{C.14}$$

The CF is obtained from the update integral.

$$\bar{\phi}_{k+2}(\nu) = \int_{-\infty}^{+\infty} G \cdot \exp\left(-\frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l \cdot \left|HC_l A_{21} C_i^T H^T\right|\right)}{|HC_i \Phi^T H^T| \cdot |He_3^T|} \left|H\Phi\mathbf{B}\Phi^{2T}H^T\right| \left|\frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T} - \eta\right| \right.$$
$$- \frac{\frac{\gamma}{b_1 H^T}}{|HC_i\Phi^T H^T|} \left|H\Phi D_i \Phi^{2T} H^T\right| \left|\frac{H\Phi D_i \Phi^{2T}\nu}{H\Phi D_i \Phi^{2T} H^T} - \eta\right|$$
$$- \frac{\beta}{|HC_i\Phi^T H^T|} \left|H\Phi E_i \Phi^{2T} H^T\right| \left|\frac{H\Phi E_i \Phi^{2T}\nu}{H\Phi E_i \Phi^{2T} H^T} - \eta\right|$$
$$- \frac{\gamma}{|HC_i\Phi^T H^T|} \left|HC_i \Phi^{2T} H^T\right| \left|\frac{HC_i \Phi^{2T}\nu}{HC_i \Phi^{2T} H^T} - \eta\right|$$
$$\left. -\beta \left|\Gamma^T H^T\right| \left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \eta\right| - \gamma|\eta| + jz_{k+2}\eta + j\zeta_i^{k+2|k+2}\left(\nu - H^T\eta\right)\right) d\eta \quad \text{(C.15)}$$

Then the first child term at step $k+2$ has the exponential term expressed as follow.

$$\mathcal{E}_{i,1}^{k+2|k+2} = \exp\left(-\frac{\frac{\gamma}{b_1 H^T}}{|HC_i\Phi^T H^T|} \left|H\Phi D_i \Phi^{2T} H^T\right| \left|\frac{H\Phi D_i \Phi^{2T}\nu}{H\Phi D_i \Phi^{2T} H^T} - \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right| \right.$$
$$- \frac{\beta}{|HC_i\Phi^T H^T|} \left|H\Phi E_i \Phi^{2T} H^T\right| \left|\frac{H\Phi E_i \Phi^{2T}\nu}{H\Phi E_i \Phi^{2T} H^T} - \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right|$$
$$- \frac{\gamma}{|HC_i\Phi^T H^T|} \left|HC_i \Phi^{2T} H^T\right| \left|\frac{HC_i \Phi^{2T}\nu}{HC_i \Phi^{2T} H^T} - \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right|$$
$$\left. -\beta \left|\Gamma^T H^T\right| \left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right| - \gamma \left|\frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right| + j\zeta_{i,1}^{k+2|k+2}\nu\right) \quad \text{(C.16)}$$

Note that for two 3-dim row vectors $b_1 = HC_1$ and $b_2 = HC_2$, $b_2^T b_1 - b_1^T b_2 = \sigma\mathbf{B}$, and $\sigma = \frac{b_1 A_{21} b_2^T}{He_3^T}$. In addition, simple algebra shows that

$$A_{21} H^T H\Phi\mathbf{B} - \mathbf{B}^T \Phi^T H^T H A_{12} = (H\Phi A_{21} H^T)\mathbf{B} \quad \text{(C.17)}$$

where $\mathbf{B}$ is the fundamental basis.

Because of linearity, it must hold that for any 3 by 3 skew-symmetric matrix C

$$C^T H^T H\Phi\mathbf{B} - \mathbf{B}^T \Phi^T H^T H C = (H\Phi C^T H^T)\mathbf{B} \quad \text{(C.18)}$$

which has also been verified numerically.

Therefore, this exponential term can be written as,

$$
\begin{aligned}
\mathcal{E}_{i,1}^{k+2|k+2} = \exp\Bigg(& -\frac{\frac{\gamma}{b_1 H^T}}{|HC_i\Phi^T H^T| \cdot |H\Phi\mathbf{B}\Phi^{2T}H^T|} \left| H\Phi \left(\frac{H\Phi D_i \Phi^T A_{21} \Phi \mathbf{B}^T \Phi^T H^T}{He_3^T} \right) \mathbf{B}\Phi^T \nu \right| \\
& - \frac{\beta}{|HC_i\Phi^T H^T| \cdot |H\Phi B\Phi^{2T}H^T|} \left| H\Phi \left(\frac{H\Phi E_i \Phi^T A_{21} \Phi \mathbf{B}^T \Phi^T H^T}{He_3^T} \right) \mathbf{B}\Phi^T \nu \right| \\
& - \frac{\gamma}{|HC_i\Phi^T H^T| \cdot |H\Phi\mathbf{B}\Phi^{2T}H^T|} \left| HC_i\Phi^T H^T \cdot H\Phi^2 \mathbf{B}\Phi^{2T}\nu \right| \\
& - \frac{\beta}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} \left| H \left(\Phi^2 \mathbf{B}^T \Phi^T H^T \Gamma^T - \Gamma H\Phi\mathbf{B}\Phi^{2T} \right) \nu \right| \\
& - \frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} \left| H\Phi\mathbf{B}\Phi^{2T}\nu \right| + j\zeta_{i,1}^{k+2|k+2}\nu \Bigg)
\end{aligned} \tag{C.19}
$$

Combine the first 2 elements.

$$
\begin{aligned}
\mathcal{E}_{i,1}^{k+2|k+2} = \exp\Bigg(& -\frac{\frac{\gamma}{|b_1 H^T|} |H\Phi D_i \Phi^T A_{21} \Phi \mathbf{B}^T \Phi^T H^T| + \beta |H\Phi E_i \Phi^T A_{21} \Phi \mathbf{B}^T \Phi^T H^T|}{|HC_i\Phi^T H^T| \cdot |H\Phi\mathbf{B}\Phi^{2T}H^T| \cdot |He_3^T|} |H\Phi\mathbf{B}\Phi^T\nu| \\
& - \frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} |H\Phi^2\mathbf{B}\Phi^{2T}\nu| - \frac{\beta}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} |H \left(\Phi^2 \mathbf{B}^T \Phi^T H^T \Gamma^T - \Gamma H\Phi\mathbf{B}\Phi^{2T} \right) \nu| \\
& - \frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} |H\Phi\mathbf{B}\Phi^{2T}\nu| + j\zeta_{i,1}^{k+2|k+2}\nu \Bigg)
\end{aligned} \tag{C.20}
$$

D_i and E_i are defined earlier.

C.2.2 From $\mathcal{E}_i^{k+1|k+1}$, $1 \le i \le m-1$ to $\mathcal{E}_{i,2}^{k+2|k+2}$

Consider the CF of step $k+2$ in equation (C.15). Then the second child term at step $k+2$ has the argument of the exponential,

$$
\begin{aligned}
\mathcal{E}_{i,2}^{k+2|k+2} = \exp\Bigg(& -\frac{\left(\sum_{l=1, l\neq i}^{m-1} P_l \cdot \left| HC_l A_{21} C_i^T H^T \right| \right)}{|HC_i\Phi^T H^T| \cdot \left| He_3^T \right|} \left| H\Phi\mathbf{B}\Phi^{2T}H^T \right| \left| \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T} - \frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T} \right| \\
& - \frac{\beta}{|HC_i\Phi^T H^T|} \left| H\Phi E_i\Phi^{2T}H^T \right| \left| \frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T} - \frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T} \right| \\
& - \frac{\gamma}{|HC_i\Phi^T H^T|} \left| HC_i\Phi^{2T}H^T \right| \left| \frac{HC_i\Phi^{2T}\nu}{HC_i\Phi^{2T}H^T} - \frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T} \right| \\
& - \beta \left| \Gamma^T H^T \right| \left| \frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T} \right| - \gamma \left| \frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T} \right| + j\zeta_{i,2}^{k+2|k+2}\nu \Bigg)
\end{aligned} \tag{C.21}
$$

Because

$$
\left| \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T} - \frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T} \right| = \frac{\left| H\Phi D_i \Phi^T A_{21} \Phi \mathbf{B}^T \Phi^T H^T \right|}{|He_3^T| \cdot |H\Phi\mathbf{B}\Phi^{2T}H^T| \cdot |H\Phi D_i\Phi^{2T}H^T|} \left| H\Phi\mathbf{B}\Phi^T\nu \right| \tag{C.22}
$$

$$\left|\frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T} - \frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T}\right| = \frac{\left|H\Phi D_i\Phi^T A_{21}\Phi E_i^T\Phi^T H^T\right|}{\left|He_3^T\right|\cdot\left|H\Phi E_i\Phi^{2T}H^T\right|\cdot\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| \tag{C.23}$$

And from simple algebra,

$$D_i^T\Phi^T H^T H C_i - C_i^T H^T H\Phi D_i = -\left(HC_i\Phi^T H^T\right)\cdot D_i \tag{C.24}$$

Then,

$$\begin{aligned}\mathcal{E}_{i,2}^{k+2|k+2} = \exp\Bigg(& -\frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l\cdot\left|HC_lA_{21}C_i^TH^T\right|\right)}{\left|HC_i\Phi^TH^T\right|\cdot\left|He_3^T\right|}\frac{\left|H\Phi D_i\Phi^TA_{21}\Phi B^T\Phi^TH^T\right|}{\left|He_3^T\right|\cdot\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ & -\frac{\beta}{\left|HC_i\Phi^TH^T\right|}\frac{\left|H\Phi D_i\Phi^TA_{21}\Phi E_i^T\Phi^TH^T\right|}{\left|He_3^T\right|\cdot\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ & -\frac{\gamma}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi^2D_i\Phi^{2T}\nu\right| \\ & -\frac{\beta}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\left(\Phi^2D_i^T\Phi^TH^T\Gamma^T - \Gamma H\Phi D_i\Phi^{2T}\right)\nu\right| \\ & -\frac{\gamma}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi D_i\Phi^{2T}\nu\right| + j\zeta_{i,2}^{k+2|k+2}\nu\Bigg)\end{aligned} \tag{C.25}$$

Combine the first 2 elements.

$$\begin{aligned}\mathcal{E}_{i,2}^{k+2|k+2} = \exp\Big(& -\rho_1\left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ & -\frac{\gamma}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi^2D_i\Phi^{2T}\nu\right| \\ & -\frac{\beta}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\left(\Phi^2D_i^T\Phi^TH^T\Gamma^T - \Gamma H\Phi D_i\Phi^{2T}\right)\nu\right| \\ & -\frac{\gamma}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi D_i\Phi^{2T}\nu\right| + j\zeta_{i,2}^{k+2|k+2}\nu\Big)\end{aligned} \tag{C.26}$$

where

$$\begin{aligned}\rho_1 = & \frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l\cdot\left|HC_lA_{21}C_i^TH^T\right|\right)}{\left|HC_i\Phi^TH^T\right|\cdot\left|He_3^T\right|}\frac{\left|H\Phi D_i\Phi^TA_{21}\Phi\mathbf{B}^T\Phi^TH^T\right|}{\left|He_3^T\right|\cdot\left|H\Phi D_i\Phi^{2T}H^T\right|} \\ & + \frac{\beta}{\left|HC_i\Phi^TH^T\right|}\frac{\left|H\Phi D_i\Phi^TA_{21}\Phi E_i^T\Phi^TH^T\right|}{\left|He_3^T\right|\cdot\left|H\Phi D_i\Phi^{2T}H^T\right|}\end{aligned} \tag{C.27}$$

and as defined earlier,

$$D_i = C_i^TH^Tb_1 - b_1^THC_i \quad \text{and} \quad E_i = C_i^TH^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma HC_i \tag{C.28}$$

C.2.3 From $\mathcal{E}_i^{k+1|k+1}$, $1 \le i \le m-1$ to $\mathcal{E}_{i,3}^{k+2|k+2}$

Again, consider the CF of step $k+2$ in equation (C.15). Then the third child term at step $k+2$ has the argument of the exponential,

$$\begin{aligned}
\mathcal{E}_{i,3}^{k+2|k+2} = &\exp\Bigg(-\frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l \cdot \left|HC_l A_{21} C_i^T H^T\right|\right)}{\left|HC_i\Phi^T H^T\right| \cdot \left|He_3^T\right|} \left|H\Phi\mathbf{B}\Phi^{2T}H^T\right| \left|\frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T} - \frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T}\right| \\
&- \frac{\frac{\gamma}{|b_1 H^T|}}{\left|HC_i\Phi^T H^T\right|} \left|H\Phi D_i\Phi^{2T}H^T\right| \left|\frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T} - \frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T}\right| \\
&- \frac{\gamma}{\left|HC_i\Phi^T H^T\right|} \left|HC_i\Phi^{2T}H^T\right| \left|\frac{HC_i\Phi^{2T}\nu}{HC_i\Phi^{2T}H^T} - \frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T}\right| \\
&-\beta\left|\Gamma^T H^T\right| \left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T}\right| - \gamma\left|\frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T}\right| + j\zeta_{i,3}^{k+2|k+2}\nu\Bigg)
\end{aligned} \tag{C.29}$$

