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Trees, Refining, and Combinatorial Characteristics

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ABSTRACT OF THE DISSERTATION

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The analysis of trees and the study of cardinal characteristics are both of historical and contemporary importance to set theory. In this thesis we consider each of these topics as well as questions relating to (almost) disjoint refinements. We show how structural information about trees and other similar objects is revealed by investigating the determinacy of certain two player games played on them. The games we investigate have classical analogues and can be used to prove structural dichotomies and related results. We also use them to find generalizations of the topological notions of perfectness and scatteredness for spaces like 2^κ and $P_\kappa\lambda$ and form connections to when a submodel is e.g. "T-guessing" for a certain tree T. Questions surrounding generalizations of the cardinal characteristics $\mathfrak{t}$ (the tower number), $\mathfrak{h}$ (the distributivity number), and $\mathbf{non}(\mathcal{M})$ (the uniformity number for category) in particular are considered. For example, we ask whether or not $\mathfrak{h}(\kappa)$ can be defined in a reasonable way. We give several impediments. Generalizations of a combinatorial characterization of $\mathbf{non}(\mathcal{M})$ in terms of countably matching families of functions become central for our investigation, and we show how characteristics relating to these generalizations can be manipulated by forcing. Similarly, the question of in which contexts can outer models can add strongly disjoint functions is considered. While Larson has shown [45] that this is possible with a proper forcing at ω_1, and it is a corollary of a result of Abraham and Shelah [2] that it is consistently impossible at ω_2, we note with Radin forcing that if κ has a sufficient amount of measurable reflection, then it can be done at κ. Turning to the theory of disjoint refinements, we generalize a recent result of Brendle [62], and independently Balcar and Pazák [4], that any time a real is added in an extension, the set of ground model reals can be almost disjointly refined to the setting of adding subsets of κ, and consider related topics.

Introduction

Set theory is a broad and rapidly expanding branch of mathematics, including as any sufficiently developed field does a host of sub-fields and specializations, recurring themes and unifying ideas, sophisticated arguments, and deep connections to diverse areas. The investigation of independence phenomena in various contexts—that is of mathematical statements which cannot be settled in a particular mathematical theory—remains of central importance to the field. An important tool in the investigation of independence phenomena is forcing, a technique discovered by Cohen [14] and subsequently developed by Solovay and many others for producing "generic" sets. Cohen's first use of forcing was to generate new reals, resulting in the consistency of the negation of the Continuum Hypothesis (CH) from the consistency of the Zermelo-Fraenkel axioms with choice, ZFC. Because Gödel had previously established the consistency of the CH with ZFC [32], ZFC does not settle the CH. However, it would be a woefully inadequate characterization to equate this sort of investigation with contemporary set theory. Indeed, some of the most exacting and subtle arguments and techniques of the modern theory have been used to establish results of ZFC. For example, while it was realized early on that much of the combinatorics of regular cardinals is easily manipulated by forcing (see for example Easton's theorem [21]), with the development of Shelah's pcf theory (initiated in [58] but spread over many papers) especially, the combinatorics of singular cardinals has been seen to more immutable. This thesis includes independence results established via forcing as well as ZFC results which are proven by purely combinatorial means. However, we will see in some cases that while a particular result is true in ZFC, the proof we find may easily be viewed through a forcing-theoretic lens.

Trees—partial orders where the set of predecessors to every element is well-ordered—are a central combinatorial object. In chapter 1 we investigate trees and a natural generalization, $P_\kappa\lambda$-forests, through the lens of two-player games. In particular, we formulate games that can be played on these objects, some of which may be viewed as natural generalizations of the classical cut-and-choose $*$-Game of Davis [18], and use these games to extract structural information about these objects, give new ways of looking at old constructions, and provide a platform for independence results. These things are often accomplished by investigating when one of the players has a winning strategy in a game of a particular length. When the games are played to lengths $\delta > \omega$, it is often the case that they can be non-determined, in that neither player has a winning strategy. We will see that while sometimes games may appear similarly formulated, they can behave quite differently, and that looking from the perspective of one game over another may be advantageous. For example, it is not difficult to observe through the lens of our cut-and-choose type game that only an inaccessible is required to establish the consistency of Väänänen's generalized Cantor-Bendixson theorem for the space ${}^{\omega_1}\omega_1$, which was originally achieved with a measurable in [69]. Because the bodies of trees and forests are closed in certain topologies, trees and forests can sometimes be said to code closed subsets in these settings. Questions and topics motivated by this observation, such as the generalized Cantor-Bendixson theorem just mentioned, will provide a starting point to our investigation in chapter 1, but much of the material may be viewed independently of this context.

In chapter 2 we consider several related topics largely surrounding (almost) disjoint refinements. The theory of disjoint refinements has an extensive literature and questions regarding refinements by countable sets in particular, starting of course with the Boolean algebra $P(\omega)/<\omega$, have received considerable attention. In this chapter we investigate some analogous questions for spaces like $P(\kappa)/<\kappa$. For example, for many years it was known that adding certain types of reals, e.g. Cohen reals or more generally unbounded reals, means that the collection of ground model reals can be almost disjointly refined in the extension [33]. In the past decade, two independent proofs, one due to Brendle [62] and the other to Balcar and Pazák [4], were given to show that the set of ground model reals can be almost disjointly refined as long as *any* real is added in the extension. Here we give partial generalizations to this result for adding subsets of κ. For example, for κ regular we show that if $2^{<\kappa}=\kappa$ and the extension $V[G]$ is obtained by a κ-strategically closed forcing, then $([\kappa]^\kappa)^V$ can be almost disjointly refined in $V[G]$. If additionally e.g. $2^\kappa=\kappa^+$, then generally in any outer model M adding a new subset of κ (not adding subsets of κ of smaller size), $([\kappa]^\kappa)^V$ can be almost disjointly refined in M.

Throughout this thesis several generalized cardinal characteristics appear. For example, in a minor proposition in chapter 1 the (un)bounding number for κ, $\mathfrak{b}(\kappa)$, is used to compare two different topologies defined relative to κ over $P_\mu\lambda$. For a comprehensive outline of much of the classical material on cardinal characteristics on ω, a good resource is [10]. The study of generalized cardinal characteristics is a rapidly expanding one (see e.g. [17],[29],[28],[12],[56] and many others), and in the final two chapters topics which can be motivated partially by this study are investigated.

In chapter 3 a starting point is the generalization to spaces like $[\kappa]^\kappa/<\kappa$ of two cardinal characteristics, the tower number $\mathfrak{t}$ and the distributivity number $\mathfrak{h}$ for subsets of ω. $\mathfrak{t}$ and $\mathfrak{h}$ are characteristics that one could say are both related to how structures determined by the processes of thinning infinite subsets of ω look. Here we show that these characteristics behave quite differently when looking at subsets of κ for $\kappa>\omega$ (and related settings) and consider the question of whether or not $\mathfrak{h}(\kappa)$ can be defined in a useful way. In unpublished works it has been asserted that it can be [44], [12], however we believe that this question is as yet unresolved. Impediments to this include the facts that if κ is uncountable and $\mathrm{cf}(\kappa)>\omega$, then there exist countably many open dense sets in $[\kappa]^\kappa / < \kappa$ with empty intersection, while if $\mathrm{cf}(\kappa) = \omega$ there exist ω_1-many open dense sets in $[\kappa]^\kappa / < \kappa$ with empty intersection. These facts were proven independently by Balcar and Simon [5] using purely combinatorial arguments, while the arguments we give here use forcing terminology. Several related topics are also considered. For example, we give other impediments, prove the existence of base trees in certain situations, and prove some ZFC results about e.g. towers in $[\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2$ for κ singular, where $\mathcal{I}_2$ is the ideal of bounded subsets of κ. The existence of base trees in certain situations is used in proving some of the results about disjoint refinements in chapter 2.

A guiding topic in chapter 4 is the consideration of generalizations of a purely combinatorial characterization of $\mathbf{non}(\mathcal{M})$, the uniformity number for category, due to Bartoszyński [8], namely the minimal cardinality of a collection of functions $F \subseteq {}^\omega\omega$ which is countably matching. That is such that for every $g \in {}^\omega\omega$ there exits $f \in F$ with $|\{n \in \omega : f(n) = g(n)\}| = \omega$. We give some independence results about generalizations of this quantity. The question of when cardinal-preserving outer models can add functions in ${}^\kappa\kappa$ which are strongly disjoint from all ground model functions becomes central, and we will see that it depends highly on the cardinal. For example, it was shown by Larson [45] that there's a proper forcing which adds a function in ${}^{\omega_1}\omega_1$ which is modulo-finite disjoint from all ground model functions, while it is a corollary of a result of Abraham and Shelah [2] that it's consistent (forcing over L) that there's a collection of functions in ${}^{\omega_2}\omega_2$ of size ω_2 which is ω_1-matching in any outer model not collapsing ω_2. While this sort of consistent behavior may occur for small accessible cardinals other than ω_1, we show here that for certain large cardinals it cannot. For example, if κ has a sufficient amount of measurable reflection to ensure that performing Radin forcing at κ preserves its regularity, then Radin forcing adds a function $f \in {}^\kappa\kappa$ which is modulo-finite disjoint from every ground model function in ${}^\kappa\kappa$. We also consider briefly how guessing principles weaker than $\diamondsuit$, such as $\clubsuit$ and $\dagger$, interact with these function families.

The preliminary sections of each chapter often include notation and some classical background. While the chapters can largely be read separately, it is the case that some of the preliminary notation fixed in earlier chapters will be used in later chapters. Not being able to investigate all potentially interesting questions which arose during the writing of this thesis, at the end of each chapter is included an "unconsidered directions" section. Being listed as a question in one of these sections does not indicate difficulty or central importance, and the reader will undoubtedly notice some obvious unlisted questions. However, these sections may still be useful in providing future directions.

Chapter 1

Trees and forests via games

1.1 Background and initial observations

In this section we give some essential background definitions, introduce and fix notation, and review relevant aspects of the classical setting as well as some straightforward generalizations to other settings. Proofs of standard results are often not given. However, this chapter is meant to be largely self-contained so some proofs of standard results, especially when they provide contextual support, will be explained in at least some detail. A standard reference for any background material which has been omitted is [35].

1.1.1 Notation and conventions

- Typically $\theta, \kappa, \lambda, \mu$, etc. denote cardinals, while $\alpha, \beta, \gamma, \delta$, etc. denote ordinals, however this is not an absolute convention. Ord will denote the class of all ordinal. κ will typically denote a regular cardinal in this chapter, and this distinction may sometimes be omitted.

- In this chapter $P_\kappa\lambda$ indicates $[\lambda]^{<\kappa}$, that is the collection of subsets of λ of size $< \kappa$, not necessarily those that are, for example, transitive below λ, as it is sometimes used to indicate.

- ${}^\delta\gamma$ denotes the set of functions from δ to γ. We will sometimes use, e.g. 2^κ and ${}^\kappa 2$ interchangeably—the former often, however, also refers to $|{}^\kappa 2|$.

- If $x \in {}^\gamma\delta$, we write $\text{length}(x) = \text{lh}(x) = \gamma$. Often in this chapter $\delta = 2$. We call ${}^\kappa 2$ the κ-Cantor space and ${}^\kappa\kappa$ the κ-Baire space. An element of either may be referred to as a κ-real.

- If $f : X \to Y$ is a function from X to Y and $A \subseteq X$, we denote the pointwise image of A by f as $f''A = \{y \in Y : \text{there exists } x \in X \text{ such that } f(x) = y\}$. For $y \in Y$, we use $f^{-1}[y] = \{x \in X : f(x) = y\}$ to denote the complete f-preimage of y.

- For an ordinal α, let $\text{cf}(\alpha)$ denote the cofinality of α. Let $\text{succ}(\alpha)$ denote the collection of successor ordinals in α, that is $\text{succ}(\alpha) = \{\gamma \in \alpha : \text{there exists } \beta \in \alpha \text{ with } \beta + 1 = \gamma\} = \{\gamma \in \alpha : \text{cf}(\gamma) = 1\}$. Let $\lim(\alpha)$ or $\text{acc}(\alpha)$ denote the set of limit ordinals in α, that is $\lim(\alpha) = \text{acc}(\alpha) = \{\gamma \in \alpha : \text{cf}(\gamma) \geq \omega\}$. More generally, if $X \subseteq \text{Ord}$ then $\text{acc}(X) = \lim(X) = \{\gamma \in X : X \cap \gamma \subseteq \gamma \text{ is cofinal in } \gamma\}$ and $\text{nacc}(X) = \{\gamma \in X : X \cap \gamma \subseteq \gamma \text{ is bounded in } \gamma\}$. These need not be sets if X isn't, but the meaning here is clear. For μ a regular cardinal, let $\text{Cof}(\mu)$ denote the class of ordinals of cofinality μ. Here of course then by $\text{Cof}(\mu) \cap \kappa$, for example, we mean $\{\alpha \in \kappa : \text{cf}(\alpha) = \mu\}$. We frequently use interval notation as is typical for certain sets of ordinals, for example if $\beta \in \gamma$, $[\beta, \gamma) = \{\delta : \beta \leq \delta < \gamma\}$, $(\beta, \gamma) = \{\delta : \beta \in \delta \in \gamma\}$, etc.

1.1.2 General definitions

Definition 1.1.1. Say that a cardinal κ is a (strongly) inaccessible cardinal, or κ is an inaccessible cardinal, if and only if κ is a regular uncountable limit cardinal such that for every cardinal $\mu < \kappa$, $2^\mu < \kappa$. We often omit the "strongly" and just write inaccessible. Sometimes-called "weakly" inaccessible cardinals will be referred to as (uncountable) regular limit cardinals.

Definition 1.1.2. Say that κ is a weakly compact cardinal, or κ is weakly compact, if and only if κ is uncountable and satisfies the partition property $\kappa \to (\kappa)^2$, that is the property that for every partition of $[\kappa]^2$ into two pieces, there is a homogeneous set of size κ, i.e. every $F : [\kappa]^2 \to \{0, 1\}$ is constant on some $[A]^2$ for $A \in [\kappa]^\kappa$.

Definition 1.1.3. Let $\mathbb{P}$ be a forcing poset and let κ be an uncountable cardinal. Say that $\mathbb{P}$ has the κ-chain condition ($\mathbb{P}$ is κ-c.c.) if and only if all antichains in $\mathbb{P}$ have size $< \kappa$. If $\kappa = \omega_1$ we often say that $\mathbb{P}$ has the countable chain condition, and write $\mathbb{P}$ is c.c.c.. Say that $\mathbb{P}$ is κ-closed if and only if every decreasing sequence of conditions in $\mathbb{P}$ of length $< \kappa$ has a lower bound. In the particular case where $\kappa = \omega_1$, we also say that $\mathbb{P}$ is countably closed, or σ-closed. Say that $\mathbb{P}$ is (κ, ∞)-distributive if and only if forcing with $\mathbb{P}$ adds no new sequences of ordinals of length $< \kappa$. If $\mathbb{P}$ is separative, this is equivalent to saying that every collection of $< \kappa$-many open dense subsets of $\mathbb{P}$ has nonempty intersection. Say that $\mathbb{P}$ is κ-directed closed if and only if every directed set of size $< \kappa$ in $\mathbb{P}$ has a lower bound.

Definition 1.1.4. Let $\mathbb{P}$ be a forcing poset and α be an ordinal. Define a two-player game $G_\alpha(\mathbb{P})$ as follows. Players Even and Odd take turns playing conditions $p_\beta \in \mathbb{P}$ for every stage $\beta \in \alpha$. Even plays p_β at all limit stages (including $\beta = 0$, where Even must play $p_0 = \mathbf{1}_\mathbb{P}$) as well as at all stages of the form $\beta + 2$, while Odd plays at all other stages. At round β, p_β is a legal move if and only if $p_\beta \leq p_\gamma$ for every $\gamma \in \beta$. Say that Even wins a run of the game if she can play legally at stage β for every $\beta \in \alpha$.

Definition 1.1.5. Let $\mathbb{P}$ be a forcing poset and δ be an ordinal. Say that $\mathbb{P}$ is $(< \delta)$-strategically closed if and only if Even has a winning strategy in $G_\alpha(\mathbb{P})$ for every $\alpha \in \delta$ and say that $\mathbb{P}$ is δ-strategically closed if and only if Even has a winning strategy in $G_\delta(\mathbb{P})$.

Fact (Folklore) 1.1.6. Let κ be a regular cardinal. Then all κ-directed closed posets are

κ-closed, all κ-closed posets are κ-strategically closed, all κ-strategically closed posets are $(<\kappa)$-strategically closed, and all $(<\kappa)$-strategically closed posets are (κ, ∞)-distributive.

Definition 1.1.7. For sets X, Y, let $\mathrm{Fn}(X, Y, <\mu)$ denote the forcing consisting of partial functions from X to Y of cardinality $<\mu$, ordered by reverse inclusion. If $\kappa \leq \lambda$ are cardinals, let $\mathrm{Col}(\kappa, \lambda)$ denote the forcing consisting of partial functions from κ to λ of size $<\kappa$. In the case (typically) where λ is inaccessible, let $\mathrm{Col}(\kappa, <\lambda)$ denote the Lévy collapse. That is, $\mathrm{Col}(\kappa, <\lambda) = \{p : |p| < \kappa,\ p \text{ is a function, } \mathrm{dom}(p) \subseteq \lambda \times \kappa, \text{ and for every } \langle \xi, \beta \rangle \in \mathrm{dom}(p),\ p(\langle \xi, \beta \rangle) \in \xi\}$.

Note 1.1.8. In the case where κ is regular and $\lambda > \kappa$ is inaccessible, $\mathrm{Col}(\kappa, <\lambda)$ is κ-closed and λ-c.c., so it is straightforward to see that cardinals below κ are preserved, all cardinals in $[\kappa, \lambda)$ are collapsed to κ, and λ is κ^+ in the extension. For more basic facts about this and the other forcings in 1.1.7, see for example [35].

Definition 1.1.9. For an infinite cardinal κ and a stationary subset $S \subseteq \kappa^+$, $\Diamond_{\kappa^+}(S)$ asserts the existence of a sequence $\langle A_\alpha : \alpha \in S \rangle$ such that for every $\alpha \in S$, $A_\alpha \subseteq \alpha$, and if $Z \in P(\kappa^+)$ then $\{\alpha \in S : Z \cap \alpha = A_\alpha\}$ is stationary. If S is omitted, we usually take S to be $\lim(\kappa^+)$.

Definition 1.1.10. Let κ be regular. We say that a set M with $|M| = \kappa$ is internally approachable of length κ if and only if there exists a sequence $\langle M_\alpha : \alpha \in \kappa \rangle$ such that $|M_\alpha| < \kappa$ for every α, if $\alpha \in \beta \in \kappa$ then $M_\alpha \subseteq M_\beta$, for every $\alpha \in \kappa$, $\langle M_\gamma : \gamma \in \alpha \rangle \in M$, and $M = \bigcup_{\alpha \in \kappa} M_\alpha$. In the context of submodels of H_θ with $M \prec H_\theta$ we will assume that $M_\alpha \prec H_\theta$ for every α and we will also talk about an internally approachable chain, which will refer to the $\langle M_\alpha : \alpha \in \kappa \rangle$ sequence itself (with $M_\alpha \prec H_\theta$), and in this case we will insist that it is continuous and may insist that for every $\alpha \in \kappa$, $\langle M_\gamma : \gamma \in \alpha \rangle \in M_{\alpha+1}$.

Definition 1.1.11. Let κ be regular. If we weaken the requirement in 1.1.10 that for every $\alpha \in \kappa$, $\langle M_\gamma : \gamma \in \alpha \rangle \in M$, and instead insist only that for every $\alpha \in \kappa$, $M_\alpha \in M$, then M is said to be internally unbounded, and if additionally we insist that $\langle M_\alpha : \alpha \in \kappa \rangle$ is continuous, then say that M is internally club.

1.1.3 Topology, box topologies, and trees

Definition 1.1.12. A topological space (X, τ) is perfect if and only if it contains no isolated points. We also say that (X, τ) is dense-in-itself in this case. A subset of a topological space is perfect if and only if it is closed and contains no isolated points. We often suppress the collection of open sets τ when discussing a topological space and write simply, for example, X. We will only deal with Hausdorff topological spaces.

Definition 1.1.13. A topological space X is scattered if and only if every nonempty subspace contains an isolated point. A subset of a topological space is scattered if and only if every nonempty subset of this set contains an isolated point.

Definition 1.1.14. Let X be a topological space and $E \subseteq X$. Say that E is κ-compact if and only if every open cover of E in X has a subcover of size $< \kappa$. So for example, compactness is ω-compactness and Lindelöff-ness is ω_1-compactness.

Notation 1.1.15. Let X be a topological space with $A \subseteq X$. Denote the topological closure of A as $\overline{A}$, that is the intersection of all closed sets $A' \subseteq X$ with $A \subseteq A'$. Let A^c denote the complement of A in X, $X \setminus A$. We will sometimes also use $\overline{A}$ to denote a set other than the topological closure of A, however each time this is done it will be clear what is meant.

Definition 1.1.16. If $\langle (X_\alpha, \tau_\alpha) : \alpha \in \lambda \rangle$ is a sequence of topological spaces, then for $\kappa \leq \lambda^+$, the κ-box topology over $\prod_{\alpha\in\lambda} X_\alpha$ is the topology for which $\mathcal{O} = \{\prod_{\alpha\in\lambda} O_\alpha : O_\alpha \in \tau_\alpha \text{ and } |\{\alpha : O_\alpha \neq X_\alpha\}| < \kappa\}$ is a base. Typically we are interested in the case where λ is indeed a cardinal. The reader interested in cardinal invarients (which are considered in other parts of this thesis) associated with κ-box products in a general setting may consult [15].

Observations 1.1.17.

1. The ω-box topology over $\langle (X_\alpha, \tau_\alpha) : \alpha \in \lambda \rangle$ is the usual product topology.

2. The λ^+-box topology over $\langle (X_\alpha, \tau_\alpha) : \alpha \in \lambda \rangle$ is the (full) box topology.

3. If κ is regular then the κ-box topology over 2^κ is generated by basic open sets of the form $O_s = \{x \in 2^\kappa : x \upharpoonright \mathrm{lh}(s) = s\}$ for $s \in {}^\alpha 2$ with $\alpha \in \kappa$.

4. The κ box topology over 2^κ is zero-dimensional. That is, it has a basis of clopen sets (sets which are both closed and open).

Definition 1.1.18. By identifying each $x \in P(\kappa)$ with its characteristic function, we can define topologies over $P(\kappa)$ via topologies over ${}^\kappa 2$, in particular we can define the κ-box topology over $P(\kappa)$. Specifically, for each $x \in P(\kappa)$, let $\chi_x \in {}^\kappa 2$ denote the characteristic function of x. Then say that $A \subseteq P(\kappa)$ is open if and only if $\{\chi_x : x \in A\} \subseteq {}^\kappa 2$ is open. We will sometimes conflate e.g. ${}^\kappa 2$ and $P(\kappa)$, for example by writing things like for $x \in {}^\kappa 2$ and $x(\beta) = 1$, $\beta \in x$.

Definition 1.1.19. A tree is a partial order $\mathbb{T} = \langle T, \leq_T \rangle$ such that for every $s \in T$, $\overline{s} = \{s' \in T : s' <_T x\}$ is well-ordered by $\leq_T$. We will usually confuse $\mathbb{T}$ with T when $\leq_T$ is understood. Let the height of s in T indicate the order type of $\overline{s}$, denoted $\mathrm{ht}_T(s)$. Let the α^{th} level of T be denoted by $\mathrm{Lev}_\alpha(T) = \{s \in T : \mathrm{ht}_T(s) = \alpha\}$. Let the height of the tree T be the supremum of the height of its nodes, that is $\mathrm{ht}(T) = \sup\{\mathrm{ht}_T(s) : s \in T\}$.

Definition 1.1.20. Let $(T, \leq_T)$ be a tree. Say that a tree $(T_1, \leq_1)$ is a subtree of $(T, \leq_T)$ if and only if $T_1 \subseteq T$ and $\leq_1 = \leq_T \cap (T_1 \times T_1)$, that is if $\leq_1$ is the ordering induced by $\leq_T$ over T_1.

Definition 1.1.21. Let $(T_1, \leq_1)$ and $(T_2, \leq_2)$ be trees. Say that an injection $F : T_1 \to T_2$ is a tree embedding, or just an embedding, if and only if f preserves the tree structure of T_1, i.e. if and only if for every $s_1, s_2 \in T_1$, $s_1 \leq_1 s_2$ if and only if $f(s_1) \leq_2 f(s_2)$. Note that if F is an embedding from T_1 into T_2 then $F''T_1$ is a subtree of T_2 which is isomorphic to T_1.

For concreteness and notational consistency and simplicity, in what follows we almost exclusively deal with trees which are subsets of ${}^{<\kappa}2$, the complete binary tree of height κ, and phrase definitions and results in these terms. So for example, if $T \subseteq {}^{<\kappa}2$ is a tree,

$\mathrm{Lev}_\alpha(T) = \{s \in T : \mathrm{lh}(s) = \alpha\}$ and $\mathrm{ht}(T) = \sup\{\alpha \in \kappa : \mathrm{Lev}_\alpha(T) \neq \emptyset\}$. This limitation, for example, means that we will often formally not be working with trees with nodes that have more than two successors, or which have splitting at limits. Trees which do not have splitting at limits are sometimes said to have unique limits. Examples of naturally occurring trees which don't have binary splitting are subtrees of ${}^{<\kappa}\kappa$, which can sometimes be said to code closed subsets of the κ-Baire space. However, suitable modifications to definitions, arguments, and results via for example embedding considerations will typically be possible to give and evident to the reader.

Definition 1.1.22. Let $T \subseteq {}^{<\kappa}2$ be a tree with $s \in T$. Let $T \restriction s = \{s' \in T : \exists \alpha \in \mathrm{lh}(s)$ such that $s' = s \restriction \alpha$ or $\mathrm{lh}(s') \geq \mathrm{lh}(s)$ and $s' \restriction \mathrm{lh}(s) = s\}$ denote the natural restriction of T to s. If $\alpha \in \kappa$, let $T \restriction \alpha = \{s \in T : \mathrm{lh}(s) \in \alpha\} \subseteq {}^{<\alpha}2$ denote the restriction of T up to level α.

Definition 1.1.23. Let $T \subseteq {}^{<\kappa}2$ be a tree. Say that T is pruned if and only if for every $s \in T$ and $\alpha \in \kappa$ with $\mathrm{lh}(s) \in \alpha$, there exists $s' \in T$ with $\mathrm{lh}(s') = \alpha$ and $s' \restriction \mathrm{lh}(s) = s$.

Definition 1.1.24. Let $T \subseteq {}^{<\kappa}2$ be a tree. Say that a node $s \in T$ is cofinally splitting in T if and only if for every $\alpha > \mathrm{lh}(s)$, there exists $\{s', s''\} \subseteq T$ such that $\mathrm{lh}(s') > \alpha$, $\mathrm{lh}(s'') > \alpha$, $s' \restriction \alpha = s'' \restriction \alpha$, $s' \restriction \mathrm{lh}(s) = s'' \restriction \mathrm{lh}(s) = s$, and neither s' nor s'' is an initial segment of the other node. Say that a tree T is cofinally splitting if and only if every $s \in T$ is cofinally splitting in T. Cofinally splitting trees are necessarily pruned, of course. Say that T is (locally) everywhere splitting if and only if for each $s \in T$, $s \frown 0 \in T$ and $s \frown 1 \in T$. If T is everywhere splitting and pruned, then T is of course cofinally splitting.

Definition 1.1.25. Let $T \subseteq {}^{<\kappa}2$ be a tree. Denote the body of T by $[T] = \{b \in {}^{\kappa}2 : b \restriction \alpha \in T$ for every $\alpha \in \kappa\}$. If $b \in {}^{\kappa}2$ is such that for every $\alpha \in \kappa$, $b \restriction \alpha \in T$, then we say that b is a branch through T. If $c \in {}^{\gamma}2$ for some $\gamma \leq \kappa$ is such that for every $\alpha \in \gamma$, $c \restriction \alpha \in T$, then c is said to be a path through T. So with this nomenclature, not all paths are branches (only the cofinal paths are branches), but every branch is a (cofinal) path. Sometimes we may also write partial branch for path.

Definition 1.1.26. Let $T \subseteq {}^{<\kappa}2$ be a tree. For $\{s, s'\} \subseteq T$, define the meet of s and s', that is $s \wedge s'$, to be the unique node of maximal length at most $\min\{\mathrm{lh}(s), \mathrm{lh}(s')\}$ in T which is comparable with both s and s'. There is such a node because T has no splitting at limit levels.

Definition 1.1.27. Let $T \subseteq {}^{<\kappa}2$ be a tree. Say that T is a κ-tree if and only if $0 < |\mathrm{Lev}_\alpha(T)| < \kappa$ for every $\alpha \in \kappa$. To avoid confusion, note that some authors instead use the term κ-tree (in particular for $\kappa = \omega_1$) to indicate just a tree of cardinality and height κ (see for example [38]).

Definition 1.1.28. Say that a regular uncountable cardinal κ has the tree property if and only if every κ-tree has a branch. A κ-tree with no branches is called a κ-Aronszajn tree.

Definition (Todorčević [66]) 1.1.29. Let $T \subseteq {}^{<\kappa}2$ be a tree. A function $f : T \to T$ is regressive if and only if for every $s \in T$, $f(s) = s \restriction \alpha$ for some $\alpha \in \mathrm{lh}(s)$. T is called a special tree if and only if there exists a regressive function f on T such that for every $s \in T$, $f^{-1}[s]$, that is the complete f-preimage of s, is the union of $(< \kappa)$-many antichains in T (collections of incomparable nodes).

Fact 1.1.30. If $\kappa = \mu^+$ for some cardinal μ, then $T \subseteq {}^{<\kappa}2$ is special if and only if T is the union of $(< \kappa)$-many antichains in T. This is also equivalent in this case to the existence of a function $g : T \to \mu$ such that for any path $c \subseteq T$, $g \restriction c$ is injective.

Definition 1.1.31. Let $T \subseteq {}^{<\kappa}2$ be a tree and κ be a regular non-inaccessible cardinal. Say that T is a κ-Kurepa tree (sometimes we ignore the κ if it is clear from context) if and only if T is a κ-tree with $|[T]| \geq \kappa^+$. If κ is inaccessible, the complete binary tree of height κ would satisfy this requirement, so additionally in this case we insist that for every $\alpha \in \kappa$, $|\mathrm{Lev}_\alpha(T)| \leq |\alpha| + \omega$.

Definition 1.1.32. Let $T \subseteq {}^{<\kappa}2$ be a tree and κ be a regular non-inaccessible cardinal. Say that T is a weak κ-Kurepa tree (again sometimes we ignore the κ if it is clear from context) if and only if for every $\alpha \in \kappa$, $|\mathrm{Lev}_\alpha(T)| \leq \kappa$ and $|[T]| \geq \kappa^+$.

Definition 1.1.33. Let $T \subseteq {}^{<\kappa}2$ be a tree. Say that T is a Jech-Kunen tree if and only if T is a κ-Kurepa tree and $|[T]| \in [\kappa^+, 2^\kappa)$. That is, Jech-Kunen trees are Kurepa trees whose bodies have cardinality strictly between κ and 2^κ.

Fact (Erdös-Tarski [22]) 1.1.34. Let κ be a regular uncountable cardinal. Then κ is inaccessible and has the tree property if and only if κ is weakly compact.

Proposition 1.1.35. If κ is regular and $T \subseteq {}^{<\kappa}2$ is a tree, $[T] \subseteq 2^\kappa$ is closed in the κ-box topology. On the other hand, if $E \subseteq 2^\kappa$ is closed in the κ-box topology then the tree induced by E, $T_E = \{s \in {}^{<\kappa}2 : \exists \alpha \in \kappa, x \in E \text{ such that } x \restriction \alpha = s\} \subseteq {}^{<\kappa}2$ is a tree and $[T_E] = E$.

Proof. Let $x \notin [T]$. Then for some α, $x \restriction \alpha \notin T$. Then $O_{x\restriction\alpha} \cap [T] = \emptyset$, so $[T]^c$ is open and $[T]$ is closed. Next, for any $E \subseteq 2^\kappa$ it is clear that T_E is a tree and $E \subseteq [T_E]$. Suppose E is closed. If $x \in [T_E]$ then for every $\alpha \in \kappa$, $x \restriction \alpha \in T_E = \{s \in {}^{<\kappa}2 : \exists \alpha \in \kappa, x \in E \text{ such that } x \restriction \alpha = s\}$. That is, $x \restriction \alpha = y_\alpha \restriction \alpha$ for some $y_\alpha \in E$. If $x \notin E$, then for some $A \in P_\kappa \kappa$ and $f \in {}^A2$, $O_f = \{x \in {}^\kappa 2 : x(\alpha) = f(\alpha) \text{ for every } \alpha \in A\} \cap E = \emptyset$. However because κ is regular, for some $\gamma \in \kappa$, $\sup\{A\} \subseteq \gamma$, but $x \in E \cap O_{x\restriction\gamma} \subseteq E \cap O_f$, a contradiction. $\square$

Definition 1.1.36. Let $T \subseteq {}^{<\kappa}2$ be a tree and κ be regular. Say that T codes a closed subset of 2^κ if and only if $T_{[T]} = T$. This is true if and only if for every $s \in T$, there exists $x_s \in [T]$ such that $x_s \restriction \mathrm{lh}(s) = s$.

Observation 1.1.37. Let $T \subseteq {}^{<\kappa}2$ and $T' \subseteq {}^{<\kappa}2$ be trees coding closed subsets of 2^κ. If $[T] = [T']$ then $T = T'$.

Proposition 1.1.38. If κ is either ω or a weakly compact cardinal and $T \subseteq {}^{<\kappa}2$ is a tree, then the definitions for T being pruned and T coding a closed subset of κ are equivalent. This is not necessarily the case if κ is not weakly compact.

Proof. Suppose first that κ is either ω or a weakly compact cardinal. If $T \subseteq {}^{<\kappa}2$ is pruned then for any $s \in T$, $T \upharpoonright s$ is a κ-tree of height κ, so $[T \upharpoonright s] \neq \emptyset$. Thus the definitions for T being pruned and T coding a closed subset of κ are equivalent in this case. For examples of where this equivalence can fail for other κ, see 1.1.62 below. □

1.1.4 Sequential convergence

Definition 1.1.39. Let κ be a regular cardinal and X be a set. Say that $\overline{x} = \langle x_\alpha : \alpha \in \kappa\rangle \subseteq {}^X2$ converges if and only if $\operatorname{limsup}(\overline{x}) = \operatorname{liminf}(\overline{x}) = \lim(\overline{x})$. Identifying ${}^X2$ with $P(X)$, as usual $\operatorname{limsup}(\overline{x}) = \{x \in X : |\{\beta \in \kappa : x \in x_\beta\}| = \kappa\}$ and $\operatorname{liminf}(\overline{x}) = \{x \in X : |\{\beta \in \kappa : x \notin x_\beta\}| < \kappa\}$.

For κ regular, it is clear that $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq P(X)$ converges in the sense of 1.1.39 if and only if for every $x \in X$, taking $x_\alpha(x) = 0$ to mean $x \notin x_\alpha$ and $x_\alpha(x) = 1$ to mean $x \in x_\alpha$, there exists $\beta \in \kappa$ such that for every $\beta' \in [\beta, \kappa)$, $x_{\beta'}(x)$ is a constant, either 0 or 1. Because κ is regular, this means that for any $Y \in P_\kappa X$, there exists $\beta \in \kappa$ such that for every $\beta' \in [\beta, \kappa)$, $x_\beta \upharpoonright Y$ is constant. If $|X| = \kappa$, then in particular if $\langle y_\gamma : \gamma \in \kappa\rangle = X$ is an enumeration of X, all initial segments of X according to this enumeration are fixed. That is, for every $\alpha \in \kappa$ there exists $\gamma \in \kappa$ such that if $\beta, \beta' \geq \gamma$, $x_\beta \upharpoonright \alpha = x_\beta \cap \{y_\delta : \delta \in \alpha\} = x_{\beta'} \upharpoonright \alpha = x_{\beta'} \cap \{y_\delta : \delta \in \alpha\}$. This criterion of fixing all initial segments of X is clearly equivalent to convergence in the sense of 1.1.39, and is invariant under re-enumeration by regularity. To summarize, identifying X with κ in this setting, $\overline{x} = \langle x_\alpha : \alpha \in \kappa\rangle \subseteq {}^\kappa 2$ converges if and only if for every $\alpha \in \kappa$ there exists $\gamma \in \kappa$ such that if $\beta, \beta' \geq \gamma$, $x_\beta \upharpoonright \alpha = x_{\beta'} \upharpoonright \alpha$. In this case $\lim(\overline{x})$ can be described explicitly via first defining by recursion $f \in {}^\kappa\kappa$ by setting $f(\alpha)$ to be minimal with the property that for every $\beta, \beta' \geq f(\alpha)$, $x_\beta \upharpoonright (\alpha + 1) = x_{\beta'} \upharpoonright (\alpha + 1)$ and $f(\alpha) > f(\eta)$ for every $\eta \in \alpha$ and then defining $\lim(\overline{x}) \in {}^\kappa 2$ by setting $\lim(\overline{x})(\alpha) = x_{f(\alpha)}(\alpha)$. Note that $\lim(\overline{x}) \upharpoonright (\alpha + 1) = x_\delta \upharpoonright (\alpha + 1)$ for every $\delta \geq f(\alpha)$.

Proposition 1.1.40. Let κ be a regular cardinal. Then $E \subseteq 2^\kappa$ is closed in the κ-box topology if and only if every convergent sequence $\langle x_\alpha : \alpha \in \kappa \rangle \subseteq E$ converges to $x \in E$.

Proof. Suppose first that E is closed. Let $\langle x_\alpha : \alpha \in \kappa \rangle \subseteq E$ be a convergent sequence with limit x. Towards a contradiction, suppose $x \notin E$. Because E is closed and κ is regular, there exists $\gamma \in \kappa$ such that $O_{x \restriction \gamma} \cap E = \emptyset$. However, by definition $x_{f(\gamma)} \restriction (\gamma + 1) = x \restriction (\gamma + 1)$, and $x_{f(\gamma)} \in E$, a contradiction. Next, suppose that E is closed under convergent sequences. Let $x \notin E$. If for every $\alpha \in \kappa$ there existed $y_\alpha \in E$ such that $y_\alpha \restriction \alpha = x \restriction \alpha$, then $\langle y_\alpha : \alpha \in \kappa \rangle \subseteq E$ would be a convergent sequence in E with a limit outside of E (namely x). So there exists $\alpha \in \kappa$ so that for no $y \in E$ is $y \restriction \alpha = x \restriction \alpha$. That is, $O_{x \restriction \alpha} \cap E = \emptyset$. So E^c is open, and E is closed. □

1.1.5 Topological Cantor-Bendixson process

Definition 1.1.41. For X a topological space, let X_α denote the α^{th} derived set (or α^{th} Cantor-Bendixson derivative) of X. This is defined by recursion on α as follows. First, $X_0 = X$. For successors, $X_{\alpha+1} = \{x \in X_\alpha : x \text{ is a limit point of } X_\alpha\}$. By a limit point in this context, we mean that for every open set $O \subseteq X_\alpha$, if $x \in O$ then $O \cap X_\alpha \neq \{x\}$. For α a limit, $X_\alpha = \bigcap_{\beta \in \alpha} X_\beta$. There must exist a minimal α_0 such that $X_{\alpha_0} = X_{\alpha_0+1}$, and this α_0 is called the (Cantor-Bendixson) height of X, $\text{ht}_{CB}(X)$. Let $I_\alpha(X) = X_\alpha \setminus X_{\alpha+1}$. $I_\alpha(X)$ is called the α^{th} (Cantor-Bendixson) level of X. If $x \in I_\alpha$, then the Cantor-Bendixson rank of x, $\text{rank}_{CB}(x)$, is α.

Proposition 1.1.42. For every α, X_α is closed. If $\alpha_0 = \text{ht}_{CB}(X)$, then X_{α_0} has no isolated points, so X_{α_0} is a perfect subset of X. Furthermore, $X_{\alpha_0} = \emptyset$ if and only if X is scattered. More generally, $X \setminus X_{\alpha_0}$ is scattered, and in this way one may verify that every topological space can be written as the disjoint union of two sets, the perfect kernel of X, $\text{Ker}(X) = X_{\alpha_0}$, and the scattered part of X, $\text{Sc}(X) = X \setminus X_{\alpha_0}$.

Proof. It is not difficult to see that every X_α is closed, and X_{α_0} has no isolated points, so $\mathrm{Ker}(X)$ is perfect. Furthermore, if X is scattered then every $I_\alpha \neq \emptyset$, so $X_{\alpha_0} = \emptyset$. On the other hand, suppose $X_{\alpha_0} = \emptyset$, and assume towards a contradiction that we could find a nonempty $Z \subseteq X$ where Z had no isolated points. By induction, no points of Z would ever be removed in the Cantor-Bendixson process, so $Z \subseteq X_{\alpha_0}$. Thus $X_{\alpha_0} = \emptyset$ if and only if X is scattered. Also, $\mathrm{Sc}(X)$ is a scattered subset of X, because again any $A \subseteq X$ with no isolated points never has any points removed, i.e. $A \subseteq \mathrm{Ker}(X)$, so no such A could be a subset of $\mathrm{Sc}(X)$. □

1.1.6 Cantor-Bendixson process on trees $T \subseteq {}^{<\omega}2$ with two applications

We give here a slightly modified version of the typical (see [42], for example) Cantor-Bendixson process on trees $T \subseteq {}^{<\omega}2$ in order to make the connection with later material more apparent and to highlight the connection that these trees have with closed subsets of 2^ω. The modifications are not very substantial, but do lead to differences (for example 1.1.59 wouldn't necessarily hold with the typical version).

Definition 1.1.43. Let $T \subseteq {}^{<\omega}2$ be a tree. Define the pruned part of T to be the subtree of T formed by removing any node which does not have extensions to every level in ω. That is, $T' = \{s \in T : \mathrm{Lev}_n(T \restriction s) \neq \emptyset \text{ for every } n \in \omega\}$.

Proposition 1.1.44. Let $T \subseteq {}^{<\omega}2$ be a tree. Then if T' denotes the pruned part of T, T' is a pruned subtree of T.

Proof. Clearly T' is a subtree of T. For trees $T \subseteq {}^{<\omega}2$, being pruned is equivalent to not having terminal nodes, so suppose towards a contradiction that there exists a terminal node $s \in T'$, that is $s ^\frown 0 \notin T'$ and $s ^\frown 1 \notin T'$. Because $s \in T'$, $\mathrm{Lev}_n(T \restriction s) \neq \emptyset$ for every $n \in \omega$.

But then necessarily either $\mathrm{Lev}_n(T \restriction s \frown 0) \neq \emptyset$ for every $n \in \omega$ or $\mathrm{Lev}_n(T \restriction s \frown 1) \neq \emptyset$ for every $n \in \omega$, a contradiction. $\square$

Definition 1.1.45. If $T \subseteq {}^{<\omega}2$ is a tree, let T_α denote the α^{th} derived tree (or α^{th} Cantor-Bendixson derivative) of T. This is defined by recursion on α. First, let T_0 be the pruned part of T. So, $T_0 = T$ if and only if T codes a closed subset of 2^ω. For successors, first let $T'_{\alpha+1}$ be the collection of nodes in T_α which are cofinally splitting in T_α, that is $T'_{\alpha+1} = \{s \in T_\alpha : s \text{ is cofinally splitting in } T_\alpha\}$. Clearly $T'_{\alpha+1}$ is a tree. Then let $T_{\alpha+1}$ be the pruned part of $T'_{\alpha+1}$. For α limit, let $T_\alpha = \bigcap\limits_{\beta \in \alpha} T_\beta$. There must exist a minimal $\alpha_0 \in \omega_1$ such that $T_{\alpha_0} = T_{\alpha_0+1}$, and this is called the Cantor-Bendixson height of the tree T, $\alpha_0 = \mathrm{ht}_{CB}(T)$. If $T_{\alpha_0} \neq \emptyset$, we call T_{α_0} a splitting or perfect tree, while if $T_{\alpha_0} = \emptyset$ we call T a scattered tree. Let $\mathrm{Ker}(T) = T_{\alpha_0}$ denote the kernel of the tree T, and let $\mathrm{Sc}(T) = T \setminus T_{\alpha_0}$ denote the scattered part of the tree T.

Observation 1.1.46. Let $T \subseteq {}^{<\omega}2$ be a tree. Then $\mathrm{Ker}(T) \subseteq T$ is a cofinally splitting subtree of T and $\mathrm{Sc}(T)$ is a disjoint union of trees with varying roots formed by "upward cones" in T (if we imagine that T is growing upwards). This is because for $s, s' \in T$, if $s' \restriction \mathrm{lh}(s) = s$, and $s \in \mathrm{Sc}(T)$, then $s' \in \mathrm{Sc}(T)$. So we may say that $T = \mathrm{Ker}(T) \cup \mathrm{Sc}(T)$ is a decomposition of T into a cofinally splitting subtree (the kernel) and a scattered part.

Observation 1.1.47. Let $T \subseteq {}^{<\omega}2$ be a tree. Then T_α is a pruned tree for every ordinal α.

Proof. For $\alpha = 0$ or $\alpha = \gamma + 1$ for some ordinal γ, this is assured by construction. So, let α be a limit ordinal, and suppose (by recursion) that T_γ is pruned for each $\gamma \in \alpha$. We need to show that T_α is pruned. Fix $s \in T_\alpha$ and $n > \mathrm{lh}(s)$. Note that $\langle \mathrm{Lev}_n(T_\gamma \restriction s) : \gamma \in \alpha \rangle$ is an α-length $\subseteq$-decreasing sequence of finite sets with $\mathrm{Lev}_n(T_\alpha \restriction s) = \bigcap\limits_{\gamma \in \alpha} \mathrm{Lev}_n(T_\gamma \restriction s)$. Then if $\mathrm{Lev}_n(T_\alpha \restriction s) = \emptyset$, that is if there are no nodes in T_α on level n extending s, we must have that for some $\gamma \in \alpha$, $\mathrm{Lev}_n(T_\gamma \restriction s) = \emptyset$, which is a contradiction. $\square$

Proposition 1.1.48. Let $T \subseteq {}^{<\omega}2$ be a pruned tree. Then for every ordinal α, $[T_\alpha] = [T]_\alpha$. As a consequence, it is not then difficult to see that $\mathrm{ht}_{CB}(T) = \mathrm{ht}_{CB}([T])$, $\mathrm{Ker}([T]) =$

[Ker(T)], and Sc([T]) = [Sc(T)]. This final equality involves an abuse of notation as Sc(T) is not typically a tree. By [Sc(T)] we mean $\{b \in [T] : \exists n \in \omega \text{ such that } b \upharpoonright n \in \mathrm{Sc}(T)\}$, that is the set of branches through T which eventually include a node in Sc(T). In particular, $[T]$ is perfect if and only if $T_0 = T_{\mathrm{ht}_{CB}(T)} = T$ and $[T]$ is scattered if and only if $T_{\mathrm{ht}_{CB}(T)} = \emptyset = [T]_{\mathrm{ht}_{CB}([T])}$.

Proof. In 1.1.59 we prove this as well as the extension of the result to where κ can also be a weakly compact cardinal. □

Observation 1.1.49. As a corollary to this proposition, it is easy to see that $\alpha_0 = \mathrm{ht}_{CB}([T]) = \mathrm{ht}_{CB}(T) \in \omega_1$ because T is countable. However, one may also argue abstractly that scattered subsets of a topological space X are always of cardinality less than or equal to the weight of X, $w(X)$—that is the minimal cardinality of a basis for X's topology—and so the Cantor-Bendixson height of such a scattered subset must be less than $w(x)^+$. For example, if $E \subseteq 2^\omega$ is scattered then $|E| \leq \omega$ because the ω-box topology over 2^ω is second countable, witnessed by the countable collection $\{O_s : s \in {}^{<\omega}2\}$. Explicitly, let $E \subseteq 2^\omega$ with $|E| = \omega_1$. Let $E' = E \setminus \bigcup \mathcal{O}$, where $\mathcal{O} = \{O_s : s \in {}^{<\omega}2 \text{ and } |E \cap O_s| \leq \omega\}$. Because there are only countably many O_s's, $|E'| = \omega_1$. We argue that E' has no isolated points. Let $x \in E'$. For any O_s with $x \in O_s$, we find $y \in E' \cap O_s$ with $y \neq x$. By the definition of $\mathcal{O}$, $|E \cap O_s| = \omega_1$, so $|(E \cap O_s) \setminus \bigcup \mathcal{O}| = \omega_1$, and $(E \cap O_s) \setminus \bigcup \mathcal{O} = E' \cap (O_s \setminus \bigcup \mathcal{O})$, so there must exist some $y \neq x$ with $y \in E' \cap O_s$.

Theorem (Cantor-Bendixson [13]) 1.1.50. *Let $E \subseteq 2^\omega$ be closed in the ω-box topology. Then $E = Ker(E) \cup Sc(E)$, where $Ker(E)$, if nonempty, is a perfect subset of 2^ω of cardinality continuum and $Sc(E)$ is an at most countably infinite (scattered) subset of 2^ω.*

Proof. We have seen that perfect subsets of 2^ω are exactly those sets which can be written as the bodies of cofinally splitting trees $T \subseteq {}^{<\omega}2$. It is then not difficult to see that if $E \subseteq 2^\omega$ is perfect and nonempty, $|E| = |[T_E]| = 2^\omega$. Let $E \subseteq 2^\omega$ be closed. We know generally that $|\mathrm{Sc}(E)| \leq \omega$ and so for closed $E \subseteq 2^\omega$, $E = \mathrm{Ker}(E) \cup \mathrm{Sc}(E)$ is a partition of E into a countable scattered component and a perfect subset of 2^ω, which is necessarily of size 2^ω if it's nonempty. Note that closure is necessary here because for closed E, perfect subsets of E in the induced topology are, in fact, perfect subsets of 2^ω, but generally this is false. $\square$

Theorem (Mansfield [48]) 1.1.51. *Let $V \subseteq M$ be models of ZFC, $T \subseteq {}^{<\omega}2$ be a tree with $T \in V$, and $(T_{ht_{CB}(T)} = \emptyset)^V$. Then $([T])^V = ([T])^M$. That is, M cannot add branches to trees whose bodies are scattered in V.*

Proof. Suppose towards a contradiction that there exists $b \in ([T])^M \setminus ([T])^V$. Note that $T_\alpha^V = T_\alpha^M$ for every α, because the Cantor-Bendixson derivative process on trees as we have defined it is an absolute process. Working in M, we prove by induction that $b \in [T_\alpha]$ for every α, which is a contradiction if $T_{\mathrm{ht}_{CB}(T)} = \emptyset$. First, b is in the body of the pruned part of T, that is $b \in [T_0]$. For limit stages, $b \in [T_\alpha]$ if and only if $b \restriction n \in T_\alpha$ for every $n \in \omega$, if and only if $b \restriction n \in T_\beta$ for every $n \in \omega$, $\beta \in \alpha$, i.e. $b \in [T_\alpha]$ if $b \in [T_\beta]$ for every $\beta \in \alpha$. For successors, suppose $b \in [T_\alpha]$. Towards a contradiction, if $b \notin [T_{\alpha+1}]$, then if $T'_{\alpha+1}$ is the cofinally splitting part of T_α, $b \notin [T'_{\alpha+1}]$ and so for some $n \in \omega$, $b \restriction n \notin T'_{\alpha+1}$, which means that $b \restriction n$ is not cofinally splitting in T_α. Then for some $m \geq n$ no splitting along b occurs above m, that is there do not exist $s_1 \neq s_2 \in T_\alpha$ with $s_1 \restriction m = s_2 \restriction m = b \restriction m$ and neither s_1 nor s_2 an initial segment of the other. However, in this case $b \in V$ because it is definable from $\{b \restriction m, T_\alpha\} \subseteq V$. $\square$

Remark 1.1.52. If there exists $x \in ({}^\omega 2)^M \setminus V$, for any $T \subseteq {}^{<\omega}2$ a tree with $T \in V$, if $T_{\mathrm{ht}_{CB}(T)} \neq \emptyset$, then $|([T])^M \setminus ([T])^V| = 2^\omega$. That is, continuum-many branches are added to trees whose bodies contain perfect subsets. On the other hand, no branches are added to trees whose bodies are scattered.

Proof. Working in M, if $|{}^\omega 2| > (|{}^\omega 2|)^V$, then clearly $|[T] \setminus ([T])^V| = 2^\omega$. On the other hand, suppose $|{}^\omega 2| = (|{}^\omega 2|)^V$ and fix $y \in {}^\omega 2 \setminus ({}^\omega 2)^V$. For for each $x \in {}^\omega 2 \cap V$, define $x_y \in {}^\omega 2$

by $x_y(n) = x(n) + y(n)$ (modulo 2). Clearly $x_y \notin V$, and viewing x_y as a prescription for defining a branch through T in the natural way, distinct x's give rise to distinct x_y's, none of which can be in V, so M contains 2^ω many branches through T which aren't in V. □

1.1.7 Cantor-Bendixson process on trees $T \subseteq {}^{<\kappa}2$

Here we give a natural extension of the Cantor-Bendixson process given above for trees $T \subseteq {}^{<\omega}2$ to trees $T \subseteq {}^{<\kappa}2$.

Definition 1.1.53. Let $T \subseteq {}^{<\kappa}2$ be a tree. Define the pruned part of T to be the subtree of T formed by successively removing nodes which do not have extensions to every level in κ until a stabilization point is reached. That is, let T^α be defined by recursion so that $T^0 = T$, $T^{\alpha+1} = \{s \in T^\alpha : \mathrm{Lev}_\beta(T^\alpha \restriction s) \neq \emptyset \text{ for every } \beta \in \kappa\}$, and $T^\alpha = \bigcap\limits_{\gamma\in\alpha} T^\gamma$ for α a limit. Then for some minimal α_1, $T^{\alpha_1+1} = T^{\alpha_1}$, and call $T' = T^{\alpha_1}$ the pruned part of T.

Observation 1.1.54. Let $T \subseteq {}^{<\kappa}2$ be a tree. If T' denotes the pruned part of T as defined in 1.1.53, then it is clear that T' is indeed a pruned subtree of T, and it is not difficult to see that any branch through T is a branch through T', i.e. $[T] = [T']$. Furthermore, by a pigeonhole argument, it is also not difficult to see that if T is a κ-tree for regular κ, then this process only takes one step, that is $T' = T^1 = \{s \in T : \mathrm{Lev}_\beta(T \restriction s) \neq \emptyset \text{ for every } \beta \in \kappa\}$.

Definition 1.1.55. For $T \subseteq {}^{<\kappa}2$ a tree, let T_α denote the α^{th} derived tree (or α^{th} derivative) of T. This is defined by recursion on α. First, let $T_0 = T'$ denote the pruned part of T. For successors, first let $T'_{\alpha+1}$ denote the cofinally splitting part of T_α. Then let $T_{\alpha+1}$ be the pruned part of $T'_{\alpha+1}$. So at each stage we first remove all nodes in the tree which do not have cofinal splitting above themselves, then take the pruned part of the resulting tree. For α limit, let T_α denote the pruned part of $T'_\alpha = \bigcap\limits_{\beta\in\alpha} T_\beta$. There must exist a minimal α_0 such that $T_{\alpha_0} = T_{\alpha_0+1}$, and this is called the height of the tree T, $\alpha_0 = \mathrm{ht}_{CB}(T)$. If, as is often assumed when working with the κ-Cantor space, $2^{<\kappa} = \kappa$, then $\alpha_0 \in \kappa^+$ of course.

Definition 1.1.56. If $T \subseteq {}^{<\kappa}2$ is a tree, let $\mathrm{Ker}(T) = \{s \in T : s \in T_\alpha \text{ for every } \alpha\}$ and let $\mathrm{Sc}(T) = \{s \in T : \text{ there exists } \alpha \text{ such that } s \in T_\alpha \setminus T_{\alpha+1}\}$. For any $s \in \mathrm{Sc}(T)$, let $\mathrm{rank}_{CB}(s) = \alpha$ denote the unique α such that $s \in T_\alpha \setminus T_{\alpha+1}$.

Observation 1.1.57. Let $T \subseteq {}^{<\kappa}2$ be a tree. Then $\mathrm{Ker}(T) \subseteq T$ is a cofinally splitting subtree of T and $\mathrm{Sc}(T)$ is a disjoint union of trees with varying roots formed by "upward cones" in T (if we imagine that T is growing upwards). This is because for $s, s' \in T$, if $s' \restriction \mathrm{lh}(s) = s$, and $s \in \mathrm{Sc}(T)$, then $s' \in \mathrm{Sc}(T)$. So we may say that $T = \mathrm{Ker}(T) \cup \mathrm{Sc}(T)$ is a decomposition of T into a cofinally splitting subtree (the kernel) and a scattered part.

Observation 1.1.58. Let $T \subseteq {}^{<\kappa}2$ be a tree. By construction we have ensured that T_α is a pruned tree for every ordinal α.

1.1.8 Comparing the topological and tree processes

If $T \subseteq {}^{<\kappa}2$ is a tree coding a closed subset of 2^κ and κ is either ω or a weakly compact cardinal then we have a strong correspondence between the Cantor-Bendixson process on T and the topological Cantor-Bendixon process on $[T]$.

Proposition 1.1.59. Let κ be either ω or a weakly compact cardinal and let $T \subseteq {}^{<\kappa}2$ be a tree coding a closed subset of 2^κ. Then $[T]_\alpha = [T_\alpha]$ for every ordinal α.

Proof. Let κ be either ω or a weakly compact cardinal. First, T_0 denotes the pruned part of T in either case, and so $[T]_0 = [T] = [T_0]$. Next, suppose that $[T]_\alpha = [T_\alpha]$. We need to see that $[T]_{\alpha+1} = [T_{\alpha+1}]$. Suppose $[T]_\alpha \setminus [T]_{\alpha+1} \neq \emptyset$ and let $x \in [T]_\alpha \setminus [T]_{\alpha+1}$. Then for some $\beta \in \kappa$, $O_{x\restriction\beta} \cap [T]_\alpha = \{x\}$. Because T_α is pruned and κ is either ω or a weakly compact cardinal, T_α codes a closed subset of 2^κ. Therefore $x \restriction \beta \in T_\alpha$ is not cofinally splitting in T_α, that is $x \restriction \beta \in T_\alpha \setminus T_{\alpha+1}$, so $x \in [T_\alpha] \setminus [T_{\alpha+1}]$. Thus $[T]_\alpha \setminus [T]_{\alpha+1} \subseteq [T_\alpha] \setminus [T_{\alpha+1}]$. On the other hand, suppose $x \in [T_\alpha] \setminus [T_{\alpha+1}]$. Then for some $\beta \in \kappa$, $x \restriction \beta \in T_\alpha$ but $x \restriction \beta \notin T_{\alpha+1}$.

Suppose first that $x \restriction \beta \notin T'_{\alpha+1}$, then $x \restriction \beta$ is not cofinally splitting in T_α, which means necessarily that for some $\gamma \in \kappa$, $O_{x\restriction\gamma} \cap [T]_\alpha = \{x\}$, i.e. $x \in [T]_\alpha \setminus [T]_{\alpha+1}$. On the other hand, if $x \restriction \beta \in T'_{\alpha+1} \setminus T_{\alpha+1}$, then for some $\gamma \geq \beta$, $x \restriction \gamma \notin T_\alpha$, so again for some $\gamma' \geq \gamma$, $O_{x\restriction\gamma'} \cap [T]_\alpha = \{x\}$, i.e. $x \in [T]_\alpha \setminus [T]_{\alpha+1}$. Thus $[T_{\alpha+1}] = [T]_{\alpha+1}$. Next, let α be a limit, and suppose that $[T_\gamma] = [T]_\gamma$ for every $\gamma \in \alpha$. If $x \in [T]_\alpha$, then $x \in [T]_\gamma = [T_\gamma]$ for every $\gamma \in \alpha$, so $x \in [\bigcap_{\gamma\in\alpha} T_\gamma = T'_\alpha] = [T_\alpha]$. On the other hand, if $x \in [T_\alpha]$, then $x \in [T'_\alpha]$, so $x \restriction \beta \in \bigcap_{\gamma\in\alpha} T_\gamma$ for every β. Then $x \in [T_\gamma] = [T]_\gamma$ for every $\gamma \in \alpha$, and so $x \in [T]_\alpha$. □

Corollary 1.1.60. Let κ be either ω or a weakly compact cardinal and let $T \subseteq {}^{<\kappa}2$ be a tree coding a closed subset of 2^κ. Then $\mathrm{ht}_{CB}(T) = \mathrm{ht}_{CB}([T])$, $\mathrm{Ker}([T]) = [\mathrm{Ker}(T)]$, and $\mathrm{Sc}([T]) = [\mathrm{Sc}([T])]$. This final equality involves an abuse of notation as $\mathrm{Sc}(T)$ is not typically a tree. So by $[\mathrm{Sc}(T)]$ we mean $\{b \in [T] : \exists\alpha \in \kappa \text{ such that } b \restriction \alpha \in \mathrm{Sc}(T)\}$, that is the set of branches through T which eventually include a node in $\mathrm{Sc}(T)$.

Proof. Let T be as in the statement of the corollary. By definition $\mathrm{ht}_{CB}([T])$ is the minimal ordinal α_0 such that $[T]_{\alpha_0} = [T]_{\alpha_0+1}$. By 1.1.59, $[T_{\alpha_0}] = [T_{\alpha_0+1}]$. Because both T_{α_0} and T_{α_0+1} are pruned, and so code closed subsets of 2^κ, we must then have that $T_{\alpha_0} = T_{\alpha_0+1}$, so $\alpha_0 \leq \mathrm{ht}_{CB}(T)$. The argument is similar in the other direction, so $\mathrm{ht}_{CB}(T) = \mathrm{ht}_{CB}([T])$. Next, $\mathrm{Ker}([T]) = [T]_{\mathrm{ht}_{CB}([T])} = [T_{\mathrm{ht}_{CB}(T)}] = [\mathrm{Ker}(T)]$. And so also $\mathrm{Sc}([T]) = \{x \in [T] : x \notin \mathrm{Ker}([T])\} = \{x \in [T] : x \notin [\mathrm{Ker}(T)]\} = \{x \in [T] : \exists\beta \in \kappa \text{ such that } x \restriction \beta \in \mathrm{Sc}(T)\} = [\mathrm{Sc}(T)]$. □

If a regular κ is neither ω nor a weakly compact cardinal, then there isn't necessarily, for example, the correspondence between the Cantor-Bendixson process on trees and the topological Cantor-Bendixson process that we've observed in 1.1.59. We give some examples to illustrate this point.

Example 1.1.61. For $T \subseteq {}^{<\kappa}2$ a pruned tree, not necessarily is $\mathrm{ht}_{CB}([T]) = \mathrm{ht}_{CB}(T)$, as is the case for trees $T \subseteq {}^{<\omega}2$ or where κ is weakly compact.

Proof. Suppose $\kappa = \omega_1$ for example, and consider the subtree T of ${}^{<\omega_1}2$ where we fix a single $x_0 \in {}^{\omega_1}2$ and let T be formed by initial segments of this x_0 along with pruned Aronszajn trees splitting off from initial segments of x_0 cofinally. It is clear then that $T_0 = T_1$, i.e. $\mathrm{ht}_{CB}(T) = 0$, but also that $[T] = \{x_0\}$, i.e. $\mathrm{ht}_{CB}([T]) = 1$. $\square$

Example 1.1.62. For $T \subseteq {}^{<\kappa}2$ a tree, not necessarily does $[T_{\mathrm{ht}_{CB}(T)}] = \emptyset$ imply that $T_{\mathrm{ht}_{CB}(T)} = \emptyset$. If κ is weakly compact or ω, however, then this implication does hold.

Proof. It is clear that if κ is either ω or a weakly compact cardinal, then $[T_{\mathrm{ht}_{CB}(T)}] = \emptyset$ implies that $T_{\mathrm{ht}_{CB}(T)} = \emptyset$. On the other hand, suppose that κ is not weakly compact. Suppose first that κ does not have the tree property. Then there exists a pruned κ-Aronszajn tree $T \subseteq {}^{<\kappa}2$. Note that $\mathrm{ht}_{CB}(T) = 0$, $[T_0] = \emptyset$, but $T_0 = T \neq \emptyset$. Next, suppose κ has the tree property, but is not strongly inaccessible. There are two cases, either $\kappa = \mu^+$ for some cardinal μ, or κ is a regular limit where for some $\mu \in \kappa$, $2^\mu \geq \kappa$.

First assume that $\kappa = \mu^+$. We produce a subtree of ${}^{<\kappa}2$ which is cofinally splitting but has no branches. The nodes in each level of our tree (which is a multiple of μ) will be identified with properly decreasing sequences (in $\supseteq$) of elements of $[\mu]^\mu$. Consider all sequences of length 1. That is, consider all elements of $[\mu]^\mu \setminus \{\mu\}$. There are $2^\mu = |[\mu]^\mu|$-many of these. Let each of these sequences be identified with a node in the μ^{th}-level of ${}^{<\kappa}2$. Proceed in this manner, putting each proper subset of size μ of each node as a "successor" on the $(\mu+\mu)^{\mathrm{th}}$-level. Then look at the induced tree. At limit levels which are multiples of μ, simply take the intersections of nodes along paths of the relevant length. If this intersection is of size $< \mu$, then there is no node extending the nodes along that path. Because all proper decreasing sequences in $[\mu]^\mu$ must be of length $< \kappa$, this tree has no branch. On the other hand, it is not hard to see that if $\alpha \in \mu^+$ and $x \in [\mu]^\mu$, then there is a sequence $\langle x_\beta : \beta \in \alpha\rangle \subseteq [\mu]^\mu$ such that if $\xi \in \zeta$, then x_ζ is a proper subset of x_ξ, and $x_0 = x$. It is also clear that if $x \in [\mu]^\mu$, then there exist $x_1, x_2 \in [x]^\mu$ so that $x_1 \neq x_2 \neq x$. So the tree we've constructed is cofinally splitting and has no branches.

Next, consider the case where κ is a regular limit, has the tree property, but there exists $\mu \in \kappa$ such that $2^\mu \geq \kappa$. A literal translation of the construction above will not work here, because decreasing subsets of μ of size μ have their order types bounded by $\mu^+ < \kappa$. However, a similar construction works, we just have to send nodes from each level higher up. So, take μ such that $2^\mu \geq \kappa$. Let $\langle x_\alpha : \alpha \in \kappa \rangle \subseteq [\mu]^\mu \setminus \{\mu\}$. Identify as before each x_α with a node in the μ^{th}-level of ${}^{<\kappa}2$. However, instead of proceeding as above here, further pair each x_α with a node on the α^{th}-level above the previously specified node. So now there are κ nodes in the tree on level μ, and above each of these x_α nodes is a unique path to another node (which we may also think of as an x_α-node) on the $(\mu + \alpha)^{\text{th}}$-level. So even now at the first stage, the tree is of height κ. Note, however, that there are no branches. This feature will persist through all μ^+-many steps of the construction. Next, we proceed as above, working at each node x_β (on level $\mu + \beta$), finding $\langle y_\alpha : \alpha \in \kappa \rangle \subseteq [x_\beta]^\mu \setminus \{x_\beta\}$ and identifying them with κ-many nodes on level $\mu+\beta+\mu$, then extending each by a unique path of length α to a node on level $\mu + \beta + \mu + \alpha$. Now just continue this process. At limits take intersections when possible. As mentioned, this requires μ^+-many steps. It is clear that the tree constructed will be cofinally splitting, and that the tree T constructed has no branches may be seen as follows. Suppose we had a branch $x \in [T]$. Then in particular, $x \restriction \mu \in T$, so is some x_α. It must then by that $x \restriction (\mu + \alpha)$ is also then the unique node on level $\mu + \alpha$ above x_α. Proceeding, $x \restriction (\mu+\alpha+\mu) \in T$, so is identified with some $y_\beta \in [x_\alpha]^\mu \setminus \{x_\alpha\}$. And again, $x \restriction (\mu + \alpha + \mu + \beta)$ is also then the unique node on level $\mu + \alpha + \mu + \beta$ above x_α and y_β. We can continue in this manner, noting that at limits we have taken intersections. However, this is a contradiction because we can only travel less than a sum of $(< \mu^+)$-many ordinals each which is less than κ, which is less than κ by regularity. $\square$

Example 1.1.63. Let $T \subseteq {}^{<\kappa}2$ be a tree coding a closed subset of 2^κ. Not necessarily does $[T]$ scattered, i.e. $\mathrm{Sc}([T]) = [T]$, imply that $[T_{\mathrm{ht}_{CB}(T)}] = \emptyset$, as was the case when $\kappa = \omega$. However, the converse does always hold, that is if $[T_{\mathrm{ht}_{CB}(T)}] = \emptyset$, then $[T]$ is scattered. Moreover, if κ is ω or a weakly compact cardinal then if $[T]$ is scattered, $[T_{\mathrm{ht}_{CB}(T)}] = \emptyset$

(which is also equal to $T_{\mathrm{ht}_{CB}(T)}$ by 1.1.62).

Proof. We saw in 1.1.61 a case where $\kappa = \omega_1$ where $[T] = \{x_0\}$, so in particular $[T]$ is scattered, but $T = T_{\mathrm{ht}_{CB}(T)}$, and so $[T_{\mathrm{ht}_{CB}(T)}] \neq \emptyset$. The same construction works as long as κ is not weakly compact by positioning cofinally splitting trees with empty bodies cofinally along a single branch (we constructed such trees in Example 1.1.62). On the other hand, if κ is either ω or a weakly compact cardinal, by a consequence of 1.1.59, if $[T]$ is scattered then $[T_{\mathrm{ht}_{CB}(T)}] = \emptyset$. Next, for general regular κ, suppose $[T_{\mathrm{ht}_{CB}(T)}] = \emptyset$. If $[T]$ were not scattered, let $A \subseteq [T]$ with A dense-in-itself and nonempty. Then $\overline{A} \subseteq [T]$ is also dense-in-itself, and so perfect. It is not difficult to see that $T_{\overline{A}} \subseteq T$ is a cofinally splitting subtree of T, so no nodes from $T_{\overline{A}}$ will ever be removed in the tree Cantor-Bendixson process, i.e. $T_{\overline{A}} \subseteq T_{\mathrm{ht}_{CB}(T)}$, so $\overline{A} \subseteq [T_{\mathrm{ht}_{CB}(T)}]$, a contradiction if the latter is empty. $\square$

It is also the case that some generalizations to the κ uncountable setting of aspects of the situation when $\kappa = \omega$ never hold, no matter what $\kappa > \omega$ is. Consider in particular the perfect set property for closed subsets of 2^ω, that is the Cantor-Bendixson theorem 1.1.50. We have the following impediment to any (direct topological) generalization in the κ-Cantor space.

Example 1.1.64. Let $T \subseteq {}^{<\kappa}2$ be a tree. Not necessarily does $[T]$ not being scattered imply that $|[T]| = 2^\kappa$, as was the case with $\kappa = \omega$. In particular, there always exist topologically perfect subsets of 2^κ of size κ. This is even true for singular κ (still with the κ-box topology).

Proof. Let $T \subseteq {}^{<\kappa}2$ consist of nodes with only finitely many 1's. That is, $T = \{s \in {}^{<\kappa}2 : |\{\alpha \in \mathrm{lh}(s) : s(\alpha) = 1\}| < \omega\}$. One may observe that $[T] = \{x \in 2^\kappa : |\{\alpha \in \kappa : x(\alpha) = 1\}| < \omega\}$, so $|[T]| = \kappa$. Furthermore, $[T]$ is closed and dense-in-itself. This is true generally in the κ-box topology even for κ singular with $E = \{x \in 2^\kappa : |\{\alpha \in \kappa : x(\alpha) = 1\}| < \omega\} \subseteq 2^\kappa$, as follows. If $x \notin E$, then $|\{\alpha \in \kappa : x(\alpha) = 1\}| \geq \omega$. Let $A \in [\kappa]^\omega$ be such that $x(\alpha) = 1$ for every $\alpha \in A$. Then $O_{x\restriction A} \cap E = \emptyset$, so E is closed. Furthermore, if $A \in P_\kappa\kappa$ and $x \in E$, then $O_{x\restriction A} \cap E \neq \{x\}$, so E is dense-in-itself. $\square$

1.2 Games played on subsets of 2^κ and on trees $T \subseteq {}^{<\kappa}2$

We describe two games of two players which can be used, in particular, to characterize perfectness and scatteredness for (closed) subsets of 2^κ and their trees, respectively. One of the games is played on arbitrary subsets $E \subseteq 2^\kappa$ while the other is played on trees $T \subseteq {}^{<\kappa}2$. The former game is the straightforward generalization of the one given by Väänänen originally in [69], while the later game is reminiscent of the classical cut-and-choose-type $*$-Game of Davis [18], but played on trees and generalized to the κ-Cantor space. Generally speaking, Väänänen's game when played to length ω has a close affinity with the topological Cantor-Bendixson process, while the cut-and-choose-type game when played to length ω has a close affinity with the Cantor-Bendixson process on trees. As we have seen, when κ is ω or a weakly compact cardinal we have a correspondence between these processes, and so between the games. In this and the following sections, we explore also specific ways that these games can diverge, structural information that winning strategies for the different players in the cut-and-choose-type game give us about the underlying tree, generalized dichotomy theorems in the spirit of the Cantor-Bendixson theorem for closed subsets of 2^ω, the perspective that the games can give to old results, generalizations of these games to spaces other than 2^κ and to objects other than trees, and assorted other topics.

1.2.1 Väänänen's game

Definition (Väänänen [69]) 1.2.1. Let κ be regular. For $E \subseteq 2^\kappa$, $x_0 \in E$, and $\delta \leq \kappa$, define the game of length δ starting at x_0 played on E, $G(E, x_0, \delta)$. There are two players, I

and II. Player I plays increasing ordinals in κ and player II plays points in E. The game is initialized at $x_0 \in E$. Player I plays first $\alpha_1 \in \kappa$ and player II must respond and play $x_1 \in E$ such that $x_1 \neq x_0$, but $x_1 \restriction \alpha_1 = x_0 \restriction \alpha_1$. Player I then plays $\alpha_2 > \alpha_1$, and player II must respond with $x_2 \neq x_1, x_0$ with $x_2 \restriction \alpha_2 = x_1 \restriction \alpha_2$. The game proceeds in this manner, with the additional requirement that at limit stages player I plays first and must play the supremum of the ordinals he has played already. So, at stage β, player I plays α_β and player II must play $x_\beta \in E$ such that $x_\beta \neq x_\gamma$ and $x_\beta \restriction \alpha_{\gamma+1} = x_\gamma \restriction \alpha_{\gamma+1}$ for every $\gamma \in \beta$. Player II wins a run of the game if she can play legally at stage β for every $\beta \in \delta$.

Definition (From [69]) 1.2.2. Say that a closed $E \subseteq 2^\kappa$ is δ-perfect if and only if player II has a winning strategy in $G(E, x_0, \delta)$ for every $x_0 \in E$. Say that $E \subseteq 2^\kappa$ is δ-scattered if and only if player I has a winning strategy in $G(E, x_0, \delta)$ for every $x_0 \in E$. Define the δ-kernel of E, $\mathrm{Ker}(E, \delta)$, to be the set of $x_0 \in E$ such that player II has a winning strategy in $G(E, x_0, \delta)$, and the δ-scattered part of E, $\mathrm{Sc}(E, \delta)$, to be the set of $x_0 \in E$ such that player I has a winning strategy in $G(E, x_0, \delta)$.

Fact 1.2.3. By the Gale-Stewart theorem (see [27]), $G(E, x_0, \omega)$ is determined—that is, either player I or player II has a winning strategy in $G(E, x_0, \omega)$. This is because the game can be reformulated in such a way that membership in the payoff set for player I is decided in a finite number of steps, i.e. the payoff set for player I is open. It may be however that for $\delta \geq \omega + 1$, $G(T, x_0, \delta)$ is undetermined. We will elaborate on this in later sections.

Proposition 1.2.4. The game of length ω characterizes perfectness and scatteredness: for $E \subseteq 2^\kappa$, E is dense-in-itself if and only if player II has a winning strategy in $G(E, x_0, \omega)$ for every $x_0 \in E$ and E is scattered if and only if player I has a winning strategy in $G(E, x_0, \omega)$ for every $x_0 \in E$. That is, E is ω-scattered if and only if E is scattered, and for E closed, E is ω-perfect if and only if E is perfect. Moreover, for $E \subseteq 2^\kappa$, $\mathrm{Ker}(E) = \mathrm{Ker}(E, \omega)$ and $\mathrm{Sc}(E) = \mathrm{Sc}(E, \omega)$.

Proof. For all of these statements, it suffices to show that if $E \subseteq 2^\kappa$, $\mathrm{Ker}(E) = \mathrm{Ker}(E, \omega)$

and $\mathrm{Sc}(E) = \mathrm{Sc}(E, \omega)$. First, note that $\mathrm{Ker}(E)$ is dense-in-itself and it is not difficult to see then that player II can play all ω-many moves in $G(\mathrm{Ker}(E), x_0, \omega)$ for any $x_0 \in \mathrm{Ker}(E)$ regardless of the ω-many ordinals that player I picks along the way. So $\mathrm{Ker}(E) \subseteq \mathrm{Ker}(E, \omega)$. On the other hand, suppose that $x_0 \in E \setminus \mathrm{Ker}(E)$. Then $x_0 \in \mathrm{Sc}(E)$. It suffices to see then that $x_0 \in \mathrm{Sc}(E, \omega)$. Because $x_0 \in \mathrm{Sc}(E)$, there exists α_0 such that $x_0 \in E_{\alpha_0} \setminus E_{\alpha_0+1}$, i.e. $\mathrm{rank}_{CB}(x_0) = \alpha_0$. Recall that here we are referring to the topological Cantor-Bendixson process with respect to E. Then there must exist $\beta_1 \in \kappa$ such that $O_{x_0 \restriction \beta_1} \cap E_{\alpha_0} = \{x_0\}$. Let I play β_1 as his first move. If II is able to play, she has to play some $x_1 \in E$ with $\mathrm{rank}_{CB}(x_1) < \alpha_0$. Proceed in this manner, and observe that because there does not exist an infinitely descending sequence of ordinals, after finitely many steps player II is not able to play, i.e. this describes a winning strategy for player I and so $x_0 \in \mathrm{Sc}(E, \omega)$. □

Proposition (from [69]) 1.2.5. For closed $E \subseteq 2^\kappa$, if $\mathrm{Ker}(E, \kappa) \neq \emptyset$, then $|E| = 2^\kappa$. This is by virtue of ${}^{<\kappa}2$ embedding into T_E, and $E = [T_E]$. For any $\delta \in \kappa$, if $\mathrm{Ker}(E, \delta + 1) \neq \emptyset$, then ${}^{<(\delta+1)}2$ can be embedded into T_E, so $|E| \geq 2^\delta$.

Proof. We sketch the proof (for some details see [69]). For the second statement, first fix $\delta \in \kappa$. Find $x_0 \in E$ so that player II has a winning strategy, τ, for $G(E, x_0, \delta + 1)$. Repeatedly apply τ using different stimuli from player I to build an embedding of ${}^{\leq\delta}2$ into T_E. If $\mathrm{Ker}(E, \kappa) \neq \emptyset$ then we can do the same thing, and we'll be building an embedding of ${}^{<\kappa}2$ into T_E, and so because E is closed, we'll have $|[T_E]| = 2^\kappa$. □

1.2.2 A cut-and-choose game played on trees $T \subseteq {}^{<\kappa}2$

The Väänänen game when played on closed subsets $E \subseteq 2^\kappa$ has player II choosing elements from the body of the tree T_E at each stage. We define here a similar game which intuitively also captures something of what it might mean for a tree to be κ-perfect or κ-scattered, for example, but which is not played on the body of a tree but on the tree itself. While there are close connections between the Väänänen game, this game, and perfectness and scatteredness, especially when $\kappa = \omega$, the games can be dramatically different when $\kappa, \delta > \omega$ as we will see. The game may be viewed as a sort of generalization of the classical cut-and-choose-type $*$-Game of Davis [18].

Definition 1.2.6. Let κ be regular, $T \subseteq {}^{<\kappa}2$ be a tree, and $s_0 \in T$. Define the two player game of length $\delta \leq \kappa$ starting at s_0 played on T, $G(T, s_0, \delta)$, as follows. It is notationally convenient to call the first round the 0^{th} round. Player I starts and plays a level of the tree $\alpha_0 \geq \mathrm{lh}(s_0)$. Player II then responds with a splitting pair of nodes on the same level $\{s_0^0, s_1^0\} \subseteq T$ extending s_0 which agree up to level α_0. That is, player II plays $\{s_0^0, s_1^0\} \subseteq T$ such that $\mathrm{lh}(s_0^0) = \mathrm{lh}(s_1^0) > \alpha_0$, $s_0^0 \restriction \alpha_0 = s_1^0 \restriction \alpha_0$, $s_0^0 \restriction \mathrm{lh}(s_0) = s_1^0 \restriction \mathrm{lh}(s_0) = s_0$, and $s_0^0 \neq s_1^0$. At all successor rounds, player I both plays a level and chooses a node. So, at the next round player I chooses $s_1 \in \{s_0^0, s_1^0\}$ and a level $\alpha_1 \geq \mathrm{lh}(s_1)$. The game proceeds in this manner. At limit stages β, player I plays first and plays $\alpha_\beta \geq \sup\{\alpha_\gamma : \gamma \in \beta\}$ and player II must respond with a splitting pair of nodes $\{s_0^\beta, s_1^\beta\} \subseteq T$ extending the path through the tree constructed so far which agree up to level α_β. That is, such that $\mathrm{lh}(s_0^\beta) = \mathrm{lh}(s_1^\beta) > \alpha_\beta$, $s_0^\beta \restriction \alpha_\beta = s_1^\beta \restriction \alpha_\beta$, $s_0^\beta \neq s_1^\beta$, and $s_0^\beta \restriction \mathrm{lh}(s_\gamma) = s_1^\beta \restriction \mathrm{lh}(s_\gamma) = s_\gamma$ for every $\gamma \in \beta$. Player II wins a run of the game if she can play legally at stage β for every $\beta \in \delta$.

Definition 1.2.7. For $T \subseteq {}^{<\kappa}2$ a tree and $\delta \leq \kappa$, say that T is δ-perfect if and only if player II has a winning strategy in $G(T, s_0, \delta)$ for every $s_0 \in T$. Say that T is δ-scattered if and only if player I has a winning strategy in $G(T, s_0, \delta)$ for every $s_0 \in T$. Define the δ-kernel of T, $\mathrm{Ker}(T, \delta)$, to be the set of $s_0 \in T$ such that player II has a winning strategy in $G(T, s_0, \delta)$, and the δ-scattered part of T, $\mathrm{Sc}(T, \delta)$, to be the set of $s_0 \in T$ such that player I has a winning strategy in $G(T, s_0, \delta)$.

Fact 1.2.8. By the Gale-Stewart theorem, for every $s_0 \in T$, $G(T, s_0, \omega)$ is determined.

Remark 1.2.9. As with Väänänen's game, it may be however that for $\delta \geq \omega+1$, $G(T, s_0, \delta)$ is undetermined.

Observation 1.2.10. $\mathrm{Ker}(T,\delta) \subseteq T$ is a tree while if $s \in \mathrm{Sc}(T,\delta)$, then $s' \in \mathrm{Sc}(T,\delta)$ for every $s' \in T$ with $s' \restriction \mathrm{lh}(s) = s$. So in the case where $\delta = \omega$ for example, $T = \mathrm{Ker}(T,\omega) \cup \mathrm{Sc}(T,\omega)$ is a decomposition of the tree T into an ω-perfect subtree and an ω-scattered part.

Proof. First observe that for $s_1, s_2 \in T$, if $s_2 \restriction \mathrm{lh}(s_1) = s_1$, and $s_2 \in \mathrm{Ker}(T,\delta)$, then $s_1 \in \mathrm{Ker}(T,\delta)$, because player II can use a winning strategy in $G(T, s_2, \delta)$ to generate a winning strategy in $G(T, s_1, \delta)$. So $\mathrm{Ker}(T,\delta)$ is a tree. On the other hand, if player I has a winning strategy in $G(T, s, \delta)$ for some $s \in T$, then it's not difficult to see that player I has a winning strategy in $G(T, s', \delta)$ for every $s' \in T$ extending s. This is because, for example, the first move by II in the latter game is a legal move for II in the former game, and then the games may be played in parallel. $\square$

Proposition 1.2.11. Let $T \subseteq {}^{<\kappa}2$ be a tree. Then similarly to how Väänänen's game can be used to characterize perfectness and scatteredness of $[T]$, $G(T, s_0, \omega)$ can be used to characterize perfectness and scatteredness of T. That is, $\mathrm{Ker}(T,\omega) = \mathrm{Ker}(T)$, so also $\mathrm{Sc}(T,\omega) = \mathrm{Sc}(T)$. If κ is either ω or a weakly compact cardinal and T is pruned then, by 1.1.60, $[\mathrm{Ker}(T,\omega)] = \mathrm{Ker}([T])$ and $[\mathrm{Sc}(T,\omega)] = \mathrm{Sc}([T])$. By $[\mathrm{Sc}(T,\omega)]$ we mean of course as before $\{b \in [T] : \exists \alpha \in \kappa \text{ such that } b \restriction \alpha \in \mathrm{Sc}(T,\omega)\}$.

Proof. We show that $\mathrm{Ker}(T,\omega) = \mathrm{Ker}(T)$ and $\mathrm{Sc}(T,\omega) = \mathrm{Sc}(T)$. Note that $\mathrm{Ker}(T)$ is a cofinally splitting subtree of T, so for any $s \in \mathrm{Ker}(T)$, player II has a winning strategy in $G(\mathrm{Ker}(T), s, \omega)$. Thus $\mathrm{Ker}(T) \subseteq \mathrm{Ker}(T,\omega)$. On the other hand, suppose $s \in \mathrm{Sc}(T)$. We need to show that $s \in \mathrm{Sc}(T,\omega)$. Suppose that $\mathrm{rank}_{CB}(s) = \gamma_0$, so γ_0 is minimal such that $s \in T_{\gamma_0} \setminus T_{\gamma_0+1}$. Suppose first that $s \notin T'_{\gamma_0+1}$. Then for some $\beta_0 \in \kappa$, $s \in T_{\gamma_0}$ is not cofinally splitting above β_0. Let player I play $\alpha_0 = \beta_0$ in this case. Then player II responds with $\{s^0_0, s^0_1\}$ extending s splitting above β_0, and so necessarily not both of s^0_0 and s^0_1 are in T_{γ_0}. Let player I choose such a node and proceed as above, noting that

$\mathrm{rank}_{CB}(s_1) < \mathrm{rank}_{CB}(s)$. If, on the other hand, $s \in T'_{\gamma_0+1}$, then necessarily for some $\beta_0 \in \kappa$ with $\beta_0 \geq \mathrm{lh}(s)$, $\mathrm{Lev}_{\beta_0}(T'_{\gamma_0+1} \restriction s) = \emptyset$, and let player I play β_0. Then if player II plays $\{s_0^0, s_1^0\}$ extending s splitting past β_0, neither s_0^0 nor s_1^0 is in T'_{γ_0+1}. If one of them is not in T_{γ_0}, choose this node and proceed as above, noting that $\mathrm{rank}_{CB}(s_1) < \mathrm{rank}_{CB}(s)$. If both s_0^0 and s_1^0 are in T_{γ_0}, then have player I choose one of them as s_1 and play β_1 such that s_1 is not splitting above β_1 in T_{γ_0}. Then in the next round if player II plays $\{s_0^1, s_1^1\}$ extending s_1 splitting above β_1, necessarily one of s_0^1 or s_1^1 is not in T_{γ_0}. Choose such a node as s_2 and note that $\mathrm{rank}_{CB}(s_2) < \gamma_0$, and proceed as above. If player II were able to play all of her moves, then $\langle \mathrm{rank}_{CB}(s_n) : n \in \omega \rangle$ would be a non-increasing sequence of ordinals which never stabilizes, which is impossible. So this describes a winning strategy for player I. Thus $\mathrm{Ker}(T, \omega) = \mathrm{Ker}(T)$ and $\mathrm{Sc}(T, \omega) = \mathrm{Sc}(T)$. □

1.2.3 The behavior of the two games when $\delta = \kappa$

We observed in 1.2.4 that Väänänen's game when played to length ω characterizes the end result of the topological Cantor-Bendixson process, while we observed in 1.2.11 that our cut-and-choose-type game when played to length ω characterizes the end result of the Cantor-Bendixson process on trees. We shall see here that when the games are played to length $\delta = \kappa$, there are some easy conditions in which players I or II have winning strategies.

