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Operator Theory for Analysis of Convex Optimization Methods in Machine Learning

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Operator Theory for Analysis of Convex Optimization Methods in Machine Learning

A dissertation submitted in partial satisfaction of the
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in

Cognitive Science

by

Patrick W. Gallagher

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2014

The dissertation of Patrick W. Gallagher is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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University of California, San Diego

2014

EPIGRAPH

...nothing at all takes place in the universe in which some rule of maximum or minimum does not appear. — Leonhard Euler

...between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain. — Paul Painlevé

You can see everything with one eye, but looking with two eyes is more convenient. — Jan Brinkhuis and Vladimir Tikhomirov

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Chapter 17, in part, is currently being prepared for submission for publication of the material. Gallagher, Patrick; Tu, Zhuowen. The dissertation author was the primary author of this material.

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ABSTRACT OF THE DISSERTATION

Operator Theory for Analysis of Convex Optimization Methods in Machine Learning

by

Patrick W. Gallagher

Doctor of Philosophy in Cognitive Science

University of California, San Diego, 2014

Professor Virginia de Sa, Chair
Professor Philip Gill, Co-Chair

As machine learning has more closely interacted with optimization, the concept of convexity has loomed large. Two properties beyond simple convexity have received particularly close attention: strong smoothness and strong convexity. These properties (and their relatives) underlie machine learning analyses from convergence rates to generalization bounds — they are central and fundamental.

This thesis takes as its focus properties from operator theory that, in specific instances, relate to broadened conceptions of convexity, strong smoothness, and strong convexity. Some of the properties we consider coincide with strong smoothness and strong convexity in some settings, but represent broadenings of these concepts in other

situations of interest. Our intention throughout is to take an approach that balances theoretical generality with ease of use and subsequent extension.

Through this approach we establish a framework, novel in its scope of application, in which a single analysis serves to recover standard convergence rates (typically established via a variety of separate arguments) for convex optimization methods prominent in machine learning.

The framework is based on a perspective in which the iterative update for each convex optimization method is regarded as the application of some operator. We establish a collection of correspondences, novel in its comprehensiveness, that exist between "contractivity-type" properties of the iterative update operator and "monotonicity-type" properties of the associated displacement operator. We call particular attention to the comparison between the broader range of properties that we discuss and the more restricted range considered in the contemporary literature, demonstrating as well the relationship between the broader and narrower range.

In support of our discussion of these property correspondences and the optimization method analyses based on them, we relate operator theory concepts that may be unfamiliar to a machine learning audience to more familiar concepts from convex analysis. In addition to grounding our discussion of operator theory, this turns out to provide a fresh perspective on many touchstone concepts from convex analysis.

Chapter 1

Introduction

1.1 Introduction

As machine learning has more and more closely interacted with optimization theory and optimization practice, the concept of convexity has taken center stage.

When applied to the analysis of algorithm performance, two properties beyond simple convexity have received particularly close attention: strong smoothness and strong convexity. These properties (and their relatives) underlie machine learning analyses from convergence rates to generalization bounds — they are central and fundamental.

This thesis takes as its focus properties from operator theory that, in specific instances, relate to broadened conceptions of convexity, strong smoothness, and strong convexity. Some of the specific properties we consider coincide with strong smoothness and strong convexity in the most basic settings, but represent broadenings of these concepts when applied to other situations of interest. Our intention throughout is to take an approach that balances theoretical generality with ease of use and subsequent extension. Through this approach we establish a framework, novel in its generality, in which a single analysis serves to recover standard convergence rates (typically established via a variety of separate arguments) for many of the most prominent convex optimization methods in machine learning. We also touch on a more restrictive setting in which we can say more.

The framework mentioned above is based on a perspective in which the iterative update for each convex optimization method under consideration is regarded as the application of some operator. We establish a collection of correspondences, novel in its comprehensiveness, that exist between "contractivity-type" properties of the iterative update operator and "monotonicity-type" properties of the associated displacement operator. We call particular attention to the comparison between the broader range of properties that we discuss and the more restricted range considered in the contemporary literature, demonstrating as well the relationship between the broader range and the narrower range.