Because

$$\left|\frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T} - \frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T}\right| = \frac{\left|H\Phi E_i\Phi^T A_{21}\Phi\mathbf{B}^T\Phi^T H^T\right|}{\left|He_3^T\right| \cdot \left|H\Phi\mathbf{B}\Phi^{2T}H^T\right| \cdot \left|H\Phi E_i\Phi^{2T}H^T\right|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| \tag{C.30}$$

$$\left|\frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T} - \frac{H\Phi E_i\Phi^{2T}\nu}{H\Phi E_i\Phi^{2T}H^T}\right| = \frac{\left|H\Phi D_i\Phi^T A_{21}\Phi E_i^T\Phi^T H^T\right|}{\left|He_3^T\right| \cdot \left|H\Phi E_i\Phi^{2T}H^T\right| \cdot \left|H\Phi D_i\Phi^{2T}H^T\right|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| \tag{C.31}$$

And

$$E_i^T\Phi^T H^T HC_i - C_i^T H^T H\Phi E_i = -\left(HC_i\Phi^T H^T\right) \cdot E_i \tag{C.32}$$

Substitute equation (C.30) (C.31) and (C.32) back to equation (C.29).

$$\begin{aligned}
\mathcal{E}_{i,3}^{k+2|k+2} = &\exp\Bigg(-\frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l \cdot \left|HC_l A_{21} C_i^T H^T\right|\right)}{\left|HC_i\Phi^T H^T\right| \cdot \left|He_3^T\right|} \frac{\left|H\Phi E_i\Phi^T A_{21}\Phi\mathbf{B}^T\Phi^T H^T\right|}{\left|He_3^T\right| \cdot \left|H\Phi E_i\Phi^{2T}H^T\right|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| \\
&- \frac{\frac{\gamma}{|b_1 H^T|}}{\left|HC_i\Phi^T H^T\right|} \frac{\left|H\Phi D_i\Phi^T A_{21}\Phi E_i^T\Phi^T H^T\right|}{\left|He_3^T\right| \cdot \left|H\Phi E_i\Phi^{2T}H^T\right|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| \\
&- \frac{\gamma}{\left|H\Phi E_i\Phi^{2T}H^T\right|} \left|H\Phi^2 E_i\Phi^{2T}\nu\right| \\
&- \frac{\beta}{\left|H\Phi E_i\Phi^{2T}H^T\right|} \left|H\left(\Phi^2 E_i^T\Phi^T H^T\Gamma^T - \Gamma H\Phi E_i\Phi^{2T}\right)\nu\right| \\
&- \frac{\gamma}{\left|H\Phi E_i\Phi^{2T}H^T\right|} \left|H\Phi E_i\Phi^{2T}\nu\right| + j\zeta_{i,3}^{k+2|k+2}\nu\Bigg)
\end{aligned} \tag{C.33}$$

Combine the first 2 elements, then,

$$\begin{aligned}\mathcal{E}_{i,3}^{k+2|k+2} = \exp\Big(&-\rho_1 \left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ &- \frac{\gamma}{|H\Phi E_i\Phi^{2T}H^T|}\left|H\Phi^2 E_i\Phi^{2T}\nu\right| \\ &- \frac{\beta}{|H\Phi E_i\Phi^{2T}H^T|}\left|H\left(\Phi^2 E_i^T\Phi^T H^T\Gamma^T - \Gamma H\Phi E_i\Phi^{2T}\right)\nu\right| \\ &- \frac{\gamma}{|H\Phi E_i\Phi^{2T}H^T|}\left|H\Phi E_i\Phi^{2T}\nu\right| + j\zeta_{i,3}^{k+2|k+2}\nu\Big)\end{aligned} \tag{C.34}$$

where

$$\begin{aligned}\rho_1 =& \frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l\cdot\left|HC_lA_{21}C_i^TH^T\right|\right)}{|HC_i\Phi^TH^T|\cdot|He_3^T|}\frac{\left|H\Phi E_i\Phi^TA_{21}\Phi\mathbf{B}^T\Phi^TH^T\right|}{|He_3^T|\cdot|H\Phi E_i\Phi^{2T}H^T|} \\ &+ \frac{\frac{\gamma}{|b_1H^T|}}{|HC_i\Phi^TH^T|}\frac{\left|H\Phi D_i\Phi^TA_{21}\Phi E_i^T\Phi^TH^T\right|}{|He_3^T|\cdot|H\Phi E_i\Phi^{2T}H^T|}\end{aligned} \tag{C.35}$$

As defined earlier,

$$D_i = C_i^TH^Tb_1 - b_1^THC_i \quad \text{and} \quad E_i = C_i^TH^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma HC_i \tag{C.36}$$

C.2.4 From $\mathcal{E}_m^{k+1|k+1}$ to $\mathcal{E}_{m,i}^{k+2|k+2}$ for $1 \leq i \leq m-1$

At step $k+1$, consider the exponential term $\mathcal{E}_m^{k+1|k+1}$ expressed in equation (C.10). After time propagation, this term becomes,

$$\begin{aligned}\mathcal{E}_m^{k+2|k+1} = \exp\Bigg(&-\frac{P_1}{|b_1\Phi^TH^T|}\left|H\Phi D_1\Phi^{2T}\nu\right| - \dots - \frac{P_{m-1}}{|b_1\Phi^TH^T|}\left|H\Phi D_{m-1}\Phi^{2T}\nu\right| \\ &- \frac{\beta}{|b_1\Phi^TH^T|}\left|H\Phi D_{gb}\Phi^{2T}\nu\right| - \frac{\gamma}{|b_1\Phi^TH^T|}\left|b_1\Phi^{2T}\nu\right| - \beta\left|\Gamma^T\nu\right| + j\zeta_m^{k+2|k+1}\nu\Bigg)\end{aligned} \tag{C.37}$$

Then at the $(k+2)^{th}$ measurement update, the CF is obtained from the update integral.

$$\begin{aligned}\bar{\phi}_{k+2}(\nu) = &\int_{-\infty}^{+\infty} G \cdot \exp\Bigg(-\frac{P_1}{|b_1\Phi^T H^T|}\left|H\Phi D_1\Phi^{2T}H^T\right|\left|\frac{H\Phi D_1\Phi^{2T}\nu}{H\Phi D_1\Phi^{2T}H^T}-\eta\right| - \cdots \\ &- \frac{P_{m-1}}{|b_1\Phi^T H^T|}\left|H\Phi D_{m-1}\Phi^{2T}H^T\right|\left|\frac{H\Phi D_{m-1}\Phi^{2T}\nu}{H\Phi D_{m-1}\Phi^{2T}H^T}-\eta\right| \\ &- \frac{\beta}{|b_1\Phi^T H^T|}\left|H\Phi D_{gb}\Phi^{2T}H^T\right|\left|\frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}-\eta\right| \\ &- \frac{\gamma}{|b_1\Phi^T H^T|}\left|b_1\Phi^{2T}H^T\right|\left|\frac{b_1\Phi^{2T}\nu}{b_1\Phi^{2T}H^T}-\eta\right| - \beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T}-\eta\right| \\ &-\gamma|\eta| + jz_{k+2}\eta + j\zeta_m^{k+2|k+1}\left(\nu - H^T\eta\right)\Bigg)\,d\eta \end{aligned} \tag{C.38}$$

Next, find the exponential term of the i^{th} child term at step $k+2$ for $1 \le i \le m-1$, and denoted as $\mathcal{E}_{m,i}^{k+2|k+2}$.

$$\begin{aligned}\mathcal{E}_{m,i}^{k+2|k+2} = \exp\Bigg(&-\frac{P_1}{|b_1\Phi^T H^T|}\left|H\Phi D_1\Phi^{2T}H^T\right|\left|\frac{H\Phi D_1\Phi^{2T}\nu}{H\Phi D_1\Phi^{2T}H^T}-\frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T}\right| - \cdots \\ &- \frac{P_{m-1}}{|b_1\Phi^T H^T|}\left|H\Phi D_{m-1}\Phi^{2T}H^T\right|\left|\frac{H\Phi D_{m-1}\Phi^{2T}\nu}{H\Phi D_{m-1}\Phi^{2T}H^T}-\frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T}\right| \\ &- \frac{\beta}{|b_1\Phi^T H^T|}\left|H\Phi D_{gb}\Phi^{2T}H^T\right|\left|\frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}-\frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T}\right| \\ &- \frac{\gamma}{|b_1\Phi^T H^T|}\left|b_1\Phi^{2T}H^T\right|\left|\frac{b_1\Phi^{2T}\nu}{b_1\Phi^{2T}H^T}-\frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T}\right| \\ &-\beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T}-\frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T}\right| - \gamma\left|\frac{H\Phi D_i\Phi^{2T}\nu}{H\Phi D_i\Phi^{2T}H^T}\right| + j\zeta_{m,i}^{k+2|k+2}\nu\Bigg)\end{aligned} \tag{C.39}$$

Because for $l = 1, 2, ..., m-1$, $l \neq i$,

$$\Phi D_i^T\Phi^T H^T H\Phi D_l\Phi^T - \Phi D_l^T\Phi^T H^T H\Phi D_i\Phi^T = \frac{H\Phi D_l\Phi^T A_{21}\Phi D_i^T\Phi^T H^T}{He_3^T}\cdot\mathbf{B} \tag{C.40}$$

And

$$\Phi D_i^T\Phi^T H^T H\Phi D_{gb}\Phi^T - \Phi D_{gb}^T\Phi^T H^T H\Phi D_i\Phi^T = \frac{H\Phi D_{gb}\Phi^T A_{21}\Phi D_i^T\Phi^T H^T}{He_3^T}\cdot\mathbf{B} \tag{C.41}$$

In addition,

$$\begin{aligned}D_i^T\Phi^T H^T b_1 - b_1^T H\Phi D_i &= \left(C_i^T H^T b_1 - b_1^T HC_i\right)\Phi^T H^T b_1 - b_1^T H\Phi\left(b_1^T HC_i - C_i^T H^T b_1\right) \\ &= -\left(b_1\Phi^T H^T\right)\cdot D_i\end{aligned} \tag{C.42}$$

Substitute equation (C.40) (C.41) and (C.42) back into equation (C.39), and combine the first $(m-1)$ elements.

$$\begin{aligned}\mathcal{E}_{m,i}^{k+2|k+2} = \exp\Big(&-\rho_2 \left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ &- \frac{\gamma}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi^2 D_i\Phi^{2T}\nu\right| \\ &- \frac{\beta}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\left(\Phi^2 D_i^T\Phi^T H^T\Gamma^T - \Gamma H\Phi D_i\Phi^{2T}\right)\nu\right| \\ &- \frac{\gamma}{\left|H\Phi D_i\Phi^{2T}H^T\right|}\left|H\Phi D_i\Phi^{2T}\nu\right| + j\zeta_{m,i}^{k+2|k+2}\nu\Big)\end{aligned} \tag{C.43}$$

where

$$\rho_2 = \frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l\cdot\left|H\Phi D_l\Phi^T A_{21}\Phi D_i^T\Phi^T H^T\right|\right)}{\left|b_1\Phi^T H^T\right|\left|H\Phi D_i\Phi^{2T}H^T\right|\left|He_3^T\right|} + \frac{\beta\cdot\left|H\Phi D_{gb}\Phi^T A_{21}\Phi D_i^T\Phi^T H^T\right|}{\left|b_1\Phi^T H^T\right|\left|H\Phi D_i\Phi^{2T}H^T\right|\left|He_3^T\right|} \tag{C.44}$$

C.2.5 From $\mathcal{E}_m^{k+1|k+1}$ to $\mathcal{E}_{m,m}^{k+2|k+2}$

At step $k+1$, consider the CF expressed in equation (C.38). The m^{th} child term at step $k+2$ has the exponential term as,

$$\begin{aligned}\mathcal{E}_{m,m}^{k+2|k+2} = \exp\Bigg(&-\frac{P_1}{\left|b_1\Phi^T H^T\right|}\left|H\Phi D_1\Phi^{2T}H^T\right|\cdot\left|\frac{H\Phi D_1\Phi^{2T}\nu}{H\Phi D_1\Phi^{2T}H^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| - \cdots \\ &- \frac{P_{m-1}}{\left|b_1\Phi^T H^T\right|}\left|H\Phi D_{m-1}\Phi^{2T}H^T\right|\cdot\left|\frac{H\Phi D_{m-1}\Phi^{2T}\nu}{H\Phi D_{m-1}\Phi^{2T}H^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| \\ &- \frac{\gamma}{\left|b_1\Phi^T H^T\right|}\left|b_1\Phi^{2T}H^T\right|\cdot\left|\frac{b_1\Phi^{2T}\nu}{b_1\Phi^{2T}H^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| \\ &- \beta\left|\Gamma^T H^T\right|\cdot\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| \\ &-\gamma\left|\frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| + j\zeta_{m,m}^{k+2|k+2}\nu\Bigg)\end{aligned} \tag{C.45}$$

Because

$$\left|\frac{H\Phi D_l\Phi^{2T}\nu}{H\Phi D_l\Phi^{2T}H^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| = \frac{\left|H\Phi D_l\Phi^T A_{21}\Phi D_{gb}^T\Phi^T H^T\right|}{\left|He_3^T\right|\cdot\left|H\Phi D_{gb}\Phi^{2T}H^T\right|\cdot\left|H\Phi D_l\Phi^{2T}H^T\right|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| \tag{C.46}$$

$$D_{gb}^T\Phi^T H^T b_1 - b_1^T H\Phi D_{gb} = -\left(b_1\Phi^T H^T\right)\cdot D_{gb} \tag{C.47}$$

