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A COMPUTER PROGRAM FOR THE DYNAMIC STRESS ANALYSIS OF UNDERGROUND STRUCTURES

BY

EDWARD L. WILSON

REPORT TO
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STRUCTURAL ENGINEERING LABORATORY
COLLEGE OF ENGINEERING
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ABSTRACT

The finite element method coupled with a stable step-by-step integration procedure is used to evaluate the dynamic response of linearly elastic two-dimensional stress structures. Linear strain quadrilateral elements and one-dimensional elements are combined to represent complex structural systems of arbitrary geometry. The computer program included in this report is suitable for the earthquake or blast analysis of underground structures. Arbitrary time-dependent displacements or loads may be specified at any point in the system. The method is applied to the blast analysis of a cylinder buried underground and the results are compared to an experimental study. The use of the program and a listing of the FORTRAN IV program for the CDC 6400 are given in the Appendices.

ACKNOWLEDGMENTS

The investigation reported here was performed at Berkeley under the terms of DACA 39-67-C-0020 between the Waterways Experiment Station of the U.S. Army Corps of Engineers and the Regents of the University of California. The project was under the general coordination of Engineer Jesse L. Kirkland of the Waterways Experiment Station. The work performed at Berkeley was under the general direction of the faculty investigators, Professors E. L. Wilson, R. W. Clough and J. Lysmer. The computer program was written by Junior Specialist J. Schujman. The development of the constant strain triangular element, Appendix B, is due to Dr. Carlos A. Felippa.

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I. INTRODUCTION

Dynamic response analyses of underground structures by formal analytical techniques have been restricted to linearly elastic bodies of idealized geometry and subjected to a restricted form of loading. Therefore, for the analysis of practical structures including realistic material properties and loading, approximate numerical methods must be employed.

The finite element method is a recently developed technique that has been extremely successful in the static and dynamic analysis of continuous structures.[1,2,3,4] The advantages of the finite element method, as compared to other numerical approaches, are many. The method is completely general with respect to geometry and material properties. Since each element in the system may have different properties, complex bodies composed of many different layered, anisotropic materials are easily represented. Displacement or stress boundary conditions may be specified at any point in the finite element system. Mathematically, it can be shown that the method converges to the exact solution as the number of elements is increased; therefore, any desired degree of accuracy may be obtained. In addition, for both static and dynamic analyses, the finite element approach generates equilibrium equations which produce a symmetric positive-definite matrix that may be placed in a band form and solved with a minimum of computer storage and time.

Recently, several research programs involving the development of finite element methods have been conducted at the University of California at Berkeley.[5,6] As a result of these previous studies, considerable experience has been gained in problems associated with the dynamic analysis of finite element systems. The mode superposition approach and the step-by-step integration procedure[7] have been applied to the dynamic response analysis of two-dimensional systems. Both of these techniques yield the same results and involve equivalent amounts of computer effort for linearly elastic materials; however, for the proposed study, the step-by-step method will be used since the mode-superposition approach is appropriate only for elastic structures subjected to small displacements.

Another investigator[8] has used a step-by-step technique in the evaluation of the dynamic response of elastic finite element systems. However, the element used was the constant strain triangle which is extremely inaccurate compared to the linear strain quadrilateral presented in this report. Also, the step-by-step integration method which was used does not appear to be a stable procedure. The resulting computer program was difficult to use for the blast analysis of underground structures and required large amounts of computer time.

This proposed investigation will be restricted to two-dimensional plane structures. However, an important advantage of the finite element method and the step-by-step procedure is that they may be readily extended to other classes of structures. For example, the resulting computer program for the dynamic analysis of two-dimensional structures may be easily modified to perform axisymmetric analyses.

The present report will be concerned with the development of a general computer program for the dynamic analysis of two-dimensional plane strain structures. The structures which may be analyzed are of arbitrary geometry and the material properties are assumed to be linearly elastic. Automatic load generation procedures make the program well suited for the consideration of traveling blast pressure and earthquake loads. One-dimensional truss elements are included in the program; therefore, a certain class of foundation structure interaction behavior may be studied.

II. METHOD OF ANALYSIS

In the finite element idealization of solids the continuous structure is replaced by a finite number of elements, or regions, which have common joints, or nodal points. For the purpose of describing the behavior of the finite element system an approximate displacement field is assumed within each element. In the case of two-dimensional solids, expressions for both the x and y displacement fields are required. These fields within each element are expressed in terms of a discrete number of unknown displacements associated with the connecting nodal points. Therefore, for the dynamic response a lumped parameter idealization of the actual structure is possible, in which the mass properties of the system are separated from the elastic properties of the system. The advantage of this discrete mathematical formulation is that the force equilibrium of the system may be expressed by a set of ordinary differential equations rather than the partial differential equation required to describe the actual continuous structure.

These simultaneous differential equations representing the equilibrium of the system may be expressed most conveniently as a matrix equation. Then either of the two different approaches to the solution of this equation may be adopted. The mode-superposition method involves the solution of the characteristic value problem represented by the free vibration response of the system, followed by the transformation to the principal coordinates determined as the characteristic shapes of the system. This procedure uncouples the response of the system, so that the response of each coordinate may be evaluated independently of the others. The second method of dynamic analysis is called the step-by-step method, and involves the direct numerical integration of the equilibrium equations in their original form, without transformation to the principal coordinates of the system.

One of the principal advantages of the mode-superposition method lies in the fact that the response of the system is largely expressed by the first few modes of vibration; thus, good accuracy may be obtained by this method from an analysis involving only a few of the principal coordinates, while all coordinates must be retained in the step-by-step method. On the other hand, the evaluation of the characteristic value problem and transformation to the principal coordinates are major computational problems not required of the step-by-step method. Furthermore, the mode-superposition method is based on the assumption of linear structural behavior, whereas the step-by-step method may be applied to nonlinear systems simply by modifying the assumed linear properties approximately at each successive step of integration.

A. Dynamic Equilibrium Equations for a Finite Element System

If a viscous form of damping is assumed, the force equilibrium equations for a linear elastic finite element system may be expressed by the following matrix equations:

$$[M][\ddot{U}_t] + [C][\dot{U}_t] + [K][U_t] = [P_t] \qquad (1)$$

where

$[U_t]$ is a vector of nodal point displacements at time "t"

$[\dot{U}_t]$ is a vector of nodal point velocities

$[\ddot{U}_t]$ is a vector of nodal point accelerations

$[P_t]$ is a vector of nodal point forces

$[M]$ is the mass matrix

[C] is the damping matrix

[K] is the stiffness matrix

In the following section the development of these matrices for a finite element system will be discussed. Also, the step-by-step solution procedure will be given.

B. Finite Element Stiffness Matrix

The stiffness matrix [K] used in the dynamic equilibrium relationship, Eq. (1), is independent of time and is identical to the stiffness matrix used in static analysis. The particular type of element used in this report is a quadrilateral element composed of two triangular elements and is shown in Fig. 1.

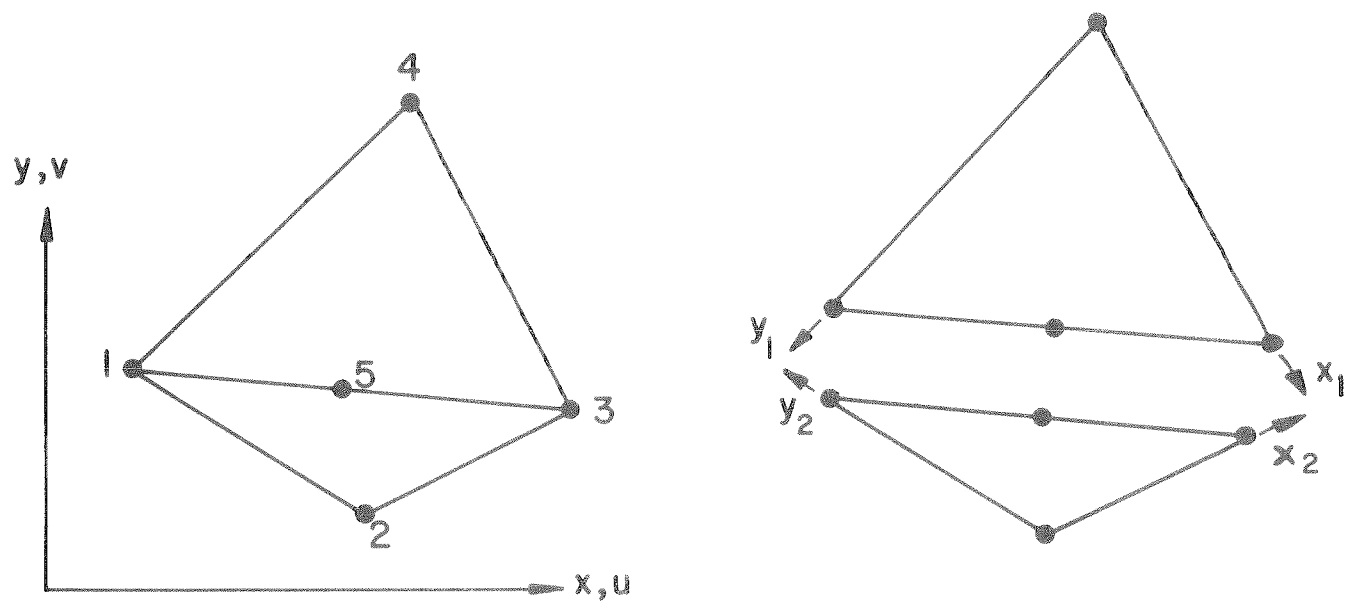
Within each triangular element the displacements in the x and y coordinate system may be expressed in the local skewed coordinate system by the following equations:

$$u = \alpha_1 + \alpha_2 X_i + \alpha_3 Y_i + \alpha_4 X_i Y_i$$

$$v = \beta_1 + \beta_2 X_i + \beta_3 Y_i + \beta_4 X_i Y_i$$

This displacement field assumption forces a linear (and compatible) variation of the displacements along sides 1-2, 2-3, 3-4 and 4-1. The displacements along line 1-5-3 vary parabolically; however, full compatibility is maintained since the three points along the straight line are forced to have common displacements. The details of the stiffness matrix development for this type of element is given in Appendix B.

After the 8 x 8 stiffness matrices for the two triangular elements are developed they are combined to form the 10 x 10 quadrilateral element stiffness matrix. The unknowns associated with point 5 are then



a- QUADRILATERAL ELEMENT

b- TWO LINEAR STRAIN TRIANGLES

FIG. 1 LINEAR STRAIN QUADRILATERAL ELEMENT expressed in terms of the unknowns of points 1 through 4 and eliminated from the system; this results in the development of the 8 x 8 quadrilateral element stiffness matrix. The quadrilateral element stiffness matrices are combined by direct stiffness procedures to form the complete stiffness matrix for the finite element system.

C. Lumped Mass Approximation

A formal mathematical development of the mass matrix is possible. Such an approach would result in a mass matrix with the same coupling properties as the stiffness matrix. However, if the physical lumped mass approximation is made the mass matrix will be diagonal. The lumped mass approximation results in a small reduction in accuracy and a considerable saving in computer storage and time. In this investigation one-fourth the mass of each quadrilateral is assumed to be concentrated at each of the four nodal points.

D. Damping Matrix

For most structures the exact form of the damping is unknown. Since its effect on the transient response of a structure is generally small, a simplifying assumption as to its form is justifiable. In the step-by-step solution procedure the damping matrix may be completely arbitrary; however, there is little experimental justification for selecting specific damping coefficients. A form of viscous damping, which is sufficiently general for most structures, is given by the following matrix equation:

$$[C] = \alpha[M] + \beta[K] \tag{2}$$

By making the damping matrix proportional to the mass and stiffness matrices no additional storage is required within the computer program.

A significant portion of our experience with damping has been related to the frequencies and mode shapes of the system; therefore, it is important that α and β be interpreted in terms of equivalent modal damping. Since the determination of mode shapes and frequencies is not an essential part of the step-by-step method, modal damping cannot be used directly.