Proposition (converse to 1.2.5, see [69]) 1.2.12. Let κ be regular, $E \subseteq 2^\kappa$ be closed, and $x_0 \in E$. Then player II has a winning strategy in $G(E, x_0, \kappa)$ if and only if there exists an embedding of ${}^{<\kappa}2$ into T_E such that x_0 is a branch through the subtree of T_E induced by this embedding. That is, player II has a winning strategy in $G(E, x_0, \kappa)$ if and only if T_E contains a subtree isomorphic to ${}^{<\kappa}2$ with x_0 as a branch.

Proof. A winning strategy for player II in $G(E, x_0, \kappa)$ can be used by repeated application to build a copy of ${}^{<\kappa}2$ in T_E having x_0 as a branch, as in 1.2.5 (essentially noted in [69]). On the other hand, it is clear that player II has a winning strategy in $G(E, x_0, \kappa)$ if there exists an embedding of ${}^{<\kappa}2$ into T_E such that x_0 is a branch through the subtree of T_E induced by this embedding, by playing branches though T_E which are also branches through the pointwise image of this embedding. $\square$

Proposition 1.2.13. Let κ be regular, $T \subseteq {}^{<\kappa}2$ be a tree, and $s_0 \in T$. Then player II has a winning strategy in $G(T, s_0, \kappa)$ if and only if there exists an embedding of ${}^{<\kappa}2$ into T with s_0 contained in the image of this embedding.

Proof. Suppose first that player II has a winning strategy τ in $G(T, s_0, \kappa)$. An embedding from f from ${}^{<\kappa}2$ to T with $f(\emptyset) = s_0$ can be built by recursion via iterated application of τ. For example, suppose player I plays $\alpha_0 = \mathrm{lh}(s_0)$, so $\tau(\langle \alpha_0 \rangle) = \{s_0^0, s_1^0\}$. Then let $f(\langle 0 \rangle) = s_0^0$ and $f(\langle 1 \rangle) = s_1^0$. Allow two different games to then be played, where player I chooses $s_1 = s_0^0$ and plays $\alpha_1 = \mathrm{lh}(s_1)$ and where player I chooses $s_1 = s_1^0$ and plays $\alpha_1 = \mathrm{lh}(s_1)$, and player II responds with $\tau(\langle \alpha_0, \{s_0^0, s_1^0\}, (s_0^0, \alpha_1)\rangle) = \{s_0^1, s_1^1\}$ and $\tau(\langle \alpha_0, \{s_0^0, s_1^0\}, (s_1^0, \alpha_1)\rangle) = \{s_0'^1, s_1'^1\}$, respectively. Then set $f(\langle 00 \rangle) = s_0^1$, $f(\langle 01 \rangle) = s_1^1$, $f(\langle 10 \rangle) = s_1'^0$, and $f(\langle 11 \rangle) = s_1'^1$. There are now four runs of the game to be considered. It is clear that f can continue to be built in this manner so that it is a total embedding of ${}^{<\kappa}2$ into T with $f(\emptyset) = s_0$. On the other hand, clearly if T contains a copy of ${}^{<\kappa}2$ containing s_0, then player II has a winning strategy in $G(T, s_0, \kappa)$. $\square$

Proposition 1.2.14. Let κ be regular, $T \subseteq {}^{<\kappa}2$ be a tree coding a closed subset of 2^κ, and $s_0 \in T$. Then if $|[T \upharpoonright s_0]| \leq \kappa$, player I has a winning strategy in $G(T, s_0, \kappa)$.

Proof. This proof is a simple diagonalization—at stage $\alpha + 1$ player one "takes care of" the α^{th} element in $[T \upharpoonright s_0]$. Let $\langle x_\alpha : \alpha \in \kappa \rangle = [T \upharpoonright s_0]$ be a surjective listing of elements in $[T \upharpoonright s_0]$. At each stage $\alpha + 1 \in \kappa$, player I is presented with a pair of splitting nodes $\{s_0^\alpha, s_1^\alpha\}$. At most one of these nodes is an initial segment of x_α, and let player I choose the other node to be $s_{\alpha+1}$. Suppose towards a contradiction that player II is able to play all κ-many moves.

Then the two players will have constructed a branch $x \in [T \upharpoonright s_0]$. However, then for some α, $x = x_\alpha$, but at stage $\alpha+1$ player I chose $s_{\alpha+1}$ in such a way that $x \upharpoonright \mathrm{lh}(s_{\alpha+1}) \neq x_\alpha \upharpoonright \mathrm{lh}(s_{\alpha+1})$, which is impossible. $\square$

Proposition 1.2.15. Let κ be regular, $E \subseteq 2^\kappa$ be closed, and $x_0 \in E$ such that there exists $\beta \in \kappa$ with $|O_{x_0 \upharpoonright \beta} \cap E| \leq \kappa$. Then player I has a winning strategy in $G(E, x_0, \kappa)$.

Proof. This proof is similar to the one above. First list surjectively $\langle y_\delta : \delta \in \kappa \rangle = O_{x_0 \upharpoonright \beta} \cap E$. Let player I play $\alpha_1 = \beta$ first, and generally at stage $\delta + 2$ play $\alpha_{\delta+2}$ large enough so that y_δ and $x_{\delta+1}$ disagree before $\alpha_{\delta+2}$ (unless $y_\delta = x_{\delta+1}$, in which case play arbitrarily). If player II is able to play all κ-many moves, then $\langle x_\delta \upharpoonright \alpha_{\delta+1} : \delta \in \kappa \rangle$ is a coherent set of nodes through T_E, i.e. players have built a branch $y \in [T_E]$, so there must exist some $\delta \in \kappa$ for which $y = y_\delta$. However, then at stage $\delta + 2$, player I chose $\alpha_{\delta+2}$ large enough so that $y_\delta \upharpoonright \alpha_{\delta+2} = x_{\delta+2} \upharpoonright \alpha_{\delta+2} \neq x_{\delta+1} \upharpoonright \alpha_{\delta+2}$, which is a contradiction (in the case where $y_\delta = x_{\delta+1}$, then there is an immediate contradiction as well). $\square$

1.3 Adding branches through $T \subseteq {}^{<\kappa}2$

Unlike when $\kappa = \omega$, where one can undertake an absolute combinatorial process (the tree Cantor-Bendixson process) to exactly characterize when outer models can add branches to trees $T \subseteq {}^{<\omega}2$ (see 1.1.51), when $\kappa > \omega$ the question of when outer models can add branches to trees $T \subseteq {}^{<\kappa}2$, and by what means, is significantly more complicated. Using the tools we have developed so far we make some observations in this direction and also show how using games can yield results of the folklore. The following is an analogue in the spirit of Mansfield's theorem for trees $T \subseteq {}^{<\kappa}2$.

Proposition 1.3.1. Let $T \subseteq {}^{<\kappa}2$ be a tree, $V \subseteq M$ be transitive models of ZFC, $T \in V$, and $(T_{\mathrm{ht}_{CB}(T)} = \emptyset)^V$. Then $([T])^V = ([T])^M$.

Proof. As we have noted before, the Cantor-Bendixson process on trees $T \subseteq {}^{<\kappa}2$ is absolute, so in particular $(T_\alpha)^V = (T_\alpha)^M$ for every ordinal α. This proof is then the same as in 1.1.51. Explicitly, suppose towards a contradiction that $b \in ([T])^M \setminus V$. Working in M, clearly $b \notin [T_{\mathrm{ht}_{CB}(T)}]$ because $T_{\mathrm{ht}_{CB}(T)} = \emptyset$. Find the minimal $\gamma + 1$ such that $b \notin [T_{\gamma+1}]$. This must be a successor, as follows. If γ is a limit, then $T'_\gamma = \bigcap_{\eta \in \gamma} T_\eta$ and $[T'_\gamma] = [T_\gamma]$, so $b \notin [T_\gamma]$ implies that for some $\delta \in \kappa$, $b \upharpoonright \delta \notin T'_\gamma$, but then for some $\eta_\delta \in \gamma$, $b \upharpoonright \delta \notin T_{\eta_\delta}$, a contradiction. So, suppose that $\gamma + 1$ is minimal with $b \notin [T_{\gamma+1}]$. Then $b \notin [T'_{\gamma+1}]$ so for some $\beta \in \kappa$, $b \upharpoonright \beta$ is not cofinally splitting in T_γ. Because $b \in [T_\gamma]$, without loss of generality we can then take β large enough so that no splitting along b occurs above $b \upharpoonright \beta$ in T_γ. However, then b is definable from $b \upharpoonright \beta$ and T_γ, both of which are objects in V, a contradiction. □

Note 1.3.2. In 1.3.1 we see that outer models cannot add branches to scattered trees $T \subseteq {}^{<\kappa}2$. However, this is a very limited result, and indeed unless κ is ω or weakly compact, $[T]$ being scattered is not necessarily equivalent to T being scattered. Outside of these situations then, the body of a tree being scattered often has little bearing on whether or not branches can be added to the tree. For example, we can have $[T_{\mathrm{rank}_{CB}(T)}] = \emptyset$ but branches can be added (for example if $\kappa = \omega_1$ and there exists $T \subseteq {}^{<\omega_1}2$ a pruned Suslin tree). Indeed, in this case a branch is even added by a c.c.c. forcing, the tree itself. On the other hand, we can have $[T_{\mathrm{rank}_{CB}(T)}] = \emptyset$ (with $T_{\mathrm{rank}_{CB}(T)} \neq \emptyset$), but branches can't be added any cardinal-preserving outer model. For example this will be the case if T is a special Aronszajn tree. We could also have $[T_{\mathrm{rank}_{CB}(T)}] \neq \emptyset$, but branches can't be added by any cardinal-preserving outer model. For example, take $[T] = [T_{\mathrm{rank}_{CB}(T)}] = \{x_0\}$ by forming T by taking the tree generated by x_0 along with a cofinally branching sequence of special Aronszajn trees.

By the note above, the body of a tree in the 2^κ-setting being scattered seems to have little bearing on whether branches can or cannot be added to the tree in outer models, unless κ happens to be weakly compact. However, with the more general notion of κ-scattered, we can say that certain forcing extensions can't add branches to certain trees. We prove the following proposition 1.3.3, which is a result in the folklore, using the framework of the cut-and-choose-type game. A similar argument can be used for Väänänen's game played on $[T]$, and because the existence of a winning strategy for player II in either game of length κ yields similar structural information about the tree, the conclusions are the same. We also give a natural generalization in 1.3.7, again using the cut-and-choose-type game, but using a slightly different argument which emphasizes a difference between the two games.

Proposition (folklore) 1.3.3. Let κ be regular, $T \subseteq {}^{<\kappa}2$ be a tree, and $\mathbb{P}$ be a κ-strategically closed forcing notion. Then if forcing with $\mathbb{P}$ adds a branch to T, T must contain a copy of ${}^{<\kappa}2$, i.e. $\mathrm{Ker}(T, \kappa) \neq \emptyset$.

Proof. Let κ be regular, $T \subseteq {}^{<\kappa}2$ be a tree, $\mathbb{P}$ be κ-strategically closed, and G be $(V, \mathbb{P})$-generic. Suppose that $([T])^{V[G]} \setminus ([T])^V \neq \emptyset$. In V, we show that player II has a winning strategy in $G(T, \emptyset, \kappa)$. Without loss of generality, suppose $\dot{b}$ is a $\mathbb{P}$-name for a new branch through T, i.e. $\mathbf{1}_{\mathbb{P}} \Vdash$ "$\dot{b} \in [T]$ and $\dot{b} \notin V$". The recurring idea in this version of the argument is that of "candidate" branches for $\dot{b}$, which because $\mathbb{P}$ is κ-strategically closed, can be built through T in V. First, note that for every $p \in \mathbb{P}$, there exists a unique maximal $\beta_p \in \kappa$ and corresponding $s_p \in \mathrm{Lev}_{\beta_p}(T)$ such that $p \Vdash \dot{b} \restriction \beta_p = s_p$. Such a β_p exists because T does not split at limit stages, and all conditions force that $\dot{b}$ is a branch through T. Call a $\leq$-decreasing sequence of conditions $\langle p_\alpha : \alpha \in \kappa \rangle$ a candidate sequence for $\dot{b}$ if $\langle \beta_{p_\alpha} : \alpha \in \kappa \rangle \subseteq \kappa$ is non-decreasing (this is always true, of course) and unbounded in κ. If $\langle p_\alpha : \alpha \in \kappa \rangle$ is a candidate sequence for $\dot{b}$ then $\langle s_{p_\alpha} : \alpha \in \kappa \rangle$ is a coherent collection of nodes with unbounded rank in T, and call the unique $b_p \in [T]$ such that $b_p \restriction \beta_{p_\alpha} = s_{p_\alpha}$ the candidate branch for $\dot{b}$ corresponding to $\langle p_\alpha : \alpha \in \kappa \rangle$. Because $\mathbb{P}$ is κ-strategically closed, candidate sequences and branches for $\dot{b}$ abound. For example, at every odd round $\beta + 1$ in the $G_\kappa(\mathbb{P})$-game Odd can always play $p_{\beta+1}$ which fixes the value of $\dot{b} \restriction \gamma_{\beta+1}$, and Even is always able to respond, in particular at limit stages, so as long as $\sup\{\gamma_{\beta+1} : \beta \in \kappa\} = \kappa$, the two players will have built a candidate sequence and branch. We use this observation to define a winning strategy for player II in $G(T, \emptyset, \kappa)$. Suppose player I plays α_0. Via a winning strategy for Even in the $G_\kappa(\mathbb{P})$ game, build a candidate sequence $\langle p^0_\alpha : \alpha \in \kappa \rangle$ and branch $b_0 \in T$. Find the minimal $\delta_0 \in \kappa$ which is either a limit or of the form $\xi + 2$ such that $\beta_{\delta_0} > \alpha_0$. Then, let $p^1_{\delta_0+1} \leq p^0_{\delta_0}$ be such that $s_{p^1_{\delta_0+1}} \neq b_0 \restriction \beta_{p^1_{\delta_0+1}}$, which is possible because all conditions force that $\dot{b}$ is not in V, so we can strengthen any particular condition to find one which fixes an initial segment of $\dot{b}$ which is different than the corresponding initial segment of $b_0 \in [T] \cap V$. Now, suppose that instead of playing $p^0_{\delta_0+1}$ at stage $\delta_0 + 1$, Odd plays $p^1_{\delta_0+1} \leq p^0_{\delta_0}$. Even can follow her winning strategy in this run of the game to produce then a candidate sequence $\langle p^0_\alpha : \alpha \in \delta_0 + 1 \rangle \frown \langle p^1_\alpha : \alpha \in [\delta_0 + 1, \kappa) \rangle$ and corresponding candidate branch $b_1 \in [T]$ such that $b_1 \restriction \beta_{p^i_\alpha} = s_{p^i_\alpha}$ for $i \in \{0, 1\}$, $\alpha \in \kappa$. Let player II play then $\{s^0_0 = b_0 \restriction \beta_{p^1_{\delta_0+1}}, s^0_1 = b_1 \restriction \beta_{p^1_{\delta_0+1}}\}$, which is a splitting pair that splits above α_0. Player I then chooses $s_1 = s^0_0$ or $s_1 = s^0_1$ and plays $\alpha_1 \geq \beta_{p^1_{\delta_0+1}}$. Player II can continue as before, namely if $s_1 = s^0_0$, then return to $\langle p^0_\alpha : \alpha \in \kappa \rangle$ and find the minimal δ_1 which is either a limit or of the form $\xi + 2$ such that $\beta_{\delta_1} > \alpha_1$ and proceed as above. If $s_1 = s^0_1$, then return instead to $\langle p^0_\alpha : \alpha \in \delta_0 + 1 \rangle \frown \langle p^1_\alpha : \alpha \in [\delta_0 + 1, \kappa) \rangle$ and find the minimal δ_1 which is either a limit or of the form $\xi + 2$ such that $\beta_{\delta_1} > \alpha_1$, and then again proceed as above. It is clear that player II can follow this strategy at successor stages. At limit stages, either the two players have played along a particular candidate branch which was built previously via Even's winning strategy in $G_\kappa(\mathbb{P})$, in which case it is clear that player II can offer a splitting pair whose common part extends the path that the two players have constructed so far and is of arbitrarily large height, or cofinally often player I has chosen the node corresponding to the "new" candidate branch's restriction. However, Even is still able to follow her winning strategy in $G_\kappa(\mathbb{P})$, so this is also not an issue. Specifically, suppose the two players have played up to stage $\eta \in \lim(\kappa)$ in $G(T, \emptyset, \kappa)$ in such a way that player I has chosen the "new" candidate-branch restriction cofinally often. Then for some $\xi \in \lim(\eta) \cup \{\eta\}$ we will have a descending sequence of conditions which is of the form $\langle p^0_\alpha : \alpha \in \delta_0 + 1 \rangle \frown \langle p^1_\alpha : \alpha \in [\delta_0 + 1, \delta_1 + 1) \rangle \frown \ldots \frown \langle p^\gamma_\alpha : \alpha \in [\delta_\gamma + 1, \delta_\gamma + 1) \rangle \frown \ldots$ for

$\gamma \in \xi$. Then Even plays $p^{\xi}_{\sup\{\delta_\gamma:\gamma\in\xi\}} \leq p^{\gamma}_{\alpha}$ for every $\gamma \in \xi$, $\alpha \in \sup\{\delta_\gamma : \gamma \in \xi\}$ according to her winning strategy in $G_\kappa(\mathbb{P})$, which forces in particular that $\dot{b} \restriction \sup\{\beta_{p^{\gamma}_{\delta_\gamma}} : \gamma \in \xi\}$ agrees with the path constructed so far. So player II can proceed in this case also. $\square$

Note 1.3.4. As mentioned earlier, this result can also be proved looking through the framework of Väänänen's game played on $[T]$. To illustrate, let's see how this would work in the (notationally simpler) case where $\mathbb{P}$ is κ-closed. So, let $\mathbb{P}$ be κ-closed, G be $(V, \mathbb{P})$-generic, $T \subseteq {}^{<\kappa}2$ be a tree in V, and fix a name $\dot{b}$ for an element of $([T])^{V[G]} \setminus ([T])^V$. We show that $\mathrm{Ker}([T], \kappa) \neq \emptyset$. It may be, of course, that for some particular $x_0 \in [T]$, player I even has a winning strategy in $G([T], x_0, \kappa)$, so the first step is to find a suitable x_0 at which to start the game, where actually player II will have a winning strategy in $G([T], x_0, \kappa)$. This is done as above, with a candidate branch. So in V using the κ-closure of $\mathbb{P}$, construct $\langle p_\xi : \xi \in \kappa\rangle$ and $\langle x_0(\xi) : \xi \in \omega_1\rangle$ so that $p_\xi \Vdash \dot{b}(\xi) = x_0(\xi)$. That is, $p_\xi \Vdash \dot{b} \restriction (\xi + 1) = x_0 \restriction (\xi + 1)$. Let player I play β_1. Note that $p_{\beta_1} \Vdash \dot{b} \restriction (\beta_1 + 1) = x_0 \restriction (\beta_1 + 1)$. Because $\dot{b}$ is a name for a branch not in V, there exists $\alpha > \beta_1$ and $p^1_0 \leq p_{\beta_1}$ so that $p^1_0 \Vdash \dot{b}(\alpha) \neq x_0(\alpha)$. Without loss of generality suppose p^1_0 fixes $\dot{b} \restriction (\alpha + 1)$. As before, build a descending sequence $\langle p^1_\xi : \xi \in \kappa\rangle$ below p^1_0 and a candidate branch $x_1 \in ([T])^V$ so that $p^1_\xi \Vdash \dot{b}(\xi) = x_1(\xi)$. Note that $x_1 \restriction \beta_1 = x_0 \restriction \beta_1$, but $x_1(\alpha) \neq x_0(\alpha)$. Let player II play x_2. Suppose player I plays $\beta_2 > \beta_1$. Note that $p^1_{\beta_2} \Vdash \dot{b} \restriction (\beta_2 + 1) = x_1 \restriction (\beta_2 + 1)$. We can proceed as above. At limit stages we have built a sequence of length $< \kappa$, e.g. $p \geq \ldots \geq p_{\beta_1} \geq p^1_0 \geq \ldots \geq p^1_{\beta_2} \geq \ldots \geq p^\alpha_0 \geq \ldots \geq p^\alpha_{\beta_{\alpha+1}} \geq \ldots$ for $\alpha \in \gamma$. There is a lower bound for this sequence, p^γ, and player II can construct a candidate branch in V for $\dot{b}$ below p^γ, x_γ, which necessarily will be distinct from all previous branches x_α played (because it agrees with each x_α up to level $\beta_{\alpha+1}$, and they split between successive $\beta_{\alpha+1}$ and $\beta_{\alpha+2}$ levels).

When $\kappa = \omega$, because all forcing posets have the property that a finite $\leq$-decreasing sequence has a lower bound (namely the strongest element in the sequence), the above method of proof provides an alternate proof of Mansfield's theorem 1.1.51 for the case of forcing extensions.

At no point is it necessary to use the absolute Cantor-Bendixson analysis of trees, as was required before:

Corollary (Alternate Mansfield Proof for Forcing Extensions) 1.3.5. Let $V \subseteq V[G]$ where G is $(V, \mathbb{P})$-generic for some $\mathbb{P} \in V$ and there exists $x \in ({}^{\omega}2)^{V[G]} \setminus ({}^{\omega}2)^{V}$. Then for $T \subseteq {}^{<\omega}2$ a tree in V, $([T])^V = ([T])^{V[G]}$ if and only if T contains a copy of the complete binary tree ${}^{<\omega}2$ (if and only if $[T]$ is not scattered, etc.).

Proof. By 1.3.3 if a branch is added to T then using $\dot{b}$ a name for a new branch, player II has a winning strategy in $G(T, \emptyset, \omega)$. By repeatedly applying this strategy it is not difficult to see that T contains a copy of ${}^{<\omega}2$. On the other hand, if T contains a copy of ${}^{<\omega}2$ then a new branch will be added to T in $V[G]$ (because reals are added). $\square$

In the early 1970s Silver proved that from the existence of an inaccessible cardinal, one may force a model where there are no Kurepa trees. A crucial step in this argument has since become known as Silver's lemma, which says in particular that σ-closed forcings can't add branches to ω_1-trees. The straightforward generalization is as follows, and is easily seen to be an immediate corollary of 1.3.3, which says that a larger class of forcing notions don't add branches to trees $T \subseteq {}^{<\kappa}2$ which satisfy the weaker requirement of not containing a copy of ${}^{<\kappa}2$.

Corollary (Silver's Lemma) [61] 1.3.6. Let κ be regular such that there exists $\mu \in \kappa$ with $2^{\mu} \geq \kappa$. Then κ-closed forcings do not add branches to κ-trees.

Proof. By 1.3.3 if any branches are added to T after forcing with $\mathbb{P}$ then T contains a copy of ${}^{<\kappa}2$, which is impossible if T is a κ tree and there exists $\mu \in \kappa$ with $2^{\mu} \geq \kappa$. $\square$

More generally, we can generalize 1.3.3 and observe that for any δ which is either a limit ordinal or the successor to a limit ordinal, if a forcing notion is δ-strategically closed and adds a branch to a tree, then this tree has a nonempty δ-kernel, which gives structural information about the tree. Typically δ will be either a regular cardinal μ, or of the form $\mu+1$. In terms of the consequence of the δ-kernel of a tree being nonempty that the tree contains a copy of the binary tree of height δ, the following proposition (in one form or another) is probably folklore.

Proposition 1.3.7. Let $T \subseteq {}^{<\kappa}2$ be a tree, let $\delta \leq \kappa$ be either a limit ordinal or the successor to a limit ordinal, let $\mathbb{P}$ be a δ-strategically closed forcing notion, and suppose that forcing with $\mathbb{P}$ adds a branch to T. Then $\mathrm{Ker}(T, \delta) \neq \emptyset$.

Proof. Note that if $\delta = \omega$, we have already shown this—$G(T, \emptyset, \omega)$ is determined, and if player I has a winning strategy then by 1.2.11, $T_{\mathrm{ht}_{CB}(T)} = \emptyset$, so by 1.3.1 no branches are added to T. More generally, let δ, $\mathbb{P}$, etc. be as stated. We use the same terminology as in 1.3.3. Let $\dot{b}$ be a name for a new branch through T. We describe a winning strategy for player II in $G(T, \emptyset, \delta)$. Suppose player I plays α_0 first. Find $p_0 \in \mathbb{P}$ such that $\beta_{p_0} \geq \alpha_0$. Suppose that Odd plays $q_1 = p_0$ as his first move in $G_\delta(\mathbb{P})$. Even responds following her winning strategy in $G_\delta(\mathbb{P})$ with q_2. There must exist $p_0', p_0'' \leq q_2$ such that $s_{p_0'} \restriction \min\{\beta_{p_0'}, \beta_{p_0''}\} \neq s_{p_0''} \restriction \min\{\beta_{p_0'}, \beta_{p_0''}\}$, because otherwise for G some $(V, \mathbb{P})$-generic with $q_2 \in G$, we'd have $\dot{b}^G \in V$. Let player II play $\{s_0^0 = s_{p_0'} \restriction \min\{\beta_{p_0'}, \beta_{p_0''}\}, s_1^0 = s_{p_0''} \restriction \min\{\beta_{p_0'}, \beta_{p_0''}\}\}$ as her response to α_0 in $G(T, \emptyset, \delta)$. Player I responds with $s_1 \in \{s_0^0, s_1^0\}$ and α_1. Whether or not $s_1 = s_0^0$ or $s_1 = s_1^0$, there exists $p_1' \leq p_0'$ or $p_1'' \leq p_0''$ such that $\beta_{p_1}' \geq \alpha_1$. Let Odd play this p_1' as q_3 in $G_\delta(\mathbb{P})$. Even responds following her winning strategy in $G_\delta(\mathbb{P})$ with q_4. As before, there must exist $p_1', p_1'' \leq q_4$ such that $s_{p_1'} \restriction \min\{\beta_{p_1'}, \beta_{p_1''}\} \neq s_{p_1''} \restriction \min\{\beta_{p_1'}, \beta_{p_1''}\}$. Let player II respond then with $\{s_0^1 = s_{p_1'} \restriction \min\{\beta_{p_1'}, \beta_{p_1''}\}, s_1^1 = s_{p_1''} \restriction \min\{\beta_{p_1'}, \beta_{p_1''}\}\}$ in $G(T, \emptyset, \delta)$. Note that $\{s_0^1, s_1^1\}$ is a splitting pair extending s_1 (which splits above α_1) by construction. Player II can continue to proceed in this way. Note that at limit stages $\eta \in \delta$, Even is able to play an appropriate q_η in the appropriate $G_\delta(\mathbb{P})$-game, so certainly the partial path constructed through T by the two players up to stage η in $G(T, \emptyset, \delta)$ can be extended in T, and then q_η can be strengthened to a condition which fixes the value of $\dot{b}$ up to α_η, and this condition must split into two conditions which fix different initial segments of $\dot{b}$, and then player II can use these different values to offer $\{s_0^\eta, s_1^\eta\}$ and proceed. It is clear that if δ is either a limit ordinal, or the successor to a limit ordinal, this procedure works. $\square$

1.4 A Cantor-Bendixson theorem for the κ-Cantor space

1.4.1 Väänänen's Cantor-Bendixson theorem

In [69] Väänänen proves the following theorem, which we phrase in terms of the space 2^{ω_1} instead of $\omega_1^{\omega_1}$ as he does (the proofs are identical).

Theorem (Theorem 4 in [69]) 1.4.1. *Assume $\mathcal{I}(\omega)$. Then if $E \subseteq 2^{\omega_1}$ is closed, $E = Ker(E, \omega_1) \cup Sc(E, \omega_1)$ with $|Sc(\omega_1)| \leq \omega_1$.*

Here $\mathcal{I}(\omega)$ is a hypothesis asserting the existence of a certain type of strong ideal over ω_2 (which is equiconsistent with a measurable cardinal). One may observe that it is straightforward to generalize the argument in 1.4.1 to any uncountable regular cardinal κ where the appropriately similar assertion of the existence of a certain type of strong ideal over κ^+ holds. Indeed, this is also noted in a comment by Sziráki and Väänänen (Remark 2.5 in [64]), where they extend these Cantor-Bendixson results beyond closed subsets and use the following generalization of $\mathcal{I}(\omega)$, $\mathcal{I}^-(\kappa)$:

Definition (from [64]) 1.4.2. Let $\mathcal{I}^-(\kappa)$ denote the hypothesis that there exists a $(< \kappa^+)$-complete, normal, non-principle, ideal $\mathcal{I}$ over κ^+ such that the collection of $\mathcal{I}^+$ sets has a $\subseteq$-dense subset in which every $\subseteq$-descending sequence of length less than κ has a lower bound in this subset.

Fact 1.4.3. $\mathcal{I}^-(\kappa)$ is consistent modulo large cardinals. For example, if we Lévy-collapse a measurable cardinal λ to ω_2, in the extension the poset Lévy-collapsing $j(\lambda)$ to ω_2 with countable conditions densely embeds into $P(\lambda)/\mathcal{I}$, and this poset is σ-closed. Here j is the elementary embedding witnessing the measurability of λ with respect to an ultrafilter U and $\mathcal{I}$ is the ideal generated in the extension by the dual to U. This gives us a $K \subseteq \mathcal{I}^+$ which is dense modulo $\mathcal{I}$ and has the property that every descending modulo $\mathcal{I}$ ω-sequence has a modulo $\mathcal{I}$-lower bound. This result is standard in the theory of precipitous ideals (see for example [35] for background). In this model CH holds, and indeed the existence of such an ideal implies CH. Consider $K' = \{Y : \exists X \in K \text{ s.t. } (X \setminus Y) \cup (Y \setminus X) \in I\}$. Then K' is $\subseteq$-dense, and also has the property that any countable modulo $\mathcal{I}$-descending sequence in K' has a modulo $\mathcal{I}$-lower bound, and indeed that any countable $\subseteq$-descending sequence in K' has a $\subseteq$-lower bound, because I is $(< \omega_2)$-complete (so ω_1-sized, and so in particular countable, unions of measure zero sets are of measure zero).

Using the same argument as in Väänänen's proof of 1.4.1, it is straightforward to show the following. For completeness, we describe later (see 1.9.27) how this method works:

Proposition 1.4.4. Assume $\mathcal{I}^-(\kappa)$. Then if $E \subseteq 2^\kappa$ is closed, $E = \mathrm{Ker}(E, \kappa) \cup \mathrm{Sc}(E, \kappa)$ with $|\mathrm{Sc}(E, \kappa)| \leq \kappa$.

As above, we can use a measurable cardinal to force a model where, for example, $\mathcal{I}(\omega)$ holds, and because the ideal given by $\mathcal{I}(\omega)$ is precipitous, the consistency strength of $\mathcal{I}(\omega)$ is exactly that of a measurable cardinal (see e.g. [35]). On the other hand, if $T \subseteq {}^{<\kappa}2$ is a Kurepa tree then $|[T]| \geq \kappa^+$ yet clearly player II cannot have a winning strategy in $G([T], x_0, \kappa)$ for any $x_0 \in [T]$ because then by 1.2.5 T contains a copy of ${}^{<\kappa}2$, which is not true of a Kurepa tree. So even in the case where $G([T], x_0, \kappa)$ is determined for every $x_0 \in [T]$, we would then necessarily have $|\mathrm{Sc}([T], \kappa)| \geq \kappa^+$. Therefore because the nonexistence of Kurepa trees in, for example ${}^{<\omega_1}2$, is equiconsistent with the existence of an inaccessible cardinal (see e.g. [35]), Väänänen's Cantor-Bendixson dichotomy as in 1.4.1 has consistency strength somewhere between that of an inaccessible and a measurable cardinal. Noting exactly this in [69], Väänänen asks what the exact consistency strength is. More recently, Sziráki and Väänänen again ask (in particular) what the exact consistency strength of Väänänen's Cantor-Bendixson theorem is, noting that it is somewhere between an inaccessible and a measurable cardinal (see [64]).

1.4.2 A tree decomposition

Here we resolve the above consistency question by showing in particular that for a regular cardinal κ and an inaccessible cardinal $\lambda > \kappa$, in the Lévy collapse of λ to κ^+ with conditions of size $< \kappa$, not only are there no Kurepa trees $T \subseteq {}^{<\kappa}2$ in the extension, but Väänänen's Cantor-Bendixson dichotomy holds. So the consistency strength is exactly that of an inaccessible. We phrase things in terms of our cut-and-choose game in order to emphasize that there is also a decomposition of all trees into a κ-kernel and κ-scattered part in this model, so that the Cantor-Bendixson dichotomy of Väänänen will follow as a corollary. However the same argument could be used directly to show, e.g. Väänänen's dichotomy. The outline of the argument is straightforward: Because κ-closed forcings not only do not add branches to κ-trees, but also do not add branches to trees which don't contain a copy of ${}^{<\kappa}2$, in the extension obtained by Lévy collapsing λ to κ^+, any tree which has more than κ-many branches must contain a copy of ${}^{<\kappa}2$, which means that player II will have a winning strategy in some $G(T, s_0, \kappa)$, while otherwise player I will have a winning strategy by diagonalizing against the $\leq \kappa$-many branches.

Theorem 1.4.5. *Let κ be a regular cardinal and let λ be a strong inaccessible cardinal with $\kappa < \lambda$. Let $\mathbb{P} = Col(\kappa, < \lambda)$. Then if G is $(V, \mathbb{P})$-generic and $T \subseteq {}^{<\kappa}2$ is a tree in $V[G]$, $T = Ker(T, \kappa) \cup Sc(T, \kappa)$. Furthermore, $[Ker(T, \kappa)] = Ker([T], \kappa)$ and $[Sc(T, \kappa)] = Sc([T], \kappa)$ with $|[Sc(T, \kappa)]| \leq \kappa$.*

Proof. Let κ be a regular cardinal and $\lambda > \kappa$ be an inaccessible cardinal. Let $\mathbb{P}$ be the Lévy collapse of λ to κ^+, that is $\mathbb{P} = \mathrm{Col}(\kappa, <\lambda)$. For any missing details of the following argument (e.g. about product forcing, etc.) see for example [35]. It is straightforward to see that for any $\gamma \in [\kappa, \lambda)$, $\mathbb{P}_\gamma = \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq \gamma \times \kappa\}$ and $\mathbb{P}^\gamma = \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq (\lambda \setminus \gamma) \times \kappa\}$ are such that $\mathbb{P}$ is isomorphic to the product $\mathbb{P}_\gamma \times \mathbb{P}^\gamma$. Let G be $(V, \mathbb{P})$-generic and let $T \subseteq {}^{<\kappa}2$ be a tree in $V[G]$ which does not contain a copy of ${}^{<\kappa}2$. We show that $(|[T]| \leq \kappa)^{V[G]}$. Note that $({}^{<\kappa}2)^V = ({}^{<\kappa}2)^{V[G]}$ because $\mathbb{P}$ is κ-closed. Viewing T as simply a subset of ${}^{<\kappa}2$, we can fix a nice name for T, $\dot{T}$. $\dot{T}$ is of the form $\bigcup\{\{\check{s}\} \times A_s : s \in {}^{<\kappa}2\}$ where each A_s is an antichain in $\mathbb{P}$. Because $\mathbb{P}$ is λ-c.c., every $|A_v| < \lambda$. If we want to include a tree relation or be more general, we can simply e.g. view for some $\delta \in \lambda$, $T = \langle \delta, X \rangle$ as a tree over δ where $X \subseteq \delta \times \delta$—the argument is the same. In any case, there exists $\gamma \in [\kappa, \lambda)$ such that for every $s \in {}^{<\kappa}2$ and every $p \in A_s$, $\mathrm{dom}(p) \subseteq \gamma \times \kappa$. We may view $V[G]$ as $V[G_0][G_1]$, where G_0 is $(V, \mathbb{P}_\gamma)$-generic and G_1 is $(V[G_0], \mathbb{P}^\gamma)$-generic. Note that $T \in V[G_0]$ and also that because $\mathbb{P}_\gamma$ is κ-closed, every $(<\kappa)$-decreasing sequence in $\mathbb{P}^\gamma$ in $V[G_0]$ is in V and so has a lower bound, i.e. $(\mathbb{P}^\gamma \text{ is } \kappa\text{-closed})^{V[G_0]}$. Because T does not contain a copy of ${}^{<\kappa}2$ in $V[G_0]$, we may then apply 1.3.3 and conclude that $([T])^{V[G_0]} = ([T])^{V[G_0][G_1]}$. However, λ is inaccessible in $V[G_0]$, so $(|[T]| < \lambda)^{V[G_0]}$, and so because all cardinals in $V[G_0]$ strictly between κ and λ are collapsed to κ in $V[G_0][G_1]$, $(|[T]| \leq \kappa)^{V[G_0][G_1]}$. Now working in $V[G]$, let $T \subseteq {}^{<\kappa}2$ be a tree and let $s_0 \in T$. If $|[T \restriction s_0]| \leq \kappa$, then by 1.2.14 player I has a winning strategy in $G(T, s_0, \kappa)$. On the other hand if $|[T \restriction s_0]| \geq \kappa^+$, then we have seen that $T \restriction s_0$ must contain a copy of ${}^{<\kappa}2$, so by 1.2.13 player II has a winning strategy in $G(T, s_0, \kappa)$. Therefore $T = \mathrm{Ker}(T, \kappa) \cup \mathrm{Sc}(T, \kappa)$, as desired. Next, suppose towards a contradiction that $|[\mathrm{Sc}(T, \kappa)]| \geq \kappa^+$. Then because $|{}^{<\kappa}2| = \kappa$, there must exist some $s_0 \in T$ such that $s_0 \in \mathrm{Sc}(T, \kappa)$ and $|[T \restriction s_0]| \geq \kappa^+$. However, then $T \restriction s_0$ contains a copy of ${}^{<\kappa}2$ so in fact $s_0 \in \mathrm{Ker}(T, \kappa)$, a contradiction. Finally we show that $[\mathrm{Ker}(T, \kappa)] = \mathrm{Ker}([T], \kappa)$ and $[\mathrm{Sc}(T, \kappa)] = \mathrm{Sc}([T], \kappa)$. Let $x_0 \in [\mathrm{Ker}(T, \kappa)]$. Suppose player I plays α_1 as his first move in $G([T], x_0, \kappa)$. Then $x_0 \restriction \alpha_0 \in \mathrm{Ker}(T, \kappa)$, so a copy of ${}^{<\kappa}2$ can be built inside of

$T \restriction (x_0 \restriction \alpha_1)$. By successively choosing branches through this copy, it is straightforward to see that player II has a winning strategy in $G([T], x_0, \kappa)$ when player I chooses α_1 first. So $[\mathrm{Ker}(T, \kappa)] \subseteq \mathrm{Ker}([T], \kappa)$. On the other hand, suppose that $x_0 \in [\mathrm{Sc}(T, \kappa)]$, that is suppose there exists α_1 such that $x_0 \restriction \alpha_1 \in \mathrm{Sc}(T, \kappa)$. We need to show that $x_0 \in \mathrm{Sc}([T], \kappa)$. However, because $|[\mathrm{Sc}(T, \kappa)]| \leq \kappa$, $|[T \restriction (x_0 \restriction \alpha_1)]| \leq \kappa$, so by 1.2.15, $x_0 \in \mathrm{Sc}([T], \kappa)$. So in particular, Väänänen's Cantor-Bendixson dichotomy holds in $V[G]$. $\square$

1.5 Tree structure implications of player I having a winning strategy in the cut-and-choose game

In the previous sections we have seen in several cases where Väänänen's game and our cut-and-choose game behave similarly—in suitable models they can both provide a framing for Cantor-Bendixson-type theorems, when played to length ω they both characterize perfectness and scatteredness for closed subsets of 2^ω or 2^κ for κ weakly compact, and when played to length κ player II having a winning strategy in either game is just a restatement of the condition that a tree embeds a copy of ${}^{<\kappa}2$, while player I in both games can sometimes have a winning strategy via a diagonalization process, for example. We will see in this section however that the games can in fact behave quite differently from one another, for example we will see that strong structural requirements are exerted on the tree if player I is to have a winning strategy in the cut-and-choose-game which aren't necessarily required for player I to have a winning strategy in Väänänen's game. The specific case where $\mu = \omega$ and $\kappa = \omega_1$ in 1.5.4 and 1.5.8 has been independently proven and investigated by König, see [43], using a different sort of game. We first need some definitions.

Definition 1.5.1. Let κ be regular and μ be a cardinal less than κ. Let $T^\kappa_{<\mu} \subseteq {}^{<\kappa}2$ denote the tree comprising all $s \in {}^{<\kappa}2$ such that $|\{\alpha \in \mathrm{lh}(s) : s(\alpha) = 1\}| < \mu$. Note that $[T] = \{b \in {}^\kappa 2 : |\{\alpha : b(\alpha) = 1\}| < \mu\}$ and that T is an everywhere splitting tree coding a closed subset of 2^κ.

Definition 1.5.2. Let $\mu \leq \kappa \leq \lambda^+$ be cardinals with μ and κ regular. Say that a set $X \subseteq P_\kappa\lambda$ is μ-closed if and only if for every $\subseteq$-increasing sequence $\langle x_\alpha : \alpha \in \mu\rangle \subseteq X$, $\bigcup\limits_{\beta\in\mu} x_\beta \in X$. Say that a set $X \subseteq P_\kappa\lambda$ is a μ-club if and only if it is μ-closed and cofinal in $P_\kappa\lambda$, that is for every $y \in P_\kappa\lambda$, there exists $x \in X$ with $y \subseteq X$. In the case where $\kappa = \mu^+$, we will consider μ-clubs in $[Z]^\mu$ for non-ordinal sets Z.

We are interested in the structural properties that a tree $T \subseteq {}^{<\kappa}2$ must satisfy if player I has a winning strategy in the cut-and-choose game. We have seen what the tree must satisfy if player I is to have a winning strategy in the game of length ω. The most natural game to consider is then the game of length $\omega + 1$, and more generally of length $\mu + 1$ for μ a regular cardinal less than κ. In these games there is a final round of play where player II must be able to extend the path constructed over the first μ-rounds, i.e. this path needs to not be maximal, in particular. Intuitively then, for player I to have a winning strategy in $G(T, \emptyset, \mu+1)$, he must be able to enforce locally along the way that the path constructed by the two players up to stage μ will be maximal in T. If player I is not able to do this, then player II might win that particular run of the game, because the path jointly built by the two players up to stage μ might be extendable in T. The way that we make this intuition precise involves the use of submodels (of e.g. H_θ) of a certain type, which we call guessing models, whose existence allows player I to, in a sense, "act locally but affect globally." The formulation of definition 1.5.3 came up naturally when investigating how PFA might affect the determinacy of the cut-and-choose game, because using models as side conditions is a common way to attempt to ensure properness of certain forcing notions and the apparent interaction requirements between the side conditions and the working part of the forcing conditions seemingly necessitated by these forcings led to the definition as in 1.5.3.

Definition 1.5.3. Let $T \subseteq {}^{<\kappa}2$ be a tree, let θ be a regular cardinal sufficiently larger than κ, and let $M \prec H_\theta$ be an elementary submodel with $T \in M$ and $|M| < \kappa$. Let

$\delta_M = \sup(M \cap \kappa)$. Say that M is T-guessing, or say that M guesses T, if and only if for every $s \in \mathrm{Lev}_{\delta_M}(T)$ such that $s \restriction \beta \in M$ for cofinally many $\beta \in \delta_M$, there exists $b \in [T] \cap M$ such that $b \restriction \delta_M = s$.

Theorem 1.5.4. *Let κ be regular, let $\mu \in \kappa$ be a regular cardinal, and let $T \subseteq {}^{<\kappa}2$ be a cofinally splitting tree. Then player I has a winning strategy in $G(T, \emptyset, \mu+1)$ if and only if there exists a μ-club $\mathcal{C} \subseteq [H_\theta]^\mu$ of T-guessing submodels $M \prec H_\theta$ of size μ for some sufficiently large θ.*

Proof. Suppose first that there exists a μ-club $\mathcal{C} \subseteq [H_\theta]^\mu$ of T-guessing submodels $M \prec H_\theta$. We produce a winning strategy for player I in $G(T, \emptyset, \mu+1)$. The idea is simply to have player I diagonalize against all branches contained in any relevant model, of which there will be at most μ-many. First, fix $f : \mu \to \mu \times \mu$ to be a surjection so that for every $\langle \xi, \nu \rangle \in \mu \times \mu$, $|f^{-1}[\langle \xi, \nu \rangle] \cap \mathrm{succ}(\mu)| = \mu$. For every $M \in \mathcal{C}$, let $\delta_M = \sup(M \cap \kappa)$ and fix $e_M : \mu \to \delta_M$ to be a strictly increasing function whose range is cofinal in δ_M. Also, let $g_M : \mu \to M \cap [T]$ be a surjection. In the course of defining a winning strategy for player I, we are going to construct a $\subseteq$-chain of models $M_0 \subseteq M_1 \subseteq \ldots$ in $\mathcal{C}$—at stages $\beta+1$ in the game, player I is going to check whether $\{s_0^\beta, s_1^\beta\} \subseteq M_\beta$ and define $M_{\beta+1}$ and $s_{\beta+1}, \alpha_{\beta+1}$ accordingly. At limit stages β where player I only plays α_β, this will be defined in conjunction with M_β. First, choose $M_0 \in \mathcal{C}$ arbitrarily and let player I play $\alpha_0 = e_{M_0}(0)$. Player II responds with $\{s_0^0, s_1^0\}$ splitting above α_0. Player I plays in round 1 as follows. Either $\{s_0^0, s_1^0\} \subseteq M_0$ or not. Suppose first that $\{s_0^0, s_1^0\} \subseteq M_0$, let $M_1 = M_0$ and consider $f(1)$. If $f(1) = \langle 1, \xi \rangle$ for some $\xi \in \mu$ or $f(1) = \langle 0, \xi \rangle$ for some $\xi \in \mu$, consider $g_{M_1}(\xi) \in M_1 \cap [T]$ or $g_{M_0}(\xi) \in M_1 \cap [T]$, respectively. For at least one $i \in \{0,1\}$, $g_{M_1}(\xi) \restriction \mathrm{lh}(s_i^0) \neq s_i^0$ (or in the case where $f(1) = \langle 0, \xi \rangle$, $g_{M_0}(\xi) \restriction \mathrm{lh}(s_i^0) \neq s_i^0$). Let player I choose such an $s_1 \in \{s_0^0, s_1^0\}$ and play $\alpha_1 = \sup\{e_{M_0}(1), e_{M_1}(1), \mathrm{lh}(s_1)\}$. If any of these conditions fails, let player I choose s_1 arbitrarily and play the same α_1. On the other hand, suppose that $\{s_0^0, s_1^0\}$ is not a subset of M_0. In this case find $M_1 \in C$ with $M_0 \cup \{s_0^0, s_1^0\} \subseteq M_1$ and proceed much as above: consider first $f(1)$. If $f(1)$ is of the form $\langle 1, \xi \rangle$ for some $\xi \in \mu$ or of the form $\langle 0, \xi \rangle$ for some $\xi \in \mu$, consider $g_{M_1}(\xi) \in M_1 \cap [T]$ or $g_{M_0}(\xi) \in M_1 \cap [T]$, respectively, and then for at least one $i \in \{0,1\}$, $g_{M_1}(\xi) \upharpoonright \mathrm{lh}(s_i^0) \neq s_i^0$ (or in the case where $f(1) = \langle 0, \xi \rangle$, $g_{M_0}(\xi) \upharpoonright \mathrm{lh}(s_i^0) \neq s_i^0$). Let player I choose such an $s_1 \in \{s_0^0, s_1^0\}$ and play $\alpha_1 = \sup\{e_{M_0}(1), e_{M_1}(1), \mathrm{lh}(s_1)\}$. If any of these conditions fails, let player I choose s_1 arbitrarily and play the same α_1. Player II responds with $\{s_0^1, s_1^1\}$ extending s_1 splitting above α_1.

Player I proceeds in this fashion. At limit stages $\beta \in \mu$, we have constructed a $\subseteq$-increasing sequence $\langle M_\eta : \eta \in \beta \rangle \subseteq \mathcal{C}$. Let $M_\beta \in \mathcal{C}$ be such that $\bigcup\limits_{\eta \in \beta} M_\eta \subseteq M_\beta$, and let player I play $\alpha_\beta = \sup\{e_{M_\eta}(\beta), \alpha_\xi : \xi \in \beta, \eta \in \beta + 1\}$. Player II responds with $\{s_0^\beta, s_1^\beta\}$ extending the path of length $\sup\{\alpha_\eta : \eta \in \beta\}$ constructed so far splitting above α_β. Now at successor stages $\beta + 1 \in \mu$, player I chooses an element from a splitting pair $\{s_0^\beta, s_1^\beta\}$ and plays some $\alpha_{\beta+1}$. First player I defines $M_{\beta+1}$ accordingly based on whether or not $\{s_0^\beta, s_1^\beta\} \subseteq M_\beta$. Then player I looks at $f(\beta+1)$ and if it is of the form $\langle \gamma, \xi \rangle$ for some $\gamma \in \beta+2$ and $\xi \in \mu$, player I chooses $s_{\beta+1} \in \{s_0^\beta, s_1^\beta\}$ so as to be incompatible with the ξ^{th} branch through T in M_γ under the enumeration g_{M_γ}. Player I also plays $\alpha_{\beta+1} = \sup\{e_{M_\eta}(\beta+1), \mathrm{lh}(s_{\beta+1}) : \eta \in \beta+2\}$. Now, suppose that the two players play μ-many rounds. We need to see that they have built a path through T which cannot be extended. Suppose first that for some $\beta \in \mu$, for every $\gamma \in [\beta, \mu)$, $M_\gamma = M_\beta$. By construction $\sup\{\alpha_\eta : \eta \in \mu\} = \delta_{M_\beta}$, and towards a contradiction assume that $s \in \mathrm{Lev}_{\delta_{M_\beta}}(T)$ extends the path constructed. By hypothesis $s \upharpoonright \zeta \in M_\beta$ for cofinally many $\zeta \in \mu$ and M_β is T-guessing, so there must exist $\xi \in \mu$ such that $g_{M_\beta}(\xi) \upharpoonright \delta_{M_\beta} = s$. However, at some sufficiently large round $\gamma + 1 \in [\beta, \mu)$, we have $f(\gamma+1) = \langle \beta, \xi \rangle$, and so player I chose $s_{\gamma+1}$ to not be an initial segment of $g_{M_\beta}(\xi)$, which is a contradiction. On the other hand, suppose that there are μ-many distinct M_ξ's in $\langle M_\alpha : \alpha \in \mu \rangle$. Because $\mathcal{C}$ is μ-closed, $M_\mu = \bigcup\limits_{\alpha \in \mu} M_\alpha \prec H_\theta$. Note that $\delta_{M_\mu} = \sup\{\delta_{M_\alpha} : \alpha \in \mu\} = \sup\{\alpha_\beta : \beta \in \mu\}$ in this case. Again, suppose towards a contradiction that $s \in \mathrm{Lev}_{\delta_{M_\mu}}(T)$ extends the path constructed. By hypothesis $s \upharpoonright \zeta \in M_\mu$ for cofinally many $\zeta \in \delta_{M_\mu}$ and M_μ is T-guessing, so for some $b \in M_\mu \cap [T]$, $b \upharpoonright \delta_{M_\mu} = s$. Then for some $\xi, \zeta \in \mu$, $g_{M_\xi}(\zeta) \upharpoonright \delta_{M_\mu} = s$ (that is

b is the ζ^{th} branch of M_ξ). However at some sufficiently large round $\gamma + 1 \in [\xi, \mu)$ we have $f(\gamma + 1) = \langle \xi, \zeta \rangle$, and player I chose $s_{\gamma+1}$ to not be an initial segment of $g_{M_\xi}(\zeta)$, which is a contradiction.

Next, suppose that τ is a winning strategy for player I in $G(T, \emptyset, \mu + 1)$. For simplicity initially, suppose that $\mu^{<\mu} = \mu$. Let $M \prec H_\theta$ with $\{T, \tau, \mu\} \subseteq M$, $|M| = \mu$, and ${}^{<\mu}M \subseteq M$. We show that M is T-guessing, which suffices because the collection of such models constitutes a μ-club in $[H_\theta]^\mu$. Fix $s \in \mathrm{Lev}_{\delta_M}(T)$ such that $s \restriction \beta \in M$ for cofinally many $\beta \in \delta_M$. Either $s \restriction \beta$ is a splitting node for cofinally many $\beta \in M \cap \delta_M$ or not. Now because we have assumed T is cofinally splitting, this will be true, but even if not the task is easy in this case, as follows. Note that working in M there exists $\alpha \in \kappa$ such that for every $\alpha' \geq \alpha$, there is a unique node $s_{\alpha'}$ in $\mathrm{Lev}_{\alpha'}(T)$ such that $s' \restriction \alpha = s \restriction \alpha$. So by elementarity, there exists a branch $b \in [T] \cap M$ such that $b \restriction \delta_M = s$. Therefore we may assume that $s \restriction \beta$ is a splitting node for cofinally many $\beta \in M \cap \delta_M$. Construct in V a run of the game $\overline{x}$ where player I is playing according to τ and which is maximal below s according to M. Here is what we mean by this. First,

$$\overline{x} = \langle \langle \{\alpha_0\}, \{s_0^0, s_1^0\} \rangle, \langle \{\alpha_1, s_1\}, \{s_0^1, s_1^1\} \rangle, \langle \{\alpha_2, s_2\}, \{s_0^2, s_1^2\} \rangle, \ldots, \langle \{\alpha_\xi, s_\xi\} \rangle \rangle$$

Here by player I playing according to τ in $\overline{x}$, we mean that $\alpha_0 = \tau(\emptyset)$, $\{\alpha_1, s_1\} = \tau(\langle \{\alpha_0\}, \{s_0^0, s_1^0\} \rangle)$, $\{\alpha_2, s_2\} = \tau(\langle \langle \alpha_0, \{s_0^0, s_1^0\} \rangle, \langle \{\alpha_1, s_1\}, \{s_0^1, s_1^1\} \rangle \rangle)$, etc.. It may be that the final move made by player I, which we have written here as $\{\alpha_\xi, s_\xi\}$, is simply $\{\alpha_\xi\}$, which will be the case if ξ is a limit ordinal. We also insist that all ordinals and nodes played, as well as initial segments of $\overline{x}$, are in M and that $s_\eta = s \restriction \mathrm{lh}(s_\eta)$ for every $\eta \leq \xi$—that is we insist that this is a play along s. Finally, we insist that this play is maximal along s, by which we mean that no matter what move player II makes next in M, τ dictates that player I choose a node incomparable with s. Specifically, if ξ is a successor, then

$$\{\alpha_\xi, s_\xi\} = \tau(\langle\langle\{\alpha_0\}, \{s_0^0, s_1^0\}\rangle, \ldots, \langle\{\alpha_{\xi-1}\} \text{ or } \{\alpha_{\xi-1}, s_{\xi-1}\}, \{s_0^{\xi-1}, s_1^{\xi-1}\}\rangle\rangle)$$

and no matter what pair $\{s_0^\xi, s_1^\xi\} \subseteq M$ player II plays, τ dictates that player I chooses $s_{\xi+1} \in \{s_0^\xi, s_1^\xi\}$ so that $s_{\xi+1} \neq s \restriction \mathrm{lh}(s_{\xi+1})$. Similarly if ξ is a limit, for $\{\alpha_\xi\} = \tau(\langle\langle\{\alpha_0\}, \{s_0^0, s_1^0\}\rangle, \ldots\rangle)$, no matter what pair $\{s_0^\xi, s_1^\xi\} \subseteq M$ player II plays, τ dictates that player I chooses $s_{\xi+1} \in \{s_0^\xi, s_1^\xi\}$ so that $s_{\xi+1} \neq s \restriction \mathrm{lh}(s_{\xi+1})$. Now, because τ is a winning strategy for player I in $G(T, \emptyset, \mu+1)$, we can indeed build such an $\overline{x}$ and it must be that $\xi \in \mu$, because all plays made are along s, so if $\xi = \mu$ at stage μ player II would be able to play, a contradiction. That is, by following τ playing along s, which is cofinally splitting in M below δ_M, we have ensured that player II can always play in M at limit stages. Because ${}^{<\mu}M \subseteq M$, $\overline{x} \in M$ and is maximal along s in the sense that we have described. We show that in this case a branch extending s, $b \in [T] \cap M$, can be defined from $\overline{x}, \tau, s \restriction \alpha_\xi \in M$. If $\mu^{<\mu} > \mu$ however, proceeding literally as above will not allow us to guarantee that such an $\overline{x}$ exists in M. However, this problem can be solved by assuming that $M \prec (H_\theta, \trianglelefteq, \ldots)$ where $\trianglelefteq$ is a predicate for a well-ordering of H_θ. In this case given $s \in \mathrm{Lev}_{\delta_M}(T)$, one can build the unique "$\trianglelefteq$-minimal" maximal run $\overline{x}_s$ according to τ below s, which is done as above but at each stage letting $\{s_0^\beta, s_1^\beta\}$ be the the $\trianglelefteq$-minimal splitting pair along s below level δ_M which is a legal move at round β, and τ choosing along s, etc. as before. Furthermore, even though $s \notin M$, for some sufficiently large $\zeta \in M \cap \delta_M$ (ζ larger than α_ξ, for example) we will have $s \restriction \zeta \in M$ and $\overline{x}_s = \overline{x}_{s\restriction\zeta}$. But $\overline{x}_{s\restriction\zeta} \in M$ by elementarity. Now, working in M, let b be a path of maximal length such that $b \restriction \alpha_\xi = s \restriction \alpha_\xi$ with the property that for every node in b having an immediate predecessor which splits in T, if player II were to play this splitting pair, τ would choose the node outside of b. Specifically, b has the property that for every $s' \in b$ such that $\mathrm{lh}(s') \in \mathrm{succ}(\kappa)$, $\mathrm{lh}(s') > \alpha_\xi$, and for the unique $s'' \in b$, $i \in \{0,1\}$ with $s'' \frown i = s'$, $\{s'' \frown 0, s'' \frown 1\} \subseteq T$ and if player II played $\{s_0^\xi = s'' \frown 0, s_1^\xi = s'' \frown 1\}$ as her next move in the run of the game following $\overline{x}$, that τ chooses $s_{\xi+1} \neq s'$. We need to show that $b \in [T]$ and $b \restriction \delta_M = s$. First suppose that $b \notin [T]$, that is $\sup\{\mathrm{lh}(s') : s' \in b\} \in \kappa$.

Because this ordinal must be in M, $\sup\{\mathrm{lh}(s') : s' \in b\} \in \delta_M$. The collection of $s \restriction \alpha \in M$ is cofinal below s, so if for every $\alpha \in \sup\{\mathrm{lh}(s') : s' \in b\}$, $s \restriction \alpha = b \restriction \alpha$, this is a contradiction because b is maximal. Then for some $\alpha \in \sup\{\mathrm{lh}(s') : s' \in b\}$, $b \restriction \alpha \neq s \restriction \alpha$. Find the minimal such $\alpha + 1$ (this ordinal necessarily is a successor), which is in M because M can define the splitting point between b and some sufficiently large $s \restriction \beta$. Suppose without loss of generality that $s(\alpha) = 0$, so $b(\alpha) = 1$. But then when presented with the one point extension $\{s_0^\xi = s \restriction (\alpha + 1), s_1^\xi = b \restriction (\alpha + 1)\}$, by the definition of b we have that τ chooses along s, which is a contradiction. Thus $b \in [T]$. Furthermore, by the same argument we just gave, it must be that $b \restriction \delta_M = s$, as desired. $\square$

Sometimes it is easy to see that player I has a winning strategy in $G(T, \emptyset, \mu + 1)$—for example if T is isomorphic to a subtree of $T^\kappa_{<\mu}$. In this case player I is offered a splitting pair $\{s_0^\alpha, s_1^\alpha\}$ at every round $\alpha + 1$, and can simply choose $s_{\alpha+1} \in \{s_0^\alpha, s_1^\alpha\}$ to be one so that there exists $\beta \in (\sup\{\mathrm{lh}(s_\gamma) : \gamma \in \alpha\}, \mathrm{lh}(s_\alpha))$ with $s_\alpha(\beta) = 1$, which must exist. This must be a winning strategy because if player II could play at the μ^{th} round in particular, she must play $\{s_0^\mu, s_1^\mu\}$ with $|\{\alpha \in \mathrm{lh}(s_i^\mu) : s_i^\mu(\alpha) = 1\}| \geq \mu$ for $i \in \{0, 1\}$, which is a contradiction. Somewhat surprisingly perhaps it turns out that, in the case of trees with levels of size $\leq \mu$, these are exactly the trees in which player I has a winning strategy. To prove this, we define first a game similar to $G(T, s_0, \delta)$, but which is more difficult for player I to win and does not share all of the properties of $G(T, s_0, \delta)$, for example the connection to the Cantor-Bendixson process on trees 1.2.11, that we have discussed.

Definition 1.5.5. Let κ be regular, $T \subseteq {}^{<\kappa}2$ be a tree, and $s_0 \in T$. Define the two player game of length $\delta \leq \kappa$ starting at s_0 played on T, $G_2(T, s_0, \delta)$ to be the same as $G(T, s_0, \delta)$ except that at every round $\beta \in \delta$, player I is forced to play $\alpha_\beta = \sup\{\mathrm{lh}(s_\gamma) : \gamma \in \beta\}$. That is, player I no longer chooses levels at any stage—he only chooses nodes at successor stages (and at limits e.g. plays the supremum of the levels played already). Accordingly, when describing play in this game we often ignore any ordinal plays by player I and just imagine that at successor stages player I chooses a node, and at limit stages does nothing.

Remark 1.5.6. For $\mu < \kappa$, it may be that $G_2(T, \emptyset, \mu+1)$ and $G(T, \emptyset, \mu+1)$ behave quite differently. For example, suppose $T = \{s \in {}^{<\kappa}2 : |\{\alpha \in [\mu, \kappa) : s(\alpha) = 1\}| < \omega\}$. Then T is an everywhere splitting tree coding a closed subset of 2^κ, and it is easy to see that player II has a winning strategy in $G_2(T, \emptyset, \mu+1)$, while player I has a winning strategy in $G(T, \emptyset, \mu+1)$. Indeed, player I has a winning strategy in $G(T, \emptyset, \omega+1)$.

Proposition 1.5.7. Let $\kappa = \mu^+$ with μ regular, and let $T \subseteq {}^{<\kappa}2$ be a cofinally splitting tree such that for every $\alpha \in \kappa$, $|\mathrm{Lev}_\alpha(T)| \leq \mu$. Then player I has a winning strategy in $G(T, \emptyset, \mu+1)$ if and only if player I has a winning strategy in $G_2(T, \emptyset, \mu+1)$.

Proof. Player I can use a winning strategy in $G_2(T, \emptyset, \mu+1)$ to define a winning strategy in $G(T, \emptyset, \mu+1)$, so we only need to see the other direction. Let τ be a winning strategy for player I in $G(T, \emptyset, \mu+1)$. This argument resembles the diagonalization across models method used in 1.5.4. The idea is simple: diagonalize against every node at the levels of the tree dictated by τ as long as player II is playing below these levels, then follow τ when player II plays above. Suppose $\tau(\emptyset) = \alpha_0$. Let $f_0 : \mu \to \mathrm{Lev}_{\alpha_0}(T)$ be a surjection so that $|f^{-1}[s] \cap \mathrm{succ}(\mu)| = \mu$ for every $s \in \mathrm{Lev}_{\alpha_0}(T)$. As long as player II plays $\{s_0^\xi, s_1^\xi\}$ so that $\mathrm{lh}(s_0^\xi) = \mathrm{lh}(s_1^\xi) < \alpha_0$, let player I choose $s_{\xi+1} \in \{s_0^\xi, s_1^\xi\}$ so that $s_{\xi+1}$ is not an initial segment of $f(\xi+1)$. If player II plays all of her moves below level α_0, then this describes a winning strategy for player I by diagonalization. Otherwise, let β_0 be minimal so that the splitting point of $\{s_0^{\beta_0}, s_1^{\beta_0}\}$ is at a level $\geq \alpha_0$. In this case, let player I play s_{β_0+1} corresponding to the play that τ would dictate following a first play by player II in $G(T, \emptyset, \mu+1)$ of $\{s_0^{\beta_0}, s_1^{\beta_0}\}$ following $\tau(\emptyset) = \alpha_0$. That is, suppose $\tau(\langle \emptyset, \{s_0^{\beta_0}, s_1^{\beta_0}\}\rangle) = \{\alpha_1, s_1\}$ and let $s_{\beta_0+1} = s_1$. Next, let $f_1 : \mu \to \mathrm{Lev}_{\alpha_1}(T)$ be a surjection so that $|f^{-1}[s] \cap \mathrm{succ}(\mu)| = \mu$ for every $s \in \mathrm{Lev}_{\alpha_1}(T)$. Much as before, as long as player II continues to play $\{s_0^\xi, s_1^\xi\}$ so that $\mathrm{lh}(s_0^\xi) = \mathrm{lh}(s_1^\xi) < \alpha_1$, let player I choose $s_{\xi+1} \in \{s_0^\xi, s_1^\xi\}$ so that $s_{\xi+1}$ is not an initial segment of $f(\xi+1)$. Here of course, $\xi > \beta_0 + 1$. Let $\beta_1 > \beta_0$ be minimal so that the splitting point of $\{s_0^{\beta_1}, s_1^{\beta_1}\}$

is at a level $\geq \alpha_1$. As before, let player I play s_{β_1+1} corresponding to the play that τ would dictate in $G(T, \emptyset, \mu+1)$ following $\langle\langle\emptyset, \{s_0^{\beta_0}, s_1^{\beta_0}\}\rangle, \langle s_1, \{s_0^{\beta_1}, s_1^{\beta_1}\}\rangle\rangle$. That is, suppose $\tau(\langle\langle\emptyset, \{s_0^{\beta_0}, s_1^{\beta_0}\}\rangle, \langle s_1, \{s_0^{\beta_1}, s_1^{\beta_1}\}\rangle\rangle) = \{\alpha_2, s_2\}$ and let $s_{\beta_1+1} = s_2$. If at some limit stage γ in playing $G_2(T, \emptyset, \mu+1)$ we have reached a limit stage $\gamma' \leq \gamma$ in the $G(T, \emptyset, \mu+1)$ game, player I simply proceeds to diagonalize against all nodes in $\mathrm{Lev}_{\alpha_{\gamma'}}(T)$ until player II plays a pair of nodes which splits at or above level $\alpha_{\gamma'}$. It is clear that player I can proceed in this manner. Towards a contradiction, suppose that player II is able to move, say $\{s_0^\mu, s_1^\mu\}$, at stage μ. There are two cases, either the game we have been playing in parallel in $G(T, \emptyset, \mu+1)$ has run to stage $\mu+1$ also and player II is able to play here, which is a contradiction because we assumed that τ is a winning strategy, or for some minimal $\xi \in \mu$, $\sup\{\mathrm{lh}(s_\beta) : \beta \in \mu\} \leq \alpha_\xi$. Suppose first that $\sup\{\mathrm{lh}(s_\beta) : \beta \in \mu\} = \alpha_\xi$. If ξ is a successor, then by minimality player I in $G_2(T, \emptyset, \mu+1)$ eventually has begun the stage of his strategy where he diagonalizes against all nodes in $\mathrm{Lev}_{\alpha_\xi}(T)$, which is a contradiction. So we may assume that ξ is a limit ordinal. Because μ is regular, it cannot be that $\sup\{\mathrm{lh}(s_\beta) : \beta \in \mu\} = \alpha_\xi$, so we may assume that $\sup\{\mathrm{lh}(s_\beta) : \beta \in \mu\} < \alpha_\xi$. Furthermore, by minimality it cannot be that $\sup\{\mathrm{lh}(s_\beta) : \beta \in \mu\} < \sup\{\alpha_\gamma : \gamma \in \xi\}$, so $\sup\{\mathrm{lh}(s_\beta) : \beta \in \mu\} \geq \sup\{\alpha_\gamma : \gamma \in \xi\}$, and again because μ is regular it must then be that $\sup\{\mathrm{lh}(s_\beta) : \beta \in \mu\} > \sup\{\alpha_\gamma : \gamma \in \xi\}$. However, this means also that player I in $G_2(T, \emptyset, \mu+1)$ eventually has begun the stage of his strategy where he diagonalizes against all nodes in $\mathrm{Lev}_{\alpha_\xi}(T)$, which is a contradiction. $\square$

Theorem 1.5.8. Let $\mu < \kappa$ be regular and let $T \subseteq {}^{<\kappa}2$ be a cofinally splitting tree. Then player I has a winning strategy in $G_2(T, \emptyset, \mu+1)$ if and only if T is isomorphic to a subtree of $T^\kappa_{<\mu}$.

Proof. As mentioned previously if T is isomorphic to a subtree of $T^\kappa_{<\mu}$ then player I has a winning strategy in $G_2(T, \emptyset, \mu+1)$ by choosing "a node with a new 1" at each stage. On the other hand, let τ be a winning strategy for player I in $G_2(T, \emptyset, \mu+1)$. We build an embedding f from T into $T^\kappa_{<\mu}$. This embedding will be the union of a sequence of coherent embeddings

$\langle\overline{f_\beta} : T \restriction (\beta+1) \to T^\kappa_{<\mu} \restriction (\beta+1) : \beta \in \mathrm{Cof}(\mu) \cap \kappa\rangle$ induced from a sequence of maps $\langle f_\beta : \mathrm{Lev}_\beta(T) \to \mathrm{Lev}_\beta(T^\kappa_{<\mu}) : \beta \in \mathrm{Cof}(\mu) \cap \kappa\rangle$. By coherent we mean that if $\beta \in \beta' \in \kappa$ are both of cofinality μ, then $\overline{f_{\beta'}}(s) = \overline{f_\beta}(s)$ for every $s \in T \restriction (\beta+1)$. Furthermore, these embeddings will preserve length, that is for every $s \in T$ and $\beta \in \mathrm{Cof}(\mu) \cap [\mathrm{lh}(s), \kappa)$, $\mathrm{lh}(\overline{f_\beta}(s)) = \mathrm{lh}(s)$. Clearly then if f is the union of the $\overline{f_\beta}$ for $\beta \in \mathrm{Cof}(\mu) \cap \kappa$, f will be an embedding from T into $T^\kappa_{<\mu}$. To simplify the setting, note that if τ is a winning strategy for player I in $G_2(T, \emptyset, \mu+1)$ then we can easily form a winning strategy τ' for player I in $G_2(T', \emptyset, \mu+1)$, where T' is obtained from T by adjoining a copy of $T^\kappa_{<\omega}$ splitting above any node which doesn't split in T, and so we may assume without loss of generality that T is everywhere splitting. So, let $\beta \in \mathrm{Cof}(\mu) \cap \kappa$ and let $s \in \mathrm{Lev}_\beta(T)$. We first define the canonical maximal run of the game $\overline{x_s}$ following τ below s by recursion. This is similar to the construction used in 1.5.4 but is done in a more uniform way. Depending on whether or not the final round is a successor or a limit, $\overline{x_s} = \langle\langle\emptyset, \{s_0^0, s_1^0\}\rangle, \langle s_1, \{s_0^1, s_1^1\}\rangle, \ldots, \langle s_{\xi-1}, \{s_0^{\xi-1}, s_1^{\xi-1}\}\rangle, \langle s_\xi\rangle\rangle$ or $\overline{x_s} = \langle\langle\emptyset, \{s_0^0, s_1^0\}\rangle, \langle s_1, \{s_0^1, s_1^1\}\rangle, \ldots, \langle s_\gamma, \{s_0^\gamma, s_1^\gamma\}\rangle, \ldots : \gamma \in \xi\rangle$, respectively. This $\overline{x_s}$ has several special properties:

1. Every splitting pair in this run offered by player II is on a successor level and the two nodes in the pair agree up to their predecessor, that is for every $\gamma \in \xi$, there exists $\alpha_\gamma \in \kappa$ such that $\{s_0^\gamma, s_1^\gamma\} \subseteq \mathrm{Lev}_{\alpha_\gamma+1}(T)$ and $s_0^\gamma \restriction \alpha_\gamma = s_1^\gamma \restriction \alpha_\gamma$.

2. This run is below s in the sense that for every $\gamma \in \xi$, $s \restriction (\alpha_\gamma + 1) \in \{s_0^\gamma, s_1^\gamma\}$ and $s_{\gamma+1} = s \restriction (\alpha_\gamma + 1)$.

3. This run follows τ in that for every $\gamma \in \xi$, $\tau(\langle\langle\emptyset, \{s_0^0, s_0^1\}\rangle, \ldots, \langle\emptyset \text{ or } s_\gamma, \{s_0^\gamma, s_1^\gamma\}\rangle\rangle) = s_{\gamma+1}$. This $\emptyset$ for player I's play in round γ indicates that if γ is a limit, player I does not choose a node.

4. This run is canonical in that for each $\gamma \in \xi$, α_γ is minimal with the property that there exists a splitting pair as in properties 1. and 2. on level $\alpha_\gamma + 1$ such that τ chooses

along s. That is, α_γ is minimal such that there exists $\{s_0^\gamma, s_1^\gamma\} \subseteq \mathrm{Lev}_{\alpha_\gamma+1}(T)$ as in properties 1 and 2. with $\tau(\langle\langle\emptyset, \{s_0^0, s_0^1\}\rangle, \ldots, \langle\emptyset \text{ or } s_\gamma, \{s_0^\gamma, s_1^\gamma\}\rangle) = s \restriction (\alpha_\gamma + 1) = s_{\gamma+1}$.

5. This run is maximal below s in that for every $\{s_0^\xi, s_1^\xi\} \subseteq \mathrm{Lev}_{\delta+1}(T)$ as in properties 1 and 2. for $\delta \in \beta$ offered by player II in round ξ, τ chooses $s_{\xi+1} \neq s \restriction (\delta + 1)$.

Because $\mathrm{cf}(\beta) = \mu$, similar arguments to those in 1.5.4 show that $\xi \in \mu$. Define $f_\beta(s) \in {}^\beta 2$ to be the function which maps α_γ to 1 for every $\gamma \in \xi$ and maps all other ordinals in β to 0. Note then that $f_\beta : \mathrm{Lev}_\beta(T) \to \mathrm{Lev}_\beta(T^\kappa_{<\mu})$. Next, let $s \in T \restriction \beta + 1$. T is pruned, so there exists $s' \in \mathrm{Lev}_\beta(T)$ with $s' \restriction \mathrm{lh}(s) = s$. Define $\overline{f_\beta}(s) = f_\beta(s') \restriction \mathrm{lh}(s)$. We need to see that this function is well defined. Suppose $\{s', s''\} \subseteq \mathrm{Lev}_\beta(T)$ with $s' \restriction \mathrm{lh}(s) = s'' \restriction \mathrm{lh}(s) = s$. Let $\mathrm{lh}(s' \wedge s'') = \delta$. Without loss of generality suppose that $(s' \wedge s'') ^\frown 0 = s' \restriction (\delta + 1)$ and $(s' \wedge s'') ^\frown 1 = s'' \restriction (\delta + 1)$. Let the $\alpha_\eta^{s'}$'s and $\alpha_\eta^{s''}$'s be defined in the usual way according to $\overline{x_{s'}}$ and $\overline{x_{s''}}$, respectively. Note that for some $\gamma \in \xi$, the canonical maximal runs following τ below s' and s'' agree up round γ with $\alpha_\eta^{s'} = \alpha_\eta^{s''}$ for every $\eta \in \gamma$, but differ at round γ, where exactly one of the $\{\alpha_\gamma^{s'}, \alpha_\gamma^{s''}\}$ will be equal to δ. This is because at that round, τ will choose exactly one of $\{(s' \wedge s'') ^\frown 0, (s' \wedge s'') ^\frown 1\}$. To illustrate, without loss of generality suppose that τ chooses $(s' \wedge s'') ^\frown 0$. Then $\overline{x_{s'}} = \langle\langle\emptyset, \{s_0^0, s_1^0\}\rangle, \langle s_1, \{s_0^1, s_1^1\}\rangle, \ldots, \langle s_\gamma \text{ or } \emptyset, \{s_0'^\gamma = (s' \wedge s'') ^\frown 0, s_1'^\gamma = (s' \wedge s'') ^\frown 1\}\rangle, \langle s_{\gamma+1} = (s' \wedge s'') ^\frown 0, \{s_0'^{\gamma+1}, s_1'^{\gamma+1}\}\rangle, \ldots\rangle$ while $\overline{x_{s''}} = \langle\langle\emptyset, \{s_0^0, s_1^0\}\rangle, \langle s_1, \{s_0^1, s_1^1\}\rangle, \ldots, \langle s_\gamma \text{ or } \emptyset, \{s_0''^\gamma \neq (s' \wedge s'') ^\frown 0, s_1''^\gamma \neq (s' \wedge s'') ^\frown 1\}\rangle, \langle s''_{\gamma+1} = s'' \restriction \mathrm{lh}(s_i^\gamma), \{s_0''^{\gamma+1}, s_1''^{\gamma+1}\}\rangle, \ldots\rangle$. Then $f_\beta(s') \restriction \delta = f_\beta(s'') \restriction \delta$, but $f_\beta(s')(\delta) = 1$ and $f_\beta(s'')(\delta) = 0$. However, by assumption $\mathrm{lh}(s) \leq \delta$ so $\overline{f_\beta}(s) = f_\beta(s') \restriction \mathrm{lh}(s) = f_\beta(s'') \restriction \mathrm{lh}(s)$. Therefore $\overline{f_\beta} : T \restriction (\beta+1) \to T^\kappa_{<\mu} \restriction (\beta+1)$ is well defined, and so to show that it is an embedding we need only to see that it is injective. For this it suffices to show that if $\{s', s''\} \subseteq \mathrm{Lev}_\zeta(T \restriction (\beta + 1))$ for some $\zeta \in \beta + 1$, $\overline{f_\beta}(s') \neq \overline{f_\beta}(s'')$. However, by the same argument as we just gave for why $\overline{f_\beta}$ is well defined, one may observe that $\overline{f_\beta}(s') \restriction \mathrm{lh}(s' \wedge s'') = \overline{f_\beta}(s'') \restriction \mathrm{lh}(s' \wedge s'')$ but $\overline{f_\beta}(s')(\mathrm{lh}(s' \wedge s'')) \neq \overline{f_\beta}(s'')(\mathrm{lh}(s' \wedge s''))$. Now, if $\beta \in \beta' \in \kappa$ are both of cofinality μ, then for $s \in T \restriction (\beta + 1)$, let $s' \in \mathrm{Lev}_\beta(T)$ with $s' \restriction \mathrm{lh}(s) = s$ and let $s'' \in \mathrm{Lev}_{\beta'}(T)$ with

$s'' \restriction \mathrm{lh}(s') = s'$. By the uniform way that we have defined the $\overline{x}_{s'}$ and $\overline{x}_{s''}$ runs, it is clear in particular that $\overline{f_{\beta'}}(s') = f_\beta(s')$, so in fact $\overline{f_\beta}(s) = f_\beta(s') \restriction \mathrm{lh}(s) = f_{\beta'}(s'') \restriction \mathrm{lh}(s) = \overline{f_{\beta'}}(s)$. Finally then, letting f be the union of these $\overline{f_\beta}$ embeddings, $f : T \to T^{\kappa}_{<\mu}$ is an embedding, as desired. □

Corollary 1.5.9. If $\mu \in \kappa$ are regular cardinals and $T \subseteq {}^{<\kappa}2$ is a cofinally splitting tree with $|\mathrm{Lev}_\alpha(T)| \leq \mu$ for every $\alpha \in \kappa$ (so we must then have $\kappa = \mu^+$), the following are equivalent:

1. Player I has a winning strategy in $G(T, \emptyset, \mu + 1)$.
2. Player I has a winning strategy in $G_2(T, \emptyset, \mu + 1)$.
3. There exists a μ-club $\mathcal{C} \subseteq [H_\theta]^\mu$ of T-guessing submodels $M \prec H_\theta$ for some sufficiently large θ.
4. T is isomorphic to a subtree of $T^{\kappa}_{<\mu}$.

Proof. Follows immediately from 1.5.4, 1.5.8, and 1.5.7. □

1.5.1 Determinacy of the cut-and-choose game

Let κ be a regular cardinal and let $T \subseteq {}^{<\kappa}2$ be a tree. As remarked previously, it is true that $G(T, \emptyset, \omega)$ is determined, but it may be that $G(T, \emptyset, \delta)$ is undetermined for some $\delta \geq \omega + 1$. Similarly if T codes a closed subset of 2^κ, for every $x_0 \in [T]$, $G([T], x_0, \omega)$ is determined, but it may be that $G([T], x_0, \delta)$ is undetermined for some $\delta \geq \omega + 1$. In this section we give several examples of both of these situations, and later also give examples (with trees coding closed subsets of 2^κ of course) where the determinacy of the cut-and-choose game and Väänänen's game are quite different. The structure theorems 1.5.4 and 1.5.8 make it easy to construct, for example, trees where $G(T, \emptyset, \mu + 1)$ is undetermined. However, we also give some examples in this section of (weaker) results, such as 1.5.12 and 1.5.13, in order to illustrate different methods. Our first example of an undetermined tree (and closed set which it is coding) is a slight generalization of one given by Väänänen in [69].

Example 1.5.10. Let $\mu \in \kappa$ with $\mu^{<\mu} = \mu$. Let $S \subseteq \mathrm{Cof}(\mu) \cap \kappa$ be stationary such that $(\kappa \setminus S) \cap \mathrm{Cof}(\mu)$ is also stationary. Let $T = \{s \in {}^{<\kappa}2 : \{\alpha \in \mathrm{lh}(s) : s(\alpha) = 1\} \subseteq S$ is closed under μ-sequences$\}$. Note that T is an everywhere splitting tree coding a closed subset of 2^κ. Then $G(T, \emptyset, \mu+1)$ is undetermined and for every $x_0 \in [T]$, $G([T], x_0, \mu+1)$ is undetermined.

Proof. Suppose towards a contradiction that τ is a winning strategy for player I in $G(T, \emptyset, \mu+1)$. Let $\{T, \tau, \mu\} \subseteq M \prec H_\theta$ with θ sufficiently large, ${}^{<\mu}M \subseteq M$, and $|M| = \mu$. Let $f : \mu \to \delta_M$ be a strictly increasing cofinal map. Have the two players play a run of the game in M where at every stage $\gamma \in \mu$, player II selections $\{s_0^\gamma, s_1^\gamma\} \subseteq M$ so that $\mathrm{lh}(s_0^\gamma \wedge s_1^\gamma) \geq f(\gamma)$ and there exists $\beta \in (\alpha_\gamma, \mathrm{lh}(s_0^\gamma \wedge s_1^\gamma))$ with $s_0^\gamma(\beta) = s_1^\gamma(\beta) = 1$. Clearly this is always possible and because τ is a winning strategy, it must be that $\delta_M \notin S$, as otherwise player II would win this run of the game. However, the collection of δ_M for such models M is unbounded and closed under increasing sequences of length μ, but this is impossible as we assumed S is stationary. If τ is a winning strategy for player II in $G(T, \emptyset, \mu+1)$, the proof is similar: fix a model M containing all relevant objects of size μ closed under sequences of length $< \mu$ and have player I play a sequence of ordinals in M which are cofinal in δ_M and pick a node in the splitting pair with a 1 above the path constructed so far. Then necessarily $\delta_M \in S$, which is a contradiction as $(\kappa \setminus S) \cap \mathrm{Cof}(\mu)$ is stationary. The proof to show that $G([T], x_0, \mu+1)$ is undetermined is similar—one must just make sure that the relevant players again play objects in M. $\square$

It is necessary for the above example to work that the tree has certain large levels, e.g. $|\mathrm{Lev}_{S(\mu)}(T)| = 2^\mu$. An everywhere splitting tree coding a closed subset of 2^κ has $|\mathrm{Lev}_\alpha(T)| \geq \alpha + \omega$ for every $\alpha \in \kappa$. A natural question then is if we can have $(\mu+1)$-undetermined trees with levels of the smallest possible cardinality. Suppose for concreteness that $\mu = \omega$ and $\kappa = \omega_1$. Of course if $T \subseteq {}^{<\omega_1}2$ is an ω_1-tree, then player II cannot have a winning strategy in $G(T, \emptyset, \omega + 1)$ because e.g. by repeated application of this strategy within a countable model, one can build a copy of ${}^{<\omega}2$ cofinal in $T \restriction \delta_M$, so $|\mathrm{Lev}_{\delta_M}(T)| = 2^\omega$. Similarly for Väänänen's game. So for ω_1-trees, player II never has a winning strategy in the games of length $\delta \geq \omega + 1$, and so whether or not player I has a winning strategy is of interest. In [69], Väänänen proves the following:

Example (from [69]) 1.5.11. Let $\mathbb{P}$ be the forcing from a model of $2^\omega = \omega_1$ to add a Kurepa tree T with countable conditions due to Stewart [63] and written up by Jech [37]. Then if G is $(V, \mathbb{P})$-generic and T is the Kurepa tree added, in $V[G]$ for every $x_0 \in [T]$, player I does not have a winning strategy in $G([T], x_0, \omega + 1)$.

It is not difficult to see that the tree built as above is also such that $G(T, \emptyset, \omega + 1)$ is undetermined. The idea is that at countable limit stages in the construction of the tree, certain maximal paths are extended generically, in such a way that player I cannot predict which ones will be. With this intuition, one might also expect the following.

Example 1.5.12. Suppose $(2^\omega = \omega_1)^V$ and let G be $(V, \mathrm{Fn}(\omega_1, 2, < \omega_1))$-generic. Then there exists an ω_1-tree $T \in V[G]$ such that $T \subseteq {}^{<\omega_1}2$, $V[G] = V[T]$, and $G(T, \emptyset, \omega + 1)$ is undetermined.

Proof. Let G be $(V, \mathrm{Fn}(\omega_1, 2, < \omega_1))$-generic and let $f_G \in {}^{\omega_1}2$ be defined by $f_G(\alpha) = 1$ if and only if for some $p \in G$ with $\alpha \in \mathrm{dom}(p)$, $p(\alpha) = 1$. Work in $V[G]$. Define $T \subseteq {}^{<\omega}2$ by recursion. If $s \in \mathrm{Lev}_\alpha(T)$, let $\{s \frown 0, s \frown 1\} \in \mathrm{Lev}_{\alpha+1}(T)$. If $\alpha \in \lim(\omega_1)$, let $\mathrm{Lev}_\alpha(T)$ consist of every $s \in {}^\alpha 2$ such that $s \restriction \beta \in \mathrm{Lev}_\beta(T)$ for every $\beta \in \alpha$ and either for some $\beta \in \alpha$, $s(\gamma) = 0$ for every $\gamma \in [\beta, \alpha)$, or if $s(\beta) = f_G(\alpha + \beta)$ for every $\beta \in \alpha$. It is not difficult to see that T is an everywhere splitting ω_1-tree—T is formed by taking direct limits at every limit stage except for possibly adding a single additional path, by consulting f_G. Working now in V, suppose towards a contradiction that $\dot{\tau}$ is a name for a winning strategy for player I in $G(\dot{T}, \emptyset, \omega+1)$. It is not difficult to see that we can construct by recursion a sequence $\langle\langle p_n, \alpha_n, s_n, \{s_0^n, s_1^n\}\rangle : n \in \omega\rangle$ such that

1. $\langle p_n : n \in \omega\rangle \subseteq \mathrm{Fn}(\omega_1, 2, <\omega_1)$ is $\leq$-decreasing and $\mathrm{dom}(p_n) \in \mathrm{Ord}$ such that $\{\alpha_n, \mathrm{lh}(s_n)\} \subseteq \mathrm{dom}(p_n)$ for every $n \in \omega$,
2. $\{s_0^n, s_1^n\} \subseteq T^\kappa_{<\omega}$ is a splitting pair extending $s_{n-1} \in T^\kappa_{<\omega}$ with $\mathrm{lh}(s_0^n \wedge s_1^n) > \mathrm{dom}(p_n)$, and
3. $p_n \Vdash \dot{\tau}(\langle\langle\alpha_0, \{s_0^0, s_1^0\}\rangle, \langle\{\alpha_1, s_1\}, \{s_0^1, s_1^1\}\rangle, \ldots, \langle\{\alpha_{n-1}, s_{n-1}\}, \{s_0^{n-1}, s_1^{n-1}\}\rangle\rangle) = \{\alpha_n, s_n\}$.