Then the exponential term can be rewritten as,

$$\begin{aligned}
\mathcal{E}_{m,m}^{k+2|k+2} = \exp\Bigg(& -\frac{P_1 \left|H\Phi D_1 \Phi^T A_{21} \Phi D_{gb}^T \Phi^T H^T\right|}{\left|b_1 \Phi^T H^T\right| \cdot \left|H\Phi D_{gb} \Phi^{2T} H^T\right| \cdot \left|He_3^T\right|} \left|H\Phi \mathbf{B} \Phi^T \nu\right| - \cdots \\
& - \frac{P_{m-1} \left|H\Phi D_{m-1} \Phi^T A_{21} \Phi D_{gb}^T \Phi^T H^T\right|}{\left|b_1 \Phi^T H^T\right| \cdot \left|H\Phi D_{gb} \Phi^{2T} H^T\right| \cdot \left|He_3^T\right|} \left|H\Phi \mathbf{B} \Phi^T \nu\right| \\
& - \frac{\gamma}{\left|H\Phi D_{gb} \Phi^{2T} H^T\right|} \left|H\Phi^2 D_{gb} \Phi^{2T} \nu\right| \\
& - \frac{\beta}{\left|H\Phi D_{gb} \Phi^{2T} H^T\right|} \left|H\left(\Phi^2 D_{gb}^T \Phi^T H^T \Gamma^T - \Gamma H \Phi D_{gb} \Phi^{2T}\right)\nu\right| \\
& - \frac{\gamma}{\left|H\Phi D_{gb} \Phi^{2T} H^T\right|} \left|H\Phi D_{gb} \Phi^{2T} \nu\right| + j\zeta_{m,m}^{k+2|k+2} \nu \Bigg)
\end{aligned} \tag{C.48}$$

Combine the first (m-1) elements, then,

$$\begin{aligned}
\mathcal{E}_{m,m}^{k+2|k+2} = \exp\Bigg(& -\frac{\sum_{l=1}^{m-1} \left(P_l \left|H\Phi D_l \Phi^T A_{21} \Phi D_{gb}^T \Phi^T H^T\right|\right)}{\left|b_1 \Phi^T H^T\right| \cdot \left|H\Phi D_{gb} \Phi^{2T} H^T\right| \cdot \left|He_3^T\right|} \left|H\Phi \mathbf{B} \Phi^T \nu\right| \\
& - \frac{\gamma}{\left|H\Phi D_{gb} \Phi^{2T} H^T\right|} \left|H\Phi^2 D_{gb} \Phi^{2T} \nu\right| \\
& - \frac{\beta}{\left|H\Phi D_{gb} \Phi^{2T} H^T\right|} \left|H\left(\Phi^2 D_{gb}^T \Phi^T H^T \Gamma^T - \Gamma H \Phi D_{gb} \Phi^{2T}\right)\nu\right| \\
& - \frac{\gamma}{\left|H\Phi D_{gb} \Phi^{2T} H^T\right|} \left|H\Phi D_{gb} \Phi^{2T} \nu\right| + j\zeta_{m,m}^{k+2|k+2} \nu \Bigg)
\end{aligned} \tag{C.49}$$

where

$$D_l = -b_1^T H C_l + C_l^T H^T b_1, \quad l = 1, ..., m-1 \tag{C.50}$$

$$D_{gb} = b_1^T \Gamma^T \Phi^{-T} - \Phi^{-1} \Gamma b_1 \tag{C.51}$$

C.2.6 From $\mathcal{E}_{m+1}^{k+1|k+1}$ to $\mathcal{E}_{m+1,i}^{k+2|k+2}$ for $1 \leq i \leq m-1$

At step $k+1$, consider the exponential term $\mathcal{E}_{m+1}^{k+1|k+1}$ expressed in equation (C.13). After time propagation, this term becomes,

$$\begin{aligned}
\mathcal{E}_{m+1}^{k+2|k+1} = \exp\Bigg(& -\frac{P_1}{\left|\Gamma^T H^T\right|} \left|H\Phi E_1 \Phi^{2T} \nu\right| - \cdots - \frac{P_{m-1}}{\left|\Gamma^T H^T\right|} \left|H\Phi E_{m-1} \Phi^{2T} \nu\right| \\
& - \frac{\gamma}{\left|b_1 H^T\right| \cdot \left|\Gamma^T H^T\right|} \left|H\Phi D_{gb} \Phi^{2T} \nu\right| - \frac{\gamma}{\left|\Gamma^T H^T\right|} \left|\Gamma^T \Phi^T \nu\right| - \beta \left|\Gamma^T \nu\right| + j\zeta_{m+1}^{k+2|k+1} \nu \Bigg)
\end{aligned} \tag{C.52}$$

Then at the $(k+2)^{th}$ measurement update, the CF is,

$$\begin{aligned}\bar{\phi}_{k+2}(\nu) = \int_{-\infty}^{+\infty} G \cdot \exp\Bigg(&-\frac{P_1}{|\Gamma^T H^T|} \left|H\Phi E_1 \Phi^{2T} H^T\right| \left|\frac{H\Phi E_1 \Phi^{2T}\nu}{H\Phi E_1 \Phi^{2T} H^T} - \eta\right| - \cdots \\ &- \frac{P_{m-1}}{|\Gamma^T H^T|} \left|H\Phi E_{m-1} \Phi^{2T} H^T\right| \left|\frac{H\Phi E_{m-1} \Phi^{2T}\nu}{H\Phi E_{m-1} \Phi^{2T} H^T} - \eta\right| \\ &- \frac{\gamma}{|b_1 H^T| \cdot |\Gamma^T H^T|} \left|H\Phi D_{gb} \Phi^{2T} H^T\right| \left|\frac{H\Phi D_{gb} \Phi^{2T}\nu}{H\Phi D_{gb} \Phi^{2T} H^T} - \eta\right| \\ &- \frac{\gamma}{|\Gamma^T H^T|} \left|\Gamma^T \Phi^T H^T\right| \left|\frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T} - \eta\right| \\ &-\beta \left|\Gamma^T H^T\right| \left|\frac{\Gamma^T \nu}{\Gamma^T H^T} - \eta\right| - \gamma\,|\eta| + j z_{k+2}\eta + j\zeta_{m+1}^{k+2|k+1}\left(\nu - H^T\eta\right)\Bigg)\, d\eta \end{aligned} \tag{C.53}$$

Find the exponential term of the i^{th} child term at step $k+2$ when $1 \leq i \leq m-1$, and denoted as $\mathcal{E}_{m+1,i}^{k+2|k+2}$.

$$\begin{aligned}\mathcal{E}_{m+1,i}^{k+2|k+2} = \exp\Bigg(&-\frac{P_1}{|\Gamma^T H^T|} \left|H\Phi E_1 \Phi^{2T} H^T\right| \left|\frac{H\Phi E_1 \Phi^{2T}\nu}{H\Phi E_1 \Phi^{2T} H^T} - \frac{H\Phi E_i \Phi^{2T}\nu}{H\Phi E_i \Phi^{2T} H^T}\right| - \cdots \\ &- \frac{P_{m-1}}{|\Gamma^T H^T|} \left|H\Phi E_{m-1} \Phi^{2T} H^T\right| \left|\frac{H\Phi E_{m-1} \Phi^{2T}\nu}{H\Phi E_{m-1} \Phi^{2T} H^T} - \frac{H\Phi E_i \Phi^{2T}\nu}{H\Phi E_i \Phi^{2T} H^T}\right| \\ &- \frac{\gamma}{|b_1 H^T| \cdot |\Gamma^T H^T|} \left|H\Phi D_{gb} \Phi^{2T} H^T\right| \left|\frac{H\Phi D_{gb} \Phi^{2T}\nu}{H\Phi D_{gb} \Phi^{2T} H^T} - \frac{H\Phi E_i \Phi^{2T}\nu}{H\Phi E_i \Phi^{2T} H^T}\right| \\ &- \frac{\gamma}{|\Gamma^T H^T|} \left|\Gamma^T \Phi^T H^T\right| \left|\frac{\Gamma^T \Phi^T \nu}{\Gamma^T \Phi^T H^T} - \frac{H\Phi E_i \Phi^{2T}\nu}{H\Phi E_i \Phi^{2T} H^T}\right| \\ &-\beta \left|\Gamma^T H^T\right| \left|\frac{\Gamma^T \nu}{\Gamma^T H^T} - \frac{H\Phi E_i \Phi^{2T}\nu}{H\Phi E_i \Phi^{2T} H^T}\right| - \gamma \left|\frac{H\Phi E_i \Phi^{2T}\nu}{H\Phi E_i \Phi^{2T} H^T}\right| + j\zeta_{m+1,i}^{k+2|k+2}\nu\Bigg) \end{aligned} \tag{C.54}$$

When $l = 1, 2, ..., m-1$, $l \neq i$,

$$\Phi E_i^T \Phi^T H^T H\Phi E_l \Phi^T - \Phi E_l^T \Phi^T H^T H\Phi E_i \Phi^T = \frac{H\Phi E_l \Phi^T A_{21} \Phi E_i^T \Phi^T H^T}{He_3^T} \cdot \mathbf{B} \tag{C.55}$$

And

$$\Phi E_i^T \Phi^T H^T H\Phi D_{gb} \Phi^T - \Phi D_{gb}^T \Phi^T H^T H\Phi E_i \Phi^T = \frac{H\Phi D_{gb} \Phi^T A_{21} \Phi E_i^T \Phi^T H^T}{He_3^T} \cdot \mathbf{B} \tag{C.56}$$

In addition,

$$\begin{aligned} &E_i^T \Phi^T H^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma H\Phi E_i \\ &= \left(C_i^T H^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma H C_i\right) \Phi^T H^T b_1 - b_1^T H\Phi \left(\Phi^{-1}\Gamma H C_i - C_i^T H^T \Gamma^T \Phi^{-T}\right) \\ &= \left(\Gamma^T H^T\right) \cdot \left(C_i^T H^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma H C_i\right) \\ &= -\left(\Gamma^T H^T\right) \cdot E_i \end{aligned} \tag{C.57}$$

Substitute equation (C.55) (C.56) and (C.57) back into equation (C.54), and combine the first $(m-1)$ elements.

$$\begin{aligned}\mathcal{E}_{m+1,i}^{k+2|k+2} = \exp\Big(&-\rho_2 \left|H\Phi B\Phi^T\nu\right| \\ &- \frac{\gamma}{|H\Phi E_i\Phi^{2T}H^T|}\left|H\Phi^2E_i\Phi^{2T}\nu\right| \\ &- \frac{\beta}{|H\Phi E_i\Phi^{2T}H^T|}\left|H\left(\Phi^2E_i^T\Phi^TH^T\Gamma^T - \Gamma H\Phi E_i\Phi^{2T}\right)\nu\right| \\ &- \frac{\gamma}{|H\Phi E_i\Phi^{2T}H^T|}\left|H\Phi E_i\Phi^{2T}\nu\right| + j\zeta_{m+1,i}^{k+2|k+2}\nu\Big)\end{aligned} \tag{C.58}$$

where

$$\rho_2 = \frac{\left(\sum_{l=1,l\neq i}^{m-1} P_l \cdot \left|H\Phi E_l\Phi^TA_{21}\Phi E_i^T\Phi^TH^T\right|\right)}{|\Gamma^TH^T|\,|H\Phi E_i\Phi^{2T}H^T|\,|He_3^T|} + \frac{\gamma\cdot\left|H\Phi D_{gb}\Phi^TA_{21}\Phi E_i^T\Phi^TH^T\right|}{|b_1H^T|\cdot|\Gamma^TH^T|\cdot|H\Phi E_i\Phi^{2T}H^T|\cdot|He_3^T|} \tag{C.59}$$

C.2.7 From $\mathcal{E}_{m+1}^{k+1|k+1}$ to $\mathcal{E}_{m+1,m}^{k+2|k+2}$

At step $k+1$, consider the CF expressed in equation (C.53). The m^{th} child term at step $k+2$ has the exponential term as,

$$\begin{aligned}\mathcal{E}_{m+1,m}^{k+2} = \exp\Bigg(&-\frac{P_1}{|\Gamma^TH^T|}\left|H\Phi E_1\Phi^{2T}H^T\right|\left|\frac{H\Phi E_1\Phi^{2T}\nu}{H\Phi E_1\Phi^{2T}H^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| - \cdots \\ &- \frac{P_{m-1}}{|\Gamma^TH^T|}\left|H\Phi E_{m-1}\Phi^{2T}H^T\right|\left|\frac{H\Phi E_{m-1}\Phi^{2T}\nu}{H\Phi E_{m-1}\Phi^{2T}H^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| \\ &- \frac{\gamma}{|\Gamma^TH^T|}\left|\Gamma^T\Phi^TH^T\right|\left|\frac{\Gamma^T\Phi^T\nu}{\Gamma^T\Phi^TH^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| \\ &-\beta\left|\Gamma^TH^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^TH^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| - \gamma\left|\frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| + j\zeta_{m+1,m}^{k+2|k+2}\nu\Bigg)\end{aligned} \tag{C.60}$$

Because,

$$\left|\frac{H\Phi E_l\Phi^{2T}\nu}{H\Phi E_l\Phi^{2T}H^T} - \frac{H\Phi D_{gb}\Phi^{2T}\nu}{H\Phi D_{gb}\Phi^{2T}H^T}\right| = \frac{\left|H\Phi E_l\Phi^TA_{21}\Phi D_{gb}^T\Phi^TH^T\right|}{|He_3|\cdot|H\Phi D_{gb}\Phi^{2T}H^T|\cdot|H\Phi E_l\Phi^{2T}H^T|}\left|H\Phi\mathbf{B}\Phi^T\nu\right| \tag{C.61}$$

and

$$D_{gb}^T\Phi^TH^T\Gamma^T\Phi^{-T} - \Phi^{-1}\Gamma H\Phi D_{gb} = -\left(\Gamma^TH^T\right)\cdot D_{gb} \tag{C.62}$$