It can be shown that the modal damping ratio for the i^{th} mode is given in terms of α and β by

$$\lambda_i = \frac{\alpha}{2\omega_i} + \beta\frac{\omega_i}{2} \qquad (2a)$$

where ω_i is the frequency of the i^{th} mode. For given values of α and β the frequency $\bar{\omega}$ which yields a minimum value of damping ratio $\bar{\lambda}$ is given by

$$\bar{\omega} = \sqrt{\frac{\alpha}{\beta}}$$

If the minimum damping ratio, $\bar{\lambda}$, and the frequency, $\bar{\omega}$, are given, the damping coefficients α and β are calculated from the following equations:

$$\alpha = \bar{\lambda}\bar{\omega} \qquad (2b)$$

$$\beta = \bar{\lambda}/\bar{\omega} \qquad (2c)$$

Equation (2a) can now be rewritten as

$$\lambda_i = \left(\frac{\bar{\omega}}{\omega_i} + \frac{\omega_i}{\bar{\omega}}\right)\frac{\bar{\lambda}}{2}$$

or in terms of the periods of the system

$$\lambda_i = \left(\frac{T_i}{\bar{T}} + \frac{\bar{T}}{T_i}\right)\frac{\bar{\lambda}}{2}$$

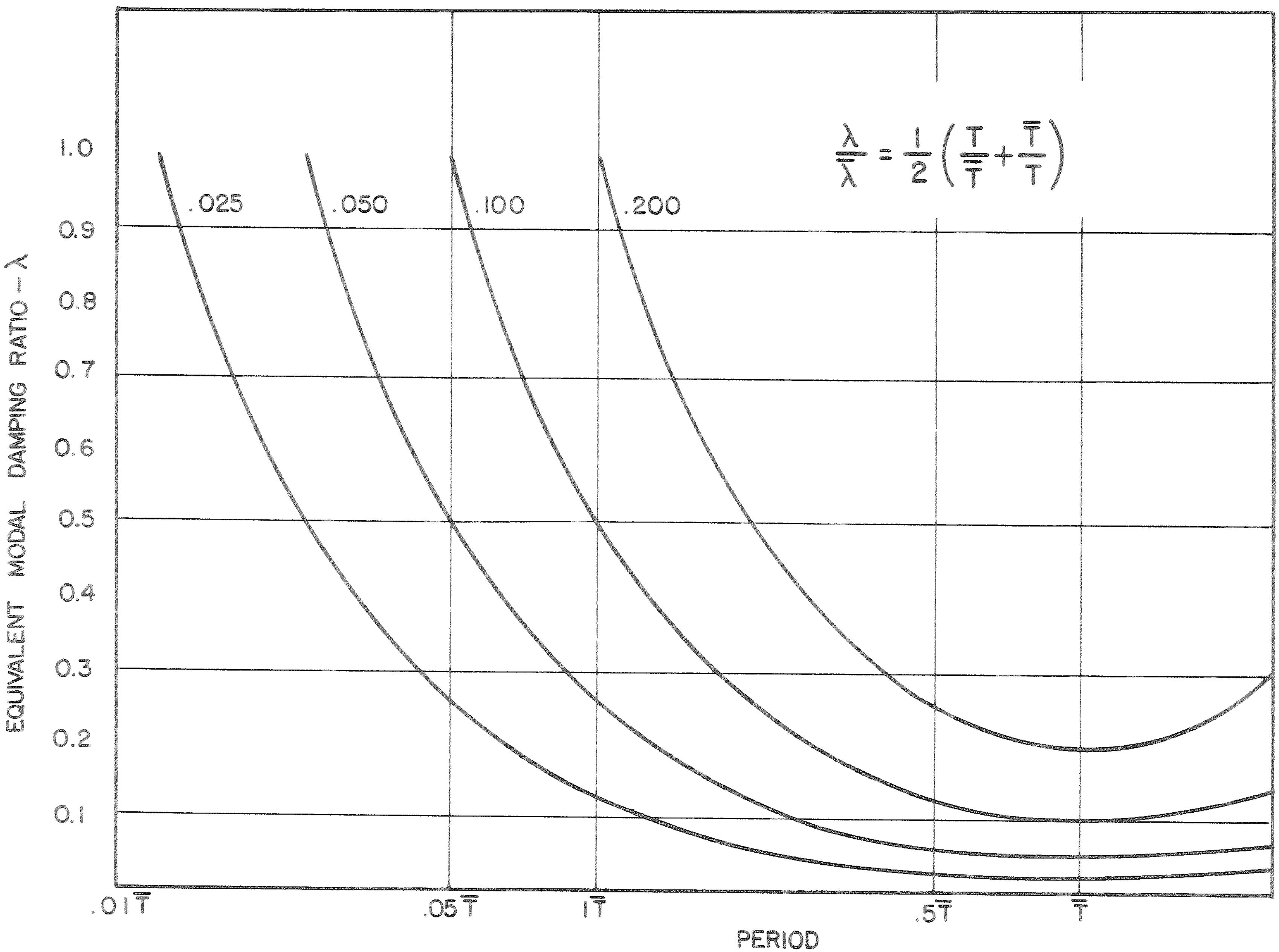


FIG. 2 EQUIVALENT MODAL DAMPING AS A FUNCTION OF PERIOD

A graphical representation of this equation is shown in Fig. 2. Therefore, if the significant frequency range is established for a structure and the equivalent modal damping is selected, the constants α and β can be calculated directly from Eqs. (2b) and (2c).

E. Step-by-Step Integration of Equilibrium Equations

The dynamic equilibrium of the finite element system is given by Eq. (1). The solution of this set of second order differential equations is accomplished by a step-by-step procedure. The only approximation which is made is that the acceleration of each point in the system varies linearly within a small time interval, Δt.

This assumption, which is illustrated in Fig. (3), leads to a parabolic variation of velocity and a cubic variation of displacement within the time interval.

A direct integration over the interval gives the following equations for acceleration and velocity at the end of the time interval:

$$\ddot{u}_t = \frac{6}{\Delta t^2} u_t - A_t \tag{3}$$

$$\dot{u}_t = \frac{3}{\Delta t} u_t - B_t \tag{4}$$

where

$$A_t = \frac{6}{\Delta t^2} u_{t-\Delta t} + \frac{6}{\Delta t} \dot{u}_{t-\Delta t} + 2\ddot{u}_{t-\Delta t} \tag{5}$$

$$B_t = \frac{3}{\Delta t} u_{t-\Delta t} + 2 \dot{u}_{t-\Delta t} + \frac{\Delta t}{2} \ddot{u}_{t-\Delta t} \tag{6}$$

The substitution of the matrix form of Eqs. (3) and (4) into the equation of equilibrium, Eq. (1)

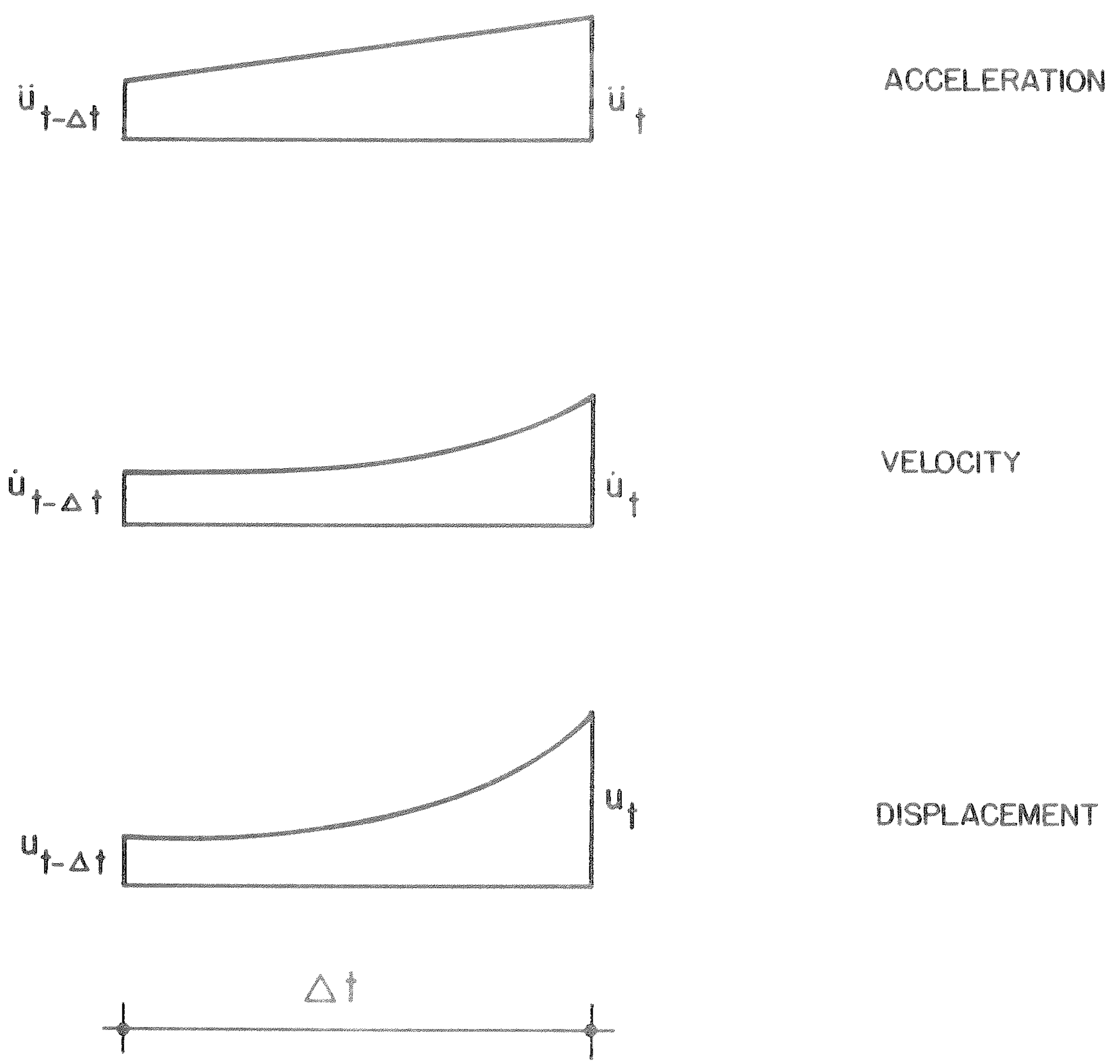


FIG. 3 ASSUMED BEHAVIOR OF TYPICAL DISPLACEMENT COMPONENT

$$[K]\{u\}_t + [C]\{\dot{u}\}_t + [M]\{\ddot{u}\}_t = \{P\}_t \tag{7}$$

yields the following equation for the displacement at the end of the time interval:

$$[\bar{K}]\{u\}_t = \{\bar{P}\}_t \tag{8}$$

where

$$[\bar{K}] = [K] + \frac{3}{\Delta t}[C] + \frac{6}{\Delta t^2}[M] \tag{9}$$

$$\{\bar{P}\}_t = \{P\}_t + [C]\{B\}_t + [M]\{A\}_t \tag{10}$$

The solution technique for Eq. (8) is discussed in Appendix A.

The acceleration and velocity at the end of the time interval is obtained from the matrix form of Eqs. (3) and (4):

$$\{\ddot{u}\}_t = \frac{6}{\Delta t^2}\{u\}_t - \{A\}_t \tag{11}$$

$$\{\dot{u}\}_t = \frac{3}{\Delta t}\{u\}_t - \{B\}_t \tag{12}$$

In order to simplify computations and to minimize computer storage requirements the damping matrix is assumed to be a linear combination of the mass matrix and stiffness matrix, as indicated by Eq. (2). The substitution of Eqs. (2), (11) and (12) into the dynamic equilibrium relation, Eq. (7), yields the following set of linear equations in terms of some unknown "effective" displacement

$$[\bar{K}][\bar{u}]_t = [\bar{P}]_t \tag{13}$$

where

$$[\bar{K}] = [K] + C_2[M] \tag{14}$$

$$[\bar{P}]_t = [P]_t + [M]\Big[[A]_t + C_3[B]_t\Big] \tag{15}$$

$$[\bar{u}]_t = \frac{1}{C_1}[u]_t - \beta[B]_t \tag{16}$$

$$C_0 = \frac{3\alpha}{\Delta t} + \frac{6}{\Delta t^2} \tag{17}$$

$$C_1 = \frac{1}{1 + \frac{3}{\Delta t}\beta} \tag{18}$$

$$C_2 = C_0 C_1 \tag{19}$$

$$C_3 = \alpha - C_2\beta \tag{20}$$

The complete step-by-step method is summarized in Table I.

F. Stability of the Step-by-Step Method

The previously described step-by-step integration technique is accurate if the time step is small compared to the shortest period of the finite element system. If the time step is long compared to the shortest period, the method will become unstable and fail to produce realistic results. Newmark[9] has studied this instability and has suggested a constant acceleration method. Newmark's procedure was found to be stable when applied to finite element systems; however, spurious finite oscillations associated with the high frequencies of the system were still present in the results. Several other stable step-by-step methods were investigated with respect to finite element systems; the method found to be completely stable without the addition of damping was a modification of the previously described linear acceleration method.

The instability in the linear acceleration method is first initiated by an oscillation of the displacements about the true solution. This oscillation can be eliminated by a simple modification of the methods. In the early stages of the instability it is apparent that the displacements at the center of the time interval are a good approximation of the

TABLE I. SUMMARY OF STEP-BY-STEP METHOD

1. Initialization

   a. Form stiffness matrix $[K]$ and mass matrix $[M]$

   b. Form "effective" stiffness matrix

$$[\bar{K}] = [K] + C_2[M]$$

   c. Triangularize $[\bar{K}]$

2. For Each Time Increment

   a. Form $[A]_t$ and $[B]_t$

$$[A]_t = \frac{6}{\Delta t^2}[u]_{t-\Delta t} + \frac{6}{\Delta t}[\dot{u}]_{t-\Delta t} + 2[\ddot{u}]_{t-\Delta t}$$

$$[B]_t = \frac{3}{\Delta t}[u]_{t-\Delta t} + 2[\dot{u}]_{t-\Delta t} + \frac{\Delta t}{2}[\ddot{u}]_{t-\Delta t}$$

   b. Form "effective" load

$$[\bar{P}]_t = [P]_t + [M]\left[[A]_t + C_3[B]_t\right]$$

   c. Solve for "effective" displacements

$$[\bar{u}]_t = [\bar{K}]^{-1}[\bar{P}]_t$$

   d. Calculate Displacements, Velocities and Accelerations at time t

$$[u]_t = C_1[\bar{u}]_t + C_1\beta[B]_t$$

$$[\dot{u}]_t = \frac{3}{\Delta t}[u]_t - [B]_t$$

$$[\ddot{u}]_t = \frac{6}{\Delta t^2}[u]_t - [A]_t$$

   e. Repeat for next time increment

true solution. Therefore, if this mid-point solution is utilized, the tendency for oscillations to develop is eliminated. In order to modify the previous step-by-step equations to reflect this approach a time increment of $2\Delta t$ is introduced and the acceleration, $\ddot{u}_{t+\Delta t}$, at the end of the time interval is calculated. The mid-point acceleration is calculated as

$$\ddot{u}_t = \frac{1}{2} (\ddot{u}_{t-\Delta t} + \ddot{u}_{t+\Delta t}) \tag{21}$$

The velocity and displacements at time "t" are calculated from

$$\dot{u}_t = \dot{u}_{t-\Delta t} + \frac{\Delta t}{2} \ddot{u}_{t-\Delta t} + \frac{\Delta t}{2} \ddot{u}_t \tag{22}$$

$$u_t = u_{t-\Delta t} + \Delta t \, \dot{u}_{t-\Delta t} + \frac{\Delta t^2}{3} \ddot{u}_{t-\Delta t} + \frac{\Delta t^2}{6} \ddot{u}_t \tag{23}$$

This approach has eliminated all stability problems from the step-by-step method and is the technique which has been incorporated into the computer program included with this report.