Let $p_\omega = \bigcup_{n\in\omega} p_n \in \mathrm{Fn}(\omega_1, 2, <\omega_1)$, suppose $\alpha = \mathrm{dom}(p_\omega)$, and let $s = \bigcup_{n\in\omega} s_n \in {}^{\alpha}2$. If $p'_\omega \leq p_\omega$ is defined by $p'_\omega(\beta) = p_\omega(\beta)$ for every $\beta \in \alpha$, and $p'_\omega(\alpha+\beta) = s(\beta)$ for every $\beta \in \alpha$ (so that $\mathrm{dom}(p'_\omega) = \alpha + \alpha$), then one may observe that in fact p'_ω forces that $\dot{\tau}$ is no longer a winning strategy for player I in $G(\dot{T}, \emptyset, \omega+1)$, because it fixes an initial ω-run of the sequence constructing a path of length α which is made no longer maximal in $\dot{T}$ by consulting f_G, i.e. player II will be able to play in the ω^{th} round. It is also clear that $V[T] = V[G]$ in this case. $\square$

It is well known that for κ a successor, adding a subset of κ with functions of size $< \kappa$, i.e. forcing with $\mathrm{Fn}(\kappa, 2, <\kappa)$, forces $\diamondsuit_\kappa$ (and indeed forces $\diamondsuit_\kappa(S)$ for any $S \subseteq \kappa$ stationary in V). Working again in the specific case where $\kappa = \omega_1$, if $2^\omega = \omega_1$ then a strategy for player I in $G(T, \emptyset, \omega+1)$ may be coded by a subset of ω_1. $\diamondsuit_{\omega_1}$ then allows one to act in much the same way as in 1.5.12 to decide at limit stages in the construction of an undetermined tree which cofinal paths to make no longer maximal.

Example 1.5.13. Suppose that $\diamondsuit_{\omega_1}$ holds. Then there exists an everywhere splitting ω_1-tree $T \subseteq {}^{<\omega_1}2$ such that $G(T, \emptyset, \omega+1)$ is undetermined.

Proof. Suppose that $\diamondsuit_{\omega_1}$ holds (so also $|{}^{<\omega_1}2| = \omega_1$) and let $\langle A_\alpha : \alpha \in \lim(\omega_1)\rangle$ witness this. Note that for any tree $T \subseteq {}^{<\omega_1}2$, a strategy τ for player I in $G(T, \emptyset, \omega+1)$ is a function which takes a countable (and if the tree is cofinally splitting and the strategy is winning, finite—because in this case player I winning cannot occur if the path constructed to stage ω is not maximal) string of objects which are either ordinals or nodes or pairs of these objects and returns a pair (except at stages 0 and ω) consisting of a node and an ordinal. The collection of all such strings of objects then has size ω_1, and so τ may be coded as a partial function from ω_1 to ω_1 via bijections g, h from $\mathrm{dom}(\tau) \to \omega_1$ and $\mathrm{rng}(\tau) \to \omega_1$. Via a further bijection f from $\omega_1 \times \omega_1 \to \omega_1$, τ may be coded as a subset of ω_1. First, build $T \subseteq {}^{<\omega_1}2$ by recursion. If $s \in \mathrm{Lev}_\alpha(T)$, let $\{s \frown 0, s \frown 1\} \subseteq \mathrm{Lev}_{\alpha+1}(T)$. At limit stages α, if $A_\alpha \subseteq \alpha$ is the code (via f, g, h, etc.) for a partial strategy for player I in the natural game on the tree constructed so far, $G(T \restriction \alpha, \emptyset, \omega+1)$, and furthermore this partial strategy is sufficient to construct an ω-length run of the game cofinal in $T \restriction \alpha$ with player I following τ_{A_α}, the partial strategy corresponding to this code, then choose one such run of the game and if $s \subseteq T \restriction \alpha$ is the resulting path of length α, let $s \in \mathrm{Lev}_\alpha(T)$ too. Just as in 1.5.12 then, this tree is constructed by splitting everywhere and taking direct limits at every limit stage except for possibly adding a single additional path, by consulting $\langle A_\alpha : \alpha \in \lim(\omega_1)\rangle$. Let τ be a strategy for player I in $G(T, \emptyset, \omega+1)$. Then via f, g, h, etc. τ may be viewed as a subset of ω_1, X_τ. By elementarity, it is not difficult to see that for any $M \prec H_\theta$ a countable submodel with $\{\tau, f, g, h, \text{ etc.}\} \subseteq M$, $X_\tau \cap \delta_M$ is a code for a partial strategy for player I in $G(T \restriction \delta_M, \emptyset, \omega+1)$ which is capable of constructing a cofinal (in $T \restriction \delta_M$) ω-length run of the game with player I following $\tau_{X_\tau \cap \delta_M}$. The collection of δ_M for such M is club in ω_1, so there must exist some such δ_M so that $A_{\delta_M} = X_\tau \cap \delta_M$. However, then an ω-length run of the game with player I following τ is possible which ends up along a cofinal path $s \subseteq T \restriction \delta_M$ which is not maximal, i.e. $s \in \mathrm{Lev}_{\delta_M}(T)$. So because T is cofinally splitting, no matter what τ chooses as α_ω, player I loses.

One may also construct such a tree from $\diamondsuit_{\omega_1}$ in a slightly different way, using the same idea that is used to construct from $\diamondsuit^+_{\omega_1}$ a Kurepa tree. In that case, the $\diamondsuit^+_{\omega_1}$-sequence is used to build a tree where on a club of limit levels δ, "so many" cofinal paths in $T \upharpoonright \delta$ are made no longer maximal (but still countably many) that in the end $|[T]| \geq \omega_2$. One way to be precise about this is with a discontinuous sequence of countable elementary submodels, each containing certain relevant objects, which is what we do here. While it is not the case that $\diamondsuit_{\omega_1}$ is sufficient to construct a Kurepa tree (for example, $\diamondsuit_{\omega_1}$ holds in the Lévy model where an inaccessible is collapsed to ω_2 with countable conditions, but there are no Kurepa trees), we can use a $\diamondsuit_{\omega_1}$ sequence to build a tree where on a club of limit levels δ "enough" cofinal paths are made no longer maximal. So, as before fix our $\diamondsuit_{\omega_1}$ sequence $\langle A_\alpha : \alpha \in \lim(\omega_1)\rangle$ and functions to code strategies as subsets of ω_1, f, g, h, etc.. Form a sequence of countable submodels $\langle M_\alpha : \alpha \in \lim(\omega_1)\rangle$ such that for every $\alpha \in \lim(\omega_1)$, $\{f, g, h, \langle A_\alpha : \alpha \in \lim(\omega_1)\rangle, \alpha+1, \text{ etc.}\} \subseteq M_\alpha \prec H_\theta$ and $\langle M_\beta : \beta \in \alpha \cap \lim(\omega_1)\rangle \in M_\alpha$. Let $T \subseteq {}^{<\omega_1}2$ be the everywhere splitting ω_1-tree formed by recursion on limit levels by letting $\mathrm{Lev}_\alpha(T) = \{\chi_x : x \in M_\alpha \cap P(\alpha) \text{ and } \chi_x \upharpoonright \beta \in \mathrm{Lev}_\beta(T) \text{ for } \beta \in \lim(\omega_1) \cap \alpha\}$ for $\alpha \in \lim(\omega_1)$. Let τ be a strategy for player I in $G(T, \emptyset, \omega+1)$, corresponding to $X_\tau \subseteq \omega_1$. Then in particular for some stationary $S \subseteq \lim(\lim(\omega_1))$, $X_\tau \cap \alpha = A_\alpha$. However, $\{T \upharpoonright \alpha, A_\alpha, \ldots\} \subseteq M_\alpha$ and so M_α can construct within itself an ω-length run of the game with player I following τ cofinal in $T \upharpoonright \alpha$ such that the path built by the two players is an element of M_α. But then this path is not maximal in T, so τ is not a winning strategy. $\square$

Remark 1.5.14. The same ideas as in 1.5.13 can be used to build, for μ regular and a $\diamondsuit_{\mu^+}$-sequence, a μ^+-tree $T \subseteq {}^{<\mu^+}2$ where player I does not have a winning strategy in $G(T, \emptyset, \mu+1)$.

In 1.5.12 and 1.5.13, it may have seemed important that at limit stages of the tree's construction we had some device to decide which cofinal paths to extend, either a sequence built generically in a certain way (as in 1.5.12) or a sequence which is able to guess partial strategies for player I for the tree built so far and make sure that they cannot be partial strategies of a winning strategy (as in 1.5.13). Indeed, even if the $\Diamond_{\omega_1}$ sequence is weakened to something like $\clubsuit_{\omega_1}$, the proofs as stated in 1.5.13 will apparently no longer work. However as mentioned, due to the structure theorems 1.5.4 and 1.5.8 it is, in fact, very easy to construct trees where, e.g. player I does not have a winning strategy. We give some examples.

Example 1.5.15. There exists a pruned everywhere splitting ω_1-tree $T \subseteq {}^{<\omega_1}2$ such that player I does not have a winning strategy in $G(T, \emptyset, \omega + 1)$.

Proof. Fix $\langle C_\alpha : \alpha \in \lim(\omega_1)\rangle$ such that $C_\alpha \subseteq \alpha$ is cofinal and $\mathrm{otp}(C_\alpha) = \omega$. Let $T' \subseteq {}^{<\omega_1}2$ be the tree induced by the characteristic functions for these C_α viewed as subsets of ω_1, that is $T' = \{s \in {}^{<\omega_1}2 : \exists\alpha \in \lim(\omega_1)$ such that $\chi_{C_\alpha} \restriction \mathrm{lh}(s) = s\}$. Let T be formed from T' by adjoining a copy of $T^{\omega_1}_{<\omega}$ to every non-splitting node. Because the order type of every C_α is ω, for every $\alpha \in \omega_1$, $|\{\chi_{C_\beta} \restriction \alpha : \beta \in \lim(\omega_1)\}| \leq \omega$, so it is not difficult to see that T is a pruned everywhere splitting ω_1 tree. Note that for any countable model $M \prec H_\theta$ with $T \in M$, $T \restriction \delta_M \subseteq M$ and $\chi_{C_{\delta_M}} \restriction \delta_M \in \mathrm{Lev}_{\delta_M}$. However if for some $b \in M \cap [T]$, $b \restriction \delta_M = \chi_{C_{\delta_M}} \restriction \delta_M$, then the unique ordinal β such that $\mathrm{otp}(\{\alpha \in \beta : b(\alpha) = 1\}) = \omega$ must be in M, but this ordinal is δ_M, a contradiction. Thus M is not T-guessing, so by 1.5.4, player I does not have a winning strategy. If one wants to define such a tree by recursion on levels, this is easy also: if $\alpha \in \mathrm{Lev}_\alpha(T)$, let $\{s \frown 0, s \frown 1\} \subseteq \mathrm{Lev}_{\alpha+1}(T)$ and for $\alpha \in \lim(\omega_1)$, let $\mathrm{Lev}_\alpha(T)$ consist of a single $s \in {}^\alpha 2$ such that $s \restriction \beta \in \mathrm{Lev}_\beta(T)$ for every $\beta \in \alpha$ and $\mathrm{otp}(\{\beta \in \alpha : s(\beta) = 1\}) = \omega$, which we will always be able to find, along with every $s \in {}^\alpha 2$ such that $s \restriction \beta \in \mathrm{Lev}_\beta(T)$ for every $\beta \in \alpha$ and for some $\beta \in \alpha$, $s(\gamma) = 0$ for every $\gamma \in [\beta, \alpha)$. $\square$

Example 1.5.16. Let $\mu^{<\mu} = \mu$ and let $\kappa = \mu^+$. Then there exists a pruned everywhere splitting tree $T \subseteq {}^{<\kappa}2$ with $|\mathrm{Lev}_\alpha(T)| \leq \mu$ such that player I does not have a winning strategy in $G(T, \emptyset, \mu + 1)$.

Proof. Fix $\langle C_\alpha : \alpha \in \mathrm{Cof}(\mu) \cap \kappa \rangle$ such that $C_\alpha \subseteq \alpha$ is a club in α of order type μ. Let $T' = \{s \in {}^{<\kappa}2 : \exists \alpha \in \mathrm{Cof}(\mu) \cap \kappa$ such that $\chi_{C_\alpha} \restriction \mathrm{lh}(s) = s\}$. Note that $|\mathrm{Lev}_\delta(T')| \leq \mu$ for every $\delta \in \kappa$ because $|\delta|^{<\mu} \leq \mu$. Let T be formed from T' by adjoining a copy of $T^\kappa_{<\omega}$ to every non-splitting node. If $M \prec H_\theta$ with $\{\mu, T, \mathrm{etc.}\} \subseteq M$, $|M| = \mu$, and ${}^{<\mu}M \subseteq M$, note that $(\chi_{C_{\delta_M}} \restriction \delta_M) \restriction \beta \in M$ for every $\beta \in \delta_M$, but clearly no branch in M can extend $\chi_{C_{\delta_M}} \restriction \delta_M$. So by 1.5.4, player I does not have a winning strategy in $G(T, \emptyset, \mu+1)$. □

Example 1.5.17. Let $\mu^{<\mu} = \mu$ and $\mu < \kappa$. Then there exists a pruned everywhere splitting tree $T \subseteq {}^{<\kappa}2$ with $|\mathrm{Lev}_\delta(T)| \leq |\delta|^\mu$ for every $\delta \in [\mu^+, \kappa)$ such that player I does not have a winning strategy in $G(T, \emptyset, \mu+1)$.

Proof. Fix a μ-stationary set $S \subseteq [H_\theta]^\mu$ of models $M \prec H_\theta$ with $\{\mu, \kappa, \mathrm{etc.}\} \subseteq M$, $|M| = \mu$, and ${}^{<\mu}M \subseteq M$. By μ-stationary, we mean that S has nonempty intersection with every μ-club $C \subseteq [H_\theta]^\mu$. For every $\delta \in \mathrm{Cof}(\mu) \cap \kappa$, let $D^S_\delta = \{M \cap \kappa : \delta_M = \delta$ and $M \in S\}$. Fix $\langle \mathcal{C}_\alpha : \alpha \in \mathrm{Cof}(\mu) \cap \kappa \rangle$ such that $\mathcal{C}_\alpha = \{C^M_\alpha : M \in D^S_\alpha\} \subseteq [\alpha]^\mu$ where for each $C^M_\alpha \in \mathcal{C}_\alpha$, $C^M_\alpha \subseteq M \cap \kappa$ is a club in α of order type μ. Let $T' = \{s \in {}^{<\kappa}2$: for some $\alpha \in \mathrm{Cof}(\mu) \cap \kappa$ and $M \in D^S_\alpha, \chi_{C^M_\alpha} \restriction \mathrm{lh}(s) = s\}$, and let T be formed from T' by adjoining a copy of $T^\kappa_{<\omega}$ to every non-splitting node. For every $\delta \in [\mu^+, \kappa)$, because $|D^S_\delta| \leq |\delta|^\mu$ and $\mu^{<\mu} = \mu$, $|\mathrm{Lev}_\delta(T)| \leq |\delta|^\mu$. Now if $M \in S$, then $(\chi_{C_{\delta_M}} \restriction \delta_M) \restriction \beta \in M$ for cofinally many $\beta \in \delta_M$, but no branch in M can extend $\chi_{C_{\delta_M}} \restriction \delta_M$, so by 1.5.4 player I does not have a winning strategy in $G(T, \emptyset, \mu+1)$. The reader interested in whether or not $|D^S_\delta|$ can be made to be of small size, depending on S, will be led to the notion of skinny stationary subsets of $P_\kappa\lambda$, which has been recently explored in [49]. □

For $T \subseteq {}^{<\kappa}2$ a tree, a node $s \in T$ on a level of cofinality $\geq \delta$, and a winning strategy τ for player I in $G(T, \emptyset, \delta+1)$, the idea of a $\trianglelefteq$-minimal maximal run of the game following τ below s as used in 1.5.4 can be used to show that player I never has a winning strategy in trees with many branches.

Proposition 1.5.18. Let κ be regular and $T \subseteq {}^{<\kappa}2$ be a tree with $|[T]| > |T|$. Then player I does not have a winning strategy in $G(T, \emptyset, \kappa)$.

Proof. If $|[T]| > |T|$, we can find a cofinally splitting subtree $T' \subseteq T$ with $|[T']| > |T|$, so assume without loss of generality that T is cofinally splitting. Suppose towards a contradiction that player I has a winning strategy τ in $G(T, \emptyset, \kappa)$. Fix a sufficiently large θ and fix $(H_\theta, \trianglelefteq, \ldots)$, where $\trianglelefteq$ is a well-ordering of H_θ so that $\{T, \tau, \text{etc.}\} \subseteq H_\theta$. For every $b \in [T]$ as in the proof of 1.5.4, we can construct the $\trianglelefteq$-minimal maximal run of the game $\overline{x_b}$ according to τ along b. Specifically, $\overline{x_b} = \langle\langle\{\alpha_0\}, \{s_0^0, s_1^0\}\rangle, \langle\{\alpha_1, s_1\}, \{s_0^1, s_1^1\}\rangle, \ldots, \langle\{\alpha_\xi, s_\xi\}\rangle\rangle$, where $\alpha_0 = \tau(\emptyset)$, $\{\alpha_1, s_1\} = \tau(\langle\{\alpha_0\}, \{s_0^0, s_1^0\}\rangle)$, etc.. It may be that the final move made by player I, which we have written here as $\{\alpha_\xi, s_\xi\}$, is simply $\{\alpha_\xi\}$, which will be the case if ξ is a limit ordinal. As before, $s_\eta = b \restriction \mathrm{lh}(s_\eta)$ for every $\eta \leq \xi$—that is we insist that this is a play along b. We also insist that this play is maximal along b, by which we mean that no matter what move player II makes next, τ dictates that player I choose a node incomparable with b. Specifically, if ξ is a successor, then $\{\alpha_\xi, s_\xi\} = \tau(\langle\langle\{\alpha_0\}, \{s_0^0, s_1^0\}\rangle, \ldots, \langle\{\alpha_{\xi-1}\} \text{ or } \{\alpha_{\xi-1}, s_{\xi-1}\}, \{s_0^{\xi-1}, s_1^{\xi-1}\}\rangle\rangle)$ and no matter what pair $\{s_0^\xi, s_1^\xi\} \subseteq M$ player II plays, τ dictates that player I chooses $s_{\xi+1} \in \{s_0^\xi, s_1^\xi\}$ so that $s_{\xi+1} \neq b \restriction \mathrm{lh}(s_{\xi+1})$. Similarly if ξ is a limit, for $\{\alpha_\xi\} = \tau(\langle\langle\{\alpha_0\}, \{s_0^0, s_1^0\}\rangle, \ldots\rangle)$, no matter what pair $\{s_0^\xi, s_1^\xi\}$ player II plays, τ dictates that player I chooses $s_{\xi+1} \in \{s_0^\xi, s_1^\xi\}$ so that $s_{\xi+1} \neq b \restriction \mathrm{lh}(s_{\xi+1})$. Finally, we insist that $\overline{x_b}$ is $\trianglelefteq$-minimal in the sense that at every round $\eta \in \xi$, player II plays the $\trianglelefteq$-minimal splitting pair $\{s_0^\eta, s_1^\eta\}$ so that player I following τ continues to play along b. Note that $\xi \in \kappa$ because T is cofinally splitting, κ is regular, and we have assumed that τ is a winning strategy for player I in $G(T, \emptyset, \kappa)$. Now to each $b \in [T]$, consider $\overline{x_b}$ and let s_b be the maximal path along b determined by $\overline{x_b}$ (that is if ξ is a successor, $s_b = s_\xi$, and if ξ is a limit, $s_b = \bigcup_{\eta\in\xi} s_\eta$) and $\alpha_b = \alpha_\xi$, that is the final ordinal played by player I in $\overline{x_b}$. Because $|[T]| > |T|$ and $|\{\{s, \alpha\} : s \in T \text{ and } \alpha \in \kappa\}| = |T|$, there must exist some $\{s, \alpha\}$ and a subset $A \subseteq [T]$ with $|A| > |T|$ such that for every $b \in A$,

$\{s_b, \alpha_b\} = \{s, \alpha\}$. However, suppose that $\{b, b'\} \subseteq A$. By $\trianglelefteq$-minimality, it is clear then that in fact $\overline{x_b} = \overline{x_{b'}}$ (the $\trianglelefteq$-minimality is necessary, because without it two plays of the game could end up having the same final round but different intermediate rounds). By maximality, we must then have $\mathrm{lh}(b \wedge b') \in [\mathrm{lh}(s), \alpha)$. However, because $|\mathrm{Lev}_\alpha(T)| \leq |T| < |A|$, we must have for some $A' \subseteq A$ with $|A'| > |T|$ that for every $b \in A'$, $b \restriction \alpha = s'$ for some $s' \in \mathrm{Lev}_\alpha(T)$, which is a contradiction. $\square$

1.5.2 Digression on the determinacy of trees without branches

Suppose that $\mu \in \kappa$ are regular cardinals and $T \subseteq {}^{<\kappa}2$ is a cofinally splitting tree. By 1.5.4, if player I has a winning strategy in $G(T, \emptyset, \mu+1)$ then there exist some T-guessing models M of size μ. For any such model M, any $s \in \mathrm{Lev}_{\delta_M}(T)$ such that $s \restriction \beta \in M$ cofinally can be extended to a branch in $[T]$. So in particular, $[T] \neq \emptyset$. Therefore, if $[T] = \emptyset$ then player I does not have a winning strategy in $G(T, \emptyset, \mu+1)$ for any regular $\mu \in \kappa$. Examples of such trees were given in 1.1.62. Here we see then that the examples in 1.5.15 and 1.5.16 are redundant because if T is a κ-Aronszajn tree then player I does not have a winning strategy in $G(T, \emptyset, \mu+1)$, and if $\mu^{<\mu} = \mu$ and $\kappa = \mu^+$, then there exists a κ-Aronszajn tree (for a construction using Todorčević's method of minimal walks, see [66]).

Also if $|\mathrm{Lev}_\alpha(T)| \leq \mu$ for every $\alpha \in \kappa$, then if player I does not have a winning strategy in $G(T, \emptyset, \mu+1)$, by 1.5.8 and 1.5.7, T does not embed into $T^\kappa_{<\mu}$. For example, if $\kappa = \mu^+$ and T is a κ-Aronszajn tree, then T does not embed into $T^\kappa_{<\mu}$. We originally observed this by another method, as follows. We first need a preliminary proposition.

Proposition (From [47], Anticipated by [39]) 1.5.19. Let κ be regular and let $T \subseteq {}^{<\kappa}2$ be a tree coding a closed subset of 2^κ. Then $[T] \subseteq {}^{\kappa}2$ is κ-compact if and only if T is a κ-tree which does not contain any κ-Aronszajn subtrees. The proof is the same in the κ-Baire space, that is if $T \subseteq {}^{<\kappa}\kappa$ instead.

Proof. We observed this proposition independently. We give the relevant direction for us now, that is that if T isn't a κ-tree or contains a κ-Aronszjan subtree then $[T]$ isn't κ-compact. The interested reader may consult [47] for the reverse direction. Note first that if T isn't a κ-tree then the open sets generated by the nodes on a level of size $\geq \kappa$ constitute an open cover of $[T]$ of size $\geq \kappa$ with no $(< \kappa)$-sized subcover. Next, suppose towards a contradiction that there exists $S \subseteq T$ a κ-Aronszjan tree. Let $B = \{t \in T : t \notin S \text{ and for every } \alpha \in \mathrm{lh}(t), t \restriction \alpha \in S\}$. Because $[S] = \emptyset$, for each $x \in [T]$ there exists a unique $\alpha_x \in \kappa$ so that $x \restriction \alpha_x \in B$. Furthermore, if $x_1, x_2 \in [T]$ with $x_1 \neq x_2$, then $x_1 \restriction \alpha_{x_1}$ and $x_2 \restriction \alpha_{x_2}$ are either incomparable or the same. Therefore $\{O_{x \restriction \alpha_x} : x \in [T]\}$ is a disjoint open cover of $[T]$. However, because S has height κ and $T_{[T]} = T$, $\{\alpha_x : x \in [T]\} \subseteq \kappa$ is unbounded. So $\{O_{x \restriction \alpha_x} : x \in [T]\}$ is then an open cover of $[T]$ of size κ which has no $(< \kappa)$-sized subcover. $\square$

Proposition 1.5.20. Let $\mu \in \kappa$ be regular cardinals. Then $T^{\kappa}_{<\mu}$ has no κ-Aronszajn subtrees.

Proof. We show that if $T' \subseteq T^{\kappa}_{<\mu}$ is a κ-tree coding a closed subset of 2^{κ} such that for every $t \in T'$, $b_t \in [T']$ where $b_t(\alpha) = t(\alpha)$ for every $\alpha \in \mathrm{lh}(t)$ and $b_t(\alpha) = 0$ for every $\alpha \in [\mathrm{lh}(t), \kappa)$, then $[T']$ is κ-compact. This suffices by 1.5.19, because any κ-Aronszajn subtree of T is a subtree of such a κ-tree T' coding a closed subset of 2^{κ}. Let $\mathbb{O} = \{O_{t_\alpha} : \alpha \in \kappa\}$ be an open cover of $[T']$, where for every $\alpha \in \kappa$, $t_\alpha \in T'$ and $O_{t_\alpha} = \{x \in {}^{\kappa}2 : x \restriction \mathrm{lh}(t_\alpha) = t_\alpha\}$. For each $\alpha \in \kappa$, let $\mathbb{O}_\alpha = \{O_t \in \mathbb{O} : \mathrm{lh}(t) \leq \alpha\}$. Because T' is a κ-tree, $|\mathbb{O}_\alpha| \in \kappa$. Suppose towards a contradiction that no $\mathbb{O}_\alpha$ is an open cover of $[T']$. Then in particular, for every $\alpha \in \mathrm{Cof}(\mu) \cap \kappa$, there exists $b_\alpha \in [T']$ such that $b_\alpha \notin O_t$ for every $O_t \in \mathbb{O}_\alpha$. Define $b'_\alpha \in {}^{\kappa}2$ by $b'_\alpha(\delta) = b_\alpha(\delta)$ for every $\delta \in \alpha$ and $b'_\alpha(\delta) = 0$ for every $\delta \in [\alpha, \kappa)$. Then $b'_\alpha \in [T']$ and it must also be that $b'_\alpha \notin O_t$ for every $O_t \in \mathbb{O}_\alpha$. Define $f : \mathrm{Cof}(\mu) \cap \kappa \to \kappa$ by $f(\alpha) = \sup\{\beta + 1 \in \kappa : b'_\alpha(\beta) = 1\} \in \alpha$, and the corresponding $\overline{f} : \{b'_\alpha : \alpha \in \mathrm{Cof}(\mu) \cap \kappa\} \to T'$ by $\overline{f}(b'_\alpha) = b'_\alpha \restriction f(\alpha)$. Because f is regressive on a stationary set, for some stationary $S \subseteq \mathrm{Cof}(\mu) \cap \kappa$, $f''S = \{\xi\}$ for some $\xi \in \kappa$. But $|\mathrm{Lev}_\xi(T')| \in \kappa$ so for some $A \in [S]^{\kappa}$ and $t \in \mathrm{Lev}_\xi(T')$, $\overline{f}(b'_\alpha) = t$ for every $\alpha \in A$. However, then for some $b \in [T]$, for every $\alpha \in A$, $b'_\alpha = b$, but then $b \notin \bigcup_{O_t \in \mathbb{O}} O_t$, a contradiction. $\square$

Remark 1.5.21. Suppose κ is regular. An argument similar to the one in 1.5.20 can be used to show that very thin trees always have branches. Specifically, if $T \subseteq {}^{<\kappa}2$ is a tree of height κ such that for some $\mu \in \kappa$, $|\mathrm{Lev}_\alpha(T)| < \mu$ for every $\alpha \in \kappa$, then $[T] \neq \emptyset$.

Proof. Assume without loss of generality that T is pruned, so that $\langle |\mathrm{Lev}_\alpha(T)| : \alpha \in \kappa \rangle \subseteq \kappa$ is a non-decreasing sequence of cardinals. It must then be that if μ is minimal such that $|\mathrm{Lev}_\alpha(T)| < \mu$ for every $\alpha \in \kappa$, μ is a regular cardinal. For every $\alpha \in \mathrm{Cof}(\mu) \cap \kappa$, fix $s_\alpha \in \mathrm{Lev}_\alpha(T)$. Because $|\mathrm{Lev}_\alpha(T)| \in \mu$, there exists $\beta \in \alpha$ such that $\mathrm{lh}(s_\alpha \wedge s) \in \beta$ for every $s \in \mathrm{Lev}_\alpha(T) \setminus \{s_\alpha\}$. Let $f(\alpha) = \beta$, and so there must exist a stationary $S \subseteq \mathrm{Cof}(\mu) \cap \kappa$ such that $f''S = \{\xi\}$ for some ξ. But then for some $A \in [S]^\kappa$ and $\overline{s}_\xi \in \mathrm{Lev}_\xi(T)$, $s_\alpha \restriction \xi = \overline{s}_\xi$. However, if $\{\alpha, \beta\} \subseteq A$ with $\alpha \in \beta$, then we must have $s_\beta \restriction \alpha = s_\alpha$, because otherwise $\xi > \mathrm{lh}(s_\alpha \wedge (s_\beta \restriction \alpha))$, but this is impossible because $\overline{s}_\xi$ is an initial segment of both s_α and $s_\beta \restriction \alpha$. Therefore $b_A = \{s \in {}^{<\kappa}2 : \text{ for some } \alpha \in A, s = s_\alpha \restriction \mathrm{lh}(s)\} \in [T]$, as desired. $\square$

1.5.3 Comparison with Väänänen's game

Because strategies for player I in Väänänen's game involve objects which are large—player II is playing branches through a tree instead of splitting pairs in the tree—player I sometimes can have a winning strategy in Väänänen's game but not in our cut-and-choose game. Of course, we have also seen some examples where this doesn't happen, like in the model obtained by Lévy collapsing λ to κ^+ with conditions of size $< \kappa$ where the two players play games of length κ, or for shorter games as in 1.5.10 and 1.5.11. On the other hand, in 1.5.18, we saw that in particular for κ-Kurepa trees, or even weak κ-Kurepa trees, T, player I does not have a winning strategy in $G(T, \emptyset, \kappa)$. Consistently much different behavior can be exhibited by Väänänen's game:

Theorem (from [69], method due to Woodin) 1.5.22. *The existence of a Kurepa tree $T \subseteq {}^{<\omega_1}2$ coding a closed subset of 2^{ω_1} such that player I has a winning strategy in $G([T], x_0, \omega+1)$ for every $x_0 \in [T]$ can be forced from a model of $2^\omega = \omega_1$.*

The proof of 1.5.22 uses an augmented version of the forcing to add a Kurepa tree with countable conditions as in 1.5.11. This augmentation takes the form of forcing ordinal-valued "rank" functions, which are strictly decreasing along initial plays along branches when following a winning strategy, and in a sense these rank functions may be viewed as having an affinity with the maximal plays along nodes and branches in our cut-and-choose game which were used to show e.g. 1.5.4, 1.5.8, etc.. However as mentioned earlier, an analogous argument to 1.5.22 clearly cannot work for the cut-and-choose game, and indeed upon careful inspection one will notice that when using countable conditions one may no longer, for example, be able to ensure that the forcing is σ-closed, while if attempting to use finite conditions one may not be able to ensure using a c.c.c. or properness argument that e.g. ω_1 is not collapsed either.

1.6 κ-topologies over 2^λ

By identifying elements of $P(\kappa)$ with their characteristic functions as in 1.1.18 it makes sense to consider δ-scattered subsets of $P(\kappa)$, δ-perfect subsets of $P(\kappa)$, etc.. We now discuss one way of extending these definitions and associated analysis to subsets of $P_\kappa\lambda$ (in particular).

1.6.1 $P_\kappa\lambda$-forests and the κ-box topology over 2^λ

In the 2^κ context of previous sections it was natural to consider the κ-box topology. In 1.1.35, as long as κ is regular, we saw that the κ-box topology over 2^κ is characterized by saying that closed sets are exactly those sets which are bodies of trees in ${}^{<\kappa}2$. More generally if $\kappa \leq \lambda^+$, it is natural to consider the κ-box topology over 2^λ, and we can observe an analogy to this characterization in these settings (even if κ is singular), with the tree being replaced by another type of object. When λ (or more generally an arbitrary set X) is clear from context, we use $\tau_\kappa^{\mathrm{BOX}}$ to denote the κ-box topology over 2^λ, and will use notation which relies on the understanding that formally $\tau_\kappa^{\mathrm{BOX}}$ is a collection of open subsets of 2^λ, i.e. $\tau_\kappa^{\mathrm{BOX}} \subseteq P(P(\lambda))$. Here of course λ is always fixed or clear from context, and again we often identify 2^λ with ${}^\lambda 2$ with $P(\lambda)$ for notational convenience or where most appropriate. A natural generalization of a tree to an object whose levels do not need to be identifiable with ordinals but can be arbitrary elements in $P_\kappa\lambda$ is the following, which seems to have been originally termed a (binary) mess by Jech ([36]). It, or related objects, have also been called (κ, λ)-trees ([25]), $(\kappa, \lambda, 2)$-forests ([23]), or (the downward closures of) $P_\kappa\lambda$-lists ([70]). Not being initially aware of any of this nomenclature, we chose the term $P_\kappa\lambda$-forest, which we will adopt here. Again, in most of the following, λ could be replaced with an arbitrary set X.

Definition 1.6.1. Given a cardinal κ and an ordinal λ, define a $P_\kappa\lambda$-forest to be a set of functions F satisfying each of the following conditions:

1. For every $f \in F$, $f : \mathrm{dom}(f) \to 2$ with $\mathrm{dom}(f) \in P_\kappa\lambda$.

2. If $f \in F$ and $z \subseteq \mathrm{dom}(f)$, then $f \upharpoonright z \in F$.

For F a $P_\kappa\lambda$-forest and $z \in P_\kappa\lambda$, let $\mathrm{Lev}_z(F) = \{f \in F : \mathrm{dom}(f) = z\}$. Say that F is a pruned $P_\kappa\lambda$-forest if for every $f \in F$ and $z \in P_\kappa\lambda$ such that $\mathrm{dom}(f) \subseteq z$, there exists $g \in \mathrm{Lev}_z(F)$ with $g \upharpoonright \mathrm{dom}(f) = f$. As with trees, we will often deal only with pruned $P_\kappa\lambda$-forests, and may even omit this adjective. We also will often only deal with $P_\kappa\lambda$-forests where for every $z \in P_\kappa\lambda$, $F_z \neq \emptyset$. If $f \in F$, let $F \upharpoonright f$ be the $P_\kappa\lambda$ forest defined by taking all extensions of f and closing downward. That is, $h \in F \upharpoonright f$ if and only if for some $g \in \mathrm{Lev}_z(F)$

with $\mathrm{dom}(f) \subseteq z \in P_\kappa\lambda$, $\mathrm{dom}(h) \subseteq z$, and $g \restriction \mathrm{dom}(f) = f$, $g \restriction \mathrm{dom}(h) = h$. So F is pruned if and only if $\mathrm{Lev}_z(F \restriction f) \neq \emptyset$ for every $f \in F$ and $z \in P_\kappa\lambda$.

Definition 1.6.2. Let F be a $P_\kappa\lambda$-forest. Say that a node $f \in F$ is cofinally splitting in F if and only if for every $z \in P_\kappa\lambda$ with $\mathrm{dom}(f) \subseteq z$, there exists $\{g_1, g_2\} \subseteq F$ such that $\mathrm{dom}(g_1) \supseteq z$, $\mathrm{dom}(g_2) \supseteq z$, $g_1 \restriction z = g_2 \restriction z$, $g_1 \restriction \mathrm{dom}(f) = g_2 \restriction \mathrm{dom}(f) = f$, and there exists $\beta \in \mathrm{dom}(g_1) \cap \mathrm{dom}(g_2)$ with $g_1(\beta) \neq g_2(\beta)$. Say that F is cofinally splitting if and only if every $f \in F$ is cofinally splitting in F.

Definition 1.6.3. For F a $P_\kappa\lambda$-forest, let $[F] \subseteq 2^\lambda$ denote the body of F, $[F] = \{b \in {}^\lambda 2 : \forall z \in P_\kappa\lambda, b \restriction z \in F_z\}$.

Definition 1.6.4. For F a $P_\kappa\lambda$-forest, say that F codes a closed subset of 2^λ in $\tau_\kappa^{\mathrm{BOX}}$ if and only if $F_{[F]} = F$. Here $F_{[F]}$ is the $P_\kappa\lambda$-forest generated by $[F]$, that is $F_{[F]} = \{f : \text{ for some } b \in [F], b \restriction \mathrm{dom}(f) = f\}$. This is true if and only if for every $f \in F$, there exists $x_f \in [F]$ such that $x_f \restriction \mathrm{dom}(f) = f$. When the context is clear, we will abbreviate this to just "codes a closed subset" or "codes a κ-closed subset."

Definition 1.6.5. Let F be a $P_\kappa\lambda$-forest. Define the pruned part of F in a similar way to as was done for trees in 1.1.53. That is, the pruned part of F will be the subforest of F formed by successively removing nodes which do not have extensions to every level in $P_\kappa\lambda$ until a stabilization point is reached. So F^α is defined by recursion with $F^0 = F$, $F^{\alpha+1} = \{f \in F^\alpha : \mathrm{Lev}_z(F^\alpha \restriction f) \neq \emptyset \text{ for every } z \in P_\kappa\lambda\}$, and $F^\alpha = \bigcap F^\gamma$ for $\gamma \in \alpha$. Then for some minimal α_1, $F^{\alpha_1+1} = F^{\alpha_1}$, and call $F' = F^{\alpha_1}$ the pruned part of F.

Observation 1.6.6. If F' is the pruned part of F, it is clear that F' is a pruned $P_\kappa\lambda$ forest. Furthermore, by an induction on the F^α's, it is not difficult to see that any branch through F is a branch through F', i.e. $[F] = [F']$. In the special case where κ is regular and $|\mathrm{Lev}_z(F)| < \kappa$ for every $z \in P_\kappa\lambda$, just as with trees the pruning process terminates after one step, that is $F^1 = F'$. This may be seen as follows. Suppose otherwise, and choose $f \in F^1$ such that there exists $z_f \in P_\kappa\lambda$ with $\mathrm{dom}(f) \subseteq z_f$ and $\mathrm{Lev}_{z_f}(F^1 \restriction f) = \emptyset$. Let the set of every extension of f to level z_f in F be written as $\{g_\xi : \xi \in \mu\} \subseteq \mathrm{Lev}_{z_f}(F \upharpoonright f)$. Because none of these extensions remain in F^1, it must be that for every $\xi \in \mu \in \kappa$, there exists $z_\xi \supseteq z_f$ with $\mathrm{Lev}_{z_\xi}(F \upharpoonright g_\xi) = \emptyset$. However, let $z = \bigcup_{\xi \in \mu} z_\xi \in P_\kappa\lambda$. If $\overline{f} \in \mathrm{Lev}_z(F \upharpoonright F)$, then for some $\xi \in \mu$, $\overline{f} \upharpoonright z_f = g_\xi$. But then necessarily $\overline{f} \upharpoonright z_\xi \in \mathrm{Lev}_{z_\xi}(F \upharpoonright g_\xi)$, a contradiction.

Just as with trees in the case where $\lambda^+ = \kappa$ (but now even including the case where κ is singular), we can characterize the κ-box topology over 2^λ by means of forests.

Proposition 1.6.7. Let $\kappa \leq \lambda^+$. Then $E \subseteq 2^\lambda$ is closed in the κ-box topology if and only if for some $P_\kappa\lambda$-forest F, $E = [F]$. In particular, if F_E denotes the $P_\kappa\lambda$-forest generated by E, then $[F_E] = E$. More generally, let $\mu \leq \lambda^+$. Then a set $E \subseteq P_\mu\lambda$ is closed in the topology over $P_\mu\lambda$ induced by the κ-box topology over 2^λ, $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_\mu\lambda$, if and only if $E = [F_E] \cap P_\mu\lambda$.

Proof. This is purely definitional. First, let $E \subseteq 2^\lambda$ be closed in the κ-box topology. Consider F_E as above. If $x \notin E$, for some $z \in P_\kappa\lambda$, $O_{x\upharpoonright z} \cap E = \emptyset$. But then $x \upharpoonright z \notin F_E$, so $x \notin [F_E]$. On the other hand, suppose that $E = [F_E]$. Then if $x \notin [F_E]$, for some $z \in P_\kappa\lambda$, $x \upharpoonright z \notin F_E$, so $O_{x\upharpoonright z} \cap E = \emptyset$. More generally, suppose $E \subseteq P_\mu\lambda$ is closed in $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_\mu\lambda$. Then for some $\overline{E} \subseteq P(\lambda)$ closed in $\tau_\kappa^{\mathrm{BOX}}$, $E = \overline{E} \cap P_\mu\lambda = [F_{\overline{E}}] \cap P_\mu\lambda$. However clearly $F_E \subseteq F_{\overline{E}}$, so if $b \notin E$ with $b \in P_\mu\lambda$ then $b \notin [F_{\overline{E}}]$ and so $b \notin [F_E]$. Thus $E = [F_E] \cap P_\mu\lambda$. On the other hand, if $E = [F_E] \cap P_\mu\lambda$, then E is closed in $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_\mu\lambda$ because F_E is a $P_\kappa\lambda$ forest and so $[F_E]$ is closed in $\tau_\kappa^{\mathrm{BOX}}$. $\square$

Here we give some examples and make some observations about the κ-box topologies over 2^λ. For illustration, we often use the characterization in 1.6.7.

Observation 1.6.8. A union of less than $\mathrm{cf}(\kappa)$-many closed sets in $\tau_\kappa^{\mathrm{BOX}}$ is closed.

Proof. Let $\delta \in \mathrm{cf}(\kappa)$, and let $\langle E_\alpha = [F_\alpha] : \alpha \in \delta\rangle$ be a sequence of closed subsets of 2^λ in $\tau_\kappa^{\mathrm{BOX}}$. Let $E = \bigcup_{\alpha \in \delta} E_\alpha$ and consider F_E. As usual F_E is a $P_\kappa\lambda$-forest, and $E \subseteq [F_E]$. So let

$b \in [F_E]$. If $b \notin E$, then for every $\alpha \in \delta$, $b \notin [F_\alpha]$, so there exists $z_\alpha \in P_\kappa\lambda$ with $b \restriction z_\alpha \notin F_\alpha$. However, if $z = \bigcup_{\alpha\in\delta} z_\alpha$, then $z \in P_\kappa\lambda$ and $b \restriction z \notin F_\alpha$ for every $\alpha \in \delta$. But then $b \notin [F_E]$, a contradiction. $\square$

Observation 1.6.9. If $\kappa_1 < \kappa_2 \leq \lambda^+$ then $\tau^{\mathrm{BOX}}_{\kappa_1}$ is a proper subset of $\tau^{\mathrm{BOX}}_{\kappa_2}$.

Proof. Let $\kappa_1 < \kappa_2 \leq \lambda^+$ and let $E_1 = [F_{E_1}]$ be closed in $\tau^{\mathrm{BOX}}_{\kappa_1}$. Here we use F_{E_1} to indicate the $P_{\kappa_1}\lambda$-forest generated by E_1. Let $F_2 = \{b \restriction z : b \in E_1, z \in P_{\kappa_2}\lambda\}$ be the $P_{\kappa_2}\lambda$-forest generated by E_1 and suppose that $b \in [F_2]$. We need to see that $b \in [F_{E_1}]$. However, if $y \in P_{\kappa_1}\lambda$ then $y \in P_{\kappa_2}\lambda$ so there exists $b_y \in E$ such that $b_y \restriction y = b \restriction y$. But then $b \restriction y \in F_{E_1}$ as well, so $b \in [F_{E_1}]$. Thus $\tau^{\mathrm{BOX}}_{\kappa_1} \subseteq \tau^{\mathrm{BOX}}_{\kappa_2}$. Next, let $E_2 = \{b \in {}^\lambda 2 : |\{\alpha \in \lambda : b(\alpha) = 1\}| < \kappa_1\}$. That is, under the identification of 2^λ with $P(\lambda)$, $E_2 = P_{\kappa_1}\lambda$. We argue that E_2 is closed in $\tau^{\mathrm{BOX}}_{\kappa_2}$, but not in $\tau^{\mathrm{BOX}}_{\kappa_1}$. If F_{E_2} is the $P_{\kappa_2}\lambda$-forest generated by E_2, then if $b \in [F_{E_2}]$ necessarily $|\{\alpha \in \lambda : b(\alpha) = 1\}| < \kappa_1$, as otherwise for some $z \in P_{\kappa_2}\lambda$ with $|z| = \kappa_1$ we would have $b(\alpha) = 1$ for every $\alpha \in z$, but then there couldn't exist $b' \in E_2$ with $b' \restriction z = b \restriction z$. So $E_2 = [F_{E_2}]$. On the other hand, if F_1 is the $P_{\kappa_1}\lambda$-forest generated by E_2, that is $F_1 = \{f : \mathrm{dom}(f) \in P_{\kappa_1}\lambda \text{ and } f \in {}^{\mathrm{dom}(f)}2\}$, then $[F_1] = 2^\lambda \neq E_2$. So E_2 is not closed in $\tau^{\mathrm{BOX}}_{\kappa_1}$, but closed in $\tau^{\mathrm{BOX}}_{\kappa_2}$. $\square$

Observation 1.6.10. If κ is a limit cardinal and E is closed in $\tau^{\mathrm{BOX}}_{\kappa}$, then E can be written as the intersection of $\mathrm{cf}(\kappa)$-many subsets of 2^λ, each of which is closed in a $\tau^{\mathrm{BOX}}_{\mu}$ for some $\mu \in \kappa$.

Proof. Let $E = [F]$ be closed in $\tau^{\mathrm{BOX}}_{\kappa}$. Let $\langle \mu_\alpha : \alpha \in \mathrm{cf}(\kappa)\rangle$ be an increasing cofinal sequence of cardinals in κ. For each α, let $F_\alpha = \{b \restriction z : b \in E, z \in P_{\mu_\alpha}\lambda\}$. By the same argument as in 1.6.9, for $\alpha \in \beta$, $[F_\beta] \subseteq [F_\alpha]$ and $[F] \subseteq [F_\alpha]$ for every $\alpha \in \mathrm{cf}(\kappa)$. So clearly $[F] \subseteq \bigcap_{\alpha\in\mathrm{cf}(\kappa)} [F_\alpha]$. On the other hand, suppose $b \in \bigcap_{\alpha\in\mathrm{cf}(\kappa)} [F_\alpha]$ and $z \in P_\kappa\lambda$. Then for some α, $z \in P_{\mu_\alpha}\lambda$, and $b \in [F_\alpha]$, so there exists $b_z \in E$ with $b_z \restriction z = b \restriction z$. Then $b \restriction z \in F$, so $[F] = \bigcap_{\alpha\in\mathrm{cf}(\kappa)} [F_\alpha]$. $\square$

Proposition 1.6.11. In 1.6.9 we saw that for $\kappa_1 < \kappa_2 \leq \lambda^+$, there exists $E \subseteq 2^\lambda$ which is closed in $\tau_{\kappa_2}^{\mathrm{BOX}}$ and not closed in $\tau_{\kappa_1}^{\mathrm{BOX}}$. So for κ a successor cardinal, this shows that we can find subsets closed in $\tau_\kappa^{\mathrm{BOX}}$ which aren't closed in τ_μ^{BOX} for any $\mu \in \kappa$. This is also true for limit cardinals κ, that is there exist $E \subseteq 2^\lambda$ which are closed in $\tau_\kappa^{\mathrm{BOX}}$ but not closed in τ_μ^{BOX} for any $\mu \in \kappa$.

Proof. Let κ be a limit cardinal, and let $\langle \mu_\alpha : \alpha \in \mathrm{cf}(\kappa) \rangle$ be an increasing cofinal sequence of cardinals in κ. Partition κ into $\mathrm{cf}(\kappa)$-many sets of size κ, $\langle A_\alpha : \alpha \in \mathrm{cf}(\kappa) \rangle$. Define $E = \{b \in {}^\lambda 2 : \exists \alpha \in \mathrm{cf}(\kappa) \text{ s.t. } b(\beta) = 0 \text{ for } \beta \notin A_\alpha \text{ and } |\{\beta \in A_\alpha : b(\beta) = 1\}| < \mu_\alpha\}$. Note that $|E| = \kappa^{<\kappa}$. Let $F = \{b \restriction z : b \in E, z \in P_\kappa \lambda\}$. If $b \in [F]$, then suppose $b \notin E$. Then either $b(\beta_1) = 1$ and $b(\beta_2) = 1$ for $\beta_1 \in A_{\alpha_1}$ and $\beta_2 \notin A_{\alpha_1}$, in which case there cannot exist $b_{\{\beta_1,\beta_2\}} \in E$ such that $b_{\{\beta_1,\beta_2\}} \restriction \{\beta_1, \beta_2\} = b \restriction \{\beta_1, \beta_2\}$, or for some $\alpha \in \mathrm{cf}(\kappa)$, $|\{\beta \in A_\alpha : b(\beta) = 1\}| \geq \mu_\alpha$, in which case for some $z \in P_\kappa\lambda$ with $|z| = \mu_\alpha$ and $b(\beta) = 1$ for every $\beta \in z$, there cannot exist $b_z \in E$ with $b_z \restriction z = b \restriction z$. Thus $b \in E$, so E is closed in $\tau_\kappa^{\mathrm{BOX}}$. On the other hand, if $\mu \in \kappa$ then for some α, $\mu < \mu_\alpha$, and if we set $F_\alpha = \{b \restriction y : b \in E, y \in P_{\mu_\alpha}\lambda\}$ then for any $\beta \geq \alpha$, $b_\beta \in [F_\alpha]$ where $b_\beta(\gamma) = 0$ for $\gamma \notin A_\beta$ and $b_\beta(\gamma) = 1$ for $\gamma \in A_\beta$, i.e. $b_\beta = \mathbf{1}_{A_\beta}$. However, clearly $b_\beta \notin E$. So E is not closed in τ_μ^{BOX} for any $\mu \in \kappa$. □

Observation 1.6.12. If $E \subseteq 2^\lambda$ is such that $|E| < \mu$ for any cardinal μ, then E is closed in τ_μ^{BOX}.

Proof. Enumerate $E = \langle b_\alpha : \alpha \in \delta \rangle$ for some $\delta \in \mu$. Let $F_\mu = \{b \restriction y : y \in P_\mu \lambda\}$. As usual, $E \subseteq [F_\mu]$. On the other hand, if there were a $b \in [F_\mu] \setminus E$, then for every $\alpha \in \delta$, there exists $\beta_\alpha \in \lambda$ with $b(\beta_\alpha) \neq b_\alpha(\beta_\alpha)$. If $z = \{\beta_\alpha : \alpha \in \delta\}$, then $z \in P_\mu\lambda$, so $b \restriction z \in F_\mu$. However, this is impossible. So $[F_\mu] = E$. □

Observation 1.6.13. The observation in 1.6.12 is sharp— that is for every cardinal $\mu \leq \lambda$, there exists $E \subseteq 2^\lambda$ of size μ which is not closed in τ_μ^{BOX}.

Proof. For regular cardinals μ, one can consider $E = \{b \in {}^{\lambda}2 : \exists \beta \in \mu \text{ s.t. } b(\gamma) = 1 \text{ if } \gamma \in \beta \text{ and } b(\gamma) = 0 \text{ if } \beta \leq \gamma\}$. This set is of size μ, and if F is the $P_{\mu}\lambda$-forest generated by E, $b \in [F] \setminus E$ where $b(\beta) = 1$ for every $\beta \in \mu$ and $b(\beta) = 0$ for $\mu \leq \beta$. However, this argument does not work if μ is singular, because if μ is singular this b is not an accumulation point of E in $\tau_{\mu}^{\mathrm{BOX}}$. However, one can consider instead more generally for any μ, $E = \{b \in {}^{\lambda}2 : \exists \beta \in \mu \text{ s.t. } b(\beta) = 1 \text{ and } b(\gamma) = 0 \text{ if } \gamma \neq \beta\}$. Then $|E| = \mu$, and one may easily verify that if F is the $P_{\mu}\lambda$-forest generated by E, that $b \in [F] \setminus E$, where $b \in {}^{\lambda}2$ is such that $b(\beta) = 0$ for every $\beta \in \lambda$. $\square$

1.6.2 The κ-sequentially closed topology over 2^{λ}

We saw in 1.1.40 that for κ regular the κ-box topology over 2^{κ} is characterized by saying that closed sets are exactly those which are closed under the limits of κ-sequences of elements. Here we define for κ regular, analogous κ-sequentially closed topologies over 2^{λ} and give some examples of how they can be both the same, or different, from the κ-box topologies. The κ-box topology is also defined for κ singular, but of course because here we are only defining the κ-sequentially closed topology for κ regular, when the two are discussed simultaneously it is assumed that κ is regular.

Definition 1.6.14. Let κ be regular and $\kappa \leq \lambda^{+}$. Define the κ-sequentially closed topology $\tau_{\kappa}^{\mathrm{SC}}$ over 2^{λ} by saying that $E \subseteq 2^{\lambda}$ is (κ-sequentially) closed in $\tau_{\kappa}^{\mathrm{SC}}$ if and only if E is closed under all convergent κ sequences, i.e. for every convergent $\langle x_{\alpha} : \alpha \in \kappa \rangle \subseteq E$, $\lim(\langle x_{\alpha} : \alpha \in \kappa \rangle) \in E$. If $A \subseteq P(\lambda)$ is closed in $\tau_{\kappa}^{\mathrm{SC}}$, we often say that A is κ-sequentially closed.

Proposition 1.6.15. The κ-sequentially closed topology $\tau_{\kappa}^{\mathrm{SC}}$ over 2^{λ} is indeed a topology and it has the following properties.

1. If $\kappa = \lambda^{+}$, then $\tau_{\kappa}^{\mathrm{SC}}$ is the (full) box topology over 2^{λ}, that is $\tau_{\kappa}^{\mathrm{SC}} = P(P(\lambda)) = \tau_{\kappa}^{\mathrm{BOX}}$.

2. If $\kappa = \lambda$, then $\tau_\kappa^{\mathrm{SC}} = \tau_\kappa^{\mathrm{BOX}}$.

3. If $A \subseteq P(\lambda)$ with $|A| < \kappa$, then A is κ-sequentially closed, and indeed the union of fewer than κ-many κ-sequentially closed sets is κ-sequentially closed.

Proof. It is straightforward to see that the union of two κ-sequentially closed sets is κ-sequentially closed, the arbitrary intersection of κ-sequentially closed sets is κ-sequentially closed, and $\emptyset$ and 2^λ are both κ-sequentially closed, so $\tau_\kappa^{\mathrm{SC}}$ is a topology. Suppose $\kappa = \lambda^+$ and take $A \subseteq 2^\lambda$. Then any κ-convergent sequence $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq A$ eventually has every coordinate in λ fixed to a particular value, so $\lim(\langle x_\alpha : \alpha \in \kappa\rangle) = x_\beta \in A$ for some sufficiently large β. So in this case $\tau_\kappa^{\mathrm{SC}} = P(P(\lambda))$. In this case it is clear that $\tau_\kappa^{\mathrm{BOX}} = P(P(\lambda))$. Next, suppose that $\kappa = \lambda$. In 1.1.40 we saw that $\tau_\kappa^{\mathrm{SC}} = \tau_\kappa^{\mathrm{BOX}}$. Finally, note that clearly any singleton $\{x\} \subseteq 2^\lambda$ is κ-sequentially closed, and if $\langle A_\alpha : \alpha \in \gamma\rangle \subseteq P(P(\lambda))$ with $\gamma \in \kappa$ is a sequence of κ-sequentially closed subsets of 2^λ, then $\bigcup_{\alpha\in\gamma} A_\alpha$ is κ-sequentially closed. This is because for any convergent $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq \bigcup_{\alpha\in\gamma} A_\alpha$, there must exist some $\beta \in \gamma$ and a cofinal sub-sequence $\langle x_{\alpha_\xi} : \xi \in \kappa\rangle \subseteq A_\beta$ by regularity, which must then converge in A_β. So then also, if $|A| < \kappa$, A is κ-sequentially closed. □

Proposition 1.6.16. If $\kappa < \lambda$, then the κ-box topology over 2^λ is a proper subset of the κ-sequentially closed topology over 2^λ (that is, $\tau_\kappa^{\mathrm{BOX}} \subsetneq \tau_\kappa^{\mathrm{SC}}$). More generally, for any $\mu \in (\kappa^+, \lambda^+]$, $\tau_\kappa^{\mathrm{BOX}} \restriction P_\mu\lambda$ is a proper subset of the topology over $P_\mu\lambda$ induced by the κ-sequentially closed topology over 2^λ, $\tau_\kappa^{\mathrm{SC}} \restriction P_\mu\lambda$ (that is, $\tau_\kappa^{\mathrm{BOX}} \restriction P_\mu\lambda \subsetneq \tau_\kappa^{\mathrm{SC}} \restriction P_\mu\lambda$).

Proof. It suffices to show the more general statement, so let $\mu \in (\kappa^+, \lambda^+]$. To show $\tau_\kappa^{\mathrm{BOX}} \restriction P_\mu\lambda \subseteq \tau_\kappa^{\mathrm{SC}} \restriction P_\mu\lambda$, definitionally it suffices to show $\tau_\kappa^{\mathrm{BOX}} \subseteq \tau_\kappa^{\mathrm{SC}}$. So take $E \subseteq P(\lambda)$ closed in $\tau_\kappa^{\mathrm{BOX}}$. Then $E = [F_E]$. Suppose $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq [F_E]$ is convergent with limit x. Then for any $z \in P_\kappa\lambda$, $x_\alpha \restriction z$ equal to $x \restriction z$ for all sufficiently large α. But then $x \restriction z \in F_E$ for every $z \in P_\kappa\lambda$, so $x \in [F_E]$. Thus E is closed in $\tau_\kappa^{\mathrm{SC}}$. Next, consider $E = P_{\kappa^+}\lambda \subsetneq P_\mu\lambda$. We show that E is closed in $\tau_\kappa^{\mathrm{SC}} \restriction P_\mu\lambda$, but not in $\tau_\kappa^{\mathrm{BOX}} \restriction P_\mu\lambda$. First, $F_E = \{f : \mathrm{dom}(f) \in$

$P_\kappa\lambda$ and $f \in {}^{\mathrm{dom}(f)}2\}$, so $[F_E] \cap P_\mu\lambda = P_\mu\lambda \neq E$, i.e. E is not closed in $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_\mu\lambda$. On the other hand, $E \subseteq P_\mu\lambda$, so to show E is closed in $\tau_\kappa^{\mathrm{SC}} \upharpoonright P_\mu\lambda$ it suffices to show that E is closed in $\tau_\kappa^{\mathrm{SC}}$. Suppose $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq E$ is convergent with limit x. If $|x| \geq \kappa^+$, then because $x = \mathrm{liminf}(\langle x_\alpha : \alpha \in \kappa\rangle)$, there must be some $\gamma \in \kappa$ so that a κ^+-sized subset of x is a subset of x_γ, which is impossible. Thus $x \in P_{\kappa^+}\lambda = E$. $\square$

Proposition 1.6.17. In 1.6.16 we saw that as long as $\mu \in (\kappa^+, \lambda^+]$, $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_\mu\lambda \subsetneq \tau_\kappa^{\mathrm{SC}} \upharpoonright P_\mu\lambda$. In contrast, if $\mu \leq \kappa^+$, then $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_\mu\lambda = \tau_\kappa^{\mathrm{SC}} \upharpoonright P_\mu\lambda$. This may be viewed as a generalization of 1.1.40.

Proof. Let $\mu \leq \kappa^+$ and let $E \subseteq P_\mu\lambda$ be closed in $\tau_\kappa^{\mathrm{SC}} \upharpoonright P_\mu\lambda$. We need to see that E is closed in $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_\mu\lambda$. So, let $b \in [F_E] \cap P_\mu\lambda$. Fix $M \prec H_\theta$ to be an internally unbounded submodel with $|M| = \kappa$, $\kappa \subseteq M$, witnessed by $\langle M_\alpha : \alpha \in \kappa\rangle$ such that for every $\alpha \in \kappa$, $M_\alpha \prec H_\theta$ with $\{\kappa, b, E, F_E, \mu, \lambda, \mathrm{etc.}\} \subseteq M_\alpha$, $M_\alpha \in M$, $|M_\alpha| < \kappa$, if $\alpha \in \beta$ then $M_\alpha \subseteq M_\beta$, and $M = \bigcup\limits_{\alpha\in\kappa} M_\alpha$. Note that $\langle A_\alpha = M_\alpha \cap \lambda : \alpha \in \kappa\rangle \subseteq M$, and then by elementarity because $b \in [F_E]$, for every $\alpha \in \kappa$ there exists $b_\alpha \in E \cap M$ with $b_\alpha \upharpoonright A_\alpha = b \upharpoonright A_\alpha$. We show that $\langle b_\alpha : \alpha \in \kappa\rangle$ converges to b. If $\beta \in b$, then $\beta \in M$ (as $b \in E \cap M \subseteq P_\mu\lambda$ and $\kappa \subseteq M$, $\kappa \in M$) and so for all sufficiently large M_α, $\beta \in M_\alpha$, so $\beta \in b_\alpha$ for these α. On the other hand, if $\beta \notin b$, then either $\beta \notin M$ in which case because $b_\alpha \subseteq M$ (as $b_\alpha \in E \cap M \subseteq P_\mu\lambda$ as above), $\beta \notin b_\alpha$ for every α, or $\beta \in M$, in which case for all sufficiently large M_α, $\beta \in M_\alpha$ so $\beta \notin b_\alpha$ for all these α. It is clear then that $\lim(\langle b_\alpha : \alpha \in \kappa\rangle) = b$ and so because E is of the form $\overline{E} \cap P_\mu\lambda$ for $\overline{E}$ closed in $\tau_\kappa^{\mathrm{SC}}$, $b \in E$. So E is closed in $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_\mu\lambda$, as desired. This argument as written only of course makes sense for $\kappa > \omega$, however if $\kappa = \omega$ then every countable elementary submodel $M \prec H_\theta$ is such that ${}^{<\omega}M \subseteq M$, so we can proceed as above. $\square$

Observation 1.6.18. Let $\kappa_1 < \kappa_2 \leq \lambda^+$ be regular. If $\mu \leq \kappa_1^+$, by 1.6.17 and 1.6.9, $\tau_{\kappa_1}^{\mathrm{SC}} \upharpoonright P_\mu\lambda \subseteq \tau_{\kappa_2}^{\mathrm{SC}} \upharpoonright P_\mu\lambda$. However in general, and in strong contrast to $\tau_{\kappa_1}^{\mathrm{BOX}}$ and $\tau_{\kappa_2}^{\mathrm{BOX}}$, neither $\tau_{\kappa_1}^{\mathrm{SC}} \subseteq \tau_{\kappa_2}^{\mathrm{SC}}$ nor $\tau_{\kappa_2}^{\mathrm{SC}} \subseteq \tau_{\kappa_1}^{\mathrm{SC}}$.

Proof. Let $E = P_{\kappa_2}\lambda$. Unless $\kappa_2 = \lambda^+$, in which case $\tau^{\mathrm{SC}}_{\kappa_1} \subseteq \tau^{\mathrm{SC}}_{\kappa_2} = P(P(\lambda))$, E is not closed in $\tau^{\mathrm{SC}}_{\kappa_2}$, witnessed by $\langle \alpha : \alpha \in \kappa_2 \rangle \subseteq E$, which converges to $\kappa_2 \notin E$. On the other hand, E is closed in $\tau^{\mathrm{SC}}_{\kappa_1}$, as follows. Let $\langle x_\alpha : \alpha \in \kappa_1 \rangle \subseteq E$ converge to x. It must be that $|x| < \kappa_2$, because otherwise by the pigeonhole principle there would exist $\alpha \in \kappa_1$ with $|x_\alpha| \geq \kappa_2$, which is impossible. So it is not the case that $\tau^{\mathrm{SC}}_{\kappa_1} \subseteq \tau^{\mathrm{SC}}_{\kappa_2}$. Similarly, $\{\alpha : \alpha \in \kappa_1\}$ is not closed in $\tau^{\mathrm{SC}}_{\kappa_1}$, however it is closed in $\tau^{\mathrm{SC}}_{\kappa_2}$, because any convergent κ_2-length sequence consisting of elements of $\{\alpha : \alpha \in \kappa_1\}$ must be eventually constant. $\square$

In 1.6.7, we saw that closed sets in $\tau^{\mathrm{BOX}}_{\kappa} \restriction P_\mu\lambda$ can be characterized a similar way as closed sets in $\tau^{\mathrm{BOX}}_{\kappa}$ can be. Considering $\tau^{\mathrm{SC}}_{\kappa} \restriction P_\mu\lambda$, we have the following proposition.

Proposition 1.6.19. Let $\kappa \leq \lambda^+$ be regular and $\mu \leq \lambda^+$ such that either $\mathrm{cf}(\mu) \neq \kappa$ or $\mathrm{cf}(\mu) = \kappa$ and $\mathfrak{b}({}^\kappa\kappa/ < \kappa) > \mu$. Here $\mathfrak{b}({}^\kappa\kappa/ < \kappa)$ denotes the (un)bounding number for ${}^\kappa\kappa$ modulo the ideal of subsets of size $< \kappa$. That is, the smallest cardinality of a collection of functions $F \subseteq {}^\kappa\kappa$ such that for any $g \in {}^\kappa\kappa$, there exists $f \in F$ such that $|\{\alpha \in \kappa : f(\alpha) \geq g(\alpha)\}| = \kappa$. Then $E \subseteq P_\mu\lambda$ is closed in $\tau^{\mathrm{SC}}_{\kappa} \restriction P_\mu\lambda$ if and only if for every convergent $\langle x_\alpha : \alpha \in \kappa \rangle \subseteq E$ with $\lim(\langle x_\alpha : \alpha \in \kappa \rangle) \in P_\mu\lambda$, $\lim(\langle x_\alpha : \alpha \in \kappa \rangle) \in E$.

Proof. It is always true that if $E \subseteq P_\mu\lambda$ is closed in $\tau^{\mathrm{SC}}_{\kappa} \restriction P_\mu\lambda$, for every convergent $\langle x_\alpha : \alpha \in \kappa \rangle \subseteq E$ with $\lim(\langle x_\alpha : \alpha \in \kappa \rangle) \in P_\mu\lambda$, $\lim(\langle x_\alpha : \alpha \in \kappa \rangle) \in E$. For the other direction, first suppose that $\mathrm{cf}(\mu) > \kappa$. We argue that if $\langle x_\alpha : \alpha \in \kappa \rangle \subseteq P_\mu\lambda$ converges to x, then $x \in P_\mu\lambda$. However this is clear because $x \subseteq \bigcup_{\alpha\in\kappa} x_\alpha$, $|x_\alpha| < \mu$, and $\mathrm{cf}(\mu) > \kappa$ so $|\bigcup_{\alpha\in\kappa} x_\alpha| < \mu$. Next suppose that $\mathrm{cf}(\mu) < \kappa$. Similarly, we can argue that if $\langle x_\alpha : \alpha \in \kappa \rangle \subseteq P_\mu\lambda$ converges to x, then $x \in P_\mu\lambda$. To every $\gamma \in x$, let $\alpha_\gamma \in \kappa$ be minimal such that for every $\beta \in [\alpha_\gamma, \kappa)$, $\gamma \in x_\beta$. For each $\alpha \in \kappa$, let $A_\alpha \subseteq x_\alpha$ be the collection of all $\gamma \in x$ such that $\alpha_\gamma = \alpha$. Then $\langle A_\alpha : \alpha \in \kappa \rangle$ is $\subseteq$-increasing, $\bigcup_{\alpha\in\kappa} A_\alpha = x$, and $|A_\alpha| < \mu$ for every α. Note that $\langle \mathrm{otp}(A_\alpha) : \alpha \in \kappa \rangle \subseteq \mu$ is then $\leq$-increasing. However, because $\mathrm{cf}(\mu) < \kappa$, it cannot be that $\sup\{\mathrm{otp}(A_\alpha) : \alpha \in \kappa\} = \mu$. But then $\mathrm{otp}(\bigcup_{\alpha\in\kappa} A_\alpha) = \mathrm{otp}(x) < \mu$. So, suppose $\mathrm{cf}(\mu) = \kappa$ and $\mathfrak{b}({}^\kappa\kappa/ < \kappa) > \mu$ (for example this will always be true for $\mu = \kappa$). Let $E \subseteq P_\mu\lambda$ be such that for every convergent $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq E$ with $\lim(\langle x_\alpha : \alpha \in \kappa\rangle) \in P_\mu\lambda$, $\lim(\langle x_\alpha : \alpha \in \kappa\rangle) \in E$. Let $E' = E \cup \lim(E)$, where $\lim(E) = \{x \in 2^\lambda \setminus E :$ for some $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq E, \lim(\langle x_\alpha : \alpha \in \kappa\rangle) = x\}$. Equivalently then, $\lim(E) = \{x \in [\lambda]^\mu :$ for some $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq E, \lim(\langle x_\alpha : \alpha \in \kappa\rangle) = x\}$. If we show that for any convergent $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq E'$ with limit x, there exists a convergent $\langle x'_\alpha : \alpha \in \kappa\rangle \subseteq E$ with limit x, then we will be done, because not only then will E' be closed in $\tau_\kappa^{\mathrm{SC}}$, but $E' \cap P_\mu\lambda = E$. So, suppose that $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq E'$ is convergent with limit x. Then any cofinal subsequence is also convergent with limit x, and without loss of generality we may therefore assume that $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq \lim(E)$. So for every $\alpha \in \kappa$, there exists a convergent $\langle y^\alpha_\beta : \beta \in \kappa\rangle \subseteq E$ with limit x_α. Note that $x \subseteq \bigcup\limits_{\alpha,\beta\in\kappa} y^\alpha_\beta$. For every $\gamma \in \bigcup\limits_{\alpha,\beta\in\kappa} y^\alpha_\beta \setminus x$, there exists $\delta_\gamma \in \kappa$ such that for every $\delta \in [\delta_\gamma, \kappa)$, $\gamma \notin x_\delta$. And similarly then, for every such δ, there exists $\xi_\delta \in \kappa$ such that for every $\xi \in [\xi_\delta, \kappa)$, $\gamma \notin y^\delta_\xi$. Define $f_\gamma \in {}^\kappa\kappa$ by $f_\gamma(\eta) = 0$ if $\eta \in \delta_\gamma$ and $f_\gamma(\eta) = \xi_\eta$ for $\eta \in [\delta_\gamma, \kappa)$. Suppose that $g \in {}^\kappa\kappa$ such that $f_\gamma <^* g$, that is so that for some $\nu \in \kappa$ for every $\nu' \geq \nu$, $f_\gamma(\nu') < g(\nu')$. Consider $\langle y^\alpha_{g(\alpha)} : \alpha \in \kappa\rangle \subseteq E$. Then $\gamma \notin \operatorname{limsup}(\langle y^\alpha_{g(\alpha)} : \alpha \in \kappa\rangle)$. Similarly, suppose $\delta \in x$. Then there exists $\gamma_\delta \in \kappa$ such that for every $\gamma \in [\gamma_\delta, \kappa)$, $\delta \in x_\gamma$. And again then, for each such γ, there exists $\zeta_\gamma \in \kappa$ such that for every $\zeta \in [\zeta_\gamma, \kappa)$, $\delta \in y^\gamma_\zeta$. Define $f_\delta \in {}^\kappa\kappa$ by $f_\delta(\eta) = 0$ if $\eta \in \gamma_\delta$ and $f_\delta(\eta) = \zeta_\eta$ for $\eta \in [\gamma_\delta, \kappa)$. Again, if $g \in {}^\kappa\kappa$ is such that $f_\delta <^* g$, then $\delta \in \operatorname{liminf}(\langle y^\alpha_{g(\alpha)} : \alpha \in \kappa\rangle)$. Let $F = \{f_\gamma : \gamma \in \bigcup\limits_{\alpha,\beta\in\kappa} y^\alpha_\beta \setminus x\} \cup \{f_\delta : \delta \in x\}$. Note that $|F| \leq \mu$, and so because $\mathfrak{b}({}^\kappa\kappa/<\kappa) > \mu$, there exists $g \in {}^\kappa\kappa$ such that for every $f \in F$, $f <^* g$. However, it is then not difficult to verify that $\lim(\langle y^\alpha_{g(\alpha)} : \alpha \in \kappa\rangle) = x$, as desired. $\square$

Remark 1.6.20. The extra conditions in the characterization of closed sets in $\tau_\kappa^{\mathrm{SC}} \restriction P_\mu\lambda$ in 1.6.19 are not superfluous—that is if, for example, $\mathrm{cf}(\mu) = \kappa$ and $\kappa^\kappa < \mu$, then there exists $E \subseteq P_\mu\lambda$ such that for every convergent $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq E$ with $\lim(\langle x_\alpha : \alpha \in \kappa\rangle) \in P_\mu\lambda$, $\lim(\langle x_\alpha : \alpha \in \kappa\rangle) \in E$, but E is not closed in $\tau_\kappa^{\mathrm{SC}} \restriction P_\mu\lambda$.

Proof. Let $\mathrm{cf}(\mu) = \kappa$ and $\kappa^\kappa < \mu$. Fix $A \subseteq \mu$ and $B \subseteq \mu$ such that $|A| = |B| = \kappa^\kappa$,

$A \cap B = \emptyset$, and via bijections between ${}^\kappa\kappa$ and A and B, identify ordinals in A and B with functions in ${}^\kappa\kappa$, i.e. for each $\gamma \in A$, $f_\gamma \in {}^\kappa\kappa$, if $\gamma \neq \zeta$ then $f_\gamma \neq f_\zeta$, and $\{f_\gamma : \gamma \in A\} = {}^\kappa\kappa$, and similarly for each $\delta \in B$, $f'_\delta \in {}^\kappa\kappa$, if $\delta \neq \xi$ then $f_\delta \neq f_\xi$, and $\{f_\delta : \delta \in B\} = {}^\kappa\kappa$. Fix $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq [\mu]^\mu$ such that if $\alpha \in \beta \in \mu$, then $|x_\alpha \setminus x_\beta| = \mu$, $x_\beta \subseteq x_\alpha$, and $\bigcap_{\alpha\in\kappa} x_\alpha = A$. For every $\alpha, \beta \in \kappa$, let $y'^\alpha_\beta = \{\gamma \in A : f_\gamma(\alpha) \leq \beta\}$. Note that that for each $\alpha \in \kappa$, $\lim(\langle y'^\alpha_\beta : \beta \in \kappa\rangle) = A \subseteq x_\alpha$. For each $\alpha \in \kappa$, let $\langle z^\alpha_\beta : \beta \in \kappa\rangle \subseteq P_\mu(x_\alpha \setminus A)$ be $\subseteq$-increasing such that $\lim(\langle z^\alpha_\beta : \beta \in \kappa\rangle) = x_\alpha \setminus A$. This is possible because $\mathrm{cf}(\mu) = \kappa$. Also for $\beta \in \kappa$, let $y''^0_\beta = \{\delta \in B : f'_\delta(\beta) > \beta\}$. Finally, for $\alpha \in (0, \kappa)$ and $\beta \in \kappa$, let $y^\alpha_\beta = y'^\alpha_\beta \cup z^\alpha_\beta$ and for $\alpha = 0$ and $\beta \in \kappa$ let $y^0_\beta = y'^0_\beta \cup y''^0_\beta \cup z^0_\beta$. Let $E = \{y^\alpha_\beta : \alpha, \beta \in \kappa\}$. Note that for each $\alpha \in \kappa$, $\langle y^\alpha_\beta : \beta \in \kappa\rangle \subseteq P_\mu\lambda$ is convergent with limit x_α, and $\lim(\langle x_\alpha : \alpha \in \kappa\rangle) = A \in P_\mu\lambda \setminus E$. Thus E is not closed in $\tau^{\mathrm{SC}}_\kappa \restriction P_\mu\lambda$. We need therefore only to see that E contains the limit of all of its κ-convergent sequences which happens to be in $P_\mu\lambda$. Suppose that $\langle w_\alpha : \alpha \in \kappa\rangle \subseteq E$. Then either a cofinal subsequence of $\langle w_\alpha : \alpha \in \kappa\rangle$ is a cofinal subsequence of $\langle y^\alpha_\beta : \beta \in \kappa\rangle$ for some α, in which case $\lim(\langle w_\alpha : \alpha \in \kappa\rangle) = x_\alpha \notin P_\mu\lambda$, or a cofinal subsequence of $\langle w_\alpha : \alpha \in \kappa\rangle$ is of the form $\langle y^\alpha_{g(\alpha)} : \alpha \in \mathrm{dom}(g)\rangle$ for some $g \in {}^{\mathrm{dom}(g)}\kappa$ with $\mathrm{dom}(g) \in [\kappa]^\kappa$. There are two cases, either $g =^* 0$, that is on a co-bounded subset of $\mathrm{dom}(g)$, $g(\alpha) = 0$, or there exists $\gamma \in A$ with $|\{\beta \in \mathrm{dom}(g) : f_\gamma(\beta) < g(\beta)\}| = \kappa$ and $|\{\beta \in \mathrm{dom}(g) : f_\gamma(\beta) > g(\beta)\}| = \kappa$. In the former case, without loss of generality we may assume that $\langle w_\alpha : \alpha \in \kappa\rangle$ is a cofinal subsequence of $\langle y^0_\beta : \beta \in \kappa\rangle$, that is for some $g \in {}^\kappa\kappa$ increasing with $g(\alpha) \geq \alpha$, we have $w_\alpha = y^0_{g(\alpha)}$ for every $\alpha \in \kappa$. Then there exists $\delta \in B$ such that $|\{\alpha \in \kappa : f_\delta(\alpha) < g(\alpha)\}| = |\{\alpha \in \kappa : f_\delta(\alpha) > g(\alpha)\}| = \kappa$. But then $\delta \in \mathrm{limsup}(\langle w_\alpha : \alpha \in \kappa\rangle) \setminus \mathrm{liminf}(\langle w_\alpha : \alpha \in \kappa\rangle)$, so $\langle w_\alpha : \alpha \in \kappa\rangle$ is not convergent. In the latter case, choose $\gamma \in A$ with $|\{\beta \in \mathrm{dom}(g) : f_\gamma(\beta) < g(\beta)\}| = \kappa$ and $|\{\beta \in \mathrm{dom}(g) : f_\gamma(\beta) > g(\beta)\}| = \kappa$. But then similarly $\gamma \in \mathrm{limsup}(\langle w_\alpha : \alpha \in \kappa\rangle) \setminus \mathrm{liminf}(\langle w_\alpha : \alpha \in \kappa\rangle)$, which is impossible. $\square$

1.6.3 Cantor-Bendixson process on $P_\kappa\lambda$-forests

Just as we did for trees, here we give a Cantor-Bendixson process on $P_\kappa\lambda$ forests.

Definition 1.6.21. Let F be a $P_\kappa\lambda$-forest. Let F_α denote the α^{th} derived forest (or α^{th} derivative) of F. This is defined by recursion on α. First, let $F_0 = F'$ denote that pruned part of F (as in 1.6.5). For successors, let $F'_{\alpha+1}$ denote the cofinally splitting part of F_α, that is $F'_{\alpha+1} = \{f \in F_\alpha : f \text{ is cofinally splitting in } F_\alpha\}$, and let $F_{\alpha+1}$ be the pruned part of $F'_{\alpha+1}$. So at each stage, we remove all nodes which aren't cofinally splitting, and then look at the pruned part of the resulting forest. If α is a limit, let F_α be the pruned part of $\bigcap_{\gamma\in\alpha} F_\gamma$. There must exist a minimal α_0 such that $F_{\alpha_0} = F_{\alpha_0+1}$, and we call this the (Cantor-Bendixson) height of the forest, $\alpha_0 = \mathrm{ht}_{CB}(F)$. Note that by construction we have ensured that F_α is pruned for every α.

Definition 1.6.22. For F a $P_\kappa\lambda$-forest, let $\mathrm{Ker}(F) = \{f \in F : f \in F_\alpha \text{ for every } \alpha\}$ and let $\mathrm{Sc}(F) = \{f \in F : \text{ there exists } \alpha \text{ such that } f \in F_\alpha \setminus F_{\alpha+1}\}$. For any $f \in \mathrm{Sc}(F)$, let $\mathrm{rank}_{CB}(f) = \alpha$ denote the unique α such that $f \in F_\alpha \setminus F_{\alpha+1}$, that is the Cantor-Bendixson rank of f in F.

Observation 1.6.23. Let F be a $P_\kappa\lambda$-forest. Then $\mathrm{Ker}(F) \subseteq F$ is a cofinally splitting subforest of F and $\mathrm{Sc}(F)$ is a disjoint union of "upward cones" in F. This is because for $f, f' \in F$, if $f' \restriction \mathrm{dom}(f) = f$, and $f \in \mathrm{Sc}(F)$, then $f' \in \mathrm{Sc}(F)$. So we may say that $F = \mathrm{Ker}(F) \cup \mathrm{Sc}(F)$ is a decomposition of F into a cofinally splitting subforest (the kernel) and a scattered part.

1.6.4 Comparing the topological and forest processes

Just as for inaccessible κ, κ is weakly compact if and only if κ has the tree property, modulo inaccessibility we can characterize a stronger type of compactness with a stronger type of tree property.

Definition 1.6.24. Let κ be a regular uncountable cardinal. Say that κ is λ-strongly compact if and only if every κ-complete filter generated by a collection of size at most λ over a set S can be extended to a κ-complete ultrafilter over S. This property is often called λ-compact instead of λ-strongly compact. See for example [36] and [41]. Say that κ is strongly compact if and only if κ is λ-strongly compact for every $\lambda \geq \kappa$. Equivalently, if S is an arbitrary set, every κ-complete filter over S can be extended to a κ-complete ultrafilter over S.

Definition 1.6.25. Let κ be a regular uncountable cardinal. Say that κ has the λ-strong tree property if and only if for every $P_\kappa\lambda$-forest F with $0 < |\mathrm{Lev}_z(F)| < \kappa$ for every $z \in P_\kappa\lambda$, $[F] \neq \emptyset$. Say that κ has the strong tree property if and only if κ has the λ-strong tree property for every $\lambda \geq \kappa$.

Observation 1.6.26. If κ has the λ-strong tree property, then if F is a pruned $P_\kappa\lambda$ tree with $|\mathrm{Lev}_z(F)| < \kappa$ for every $z \in P_\kappa\lambda$, F codes a κ-closed subset of 2^λ.

Fact (see [36], [25], or [70]) 1.6.27. If an inaccessible κ is $\lambda^{<\kappa}$-strongly compact, then κ has the λ-strong tree property. Similarly, if κ has the λ-strong tree property, then κ is λ-strongly compact. So also for κ inaccessible, κ is strongly compact if and only if κ has the strong tree property. In typical cases also then, where $\lambda^{<\kappa} = \lambda$, κ is λ-strongly compact if and only if κ has the λ-strong tree property.

Remark 1.6.28. Examining any proof of the equivalence for inaccessible κ and $\lambda \geq \kappa$ with $\lambda^{<\kappa} = \lambda$ of κ having the λ-strong tree property and κ being λ-strongly compact, using for example the filter characterization of strong compactness as in [25] or the infinitary language compactness-theorem characterization as in [36], one may observe that ω has the strong tree property in that every pruned $P_\omega\lambda$ forest has a branch. This follows, for example because all filters can be extended to ultrafilters, or from the usual compactness theorem, etc. This fact seems to have been proven several times independently, the earliest of which as noted by Jech [36] might be due to Rado [55].

In analogy to the case with trees, if F is a $P_\kappa\lambda$-forest coding a κ-closed subset of 2^λ and κ is $\lambda^{<\kappa}$-compact or ω, then we have a strong correspondence between the Cantor-Bendixson process on $[F]$ (the topological process with respect to $\tau_\kappa^{\mathrm{BOX}}$) and the Cantor-Bendixson process on F.

Proposition 1.6.29. Let κ be either ω or $\lambda^{<\kappa}$-strongly compact and let F be a $P_\kappa\lambda$-forest coding a κ-closed subset of 2^λ. Then $[F]_\alpha = [F_\alpha]$ for every ordinal α.

Proof. Note that F_0 denotes the pruned part of F, so $[F_0] = [F] = [F]_0$. Next, suppose that $[F]_\alpha = [F_\alpha]$. We need to see that $[F]_{\alpha+1} = [F_{\alpha+1}]$. First suppose $x \in [F]_\alpha \setminus [F]_{\alpha+1}$. Then for some $z \in P_\kappa\lambda$, $O_{x\restriction z} \cap [F]_\alpha = \{x\}$. Because F_α is pruned and κ is λ-compact, F_α codes a κ-closed subset of 2^λ. Therefore it must be that $x \restriction z \in F_\alpha$ is not cofinally splitting in F_α, so $x \restriction z \in F_\alpha \setminus F_{\alpha+1}$. Consequently $x \in [F_\alpha] \setminus [F_{\alpha+1}]$. On the other hand, suppose $x \in [F_\alpha] \setminus [F_{\alpha+1}]$. Then for some $z \in P_\kappa\lambda$, $x \restriction z \in F_\alpha \setminus F_{\alpha+1}$. Suppose first that $x \restriction z \notin F'_{\alpha+1}$. Then $x \restriction z$ is not cofinally splitting in F_α, and so for some $z' \in P_\kappa\lambda$ with $z' \supseteq z$, $O_{x\restriction z'} \cap [F_\alpha] = O_{x\restriction z'} \cap [F]_\alpha = \{x\}$, so $x \in [F]_\alpha \setminus [F]_{\alpha+1}$. Next suppose that $x \restriction z \in F'_{\alpha+1} \setminus F_{\alpha+1}$. Then for some $z' \in P_\kappa\lambda$ with $z' \supseteq z$, $\mathrm{Lev}_{z'}(F'_{\alpha+1} \restriction (x \restriction z')) = \emptyset$, which means that $x \restriction z'$ is not cofinally splitting in F_α, so again for some $z'' \in P_\kappa\lambda$ with $z'' \supseteq z'$, $O_{x\restriction z''} \cap [F_\alpha] = O_{x\restriction z''} \cap [F]_\alpha = \{x\}$, and so $x \in [F]_\alpha \setminus [F]_{\alpha+1}$. Next, let α be a limit and suppose that $[F_\gamma] = [F]_\gamma$ for every $\gamma \in \alpha$. If $x \in [F]_\alpha$, then $x \in [F_\gamma]$ for every $\gamma \in \alpha$, so $x \in [\bigcap_{\gamma\in\alpha} F_\gamma = F'_\alpha] = [F_\alpha]$. On the other hand, if $x \in [F_\alpha]$ then $x \in [F'_\alpha]$ so for every $z \in P_\kappa\lambda$, $x \restriction z \in \bigcap_{\gamma\in\alpha} F_\alpha$. But then $x \in [F_\gamma] = [F]_\gamma$ for every $\gamma \in \alpha$, so $x \in [F]_\alpha$. □

Corollary 1.6.30. Let κ be either ω or $\lambda^{<\kappa}$-strongly compact and let F be a $P_\kappa\lambda$-forest coding a κ-closed subset of 2^λ. Then $\mathrm{ht}_{CB}(F) = \mathrm{ht}_{CB}([F])$, $\mathrm{Ker}([F]) = [\mathrm{Ker}(F)]$, and $\mathrm{Sc}([F]) = [\mathrm{Sc}(F)]$. By $[\mathrm{Sc}(F)]$, we mean of course $\{b \in [F] : \exists z \in P_\kappa\lambda$ such that $b \restriction z \in \mathrm{Sc}(F)\}$.

Proof. Let $\alpha_0 = \mathrm{ht}_{CB}([F])$, so $[F]_{\alpha_0} = [F]_{\alpha_0+1}$. By 1.6.29, $[F_{\alpha_0}] = [F_{\alpha_0+1}]$, and so because both F_{α_0} and F_{α_0+1} code κ-closed subsets of 2^λ, we must have that $F_{\alpha_0} = F_{\alpha_0+1}$. So $\alpha_0 \leq \mathrm{ht}_{CB}(F)$. On the other hand, if $F_{\alpha_0} = F_{\alpha_0+1}$ then by 1.6.29, $[F]_{\alpha_0} = [F]_{\alpha_0+1}$. Thus $\mathrm{ht}_{CB}(F) = \mathrm{ht}_{CB}([F])$. Then $\mathrm{Ker}([F]) = [F]_{\mathrm{ht}_{CB}([F])} = [F_{\mathrm{ht}_{CB}([F])}] = [F_{\mathrm{ht}_{CB}(F)}] = [\mathrm{Ker}(F)]$. Similarly, $\mathrm{Sc}([F]) = \{x \in [F] : x \notin \mathrm{Ker}([F])\} = \{x \in [F] : x \notin [\mathrm{Ker}(F)]\} = \{x \in [F] : \exists z \in P_\kappa\lambda$ such that $x \restriction z \in \mathrm{Sc}(F)\} = [\mathrm{Sc}(F)]$. □

1.7 Games played on $P_\kappa\lambda$-forests

Here we describe a natural analogue to the cut-and-choose game played on a tree $T \subseteq {}^{<\kappa}2$ which is played on a $P_\kappa\lambda$-forest F.