Then the exponential term can be rewritten as,

$$\begin{aligned}\mathcal{E}_{m+1,m}^{k+2|k+2} = \exp\Bigg(& -\frac{P_1 \left|H\Phi E_1 \Phi^T A_{21} \Phi D_{gb}^T \Phi^T H^T\right|}{|\Gamma^T H^T| \cdot |He_3| \cdot |H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| - \cdots \\ & - \frac{P_{m-1} \left|H\Phi E_{m-1} \Phi^T A_{21} \Phi D_{gb}^T \Phi^T H^T\right|}{|\Gamma^T H^T| \cdot |He_3| \cdot |H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ & - \frac{\gamma}{|H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\Phi^2 D_{gb}\Phi^{2T}\nu\right| \\ & - \frac{\beta}{|H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\left(\Phi^2 D_{gb}^T \Phi^T H^T \Gamma^T - \Gamma H \Phi D_{gb}\Phi^{2T}\right)\nu\right| \\ & - \frac{\gamma}{|H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\Phi D_{gb}\Phi^{2T}\nu\right| \end{aligned} \tag{C.63}$$

Combine the first $(m-1)$ elements.

$$\begin{aligned}\mathcal{E}_{m+1,m}^{k+2|k+2} = \exp\Bigg(& -\frac{\sum_{l=1}^{m-1}\left(P_l \left|H\Phi E_l \Phi^T A_{21} \Phi D_{gb}^T \Phi^T H^T\right|\right)}{|\Gamma^T H^T| \cdot |He_3| \cdot |H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\Phi\mathbf{B}\Phi^T\nu\right| \\ & - \frac{\gamma}{|H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\Phi^2 D_{gb}\Phi^{2T}\nu\right| \\ & - \frac{\beta}{|H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\left(\Phi^2 D_{gb}^T \Phi^T H^T \Gamma^T - \Gamma H \Phi D_{gb}\Phi^{2T}\right)\nu\right| \\ & - \frac{\gamma}{|H\Phi D_{gb}\Phi^{2T} H^T|} \left|H\Phi D_{gb}\Phi^{2T}\nu\right| \end{aligned} \tag{C.64}$$

where

$$E_l = \Phi^{-1}\Gamma H C_l - C_l^T H^T \Gamma^T \Phi^{-T}, \quad l = 1, ..., m-1 \tag{C.65}$$

and

$$D_{gb} = \Phi^{-1}\Gamma b_1 - b_1^T \Gamma^T \Phi^{-T} \tag{C.66}$$

C.3 Terms at Step $k+3$

C.3.1 From $\mathcal{E}_{i,1}^{k+2|k+2}$ to $\mathcal{E}_{i,1,1}^{k+3|k+3}$

Consider the term $\mathcal{E}_{i,1}^{k+2|k+2}$ at step $k+2$, and find its first child term $\mathcal{E}_{i,1,1}^{k+3|k+3}$ at step $k+3$. For convenience, rewrite $\mathcal{E}_{i,1}^{k+2|k+2}$ as,

$$\mathcal{E}_{i,1}^{k+2|k+2} = \exp\left(-\rho_1|H\Phi\mathbf{B}\Phi^T\nu| - \rho_2|H\Phi^2\mathbf{B}\Phi^{2T}\nu| - \rho_3|HC\nu| - \rho_4|H\Phi\mathbf{B}\Phi^{2T}\nu| + j\zeta_{i,1}^{k+2|k+2}\nu\right) \tag{C.67}$$

where

$$\rho_1 = \frac{\frac{\gamma}{|b_1 H^T|}|H\Phi D_i \Phi^T A_{21}\Phi\mathbf{B}^T\Phi^T H^T| + \beta|H\Phi E_i\Phi^T A_{21}\Phi\mathbf{B}^T\Phi^T H^T|}{|HC_i\Phi^T H^T|\cdot|H\Phi\mathbf{B}\Phi^{2T}H^T|\cdot|He_3^T|} \tag{C.68}$$

$$\rho_2 = \rho_4 = \frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} \tag{C.69}$$

$$\rho_3 = \frac{\beta}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} \tag{C.70}$$

$$C = \Phi^2\mathbf{B}^T\Phi^T H^T\Gamma^T - \Gamma H\Phi\mathbf{B}\Phi^{2T} \tag{C.71}$$

After the propagation, the exponential term becomes,

$$\begin{aligned}\mathcal{E}_{i,1}^{k+3|k+2} = \exp\big(&-\rho_1|H\Phi\mathbf{B}\Phi^{2T}\nu| - \rho_2|H\Phi^2\mathbf{B}\Phi^{3T}\nu| - \rho_3|HC\Phi^T\nu|\\ &-\rho_4|H\Phi\mathbf{B}\Phi^{3T}\nu| - \beta|\Gamma^T\nu| + j\zeta_{i,1}^{k+3|k+2}\nu\big)\end{aligned} \tag{C.72}$$

The CF can be obtained from the update integral.

$$\begin{aligned}\bar{\phi}_{k+3}(\nu) = \int_{-\infty}^{+\infty} G\cdot\exp\Bigg(&-\rho_1\left|H\Phi\mathbf{B}\Phi^{2T}H^T\right|\left|\frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T} - \eta\right|\\ &-\rho_2\left|H\Phi^2\mathbf{B}\Phi^{3T}H^T\right|\left|\frac{H\Phi^2\mathbf{B}\Phi^{3T}\nu}{H\Phi^2\mathbf{B}\Phi^{3T}H^T} - \eta\right| - \rho_3\left|HC\Phi^T H^T\right|\left|\frac{HC\Phi^T\nu}{HC\Phi^T H^T} - \eta\right|\\ &-\rho_4\left|H\Phi\mathbf{B}\Phi^{3T}H^T\right|\left|\frac{H\Phi\mathbf{B}\Phi^{3T}\nu}{H\Phi\mathbf{B}\Phi^{3T}H^T} - \eta\right| - \beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \eta\right|\\ &-\gamma|\eta| + jz_{k+3}\eta + j\zeta_{i,1}^{k+3|k+2}\left(\nu - H^T\eta\right)\Bigg)\,d\eta\end{aligned} \tag{C.73}$$

Then, the first child term at step $k+3$ will have the exponential term expressed as,

$$\begin{aligned}\mathcal{E}_{i,1,1}^{k+3|k+3} = \exp\Bigg(&-\rho_2\left|H\Phi^2\mathbf{B}\Phi^{3T}H^T\right|\left|\frac{H\Phi^2\mathbf{B}\Phi^{3T}\nu}{H\Phi^2\mathbf{B}\Phi^{3T}H^T} - \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right|\\ &-\rho_3\left|HC\Phi^T H^T\right|\left|\frac{HC\Phi^T\nu}{HC\Phi^T H^T} - \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right|\\ &-\rho_4\left|H\Phi\mathbf{B}\Phi^{3T}H^T\right|\left|\frac{H\Phi\mathbf{B}\Phi^{3T}\nu}{H\Phi\mathbf{B}\Phi^{3T}H^T} - \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right|\\ &-\beta\left|\Gamma^T H^T\right|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right| - \gamma\left|\frac{H\Phi\mathbf{B}\Phi^{2T}\nu}{H\Phi\mathbf{B}\Phi^{2T}H^T}\right| + j\zeta_{i,1}^{k+3|k+3}\nu\Bigg)\end{aligned} \tag{C.74}$$

Then,

$$\mathcal{E}_{i,1,1}^{k+3|k+3} = \exp\Bigg(-\frac{\rho_2}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}\left|H\Phi\left(\Phi\mathbf{B}^T\Phi^TH^TH\Phi^2\mathbf{B}\Phi^{2T}-\Phi^2\mathbf{B}^T\Phi^{2T}H^TH\Phi\mathbf{B}\Phi^T\right)\Phi^T\nu\right|$$
$$-\frac{\rho_3}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}\left|H\Phi\left(\Phi\mathbf{B}^T\Phi^TH^THC-C^TH^TH\Phi\mathbf{B}\Phi^T\right)\Phi^T\nu\right|$$
$$-\frac{\rho_4}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}\left|H\Phi^2\left(\mathbf{B}^T\Phi^TH^TH\Phi\mathbf{B}\Phi^T-\Phi\mathbf{B}^T\Phi^TH^TH\Phi\mathbf{B}\right)\Phi^{2T}\nu\right|$$
$$-\frac{\beta}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}\left|H\left(\Phi^2\mathbf{B}^T\Phi^TH^T\Gamma^T-\Gamma H\Phi\mathbf{B}\Phi^{2T}\right)\nu\right|$$
$$-\frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}\left|H\Phi\mathbf{B}\Phi^{2T}\nu\right|+j\zeta_{i,1}^{k+3|k+3}\nu\Bigg) \tag{C.75}$$

Because

$$\Phi\mathbf{B}^T\Phi^TH^TH\Phi^2\mathbf{B}\Phi^{2T}-\Phi^2\mathbf{B}^T\Phi^{2T}H^TH\Phi\mathbf{B}\Phi^T=\frac{H\Phi^2\mathbf{B}\Phi^{2T}A_{21}\Phi\mathbf{B}^T\Phi^TH^T}{He_3^T}\cdot\mathbf{B} \tag{C.76}$$

$$\Phi\mathbf{B}^T\Phi^TH^THC-C^TH^TH\Phi\mathbf{B}\Phi^T=\frac{HCA_{21}\Phi\mathbf{B}^T\Phi^TH^T}{He_3^T}\cdot\mathbf{B} \tag{C.77}$$

And let C in equation (C.18) equal to $\Phi\mathbf{B}\Phi^T$. Hence,

$$\mathbf{B}^T\Phi^TH^TH\Phi\mathbf{B}\Phi^T-\Phi\mathbf{B}^T\Phi^TH^TH\Phi\mathbf{B}=-\left(H\Phi\mathbf{B}\Phi^{2T}H^T\right)\cdot\mathbf{B} \tag{C.78}$$

Substitute equation (C.76) (C.77) and (C.78) back to the exponential term expressed in equation (C.75), and combine the first two elements.

$$\mathcal{E}_{i,1,1}^{k+3|k+3} = \exp\Bigg(-\frac{\rho_2\left|H\Phi^2\mathbf{B}\Phi^{2T}A_{21}\Phi\mathbf{B}^T\Phi^TH^T\right|+\rho_3\left|HCA_{21}\Phi\mathbf{B}^T\Phi^TH^T\right|}{|H\Phi\mathbf{B}\Phi^{2T}H^T|\cdot|He_3^T|}\left|H\Phi\mathbf{B}\Phi^T\nu\right|$$
$$-\rho_4\left|H\Phi^2\mathbf{B}\Phi^{2T}\nu\right|-\frac{\beta}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}\left|H\left(\Phi^2\mathbf{B}^T\Phi^TH^T\Gamma^T-\Gamma H\Phi\mathbf{B}\Phi^{2T}\right)\nu\right|$$
$$-\frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|}\left|H\Phi\mathbf{B}\Phi^{2T}\nu\right|+j\zeta_{i,1}^{k+3|k+3}\nu\Bigg) \tag{C.79}$$

where

$$\rho_2=\rho_4=\frac{\gamma}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} \tag{C.80}$$

$$\rho_3=\frac{\beta}{|H\Phi\mathbf{B}\Phi^{2T}H^T|} \tag{C.81}$$

APPENDIX D

Proof of Lemma 4.3.2

When the top right corner of S_{k+1} is discussed to derive the recursion of S matrix, Lemma 4.3.2 is introduced. This lemma states that,

Lemma 4.3.2 *Consider the exponential part of the i^{th} term, $\mathcal{E}_i^{k-1|k-1}$, at step $k-1$ and the m^{th} term, $\mathcal{E}_m^{k-1|k-1}$ at step $k-1$, then the l^{th} child term at step k of $\mathcal{E}_i^{k-1|k-1}$ and the p^{th} child term at step k of $\mathcal{E}_m^{k-1|k-1}$ can be combined if and only if the l^{th} grandchild term at step $k+1$ of the old child term $\mathcal{E}_{i,old}^{k|k}$ at step k and the p^{th} grandchild term at step $k+1$ of the old child term $\mathcal{E}_{m,old}^{k|k}$ at step k can be combined, i.e.*

$$\mathcal{E}_{i,l}^{k|k} = \mathcal{E}_{m,p}^{k|k} \quad \textit{if and only if} \quad \mathcal{E}_{i,old,l}^{k+1|k+1} = \mathcal{E}_{m,old,p}^{k+1|k+1} \tag{D.1}$$

Proof. At step $k-1$, consider the i^{th} exponential term,

$$\mathcal{E}_i^{k-1|k-1} = \exp\left(-P_1|B_1\nu| - ... - P_{N_{ei}^{k-1|k-1}}|B_{N_{ei}^{k-1|k-1}}\nu| + j\zeta_i^{k-1|k-1}\nu\right) \tag{D.2}$$

where $N_{ei}^{k-1|k-1}$ is the number of elements in the argument of the exponential term.