G. Blast Load Forces

In the case of blast analysis of underground structures the loads may take one or both of the following forms: First, the loading may be due to time dependent blast pressures applied at the free surface immediately above the underground structure. Second, the loading may be in the form of a ground shock which was initiated at some remote location in reference to the structure. The method of analysis and the resulting computer program which is presented in this report will consider both of these types of loading.

Pressure Surface Loading - The method of analysis, in general, is capable of considering arbitrary time-dependent forces at the nodal points of the finite element system. To allow for this generality in a computer program would require a prohibitive amount of input data. In order to minimize data input and to provide sufficient flexibility for the solution of most problems the loading is restricted to the form shown in Fig. 4. Only two different load forms $f_x(\tau)$ and $f_y(\tau)$ are defined and supplied to the computer program as input. In addition, at each nodal point in the system the constants F_x, F_y and t_o are specified. The time dependent nodal forces in the x and y directions are calculated within the program by evaluating the following equations:

$$R_x(t) = F_x f_x(t-t_o) \tag{24}$$

$$R_y(t) = F_y f_y(t-t_o) \tag{25}$$

where t is the absolute time and t_o is the arrival time of the load function. The constant F is related to the loaded surface area which contributes to the total load at the nodal point.

Displacement Boundary Conditions - The problem of specifying displacements at nodal points as a function of time can also be accomplished in the same computer program. As an example, consider the structure shown in Fig. 5 which is subjected to a horizontal displacement at the left vertical boundary of $U_x(t)$. A typical nodal point on this boundary may be idealized by a one-dimensional spring element connected to the finite element system as shown below.

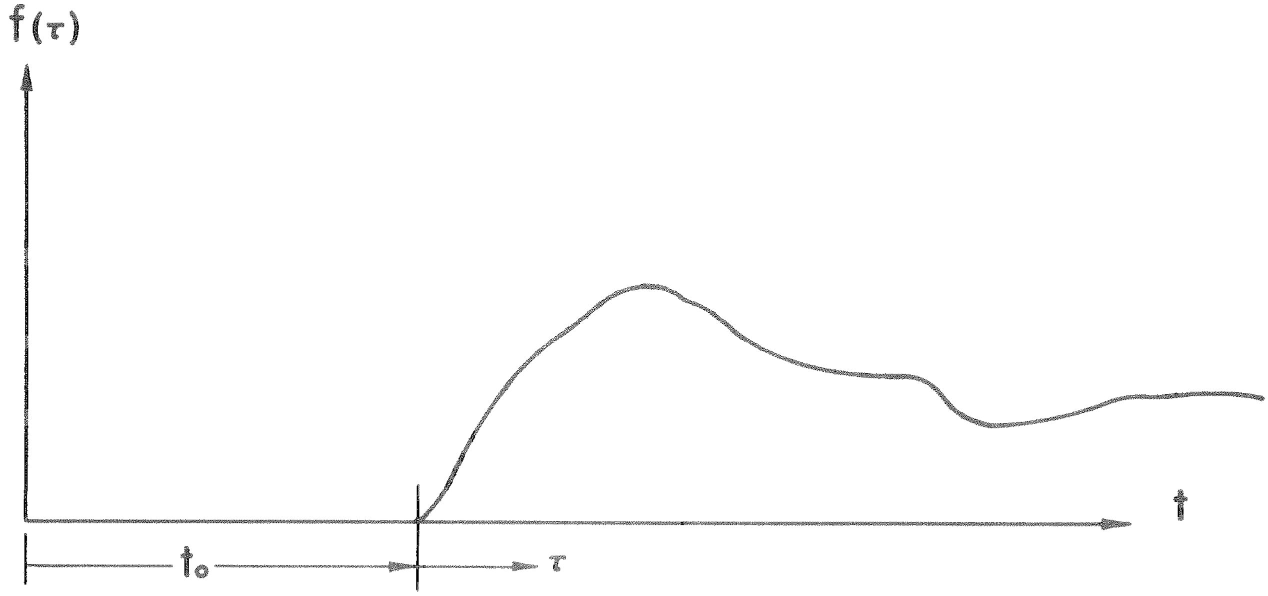


FIG. 4 GENERAL TIME-DEPENDENT LOAD FUNCTION

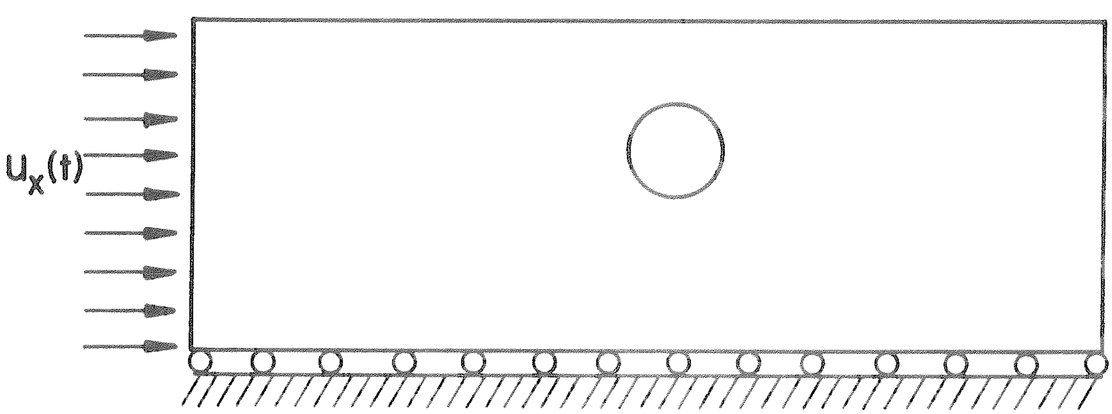


FIG. 5 TIME-DEPENDENT DISPLACEMENTS

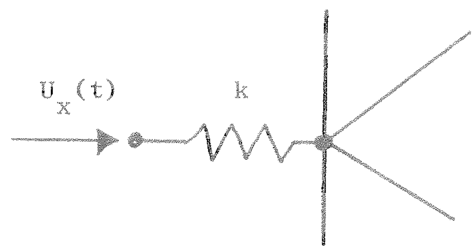


FIGURE 6

The stiffness of the spring k represents the stiffness of a small amount of soil material (i.e. one foot). For the purpose of computer program input the above model is equivalent to the system shown below.

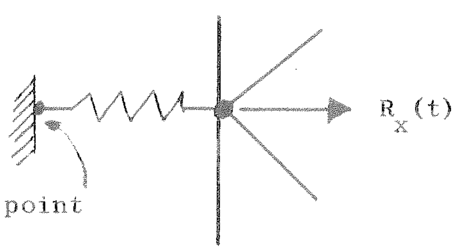


FIGURE 7

in which the force at the nodal point is given by

$$R_x(t) = k\ u_x(t) \tag{26}$$

or in terms of using the computer program

$$F_x = k \tag{27}$$

$$f_x(t) = u_x(t) \tag{28}$$

The creation of this pseudo spring is necessary only to conform with the existing computer program input format; a formal inclusion of the displacement boundary condition could have been accomplished in a more direct manner.

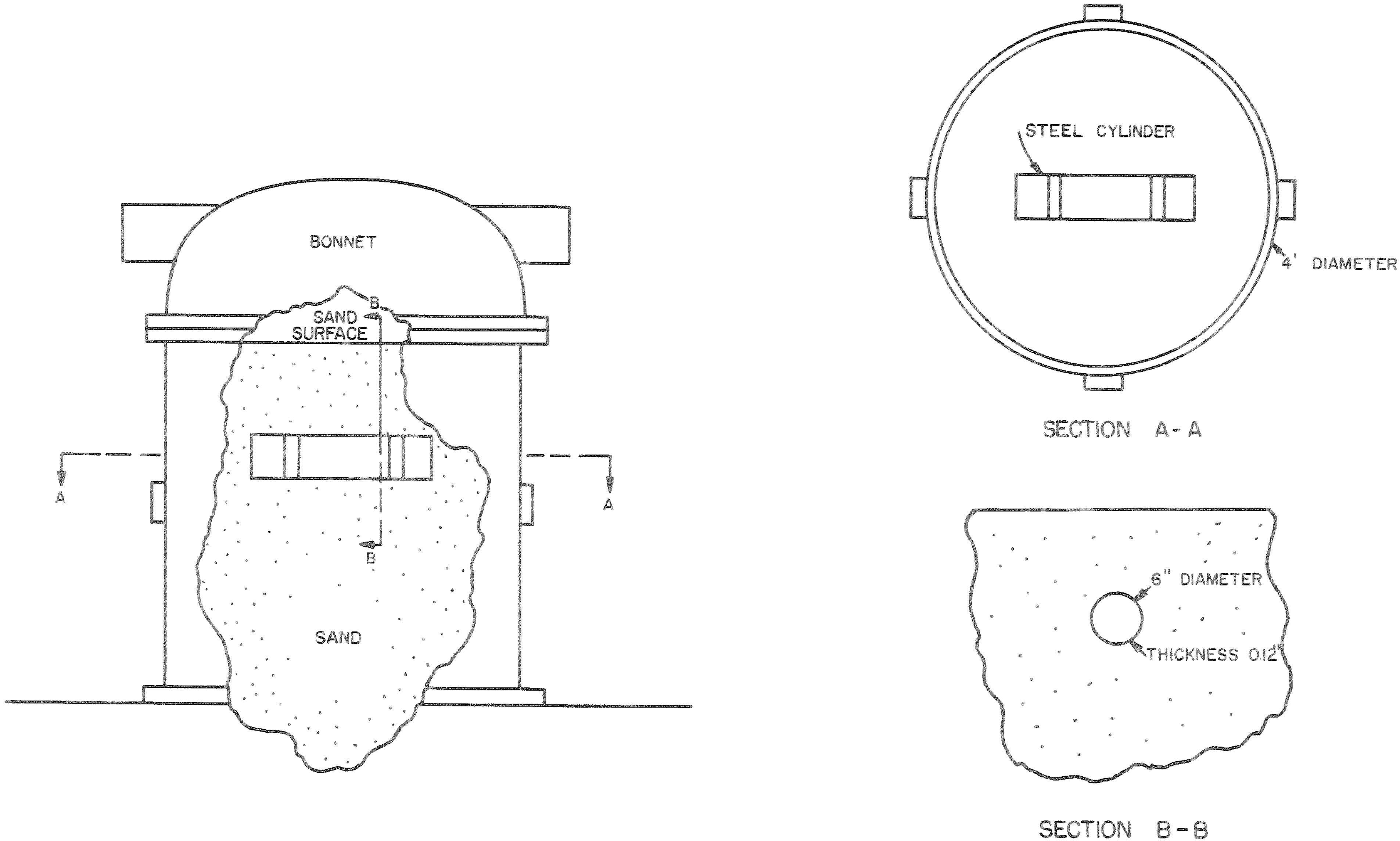


FIG. 8 TEST CONFIGURATION

III. EXAMPLE

Several examples with known exact solutions were solved to verify the method of analysis and to establish the validity of the computer program. In all cases, as the time interval and the space mesh size were reduced, the results approached the known values.

In order to illustrate the application of the program to a practical problem, the blast load analysis of an underground cylinder buried in a sand material was selected. This particular structure was tested at W.E.S. The experimental layout for the system is shown in Fig. 8. The measured surface pressure as a function of time is plotted in Fig. 9. In this case the analysis of the actual structure is a complex three-dimensional problem. The two-dimensional plane strain, finite element analysis represents only an approximate solution. The material properties used in the analysis were

Modulus of Elasticity of Cylinder	29×10^6 psi
Modulus of Elasticity of Soil	25000 psi
Poisson's Ratio of Soil	0.3
Mass Density of Soil	$.000164\ \#sec^2/in^4$

The structure was analyzed with two different finite element idealizations -- a fine mesh which is shown in Fig. 10 and a coarse mesh which is not shown. The fine mesh contained 176 elements and 187 nodal points. The course mesh contained 108 elements and 117 nodal points. The cylinder was idealized by one-dimensional elements with axial stiffness only -- the bending stiffness of the cylinder has been neglected.