In support of our discussion of these operator property correspondences and the optimization method analyses based on them, we provide necessary background in convex analysis, optimization theory, and operator theory. As we proceed, we relate operator theory concepts that may be unfamiliar to a machine learning audience to more familiar concepts from convex analysis. In addition to grounding our discussion of operator theory, this turns out to have the added benefit of providing a fresh perspective on many of the touchstone concepts from convex analysis. Underlying our approach throughout is the goal of establishing each definition, concept, or result through a verbal description, an explicit mathematical expression, and (whenever possible) a visualization.

1.2 Context

This is a thesis about convex optimization methods.

At the heart of our approach are ideas from operator theory concerning correspondences between "contractivity-type" properties of an operator representing the iterative update of an optimization method and "monotonicity-type" properties of an associated "displacement" operator.

We show how these ideas are fundamentally connected to central ideas in convex analysis, duality, and optimization.

Most of our attention will be directed to algorithms for convex optimization.

Focusing attention on the iterative update from the kth iterate, x^k, to the next iterate, x^{k+1}, we provide the update with a generic name: $T(\cdot)$. Thus, we say that an optimization method gets from x^k to x^{k+1} by applying the iterative update operator $T(\cdot)$, so that $x^{k+1} \overset{\text{set}}{=} T\left(x^k\right)$.

Our investigations will relate properties of the iterative update operator (or, equivalently, of the associated displacement operator $G \overset{\text{set}}{=} I - T$) to the behavior of the optimization methods in question.

In particular, we will see that monotonicity-type properties of the displacement operator, $G \overset{\text{set}}{=} I - T$, associated with the iterative update T correspond to contractivity-type properties of T; these contractivity properties can then be leveraged to yield analysis of convergence and of convergence rates. From a discussion of correspondences between "contractivity-type" properties and "monotonicity-type" properties that is novel in its comprehensiveness, we arrive at a framework in which a single analysis suffices to establish standard convergence rates (typically established via a variety of separate arguments) for many of the most prominent convex optimization methods.

1.2.1 Convexity, strong smoothness, and strong convexity

Our discussion throughout relies on properties of functions and operators. While we will go into formal discussions of each in subsequent chapters, for now our needs can be met with relatively simple concepts of each: each point in the graph of a function is of the form $(\mathbb{R}^n, \mathbb{R})$; each point in the graph of an operator is of the form $(\mathbb{R}^n, \mathbb{R}^n)$[1].

Throughout this thesis, we will repeatedly discuss a particular familiar pairing of function and operator: the function will be some convex objective function; the operator will be the gradient (or the subdifferential) of that function. For this pairing of function and operator, we can motivate our operator theoretic approach by comparing function properties to the corresponding properties satisfied by the gradient operator.

The correspondence that we illustrate below relates convexity of a function to monotonicity of its gradient operator, strong convexity of a function to strong monotonicity of its gradient operator, and strong smoothness of a function to inverse strong monotonicity of its gradient operator.

When we say that a function is convex, we are making a statement about the

[1]Strictly speaking, our discussion will consider points in the graph of an operator that are of the form $(\mathbb{R}^n, \mathbb{R}^{n*})$ — primal space, dual space.