After time propagation and measurement update, at step k, the exponential part of the l^{th} child term can be derived as,

$$\begin{aligned}\mathcal{E}_{i,l}^{k|k} = \exp\Bigg(&-P_1|B_1\Phi^T H^T|\left|\frac{B_1\Phi^T\nu}{B_1\Phi^T H^T} - \frac{B_l\Phi^T\nu}{B_l\Phi^T H^T}\right| - ... \\ &- P_{N_{ei}^{k-1|k-1}}|B_{N_{ei}^{k-1|k-1}}\Phi^T H^T|\left|\frac{B_{N_{ei}^{k-1|k-1}}\Phi^T\nu}{B_1\Phi^T H^T} - \frac{B_l\Phi^T\nu}{B_l\Phi^T H^T}\right| \\ &-\beta|\Gamma^T H^T|\left|\frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{B_l\Phi^T\nu}{B_l\Phi^T H^T}\right| - \gamma\left|-\frac{B_l\Phi^T\nu}{B_l\Phi^T H^T}\right| + j\zeta_{i,l}^{k|k}\nu\Bigg)\end{aligned} \tag{D.3}$$

and the first $N_{ei}^{k-1|k-1}$ child directions in equation (D.3) are co-aligned with $H\mathbf{A}$. According to term combination rules, the last two child terms of $\mathcal{E}_i^{k-1|k-1}$ never combine. Thus one only needs to look at the first $N_{ei}^{k-1|k-1}$ child terms, where only the parent directions in the form of $\frac{B_l \Phi^T \nu}{B_l \Phi^T H^T}$ for $1 \leq l \leq N_{ei}^{k-1|k-1}$ are being considered.

Define $B_l^T B_i - B_i^T B_l = \theta_i \cdot \mathbf{A}$, $B_l^T \Gamma^T \Phi^{-T} - \Phi^{-1} \Gamma B_l = \theta_{N_{ei}^{k-1|k-1}+1} \cdot H\mathbf{A}$, and define $P_{N_{ei}^{k-1|k-1}+1} = \beta$. From simple algebra, we know that $H\Phi\mathbf{A}\Phi^T = \det(\Phi) \cdot H\mathbf{A}$. Combine the directions of (D.3) except the last one, the exponential part then can be expressed as,

$$\mathcal{E}_{i,l}^{k|k} = \exp\left(-Q_1|H\mathbf{A}\nu| - \gamma\left|-\frac{B_l\Phi^T\nu}{B_l\Phi^T H^T}\right| + j\zeta_{i,l}^{k|k}\nu\right) \tag{D.4}$$

where

$$Q_1 = \frac{|\det(\Phi)|}{|B_l\Phi^T H^T|}\left\{\sum_{q=1,q\neq l}^{N_{ei}^{k-1|k-1}+1} P_q \cdot |\theta_q|\right\} \tag{D.5}$$

Similarly, the m^{th} exponential term at step $k-1$ is expressed as,

$$\mathcal{E}_m^{k-1|k-1} = \exp\left(-M_1|C_1\nu| - ... - M_{N_{em}^{k-1|k-1}}|C_{N_{em}^{k-1|k-1}}\nu| + j\zeta_m^{k-1|k-1}\nu\right) \tag{D.6}$$

Write down the exponential part of the p^{th} child term at step k of the m^{th} term at step $k-1$ as,

$$\mathcal{E}_{m,p}^{k|k} = \exp\left(-Q_2|H\mathbf{A}\nu| - \gamma\left|-\frac{C_l\Phi^T\nu}{C_l\Phi^T H^T}\right| + j\zeta_{m,p}^{k|k}\nu\right) \tag{D.7}$$

where

$$Q_2 = \frac{|\det(\Phi)|}{|C_l\Phi^T H^T|}\left\{\sum_{q=1,q\neq l}^{N_{em}^{k-1|k-1}+1} M_q \cdot |\vartheta_q|\right\} \tag{D.8}$$

and $C_l^T C_i - C_i^T C_l = \vartheta_i \cdot \mathbf{A}$, $C_l^T \Gamma^T \Phi^{-T} - \Phi^{-1}\Gamma C_l = \vartheta_{N_{em}^{k-1|k-1}+1} \cdot H\mathbf{A}$, and define $M_{N_{em}^{k-1|k-1}+1} = \beta$.

There are in total $N_{ei}^{k-1|k-1} + 2$ child terms at step k. The last child term is the old child term, whose exponential part is represented as $\mathcal{E}_{i,old}^{k|k}$. And,

$$\mathcal{E}_{i,old}^{k|k} = \exp\left(-P_1|B_1\Phi^T\nu| - ... - P_{N_{ei}^{k-1|k-1}}|B_{N_{ei}^{k-1|k-1}}\Phi^T\nu| - \beta|\Gamma^T\Phi^T\nu| + j\zeta_{i,old}^{k|k}\nu\right) \tag{D.9}$$

Then after time propagation and measurement, at step $k+1$, the exponential part of the l^{th} child term is,

$$\begin{aligned}
\mathcal{E}_{i,old,l}^{k+1|k+1} = \exp\Bigg(& -P_1|B_1\Phi^{2T}H^T| \left| \frac{B_1\Phi^{2T}\nu}{B_1\Phi^{2T}H^T} - \frac{B_l\Phi^{2T}\nu}{B_l\Phi^{2T}H^T} \right| - \ldots \\
& - P_{N_{ei}^{k-1|k-1}}|B_{N_{ei}^{k-1|k-1}}\Phi^{2T}H^T| \left| \frac{B_{N_{ei}^{k-1|k-1}}\Phi^{2T}\nu}{B_1\Phi^{2T}H^T} - \frac{B_l\Phi^{2T}\nu}{B_l\Phi^{2T}H^T} \right| \\
& - \beta|\Gamma^T\Phi^T H^T| \left| \frac{\Gamma^T\Phi^T\nu}{\Gamma^T\Phi^T H^T} - \frac{B_l\Phi^{2T}\nu}{B_l\Phi^{2T}H^T} \right| \\
& -\beta|\Gamma^T H^T| \left| \frac{\Gamma^T\nu}{\Gamma^T H^T} - \frac{B_l\Phi^{2T}\nu}{B_l\Phi^{2T}H^T} \right| - \gamma \left| -\frac{B_l\Phi^{2T}\nu}{B_l\Phi^{2T}H^T} \right| + j\zeta_{i,old,l}^{k+1|k+1}\nu \Bigg)
\end{aligned} \tag{D.10}$$

Combine the first $N_{ei}^{k-1|k-1}+1$ elements,

$$\mathcal{E}_{i,old,l}^{k+1|k+1} = \exp\left(-N_1|H\mathbf{A}\nu| - \gamma \left| -\frac{B_l\Phi^{2T}\nu}{B_l\Phi^{2T}H^T} \right| + j\zeta_{i,old,l}^{k+1|k+1}\nu \right) \tag{D.11}$$

where

$$\begin{aligned}
N_1 &= \frac{|\det(\Phi)|^2}{|B_l\Phi^{2T}H^T|} \left\{ \sum_{q=1,q\neq l}^{N_{ei}^{k-1|k-1}+1} P_q \cdot |\theta_q| \right\} + \frac{\beta|\det(\Phi)| \cdot |\theta_{N_{ei}^{k-1|k-1}+1}|}{|B_l\Phi^{2T}H^T|} \\
&= \frac{|\det(\Phi)| \cdot |B_l\Phi^T H^T|}{|B_l\Phi^{2T}H^T|} \cdot Q_1 + \frac{\beta|\det(\Phi)| \cdot |\theta_{N_{ei}^{k-1|k-1}+1}|}{|B_l\Phi^{2T}H^T|}
\end{aligned} \tag{D.12}$$

And the grandchild term $\mathcal{E}_{m,old,p}^{k+1|k+1}$ is,

$$\mathcal{E}_{m,old,p}^{k+1|k+1} = \exp\left(-N_2|H\mathbf{A}\nu| - \gamma \left| -\frac{C_l\Phi^{2T}\nu}{C_l\Phi^{2T}H^T} \right| + j\zeta_{m,old,p}^{k+1|k+1}\nu \right) \tag{D.13}$$

where

$$\begin{aligned}
N_2 &= \frac{|\det(\Phi)|^2}{|C_l\Phi^{2T}H^T|} \left\{ \sum_{q=1,q\neq l}^{N_{em}^{k-1|k-1}+1} M_q \cdot |\vartheta_q| \right\} + \frac{\beta|\det(\Phi)| \cdot |\theta_{N_{em}^{k-1|k-1}+1}|}{|C_l\Phi^{2T}H^T|} \\
&= \frac{|\det(\Phi)| \cdot |C_l\Phi^T H^T|}{|C_l\Phi^{2T}H^T|} \cdot Q_2 + \frac{\beta|\det(\Phi)| \cdot |\vartheta_{N_{em}^{k-1|k-1}+1}|}{|C_l\Phi^{2T}H^T|}
\end{aligned} \tag{D.14}$$

Now, look at (D.4), (D.7), (D.11) and (D.13). As mentioned before, we are only interested in the real part of the exponential terms, because the imaginary part will match whenever the real part of two exponential terms are identical.

If $\mathcal{E}_{i,l}^{k|k} = \mathcal{E}_{m,p}^{k|k}$, then $Q_1 = Q_2$, and B_l and C_l are parallel, i.e. there exists a scalar s such that $B_l = sC_l$. Then $N_1 = N_2$, and hence $\mathcal{E}_{i,old,l}^{k+1|k+1} = \mathcal{E}_{m,old,p}^{k+1|k+1}$.

If $\mathcal{E}_{i,old,l}^{k+1|k+1} = \mathcal{E}_{m,old,p}^{k+1|k+1}$, then $N_1 = N_2$, and B_l and C_l are parallel. Therefore $Q_1 = Q_2$, and hence $\mathcal{E}_{i,l}^{k|k} = \mathcal{E}_{m,p}^{k|k}$. □

APPENDIX E

Updating R in Each Layer of G Terms for 2-State Case

The recursion of R in the G terms in equation (5.1) for n-state systems has been presented in general in Chapter 2. For two-state case, this appendix chapter provides a clear algorithm for the recursion explicitly. It is obtained utilizing the directions co-alignment and terms combination properties that has been discussed in previous chapters. It is interesting to notice that a large part of the recursion for two-state cases will preserve when extended to higher-order cases. The only major difference of this recursive structure among different system dimensions is how to combine directions onto the fundamental basis. And this only effects the sign function part of the structure in each layer.

The implementation of layer update will be initialized at the first measurement update, and then be generalized to step k in a recursive manner.

E.1 1^{st} Measurement Update

At the 1^{st} measurement update, there are 3 terms, initialized as follows.

E.1.1 The First Term i.e. $i = 1$

$$\rho_{o1}^{(1)}(i=1) = \left[\alpha_1 \left|e_1 H^T\right|\right], \quad \rho_{c1}^{(1)}(i=1) = \left[\alpha_2 \left(e_2 H^T\right) \cdot sgn(e_1 H^T), \quad \gamma\right] \tag{E.1}$$

$$F_{o1}^{(1)}(i=1) = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c1}^{(1)}(i=1) = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{E.2}$$

where $s_1 = sgn(H\mathbf{A}\nu)$, $s_2 = sgn(-\frac{e_1\nu}{e_1 H^T})$.

E.1.2 The Second Term i.e. $i = 2$

$$\rho_{o1}^{(1)}(i=2) = \left[\alpha_2 \left|e_2 H^T\right|\right], \quad \rho_{c1}^{(1)}(i=2) = \left[\alpha_1 \left(e_1 H^T\right) \cdot sgn(-e_2 H^T), \quad \gamma\right] \tag{E.3}$$

$$F_{o1}^{(1)}(i=2) = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c1}^{(1)}(i=2) = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{E.4}$$

where $s_1 = sgn(H\mathbf{A}\nu)$, $s_2 = sgn(-\frac{e_2\nu}{e_2 H^T})$.

E.1.3 The Third Term i.e. $i = 3$

$$\rho_{o1}^{(1)}(i=3) = [\gamma], \quad \rho_{c1}^{(1)}(i=3) = \left[\alpha_1 \left(e_1 H^T\right), \quad \alpha_2 \left(e_2 H^T\right)\right] \tag{E.5}$$

$$F_{o1}^{(1)}(i=3) = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c1}^{(1)}(i=3) = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{E.6}$$

where $s_1 = sgn(e_1\nu)$, $s_2 = sgn(e_2\nu)$.

E.2 From Step k to Step $k+1$

At step k, assume the general form of the exponential part as follow. This term can be either new term or an old term. The number of directions is notated as N_e.

$$\mathcal{E}^{k|k} = \exp\left(-P_1^{k|k}\left|B_1^{k|k}\nu\right| - P_2^{k|k}\left|B_2^{k|k}\nu\right| - \cdots - P_{N_e}^{k|k}\left|B_{N_e}^{k|k}\nu\right| + j\zeta^{k|k}\nu\right) \tag{E.7}$$

At each layer of the G part, the sequence ρ and the matrix F are split into two components. $\rho_{ok}^{(m)}$ and $F_{ok}^{(m)}$ are associated with the offsets, and $\rho_{ck}^{(m)}$ and $F_{ck}^{(m)}$ are associated with the coefficients of the sign functions.

$$R_k^{(m)} = \left[\rho_{ok}^{(m)} \quad | \quad \rho_{ck}^{(m)}\right] \times \begin{bmatrix} F_{ok}^{(m)} \\ ---- \\ F_{ck}^{(m)} \end{bmatrix}, \quad 1 \le m \le k \tag{E.8}$$

Suppose $\rho_{ok}^{(m)} \in \mathbb{R}^{1\times r}$ and $\rho_{ck}^{(m)} \in \mathbb{R}^{1\times q}$, then

$$F_{ok}^{(m)} = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots \\ 1 & 1 & -1 & -1 & \cdots \\ \vdots & & & & \\ 1 & 1 & 1 & 1 & \cdots \end{bmatrix} \in \mathbb{R}^{r\times 2^{k+1-m}} \tag{E.9}$$

$$F_{ck}^{(m)} = \begin{bmatrix} s_1 & \cdots & s_1 \\ \vdots & & \vdots \\ s_q & \cdots & s_q \end{bmatrix} \in \mathbb{R}^{q\times 2^{k+1-m}}, \quad q \leq N_e \tag{E.10}$$

At step $k+1$, there will be $(k+1)$ layers in the G part of the child terms. The top k layers can be expressed in terms of the layers of its parent term at step k, while the bottom layer directly comes from the exponential parts.