Fig. 11 illustrates the axial stress in the cylinder at point A as a function of time. Results of the two analyses with different

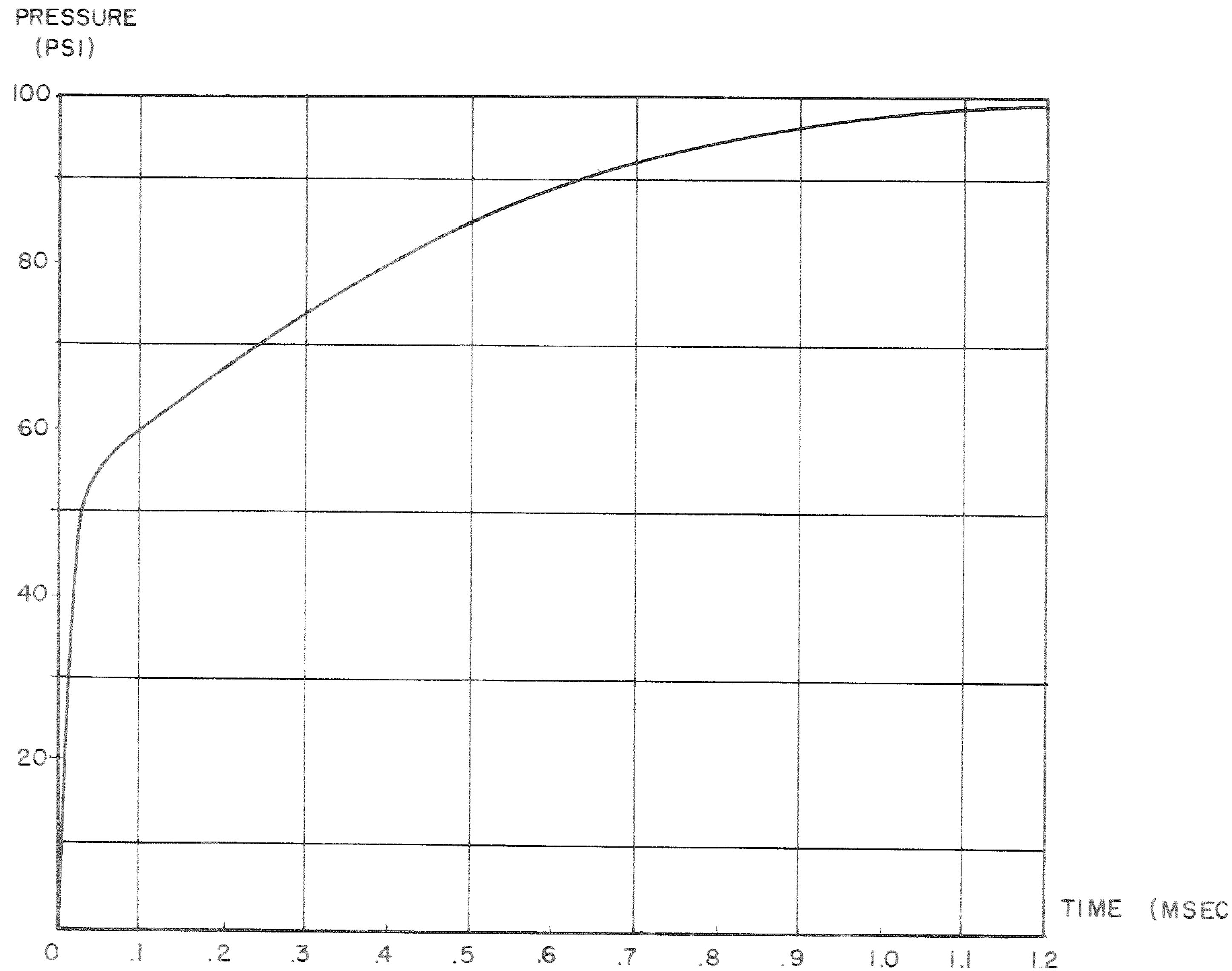


FIG. 9 SURFACE BLAST PRESSURE VS TIME

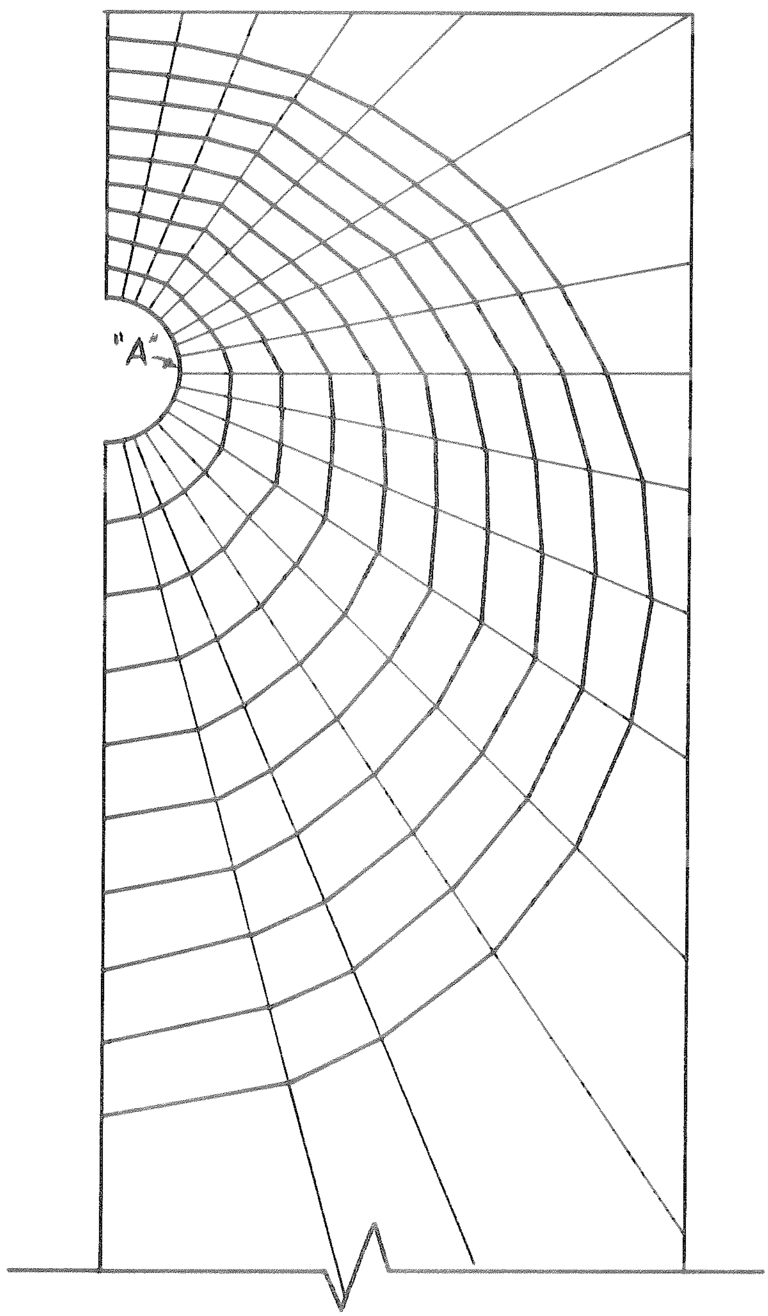


FIG. 10 FINITE ELEMENT REPRESENTATION OF UNDERGROUND STRUCTURAL SYSTEM

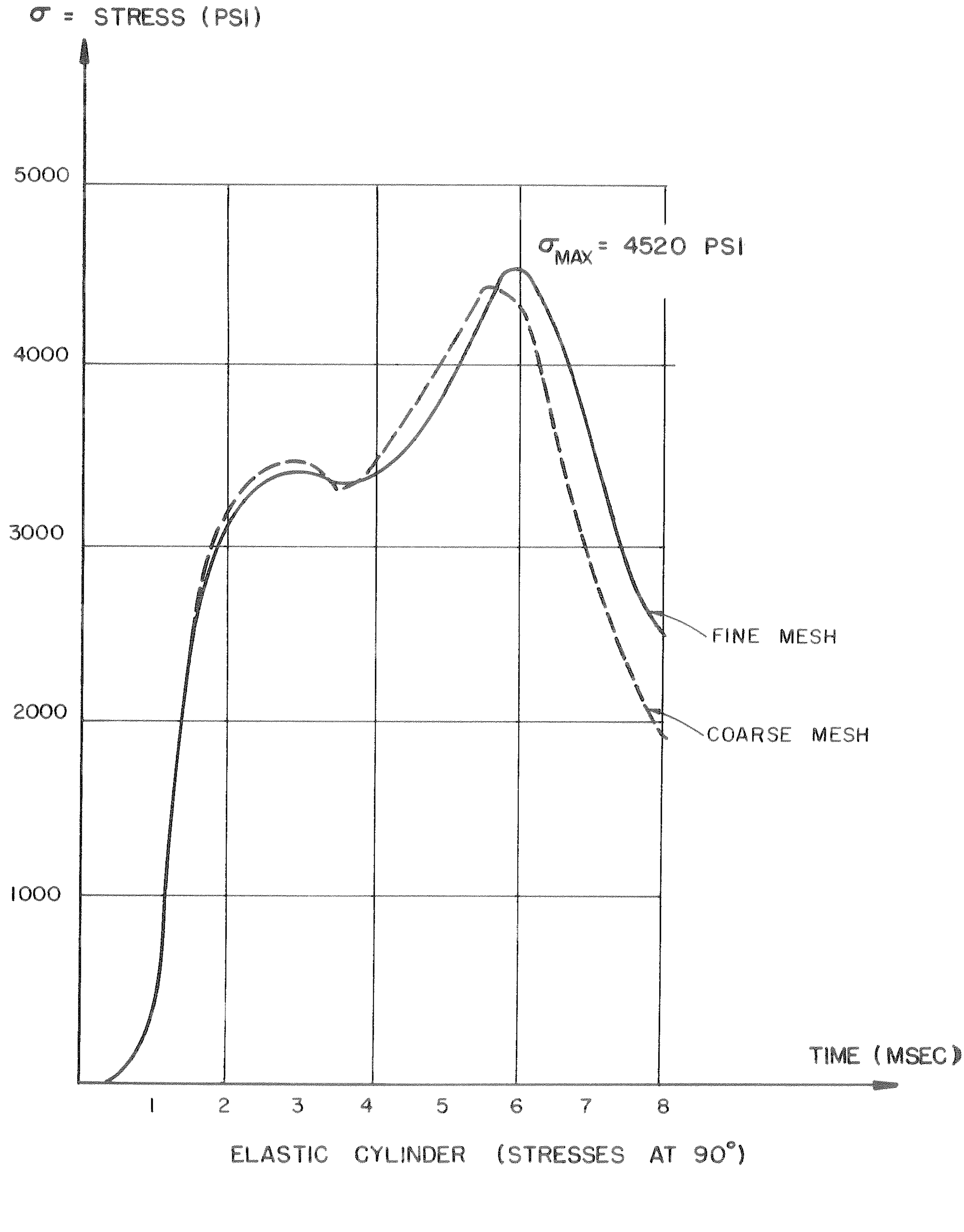


FIG. 11 STRESS VS TIME IN CYLINDER

mesh size are in reasonable agreement. The fine mesh analysis yields a maximum stress of 4520 psi which compares with an experimental value of 5450 psi obtained by W.E.S. Considering the many approximations involved, these results are realistic.

The stresses in the soil as a function of time for points at different distances below the surface are plotted in Fig. 12. Point 1, which is very near the surface, correctly reflects the applied surface pressure. Points 2, 3 and 4 illustrate a time lag in their response to the surface loading as one would expect.

A description of the specific computer program used for this analysis is given in Appendix C. Appendix D contains a FORTRAN IV listing of the program. The program can be used directly on a CDC 6400 with 32k storage. It can be converted to other computers with a minimum of effort. Of course, the capacity and speed of the program will depend on the computer used. For the fine mesh analysis used in this example, approximately five minutes of CDC 6400 computer time was required.

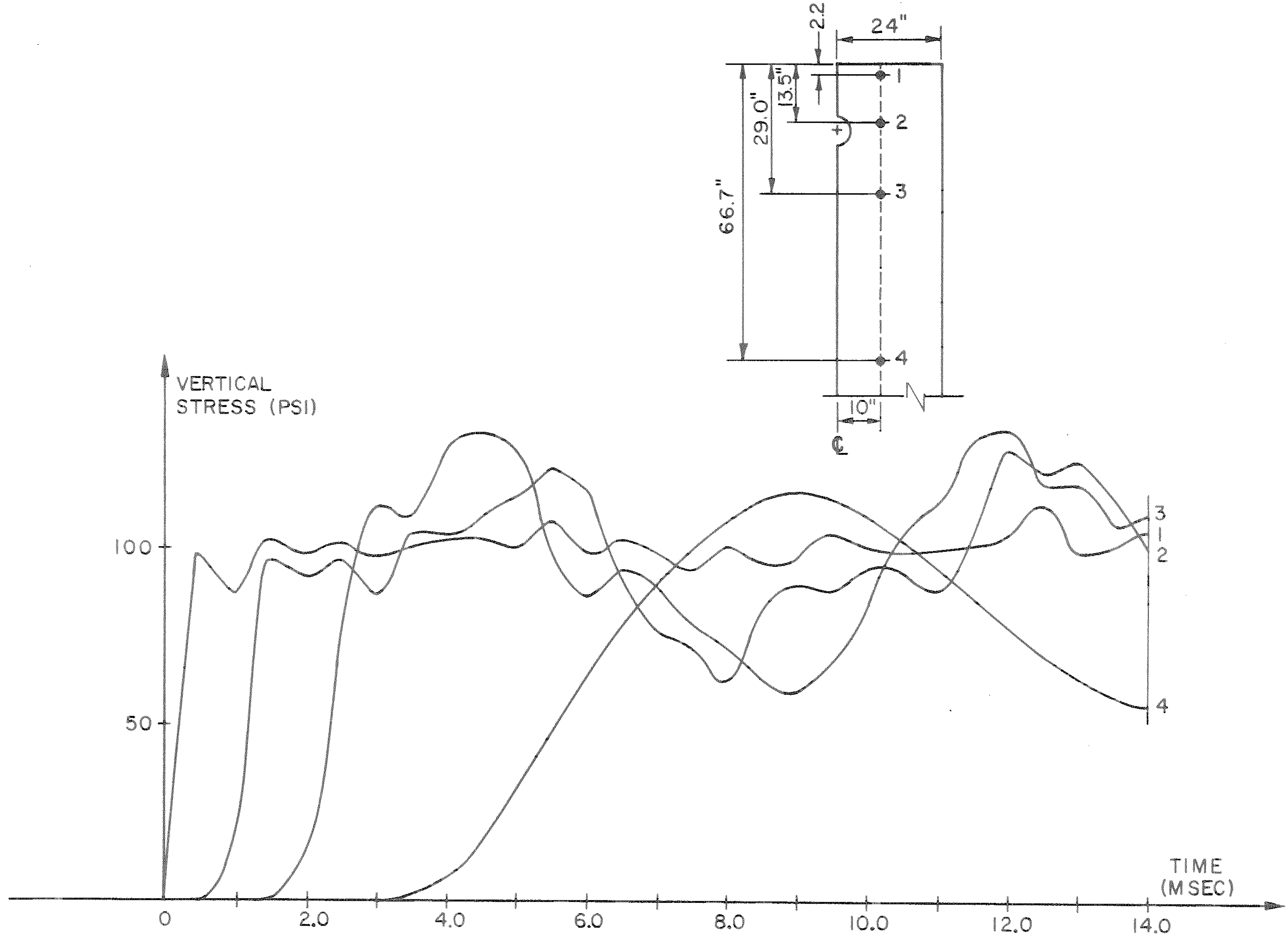


FIG. 12 STRESSES VS TIME IN SOIL MATERIAL

IV. FINAL REMARKS

In this report the theory and computer program for dynamic, elastic analysis of two-dimensional structures of arbitrary shape and material properties are presented. The program is especially suitable for the analysis of underground structures subject to surface blast loading.