Definition 1.7.1. Let κ be regular, F be a $P_\kappa\lambda$-forest, and $f_0 \in F$. Define the two player game of length $\delta \leq \kappa$ starting at f_0 played on F, $G(F, f_0, \delta)$ as follows. Player I starts and plays a level of the tree $A_0 \in P_\kappa\lambda$ with $\mathrm{dom}(f_0) \subseteq A_0$. Player II then responds with a splitting pair of nodes on the same level $\{f_0^0, f_1^0\} \subseteq F$ extending f_0 which agree up to level A_0. That is, player II plays $\{f_0^0, f_1^0\} \subseteq F$ such that $\mathrm{dom}(f_0^0) = \mathrm{dom}(f_1^0) \supsetneq A_0$, $f_0^0 \restriction A_0 = f_1^0 \restriction A_0$, $f_0^0 \restriction \mathrm{dom}(f_0) = f_1^0 \restriction \mathrm{dom}(f_0) = f_0$, and $f_0^0 \neq f_1^0$. At successor rounds, player I both plays a level and chooses a node. So at the next round player I chooses $f_1 \in \{f_0^0, f_1^0\}$ and plays a level $A_1 \supseteq \mathrm{dom}(f_1)$. The game proceeds in this manner, and at limit stages β, player I plays first and plays $A_\beta \in P_\kappa\lambda$ with $A_\beta \supseteq \bigcup_{\gamma\in\beta} A_\gamma$ and player II must respond with a splitting pair of nodes $\{f_0^\beta, f_1^\beta\} \subseteq F$ extending the path through the tree constructed so far which agree up to level A_β. That is, such that $\mathrm{dom}(f_0^\beta) = \mathrm{dom}(f_1^\beta) \supsetneq A_\beta$, $f_0^\beta \restriction A_\beta = f_1^\beta \restriction A_\beta$, $f_0^\beta \neq f_1^\beta$, and $f_0^\beta \restriction \mathrm{dom}(f_\gamma) = f_1^\beta \restriction \mathrm{dom}(f_\gamma) = f_\gamma$ for every $\gamma \in \beta$. Player II wins a run of the game if she can play legally at stage β for every $\beta \in \delta$.

Definition 1.7.2. Let F be a $P_\kappa\lambda$ forest and $\delta \leq \kappa$. Say that F is δ-perfect if and only if player II has a winning strategy in $G(F, f_0, \delta)$ for every $f_0 \in F$. Say that F is δ-scattered if and only if player I has a winning strategy in $G(F, f_0, \delta)$ for every

$f_0 \in F$. Let $\mathrm{Ker}(F,\delta) = \{f_0 \in F :$ Player II has a winning strategy in $G(F, f_0, \delta)\}$ and $\mathrm{Sc}(F,\delta) = \{f_0 \in F :$ Player I has a winning strategy in $G(F, f_0, \delta)\}$ be the δ-kernel and δ-scattered parts of F, respectively.

Remark 1.7.3. As usual by the Gale-Stewart theorem, for every $f_0 \in F$, $G(F, f_0, \omega)$ is determined. $\mathrm{Ker}(F,\delta) \subseteq F$ is a $P_\kappa\lambda$-forest, while if $f_0 \in \mathrm{Sc}(F,\delta)$, then $f_0' \in \mathrm{Sc}(F,\delta)$ for every $f_0' \in F$ with $\mathrm{dom}(f_0') \supseteq \mathrm{dom}(f_0)$ and $f_0' \restriction f_0 = f_0$. So in particular, if $\delta = \omega$, then $F = \mathrm{Ker}(F,\omega) \cup \mathrm{Sc}(F,\omega)$ is a decomposition of F into an ω-perfect subforest and an ω-scattered part (which is not, of course, strictly speaking a forest).

Proposition 1.7.4. Let F be a $P_\kappa\lambda$ forest. Then $\mathrm{Ker}(F,\omega) = \mathrm{Ker}(F)$ and $\mathrm{Sc}(F,\omega) = \mathrm{Sc}(F)$.

Proof. Because $\mathrm{Ker}(F)$ is a cofinally splitting subforest of F, it is clear that $\mathrm{Ker}(F) \subseteq \mathrm{Ker}(F,\omega)$. On the other hand, suppose $f \in \mathrm{Sc}(F)$. We need to see that $f \in \mathrm{Sc}(F,\omega)$. Suppose $\mathrm{rank}_{CB}(f) = \gamma_0$. Suppose first that $f \notin F'_{\gamma_0+1}$. Then for some $z_0 \in P_\kappa\lambda$, $f \in F_{\gamma_0}$ is not cofinally splitting above z_0. Let player I play $A_0 = z_0$ in this case. Then player II responds with $\{f^0_0, f^0_1\}$ extending f splitting above z_0, and so necessarily not both of f^0_0 and f^0_1 are in F_{γ_0}. Let player I choose such a node and proceed as above, noting that $\mathrm{rank}_{CB}(f_1) < \mathrm{rank}_{CB}(f)$. If, on the other hand, $f \in F'_{\gamma_0+1}$, then necessarily for some $z_0 \in P_\kappa\lambda$ with $z_0 \supseteq \mathrm{dom}(f)$, $\mathrm{Lev}_{z_0}(F'_{\gamma_0+1} \restriction f) = \emptyset$. Let player I play $A_0 = z_0$ in this case. Then if player II plays $\{f^0_0, f^0_1\}$ extending f splitting above z_0, neither f^0_0 nor f^0_1 is in F'_{γ_0+1}. If one of them is not in F_{γ_0} have player I choose this node and proceed as above, again noting that $\mathrm{rank}_{CB}(f_1) < \mathrm{rank}_{CB}(f)$. If both f^0_0 and f^0_1 are in F_{γ_0}, then have player I choose randomly and play A_1 such that f_1 is not splitting above A_1 in F_{γ_0}. Then in the next round if player II plays $\{f^1_0, f^1_1\}$ extending f_1 splitting above A_1, necessarily one of f^1_0 or f^1_1 is not in F_{γ_0}. Have player I choose such a node as f_2, and so again $\mathrm{rank}_{CB}(f_2) < \mathrm{rank}_{CB}(f)$. Player I can proceed in this fashion, and if player II were able to play ω-many moves, then $\langle \mathrm{rank}_{CB}(f_n) : n \in \omega\rangle$ would be a non-increasing sequence of ordinals which never stabilizes (ordinals repeat at most once), which is impossible. So $f \in \mathrm{Sc}(F,\omega)$, as desired. $\square$

1.7.1 The behavior of the game when $\delta = \kappa$

Just as in the case with trees $T \subseteq {}^{<\kappa}2$, if F is a $P_\kappa\lambda$-forest then winning strategies in $G(F, \emptyset, \kappa)$ can give structural information about F. We need some preliminary definitions.

Definition 1.7.5. Let κ be regular and F be a $P_\kappa\lambda$-forest. If $B \in P(\lambda) \setminus P_\kappa\lambda$, let $F \restriction B$ denote the $P_\kappa B$-forest generated by F. That is, $F \restriction B = \{f \in F : \mathrm{dom}(f) \in P_\kappa B\}$. Let $[F \restriction B]$ denote the body of $F \restriction B$, that is $\{x \in {}^B2 : x \restriction z \in F \restriction B \text{ for every } z \in P_\kappa B\}$.

Definition 1.7.6. Let κ be regular, F be a $P_\kappa\lambda$-forest, θ be a regular cardinal sufficiently larger than κ, λ, and $M \prec H_\theta$ with $\{\kappa, \lambda, F, \text{etc.}\} \subseteq M$. Say that M is F-guessing, or say that M guesses F, if and only if $M \cap P_\kappa(M \cap \lambda)$ is cofinal in $P_\kappa(M \cap \lambda)$ and for every $\bar{b} \in [F \restriction (M \cap \lambda)]$ such that $\bar{b} \restriction z \in M$ for cofinally many $z \in P_\kappa(M \cap \lambda)$, there exists $b \in [F] \cap M$ such that $b \restriction (M \cap \lambda) = \bar{b}$.

Theorem 1.7.7. *Let κ be regular and F be a $P_\kappa\lambda$-forest. Then player I has a winning strategy in $G(F, \emptyset, \kappa)$ if and only if there exists a κ-club $\mathcal{C} \subseteq [H_\theta]^\kappa$ of F-guessing submodels $M \prec H_\theta$ of size κ for some sufficiently large θ.*

Proof. This proof is very similar to that of 1.5.4. Suppose first that there exists a κ-club $\mathcal{C} \subseteq [H_\theta]^\kappa$ of F-guessing submodels $M \prec H_\theta$. Without loss of generality, assume that these are in fact elementary submodels of $(H_\theta, \trianglelefteq, F, \kappa, \ldots)$ where $\trianglelefteq$ is a predicate for a well-ordering of H_θ and $\kappa \subseteq M$ for every $M \in \mathcal{C}$. Fix $f : \kappa \to \kappa \times \kappa$ a surjection so that for every $\langle \xi, \nu \rangle \in \kappa \to \kappa$, $|f^{-1}[\langle \xi, \nu \rangle] \cap \mathrm{succ}(\kappa)| = \kappa$. Also for every $M \in \mathcal{C}$, let $e_M : \kappa \to M \cap \lambda$ and $g_M : \kappa \to M \cap [F]$ be surjections. In the course of defining a winning strategy for player I, we are going to construct a $\subseteq$-chain of models $M_0 \subseteq M_1 \subseteq \ldots$ in $\mathcal{C}$, where at stages $\beta + 1$ in the game, player I is going to check whether $\{f_0^\beta, f_1^\beta\} \subseteq M_\beta$ and define $M_{\beta+1}$ and $f_{\beta+1}, A_{\beta+1}$ accordingly. At limit stages β, A_β will be determined from M_β. First, choose $M_0 \in \mathcal{C}$ and let player I play $A_0 \in M_0$ such that $e_{M_0}(0) \in A_0$. Player II responds with $\{f_0^0, f_0^1\}$ splitting above A_0. Either $\{f_0^0, f_0^1\} \subseteq M_0$ or not. If $\{f_0^0, f_0^1\} \subseteq M_0$, let $M_1 = M_0$ and consider $f(1)$. If

$f(1) = \langle 1, \xi \rangle$ or $f(0) = \langle 0, \xi \rangle$ for some $\xi \in \kappa$, consider $g_{M_1}(\xi) \in M_1 \cap [F]$ or $g_{M_0}(\xi) \in M_1 \cap [F]$, respectively. Let player I choose $f_1 \in \{f_0^0, f_1^0\}$ so that e.g. $g_{M_1}(\xi) \restriction \mathrm{dom}(f_1) \neq f_1$. Let player I play $A_1 \in M_1$ such that $\{e_{M_0}(1), e_{M_1}(1)\} \cup \mathrm{dom}(f_1) \subseteq A_1$. If $\{f_0^0, f_1^0\}$ is not a subset of M_0, find $M_1 \in \mathcal{C}$ such that $M_0 \cup \{f_0^0, f_1^0\} \subseteq M_1$ and proceed as above, choosing f_1 to be incompatible with $g_{M_1}(\xi)$ or $g_{M_0}(\xi)$ as dependent on $f(1)$. Otherwise, just let player I play f_1 arbitrarily. Play proceeds in this fashion. At limit stages $\beta \in \kappa$, we have constructed a $\subseteq$-increasing sequence $\langle M_\eta : \eta \in \beta \rangle \subseteq \mathcal{C}$. Let $M_\beta \in \mathcal{C}$ such that $\bigcup_{\eta \in \beta} M_\eta \subseteq M_\beta$ and let player I play $A_\beta \in M_\beta$ containing all ordinals in $\{e_{M_\eta}(\beta) : \eta \in \beta + 1\}$ and $\bigcup_{\xi \in \beta} A_\xi$. At successor stages $\beta + 1 \in \kappa$, player I defines $M_{\beta+1}$ based on whether or not $\{f_0^\beta, f_1^\beta\} \subseteq M_\beta$, considers $f(\beta+1)$, and if via some g_{M_γ} for $\gamma \leq \beta$, $f(\beta+1)$ labels the ξ^{th} branch through F in M_γ, chooses $f_{\beta+1}$ to be incompatible with this branch. Player 1 also plays $A_{\beta+1}$ sufficiently large in $M_{\beta+1}$ to ensure that, in particular, the first $(\beta+1)$-many ordinals in every M_γ for $\gamma \leq \beta+1$ as determined by the e_{M_γ} maps are covered by $A_{\beta+1}$. Suppose towards a contradiction that the two players play κ-many rounds. We have a $\subseteq$-increasing sequence $\langle M_\alpha : \alpha \in \kappa \rangle \subseteq \mathcal{C}$. If for some $\beta \in \kappa$, for every $\gamma \in [\beta, \kappa)$, $M_\gamma = M_\beta$, then M_β is F-guessing and by construction we have built some $\bar{b} \in [F \restriction (M \cap \lambda)]$. However, then because M_β is F-guessing, there must be some $b \in [F] \cap M_\beta$ with $b \restriction (M \cap \lambda) = \bar{b}$. However, we would have chosen a node incompatible with b at some sufficiently large successor stage, which is impossible. On the other hand, suppose that there are κ-many distinct M_ξ's in $\langle M_\alpha : \alpha \in \kappa \rangle$. Then because $\mathcal{C}$ is κ-club, $M_\kappa = \bigcup_{\alpha \in \kappa} M_\alpha \prec (H_\theta, \trianglelefteq, F, \kappa, \ldots)$. Note that $M_\kappa \cap \lambda = \bigcup_{\alpha \in \kappa} (M_\alpha \cap \lambda)$. By construction, we will then have built some $\bar{b} \in [F \restriction (M_\kappa \cap \lambda)]$, which because M_κ is F-guessing must be an $(M_\kappa \cap \lambda)$-segment of some $b \in [F] \cap M_\kappa$, but then this b is in every sufficiently large M_α and we would have chosen a node incompatible with b at some sufficiently large successor stage, which is impossible.

Next, suppose that τ is a winning strategy for player I in $G(F, \emptyset, \kappa)$. Let $M \prec (H_\theta, \tau, \trianglelefteq, F, \kappa, \ldots)$ such that $|M| = \kappa$, $\kappa \subseteq M$, and M is internally unbounded. We show that M is F-guessing. Note first that the collection of such M is κ-club in $[H_\theta]^\kappa$, as follows.

Suppose that $\langle M_\alpha : \alpha \in \kappa \rangle \subseteq [H_\theta]^\kappa$ is a $\subsetneq$-increasing sequence of internally unbounded submodels, witnessed by $\{\langle M^\alpha_\beta : \beta \in \kappa \rangle : \alpha \in \kappa\}$, as indicated. We need to see that $M = \bigcup_{\alpha \in \kappa} M_\alpha$ is internally unbounded in particular. For each $\alpha \in \kappa$, let $N_\alpha \in M_{\alpha+1}$ be of the form $M^{\alpha+1}_\beta$ for some $\beta \in \kappa$ such that $\bigcup_{\gamma,\delta \in \alpha} M^\gamma_\delta \subseteq M^{\alpha+1}_\beta$. Note that $\langle N_\alpha : \alpha \in \kappa \rangle \subseteq M$ is $\subseteq$-increasing, $|N_\alpha| < \kappa$ for every α, and if $x \in M$, then there exists α' such that $x \in M_\alpha$ for every $\alpha \geq \alpha'$, and so for each such α there exists α_β such that $x \in M^\alpha_{\alpha_\beta}$. Let $f \in {}^\kappa\kappa$ be defined by $f(\alpha) = \alpha_\beta$ for every $\alpha \geq \alpha'$ (and be 0 otherwise). Find a $\gamma \in (\alpha', \kappa)$ such that $f''\gamma \subseteq \gamma$. Then by construction, $x \in N_\gamma$. So M is internally unbounded. Now, in order to show that our $M \prec (H_\theta, \tau, \trianglelefteq, F, \kappa, \ldots)$ is F-guessing, similar to as in 1.5.4, we use the idea of the $\trianglelefteq$-minimal maximal run of the game below a path following τ. Specifically, fix $\bar{b} \in [F \restriction (M \cap \lambda)]$ such that $\bar{b} \restriction z \in M$ for cofinally many $z \in P_\kappa(M \cap \lambda)$ (of course because M is internally unbounded, $M \cap P_\kappa(M \cap \lambda)$ is cofinal in $P_\kappa(M \cap \lambda)$). Without loss of generality, we may assume that $\bar{b} \restriction z$ is cofinally splitting below $M \cap \lambda$ for every $z \in P_\kappa(M \cap \lambda)$, because otherwise within M we could define a branch through F agreeing with $\bar{b}$ on $M \cap \lambda$ by elementarity. Construct a run of the game $\overline{x}$ where player I is playing according to τ and player II plays the $\trianglelefteq$-minimal pair at all times and which is maximal below $\bar{b}$. So, $\overline{x} = \langle \langle \{A_0\}, \{f^0_0, f^0_1\} \rangle, \langle \{A_1, f_1\}, \{f^1_0, f^1_1\} \rangle, \langle \{A_2, f_2\}, \{f^2_0, f^2_1\} \rangle, \ldots, \langle \{A_\xi, f_\xi\} \rangle \text{ or } \langle \{A_\xi\} \rangle \rangle$ satisfies the following conditions. $A_0 = \tau(\emptyset)$, $\{A_1, f_1\} = \tau(\langle \{A_0\}, \{f^0_0, f^0_1\} \rangle)$, etc.. That is, player I is following τ. Furthermore, $\{f^0_0, f^1_1\}$ is the $\trianglelefteq$-minimal pair splitting above A_0 (in $F \restriction (M \cap \lambda)$) where $f^0_i = \bar{b} \restriction \mathrm{dom}(f^0_i)$ for some $i \in \{0, 1\}$ with the property that with $\{A_1, f_1\} = \tau(\langle \{A_0\}, \{f^0_0, f^0_1\} \rangle)$, $f_1 = \bar{b} \restriction \mathrm{dom}(f_1)$, etc.. And more generally for $\eta \in \xi$, $\{f^\eta_0, f^\eta_1\} \subseteq F \restriction (M \cap \lambda)$ is the $\trianglelefteq$-minimal pair splitting above A_η extending the path constructed where $f^\eta_i = \bar{b} \restriction \mathrm{dom}(f^\eta_i)$ for some $i \in \{0, 1\}$ so that with $\{A_{\eta+1}, f_{\eta+1}\} = \tau(\langle \langle \{A_0\}, \{f^0_0, f^0_1\} \rangle, \ldots, \langle \{A_\eta, f_\eta\}, \{f^\eta_0, f^\eta_1\} \rangle \rangle)$, $f_{\eta+1} = \bar{b} \restriction \mathrm{dom}(f_{\eta+1})$. Finally, $\overline{x}$ is maximal below $\bar{b}$ in that whichever $\{f^\xi_0, f^\xi_1\} \subseteq F \restriction (M \cap \lambda)$ player II chooses at stage ξ extending f_ξ (or the path constructed, if ξ is a limit ordinal) splitting above A_ξ where $f^\eta_i = \bar{b} \restriction \mathrm{dom}(f^\eta_i)$ for some $i \in \{0, 1\}$, player I chooses $f_{\xi+1} \in \{f^\xi_0, f^\xi_1\}$ so that

$\bar{b} \upharpoonright \mathrm{dom}(f_{\xi+1}) \neq f_{\xi+1}$. Of course because τ is a winning strategy for player I in $G(F, \emptyset, \kappa)$, we must have $\xi \in \kappa$.

It is not immediately apparent that $\overline{x} \in M$, because it is defined in particular using $\bar{b} \notin M$. However, let $z \in M \cap P_\kappa(M \cap \lambda)$ so that $A_\xi \subseteq z$ and $\bar{b} \upharpoonright z \in M$. Indeed, we have by recursion, for example, that each of the A_η for $\eta \leq \xi$ is in M because $\{\tau, \bar{b} \upharpoonright z, \trianglelefteq\} \subseteq M$. Then $\overline{x}$ is definable from $\{\tau, \bar{b} \upharpoonright z, \trianglelefteq\} \subseteq M$ by letting it be the $\trianglelefteq$-minimal maximal run of the game below $\bar{b} \upharpoonright z$ following τ. It is clear that $\overline{x}$ can be defined this way and that all relevant parameters are in M, so that $\overline{x} \in M$. Now, let $b \in {}^\lambda 2 \cap M$ be defined from $\overline{x}$ in the usual way. That is, first for $\gamma \in z$ let $b(\gamma) = (\bar{b} \upharpoonright z)(\gamma)$. If $\gamma \notin z$, suppose first that $\mathrm{Lev}_{z \cup \{\gamma\}}(F \upharpoonright (\bar{b} \upharpoonright z)) = \{f_\gamma\}$. Let $b(\gamma) = f_\gamma(\gamma)$ in this case. Otherwise, $\mathrm{Lev}_{z \cup \{\gamma\}}(F \upharpoonright (\bar{b} \upharpoonright z)) = \{f^0_\gamma, f^1_\gamma\}$ where we may assume $f^0_\gamma(\gamma) = 0$ and $f^1_\gamma(\gamma) = 1$. Suppose that player II offers $\{f^\xi_0 = f^0_\gamma, f^\xi_1 = f^1_\gamma\}$ as his next move following $\overline{x}$. Then let $b(\gamma) = i \in \{0, 1\}$ such that $f^i_\gamma \notin \tau(\overline{x}^\frown \{f^\xi_0, f^\xi_1\})$. That is, choose b so that it agrees with the function that τ does not pick when offered a splitting one point extension following $\overline{x}$ from $\bar{b} \upharpoonright z$ to level $z \cup \{\gamma\}$. By elementarity, $b \in M$ and clearly $b \in {}^\lambda 2$. First we show that $b \upharpoonright (M \cap \lambda) = \bar{b}$. If not, then choose the minimal $\gamma \in M \cap \lambda$ such that $b(\gamma) \neq \bar{b}(\gamma)$. But then $\{f^\xi_0 = b \upharpoonright (z \cup \{\gamma\}), f^\xi_1 = \bar{b} \upharpoonright (z \cup \{\gamma\})\}$ is a splitting pair in M along $\bar{b}$ above A_ξ which when offered by player II following $\overline{x}$ is such that τ chooses away from b by definition, but then chooses along $\bar{b}$, which contradicts the maximality of $\overline{x}$. Next if $b \notin [F]$, by elementarity there exits $z \in M \cap P_\kappa(M \cap \lambda)$ such that $b \upharpoonright z \notin F$. However, we have $\bar{b} \upharpoonright z = b \upharpoonright z$, so this is impossible. Therefore M is F-guessing. $\square$

1.8 Adding branches through $P_\kappa\lambda$-forests

In much the same way as we proved Mansfield's theorem 1.1.51 for trees $T \subseteq {}^{<\omega}2$, we can show that outer models cannot add branches to $P_\omega\lambda$-forests if they (equivalently their bodies) are scattered.

Proposition 1.8.1. Let $V \subseteq M$ be two models of ZFC, F be a $P_\omega\lambda$ forest with $F \in V$, and $(F_{\mathrm{ht}_{CB}(F)} = \emptyset)^V$. Then $([F])^V = ([F])^M$. That is, M cannot add branches to $P_\omega\lambda$ forests which are scattered in V.

Proof. Note that $(P_\omega\lambda)^V = (P_\omega\lambda)^M$, so $F \in M$ is indeed still a $P_\omega\lambda$-forest. It is not difficult to see that as we defined it in 1.6.21, the Cantor-Bendixson derivative process on F is absolute, so in particular $(F_\alpha)^V = (F_\alpha)^M$. Suppose towards a contradiction that $b \in ([F])^M \setminus V$. Working in M, clearly $b \notin [F_{\mathrm{ht}_{CB}(F)}]$, because $F_{\mathrm{ht}_{CB}(F)} = 0$. Find the minimal $\gamma + 1$ such that $b \notin [F_{\gamma+1}]$. Note that this is indeed a successor, because if $b \in \bigcap_{\delta \in \gamma} [F_\delta]$, then $b \restriction z \in F_\gamma$ for every $\delta \in \gamma$ and $z \in P_\kappa\lambda$, but then $b \in F'_\gamma$, so necessarily $b \in [F_\gamma]$. In any case, $b \notin [F'_{\gamma+1}]$, so for some $z \in P_\kappa\lambda$, $b \restriction z$ is not cofinally splitting in F_γ. Because $b \in [F_\gamma]$ then, there must exist $z' \supseteq z$ with $z' \in P_\kappa\lambda$ such that $b \restriction z'$ has a unique extension in F_γ to every level $z'' \supseteq z'$, namely $b \restriction z''$. But then b is definable from $\{b \restriction z', F_\gamma\} \subseteq V$, a contradiction. □

In a similar fashion to how we were able to show for more general scattered trees $T \subseteq {}^{<\kappa}2$ that branches can't be added by outer models, we can show that outer models cannot add branches to $P_\kappa\lambda$-forests F if they are scattered.

Proposition 1.8.2. Let $V \subseteq M$ be models of ZFC, F be a $P_\kappa\lambda$-forest with $F \in V$, and $(F_{\mathrm{rank}_{CB}(F)} = \emptyset)^V$. Then $([F])^V = ([F])^M$. That is, M cannot add branches to $P_\kappa\lambda$ forests which are scattered in V. However, unless $(P_\kappa\lambda)^V = (P_\kappa\lambda)^M$, in M, F will no longer be a $P_\kappa\lambda$-forest in M, but we can still interpret $([F])^M$ as $\{b \in {}^\lambda 2 : b \restriction z \in F \text{ for every } z \in (P_\kappa\lambda)^V\}$.

Proof. In M, first let F be any $(V, P_\kappa\lambda)$-forest, by which we mean that, relative to $(P_\kappa\lambda)^V$, F is a $P_\kappa\lambda$-forest. So, F is a collection of functions with domains in $(P_\kappa\lambda)^V$ to $\{0,1\}$ such that if $f \in F$ and $z \subseteq \mathrm{dom}(f)$ with $z \in (P_\kappa\lambda)^V$, $f \restriction z \in F$. Say that F is V-pruned if and only if every $f \in F$ has an extension to every level $z \in (P_\kappa\lambda)^V$. Say that a node is $f \in F$ is V-cofinally splitting in F if and only if for every $z \subseteq (P_\kappa\lambda)^V$ with $z \supseteq \mathrm{dom}(f)$, there exists

$\{g_1, g_2\} \subseteq F$ such that $\mathrm{dom}(g_1) \supseteq z$, $\mathrm{dom}(g_2) \supseteq z$, $g_1 \restriction \mathrm{dom}(f) = g_2 \restriction \mathrm{dom}(f) = f$, and there exists $\beta \in \mathrm{dom}(g_1) \cap \mathrm{dom}(g_2)$ with $g_1(\beta) \neq g_2(\beta)$. Say that F is V-cofinally splitting if and only if every $f \in F$ is V-cofinally splitting in F. Define the V-pruned part of F by recursion. Let $F^0 = F$, $F^{\alpha+1} = \{f \in F^\alpha : f$ has an extension to every level $z \in (P_\kappa\lambda)^V$ of $F^\alpha\}$, and $F^\alpha = \bigcap_{\gamma \in \alpha} F^\gamma$. For some minimal α_1, $F^{\alpha_1} = F^{\alpha_1+1}$, and call this $F^{\alpha_1} = F'$ the pruned part of F. It is clear that, for example, $([F])^M = ([F'])^M$. Define the Cantor-Bendixson process on F relativized to V in the natural way. So let $F_0 = F'$ denote the V-pruned part of F. Let $F'_{\alpha+1} = \{f \in F_\alpha : f$ is V-cofinally splitting in $F_\alpha\}$, and let $F_{\alpha+1}$ be the V-pruned part of $F'_{\alpha+1}$. For limits, let F_α be the V-pruned part of $\bigcap_\gamma F_\gamma = F'_\alpha$. Call the minimal α_0 with $F_{\alpha_0} = F_{\alpha_0+1}$ the V-Cantor-Bendixson height of F, that is $\alpha_0 = \mathrm{ht}_{CB}(F)$. By construction, every F_α is V-pruned. Throughout, when context is clear, we may omit the V-prefix. It is not difficult to see that this process is absolute between V and M, in the sense that $(F_\alpha)^V = (F_\alpha)^M$ for every α, where this latter $(F_\alpha)^M$ denotes the V-Cantor-Bendixson derivative as above. Now the proof proceeds as usual. Working in M, suppose towards a contradiction that we have some $b \in [F] \setminus V$. By recursion, we argue $b \in [F_\alpha]$ for every α, which is impossible as $F_\alpha = \emptyset$ eventually. For limit stages, if $b \in [F_\gamma]$ for every $\gamma \in \alpha$, then $b \restriction z \in F_\gamma$ for every $\gamma \in \alpha$, $z \in (P_\kappa\lambda)^V$, so $b \restriction z \in \bigcap_{\gamma \in \alpha} F_\gamma$, i.e. $b \in [F'_\alpha] = [F_\alpha]$. Next, suppose $b \in [F_\alpha]$. We want to show $b \in [F_{\alpha+1}]$. If not, then $b \notin [F'_{\alpha+1}]$ so for some $z \in (P_\kappa\lambda)^V$, $b \restriction z \notin F'_{\alpha+1}$, i.e. there exists $z' \supseteq z$, $z' \in (P_\kappa\lambda)^V$ such that for every $g_1, g_2 \in F_\alpha$ with $g_1 \restriction z' = g_2 \restriction z'$ and $g_1 \restriction z = g_2 \restriction z = b \restriction z$, for every $\gamma \in \mathrm{dom}(g_1) \cap \mathrm{dom}(g_2)$, $g_1(\gamma) = g_2(\gamma)$. However, then in V we can define $\bar{b} \in {}^\lambda 2$ via $\bar{b}(\gamma) = (b \restriction z')(\gamma)$ for $\gamma \in z'$, and $\bar{b}(\gamma) = b_\gamma(\gamma)$, where $b_\gamma \in F_{z' \cup \{\gamma\}}$, for $\gamma \notin z'$. Because $b \restriction (z' \cup \{\gamma\}) \in F_{z' \cup \{\gamma\}}$ for every γ, we must have $b(\gamma) = b_\gamma(\gamma)$ for every such γ, i.e. $\bar{b} = b$, a contradiction. □

Remark 1.8.3. If $(P_\kappa\lambda)^V = (P_\kappa\lambda)^M$, then F is actually a $P_\kappa\lambda$-forest in M. Additionally, in M if $(P_\kappa\lambda)^V$ is cofinal in $(P_\kappa\lambda, \subseteq)$, i.e. if every subset of size $< \kappa$ of λ in M is covered by a subset of size $< \kappa$ of λ in V, then we can consider the induced $P_\kappa\lambda$-forest, $\overline{F}$, in M, from F. That is, $\overline{F} = \{f : f \in {}^z 2$, $z \in P_\kappa\lambda$, and there exists $z' \in (P_\kappa\lambda)^V$ with $z \subseteq z'$ and $f_{z'} \in$

$\mathrm{Lev}_{z'}(F)$ with $f = f_{z'} \restriction z\}$. So $\overline{F}$ is formed by closing F downwards in the natural sense. It is straightforward to see that $\overline{F}$ is indeed a $P_\kappa\lambda$-forest, and in fact $[\overline{F}] = [F]$, where the latter is interpreted again as the set of functions from λ to 2 where the restrictions to all $z \in (P_\kappa\lambda)^V$ lie in F. In this case then we also have that if $(F_{\mathrm{rank}_{CB}(F)} = \emptyset)^V$, $([\overline{F}])^M \setminus V = \emptyset$.

As in the case for trees, unless κ is $\lambda^{<\kappa}$-strongly compact or ω, whether or not the body of a $P_\kappa\lambda$-forest F is topologically scattered has potentially little bearing on whether or not branches can be added through F in outer models. However, just as we did for trees in 1.3.7, using the cut-and-choose game, we can say that certain forcing extensions can't add branches to certain forests.

Proposition 1.8.4. Let F be a $P_\kappa\lambda$ forest, let $\delta \leq \kappa$ be either a limit ordinal or the successor to a limit ordinal, let $\mathbb{P}$ be δ-strategically closed, and suppose that forcing with $\mathbb{P}$ adds a branch through F. Then $\mathrm{Ker}(F, \delta) \neq \emptyset$.

Proof. Due to the equivalence of $\mathrm{Ker}(F, \omega)$ with $\mathrm{Ker}(F)$ and $\mathrm{Sc}(F, \omega)$ with $\mathrm{Sc}(F)$, by 1.8.2 we are done if $\delta = \omega$. More generally, let $\dot{b}$ be a name for a new branch through F. Working in V, we describe a winning strategy for player II in $G(F, \emptyset, \delta)$. Suppose player I plays A_0 first. Find $p_0 \in \mathbb{P}$ such that for some $B_{p_0} \in P_\kappa\lambda$ with $B_{p_0} \supseteq A_0$, there exists $f_{p_0} \in \mathrm{Lev}_{B_{p_0}}(F)$ such that $p_0 \Vdash \dot{b} \restriction B_{p_0} = f_{p_0}$. Suppose that Odd plays $q_1 = p_0$ as his first move in $G_\delta(\mathbb{P})$. Even responds following her winning strategy in $G_\delta(\mathbb{P})$ with q_2. Because it is forced that $\dot{b}$ is not in V, there must exist $p_0', p_0'' \leq q_2$ such that for some $B_{p_0', p_0''} \in P_\kappa\lambda$ with $B_{p_0', p_0''} \supseteq B_{p_0}$ there exist $f_{p_0'} \in \mathrm{Lev}_{B_{p_0', p_0''}}(F)$ and $f_{p_0''} \in \mathrm{Lev}_{B_{p_0', p_0''}}(F)$ with $p_0' \Vdash \dot{b} \restriction B_{p_0', p_0''} = f_{p_0'}$, $p_0'' \Vdash \dot{b} \restriction B_{p_0', p_0''} = f_{p_0''}$, and $f_{p_0'} \neq f_{p_0'}$. Let player II play $\{f_0^0 = f_{p_0'} \restriction B_{p_0', p_0''}, f_1^0 = f_{p_0''} \restriction B_{p_0', p_0''}\}$ as her response to A_0 in $G(F, \emptyset, \delta)$. Player I responds with $f_1 \in \{f_0^0, f_1^0\}$ and A_1. Whether or not $f_1 = f_0^0$ or $f_1 = f_1^0$, there exists $\overline{p_1'} \leq p_0'$ or $\overline{p_1''} \leq p_0''$ such that e.g. for some $B_{\overline{p_1'}} \in P_\kappa\lambda$ with $B_{\overline{p_1'}} \supseteq A_1$, there exists $f_{\overline{p_1'}}$ with $\overline{p_1'} \Vdash \dot{b} \restriction B_{\overline{p_1'}} = f_{\overline{p_1'}}$. Let Odd play e.g. $\overline{p_1'}$ as q_3 in $G_\delta(\mathbb{P})$. Even responds following her winning strategy in $G_\delta(\mathbb{P})$ with q_4. As before, there must exist $p_1', p_1'' \leq q_4$

such that for some $B_{p_1',p_1''} \in P_\kappa\lambda$ with $B_{p_1',p_1''} \supseteq B_{\overline{p_1'}}$, there exist $f_{p_1'} \in \mathrm{Lev}_{B_{p_1',p_1''}}(F)$ and $f_{p_1''} \in \mathrm{Lev}_{B_{p_1',p_1''}}(F)$ with $p_1' \Vdash \dot{b} \restriction B_{p_1',p_1''} = f_{p_1'}$, $p_1'' \Vdash \dot{b} \restriction B_{p_1',p_1''} = f_{p_1''}$, and $f_{p_1'} \neq f_{p_1'}$. Let player II respond then with $\{f_0^1 = f_{p_1'} \restriction B_{p_1',p_1''}, f_1^1 = f_{p_1''} \restriction B_{p_1',p_1''}\}$. It is clear that player II can proceed in this fashion. At any limit stage $\eta \in \delta$, Even is able to play a q_η in the appropriate run of $G_\delta(\mathbb{P})$, and so the partial function constructed by the two players up to stage η can be extended to the limit level. Then q_η can be strengthened to a condition which fixes the value of $\dot{b}$ up to any level A_η played by player I, and this condition must split into two conditions fixing different values of $\dot{b}$ beyond level A_η, and player II can offer these different values as $\{f_0^\eta, f_1^\eta\}$ and proceed (i.e. depending on which player I picks, have Odd play the appropriate splitting condition, etc.). It is clear that if δ is a limit ordinal, or the successor to a limit ordinal, this describes a winning strategy for player II in $G(F, \emptyset, \delta)$, as desired. □

As a corollary to 1.8.4, we have the following weakening, which has been identified several times and has been called a generalized Silver's lemma (see Proposition 2.1.12 in [70], or [25], for example), because it says that sufficient closure of a forcing notion means in particular that branches cannot be added through forests with small enough levels.

Corollary 1.8.5. Let κ be regular, $\eta \in \kappa$ be minimal such that $2^\eta \geq \kappa$, F be a $P_\kappa\lambda$-forest, and $\mathbb{P}$ be an $(\eta+1)$-strategically closed forcing. Then if forcing with $\mathbb{P}$ adds a branch through F, for some $z \in P_\kappa\lambda$, $|\mathrm{Lev}_z(F)| \geq 2^\eta$.

Proof. Without loss of generality, we may assume that F is pruned. By 1.8.4 there exists a winning strategy τ for player II in $G(F, \emptyset, \eta+1)$. Because κ is regular, and $\eta \in \kappa$ is minimal such that $2^\eta \geq \kappa$, we must have that $2^{<\eta} \in \kappa$. Let $M \prec H_\theta$ be such that ${}^{<\eta}2 \subseteq M$, $\{\tau, F, \kappa, \lambda, \text{etc.}\} \subseteq M$, and $|M| < \kappa$. The idea is simple: By having player I provide different stimuli corresponding to elements in ${}^{<\eta}2$, we can construct in M sequences of plays with player II following τ such that because τ is a winning strategy in $G(F, \emptyset, \eta+1)$, every branch through ${}^{<\eta}2$ corresponds to a game where player II is following τ, and so the resulting functions will be able to be extended, and then by elementarity we will be able to witness incompatibility within $M \cap \lambda$, so that $|\mathrm{Lev}_{M\cap\lambda}(F)| \geq 2^\eta$. More specifically, build by recursion in M a sequence of nodes in F, $\langle f_s : s \in {}^{<\eta}2\rangle \subseteq M$ which correspond to the choices offered by player II in $G(F, \emptyset, \eta+1)$. Player I will always play the minimal possible level, i.e. player I just chooses nodes. So, $f_\emptyset = \emptyset$. If $\tau(\emptyset) = \{f_0^0, f_1^0\}$, let $f_0 = f_0^0$ and $f_1 = f_1^0$. If $\tau(\langle \emptyset, \{f_0^0, f_1^0\}\rangle, \langle f_1 = f_0^0\rangle) = \{f_0^1, f_1^1\}$, let $f_{00} = f_0^1$, $f_{01} = f_1^1$, and if $\tau(\langle \emptyset, \{f_0^0, f_1^0\}\rangle, \langle f_1 = f_0^1\rangle) = \{f_0'^1, f_1'^1\}$, let $f_{10} = f_0'^1$, $f_{11} = f_1'^1$. Proceed in this manner, and note that because ${}^{<\eta}2 \subseteq M$, every $f_s \in M$. $\langle f_s : s \in {}^{<\eta}2\rangle \subseteq M$ satisfies the properties that if s' is an initial segment of s, $f_s \restriction \mathrm{dom}(f_{s'}) = f_{s'}$, for every $s \in {}^{<\eta}2$ $\mathrm{dom}(f_{s^\frown 0}) = \mathrm{dom}(f_{s^\frown 1})$ and there exists $\gamma \in \mathrm{dom}(f_{s^\frown 0})$ such that $f_{s^\frown 0}(\gamma) \neq f_{s^\frown 1}(\gamma)$, and if $\mathrm{lh}(s) \in \lim(\eta)$, $f_s = \bigcup_{\gamma\in \mathrm{lh}(s)} f_{s\restriction\gamma}$. Fix $b \in {}^\eta 2$. Because τ is a winning strategy, if $A_b = \bigcup_{\gamma\in\eta} \mathrm{dom}(f_{b\restriction\gamma})$, there exists $f_b \in \mathrm{Lev}_{A_b}(F)$ such that $f_b \restriction \mathrm{dom}(f_{b\restriction\gamma}) = f_{b\restriction\gamma}$ for every $\gamma \in \eta$. Furthermore, if $b_1 \neq b_2$ are both in ${}^\eta 2$, then for some minimal $\gamma+1$, $b_1 \restriction \gamma = b_2 \restriction \gamma$ but $b_1(\gamma) \neq b_2(\gamma)$. Then $\{f_{b_1\restriction(\gamma+1)}, f_{b_1\restriction(\gamma+1)}\} \subseteq M$ and so by elementarity there exists $\xi \in \mathrm{dom}(f_{b_1\restriction(\gamma+1)})(= \mathrm{dom}(f_{b_1\restriction(\gamma+1)})) \cap M$ such that $f_{b_1\restriction(\gamma+1)}(\xi) \neq f_{b_2\restriction(\gamma+1)}(\xi)$. But then if we consider $\{g_b : b \in {}^\eta 2\} \subseteq \mathrm{Lev}_{M\cap\lambda}(F)$ where for each $b \in {}^\eta 2$, $g_b \in \mathrm{Lev}_{M\cap\lambda}(F)$ is such that $g_b \restriction (A_b \cap M) = f_b \restriction (A_b \cap M)$, if $b_1 \neq b_2$ then $g_{b_1} \neq g_{b_2}$, so $|\mathrm{Lev}_{M\cap\lambda}(F)| \geq 2^\eta$, as desired. $\square$

1.9 Localized games and a Cantor-Bendixson theorem for $P_{\kappa^+}\lambda$

If $E \subseteq P_{\kappa^+}\lambda$ (and typically closed in $\tau_\kappa^{\mathrm{BOX}}$) then it is straightforward to consider Väänänen's game played on a localized subset of E by restricting play to a κ-sized domain set.

Definition 1.9.1. Let κ be regular, $\delta \leq \kappa$, $E \subseteq P_{\kappa^+}\lambda$, $B \in [\lambda]^\kappa$ and $a_0 \in E \cap P(B)$. Define the two player game of length δ starting at a_0 played on $E \cap P(B)$, $G(E, B, a_0, \delta)$, as follows.

Enumerate $B = \langle \beta_\gamma : \gamma \in \kappa \rangle$. The game is initialized at a_0. Player I plays an increasing sequence of ordinals in κ, while player II plays points in $E \cap P(B)$. At limit stages player I must play the supremum of the ordinals he's played already, and at stage β generally player I plays α_β and player II must play $a_\beta \in E \cap P(B)$ such that $a_\beta \neq a_\eta$ for every $\eta \in \beta$ and $a_\beta \cap \{\beta_\gamma : \gamma \in \alpha_{\eta+1}\} = a_\eta \cap \{\beta_\gamma : \gamma \in \alpha_{\eta+1}\}$. Player II wins a run of the game if she can play legally at stage β for every $\beta \in \delta$.

Note 1.9.2. Whether or not player I or II has a winning strategy in $G(E, B, a_0, \delta)$ is independent of the enumeration chosen for B.

Proof. Consider two enumerations of B, $\overline{B} = \langle \beta_\gamma : \gamma \in \kappa \rangle$ and $\underline{B} = \langle \delta_\gamma : \gamma \in \kappa \rangle$. Suppose without loss of generality that player I has a winning strategy $\overline{\tau}$ in $G(E, \overline{B}, a_0, \kappa)$. We describe a winning strategy $\underline{\tau}$ for player I in $G(E, \underline{B}, a_0, \delta)$. Suppose $\overline{\alpha_1} = \overline{\tau}(\langle a_0 \rangle)$. Find $\underline{\alpha_1} \in \kappa$ such that $\{\beta_\gamma : \gamma \in \overline{\alpha_1}\} \subseteq \{\delta_\gamma : \gamma \in \underline{\alpha_1}\}$ and let $\underline{\tau}(\langle a_0 \rangle) = \underline{\alpha_1}$. Suppose player II responds with a_1. Then $a_0 \cap \{\delta_\gamma : \gamma \in \underline{\alpha_1}\} = a_1 \cap \{\delta_\gamma : \gamma \in \underline{\alpha_1}\}$, so $a_0 \cap \{\beta_\gamma : \gamma \in \overline{\alpha_1}\} = a_1 \cap \{\beta_\gamma : \gamma \in \overline{\alpha_1}\}$, and so $\langle a_0, \overline{\alpha_1}, a_1 \rangle$ is a valid sequence of moves in $G(E, \overline{B}, a_0, \delta)$. Suppose $\overline{\tau}(\langle a_0, \overline{\alpha_1}, a_1 \rangle) = \overline{\alpha_2}$. As before, find $\underline{\alpha_2} > \underline{\alpha_1}$ so that $\{\beta_\gamma : \gamma \in \overline{\alpha_2}\} \subseteq \{\delta_\gamma : \gamma \in \underline{\alpha_2}\}$ and let $\underline{\tau}(\langle a_0, \underline{\alpha_1}, a_1 \rangle) = \underline{\alpha_2}$. Player I can proceed at successors in $G(E, \underline{B}, a_0, \kappa)$ in this manner. Similarly, at limit stages ξ, if player II can respond in $G(E, \underline{B}, a_0, \kappa)$ to $\langle a_0, \underline{\alpha_1}, a_1, \underline{\alpha_2}, \ldots, \underline{\alpha_\xi} = \sup\{\underline{\alpha_\eta} : \eta \in \xi\} \rangle$ with a_ξ then a_ξ is a legal move for player II following the $\langle a_0, \overline{\alpha_0}, a_1, \ldots, \overline{\alpha_\xi} = \sup\{\overline{\alpha_\eta} : \eta \in \xi\} \rangle$ sequence in $G(E, \overline{B}, a_0, \delta)$. So, because $\overline{\tau}$ is a winning strategy in $G(E, \overline{B}, a_0, \delta)$, $\underline{\tau}$ must be a winning strategy in $G(E, \underline{B}, a_0, \delta)$. Next, suppose player II has a winning strategy $\overline{\tau}$ in $G(E, \overline{B}, a_0, \delta)$. We describe a winning strategy for player II in $G(E, \underline{B}, a_0, \omega_1)$. Suppose player I plays $\underline{\alpha_1}$ first. Find $\overline{\alpha_1}$ such that $\{\beta_\gamma : \gamma \in \overline{\alpha_1}\} \supseteq \{\delta_\gamma : \gamma \in \underline{\alpha_1}\}$. Let I play $\overline{\alpha_1}$ in $G(E, \overline{B}, a_0, \omega_1)$ as his first move. Then if $\overline{\tau}(\langle a_0, \overline{\alpha_1} \rangle) = a_1$, a_1 is a legal move for player II following $\langle a_0, \underline{\alpha_1} \rangle$ in $G(E, \underline{B}, a_0, \delta)$. Let $\underline{\tau}(\langle a_0, \underline{\alpha_1} \rangle) = a_1$. As before, we can proceed in this manner, using the moves that player II makes following $\overline{\tau}$ in $G(E, \overline{B}, a_0, \delta)$ to yield legal moves for player II in $G(E, \underline{B}, a_0, \delta)$. So because player II can play a δ-sequence of moves in $G(E, \overline{B}, a_0, \delta)$, she can do so no matter how player I plays in $G(E, \underline{B}, a_0, \delta)$. $\square$

Due to the nature of $\tau_\kappa^{\mathrm{BOX}} \restriction P_{\kappa^+}\lambda$ (which is equal to $\tau_\kappa^{\mathrm{SC}} \restriction P_{\kappa^+}\lambda$ by 1.6.17), many of the results about Väänänen's game which we explored in the early sections of this chapter generalize to the $P_{\kappa^+}\lambda$ setting. In order to proceed, we need some natural definitions.

Definition 1.9.3. For $E \subseteq P_{\kappa^+}\lambda$, define the local δ-kernel of E, $\mathrm{Ker}_l(E, \delta)$, to be the set of $a_0 \in E$ such that player II has a winning strategy in $G(E, B, a_0, \delta)$ for some $B \in [\lambda]^\kappa$ with $a_0 \in E \cap P(B)$. Define the local δ-scattered part of E, $\mathrm{Sc}_l(E, \delta)$, to be the set of $a_0 \in E$ such that player I has a winning strategy in $G(E, B, a_0, \delta)$ for every $B \in [\lambda]^\kappa$ with $a_0 \in E \cap P(B)$. Say that a κ-closed E is locally δ-perfect if and only if $E = \mathrm{Ker}_l(E, \delta)$ and say that E is locally δ-scattered if and only if $E = \mathrm{Sc}_l(E, \delta)$.

As we might expect, if $\omega = \delta \leq \kappa$ we have the usual correspondence between topological perfectness and scatteredness and local ω-perfectness and ω-scatteredness:

Proposition 1.9.4. If $E \subseteq P_{\kappa^+}\lambda$ then $\mathrm{Ker}_l(E, \omega) = \mathrm{Ker}(E)$ and $\mathrm{Sc}_l(E, \omega) = \mathrm{Sc}(E)$. Here $P_{\kappa^+}\lambda$ is equipped with $\tau_\kappa^{\mathrm{BOX}} \restriction P_{\kappa^+}\lambda$.

Proof. Note that $\mathrm{Ker}(E)$ is dense-in-itself. Fix $a_0 \in \mathrm{Ker}(E)$. Let $M \prec H_\theta$ such that $|M| = \kappa$, M is internally unbounded, $\kappa \subseteq M$, and $\{a_0, \kappa, \lambda, E, \text{etc.}\} \subseteq M$. Let $B = M \cap \lambda$. We show that player II has a winning strategy in $G(E, B, a_0, \omega)$. Let $\langle \beta_\gamma : \gamma \in \kappa \rangle$ be an enumeration of B. Suppose player I plays α_1. Because M is internally unbounded, there exists $A_1 \in M \cap P_\kappa(B)$ such that $\{\beta_\gamma : \gamma \in \alpha_1\} \subseteq A_1$. By elementarity and because $\mathrm{Ker}(E)$ is dense-in-itself, there exists $a_1 \in M \cap O_{a_0 \restriction A_1} \cap E$ with $a_1 \neq a_0$. Let player II play such an a_1. Player I responds with α_2, and again there exists $A_2 \in M \cap P_\kappa(B)$ such that $\{\beta_\gamma : \gamma \in \alpha_2\} \subseteq A_2$ and also such that for some $\delta \in A_2$, $a_1(\delta) \neq a_2(\delta)$. As before, by elementarity and because $\mathrm{Ker}(E)$ is dense-in-itself, there exists $a_2 \in M \cap O_{a_1 \restriction A_2} \cap E$ with

$a_2 \neq a_1$ (and necessarily $a_2 \neq a_0$). By proceeding in this manner, it is clear that player II has a winning strategy in $G(E, B, a_0, \omega)$. So $\mathrm{Ker}(E) \subseteq \mathrm{Ker}_l(E, \omega)$. On the other hand, let $a_0 \in \mathrm{Sc}(E)$. Fix $B \in [\lambda]^\kappa$ such that $a_0 \in P(B)$. We need to see that player I has a winning strategy in $G(E, B, a_0, \omega)$. Suppose $\xi_0 = \mathrm{rank}_{CB}(a_0)$, i.e. $a_0 \in E_{\xi_0} \setminus E_{\xi_0+1}$. Find $A_0 \in P_\kappa\lambda$ such that $E_{\xi_0} \cap O_{A_0} = \{a_0\}$. Let $\alpha_1 \in \kappa$ be such that $A_0 \cap B \subseteq \{\beta_\gamma : \gamma \in \alpha_1\}$ and let player I play α_1. Player II responds with $a_1 \in E \cap P(B)$ such that $a_1 \neq a_0$ but $a_1 \cap \{\beta_\gamma : \gamma \in \alpha_1\} = a_0 \cap \{\beta_\gamma : \gamma \in \alpha_1\}$. But then necessarily $\mathrm{rank}_{CB}(a_1) = \xi_1 < \xi_0$. Player I can proceed in this manner, and so this describes a winning strategy. $\square$

Remark 1.9.5. In the case where $\kappa = \omega$, if $E \subseteq P_{\omega_1}\lambda$ is closed in $\tau_\omega^{\mathrm{BOX}} \restriction P_{\omega_1}\lambda$, $E = [F_E] \cap P_{\omega_1}\lambda$ where F_E is the $P_\omega\lambda$-forest induced from E. Then $\mathrm{Ker}(E) = \mathrm{Ker}(E, \omega)$ by 1.9.4. Additionally, $\mathrm{Ker}(F_E) = \mathrm{Ker}(F_E, \omega)$ by 1.7.4. Moreover, because $\mathrm{Ker}(F_E)$ is a cofinally splitting subforest of F_E, which codes an ω-closed subset of $P_{\omega_1}\lambda$, $[\mathrm{Ker}(F_E)] \cap P_{\omega_1}\lambda$ is a closed dense-in-itself subset of E, so $[\mathrm{Ker}(F_E)] \cap P_{\omega_1}\lambda \subseteq \mathrm{Ker}(E)$. On the other hand, it is straightforward to see that $F_{\mathrm{Ker}(E)} \subseteq F$ is a cofinally splitting subforest, so $F_{\mathrm{Ker}(E)} \subseteq \mathrm{Ker}(F_E)$. To summarize then, we have $\mathrm{Ker}(E) = \mathrm{Ker}(E, \omega) = [\mathrm{Ker}(F_E)] \cap P_{\omega_1}\lambda = [\mathrm{Ker}(F_E, \omega)] \cap P_{\omega_1}\lambda$. Similarly then, $\mathrm{Sc}(E) = \mathrm{Sc}(E, \omega) = [\mathrm{Sc}(F_E)] \cap P_{\omega_1}\lambda = \mathrm{Sc}(F_E, \omega)] \cap P_{\omega_1}\lambda$, where e.g. $[\mathrm{Sc}(F_E)]$ is the set of $b \in [F_E]$ such that for some $z \in P_\omega\lambda$, $b \restriction z \in \mathrm{Sc}(F_E)$.

If F is a $P_\kappa\lambda$-forest we can also define the local δ-kernel and the local δ-scattered parts of F in the natural way.

Definition 1.9.6. Let F be a $P_\kappa\lambda$ forest with $\delta \leq \kappa$. Let the local δ-kernel of F, $\mathrm{Ker}_l(F, \delta)$, be the set of $f \in F$ such that for some $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, player II has a winning strategy in $G(F \restriction B, f, \delta)$. Similarly let the local δ-scattered part of f, $\mathrm{Sc}_l(F, \delta)$, be the set of $f \in F$ such that for every $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, player I has a winning strategy in $G(F \restriction B, f, \delta)$.

Remark 1.9.7. It is straightforward to see that $\mathrm{Ker}_l(F, \delta)$ is a $P_\kappa\lambda$-forest and if $f \in \mathrm{Sc}_l(F, \delta)$, every $g \in F$ with $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $g \restriction \mathrm{dom}(f) = f$ is also in $\mathrm{Sc}_l(F, \delta)$.

Just as with the tree decomposition 1.4.5 which holds in the Lévy model obtained by collapsing an inaccessible $\theta > \kappa$ to κ^+ with conditions of size $< \kappa$, in this model we also have a (local) forest decomposition.

Proposition 1.9.8. Let κ be a regular cardinal and let θ be a strong inaccessible cardinal with $\kappa < \theta$. Let $\mathbb{P} = \mathrm{Col}(\kappa, < \theta)$. Then if G is $(V, \mathbb{P})$-generic, in $V[G]$ if F is a $P_\kappa\lambda$-forest, $F = \mathrm{Ker}_l(F, \kappa) \cup \mathrm{Sc}_l(F, \kappa)$.

Proof. Let $f \in F$ and suppose that for some $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, $|[(F \restriction f) \restriction B]| \geq \kappa^+$. Let $B = \langle \beta_\gamma : \gamma \in \kappa \rangle$ be an enumeration of B via $h \in {}^\kappa B$ a bijection with $h(\gamma) = \beta_\gamma$ for every $\gamma \in \kappa$. Let $g : [(F \restriction f) \restriction B] \to {}^\kappa 2$ be defined via this bijection, that is $g(b)(\gamma) = 0$ if $b(\beta_\gamma) = 0$ and $g(b)(\gamma) = 1$ if $b(\beta_\gamma) = 1$. Let $\overline{f}$ be the natural partial function on κ corresponding to f via h. That is, $\mathrm{dom}(\overline{f}) = (h^{-1})''\mathrm{dom}(f)$ and $\overline{f}(\gamma) = 0$ if $f(h(\gamma)) = 0$ and $\overline{f}(\gamma) = 1$ if $f(h(\gamma)) = 1$ for every $\gamma \in \mathrm{dom}(\overline{f})$. Consider $T_{g''[(F\restriction f)\restriction B]} \subseteq {}^{<\kappa}2$. That is, $s \in T_{g''[(F\restriction f)\restriction B]}$ if and only if for some $b \in [(F \restriction f) \restriction B]$, $g(b) \restriction \mathrm{lh}(s) = s$. It is clear that $|[T_{g''[(F\restriction f)\restriction B]}]| \geq \kappa^+$, so by 1.4.5, there exists an embedding $e : {}^{<\kappa}2 \to T_{g''[(F\restriction f)\restriction B]}$. By restricting to an upward cone of ${}^{<\kappa}2$ if necessary, we may assume that this embedding is such that $e(\emptyset) \supseteq \overline{f}$. We show that player II has a winning strategy in $G(F \restriction B, f, \kappa)$. The idea is simple: player II plays nodes in $F \restriction B$ corresponding to nodes coming from $e''{}^{<\kappa}2$, which is possible because the latter is a subtree of $T_{g''[(F\restriction f)\restriction B]}$, and will so be able to play κ-many rounds. Specifically, suppose first that player I plays $A_0 \in P_\kappa B$ such that $\mathrm{dom}(f) \subseteq A_0$. Let $\gamma_0 \in \kappa$ be minimal such that $A_0 \subseteq \{\beta_{\gamma'} : \gamma \in \gamma_0\}$. Find $\{\overline{s_0^0}, \overline{s_1^0}\} \subseteq e''{}^{<\kappa}2$ such that $\overline{s_0^0} \restriction \gamma_0 = \overline{s_1^0} \restriction \gamma_0$, $\mathrm{lh}(\overline{s_0^0}) = \mathrm{lh}(\overline{s_1^0})$, and there exists $\beta \in \kappa$ such that $\overline{s_0^0}(\beta) \neq \overline{s_1^0}(\beta)$. Define $\{f_0^0, f_1^0\} \subseteq F \restriction B$ from $\{\overline{s_0^0}, \overline{s_1^0}\}$ via h in the natural way. So for example, $\mathrm{dom}(f_0^0) = h''\mathrm{dom}(\overline{s_0^0}) = \mathrm{dom}(f_1^0) = h''\mathrm{dom}(\overline{s_1^0})$ and $f_0^0(\beta_\gamma) = 0$ if $\overline{s_0^0}(\gamma) = 0$ along with $f_0^0(\beta_\gamma) = 1$ if $\overline{s_0^0}(\gamma) = 1$ for every $\gamma \in \mathrm{dom}(\overline{s_0^0})$. Then $\{f_0^0, f_1^0\}$ is a splitting pair of nodes in $F \restriction B$ extending f splitting above A_0. Have player II play this pair. Player I responds next with $A_1 \in P_\kappa(B)$ and chooses $f_1 \in \{f_0^0, f_1^0\}$. Let γ_1 be minimal such that $A_1 \subseteq \{\beta_\gamma : \gamma \in \gamma_1\}$. Player II can find a splitting pair

$\{\overline{s_0^1}, \overline{s_0^1}\} \subseteq e''{}^{<\kappa}2$ extending e.g. $\overline{s_0^0}$ (in the case where $f_1 = f_0^0$) splitting above γ_1. Let player II play $\{f_0^1, f_1^1\} \subseteq F \upharpoonright B$ corresponding to $\{\beta_\gamma : \gamma \in \gamma_1\}$ via h in the natural way, i.e. $\mathrm{dom}(f_0^1) = h''\mathrm{dom}(\overline{s_0^1})$, etc. Because player II can always choose nodes in $F \upharpoonright B$ corresponding to nodes in $e''{}^{<\kappa}2$, player II can continue playing in this fashion (at limits too, of course). So $f \in \mathrm{Ker}_l(F, \kappa)$. On the other hand, suppose that for every $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, $|[(F \upharpoonright f) \upharpoonright B]| \leq \kappa$. Choose such a B. We show that player I has a winning strategy in $G(F \upharpoonright B, f, \kappa)$. As before, let $B = \langle \beta_\gamma : \gamma \in \kappa \rangle$ be an enumeration of B via $h \in {}^\kappa B$ a bijection with $h(\gamma) = \beta_\gamma$ for every $\gamma \in \kappa$. Let $g \in {}^\kappa[(F \upharpoonright f) \upharpoonright B]$ be a surjection. Player I simply has to diagonalize against the ξ^{th} element in $[(F \upharpoonright f) \upharpoonright B]$ at stage $\xi + 1$, and also insist that if the two players do play κ-many rounds, that they construct a function with domain B. Specifically, at successor rounds $\xi + 1$, let player I play $A_{\xi+1} = \mathrm{dom}(f_\xi) \cup \{h(\xi)\}$ and choose $f_{\xi+1} \in \{f_0^\xi, f_1^\xi\}$ such that for some $\beta \in \mathrm{dom}(f_{\xi+1})$, $f_{\xi+1}(\beta) \neq g(\xi)(\beta)$. At limit rounds ξ, let player I play $A_\xi = \{h(\xi)\} \cup \bigcup_{\zeta \in \xi} A_\zeta$. If the game lasts κ-many rounds then $f' = \bigcup_{\zeta \in \kappa} f_\zeta \in [(F \upharpoonright f) \upharpoonright B]$, but this is impossible because then for some $\xi \in \kappa$, $f' = g(\xi)$, and so for some $\beta \in \mathrm{dom}(f_{\xi+1}) \subseteq B$, $f'(\beta) = f_{\xi+1}(\beta) \neq g(\xi)(\beta)$. □

We also have the natural analogue to Väänänen's dichotomy 1.4.4 in the Lévy model for closed subsets of $P_{\kappa^+}\lambda$.

Proposition 1.9.9. Let κ be a regular cardinal and let θ be a strong inaccessible cardinal with $\kappa < \theta$. Let $\mathbb{P} = \mathrm{Col}(\kappa, <\theta)$. Then if G is $(V, \mathbb{P})$-generic, in $V[G]$ if $E \subseteq P_{\kappa^+}\lambda$ is closed in $\tau_\kappa^{\mathrm{BOX}} \upharpoonright P_{\kappa^+}\lambda$, $E = \mathrm{Ker}_l(E, \kappa) \cup \mathrm{Sc}_l(E, \kappa)$ where for every $B \in [\lambda]^\kappa$, $|\mathrm{Sc}_l(E, \kappa) \cap P(B)| \leq \kappa$.

Proof. Suppose $a \in E$ and there exists a $B \in [\lambda]^\kappa$ such that $a \subseteq B$ and for every $A \in P_\kappa B$, $|O_{\chi_a \upharpoonright A} \cap E \cap P(B)| \geq \kappa^+$. Here χ_a indicates the characteristic function of a, viewed as a subset of λ. So by $O_{\chi_a \upharpoonright A} \cap E \cap P(B)$, we mean $\{b \in E \cap P(B) : b \cap A = a \cap A\}$. We show that player II has a winning strategy in $G(E, B, a, \kappa)$. The idea here is simple: because $E \cap P(B)$ is closed in $\tau_\kappa^{\mathrm{BOX}} \upharpoonright B$, which is isomorphic to $\tau_\kappa^{\mathrm{BOX}}$ over 2^κ via some bijection of κ with B, no matter what player I makes as his first move in $G(E, B, a, \kappa)$, for some sufficiently large A, $|O_{\chi_a \restriction A} \cap E \cap P(B)| \geq \kappa^+$, and so we can embed a copy of ${}^{<\kappa}2$ into the tree induced by $E \cap P(B)$ above $a \restriction A$. Specifically, let $h : \kappa \to B$ be a bijection, so $h(\gamma) = \beta_\gamma$ for every $\gamma \in \kappa$ and $B = \langle \beta_\gamma : \gamma \in \kappa \rangle$. Suppose player I plays α_1 first. Note that $|O_{\chi_a \restriction \{\beta_\gamma : \gamma \in \alpha_1\}} \cap E \cap P(B)| \geq \kappa^+$. Also, $O_{\chi_a \restriction \{\beta_\gamma : \gamma \in \alpha_1\}} \cap E \cap P(B) \subseteq {}^B2$ is closed in $\tau_\kappa^{\text{BOX}} \restriction {}^B2$, and so by 1.4.5 there exists an embedding $e : {}^{<\kappa}2 \to T_{O_{\chi_a \restriction \{\beta_\gamma : \gamma \in \alpha_1\}} \cap E \cap P(B)} \subseteq {}^{<B}2$. Here by ${}^{<B}2$ we mean the collection of functions which have domain $\{\beta_\gamma : \gamma \in \gamma'\}$ for some $\gamma' \in \kappa$ and codomain $\{0, 1\}$. However, it is then not difficult to see that player II has a winning strategy in $G(E, B, a, \kappa)$ following I's initial move of α_1, because a is a branch through a subtree of $T_{O_{\chi_a \restriction \{\beta_\gamma : \gamma \in \alpha_1\}} \cap E \cap P(B)} \subseteq {}^{<B}2$ isomorphic to ${}^{<\kappa}2$. On the other hand, suppose that for every $B \in [\lambda]^\kappa$ such that $a \subseteq B$ there exists $A \in P_\kappa B$ such that $|O_{\chi_a \restriction A} \cap E \cap P(B)| \leq \kappa$. We show that player I has a winning strategy in $G(E, B, a, \kappa)$. Enumerate $B = \langle \beta_\gamma : \gamma \in \kappa \rangle$. Let player I play α_1 such that $|O_{\chi_a \restriction \{\beta_\gamma : \gamma \in \alpha_1\}} \cap E \cap P(B)| \leq \kappa$. Let $g : \kappa \to O_{\chi_a \restriction \{\beta_\gamma : \gamma \in \alpha_1\}} \cap E \cap P(B)$ be a surjection. At stages $\xi + 2$, let player I choose $\alpha_{\xi+2} > \alpha_{\xi+1}$ large enough so that $a_{\xi+1} \cap \{\beta_\gamma : \gamma \in \alpha_{\xi+2}\} \neq g(\xi) \cap \{\beta_\gamma : \gamma \in \alpha_{\xi+2}\}$, unless $\alpha_{\xi+1} = g(\xi)$, in which case let player I play arbitrarily. If the two players play κ-many rounds, then they will have built a branch through $T_{O_{\chi_a \restriction \{\beta_\gamma : \gamma \in \alpha_1\}} \cap E \cap P(B)} \subseteq {}^{<B}2$ not equal to any element in $O_{\chi_a \restriction \{\beta_\gamma : \gamma \in \alpha_1\}} \cap E \cap P(B)$, which is a contradiction as this set is closed in $\tau_\kappa^{\text{BOX}}(B)$. Thus $E = \text{Ker}_l(E, \kappa) \cup \text{Sc}_l(E, \kappa)$. Furthermore, if $B \in [\lambda]^\kappa$, then $|\text{Sc}_l(E, \kappa) \cap P(B)| \leq \kappa$ as follows. Suppose otherwise, so that $|\text{Sc}_l(E, \kappa) \cap P(B)| \geq \kappa^+$. For every $a \in \text{Sc}_l(E, \kappa) \cap P(B)$, there exists $A_a \in P_\kappa B$ such that $|O_{\chi_a \restriction A_a} \cap E \cap P(B)| \leq \kappa$. However, $2^{<\kappa} = \kappa$ so there must exist some $A \in P_\kappa B$ such that $|\{a \in \text{Sc}_l(E, \kappa) \cap P(B) : A_a = A\}| \geq \kappa^+$. But then $|O_{\chi_a \restriction A} \cap E \cap P(B)| \geq |O_{\chi_a \restriction A} \cap \text{Sc}_l(E, \kappa) \cap P(B)| \geq \kappa^+$, a contradiction. $\square$

We might expect that there is a further relationship between the local forest decomposition in 1.9.8 and the dichotomy in 1.9.9, in particular in the Lévy model. The reader might also question why in the definition of, for example, the δ-local kernel of a $P_\kappa\lambda$-forest F, we only require that there exists $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$ such that player II has a winning strategy in $G(F \restriction B, f, \kappa)$, not that there exist many such B. Indeed, the following definition (and its natural generalizations) is natural:

Definition 1.9.10. Let F be a $P_\kappa\lambda$ forest with $\delta \leq \kappa$. Let $\mathrm{Ker}_l^\exists(F, \delta)$ be the set of $f \in F$ such that for some $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, player II has a winning strategy in $G(F \restriction B, f, \delta)$. So $\mathrm{Ker}_l^\exists(F, \delta) = \mathrm{Ker}_l(F, \delta)$ with the notation from 1.9.6. Similarly, let $\mathrm{Sc}_l^\forall(F, \delta)$ be the set of $f \in F$ such that for every $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, player I has a winning strategy in $G(F \restriction B, f, \delta)$. So $\mathrm{Sc}_l^\forall(F, \delta) = \mathrm{Sc}_l(F, \delta)$ with the notation from 1.9.6. More generally, let $\mathrm{Ker}_l^{\kappa\text{-stat}}(F, \delta)$ be the set of $f \in F$ such that for a κ-stationary set of $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, player II has a winning strategy in $G(F \restriction B, f, \delta)$, let $\mathrm{Ker}_l^{\kappa\text{-club}}(F, \delta)$ be the set of $f \in F$ such that for a κ-club set of $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, player II has a winning strategy in $G(F \restriction B, f, \delta)$, and $\mathrm{Ker}_l^\forall(F, \delta)$ be the set of $f \in F$ such that for every $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, player II has a winning strategy in $G(F \restriction B, f, \delta)$. Define $\mathrm{Sc}_l^{\kappa\text{-club}}(F, \delta)$, $\mathrm{Sc}_l^{\kappa\text{-stat}}(F, \delta)$, and $\mathrm{Sc}_l^\exists(F, \delta)$ similarly.

Remark 1.9.11. By definition, $\mathrm{Sc}_l^\exists(F, \delta) \subseteq \mathrm{Sc}_l^{\kappa\text{-stat}}(F, \delta) \subseteq \mathrm{Sc}_l^{\kappa\text{-club}}(F, \delta) \subseteq \mathrm{Sc}_l^\forall(F, \delta)$ and $\mathrm{Ker}_l^\exists(F, \delta) \subseteq \mathrm{Ker}_l^{\kappa\text{-stat}}(F, \delta) \subseteq \mathrm{Ker}_l^{\kappa\text{-club}}(F, \delta) \subseteq \mathrm{Ker}_l^\forall(F, \delta)$. Furthermore, $\mathrm{Ker}_l^i(F, \delta)$ is a $P_\kappa\lambda$ forest for every $i \in \{\exists, \kappa\text{-stat}, \kappa\text{-club}, \forall\}$ and also for every $i \in \{\kappa\text{-stat}, \kappa\text{-club}, \forall\}$, if $f \in \mathrm{Sc}_l^i(F, \delta)$, every $g \in F$ with $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $g \restriction \mathrm{dom}(f) = f$ is also in $\mathrm{Sc}_l^i(F, \delta)$. Also, $\mathrm{Sc}_l^\exists(F, \delta) \cap \mathrm{Ker}_l^\forall(F, \delta) = \mathrm{Sc}_l^\forall(F, \delta) \cap \mathrm{Ker}_l^\exists(F, \delta) = \emptyset$ and $\mathrm{Sc}_l^{\kappa\text{-stat}}(F, \delta) \cap \mathrm{Ker}_l^{\kappa\text{-club}}(F, \delta) = \mathrm{Sc}_l^{\kappa\text{-club}}(F, \delta) \cap \mathrm{Ker}_l^{\kappa\text{-stat}}(F, \delta) = \emptyset$.

One may also of course define the similar analogues for the Väänänen-type game.

Definition 1.9.12. Let $E \subseteq P_{\kappa^+}\lambda$. Let $\mathrm{Ker}_l^\exists(E, \delta)$ be the set of $a_0 \in E$ such that for some $B \in [\lambda]^\kappa$ with $a_0 \in E \cap P(B)$, player II has a winning strategy in $G(E, B, a_0, \delta)$. So $\mathrm{Ker}_l^\exists(E, \delta) = \mathrm{Ker}_l(E, \delta)$ with the notation from 1.9.3. Similarly, let $\mathrm{Sc}_l^\forall(E, \delta)$ be the set of $a_0 \in E$ such that for every $B \in [\lambda]^\kappa$ with $a_0 \in E \cap P(B)$, player I has a winning strategy in $G(E, B, a_0, \delta)$. So $\mathrm{Sc}_l^\forall(E, \delta) = \mathrm{Sc}_l(E, \delta)$ with the notation from 1.9.3. More generally, let $\mathrm{Ker}_l^{\kappa\text{-stat}}(E,\delta)$ be the set of $a_0 \in E$ such that for a κ-stationary set of $B \in [\lambda]^\kappa$ with $a_0 \in P(B)$, player II has a winning strategy in $G(E,B,a_0,\delta)$, let $\mathrm{Ker}_l^{\kappa\text{-club}}(E,\delta)$ be the set of $a_0 \in E$ such that for a κ-club set of $B \in [\lambda]^\kappa$ with $a_0 \in P(B)$, player II has a winning strategy in $G(E,B,a_0,\delta)$, and $\mathrm{Ker}_l^{\forall}(E,\delta)$ be the set of $a_0 \in E$ such that for every $B \in [\lambda]^\kappa$ with $a_0 \in P(B)$, player II has a winning strategy in $G(E,B,a_0,\delta)$. Define $\mathrm{Sc}_l^{\kappa\text{-club}}(E,\delta)$, $\mathrm{Sc}_l^{\kappa\text{-stat}}(E,\delta)$, and $\mathrm{Sc}_l^{\exists}(E,\delta)$ similarly.

Remark 1.9.13. By definition, $\mathrm{Sc}_l^{\exists}(E,\delta) \subseteq \mathrm{Sc}_l^{\kappa\text{-stat}}(E,\delta) \subseteq \mathrm{Sc}_l^{\kappa\text{-club}}(E,\delta) \subseteq \mathrm{Sc}_l^{\forall}(E,\delta)$ and $\mathrm{Ker}_l^{\exists}(E,\delta) \subseteq \mathrm{Ker}_l^{\kappa\text{-stat}}(E,\delta) \subseteq \mathrm{Ker}_l^{\kappa\text{-club}}(E,\delta) \subseteq \mathrm{Ker}_l^{\forall}(E,\delta)$. Also, $\mathrm{Sc}_l^{\exists}(E,\delta) \cap \mathrm{Ker}_l^{\forall}(E,\delta) = \mathrm{Sc}_l^{\forall}(E,\delta) \cap \mathrm{Ker}_l^{\exists}(E,\delta) = \emptyset$ and $\mathrm{Sc}_l^{\kappa\text{-stat}}(E,\delta) \cap \mathrm{Ker}_l^{\kappa\text{-club}}(E,\delta) = \mathrm{Sc}_l^{\kappa\text{-club}}(E,\delta) \cap \mathrm{Ker}_l^{\kappa\text{-stat}}(E,\delta) = \emptyset$.

Observation 1.9.14. The proof of 1.9.4 actually shows that $\mathrm{Ker}(E) \subseteq \mathrm{Ker}_l^{\kappa\text{-club}}(E,\omega)$. So in fact $\mathrm{Ker}_l^{\exists}(E,\omega) = \mathrm{Ker}_l^{\kappa\text{-stat}}(E,\omega) = \mathrm{Ker}_l^{\kappa\text{-club}}(E,\omega) = \mathrm{Ker}(E)$. And similarly then, $\mathrm{Sc}_l^{\kappa\text{-stat}}(E,\omega) = \mathrm{Sc}_l^{\kappa\text{-club}}(E,\omega) = \mathrm{Sc}_l^{\forall}(E,\omega) = \mathrm{Sc}(E)$.

Because the determinacy results in 1.9.8 and 1.9.9 which hold in the Lévy model rely simply on limitations on the structure of trees $T \subseteq {}^{<\kappa}2$ in the Lévy model, the following natural analogues follow by the same arguments.

Proposition 1.9.15. Let κ be a regular cardinal and let θ be a strong inaccessible cardinal with $\kappa < \theta$. Let $\mathbb{P} = \mathrm{Col}(\kappa, <\theta)$. Then if G is $(V,\mathbb{P})$-generic, in $V[G]$ if $E \subseteq P_{\kappa^+}\lambda$ is closed in $\tau_\kappa^{\mathrm{BOX}} \restriction P_{\kappa^+}\lambda$, $E = \mathrm{Ker}_l^{\exists}(E,\kappa) \cup \mathrm{Sc}_l^{\forall}(E,\kappa) = \mathrm{Ker}_l^{\kappa\text{-stat}}(E,\kappa) \cup \mathrm{Sc}_l^{\kappa\text{-club}}(E,\kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(E,\kappa) \cup \mathrm{Sc}_l^{\kappa\text{-stat}}(E,\kappa) = \mathrm{Ker}_l^{\forall}(E,\kappa) \cup \mathrm{Sc}_l^{\exists}(E,\kappa)$.

Proof. The proof here is identical to that in 1.9.9, using the fact that for any $a_0 \in E$ and $B \in [\lambda]^\kappa$ such that $a_0 \subseteq B$, player II has a winning strategy in $G(E,B,a_0,\kappa)$ if and only if for every $A \in P_\kappa B$, $|O_{\chi_a \restriction A} \cap E \cap P(B)| \geq \kappa^+$, and player I has a winning strategy in $G(E,B,a_0,\kappa)$ if and only if for some $A \in P_\kappa B$, $|O_{\chi_a \restriction A} \cap E \cap P(B)| \leq \kappa$. So if there is a single $B \in [\lambda]^\kappa$ such that $a_0 \subseteq B$ and for every $A \in P_\kappa B$, $|O_{\chi_a \restriction A} \cap E \cap P(B)| \geq \kappa^+$, then $a_0 \in \mathrm{Ker}_l^{\exists}(E, \kappa)$, if there is a κ-stationary set of such B, $a_0 \in \mathrm{Ker}_l^{\kappa\text{-stat}}(E, \kappa)$, if there is a κ-club of such B, $a_0 \in \mathrm{Ker}_l^{\kappa\text{-club}}(E, \kappa)$, and if for every $B \in [\lambda]^\kappa$ with $a_0 \in P(B)$, $|O_{\chi_a \restriction A} \cap E \cap P(B)| \geq \kappa^+$ for every $A \in P_\kappa B$, $a_0 \in \mathrm{Ker}_l^{\forall}(E, \kappa)$. Furthermore, because e.g. if there is not a κ-club of $B \in [\lambda]^\kappa$ with $a_0 \in P(B)$ and $|O_{\chi_a \restriction A} \cap E \cap P(B)| \geq \kappa^+$ for every $A \in P_\kappa B$, the set of $B \in [\lambda]^\kappa$ with $a_0 \in P(B)$ and $|O_{\chi_a \restriction A} \cap E \cap P(B)| \leq \kappa$ for some $A \in P_\kappa B$ is κ-stationary in $[\lambda]^\kappa$, so $a_0 \in \mathrm{Sc}_l^{\kappa\text{-stat}}(E, \kappa)$. □

Proposition 1.9.16. Let κ be a regular cardinal and let θ be a strong inaccessible cardinal with $\kappa < \theta$. Let $\mathbb{P} = \mathrm{Col}(\kappa, < \theta)$. Then if G is $(V, \mathbb{P})$-generic, in $V[G]$ if F is a $P_\kappa\lambda$-forest, $F = \mathrm{Ker}_l^{\exists}(F, \kappa) \cup \mathrm{Sc}_l^{\forall}(F, \kappa) = \mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa) \cup \mathrm{Sc}_l^{\kappa\text{-club}}(F, \kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(F, \kappa) \cup \mathrm{Sc}_l^{\kappa\text{-stat}}(F, \kappa) = \mathrm{Ker}_l^{\forall}(F, \kappa) \cup \mathrm{Sc}_l^{\exists}(F, \kappa)$

Proof. This proof is identical to that in 1.9.8, using the fact that for any $f \in F$ and $B \in [\lambda]^\kappa$ with $\mathrm{dom}(f) \subseteq B$, player II has a winning strategy in $G(F \restriction B, f, \kappa)$ if and only if $|[(F \restriction f) \restriction B]| \geq \kappa^+$ and player I has a winning strategy in $G(F \restriction B, f, \kappa)$ if and only if $|[(F \restriction f) \restriction B]| \leq \kappa$. So the prevalence of B's of these types determines whether or not $f \in \mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa), \mathrm{Ker}_l^{\kappa\text{-club}}(F, \kappa)$, etc.. And again, if e.g. $f \notin \mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa)$, then necessarily $f \in \mathrm{Sc}_l^{\kappa\text{-club}}(F, \kappa)$. □

In the remainder of this section, we explore some connections that these games have to one another. A few more definitions are useful.

Definition 1.9.17. Let F be a $P_\kappa\lambda$ forest and $B \in P(\lambda) \setminus P_\kappa\lambda$. Say that B is a reflecting set (for F) if and only if for every $\bar{b} \in [F \restriction B]$, there exists $b \in [F]$ such that $b \restriction B = \bar{b}$. Say that B is a 0-reflecting set (for F) if and only if for every $\bar{b} \in [F \restriction B]$, there exists $b \in [F]$ such that $b(\gamma) = \bar{b}(\gamma)$ for $\gamma \in B$ and $b(\gamma) = 0$ for $\gamma \notin B$.

Observation 1.9.18. Let F be a $P_\kappa\lambda$ forest such that $|\mathrm{Lev}_z(F)| \leq \kappa$ for every $z \in P_\kappa\lambda$. Then if $M \prec H_\theta$ is F-guessing with $\kappa \subseteq M$, $B = M \cap \lambda$ is a reflecting set for F.

Proof. Let $B = M \cap \lambda$ and suppose $\bar{b} \in [F \upharpoonright B]$. By definition $M \cap P_\kappa B$ is cofinal in $P_\kappa B$ and because $|\mathrm{Lev}_z(F)| \leq \kappa$ for every $z \in P_\kappa\lambda$ and $\kappa \subseteq M$, if $z \in M \cap P_\kappa B$, $\mathrm{Lev}_z(F) \subseteq M$. So necessarily for every $\bar{b} \in [F \upharpoonright B]$, there exists $b \in [F] \cap M$ such that $b \upharpoonright B = \bar{b}$. $\square$

Proposition 1.9.19. Let F be a $P_\kappa\lambda$ forest coding a closed subset of $P_{\kappa^+}\lambda$ with $\kappa^{<\kappa} = \kappa$. Then there exists a κ-club of 0-reflecting sets for F in $[\lambda]^\kappa$.

Proof. Let $M \prec H_\theta$ such that ${}^{<\kappa}M \subseteq M$, $|M| = \kappa$, and $\{\kappa, \lambda, F, \text{etc.}\} \subseteq M$. We show that $B = M \cap \lambda$ is a 0-reflecting set for F, which suffices. Let $\bar{b} \in [F \upharpoonright B]$. Then for every $A \in P_\kappa B$, by elementarity there exists $b_A \in M \cap [F] \cap P_{\kappa^+}\lambda$ such that $b_A \upharpoonright A = \bar{b} \upharpoonright A$. Define $b \in {}^\lambda 2$ by $b(\gamma) = \bar{b}(\gamma)$ for every $\gamma \in B$ and $b(\gamma) = 0$ for $\gamma \notin B$. If $b \notin [F]$, then there exists $z \in P_\kappa\lambda$ such that $b \upharpoonright z \notin F$. However, $b \upharpoonright (z \cap B) = \bar{b} \upharpoonright (z \cap B) \in F$, and so there exists $b_{z\cap B} \in M \cap [F] \cap P_{\kappa^+}\lambda$ with $b_{z\cap B} \upharpoonright (z \cap B) = b \upharpoonright (z \cap B)$. However, $\mathrm{supp}(b_{z\cap B}) \subseteq B$ and $\mathrm{supp}(b) \subseteq B$, so in fact $b_{z\cap B} \upharpoonright z = b \upharpoonright z$, which is a contradiction because $b_{z\cap B} \in [F]$. $\square$

Observation 1.9.20. If $\kappa^{<\kappa} = \kappa$ and F is a $P_\kappa\lambda$-forest, then $\mathrm{Ker}(F, \kappa) \subseteq \mathrm{Ker}_l^{\kappa\text{-club}}(F, \kappa)$.

Proof. Let $f \in \mathrm{Ker}(F, \kappa)$. Suppose $M \prec H_\theta$ is such that ${}^{<\kappa}M \subseteq M$, $\{f, \kappa, \lambda, F, \text{etc.}\} \subseteq M$, and $|M| = \kappa$. Let $B = M \cap \lambda$. By elementarity, there exists τ a winning strategy for player II in $G(F, f, \kappa)$ in M. It is not difficult to see then that player II has a winning strategy in $G(F \upharpoonright B, f, \kappa)$ by following τ (because ${}^{<\kappa}M \subseteq M$). The collection of such B is κ-club. $\square$

Proposition 1.9.21. Let F be a $P_\kappa\lambda$ forest coding a closed subset of $P_{\kappa^+}\lambda$ with $\kappa^{<\kappa} = \kappa$. Then $\mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa) \subseteq \mathrm{Ker}(F, \kappa)$.

Proof. Let $f \in \mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa)$. By 1.9.19, there must exist $B \in [\lambda]^\kappa$ such that $\mathrm{dom}(f) \subseteq B$, player II has a winning strategy τ in $G(F \upharpoonright B, f, \kappa)$, and B is a 0-reflecting set for F. We argue that player II has a winning strategy in $G(F, f, \kappa)$ which we define from τ. Suppose player I plays $A_0 \in P_\kappa\lambda$ such that $\mathrm{dom}(f) \subseteq A_0$. Let $\overline{A_0} = A_0 \cap B$. Let $\tau(\langle \overline{A_0} \rangle) = \{\overline{f_0^0}, \overline{f_1^0}\}$. Because F codes a closed subset of $P_{\kappa^+}\lambda$ and B is 0-reflecting, there exists $\{b_0^0, b_1^0\} \subseteq [F] \cap P_{\kappa^+}\lambda$ with $b_0^0 \restriction \mathrm{dom}(\overline{f_0^0}) = \overline{f_0^0}$, $b_1^0 \restriction \mathrm{dom}(\overline{f_1^0}) = \overline{f_1^0}$ and $b_0^0(\gamma) = b_1^0(\gamma) = 0$ for every $\gamma \notin B$. Let player II play $\{f_0^0 = b_0^0 \restriction A_0 \cup \mathrm{dom}(\overline{f_0^0}), f_1^0 = b_1^0 \restriction A_0 \cup \mathrm{dom}(\overline{f_1^0})\}$. This is a splitting pair extending f which splits above A_0. Player I then chooses $f_1 \in \{f_0^0, f_1^0\}$ and plays A_1. Without loss of generality assume that $f_1 = f_0^0$. Let $\overline{A_1} = A_1 \cap B$. Suppose that $\tau(\langle\langle \overline{A_0}, \{\overline{f_0^0}, \overline{f_1^0}\}\rangle, \langle\{\overline{f_1} = \overline{f_0^0}, \overline{A_1}\}\rangle\rangle) = \{\overline{f_0^1}, \overline{f_1^1}\}$. Again there exists $\{b_0^1, b_1^1\} \subseteq [F] \cap P_{\kappa^+}\lambda$ with $b_0^1 \restriction \mathrm{dom}(\overline{f_0^1}) = \overline{f_0^1}$, $b_1^1 \restriction \mathrm{dom}(\overline{f_1^1}) = \overline{f_1^1}$ and $b_0^1(\gamma) = b_1^1(\gamma) = 0$ for every $\gamma \notin B$. So player II can play $\{f_0^1 = b_0^1 \restriction A_1 \cup \mathrm{dom}(\overline{f_0^1}), f_1^1 = b_1^1 \restriction A_1 \cup \mathrm{dom}(\overline{f_1^1})\}$. This is a splitting pair extending f_1 which splits above A_1. It is clear that player II can continue in this manner, and at limit stages $\delta \in \kappa$ if player I plays $A_\delta \in P_\kappa\lambda$ extending $\bigcup_{\xi\in\delta} \mathrm{dom}(f_\xi)$, then in the $F \restriction B$-game, we need only consider $\tau(\langle\langle \overline{A_0}, \{\overline{f_0^0}, \overline{f_1^0}\}\rangle, \langle\{\overline{f_1}, \overline{A_1}\}\rangle, \ldots, \langle \overline{A_\delta} = A_\delta \cap B\rangle\rangle) = \{\overline{f_0^\delta}, \overline{f_1^\delta}\}$, where in particular $\overline{f_\delta'} = \bigcup_{\xi\in\delta} \overline{f_\xi} \in F \restriction B$, so necessarily $f_\delta' = \bigcup_{\xi\in\delta} f_\xi \in F$ because $\overline{f_\delta'}$ can be extended to a branch through F which is 0 outside of B, and this branch can then be restricted to level $\bigcup_{\xi\in\delta} A_\xi$, extend $\overline{f_0^\delta}$ and $\overline{f_1^\delta}$ to branches through F which are 0 outside B, and restrict these branches to level A_δ forming $\{f_0^\delta, f_1^\delta\}$ which extend f_δ' and split above A_δ. $\square$

Observation 1.9.22. Let F be a $P_\kappa\lambda$ forest coding a closed subset of $P_{\kappa^+}\lambda$ with $\kappa^{<\kappa} = \kappa$. By 1.9.20 and 1.9.21, $\mathrm{Ker}_l^{\kappa\text{-stat}}(F,\kappa) \subseteq \mathrm{Ker}(F,\kappa) \subseteq \mathrm{Ker}_l^{\kappa\text{-club}}(F,\kappa) \subseteq \mathrm{Ker}_l^{\kappa\text{-stat}}(F,\kappa)$. So $\mathrm{Ker}_l^{\kappa\text{-stat}}(F,\kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(F,\kappa) = \mathrm{Ker}(F,\kappa)$.