E.2.1 The Top k Layer i.e. $1 \leq m \leq k$

The top k layers at step $k+1$ are updated from the layers at step k. The offset part of $\rho_k^{(m)}$, i.e. $\rho_{ok}^{(m)}$ will either stay the same or add a new offset to the end of the sequence. For two-state systems, all the child directions that are produced from any two non-zero parent directions is co-aligned with the $H\mathbf{A}$ direction. Therefore, the sign function component of $\rho_k^{(m)}$, i.e. $\rho_{ck}^{(m)}$ for new child terms will be either empty or contain only one element.

For convenience of direction combination, define

$$t_l = sgn\left[\frac{\left(\frac{B_l\Phi^T}{B_l\Phi^T H^T} - \frac{B_i\Phi^T}{B_i\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right], \quad l < i \tag{E.11}$$

and

$$t_l = sgn\left[\frac{\left(\frac{B_{l+1}\Phi^T}{B_{l+1}\Phi^T H^T} - \frac{B_i\Phi^T}{B_i\Phi^T H^T}\right)e_1^T}{(H\mathbf{A})e_1^T}\right], \quad i \leq l \leq N_e \tag{E.12}$$

Due to the propagation, $B_{N_e+1} = \Gamma^T\Phi^{-T}$, $P_{N_e+1} = \beta$.

E.2.1.1 The child terms when $1 \leq i \leq q$

Consider the first q child terms. In $F_{ck}^{(m)}$, there are q sign functions, from s_1 to s_q. Hence for the i^{th} child term, the coefficient of s_i, i.e. $\rho_{ck,i}^{(m)}$ will be pulled out to compute the new offset. The rest coefficients of sign functions will be combined.

For two-state system, consider the any layer except the bottom one. When $q = 1$ this is a new parent term. The corresponding layers of the first child term will not contain sign function component of ρ, i.e. $\rho_{c(k+1)}^{(m)}$ will be empty. When $q > 1$, this is an old parent term. To produce new child terms, the element in $\rho_{ck}^{(m)}$ will collapse and $\rho_{c(k+1)}^{(m)}$ should only contain one element.

$$\rho_{o(k+1)}^{(m)} = \left[\rho_{ok}^{(m)}, \quad \rho_{ck,i}^{(m)} \cdot sgn\left(B_i \Phi^T H^T\right)\right], \quad \rho_{c(k+1)}^{(m)} = q_1 \tag{E.13}$$

where

$$\begin{aligned} q_1 = & \rho_{ck,1}^{(m)} sgn\left(B_1 \Phi^T H^T\right) \cdot t_1 + \cdots + \rho_{ck,i-1}^{(m)} sgn\left(B_{i-1} \Phi^T H^T\right) \cdot t_{i-1} + \rho_{ck,i+1}^{(m)} sgn\left(B_{i+1} \Phi^T H^T\right) \cdot t_i \\ & + \cdots + \rho_{ck,q}^{(m)} sgn\left(B_q \Phi^T H^T\right) \cdot t_{q-1} \end{aligned} \tag{E.14}$$

If $q = 1$, q_1 is not valid. Hence $\rho_{c(k+1)}^{(m)}$ is empty, i.e. $\rho_{c(k+1)}^{(m)} = [\quad]$.

The F matrix is straightforward.

$$F_{o(k+1)}^{(m)} = \begin{bmatrix} F_{ok}^{(m)} & | & F_{ok}^{(m)} \\ ---- & + & ---- \\ 1 \cdots 1 & | & -1 \cdots -1 \end{bmatrix} \in \mathbb{R}^{(r+1) \times 2^{k+2-m}} \tag{E.15}$$

and

$$F_{c(k+1)}^{(m)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \end{bmatrix} \in \mathbb{R}^{1 \times 2^{k+2-m}} \tag{E.16}$$

E.2.1.2 The child terms when $q + 1 \leq i \leq N_e + 1$

Because the sign function only goes from s_1 to s_q, the i^{th} child term at step $(k + 1)$ when $q + 1 \leq i \leq N_e + 1$ will not change the offset component of ρ at step k.

$$\rho_{o(k+1)}^{(m)} = \rho_{ok}^{(m)} \tag{E.17}$$

and

$$F_{o(k+1)}^{(m)} = \begin{bmatrix} F_{ok}^{(m)} & | & F_{ok}^{(m)} \end{bmatrix} \in \mathbb{R}^{r \times 2^{k+2-m}} \tag{E.18}$$

The sign function component of ρ involves direction combination.

$$\rho_{c(k+1)}^{(m)} = \rho_{ck,1}^{(m)} sgn\left(B_1 \Phi^T H^T\right) \cdot t_1 + \cdots + \rho_{ck,q}^{(m)} sgn\left(B_q \Phi^T H^T\right) \cdot t_q \tag{E.19}$$

And

$$F_{c(k+1)}^{(m)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \end{bmatrix} \in \mathbb{R}^{1 \times 2^{k+2-m}} \tag{E.20}$$

E.2.1.3 The child terms when $i = N_e + 2$ (old)

The offset component of ρ stays the same.

$$\rho_{o(k+1)}^{(m)} = \rho_{ok}^{(m)}, \quad F_{o(k+1)}^{(m)} = \begin{bmatrix} F_{ok}^{(m)} & | & F_{ok}^{(m)} \end{bmatrix} \in \mathbb{R}^{r \times 2^{k+2-m}} \tag{E.21}$$

The sign function component of ρ also remain the same structure. In this case, only the value of s_1, ..., s_q change.

$$\rho_{c(k+1)}^{(m)} = \rho_{ck}^{(m)}, \quad F_{c(k+1)}^{(m)} = \begin{bmatrix} F_{ck}^{(m)} & | & F_{ck}^{(m)} \end{bmatrix} \in \mathbb{R}^{q \times 2^{k+2-m}} \tag{E.22}$$

where the new sign functions are $s_1 = sgn\left(B_1 \Phi^T \nu\right)$, $s_2 = sgn\left(B_2 \Phi^T \nu\right)$, ..., $s_q = sgn\left(B_q \Phi^T \nu\right)$.

E.2.2 The Bottom Layer i.e $m = k + 1$

For the bottom layer, we consider two scenarios separately. When $1 \leq i \leq N_e + 1$, the i^{th} child term is a new term. The bottom layer will contain only three elements in ρ, among which one is offset, the other two of them are coefficients of sign functions. When $i = N_e + 2$, the i^{th} child term at step $k + 1$ is an old term. The number of elements in ρ for the bottom layer will be $N_e + 2$.

E.2.2.1 New child terms when $1 \leq i \leq N_e + 1$

$$\rho_{o(k+1)}^{(k+1)} = \left[P_i \left| B_i \Phi^T H^T \right| \right], \quad \rho_{c(k+1)}^{(k+1)} = \left[q_3, \quad \gamma \right] \tag{E.23}$$

$$F_{o(k+1)}^{(k+1)} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c(k+1)}^{(k+1)} = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \tag{E.24}$$

where the sign functions are $s_1 = sgn\left(H\mathbf{A}\nu\right)$, and $s_2 = sgn\left(-\frac{B_i \Phi^T \nu}{B_i \Phi^T H^T}\right)$. And

$$\begin{aligned} q_3 =& P_1 \left| B_1 \Phi^T H^T \right| \cdot t_1 + \cdots + P_{i-1} \left| B_{i-1} \Phi^T H^T \right| \cdot t_{i-1} \\ &+ P_{i+1} \left| B_{i+1} \Phi^T H^T \right| \cdot t_i + \cdots + P_{N_e+1} \left| B_{N_e+1} \Phi^T H^T \right| \cdot t_{N_e} \end{aligned} \tag{E.25}$$

E.2.2.2 Old child terms when $i = N_e + 2$

$$\rho_{o(k+1)}^{(k+1)} = \gamma, \quad \rho_{c(k+1)}^{(k+1)} = \left[P_1 \left(B_1 \Phi^T H^T \right), \quad \cdots, \quad P_{N_e} \left(B_{N_e} \Phi^T H^T \right), \quad \beta \left(\Gamma^T H^T \right) \right] \tag{E.26}$$

$$F_{o(k+1)}^{(k+1)} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c(k+1)}^{(k+1)} = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \\ \vdots & \\ s_{N_e+1} & s_{N_e+1} \end{bmatrix} \tag{E.27}$$

where the sign functions are $s_1 = sgn\left(B_1 \Phi^T \nu\right)$, $s_2 = sgn\left(B_2 \Phi^T \nu\right)$, ..., $s_{N_e} = sgn\left(B_{N_e} \Phi^T \nu\right)$, $s_{N_e+1} = sgn\left(\Gamma^T \nu\right)$.

APPENDIX F

Updating R in Each Layer of G Terms for 3-State Case

In this appendix chapter, an explicit, analytic structure for updating R in each layer of G terms in equation (5.1) for three-state systems is presented. The derivation is based on the understanding of fundamental properties of the Cauchy estimator that have been elaborated in main chapters. In following sections, the G layer update will be initialized at the first update, and then be generalized to step k in a recursive manner.

F.1 1^{st} Measurement Update

The characteristic function of the unconditioned initial states $\phi_{x_1}(\nu)$ is,

$$\phi_{X_1}(\nu) = \exp\left[-\alpha_1 \left|e_1\nu\right| - \alpha_2 \left|e_2\nu\right| - \alpha_3 \left|e_3\nu\right|\right] \tag{F.1}$$

where $e_1 = [1 \quad 0 \quad 0]$, $e_2 = [0 \quad 1 \quad 0]$, and $e_2 = [0 \quad 0 \quad 1]$.

At the 1^{st} measurement update, the CF is,

$$\begin{aligned}
\bar{\phi}_{X_1|Z_1}(\nu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{X_1}(z_1 - Hx_1)\phi_V(-\eta)e^{jz_1\eta}d\eta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\alpha_1 \left|e_1H^T\right| \left|\frac{e_1\nu}{e_1H^T} - \eta\right| - \alpha_2 \left|e_2H^T\right| \left|\frac{e_2\nu}{e_2H^T} - \eta\right|\right. \\
&\qquad \left. -\alpha_3 \left|e_3H^T\right| \left|\frac{e_3\nu}{e_3H^T} - \eta\right| - \gamma\left|-\eta\right| + jz_1\eta\right] d\eta \\
&= \sum_{i=1}^{4} G_i^{1|1}(\nu)\cdot\mathcal{E}^{1|1}(\nu)
\end{aligned} \tag{F.2}$$

There are four terms in the sum. The exponential part of these four terms are,

$$\mathcal{E}_1^{1|1}(\nu) = \exp\left[-\frac{\alpha_2}{|e_1 H^T|}|HA_1\nu| - \frac{\alpha_3}{|e_1H^T|}|HA_2\nu| - \gamma\left|-\frac{e_1\nu}{e_1H^T}\right| + jz_1\frac{e_1\nu}{e_1H^T}\nu\right] \tag{F.3}$$

$$\mathcal{E}_2^{1|1}(\nu) = \exp\left[-\frac{\alpha_1}{|e_2 H^T|}|HA_1\nu| - \frac{\alpha_3}{|e_2H^T|}|HA_3\nu| - \gamma\left|-\frac{e_2\nu}{e_2H^T}\right| + jz_1\frac{e_2\nu}{e_2H^T}\nu\right] \tag{F.4}$$

$$\mathcal{E}_3^{1|1}(\nu) = \exp\left[-\frac{\alpha_1}{|e_3 H^T|}|HA_2\nu| - \frac{\alpha_2}{|e_3H^T|}|HA_3\nu| - \gamma\left|-\frac{e_3\nu}{e_3H^T}\right| + jz_1\frac{e_3\nu}{e_3H^T}\nu\right] \tag{F.5}$$

$$\mathcal{E}_4^{1|1}(\nu) = \exp\left[-\alpha_1|e_1\nu| - \alpha_2|e_2\nu| - \alpha_3|e_3\nu|\right] \tag{F.6}$$

where A_i is produced from the initial directions e_1, e_2, and e_3.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{F.7}$$

Consider the G part of the first term.

$$\begin{aligned} G_1^{1|1}(\nu) &= \frac{1}{2\pi}\left\{\frac{1}{jz_1 + \alpha_1|e_1H^T| + \alpha_2|e_2H^T|\,sgn\left(\frac{e_2\nu}{e_2H^T} - \frac{e_1\nu}{e_1H^T}\right) + \alpha_3|e_3H^T|\,sgn\left(\frac{e_3\nu}{e_3H^T} - \frac{e_1\nu}{e_1H^T}\right) + \gamma sgn\left(0 - \frac{e_1\nu}{e_1H^T}\right)}\right. \\ &\left. - \frac{1}{jz_1 - \alpha_1|e_1H^T| + \alpha_2|e_2H^T|\,sgn\left(\frac{e_2\nu}{e_2H^T} - \frac{e_1\nu}{e_1H^T}\right) + \alpha_3|e_3H^T|\,sgn\left(\frac{e_3\nu}{e_3H^T} - \frac{e_1\nu}{e_1H^T}\right) + \gamma sgn\left(0 - \frac{e_1\nu}{e_1H^T}\right)}\right\} \end{aligned} \tag{F.8}$$

$$\begin{aligned} &= \frac{1}{2\pi}\left\{\frac{1}{jz_1 + \alpha_1|e_1H^T| + \alpha_2(e_2H^T)sgn\left(e_1H^T\right)sgn\left(HA_1\nu\right) + \alpha_3(e_3H^T)sgn\left(e_1H^T\right)sgn\left(HA_2\nu\right) + \gamma sgn\left(-\frac{e_1\nu}{e_1H^T}\right)}\right. \\ &\left. - \frac{1}{jz_1 - \alpha_1|e_1H^T| + \alpha_2(e_2H^T)sgn\left(e_1H^T\right)sgn\left(HA_1\nu\right) + \alpha_3(e_3H^T)sgn\left(e_1H^T\right)sgn\left(HA_2\nu\right) + \gamma sgn\left(-\frac{e_1\nu}{e_1H^T}\right)}\right\} \end{aligned} \tag{F.9}$$