The finite element method is used to reduce the continuous structure into a discrete system. A stable step-by-step integration procedure is used to evaluate the response of the finite element system. The step-by-step method was selected, in preference to the mode superposition approach, because it may be directly extended to include the effects of non-linear material properties and large displacements.

The next phase of this investigation will involve the incorporation of nonlinear materials. This will require the calculation of an incremental stiffness matrix at each step in the solution procedure. The incremental stiffness will be based on the state of stress in the element at the beginning of the time step. It is apparent that this approach will require a tremendous increase in computation effort for each time step. However, it is hoped that large time steps can be used; since a stable step-by-step will be employed. An important part of this phase will be the selection of a realistic nonlinear model for the soil material.

After the incorporation of nonlinear material behavior, the effects of large deformations can be included without a significant increase in computational time. This will involve the formulation of the incremental stiffness in the deformed coordinate system and the evaluation of the unbalanced forces due to the violation of the equilibrium in the deformed position. These unbalanced forces may be reduced by the application of equal and opposite loads in the next time increment. In this way it should be possible to satisfy total equilibrium of the system regardless of the magnitudes of the displacements.

The extension of the Method of Analysis and the computer program to the dynamic response of Axisymmetric Solids is a very simple procedure. This would require a slight modification in the calculation of the element stiffness matrix to reflect the addition of the hoop stiffness in a triangular ring element. The step-by-step method, the incorporation of nonlinear material and large deformation would be identical to the plane strain problem.

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APPENDIX A. SOLUTION OF LINEAR EQUATIONS

The equilibrium equations for a structural system can be written in the following form:

$$A_{11}X_1 + A_{12}X_2 + A_{13}X_3 \ldots\ldots + A_{1N}X_N = B_1 \quad \text{(A-1a)}$$

$$A_{21}X_1 + A_{22}X_2 + A_{23}X_3 \ldots\ldots + A_{2N}X_N = B_2 \quad \text{(A-1b)}$$

$$A_{31}X_1 + A_{32}X_2 + A_{33}X_3 \ldots\ldots + A_{3N}X_N = B_3 \quad \text{(A-1c)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{N1}X_1 + A_{N2}X_2 + A_{N3}X_3 \ldots\ldots + A_{NN}X_N = B_N \quad \text{(-)}$$

or, symbolically,

$$[A][X] = [B] \quad \text{(A-1)}$$

where

$[A]$ = the stiffness matrix

$[X]$ = the unknown displacements

$[B]$ = the applied loads

A.1 Gaussian Elimination

The first step in the solution of the above set of equations is to solve Eq. (A-1a) for X_1 :

$$X_1 = B_1/A_{11} - (A_{12}/A_{11})X_2 - (A_{13}/A_{11})X_3 \ldots (A_{1N}/A_{11})X_N \quad \text{(A-2)}$$

If Eq. (A-2) is substituted into Eqs. (A-1b, c,..., N), a modified set of N-1 equations is obtained:

$$A^1_{22}X_2 + A^1_{23}X_3 \ldots\ldots + A^1_{2N}X_N = B^1_2 \quad \text{(A-3a)}$$

$$A^1_{32}X_2 + A^1_{33}X_3 \ldots\ldots + A^1_{3N}X_N = B^1_3 \quad \text{(A-3b)}$$

$$A^{1}_{N2}X_2 + A^{1}_{N3}X_3 \ldots\ldots + A^{1}_{NN}X_N = B^{1}_{N} \qquad (--)$$

where

$$A^{1}_{ij} = A_{ij} - A_{i1}A_{1j}/A_{11} \quad i,\ j = 2, \ldots, N \qquad \text{(A-4a)}$$

$$B^{1}_{i} = B_i - A_{i1}B_1/A_{11} \quad i = 2, \ldots, N \qquad \text{(A-4b)}$$

A similar procedure is used to eliminate X_2 from Eq. (A-3), etc. A general algorithm for the elimination of X_n can be written as

$$X_n = (B^{n-1}_{n}/A^{n-1}_{nn}) - \sum (A^{n-1}_{nj}/A^{n-1}_{nn})\ X_j \quad j = n + 1, \ldots, N \qquad \text{(A-5)}$$

$$A^{n}_{ij} = A^{n-1}_{ij} - A^{n-1}_{in}(A^{n-1}_{nj}/A^{n-1}_{nn}) \quad i,\ j = n + 1, \ldots, N \qquad \text{(A-6)}$$

$$B^{n}_{i} = B^{n-1}_{i} - A^{n-1}_{in}(B^{n-1}_{n}/A^{n-1}_{nn}) \quad i = n + 1, \ldots, N \qquad \text{(A-7)}$$

Equations A-5, A-6, and A-7 can be rewritten in compact form:

$$X_n = D_n - \sum H_{nj}X_j \quad j = n + 1, \ldots, N \qquad \text{(A-8)}$$

$$A^{n}_{ij} = A^{n-1}_{ij} - A^{n-1}_{in}\ H_{nj} \quad i,\ j = n + 1, \ldots, N \qquad \text{(A-9)}$$

$$B^{n}_{i} = B^{n-1}_{i} - A^{n-1}_{in}\ D_n \quad i = n + 1, \ldots, N \qquad \text{(A-10)}$$

where

$$D_n = B^{n-1}_{n}/A^{n-1}_{nn}$$

$$H_{nj} = A^{n-1}_{nj}/A^{n-1}_{nn}$$

After the above procedure is applied $N-1$ times, the original set of equations is reduced to the single equation

$$A^{N-1}_{NN}X_N = B^{N-1}_{N}$$

which is solved directly for X_N :

$$X_N = B_N^{N-1}/A_{NN}^{N-1}$$

In terms of the previous notation, this is

$$X_N = D_N \tag{A-11}$$

The remaining unknowns are determined in reverse order by the repeated application of Eq. (A-8).

A.2 Simplification for Band Matrices

For many structural systems, the stiffness matrix occurs in a "band" form which results in the concentration of the elements of the stiffness matrix along the main diagonal. Therefore, the following simplifications in the general algorithm (Eqs. [A-8], [A-9], and [A-10] are possible:

$$X_n = D_n - \sum H_{nj} X_j \qquad j = n+1, \ldots, n+M-1 \tag{A-12}$$

$$A_{ij}^n = A_{ij}^{n-1} - A_{in}^{n-1} H_{nj} \quad i, j = n+1, \ldots, n+M-1 \tag{A-13}$$

$$B_i^n = B_i^{n-1} - A_{in}^{n-1} D_n \qquad i = n+1, \ldots, n+M-1 \tag{A-14}$$

where M is the band width of the matrix.

The number of numerical operations can further be reduced by recognizing that the reduced matrix at any stage of the procedure is symmetric. Accordingly, since

$$A_{ji}^n = A_{ij}^n$$

Eq. (A-13) can be replaced by

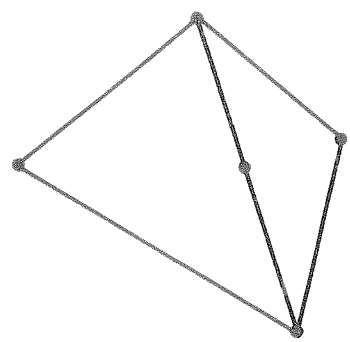
$$A_{ij}^n = A_{ij}^{n-1} - A_{in}^{n-1} H_{nj} \qquad \begin{matrix} i = n+1, \ldots, N+M-1 \\ j = i, \ldots, n+M-1 \end{matrix} \tag{A-15}$$

The number of numerical operations required for the solution of a band matrix is proportional to NM^2 as compared to N^3 which is required for the solution of a full matrix. Also, the computer storage required by the band matrix procedure is NM as compared to N^2 required by a set of N arbitrary equations. This is the technique used within the computer program presented in this report.

APPENDIX B. STIFFNESS OF LINEAR STRAIN QUADRILATERAL

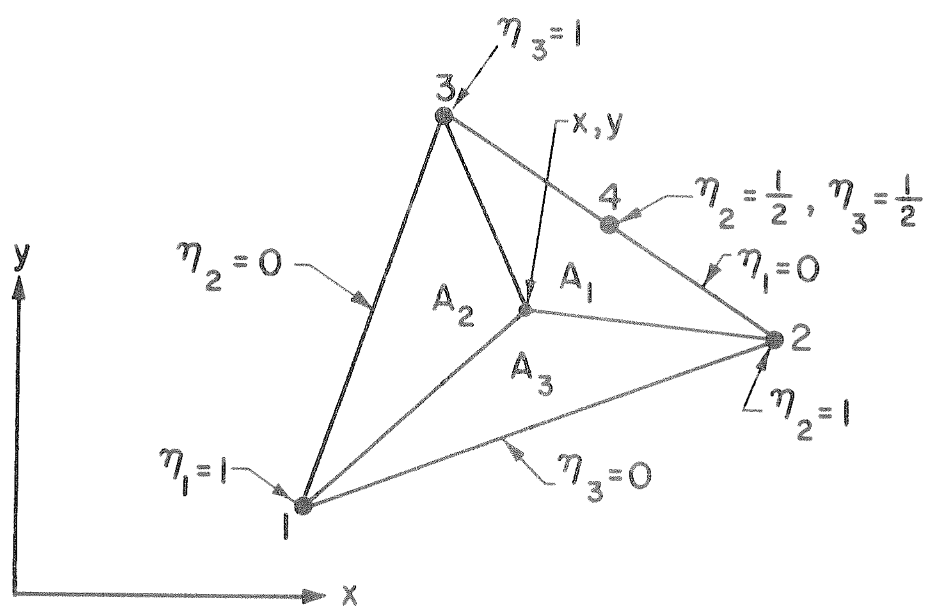
The Quadrilateral Element

The linear strain quadrilateral is composed of two four nodal point triangles as shown below.



The Area Coordinate System

Within a triangular element the displacements may be expressed as a function of space and the values of the displacements at the four nodes. A convenient form in which these displacements may be expressed directly is in the area coordinate system shown below.



The dimensionless area coordinate is the ratio of the subarea to the total area of the element. Or,

$$\eta_i = \frac{A_i}{A} \qquad \text{(B-1)}$$

where

$$A = A_1 + A_2 + A_3$$

Therefore, a point within the triangle may be defined in terms of the global system x and y or in terms of the local area coordinates A_1, A_2 and A_3 or in terms of the dimensionless coordinates η_1, η_2 and η_3.

The Displacement Field Approximation

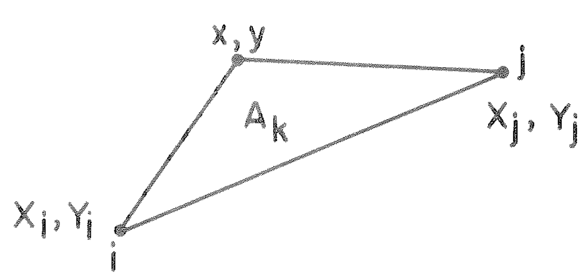
The advantage of the area coordinate system is that the compatible displacement functions can be written directly in a simple form. Hence, the x and y components of the displacement field are

$$u = \eta_1 u_1 + \eta_2(1-2\eta_3)\ u_2 + \eta_3(1-2\eta_2)\ u_3 + 4\eta_2\eta_3\ u_4 \qquad \text{(B-2a)}$$

$$v = \eta_1 v_1 + \eta_2(1-2\eta_3)\ v_2 + \eta_3(1-2\eta_2)\ v_3 + 4\eta_2\eta_3\ V_4 \qquad \text{(B-2b)}$$

Relationship Between Coordinate Systems

A typical subarea region of the triangular element is shown below.