Proposition 1.9.23. Let $E \subseteq P_{\kappa^+}\lambda$. The $\mathrm{Ker}_l^{\exists}(E,\kappa) \subseteq [\mathrm{Ker}_l^{\kappa\text{-club}}(F_E,\kappa)] \cap P_{\kappa^+}\lambda$. Here F_E is the $P_\kappa\lambda$-forest induced by E.

Proof. We give a sketch of the proof. Let $a \in \mathrm{Ker}_l^{\exists}(E,\kappa)$. So for some $B \in [\lambda]^\kappa$ with $a \in P(B)$, player II has a winning strategy in $G(E,B,a,\kappa)$. Let $B' \in [\lambda]^\kappa$ with $B \subseteq B'$. We argue that player II has a winning strategy in $G(F_E \restriction B', \chi_a \restriction z, \kappa)$ for every $z \in P_\kappa B'$, which suffices. This is because by repeated application of a winning strategy for player II in $G(E,B,a,\kappa)$, if we assume that player I plays $\alpha_1 \in \kappa$ large enough so that $\{\beta_\gamma : \gamma \in \alpha_1\} \supseteq a \cap z$, where $B = \langle \beta_\gamma : \gamma \in \kappa\rangle$ is an enumeration of B, we can define an embedding $e : {}^{<\kappa}2 \to T_{[F_E \restriction B']} \subseteq {}^{<B'}2$ such that $e(\emptyset) \supseteq \chi_a \restriction z$. But then it is not difficult to see by

always choosing splitting pairs from the pointwise image of this embedding, that player II has a winning strategy in $G(F \upharpoonright B', \chi_a \upharpoonright z, \kappa)$. $\square$

Proposition 1.9.24. Let F be a $P_\kappa\lambda$-forest coding a closed subset $E \subseteq P_{\kappa^+}\lambda$ with $\kappa^{<\kappa} = \kappa$. Then $[\mathrm{Ker}(F, \kappa)] \cap P_{\kappa^+}\lambda \subseteq \mathrm{Ker}_l^{\kappa\text{-club}}(E, \kappa)$.

Proof. We give a sketch of the proof. Let $a \in [\mathrm{Ker}(F, \kappa)] \cap P_{\kappa^+}\lambda$. Suppose $M \prec H_\theta$ is such that ${}^{<\kappa}M \subseteq M$, $\{a, f, \kappa, \lambda, F, \text{etc.}\} \subseteq M$, and $|M| = \kappa$. Let $B = M \cap \lambda$, and $\langle \beta_\gamma : \gamma \in \kappa \rangle$ be an enumeration of B. We show that player II has a winning strategy in $G(E, B, a, \kappa)$, which suffices. Suppose player I plays α_1. Then $\chi_a \upharpoonright \{\beta_\gamma : \gamma \in \alpha_1\} \in \mathrm{Ker}(F, \kappa)$, so by elementarity player II has a winning strategy in $G(F, \chi_a \upharpoonright \{\beta_\gamma : \gamma \in \alpha_1\}, \kappa)$, $\tau \in M$. By repeated application of this strategy, where player I plays so as to ensure at each stage ξ that $\beta_\xi \in \mathrm{dom}(f_{\xi+1})$, it is not difficult to see that there exists an embedding $e : {}^{<\kappa}2 \to F \upharpoonright M$ such that $e(\emptyset) \supseteq \chi_a \upharpoonright \{\beta_\gamma : \gamma \in \alpha_1\}$ with the additional property that for every $b \in {}^\kappa 2$, if $f_b = \bigcup_{\alpha \in \kappa} e(b \upharpoonright \alpha)$, $\mathrm{dom}(f_b) = B$. That is, $f_b \in [F \upharpoonright B]$. Because B is 0-reflecting for F, each of these f_b branches is the B-characteristic function for a subset $\bar{b} \subseteq E \cap B$. But then it is not too difficult to see that if player II plays elements of the form $\bar{b} \in E \cap P(B)$, player II has a winning strategy in $G(E, B, a, \kappa)$, as long as player I plays α_1 first. $\square$

To summarize, if $\kappa^{<\kappa} = \kappa$ and F is a $P_\kappa\lambda$-forest coding a closed set $E \subseteq P_{\kappa^+}\lambda$, then $\mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(F, \kappa) = \mathrm{Ker}(F, \kappa)$ and $[\mathrm{Ker}(F, \kappa)] \cap P_{\kappa^+}\lambda \subseteq \mathrm{Ker}_l^{\kappa\text{-club}}(E, \kappa) \subseteq \mathrm{Ker}_l^{\exists}(E, \kappa) \subseteq [\mathrm{Ker}(F, \kappa)] \cap P_{\kappa^+}\lambda$. That is, $\mathrm{Ker}_l^{\exists}(E, \kappa) = \mathrm{Ker}_l^{\kappa\text{-stat}}(E, \kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(E, \kappa) = [\mathrm{Ker}(F, \kappa)] \cap P_{\kappa^+}\lambda$.

Remark 1.9.25. In the Lévy model, if F is a $P_\kappa\lambda$-forest coding a closed subset $E \subseteq P_{\kappa^+}\lambda$ then $E = \mathrm{Ker}_l^{\exists}(E, \kappa) \cup \mathrm{Sc}_l^{\forall}(E, \kappa) = \mathrm{Ker}_l^{\kappa\text{-stat}}(E, \kappa) \cup \mathrm{Sc}_l^{\kappa\text{-club}}(E, \kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(E, \kappa) \cup \mathrm{Sc}_l^{\kappa\text{-stat}}(E, \kappa)$ with $\mathrm{Ker}_l^{\exists}(E, \kappa) = \mathrm{Ker}_l^{\kappa\text{-stat}}(E, \kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(E, \kappa) = [\mathrm{Ker}(F, \kappa)] \cap P_{\kappa^+}\lambda = [\mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa)] \cap P_{\kappa^+}\lambda = [\mathrm{Ker}_l^{\kappa\text{-club}}(F, \kappa)] \cap P_{\kappa^+}\lambda$ and $F = \mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa) \cup \mathrm{Sc}_l^{\kappa\text{-club}}(F, \kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(F, \kappa) \cup \mathrm{Sc}_l^{\kappa\text{-stat}}(F, \kappa)$, so necessarily then, for example, $[\mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa)] \cap P_{\kappa^+}\lambda =$

$\mathrm{Ker}_l^{kappa\text{-stat}}(E,\kappa)$ and $[\mathrm{Sc}_l^{\kappa\text{-club}}(F,\kappa)] \cap P_{\kappa^+}\lambda = \mathrm{Sc}_l^{\kappa\text{-club}}(E,\kappa)$. By $[\mathrm{Sc}_l^{\kappa\text{-club}}(F,\kappa)] \cap P_{\kappa^+}\lambda$ we mean of course the collection of $b \in [F] \cap P_{\kappa^+}\lambda$ such that for some $z \in P_\kappa\lambda$, $b \restriction z \in \mathrm{Sc}_l^{\kappa\text{-club}}(F,\kappa)$. These equalities may be viewed as natural generalizations of 1.4.5. Additionally, just as in 1.4.5, one could also prove these statements in 1.9.16 as a consequence of the corresponding ones in 1.9.15, or vice versa.

1.9.1 How $\mathcal{I}^-(\kappa)$ could be used : an illustration

When we proved 1.4.5, we did not follow Väänänen's method of using $\mathcal{I}^-(\kappa)$, though we noted in 1.4.4 that one could. For completeness, here is an example (following the idea in [69]) to illustrate how this method works, in the more general setting we have described here. The hypothesis of a strong ideal over κ^+ is a natural one to consider as we will see, because it allows one to "take intersections" at stages of the game $< \kappa$, yielding a winning strategy for player II.

Observation 1.9.26. If $E \subseteq P_{\kappa^+}\lambda$ is closed and $|E| \leq \kappa$, then $\mathrm{Sc}_l^{\forall}(E,\kappa) = E$.

Proof. This is the usual diagonalization argument. Fix $a_0 \in E$ and any $B \in [\lambda]^\kappa$ with $a_0 \in P(B)$. Enumerate $B = \langle \beta_\gamma : \gamma \in \kappa \rangle$. We need to see that I has a winning strategy in $G(E, B, a_0, \omega_1)$. Let $g : \kappa \to E \cap P(B)$ be a surjection. At stage $\xi + 1 \in \kappa$ let player I play $\alpha_{\xi+1} > \alpha_\xi$ such that $a_\xi \cap \{\beta_\gamma : \gamma \in \alpha_{\xi+1}\} \neq a_\zeta \cap \{\beta_\gamma : \gamma \in \alpha_{\xi+1}\}$ for every $\zeta \in \xi$ and if $a_\xi \neq b(\xi)$ also such that $a_\xi \cap \{\beta_\gamma : \gamma \in \alpha_{\xi+1}\} \neq g(\xi) \cap \{\beta_\gamma : \gamma \in \alpha_{\xi+1}\}$. If player II can respond to every move, then the two players will have built a convergent sequence $\langle a_\xi : \xi \in \kappa \rangle \subseteq E \cap P(B)$. Suppose this sequence converges to an element in $E \cap P(B)$, so for some $\delta \in \kappa$, it converges to $g(\delta)$ such that $g(\delta) \cap \{\beta_\gamma : \gamma \in \alpha_{\xi+1}\} = a_\xi \cap \{\beta_\gamma : \gamma \in \alpha_{\xi+1}\}$ for every $\xi \in \kappa$. However, at stage δ if $a_\delta \neq g(\delta)$ then $a_\delta \cap \{\beta_\gamma : \gamma \in \alpha_{\delta+1}\} \neq g(\delta) \cap \{\beta_\gamma : \gamma \in \alpha_{\delta+1}\}$, which is a contradiction. On the other hand, if $a_\delta = g(\delta)$, then at stage $\delta + 2$, $a_{\delta+1} \cap \{\beta_\gamma : \gamma \in \alpha_{\delta+2}\} = g(\delta) \cap \{\beta_\gamma : \gamma \in \alpha_{\delta+2}\} \neq a_\delta \cap \{\beta_\gamma : \gamma \in \alpha_{\delta+2}\}$, which is also a contradiction. $\square$

Lemma 1.9.27. Suppose $\mathcal{I}^-(\kappa)$ holds. If $E \subseteq P_{\kappa^+}\lambda$ is such that for some $B \in [\lambda]^\kappa$ we have $|E \cap P(B)| \geq \kappa^+$, then $E \cap P(B) \cap \mathrm{Ker}_l^\exists(E,\kappa) \neq \emptyset$. In particular, player II has a winning strategy in $G(E, B, a_0, \kappa)$ for some $a_0 \in E \cap P(B)$. Note that we do not require that E is closed.

Proof. Fix $B \in [\lambda]^\kappa$ such that $|E \cap P(B)| \geq \kappa^+$ and choose $E' \subseteq E$ such that $|E' \cap P(B)| = \kappa^+$. Enumerate $B = \langle \beta_\gamma : \gamma \in \kappa \rangle$. Via $\mathcal{I}^-(\kappa)$, let $\mathcal{I}$ be a normal $(< \kappa^+)$-complete ideal over $E' \cap P(B)$ such that the collection of $\mathcal{I}^+$-sets has a $\subseteq$-dense subset, $K \subseteq \mathcal{I}^+$, in which every $\subseteq$-descending $< \kappa$-length sequence has a $\subseteq$-lower bound in K. We first argue that if $X \in \mathcal{I}^+$, there exists $a \in X$ such that for every $O_{a,\eta}$ a basic open neighborhood (relative to B) containing a, $O_{a,\eta} \cap X \in \mathcal{I}^+$. Such an a is called an $\mathcal{I}$-point for X. By $O_{a,\eta}$, we mean the set of $b \in P(B)$ of the form $(a \cap \{\beta_\gamma : \gamma \in \eta\}) \cup c$ where $c \in P(B \setminus \{\beta_\gamma : \gamma \in \eta\})$, that is the set of $b \in P(B)$ which have the same intersection with $\{\beta_\gamma : \gamma \in \eta\}$ as a does. Because $\mathcal{I}^-(\kappa)$ implies that $2^{<\kappa} = \kappa$, there are only κ-many basic open neighborhoods. So if there were no such $a \in X$, to every a we could assign $O_{a,\eta}$ so that $O_{a,\eta} \cap X \in \mathcal{I}$. However, then we would have written X as a κ-sized union of sets in $\mathcal{I}$, which is impossible because $\mathcal{I}$ is $(< \kappa^+)$-complete. So, let $a_0 \in E' \cap P(B)$ be an $\mathcal{I}^+$-point for $E' \cap P(B)$. We show that player II has a winning strategy in $G(E, B, a_0, \kappa)$. Suppose player I plays α_1. By assumption on a_0, $O_{a_0,\alpha_1} \cap E' \cap P(B) \in \mathcal{I}^+$. Find $X_1 \in K$ so that $X_1 \subseteq O_{a_0,\alpha_1} \cap E' \cap P(B)$. Because $X_1 \in \mathcal{I}^+$, there exists an $\mathcal{I}$-point for X_1 in X_1. Let player II play such a point, a_1. Next, suppose player I plays $\alpha_2 > \alpha_1$. We may proceed as above. Namely, because a_1 is an $\mathcal{I}$-point for X_1, $O_{a_1,\alpha_2} \cap X_1 \in \mathcal{I}^+$, so we can find $X_2 \in K$ with $X_2 \subseteq O_{a_1,\alpha_2} \cap X_1$. Player II may then play a_2 an $\mathcal{I}$-point for X_2. At limit stages μ, player I must play $\alpha_\mu = \sup\{\alpha_\nu : \nu \in \mu\}$. Fix $\langle \mu_\xi : \xi \in \mathrm{cf}(\mu) \rangle \subseteq \mu$ cofinally increasing such that μ_ξ is a successor ordinal for every $\xi \in \mathrm{cf}(\mu)$. By construction we will have $\langle X_{\mu_\xi} : \xi \in \mathrm{cf}(\mu) \rangle$ a descending $\mathrm{cf}(\mu)$-sequence of $\mathcal{I}^+$-sets in K, having been constructed by II during the course of play. So there exists a $\subseteq$-lower bound in

K for this sequence, call it X_μ. By construction $X_{\mu_\xi} \subseteq O_{a_{\mu_{xi}-1},\alpha_{\mu_\xi}}$ for every ξ, so player II can proceed by choosing a_μ an $\mathcal{I}$-point for X_μ. This describes a winning strategy for II. $\square$

Lemma 1.9.28. Suppose $\mathcal{I}^-(\kappa)$ holds. Let $E \subseteq P_{\kappa^+}\lambda$ be closed. If $B \in [\lambda]^\kappa$ and $a \in E \cap P(B)$, then either player I has a winning strategy in $G(E, B, a, \kappa)$ or player II has a winning strategy in $G(E, B, a, \kappa)$. Furthermore, set of points $a \in E \cap P(B)$ such that player I has a winning strategy in $G(E, B, a, \kappa)$ is of cardinality $\leq \kappa$.

Proof. Fix $B \in [\lambda]^\kappa$. First, suppose that $|E \cap P(B)| \leq \kappa$. Because $E \cap P(B) \subseteq P_{\kappa^+}\lambda$ is closed, by 1.9.26, $\mathrm{Sc}_l^\forall(E \cap P(B), \kappa) = E \cap P(B)$. So we may assume that $|E \cap P(B)| \geq \kappa^+$. Suppose towards a contradiction that the cardinality of the set of $a \in E \cap P(B)$ such that player I has a winning strategy in $G(E, B, a, \kappa)$ is $\geq \kappa^+$. By 1.9.27, there then exists $a \in E \cap P(B)$ in this set such that player II has a winning strategy in $G(E, B, a_0, \kappa)$, which is a contradiction. Therefore the cardinality of the set of $a \in E \cap P(B)$ such that player I has a winning strategy in $G(E, B, a, \kappa)$ is $\leq \kappa$. Then let $A \subseteq E \cap P(B)$ be the collection of a such that player I does not have a winning strategy in $G(E, B, a_0, \kappa)$, so $|A| \geq \kappa^+$. First, note that A is closed in $P_{\kappa^+}\lambda$, as follows. Suppose $\langle a_\alpha : \alpha \in \kappa \rangle \subseteq A$ is a convergent sequence with limit a. We need to argue that $a \in A$, so we need to see that player I does not have a winning strategy in $G(E, B, a, \kappa)$. Enumerate $B = \langle \beta_\gamma : \gamma \in \kappa \rangle$ and suppose towards a contradiction that player I does have a winning strategy, starting with α_1. Because $\langle a_\alpha : \alpha \in \kappa \rangle \subseteq A$ converges to a, there exists $a' \in A$ such that $a \cap \{\beta_\gamma : \gamma \in \alpha_1\} = a' \cap \{\beta_\gamma : \gamma \in \alpha_1\}$. But then it is not difficult to see that player I also has a winning strategy in $G(E, B, a', \kappa)$, which is a contradiction by definition of A. So $A \subseteq P_{\kappa^+}\lambda$ is closed. Let $a_0 \in A$. We show that player II has a winning strategy in $G(E, B, a_0, \kappa)$, which suffices. Suppose player I plays α_1. Let $\overline{A} = O_{a_0,\alpha_1} \cap A$, that is $\overline{A}$ is the subset of A consisting of $\overline{a}$ such that $\overline{a} \cap \{\beta_\gamma : \gamma \in \alpha_1\} = a \cap \{\beta_\gamma : \gamma \in \alpha_1\}$. Note that $\overline{A}$ is a closed subset of $P_{\kappa^+}\lambda$. Suppose first that $|\overline{A}| \geq \kappa^+$. By 1.9.27, player II has a winning strategy in $G(\overline{A} \setminus \{a_0\}, B, a_1, \kappa)$ for some $a_1 \in \overline{A} \setminus \{a_0\}$. But then player II can play a_1 and then proceed in $G(E, B, a_0, \kappa)$ according to her winning strategy in

$G(\overline{A}\setminus\{a_0\}, B, a_1, \kappa)$. So suppose $|\overline{A}| \leq \kappa$. Then by 1.9.26, player I has a winning strategy in $G(\overline{A}, B, a_0, \kappa)$. We argue that this is impossible, because it gives rise to a winning strategy for player I in $G(E, B, a_0, \kappa)$, as follows. Suppose that player II plays a_1 in response to α_1 in $G(E, B, a_0, \kappa)$. There are two cases, either $a_1 \in \overline{A}$, i.e. $a_1 \in A$, or $a_1 \notin A$. If $a_1 \notin A$, then by definition player I has a winning strategy in $G(E, B, a_1, \kappa)$, and player I can proceed according to this strategy in $G(E, B, a_0, \kappa)$. So we may suppose that $a_1 \in A$. Indeed, if at any stage $\xi \in \kappa$ player II plays $a_\xi \notin A$, player I will be able to win the game as above. So player II must play every $a_\xi \in A$. But then this run of the game $G(E, B, a_0, \kappa)$ corresponds to a run of the game $G(\overline{A}, B, a_0, \kappa)$, and player I can play according to his winning strategy in this latter game. □

Proposition 1.9.29. Suppose $\mathcal{I}^-(\kappa)$ holds. Let $E \subseteq P_{\kappa^+}\lambda$ be closed. Then $E = \mathrm{Ker}_l^{\exists}(E,\kappa) \cup \mathrm{Sc}_l^{\forall}(E,\kappa) = \mathrm{Ker}_l^{\kappa\text{-stat}}(E,\kappa) \cup \mathrm{Sc}_l^{\kappa\text{-club}}(E,\kappa) = \mathrm{Ker}_l^{\kappa\text{-club}}(E,\kappa) \cup \mathrm{Sc}_l^{\kappa\text{-stat}}(E,\kappa) = \mathrm{Ker}_l^{\forall}(E,\kappa) \cup \mathrm{Sc}_l^{\exists}(E,\kappa)$. Moreover, for every $B \in [\lambda]^\kappa$, $|\mathrm{Sc}_l^{\kappa\text{-stat}}(E,\kappa) \cap P(B)| \leq \kappa$ (for example).

Proof. By 1.9.28 if $B \in [\lambda]^\kappa$ and $a \in E \cap P(B)$, then either player I or player II has a winning strategy in $G(E, B, a, \kappa)$. If player II has a winning strategy in $G(E, B, a, \kappa)$ then player II also has a winning strategy in $G(E, B', a, \kappa)$ for every $B' \in [\lambda]^\kappa$ such that $B \subseteq B'$. So $\mathrm{Ker}_l^{\kappa\text{-club}}(E,\kappa) = \mathrm{Ker}_l^{\exists}(E,\kappa)$ (in particular). So for example, if $a \in E \cap P(B)$ is then such that for a κ-stationary set of $B \in [\lambda]^\kappa$, player II does not have a winning strategy, then $a \in \mathrm{Sc}_l^{\kappa\text{-stat}}(E,\kappa)$. Of course in fact here $a \in \mathrm{Sc}_l^{\subseteq\text{-cofinal}}(E,\kappa)$, etc.. It is clear then by 1.9.27 that, e.g. for every $B \in [\lambda]^\kappa$, $|\mathrm{Sc}_l^{\kappa\text{-stat}}(E,\kappa) \cap P(B)| \leq \kappa$. □

1.10 Unconsidered directions

1. Does there exist an $(\omega+1)$-undetermined closed subset $[T] \subseteq 2^{\omega_1}$ with T an ω_1-tree in ZFC? We have seen in the cut-and-choose game that this is easy to produce.

2. A natural generalization of a of $P_\kappa\lambda$-forest is a coherent functional system where the codomain is some ν larger than $\{0,1\}$. Is it useful to analyze these objects with games also?

3. Is it possible to give more structure theorems for when player I or II have winning strategies in $G(F, \emptyset, \mu)$ or $G(F, \emptyset, \mu{+}1)$? For player II, what about reasonable structure statements that aren't too closely tied to the game for $G(F, \emptyset, \kappa)$?

4. What are the canonical constructive examples of e.g. $P_\kappa\lambda$ forests which are undetermined (in various ways)?

5. Can the different definitions of local games can be separated nontrivially (e.g. $\mathrm{Ker}_l^{\kappa\text{-stat}}(F, \kappa) \supsetneq \mathrm{Ker}_l^{\kappa\text{-club}}(F, \kappa)$, etc.)?

6. Are there things to be said in these contexts for games where player II offers more than just a splitting pair (and player I can then choose e.g. more than just a single node)?

Chapter 2

Disjoint refinements

2.1 Background and initial Observations

2.1.1 Definitions and background

Definition 2.1.1. Let κ be a cardinal and $\mathcal{I}$ be an ideal over κ. Say that $\mathcal{A} \subseteq \mathcal{I}^+$ is $\mathcal{I}$-almost disjoint if and only if $A \cap B \in \mathcal{I}$ for every distinct A, B in $\mathcal{A}$. If $A \setminus B \in \mathcal{I}$, say that A is modulo $\mathcal{I}$-contained in B, and write $A \subseteq_{\mathcal{I}} B$, or $A \subseteq^* B$ when $\mathcal{I}$ is clear. For $S \subseteq \mathcal{I}^+$, say that S can be $\mathcal{I}$-injectively almost disjointly refined if and only if there exits $\mathcal{A}_S = \{a_s : s \in S\} \subseteq \mathcal{I}^+$ such that $a_s \in P(s)$ for every $s \in S$ and if $s \neq s'$ then $a_s \cap a_{s'} \in \mathcal{I}$. We will often refer to e.g. an $\mathcal{I}$-almost disjoint refinement and mean an $\mathcal{I}$-injective almost disjoint refinement. When $\mathcal{I}$ is understood, we will also refer just to almost disjoint refinements, etc.. Often in this case $\mathcal{I} =< \kappa$, that is the ideal of subsets of κ of size $< \kappa$, and $\mathcal{A} \subseteq [\kappa]^\kappa$.

Definition 2.1.2. Let κ be a cardinal and $\mathcal{I}$ be an ideal over κ. Let $\mathbf{ref}(\kappa/\mathcal{I})$ denote the smallest cardinality of a set $X \subseteq [\kappa]^\kappa$ which cannot be $\mathcal{I}$-injectively almost disjointly refined.

Again, we will often say just refined, almost disjointly refined, etc. when $\mathcal{I}$ is clear from context. Let $\mathbf{ref}(\kappa) = \mathbf{ref}(\kappa/ < \kappa)$. More generally, if $A \subseteq \mathcal{I}^+$ does not have an almost disjoint refinement comprising a set of elements from A, let $\mathbf{ref}(A/\mathcal{I})$ denote the smallest cardinality of a subset of A which does not have such an almost disjoint refinement.

Definition 2.1.3. Let κ be a cardinal and $\mathcal{I}$ be an ideal over κ. Let $\mathrm{MAD}(\kappa/\mathcal{I})$ denote the spectrum of cardinalities of maximal $\mathcal{I}$-almost disjoint subsets of κ. That is, $\mathrm{MAD}(\kappa/\mathcal{I}) = \{\lambda : \exists \mathcal{A} \subseteq [\kappa]^\kappa\ \mathcal{I}\text{-maximal almost disjoint with } |\mathcal{A}| = \lambda\}$. Similarly, let $\mathrm{AD}(\kappa/\mathcal{I})$ denote the spectrum of cardinalities of $\mathcal{I}$-almost disjoint subsets of κ. Let $\mathrm{MAD}(\kappa/ < \kappa) = \mathrm{MAD}(\kappa)$ and $\mathrm{AD}(\kappa/ < \kappa) = \mathrm{AD}(\kappa)$. Note of course that, for example, $\mathrm{AD}(\kappa) = \{\mu : \text{for some } \lambda \geq \mu, \lambda \in \mathrm{MAD}(\kappa)\}$.

Observation 2.1.4. There exists an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ of cardinality 2^ω. More generally if $2^{<\kappa} = \kappa$ then there exists an almost disjoint family $\mathcal{A} \subseteq [\kappa]^\kappa$ of cardinality 2^κ.

Proof. Let $f : {}^{<\kappa}2 \to \kappa$ be an injection and note that $\mathcal{A} = \{\{f(b \restriction \alpha) : \alpha \in \kappa\} : b \in {}^\kappa 2\}$ is almost disjoint. For a less standard argument in the e.g. the ω-case, say, let $f : \omega \to \omega \times \omega$ be a bijection. For every real slope $\theta \in \mathbb{R}$, let a_θ denote the set of n such that $f(n)$ is less distance 2 from the line $y = \theta x$. It is clear that $\{a_\theta : \theta \in \mathbb{R}\} \subseteq [\omega]^\omega$ is an almost disjoint collection of size 2^ω. $\square$

Note 2.1.5. While $2^{<\kappa} = \kappa$ is sufficient to guarantee that $2^\kappa \in \mathrm{AD}(\kappa)$, it is neither necessary nor optimal. For example, if $2^{\omega_1} > 2^\omega$ and $2^\omega < \aleph_{\omega_1}$, then $2^{\omega_1} \in \mathrm{AD}(\omega_1)$. See [9], for example, for a proof. Also in [9], Baumgartner gives methods to manipulate e.g. $\mathrm{MAD}(\kappa)$. For example, if $V \models GCH$ and G is $(V, \mathrm{Fn}(\omega_4, 2, < \omega))$-generic, then in $V[G]$, $\mathrm{MAD}(\omega_1) \cap (\omega_1, 2^{\omega_1}] = \{\omega_2\}$. It is known that if μ is any singular limit of cardinals in $\mathrm{MAD}(\kappa)$ then $\mu \in \mathrm{MAD}(\kappa)$, see [52] for example. So in particular, if $\sup(\mathrm{AD}(\kappa))$ is singular then $\sup(\mathrm{AD}(\kappa)) \in \mathrm{AD}(\kappa)$. We do not know, however, whether it is consistent that e.g. $\sup(\mathrm{AD}(\kappa)) \notin \mathrm{AD}(\kappa)$ if $\sup(\mathrm{AD}(\kappa))$ is a regular cardinal. For a more recent detailed investigation into the size of (maximal) almost disjoint families in a general setting, see [53].

2.2 Almost disjoint refinement in $[\kappa]^\kappa / < \kappa$

2.2.1 Almost disjoint refinement by countable sets

Aspects of the basic question of which collections of subsets (in particular) admit an almost disjoint refinement (in whatever the relevant sense is) have been studied by many individuals over several decades. We will not give a complete, or even a partial, survey of these developments here. However, the following early theorem and still-open question 2.2.3 have received significant attention.

Theorem (Balcar, Vojtáš, [7]) 2.2.1. *If U is a uniform ultrafilter over ω, then U has an almost disjoint refinement.*

Definition 2.2.2. Say that an ideal $\mathcal{I}$ over ω is tall if and only if for every $x \in [\omega]^\omega$, there exists $y \in \mathcal{I} \cap [\omega]^\omega$ with $y \subseteq x$.

Open question 2.2.3. Let $\mathcal{I}$ be a tall ideal over ω. Does $\mathcal{I}^+$ have an almost disjoint refinement?

Note 2.2.4. It is not too difficult to see that the statement in 2.2.3 is consistent, as follows. Suppose that $\mathrm{MAD}(\omega) \cap (\omega, 2^\omega] = \{2^\omega\}$. Let $\mathcal{I}$ be a tall ideal, and enumerate $\mathcal{I}^+ = \langle x_\alpha : \alpha \in 2^\omega \rangle$. At stage α, suppose we have constructed $\langle a_\gamma : \gamma \in \alpha \rangle$. Because $\mathrm{MAD}(\omega) \cap (\omega, 2^\omega] = \mathrm{MAD}([x_\alpha]^\omega / < \omega) \cap (\omega, 2^\omega] = \{2^\omega\}$, there exists some $z_\alpha \in [x_\alpha]^\omega$ with $|z_\alpha \cap (x_\alpha \cap a_\gamma)| < \omega$ for every $\gamma \in \alpha$. Let $a_\alpha \subseteq z_\alpha$ with $a_\alpha \in \mathcal{I} \cap [\omega]^\omega$. We can proceed in this fashion, so that $\langle a_\alpha : \alpha \in 2^\omega \rangle$ is an almost disjoint refinement of $\mathcal{I}^+$.

2.2.2 Observations regarding ref(κ)

The following proposition 2.2.6 is folklore, and has seemingly been proven by many individuals independently. In order to motivate the diagonalization technique used, first consider the following (perhaps more apparent) observation.

Observation 2.2.5. Let $S \subseteq [\omega]^\omega$ be such that $|S| < \min\{2^\omega, \mathfrak{t}^+\}$. Then S has an almost disjoint refinement. Here $\mathfrak{t}$ denotes the tower number, that is the smallest cardinality of a $\subseteq^*$-decreasing chain of elements of $[\omega]^\omega$ with no pseudointersection (that is with no $\subseteq^*$-lower bound).

Proof. Let $S = \langle s_\alpha : \alpha \in \lambda \rangle$ for some $\lambda \in \min\{2^\omega, \mathfrak{t}^+\}$. Define $\langle a_\alpha : \alpha \in \lambda \rangle$ by recursion. At stage ξ, we have defined $\langle a_\gamma : \gamma \in \xi \rangle$ and $\langle s_\alpha^\gamma : \gamma \in \xi \rangle \subseteq s_\alpha$ for every $\alpha \in [\xi, \lambda)$ a $\subseteq^*$-decreasing sequence of subsets of s_α. First, because $\lambda \leq \mathfrak{t}$, for every $\alpha \in [\xi, \lambda)$ we can find $s_\alpha'^\xi \subseteq s_\alpha$ such that $s_\alpha'^\xi$ is a $\subseteq^*$-lower bound for $\langle s_\alpha^\gamma : \gamma \in \xi \rangle$. Let $\langle a_\xi^\eta : \eta \in 2^\omega \rangle$ be an almost disjoint collection of infinite subsets of $s_\xi'^\xi$. For every $\alpha \in [\xi + 1, \lambda)$, for at most one η is $s_\alpha'^\xi \subseteq^* a_\xi^\eta$, so because $\lambda < 2^\omega$, for some η_ξ, for every $\alpha \in [\xi+1, \lambda)$, $|s_\alpha'^\xi \setminus a_\xi^{\eta_\xi}| = \omega$. Let $a_\xi = a_\xi^{\eta_\xi}$ and $s_\alpha^\xi = s_\alpha'^\xi \setminus a_\xi^{\eta_\xi}$ for every $\alpha \in [\xi + 1, \lambda)$. It is clear that we can proceed in this fashion, so defining $\langle a_\alpha : \alpha \in \lambda \rangle$ such that $a_\alpha \subseteq s_\alpha$ for every $\alpha \in \lambda$. We need to see that this collection is almost disjoint. Fix $\gamma \in \xi \in \lambda$. By construction, $a_\gamma = a_\gamma^{\eta_\gamma}$, while $a_\xi \subseteq s_\xi'^\xi \subseteq^* s_\xi^\gamma = s_\xi'^\gamma \setminus a_\gamma^{\eta_\gamma}$. So $|a_\gamma \cap a_\xi| < \omega$. □

Proposition 2.2.6. Let $S \subseteq [\omega]^\omega$ be such that $|S| < 2^\omega$. Then S has an almost disjoint refinement.

Proof. Suppose $\lambda \in 2^\omega$ and $\langle x_\alpha : \alpha \in \lambda \rangle \subseteq [\omega]^\omega$. Fix $\langle \mathcal{A}_\alpha : \alpha \in \lambda \rangle$ so that for every $\alpha \in \lambda$, $\mathcal{A}_\alpha \subseteq [x_\alpha]^\omega$ is an almost disjoint family of size λ^+ (for the following argument we could also choose $\mathcal{A}_\alpha$ to be an almost disjoint family of maximal size size (i.e. 2^ω) as long as $\mathrm{cf}(2^\omega) > \lambda$). To each $\alpha \in \lambda$, remove from $\mathcal{A}_\alpha$ any a which for some $\beta \neq \alpha$ is an element of $\{a \in \mathcal{A}_\alpha : |a \cap x_\beta| = \omega\}$ provided that $|\{a \in \mathcal{A}_\alpha : |a \cap x_\beta| = \omega\}| < \lambda^+$. Note that there are fewer than λ^+-many of these a which are removed. So without loss of generality, assume that if $a \in \mathcal{A}_\alpha$ then for $\beta \neq \alpha$, if $|a \cap x_\beta| = \omega$ then $|\{a \in \mathcal{A}_\alpha : |a \cap x_\beta| = \omega\}| = \lambda^+$. We construct an injective almost disjoint refinement by recursion. At stage $\alpha \in \lambda$, choose the minimal $\delta_\alpha \leq \alpha$ such that $|\{a \in \mathcal{A}_{\delta_\alpha} : |a \cap x_\alpha| = \omega\}| = \lambda^+$. Then let $a'_\alpha \in \mathcal{A}_{\delta_\alpha}$ such that $|a'_\alpha \cap x_\alpha| = \omega$ and $a'_\alpha \neq a'_\xi$ for every $\xi \in \alpha$. Set $a_\alpha = x_\alpha \cap a'_\alpha$. Because $\lambda^+ > \lambda$, we can build $\langle a_\alpha : \alpha \in \lambda \rangle$ this way. We need to see that for $\alpha \in \beta \in \lambda$, $|a_\alpha \cap a_\beta| < \omega$. First, if $\delta_\beta = \delta_\alpha$ then $a'_\beta \neq a'_\alpha$, so $|a_\alpha \cap a_\beta| < \omega$. Suppose $\delta_\beta \in \delta_\alpha$. Then by minimality of δ_α, we must have $|\{a \in \mathcal{A}_{\delta_\beta} : |a \cap x_\alpha| = \omega\}| < \lambda^+$. Then by hypothesis $|a'_\beta \cap x_\alpha| < \omega$, so necessarily $|a_\alpha \cap a_\beta| < \omega$. Similarly, if $\delta_\alpha \in \delta_\beta$ then $|\{a \in \mathcal{A}_{\delta_\alpha} : |a \cap x_\beta| = \omega\}| < \lambda^+$, so $|a'_\alpha \cap x_\beta| < \omega$ and as a result $|a_\alpha \cap a_\beta| < \omega$. □

Corollary 2.2.7. Let κ be a cardinal. Then $\mathbf{ref}([\kappa]^\omega / < \omega) = \mathbf{ref}([\kappa]^\omega) = 2^\omega$. More generally for any $\mu \geq \kappa$, $\mathbf{ref}([\mu]^\kappa) = \mathbf{ref}(\kappa)$.

Proof. Because $[\kappa]^\kappa \subseteq [\mu]^\kappa$, $\mathbf{ref}([\mu]^\kappa) \leq \mathbf{ref}(\kappa)$. On the other hand, suppose $\lambda \in \mathbf{ref}(\kappa)$ and suppose $X = \langle x_\alpha : \alpha \in \lambda \rangle \subseteq [\mu]^\kappa$. Let $X_0 = \{x_\alpha \in X : |x_\alpha \cap x_0| = \kappa\}$. We can almost disjointly refine $\{x_\alpha \cap x_0 : x_\alpha \in X_0\}$ in $[x_0]^\kappa$. Then take the minimal β_1 such that $x_{\beta_1} \notin X_0$ and let $X_1 = \{x_\alpha \in X \setminus X_0 : |x_\alpha \cap x_{\beta_1}| = \kappa\}$. As before, we can almost disjointly refine $\{x_\alpha \cap x_{\beta_1} : x_\alpha \in X_1\}$ in $[x_{\beta_1}]^\kappa$. It is clear that we can proceed in this fashion, and the resulting $\langle a_\alpha : \alpha \in \lambda \rangle$ will be an almost disjoint refinement of $\langle x_\alpha : \alpha \in \lambda \rangle \subseteq [\mu]^\kappa$. □

Proposition 2.2.8. For any cardinal κ, $\sup(\mathrm{AD}(\kappa)) \leq \mathbf{ref}(\kappa)$. If $\sup(\mathrm{AD}(\kappa)) \notin \mathrm{AD}(\kappa)$ or $\sup(\mathrm{AD}(\kappa)) = 2^\kappa$ then $\sup(\mathrm{AD}(\kappa)) = \mathbf{ref}(\kappa)$.

Proof. Trivially $\mathbf{ref}(\kappa) \leq 2^\kappa$ and $\mathbf{ref}(\kappa) \leq \lambda$ if $\lambda \notin \mathrm{AD}(\kappa)$, so if $\sup(\mathrm{AD}(\kappa)) \leq \mathbf{ref}(\kappa)$, then of course if $\sup(\mathrm{AD}(\kappa)) \notin \mathrm{AD}(\kappa)$ or $\sup(\mathrm{AD}(\kappa)) = 2^\kappa$, $\sup(\mathrm{AD}(\kappa)) = \mathbf{ref}(\kappa)$. The proof here is identical to that in 2.2.6. Let $\lambda = \sup(\mathrm{MAD}(\kappa))$, let $\mu < \lambda$ be a cardinal, and let $\langle x_\alpha : \alpha \in \mu \rangle \subseteq [\kappa]^\kappa$. Because $\mu < \lambda$, there exists $\langle \mathcal{A}_\alpha : \alpha \in \mu \rangle$ such that for every $\alpha \in \mu$, $\mathcal{A}_\alpha \subseteq [x_\alpha]^\kappa$ is an almost disjoint family of size μ^+. Next, for any $\alpha \neq \beta$ if $a \in \mathcal{A}_\alpha$ is such that $|a \cap x_\beta| = \kappa$ and $|\{a \in \mathcal{A}_\alpha : |a \cap x_\beta| = \kappa\}| < \mu^+$, remove a from $\mathcal{A}_\alpha$. So without loss of generality we may assume that for any $a \in \mathcal{A}_\alpha$ and $\beta \neq \alpha$, if $|a \cap x_\beta| = \kappa$ then $|\{a \in \mathcal{A}_\alpha : |a \cap x_\beta| = \kappa\}| = \mu^+$. Build our almost disjoint refinement

$\langle a_\alpha : \alpha \in \mu \rangle$ by induction as in 2.2.6. At stage α choose the minimal $\delta_\alpha \leq \alpha$ such that $|\{a \in \mathcal{A}_{\delta_\alpha} : |a \cap x_\alpha| = \kappa\}| = \mu^+$, pick some $a'_\alpha \in \mathcal{A}_{\delta_\alpha}$ which hasn't been chosen before, and set $a_\alpha = a'_\alpha \cap x_\alpha$. As before, $\langle a_\alpha : \alpha \in \mu \rangle$ is an almost disjoint refinement. □

2.2.3 Almost disjoint refinement when adding κ-reals

For $V \subseteq M$ two models of ZFC, a natural question that one can ask (as long as $[\kappa]^\kappa \setminus V \neq \emptyset$ in M) is whether or not $[\kappa]^\kappa \cap V$ admits an almost disjoint refinement in M. For the case where $\kappa = \omega$, this question was asked by L. Soukup and answered by Brendle, and independently by Balcar and Pazák, in the affirmative. The methods used are quite different, and we generalize both of them in this chapter.

Theorem (Brendle [62], Balcar and Pazák [4]) 2.2.9. *If $V \subseteq M$ are models of ZFC and there exists $x \in ([\omega]^\omega)^M \setminus V$, then $[\omega]^\omega \cap V$ has an almost disjoint refinement in M.*

Because $P_\omega\omega \cap V = P_\omega\omega \cap M$, $(P(\omega)/<\omega)^V$ is a subalgebra of $(P(\omega)/<\omega)^M$. We investigate the following analogous question and several related ones:

Question 2.2.10. Let $V \subseteq M$ be models of ZFC with $(P_\kappa\kappa)^V = (P_\kappa\kappa)^M$ and $([\kappa]^\kappa)^M \setminus V \neq \emptyset$. Under what circumstances does $([\kappa]^\kappa)^V$ have an almost disjoint refinement in M?

2.3 Regular subalgebras and semidistributivity

2.3.1 Definitions and basic observations

Suppose $V \subseteq M$ are models of ZFC. If $([\kappa]^\kappa)^V$ is to be almost disjointly refined in M, then certainly it must be the case that not for every $y \in ([\kappa]^\kappa)^M$ does there exist $x_y \in ([\kappa]^\kappa)^V$ with $x_y \subseteq y$, because then if $a \in ([\kappa]^\kappa)^M$ is an element of any almost disjoint collection $\mathcal{A} \in M$, for some $x_a \in [a]^\kappa \cap V$, there exists $\{x_a^0, x_a^1\} \subseteq ([x_a]^\kappa)^V$ with $x_a^0 \cap x_a^1 = \emptyset$. But then there cannot exist $a' \in \mathcal{A}$ with (for example) $a' \subseteq x_a^0$. One may also observe that any in extension M with this property, so long as $(P_\kappa\kappa)^V = (P_\kappa\kappa)^M$, $(P(\kappa)/<\kappa)^V$ is a regular subalgebra of $(P(\kappa)/<\kappa)^M$. Indeed, in any extension M with this property, there is a dense embedding $(P(\kappa)/<\kappa)^V \to (P(\kappa)/<\kappa)^M$. The following definitions are relevant.

Definition (Hrušák [34]) 2.3.1. Suppose $V \subseteq M$ are models of ZFC. Say that M is (κ,λ)-semidistributive over V, or just (κ,λ)-semidistributive if V is clear, if and only if for every $y \in ([\lambda]^\lambda)^M$, there exists $x_y \in ([\lambda]^\kappa)^V$ such that $x_y \subseteq y$. If $\mathbb{P}$ is a forcing notion, similarly say that $\mathbb{P}$ is (κ,λ)-semidistributive if and only if (necessarily) $V[G]$ is (κ,λ)-semidistributive over V.

Definition 2.3.2. Let $\mathbb{A}$ and $\mathbb{B}$ be Boolean algebras with $\mathbb{A}$ a subalgebra of $\mathbb{B}$. Say that $\mathbb{A}$ is a regular subalgebra of $\mathbb{B}$ if and only if any of the following equivalent conditions holds:

1. Every maximal antichain in $\mathbb{A}$ remains maximal in $\mathbb{B}$.
2. For every $b \in \mathbb{B}^+$, there exists $a_b \in \mathbb{A}^+$ such that for every $x \in \mathbb{A}^+$, if $x \leq_{\mathbb{A}} a_b$, then $x \wedge b \neq 0$. This a_b is called a pseudoprojection or a reduct of b.
3. Every predense subset in $\mathbb{A}$ is a predense subset in $\mathbb{B}$.
4. If G is a $(V,\mathbb{B})$-generic filter, then $G \cap \mathbb{A}$ is a $(V,\mathbb{A})$-generic filter.

If $j : \mathbb{A} \to \mathbb{B}$ is an embedding, say that j is a regular embedding if and only if $j''\mathbb{A} \subseteq \mathbb{B}$ is a regular subalgebra of $\mathbb{B}$. Similarly, say that j is a dense embedding if and only if $j''\mathbb{A}$ is a dense subalgebra of $\mathbb{B}$. If $\mathbb{A} \subseteq \mathbb{B}$ is dense, it is clear that $\mathbb{A}$ is a regular subalgebra of $\mathbb{B}$.

Note 2.3.3. As noted, if M is (κ,κ)-semidistributive over V then there exists $j : (P(\kappa)/<\kappa)^V \to (P(\kappa)/<\kappa)^M$ a dense embedding, via $j(([a]_{<\kappa})^V) = ([a]_{<\kappa})^M$ for every $a \in (P(\kappa))^V$. So in particular, if M is (κ,κ)-semidistributive over V and $(P_\kappa\kappa)^V = (P_\kappa\kappa)^M$, then $(P(\kappa)/<\kappa)^V$ is a dense, and so regular, subalgebra of $(P(\kappa)/<\kappa)^M$.

Example 2.3.4. $\mathbb{P} = \text{Fn}(\omega, 2, < \omega)$, i.e. Cohen forcing, is (ω_1, ω_1)-semidistributive.

Proof. This is a simple consequence of $\mathbb{P}$ being countable. If $y \in ([\omega_1]^{\omega_1})^{V[G]}$ then fixing $\dot{y}$ a $\mathbb{P}$-name for y, for every $\alpha \in y$ there exists $p \in G$ so that $p \Vdash \alpha \in \dot{y}$, so because $|\mathbb{P}| = \omega$, there exists $p \in G$ so that for uncountably many $\alpha \in \omega_1$, $p \Vdash \alpha \in \dot{y}$. And so $x_y = \{\alpha : p \Vdash \alpha \in \dot{y}\} \subseteq y$ with $x_y \in ([\omega_1]^{\omega_1})^V$. $\square$

Example 2.3.5. Many common forcings to add reals are (ω, ω_1)-semidistributive, and in certain models are even (ω_1, ω_1)-semidistributive. For example, Hechler forcing, Mathias forcing, Laver forcing, the Random Real forcing, and Sacks forcing are all (ω, ω_1)-semidistributive. In some models these forcings may be (ω_1, ω_1)-semidistributive. For example, if PFA holds then Sacks forcing is (ω_1, ω_1)-semidistributive (see [34] for details).

2.3.2 $(P(\kappa)/ < \kappa)^V$ a regular subalgebra of $(P(\kappa)/ < \kappa)^M$

Suppose $V \subseteq M$ are two models of ZFC with $(P_\kappa \kappa)^V = (P_\kappa \kappa)^M$ and $([\kappa]^\kappa)^M \setminus V \neq \emptyset$. Balcar and Pazák's argument in [4] for 2.2.9 begins by observing that in this case for $\kappa = \omega$, $(P(\omega)/ < \omega)^V$ is a not a regular subalgebra of $(P(\omega)/ < \omega)^M$. Assuming $(2^{<\kappa} = \kappa)^V$, this also holds for κ generally, as well as the analogous statement with the bounded ideal.

Proposition 2.3.6. Let κ be a cardinal in V with $(2^{<\kappa} = \kappa)^V$. Let $\mathcal{I}_1 = P_\kappa \kappa$, that is the ideal of subsets of κ of size $< \kappa$. Let $\mathcal{I}_2 = \{z \in P(\kappa) : \sup(z) < \kappa\}$, that is the ideal of bounded subsets of κ. Note that κ is regular if and only if $\mathcal{I}_1 = \mathcal{I}_2$. Let $M \supseteq V$ be any outer model. Then if $(\mathcal{I}_i)^V = (\mathcal{I}_i)^M$ but $(\mathcal{I}_i^+)^M \setminus V \neq \emptyset$, $(P(\kappa)/\mathcal{I}_i)^V$ is not a regular subalgebra of $(P(\kappa)/\mathcal{I}_i)^M$, for each $i \in \{0, 1\}$.

Proof. Fix $i \in \{0, 1\}$. Work initially in V. Fix a bijection $f : {}^{<\kappa}2 \to \kappa$ with the properties that for any $t \in {}^{<\kappa}2$, $\sup(\{f(t \restriction \alpha) : \alpha \leq \text{lh}(t)\}) < \kappa$, and for every $\alpha \in \kappa$, there exits $\beta \in \kappa$ such that for every $t \in {}^{<\kappa}2$ with $\text{lh}(t) \geq \beta$, $f(t) > \alpha$. First note that this is possible, as follows. Suppose first that κ is regular. Then both conditions are satisfied by any bijection f automatically, the first because if $x \in P_\kappa({}^{<\kappa}2)$, then $\sup\{f''x\} < \kappa$ and the second because if $\alpha \in \kappa$, $\sup\{\gamma : \text{for some } \beta \leq \alpha, \mathrm{lh}(f^{-1}(\beta)) = \gamma\} < \kappa$, both by regularity. On the other hand, suppose κ is singular. Then not necessarily are both conditions automatically satisfied, though we can ensure that they are for a particular bijection f. Because $2^{<\kappa} = \kappa$ and κ is singular, for every $\gamma \in \kappa$, $2^\gamma < \kappa$, i.e. κ is a strong limit. Fix $\langle \mu_\alpha : \alpha \in \lambda \rangle$ a continuous increasing cofinal sequence of cardinals in κ such that for every $\alpha \in \lambda$, $2^{\mu_\alpha} = \mu_{\alpha+1}$. By recursion define f as follows. Let $f \restriction {}^{\leq\mu_0}2 : {}^{\leq\mu_0}2 \to \mu_{\alpha+1}$ be a bijection. And generally at stage $\xi + 1 \in \lambda$, having defined $f \restriction {}^{\leq\mu_\xi}2 \to \mu_{\xi+1}$ a bijection, let $f \restriction ({}^{\leq\mu_{\xi+1}}2 \setminus {}^{\leq\mu_\xi}2) : {}^{\leq\mu_{\xi+1}}2 \setminus {}^{\leq\mu_\xi}2 \to \mu_{\xi+2} \setminus \mu_{\xi+1}$ be a bijection. At limit stages $\xi \in (0, \lambda)$, having defined $f \restriction {}^{<\mu_\xi}2 : {}^{<\mu_\xi}2 \to \mu_\xi$ a bijection, let $f \restriction {}^{\mu_\xi}2 : {}^{\mu_\xi}2 \to \mu_{\xi+1} \setminus \mu_\xi$ be a bijection. Such an f satisfies our two requirements. Next, let $x \in (\mathcal{I}_i^+)^M \setminus V$. Identify x with its characteristic function $\chi_x \in {}^\kappa 2$, i.e. $x \in {}^\kappa 2$. Let $y = \{f(x \restriction \alpha) : \alpha \in \kappa\} \in (\mathcal{I}_i^+)^M$. We show that y has no reduct in V. Let $z \in (\mathcal{I}_i^+)^V$. Let $T \subseteq {}^{<\kappa}2$ be the tree induced by $f^{-1}[z]$, that is $T = \{t \in {}^{<\kappa}2 : \text{for some } s \in f^{-1}[z] \text{ and } \alpha \in (\mathrm{lh}(s)+1),\ s \restriction \alpha = t\}$. If for any $\alpha \in \kappa$, $\mathrm{Lev}_\alpha(T) = \emptyset$, then necessarily $z \cap y \in \mathcal{I}_i$, because of the first property of f. On the other hand, if for any $\alpha \in \kappa$, $|\mathrm{Lev}_\alpha(T)| = \kappa$, then there exists $z' \in ([f^{-1}[z]]^\kappa)^V$ such that z' is an antichain and $\mathrm{lh}(s) \geq \alpha$ for every $\alpha \in z'$. However, then $|f''z' \cap y| \leq 1$, so z cannot be a reduct in this case either (as $f''z' \in (\mathcal{I}_i^+)^V$ and $f''z' \subseteq z$). So we may assume that $|\mathrm{Lev}_\alpha(T)| \in (0, \kappa)$ for every $\alpha \in \kappa$. Suppose first that $x \notin [T]$. Then for some $\alpha \in \kappa$, $x \restriction \alpha \notin T$. Let $z' = \{s \in T : \mathrm{lh}(s) \geq \alpha\}$. Necessarily $f''z' \in (\mathcal{I}_i^+)^V$, and $|f''z' \cap y| = 0$. So we may assume that $x \in [T]$. Then necessarily for every $\alpha \in \kappa$, there exists $a_\alpha \in \mathrm{Lev}_\alpha(T)$ such that $f''\{s \in f^{-1}[z] : \mathrm{lh}(s) \geq \alpha \text{ and } s \restriction \alpha = a_\alpha\} \in (\mathcal{I}_i^+)^V$, because $x \restriction \alpha \in \mathrm{Lev}_\alpha(T)$, $y \setminus \{f(x \restriction \beta) : \beta \in \alpha\} \in (\mathcal{I}_i^+)^M$, $y \setminus \{f(x \restriction \beta) : \beta \in \alpha\} \subseteq f''\{s \in f^{-1}[z] : \mathrm{lh}(s) \geq \alpha \text{ and } s \restriction \alpha = a_\alpha\}$, and $(\mathcal{I}_i)^V = (\mathcal{I}_i)^M$. If for any $\alpha \in \kappa$ there exists distinct $a, b \in \mathrm{Lev}_\alpha(T)$ such that $f''\{s \in f^{-1}[z] : \mathrm{lh}(s) \geq \alpha \text{ and } s \restriction \alpha = a\} \in (\mathcal{I}_i^+)^V$ and $f''\{s \in f^{-1}[z] : \mathrm{lh}(s) \geq \alpha \text{ and } s \restriction \alpha = b\} \in (\mathcal{I}_i^+)^V$, then without loss of generality e.g. $x \restriction \alpha \neq a$, so $|f''\{s \in f^{-1}[z] : \mathrm{lh}(s) \geq \alpha \text{ and } s \restriction \alpha = a\} \cap y| = 0$. Then for each $\alpha \in \kappa$, there exists a unique $a_\alpha \in \mathrm{Lev}_\alpha(T)$ such that $f''\{s \in f^{-1}[z] : \mathrm{lh}(s) \geq \alpha \text{ and } s \restriction \alpha = a_\alpha\} \in (\mathcal{I}_i^+)^V$. Let $z \in ({}^\kappa 2)^V$ be defined by $z \restriction \alpha = a_\alpha$ (it is clear that for $\alpha \in \beta \in \kappa$, $a_\beta \restriction \alpha = a_\alpha$). Necessarily of course we have $z = x$ which is a contradiction, though to be explicit because $x \notin V$, for some $\alpha \in \kappa$, $x \restriction \alpha \neq z \restriction \alpha$. But then $f''\{s \in f^{-1}[z] : \mathrm{lh}(s) \geq \alpha \text{ and } s \restriction \alpha = z \restriction \alpha\} \in (\mathcal{I}_i^+)^V$ and $f''\{s \in f^{-1}[z] : \mathrm{lh}(s) \geq \alpha \text{ and } s \restriction \alpha = z \restriction \alpha\} \cap y \in \mathcal{I}_i$, so y has no reduct in V. $\square$

2.3.3 Density observations

We saw in 2.3.6 that if $2^{<\kappa} = \kappa$ then in particular no outer model preserving the ideal of subsets of κ of size $< \kappa$ but adding a new subset of κ can be (κ, κ)-semidistributive, because $(P(\kappa)/ < \kappa)^V$ is not a regular subalgebra of $(P(\kappa)/ < \kappa)^M$. In this section we make some observations about how the density properties of a forcing notion $\mathbb{P}$ can influence whether or not $\mathbb{P}$ is (κ, κ)-semidistributive. We first need a definition.

Definition 2.3.7. For a forcing notion $\mathbb{P}$, let the density of $\mathbb{P}$, $d(\mathbb{P})$, denote the smallest cardinality of a dense subset of $\mathbb{P}$.

Proposition 2.3.8. If $d(\mathbb{P}) = \kappa$, then $\mathbb{P}$ is (κ^+, κ^+)-semidistributive. More generally, if G is $(V, \mathbb{P})$-generic and there exists $p \in \mathbb{P} \cap G$ such that $d(\mathbb{P} \restriction p) = \kappa$, then $V[G]$ is an (κ^+, κ^+)-semidistributive extension of V.

Proof. This proof is much the same as 2.3.4. Suppose G is $(V, \mathbb{P})$-generic and there exists $p \in \mathbb{P} \cap G$ so that $d(\mathbb{P} \restriction p) = \kappa$. Choose $D \subseteq \mathbb{P} \restriction p$ dense with $|D| = \kappa$. Let $\dot{x}$ be a name for an element $x \in ([\kappa^+]^{\kappa^+})^{V[G]}$. Then for every $\alpha \in x$, there exists $p_1 \in G$ with $p_1 \Vdash \alpha \in \dot{x}$, so there exists $p_2 \in G$ with $p_2 \leq p_1$ and $p_2 \leq p$. But D is dense below p_2 in particular, so there exists $p_3 \leq p_2$ with $p_3 \in D \cap G$. Therefore $x = \{\alpha \in \kappa^+ : \exists q \in D \cap G \text{ with } q \Vdash \alpha \in \dot{x}\}$. Because $|x| = \kappa^+$ and $|D| = \kappa$, there must be $q \in D \cap G$ with $|\{\alpha : q \Vdash \alpha \in \dot{x}\}| = \kappa^+$. However, this set is in V. $\square$

Proposition 2.3.9. Let κ be a cardinal and $\mathbb{P}$ be a separative forcing notion. Then if G is $(V, \mathbb{P})$-generic and there exists $p \in \mathbb{P} \cap G$ such that for every $q \leq p$, $d(\mathbb{P} \restriction q) = \kappa$, and $(\kappa \text{ is regular})^{V[G]}$, then $V[G]$ is not a (κ, κ)-semidistributive extension.

Proof. Let κ be regular, $\mathbb{P}$ be separative, and let G be $(V, \mathbb{P})$-generic with $p \in \mathbb{P} \cap G$ such that for every $q \leq \mathbb{P}$, $d(\mathbb{P} \restriction q) = \kappa$. Let $D \subseteq \mathbb{P} \restriction p$ be dense of size κ and enumerate $D = \langle p_\alpha : \alpha \in \kappa \rangle$. By recursion, define $E \subseteq D$ as follows. First let $p_0 \in E$. At stage β, having determined whether or not each of $\langle p_\gamma : \gamma \in \beta \rangle$ is in E, if for some $\gamma \in \beta$, $p_\gamma \in E$ and $p_\gamma \leq p_\beta$, let $p_\beta \notin E$. Otherwise let $p_\beta \in E$. It is not difficult to see that E is dense below p, and so of size κ. Enumerate $E = \langle p'_\alpha : \alpha \in \kappa \rangle$. By construction if $\alpha \in \beta \in \kappa$ then $\neg(p'_\alpha \leq p'_\beta)$. Let $\dot{x}$ be a name for the set $x = \{\alpha : p'_\alpha \in E \cap G\}$. We argue that $x \in ([\kappa]^\kappa)^{V[G]}$ and for no $y \in ([\kappa]^\kappa)^V$ is $y \subseteq x$. First, suppose towards a contradiction that $x \in (P_\kappa \kappa)^{V[G]}$. By regularity of κ in $V[G]$, for some $\beta \in \kappa$, $x \subseteq \beta$, and the set of conditions $s \in E$ such that for some $\gamma \in \kappa \setminus \beta$, $s \leq p'_\gamma$ or for every $\gamma \in \kappa \setminus \beta$, $s \perp p'_\gamma$, is dense below p. So there exists some such $s \in G$, and so then necessarily for some $\alpha \in x$, $s = p'_\alpha$. Either $p'_\alpha \leq p'_\gamma$ for some $\gamma \in \kappa \setminus \beta$ or for every $\gamma \in \kappa \setminus \beta$, $p'_\alpha \perp p'_\gamma$. We cannot have $p'_\alpha \leq p'_\gamma$ for some $\gamma \in \kappa \setminus \beta$, because $x \subseteq \beta$ and so $\alpha < \gamma$, which means that $\neg(p'_\alpha \leq p'_\gamma)$. Therefore for every $\gamma \in \kappa \setminus \beta$, $p'_\alpha \perp p'_\gamma$. However, then $E \cap (\mathbb{P} \restriction p'_\alpha) \subseteq \{p'_\delta : \delta \in \beta\} \subseteq \mathbb{P} \restriction p'_\alpha$ is dense, a contradiction because we assumed $d(\mathbb{P} \restriction p'_\alpha) = \kappa$. Thus $x \in ([\kappa]^\kappa)^{V[G]}$. Next, suppose towards a contradiction that for some $y \in P(x) \cap V$ with y unbounded in κ, $y \subseteq x$. Then for some $r_1 \in G$, $r_1 \Vdash y \subseteq \dot{x}$. There exists $r_2 \in G$ with $r_2 \leq r_1$ and $r_2 \leq p$. Because E is dense below r_2, there exists some $q \in E \cap G$ with $q \leq r_2, r_1, p$. So in particular, $q \Vdash y \subseteq \dot{x}$. But then $q \Vdash p'_\gamma \in G$ for every $\gamma \in y$. Because $\mathbb{P}$ is separative, we must have that $q \leq p'_\gamma$ for every $\gamma \in y$. However, $q \in E$ so for some $\alpha \in \kappa$, $q = p'_\alpha$. However, because y is unbounded in κ there exists $\gamma \in y \setminus (\alpha + 1)$, and by construction of E, $\neg(p'_\alpha \leq p'_\gamma)$, a contradiction. $\square$

Corollary 2.3.10. Suppose $|\mathbb{P}| = \omega_1$. Passing to $\mathbb{P}$'s separative quotient we may assume that $\mathbb{P}$ is separative. Let G be $(V, \mathbb{P})$-generic. The set of conditions p such that there does not exist $q \leq p$ with $d(\mathbb{P} \restriction q) < d(\mathbb{P} \restriction p)$ is dense because there are no infinitely descending sequences of ordinals. So there exists such a $p \in G$. If $d(\mathbb{P} \restriction p) = \omega$, then necessarily $V[G]$ contains a Cohen real over V. If $d(\mathbb{P} \restriction p) = \omega_1$, then as long as ω_1 remains regular in $V[G]$, $V[G]$ is not (ω_1, ω_1)-semidistributive by 2.3.9. In 2.3.5 we saw that e.g. if PFA holds in V then Sacks forcing is (ω_1, ω_1)-semidistributive (and doesn't collapse ω_1 and doesn't add a Cohen real), so necessarily we must have that 2^ω, the size of Sacks forcing, is greater than ω_1, as indeed is the case here. Additionally, in the context of refining $(P(\omega_1)/ < \omega_1)^V$ in $V[G]$, as 2.3.9 shows that any (ω_1, ω_1)-semidistribtive forcing of size ω_1 not collapsing ω_1 must add a Cohen real, there are no (ω_1, ω_1)-semidistributive forcings adding a subset of ω_1 of size ω_1 which don't add reals. This improves that consequence of 2.3.6 in the particular case where $|\mathbb{P}| = \omega_1$ and the CH holds in V, because the CH is not required to hold in V here.

2.3.4 Using a chain condition

We saw in 2.3.10 that for V a model of ZFC, there cannot exist a forcing notion $\mathbb{P} \in V$ such that $|\mathbb{P}| = \omega_1$ and for some G $(V, \mathbb{P})$-generic, $(P(\omega_1) \setminus V \neq \emptyset)^{V[G]}$, $(P_{\omega_1}\omega_1)^V = (P_{\omega_1}\omega_1)^{V[G]}$, and $V[G]$ is (ω_1, ω_1)-semidistributive over V. Note of course that if $(P_{\omega_1}\omega_1)^V = (P_{\omega_1}\omega_1)^{V[G]}$ then ω_1 remains uncountable in $V[G]$. Using a different method, we can strengthen this result and show that this holds for any $\mathbb{P}$ which is ω_2-c.c. as follows.

Proposition 2.3.11. Let $\mathbb{P}$ be ω_2-c.c., add a new uncountable subset of ω_1, and not add reals. Then $\mathbb{P}$ is not (ω_1, ω_1)-semidistributive. More generally, if $\mathbb{P}$ is κ^+-c.c., adds a new subset of κ of size κ, but doesn't add any subsets of κ of size $< \kappa$, then $\mathbb{P}$ is not (κ, κ)-semidistributive.

Proof. We prove something more general. Let $\dot{x}$ be a name for a new element of ${}^\kappa 2$. Let λ be sufficiently larger than κ. Because $\mathbb{P}$ is κ^+-c.c, if $N \prec (H_\lambda, \mathbb{P}, \dot{x}, \ldots)$ with $|N| = \kappa$ and

$\kappa \subseteq N$, if G is $(V,\mathbb{P})$-generic, then G is $(N,\mathbb{P})$-generic (because $\mathbf{1}_{\mathbb{P}}$ is a master condition for every such N). So $N[G] \cap V = N \cap V$. Consider the tree $T \subseteq {}^{<\kappa}2$ in V where $\mathrm{Lev}_\alpha(T)$ consists of the functions $f \in {}^{\alpha}2 \cap N$. This tree is indeed a tree and has cardinality and height κ. Fix a bijection $g : T \to \kappa$ in V. Because $\dot{x}^G \in N[G]$, $\dot{x}^G \restriction \alpha \in N[G]$ for every $\alpha \in \kappa$, and so because $\mathbb{P}$ doesn't add subsets of κ of size $< \kappa$, $\dot{x}^G \restriction \alpha \in V$, so $\dot{x}^G \restriction \alpha \in N$. Therefore $\dot{x}^G \in [T]$. Consider $y = \{g(\dot{x}^G \restriction \alpha) : \alpha \in \kappa\} \in ([\kappa]^\kappa)^{V[G]}$. If there existed $x \subseteq y$ with $x \in ([\kappa]^\kappa)^V$, then $g^{-1}[x] \subseteq T$ is a set of comparable nodes with unbounded height. But then $\dot{x}^G \in V$ as it can be defined from $g^{-1}[x]$, a contradiction. $\square$

2.3.5 No (ω_1, ω_1)-semidistributive extensions and $\neg CH$

We saw in particular in 2.3.6 that if the CH holds in V then if $M \supseteq V$ is any extension not adding reals but adding a new subset of ω_1, $(P(\omega_1)/<\omega_1)^V$ is not a regular subalgebra of $(P(\omega_1)/<\omega_1)^M$. Consistently we can also have this result for generic extensions if $V \models 2^\omega = 2^{\omega_1} = \omega_2$, in particular if $V \models PFA$.

Observation 2.3.12. Let $\mathfrak{a}(\kappa)$ denote the minimal cardinality of a (nontrivial) maximal antichain in $P(\omega_1)/<\kappa$, that is $\mathfrak{a}(\kappa) = \min((MAD(\kappa) \cap (\kappa, 2^\kappa])$. Then for $V \subseteq M$ transitive models of ZFC with $(P_\kappa \kappa)^V = (P_\kappa \kappa)^M$, if $(|\mathfrak{a}(\kappa)^V| < \mathfrak{a}(\kappa))^M$, then $(P(\kappa)/<\kappa)^V$ is not a regular subalgebra of $(P(\kappa)/<\kappa)^M$. So also, of course, M is not a (κ,κ)-semidistributive extension.

Proof. By hypothesis some maximal antichain in $(P(\kappa)/<\kappa)^V$ is made no longer maximal in M. $\square$

Fact (Originally [67]) 2.3.13. If PFA holds in V, then if M is a generic extension of V which contains a new subset of ω_1, either M contains a new subset of ω or ω_2^V is not a cardinal in M.

For a recent nice, short, proof of 2.3.13 using guessing models, see [16]. So if PFA holds in V, by 2.3.13 because $(\mathfrak{a}(\omega_1) = \omega_2)^V$, any forcing not adding a real but adding a new subset of ω_1 must not yield in particular an (ω_1, ω_1)-semidistributive extension, because $(P(\kappa)/<\kappa)^V$ will not be a regular subalgebra of $(P(\kappa)/<\kappa)^{V[G]}$.

2.4 Refining $([\kappa]^\kappa)^V$ in $V[G]$

2.4.1 A method using diagonalization

In this section we generalize 2.2.9 to cardinals $\kappa > \omega$ in certain specific contexts. In order to motivate the general idea behind the proof of 2.4.10, an initial discussion of some aspects of our proof for 2.2.6 is perhaps helpful. In our proof for 2.2.6, we first split up every set we wanted to refine into an almost disjoint family of larger size than the collection of sets to be refined, then used a cardinality argument to thin out each of these families so that if any element of one of these families intersected nontrivially an element of the set we wanted to refine, then in fact a large collection of elements of that family did this also. Once this was done, we were able to assign sequentially to each element of the set we wanted to refine some element which hadn't been assigned at any previous stage from an almost disjoint family from the earliest such coordinate whose family contained many elements intersecting this element nontrivially. It is readily seen that an important feature of this process is the dichotomy present in an element in an almost disjoint family intersecting a set which is to be refined: either it does so trivially, or a large collection of other elements in that almost disjoint family also intersect that set nontrivially. The way this dichotomy was able to be ensured in 2.2.6 was because of the presence of almost disjoint subsets of ω larger than the length of the process. If, however, such a situation is impossible because for example we want to refine e.g. 2^ω-many sets, the question becomes how to ensure that we have a similar sort of dichotomy. In the context of outer models adding reals, this is indeed exactly what happens in certain settings, as we saw in our chapter on trees and forests: Mansfield's theorem in particular says that branches can only be added to trees $T \subseteq {}^{<\omega}2$ if those trees contain a copy of the complete binary tree ${}^{<\omega}2$, and if that is the case, then if one branch is added to T, in fact 2^{ω}-many branches are added, while otherwise the body of the tree remains in the ground model. Indeed, this dichotomy (more specifically the perfect set property for G_δ subsets of 2^{ω}) and associated diagonalization is exactly what Brendle uses in his proof of 2.2.9. Our proof of 2.4.10 is a "worked-out" version of Brendle's argument. We need a few preliminaries. Unless otherwise stated, we assume that κ is regular, and this may sometimes not be noted explicitly.

Lemma 2.4.1. Let κ be a regular cardinal with $2^{<\kappa} = \kappa$. Then there exists $A \subseteq [\kappa]^{\kappa}$ a collection of almost disjoint subsets of κ such that A is closed in the κ-box topology and $T_A \cong {}^{<\kappa}2$. Here as usual $T_A = \{s \in {}^{<\kappa}2 : \text{for some } a \in A, a \restriction \mathrm{lh}(s) = s\}$. So in particular, $[T_A] = A$ and $|A| = 2^{\kappa}$.

Proof. If we fix any injection $f : {}^{<\kappa}2 \to \kappa$, we will be able to produce $A \subseteq [\kappa]^{\kappa}$ an almost disjoint family of size 2^{κ} by letting $A = \{x_b \in [\kappa]^{\kappa} : b \in {}^{\kappa}2\}$ where if $b \in {}^{\kappa}2$, $x_b = \{f(b \restriction \alpha) : \alpha \in \kappa\}$. However, we have no guarantee that A is κ-closed and $T_A \cong {}^{<\kappa}2$. In order to do this, we need to choose our injection f carefully. Partition κ into κ-many disjoint subsets of κ of size κ, $\langle x_\alpha : \alpha \in \kappa\rangle \subseteq [\kappa]^{\kappa}$. Construct f by recursion. Having built $f \restriction {}^{<\alpha}2 : {}^{<\alpha}2 \to \bigcup_{\beta\in\alpha} x_\beta$ an injection, enumerate ${}^{\alpha}2 = \langle s_\gamma : \gamma \in \lambda\rangle$ for $\lambda \leq \kappa$. At stage $\gamma \in \lambda$, let $f(s_\gamma) \in x_\alpha$ be such that $f(s_\gamma) > f(s_\gamma \restriction \eta)$ for every $\eta \in \alpha$ and $f(s_\gamma) > f(s_\delta)$ for every $\delta \in \gamma$. Because κ is regular, this is possible, and we can proceed to build $f : {}^{<\kappa}2 \to \kappa$ an injection in this fashion, which has the property that if $\{s, t\} \subseteq {}^{<\kappa}2$, $\mathrm{lh}(s) < \mathrm{lh}(t)$, and $t \restriction \mathrm{lh}(s) = s$, then $f(s) < f(t)$. In other words, f is increasing along paths. Now, as before for $b \in {}^{\kappa}2$ let $x_b = \{f(b \restriction \alpha) : \alpha \in \kappa\}$ and let $A = \{x_b : b \in {}^{\kappa}2\} \subseteq [\kappa]^{\kappa}$. We show that A is κ-closed and $T_A \cong {}^{<\kappa}2$. To see that $T_A \cong {}^{<\kappa}2$, it suffices to show that T_A is cofinally splitting and has no maximal paths of length

$\mu \in \kappa$. It is not difficult to see that T_A is cofinally splitting, so we need only to see that T_A has no maximal paths of length $\mu \in \kappa$. Take $\langle s_\alpha : \alpha \in \mu \rangle \subseteq T_A$ with $\langle \delta_\alpha : \alpha \in \mu \rangle$ such that for every $\alpha \in \mu$, $\mathrm{lh}(s_\alpha) = \delta_\alpha$, and for $\alpha \in \beta \in \mu$, $\delta_\alpha < \delta_\beta$ and $s_\beta \restriction \delta_\alpha = s_\alpha$. We need to see that $\bigcup_{\alpha \in \mu} s_\alpha \in T_A$. For every $\alpha \in \mu$, there exists $x_{b_\alpha} \in A$ such that $x_{b_\alpha} \restriction \delta_\alpha = s_\alpha$. So for $\alpha \in \beta \in \mu$, $x_{b_\beta} \restriction \delta_\alpha = x_{b_\alpha} \restriction \delta_\alpha$. That is, $\{f(b_\beta \restriction \gamma) : \gamma \in \kappa\} \restriction \delta_\alpha = \{f(b_\alpha \restriction \gamma) : \gamma \in \kappa\} \restriction \delta_\alpha$. Because f is increasing along paths, $\langle f(b_\beta \restriction \gamma) : \gamma \in \kappa \rangle$ is an increasing sequence of ordinals, so for some minimal γ', we have $f(b_\beta \restriction \gamma_1') \geq \delta_\alpha$ for every $\gamma_1' \geq \gamma'$ and so for every $\gamma \in \gamma'$, $f(b_\beta \restriction \gamma) \in \delta_\alpha$. Similarly, for some minimal γ'', we have $f(b_\alpha \restriction \gamma_1'') \geq \delta_\alpha$ for every $\gamma_1'' \geq \gamma''$ and so for every $\gamma \in \gamma''$, $f(b_\alpha \restriction \gamma) \in \delta_\alpha$. But then because $\{f(b_\beta \restriction \gamma) : \gamma \in \kappa\} \restriction \delta_\alpha = \{f(b_\alpha \restriction \gamma) : \gamma \in \kappa\} \restriction \delta_\alpha$, we must have $\gamma' = \gamma''$. So, for every $\alpha \in \mu$, let $\gamma_\alpha \in \kappa$ be minimal such that $f(b_\alpha \restriction \gamma_1) \geq \delta_\alpha$ for every $\gamma_1 \geq \gamma_\alpha$ and for every $\gamma \in \gamma_\alpha$, $f(b_\alpha \restriction \gamma) \in \delta_\alpha$. Then for every $\alpha \in \beta \in \mu$, for every $\gamma \in \gamma_\alpha$, $f(b_\alpha \restriction \gamma) = f(b_\beta \restriction \gamma)$. So in fact, $b_\alpha \restriction \gamma = b_\beta \restriction \gamma$ for every $\gamma \in \gamma_\alpha$. Because $\langle \gamma_\alpha : \alpha \in \mu \rangle$ is $\leq$-increasing, if $s_\alpha' = b_\alpha \restriction \gamma_\alpha$ (or $s_\alpha' = b_\alpha \restriction (\gamma_\alpha - 1)$ in case γ_α is a successor) we have that $\langle s_\alpha' : \alpha \in \mu \rangle \subseteq {}^{<\kappa}2$ is $\subseteq$-increasing, and if $s = \bigcup_{\alpha \in \mu} s_\alpha'$, then for any $b \in {}^{\kappa}2$ with $b \restriction \mathrm{lh}(s) = s$, it is not difficult to see that $x_b \restriction \bigcup_{\alpha \in \mu} \delta_\alpha = \bigcup_{\alpha \in \mu} s_\alpha \in T_A$, as desired. In order to see that A is κ-closed, it suffices to show that $[T_A] \subseteq A$. The argument is very similar to the above, except $\mu = \kappa$ in this case. Specifically, let $\bar{b} \in [T_A]$. So for every $\alpha \in \kappa$, there exists $x_{b_\alpha} \in A$ such that $x_{b_\alpha} \restriction \alpha = \bar{b} \restriction \alpha$. Then for every $\alpha \in \beta \in \kappa$, $x_{b_\beta} \restriction \alpha = x_{b_\alpha} \restriction \alpha$. That is, $\{f(b_\beta \restriction \gamma) : \gamma \in \kappa\} \restriction \alpha = \{f(b_\alpha \restriction \gamma) : \gamma \in \kappa\} \restriction \alpha$. Because f is increasing along paths as before, we must have that for some γ_α, $f(b_\beta \restriction \gamma) \in \alpha$ for every $\gamma \in \gamma_\alpha$, $f(b_\beta \restriction \gamma) \geq \alpha$ for every $\gamma \geq \gamma_\alpha$, $f(b_\alpha \restriction \gamma) \in \alpha$ for every $\gamma \in \gamma_\alpha$, and $f(b_\alpha \restriction \gamma) \geq \alpha$ for every $\gamma \geq \gamma_\alpha$. Note that $\langle s_\alpha' : \alpha \in \kappa \rangle$ as defined before is a (cofinal in this case) subset of a branch $b \in [{}^{<\kappa}2]$, and it is not difficult to see that $x_b = \bar{b}$. $\square$

Definition 2.4.2. In standard terminology, a G_δ subset of e.g. the Cantor space 2^ω refers to a set which is a countable intersection of open sets (in the ω-box topology). Analogously, say that $B \subseteq {}^{\kappa}2$ is G_{δ_κ} if and only if B is formed taking the intersection of κ-many open sets. Here of course we mean open in the κ-box topology.

Observation 2.4.3. Let κ be regular. Let $O \subseteq {}^{\kappa}2$ be open in the κ-box topology. Then there exists an antichain $\mathcal{A}_O \subseteq {}^{<\kappa}2$ such that $O_{\mathcal{A}_O} = O$. Here of course for $X \subseteq {}^{<\kappa}2$, $O_X = \{b \in {}^{\kappa}2 : \text{for some } s \in X, b \restriction \mathrm{lh}(s) = s\}$. Such an $\mathcal{A}_O$ is not unique, and we may take, for example, if $\alpha \in \kappa$, some $\mathcal{A}_O$ such that $\mathrm{lh}(s) \geq \alpha$ for every $s \in \mathcal{A}_O$. Furthermore, if $O_2 \subseteq O_1$ are open subsets of ${}^{<\kappa}2$, then we may take $\mathcal{A}_{O_2}$ to be a refinement of $\mathcal{A}_{O_1}$, namely if $s \in \mathcal{A}_{O_2}$, then there exists $t \in \mathcal{A}_{O_1}$ such that $\mathrm{lh}(t) \leq s$ and $s \restriction \mathrm{lh}(t) = t$.

Proof. Because κ is regular, $\{O_s : s \in {}^{<\kappa}2\}$ forms a basis for the κ-box topology over ${}^{\kappa}2$, where $O_s = \{b \in {}^{\kappa}2 : b \restriction \mathrm{lh}(s) = s\}$. For $\{s_1, s_2\} \subseteq {}^{<\kappa}2$, either $s_1 \restriction \mathrm{lh}(s_2) = s_2$, $s_2 \restriction \mathrm{lh}(s_1) = s_1$, or for some $\beta \in \mathrm{lh}(s_1) \cap \mathrm{lh}(s_2)$, $s_1(\beta) \neq s_2(\beta)$. It is not then difficult to see that if $\mathcal{O} = \bigcup\limits_{s \in \mathcal{A}} O_s$ for some $\mathcal{A} \subseteq {}^{<\kappa}2$, we can take $\mathcal{A}$ to be an antichain. Furthermore if $\alpha \in \kappa$, because for any $s \in {}^{<\kappa}2$ with $\alpha > \mathrm{lh}(s)$, $O_s = \bigcup\limits_{s' \in A'} O_{s'}$ where $A' = \{s' \in {}^{<\kappa}2 : \mathrm{lh}(s') = \alpha \text{ and } s' \restriction \mathrm{lh}(s) = s\}$, we may assume that e.g. $\mathrm{lh}(s) \geq \alpha$ for every $\alpha \in \mathcal{A}$. It is also straightforward to see that if $O_2 \subseteq O_1$ are open subsets of ${}^{<\kappa}2$, then we may take $\mathcal{A}_{O_2}$ to be a refinement of $\mathcal{A}_{O_1}$ such that, for example, $\mathcal{A}_{O_2} \cap \mathcal{A}_{O_2} = \emptyset$ (that is if, for example we have for some $s \in \mathcal{A}_{O_1}$, $O_s \subseteq O_2$, we will have, e.g. $\{s ^\frown 0, s ^\frown 1\} \subseteq \mathcal{A}_{O_2}$ and $s \notin \mathcal{A}_{O_2}$). $\square$

Definition 2.4.4. Say that a tree $T \subseteq {}^{<\kappa}({}^{<\kappa}2)$ codes a G_{δ_κ} subset $B \subseteq {}^{\kappa}2$ if and only if

1. $\langle \mathcal{A}_\alpha = \{s(\alpha) : s \in \mathrm{Lev}_{\alpha+1}(T)\} : \alpha \in \kappa \rangle$ is a sequence of antichains in ${}^{<\kappa}2$ such that if $\alpha \in \beta \in \kappa$ then $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset$. Note that this implies in particular in conjunction with the other requirements that $\mathrm{lh}(s) \geq \alpha$ for every $s \in \mathcal{A}_\alpha$.

2. For $\alpha \in \kappa$ and $\overline{s} \in \mathrm{Lev}_\alpha(T)$, let $\overline{s} ^\frown s \in \mathrm{Lev}_{\alpha+1}(T)$ if and only if $s \in \mathcal{A}_\alpha$ and for every $\beta \in \alpha$, $s \restriction \mathrm{lh}(\overline{s}(\beta)) = \overline{s}(\beta)$.

3. For $\alpha \in \lim(\kappa)$, $\mathrm{Lev}_\alpha(T) = \{\overline{s} \in {}^{\alpha}({}^{<\kappa}2) : \overline{s} \restriction \gamma \in \mathrm{Lev}_\gamma(T) \text{ for every } \gamma \in \alpha\}$. That is, $\mathrm{Lev}_\alpha(T) = [T \restriction \alpha]$.

4. $B = \bigcap_{\alpha\in\kappa} O_{\mathcal{A}_\alpha}$. Because the $\mathcal{A}_\alpha$'s are refining each other, $\langle O_{\mathcal{A}_\alpha} : \alpha \in \kappa\rangle$ is a $\subseteq$-descending sequence of open sets.

Note 2.4.5. As we have defined it in 2.4.4, any tree $T \subseteq {}^{<\kappa}({}^{<\kappa}2)$ coding a G_{δ_κ} subset $B \subseteq {}^{<\kappa}2$ is quite simple, in the sense that there are no maximal paths of limit length $< \kappa$. Any maximal path of length $< \kappa$ is of successor length $\alpha + 1$, and the final coordinate in this path corresponds to some $s \in \mathcal{A}_\alpha$ such that $O_s \cap O_{\mathcal{A}_{\alpha+1}} = \emptyset$.

Definition 2.4.6. Let $T \subseteq {}^{<\kappa}({}^{<\kappa}2)$ code a G_{δ_κ} subset $B \subseteq {}^{\kappa}2$. Define the coded body of T, $[[T]] \subseteq {}^{\kappa}2$, by $b \in [[T]]$ if and only if there exists $\bar{b} \in [T]$ such that $b \in \bigcap_{\alpha\in\kappa} O_{\bar{b}(\alpha)}$. In this case in fact $\{b\} = \bigcap_{\alpha\in\kappa} O_{\bar{b}(\alpha)}$.

Lemma 2.4.7. Let κ be regular and let $B \subseteq {}^{\kappa}2$ be G_{δ_κ}. Then there exists a tree $T \subseteq {}^{<\kappa}({}^{<\kappa}2)$ coding B such that $[[T]] = B$.

Proof. Suppose for some $\langle O_\alpha : \alpha \in \kappa\rangle$ a sequence of open sets, $B = \bigcap_{\alpha\in\kappa} O_\alpha$. Because a $(< \kappa)$-sized intersection of a collection of open sets is open, without loss of generality we may assume that for $\alpha \in \beta$, $O_\beta \subseteq O_\alpha$. It is not difficult to see by recursion along the lines of 2.4.3 that we can choose a sequence $\langle \mathcal{A}_\alpha : \alpha \in \kappa\rangle \subseteq {}^{<\kappa}2$ of antichains such that if $\alpha \in \beta \in \kappa$ then $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset$ such that $O_\alpha = O_{\mathcal{A}_\alpha}$ for every $\alpha \in \kappa$. It is not then difficult to see then that we can form a tree $T \subseteq {}^{<\kappa}({}^{<\kappa}2)$ satisfying all the conditions in 2.4.4. If $b \in B$, then to every $\mathcal{A}_\alpha$ there exists a unique $s_\alpha \in \mathcal{A}_\alpha$ such that $b \restriction \mathrm{lh}(s_\alpha) = s_\alpha$, but then $\bar{b} \in [T]$ where $\bar{b}(\alpha) = s_\alpha$ for every $\alpha \in \kappa$, and so $b \in [[T]]$. On the other hand, if $b \in [[T]]$ then for some $\bar{b} \in [T]$, $\{b\} = \bigcap_{\alpha\in\kappa} O_{\bar{b}(\alpha)}$. But then clearly $b \in \bigcap_{\alpha\in\kappa} O_\alpha$, i.e. $b \in B$. $\square$

Note 2.4.8. Let $B \subseteq {}^{\kappa}2$ be closed, so $[T_B] = T$, where $T_B \subseteq {}^{<\kappa}2$ is the tree generated by B as usual. Then B is G_{δ_κ} and we may take the tree $T \subseteq {}^{<\kappa}({}^{<\kappa}2)$ with $[[T]] = B$ to be canonical in the sense that we can take $\mathcal{A}_\alpha = \mathrm{Lev}_\alpha(T_B)$.