The G part of other terms can also be derived via the same procedure.

$$\begin{aligned} G_2^{1|1}(\nu) = \frac{1}{2\pi}&\left\{\frac{1}{jz_1 + \alpha_2|e_2H^T| + \alpha_1(e_1H^T)sgn\left(-e_2H^T\right)sgn\left(HA_1\nu\right) + \alpha_3(e_3H^T)sgn\left(e_2H^T\right)sgn\left(HA_3\nu\right) + \gamma sgn\left(-\frac{e_2\nu}{e_2H^T}\right)}\right. \\ &\left. - \frac{1}{jz_1 - \alpha_2|e_2H^T| + \alpha_1(e_1H^T)sgn\left(-e_2H^T\right)sgn\left(HA_1\nu\right) + \alpha_3(e_3H^T)sgn\left(e_2H^T\right)sgn\left(HA_3\nu\right) + \gamma sgn\left(-\frac{e_2\nu}{e_2H^T}\right)}\right\} \end{aligned} \tag{F.10}$$

$$\begin{aligned} G_3^{1|1}(\nu) = \frac{1}{2\pi}&\left\{\frac{1}{jz_1 + \alpha_3|e_3H^T| + \alpha_1(e_1H^T)sgn\left(-e_3H^T\right)sgn\left(HA_2\nu\right) + \alpha_2(e_2H^T)sgn\left(-e_3H^T\right)sgn\left(HA_3\nu\right) + \gamma sgn\left(-\frac{e_3\nu}{e_3H^T}\right)}\right. \\ &\left. - \frac{1}{jz_1 - \alpha_3|e_3H^T| + \alpha_1(e_1H^T)sgn\left(-e_3H^T\right)sgn\left(HA_2\nu\right) + \alpha_2(e_2H^T)sgn\left(-e_3H^T\right)sgn\left(HA_3\nu\right) + \gamma sgn\left(-\frac{e_3\nu}{e_3H^T}\right)}\right\} \end{aligned} \tag{F.11}$$

$$G_4^{1|1}(\nu) = \frac{1}{2\pi}\left\{\frac{1}{jz_1 + \gamma + \alpha_1(e_1H^T)sgn\,(e_1\nu) + \alpha_2(e_2H^T)sgn\,(e_2\nu) + \alpha_3(e_3H^T)sgn\,(e_3\nu)}\right.$$
$$\left. - \frac{1}{jz_1 - \gamma + \alpha_1(e_1H^T)sgn\,(e_1\nu) + \alpha_2(e_2H^T)sgn\,(e_2\nu) + \alpha_3(e_3H^T)sgn\,(e_3\nu)}\right\} \tag{F.12}$$

Rewrite the G parts in the following form.

$$G_i^{1|1}(\nu) = \frac{1}{2\pi}\left\{\frac{1}{jz_1 + R_{1,1}^{(1)}(i)} - \frac{1}{jz_1 + R_{1,2}^{(1)}(i)}\right\} \tag{F.13}$$

Using the structure provided in Chapter 5, the R's in each layer of G are expressed as follows.

F.1.1 The First Term i.e. $i = 1$

$$\rho_{o1}^{(1)}(i=1) = \alpha_1\left|e_1H^T\right| \tag{F.14}$$

$$\rho_{c1}^{(1)}(i=1) = \left[\alpha_2\left(e_2H^T\right)\cdot sgn(e_1H^T), \quad \alpha_3\left(e_3H^T\right)\cdot sgn(e_1H^T), \quad \gamma\right] \tag{F.15}$$

$$F_{o1}^{(1)}(i=1) = \begin{bmatrix}1 & -1\end{bmatrix}, \quad F_{c1}^{(1)}(i=1) = \begin{bmatrix}s_1 & s_1\\ s_2 & s_2\\ s_3 & s_3\end{bmatrix} \tag{F.16}$$

where $s_1 = sgn(HA_1\nu)$, $s_2 = sgn(HA_2\nu)$, $s_3 = sgn(-\frac{e_1\nu}{e_1H^T})$.

F.1.2 The Second Term i.e. $i = 2$

$$\rho_{o1}^{(1)}(i=2) = \alpha_2\left|e_2H^T\right| \tag{F.17}$$

$$\rho_{c1}^{(1)}(i=2) = \left[\alpha_1\left(e_1H^T\right)\cdot sgn(-e_2H^T), \quad \alpha_3\left(e_3H^T\right)\cot sgn(e_2H^T), \quad \gamma\right] \tag{F.18}$$

$$F_{o1}^{(1)}(i=2) = \begin{bmatrix}1 & -1\end{bmatrix}, \quad F_{c1}^{(1)}(i=2) = \begin{bmatrix}s_1 & s_1\\ s_2 & s_2\\ s_3 & s_3\end{bmatrix} \tag{F.19}$$

where $s_1 = sgn(HA_1\nu)$, $s_2 = sgn(HA_3\nu)$, $s_2 = sgn(-\frac{e_2\nu}{e_2H^T})$.

F.1.3 The Third Term i.e. $i = 3$

$$\rho_{o1}^{(1)}(i=3) = \alpha_3 \left| e_3 H^T \right| \tag{F.20}$$

$$\rho_{c1}^{(1)}(i=3) = \left[\alpha_1 \left(e_1 H^T\right) \cdot sgn(-e_3 H^T), \quad \alpha_2 \left(e_2 H^T\right) \cdot sgn(-e_3 H^T), \quad \gamma\right] \tag{F.21}$$

$$F_{o1}^{(1)}(i=3) = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c1}^{(1)}(i=3) = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \\ s_3 & s_3 \end{bmatrix} \tag{F.22}$$

where $s_1 = sgn(HA_2\nu)$, $s_2 = sgn(HA_3\nu)$, $s_2 = sgn(-\frac{e_3\nu}{e_3 H^T})$.

F.1.4 The Fourth Term i.e. $i = 4$

$$\rho_{o1}^{(1)}(i=4) = [\gamma], \quad \rho_{c1}^{(1)}(i=4) = \left[\alpha_1 \left(e_1 H^T\right), \quad \alpha_2 \left(e_2 H^T\right), \quad \alpha_3 \left(e_3 H^T\right)\right] \tag{F.23}$$

$$F_{o1}^{(1)}(i=4) = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad F_{c1}^{(1)}(i=4) = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \\ s_3 & s_3 \end{bmatrix} \tag{F.24}$$

where $s_1 = sgn(e_1\nu)$, $s_2 = sgn(e_2\nu)$, $s_3 = sgn(e_3\nu)$.

F.2 From Step k to Step $k + 1$

At step k, assume the general form of the exponential part as follow. The number of directions is denoted as N_e.

$$\mathcal{E}^{k|k} = \exp\left(-P_1^{k|k} \left|B_1^{k|k}\nu\right| - P_2^{k|k} \left|B_2^{k|k}\nu\right| - \cdots - P_{N_e}^{k|k} \left|B_{N_e}^{k|k}\nu\right| + j\zeta^{k|k}\nu\right) \tag{F.25}$$

The coefficient term G is expressed in equation (5.1). And

$$R_k^{(m)} = \left[\rho_{ok}^{(m)} \quad | \quad \rho_{ck}^{(m)}\right] \times \begin{bmatrix} F_{ok}^{(m)} \\ ---- \\ F_{ck}^{(m)} \end{bmatrix}, \quad 1 \le m \le k \tag{F.26}$$

where the sign function component of the matrix F, $F_{ck}^{(m)}$, is expressed as,

$$F_{ck}^{(m)} = \begin{bmatrix} s_1 & \cdots & s_1 \\ \vdots & & \vdots \\ s_q & \cdots & s_q \end{bmatrix}, \quad q \leq N_e \tag{F.27}$$

At step $k+1$, there will be $(k+1)$ layers in the G part of the child terms. The top k layers can be expressed in terms of the layers of its parent term at step k, while the bottom layer directly comes from the exponential parts.

There are three type of terms at step k. Firstly, new terms at step k have directions in the form of $[HC_1, HC_2, ..., HC_{N_e-1}, b]$, where the 3-dim matrices C_i are anti-symmetric. Secondly, a general old term has the directions as $[HC_1\Phi^{T\theta}, HC_2\Phi^{T\theta}, ..., HC_{N_e-\theta-1}\Phi^{T\theta}, b\Phi^{T\theta}, \Gamma^T\Phi^{T(\theta-1)}, ..., \Gamma^T]$. The scalar θ indicates how old the term is. For example, if $\theta = 2$, this term is a 2-step old term at step k. Its parent term at step $k-1$ is an 1-step old term, and its grandparent term at step $k-2$ is a new term. These forms of new and old terms can cover all the cases, except the oldest one, which does not contain directions in a HC format. This special old term at step k is produced from the 4^{th} term at the 1^{st} measurement update, and has the directions expressed as $[e_1\Phi^{T(k-1)}, e_2\Phi^{T(k-1)}, e_3\Phi^{T(k-1)}, \Gamma^T\Phi^{T(k-2)}, ..., \Gamma^T]$ for $k \geq 2$. This is the third type of terms.

To make it simpler, we do not distinguish the case between new terms and general old terms in the following derivation. In fact, a new term at step k can be expressed by the form of a general old term, when letting the index θ to be zero. Also notice that the last few directions in a general old term involves Γ^T, which does not appear in the direction in a new term. We can still merge these two cases together by stating that no inverse of Φ are allowed in the directions; when $\theta - 1$ is negative, i.e. $\theta = 0$, no Γ^T will appear in the directions. After all, the first several directions in the HC forms are much more important than the last few directions in the Γ^T forms. The HC directions are pivotal for term combination without comparison, hence is one of the keys that can enhance the algorithm efficiency.

F.2.1 The Top k Layer i.e. $1 \leq m \leq k$

The top k layers at step $k+1$ are updated from the layers at step k. The offset part of $\rho_k^{(m)}$, i.e. $\rho_{ok}^{(m)}$ will either stay the same or add a new offset to the end of the sequence. For three-state systems, some child directions that are produced from 2 parent directions will be aligned onto the $H\Phi^{\theta+1}\mathbf{B}\Phi^{T(\theta+1)}$ direction and others are not. It is completely predictable when the direction combination will happen based on the previous studies on directions. Here, let us define,

$$t_l = sgn\left[\frac{\left(\frac{B_l\Phi^T}{B_l\Phi^T H^T} - \frac{B_i\Phi^T}{B_i\Phi^T H^T}\right)e_1^T}{(H\Phi^{\theta+1}\mathbf{B}\Phi^{T(\theta+1)})e_1^T}\right], \quad l < i \tag{F.28}$$

and

$$t_l = sgn\left[\frac{\left(\frac{B_{l+1}\Phi^T}{B_{l+1}\Phi^T H^T} - \frac{B_i\Phi^T}{B_i\Phi^T H^T}\right)e_1^T}{(H\Phi^{\theta+1}\mathbf{B}\Phi^{T(\theta+1)})e_1^T}\right], \quad i \leq l \leq N_e \tag{F.29}$$

Due to the propagation, $B_{N_e+1} = \Gamma^T\Phi^{-T}$, $P_{N_e+1} = \beta$.

This definition provides the convenience to combine elements in one term. This form has a meaning only if this parent term at step k will produce some repeated child directions at step $k+1$. In addition, in order for t_l to be meaningful, the two involved parent directions should be both in HC form. Nevertheless, the above expression of t_l is a superset of what the algorithm requires and does not effect the validity of the rest parts of the program.

F.2.1.1 The child terms when $1 \leq i \leq q$

Consider the first q child terms. In $F_{ck}^{(m)}$, there are q sign functions, from s_1 to s_q. Hence for the i^{th} child term, the coefficient of s_i, i.e. $\rho_{ck,i}^{(m)}$ will be pulled out to compute the new offset. And the offset part of the F matrix at step $k+1$ is will be simply replication of F_o at step k added by another one row at the bottom, representing that new offset at step $k+1$.

The offset part of ρ and F can be directly transferred from two-state case. In fact, anything in this algorithm that does not involve direction combination are general forms and are valid for n-state case.

The offset component of ρ and F for the top k layers of the first q child terms at step $k+1$ can be expressed as follows.

$$\rho_{o(k+1)}^{(m)} = \left[\rho_{ok}^{(m)}, \quad \rho_{ck,i}^{(m)} \cdot sgn\left(B_i \Phi^T H^T\right)\right], \quad F_{o(k+1)}^{(m)} = \begin{bmatrix} F_{ok}^{(m)} & | & F_{ok}^{(m)} \\ ---- & + & ---- \\ 1 \cdots 1 & | & -1 \cdots -1 \end{bmatrix} \tag{F.30}$$

For system of different dimensions, the sign function component of ρ and F are different. In particular, for three-state systems, we consider three scenarios respectively.