The area of the region is given by

$$A_k = \left(\frac{y+y_j}{2}\right)(x_j-x) + \left(\frac{y+y_i}{2}\right)(x-x_i) - \left(\frac{y_j+y_i}{2}\right)(x_j-x_i) \qquad \text{(B-3)}$$

Or in dimensionless form

$$\eta_k = \frac{1}{2A}\left[(y+y_j)(x_j-x) + (y+y_i)(x-x_i) - (y_j+y_i)(x_j-x_i)\right] \qquad \text{(B-4)}$$

The derivatives with respect to the global coordinates are in general

$$\frac{\partial \eta_k}{\partial x} = \frac{y_i-y_j}{2A} \quad \text{and} \quad \frac{\partial \eta_k}{\partial y} = \frac{x_j-x_i}{2A} \qquad \text{(B-5)}$$

These evaluated for the three coordinates are

$$\begin{aligned} \frac{\partial \eta_1}{\partial x} &= \frac{b_1}{2A} & \frac{\partial \eta_1}{\partial y} &= \frac{a_1}{2A} \\ \frac{\partial \eta_2}{\partial x} &= \frac{b_2}{2A} & \frac{\partial \eta_2}{\partial y} &= \frac{a_2}{2A} \\ \frac{\partial \eta_3}{\partial x} &= \frac{b_3}{2A} & \frac{\partial \eta_3}{\partial y} &= \frac{a_3}{2A} \end{aligned} \qquad \text{(B-6)}$$

where

$$\begin{aligned} a_1 &= x_3 - x_2 \\ a_2 &= x_1 - x_3 \\ a_3 &= x_2 - x_1 \\ b_1 &= y_2 - y_3 \\ b_2 &= y_3 - y_1 \\ b_3 &= y_1 - y_2 \end{aligned} \qquad \text{(B-7)}$$

Expressions for Element Strains

The element strains may be derived from the displacement field by the use of the chain rule. Or

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta_1} \cdot \frac{\partial \eta_1}{\partial x} + \frac{\partial u}{\partial \eta_2} \cdot \frac{\partial \eta_2}{\partial x} + \frac{\partial u}{\partial \eta_3} \cdot \frac{\partial \eta_3}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta_1} \cdot \frac{\partial \eta_1}{\partial y} + \frac{\partial v}{\partial \eta_2} \cdot \frac{\partial \eta_2}{\partial y} + \frac{\partial v}{\partial \eta_3} \cdot \frac{\partial \eta_3}{\partial y}$$

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{B-8}$$

$$= \frac{\partial u}{\partial \eta_1} \cdot \frac{\partial \eta_1}{\partial y} + \frac{\partial u}{\partial \eta_2} \cdot \frac{\partial \eta_2}{\partial y} + \frac{\partial u}{\partial \eta_3} \cdot \frac{\partial \eta_3}{\partial y}$$

$$+ \frac{\partial v}{\partial \eta_1} \cdot \frac{\partial \eta_1}{\partial x} + \frac{\partial v}{\partial \eta_2} \cdot \frac{\partial \eta_2}{\partial x} + \frac{\partial v}{\partial \eta_3} \cdot \frac{\partial \eta_3}{\partial x}$$

Therefore, the three global components of strain are given in terms of the local dimensionless coordinate system as

$$\epsilon_x = \frac{1}{2A} \Big[b_1 u_1 + (b_2 - 2b_3\eta_2 - 2b_2\eta_3)\, u_2$$

$$+ (b_3 - 2b_2\eta_3 - 2b_3\eta_2)\, u_3 + (4b_3\eta_2 + 4b_2\eta_3)\, u_4 \Big]$$

$$\epsilon_y = \frac{1}{2A} \Big[a_1 v_1 + (a_2 - 2a_3\eta_2 - 2a_2\eta_3)\, v_2$$

$$+ (a_3 - 2a_2\eta_3 - 2a_3\eta_2)\, v_3 + (4a_3\eta_2 + 4a_2\eta_3)\, v_4 \Big]$$

$$\gamma = \frac{1}{2A} \Big[a_1 u_1 + (a_2 - 2a_3\eta_2 - 2a_2\eta_3)\, u_2 \tag{B-9}$$

$$+ (a_3 - 2a_2\eta_3 - 2a_3\eta_2)\, u_3 + (4a_3\eta_2 + 4a_2\eta_3)\, u_4$$

$$+ b_1 v_1 + (b_2 - 2b_3\eta_2 - 2b_2\eta_3)\, v_2$$

$$+ (b_3 - 2b_2\eta_3 - 2b_3\eta_2)\, v_3 + (4b_3\eta_2 + 4b_2\eta_3)\, v_4 \Big]$$

Expression for Strain Energy

The total strain energy stored in the element of constant thickness t is

$$\Phi = \frac{t}{2} \int [\epsilon]^T \; [c][\epsilon] \cdot dA \qquad \text{(B-10)}$$

In this case, $[c]$ is a 3 x 3 matrix of material properties and $[\epsilon]$ is a 3 x 1 matrix of the three components of strain which are linear functions of space. A closed form expression for the evaluation of a product of two linear functions has been developed and is given as

$$\int f \cdot g \cdot dA = \frac{A}{12} \langle f_1 f_2 f_3 \rangle \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}, \qquad \text{(B-11)}$$

where the subscripts 1, 2 and 3 indicate the values of the functions evaluated at the corners of the triangular element.

The application of this integration formula to Eq. 10 results in the following expression for the strain energy in terms of the strains at the corner nodal points:

$$\Phi = \frac{At}{24} \begin{bmatrix} \bar{\epsilon}_x^T & \bar{\epsilon}_y^T & \bar{\gamma}^T \end{bmatrix} \begin{bmatrix} C_{11}Q & C_{12}Q & C_{13}Q \\ C_{21}Q & C_{22}Q & C_{23}Q \\ C_{31}Q & C_{32}Q & C_{33}Q \end{bmatrix} \begin{bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma} \end{bmatrix} \qquad \text{(B-12)}$$

where the corner strain submatrices are defined as

$$\bar{\epsilon}_x = \begin{Bmatrix} \epsilon_{x1} \\ \epsilon_{x2} \\ \epsilon_{x3} \end{Bmatrix} \; ; \quad \bar{\epsilon}_y = \begin{Bmatrix} \epsilon_{y1} \\ \epsilon_{y2} \\ \epsilon_{y3} \end{Bmatrix} \; ; \quad \bar{\gamma} = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{Bmatrix} \qquad \text{(B-13a,b,c)}$$

and

$$Q = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \text{(B-14)}$$

The evaluation of Eq. 9 results in the following expression for corner strains in terms of global displacements at the four element nodal points:

$$\begin{bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma} \end{bmatrix} = \begin{bmatrix} U & \cdot \\ \cdot & V \\ V & U \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \text{(B-15)}$$

where the submatrices are defined as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \qquad \text{(B-16a, b)}$$

$$U = \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 & \cdot \\ b_1 & b_2 - 2b_3 & -b_3 & 4b_3 \\ b_1 & -b_2 & b_3 - 2b_2 & 4b_2 \end{bmatrix} \qquad \text{(B-16c)}$$

$$V = \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 & \cdot \\ a_1 & a_2 - 2a_3 & -a_3 & 4a_3 \\ a_1 & -a_3 & a_3 - 2a_2 & 4a_2 \end{bmatrix} \qquad \text{(B-16d)}$$

The substitution of Eq. 15 into Eq. 12 yields the following equation for the strain energy of the element:

$$\Phi = \frac{1}{2} [u^T \; v^T] \begin{bmatrix} k \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \qquad \text{(B-17)}$$

With the strain energy written in this form it is apparent that $[k]$ is the element stiffness matrix and is given by

$$[k] = \frac{At}{12} \begin{bmatrix} U^T & \cdot & V^T \\ & & \\ \cdot & V^T & U^T \end{bmatrix} \begin{bmatrix} C_{11}Q & C_{12}Q & C_{13}Q \\ C_{21}Q & C_{22}Q & C_{23}Q \\ C_{31}Q & C_{32}Q & C_{33}Q \end{bmatrix} \begin{bmatrix} U & \cdot \\ \cdot & V \\ V & U \end{bmatrix} \qquad (B-18)$$

within the computer program given in this report, the submatrices U and V are formed and then the element stiffness matrix is formed directly in a series of operations which minimize programming effort and optimizes computer execution time.

APPENDIX C. COMPUTER PROGRAM USAGE

Purpose

The purpose of this computer program is to determine the time-dependent displacements and stresses within two-dimensional plane strain structures of arbitrary shape. The effects of displacement or stress boundary conditions are included.

Input Data

The first step in the dynamic analysis of a two-dimensional plane strain structure is to select a finite element representation of the cross-section of the body. Elements and nodal points are then numbered in two numerical sequences, each starting with one. The following group of punched cards numerically define the two-dimensional structure to be analyzed.

A. IDENTIFICATION CARD - (72H)

Columns 1 to 72 of this card contain information to be printed with results.

B. CONTROL CARD (6I5, 3F10.0)

Columns	1-5	Number of nodal points (200 maximum)
	6-10	Number of elements (180 maximum)
	11-15	Number of different materials (12)
	16-20	Number of time steps (no limit)
	21-25	Number of intervals for which we want to have the results printed
	26-30	Number of load cards (100 maximum)
	31-40	Damping coefficient
	41-50	Damping coefficient
	51-60	Time increment

C. MATERIAL PROPERTY INFORMATION

The following card must be supplied for each different material (I5, 3F10.0)

Columns	1-5	Material identification - any number from 1 to 12
	6-15	Modulus of elasticity
	16-25	Poisson's ratio or the area of a bar element
	26-35	Mass density of material

D. NODAL POINT CARDS (I5, F5.0, 6F10.0)

One card for each nodal point with the following information

Columns	1-5	Nodal point number
	6-10	Boundary condition code "k"
	11-20	x-ordinate
	21-30	y-ordinate
	31-40	Coefficient which affects horizontal loads
	41-50	Coefficient which affects vertical loads
	51-60	Arrival time of loads
	61-70	Initial condition coefficient (I)

If $I = 0$ the program generates zero initial conditions for this nodal point

If $I \neq 0$ it is necessary to add another card with the following information (4E20.10)

Columns	1-20	Initial x-displacement
	21-40	Initial y-displacement
	41-60	Initial x-velocity
	61-80	Initial y-velocity

Specifications for code "k"

If

k = 00	load in the x-direction load in the y-direction
k = 10	zero displacement in the x-direction load in the y-direction
k = 01	load in the x-direction zero displacement in the y-direction
k = 11	zero displacement in the x-direction zero displacement in the y-direction

Nodal point cards must be in numerical sequence. If cards are omitted, the omitted nodal points are generated at equal intervals along a straight line between the defined nodal points. The initial displacements and velocities and the arrival time of loads are set equal to zero, while the boundary condition code and the coefficients for horizontal and vertical loads are set equal to the values on the last card.

E. LOAD IDENTIFICATION (I5)

Columns 1-5 Load code, Q

If Q = 0, loads will be specified

If Q ≠ 0, load cards represent ground accelerations and the values of F_x and F_y at the nodes are set equal to the nodal masses within the program.

F. LOAD CARDS (3F10.0)

One card for each time with the following information

Columns	1-10	Time
	11-20	Horizontal load or acceleration
	21-30	Vertical load or acceleration

G. ELEMENT CARDS

One card for each element (6I5)

Columns	1-5	Element number	
	6-10	Nodal point I	The maximum difference in nodal point numbers is 12
	11-15	Nodal point J	
	16-20	Nodal point K	
	21-25	Nodal point L	
	26-30	Material identification	

For a right-hand coordinate system, order nodal points counter-clockwise around.

Element cards must be in element number sequence. If element cards are omitted, the program automatically generates the omitted information by incrementing by one the preceding I, J, K and L. The material identification for the generated cards is set equal to the corresponding value on the last card. The last element card must always be supplied. One-dimensional bar elements are identified by a nodal point numbering sequence of the form I, J, J, I.