Observation 2.4.9. Fix $y \in [\kappa]^\kappa$. Then $B^y = \{a \in {}^{\kappa}2 : |a \cap y| = \kappa\}$ is G_{δ_κ}.

Proof. Note that if $\alpha \in \kappa$, $O_\alpha = \{a \in {}^\kappa 2 : \mathrm{otp}(a \cap y) \geq \alpha\}$ is open. This is because if $x \notin O_\alpha$, then for some $\beta \in \kappa$, $(x \cap [\beta, \gamma)) \cap (y \cap [\beta, \gamma)) = \emptyset$, i.e. $O_{x \restriction \beta} \cap O_\alpha = \emptyset$. Next, $B^y = \bigcap_{\alpha \in \kappa} O_\alpha$, so B^y is G_{δ_κ}. □

We are now ready to generalize 2.2.9 to the context of adding new subsets of κ.

Theorem 2.4.10. *Let κ be a regular cardinal with $2^{<\kappa} = \kappa$. If $\mathbb{P}$ is κ-strategically closed, G is $(V, \mathbb{P})$-generic, and $([\kappa]^\kappa)^{V[G]} \setminus V \neq \emptyset$, then $([\kappa]^\kappa)^V$ can be almost disjointly refined in $V[G]$.*

Proof. First, if $(2^\kappa)^V$ is no longer a cardinal in $V[G]$, by 2.2.8 $([\kappa]^\kappa)^V$ can be almost disjointly refined in $V[G]$. So assume that $(2^\kappa)^V = (2^\kappa)^{V[G]}$. In $V[G]$, enumerate $([\kappa]^\kappa)^V = \langle x_\alpha : \alpha \in \kappa \rangle$. In V, for every $\alpha \in \kappa$, fix a bijection $f_\alpha : x_\alpha \to \kappa$. Identifying x_α with κ via f_α, choose $A_\alpha \subseteq [x_\alpha]^\kappa$ to be a closed in the κ-box topology with $T_A \cong {}^{<x_\alpha}2$, which is possible by 2.4.1. So in particular, $[T_A] = A \subseteq {}^{x_\alpha}2$ and $|A| = 2^\kappa$. For every $y \in ([\kappa]^\kappa)^V$, let $B^y_{x_\alpha} = \{a \in A_\alpha : |a \cap y| = \kappa\}$. Because $B^y_{x_\alpha} = \{a \in {}^{x_\alpha}2 : |a \cap y| = \kappa\} \cap A_\alpha$, by 2.4.9 $B^y_{x_\alpha}$ is the intersection of a G_{δ_κ} set with a closed set, so $B^y_{x_\alpha}$ is G_{δ_κ}. By 2.4.7, there exists $T^y_{x_\alpha} \subseteq {}^{<\kappa}({}^{<x_\alpha}2)$ a tree coding $B^y_{x_\alpha}$ with $[[T^y_{x_\alpha}]] = B^y_{x_\alpha}$. Note that we have $([[T^y_{x_\alpha}]] = B^y_{x_\alpha})^V$ and $([[T^y_{x_\alpha}]] = B^y_{x_\alpha})^{V[G]}$. Because $\mathbb{P}$ is κ-strategically closed, by the discussion in the previous chapter, $([T^y_{x_\alpha}])^{V[G]} \neq ([T^y_{x_\alpha}])^V$ if and only if $T^y_{x_\alpha}$ contains a copy of ${}^{<\kappa}2$. And in that case, $|([T^y_{x_\alpha}])^{V[G]} \setminus V| = 2^\kappa$. Otherwise, $([T^y_{x_\alpha}])^{V[G]} = ([T^y_{x_\alpha}])^V$. It is not difficult to see that this implies also that $([[T^y_{x_\alpha}]])^{V[G]} \neq ([[T^y_{x_\alpha}]])^V$ if and only if $T^y_{x_\alpha}$ contains a copy of ${}^{<\kappa}2$. And in that case, $|([[T^y_{x_\alpha}]])^{V[G]} \setminus V| = 2^\kappa$. Otherwise, $([[T^y_{x_\alpha}]])^{V[G]} = ([[T^y_{x_\alpha}]])^V$. This is exactly the sort of dichotomy which is required to run the analogous argument to that in 2.2.6 or 2.2.8. So, by recursion we define in $V[G]$ a sequence $\langle a_\alpha : \alpha \in 2^\kappa \rangle \subseteq [\kappa]^\kappa$ such that if $\alpha \in \beta \in 2^\kappa$ then $|a_\alpha \cap a_\beta| < \kappa$ and for every $\alpha \in 2^\kappa$, $a_\alpha \in [x_\alpha]^\kappa$. At stage α, find the minimal γ_α such that $\neg([[T^{x_\alpha}_{x_{\gamma_\alpha}}]] \subseteq V)$. It is clear that we can do this, because if $\alpha = \gamma_\alpha$ then $B^{x_\alpha}_{x_\alpha} = A_\alpha$ and $T_{A_\alpha} \cong {}^{<\kappa}2$, so by 2.4.8 it is not difficult to see that $T^{x_\alpha}_{x_\alpha}$ contains a copy of ${}^{<\kappa}2$, and so $|([[T^{x_\alpha}_{x_\alpha}]])^{V[G]} \setminus V| = 2^\kappa$. Next, choose some $a'_\alpha \in [[T^{x_\alpha}_{x_{\gamma_\alpha}}]] \setminus V$ such that $a'_\alpha \neq a'_\gamma$ for any $\gamma \in \alpha$, and set $a_\alpha = x_\alpha \cap a'_\alpha$. Again, because if $\neg([[T^{x_\alpha}_{x_{\gamma_\alpha}}]] \subseteq V)$ then $|([[T^{x_\alpha}_{x_{\gamma_\alpha}}]])^{V[G]} \setminus V| = 2^\kappa$, it is clear that we can continue this procedure and define $\langle a_\alpha : \alpha \in 2^\kappa \rangle$ such that for every $\alpha \in 2^\kappa$, $a_\alpha \in [x_\alpha]^\kappa$. Let $\alpha \in \beta \in 2^\kappa$. If $\gamma_\alpha = \gamma_\beta = \gamma$, then because at stage β we set $a'_\beta \neq a'_\alpha$ and A_γ consists of almost disjoint sets, $|a'_\beta \cap a'_\alpha| < \kappa$ so necessarily $|a_\beta \cap a_\alpha| < \kappa$. On the other hand, suppose first instead that $\gamma_\alpha \in \gamma_\beta$. Then $a'_\alpha \in [[T^{x_\alpha}_{x_{\gamma_\alpha}}]] \setminus V$ and $[[T^{x_\beta}_{x_{\gamma_\alpha}}]] \subseteq V$. So $a'_\alpha \notin [[T^{x_\beta}_{x_{\gamma_\alpha}}]]$, but then $|a'_\alpha \cap x_\beta| < \kappa$ so necessarily $|a_\beta \cap a_\alpha| < \kappa$ too. Similarly, if $\gamma_\beta \in \gamma_\alpha$ then $a'_\beta \in [[T^{x_\beta}_{x_{\gamma_\beta}}]] \setminus V$ and $[[T^{x_\alpha}_{x_{\gamma_\beta}}]] \subseteq V$. So $a'_\beta \notin [[T^{x_\alpha}_{x_{\gamma_\beta}}]]$, i.e. $|a'_\beta \cap x_\alpha| < \kappa$, so necessarily $|a_\beta \cap a_\alpha| < \kappa$. $\square$

Note 2.4.11. An inspection of the proof of 2.4.10 shows that $2^{<\kappa} = \kappa$ itself is not necessary, it is only used to guarantee the existence of an almost disjoint collection $A \subseteq [\kappa]^\kappa$ which is κ-closed and such that T_A contains a copy of ${}^{<\kappa}2$. Indeed, in the case where $2^{<\kappa} = \kappa$ we could identify ${}^{<\kappa}2$ with κ and so consider trees $T \subseteq {}^{<\kappa}\kappa$ in place of $T \subseteq {}^{<\kappa}({}^{<\kappa}2)$. Similarly, the property that $M = V[G]$ adding a new subset of κ but not adding subsets of κ of size $< \kappa$ needed to have over V which is satisfied if G is $(V, \mathbb{P})$-generic for $\mathbb{P}$ a κ-strategically closed forcing, is that for a tree $T \subseteq {}^{<\kappa}({}^{<\kappa}2) \in V$, branches are added to T only if T contains a copy of ${}^{<\kappa}2$, in which case 2^κ-many branches are added.

Note 2.4.12. As indicated previously, in the context of an outer model M almost disjointly refining $([\kappa]^\kappa)^V$, we would want M to be $(<\kappa, \kappa)$-distributive. Because $\mathbb{P}$ is (κ, ∞)-distributive if and only if Odd does not have a winning strategy in $G_\kappa(\mathbb{P})$, while $\mathbb{P}$ is κ-strategically closed if and only if Even does have a winning strategy in $G_\kappa(\mathbb{P})$, the result in 2.4.10 is in some sense not so far from being sharp.

Note 2.4.13. The coding of G_{δ_κ} subsets of 2^κ by trees $T \subseteq {}^{<\kappa}({}^{<\kappa}2)$ as used in 2.2.9 works in the case where $\kappa = \omega$, in which case $2^{<\omega} = \omega$ and so we can consider $T \subseteq {}^{<\omega}\omega$. Because any such tree either contains a copy of ${}^{<\omega}2$, in which case $|[[T]]| = 2^\omega$ or not, in which case $|[[T]]| \leq \omega$, this establishes the perfect set property for G_δ subsets of 2^ω "from the ground up."

2.4.2 A method using base trees

As mentioned, Balcar and Pazák's argument in [4] for 2.2.9 begins by observing that $(P(\omega)/<\omega)^V$ is a not a regular subalgebra of $(P(\omega)/<\omega)^M$. In 2.3.6 we generalized this and showed that if $(2^{<\kappa}=\kappa)^V$ then $(P(\kappa)/<\kappa)^V$ is not a regular subalgebra of $(P(\kappa)/<\kappa)^M$. We also showed that this held with $\mathcal{I}_2$, the bounded ideal. Balcar and Pazák's argument for 2.2.9 proceeds by using the fact that a large maximal antichain from V must be made no longer maximal in M, and refines a base tree in V. In order to imitate this argument in a general setting with κ, we need then to have the existence of base trees in this setting. In this thesis' chapter on tower and distributivity numbers, we prove the following, in particular:

Proposition 2.4.14. Let $\kappa>\omega$ be regular with $\mathrm{MAD}(\kappa)\cap(\kappa,2^\kappa]=\{2^\kappa\}$. Then there exists a tree $(T,\subseteq)$ such that $T\subseteq[\kappa]^\kappa$, $\mathrm{ht}(T)=\omega$, $\mathrm{Lev}_n(T)$ is a MAD family in $[\kappa]^\kappa/<\kappa$ for every $n\in\omega$ (with $\mathrm{Lev}_0(T)=\{\kappa\}$ for concreteness) and such that for every $x\in[\kappa]^\kappa$, there exists $t\in T$ with $t\subseteq x$.

Proposition 2.4.15. Let κ be singular with $\mathrm{cf}(\kappa)=\omega$, $2^\omega>\kappa$, and $\mathrm{MAD}(\kappa)\cap[2^\omega,2^\kappa]=\{2^\kappa\}$. Then there exists a tree $(T,\subseteq^*)$ such that $T\subseteq[\kappa]^\kappa$, $\mathrm{ht}(T)=\omega_1$, $\mathrm{Lev}_\alpha(T)$ is a MAD family in $[\kappa]^\kappa/<\kappa$ for every $\alpha\in\omega_1$ (with $\mathrm{Lev}_0(T)=\{\kappa\}$ for concreteness) and such that for every $x\in[\kappa]^\kappa$, there exists $t\in T$ with $t\subseteq x$. Here we could replace the ideal $<\kappa$ with $\mathcal{I}_2$.

Using 2.4.14, it is straightforward to generalize Balcar and Pazák's argument to prove the following.

Theorem 2.4.16. *Let $V\subseteq M$ be models of ZFC where in V, $\kappa>\omega$ is a regular cardinal with $2^{<\kappa}=\kappa$ and $MAD(\kappa)\cap(\kappa,2^\kappa]=\{2^\kappa\}$. Then if there exists $x\in([\kappa]^\kappa)^M\setminus V$, $([\kappa]^\kappa)^V$ can be almost disjointly refined in M. As usual, assume $(P_\kappa\kappa)^V=(P_\kappa\kappa)^M$.*

Proof. By 2.2.8, we may assume that $(2^\kappa)^V = (2^\kappa)^M$. By 2.3.6, $(P(\kappa)/<\kappa)^V$ is not a regular subalgebra of $(P(\kappa)/<\kappa)^M$, so there exists in V a maximal almost disjoint family $\mathcal{A} \subseteq ([\kappa]^\kappa)^V$ such that $\mathcal{A}$ is no longer maximal in M, witnessed by $z \in ([\kappa]^\kappa)^M$. That is, for every $a \in \mathcal{A}$, $|z \cap a| < \kappa$. Because in V, $\mathrm{MAD}(\kappa) \cap (\kappa, 2^\kappa] = \{2^\kappa\}$, and any maximal almost disjoint family of size $< \kappa$ in V remains maximal almost disjoint in M, $|\mathcal{A}| = 2^\kappa$. In V, let $(T, \subseteq)$ be a base tree as in 2.4.14. We construct from T another base tree $(T', \subseteq)$ in V which we can almost disjointly refine in M, which suffices to show that $([\kappa]^\kappa)^V$ can be almost disjointly refined in M. This is because for every $x \in ([\kappa]^\kappa)^V$, there exist 2^κ-many nodes in T' which are subsets of x, so we can define by recursion in V an injection $f : [\kappa]^\kappa \to T'$ so that $f(x) \in [x]^\kappa$ for every x, and then pass via this injection from an almost disjoint refinement of T' to an almost disjoint refinement of $([\kappa]^\kappa)^V$. First, fix in V for every $t \in [\kappa]^\kappa$ a bijection $b_t \in {}^t\kappa$, with $b_\kappa(\alpha) = \alpha$ for every $\alpha \in \kappa$. Note in particular that $b_t^{-1}[\mathcal{A}] \subseteq [t]^\kappa$ is not a maximal almost disjoint family in $[t]^\kappa$ in M, witnessed by $b_t^{-1}[z]$. Begin by setting $\mathrm{Lev}_0(T') = \bigcup_{t \in \mathrm{Lev}_0(T)} b_t^{-1}[\mathcal{A}] = \mathcal{A}$. If $\mathrm{Lev}_n(T')$ has been defined, let X_{n+1} be a common maximal almost disjoint refinement of $\mathrm{Lev}_n(T')$ and $\mathrm{Lev}_{n+1}(T)$, and let $\mathrm{Lev}_{n+1}(T') = \bigcup_{t \in X_{n+1}} b_t^{-1}[\mathcal{A}]$. Because X_{n+1} and $\mathcal{A}$ are both MAD, it is not difficult to see that $\mathrm{Lev}_{n+1}(T')$ is MAD. We can proceed by recursion to define $(T', \subseteq)$, whose levels are all in particular maximal almost disjoint families in $[\kappa]^\kappa/<\kappa$ in V which refine the levels of $(T, \subseteq)$, so $(T', \subseteq)$ is a base tree. In M, for every $t \in T'$ let $a_t = b_t^{-1}[z] \in [t]^\kappa$. If t_1 and t_2 are incomparable in T', then already $|t \cap t'| < \kappa$, so we may assume without loss of generality that $t_2 \restriction \mathrm{lh}(t_1) = t_1$, i.e. $t_2 \subseteq t_1$. We may assume in fact that t_2 is actually a direct successor to t_1 in T'. However, $a_{t_1} = b_{t_1}^{-1}[z]$ is almost disjoint from every element in $b_{t_1}^{-1}[\mathcal{A}]$, and by construction every successor to t_1 in T' is a subset of such an element, so $|a_{t_1} \cap t_2| < \kappa$. Then $|a_{t_1} \cap a_{t_2}| < \kappa$, and so $\langle a_t : t \in T' \rangle \subseteq [\kappa]^\kappa$ is an almost disjoint refinement of T' in M. $\square$

Note 2.4.17. If in V, for example $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, then the hypotheses of 2.4.16 hold and so we see that the ground model κ-reals can be almost disjointly refined in any extension adding a new subset of κ (not adding a subset of smaller size). This is a more powerful result in this setting than e.g. in 2.4.10 where we insisted that the outer model be obtained by a κ-strategically closed forcing.

2.5 Strongly splitting and unbounded κ-reals

2.5.1 Strongly splitting κ-reals

In 2.4.10 and 2.4.16 we saw that under some cardinal arithmetic assumptions, the almost disjoint refinement of all ground model κ-reals can be carried out in a large class of extensions. In the case where $\kappa = \omega$, much before 2.2.9 was proven in its general form, it was known that adding certain types of reals allows an almost disjoint refinement of the set of ground model reals to be given in the extension. Perhaps the first example of this is due the Hechler [33] where he observes that anytime $M \supseteq V$ contains a Cohen real over V, then $([\omega]^\omega)^V$ can be almost disjointly refined in M. In this section we note that this argument generalizes to other κ, isolate the relevant combinatorial property that the real must have, and consider some of its basic properties. This line of questioning has independently recently been considered by Farkas, Khomskii, and Vidnyánsky [24] (in the case where $\kappa = \omega$), who use the terms e.g. "mixing real" and "injective mixing real" instead of "strongly splitting" as we use here. They also observe, for example, 2.5.5.

Proposition (Hechler [33]) 2.5.1. Let κ be regular and $2^{<\kappa} = \kappa$. If G is $(V, \mathrm{Fn}(\kappa, 2, <\kappa))$-generic then $([\kappa]^\kappa)^V$ can be almost disjointly refined in $V[G]$

Proof. Let κ be regular and $2^{<\kappa} = \kappa$. Let $\mathbb{P} = \mathrm{Fn}(\kappa, 2, <\kappa)$ and let G be $(V, \mathbb{P})$-generic. In $V[G]$, let $r = \bigcup_{p \in G} p \in [\kappa]^\kappa$ be our κ-Cohen real. We first argue that if $\{x, y\} \subseteq ([\kappa]^\kappa)^V$, then $|f_r'' x \cap y| = \kappa$. Here $f_r \in {}^\kappa\kappa$ is the unique order preserving bijection from κ to r, i.e. the enumerating function for r. Work in V and let $\{x, y\} \subseteq [\kappa]^\kappa$. For every $\alpha \in \kappa$, let

$D_\alpha = \{q \in \mathbb{P} :$ for some $\beta \geq \alpha, q \Vdash \beta \in y \cap \dot{f}_r{}''x\}$. We argue that D_α is dense. If $p \in \mathbb{P}$, without loss of generality assume $p \in {}^\gamma 2$ for some $\gamma \in \kappa$. Find $\delta \geq \gamma$ such that $\delta \in x$, and then find $\eta > \alpha$ with $\eta \in y$ where we can extend p to q such that $\mathrm{dom}(q) \in \kappa$ and the δ^{th} element of the domain of q is η. Then $q \Vdash \eta \in y \cap \dot{f}_r{}''x$. Next, because $2^{<\kappa} = \kappa$ in particular, $2^\kappa \in \mathrm{AD}(\kappa)$, so we can fix in V $\langle a_\alpha : \alpha \in 2^\kappa\rangle \subseteq [\kappa]^\kappa$ an almost disjoint family. In $V[G]$, enumerate $([\kappa]^\kappa)^V = \langle z_\alpha : \alpha \in 2^\kappa\rangle$. For every $\alpha \in 2^\kappa$, $\{a_\alpha, z_\alpha\} \subseteq ([\kappa]^\kappa)^V$, so $z_\alpha \cap f_r''a_\alpha \in [z_\alpha]^\kappa$. However, for $\alpha \neq \beta$, $|a_\alpha \cap a_\beta| \in \kappa$, so because f_r is in particular an increasing injection from κ to κ, $|f_r''a_\alpha \cap f_r''a_\beta| \in \kappa$, and so $\langle z_\alpha \cap f_r''a_\alpha : \alpha \in 2^\kappa\rangle$ is an almost disjoint refinement of $\langle z_\alpha : \alpha \in 2^\kappa\rangle = ([\kappa]^\kappa)^V$. $\square$

Observation 2.5.2. In 2.5.1 the relevant combinatorial feature of the κ-Cohen real r is that for every $\{x, y\} \subseteq ([\kappa]^\kappa)^V$, $|y \cap f_r''x| = \kappa$. The other necessary ingredient in the proof is that $(|([\kappa]^\kappa)^V| \in \mathrm{AD}(\kappa)$. That is, as long as we have a κ-real r with that property and $(|([\kappa]^\kappa)^V| \in \mathrm{AD}(\kappa)$, $([\kappa]^\kappa)^V$ will be able to be almost disjointly refined in e.g. $V[G]$. Accordingly, consider the following definition.

Definition 2.5.3. Let κ be a regular cardinal and $V \subseteq M$ be models of ZFC. In M, say that $r \in [\kappa]^\kappa$ is a strongly splitting κ-real over V (or just strongly splitting if the context is clear) if and only if for every $\{x, y\} \subseteq ([\kappa]^\kappa)^V$, $|y \cap f_r''x| = \kappa$.

Proposition 2.5.4. Let κ be a regular cardinal, $V \subseteq M$ be models of ZFC, and $r \in M$ be a strongly splitting κ-real over V. Then if $z \in ([\kappa]^\kappa)^V$, $f_r''z \in [\kappa]^\kappa$ is also a strongly splitting κ-real over V.

Proof. In the following we sometimes identify sets in $[\kappa]^\kappa$ with their enumerating functions, so for example $x(\alpha) = f_x(\alpha)$. For two sets $\{x, y\} \subseteq [\kappa]^\kappa$, let $g(x, y) = f_x''y \in [\kappa]^\kappa$. Let $z \in ([\kappa]^\kappa)^V$, $\{x, y\} \subseteq ([\kappa]^\kappa)^V$, and $r \in [\kappa]^\kappa$ be a strongly splitting κ-real over V. We need to see that $|y \cap f_{f_r''z}''x| = \kappa$. Note that $f_{f_r''z}''x = g(g(r, z), x)$. First, it is not too difficult to see that $g(g(r, z), x) = g(r, g(z, x)) = f_r''g(z, x) = f_r''(f_z''x)$. This is because $g(r, g(z, x))(\alpha) =$

$r(g(z,x)(\alpha)) = r(z(x(\alpha))) = g(r,z)(x(\alpha)) = g(g(r,z),x)(\alpha)$ for every $\alpha \in \kappa$. Then note that $f_z''x \in ([\kappa]^\kappa)^V$ so because r is strongly splitting, $|y \cap f_r''(f_z''x)| = |y \cap f_{f_r''z}''x| = \kappa$, as desired. $\square$

Intuitively, because a strongly splitting κ-real has the property that the subset of its points prescribed by any ground-model subset of κ of size κ intersects any other ground model subset in a set of size κ, it must be that these points contain a gap structure between them which is unanticipatable by ground model functions, in particular one which can not be dominated by a ground model function. This is indeed the case, as follows.

Proposition 2.5.5. Let κ be a regular cardinal, $V \subseteq M$ be models of ZFC, and $r \in M$ be a strongly splitting κ-real over V. Then $f_r \in {}^\kappa\kappa$ is unbounded with respect to $({}^\kappa\kappa)^V$. That is, there does not exist $g \in ({}^\kappa\kappa)^V$ such that $f_r \leq^* g$, i.e. such that $|\{\alpha \in \kappa : f_r(\alpha) > g(\alpha)\}| < \kappa$.

Proof. Suppose towards a contradiction otherwise. Because κ is regular, we may assume without loss of generality that there exists $g \in ({}^\kappa\kappa)^V$ such that for every $\alpha \in \kappa$, $f_r(\alpha) < g(\alpha)$ and g is strictly increasing. The idea here is to use g to dictate the gap structure of two subsets of κ in such a way that f_r could never use one subset to intersect the other nontrivially. Working in V, define $\overline{g} \in {}^\kappa\kappa$ by setting $\overline{g}(0) = g(0)$, for $\alpha \in \lim(\kappa)$, $\overline{g}(\alpha) = \sup(\{\overline{g}(\beta) : \beta \in \alpha\})$, and for successors $\alpha+1$, $\overline{g}(\alpha+1) = g(\overline{g}(\alpha))$. Let $x = \{\overline{g}(\alpha+2k) : \alpha \in \lim(\kappa) \text{ and } k \in \omega\} \in [\kappa]^\kappa$ and $y = \{\overline{g}(\alpha+2k+1) : \alpha \in \lim(\kappa) \text{ and } k \in \omega\} \in [\kappa]^\kappa$. We argue that $y \cap f_r''x = \emptyset$. Suppose $\overline{g}(\alpha+2k) \in x$. Then $f_r(\overline{g}(\alpha+2k)) \in [\overline{g}(\alpha+2k), g(\overline{g}(\alpha+2k)))$. However, $g(\overline{g}(\alpha+2k)) = \overline{g}(\alpha+2k+1))$, and it is clear that $y \cap [\delta, \overline{g}(\alpha+2k+1)) = \emptyset$. So r cannot be a strongly splitting κ-real, a contradiction. $\square$

Observation 2.5.6. Let κ be regular and r be a strongly splitting κ-real. Then r is a splitting κ-real, in that for every $x \in ([\kappa]^\kappa)^V$, $|x \cap r| = \kappa$ and $|x \setminus r| = \kappa$

Proof. Let r be a strongly splitting κ-real and let $x \in ([\kappa]^\kappa)^V$. Then in particular $|f_r''\kappa \cap x| = \kappa = |r \cap x|$. On the other hand, suppose for some $\beta \in \kappa$, $x \setminus \beta \subseteq r$. Then $f_x' \in ({}^\kappa\kappa)^V$ defined by $f'_x(\alpha) = f_x(\alpha + \beta) + 1$ is such that for every $\alpha \in \kappa$, $f'_x(\alpha) > f_r(\alpha)$, which is impossible by 2.5.5 because f_r is unbounded. So $|x \setminus r| = \kappa$ too. $\square$

Note 2.5.7. Because strongly splitting κ-reals have enumeration functions which are unbounded in ${}^\kappa\kappa$ by 2.5.5, in extensions by e.g. forcings which are ${}^\kappa\kappa$-bounding, meaning that no new unbounded functions in ${}^\kappa\kappa$ are added, the collection of ground model subsets of κ of size κ could be not be shown to be almost disjointly refined by the argument of 2.5.2. An example of this sort of forcing is Sacks(κ) for inaccessible κ, which is a κ-closed, ${}^\kappa\kappa$-bounding forcing adding a new subset of κ (see [40] for where this forcing was first formulated as a natural analogue of the Sacks forcing on ω). However, by 2.4.10, we know of course that even in an extension by Sacks(κ), we will be able to almost disjointly refine $([\kappa]^\kappa)^V$.

Note 2.5.8. Hechler also notes in [33] that the union of $< \mathfrak{t}$-many subsets of $[\omega]^\omega$ which can each be almost disjointly refined can be almost disjointly refined. It is straightforward to show using his argument that finite unions of subsets of $[\kappa]^\kappa$ which can each be almost disjointly refined can be almost disjointly refined, and indeed because if $\mathrm{cf}(\kappa) = \omega$ there are no countable towers, countable unions of subsets of $[\kappa]^\kappa$ which can each be almost disjointedly refined can be almost disjointly refined.

2.5.2 Unbounded κ-reals

By Hechler's argument, as long as in an outer model with a strongly splitting κ-real $(|([\kappa]^\kappa)^V| \in \mathrm{AD}(\kappa)$, $([\kappa]^\kappa)^V$ will be able to be almost disjointly refined. In the case where $\kappa = \omega$, by a separate combinatorial argument to those establishing 2.2.9, this can be improved by arguing directly that if an unbounded function in ${}^\omega\omega$ is added, then $([\omega]^\omega)^V$ can be almost disjointly refined, as follows. In [7] the following fact is established: If $Q = \{q_n : n \in \omega\}$ is a partition of ω into infinitely many finite or infinite pieces such that there are an infinite number of them of cardinality $\geq k$ for every $k \in \omega$, then if $\mathcal{I}$ is the ideal generated by

$\{x \in P(\omega) :$ for some k for every $n \in \omega, |x \cap q_n| \leq k\} \cup \{q_n : n \in \omega\}$, $\mathcal{I}^+$ can be almost disjointly refined. If in some outer model a strictly increasing unbounded $f \in {}^\omega\omega$ with $f(0) = 0$ is added, it is not difficult to see that $Q = \{[f(n), f(n+1)) : n \in \omega\}$ is such a partition as above, and for any $x \in ([\omega]^\omega)^V$, because f is unbounded, it must be that e.g. $\limsup(\{|x \cap q_n| : n \in \omega\}) = \omega$. So by the fact, $([\omega]^\omega)^V$ can be almost disjointly refined in the extension. Balcar and Vojtáš' proof of this fact uses in a critical way the existence of a Base Tree for the collection of sets $[\omega]^\omega$. Because we have analogues to this under certain cardinal arithmetic assumptions, we can prove something similar.

Proposition 2.5.9. Let $\kappa > \omega$ be regular with $\mathrm{MAD}(\kappa) \cap (\kappa, 2^\kappa] = \{2^\kappa\}$. Then if $\{q_\beta : \beta \in \kappa\}$ is a partition of κ into sets of size $< \kappa$ such that $\limsup(\langle \mathrm{otp}(q_\beta) : \beta \in \kappa \rangle) \geq \omega$, $\{x \in [\kappa]^\kappa : \limsup(\langle \mathrm{otp}(x \cap q_\beta) : \beta \in \kappa \rangle) \geq \omega\}$ can be almost disjointly refined.

Proof. This proof follows Balcar and Vojtáš's proof for the case where $\kappa = \omega$, see [7]. As in 2.4.14, let $(T, \subseteq)$ be a tree such that $T \subseteq [\kappa]^\kappa$, $\mathrm{ht}(T) = \omega$, $\mathrm{Lev}_n(T)$ is a MAD family in $[\kappa]^\kappa / < \kappa$ for every $n \in \omega$ (with $\mathrm{Lev}_0(T) = \{\kappa\}$ for concreteness) such that for every $x \in [\kappa]^\kappa$, there exists $t \in T$ with $t \subseteq x$. We may assume that every node in T has 2^κ many immediate successors. Let $\langle x_\alpha : \alpha \in 2^\kappa \rangle = \{x \in [\kappa]^\kappa : \limsup(\langle \mathrm{otp}(x \cap q_\beta) : \beta \in \kappa \rangle) \geq \omega\}$. First, note that if $B_0 \subseteq T$ with $|B_0| < 2^\kappa$, then for every $x \in [\kappa]^\kappa$ there exits $t \in T \setminus B_0$ such that $t \subseteq x$ and for every $s \in B_0$, either $|t \cap s| < \kappa$ or $t \subseteq s$. This can be seen by first finding $t' \in T$ with $t' \subseteq x$, then looking at the 2^κ-many successors of t' in T and eliminating any successor which has some element of B_0 at or below it. Next, for every $\alpha \in \kappa$, let $c_\alpha \in [\kappa]^\kappa$ be such that for every $\delta \in c_\alpha$, $\omega \leq |x_\alpha \cap q_\delta|$. This is possible by regularity of κ and $\limsup(\langle \mathrm{otp}(x \cap q_\beta) : \beta \in \kappa \rangle) \geq \omega$. By recursion we define $a_\alpha \in [x_\alpha]^\kappa$ and $t_\alpha \subseteq c_\alpha$ with $t_\alpha \in T$ such that $a_\alpha \subseteq \bigcup_{\beta \in t_\alpha} (q_\beta \cap x_\alpha)$ with $|a_\alpha \cap q_\beta| \leq 1$ for every $\beta \in \kappa$ and if $\alpha \in \beta \in \kappa$ then $|a_\alpha \cap a_\beta| < \kappa$ and $t_\alpha \neq t_\beta$. First find $t_0 \in T$ such that $t_0 \subseteq c_0$ and let $a_0 \in [\bigcup_{\beta \in t_0} (q_\beta \cap x_\alpha)]^\kappa$ with $|a_\alpha \cap q_\beta| \leq 1$ for every $\beta \in \kappa$. At stage $\alpha \in 2^\kappa$, for every $\beta \neq \gamma$ in α we have defined a_β, a_γ, t_β, and t_γ. Let $B_0 = \{t_\beta : \beta \in \alpha\} \in P_{2^\kappa} T$, and find $t_\alpha \in T \setminus B_0$ such that $t_\alpha \subseteq c_\alpha$

and for every $\beta \in \alpha$, either $|t_\alpha \cap t_\beta| < \kappa$ or $t_\alpha \subseteq t_\beta$. Because $\mathrm{ht}(T) = \omega$, and $\beta \neq \gamma$ implies that $t_\beta \neq t_\gamma$, $X_\alpha = \{\beta \in \alpha : t_\alpha \subseteq t_\beta\}$ is such that $|X_\alpha| < \omega$. Then for every $\delta \in c_\alpha$, $|(x_\alpha \cap q_\delta) \setminus \bigcup_{\beta \in X_\alpha} a_\beta| = \omega$, because $|a_\beta \cap q_\delta| \leq 1$ for every $\beta \in X_0$ and $\delta \in \kappa$. So, let $a_\alpha \in [\bigcup_{\beta \in t_\alpha} ((q_\beta \cap x_\alpha) \setminus \bigcup_{\xi \in X_\alpha} a_\xi)]^\kappa$ such that for every $\delta \in t_\alpha$, $|a_\alpha \cap q_\delta| \leq 1$. Then if $\beta \in X_\alpha$, by construction $a_\beta \cap a_\alpha = \emptyset$, while if $\beta \in \alpha \setminus X_\alpha$, then $|t_\alpha \cap t_\beta| < \kappa$, and so because a_α and a_β intersect each q_δ interval in at set of size at most 1, $|a_\alpha \cap a_\beta| < \kappa$. Proceeding, we can produce $\langle a_\alpha : \alpha \in 2^\kappa \rangle$ an almost disjoint refinement of $\langle x_\alpha : \alpha \in 2^\kappa \rangle$. $\square$

Corollary 2.5.10. Let $V \subseteq M$ be models of ZFC where in V and M, $\kappa > \omega$ is regular with $\mathrm{MAD}(\kappa) \cap (\kappa, 2^\kappa] = \{2^\kappa\}$. Then in M if there exists $f \in {}^\kappa\kappa$ such that for no $g \in ({}^\kappa\kappa)^V$ is $|\{\alpha \in \kappa : f(\alpha) > g(\alpha)\}| < \kappa$, $([\kappa]^\kappa)^V$ can be almost disjointly refined.

Proof. Work in M. Without loss of generality, assume that f is strictly increasing and $f(0) = 0$. For every $\alpha \in \kappa$, let $q_\alpha = [f(\alpha), f(\alpha+1))$. Because f is unbounded with respect to $({}^\kappa\kappa)^V$, it is not difficult to see that $\{q_\alpha : \alpha \in \kappa\}$ is a partition of κ into sets of size $< \kappa$ such that $\limsup(\langle \mathrm{otp}(q_\beta) : \beta \in \kappa \rangle) = \kappa$, so in particular $\limsup(\langle \mathrm{otp}(q_\beta) : \beta \in \kappa \rangle) \geq \omega$. Next, fix $x \in ([\kappa]^\kappa)^V$. Towards a contradiction, if $\limsup(\langle \mathrm{otp}(x \cap q_\beta) : \beta \in \kappa \rangle) < \mu < \kappa$, then $f_x \in ({}^\kappa\kappa)^V$ defined by $f_x(\delta) = x(\delta + \mu)$ is such that for every α, $f(\alpha) < f_x(\alpha)$, which is a contradiction. Thus $\limsup(\langle \mathrm{otp}(x \cap q_\beta) : \beta \in \kappa \rangle) = \kappa$, so in particular $\limsup(\langle \mathrm{otp}(x \cap q_\beta) : \beta \in \kappa \rangle) \geq \omega$ for every $x \in ([\kappa]^\kappa)^V$. Then by 2.5.9, $x \in ([\kappa]^\kappa)^V$ can be almost disjointly refined in M. $\square$

2.6 Unconsidered directions

1. Can the approach in 2.3.10 be improved using e.g. a classification of posets of size ω_1 to push the result further up?

2. If $\mathrm{AD}(\kappa)$ has a maximum element ($< 2^\kappa$), is $\mathbf{ref}(\kappa)$ equal to this element necessarily?

In other words, is $\mathbf{ref}(\kappa)$ deducible from $\mathrm{AD}(\kappa)$ in ZFC? Can there be a difference between injective almost disjoint refinement and (possibly non-injective) almost disjoint refinement in certain settings?

3. Does adding a strongly splitting κ-real imply that we have added a κ-Cohen real?

4. Are we ever not able to carry out an almost disjoint refinement of ground model κ-reals in some $(<\kappa, \kappa)$-distributive extension adding a new subset of κ? Can we even have such an extension $V \subseteq M$ which is (κ, κ)-semidistributive or where $(P(\kappa)/<\kappa)^V$ is a regular subalgebra of $(P(\kappa)/<\kappa)^M$? What about when $\kappa = \omega_1$?

Chapter 3

Tower and distributivity numbers

3.1 Towers over κ: initial observations

Definition 3.1.1. Let κ be a cardinal, $A \subseteq P(\kappa)$, and $\mathcal{I}$ an ideal over κ. Typically $\mathcal{A} \subseteq \mathcal{I}^+$. Say that $\langle x_\alpha : \alpha \in \lambda \rangle \subseteq A$ is a tower in A modulo $\mathcal{I}$ if and only if for $\alpha \in \beta \in \lambda$, $x_\beta \setminus x_\alpha \in \mathcal{I}$ and there does not exist $x \in A \cap \mathcal{I}^+$ such that for every $\alpha \in \lambda$, $x \setminus x_\alpha \in \mathcal{I}$. In a slight abuse of notation we also refer to these as towers being in $A/\mathcal{I}$. As usual, we write $x_\beta \subseteq^*_I x_\alpha$, or $x_\beta \subseteq^* x_\alpha$ when there is no confusion, if $x_\beta \setminus x_\alpha \in \mathcal{I}$. Such an x is called a pseudointersection of $\langle x_\alpha : \alpha \in \lambda \rangle$ with respect to $\mathcal{I}$. As an example, if $\kappa = \omega$, $A = [\omega]^\omega$, and $\mathcal{I}$ is the ideal of finite subsets of ω, then these towers are the usual towers considered in the theory of cardinal characteristics of the continuum, the study of $P(\omega)/ < \omega$, etc..

Observation 3.1.2. If $\langle x_\alpha : \alpha \in \lambda \rangle \subseteq A$ is a tower, then if $\langle \alpha_\beta : \beta \in \mathrm{cf}(\lambda) \rangle$ is increasing and cofinal in λ, $\langle x_{\alpha_\beta} : \beta \in \mathrm{cf}(\lambda) \rangle \subseteq A$ is also a tower.

Observation 3.1.3. If $\lambda \in \mathrm{cf}(\kappa)$ is regular then there is a tower of length λ in $[\kappa]^\kappa$ modulo $\mathcal{I}$ the ideal of sets of cardinality $< \kappa$, that is a tower in $[\kappa]^\kappa / < \kappa$. Note that by 3.1.2, if λ is not regular than any tower of length λ contains a sub-tower of length $\mathrm{cf}(\lambda)$.

Proof. Let $\lambda \in \mathrm{cf}(\kappa)$ be regular. Partition κ into a λ-sized disjoint collection of κ-sized subsets, $\langle A_\alpha : \alpha \in \lambda\rangle$. Let $x_\alpha = \kappa \setminus \bigcup_{\gamma\in\alpha} A_\gamma$. We show that $\langle x_\alpha : \alpha \in \lambda\rangle \subseteq [\kappa]^\kappa$ is a tower. First, if $\alpha \in \beta \in \lambda$ then $x_\beta \subseteq x_\alpha$. Suppose towards a contradiction that there exists $x \in [\kappa]^\kappa$ with $x \subseteq^* x_\alpha$ for every α. Because $\lambda \in \mathrm{cf}(\kappa)$, there exists $\alpha \in \lambda$ with $|x \cap A_\alpha| = \kappa$. However, then $\neg(x \subseteq^* x_{\alpha+1})$, a contradiction. □

Observation 3.1.4. There does not exist a tower of length $\mathrm{cf}(\kappa)$ in $[\kappa]^\kappa / < \kappa$.

Proof. Let $\langle \delta_\alpha : \alpha \in \mathrm{cf}(\kappa)\rangle$ be an increasing cofinal sequence of ordinals in κ and let $\langle x_\alpha : \alpha \in \mathrm{cf}(\kappa)\rangle \subseteq [\kappa]^\kappa$ be $\subseteq^*$-decreasing. Note that for every $\alpha \in \mathrm{cf}(\kappa)$, $|\bigcap_{\gamma\in\alpha} x_\gamma| = \kappa$. For each such α, let $A_\alpha \subseteq \bigcap_{\gamma\in\alpha} x_\gamma$ be a set of ordinals of order type δ_α. Let $x = \bigcup_{\alpha\in\mathrm{cf}(\kappa)} A_\alpha$. Note that $x \in [\kappa]^\kappa$ and for any $\alpha \in \mathrm{cf}(\kappa)$, $x \setminus x_\alpha \subseteq \bigcup_{\gamma\in\alpha} A_\gamma$ so that $x \subseteq^* x_\alpha$. □

Proposition 3.1.5. Let $\lambda \in (\mathrm{cf}(\kappa), \kappa)$ be regular. Then there exists a tower of length λ in $[\kappa]^\kappa / < \kappa$.

Proof. Let $\lambda \in (\mathrm{cf}(\kappa), \kappa)$ be regular and fix $\langle \mu_\alpha : \alpha \in \mathrm{cf}(\kappa)\rangle \subseteq \kappa$ an increasing cofinal sequence of regular cardinals with $\mu_0 = 0$ and $\mu_1 > \lambda$. For every $\alpha \in \mathrm{cf}(\kappa)$, let $A_\alpha = [\mu_\alpha, \mu_\alpha \cdot \lambda)$. To avoid confusion, we write $\mu_\alpha \cdot \lambda$ to indicate the ordinal comprising λ-many copies of μ_α placed one after another. Now, for every $\beta \in \lambda$, let $x_\beta = \bigcup_{\alpha\in\mathrm{cf}(\kappa)} (A_\alpha \setminus [\mu_\alpha, \mu_\alpha \cdot \beta))$. So x_β is the union of all the A_α's each without an initial segment up to $\mu_\alpha \cdot \beta$. Note that $\langle x_\beta : \beta \in \lambda\rangle \subseteq [\kappa]^\kappa$ is $\subseteq$-decreasing. We argue that $\langle x_\beta : \beta \in \lambda\rangle$ has no pseudointersection. Suppose towards a contradiction otherwise, and fix $x \in [\kappa]^\kappa$ so that $x \subseteq^* x_\beta$ for every $\beta \in \lambda$. Without loss of generality we may assume that $x \subseteq x_0 = \bigcup_{\alpha\in\mathrm{cf}(\kappa)} A_\alpha$. Because $\mu_1 > \lambda > \mathrm{cf}(\kappa)$ is regular, there must exist some $\alpha \in \mathrm{cf}(\kappa)$ with $|x \cap A_\alpha| \geq \mu_1$. Let α_1 be the minimal ordinal with this property. Because $\lambda < \mu_1$, there exists $\beta_1 \in \lambda$ such that $|x \cap [\mu_{\alpha_1}, \mu_{\alpha_1} \cdot \beta_1)| \geq \mu_1$. Again, let β_1 be minimal with this property. We proceed in this manner, defining by recursion increasing sequences $\langle \alpha_\gamma : \gamma \in \mathrm{cf}(\kappa)\rangle \subseteq \mathrm{cf}(\kappa)$ and $\langle \beta_\gamma : \gamma \in \mathrm{cf}(\kappa)\rangle \subseteq \lambda$. For example, at the next step we choose $\alpha_2 \in \mathrm{cf}(\kappa)$ to be the minimal ordinal with the property that $|x \cap A_{\alpha_2}| \geq \mu_2$

and $\alpha_2 > \alpha_1$, and β_2 to be the minimal ordinal with the property that $|x \cap [\mu_{\alpha_2}, \mu_{\alpha_2} \cdot \beta_2)| \geq \mu_2$ and $\beta_2 > \beta_1$. Similarly, at limits $\gamma \in \mathrm{cf}(\kappa)$ we can choose α_γ to be the minimal ordinal with the property that $\alpha_\gamma > \alpha_\nu$ for every $\nu \in \gamma$ and $|x \cap A_{\alpha_\gamma}| \geq \mu_\gamma$, and then choose $\beta_\gamma \in \lambda$ to be minimal with the property that $\beta_\gamma > \beta_\nu$ for every $\nu \in \gamma$ and $|x \cap [\mu_{\alpha_\gamma}, \mu_{\alpha_\gamma} \cdot \beta_\gamma)| \geq \mu_\gamma$. Because $\mathrm{cf}(\kappa) < \lambda$, $\beta = \sup\{\beta_\gamma : \gamma \in \mathrm{cf}(\kappa)\} \in \lambda$. However, in this case $|x \setminus x_\beta| \geq \mu_\gamma$ for every $\gamma \in \mathrm{cf}(\kappa)$ so $\neg(x \subseteq^* x_\beta)$, a contradiction. $\square$

3.2 Tower number definitions

Taken together, 3.1.3, 3.1.4, and 3.1.5 mean that for $\kappa > \omega$ in order to define a non-degenerate tower number for $[\kappa]^\kappa / < \kappa$, we need to limit our consideration only to towers of length $> \kappa$. For example, if $\mathrm{cf}(\kappa) > \omega$ then 3.1.2 implies that there are countable towers in $[\kappa]^\kappa / < \kappa$, while if $\mathrm{cf}(\kappa) = \omega$ then 3.1.5 implies that there are towers of length ω_1 in $[\kappa]^\kappa / < \kappa$. There are two natural ways then to define the tower number, $\mathfrak{t}(\kappa)$, one which is similar to the natural generalization of the pseudointersection number over ω, $\mathfrak{p}$, to the pseudointersection number over κ, $\mathfrak{p}(\kappa)$, and one which simply asserts that the towers must be of a certain minimum length.

Definition 3.2.1. Let $\mathfrak{t}_1([\kappa]^\kappa / < \kappa) = \mathfrak{t}_1(\kappa) = \mathfrak{t}(\kappa)$ denote the shortest regular length $\geq \kappa^+$ of a tower in $[\kappa]^\kappa / < \kappa$.

Definition 3.2.2. Let $\mathfrak{p}([\kappa]^\kappa / < \kappa) = \mathfrak{p}(\kappa)$ denote the smallest cardinal λ such that there exists a collection $\{x_\alpha : \alpha \in \lambda\} \subseteq [\kappa]^\kappa$ with the κ-strong intersection property (κ-SIP), and no pseudointersection. Here the κ-SIP means that for any $A \in P_\kappa \lambda$, $|\bigcap_{\alpha \in A} x_\alpha| = \kappa$.

Definition 3.2.3. Let $\mathfrak{t}_2([\kappa]^\kappa / < \kappa) = \mathfrak{t}_2(\kappa)$ denote the smallest cardinal λ so that there is a tower with the κ-SIP in $[\kappa]^\kappa / < \kappa$ of length λ.

Note 3.2.4. Note that $\mathfrak{t}_1(\omega) = \mathfrak{t}_2(\omega) = \mathfrak{t}$ and $\mathfrak{p}(\omega) = \mathfrak{p}$. Furthermore if κ is regular, because generally every tower in $[\kappa]^\kappa/<\kappa$ with the κ-SIP has length $\geq \kappa^+$, and because for regular κ any tower of regular length $\geq \kappa^+$ has the κ-SIP, $\mathfrak{t}_1(\kappa) = \mathfrak{t}_2(\kappa)$.

Note 3.2.5. For regular κ, both $\mathfrak{t}_1(\kappa)$ and $\mathfrak{t}_2(\kappa)$ have been defined in the literature (and as in 3.2.4, they are equivalent). See [60] and [28],[12], respectively, for examples.

Definition 3.2.6. Let the (un)bounding number at κ, $\mathfrak{b}(\kappa)$, be defined as the minimal cardinality of a set F of functions $f : \kappa \to \kappa$ such that there does not exist $g \in {}^\kappa\kappa$ with $f <^* g$ for every $f \in F$, where $f <^* g$ if and only if $|\{\alpha \in \kappa : f(\alpha) \geq g(\alpha)\}| < \kappa$. Let the dominating number at κ, $\mathfrak{d}(\kappa)$, be defined as the minimal cardinality of a set F of functions $f : \kappa \to \kappa$ such that for every $g \in {}^\kappa\kappa$, there exists $f \in F$ with $g <^* f$. Cummings and Shelah [17] were perhaps the first to consider these characteristics in detail.

Definition 3.2.7. If κ is singular it is also reasonable to let $\mathcal{I}_2$ be the ideal of bounded subsets of κ and consider tower numbers relative to this ideal. In this setting, we can let $\mathfrak{t}([\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2)$ denote the tower number for cofinal $\mathrm{cf}(\kappa)$-sized subsets of κ (which are not necessarily of order type $\mathrm{cf}(\kappa)$) modulo $\mathcal{I}_2$. Similarly, in an abuse of notation, we can let $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2)$ denote the tower number for $\mathrm{cf}(\kappa)$-sized subsets of κ which have order type $\mathrm{cf}(\kappa)$ (here we are identifying the functions with their images). Just as in 3.2.1 to avoid degenerate cases we insist that the length of these towers is regular and $\geq \mathrm{cf}(\kappa)^+$. Similarly, one can consider $\mathfrak{t}(P(\kappa)/\mathcal{I}_2)$, $\mathfrak{t}([\kappa]^\mu/\mathcal{I}_2)$, where $\mu \in [\mathrm{cf}(\kappa^+), \kappa)$ is a cardinal, etc..

3.3 Tower number results

The following proposition has also been observed independently by several individuals ([28], [60], for example).

Proposition 3.3.1. If κ is regular then $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$.

Proof. The case where $\kappa = \omega$, that is $\mathfrak{t} \leq \mathfrak{b}$, is a standard result in the theory of cardinal characteristics of the continuum. In what follows we discuss the distributivity number $\mathfrak{h}$ (the smallest cardinality of a set of open dense subsets of $[\omega]^\omega$ with empty intersection), and it is straightforward to see that $\mathfrak{t} \leq \mathfrak{h}$. For completeness, we show that $\mathfrak{h} \leq \mathfrak{b}$, implying in particular that $\mathfrak{t} \leq \mathfrak{b}$. Let $\lambda \in \mathfrak{h}$ and let $\langle f_\alpha : \alpha \in \lambda\rangle \subseteq {}^\omega\omega$. Without loss of generality, assume that every f_α is increasing. For each $\alpha \in \lambda$, let $D_\alpha \subseteq [\omega]^\omega$ be the set of all $x \in [\omega]^\omega$ where there exists $N \in \omega$ such that for every $n \geq N$ with $n \in x$, for every $m \in x$ with $m > n$, $m > f(n)$. That is, D_α is the collection of infinite subsets of ω where eventually always is it the case that $x(n+1)$ (the $(n+1)^{\text{st}}$ element of x) is greater than $f_\alpha(x(n))$. Because any infinite subset of ω can be thinned out to an element of D_α, and D_α is open by construction, D_α is an open dense subset of $[\omega]^\omega$. Because $\lambda \in \mathfrak{h}$, there exists $x \in \bigcap_{\alpha\in\lambda} D_\alpha$. Let $g \in {}^\omega\omega$ be defined by $g(n) = x(n+1)$. Then for each $\alpha \in \lambda$, $x \in D_\alpha$ so for some $N \in \omega$, for every $n \in \omega$ such that $x(n) > N$, $g(n) = x(n+1) > f(x(n)) \geq f(n)$. So $f_\alpha <^* g$ for every α, so $\lambda \in \mathfrak{b}$, and so $\mathfrak{h} \leq \mathfrak{b}$. Next, let $\kappa > \omega$ be regular. We prove by induction that every regular $\lambda \in (\kappa, \mathfrak{t}(\kappa))$ (so there do not exist towers of length λ) is such that $\lambda \in \mathfrak{b}(\kappa)$. Because it is not difficult to see that $\kappa \in \mathfrak{b}(\kappa)$, this implies that $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$. So, let $\langle f_\alpha : \alpha \in \lambda\rangle \subseteq {}^\kappa\kappa$. Without loss of generality assume that each f_α is increasing and furthermore by induction because $\mu \in \mathfrak{b}(\kappa)$ for every $\mu \in \lambda$ that if $\alpha \in \beta \in \lambda$ that $f_\alpha <^* f_\beta$. That is, there exists $\overline{\gamma} \in \kappa$ such that for every $\gamma \geq \overline{\gamma}$, $f_\alpha(\gamma) < f_\beta(\gamma)$. Define $C_\alpha \subseteq \kappa$ to be the set of closure points for f_α, that is $C_\alpha = \{\gamma \in \kappa : f_\alpha''\gamma \subseteq \gamma\}$. Note that C_α is a club in κ, and if $\alpha \in \beta$, then $f_\alpha <^* f_\beta$, so in fact $C_\beta \subseteq^* C_\alpha$. Because there do not exist towers of length λ there is some $C \subseteq^* C_\alpha$ for every $\alpha \in \lambda$. We may assume without loss of generality that C is also a club, because by taking its closure we would still have a set almost contained in every C_α. Define $g \in {}^\kappa\kappa$ by $g(\beta) = C(\beta + 1)$ for every $\beta \in \kappa$. Note that for $\alpha \in \lambda$, $C \subseteq^* C_\alpha$ so there exists $\overline{\gamma} \in \kappa$ such that if $\gamma \geq \overline{\gamma}$ and $\gamma \in C$, then $\gamma \in C_\alpha$. Find $\overline{\eta} \geq \overline{\gamma}$ so that $C(\overline{\eta}) = \overline{\eta}$ and $C_\alpha(\overline{\eta}) = \overline{\eta}$. Note that if $\eta \geq \overline{\eta}$ then $f_\alpha(\eta) \leq C_\alpha(\eta) \leq C(\eta) < C(\eta + 1) = g(\eta)$. So $f_\alpha <^* g$ for every $\alpha \in \lambda$. Thus $\lambda < \mathfrak{b}(\kappa)$. $\square$

Proposition 3.3.2. Let $\mathcal{I}_2$ be the ideal of bounded subsets of κ. Then $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa)) = \mathfrak{t}([\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2)$. So in particular if $\mathrm{cf}(\kappa) = \omega$, then $\mathfrak{t} = \mathfrak{t}([\kappa]^{\omega}/\mathcal{I}_2) = \mathfrak{t}({}^{\omega}\kappa/\mathcal{I}_2)$.

Proof. It is straightforward to see that $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$, as follows. First, suppose $\lambda \in (\mathrm{cf}(\kappa), \mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2))$ is regular. Fix $\langle \mu_\beta : \beta \in \mathrm{cf}(\kappa)\rangle \subseteq \kappa$ a cofinal sequence of ordinals. Let $\langle x_\alpha : \alpha \in \lambda\rangle \subseteq [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ be $\subseteq^*$-decreasing. To each x_α, let $\overline{x_\alpha} \in {}^{\mathrm{cf}(\kappa)}\kappa$ be defined by $\beta \in x_\alpha$ if and only if $\mu_\beta \in \overline{x_\alpha}$, and $\overline{x_\alpha} \subseteq \{\mu_\beta : \beta \in \mathrm{cf}(\kappa)\}$. Then $\langle \overline{x_\alpha} : \alpha \in \lambda\rangle \subseteq {}^{\mathrm{cf}(\kappa)}\kappa$ is $\subseteq^*_2$-decreasing, and there exists $\overline{x} \in {}^{\mathrm{cf}(\kappa)}\kappa$ such that $\overline{x} \subseteq^*_2 \overline{x}_\alpha$ for every α. Then define $x \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ by $\beta \in x$ if and only if $\mu_\beta \in \overline{x}$, and note that $x \subseteq^* x_\alpha$ for every α. So $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2) \leq \mathfrak{t}(\mathrm{cf}(\kappa))$. On the other hand if $\lambda \in (\mathrm{cf}(\kappa), \mathfrak{t}(\mathrm{cf}(\kappa)))$ is regular, let $\langle \overline{x_\alpha} : \alpha \in \lambda\rangle \subseteq {}^{\mathrm{cf}(\kappa)}\kappa$ be $\subseteq^*_2$-decreasing. Without loss of generality $\overline{x_\alpha} \subseteq \overline{x_0}$ for every α. Then define $\langle x_\alpha : \alpha \in \lambda\rangle \subseteq [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ by $\beta \in x_\alpha$ if and only if $\overline{x_0}(\beta) \in \overline{x_\alpha}$ (that is we're just using $\overline{x_0}$ as a coordinate system). Then $\langle x_\alpha : \alpha \in \lambda\rangle$ is $\subseteq^*$-decreasing, and so there exists $x \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ with $x \subseteq^* x_\alpha$ for each α. Letting $\overline{x} \subseteq \overline{x_0}$ be defined by $\overline{x_0}(\beta) \in \overline{x}$ if and only if $\beta \in x$, we see that $\overline{x} \subseteq^*_2 \overline{x_\alpha}$ for each α. So $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$. The proof that $\mathfrak{t}([\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$ is a little less straightforward—when translating between the $[\kappa]^{\mathrm{cf}(\kappa)}$ and $[\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ settings, we can no longer use without modification a fixed coordinate system as we did in the previous argument because elements in $[\kappa]^{\mathrm{cf}(\kappa)}$ do not need to have order type $\mathrm{cf}(\kappa)$. To see that $\mathfrak{t}([\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2) \leq \mathfrak{t}(\mathrm{cf}(\kappa))$ is easy, however, because towers comprising elements in $[\kappa]^{\mathrm{cf}(\kappa)}$ which all happen to have order type $\mathrm{cf}(\kappa)$ are still towers in $[\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2$, and $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$. To show that $\mathfrak{t}([\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$, we first prove a lemma, which is that for any $\alpha \in \mathrm{Cof}(\mathrm{cf}(\kappa)) \cap [\mathrm{cf}(\kappa), \mathrm{cf}(\kappa)^+)$, $\mathfrak{t}([\alpha]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$. Here of course $\mathcal{I}_2$ when used in this setting refers to the ideal of bounded subsets of α. First, let's see that this lemma suffices. Suppose we've shown that $\mathfrak{t}([\alpha]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$ for every $\alpha \in \mathrm{Cof}(\mathrm{cf}(\kappa)) \cap [\mathrm{cf}(\kappa), \mathrm{cf}(\kappa)^+)$. Let $\lambda \in (\mathrm{cf}(\kappa), \mathfrak{t}(\mathrm{cf}(\kappa)))$ be regular and suppose that $\langle x_\alpha : \alpha \in \lambda\rangle \subseteq [\kappa]^{\mathrm{cf}(\kappa)}$ is $\subseteq^*_2$-decreasing. Note that $\mathrm{otp}(x_0) = \delta \in \mathrm{cf}(\kappa)^+$ with $\mathrm{cf}(\delta) = \mathrm{cf}(\kappa)$ (because x_0 is unbounded in κ). Consider $\langle x_\alpha \cap x_0 : \alpha \in \lambda\rangle$. This is still $\subseteq^*_2$-decreasing, and to each $x_\alpha \cap x_0$ we associate $\overline{x_\alpha} \in [\delta]^{\mathrm{cf}(\kappa)}$

in the natural way. That is, $\overline{x_\alpha} = \{\pi''(x_0 \cap \beta) : \beta \in x_\alpha \cap x_0\}$ where π is the Mostowski collapse for x_0. This gives an isomorphism for our purposes between $\langle x_\alpha \cap x_0 : \alpha \in \lambda\rangle$ and $\langle \overline{x_\alpha} : \alpha \in \lambda\rangle \subseteq [\delta]^{\mathrm{cf}(\kappa)}$, which is $\subseteq_2^*$-decreasing, so by the lemma there exists $\overline{x} \in [\delta]^{\mathrm{cf}(\kappa)}$ with $x \subseteq_2^* x_\alpha$ for every $\alpha \in \lambda$. We may assume $\overline{x} \subseteq \overline{x_0}$. Then simply let $x = (\pi^{-1})''\overline{x} \in [\kappa]^{\mathrm{cf}(\kappa)}$ and note that $x \subseteq_2^* x_\alpha$ for each $\alpha \in \lambda$ as well.

So, it suffices to prove the lemma. First, is easy to see that $\mathfrak{t}([\alpha]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2) \leq \mathfrak{t}(\mathrm{cf}(\kappa))$. For the other direction, let $\lambda \in (\mathrm{cf}(\kappa), \mathfrak{t}(\mathrm{cf}(\kappa)))$ be regular and let $\langle x_\beta : \beta \in \lambda\rangle \subseteq [\alpha]^{\mathrm{cf}(\kappa)}$ be $\subseteq_2^*$-decreasing. If α is of the form $\beta+\mathrm{cf}(\kappa)$ for some β, then $x_0 \setminus \beta$ is of order type $\mathrm{cf}(\kappa)$ in α, so by a similar argument as for why $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$, we'll be done. So suppose α is not of this form. Because $\mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)$, we can fix $f \in {}^{\mathrm{cf}(\kappa)}\alpha$ an increasing continuous cofinal function so that $\mathrm{cf}(f(\gamma+1)) = \mathrm{cf}(\kappa)$ for every $\gamma \in \mathrm{cf}(\kappa)$ and $f(0) = 0$. Let $A_\gamma = [f(\gamma), f(\gamma+1))$. Note that in particular, the A_γ partition α into $\mathrm{cf}(\kappa)$-sized blocks. To each x_β associate $\overline{x_\beta} \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ via $\gamma \in \overline{x_\beta}$ if and only if $x_\beta \cap A_\gamma \neq \emptyset$. Note that $\langle \overline{x_\beta} : \beta \in \lambda\rangle \subseteq [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ is $\subseteq^*$-decreasing, so there exists $\overline{x} \subseteq^* x_\beta$ for every $\beta \in \lambda$. This $\overline{x}$ corresponds to a cofinal sub-collection of intervals A_γ where modulo bounded, every x_β contains an ordinal in every interval. By changing each x_β on a bounded set, we may assume that each x_β has non-empty intersection with every interval labeled by $\overline{x}$. We are going to produce $\mathrm{cf}(\kappa)$-sub-sequences of each x_β which have $< \mathrm{cf}(\kappa)$-many elements in each relevant interval, and which are $\subseteq_2^*$-decreasing. So, for simplicity, by ignoring all intervals not labeled by $\overline{x}$, we may assume not only that each x_β has nonempty intersection with every interval labeled by $\overline{x}$, but also that $\overline{x}$ labels all intervals. That is, without loss of generality suppose that for every $\beta \in \lambda$ and $\gamma \in \mathrm{cf}(\kappa)$, $x_\beta \cap A_\gamma \neq \emptyset$. Choose for each $\gamma \in \mathrm{cf}(\kappa)$ a bijection $e_\gamma \in {}^{A_\gamma}\mathrm{cf}(\kappa)$. For every $\beta \in \lambda$, let $f_\beta^0 \in {}^{\mathrm{cf}(\kappa)}\mathrm{cf}(\kappa)$ be defined by $f_\beta^0 = \min\{e_\gamma''(A_\gamma \cap x_\beta)\}$, i.e. f_β^0 picks the minimal labeling of an element in $A_\gamma \cap x_\beta$ by e_γ at each coordinate γ. Because the x_β's are $\subseteq_2^*$-decreasing, the f_β^0's are $\leq^*$-increasing. Specifically, if $\beta \in \delta \in \lambda$, there exists $\overline{\gamma} \in \mathrm{cf}(\kappa)$ such that for every $\gamma \geq \overline{\gamma}$, $x_\delta \cap A_\gamma \subseteq x_\beta \cap A_\gamma$, so $f_\beta^0(\gamma) \leq f_\delta^0(\gamma)$, i.e. $\langle f_\beta^0 : \beta \in \lambda\rangle \subseteq {}^{\mathrm{cf}(\kappa)}\mathrm{cf}(\kappa)$ is $\leq^*$-increasing. By 3.3.1 for every regular μ, $\mathfrak{t}(\mu) \leq \mathfrak{b}(\mu)$, so there exists $g \in {}^{\mathrm{cf}(\kappa)}\mathrm{cf}(\kappa)$ such that $f_\beta^0 <^* g$ for every $\beta \in \lambda$. Now, to each x_β associate $\overline{x_\beta} \subseteq x_\beta$ by $\overline{x_\beta} = \bigcup_{\gamma \in \mathrm{cf}(\kappa)} e_\gamma^{-1}[(e''_\gamma(x_\beta \cap A_\gamma)) \cap g(\gamma)]$. That is, for each γ, $\overline{x_\beta} \cap A_\gamma$ is the collection of ordinals in $x_\beta \cap A_\gamma$ such that e_γ's label is less than $g(\gamma)$. Because $f^0_\beta <^* g$, $|x_\beta| = \mathrm{cf}(\kappa)$, and $|x_\beta \cap A_\gamma| < \mathrm{cf}(\kappa)$ for each $\gamma \in \mathrm{cf}(\kappa)$, so $\mathrm{otp}(x_\beta) = \mathrm{cf}(\kappa)$. Furthermore, if $\beta \in \delta \in \lambda$, then for every $\gamma \in \mathrm{cf}(\kappa)$ sufficiently large, $x_\delta \cap A_\gamma \subseteq x_\beta \cap A_\gamma$, and it follows then that $\overline{x_\delta} \cap A_\gamma \subseteq \overline{x_\beta} \cap A_\gamma$. That is, $\langle \overline{x_\beta} : \beta \in \lambda \rangle$ is $\subseteq^*_2$-decreasing. The same argument for why $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$ can be used to show that $\mathfrak{t}({}^{\mathrm{cf}(\kappa)}\alpha/\mathcal{I}_2) = \mathfrak{t}(\mathrm{cf}(\kappa))$ for $\mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)$, and so by hypothesis there exists $\overline{x} \in {}^{\mathrm{cf}(\kappa)}\alpha \subseteq [\alpha]^{\mathrm{cf}(\kappa)}$ with $\overline{x} \subseteq^*_2 \overline{x_\beta} \subseteq x_\beta$ for every β, as desired. □

Proposition 3.3.3. Let κ be singular. If $\mathfrak{t}(\mathrm{cf}(\kappa)) \geq \kappa^+$, then $\mathfrak{t}(\kappa) \leq \mathfrak{t}(\mathrm{cf}(\kappa))$ and $\mathfrak{t}([\kappa]^\kappa/\mathcal{I}_2) \leq \mathfrak{t}(\mathrm{cf}(\kappa))$. Here just as for $\mathfrak{t}(\kappa)$, we must insist in the definition of $\mathfrak{t}([\kappa]^\kappa/\mathcal{I}_2)$ that the towers must be of length $> \kappa$ to avoid degenerate cases.

Proof. Suppose $\mathfrak{t}(\mathrm{cf}(\kappa)) \geq \kappa^+$. The proof for $\mathfrak{t}([\kappa]^\kappa/\mathcal{I}_2) \leq \mathfrak{t}(\mathrm{cf}(\kappa))$ is straightforward. Specifically, let $\lambda \in (\kappa, \mathfrak{t}([\kappa]^\kappa/\mathcal{I}_2))$ be regular. We need to see that $\lambda \in \mathfrak{t}(\mathrm{cf}(\kappa))$. Let $\langle x_\alpha : \alpha \in \lambda \rangle \subseteq [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ be $\subseteq^*$-decreasing. Let $\langle \mu_\beta : \beta \in \mathrm{cf}(\kappa) \rangle \subseteq \kappa$ be a continuous cofinal sequence of cardinals with $\mu_0 = 0$. For each $\alpha \in \lambda$, define $\overline{x_\alpha} \in [\kappa]^\kappa$ by $\overline{x_\alpha} = \bigcup_{\beta \in x_\alpha} [\mu_\beta, \mu_{\beta+1})$. Then $\langle \overline{x_\alpha} : \alpha \in \lambda \rangle \subseteq [\kappa]^\kappa$ is $\subseteq^*_2$-decreasing. By hypothesis, there exists $\overline{x} \subseteq^*_2 \overline{x_\alpha}$ for every α. Let $x = \{\beta \in \mathrm{cf}(\kappa) : \overline{x} \cap [\mu_\beta, \mu_{\beta+1}) \neq \emptyset\} \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$. Then $x \subseteq^* x_\alpha$ for every $\alpha \in \lambda$, as desired. The argument for why $\mathfrak{t}(\kappa) \leq \mathfrak{t}(\mathrm{cf}(\kappa))$ is slightly more complicated, as follows. Let $\lambda \in (\kappa, \mathfrak{t}(\kappa))$ be regular and fix $\langle x_\alpha : \alpha \in \lambda \rangle \subseteq [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ $\subseteq^*$-decreasing. Define as before for each $\alpha \in \lambda$, $\overline{x_\alpha} \in [\kappa]^\kappa$ by $\overline{x_\alpha} = \bigcup_{\beta \in x_\alpha} [\mu_\beta, \mu_{\beta+1})$. The $\overline{x_\alpha}$'s are $\subseteq^*_2$-decreasing, so they're $\subseteq^*$-decreasing and so there exists $\overline{x} \in [\kappa]^\kappa$ such that $\overline{x} \subseteq^* x_\alpha$ for every α. However, it is not necessarily the case that $\overline{x} \subseteq^*_2 x_\alpha$ so we can't define x directly from $\overline{x}$ as we did above. Instead, define $\langle z_\beta : \beta \in \mathrm{cf}(\kappa) \rangle \subseteq [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ by $z_\beta = \{\gamma \in \mathrm{cf}(\kappa) : |\overline{x} \cap [\mu_\gamma, \mu_{\gamma+1})| \geq \mu_\beta\}$. It is straightforward to see that indeed $|z_\beta| = \mathrm{cf}(\kappa)$ and clearly if $\beta \in \delta \in \mathrm{cf}(\kappa)$, $z_\delta \subseteq z_\beta$. So, because there are no towers in $[\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}/ < \mathrm{cf}(\kappa)$ of length $\mathrm{cf}(\kappa)$, there exists $z \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ with $z \subseteq^* z_\beta$ for every $\beta \in \mathrm{cf}(\kappa)$. We show that in fact $z \subseteq^* x_\alpha$ for every $\alpha \in \lambda$. So, fix

$\alpha \in \lambda$. $\overline{x} \subseteq^* \overline{x_\alpha}$, so there exists $A \in P_\kappa \kappa$ such that $\overline{x} \setminus A \subseteq \overline{x_\alpha}$. Let $\beta \in \mathrm{cf}(\kappa)$ be minimal such that $|A| < \mu_\beta$. Note that $z \subseteq^* z_\beta$, so there exists $\overline{\beta} \in \mathrm{cf}(\kappa)$ such that $z \setminus \overline{\beta} \subseteq z_\beta$. Note as follows that in fact $z \setminus \overline{\beta} \subseteq x_\beta$. Let $\gamma \in z$ with $\gamma \geq \overline{\beta}$. Then $\gamma \in z_\beta$, so $|\overline{x} \cap [\mu_\gamma, \mu_{\gamma+1})| \geq \mu_\beta$. And $|A| < \mu_\beta$, so $(\overline{x} \setminus A) \cap [\mu_\gamma, \mu_{\gamma+1}) \neq \emptyset$. However, then $\overline{x_\alpha} \cap [\mu_\gamma, \mu_{\gamma+1}) \neq \emptyset$, so $\gamma \in x_\alpha$. □

3.4 Distributivity number definitions

Definition 3.4.1. Let $A \subseteq P(\kappa)$ and $\mathcal{I}$ be an ideal over κ. Say that $D \subseteq A$ is open dense (or dense open) in $A/\mathcal{I}$ if and only if for every $a \in A$ there exists $b \in D$ with $b \subseteq a$ (D is dense) and if $a \in D$ and $b \setminus a \in \mathcal{I}$, then $b \in D$ (D is open).

Definition 3.4.2. The distributivity number $\mathfrak{h}$ is defined to be the minimal cardinality of a collection $\langle D_\alpha : \alpha \in \mathfrak{h} \rangle$ where each $D_\alpha \subseteq [\omega]^\omega$ is open dense in $[\omega]^\omega / < \omega$ and $\bigcap_{\alpha \in \mathfrak{h}} D_\alpha = \emptyset$.

Definition 3.4.3. If κ is singular, just as in the case with towers we can let $\mathcal{I}_2$ be the collection of bounded subsets of κ and consider open dense collections in $[\kappa]^\kappa/\mathcal{I}_2$, $[\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2$, ${}^{\mathrm{cf}(\kappa)}\kappa/\mathcal{I}_2$, etc.

3.5 Distributivity number results

The following is analogous to 3.3.2.

Proposition 3.5.1. Suppose $\mathrm{cf}(\kappa) = \omega$. Just as in the case for $\mathfrak{t}$, $\mathfrak{h}([\kappa]^\omega/\mathcal{I}_2) = \mathfrak{h}({}^\omega\kappa/\mathcal{I}_2) = \mathfrak{h}$.

Proof. We first describe two methods (using the same idea) for showing that $\mathfrak{h}({}^\omega\kappa/\mathcal{I}_2) \leq \mathfrak{h}$. Let $\lambda < \mathfrak{h}({}^\omega\kappa/\mathcal{I}_2)$ and $\langle D_\alpha : \alpha \in \lambda \rangle$ such that $D_\alpha \subseteq [\omega]^\omega$ is open dense. Fix $\langle \mu_n : n \in \omega \rangle \subseteq \kappa$ an increasing cofinal sequence of regular cardinals. Let $E_\alpha = (\bigcup_{x \in D_\alpha} \prod_{n \in x} [\mu_n, \mu_{n+1})) \downarrow\subseteq {}^\omega\kappa$, where the $\downarrow$ indicates closure under $\subseteq_2^*$ with respect to the ${}^\omega\kappa$ space. Clearly each E_α is open. We need to see that it's dense. If $\overline{x} \in {}^\omega\kappa$, then there is some $x' \in D_\alpha$ such that for some $\overline{x'} \in \prod_{n\in x'}[\mu_n, \mu_{n+1})$, $\overline{x'} \subseteq \overline{x}$, so that $\overline{x'} \in E_\alpha$. So because $\lambda < \mathfrak{h}({}^\omega\kappa/\mathcal{I}_2)$, there exists $\overline{x} \in \bigcap_{\alpha\in\lambda} E_\alpha$. Then for every α there is $x_\alpha \in D_\alpha$ with $\overline{x} \subseteq_2^* \overline{x^\alpha} \in \prod_{n\in x_\alpha}[\mu_n, \mu_{n+1})$ for some $\overline{x^\alpha}$. Then if we let $x \in [\omega]^\omega$ be defined by $n \in x$ if and only if $\overline{x} \cap [\mu_n, \mu_{n+1}) \neq \emptyset$, $x \subseteq^* x_\alpha$, so $x \in \bigcap_{\alpha\in\lambda} D_\alpha$. Thus $\mathfrak{h}({}^\omega\kappa/I_2) \leq \mathfrak{h}$. Next, we describe another similar method for showing that $\mathfrak{h}({}^\omega\kappa/\mathfrak{h}_2) \leq \mathfrak{h}$. Let λ, $\langle D_\alpha : \alpha \in \lambda\rangle$ be as above. Fix $A \subseteq {}^\omega\kappa$ a maximal almost disjoint family. Let $E_\alpha = \{x_a : \alpha \in \lambda, x \in D_\alpha, a \in A\} \downarrow\subseteq {}^\omega\kappa$, where for $x \in D_\alpha$ and $a \in A$, $x_a \subseteq a \in {}^\omega\kappa$ is defined by $\alpha \in x_a$ if and only if $\alpha \in a$ and there exists $n \in x$ such that $a(n) = \alpha$. That is, x_a is the copy of x in a. By construction E_α is open, and it is also dense, as follows. If $\overline{x} \in {}^\omega\kappa$, then there exists $a \in A$ with $|a \cap x| = \omega$. Let $x = \{n \in \omega : \exists\alpha \in a \cap x \text{ such that } a(n) = \alpha\} \in [\omega]^\omega$. Then there is $x' \in D_\alpha$ with $x' \subseteq x$, so that $x'_a \subseteq \overline{x}$. So because $\lambda < \mathfrak{h}({}^\omega\kappa/\mathcal{I}_2)$, there exists $\overline{x} \in \bigcap_{\alpha\in\lambda} E_\alpha$. So for every $\alpha \in \lambda$, there exists $a \in A$ and $x_\alpha \in D_\alpha$ with $\overline{x} \subseteq_2^* (x_\alpha)_a$. However, all of these a's must be the same, because if $a_1 \neq a_2$, then $|x'_{a_1} \cap x_{a_2}| < \omega$ for any $x, x' \in [\omega]^\omega$. Thus for a single $a \in A$, $\overline{x} \subseteq_2^* (x_\alpha)_a$. Letting $x = \{n \in \omega : \exists\alpha \in \overline{x} \text{ such that } a(n) = \alpha\} \in [\omega]^\omega$, we see that $x \subseteq^* x_\alpha$, so $x \in D_\alpha$ for every α. The other direction, $\mathfrak{h} \leq \mathfrak{h}({}^\omega\kappa/\mathcal{I}_2)$, is straightforward. Let $\lambda < \mathfrak{h}$, and let $\langle E_\alpha : \alpha \in \lambda\rangle$ be such that $E_\alpha \subseteq {}^\omega\kappa$ is open dense. Let $E_\alpha \upharpoonright \{\mu_n : n \in \omega\} = \{x \cap \{\mu_n : n \in \omega\} : x \in E_\alpha \wedge |x \cap \{\mu_n : n \in \omega\}| = \omega\}$. As a subset of $[\{\mu_n : n \in \omega\}]^\omega$, $E_\alpha \upharpoonright \{\mu_n : n \in \omega\}$ is open dense, and so because $[\{\mu_n : n \in \omega\}]^\omega$ is isomorphic for our purposes to $[\omega]^\omega$, $\bigcap_{\alpha\in\lambda} E_\alpha \upharpoonright \{\mu_n : n \in \omega\} \neq \emptyset$, which suffices.

Next, we show $\mathfrak{h}([\kappa]^\omega/\mathcal{I}_2) = \mathfrak{h}$. Unlike as in the case for $\mathfrak{t}([\kappa]^{\mathrm{cf}(\kappa)}/\mathcal{I}_2)$, this is no more difficult than case above. If $E \subseteq [\kappa]^\omega$ is open dense, then $E \upharpoonright {}^\omega\kappa$, i.e. considering only cofinal ω-sequences, is open dense with respect to ${}^\omega\kappa$. It follows that $\mathfrak{h} = \mathfrak{h}({}^\omega\kappa/\mathcal{I}_2) \leq \mathfrak{h}([\kappa]^\omega/\mathcal{I}_2)$. For the other direction, the proof is very similar to what we've already done for ${}^\omega\kappa$. Let $\lambda < \mathfrak{h}$ and $\langle D_\alpha : \alpha \in \lambda\rangle$ where $D_\alpha \subseteq [\omega]^\omega$ is dense open. Again let $E_\alpha = (\bigcup_{x\in D_\alpha}\prod_{n\in x}[\mu_n, \mu_{n+1})) \downarrow\subseteq {}^\omega\kappa$, where here the $\downarrow$ indicates closure under $\subseteq_2^*$ in the space $[\kappa]^\omega$. If $\overline{x} \in [\kappa]^\omega$, let $\overline{x'} \subseteq \overline{x}$ be a cofinal ω-sequence, so by the same initial argument, there exists $\overline{x''} \in E_\alpha$ such that $\overline{x''} \subseteq \overline{x'}$.

Thus every E_α is open dense, and so there exists $\overline{x} \in \bigcap_{\alpha\in\lambda} E_\alpha$, but then as before there must be for every α some $x_\alpha \in D_\alpha$ such that $\overline{x} \subseteq_2^* \overline{x^\alpha} \in \prod_{n\in x_\alpha} [\mu_n, \mu_{n+1})$, so letting $x \in [\omega]^\omega$ be defined as $n \in x$ if and only if $\overline{x} \cap [\mu_n, \mu_{n+1}) \neq \emptyset$, $x \subseteq^* x_\alpha$, i.e. $\bigcap_{\alpha\in\lambda} D_\alpha \neq \emptyset$. □

We have seen that in order to define $\mathfrak{t}(\kappa)$ in a non-degenerate way we have to exclude towers of length $< \kappa$. This is because, for example, if $\mathrm{cf}(\kappa) > \omega$ there exist countable towers in e.g. $[\kappa]^\kappa / < \kappa$, while if $\mathrm{cf}(\kappa) = \omega$ there exist towers of length ω_1 in e.g. $[\kappa]^\kappa / < \kappa$. The following theorem shows that the exactly analogous behavior occurs for $\mathfrak{h}(\kappa)$. While it was not too difficult as in 3.1.5 to build, for example, a tower of length ω_1 if $\mathrm{cf}(\kappa) = \omega$ in $[\kappa]^\kappa / < \kappa$, if we want to build a sequence of ω_1-many open dense sets in $[\kappa]^\kappa / < \kappa$ with empty intersection, we will have to ensure that all possible paths of length ω_1 through the associated tree of maximal antichains need to be towers, which is perhaps more of a challenge. We prove this fact, and the slightly easier case where $\mathrm{cf}(\kappa) > \omega$ where we find an ω-sequence of open dense sets in $[\kappa]^\kappa / < \kappa$ with empty intersection, by an idea which may be motivated by the basic theory of precipitous ideals, in partcilar the "tree of maximal antichains" characterization of precipitousness. 3.5.2 has been proven independently by Balcar and Simon [5] using forcing-free purely combinatorial arguments.

Theorem 3.5.2. *If cf(κ) $> \omega$, there exist countably many open dense sets in $[\kappa]^\kappa / < \kappa$ with empty intersection. Similarly if cf(κ) $= \omega$, there exist ω_1-many open dense sets in $[\kappa]^\kappa / < \kappa$ with empty intersection. For both of these results the ideal $< \kappa$ can be replaced with $\mathcal{I}_2$, the ideal of bounded subsets.*

Proof. Let $\mathrm{cf}(\kappa) > \omega$. Let $\mathcal{I}$ denote the ideal of subsets of κ of size $< \kappa$, and $\mathcal{I}_2$ denote the ideal of bounded subsets of κ. It is a fact of the folklore that $\mathcal{I}$ and $\mathcal{I}_2$ are never precipitous. This is seen as follows. Consider first $\mathcal{I}$. For any $[Y]_\mathcal{I} \in P(\kappa)/\mathcal{I}$, there exists $[Y']_\mathcal{I} \leq [Y]_\mathcal{I}$ such that if $f_Y : Y \to \kappa$ and $f_{Y'} : Y' \to \kappa$ are the unique order preserving bijections, then $f_{Y'}(\alpha) \in f_Y(\alpha)$ for every $\alpha \in Y'$. This may be seen by removing the limit points of

Y (in its order topology) to form Y'. That is, let the β^{th} element of Y' be the $(\beta+1)^{\text{st}}$ element of Y, $Y'(\beta) = Y(\beta+1)$, for every $\beta \in \kappa$. By density then, if G is $(V, P(\kappa)/\mathcal{I})$-generic, there exists $\langle [Y_n]_{\mathcal{I}} : n \in \omega \rangle \subseteq G$ such that $f_{Y_{n+1}}(\alpha) \in f_{Y_n}(\alpha)$ for every $\alpha \in Y_{n+1}$. Therefore $\langle [f_{Y_n}]_{^\kappa V/G} : n \in \omega \rangle \subseteq {}^\kappa V/G$ is an infinite descending sequence of ordinals in the ultrapower of V by G. The argument for $\mathcal{I}_2$ is the same, the point is that in both cases when refining Y to Y', we simply have to remove κ-many elements and we will get a new element in the forcing. When viewing $P(\kappa)/\mathcal{I}$ as a forcing, we mean of course $(P(\kappa)/\mathcal{I}) \setminus \{[\emptyset]_{\mathcal{I}}\}$. Next, we we more or less follow one half of the argument establishing the "tree of maximal antichains" characterization of precipitousness. It is the same argument for $\mathcal{I}$ or for $\mathcal{I}_2$. Fix $\mathcal{I}$ for concreteness. First note that if $\mathcal{A} \subseteq [\kappa]^\kappa$ is a maximal antichain, by which we mean if $x \in [\kappa]^\kappa$ then there exists $a \in \mathcal{A}$ such that $|a \cap x| = \kappa$, and if $a_1, a_2 \in \mathcal{A}$, then $|a_1 \cap a_2| < \kappa$, then if $\mathcal{D} = \{b \in [\kappa]^\kappa : \exists a \in \mathcal{A} \text{ s.t. } b \subseteq^* a\}$, $\mathcal{D} \subseteq [\kappa]^\kappa$ is open dense. We may denote $\mathcal{D} = \mathcal{A}\downarrow$. Also for $\mathcal{A}$ as above, let $\mathcal{A}' = \{[a]_{\mathcal{I}} : a \in \mathcal{A}\}$, so $\mathcal{A}' \subseteq P(\kappa)/\mathcal{I}$. A tree of maximal antichains is a sequence $\langle \mathcal{A}'_n : n \in \omega \rangle$ of maximal antichains in $P(\kappa)/\mathcal{I}$ such that $\mathcal{A}'_{n+1}$ refines $\mathcal{A}'_n$. A branch through a tree of maximal antichains is a sequence of conditions $\langle p_n : n \in \omega \rangle \subseteq P(\kappa)/\mathcal{I}$ so that $p_{n+1} \leq p_n$ and $p_n \in \mathcal{A}'_n$ for every n. Because $\mathcal{I}$ is never precipitous, $[\kappa]_{\mathcal{I}} = \mathbf{1} \Vdash_{P(\kappa)/\mathcal{I}}$ "${}^\kappa V/G$ is ill-founded". We can then choose terms $\dot{F}_n$ so that $\Vdash \dot{F}_n : \kappa \to V$, $\Vdash \dot{F}_n \in V$, and $\Vdash [\dot{F}_{n+1}]_{^\kappa V/G} E [\dot{F}_n]_{^\kappa V/G}$. In the final statement E is the name for the $\in$-relation defined in the ${}^\kappa V/G$ ultrapower. Let $\mathcal{A}_{-1} = \{\kappa\}$. We construct by recursion a tree of maximal antichains $\mathcal{A}'_n$ from $\mathcal{A}_n \subseteq [\kappa]^\kappa$. First let $\mathcal{A}_0$ be a maximal antichain so that for every $a \in \mathcal{A}_0$, there exists $f^0_a \in V$ such that $[a]_I \Vdash f^0_a = \dot{F}_0$. Now we proceed to construct $\mathcal{A}_1$. For every $a \in \mathcal{A}_0$, form a maximal antichain of elements, b, in $[a]^\kappa$ so that for every $b \in \mathcal{A}_1$, there exists $f^1_b \in V$ so that $[b]_{\mathcal{I}} \Vdash f^1_b = \dot{F}_1$ and with the additional property that for every $\alpha \in b$, $f^1_b(\alpha) \in f^0_a(\alpha)$. This is possible by density. Because it is forced that $[\dot{F}_1]_{^\kappa V/G} E [\dot{F}_0]_{^\kappa V/G}$, and some $[b]_{\mathcal{I}} \leq [a]_{\mathcal{I}}$ fixes the values of $\dot{F}_1$ and $\dot{F}_0$, to f^1_b and f^0_a, we can refine b to some $b' \subseteq b$ which forces on a G-measure one set, which we can assume is just b', that $f^1_b(\alpha) \in f^0_a(\alpha)$. Note that if we form maximal antichains below every a for

$a \in \mathcal{A}_0$, $\mathcal{A}_1$ will constitute a maximal antichain, and $\mathcal{A}'_1$ refines $\mathcal{A}'_0$. Proceed in this manner, and construct a tree of maximal antichains $\mathcal{A}'_n$ so that for every $a \in \mathcal{A}_n$, there exists $f^n_a \in V$ with $[A]_{\mathcal{I}} \Vdash f^n_a = \dot{F}_n$ and if $b \in \mathcal{A}_{n+1}$, $a \in \mathcal{A}_n$, and $[b]_{\mathcal{I}} \leq [a]_{\mathcal{I}}$, then $b \subseteq a$ and for every $\alpha \in b$, $f^{n+1}_b(\alpha) \in f^n_a(\alpha)$. Note that $\{\mathcal{A}_n \downarrow : n \in \omega\}$ is a countable collection of open dense subsets of $[\kappa]^\kappa$. If there existed $b \in \bigcap\limits_{n\in\omega} \mathcal{A}_n \downarrow$, then for each n there exists a unique a_n so that $[b]_{\mathcal{I}} \leq [a_n]_{\mathcal{I}}$ (i.e. $\langle [a_n]_{\mathcal{I}} : n \in \omega\rangle$ is a branch through the tree). Furthermore, because $\mathrm{cf}(\kappa) \neq \omega$, there exists $\alpha \in \bigcap\limits_{n\in\omega} a_n$. But then $\langle f^n_{a_n}(\alpha) : n \in \omega\rangle$ is an infinite descending $\in$-sequence in V, which is impossible. The argument for $\mathcal{I}_2$ is the same.