- The 1^{st} scenario is when $q = 1$. At step k, if there is only one sign function in some layers of G term, then at step $k+1$, the corresponding layers of the first child term will not contain sign function component of ρ and F, i.e. $\rho_{c(k+1)}^{(m)}$ and $F_{c(k+1)}^{(m)}$ will be empty.

- The 2^{nd} scenario is when $q \neq 1, i \leq N_e - 1 - \theta$ and this term is not the oldest term. In this case, at step k, there are in total of $(Ne - 1 - \theta)$ directions in HC form in the argument of the exponential. Hence consider the i^{th} child terms at step $k+1$, the first $(Ne - 2 - \theta)$ directions in the argument of the exponential will be co-aligned onto the fundamental direction $H\Phi^{(\theta+1)}B\Phi^{(\theta+1)T}$. Here we need to consider two sub-scenarios. If $q > Ne - 1 - \theta$, it means that in that particular layer of G, there are some parent directions that are not in the HC form. At step $k+1$, the child directions will be $H\Phi^{(\theta+1)}B\Phi^{(\theta+1)T}$, and some other HC directions. The program will need to know which directions to combine explicitly to form $\rho_{c(k+1)}^{(m)}$ and $F_{c(k+1)}^{(m)}$. On the contrary, if $q \leq Ne - 1 - \theta$, then all the child directions in the corresponding layer of G at step $k+1$ are aligned with $H\Phi^{(\theta+1)}B\Phi^{(\theta+1)T}$. The program will simply combine all of them and leave the sign function component of ρ as a scalar.

- The 3^{rd} scenario is everything else other than the first two scenarios. In this case, a term could be either the oldest term, or $q \neq 1, i > Ne - 1 - \theta$. None of the child directions can be combined.

The recursion of the sign function component of ρ and F under each of the three scenarios are presented below.

Scenario 1: $q = 1$

$$\rho^{(m)}_{c(k+1)} = [\quad], \quad F^{(m)}_{c(k+1)} = [\quad] \tag{F.31}$$

Scenario 2: $q \neq 1$**,** $i \leq N_e - 1 - \theta$**, not the oldest term**

Sub-Scenario 2.1: $q > N_e - 1 - \theta$

$$\rho^{(m)}_{c(k+1)} = \left[q_1, \quad \rho^{(m)}_{ck,Ne-\theta} sgn\left(B_{Ne-\theta}\Phi^T H^T\right), \quad \cdots, \quad \rho^{(m)}_{ck,q} sgn\left(B_q \Phi^T H^T\right) \right] \tag{F.32}$$

where

$$\begin{aligned} q_1 = &\rho^{(m)}_{ck,1} sgn\left(B_1 \Phi^T H^T\right) \cdot t_1 + \cdots + \rho^{(m)}_{ck,i-1} sgn\left(B_{i-1}\Phi^T H^T\right) \cdot t_{i-1} \\ &+ \rho^{(m)}_{ck,i+1} sgn\left(B_{i+1}\Phi^T H^T\right) \cdot t_i + \cdots + \rho^{(m)}_{ck,Ne-1-\theta} sgn\left(B_{Ne-1-\theta}\Phi^T H^T\right) \cdot t_{Ne-2-\theta} \end{aligned} \tag{F.33}$$

$$F^{(m)}_{c(k+1)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \\ s_2 & s_2 & \cdots & s_2 \\ \vdots & & & \\ s_{q-Ne+\theta+2} & s_{q-Ne+\theta+2} & \cdots & s_{q-Ne+\theta+2} \end{bmatrix} \tag{F.34}$$

where s_1 through $s_{q-Ne+\theta+2}$ are the sign function of the first $(q - Ne + \theta + 2)$ child directions multiplied by the variable ν.

Sub-Scenario 2.2: $q \leq N_e - 1 - \theta$

$$\begin{aligned} \rho^{(m)}_{c(k+1)} = &\rho^{(m)}_{ck,1} sgn\left(B_1 \Phi^T H^T\right) \cdot t_1 + \cdots + \rho^{(m)}_{ck,i-1} sgn\left(B_{i-1}\Phi^T H^T\right) \cdot t_{i-1} \\ &+ \rho^{(m)}_{ck,i+1} sgn\left(B_{i+1}\Phi^T H^T\right) \cdot t_i + \cdots + \rho^{(m)}_{ck,q} sgn\left(B_q \Phi^T H^T\right) \cdot t_{q-1} \end{aligned} \tag{F.35}$$

$$F^{(m)}_{c(k+1)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \end{bmatrix} \tag{F.36}$$

where $s_1 = sgn\left(H\Phi^{(\theta+1)}\mathbf{B}\Phi^{(\theta+1)T}\nu\right)$.

<u>Scenario 3</u>: not scenario 1 and 2

$$\rho^{(m)}_{c(k+1)} = \Big[\rho^{(m)}_{ck,1} sgn\left(B_1 \Phi^T H^T\right), \quad \cdots, \quad \rho^{(m)}_{ck,i-1} sgn\left(B_{i-1}\Phi^T H^T\right), \\ \rho^{(m)}_{ck,i+1} sgn\left(B_{i+1}\Phi^T H^T\right), \quad \cdots, \quad \rho^{(m)}_{ck,q} sgn\left(B_q \Phi^T H^T\right)\Big] \tag{F.37}$$

$$F^{(m)}_{c(k+1)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \\ s_2 & s_2 & \cdots & s_2 \\ \vdots & & & \\ s_{q-1} & s_{q-1} & \cdots & s_{q-1} \end{bmatrix} \tag{F.38}$$

where s_1 through s_{q-1} are the sign function of the first $(q-1)$ child directions multiplied by the variable ν.

(End of Scenarios.)

F.2.1.2 The child terms when $q + 1 \leq i \leq N_e + 1$

Because the sign function only goes from s_1 to s_q, the i^{th} child term at step $(k+1)$ when $q + 1 \leq i \leq N_e + 1$ will not change the offset component of ρ at step k.

$$\rho^{(m)}_{o(k+1)} = \rho^{(m)}_{ok}, \quad F^{(m)}_{o(k+1)} = \left[F^{(m)}_{ok} \quad | \quad F^{(m)}_{ok}\right] \tag{F.39}$$

Again, the number of parent directions in the HC form at step k is $(N_e - 1 - \theta)$. Hence when $i \leq N_e - 1 - \theta$ and this term is not the oldest one, because $q < i \leq N_e - 1 - \theta$, all the child directions will be co-aligned onto $H\Phi^{(\theta+1)}\mathbf{B}\Phi^{(\theta+1)T}$. The sign function component of ρ will collapse to a single scalar. On the contrary, if $i > N_e - 1 - \theta$ or this is the oldest term, no direction combination will occur at step $(k+1)$ at this layer of G. The two scenarios are stated below.

<u>Scenario 1</u>: $i \leq N_e - 1 - \theta$, not the oldest term

$$\rho^{(m)}_{c(k+1)} = \rho^{(m)}_{ck,1} sgn\left(B_1 \Phi^T H^T\right) \cdot t_1 + \cdots + \rho^{(m)}_{ck,q} sgn\left(B_q \Phi^T H^T\right) \cdot t_q \tag{F.40}$$

$$F^{(m)}_{c(k+1)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \end{bmatrix} \tag{F.41}$$

where $s_1 = sgn\left(H\Phi^{(\theta+1)}\mathbf{B}\Phi^{(\theta+1)T}\nu\right)$.

Scenario 2: otherwise

$$\rho^{(m)}_{c(k+1)} = \left[\rho^{(m)}_{ck,1} sgn\left(B_1\Phi^T H^T\right), \quad \cdots, \quad \rho^{(m)}_{ck,q} sgn\left(B_q\Phi^T H^T\right)\right] \tag{F.42}$$

$$F^{(m)}_{c(k+1)} = \begin{bmatrix} s_1 & s_1 & \cdots & s_1 \\ s_2 & s_2 & \cdots & s_2 \\ \vdots & & & \\ s_q & s_q & \cdots & s_q \end{bmatrix} \tag{F.43}$$

where s_1 through s_q are the sign function of the first q child directions multiplied by the variable ν.

(End of Scenarios.)

F.2.1.3 The child terms when $i = N_e + 2$ (old)

Consider the old child term. The offset component is,

$$\rho^{(m)}_{o(k+1)} = \rho^{(m)}_{ok}, \quad F^{(m)}_{o(k+1)} = \left[F^{(m)}_{ok} \quad | \quad F^{(m)}_{ok}\right] \tag{F.44}$$

The sign function component of ρ also remain the same structure. In this case, only the value of s_1, ..., s_q change.

$$\rho^{(m)}_{c(k+1)} = \rho^{(m)}_{ck}, \quad F^{(m)}_{c(k+1)} = \left[F^{(m)}_{ck} \quad | \quad F^{(m)}_{ck}\right] \tag{F.45}$$

where the new sign functions are $s_1 = sgn\left(B_1\Phi^T\nu\right)$, $s_2 = sgn\left(B_2\Phi^T\nu\right)$, ..., $s_q = sgn\left(B_q\Phi^T\nu\right)$.

F.2.2 The Bottom Layer i.e $m = k + 1$

For the bottom layer, when $1 \le i \le N_e+1$, the i^{th} child term is a new term. When $i = N_e+2$, the i^{th} child term at step $k+1$ is an old term.

F.2.2.1 New child terms when $1 \le i \le q+1$

The offset component is straightforward. Define $P_{N_e+1} = \beta$ and $B_{N_e+1} = \Gamma^T \Phi^{-T}$. Then,

$$\rho_{o(k+1)}^{(k+1)} = P_i \left| B_i \Phi^T H^T \right|, \quad F_{o(k+1)}^{(k+1)} = \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{F.46}$$

When this term is not the oldest term, in the meanwhile $i \le N_e - 1 - \theta$, the first $(N_e - 2 - \theta)$ child directions at step $(k+1)$ will be co-aligned with the fundamental direction $H\Phi^{(\theta+1)}\mathbf{B}\Phi^{(\theta+1)T}$. Then the number of sign functions in the bottom layer of G will be $(N_e + 1) - (N_e - 2 - \theta) + 1 = \theta + 4$. If this is the oldest term, or $i > N_e - 1 - \theta$, child directions will not be co-aligned.

Scenario 1: $i \le N_e - 1 - \theta$, not the oldest term

$$\rho_{c(k+1)}^{(m)} = \left[q, \quad P_{N_e-\theta} \left| B_{N_e-\theta} \Phi^T H^T \right|, \quad \cdots, \quad P_{N_e+1} \left| B_{N_e+1} \Phi^T H^T \right|, \quad \gamma\right] \tag{F.47}$$

where

$$\begin{aligned} q =& P_1 \left| B_1 \Phi^T H^T \right| \cdot t_1 + \cdots + P_{i-1} \left| B_{i-1} \Phi^T H^T \right| \cdot t_{i-1} \\ &+ P_{i+1} \left| B_{i+1} \Phi^T H^T \right| \cdot t_i + \cdots + P_{N_e-\theta-1} \left| B_{N_e-\theta-1} \Phi^T H^T \right| \cdot t_{N_e-\theta-2} \end{aligned} \tag{F.48}$$

$$F_{c(k+1)}^{(m)} = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \\ \vdots & \\ s_{\theta+4} & s_{\theta+4} \end{bmatrix} \tag{F.49}$$

where s_1 through $s_{\theta+4}$ are the sign function of the first $\theta + 4$ child directions multiplied by the variable ν.

Scenario 2: otherwise

$$\begin{aligned} \rho_{c(k+1)}^{(m)} = \Big[& P_1 \left| B_1 \Phi^T H^T \right|, \quad \cdots, \quad P_{i-1} \left| B_{i-1} \Phi^T H^T \right|, \quad P_{i+1} \left| B_{i+1} \Phi^T H^T \right|, \\ & \cdots, \quad P_{N_e+1} \left| B_{N_e+1} \Phi^T H^T \right|, \quad \gamma \Big] \end{aligned} \tag{F.50}$$

$$F_{c(k+1)}^{(m)} = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \\ \vdots & \\ s_{N_e+1} & s_{N_e+1} \end{bmatrix} \tag{F.51}$$

where s_1 through s_{N_e+1} are the sign function of the first N_e+1 child directions multiplied by the variable ν. In particular, these sign functions are $s_l = sgn\left(B_l \Phi^T \nu\right)$, for $1 \le l \le N_e + 1$.

(End of Scenarios.)

F.2.2.2 Old child terms when $i = N_e + 2$

The offset component is,

$$\rho_{o(k+1)}^{(k+1)} = \gamma, \quad F_{o(k+1)}^{(k+1)} = \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{F.52}$$

And the sign function component is,

$$\rho_{c(k+1)}^{(k+1)} = \left[P_1 \left(B_1 \Phi^T H^T\right), \quad \cdots, \quad P_{N_e} \left(B_{N_e} \Phi^T H^T\right), \beta \left(\Gamma^T H^T\right) \right] \tag{F.53}$$

$$F_{c(k+1)}^{(k+1)} = \begin{bmatrix} s_1 & s_1 \\ s_2 & s_2 \\ \vdots & \\ s_{N_e+1} & s_{N_e+1} \end{bmatrix} \tag{F.54}$$

where the sign functions are $s_1 = sgn\left(B_1 \Phi^T \nu\right)$, $s_2 = sgn\left(B_2 \Phi^T \nu\right)$, ..., $s_{N_e} = sgn\left(B_{N_e} \Phi^T \nu\right)$, $s_{N_e+1} = sgn\left(\Gamma^T \nu\right)$.

Till now, the recursive structure for R in each layer of G terms for three-state systems is complete.

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