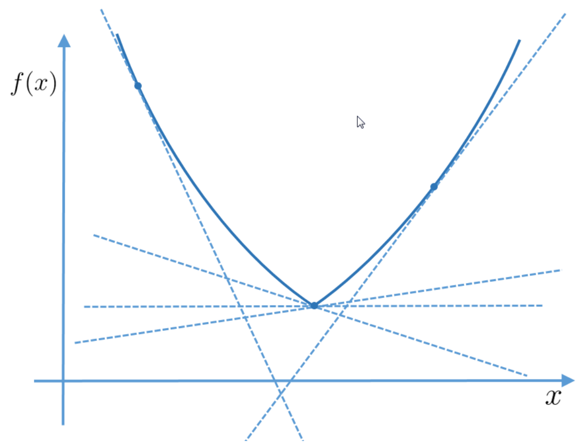

Figure 1.1: For a convex function, we can produce a supporting affine minorant to the function at any argument.

existence of supporting affine minorants. We could say that a function is "convex at a point" when there exists a supporting affine minorant to the function at that point. When there is more than one affine minorant supporting the function at a point, the function is nonsmooth at that point. See Figure 1.1.

When we say that an operator is monotone (more specifically, monotone non-decreasing), we are making a statement about "quadrantal" bounds on the graph of the operator. We could say that an operator is "monotone at a point" when the rest of the graph lies in the "first" and "third" quadrants with respect to that point. See Figure 1.2.

When we say that an operator is Lipschitz, we are making a statement about a restriction on the graph of the operator corresponding to upper-and-lower "wedge"-type bounds on the graph of the operator. Using the notation of Figure 1.3, these bounds can be stated as We could say that an operator is Lipschitz at a point when the rest of the graph lies in a wedge with respect to that point. Note that Lipschitz is distinct from monotone: the graph of a Lipschitz operator can venture outside the "relative" first and third quadrants to which a monotone operator is restricted. See Figure 1.3.

When we say that an operator is "inverse Lipschitz", we are making a statement about a restriction on the graph of the operator corresponding (analogously to the situation for Lipschitz) to upper-and-lower "wedge"-type bounds on the graph of the operator; more specifically, the "inverse operator" satisfies a Lipschitz condition. We could say that an operator is inverse Lipschitz at a point when the rest of the graph lies

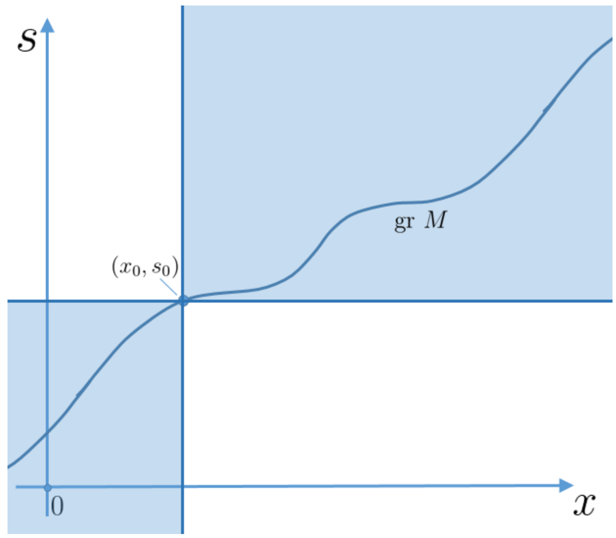

Figure 1.2: An operator is “monotone at a point” when the rest of the graph lies in the “first” and “third” quadrants with respect to that point (shaded). (After [AN09]).

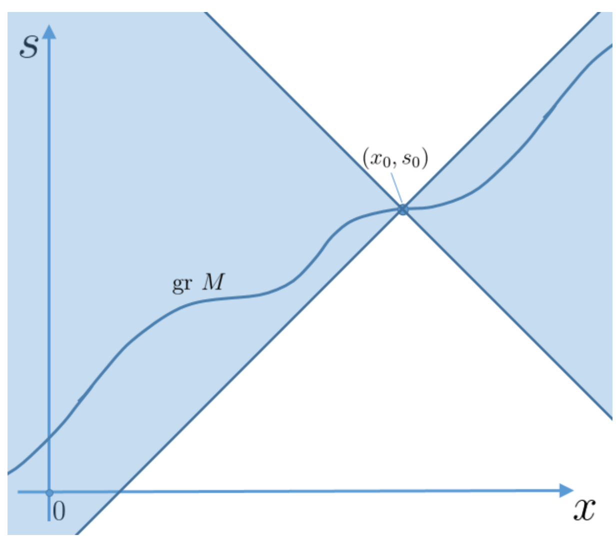

Figure 1.3: An operator M is “Lipschitz at a point” when the rest of the graph of the operator lies in a wedge (shaded) with respect to that point: the absolute value of the change in s, must be no greater than a fixed scalar times the absolute value of the change in x. (After [AN09]).