Next, suppose that $\mathrm{cf}(\kappa) = \omega$. Because there are no countable towers in the case where $\mathrm{cf}(\kappa) = \omega$, every countable collection of open dense sets in this setting has open dense intersection. We show that there are ω_1-many open dense sets in $[\kappa]^\kappa / < \kappa$ with empty intersection. The proof is a natural extension of the above, but requires some extra work. For $A, B \in [\kappa]^\kappa$, say $A \subseteq^{**}_2 B$ if and only if there exists $\gamma \in \kappa$ such that $A \setminus \gamma \subseteq B$ and for every $\delta \in A \setminus \gamma$, $f_A(\delta) < f_B(\delta)$. First, we show that if G is $(V, P(\kappa)/\mathcal{I})$-generic, then there exists $\langle [Y_\alpha]_{\mathcal{I}} : \alpha \in \omega_1\rangle \subseteq G$ such that $\alpha \in \beta \in \omega_1$ implies that $Y_\beta \subseteq^{**}_2 Y_\alpha$. We build this ω_1-length $\subseteq^{**}_2$ descending sequence in G by recursion. Note initially that for any $[A]_{\mathcal{I}} \in [\kappa]^\kappa/\mathcal{I}$, $\{[B] \in [\kappa]^\kappa/\mathcal{I} : B \subseteq^{**}_2 A\}$ is dense below $[A]_{\mathcal{I}}$. So at successors, we can extend the sequence. Furthermore, $\subseteq^{**}_2$ is transitive, i.e. $A \subseteq^{**}_2 B \subseteq^{**}_2 C$ implies that $A \subseteq^{**}_2 C$. So, having defined $\langle [Y_\alpha]_{\mathcal{I}} : \alpha \in \beta\rangle \subseteq G$ for $\beta \in \lim(\omega_1)$, to define $\langle [Y_\alpha]_{\mathcal{I}} : \alpha \in \beta + 1\rangle \subseteq G$ for $\beta \in \lim(\omega_1)$ it suffices to show, by considering a ladder in type ω to β, that, e.g. given $\langle [Y_n]_{\mathcal{I}} : n \in \omega\rangle \subseteq G$, we can find $Y_\omega \subseteq^{**}_2 Y_n$ for every n with $[Y_\omega]_{\mathcal{I}} \in G$. We prove something stronger, namely if only $Y_{n+1} \subseteq^* Y_n$, we can find such a Y_ω. It suffices to show that the following set is dense:

$$D = \{[Y]_{\mathcal{I}} : \text{For every } n,\ Y \subseteq^{**}_2 Y_n \text{ or there exists an } n \text{ such that } |Y \cap Y_n| < \kappa\}.$$

Let $[Y']_{\mathcal{I}} \in P(\kappa)/\mathcal{I}$. If there is an $n \in \omega$ such that $|Y' \cap Y_n| < \kappa$ we're done, so suppose $|Y' \cap Y_n| = \kappa$ for every n. The sequence $\langle Y' \cap Y_n : n \in \omega\rangle$ is $\subseteq^*$-decreasing. Define $Y \in [Y']^\kappa$

as follows. First, fix $\langle \kappa_n : n \in \omega \rangle \subseteq \kappa$ a cofinal sequence of regular cardinals. Let the first κ_0-many elements of Y be the first κ_0-many successors of $Y' \cap Y_0$. Note that this collection is bounded in κ. Next, let the subsequent κ_1-many elements of Y be the first κ_1-many successors of $Y' \cap (Y_0 \cap Y_1)$ which are above $Y_1(\kappa_0)$, the κ_0^{th} element of Y_1, and also above the sup of the ordinals previously added to Y. Again, this collection of ordinals is bounded in κ. In the next step, take the first κ_2-many successors of $Y' \cap (Y_0 \cap Y_1 \cap Y_2)$ which are above $Y_2(\kappa_1)$ and also all ordinals added previously. Proceed in this manner. Note that the Y which results is a subset of Y' and is of size κ. Furthermore, for any given $n \in \omega$, if $\delta \in Y$ and $\delta > Y(\kappa_{n-1})$, then in fact $\delta \in Y_n$ and moreover $f_Y(\delta) < f_{Y_n}(\delta)$. This is because for every $\gamma \geq \kappa_{n-1}$, $Y(\gamma) > Y_n(\gamma)$, because we are only taking successors of subsets of Y_n subsequently. Therefore we can proceed at limit stages of the construction, yielding $\langle [Y_\alpha]_{\mathcal{I}} : \alpha \in \omega_1 \rangle \subseteq G$ such that $\alpha \in \beta \in \omega_1$ implies that $Y_\beta \subseteq_2^{**} Y_\alpha$. Note then that $\langle [f_{Y_\alpha}]_{^\kappa V/G} : \alpha \in \omega_1 \rangle \subseteq {}^\kappa V/G$ is an ω_1-length descending sequence in the ultrapower of V by G.

So, fix $P(\kappa)/\mathcal{I}$-terms $\dot{F}_\alpha$ for $\alpha \in \omega_1$ in V such that $\Vdash \dot{F}_\alpha : \kappa \to V$, $\Vdash \dot{\alpha} \in V$, and $\Vdash [\dot{F}_\beta]_{^\kappa V/G} E [\dot{F}_\alpha]_{^\kappa V/G}$ for every $\alpha \in \beta \in \omega_1$. Note that $P(\kappa)/\mathcal{I}$ is σ-closed so that we may do this. Now, as before let $\mathcal{A}'_{-1} = [\{\kappa\}]_{\mathcal{I}}$. We construct a tree of maximal antichains by recursion. Let $\mathcal{A}'_0$ be maximal such that for every $[a]_{\mathcal{I}} \in \mathcal{A}'_0$, there exists $f_a^0 \in V$ such that $[a]_{\mathcal{I}} \Vdash f_a^0 = \dot{F}_0$. Next, form $\mathcal{A}'_1$ by constructing maximal antichains modulo $\mathcal{I}$ in $[a]^\kappa/\mathcal{I}$ of b's so that for every $[b]_{\mathcal{I}} \in \mathcal{A}'_1$, there exists $f_b^1 \in V$ such that $[b]_{\mathcal{I}} \Vdash f_b^1 = \dot{F}_1$ and with the additional property that for almost every $\alpha \in b$, $f_b^1(\alpha) \in f_a^0(\alpha)$. Here by almost every we mean that there exists $B \in P_\kappa b$ such that for every $\alpha \in b \setminus B$, $f_b^1(\alpha) \in f_a^0(\alpha)$. This is a weaker property than we used in the case where our tree only had to be of length ω, and it allows us to get through the limit stages. For limits, by considering ladders to each $\beta \in \lim(\omega_1)$, it suffices to e.g. show how to construct $\mathcal{A}'_\omega$. $\mathcal{A}'_\omega$ will comprise maximal antichains corresponding to each branch in the tree constructed so far. Fix $\langle [a_n]_{\mathcal{I}} : n \in \omega \rangle$ a branch through the tree, with corresponding $\langle f_{a_n}^n : n \in \omega \rangle \subseteq V$ fixing the values of $\langle \dot{F}_n : n \in \omega \rangle$ with the additional property that for every $n \in m \in \omega$, for almost every $\alpha \in a_m$, $f_{a_m}^m(\alpha) < f_{a_n}^n(\alpha)$. There are no countable towers in $[\kappa]^\kappa/\mathcal{I}$, so there exists (in fact there exist many) $[a_\omega]_\mathcal{I} \leq [a_n]_\mathcal{I}$ for every n. It is forced that $[\dot{F}_\omega]_{^\kappa V/G} E [\dot{F}_n]_{^\kappa V/G}$ for every n, and by using the σ-closure again we can find $[b]_\mathcal{I} \leq [a_\omega]_\mathcal{I}$ and $f^\omega_b \in V$ such that for every $n \in \omega$, for almost every $\alpha \in b$, $f^\omega_b(\alpha) < f^n_{a_n}(\alpha)$. We can keep doing this, fixing then maximal antichains of such $[b]_\mathcal{I}$'s (meaning that if $[b]_\mathcal{I}$ is in such an antichain, then $[b]_\mathcal{I} \leq [a_n]_\mathcal{I}$ for every n and moreover that for every $[c]_\mathcal{I} \leq [a_n]_\mathcal{I}$ for every n, there exists $[b]_\mathcal{I}$ in the antichain such that $|b \cap c| = \kappa$) below every branch $\langle [a_n]_\mathcal{I} : n \in \omega \rangle$, along with corresponding f^ω_b's. Because any $c \in [\kappa]^\kappa$ will intersect at least one $a \in \mathcal{A}_n$ for every n in a set of size κ, and any branch through the resulting "non-trivial intersection tree" will have had a corresponding antichain added at level ω, the union of these antichains will be maximal in $P(\kappa)/\mathcal{I}$. Iterating this process, we can define a tree of maximal antichains, $\langle \mathcal{A}'_\alpha : \alpha \in \omega_1 \rangle$. As before, $\{\mathcal{A}_\alpha \downarrow : \alpha \in \omega_1\}$ is an ω_1-sized collection of dense open subsets of $[\kappa]^\kappa$. Suppose towards a contradiction that there existed $b \in \bigcap_{\alpha \in \omega_1} \mathcal{A}_\alpha \downarrow$. Then for each α there exists a unique a_α such that $[b]_\mathcal{I} \leq [a_\alpha]_\mathcal{I}$. Now $\langle a_\alpha : \alpha \in \omega_1 \rangle \subseteq [\kappa]^\kappa$ is $\subseteq^*$-decreasing such that if $\alpha \in \beta \in \omega_1$, then for almost every $\gamma \in a_\beta$, $f_{a_\beta}(\gamma) < f_{a_\alpha}(\gamma)$. We show that this implies that $\langle a_\alpha : \alpha \in \omega_1 \rangle$ is actually a tower in $[\kappa]^\kappa/\mathcal{I}$, which is a contradiction. First, $b \subseteq^* a_\alpha$ for every α, so fix $\mu \in \kappa$ regular so that for ω_1-many $\alpha \in \omega_1$, there exists $A_\alpha \in P_\mu\kappa$ so that $b \setminus A_\alpha \subseteq a_\alpha$. Then if $A = \bigcup A_\alpha \in [\kappa]^{\leq \omega_1 \cdot \mu}$, $b \setminus A \subseteq a_\alpha$ for all such α. Let the collection of all such α be $B \in [\omega_1]^{\omega_1}$. For every $\gamma \in \eta \in B$, let $A_{\gamma\eta} \in P_\kappa\kappa$ be such that for every $\alpha \in a_\eta \setminus A_{\gamma\eta}$, $f_{a_\eta}(\alpha) < f_{a_\gamma}(\alpha)$. Again there exists $B' \in [B]^{\omega_1}$ and $\mu' \in \kappa$ regular such that for every $\gamma \in \eta \in B'$, $|A_{\gamma\eta}| < \mu'$. Then letting $A' = \bigcup A_{\gamma\eta}$ for $\gamma \in \eta \in B'$, $A' \in [\kappa]^{\leq \omega_1 \cdot \mu'}$. However, then for each $\beta \in b \setminus (A \cup A')$, for $\gamma \in \eta \in B'$, $f_{a_\eta}(\beta) < f_{a_\gamma}(\beta)$. This is then an ω_1-length descending sequence of ordinals, which is impossible. Again as before, the argument for $\mathcal{I}_2$ in place of $\mathcal{I}$ is the same. $\square$

Note 3.5.3. Extracting some of the procedures in the above proof provides an alternate way to that in 3.1.5 of generating towers of length ω_1 in $[\kappa]^\kappa/<\kappa$ in the case where $\mathrm{cf}(\kappa) = \omega$. One simply needs to build a $\subseteq^*$-decreasing ω_1-sequence $\langle a_\alpha : \alpha \in \omega_1 \rangle \subseteq [\kappa]^\kappa$ where for each $\alpha \in \beta \in \omega_1$, $f_\beta <^* f_\alpha$, i.e. there exists $A_{\beta\alpha} \in P_\kappa\kappa$ such that for each $\delta \in a_\beta \setminus A_{\beta\alpha}$,

$f_{a_\beta}(\delta) < f_{a_\alpha}(\delta)$. This can be done by diagonalization, and ensures that $\langle a_\alpha : \alpha \in \omega_1\rangle$ is a tower.

3.6 An application to base trees

The letter $\mathfrak{h}$ was chosen to represent the distributivity number for $[\omega]^\omega / < \omega$ because it is the minimal height of a certain type of tree of maximal antichains which forms a base for the space $[\omega]^\omega$. This type of tree is called a base tree, or base matrix. The following theorem is due to Balcar, Pelant, and Simon [6].

Theorem 3.6.1. *There exists a tree $(T, \subseteq^*)$ such that $T \subseteq [\omega]^\omega$, $ht(T) = \mathfrak{h}$, $Lev_\alpha(T)$ is a MAD family in $[\omega]^\omega / < \omega$ for every $\alpha \in \mathfrak{h}$ (with $Lev_0(T) = \{\kappa\}$ for concreteness) and such that for every $x \in [\omega]^\omega$, there exists $t \in T$ with $t \subseteq x$.*

We can use 3.5.2 to prove in some cases that base trees exist for e.g. $[\kappa]^\kappa / < \kappa$.

Observation 3.6.2. The proof of 3.5.2 shows in particular that:

1. If $\mathrm{cf}(\kappa) > \omega$, there exists $\langle \mathcal{A}_n : n \in \omega\rangle$ such that for every $n \in \omega$, $\mathcal{A}_n \subseteq [\kappa]^\kappa$ is a MAD family in $[\kappa]^\kappa / < \kappa$ (or in $[\kappa]^\kappa / \mathcal{I}_2$) such that $(T = \bigcup_{n\in\omega} \mathcal{A}_n, \subseteq)$ is a tree and for every $b \in [T]$, $\bigcap_{n\in\omega} b(n) = \emptyset$.
2. If $\kappa > \omega$ and $\mathrm{cf}(\kappa) = \omega$, there exists $\langle \mathcal{A}_\alpha : \alpha \in \omega_1\rangle$ such that for every $\alpha \in \omega_1$, $\mathcal{A}_\alpha \subseteq [\kappa]^\kappa$ is a MAD family in $[\kappa]^\kappa / < \kappa$ (or in $[\kappa]^\kappa / \mathcal{I}_2$) such that $(T = \bigcup_{\alpha\in\omega_1} \mathcal{A}_\alpha, \subseteq^*)$ is a tree and for every $b \in [T]$, $\langle b(\alpha) : \alpha \in \omega_1\rangle \subseteq [\kappa]^\kappa$ is a tower.

Using these observations, we can with some additional assumptions prove the existence of base trees in other settings, in the usual way.

Proposition 3.6.3. Let $\kappa > \omega$ be regular with $\mathrm{MAD}(\kappa) \cap (\kappa, 2^\kappa] = \{2^\kappa\}$. Then there exists a tree $(T, \subseteq)$ such that $T \subseteq [\kappa]^\kappa$, $\mathrm{ht}(T) = \omega$, $\mathrm{Lev}_n(T)$ is a MAD family in $[\kappa]^\kappa / < \kappa$ for every $n \in \omega$ (with $\mathrm{Lev}_0(T) = \{\kappa\}$ for concreteness) and such that for every $x \in [\kappa]^\kappa$, there exists $t \in T$ with $t \subseteq x$.

Proof. Let $\kappa > \omega$ be regular with $\mathrm{MAD}(\kappa) \cap (\kappa, 2^\kappa] = \{2^\kappa\}$. As in 3.5.2 fix $\langle \mathcal{A}_n : n \in \omega \rangle$ such that for every $n \in \omega$, $\mathcal{A}_n \subseteq [\kappa]^\kappa$ is a MAD family in $[\kappa]^\kappa / < \kappa$ such that $T' = (\bigcup_{n \in \omega} \mathcal{A}_n, \subseteq)$ is a tree and for every $b \in [T']$, $\bigcap_{n \in \omega} b(n) = \emptyset$. We build our base tree $T \subseteq [\kappa]^\kappa$ by recursion. Let $\mathrm{Lev}_0(T) = \{\kappa\}$. At stages of the construction $2k+1$ for $k \in \omega$, let $\mathrm{Lev}_{2k+1}(T)$ be a MAD family which refines both $\mathrm{Lev}_{2k}(T)$ and $\mathcal{A}_k$. This is of course possible by e.g. choosing representatives for $\{\{[s \cap t] : t \in \mathrm{Lev}_{2k}(T)\} : s \in \mathcal{A}_k\}$. Without loss of generality, also assume that $\mathrm{Lev}_1(T)$ is a MAD family of size 2^κ. At stages of the construction $2k+2$ for $k \in \omega$, let B_{2k+1} denote the set of $x \in [\kappa]^\kappa$ such that $|\{t \in \mathrm{Lev}_{2k+1}(T) : |t \cap x| = \kappa\}| = 2^\kappa$. Let $\langle b_\alpha : \alpha \in 2^\kappa \rangle = B_{2k+1}$ be a surjection. To every $b_\alpha \in B_{2k+1}$, let $\langle t^\alpha_\beta : \beta \in 2^\kappa \rangle = \{t \in \mathrm{Lev}_{2k+1}(T) : |t \cap b_\alpha| = \kappa\}$ be an enumeration. We make sure that $\mathrm{Lev}_{2k+2}(T)$ includes elements which are subsets of each b_α. This can be done by recursion. At stage $\alpha \in 2^\kappa$, choose some $t^\alpha_{\beta_\alpha}$ which has not been chosen before and let $b_\alpha \cap t^\alpha_{\beta_\alpha} \in \mathrm{Lev}_{2k+2}(T)$. Then let $\mathrm{Lev}_{2k+2}(T)$ be an expansion of $\{b_\alpha \cap t^\alpha_{\beta_\alpha} : \alpha \in 2^\kappa\}$ to a MAD family refining $\mathrm{Lev}_{2k+1}(T)$. Proceed in this manner, to build T. It suffices to show that $\bigcup_{k \in \omega} B_{2k+1} = [\kappa]^\kappa$. Suppose towards a contradiction otherwise, and fix $x \in [\kappa]^\kappa$ so that $|\{t \in \mathrm{Lev}_{2k+1}(T) : |t \cap x| = \kappa\}| < 2^\kappa$ for every $k \in \omega$. Consider $T_x = \{t \in T : |t \cap x| = \kappa\}$. It is not difficult to see that $(T_x, \subseteq)$ is a subtree of T of height ω (because every level in T is a MAD family), and furthermore that if $T'_x = \{t \cap x : t \in T_x\}$, for each $n \in \omega$, $\mathrm{Lev}_n(T'_x)$ is a MAD family in $[x]^\kappa / < \kappa$. By assumption, because there are no MAD families in $[\kappa]^\kappa / < \kappa$ of size κ for κ regular, we must have $|\mathrm{Lev}_n(T'_x)| \in \kappa$ for every $n \in \omega$. Because κ is regular then, there must be some $\mu \in \kappa$ such that $|\mathrm{Lev}_n(T'_x)| \leq \mu$ for every $n \in \omega$. Let $y \subseteq x$ denote the set of $\xi \in x$ where for every $n \in \omega$, there exists $t' \in \mathrm{Lev}_n(T'_x)$ such that $\xi \in t'$. Note that $|y| = \kappa$ because $< \kappa$-many ordinals in x are not covered by elements of $\mathrm{Lev}_n(T'_x)$ for every $n \in \omega$, and because κ is regular, so in particular of cofinality $> \omega$, $< \kappa$-many ordinals in x are not covered by nodes on every level. Because every level is of size $< \mu$, and if $\{s', t'\} \subseteq \mathrm{Lev}_n(T'_x)$, $|s' \cap t'| < \kappa$, and because κ is again regular (so in particular of cofinality $> \mu$), there exists some $\mu' \in \kappa$ so that the cardinality of z, the set of ordinals ζ in y so that for some $n \in \omega$ there exit $s' \neq t'$ in $\mathrm{Lev}_n(T'_x)$ with $\zeta \in s' \cap t'$ is $\leq \mu'$. But then there is some $\xi \in y \setminus z$, which has the property that for every $n \in \omega$ there is a unique $t'_n \in \mathrm{Lev}_n(T'_x)$ with $\xi \in t'_n$. Then there is a branch $b \in [T]$ with $\xi \in \bigcap\limits_{n\in\omega} b(n)$. However, by the way that T has been defined (e.g. so that $\mathrm{Lev}_{2k+1}(T)$ is a refinement of $\mathcal{A}_k$), this induces a branch $b' \in [T']$ such that $\bigcap\limits_{n\in\omega} b'(n) = \emptyset$, which is a contradiction. $\square$

Note 3.6.4. If κ is regular and $2^\kappa = \kappa^+$ then $\mathrm{MAD}(\kappa) \cap (\kappa, 2^\kappa] = \{2^\kappa\}$, and so there is a base tree as described in 3.6.3. This case has also been noted in [5].

Proposition 3.6.5. Let κ be singular with $\mathrm{cf}(\kappa) = \omega$, $2^\omega > \kappa$, and $\mathrm{MAD}(\kappa) \cap [2^\omega, 2^\kappa] = \{2^\kappa\}$. Then there exists a tree $(T, \subseteq^*)$ such that $T \subseteq [\kappa]^\kappa$, $\mathrm{ht}(T) = \omega_1$, $\mathrm{Lev}_\alpha(T)$ is a MAD family in $[\kappa]^\kappa / < \kappa$ for every $\alpha \in \omega_1$ (with $\mathrm{Lev}_0(T) = \{\kappa\}$ for concreteness) and such that for every $x \in [\kappa]^\kappa$, there exists $t \in T$ with $t \subseteq x$. Here we could replace the ideal $< \kappa$ with $\mathcal{I}_2$.

Proof. Let $\kappa > \omega$ with $\mathrm{cf}(\kappa) = \omega$. As in 3.5.2 fix $\langle \mathcal{A}_\alpha : \alpha \in \omega_1 \rangle$ such that for every $\alpha \in \omega_1$, $\mathcal{A}_\alpha \subseteq [\kappa]^\kappa$ is a MAD family in $[\kappa]^\kappa / < \kappa$ such that $(T = \bigcup\limits_{\alpha\in\omega_1} \mathcal{A}_\alpha, \subseteq^*)$ is a tree and for every $b \in [T]$, $\langle b(\alpha) : \alpha \in \omega_1 \rangle \subseteq [\kappa]^\kappa$ is a tower. We build our base tree $T \subseteq [\kappa]^\kappa$ by recursion. Let $\mathrm{Lev}_0(T) = \{\kappa\}$. At a stage $\alpha \in \lim(\omega_1)$ of the construction suppose we have built $T \restriction \alpha$. For every $b \in [T \restriction \alpha]$, let $B_b = \{x \in [\kappa]^\kappa : x \subseteq^* b(\gamma) \text{ for every } \gamma \in \alpha\}$. Because there are not towers of length ω, every $B_b \neq \emptyset$. Moreover, if $b_1 \neq b_2$, then $B_{b_1} \cap B_{b_2} = \emptyset$ and every B_b is closed under $\subseteq^*$. It is not difficult to see that $\bigcap\limits_{\gamma\in\alpha} \mathrm{Lev}_\gamma(T) \downarrow = \bigcup\limits_{b\in[T\restriction\alpha]} B_b$, and because the countable intersection of open dense sets in $[\kappa]^\kappa / < \kappa$ is open dense, $\bigcup\limits_{b\in[T\restriction\alpha]} B_b$ is open dense, so we can let $\mathrm{Lev}_\alpha(T)$ be a maximal antichain in $\bigcup\limits_{b\in[T\restriction\alpha]} B_b$ (which is therefore maximal in $[\kappa]^\kappa / < \kappa$). Note that we may then have splitting at limits. At successor stages $2\alpha + 1$, let $\mathrm{Lev}_{2\alpha+1}(T)$ be a MAD family which refines both $\mathrm{Lev}_{2\alpha}(T)$ and $\mathcal{A}_\alpha$. At successor stages $2\alpha+2$, let $B_{2\alpha+1}$ denote the set of $x \in [\kappa]^\kappa$ such that $|\{t \in \mathrm{Lev}_{2\alpha+1}(T) : |t \cap x| = \kappa\}| = 2^\kappa$. Let $\langle b_\beta : \beta \in 2^\kappa \rangle = B_{2\alpha+1}$ be a surjection. To every $b_\beta \in B_{2\alpha+1}$, let $\langle t^\beta_\gamma : \gamma \in 2^\kappa \rangle = \{t \in \mathrm{Lev}_{2\alpha+1}(T) : |t \cap b_\beta| = \kappa\}$ be an enumeration. We make sure that $\mathrm{Lev}_{2\alpha+2}(T)$ includes elements which are subsets of each b_β. This can be done by recursion. At stage $\beta \in 2^\kappa$, choose some $t^\beta_{\gamma_\beta}$ which has not been chosen before and let $b_\beta \cap t^\beta_{\gamma_\beta} \in \mathrm{Lev}_{2\alpha+2}(T)$. Then let $\mathrm{Lev}_{2\alpha+2}(T)$ be an expansion of $\{b_\beta \cap t^\beta_{\gamma_\beta} : \beta \in 2^\kappa\}$ to a MAD family refining $\mathrm{Lev}_{2\alpha+1}(T)$. Proceed in this manner to build T. Again, it suffices to show that $\bigcup_{\alpha \in \omega_1} B_{2\alpha+1} = [\kappa]^\kappa$. Choose $x \in [\kappa]^\kappa$. Because $\bigcap_{\alpha \in \omega_1} \mathrm{Lev}_\alpha(T) \downarrow \subseteq \bigcap_{\alpha \in \omega_1} \mathcal{A}_\alpha \downarrow = \emptyset$, for some minimal $\alpha_0 \in \omega_1$, $x \notin \mathrm{Lev}_{\alpha_0}(T)\downarrow$. Because $\mathrm{Lev}_{\alpha_0}(T)$ is maximal, this implies that there exists $t^{\langle 0 \rangle} \neq t^{\langle 1 \rangle}$ in $\mathrm{Lev}_{\alpha_0}(T)$ such that $|t^{\langle 0 \rangle} \cap x| = |t^{\langle 1 \rangle} \cap x| = \kappa$. Continuing, there must exist some minimal $\alpha^{\langle 0 \rangle} > \alpha_0$ such that $t^{\langle 0 \rangle} \cap x \notin \mathrm{Lev}_{\alpha^{\langle 0 \rangle}}(T)\downarrow$ and some corresponding $t'^{\langle 00 \rangle} \neq t'^{\langle 01 \rangle}$ in $\mathrm{Lev}_{\alpha^{\langle 0 \rangle}}(T)$ extending $t^{\langle 0 \rangle}$ with $|t'^{\langle 00 \rangle} \cap (t^{\langle 0 \rangle} \cap x)| = |t'^{\langle 01 \rangle} \cap (t^{\langle 0 \rangle} \cap x)| = \kappa$ and some minimal $\alpha^{\langle 1 \rangle} > \alpha_0$ such that $t^{\langle 1 \rangle} \cap x \notin \mathrm{Lev}_{\alpha^{\langle 1 \rangle}}(T)\downarrow$ and some corresponding $t'^{\langle 10 \rangle} \neq t'^{\langle 11 \rangle}$ in $\mathrm{Lev}_{\alpha^{\langle 1 \rangle}}(T)$ extending $t^{\langle 1 \rangle}$ with $|t'^{\langle 11 \rangle} \cap (t^{\langle 1 \rangle} \cap x)| = |t'^{\langle 10 \rangle} \cap (t'^{\langle 1 \rangle} \cap x)| = \kappa$. For $\alpha_1 = \sup\{\alpha^{\langle 0 \rangle}, \alpha^{\langle 1 \rangle}\}$, we can then find distinct nodes $t^{\langle 00 \rangle}, t^{\langle 01 \rangle}, t^{\langle 10 \rangle}$, and $t^{\langle 11 \rangle}$ in $\mathrm{Lev}_{\alpha_1}(T)$ such that $t^{\langle 00 \rangle}, t^{\langle 01 \rangle}$ extend $t^{\langle 0 \rangle}$, $|t^{\langle 00 \rangle} \cap (t^{\langle 0 \rangle} \cap x)| = |t^{\langle 01 \rangle} \cap (t^{\langle 0 \rangle} \cap x)| = \kappa$, $t^{\langle 10 \rangle}, t^{\langle 11 \rangle}$ extend $t^{\langle 1 \rangle}$, and $|t^{\langle 10 \rangle} \cap (t^{\langle 1 \rangle} \cap x)| = |t^{\langle 11 \rangle} \cap (t^{\langle 1 \rangle} \cap x)| = \kappa$. We can proceed in this manner to build an embedding $e : {}^{<\omega}2 \to T$ with $e(\emptyset) = \{\kappa\}$ such that if $s, s' \in {}^n2$ then $\mathrm{lh}(e(s)) = \mathrm{lh}(e(s')) = \alpha_{n-1}$ and $|e(s) \cap x| = |e(s') \cap x| = \kappa$. If $\alpha = \bigcup_{n \in \omega} \alpha_n$, then it is not difficult to see then that $|\{t \in \mathrm{Lev}_\alpha(T) : |t \cap x| = \kappa\}| \geq 2^\omega$. Because $\mathrm{MAD}(\kappa) \cap [2^\omega, 2^\kappa] = \{2^\kappa\}$, we must then have, again because $\mathrm{Lev}_\alpha(T'_x)$ is MAD in $[x]^\kappa$, that $|\{t \in \mathrm{Lev}_\alpha(T) : |t \cap x| = \kappa\}| = 2^\kappa$, as desired. $\square$

3.7 Defining $\mathfrak{h}(\kappa)$

While it is straightforward to define $\mathfrak{t}(\kappa)$ in a nontrivial way as in 3.2.1 or 3.2.3, it is not as clear how one might be able to define $\mathfrak{h}(\kappa)$ in a nontrivial way. In analogy with 3.2.3, one natural possibility for a definition of $\mathfrak{h}(\kappa)$ would be to say that it's the smallest cardinality of a collection of open dense subsets of $[\kappa]^\kappa$ (with respect to the ideal $\mathcal{I} =< \kappa$, for example) such that every subset of size $< \kappa$ has open dense intersection, but the intersection of the entire collection is not open dense. Call this property that a collection of open dense subsets can have the κ-SIP. A first step in showing that this definition is reasonable is to try to see that for κ regular, $\kappa < \mathfrak{h}(\kappa)$. We will see in 3.7.4 that without further qualification, even with κ being regular this is not enough to yield a reasonable definition.

Suppose $\langle D_\alpha : \alpha \in \kappa \rangle$ is a collection of open dense sets in $[\kappa]^\kappa / < \kappa$ with the κ-SIP. We may assume that $D_\beta \subseteq D_\alpha$ for every $\alpha \in \beta \in \kappa$. It may even be that each D_α is of the form $\bigcup \mathcal{A}_\alpha \downarrow$ for some $\mathcal{A}_\alpha \subseteq [\kappa]^\kappa$ a maximal antichain. If this is the case, we may further suppose that if $\alpha \in \beta$, then $\mathcal{A}_\beta$ is a refinement of $\mathcal{A}_\alpha$, that is for every $b \in \mathcal{A}_\beta$, there exists a unique $a \in \mathcal{A}_\alpha$ with $b \subseteq^* a$. So we have a tree of maximal antichains. However, there do not exist towers of length κ, so the only way for $\bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ to be empty would be if this tree had no branches. We first give an example with $\kappa = \omega_1$ where this does not happen.

3.7.1 An example with the κ-SIP which works

Let $\kappa = \omega_1$. We build a tree of maximal antichains $\langle \mathcal{A}_\alpha : \alpha \in \omega_1 \rangle$ in $[\omega_1]^{\omega_1} / < \omega_1$ of height ω_1 such that $\bigcap_{\alpha\in\omega_1} \mathcal{A}_\alpha \downarrow$ is open dense. This tree is a natural one to consider, and corresponds to $T^{\omega_1}_{<\omega}$, so is not fully trivial in the sense that there are many maximal paths through the tree of countable length, which correspond to towers of antichain elements. Let $\mathcal{A}_0 = \{\omega_1\}$. Let $\mathcal{A}_1 = \{\lim(\omega_1), \mathrm{succ}(\omega_1)\} = \{a^{\langle 0\rangle}, a^{\langle 1\rangle}\}$. Proceeding, we have

$\mathcal{A}_2 = \{\lim(\lim(\omega_1)), \mathrm{succ}(\lim(\omega_1)), \lim(\mathrm{succ}(\omega_1)), \mathrm{succ}(\mathrm{succ}(\omega_1))\} = \{a^{\langle 00\rangle}, a^{\langle 01\rangle}, a^{\langle 10\rangle}, a^{\langle 11\rangle}\}$, etc. for every $n \in \omega$. Here when we write, for example, $\lim(\mathrm{succ}(\omega_1))$, we mean the set of ordinals α where the image of α in the Mostowski collapse of the set of successors of κ is a limit ordinal. That is, $\mathrm{succ}(\omega_1) \cap \alpha$ is cofinal in α with respect to $\mathrm{succ}(\omega_1)$.

Lemma 3.7.1. $\bigcap\limits_{n\in\omega} \mathcal{A}_n \downarrow$ is open dense, and there exits $\mathcal{A}_\omega \subseteq [\omega_1]^{\omega_1}$ which is a countable partition of ω_1 (and so is a maximal antichain because $\mathrm{cf}(\omega_1) = \omega_1 > \omega$) with $\mathcal{A}_\omega$ refining every $\mathcal{A}_n$. That is, $\langle \mathcal{A}_\alpha : \alpha \in \omega + 1\rangle$ is a tree of maximal antichains in $[\omega_1]^{\omega_1}/<\omega_1$ of height $\omega + 1$. This tree is also normal (there is no splitting at level ω).

Proof. Let $\mathcal{A}_\omega = \{\bigcap\limits_{n\in\omega} a^{\langle b\restriction n\rangle} : b \in {}^\omega 2 \text{ s.t. } |\{n : b(n) = 1\}| < \omega\}$. That is, $\mathcal{A}_\omega$ is the countable collection of the intersections of nodes along branches which are eventually 0 in the tree constructed up to stage ω. First, note that if a branch is eventually 0, this corresponds to taking, for some $A \in [\omega_1]^{\omega_1}$, $\lim(A) \cap \lim(\lim(A)) \cap \ldots$. Each of these $\lim^n(A)$ sets is a club relative to A, so this intersection is a club relative to A, that is $\bigcap\limits_{n\in\omega} a^{\langle b\restriction n\rangle} \in [\omega_1]^{\omega_1}$ if $b \in {}^\omega 2$ is such that $|\{n : b(n) = 1\}| < \omega$. On the other hand, for any $b \in {}^\omega 2$ with $|\{n : b(n) = 1\}| = \omega$, it can be seen that $\bigcap\limits_{n\in\omega} a^{\langle b\restriction n\rangle} = \emptyset$, as follows. First, we argue that if $A \in [\omega_1]^{\omega_1}$, $\alpha = A(\beta_1)$, that is α is the β_1^{th} element of A, and $\alpha = \mathrm{succ}(A)(\beta_2)$, that is α is the β_2^{th} element of $\mathrm{succ}(A)$, then $\beta_2 < \beta_1$. Clearly of course, because $\mathrm{succ}(A) \subseteq A$, $\beta_2 \leq \beta_1$. Note that this suffices, because if $b \in {}^\omega 2$ with $|\{n : b(n) = 1\}| = \omega$, as for $n \leq m$, $a^{\langle b\restriction m\rangle} \subseteq a^{\langle b\restriction n\rangle}$, because b has an infinite number of 1's, for any $\alpha \in \bigcap\limits_{n\in\omega} a^{\langle b\restriction n\rangle}$, we would have an infinite $\leq$-descending sequence of ordinals β_n such that $\alpha = a^{\langle b\restriction n\rangle}(\beta_n)$ which doesn't stabilize, a contradiction. Note that $\alpha \in \mathrm{succ}(A)$ by assumption. So if $\alpha = A(\beta_1)$, $\beta_1 = \gamma_1 + k$ for some $\gamma_1 \in \lim(\omega_1)$. Because $A(\gamma_1) \notin \mathrm{succ}(A)$, $\mathrm{succ}(A)(\gamma_1) \geq A(\gamma_1 + 1)$. But then proceeding up to k, $\mathrm{succ}(A)(\gamma_1 + (k-1)) \geq A(\beta_1)$. Thus $\beta_2 \leq \gamma_1 + (k-1) < \beta_1$, as desired. This implies that $\mathcal{A}_\omega$ is a partition of ω_1, as each $\mathcal{A}_n$ partitions ω_1, so for every $\alpha \in \omega_1$, α determines a branch through $\langle \mathcal{A}_n : n \in \omega\rangle$ via the unique $a^s \in \mathcal{A}_n$ for each n with $\alpha \in a^s$. Because α is in the intersection of the nodes along this branch, this branch must contain only finitely many

1's. Being a countable partition of ω_1, $\mathcal{A}_\omega$ is a maximal antichain, and we have seen that it refines every $\mathcal{A}_n$. It is also straightforward to show that $\bigcap_{n\in\omega} \mathcal{A}_n \downarrow = \mathcal{A}_\omega \downarrow$. $\square$

Continuing the construction, at successor stages in ω_1 as done initially we split each node in $\mathcal{A}_\beta$ into its relative limit points and relative successor points to define $\mathcal{A}_{\beta+1}$. At limit stages $\gamma \in \omega_1$, we proceed as we did for $\gamma = \omega$. That is, if we have built $\langle \mathcal{A}_\beta : \beta \in \gamma \rangle$ the tree of maximal antichains, we can define $\mathcal{A}_\gamma = \{\bigcap_{\beta\in\gamma} a^{\langle b \restriction \beta \rangle} : b \in {}^\gamma 2$ such that $|\{\beta \in \gamma : b(\beta) = 1\}| < \omega\}$. Just as in the case where $\gamma = \omega$, it is not difficult to see that $\mathcal{A}_\gamma$ is a countable partition of ω_1 refining every $\mathcal{A}_\beta$ and $\langle \mathcal{A}_\beta : \beta \leq \gamma \rangle$ constitutes a tree of maximal antichains of height $\gamma + 1$ which is normal. Proceeding in this manner, we can define $\langle \mathcal{A}_\alpha : \alpha \in \omega_1 \rangle$ a tree of maximal antichains of height ω_1. We argue that $\bigcap_{\alpha\in\omega_1} \mathcal{A}_\alpha \downarrow$ is open dense. Let $T \subseteq {}^{<\omega_1}2$ be the natural representation of $\langle \mathcal{A}_\alpha : \alpha \in \omega_1 \rangle$ as a subtree of ${}^{<\omega_1}2$. It is clear to see that $T = \{s \in {}^{<\omega_1}2 : |\{\alpha \in \mathrm{lh}(s) : s(\alpha) = 1\}| < \omega\} = T^{\omega_1}_{<\omega}$ so that $[T] = \{b \in {}^{\omega_1}2 : |\{\alpha \in \omega_1 : b(\alpha) = 1\}| < \omega\}$. For such b, $\langle b(\alpha) : \alpha \in \omega_1 \rangle \subseteq [\omega_1]^{\omega_1}$ is $\subseteq$-decreasing, and so because there are no towers of length ω_1, there exists $x \in [\omega_1]^{\omega_1}$ with $x \subseteq^* b(\alpha)$ for every α. Here of course by $b(\alpha)$ we mean, for example, $a^{b\restriction(\alpha+1)}(\alpha)$. Indeed, in this discussion T and the tree of maximal antichains will often be conflated, but it will be clear what is meant. Accordingly, define $A_b = \{x \in [\omega_1]^{\omega_1} : x \subseteq^* b(\alpha)$ for every $\alpha \in \omega_1\}$. Unlike in the case of the countable branches we considered previously, we will see that there isn't a single generator for any such A_b. For $b_1 \neq b_2$, clearly $A_{b_1} \cap A_{b_2} = \emptyset$, and every A_b is open (closed under $\subseteq^*$) by construction. Furthermore, clearly $\bigcup_{b\in[T]} A_b \subseteq \bigcap_{\alpha\in\omega_1} \mathcal{A}_\alpha \downarrow$. We argue that in fact $\bigcup_{b\in[T]} A_b = \bigcap_{\alpha\in\omega_1} \mathcal{A}_\alpha \downarrow$, and moreover that $\bigcup_{b\in[T]} A_b$ is dense. First, suppose $x \in \bigcap_{\alpha\in\omega_1} \mathcal{A}_\alpha \downarrow$. Then x determines a branch through T, and so necessarily this b is such that $|\{\alpha \in \omega_1 : b(\alpha) = 1\}| < \omega$ and $x \in A_b$. So indeed, $\bigcup_{b\in[T]} A_b = \bigcap_{\alpha\in\omega_1} \mathcal{A}_\alpha \downarrow$. To see that $\bigcup_{b\in[T]} A_b$ is dense, we will need the fact proved in this thesis' chapter on trees that T contains no Aronszajn subtrees. By definition, $\bigcup_{b\in[T]} A_b$ is open. To see that it's dense, let $x \in [\omega_1]^{\omega_1}$. For every $\alpha \in \omega_1$, because each $\mathcal{A}_\alpha$ is a maximal antichain there exists $a_\alpha \in \mathcal{A}_\alpha$

with $|a_\alpha \cap x| = \omega_1$. Furthermore, the collection of every such a_α in each level α determines a subtree of height ω_1, $T_x \subseteq T$. Because T contains no Aronszajn subtrees, there exists $b \in [T_x]$. Then $\langle x \cap b(\alpha) : \alpha \in \omega_1 \rangle \subseteq [\omega_1]^{\omega_1}$ is $\subseteq^*$-decreasing, so there exists $y \subseteq^* x \cap b(\alpha)$ for every α, but then there is $y' \in A_b$ with $y' \subseteq x$, as desired. Note that because there are no maximal antichains in $[\omega_1]^{\omega_1}/<\omega_1$ of size ω_1, and $|[T]| = \omega_1$, it is not the case that every A_b is generated by a single element of $[\omega_1]^{\omega_1}$. So if we choose a maximal antichain in $\bigcup_{b\in[T]} A_b$, and consider the resulting tree of maximal antichains, which is now of height $\omega_1 + 1$, it is no longer normal, in that on level ω_1 we have distinct nodes with the same collection of predecessors.

3.7.2 An example with the κ-SIP which doesn't work

Observation 3.7.2. Let κ be regular. Let $T \subseteq {}^{<\kappa}(2^\kappa)$ be a tree of maximal antichains in $[\kappa]/<\kappa$ of height κ. Let $\langle \mathcal{A}_\alpha : \alpha \in \kappa \rangle$ be the antichains on each level. If there exists $S \subseteq T$ which is a κ-Aronszajn tree, then $\bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ is not open dense. On the other hand, if $\bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ is not open dense, then there exists $S \subseteq T$ a subtree of height κ such that $[S] = \emptyset$.

Proof. First, suppose that there exists $S \subseteq T$ which is a κ-Aronszajn tree. Then for each $\alpha \in \kappa$, $|\mathcal{A}_\alpha \cap S| < \kappa$. Let $x_\alpha = \{\delta \in \kappa : \text{there exists } a \in S \cap \mathcal{A}_\alpha \text{ such that } \delta \in a\} \in [\kappa]^\kappa$. Note that if $\alpha \in \beta \in \kappa$, then for each $a \in S \cap \mathcal{A}_\beta$, there exists a unique $a' \in S \cap \mathcal{A}_\alpha$ with $a \subseteq^* a'$. So, because $|S \cap \mathcal{A}_\beta| < \kappa$, $x_\beta = \{\delta \in \kappa : \text{there exists } a \in S \cap \mathcal{A}_\beta \text{ such that } \delta \in a\} \subseteq^* \{\delta \in \kappa : \text{there exists } a \in S \cap \mathcal{A}_\alpha \text{ such that } \delta \in a\} = x_\alpha$. That is, $\langle x_\alpha : \alpha \in \kappa \rangle \subseteq [\kappa]^\kappa$ is $\subseteq^*$-decreasing. Because there are no towers of length κ, there exists $x \in [\kappa]^\kappa$ with $x \subseteq^* x_\alpha$ for every α. Suppose towards a contradiction that $\bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ is dense. Then there exists $y \in \bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ with $y \subseteq x$. Because $y \in \bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$, for every $\alpha \in \kappa$ there exists a unique $a_\alpha \in \mathcal{A}_\alpha$ with $y \subseteq^* a_\alpha$. However, $y \subseteq x \subseteq^* x_\alpha$, and x_α is a union of $<\kappa$-many antichain elements. So there exists some $a \in S \cap \mathcal{A}_\alpha$ with $|a \cap y| = \kappa$. We must then have $a_\alpha = a \in S$.

But then $\langle a_\alpha : \alpha \in \kappa \rangle \in [S]$ is a branch, a contradiction. Next, suppose that $\bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ is not open dense. Find $x \in [\kappa]^\kappa$ such that there does not exist $y \in \bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ with $y \subseteq^* x$. Define $T_x \subseteq T$ to be the tree comprising on each level α the nodes $a \in \mathcal{A}_\alpha$ such that $|x \cap a_\alpha| = \kappa$. It is clear that T_x is a tree, and because each $\mathcal{A}_\alpha$ is maximal, $\mathrm{ht}(T_x) = \kappa$. However, $[T_x] = \emptyset$. For if $b \in [T_x]$, then $\langle x \cap b(\alpha) \rangle \subseteq [\kappa]^\kappa$ is $\subseteq^*$-decreasing so there is $x' \subseteq^* x \cap b(\alpha)$ for each α, i.e. $x' \in \bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ and $x' \subseteq^* x$, a contradiction. $\square$

In our particular example with $\kappa = \omega_1$ and $T = T^{\omega_1}_{<\omega}$ above, where the levels of T had $< \kappa$-many elements to start with, containing a κ-Aronszajn subtree was equivalent to containing a subtree of height κ with no branches. So the two observations in 3.7.2 were full converses of one another, which allowed the argument that $\bigcap_{\alpha\in\kappa} \mathcal{A}_\alpha \downarrow$ was open dense to go through. However, it is not always the case that a κ-tree of maximal antichains doesn't contain a κ-Aronszajn subtree.

Proposition 3.7.3. Let κ be regular and suppose exists a κ-Aronszajn tree. Then there exists a κ-Aronszajn tree of maximal antichains in $[\kappa]^\kappa / < \kappa$.

Proof. Let $T \subseteq {}^{<\kappa}2$ be a normal κ-Aronszajn tree, and let $f : T \to \kappa$ be a bijection. Associate to each $s \in T$ the set $N_s = \{s' \in T : s' \restriction \mathrm{lh}(s) = s\}$, that is N_s is the collection of nodes in T extending s (including s). If $\alpha \in \kappa$, then $\mathcal{A}_\alpha = \{N_s : s \in \mathrm{Lev}_\alpha(T)\}$ is a partition of $\kappa \setminus f''(T \restriction \alpha)$ of size $< \kappa$, and so is a maximal antichain in $[\kappa]^\kappa / < \kappa$. Strictly speaking of course it is $\{[f''N_s]_{<\kappa} : s \in \mathrm{Lev}_\alpha(T)\}$ which is the maximal antichain, but can ignore this distinction. It is clear that if $\alpha \in \beta \in \kappa$ then $\{N_s : s \in \mathrm{Lev}_\beta(T)\}$ is a refinement of $\{N_s : s \in \mathrm{Lev}_\alpha(T)\}$, so $\langle \mathcal{A}_\alpha : \alpha \in \kappa \rangle$ is a tree of maximal antichains in $[\kappa]^\kappa / < \kappa$ of length κ. However, because every path through this tree corresponds uniquely to a path through T and vice-versa, there are no branches through this tree (because all paths through T are of length $< \kappa$). $\square$

Note 3.7.4. We see in 3.7.3 that for those regular κ without the tree property, there is a collection of κ-many open dense sets with empty intersection with the property that any subcollection of size $< \kappa$ has open dense intersection, i.e. a collection with the κ-SIP. So for these κ, the naive definition of $\mathfrak{h}(\kappa)$ is not suitable.

3.8 Unconsidered directions

1. Generally does exist a tree of maximal antichains T in $[\kappa]^\kappa / < \kappa$ of height κ such that $\bigcap_{\alpha \in \kappa} \mathcal{A}_\alpha \downarrow$ is not open dense? This would, of course, be a barrier to a formulation of a workable definition of $\mathfrak{h}(\kappa)$. We have seen this can be done if there exists a κ-Aronszajn tree, but this isn't necessarily the case (of particular interest here is where κ is weakly compact).

2. Can there exist a tree T of maximal antichains in $[\kappa]^\kappa / < \kappa$ such that $\bigcap_{\alpha \in \kappa} \mathcal{A}_\alpha \downarrow$ is open dense, but there exists a subtree of T of height κ with no branches? In particular, can $[T] = \emptyset$? Necessarily we have seen that T cannot contain κ-Aronszajn subtrees.

Chapter 4

Matching families of functions

4.1 Notation and background

The notation $\mathbf{nm}(\kappa)$ and $\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa)$ was introduced by Blass et al. in [11].

Definition 4.1.1. Say that a family of functions $F \subseteq {}^{\gamma}\delta$ is κ-matching if and only if for every $g \in {}^{\gamma}\delta$, there exists $f \in F$ such that $|\{\xi \in \gamma : f(\xi) = g(\xi)\}| \geq \kappa$. Similarly define $(< \kappa)$-matching. Let $\mathbf{nm}_{\kappa}({}^{\gamma}\delta)$ and $\mathbf{nm}_{<\kappa}({}^{\gamma}\delta)$, respectively, denote the minimal cardinality of such a collection. If $\gamma = \delta$, we write $\mathbf{nm}_{\kappa}(\gamma)$ or $\mathbf{nm}_{<\kappa}(\gamma)$. We also use $\mathbf{nm}(\kappa)$ to denote $\mathbf{nm}_{\kappa}(\kappa)$.

Definition 4.1.2. Let κ be regular. An interval partition of κ is a partition Π of κ into intervals $[\alpha, \beta)$. A chopped κ-sequence, or a chopped κ-real, is a pair $\langle x, \Pi \rangle$, where $x \in {}^{\kappa}2$. Say that $y \in {}^{\kappa}2$ (cofinally) matches x if and only if there are κ-many intervals $I \in \Pi$ where $x \restriction I = y \restriction I$. We often identify Π with the set of its left endpoints, $\{i_{\alpha} : \alpha \in \kappa\}$. The enumerating function for this set is normal. Let $\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa)$ be equal to the minimal cardinality of $X \subseteq {}^{\kappa}2$ such that for every $\langle y, \Pi \rangle$ there exists $x \in X$ such that x matches $\langle y, \Pi \rangle$.

Definition 4.1.3. Say that a family of functions $F \subseteq {}^{\gamma}\delta$ is κ-disjointing if and only if for every $g \in {}^{\gamma}\delta$ there exists $f \in F$ such that $|\{\xi \in \gamma : f(\xi) = g(\xi)\}| < \kappa$. Let $\mathbf{cv}_{\kappa}({}^{\gamma}\delta)$ denote the minimal family of such a collection. If $\gamma = \delta$, we write $\mathbf{cv}_{\kappa}(\gamma)$. We also use $\mathbf{cv}(\kappa)$ to denote $\mathbf{cv}_{\kappa}(\kappa)$.

Fact (Bartoszyński) 4.1.4. $\mathbf{non}(\mathcal{M})$, the uniformity number for category, that is the smallest cardinality of a non-meager set, is equal to $\mathbf{nm}_{\omega}(\omega)$. This is also equal to $\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\omega)$. See [8].

Fact (Bartoszyński) 4.1.5. $\mathbf{cov}(\mathcal{M})$, the covering number for category, that is the smallest cardinality of a set of meager sets whose union is $\mathbb{R}$, is equal to $\mathbf{cv}(\omega)$. See [8].

Note that $\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\omega)$ and $\mathbf{nm}(\omega)$ give purely combinatorial characterizations of the uniformity number for category, which can have several advantages. For example, it might allow one to better understand how certain (iterated) forcings might affect this quantity. The generalizations to other (regular) κ in 4.1.1 and 4.1.2 is natural. Similarly $\mathbf{cv}(\omega)$ gives a purely combinatorial characterization of the covering number for category, and the generalization as in 4.1.3 is natural. The following more standard definitions are useful for context.

Definition 4.1.6. Say that a family of functions $F \subseteq {}^{\gamma}\delta$ is κ-cofinal or κ-dominating, if and only if for every $g \in {}^{\gamma}\delta$, there exists $f \in F$ such that $|\{\xi \in \gamma : g(\xi) \geq f(\xi)\}| < \kappa$. Call the cardinality of the smallest such collection $\mathfrak{d}_{\kappa}({}^{\gamma}\delta)$. If $\gamma = \delta$, we also write $\mathfrak{d}_{\kappa}(\gamma)$. In the case where $\kappa = \gamma$, we write $\mathfrak{d}(\kappa)$. For κ regular, this is the cofinality of ${}^{\kappa}\kappa$ under the eventual domination relation.

Definition 4.1.7. Say that a family of functions $F \subseteq {}^{\gamma}\delta$ is κ-unbounded if and only if for every $g \in {}^{\gamma}\delta$, there exists $f \in F$ such that $|\{\xi \in \gamma : f(\xi) \geq g(\xi)\}| \geq \kappa$. Call the cardinality of the smallest such collection $\mathfrak{b}_{\kappa}({}^{\gamma}\delta)$. If $\gamma = \delta$, we also write $\mathfrak{b}_{\kappa}(\gamma)$. In the case where $\kappa = \gamma$, we write $\mathfrak{b}(\kappa)$.

4.2 Matching observations

Observation 4.2.1. Let $\lambda \geq \kappa$. Then $\mathbf{nm}_\kappa({}^\kappa\lambda) \geq \kappa^+$.

Proof. Let $F \subseteq {}^\kappa\lambda$ with $|F| \leq \kappa$. Let $F = \langle f_\alpha : \alpha \in \kappa\rangle$ be a surjection and let $g \in {}^\kappa\kappa$ be defined by $g(\xi) = \min(\lambda \setminus \{f_\zeta(\xi) : \zeta \in \xi\})$. It is clear that g is not κ-matched by F. □

Observation 4.2.2. Let $\lambda < \kappa$. Then $\mathbf{nm}_\gamma({}^\kappa\lambda) = \lambda$ for every $1 \leq \gamma \leq \kappa$.

Proof. It is clear that if $F \subseteq {}^\kappa\lambda$ is of size $< \lambda$ then there exists $g \in {}^\kappa\lambda$ with g disjoint from every $f \in F$. On the other hand, note that if $f_\xi(\alpha) = \xi$ for every $\alpha \in \kappa$ and $\xi \in \lambda$, then $F = \{f_\xi : \xi \in \lambda\} \subseteq {}^\kappa\lambda$ is κ-matching. □

Observation 4.2.3. $\mathbf{nm}_\kappa({}^\kappa\kappa^+) = \mathbf{nm}(\kappa)$

Proof. Any collection of functions in ${}^\kappa\kappa^+$ which is κ-matching must be κ-matching for ${}^\kappa\kappa \subseteq {}^\kappa\kappa^+$, so $\mathbf{nm}_\kappa({}^\kappa\kappa^+) \geq \mathbf{nm}(\kappa)$. On the other hand, let $F \subseteq {}^\kappa\kappa$ be κ-matching. We produce a κ-matching $F' \subseteq {}^\kappa\kappa^+$ of equal cardinality. Fix for every $\alpha \in \kappa^+$ a surjection $e_\alpha : \kappa \to \alpha$. For $f \in {}^\kappa\kappa$ and $\alpha \in \kappa^+$, define $f_\alpha \in {}^\kappa\alpha$ in the natural way. That is, $f_\alpha(\xi) = e_\alpha(f(\xi))$ for every $\xi \in \kappa$. Let $F' = \{f_\alpha : \alpha \in \kappa^+, f \in F\}$. Note that $|F| = |F'|$, and if $g \in {}^\kappa\kappa^+$, then for some α, $g \in {}^\kappa\alpha$. However, then because F is κ-matching in ${}^\kappa\kappa$, $\{f_\alpha : f \in F\} \subseteq {}^\kappa\alpha$ is κ-matching in ${}^\kappa\alpha$, so for some $f \in F$, $|\{\beta \in \alpha : f_\alpha(\beta) = g(\beta)\}| = \kappa$, as desired. □

Proposition 4.2.4. Let μ be an infinite cardinal. Then $\mathbf{nm}(\mu^+) = \mathfrak{b}(\mu^+)$.

Proof. Clearly $\mathbf{nm}(\mu^+) \geq \mathfrak{b}(\mu^+)$ because any cofinally matching family in ${}^{\mu^+}\mu^+$ is cofinally unbounded. For the other direction, let $F \subseteq {}^{\mu^+}\mu^+$ be cofinally unbounded. Fix for every $\alpha \in \mu^+$ a surjection $e_\alpha : \mu \to \alpha$. For every $f \in F$, form μ-many functions $\{f_\alpha : \alpha \in \mu\} \subseteq {}^{\mu^+}\mu^+$ below f by letting $f_\alpha(\beta) = e_{f(\beta)}(\alpha)$ for every $\alpha \in \mu$ and $\beta \in \mu^+$. So f_α takes β to the α^{th} element of $f(\beta)$ according to $e_{f(\beta)}$. Let $F' = F \cup \{f_\alpha : \alpha \in \mu, f \in F\}$, such that

$|F| = |F'|$. Furthermore, if $g \in {}^{\mu^+}\mu^+$, then find $f \in F$ such that $|\{\beta : f(\beta) > g(\beta)\}| = \mu^+$, and note that for every such β there exists $\alpha \in \mu$ such that $g(\beta) = f_\alpha(\beta)$. Then there exists $\alpha \in \mu$ such that for μ^+-many such β, $f_\alpha(\beta) = g(\beta)$, that is g is cofinally matched by f_α. $\square$

Proposition 4.2.5. Let μ be an infinite cardinal. Then $\mathbf{nm}(\mu) \geq \mathbf{nm}_\mu(\mu^+)$.

Proof. For every $\alpha \in [\mu, \mu^+)$, fix $e_\alpha : \mu \to \alpha$ a bijection. Let $F \subseteq {}^\mu\mu$ be μ-matching. For every $f \in F$, form $f_\alpha \in {}^\alpha\alpha$ in the natural way according to e_α. That is, $f_\alpha(\beta) = e_\alpha(f_\alpha(e_\alpha^{-1}(\beta)))$ for every $\beta \in \alpha$. Note that for every $\alpha \in [\mu, \mu^+)$, $\{f_\alpha : f \in F\} \subseteq {}^\alpha\alpha$ is μ-matching. Let $F' \subseteq {}^{\mu^+}\mu^+$ denote the set of functions h where for some $\alpha \in [\mu, \mu^+)$ and $f \in F$, $h \restriction \alpha = f_\alpha$ and for every $\beta \in \mu^+ \setminus \alpha$, $h(\beta) = 0$. Then $|F'| = |F|$ and if $g \in {}^{\mu^+}\mu^+$ then for some $\alpha \in [\mu, \mu^+)$, $g''\alpha \subseteq \alpha$, and so for some $f \in F$, f_α matches $g \restriction \alpha$ on a set of size μ. Thus F' is μ-matching. $\square$

The following two propositions are proven in [11].

Proposition 4.2.6. Let κ be regular. Then $\mathbf{nm}(\kappa) \leq \mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa)$.

Proof. Let $X \subseteq {}^\kappa 2$ be combinatorially non-meager in the sense of $\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa)$, with $|X| = \mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa)$. We produce a cofinally matching family of functions of equal cardinality in ${}^\kappa\kappa$. To each $x \in X$, let $f_x \in {}^\kappa\kappa$ be defined by $f(\alpha)$ is the order type of the segment of 1's in x directly preceding the α^{th} 0. If there is no such 0, define $f_x(\alpha) = 0$. Fix $g \in {}^\kappa\kappa$. We define a $\langle y, \Pi\rangle$ such that if x matches $\langle y, \Pi\rangle$ then f_x cofinally matches g, which suffices. Build $\langle y, \Pi\rangle$ by recursion. Let $y \restriction [i_\alpha, i_{\alpha+1})$ consist of $(i_\alpha + 1)$-many 0's, each preceded by exactly $g(i_\alpha)$-many 1's. Find $x \in X$ such that x matches $\langle y, \Pi\rangle$. Note that by construction of y, the i_α^{th} 0 occurs in $I_\alpha = [i_\alpha, i_{\alpha+1})$. For any $I_\alpha \in \Pi$ such that $x \restriction I = y \restriction I$, we must also have that the i_α^{th} 0 of x occurs in I_α (because it can't occur before then and there are $(i_\alpha + 1)$-many 0's in I_α). And all such 0's are preceded by $g(i_\alpha)$-many 1's. Thus $f_x(i_\alpha) = g(i_\alpha)$, so f_x cofinally matches g. $\square$

Proposition 4.2.7. Let κ be regular. Then $\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa) \geq 2^{<\kappa}$

Proof. In fact something stronger holds. In particular, if we define $\mathbf{non}^1_{\mathrm{comb}}(\mathcal{M})(\kappa)$ to be the minimal cardinality of an $X \subseteq {}^{\kappa}2$ such that for any $\langle y, \Pi\rangle$ there exists at least one $I \in \Pi$ such that $x \restriction I = y \restriction I$, then $\mathbf{non}^1_{\mathrm{comb}}(\mathcal{M})(\kappa) \geq 2^{<\kappa}$. For every $\delta \in \kappa$, if $z \in {}^{\delta}2$ define $\langle y_z, \Pi_z = \Pi_\delta\rangle$ as follows. Let Π_δ consist of κ-many copies of δ next to each other, and let $y_z \restriction I$ be z (shifted to interval I) for every $I \in \Pi_\delta$. To each $x \in X$ we can consider the set of x's restrictions to each of the κ-many δ-blocks in Π_z. One of these restrictions for some $x \in X$ must be (a shift of) z. Because $|X| > \kappa$ by a simple diagonalization, if we then define $Y \subseteq {}^{<\kappa}2$ from X to be the collection of the κ-many restrictions of each element $x \in X$ of length δ according to Π_δ for each $\delta \in \kappa$, then $|Y| \leq |X|$ and $Y = {}^{<\kappa}2$. □

4.2.1 Some combinatorial observations

Intuitively, for a family of functions to be matching it needs to be dense in some sense. The following condition captures something of what needs to not occur for a family to be dense.

Definition 4.2.8. Say that a family of functions $F \subseteq {}^{\gamma}\delta$ is κ-non-overlapping if and only if for every $\beta \in \gamma, \nu \in \delta$, $|\{f \in F : f(\beta) = \nu\}| < \kappa$. For example, if $F \subseteq {}^{\omega_1}\omega_1$, then F is ω_1-non-overlapping if and only if for every $\alpha, \delta \in \omega_1$, $|\{f \in F : f(\alpha) = \delta\}| \leq \omega$.

Proposition 4.2.9. Let $\kappa \leq \mu$ and suppose $F \subseteq {}^{\mu}\mu$ is κ^+-non-overlapping. Then F is not κ-matching.

Proof. First, if $\kappa = \mu$ then $|F| \leq \kappa$ so F cannot be κ-matching by the usual diagonalization. Next, suppose $\mu = \kappa^+$. We need to see that if $F \subseteq {}^{\kappa^+}\kappa^+$ and F is κ^+-non-overlapping, then F is not κ-matching. Without loss of generality we may enumerate $F = \langle f_\xi : \xi \in \kappa^+\rangle$. The idea is to stratify κ^+ into a continuous increasing collection of κ^+-many closed and separating (with respect to functions in F) sets of ordinals of size κ, $\langle A_\alpha : \alpha \in \kappa^+\rangle$ by means of an elementary chain of suitable submodels, and diagonalize. Specifically, construct a continuous κ^+-length chain of elementary substructures of $(H_\theta, \in, \prec, \kappa, \langle f_\xi : \xi \in \kappa^+ \rangle, \ldots)$ of size κ containing κ as a subset, $M_0 \in M_1 \in \ldots \in M_\alpha \in \ldots$, and let $A_\alpha = M_\alpha \cap \kappa^+$ for each $\alpha \in \kappa^+$. By elementarity, if $\xi \in A_\alpha$, then $f''_\xi A_\alpha \subseteq A_\alpha$. Furthermore, if $\{\beta, \gamma\} \subseteq A_\alpha$ then $\kappa \subseteq M_\alpha$ and $\{\nu : f_\nu(\beta) = \gamma\} \in M_\alpha$, so $\{\nu : f_\nu(\beta) = \gamma\} \subseteq M_\alpha$ because F is κ^+-non-overlapping. That is, for every $\eta \notin A_\delta$, $f''_\eta A_\delta \cap A_\delta = \emptyset$. Now, let $g'' A_0 \subseteq A_1 \setminus A_0$, with the additional requirement that for every $\xi \in A_1 \setminus A_0$, $|\{\alpha \in A_0 : f_\xi(\alpha) = g(\alpha)\}| < \kappa$, which is possible as in the case where $\kappa = \mu$, because we can diagonalize against κ-many functions in e.g. ${}^\kappa\kappa$. Proceed in this manner—because the $\langle A_\alpha : \alpha \in \kappa^+ \rangle$ sequence is continuous, g will have domain κ^+. We show that g is not κ-matched by any function in F. Take some f_α. There is a unique $\xi \in \kappa^+$ (or $\xi = -1$) such that $\alpha \in A_{\xi+1} \setminus A_\xi$ (notationally then include the case where $A_{-1} = \emptyset$). Note that $A_{\xi+1}$, and all subsequent A_η's, are closure sets for f_α and by construction of g, g will be totally disjoint from f_α on $\kappa^+ \setminus A_\xi$. If ξ is a limit, then we must have $f''_\alpha A_\xi \cap A_\xi = \emptyset$, because otherwise ξ wouldn't be minimal. So in this case g is totally disjoint from f_α. If ξ is not a limit, i.e. $\xi = \eta + 1$, then by construction we've ensured that $|\{\gamma \in A_{\eta+1} \setminus A_\eta : g(\gamma) = f_\alpha(\gamma)\}| < \kappa$. And on A_η, g is disjoint from f_α, because $g'' A_\eta \subseteq A_\xi$ and $f''_\alpha A_\xi \cap A_\xi = \emptyset$. We can proceed by induction on cardinals μ to finish the proof. For successors, suppose the statement holds for μ. We need to see that if $F \subseteq {}^{\mu^+}\mu^+$ and F is κ^+-non-overlapping, then F is not κ-matching. This is the same argument as above, namely stratify μ^+ into a continuous chain of μ^+-many closed and separating sets of ordinals (with respect to the functions in F) each of size μ by means of a suitable continuous chain of submodels, and diagonalize as before to define g, using the induction hypothesis to ensure that, for example, if $\xi \in A_1 \setminus A_0$, $|\{\alpha \in A_0 : f_\xi(\alpha) = g(\alpha)\}| < \kappa$, etc. The case with limits is similar to the previous case with limits. Suppose for every $\lambda < \mu$, the statement holds. Let $F \subseteq {}^\mu\mu$ be κ^+-non-overlapping. We need to see F is not κ-matching. Enumerate $F = \langle f_\xi : \xi < \mu \rangle$, and let $\langle \kappa_\alpha : \alpha \in \mathrm{cf}(\mu) \rangle$ be a continuous sequence of cardinals cofinal in μ such that $\kappa < \kappa_0$. Stratify μ into a continuous increasing sequence of closed and separating sets of ordinals $\langle A_\alpha : \alpha \in \mathrm{cf}(\mu)\rangle$ such that $|A_\alpha| = \kappa_\alpha$ and $\kappa_\alpha \subseteq A_\alpha$ in the usual way, by means of a suitably chosen continuous chain of submodels. Then define $g \in {}^\mu\mu$ in such a way that $g''A_0 \subseteq A_1 \setminus A_0$ with the additional requirement that for every $\xi \in A_1 \setminus A_0$, $|\{\alpha \in A_0 : f_\xi(\alpha) = g(\alpha)\}| < \kappa$, etc. This is possible because we're essentially looking at κ_1-many functions in ${}^{\kappa_0}\kappa_1$, but the density condition allows us to consider only κ_0-many in some copy of ${}^{\kappa_0}\kappa_0$, which can be taken care of by the induction hypothesis. $\square$

In more general function spaces, we have the following easy observation.

Observation 4.2.10. If $F \subseteq {}^\mu\delta$ and $\omega \leq \kappa \leq \mu \leq \delta$, then if F is κ^+-non-overlapping, F is not κ-matching. On the other hand, if $\omega \leq \delta < \mu$, then it is easy to find a disjoint collection of functions $F \subseteq {}^\mu\delta$ of size δ which is μ-matching.

Proof. Because F is κ^+-non-overlapping, $|\{f \in F : f''\mu \cap \mu \neq \emptyset\}| \leq \mu$. The first statement then follows from 4.2.9. On the other hand, if $\omega \leq \delta < \mu$ let $f_\xi := \xi$ for every $\xi \in \delta$ and note that this collection is μ-matching. $\square$

We can also reformulate statements like $\mathbf{nm}_\kappa(\lambda) \leq \lambda$ using a sort of partition matrix which codes the behavior of a matching family.

Observation 4.2.11. The existence of an $F \subseteq {}^\lambda\lambda$ with $F = \langle f_\xi : \xi \in \lambda\rangle$ which is κ-matching (that is $\mathbf{nm}_\kappa(\lambda) \leq \lambda$) is equivalent to the existence of a system $\{A_{\alpha,\beta} : \alpha, \beta \in \lambda\}$ such that for every $\alpha \in \lambda$, $\bigcup\{A_{\alpha,\beta} : \beta \in \lambda\} = \lambda$, $A_{\alpha,\beta} \cap A_{\alpha,\beta'} = \emptyset$ whenever $\beta \neq \beta'$, and if $g \in {}^\lambda\lambda$, there exists $a \in [\lambda]^\kappa$ such that $\bigcap\{A_{\alpha,g(\alpha)} : \alpha \in a\} \neq \emptyset$.

Proof. Let each $A_{\alpha,\beta}$ consists exactly of the $\xi \in \lambda$ such that $f_\xi(\alpha) = \beta$. Note also that the density considerations of 4.2.9 show that not for every α, β is $|A_{\alpha,\beta}| \leq \kappa$. This can alternatively be seen by performing the diagonalization procedure with submodels using the $\{A_{\alpha,\beta} : \alpha, \beta \in \lambda\}$ system as a predicate in place of the enumeration of F. $\square$

In 4.2.5 we observed that $\mathbf{nm}(\mu) \geq \mathbf{nm}_\mu(\mu^+)$. We will see that in many cases this inequality can be strict (for example in the Laver model, that is performing an ω_2-stage countable support iteration of Laver forcing over a model of GCH, $\mathbf{nm}_\omega(\omega) = \omega_2$, but $\mathbf{nm}_\omega(\omega_1) = \omega_1$). If a situation like this occurs, one may make observations like the following:

Observation 4.2.12. Let $\mathbf{nm}(\mu) > \mu^+$ and suppose that $F = \langle f_\xi : \xi \in \mu^+ \rangle \subseteq {}^{\mu^+}\mu^+$ is μ-matching. Then there are many $g \in {}^{\mu^+}\mu^+$ such that some $f_\xi \in F$ μ-matches g, but no $f_\xi \in F$ μ^+-matches g. For any such g, in fact μ^+-many f_ξ μ-match g.

Proof. Suppose towards a contradiction that only $\{f_{\xi_\gamma} : \gamma \in \mu\} \subseteq F$ μ-match g on sets $A_{\xi_\gamma} \in [\mu^+]^\mu$, respectively. Let $A = \bigcup_{\gamma \in \mu} A_{\xi_\gamma}$ and note that $F \restriction A = \{f \restriction A : f \in F\} \subseteq {}^A\mu^+$ cannot be μ-matching, because $\mathbf{nm}_\mu(\mu) = \mathbf{nm}_\mu({}^A\mu) > \mu^+$. So there exists $h' \in {}^A\mu^+$ such no function in $F \restriction A$ μ-matches h'. But then if $h \in {}^{\mu^+}\mu^+$ is defined by $h(\delta) = h'(\delta)$ for $\delta \in A$ and $h(\delta) = g(\delta)$ for $\delta \notin A$, then h is not μ-matched by any function in F, a contradiction. $\square$

Note 4.2.13. In this situation as in 4.2.12, not only does F have to contain μ^+-many functions which μ-match such g, but in fact the associated matching sets $\{A_{\xi_\alpha} : \alpha \in \mu^+\}$ must be such that $|\bigcup_{\alpha \in \mu^+} A_{\xi_\alpha}| = \mu^+$.

Proof. Suppose towards a contradiction that for some $\beta \in \mu^+$ for every $\xi \in \mu^+$ we have that $g \restriction [\beta, \mu^+)$ is not μ-matched by $f_\xi \restriction [\beta, \mu^+)$. But then $F \restriction \beta \subseteq {}^\beta\mu^+$ is not μ-matching, so there exists $h \in {}^{\mu^+}\mu^+$ with $h(\alpha) = g(\alpha)$ for every $\alpha \in [\beta, \mu^+)$ such that h is not μ-matched by F, which is a contradiction. $\square$

4.2.2 Cofinal and disjointing observations

Observation 4.2.14. For function spaces ${}^\kappa\kappa$ for regular κ, in terms of the cofinality, it does not matter if we use the eventual domination ordering or the everywhere domination ordering. That is, for κ regular, $\mathfrak{d}_\kappa(\kappa) = \mathfrak{d}_\mu(\gamma)$ for every $\mu \leq \gamma$ (in particular for $\mu = 1$, that is the everywhere domination ordering).

Proof. Clearly $\mathfrak{d}_\kappa(\gamma) \leq \mathfrak{d}_1(\gamma)$. Let $F \subseteq {}^\kappa\kappa$ be cofinal and to every $f \in F$ adjoin f_α for every $\alpha \in \kappa$ defined by $f_\alpha(\beta) = \max\{\alpha, f(\beta)\}$ for each $\beta \in \kappa$ to F. This new family is now everywhere dominating if F is eventually dominating, and is of the same cardinality. $\square$

Observation 4.2.15. As observed initially for regular κ, $\mathfrak{d}(\kappa) = \mathfrak{d}_1(\kappa)$. However, this observation doesn't directly apply to $\mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa)$ and $\mathfrak{d}_1({}^{\kappa^+}\kappa)$. But it is still the case that $\mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa) = \mathfrak{d}_1({}^{\kappa^+}\kappa)$

Proof. First note that $\mathfrak{d}_1({}^\kappa\kappa) = \mathfrak{d}(\kappa) \leq \mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa)$. To see this, suppose $F \subseteq {}^{\kappa^+}\kappa$ is modulo $\leq \kappa$-sized sets cofinal, and form F' by first splitting κ^+ into κ^+-many blocks of size κ, $\{[0, \kappa), [\kappa, \kappa + \kappa), \ldots\} = \{I_\alpha : \alpha \in \kappa^+\}$. Then let F' consist of $f \restriction I_\alpha$ (but viewed as a function with domain κ) for every $f \in F$ and $\alpha \in \kappa^+$. Then $|F'| = |F|$, and for $g \in {}^\kappa\kappa$ let $\overline{g} \in {}^{\kappa^+}\kappa$ be given by $\overline{g}(\beta) = g(0)$ if β is the left-endpoint of some I_α, and $\overline{g}(\beta) = g(\alpha)$ if β is of the form $i_\delta + \alpha$ for some i_δ the left-endpoint of I_δ for some $\delta \in \kappa^+$ and $\alpha \in \kappa$. Then there exists $f \in F$ such that f eventually dominates $\overline{g}$. But then there is some $f' \in F'$ which totally dominates g. On the other hand, if we have a modulo $\leq \kappa$-sized sets cofinal family in ${}^{\kappa^+}\kappa$ and add a cofinal family of functions for every initial segment of every function in this family, then we will have an everywhere cofinal family. That is, $\mathfrak{d}_1({}^{\kappa^+}\kappa) \leq \mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa) \cdot \sum_{\alpha \in \kappa^+} \mathfrak{d}_1({}^\alpha\kappa)$. However, $\mathfrak{d}_1({}^\alpha\kappa) = \mathfrak{d}_1(\kappa)$ for every $\alpha \in [\kappa, \kappa^+)$, so $\mathfrak{d}_1({}^{\kappa^+}\kappa) \leq \mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa) \cdot \mathfrak{d}_1(\kappa) \leq \mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa) \cdot \mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa) = \mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa)$. So $\mathfrak{d}_{\kappa^+}({}^{\kappa^+}\kappa) = \mathfrak{d}_1({}^{\kappa^+}\kappa)$. $\square$

Note 4.2.16. Quantities like $\mathfrak{d}({}^{\kappa^+}\kappa)$ are in some sense more interesting than e.g. $\mathfrak{d}(\kappa)$. For example, the possible consistency of $\mathfrak{d}({}^{\omega_1}\omega) < 2^{\omega_1}$ is an old open problem. However, certain things are known—for example that if $\mathfrak{d}({}^{\omega_1}\omega) < 2^{\omega_1}$ then $2^\omega \geq \omega_3$ and if $2^\omega < 2^{\omega_1}$ and $2^\omega < \aleph_{\omega_1}$ then $\mathfrak{d}({}^{\omega_1}\omega) = 2^{\omega_1}$ (see e.g. [35]). This of course is in strong contrast to e.g. $\mathfrak{d}(\kappa)$

for regular κ, which as shown in [17] can, along with $\mathfrak{b}(\kappa)$, under some mild constraints, behave in quite an arbitrary way.

Note 4.2.17. Just as with $\mathfrak{d}(\kappa)$ for regular κ, it is clear that $\mathbf{cv}_1(\kappa) = \mathbf{cv}(\kappa)$.

Note 4.2.18. A disjointing family in e.g ${}^{\kappa}2$ is also disjointing in ${}^{\kappa}\kappa$, so in particular $\mathbf{cv}({}^{\kappa}2) \geq \mathbf{cv}(\kappa)$, etc..

Note 4.2.19. As is the case with $\mathfrak{d}(\kappa)$, $\mathbf{cv}_1({}^{\kappa^+}\kappa) = \mathbf{cv}_{\kappa^+}({}^{\kappa^+}\kappa)$.

Proof. This is as in 4.2.15. Take a family in ${}^{\kappa^+}\kappa$ which is eventually disjointing. To each function in this family add a $\mathbf{cv}_1({}^{\alpha}\kappa)$ family to all initial segments. In this way, observe that $\mathbf{cv}_1({}^{\kappa^+}\kappa) \leq \mathbf{cv}_{\kappa^+}({}^{\kappa^+}\kappa) \cdot \sum_{\alpha \in \kappa^+} \mathbf{cv}_1({}^{\alpha}\kappa)$. Then because $\mathbf{cv}_1({}^{\alpha}\kappa) = \mathbf{cv}_1(\kappa) \leq \mathbf{cv}({}^{\kappa^+}\kappa)$ for every $\alpha \in [\kappa, \kappa^+)$, we have $\mathbf{cv}_1({}^{\kappa^+}\kappa) = \mathbf{cv}_{\kappa^+}({}^{\kappa^+}\kappa)$. $\square$

4.3 Some forcing observations

Observation 4.3.1. Let κ be a regular uncountable cardinal. Then the value of $\mathfrak{b}(\kappa)$ cannot be increased by κ-c.c. forcings. So by 4.2.4, if $\kappa = \mu^+$ then $\mathbf{nm}(\kappa) = \mathbf{nm}(\mu^+)$ cannot be increased by κ-c.c. forcings.

Proof. Let $\mathbb{P}$ be κ-c.c. and let G be $(V, \mathbb{P})$-generic. In V, let $F \subseteq {}^{\kappa}\kappa$ be an unbounded collection. In $V[G]$, if $g \in {}^{\kappa}\kappa$ because $\mathbb{P}$ is κ-c.c. there exists $h \in ({}^{\kappa}P_{\kappa}(\kappa))^V$ such that $g(\alpha) \in h(\alpha)$ for every $\alpha \in \kappa$. If $f_h \in ({}^{\kappa}\kappa)^V$ is defined by $f_h(\beta) = \sup(h(\beta)) + 1$ then for some $f \in F$, $|\{\beta \in \kappa : f(\beta) > f_h(\beta)\}| = \kappa$. And $\{\beta \in \kappa : f(\beta) > f_h(\beta)\} \subseteq \{\beta \in \kappa : f(\beta) > g(\beta)\}$, so F is still unbounded in $V[G]$. $\square$

Note 4.3.2. The same argument in as in 4.3.1 shows that $\mathfrak{d}(\kappa)$ cannot be increased by κ-c.c. forcings. On the other hand, $\mathfrak{d}(\kappa) \leq \mathfrak{d}({}^{\kappa^+}\kappa)$, so if $\mathfrak{d}(\kappa)$ is increased by a κ^+-c.c. forcing,

$\mathfrak{d}(\kappa^+)$ remains unchanged, but $\mathfrak{d}({}^{\kappa^+}\kappa)$ may be increased. So for example if $\kappa = \omega$, under $MA+\neg CH$, $\mathfrak{d} = \mathfrak{d}({}^{\omega_1}\omega) = 2^\omega = 2^{\omega_1}$. So it's consistent that $\mathfrak{d}({}^{\omega_1}\omega) > \mathfrak{d}({}^{\omega_1}\omega_1)$ (if, for example, we start with a model of GCH and force $MA + 2^\omega = \omega_3$). Generally, using e.g. the posets in [17] it is easy to obtain similar results at any regular κ.

The following forcing is standard, but it is useful to illustrate because of a version with side conditions that we will use later.

Proposition 4.3.3. $\mathbf{nm}_\omega(\omega) = 2^\omega$ under Martin's Axiom (MA).

Proof. Suppose MA holds and let $F \subseteq {}^\omega\omega$ with $|F| < 2^\omega$. We show that F is not countably matching. Enumerate $F = \langle f_\xi : \xi \in \mu\rangle$. Let $\mathbb{P}$ be the poset consisting of $p = (f_p, A_p)$, where $f_p \in {}^{<\omega}\omega$, i.e. f_p is a finite partial function from ω to ω and $A_p \in P_\omega(\mu)$, i.e. a finite collection of ordinals in μ. Say $q = (f_q, A_q) \leq (f_p, A_p) = p$ if and only if $f_q \restriction \mathrm{dom}(f_p) = f_p$, $A_p \subseteq A_q$, and for every $\alpha \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$, for every $\xi \in A_p$, $f_q(\alpha) \neq f_\xi(\alpha)$. Because any two conditions with the same finite function are compatible, $\mathbb{P}$ is c.c.c. (in fact it's Knaster). Furthermore, $D_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(f_p)\}$ and $B_\xi = \{p \in \mathbb{P} : f_\xi \in A_p\}$ are dense for every $n \in \omega$ and $\xi \in \mu$, so by MA there exists a filter $G \subseteq \mathbb{P}$ having nonempty intersection with each of these sets. Then if $f_G = \bigcup_{p\in G} f_p$, $f_G \in {}^\omega\omega$, and if $f_\xi \in F$ there exists $q \in B_\xi \cap G$, so that for every $n \in \mathrm{dom}(f_G) \setminus \mathrm{dom}(f_q)$, $f(n) \neq f_\xi(n)$, i.e. f_ξ doesn't countably match f_G. So F isn't countably matching. $\square$

Observation 4.3.4. For regular κ, it is consistent that $\mathbf{nm}(\kappa) < \mathbf{cv}(\kappa)$. Moreover, it is consistent that $\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa) < \mathbf{cv}(\kappa)$. In particular, if $\kappa^{<\kappa} = \kappa$, $\mathrm{Fn}(F(\kappa), 2, <\kappa)) \Vdash \mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa) = \kappa^+$, where $F(\kappa)$ is any cardinal larger than κ, while if $\mathrm{cf}(F(\kappa)) > \kappa$, $\kappa^{<\kappa} = \kappa$, and e.g. $2^\kappa = \kappa^+$, then $\mathrm{Fn}(F(\kappa), 2, <\kappa)) \Vdash \mathbf{cv}(\kappa) = 2^\kappa = F(\kappa)$. This is because in fact $\mathrm{Fn}(F(\kappa), 2, <\kappa)) \Vdash MA_{<F(\kappa)}(\mathrm{Fn}(\kappa, \kappa, <\kappa))$. That is, in the generic extension by $\mathrm{Fn}(F(\kappa), 2, <\kappa))$ there exist generic filters for any collection of fewer than 2^κ many dense subsets of $\mathrm{Fn}(\kappa, \kappa, <\kappa)$.

Proof. First let's show that $\mathrm{Fn}(F(\kappa), 2, <\kappa)) \Vdash \mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa) = \kappa^+$. View $\mathrm{Fn}(F(\kappa), 2, < \kappa))$ as $\mathrm{Fn}(F(\kappa), 2, < \kappa)) \times \mathrm{Fn}(\kappa^+, 2, < \kappa))$. Because $\mathrm{Fn}(F(\kappa), 2, < \kappa)) \Vdash \kappa^{<\kappa} = \kappa$ if this is true in V, it suffices to show $\mathrm{Fn}(\kappa^+, 2, < \kappa)) \Vdash \mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa) = \kappa^+$. View $\mathrm{Fn}(\kappa^+, 2, < \kappa))$ as a κ^+-length iteration with $<\kappa$-supports of $\mathrm{Fn}(\kappa, 2, < \kappa))$, $\mathbb{P}_{\kappa^+}$, which may be done as the former is able to be densely embedded into the latter, and list $\langle f_\xi : \xi \in \kappa^+\rangle$ the generic functions added at each stage (the generic κ-reals). We show that $\{f_\xi : \xi \in \kappa^+\}$ constitutes a combinatorially non-meager set, i.e. for any $\langle y, \Pi\rangle \in V[G]$, where G is $(V, \mathrm{Fn}(\kappa^+, 2, < \kappa))$-generic, there exists $\xi \in \kappa^+$ such that f_ξ matches $\langle y, \Pi\rangle$. Let $\langle y, \Pi\rangle$ be coded by a subset $X \subseteq \kappa$. If A_α is a maximal antichain in $\{p \in \mathbb{P}_{\kappa^+} : p \Vdash \alpha \in \dot{X}\}$ for each $\alpha \in \kappa$, then $X = \{\alpha : G \cap A_\alpha \neq \emptyset\}$. By a standard Δ-system argument which may be employed because $\kappa^{<\kappa} = \kappa$, each A_α is of size $\leq \kappa$, so the supports of all possible p are bounded below some $\xi < \kappa^+$. Then $X = \{\alpha : (G \cap \mathbb{P}_\xi) \cap A_\alpha \restriction \xi\}$ also, where $A_\alpha \restriction \xi = \{p \restriction \xi : p \in A_\alpha\}$. Thus $X \in V[G_\xi]$, so $\langle y, \Pi\rangle \in V[G_\xi]$. However, density arguments show that $f_{\xi+1}$ then matches $\langle y, \Pi\rangle$, because below every condition and $\eta \in \kappa$ we can find a condition forcing that $f_{\xi+1}$ agrees with y on some $<\kappa$-sized interval starting above η. Next, we need to see that if $\mathrm{cf}(F(\kappa)) > \kappa$, $\kappa^{<\kappa} = \kappa$, and e.g. $2^\kappa = \kappa^+$ then $\mathrm{Fn}(F(\kappa), 2, < \kappa)) \Vdash \mathbf{cv}(\kappa) = F(\kappa)$. We argue that $\mathrm{Fn}(F(\kappa), 2, < \kappa)) \Vdash MA_{<F(\kappa)}(\mathrm{Fn}(\kappa, \kappa, < \kappa)$. Let G be $(V, \mathrm{Fn}(F(\kappa), 2, < \kappa))$-generic, and let $\langle D_\xi : \xi < \delta\rangle$ for $\delta < F(\kappa)$ be a sequence of dense subsets of $\mathrm{Fn}(\kappa, \kappa, < \kappa)$ in $V[G]$. Code $\langle D_\xi : \xi < \delta\rangle$ by $X = \{(\xi, p) : \xi \in \delta, p \in D_\xi\} \subseteq \delta \times \mathrm{Fn}(\kappa, \kappa, < \kappa)$. By assumption $|\delta \times \mathrm{Fn}(\kappa, \kappa, < \kappa)| < F(\kappa)$. View $\mathrm{Fn}(F(\kappa), 2, < \kappa))$ as $\mathrm{Fn}(F(\kappa), \kappa, < \kappa))$, and furthermore view this as an $F(\kappa)$-sized product of $\mathrm{Fn}(\kappa, \kappa, < \kappa)$ factors with $<\kappa$-supports. For every $y \in \delta \times \mathrm{Fn}(\kappa, \kappa, < \kappa)$, let A_y be a maximal antichain in $\{p \in \mathrm{Fn}(F(\kappa), 2, < \kappa)) : p \Vdash y \in \dot{X}\}$. Because $\kappa^{<\kappa} = \kappa$, $|A_y| \leq \kappa$. Note that $X = \{y \in \delta \times \mathrm{Fn}(\kappa, \kappa, < \kappa) : G \cap A_y \neq \emptyset\}$. If I is the union of the supports of $p \in A_y$ for every y, There will then be a $\leq F(\kappa)$-sized sub-product which adds X. That is, there exists $I \subseteq F(\kappa)$ with $I \in V$, $|I| < F(\kappa)$, and $X \in V[G \cap \mathrm{Fn}(I, \kappa, < \kappa)]$. But then the subsequent forcing with $\mathrm{Fn}(F(\kappa) \setminus I, \kappa, < \kappa)$ adds a generic object for $\langle D_\xi : \xi < \delta\rangle$. $\square$

In the second part of the argument above for 4.3.4 we viewed $\mathrm{Fn}(F(\kappa), 2, <\kappa))$ as a product, while in the first part of the argument we viewed it as an iteration. The reason for viewing it as a product in the second case was because the set of relevant supports, which was of size $< F(\kappa)$, could be cofinal, (i.e. in the case where $F(\kappa)$ is singular). If we viewed this as an iteration it wouldn't be clear that we would have added a generic object for a set of this size, while when looking at it as a product, we're allowed to change the "order" in which we add generic objects for sub-products (which can be defined in the ground model) arbitrarily. On the other hand, in the first case we needed to localize the addition of an object of size κ in a κ^+-length iteration, which presents no such difficulty, and viewing $\mathrm{Fn}(\kappa^+, 2, <\kappa)$ as a κ^+-length iteration makes distinguishing the $\langle f_\xi : \xi \in \kappa^+ \rangle$ apparent. An alternative method to showing the result of the first argument under some additional cardinal arithmetic assumptions, i.e. $\mathrm{Fn}(F(\kappa), 2, <\kappa)) \Vdash \mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\kappa) = \kappa^+$, is to show that in any such extension, ${}^{\kappa}2 \cap V$ is combinatorially non-meager. So assuming this set is of size κ^+ in the extension, we'd be done. The way to prove this is analogous to the situation on ω. That is, argue that any $\langle y, \Pi \rangle$ in the extension is present in the generic extension by a κ-sized subproduct (which exists in V and so is isomorphic to $\mathrm{Fn}(\kappa, 2, <\kappa)$), and then argue generally that any $\langle y, \Pi \rangle$ added by $\mathrm{Fn}(\kappa, 2, <\kappa)$ is cofinally matched by some $x \in {}^{\kappa}2 \cap V$. This is done by enumerating all of the $(p, \delta) : p \in \mathrm{Fn}(\kappa, 2, <\kappa), \delta \in \kappa$ pairs in order type κ (which is possible because $\kappa^{<\kappa} = \kappa$), and then constructing an $x \in V$ such that below every p, for any $\delta \in \kappa$ there is a q which forces that x agrees with y on an interval of Π starting above δ.