H. OUTPUT INFORMATION

The following information is developed and printed by the program:

1. Reprint of input data
2. Nodal point displacements as a function of time
3. Stresses at the center of each element as a function of time

APPENDIX - D

COMPUTER PROGRAM LISTING

```
      PROGRAM MAIN(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      REAL MASS
      COMMON NUMNP,NUMEL,NUMMAT,N,VOL,MTYPE,DELT,NT,NPRINT,NP,TT,NNN,Q,
     1MMM,LL,C0,C1,C2,C3,R(200),Z(200),T(200),CH(200),CV(200),MASS(200),
     2X0(400),X1(400),X2(400),CODE(200),P(3,100),AA(400),BB(400)
      COMMON/MATARG/YMOD(12),ENU(12),HED(12),RO(12),ALFA,BETA
      COMMON/ELEARG/IX(180,5),EPS(180)
      COMMON/LS4ARG/I,J,K,S(10,10),C(3,3),D(3,3),PP(10),LM(4)
      COMMON/SYMARG/MBAND,NEQ,B(400),A(400,26)
C***********************************************************************
C     READ AND PRINT OF CONTROL INFORMATION AND MATERIAL PROPERTIES
C***********************************************************************
   50 READ  (5,1000) HED,NUMNP,NUMEL,NUMMAT,NT,NPRINT.NP,ALFA,BETA,DELT
      WRITE (6,2000) HED,NUMNP,NUMEL,NUMMAT,DELT,NT,ALFA,BETA
   56 DO 59 M=1,NUMMAT
      READ (5,1001) MTYPE,YMOD(MTYPE),ENU(MTYPE),RO(MTYPE)
      WRITE (6,2011) MTYPE,YMOD(MTYPE),ENU(MTYPE),RO(MTYPE)
   59 CONTINUE
C***********************************************************************
C     READ AND PRINT OF NODAL POINT DATA
C***********************************************************************
      WRITE (6,2004)
      L=0
   60 READ (5,1002) N,CODE(N),R(N),Z(N),CH(N),CV(N),T(N),Q
      IF (Q) 65,75,65
   75 X0(2*N-1)=0.0
      X0(2*N)=0.0
      X1(2*N-1)=0.0
      X1(2*N)=0.0
      GO TO 85
   65 READ (5,1006) X0(2*N-1),X0(2*N),X1(2*N-1),X1(2*N)
   85 NL=L+1
      ZX=N-L
      DR=(R(N)-R(L))/ZX
      DZ=(Z(N)-Z(L))/ZX
   70 L=L+1
      IF(N-L) 100,90,80
   80 CODE(L)=CODE(L-1)
      R(L)=R(L-1)+DR
      Z(L)=Z(L-1)+DZ
      X0(2*L-1)=0.0
      X0(2*L)=0.0
      X1(2*L-1)=0.0
      X1(2*L)=0.0
      T(L)=0.0
      CH(L)=CH(L-1)
      CV(L)=CV(L-1)
      GO TO 70
   90 WRITE (6,2002) (K,CODE(K),R(K),Z(K),X0(2*K-1),X0(2*K),X1(2*K-1) ,
     1X1(2*K),T(K),CH(K),CV(K),K=NL,N)
      IF(NUMNP-N) 100,110,60
  100 WRITE (6,2009) N
      CALL EXIT
  110 CONTINUE
```

```
C*****************************************************************************
C        READ AND PRINT OF LOAD DATA
C*****************************************************************************
      READ (5,1005) Q
      WRITE (6,2007)
      DO 550 M=1,NP
  550 READ(5,1004) (P(K,M),K=1,3)
      WRITE (6,2005) ((P(K,M),K=1,3),M=1,NP)
C*****************************************************************************
C        READ AND PRINT OF ELEMENT PROPERTIES
C*****************************************************************************
      WRITE (6,2001)
      N=0
  130 READ (5,1003) M,(IX(M,I),I=1,5)
  140 N=N+1
      IF (M-N) 170,170,150
  150 IX(N,1)=IX(N-1,1)+1
      IX(N,2)=IX(N-1,2)+1
      IX(N,3)=IX(N-1,3)+1
      IX(N,4)=IX(N-1,4)+1
      IX(N,5)=IX(N-1,5)
  170 WRITE (6,2003) N,(IX(N,I),I=1,5)
      IF (M-N) 180,180,140
  180 IF (NUMEL-N) 190,190,130
  190 CONTINUE
C*****************************************************************************
C        CALCULATE DAMPING CONSTANTS
C*****************************************************************************
      C0=1.5*ALFA/DELT+1.5/DELT**2
      C1=1.0+1.5*BETA/DELT
      C1=1.0/C1
      C2=C0*C1
      C3=ALFA-C2*BETA
C*****************************************************************************
C     DETERMINE BAND WIDTH
C*****************************************************************************
      J=0
      DO 340 N=1,NUMEL
      DO 340 I=1,4
      DO 325 L=1,4
      KK=IABS(IX(N,I)-IX(N,L))
      IF (KK-J) 325,325,320
  320 J=KK
  325 CONTINUE
  340 CONTINUE
      MBAND=2*J+2
      NEQ=2*NUMNP
      NA=NEQ*MBAND
      IF(NA-10400)350,350,400
  400 WRITE(6,2012)NA
      CALL EXIT
C*****************************************************************************
C     FORM EFFECTIVE STIFFNESS MATRIX
C*****************************************************************************
  350 CALL STIFF
```

```
C*******************************************************************
C     TRIANGULARIZE STIFFNESS MATRIX
C*******************************************************************
      CALL SYMSOL (1)
C*******************************************************************
C       FORM EFFECTIVE LOAD VECTOR
C*******************************************************************
      LL=1
      TT=0.0
      DO 500 NNN=1,NT
      TT=TT+DELT
      DO 900 I=1,NEQ
      AA(I)=1.5*B(I)/DELT**2+3.0*X1(I)/DELT+2.0*X2(I)
  900 BB(I)=1.5*B(I)/DELT+2.0*X1(I)+DELT*X2(I)
      MMM=NNN-LL*NPRINT
      CALL LOAD
C*******************************************************************
C     CDLCULATE DISPLACEMENTS
C*******************************************************************
      CALL SYMSOL (2)
      DO 800 I=1,NEQ
      B(I)=C1*B(I)+C1*BETA*BB(I)
      ACC=1.5*B(I)/DELT**2-AA(I)
      X0(I)=X0(I)+DELT*X1(I)+5.0*DELT**2*X2(I)/12.0+ACC*DELT**2/12.0
      X1(I)=X1(I)+0.25*DELT*(ACC+3.0*X2(I))
      X2(I)=0.5*(ACC+X2(I))
  800 B(I)=X0(I)
      IF(MMM) 500,20,500
   20 LL=LL+1
      WRITE (6,2006) TT
      WRITE (6,2008) (N,B(2*N-1),B(2*N),N=1,NUMNP)
C*******************************************************************
C     COMPUTE STRESSES
C*******************************************************************
      CALL STRESS
  500 CONTINUE
C*******************************************************************
      GO TO 50
C*******************************************************************
```

```
1000 FORMAT (12A6/6I5,3F10.0)
1001 FORMAT (I5,3F10.0)
1002 FORMAT  (I5,F5.0,6F10.0)
1003 FORMAT (6I5)
1004 FORMAT (3F10.0)
1005 FORMAT (I5)
1006 FORMAT (4E20.10)
2000 FORMAT (1H1 12A6/
    1 30H0 NUMBER OF NODAL POINTS------ I4 /
    2 30H0 NUMBER OF ELEMENTS---------- I4 /
    3 30H0 NUMBER OF DIFF. MATERIALS--- I4 /
    4 30H0 TIME INCREMENT-------------- F10.5/
    5 30H0 NUMBER OF CYCLES------------ I4 /
    6 30H0 DAMPING COEFFICIENT ALFA---- F10.5 /
    7 30H0 DAMPING COEFFICIENT BETA---- F10.5 )
2001 FORMAT (49H1ELEMENT NO.       I       J       K       L     MATERIAL  )
2002 FORMAT (I6, F11.2,2F10.3,4E15.6,3F10.4)
2003 FORMAT (1I13,4I6,1I12)
2004 FORMAT (89H1                                                    INITIAL  DISPL
    1ACEMENTS         INITIAL  VELOCITIES/
    2122HONODAL POINT TYPE  X-ORD.     Y-ORD.          X0              Y0
    3            X10             Y10        TIME LOAD     CH          CV)
2005 FORMAT (3F15.7)
2006 FORMAT (8H1TIME T=F8.4/52HONODAL POINT       X-DISPLACEMENT        Y-
    1DISPLACEMENT)
2007 FORMAT  (40H1         TIME          X-LOAD           Y-LOAD)
2008 FORMAT (I15,2F20.6)
2009 FORMAT (26H0NODAL POINT CARD ERROR N= I5)
2011 FORMAT (17H0MATERIAL NUMBER= I3,3H,E= F16.6, 4H.NU= F16.6,4H.RO=
    1F16.6)
2012 FORMAT (38H0MATRIX A EXCEEDS ALLOWABLE STORAGE A= I5)
     END
```

```
      SUBROUTINE STIFF
      REAL MASS
      COMMON NUMNP,NUMEL,NUMMAT,N,VOL,MTYPE,DELT,NT,NPRINT,NP,TT,NNN,Q,
     1MMM,LL,C0,C1,C2,C3,R(200),Z(200),T(200),CH(200).CV(200),MASS(200),
     2X0(400),X1(400),X2(400).CODE(200),P(3,100),AA(400),BB(400),
     3SIG(10),XC,YC,ST(3,10)
      COMMON/MATARG/YMOD(12),FNU(12),HED(12),RO(12),ALFA,BETA
      COMMON/ELEARG/IX(180,5),EPS(180)
      COMMON/LS4ARG/I,J,K,S(10,10),C(3,3),D(3,3),PP(10),LM(4)
      COMMON/SYMARG/MBAND,NEQ,B(400),A(400,26)
      DO 40 I=1,200
   40 MASS(I)=0.0
      DO 50 I=1,400
      DO 50 J=1,26
   50 A(I,J)=0.0
      DO 30 I=1,3
      DO 30 J=1,10
   30 ST(I,J)=0.0
      DO 210 N=1,NUMEL
C************************************************** ONE-D
   90 IF (IX(N,3)-IX(N,2)) 92,91,92
   91 CALL ONED
      GO TO 165
C************************************************** ONE-D
   92 CALL QUAD
      IF (VOL) 164,164,165
  164 WRITE (6,2003) N
      CALL EXIT
C***********************************************************************
C             ADD ELEMENT STIFFNESS TO TOTAL STIFFNESS
C***********************************************************************
  165 ELMASS=VOL*RO(MTYPE)/4.0
      DO 166 I=1,4
      K=IX(N,I)
      MASS(K)=MASS(K)+ELMASS
  166 LM(I)=2*IX(N,I)-2
      DO 200 I=1,4
      DO 200 K=1,2
      II=LM(I)+K
      KK=2*I-2+K
      DO 200 J=1,4
      DO 200 L=1,2
      JJ=LM(J)+L-II+1
      LL=2*J-2+L
      IF(JJ) 200,200,175
  175 A(II,JJ)=A(II,JJ)+S(KK,LL)
  200 CONTINUE
  210 CONTINUE
      IF (Q) 230,250,230
  230 DO 270 N=1,NUMNP
      CH(N)=MASS(N)
  270 CV(N)=MASS(N)
```

```
C***********************************************************************
C     MODIFY STIFFNESS MATRIX FOR ZERO DISPLACEMENTS
C***********************************************************************
  250 DO 600 N=1,NUMNP
      KK=CODE(N)
       KD=10
      DO 600 M=1,2
      IF (KK-KD) 580,550,550
  550 NX=2*N-2+M
      DO 575 J=2,MBAND
      A(NX,J)=0.0
       NN=NX-J+1
      IF (NN) 575,575,570
  570 A(NN,J)=0.0
  575 CONTINUE
      A(NX,1)=1.0
      KK=KK-KD
  580 KD=KD/10
  600 CONTINUE
C***********************************************************************
C     CALCULATE INITIAL ACCELERATION
C***********************************************************************
      DO 100 I=1,NEQ
      X2(I)=0.0
  100 B(I)=X0(I)+BETA*X1(I)
      DO 400 I=1,NEQ
      K=NEQ-I+1
      IF (K-MBAND) 300,300,350
  350 K=MBAND
  300 DO 320 J=1,K
      IJ=I+J-1
  320 X2(I)=X2(I)+A(I,J)*B(IJ)
      IF (I-MBAND) 340,360,360
  360 L=MBAND-1
      GO TO 450
  340 L=I-1
      IF(L) 400,400,450
  450 II=I
      DO 430 J=1,L
      II=II-1
  430 X2(I)=X2(I)+A(II,J+1)*B(II)
  400 CONTINUE
      TT=-DELT
      CALL LOAD
      DO 650 M=1,NUMNP
      K=2*M
      X2(K-1)=(B(K-1)-X2(K-1))/MASS(M)-ALFA*X1(K-1)
      X2(K)=(B(K)-X2(K))/MASS(M)-ALFA*X1(K)