We could also show directly that if κ is an infinite regular cardinal with $2^{<\kappa} = \kappa$ and $\mathbb{P} = \mathrm{Fn}(\lambda, 2, <\kappa)$, if $2^\kappa = \kappa^+$ in V, then if G is $(V, \mathbb{P})$-generic, $V[G] \models \mathbf{nm}(\kappa) = \kappa^+$. And moreover generally if $\lambda \geq \kappa^+$, then in any case $V[G] \models \mathbf{nm}(\kappa) = \kappa^+$. This is as above, namely first suppose $V \models 2^\kappa = \kappa^+$. We show that $({}^{\kappa}\kappa)^V$ is κ-matching in $V[G]$. In $V[G]$, if $g \in {}^{\kappa}\kappa$, then as a κ-sized object by usual arguments there is $I \subseteq \lambda$ with $I \in V$, $|I| \leq \kappa$, and $g \in V[G \cap \mathrm{Fn}(I, 2, <\kappa)]$. So it suffices to show that if G' is $(V, \mathrm{Fn}(\kappa, 2, <\kappa))$-generic and $g \in ({}^\kappa\kappa)^{V[G']}$, then g is κ-matched by some $f \in V$. Enumerate $\mathrm{Fn}(\kappa, 2, <\kappa) \times \kappa$ as $\langle (p_\gamma, \xi_\gamma) : \gamma \in \kappa \rangle$. We construct an $f \in V$ such that no $p \in \mathrm{Fn}(\kappa, 2, <\kappa)$ can force "$\dot{g}$ is not matched by f past ξ", for any $\xi \in \kappa$. First, consider (p_0, n_0). Find $q_0 \leq p_0$ such that q_0 fixes the value of $\dot{g} \restriction (\xi_0 + 2)$. Let $f \restriction [0, \xi_0 + 2)$ to be equal to this value. Note that $q_0 \vDash \neg(\dot{g}$ is not matched by f past $\xi_0)$. Next, consider (p_1, ξ_1). Find $q_1 \leq p_1$ such that q_1 fixes the value of $\dot{g} \restriction [\xi_0 + 2, \xi_0 + 2 + \xi_1 + 2)$. Set $f \restriction [\xi_0 + 2, \xi_0 + 2 + \xi_1 + 2)$ to be equal to this value. We can proceed in this manner, building $f \in ({}^\kappa\kappa)^V$ because κ is regular. For every $p \in \mathbb{P}$ and $\xi \in \kappa$, we have found $q \leq p$ such that q forces that f agrees with $\dot{g}$ at a location past ξ. Thus $({}^\kappa\kappa)^V$ is κ-matching in $V[G]$, and if $2^\kappa = \kappa^+$ in V is of size κ^+. Alternatively, and we may do this any time $\lambda \geq \kappa^+$, regardless of what the value of 2^κ is in V, $\mathrm{Fn}(\lambda, 2, <\kappa) \vDash \mathbf{nm}(\kappa) = \kappa^+$. This is because any set of κ^+-many κ-Cohen reals is itself a κ-matching family in $V[G]$, as follows. Let $\lambda \geq \kappa^+$. Then $\mathrm{Fn}(\lambda, 2, <\kappa)$ is isomorphic to $\mathrm{Fn}(\lambda, \kappa, <\kappa) \times \mathrm{Fn}(\kappa^+, \kappa, <\kappa)$, so it suffices to show that $\mathrm{Fn}(\kappa^+, \kappa, <\kappa) \Vdash \mathbf{nm}(\kappa) = \kappa^+$. View $\mathrm{Fn}(\kappa^+, \kappa, <\kappa)$ as the $(<\kappa)$-support product of κ^+-many copies of $\mathrm{Fn}(\kappa, \kappa, <\kappa)$. This $(<\kappa)$-support product may be densely embedded into a κ^+-length iteration of $\mathrm{Fn}(\kappa, \kappa, <\kappa)$, taking direct limits at cofinality κ (e.g. $(<\kappa)$-support). Let $\langle f_\xi : \xi \in \kappa^+ \rangle$ enumerate the κ-Cohen reals added by these factors. Let G be $(V, \mathrm{Fn}(\kappa^+, \kappa, <\kappa))$-generic, and suppose $g \in ({}^\kappa\kappa)^{V[G]}$. By usual arguments, g will appear in the extension by a generic for some stage $\mathbb{P}_\eta$ with $\eta \in \kappa^+$ of the iteration (or thinking as a product, in the extension by some $<\kappa^+$-sized sub-product). But then for $\xi > \eta$, f_ξ will match (by density considerations) every $g \in ({}^\kappa\kappa)^{V[G_\eta]}$ on κ-many coordinates. So our $\langle f_\xi : \xi \in \kappa^+ \rangle$ is κ-matching.

Observation 4.3.5. For regular μ, it is consistent that $\mathbf{nm}(\mu^+) < \mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\mu^+)$. In the case where $\mu^+ = \omega_1$, this is observed in [11].

Proof. By 4.3.1, $\mathbf{nm}(\mu^+)$ is unchanged by μ^+-c.c. forcings. On the other hand, by 4.2.7 $\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\mu^+) \geq 2^\mu$. So starting from a model with regular μ and e.g. $2^{<\mu} = \mu$, and adding many μ-Cohen reals via $\mathrm{Fn}(\lambda, 2, <\mu)$, in the extension $\mathbf{nm}(\mu^+) < \lambda \leq$

$\mathbf{non}_{\mathrm{comb}}(\mathcal{M})(\mu^+)$. $\square$

4.4 $\mathbf{Con}(\mathbf{nm}_\omega(\omega_1) > \omega_1)$

In this section we show that if PFA holds then $\mathbf{nm}_\omega(\omega_1) = \omega_2$. The poset used for this result is (in a sense which we will explain) the natural one to use, and the main difficulty is not in formulating the definition of the poset, but in verifying that it's proper. The method of doing this here is due to Paul Larson [45], which is where we believe this result first appeared. Before defining the forcing, we give some motivation. First, note that the direct translation of the poset in 4.3.3 does not work, in that forcing with this translation collapses ω_1, as follows.

Note 4.4.1. The direct translation of the $\mathbf{nm}_\omega(\omega)$ forcing to $\mathbf{nm}_\omega(\omega_1)$ collapses ω_1.

Proof. Let $\mathbb{P}$ consist of $p = (f_p, A_p)$ where $f_p \in {}^{<\omega}\omega_1$ and $A_p \in P_\omega(\omega_1)$, where here $F \subseteq {}^{\omega_1}\omega_1$ and $F = \langle f_\xi : \xi \in \omega_1 \rangle$. Say $q = (f_q, A_q) \leq (f_p, A_p) = p$ if and only if $f_q \restriction \mathrm{dom}(f_p) = f_p$, $A_p \subseteq A_q$, and for every $\alpha \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$, $f_q(\alpha) \neq f_\xi(\alpha)$ for every $\xi \in A_p$. Note that $D_\xi = \{p \in \mathbb{P} : \exists n \in \omega \cap \mathrm{dom}(f_p) \text{ s.t. } f_p(n) > \xi\}$ is dense for every $\xi \in \omega_1$. So if G is sufficiently generic, G adds a cofinal ω-sequence to $(\omega_1)^V$. $\square$

The issue in this example is that using finite conditions will yield a function in e.g. ${}^{\omega_1}\omega_1$ which grows too quickly (in that its pointwise image on any infinite ground model set is cofinal in ω_1). In this setting for the purpose of forcing particular types of structures to exist, side conditions are often used to ensure that certain cardinals aren't collapsed, stationary sets are preserved, etc. Typically for a condition $p \in \mathbb{P}$, the side conditions will limit the set of conditions $q \in \mathbb{P}$ such that $q \leq p$. This might have the effect of ensuring some sort of properness or that a chain condition holds. In the case of properness, it is natural to consider side conditions consisting of some collection of submodels (e.g. of a particular H_κ), so that the existence of a model M in the side condition of an element $p \in \mathbb{P}$ could ensure that p is $(M, \mathbb{P})$-generic or -strongly generic. This approach was first developed by Todorčević in the 1980's [65]. In initial applications, the type of side conditions used to ensure properness are typically finite $\in$-chains of countable elementary submodels of H_κ. The interaction requirements between the working part of the forcing and the side conditions serve to make sure that, for example, if we wanted to show that p is $(M, \mathbb{P})$-strongly generic, that for any extension $q \leq p$, we can define a $q \restriction M \in M \cap \mathbb{P}$ such that the effect on the forcing that q has with respect to M is the same as the effect on the forcing that $q \restriction M$ has with respect to M. Precisely, that is that for every $r \leq q \restriction M$, if $r \in M$, then $r \parallel q$. An example of this sort of interaction requirement would be to say that if we're forcing a function $f_p : A \to B$, if $a \in \mathrm{dom}(f_p) \cap M$, then we must have $f_p(a) \in M$. The addition of M to the side condition of p and the interaction requirements typically will ensure that p is $(M, \mathbb{P})$-generic. With this in mind, the first forcing to try is the following.

Example 4.4.2. $F = \langle f_\xi : \xi \in \omega_1 \rangle \subseteq {}^{\omega_1}\omega_1$. Let $\mathbb{P}$ consist of $p = (f_p, A_p, \mathcal{M}_p)$ where $f_p \in {}^{<\omega}\omega_1$, $A_p \in P_\omega(\omega_1)$, and $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $(H_{\omega_2}, \in, \prec, \langle f_\xi : \xi \in \omega_1 \rangle, \ldots)$. Also insist that if $M \in \mathcal{M}_p$ and $\alpha \in \mathrm{dom}(f_p) \cap M$, then $f_p(\alpha) \in M$. Say that $q = (f_q, A_q, \mathcal{M}_q) \leq (f_p, A_p, \mathcal{M}_p) = p$ if and only if $f_q \restriction \mathrm{dom}(f_p) = f_p$, $A_p \subseteq A_q$, $\mathcal{M}_p \subseteq \mathcal{M}_q$, and for every $\alpha \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$, $f_q(\alpha) \neq f_\xi(\alpha)$ for every $\xi \in A_p$.

If we attempt to show that the partial order $\mathbb{P}$ from 4.4.2 is proper, the standard thing to do is choose some $M \prec (H_\theta, \in, \prec, \langle f_\xi : \xi \in \omega_1 \rangle, \mathbb{P}, \ldots)$ with $p = (f_p, A_p, \mathcal{M}_p) \in M$, and define $p^M = (f_p, A_p, \mathcal{M}_p \cup \{M \cap H_{\omega_2}\}) \in \mathbb{P}$, note that $p^M \leq p$, and try to show that p^M is $(M, \mathbb{P})$-generic. So fixing $D \in M$ dense, for $q \leq p^M$, without loss of generality assume $q \in D$, we want to define $q \restriction M \in M$ and then find $r \leq q \restriction M$ with $r \in D \cap M$ such that $r \parallel q$. If we set $q \restriction M = (f_q \restriction M, A_q \cap M, \mathcal{M}_q \cap M)$, then $q \restriction M \in \mathbb{P}$. However, it is not necessarily the case that $q \leq q \restriction M$. This is because while $f_q \restriction f_{q \restriction M} = f_{q \restriction M}$, $A_{q \restriction M} \subseteq A_q$, and $\mathcal{M}_{q \restriction M} \subseteq \mathcal{M}_q$, we also need that for every $\alpha \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_{q \restriction M})$, i.e. $\alpha \in \mathrm{dom}(f_q) \setminus M$, $f_q(\alpha) \neq f_\xi(\alpha)$

for every $\xi \in A_{q\restriction M}$, which isn't guaranteed. However, M is countable so no matter what $A_{q\restriction M}$ looks like, there are only countably many possible functions f_ξ that we need f_q to avoid on coordinates not in M. So, in an attempt to fix this problem, we can additionally require that every condition $q = (f_q, A_q, \mathcal{M}_q)$, for any $\alpha \in \mathrm{dom}(f_q)$ with $\alpha \notin M$ for some $M \in \mathcal{M}_q$, that $f_q(\alpha) \neq f_\xi(\alpha)$ for every $\xi \in M$. Because this is a countable set, we'll still be able to expand the partial functions associated with conditions, and will still be able to add ordinals to A_q at will. One can verify that this condition will ensure $q \leq q \restriction M$, and so one can find $r \leq q \restriction M$ with $r \in D \cap M$ so that r "looks like" q. It is not clear however that $r' \leq q$ (though $r' \leq r$). Indeed, for $r' \leq q$ it is required that for every $\alpha \in \mathrm{dom}(f_r) \setminus \mathrm{dom}(f_q)$, $f_r(\alpha) \neq f_\xi(\alpha)$ for every $\xi \in A_q$. Certainly if $\xi \in A_{q\restriction M} = A_q \cap M$, then because $r \leq q \restriction M$ we have no problems, so assume $\xi \in A_q \setminus M$. We have no restriction over the value of $f_\xi(\alpha)$, and $\xi \notin M$, but certainly $f_\xi(\alpha)$ could be in M, and $f_r(\alpha) \in M$). While this may seem like a fundamental problem, it turns out that this definition of the forcing still works, we just need to be a bit more careful with choosing e.g. r. This is important point from Larson's proof, which we give below. The reader may notice that the key lemma here in 4.4.3 and as written by Larson differ slightly, and that is because a slightly stronger version than what Larson gives in [45] is required. The following is exactly the forcing we just described, re-written so that it adds a function from ω_1 to ω_1 which has finite intersection with every function from ω_1 to ω_1 in the ground model (instead of a fixed ω_1-sized collection of them). Of course this is immaterial for the application of PFA, but is a more natural setting to consider.

Theorem (Larson [45]) 4.4.3. Let $\mathbb{P}$ be the forcing consisting of conditions $p = (f_p, A_p, \mathcal{M}_p)$ satisfying the following conditions

1. f_p is a finite partial function from ω_1 to ω_1,

2. A_p is a finite collection of functions in ${}^{\omega_1}\omega_1$, e.g. $A_p \in P_\omega({}^{\omega_1}\omega_1)$,

3. $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of H_{ω_2},

4. if $\alpha \in M \cap \mathrm{dom}(f_p)$ for some $M \in \mathcal{M}_p$, then $f_p(\alpha) \in M$, and

5. if $M \in \mathcal{M}_p$ and $\alpha \in \mathrm{dom}(f_p) \setminus M$, then for every $f \in {}^{\omega_1}\omega_1 \cap M$, $f_p(\alpha) \neq f(\alpha)$.

The ordering on $\mathbb{P}$ is such that $q = (f_q, A_q, \mathcal{M}_q) \leq (f_p, A_p, \mathcal{M}_p) = p$ if and only if

1. $f_q \restriction \mathrm{dom}(f_p) = f_p$,

2. $A_p \subseteq A_q$,

3. $\mathcal{M}_p \subseteq \mathcal{M}_q$, and

4. for every $\alpha \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$ and $f \in A_p$, $f_q(\alpha) \neq f(\alpha)$.

Then $\mathbb{P}$ is proper (so in particular ω_1 is preserved) and if G is $(V, \mathbb{P})$-generic, there exists $f \in ({}^{\omega_1}\omega_1)^{V[G]}$ such that for every $g \in ({}^{\omega_1}\omega_1)^V$, $|\{\alpha \in \omega_1 : f(\alpha) = g(\alpha)\}| < \omega$.

Proof. It is straightforward to show by density arguments that forcing with $\mathbb{P}$ adds a function $f = \bigcup_{p \in G} f_p \in {}^{\omega_1^V}\omega_1^V$ which has finite intersection with every ground model function. We need to show that $\mathbb{P}$ is proper. Fix some large H_θ and let M be a countable elementary submodel of $(H_\theta, \prec, \mathbb{P}, \ldots)$. Let $p = (f_p, A_p, \mathcal{M}_p) \in M$ and let $q = (f_p, A_p, \mathcal{M}_p \cup \{M \cap H_{\omega_2}\})$. It is clear that $q \in \mathbb{P}$ and $q \leq \mathbb{P}$, so it suffices to show that q is $(M, \mathbb{P})$-generic. So, let $D \in M$ be dense. We need to see that $D \cap M$ is pre-dense below q. Let $r = (f_r, A_r, \mathcal{M}_r) \leq q$. Define $r \restriction M = (f_r \restriction M, A_r \cap M, \mathcal{M}_r \cap M)$. It is a standard argument to show that $r \restriction M \in \mathbb{P}$. We first prove the main lemma of the argument, which is as follows. There exists $\alpha \in \lim(\omega_1)$ such that for every $F' \in P_\omega({}^{\omega_1}\omega_1)$, there exists $r' = (f_{r'}, A_{r'}, \mathcal{M}_{r'}) \leq r \restriction M$ with $r' \in D$, $f_{r'} \subseteq \alpha \times \alpha$, and for every $\beta_{r'} \in \mathrm{dom}(f'_r) \setminus \mathrm{dom}(f_{r \restriction M})$, $f_{r'}(\beta_{r'}) \neq f(\beta_{r'})$ for every $f \in F'$. One difference between this lemma and Larson's in [45] is that we insist not only that $\mathrm{dom}(f_{r'}) \subseteq \alpha$, but also that $\mathrm{rng}(f_{r'}) \subseteq \alpha$. First, let's see that this lemma suffices—suppose we have proven it. By elementarity we may apply it in M (because $r \restriction M$ and D are both in

M), and fix such an $\alpha \in \lim(\omega_1) \cap M$. Working again outside of M, then there exists $r' \in D$ with the desired properties, with respect to $F' = A_r$. Even though r' is not necessarily in M (as A_r is not in M necessarily), because $f_{r'} \subseteq \alpha \times \alpha$, $f'_r \in M$, so again by elementarity there exists $r'' \leq r \restriction M$ in $M \cap D$ such that $f_{r''} = f_{r'}$. Then in particular, we must have $f_{r''}(\beta) = f_{r'}(\beta) \neq f(\beta)$ for every $\beta \in \mathrm{dom}(f_{r''}) \setminus \mathrm{dom}(f_{r \restriction M})$ and $f \in A_r$. This is exactly what we wanted. In particular, we show that $r'' \parallel r$. Let $s = (f_{r''} \cap f_r, A_{r''} \cup A_r, \mathcal{M}_{r''} \cup \mathcal{M}_r)$. It is not difficult to see that $s \in \mathbb{P}$, what we need to see is that $s \leq r''$ and $s \leq r$. First, note that if $\beta \in \mathrm{dom}(f_s) \setminus \mathrm{dom}(f_{r''})$ and $f \in A_{r''}$, that is $\beta \in \mathrm{dom}(f_r) \setminus \mathrm{dom}(f_{r''})$ and $f \in A_{r''}$, then $f_r(\beta) \neq f(\beta)$, because $f(\beta) \in M$ and necessarily $\beta \notin M$ in this case and we designed our conditions specifically to act in this way (note that $M \cap H_{\omega_2} \in \mathcal{M}_q \subseteq \mathcal{M}_r$). On the other hand, suppose $\beta \in \mathrm{dom}(f_{r''}) \setminus \mathrm{dom}(f_r)$ and $f \in A_r$. We need to see that $f_{r''}(\beta) \neq f(\beta)$. This is where the lemma is used, to take care specifically of such β, because if $\beta \in \mathrm{dom}(f_{r''}) \setminus \mathrm{dom}(f_r)$ then $\beta \in \mathrm{dom}(f_{r''}) \setminus \mathrm{dom}(f_{r \restriction M})$, and $f \in A_r$, so by the lemma $f_{r''}(\beta) = f_{r'}(\beta) \neq f(\beta)$. Therefore $s \leq r'', r$, so $\mathbb{P}$ is proper. It suffices then to verify the lemma.

So, let $p = (f_p, A_p, \mathcal{M}_p) \in \mathbb{P}$ with $D \subseteq \mathbb{P}$ dense. We show that there exists $\alpha \in \lim(\omega_1)$ such that for every $F' \in P_\omega({}^{\omega_1}\omega_1)$, there exits $r = (f_r, A_r, \mathcal{M}_r) \leq p$ with $r \in D$ and $f_r \subseteq \alpha \times \alpha$ such that for every $\beta \in \mathrm{dom}(f_r) \setminus \mathrm{dom}(f_p)$, $f_r(\beta) \neq f(\beta)$ for every $f \in F'$. Suppose towards a contradiction that this fails. Then for every $\alpha \in \lim(\omega_1)$, there exists $F_\alpha \in P_\omega({}^{\omega_1}\omega_1)$ such that for every $r = (f_r, A_r, \mathcal{M}_r) \leq p$, $r \in D$, with $f_r \subseteq \alpha \times \alpha$, there exists $\beta_r \in \mathrm{dom}(f_r) \setminus \mathrm{dom}(f_p)$ and $f \in F_\alpha$ such that $f_r(\beta_r) = f(\beta)$. By adding functions to the F_α's, we may assume without loss of generality that $\langle |F_\alpha| : \alpha \in \lim(\omega_1) \rangle$ is non-decreasing, so in fact we may assume that for some $n \in \omega$, $|F_\alpha| = n + 1$ for every $\alpha \in \lim(\omega_1)$. Then we can enumerate each $F_\alpha = \langle F^i_\alpha : i \in n \rangle$. So for every r as above, there exists $\beta_r \in \mathrm{dom}(f_r) \setminus \mathrm{dom}(f_p)$ and $i_r \in n$ such that $f_r(\beta_r) = f^{i_r}_\alpha(\beta_r)$. The next step of the argument is one which prevents immediate generalizations to other settings, as we will see. Let U be a uniform ultrafilter over ω_1. Because we are diagonalizing modulo finite sets, we will only need this ultrafilter to be closed under finite intersections, which of course it is. In any case, note that for every $r \leq p$ with $r \in D$, for U-measure-one many α's we have $f_r \subseteq \alpha \times \alpha$, and for each such α we have β_r and i_r as indicated, and $|\mathrm{dom}(f_r) \times n| < \omega$, so we must have that for U-measure-one many α's, (β_r, i_r) is constant. In this case, $f_r(\beta_r) = f_\alpha^{i_r}(\beta_r)$ on these α's. So if $r_1, r_2 \leq p$ with $r_1, r_2 \in D$ and $(\beta_{r_1}, i_{r_1}) = (\beta_{r_2}, i_{r_2}) = (\beta, i)$, then because the intersection of two U-measure-one sets is non-empty in particular, we have $f_{r_1}(\beta) = f_\alpha^i(\beta) = f_{r_2}(\beta)$. That is, f_{r_1} and f_{r_2} take the same value at β. The idea is to use this fact and the density of D to find a condition below a certain strengthening of p obtained through this fact in D which yields a contradiction. We add $(n+1)$-many functions to A_p. For every $i \in n$, define $h_i \in {}^{\omega_1}\omega_1$ by letting $h_i(\beta) = f_r(\beta)$ if such an $r \leq p$ with $r \in D$ exists with $\beta_r = \beta$ and $i_r = i$. By what we just argued, this is well defined, because any two such r's are such that the f_r's agree on β. Otherwise set $h_i(\beta) = 0$. Note that $p' = (f_p, A_p \cup \{h_i : i \in n\}, \mathcal{M}_p) \in \mathbb{P}$ and $p' \leq p$. However, if $r \leq p'$ with $r \in D$, then by assumption for some large enough α we'll have $f_r(\beta_r) = f_\alpha^{i_r}(\beta_r) = h_{i_r}(\beta_r)$. However, this is a contradiction because $r \leq p' \leq p$ and $\beta_r \notin \mathrm{dom}(f_{p'})$, but $h_{i_r} \in A_{p'}$, so we cannot have $f_r(\beta_r) = h_{i_r}(\beta_r)$. $\square$

Corollary 4.4.4. If PFA holds then $\mathbf{nm}_\omega(\omega_1) = 2^\omega = \omega_2$.

Proof. Let $\mathbb{P}$ be as in 4.4.3. By 4.4.3, $\mathbb{P}$ is proper. Fix $\langle f_\xi : \xi \in \omega_1 \rangle \subseteq {}^{\omega_1}\omega_1$. It is clear that $D_\xi = \{p \in \mathbb{P} : f_\xi \in A_p\}$ is dense for every $\xi \in \omega_1$. Furthermore, $E_\xi = \{q \in \mathbb{P} : \xi \in \mathrm{dom}(f_q)\}$ is also dense for every $\xi \in \omega_1$, as follows. Suppose first that for no $M \in \mathcal{M}_p$ is $\xi \in M$. Then if $M \in \mathcal{M}_p$ is maximal, just let $f_q(\xi) \notin M$ avoid every $f(\xi)$ for $f \in M$ (and avoid $f(\xi)$ for every $f \in A_p$). There are only countably many functions to consider. On the other hand, suppose for some $M \in \mathcal{M}_p$ we have $\xi \in M$. Let M be the minimal such model in $\mathcal{M}_p$. If there are no models in $\mathcal{M}_p$ below M, simply let $f_q(\xi) \in M$ avoiding $f(\xi)$ for every $f \in A_p$. On the other hand, suppose M' is the model in $\mathcal{M}_p$ directly preceding M. Note that $B = \{f(\xi) : f \in {}^{\omega_1}\omega_1 \cap M'\} \in M$, and so there exist infinitely many ordinals in $(M \cap \omega_1) \setminus B$. Simply let $f_q(\xi)$ be equal to one of these ordinals avoiding $f(\xi)$ for every

$f \in A_p$. So E_ξ is dense. By PFA, there exists a filter G with $G \cap D_\xi \neq \emptyset$ and $G \cap E_\xi \neq \emptyset$ for every $\xi \in \omega_1$. But then it is not difficult to see that if $f = \bigcup_{p \in G} f_p$, $f \in {}^{\omega_1}\omega_1$ and for every $\xi \in \omega_1$, for some $p \in G$, $f_\xi \in A_p$, so for every $\beta \in \omega_1 \setminus \mathrm{dom}(f_\xi)$, $f(\beta) \neq f_\xi(\beta)$. That is, $|\{\alpha \in \omega_1 : f(\alpha) = f_\xi(\alpha)\}| < \omega$. $\square$

Note 4.4.5. If we allow the models appearing in the side conditions of $\mathbb{P}$ to be elementry submodels of some larger H_κ, etc. then $\mathbb{P}$ is still proper, and so preserves ω_1, and adds a function $g \in {}^\kappa\kappa$ which is disjoint from every ground model function $\kappa \to \kappa$ modulo finite sets. Here of course κ is collapsed to ω_1.

The one-step forcing in 4.4.3 is a proper forcing (so preserves ω_1) which adds a function $f \in {}^{\omega_1}\omega_1$ which is disjoint modulo finitely many coordinates from every function in $({}^{\omega_1}\omega_1)^V$. One might ask whether there are other types of forcing extensions (in particular) which can accomplish this or similar things, or if in certain settings it is impossible. Here we partially address both of these questions.

4.4.1 The result at ω_1

How close to being canonically proper is the forcing in 4.4.3? That is, could similar results be achieved by forcings satisfying stronger requirements than properness?

Proposition 4.4.6. Let $\mathbb{P}$ be c.c.c.. Then forcing with $\mathbb{P}$ does not add a function in ${}^{\omega_1}\omega_1$ which is disjoint modulo finitely many coordinates from every function in $({}^{\omega_1}\omega_1)^V$.

Proof. Let G be $(V, \mathbb{P})$-generic and suppose $\dot{g}$ is a name for a function in $({}^{\omega_1}\omega_1)^{V[G]}$. Because $\mathbb{P}$ is c.c.c., for every $\alpha \in \omega_1$ we may fix a maximal antichain A_α consisting of $p \in \mathbb{P}$ such that for some β, $p \Vdash \dot{g}(\alpha) = \beta$. Because A_α is countable, working in V we may form $F : \omega_1 \to P_{\omega_1}\omega_1$ such that in $V[G]$, for every $\alpha \in \omega_1$, $g(\alpha) \in F(\alpha)$. In V, enumerate $F(\alpha) = \langle F^n_\alpha : n \in \omega \rangle$. Let $h_n \in {}^{\omega_1}\omega_1$ be defined by $h_n(\alpha) = F^n_\alpha$. For each $\alpha \in \omega_1$, there exists n_α such that $g(\alpha) = h_{n_\alpha}(\alpha)$, so for at least one $n \in \omega$, for uncountably many α we have $n_\alpha = n$. But then $|\{\alpha \in \omega_1 : g(\alpha) = h_n(\alpha)\}| = \omega_1$. $\square$

Note 4.4.7. If M is a $(< \omega_1, \omega_1)$-distributive extension of V—that is if $V \subseteq M$ and M adds no countable sequences of ordinals in ω_1^V—then in M there does not exist a function in ${}^{\omega_1}\omega_1$ which is disjoint modulo finitely many coordinates from every function in $({}^{\omega_1}\omega_1)^V$.

Proof. This is clear, because e.g. if $f \in M$, $f \restriction \omega \in V$. $\square$

Proposition 4.4.8. Let $\mathbb{P}$ be the composition of an $(< \omega_1, \omega_1)$-distributive forcing with a c.c.c.. forcing, that is of the form $((< \omega_1, \omega_1)\text{-distributive}) \star (\text{c.c.c.})$. Then forcing with $\mathbb{P}$ does not add a function in ${}^{\omega_1}\omega_1$ which is disjoint modulo finitely many coordinates from every function in $({}^{\omega_1}\omega_1)^V$.

Proof. Write the extension by $\mathbb{P}$ as $V[G_1][G_2]$ and choose $g \in V[G_1][G_2]$. By the argument as in 4.4.6, there is a function $g' \in V[G_1]$ which agrees with g on uncountably many coordinates. Because all initial segments of g' are in V, there must then exist a function $g'' \in V$ which agrees with g on an infinite set. $\square$

Proposition 4.4.9. Let $\mathbb{P}$ be of the form $(\text{c.c.c.}) \star (\omega_1\text{-strategically closed})$. Then forcing with $\mathbb{P}$ does not add a function in ${}^{\omega_1}\omega_1$ which is disjoint modulo finitely many coordinates from every function in $({}^{\omega_1}\omega_1)^V$.

Proof. Write the extension by $\mathbb{P}$ as $V[G_1][G_2]$ and let $g \in V[G_1][G_2]$. The idea here is to build a candidate function for g in $V[G_1]$, which is possible because $V[G_1][G_2]$ is an extension by an ω_1-strategically closed forcing of $V[G_1]$, and then use the observation as in 4.4.6. In particular, work in $V[G_1]$ and let $\dot{g}$ be a name for g. We show that the set of conditions forcing (over $V[G_1]$) that some function in $({}^{\omega_1}\omega_1)^V$ countably matches $\dot{g}$ is dense. So, choose any condition and using the strategic closure as explained in the chapter on trees, build a candidate sequence $\langle f_\alpha : \alpha \in \omega_1 \rangle \subseteq V[G_1]$ and a corresponding $\leq$-decreasing sequence of conditions below this condition $\langle p_\alpha : \alpha \in \omega_1 \rangle$ such that $p \Vdash_{V[G_1]} f_\alpha \restriction \alpha = \dot{g} \restriction \alpha$. Let $f(\beta) = f_{\beta+1}(\beta)$ and note that $f \in V[G_1]$. In V because $V[G_1]$ is a c.c.c. extension, as in 4.4.6 there exists $g_n \in {}^{\omega_1}\omega_1$ such that $|\{\beta \in \omega_1 : g_n(\beta) = f(\beta)\}| = \omega_1$. Then for some sufficiently large α (large enough so that g_n and f agree infinitely often before α), p_α fixes the value of $f \restriction \alpha$ to be $\dot{g} \restriction \alpha$, so we have that $p_\alpha \Vdash_{V[G_1]} |\{\beta \in \alpha : g_n(\beta) = g(\beta)\}| = \omega$, as desired. □

Note 4.4.10. Let $\mathbb{P}$ be a forcing of the form $((< \omega_1, \omega_1)$-distributive$) \star$ (c.c.c.) $\star$ (ω_1-strategically closed). Then forcing with $\mathbb{P}$ does not add a function in ${}^{\omega_1}\omega_1$ which is disjoint modulo finitely many coordinates from every function in $({}^{\omega_1}\omega_1)^V$.

Proof. Write the extension by $\mathbb{P}$ as $V[G_1][G_2][G_3]$ and let $g \in V[G_1][G_2][G_3]$. Because all initial segments of functions in $V[G_1]$ are in V, and by 4.4.9 there exists $g' \in V[G_1]$ which countably matches g, there exists $g'' \in V$ which countably matches g (i.e. g'' agrees with g' on a sufficiently large initial segment). □

4.4.2 A potential barrier at ω_2

A natural question to ask is whether a result like 4.4.3 can be pushed up to e.g. ω_2. That is, can there exist a forcing $\mathbb{P}$ preserving ω_2 which adds a function from ω_2 to ω_2 which is disjoint modulo (for example) countably many coordinates from every function in $({}^{\omega_2}\omega_2)^V$? An inspection of the proof of 4.4.3 reveals that a key component is that the uniform ultrafilter chosen over ω_1 is closed under intersections of finitely many elements. To be able to prove an analogous lemma at ω_2, something like e.g. if $p = (f_p, A_p, \mathcal{M}_p) \in \mathbb{P}$ and $D \subseteq \mathbb{P}$ is dense, then there exists $\alpha \in \omega_2 \cap \mathrm{Cof}(\omega_1)$ such that for every $F' \in P_{\omega_1}({}^{\omega_2}\omega_2)$, there exists $r \in D$ with $r \leq p$ and $f_r \subseteq \alpha \times \alpha$ where for every $\beta \in \mathrm{dom}(f_r) \setminus \mathrm{dom}(f_p)$, $f_r(\beta) \neq f(\beta)$ for every $f \in F'$, one would need something stronger. Here $\mathcal{M}_p$ might be e.g. a countable $\in$-chain of suitable submodels of size ω_1, etc.. Proving a statement like this in a way analogous to 4.4.3 would require then a σ-complete uniform ultrafilter over ω_2, which does not exist. One might also attempt to diagonalize functions in ${}^{\omega_2}\omega_2$ modulo finite sets by using finite conditions and incorporating e.g. finite side conditions comprising models of two types. However, similar problems seem to occur. In the presence of a measurable cardinal κ, one could however construct a forcing with side conditions of size $< \kappa$ which is analagous to the forcing in 4.4.3, and use the normal measure to obtain a similar combinatorial lemma. However, because κ is a limit cardinal, we will not have a cardinal "gap" and this forcing will just add a function which is disjoint from all ground model functions modulo sets of size $< \kappa$, which of course could be obtained by the usual κ-closed diagonalizing forcing without side conditions. Another immediate generalization of the forcing in 4.4.3 if κ is λ-supercompact, that is if there exists a uniform normal ultrafilter over $P_\kappa \lambda$, would be to use models of size $< \kappa$ and $< \kappa$-sized conditions to force the existence of a function from λ to λ directly which is modulo sets of size $< \kappa$ disjoint from all ground model functions. However, λ would be collapsed to κ, so this could easily just have been done with the forcing of pure functions of size $< \kappa$, so in this sense there is no advantage to that approach. The intuition that large cardinals might allow strong diagonalizations while preserving cardinals, however, is not misplaced, as we will see in the subsequent section.

With some accessible cardinals other than ω_1 though, it turns out that in strong contrast with ω_1, there are settings in which an analogous situation to 4.4.3 cannot occur. Namely, it is consistent that if an outer model has the same ω_2, then if g is a function from ω_2 to ω_2 in the outer model, there exists $f : \omega_2 \to \omega_2$ in the ground model which matches g on an uncountable set of coordinates. So not only can we not diagonalize modulo finite sets, we cannot diagonalize modulo countable sets in this setting. This observation is a corollary of a stronger result of Abraham and Shelah, using some basic combinatorial methods already considered, e.g. those used to prove 4.2.4.

Definition 4.4.11. Let κ be a regular cardinal and let $\mathcal{C}$ be a collection of closed unbounded subsets of κ. Say that $C \subseteq \kappa$ a closed unbounded set is fast with respect to $\mathcal{C}$, or is a fast club, or is a diagonalizing club, if and only if for every $D \in \mathcal{C}$, $C \subseteq^* D$, i.e. if and only if for some $\gamma \in \kappa$, if $\delta \in C \setminus \gamma$, then $\delta \in D$. Say that $\mathcal{C}$ can be diagonalized if and only if there exists $C \subseteq \kappa$ fast with respect to $\mathcal{C}$.

Observation 4.4.12. Let κ be a regular cardinal. The following are equivalent:

1. $\mathfrak{b}(\kappa) > \lambda$
2. Every collection of closed unbounded subsets of κ of size $\leq \lambda$ can be diagonalized.

Proof. Suppose first that $\mathfrak{b}(\kappa) > \lambda$ and let $\mathcal{C} = \langle C_\alpha : \alpha \leq \lambda \rangle$ be a sequence of closed unbounded subsets of κ. Identify each C_α with its (normal) enumerating function, and let $g \in {}^\kappa\kappa$ dominate every C_α, which is possible because $\lambda < \mathfrak{b}(\kappa)$. Let $C = \{\beta \in \kappa : g''\beta \subseteq \beta\}$. Note that C is a club. We show that C diagonalizes $\mathcal{C}$. Fix $C_\alpha \in \mathcal{C}$. Find $\gamma \in \kappa$ such that $g(\beta) > C_\alpha(\beta)$ for every $\beta \geq \gamma$. If ξ is such that $g''\xi \subseteq \xi$ and $\xi > g(\gamma)$, we want to see $\xi \in C_\alpha$. However $g''\xi \subseteq \xi$ and $g > C_\alpha$ after $g(\gamma)$, so $C_\alpha''\xi \subseteq \xi$ too and is necessarily cofinal, so $\xi \in C_\alpha$ by closure. On the other hand, suppose that every collection of closed unbounded subsets of κ of size $\leq \lambda$ can be diagonalized. Fix $\langle f_\alpha : \alpha \in \lambda \rangle \subseteq {}^\kappa\kappa$. Without loss of generality assume every f_α is increasing. For every $\alpha \in \lambda$, let $C_\alpha = \{\xi \in \kappa : f_\alpha''\xi \subseteq \xi\}$. Note that if $h_\alpha \in {}^\kappa\kappa$ is defined by $h_\alpha(\xi) = C_\alpha(\xi + 1)$, then it is not difficult to see that for every $\xi \in \kappa$, $f_\alpha(\xi) < h_\alpha(\xi)$. Every C_α is club, so we can find a fast club $C \subseteq \kappa$ with respect to $\langle C_\alpha : \alpha \in \lambda \rangle$. Let $g(\xi) = C(\xi + 1)$ for every $\xi \in \kappa$. Fix $\alpha \in \lambda$. There exists $\beta \in \kappa$ such that $C \setminus \beta \subseteq C_\alpha$. Find $\xi > \beta$ such that $h_\alpha''\xi \subseteq \xi$ and $g''\xi \subseteq \xi$. Then for every $\delta \in [\xi, \kappa)$, $f_\alpha(\delta) < h_\alpha(\delta) \leq g(\delta)$. □

Suppose we are in the case where $\kappa = \mu^+$. By the argument of 4.2.4, if $F \subseteq {}^\kappa\kappa$ is an unbounded collection (modulo $< \kappa$), then by adding μ-many functions below each $f \in F$, we can form a κ-matching family $F' \subseteq {}^\kappa\kappa$ of the same cardinality. Via bijections $e_\alpha : \alpha \to \kappa$

for every $\alpha \in [\kappa, \kappa^+)$, as in 4.2.5 we can form κ-matching families $F'_\alpha \subseteq {}^\alpha\alpha$. Extending these functions arbitrarily, we can form a κ-matching family $F'' \subseteq {}^{\kappa^+}\kappa^+$ of the same cardinality. That is, if $\kappa = \mu^+$, from an unbounded family of functions in ${}^\kappa\kappa$, we can form a κ-matching family in ${}^{\kappa^+}\kappa^+$ of the same cardinality. As a consequence, $\mathfrak{b}(\mu^+) \leq \mathbf{nm}_{\mu^+}(\mu^{++})$. If $\mu = \omega$, then this says that $\mathfrak{b}(\omega_1) \leq \mathbf{nm}_{\omega_1}(\omega_2)$. By the observation in 4.4.12, if there is a collection of closed unbounded subsets of κ of size $\leq \lambda$ which cannot be diagonalized, then there is an unbounded collection of functions in ${}^\kappa\kappa$ of size $\leq \lambda$, and so if $\kappa = \mu^+$, a collection of functions in ${}^{\kappa^+}\kappa^+$ of size $\leq \lambda$ which is κ-matching. So whether or not there can exist a function which is disjoint modulo sets of size $< \kappa$ from a collection of functions of a particular size in ${}^{\kappa^+}\kappa^+$ can depend on whether or not a collection of closed unbounded subsets of κ of that same size can be diagonalized. In this context, this is useful because it is sometimes possible to write down certain combinatorial properties or a device (via extra functions, etc.) which would prevent a collection of closed unbounded subsets of κ from ever being able to be diagonalized, even in outer models, assuming e.g. certain cardinals are preserved. This basic idea is a common one—for example, using the absolute nature of the combinatorial device of a specializing function for a special Aronszajn tree T on ω_1, if ω_1 is preserved no branches can ever be added to T. This method is used by Abraham and Shelah in [2] to prove 4.4.13. Here we are thinking that $\lambda \geq \omega_2$.

Theorem (Abraham-Shelah [2]) 4.4.13. It is consistent via forcing (from ZFC) that there is a collection $\langle R_e : e \in E\rangle$, where E is a set of size ω_1 and for every $e \in E$, R_e is a graph on λ satisfying the following two conditions and there are clubs $C_\xi \subseteq \omega_1$ and functions $f_\xi : C_\xi \to E$ for every $\xi \in \lambda$ such that if $\xi \neq \zeta$ and $C_\xi \cap C_\zeta$ is cofinal in some $\delta \in \omega_1$, then $f_\xi(\delta) = f_\zeta(\delta) = e$ and $R_e(\xi, \zeta)$. Thinking of R_e as a subset of $\lambda \times \lambda$, that is $(\xi, \zeta) \in R_e$.

1. For every $X \in P_{\omega_1}(\lambda)$ there exists $e \in E$ such that X is a complete subgraph in R_e.

2. If X is a complete subgraph of R_e then $|X| \leq \omega$, and this remains true in any outer model of V.

Now, it is clear that because all complete subgraphs of every R_e are countable, and if two clubs C_ξ and C_ζ share an accumulation point δ then the graphs corresponding to that accumulation point according to f_ξ and f_ζ are the same, R_e for some e, and $(\xi, \zeta) \in R_e$, that there cannot be an uncountable subfamily of $\langle C_\xi : \xi \in \lambda \rangle$ which has a common accumulation point. So their intersection must be finite, and because the property that all complete subgraphs of every R_e are countable persists in every outer model, it must be true that any uncountable subcollection of $\langle C_\xi : \xi \in \lambda \rangle$ in any outer model also has finite intersection. If κ is regular with $\lambda > \kappa$ and $\langle C_\xi : \xi \in \lambda \rangle$ is a collection of club subsets of κ which can be diagonalized, then in particular the intersection of some $\geq \kappa^+$-sized subcollection contains a club. In our case here in the context of 4.4.13, $\kappa = \omega_1$ so if $\langle C_\xi : \xi \in \lambda \rangle$ is as in 4.4.13, any uncountable subcollection of $\langle C_\xi : \xi \in \lambda \rangle$ in any outer model also has finite intersection, so certainly does not have an intersection containing a club. So as long as λ remains larger than ω_1 in the outer model, $\langle C_\xi : \xi \in \lambda \rangle$ remains a collection of clubs which cannot be diagonalized. So we have the following.

Corollary 4.4.14. Let V be the model obtained in 4.4.13 with $\lambda \geq \omega_2$. Then if $V \subseteq M$, $(\omega_1^V = \omega_1)^M$, and $(|\lambda| > \omega_1)^M$, there exists $\langle f_\xi : \xi \in \lambda \rangle \subseteq {}^{\omega_2}\omega_2$ in V which is ω_1-matching in M. So in particular, M does not contain a function from ω_2 to ω_2 which is disjoint modulo countably many coordinates from every function in $({}^{\omega_2}\omega_2)^V$.

That is, it is consistent that an analogy to 4.4.3 is impossible at ω_2. The proof of 4.4.13 uses a technique introduced by Abraham which uses a preparation of the ground model by first adding Cohen subsets, and then uses these subsets to guide a subsequent construction (to ensure a degree of distributivity of a certain forcing). Another example of where this method is used is [1]. It is not outright unreasonable to expect that an analogous forcing with suitable ground model conditions (e.g. assuming $\diamondsuit_{\mu^+}(\mathrm{Cof}(\mu))$, etc.) for regular μ could sometimes be carried out to produce in particular as was done in 4.4.13 an unbounded $B \subseteq {}^{\mu^+}\mu^+$ of size $\lambda > \mu^+$ which remains unbounded in any outer model preserving μ^+ with $|\lambda| > \mu^+$, and so we would have a μ^+-matching family of functions in ${}^{\mu^{++}}\mu^{++}$ of cardinality λ which remains μ^+-matching in the outer model.

4.4.3 Strong diagonalizations are possible at large cardinals

In this section we continue with with the question of when outer models can add a function $g \in {}^\kappa\kappa$ which is strongly disjoint from every ground model $f \in ({}^\kappa\kappa)^V$. This might mean for example that there exists $\lambda < \kappa$ such that modulo $< \lambda$-many coordinates, g is disjoint from f. We saw in 4.4.3 for $\kappa = \omega_1$ that this is possible with a proper forcing, and accordingly then if PFA holds that $\mathbf{nm}_\omega(\omega_1) > \omega_1$, but in 4.4.14 that for $\kappa = \omega_2$ this is consistently impossible. As mentioned, a uniform ultrafilter over ω_1 is used as a device to show that the forcing used in 4.4.3 is proper, and there is some intuition behind the idea that with cardinals carrying measures with a greater additivity, one might be able to carry out such a diagonalization. Indeed, we have the following basic observation about Prikry forcing. For the definition of Prikry forcing and its basic properties see e.g. [35].

Observation 4.4.15. Let κ be a measurable cardinal and let $\mathbb{P}$ be the usual Prikry forcing at κ. If G is $(V, \mathbb{P})$-generic, then there exists $f \in ({}^\omega\kappa)^{V[G]}$ which is disjoint modulo finitely many coordinates from every $g \in ({}^\omega\kappa)^V$. Indeed, if $A = \langle \alpha_n : n \in \omega \rangle \subseteq \kappa$ is the Prikry sequence added, there exists $f \in ({}^A\kappa)^{V[G]}$ such that f is disjoint modulo finitely many coordinates from every $g \in ({}^\kappa\kappa)^V$.

Proof. Let $f : A \to \kappa$ be given by $f(\alpha_n) = \alpha_{n+1}$. Fix $g \in ({}^\kappa\kappa)^V$. We argue that the set of conditions forcing that g doesn't agree with f on a cofinal segment of κ (that is on a cofinal segment of A) is dense. For any $p = \langle \beta_0, \ldots, \beta_n, A_p \rangle \in \mathbb{P}$, let $q = \langle \beta_0, \ldots, \beta_n, A_p \cap C \rangle \leq^* p$. Here let $C \subseteq \kappa$ be club such that for every $\delta \in \kappa$, $C(\delta + 1) > \sup\{g(\beta) : \beta \leq C(\delta)\}$ and $C(0) > \beta_n$. Then if G is $(V, \mathbb{P})$-generic with $q \in G$, for any $m \geq n + 1$, if $\alpha_m = C(\gamma)$ then $\alpha_{m+1} \geq C(\gamma + 1) > g(\alpha_m)$, i.e. g doesn't match F on an infinite subset of A. $\square$

This observation is not sufficient for what we want however, because it is not clear how to extend the function f in 4.4.15 to have domain κ while still remaining strongly (in this case modulo-finite) disjoint from every ground model function in ${}^{\kappa}\kappa$. In order to arrange such a situation, one might not want to add just a cofinal sequence to κ of smaller order-type, but instead add e.g. a club to κ while maintaining its regularity. This is exactly what Radin forcing carried out over κ with a sufficient degree of measurable reflection can accomplish, and we show in 4.4.19 that this works. For background in Radin forcing and for the definitions and the formalization that we use here, see Gitik's [30].

Definition 4.4.16. Let $\vec{V} = \langle \vec{V}(\alpha) : \alpha \in \mathrm{lh}(\vec{V}) \rangle$ be a j-sequence of ultrafilters in $\overline{A}$ for some $j : V \to M$ with $\mathrm{cr}(j) = \kappa$. In this setting these are ultrafilters over V_κ. Recall $\overline{A}$ is such that if $\vec{F} \in \overline{A}$, then for every $\alpha \in \mathrm{lh}(\vec{F})$, $F(\alpha)$ concentrates on $\overline{A} \cap V_{\kappa(\vec{F})}$.

Definition 4.4.17. Let $\mathbb{R}_{\vec{V}}$ be the set of finite sequences $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}, A \rangle \rangle$ such that

1. $A \in \bigcap \vec{V}$ and $A \subseteq \overline{A}$,
2. $A \cap V_{\kappa(d_n)+1} = \emptyset$,
3. For every $m \in [1, n]$, either d_m is an ordinal or $d_m = \langle \kappa(d_n), \vec{F}_n, A_n \rangle$ for some $\vec{F}_n \in \overline{A}$, $A_n \subseteq \overline{A}$, and $A_n \in \bigcap \vec{F}_n$, and
4. For every $i < j$ in $[1, n]$, $\kappa(d_i) < \kappa(d_j)$, and if d_j is $\langle \kappa(d_j), \vec{F}_j, A_j \rangle$, then $A_j \cap V_{\kappa(d_i)+1} = \emptyset$.

Every time a $\langle \kappa(d_n), \vec{F}_n, A_n \rangle$ object appears, this gives rise to a Radin forcing, $\mathbb{R}_{\langle \kappa(d_n), \vec{F}_n, A_n \rangle}$.

Definition 4.4.18. Let $q = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V}, A \rangle \rangle$ and $p = \langle e_1, \ldots, e_m, \langle \kappa, \vec{V}, B \rangle \rangle$. Say $q \le p$ if and only if the following conditions hold.

1. $A \subseteq B$.
2. $m \le n$.

3. There exist $i_1 < \ldots < i_m$ in $[1, n]$ such that for every $k \in [1, m]$, either $e_k = d_{i_k}$ or $e_k = \langle \kappa(e_k), \vec{F}_{e_k}, B_{e_k} \rangle$, and in this case $d_{i_k} = \langle \kappa(d_{i_k}) = \kappa(e_k), \vec{F}_{d_{i_k}} = \vec{F}_{e_k}, C_{d_{i_k}} \rangle$, with $C_{d_{i_k}} \subseteq B_{e_k}$.

4. If $i_1, \ldots, i_m$ are as above, then for every $j \in [1, n]$, if $i_m < j$, then $d_j \in B$ or $d_j = \langle \kappa_{d_j}, \vec{F}_{d_j}, C_{d_j} \rangle$ where $\langle \kappa_{d_j}, \vec{F}_{d_j} \rangle \in B$ and $C_{d_j} \subseteq B \cap V_{\kappa(d_j)}$. On the other hand, if $j < i_m$, then for the least k such that $j < i_k$, necessarily e_k is of the form $\langle \kappa(e_k), \vec{F}_{e_k}, B_{e_k} \rangle$ such that if d_j is an ordinal then $d_j \in B_{e_k}$, and if $d_j = \langle \kappa(d_j), \vec{F}_{d_j}, C_{d_j} \rangle$, then $\langle \kappa(d_j), \vec{F}_{d_j} \rangle \in B_{e_k}$ and $C_{d_j} \subseteq B_{e_k} \cap V_{\kappa(d_j)}$.

Say that q is a direction extension of p, $q \leq^* p$, if and only if $q \leq p$ and $n = m$, i.e. we can shrink each measure one set in each measure object but can't add new elements of the Radin club in direct extensions.

Proposition 4.4.19. If $\mathbb{R}_{\vec{V}}$ is a Radin forcing where κ remains regular (for example if $\mathrm{cf}(\mathrm{lh}(\vec{V})) \geq \kappa^+$ or if there exists a (weak) repeat point), then if G is $(V, \mathbb{R}_{\vec{V}})$-generic, there exists $f \in ({}^\kappa\kappa)^{V[G]}$ such that for every $g \in ({}^\kappa\kappa)^V$, $|\{\alpha : f(\alpha) = g(\alpha)\}| < \omega$. That is, $\mathbb{R}_{\vec{V}}$ adds a function in ${}^\kappa\kappa$ which is co-finitely different from all ground model functions in ${}^\kappa\kappa$.

Proof. In $V[G]$, let $f \in {}^\kappa\kappa$ be defined by $f(\alpha) = C_G(\alpha+1)$, where C_G is the Radin club. That is, f takes α to the $(\alpha+1)^{\text{st}}$ element of the Radin club. Suppose towards a contradiction that for some $g \in ({}^\kappa\kappa)^V$, $|\{\alpha : f(\alpha) = g(\alpha)\}| \geq \omega$. Let $A = \{\alpha_n : n \in \omega\}$ be such that for every $n \in \omega$, $\alpha_n \in \alpha_{n+1}$ and $f(\alpha_n) = g(\alpha_n)$. Let $\mu = \sup(A)$. Because C_G is closed, we have $C_G(\mu) = \sup\{f(\alpha_n) : n \in \omega\}$. First note that we must have $C_G(\mu) = \mu$. Otherwise $\mu < C_G(\mu)$, and we have that $C_G(\mu) \cap g''\mu \subseteq C_G(\mu)$ is cofinal. This is because for every n, $g(\alpha_n) = C_G(\alpha_n + 1)$ and $C_G(\alpha_n + 1) \to C_G(\mu)$. However, working in V, $C_G(\mu) \cap g''\mu \in V$ is a set of cardinality $< C_G(\mu)$, which is a measurable cardinal. This is a contradiction, so $C_G(\mu) = \mu$. Without loss of generality, by replacing $g \restriction \mu$ with $g' \in ({}^\mu\mu)^V$ where $g'(\beta) = g(\beta)$ if $g(\beta) \in \mu$ and 0 otherwise, so $g' \in V$ and because g matches f on A, g'

matches f on A, we may assume for simplicity that $g \in ({}^{\mu}\mu)^V$. Now, choose $p \in G$ such that for some $d \in p$, $\kappa(d) = \mu$, and $p \Vdash C_G(\mu) = \mu$. We show that the set of conditions $q \leq p$ such that $q \Vdash$ "g doesn't match $\dot{f}$ cofinally below μ" is dense. This will be a contradiction, because then such a condition is in G, but g does indeed match f cofinally below μ. So, fix $r = \langle d_1, \ldots, d_k, \langle \mu, \vec{F}_\mu, A_\mu \rangle, d_{k+2}, \ldots, d_n, \langle \kappa, \vec{V}, A \rangle \rangle \leq p$. Note that if $C \subseteq \mu$ is club, then $A_\mu \cap (C \cup \{\langle \rho, \vec{F}_\rho \rangle : \rho \in C\}) \in \vec{F}_\mu(\alpha)$ for every $\alpha \in \mathrm{lh}(\vec{F}_\mu)$. This is because $\vec{F}_\mu$ is j-derived. Explicitly, $\vec{F}_\mu(0)$ concentrates on ordinals and is normal, so $C \cap A_\mu \in \vec{F}_\mu(0)$, and generally for $\alpha \in \mathrm{lh}(\vec{F}_\mu)$, $\{\langle \rho, \vec{F}_\rho \rangle : \rho \in C \wedge \langle \rho, \vec{F}_\rho \rangle \in A_\mu\} \in \vec{F}_\mu(\alpha)$ if and only if $\vec{F}_\mu \restriction \alpha \in j(\{\langle \rho, \vec{F}_\rho \rangle : \rho \in C \wedge \langle \rho, \vec{F}_\rho \rangle \in A_\mu\})$. However, $\mu \in j(C)$ and $\vec{F}_\mu \restriction \alpha \in j(A_\mu)$, so this is clear.

We define a $C \subseteq \mu$ club which forces that above $\kappa(d_k)$, the Radin club outpaces g. Specifically, let $C \subseteq \mu$ be a club so that for every γ, $C(\gamma+1) > \sup\{g(\beta) : \beta \leq C(\gamma)\}$. One way to construct such a C is to build a continuous $\in$-increasing elementary chain of substructures of $(H_\theta, \in, \prec, g, \mu, \ldots)$ for some large θ, $\langle M_\alpha : \alpha \in \mu \rangle$, such that $M_\alpha \cap \mu \in \mu$ for every α, and consider $C = \{M_\alpha \cap \mu : \alpha \in \mu\}$. Then $g(\beta) \in M_{\gamma+1}$ for every $\beta \leq C(\gamma)$, so $C(\gamma+1) > \sup\{g(\beta) : \beta \leq C(\gamma)\}$. Next, let $q = \langle d_1, \ldots, d_k, \langle \mu, \vec{F}_\mu, A'_\mu \rangle, d_{k+2}, \ldots, d_n, \langle \kappa, \vec{V}, A \rangle \rangle \leq^* r$, where we have only shrunk A_μ to $A'_\mu = A_\mu \cap (C \cup \{\langle \rho, \vec{F}_\rho \rangle : \rho \in C\})$. Let G be $(V, \mathbb{R}_{\vec{V}})$-generic containing q. As before $C_G(\mu) = \mu$, and note that if $\delta \in \mu$ is such that $C_G(\delta) \in (\kappa(d_k), \mu)$, then $C_G(\delta) \in C$, because all such ordinals must be added to the $\langle \mu, \vec{F}_\mu, A'_\mu \rangle$ component. So $C_G(\delta) = C(\gamma)$ for some $\gamma \in \mu$, and $C_G(\delta) \geq \delta$. Then $C_G(\delta+1) \geq C(\gamma+1) > \sup\{g(\beta) : \beta \leq C(\gamma)\}$, so $C_G(\delta+1) \geq C(\gamma+1) > g(\delta)$. Because the $\delta \in \mu$ such that $C_G(\delta) \in (\kappa(d_k), \mu)$ constitute an end-segment of μ, and f and g disagree on this end-segment, we have a contradiction. $\square$

Note 4.4.20. If G is $(V, \mathbb{R}_{\vec{V}})$-generic as in 4.4.19, then in $V[G]$ if we let $\mathbb{P}$ be $\mathrm{Col}(\omega_1, <\kappa)$ and let H be $(V[G], \mathbb{P})$-generic, then there exists in $V[G][H]$ a function in ${}^{\omega_2}\omega_2$ which is cofinitely different than every ground model function in ${}^{\omega_2}\omega_2$. Here ω_2^V is of course collapsed, but because $\mathbb{P}$ is σ-closed no reals are added, for example.

Proof. In $V[G]$ let $\mathbb{P} = \mathrm{Col}(\omega_1, < \kappa)$, i.e. $\mathbb{P}$ is the product of $\mathbb{P}_\alpha$ consisting of the set of countable partial functions $p_\alpha : \omega_1 \to \alpha$ for $\alpha \in \kappa$ with countable support. Then $\mathbb{P}$ is σ-closed, and so (ω, ∞)-distributive, and κ-c-c. We see that $\kappa = \omega_2^{V[G][H]}$. Because any countable subset of κ which exists in $V[G][H]$ exists in $V[G]$, we retain the property that f is co-finitely disjoint from every ground model function, and in particular because $V[G]$ did not add reals (it does not add bounded subsets below the first measurable, for example), $V[G][H]$ does not add reals. $\square$

Intuitively it may seem more difficult to have a cardinal-preserving extension with a function in ${}^\kappa\kappa$ which is e.g. modulo-finite disjoint from all ground model functions in ${}^\kappa\kappa$ for some $\kappa > \omega_1$ than it is for $\kappa = \omega_1$. While this works for the ω_n's, the straightforward regressive argument gets stuck at singulars. So there isn't necessarily a conflict between situations like those in 4.4.14 and 4.4.19.

Observation 4.4.21. Suppose $V \subseteq M$ is an outer model where all cardinals in $[\omega_1, \omega_n]$ are preserved. Then if there exists $g \in ({}^{\omega_n}\omega_n)^M$ such that for every $f \in ({}^{\omega_n}\omega_n)^V$, $|\{\alpha \in \omega_n : f(\alpha) = g(\alpha)\}| < \omega$, then for every $k \in n$, there exists $g_k \in ({}^{\omega_k}\omega_k)^M$ such that for every $f \in ({}^{\omega_k}\omega_k)^V$, $|\{\alpha \in \omega_k : f(\alpha) = g_k(\alpha)\}| < \omega$.

Proof. Let $g \in {}^{\omega_n}\omega_n$ be as given. Find $\alpha \in \omega_n$ such that $g''\alpha \subseteq \alpha$ with $|\alpha| = \omega_{n-1}$. Via a bijection $e : \omega_{n-1} \to \alpha$ in V, $g \restriction \alpha$ can be transformed into a function in ${}^{\omega_{n-1}}\omega_{n-1}$, which must be modulo-finite disjoint from all functions in $({}^{\omega_{n-1}}\omega_{n-1})^V$. If $k \in n-1$, we simply proceed in this manner, and if $k = n-1$ we're done. $\square$

The argument in 4.4.21 gets stuck at $\aleph_\omega$ however: If $g \in {}^{\aleph_\omega}\aleph_\omega$ is modulo-finite disjoint from every function in $({}^{\aleph_\omega}\aleph_\omega)^V$, this does not mean automatically that for some $M \in (P_{\aleph_\omega}\aleph_\omega)^V$ we have $g''M \subseteq M$, as would be required for the regression step.

4.5 Relationship to some guessing principles

4.5.1 Preliminaries

The existence of a set of functions in ${}^{\omega_1}\omega_1$ of size ω_1 which is countably matching (that is $\mathbf{nm}_\omega(\omega_1) = \omega_1$) is implied by several common guessing principles (weakenings of $\diamondsuit$). In this section we explore this, for which we first need some definitions.

Definition 4.5.1. The "club" principle, $\clubsuit$, formulated by Ostaszewski [54], asserts the existence of a sequence $\langle A_\alpha : \alpha \in \lim(\omega_1)\rangle \subseteq P_{\omega_1}(\omega_1)$ such that for every $\alpha \in \lim(\omega_1)$, $A_\alpha \subseteq \alpha$ is cofinal and for each $x \in [\omega_1]^{\omega_1}$, there exists $\alpha \in \lim(\omega_1)$ such that $A_\alpha \subseteq x$.

Fact 4.5.2. It is equivalent in the definition of $\clubsuit$ to assert that for each $X \in [\omega_1]^{\omega_1}$, the set of $\alpha \in \lim(\omega_1)$ such that $A_\alpha \subseteq X$ is stationary.

Note 4.5.3. For a given infinite cardinal μ and stationary subset of μ^+, S, one can define $\clubsuit_{\mu^+}(S)$ in the natural way, that is asserting the existence of a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that for every $\alpha \in S$, $A_\alpha \subseteq \alpha$ is cofinal and if $x \in [\mu^+]^{\mu^+}$, $\{\alpha \in S : A_\alpha \subseteq x\}$ is stationary.

Fact [54] 4.5.4. $\clubsuit_{\mu^+}(S)$ along with $2^\mu = \mu^+$ is equivalent to $\diamondsuit_{\mu^+}(S)$.

Because $\clubsuit + \neg CH$ is consistent [59], $\clubsuit$ can be seen as a sort of "cardinal arithmetic free" version of $\diamondsuit$. However, $\clubsuit$ can exhibit quite different behavior from $\diamondsuit$. For example, Kunen's argument that $\diamondsuit^-$ is equivalent to $\diamondsuit$, where $\diamondsuit^-$ is the version of $\diamondsuit$ which allows countably many guesses at each coordinate, does not work for $\clubsuit$ and indeed the corresponding version of $\clubsuit$ is consistently different than $\clubsuit$. This also applies to other natural weakenings of $\clubsuit$ where one might expect equivalence, such as only having to guess modulo finite [20]. If we forgo the requirement in $\clubsuit$ that every A_α has to be a cofinal subset of α, and instead insist only that every uncountable subset of ω_1 is guessed by some A_α, we have the following principle $\dagger$ (read as "stick"), introduced in [3].

Definition 4.5.5. The stick principle ⫯ asserts the existence of a sequence $\langle A_\alpha : \alpha \in \omega_1 \rangle \subseteq [\omega_1]^\omega$ such that for every $x \in [\omega_1]^{\omega_1}$, there exists $\alpha \in \omega_1$ such that $A_\alpha \subseteq x$.

Observation 4.5.6. If the CH holds then $|[\omega_1]^\omega| = \omega_1$, so ⫯ is a consequence the CH as well as of $\clubsuit$.

Note 4.5.7. It is reasonable to treat ⫯ as a cardinal characteristic in the natural way, defining ⫯ to be the minimal cardinality of a collection of countable subsets of ω_1, $Z \subseteq [\omega_1]^\omega$, such that for every $x \in [\omega_1]^{\omega_1}$, there exits $z \in Z$ with $z \subseteq x$. With that meaning, ⫯ as in 4.5.5 is the assertion that ⫯ $= \omega_1$.

Note 4.5.8. In the presence of the CH, $\clubsuit$ and ⫯ are certainly not equivalent because ⫯ is a consequence of the CH, while in the presence of the CH as noted $\clubsuit$ and $\diamondsuit$ are equivalent, and $CH + \neg\diamondsuit$ is consistent (Jensen's original argument [19] kills all Suslin trees, so destroying all $\diamondsuit$ sequences, without adding reals). Even in the absence of the CH, $\clubsuit$ and ⫯ are not equivalent. For example, in [20] a model is given for $\neg CH + \neg\clubsuit + \clubsuit_{\mathrm{fin}}$, where $\clubsuit_{\mathrm{fin}}$ is the version of $\clubsuit$ where it is only required that for each $x \in [\omega_1]^{\omega_1}$ there exits $\alpha \in \lim(\omega_1)$ such that $|A_\alpha \setminus x| < \omega$. By starting with a $\clubsuit_{\mathrm{fin}}$ sequence and adding to each A_α all subsets which differ by only finitely many elements, we can produce a ⫯ sequence, so $\clubsuit_{\mathrm{fin}}$ implies ⫯.

4.5.2 Observations

Observation 4.5.9. If $MA(\mathrm{Fn}(\omega_1, 2, <\omega))$ holds then ⫯ $= 2^\omega$. So in particular, $\neg$⫯ is consistent.

Proof. Let $\mathbb{P} = \mathrm{Fn}(\omega_1, 2, <\omega)$, that is the poset for adding ω_1-many Cohen reals with finite conditions. We need to see that if there exist generics for every collection of $(< 2^\omega)$-many dense subsets of $\mathbb{P}$, ⫯ $= 2^\omega$. Fix $\kappa < 2^\omega$ and let $\langle A_\alpha : \alpha \in \kappa \rangle \subseteq [\omega_1]^\omega$. It is clear that for every $\alpha \in \kappa$, $D_\alpha = \{p \in \mathbb{P} : \exists \gamma \in \mathrm{dom}(p) \cap A_\alpha : p(\gamma) = 0\}$ and for every $\alpha \in \omega_1$,

$D'_\alpha = \{p \in \mathbb{P} : \exists \gamma \in \mathrm{dom}(p) \setminus \alpha\}$ are dense subsets of $\mathbb{P}$. So if G is a filter hitting each of these and $x = \{\gamma \in \omega_1 : p(\gamma) = 1 \text{ for some } p \in G\}$, $x \in [\omega_1]^{\omega_1}$ but $\neg(A_\alpha \subseteq x)$ for every $\alpha \in \kappa$. $\square$

We have seen that as with $⫯ = \omega_1$, $\mathbf{nm}_\omega(\omega_1) = \omega_1$ is a trivial consequence of the CH. However, it is not difficult to see that $\mathbf{nm}_\omega(\omega_1) = \omega_1$ is also an immediate consequence of $⫯$.

Observation 4.5.10. If $⫯$ holds then $\mathbf{nm}_\omega(\omega_1) = \omega_1$.

Proof. Let $⫯ = \omega_1$ be witnessed by $\langle A_\alpha : \alpha \in \omega_1 \rangle$. Fix $e : \omega_1 \times \omega_1 \to \omega_1$ a bijection. For every $\alpha \in \omega_1$, if $e^{-1}[A_\alpha] \subseteq \omega_1 \times \omega_1$ is the graph of a partial function, let $g_\alpha : \omega_1 \to \omega_1$ be an arbitrary extension of $e^{-1}[A_\alpha]$ to a total function on ω_1. Note that $\langle g_\alpha : \alpha \in \omega_1 \rangle \subseteq {}^{\omega_1}\omega_1$ is countably matching, because if $f \in {}^{\omega_1}\omega_1$ then identifying f with its graph we have $e''f \in [\omega_1]^{\omega_1}$, so for some α, $A_\alpha \subseteq e''f$, but then necessarily g_α countably matches f. $\square$

As in 4.5.10, not only does $⫯$ imply that $\mathbf{nm}_\omega(\omega_1) = \omega_1$, but that there exists a sequence of ω_1-many countable partial functions which is countably matching. While one might think perhaps then that $⫯$ has some relationship with $\mathbf{nm}$, this is false. In 4.5.9 we see that by adding e.g. ω_2-many Cohen reals we will have a model where $⫯ = \omega_2$, but as in 4.3.4 $\mathbf{nm} = \omega_1$ in the extension. On the other hand, $\clubsuit$ holds in the countable support iteration of length ω_2 of Laver forcing over a model of $\diamondsuit$ [51], while Laver forcing adds disjointing functions so $\mathbf{nm} = \omega_2$ (and even $\mathfrak{b} = \omega_2$, see [10]). With respect to quantities in Cichoń's diagram, this is as good as one could hope for, because it's an early result of Truss [68] that $⫯$ implies that either $\mathbf{cov}(\mathcal{M})$ or $\mathbf{cov}(\mathcal{L})$. Here $\mathcal{L}$ stands for Lebesgue—so if $⫯$ holds then the real line can be covered by either ω_1-many meager sets or ω_1-many Lebesgue-measure zero sets. So the Laver model (forced over a model of $\diamondsuit$) is one where in particular $\mathbf{nm} > \mathbf{nm}_\omega(\omega_1)$. Of course there are easier ways to separate these two quantities, or indeed $\mathbf{nm}_\omega(\omega_1)$ from any cardinal characteristic of the continuum implied by MA to be 2^ω.

Observation 4.5.11. It is consistent that $MA + \neg CH$ holds along with $\mathbf{nm}_\omega(\omega_1) = \omega_1$.

Proof. Start with a model of the GCH and force $MA + 2^\omega > \omega_1$ with a c.c.c. forcing, forming $V[G]$. Working in $V[G]$ if $g \in {}^{\omega_1}\omega_1$ then for some $F \in ({}^{\omega_1}P_{\omega_1}\omega_1)^V$, for every $\alpha \in \omega$ we have $g(\alpha) \in F(\alpha)$. By choosing in V surjections from ω to $F(\alpha)$ for every α and defining in the natural way the f_n functions choosing for each α the n^{th} element of $F(\alpha)$ according to the α^{th} surjection, for some $f_n \in ({}^{\omega_1}\omega_1)^V$ we have $|\{\alpha \in \omega_1 : f_n(\alpha) = g(\alpha)\}| = \omega_1$. Then for some $\gamma \in \omega_1$, $f_n \restriction \gamma$ countably matches g. But then the set of all functions f in ${}^{\omega_1}\omega_1$ such that for some $\gamma \in \omega_1$, $f \restriction \gamma = \overline{f} \restriction \gamma$ for some $\overline{f} \in ({}^{\gamma}\omega_1)^V$ and $f(\beta) = 0$ for every $\beta \geq \gamma$ is countably matching, and of size ω_1. $\square$

So, in the same way that $\clubsuit$ or ⫯ may be viewed as weakenings of $\diamondsuit$ which are consistent with $\neg CH$ or some weak forms of $MA + \neg CH$, the assertion that $\mathbf{nm}_\omega(\omega_1) = \omega_1$ may be viewed as a weakening of e.g. ⫯ which is consistent with $MA + \neg CH$.

Note 4.5.12. Just as in 4.4.9, if in V, $\mathcal{A} \subseteq P_\kappa\kappa$ has the property that for every $x \in [\kappa]^\kappa$, there exists $a \in \mathcal{A}$ such that $a \subseteq x$, then $\mathcal{A}$ retains this property in any extension by $\mathbb{P}$ a κ-strategically closed forcing. So in particular, a ⫯ sequence remains a ⫯ sequence after forcing with an ω_1-strategically closed forcing.

Proof. Let $\mathbb{P}$ be κ-strategically closed. Suppose $\dot{f}$ is the name for a new function in ${}^\kappa 2$ with κ-sized support. Build as in 4.4.9 a candidate sequence of functions $\langle f_\alpha : \alpha \in \kappa \rangle \subseteq V$ and a corresponding $\leq$-decreasing sequence of conditions $\langle p_\alpha : \alpha \in \kappa \rangle$ such that $p_\alpha \Vdash f_\alpha \restriction \alpha = \dot{f} \restriction \alpha$. Let $g(\alpha) = f_{\alpha+1}(\alpha)$ for every $\alpha \in \kappa$ and note that we may ensure by construction that $\{\beta \in \kappa : g(\beta) = 1\} = x \subseteq \kappa$ is unbounded, and $g \in V$, so for some $a \in \mathcal{A}$ we have $a \subseteq x$. But then for some sufficiently large α, $a \subseteq \alpha$, so $p_\alpha \Vdash a \subseteq \{\beta \in \alpha : \dot{f}(\beta) = 1\}$. $\square$

4.5.3 Maximal almost disjoint families of functions

In [10], the notation $\mathfrak{a}_e(\kappa)$ is used to indicate the minimal cardinality of a family of functions $F \subseteq {}^\kappa\kappa$ which is maximal with respect to being κ-almost disjoint—so if $\{f, g\} \subseteq F$, $|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| < \kappa$. If the subscript e is excluded from $\mathfrak{a}_e(\kappa)$, then $\mathfrak{a}(\kappa)$ indicates as usual the minimal cardinality $\geq \kappa^+$ of a family of κ-sized subsets of κ which is maximal with respect to being κ-almost disjoint. In analogy with our notation for **nm**, we have the following.

Definition 4.5.13. For $\mu \leq \kappa$ let $\mathfrak{a}_e^\mu(\kappa)$ denote the minimal cardinality of a family of functions $F \subseteq {}^\kappa\kappa$ which is maximal with respect to being μ-almost disjoint.

Observation 4.5.14. If $F \subseteq {}^\kappa\kappa$ is maximal with respect to being μ-almost disjoint, then clearly F is μ-matching. So necessarily $\mathbf{nm}_\mu(\kappa) \leq \mathfrak{a}_e^\mu(\kappa)$.

Just as the assertion that $\mathbf{nm}_\omega(\omega_1) = \omega_1$ is implied by significant weakenings of $\clubsuit$, so is the assertion that $\mathfrak{a}_e^\omega(\omega_1) = \omega_1$. First, let $\clubsuit^\kappa_{\mu^+}(S)$ for $S \subseteq \mu^+$ stationary denote the principle that asserts the existence of a sequence $\langle \{A^\alpha_\beta : \alpha \in \kappa\} : \beta \in S\rangle$ such that for every $\alpha \in \kappa$ and $\beta \in S$, $A^\alpha_\beta \subseteq \beta$ is cofinal and if $x \in [\mu^+]^{\mu^+}$ there exists $\beta \in S$ and $\alpha \in \kappa$ such that $A^\alpha_\beta \subseteq x$. If S is omitted it is assumed to be $\lim(\mu^+)$.

Observation 4.5.15. $\clubsuit^\omega_{\omega_1}$ implies that $\mathfrak{a}_e^\omega(\omega_1) = \omega_1$.

Proof. Let $\langle \{A^n_\beta : n \in \omega\} : \beta \in \lim(\omega_1)\rangle$ witness $\clubsuit^\omega_{\omega_1}$. Note that $\{A^n_\beta : \alpha \in \kappa\} \subseteq \beta$ can be almost disjointly refined (see the disjoint refinements chapter in this thesis for more details). So without loss of generality, if $\beta \in \lim(\omega_1)$ and $n_1 \neq n_2$, $|A^{n_1}_\beta \cap A^{n_2}_\beta| < \omega$, and we may also assume that every A^n_β has order type ω, so in fact if $\langle n_1, \beta_1\rangle \neq \langle n_2, \beta_2\rangle$, $|A^{n_1}_{\beta_1} \cap A^{n_2}_{\beta_2}| < \omega$. Fix bijections $e : \omega_1 \times \omega_1 \to \omega_1$ and $h : \omega \times \omega_1 \to \omega_1$. By recursion define $f_{h^{-1}(\alpha)} \in {}^{\omega_1}\omega_1$ by first looking at $e^{-1}[A_{h^{-1}(\alpha)}]$ and checking to see whether it is the graph of a partial function. Identify as usual $e^{-1}[A_{h^{-1}(\alpha)}]$ with this partial function here. If it is a partial function, set $f_{h^{-1}(\alpha)}(\beta) \neq f_{h^{-1}(\gamma)}(\beta)$ for every $\gamma \in \alpha$ and $\beta \notin \mathrm{dom}(e^{-1}[A_{h^{-1}(\alpha)}])$, while set $f_{h^{-1}(\alpha)}(\beta) = e^{-1}[A_{h^{-1}(\alpha)}](\beta)$ for every $\beta \in \mathrm{dom}(e^{-1}[A_{h^{-1}(\alpha)}])$. It is clear that we can proceed in this manner, and by construction $\{f_{h^{-1}(\alpha)} : \alpha \in \omega_1\} \subseteq {}^{\omega_1}\omega_1$ is ω-almost disjoint. Moreover, if $g \in {}^{\omega_1}\omega_1$, then identifying g with its graph, $e''g \in [\omega_1]^{\omega_1}$ and so for some $\beta \in \lim(\omega_1)$ and $n \in \omega$, $A^n_\beta \subseteq e''g$. But then it is not difficult to see that $f_{h(\langle n,\beta\rangle)}$ countably matches g. $\square$

The argument in 4.5.15 is possible because in the $\clubsuit$ sequence we insist that every A_β is cofinal in β, so that for distinct β_1, β_2, we may assume that $|A_{\beta_1} \cap A_{\beta_2}| < \omega$. The following definition is then natural.

Definition 4.5.16. Define $⫯^{\mathrm{ad}}$ to be the assertion that there exists an almost disjoint stick sequence, that is that there exists $\langle A_\alpha : \alpha \in \omega_1\rangle \subseteq [\omega_1]^\omega$ such that for every $x \in [\omega_1]^{\omega_1}$ there exists $\alpha \in \omega_1$ with $A_\alpha \subseteq x$ and if $\alpha \neq \beta$ are in ω_1, $|A_\alpha \cap A_\beta| < \omega$.

Observation 4.5.17. The argument in 4.5.15 shows that $⫯^{\mathrm{ad}} \implies \mathfrak{a}^\omega_e(\omega_1) = \omega_1$ and $\clubsuit^\omega_{\omega_1} \implies ⫯^{\mathrm{ad}}$. Moreover, by the results in the refining chapter of this thesis, if $2^\omega > \omega_1$ then in particular any ω_1-sized collection of countable subsets of ω can be almost disjointly refined, so it is straightforward to see that $⫯ + \neg CH \implies ⫯^{\mathrm{ad}}$. This may be done by first thinning out every element of a $⫯$ sequence to have order type ω, grouping together elements with the same supremum, then almost disjointly refining each of these collections. So the equivalence of $⫯^{\mathrm{ad}}$ with $⫯$ is equivalent to whether or not the CH implies $⫯^{\mathrm{ad}}$ as it does $⫯$.

We do not know whether the CH implies $⫯^{\mathrm{ad}}$. In any case however, it is straightforward to show that $⫯^{\mathrm{ad}}$ is closer to the CH than $\diamondsuit$ is (and so under the presence of the CH, than $\clubsuit$, $\clubsuit_{\mathrm{fin}}$, $\clubsuit^\omega_{\omega_1}$, etc. are).

Observation 4.5.18. From a model of $CH + \neg\diamondsuit$, one can force a model of $CH + ⫯^{\mathrm{ad}} + \neg\diamondsuit$.

Proof. Suppose $V \models CH + \neg\diamondsuit$, for example suppose V is Jensen's original model [19] for this. Working in V, let $\langle A_\alpha : \alpha \in \omega_1\rangle \subseteq [\omega_1]^\omega$ be a maximal almost disjoint family of countable sets, so that for every $\alpha \in \beta \in \omega_1$, $|A_\alpha \cap A_\beta| < \omega$, and if $x \in [\omega_1]^\omega$, there exists $\alpha \in \omega_1$ such that $|x \cap A_\alpha| = \omega_1$. First, note that no c.c.c. forcing can add a $\diamondsuit$ sequence, as follows. Towards a contradiction, suppose $V[G] \models \diamondsuit$ and let $p \in G$ and $\dot{f}$ be such that $p \Vdash \dot{f}$ is a diamond sequence. Working in V, for every $\alpha \in \omega_1$, let $\mathcal{A}_\alpha = \{X \subseteq \alpha : \exists q \leq p$ such that $q \Vdash \dot{f}(\alpha) = X\}$. Because $\mathbb{P}$ is ccc, $|\mathcal{A}_\alpha| \leq \omega$. Because of the Kunen equivalency of $\diamondsuit'$ (where we're allowed countably many guesses at each coordinate) and $\diamondsuit$, it suffices to argue that $\langle \mathcal{A}_\alpha : \alpha \in \omega_1 \rangle$ is a $\diamondsuit'$ sequence. So let $C \subseteq \omega_1$ be a club and $Y \subseteq \omega_1$. We need $\alpha \in C$ such that $Y \cap \alpha \in \mathcal{A}_\alpha$. In $V[G]$, Y must be guessed at an ordinal in C, so there exists $q \leq p$, $q \in G$, and $\alpha \in C$ such that $q \Vdash Y \cap \alpha = \dot{f}(\alpha)$. Then by construction, $Y \cap \alpha \in \mathcal{A}_\alpha$. So, as long as we perform a c.c.c. forcing, $\neg\diamondsuit$ will still hold in the extension. So, let $\mathbb{P}$ be any c.c.c. forcing which is (ω, ω_1)-semidistributive and force with $\mathbb{P}$, forming $V[G]$. So for example, $\mathbb{P}$ could be Hechler forcing, Random forcing, or Cohen forcing (which is even (ω_1, ω_1)-semidistributive). Working in $V[G]$, for every $\alpha \in \omega_1$, $([A_\alpha]^\omega)^V$ can be almost disjointly refined (for details see the refinement chapter in this thesis) into $\langle A_\alpha^\beta : \beta \in \omega_1 \rangle \subseteq A_\alpha$, where we may assume that every A_α^β has order type ω. So necessarily $\{A_\alpha^\beta : \alpha, \beta \in \omega_1\} \subseteq [\omega_1]^\omega$ is an almost disjoint collection. Moreover, if $x \in [\omega_1]^{\omega_1}$, because $\mathbb{P}$ is (ω, ω_1)-semidistributive there exists $x' in [x]^\omega \cap V$. But then for some $\alpha \in \omega_1$, $|A_\alpha \cap x'| = \omega$ and $A_\alpha \cap x' \in V$, so for some $\beta \in \omega_1$, $A_\alpha^\beta \subseteq A_\alpha \cap x' \subseteq x$, as desired. This argument wouldn't work with any (ω, ω_1)-semidistributive extension. For example, Sacks forcing over a model of the CH adds $\diamondsuit$ [46]. $\square$

If in the final model in 4.5.18 we wanted to strengthen $\neg\diamondsuit$ to the nonexistence of Suslin trees, we would need a ccc forcing adding a real which doesn't add a Suslin tree over a model where the CH holds and there do not exist any Suslin trees. We do not know if this is possible.

4.6 Unconsidered directions

1. Is $⫯^{\mathrm{ad}}$ equivalent to $⫯$? That is, does the CH imply $⫯^{\mathrm{ad}}$?

2. Is $\mathfrak{b}(\kappa) < \mathbf{nm}(\kappa)$ for κ a regular limit consistent? If $\kappa = \omega$ this may be accomplished in the Random model, so one might search perhaps for an analogue to Random forcing (and so perhaps for an analogue to the ideal of Lebesgue measure zero sets) at κ. In particular, we might want a forcing which is κ^+-c.c., $({}^\kappa\kappa)$-bounding, and $< \kappa$-closed. There has been recent progress in this direction, with Shelah obtaining such a forcing at weakly compact κ [57] and Friedman and Laguizzi obtaining such a forcing at κ inaccessible [26].

3. Is $\mathbf{nm}_\omega(\omega_1) < \mathfrak{a}_e^\omega(\omega_1)$ consistent?

4. Can $\mathbf{nm}_\omega(\omega_1)$ be consistently larger than any cardinal characteristic of the continuum (for example $\mathfrak{b}$)?

5. Can $\mathbf{nm}_{\omega_1}(\omega_2)$ be consistently larger than 2^ω? In particular, consistently is there a cardinal-preserving forcing which adds a function in ${}^{\omega_2}\omega_2$ which is modulo-countable disjoint from every function in $({}^{\omega_2}\omega_2)^V$?

6. Is it consistent that for some large κ, $\mathbf{nm}_\omega(\kappa) > \kappa$? Given 4.4.19, one might look at extender-based Radin forcing in the sense of Merimovich [50], as used recently by Gitik and Ben-Neria to control the splitting number at κ [31].

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