      B(K-1)=X0(K-1)
  650 B(K)=X0(K)
```

```
C*********************************************************************************
C      FORM EFFECTIVE STIFFNESS MATRIX
C*********************************************************************************
       DO 700 M=1,NUMNP
       A(2*M-1,1)=A(2*M-1,1)+C2*MASS(M)
   700 A(2*M,1)=A(2*M,1)+C2*MASS(M)
       RETURN
  2003 FORMAT (26HONEGATIVE AREA ELEMENT NO. I4)
       END
```

```
      SUBROUTINE ONED
      REAL MASS
      COMMON NUMNP,NUMEL,NUMMAT,N,VOL,MTYPE,DELT,NT,NPRINT,NP,TT,NNN,Q,
     1MMM,LL,C0,C1,C2,C3,R(200),Z(200),T(200),CH(200),CV(200),MASS(200),
     2X0(400),X1(400),X2(400),CODE(200),P(3,100),AA(400),BB(400),
     3SIG(10),XC,YC,ST(3,10)
      COMMON/MATARG/YMOD(12),FNU(12),HED(12),RO(12),ALFA,BETA
      COMMON/ELEARG/IX(180,5),EPS(180)
      COMMON/LS4ARG/I,J,K,S(10,10),C(3,3),D(3,3),PP(10),LM(4)
      DO 100 I=1,8
      DO 100 J=1,8
  100 S(I,J)=0.0
C
      MTYPE=IX(N,5)
      I=IX(N,1)
      J=IX(N,2)
      DX=R(J)-R(I)
      DY=Z(J)-Z(I)
      XL=SQRT(DX**2+DY**2)
      COSA=DX/XL
      SINA=DY/XL
      COMM=YMOD(MTYPE)*FNU(MTYPE)/XL
      VOL=FNU(MTYPE)*XL
C
      S(1,1)=COSA*COSA*COMM
      S(1,2)=COSA*SINA*COMM
      S(1,3)=-S(1,1)
      S(1,4)=-S(1,2)
      S(2,1)=S(1,2)
      S(2,2)=SINA*SINA*COMM
      S(2,3)=-S(1,2)
      S(2,4)=-S(2,2)
      S(3,1)=S(1,3)
      S(3,2)=S(2,3)
      S(3,3)=S(1,1)
      S(3,4)=S(1,2)
      S(4,1)=S(1,4)
      S(4,2)=S(2,4)
      S(4,3)=S(3,4)
      S(4,4)=S(2,2)
C
      RETURN
C
      END
```

```
      SUBROUTINE LOAD
      REAL MASS
      COMMON NUMNP,NUMEL,NUMMAT,N,VOL,MTYPE,DELT,NT,NPRINT,NP,TT,NNN,Q,
     1MMM,LL,C0,C1,C2,C3,R(200),Z(200),T(200),CH(200),CV(200),MASS(200),
     2X0(400),X1(400),X2(400),CODE(200),P(3,100),AA(400),BB(400)
      COMMON/SYMARG/MBAND,NEQ,B(400),A(400,26)
      N=1
  100 TAU=TT-T(N)+DELT
      IF (TAU) 50,150,150
  150 K=1
   60 IF(TAU.GE.P(1,K).AND.TAU.LT.P(1,K+1)) GO TO 200
      K=K+1
      GO TO 60
  200 D=P(1,K+1)-P(1,K)
      DH=P(2,K+1)-P(2,K)
      DV=P(3,K+1)-P(3,K)
      DT=TAU-P(1,K)
      FH=P(2,K)+DT*DH/D
      FV=P(3,K)+DT*DV/D
  160 IF(CH(N).EQ.0.0.AND.CV(N).EQ.0.0) GO TO 50
      IF(CH(N)) 300,250,300
  250 B(2*N-1)=0.0
  350 B(2*N)=CV(N)*FV
      GO TO 500
  300 B(2*N-1)=CH(N)*FH
      IF(CV(N)) 350,400,350
   50 B(2*N-1)=0.0
  400 B(2*N)=0.0
  500 N=N+1
      IF (NUMNP -N) 130,120,120
  120 IF(T(N)-T(N-1)) 100,140,100
  140 IF (TAU) 50,160,160
  130 IF(TT+DELT) 230,270,230
  230 DO 650 M=1,NUMNP
      B(2*M-1)=B(2*M-1)+MASS(M)*(AA(2*M-1)+C3*BB(2*M-1))
  650 B(2*M)= B(2*M) + MASS(M)*(AA(2*M)+C3*BB(2*M))
  270 RETURN
      END
```

```
      SUBROUTINE STRESS
      REAL MASS
      COMMON NUMNP,NUMEL,NUMMAT,N,VOL,MTYPE,DELT,NT,NPRINT,NP,TT,NNN,Q,
     1MMM,LL,C0,C1,C2,C3,R(200),Z(200),T(200),CH(200),CV(200),MASS(200),
     2X0(400),X1(400),X2(400),CODE(200),P(3,100),AA(400),BB(400),
     3SIG(10),XC,YC,ST(3,10)
      COMMON/MATARG/YMOD(12),ENU(12),HED(12),RO(12),ALFA,BETA
      COMMON/ELEARG/IX(180,5),EPS(180)
      COMMON/LS4ARG/I,J,K,S(10,10),C(3,3),D(3,3),PP(10),LM(4)
      COMMON/SYMARG/MBAND,NEQ,B(400),A(400,26)
C***********************************************************************
C     COMPUTE ELEMENT STRESSSES
C***********************************************************************
      MPRINT=0
C
      DO 300 M=1,NUMEL
C
      N=M
      IX(N,5)=IABS(IX(N,5))
      MTYPE=IX(N,5)
      DO 50 I=1,6
   50 SIG(I)=0.0
C*************************************************** ONE-D
      IF (IX(N,3)-IX(N,2)) 90,80,90
   80 I=IX(N,1)
      J=IX(N,2)
      DX=R(J)-R(I)
      DY=Z(J)-Z(I)
      XL=SQRT(DX**2+DY**2)
      DU=B(2*J-1)-B(2*I-1)
      DV=B(2*J)-B(2*I)
      DL=DV*DY/XL +DU*DX/XL
      SIG(1)=DL*YMOD(MTYPE)/XL
      XC=0.0
      YC=0.0
      GO TO 255
C*************************************************** ONE-D
   90 DO 100 I=1,3
      DO 100 J=1,10
  100 ST(I,J)=0.0
C
      CALL QUAD
C
      DO 120 I=1,4
      II=2*I
      JJ=2*IX(N,I)
      PP(II-1)=B(JJ-1)
  120 PP(II)=B(JJ)
C
C
      DO 170 I=1,3
      D(I,1)=0.0
      DO 170 K=1,8
  170 D(I,1)=D(I,1)+ST(I,K)*PP(K)
```

```
C
      DO 180 I=1,3
      DO 180 K=1,3
  180 SIG(I)=SIG(I)+C(I,K)*D(K,1)
C
C*****************************************************************************
C     OUTPUT STRESSES
C*****************************************************************************
C
C     CALCULATE PRINCIPAL STRESSES
C
      CC=(SIG(1)+SIG(2))/2.0
      BB=(SIG(1)-SIG(2))/2.
      CR=SQRT(BB**2+SIG(3)**2)
      SIG(4)=CC+CR
      SIG(5)=CC-CR
      IF ((BB.EQ.0.0).AND.(SIG(3).EQ.0.0)) GO TO 255
      EPS(N)=ATAN2(SIG(3),BB)/2.
      SIG(6)=57.296*EPS(N)
C
  255 IF (MPRINT) 110,105,110
  105 WRITE (6,2000)
      MPRINT=50
  110 MPRINT=MPRINT-1
C
  305 WRITE (6,2001)  N,XC,YC,(SIG(I),I=1,6)
  300 CONTINUE
C
  320 RETURN
C
 2000 FORMAT (7H1EL.NO.  7X 1HX  7X 1HY 4X 8HX-STRESS 4X 8HY-STRESS 3X
     1 9HXY-STRESS 2X 10HMAX-STRESS 2X 10HMIN-STRESS 7H  ANGLE )
 2001 FORMAT (I7,2F8.2,1P5E12.4,0P1F7.2)
C
      END
```

```
      SUBROUTINE EDLST(N1,N2,N3)
      REAL MASS
      COMMON NUMNP,NUMEL,NUMMAT,N,VOL,MTYPE,DELT,NT,NPRINT,NP,TT,NNN,Q,
     1MMM,LL,C0,C1,C2,C3,X(200),Y(200),T(200),CH(200),CV(200),MASS(200),
     2X0(400),X1(400),X2(400),CODE(200),P(3,100),AA(400),BB(400),
     3SIG(10),XC,YC,ST(3,10)
      COMMON/LS4ARG/I,J,K,S(10,10),C(3,3),D(3,3),PP(10),LM(4)
      DIMENSION BA(3,2),U(3,4),V(3,4),UV(3,4,2)
      EQUIVALENCE (UV,U),(UV(13),V)
      TH=1.0
      BA(1,1)=Y(J)-Y(K)
      BA(2,1)=Y(K)-Y(I)
      BA(3,1)=Y(I)-Y(J)
      BA(1,2)=X(K)-X(J)
      BA(2,2)=X(I)-X(K)
      BA(3,2)=X(J)-X(I)
      AREA=(X(J)*BA(2,1)+X(I)*BA(1,1)+X(K)*BA(3,1))/2.
      IF (AREA) 400,400,100
  100 VOL=VOL+AREA
      COMM=TH/(48.*AREA)
      C11=C(1,1)*COMM
      C12=C(1,2)*COMM
      C13=C(1,3)*COMM
      C22=C(2,2)*COMM
      C23=C(2,3)*COMM
      C33=C(3,3)*COMM
C
C
      DO 150 NN=1,2
      D1=BA(1,NN)
      D2=BA(2,NN)
      D3=BA(3,NN)
      UV(1,1,NN)=D1
      UV(2,1,NN)=D1
      UV(3,1,NN)=D1
      UV(1,2,NN)=D2
      UV(2,2,NN)=D2-2.*D3
      UV(3,2,NN)=-D2
      UV(1,3,NN)=D3
      UV(2,3,NN)=-D3
      UV(3,3,NN)=D3-2.*D2
      UV(1,4,NN)=0.
      UV(2,4,NN)=4.*D3
  150 UV(3,4,NN)=4.*D2
      LM(1)=N1
      LM(2)=N2
      LM(3)=N3
      LM(4)=9
C
      COMM=8.*AREA
      DO 310 I=1,4
      II=LM(I)
C
      UU=(U(2,I)+U(3,I))/COMM
```

```
      VV=(V(2,I)+V(3,I))/COMM
      ST(1,II)=ST(1,II)+UU
      ST(2,II+1)=ST(2,II+1)+VV
      ST(3,II)=ST(3,II)+VV
      ST(3,II+1)=ST(3,II+1)+UU
C
      SUM=U(1,I)+U(2,I)+U(3,I)
      SUM1=SUM+U(1,I)
      SUM2=SUM+U(2,I)
      SUM3=SUM+U(3,I)
      SUM=V(1,I)+V(2,I)+V(3,I)
      SVM1=SUM+V(1,I)
      SVM2=SUM+V(2,I)
      SVM3=SUM+V(3,I)
      DO 300 J=1,4
      JJ=LM(J)
      UQU=U(1,J)*SUM1+U(2,J)*SUM2+U(3,J)*SUM3
      VQU=V(1,J)*SUM1+V(2,J)*SUM2+V(3,J)*SUM3
      VQV=V(1,J)*SVM1+V(2,J)*SVM2+V(3,J)*SVM3
      UQV=U(1,J)*SVM1+U(2,J)*SVM2+U(3,J)*SVM3
      S(II,JJ)=S(II,JJ)+ C11*UQU+C13*(VQU+UQV)+C33*VQV
      S(II+1,JJ+1)=S(II+1,JJ+1)+ C22*VQV+C23*(VQU+UQV)+C33*UQU
      S(II,JJ+1)=S(II,JJ+1)+ C23*VQV+C13*UQU+VQU*C12+C33*UQV
  300 S(JJ+1,II)=S(II,JJ+1)
  310 CONTINUE
C
  400 RETURN
C
      END
```

```
      SUBROUTINE QUAD
      REAL MASS
      COMMON NUMNP,NUMEL,NUMMAT,N,VOL,MTYPE,DELT,NT,NPRINT,NP,TT,NNN,Q,
     1MMM,LL,C0,C1,C2,C3,R(200),Z(200),T(200),CH(200),CV(200),MASS(200),
     2X0(400),X1(400),X2(400),CODE(200),P(3,100),AA(400),BB(400),
     3SIG(10),XC,YC,ST(3,10)
      COMMON/MATARG/YMOD(12),FNU(12),HED(12),RO(12),ALFA,BETA
      COMMON/ELEARG/IX(180,5),EPS(180)
      COMMON/LS4ARG/I,J,K,S(10,10),C(3,3),D(3,3),PP(10),LM(4)
      I=IX(N,1)
      J=IX(N,2)
      K=IX(N,3)
      L=IX(N,4)
      MTYPE=IX(N,5)
C***********************************************************************
C             FORM STRESS-STRAIN RELATIONSHIP
C***********************************************************************
      F=YMOD(MTYPE)/(1.0+FNU(MTYPE))/(1.-2.*FNU(MTYPE))
      C(1,1)=F*(1.-FNU(MTYPE))
      C(1,2)=F*FNU(MTYPE)
      C(1,3)=0.0
      C(2,1)=C(1,2)
      C(2,2)=C(1,1)
      C(2,3)=0.0
      C(3,1)=0.0
      C(3,2)=0.0
      C(3,3)=0.5*F*(1.-2.*FNU(MTYPE))
C***********************************************************************
C             FORM QUADRILATERAL STIFFNESS MATRIX
C***********************************************************************
      DO 100 II=1,10
      PP(II)=0.0
      DO 100 JJ=1,10
  100 S(II,JJ)=0.0
C
      VOL=0.0
      I=IX(N,1)
      J=IX(N,2)
      K=IX(N,4)
      CALL FDLST(1,3,7)
      I=IX(N,3)
      J=IX(N,4)
      K=IX(N,2)
      XC=(R(J)+R(K))/2.
      YC=(Z(J)+Z(K))/2.
      CALL FDLST(5,7,3)
C***********************************************************************
C       ELIMINATE CENTER POINT
C***********************************************************************
      DO 500 K=1,2
      IH=10-K
      ID=IH+1
      DO 500 I=1,IH
      S(ID,I)=S(ID,I)/S(ID,ID)
```

```
      DO 490 J=1,3
  490 ST(J,I)=ST(J,I)-ST(J,ID)*S(ID,I)
      DO 500 J=1,IH
  500 S(J,I)=S(J,I)-S(J,ID)*S(ID,I)
  130 RETURN
C
      END
```

```
      SUBROUTINE SYMSOL (KKK)
      COMMON/SYMARG/MM,NN,B(400),A(400,26)
C
      GO TO (1000,2000),KKK
C
C     REDUCE MATRIX
C
 1000 DO 280 N=1,NN
      DO 260 L=2,MM
      C=A(N,L)/A(N,1)
      I = N+L-1
      IF(NN-I) 260,240,240
  240 J=0
      DO 250 K=L,MM
      J=J+1
  250 A(I,J)=A(I,J)-C*A(N,K)
  260 A(N,L)=C
  280 CONTINUE
      GO TO 500
C
C     REDUCE VECTOR
C
 2000 DO 290 N=1,NN
      DO 285 L=2,MM
      I=N+L-1
      IF(NN-I) 290,285,285
  285 B(I)=B(I)-A(N,L)*B(N)
  290 B(N)=B(N)/A(N,1)
C
C     BACK SUBSTITUTION
C
      N=NN
  300 N = N-1
      IF(N) 350,500,350
  350 DO 400 K=2,MM
      L = N+K-1
      IF(NN-L) 400,370,370
  370 B(N) = B(N) - A(N,K) * B(L)
  400 CONTINUE
      GO TO 300
C
  500 RETURN
C
      END
```