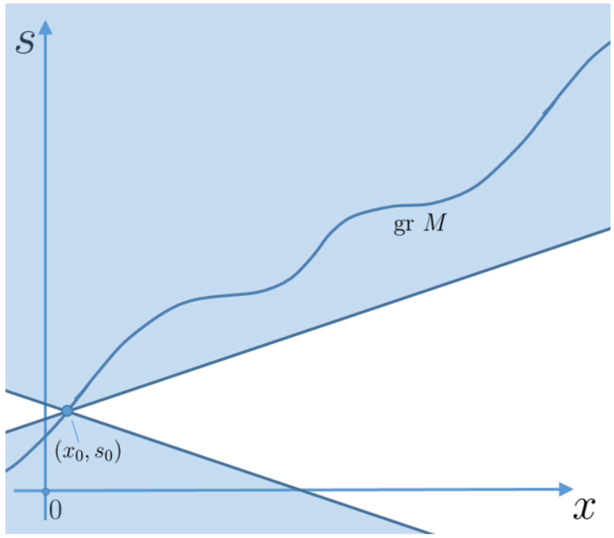

Figure 1.4: An operator *M* is "Lipschitz at a point" when the rest of the graph of the operator lies in a wedge (shaded) with respect to that point: the absolute value of the change in *x*, must be no greater than a fixed scalar times the absolute value of the change in *s*.

in a wedge with respect to that point. Note again that "inverse Lipschitz" is distinct from monotone; the graph of an "inverse Lipschitz" operator can venture outside the "relative" first and third quadrants to which a monotone operator is restricted. See Figure 1.4.

Returning our focus to functions instead of operators, when we say that a function is strongly smooth, we are making a statement about bounds on how quickly the function can change. Specifically, we are saying that the graph of the function lies between the graphs of two quadratic functions, one "curving upward" and one "curving downward". See Figure 1.5.

When we say that a *convex* function is strongly smooth, the supporting quadratic function provides "less information" than the supporting affine minorant mentioned above; thus for a convex function, strong smoothness in essence refers to the upward-curving upper bounding quadratic function. See Figure 1.6

The corresponding statement about an operator would be monotone-and-Lipschitz-at-a-point. When we say an operator is both monotone and Lipschitz and some point, we again have bounds on the graph of the operator: now one of the relevant bounds comes from the "horizontal" bound that arises from monotonicity, the other comes from

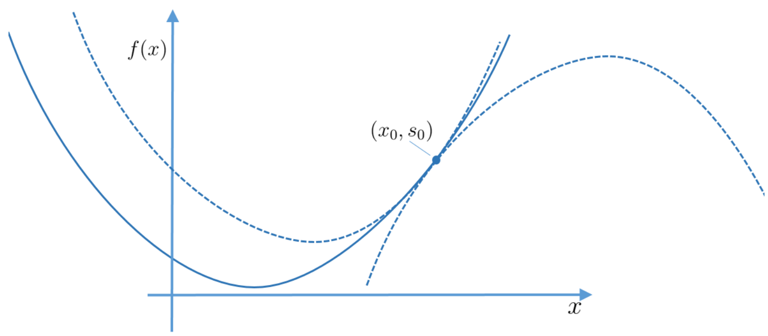

Figure 1.5: A supported-at-$(x_0, f(x_0))$ quadratic majorant to a function $f(\cdot)$ and supporting-at-$(x_0, f(x_0))$ quadratic minorant to a function $f(\cdot)$ characterization of strong smoothness.

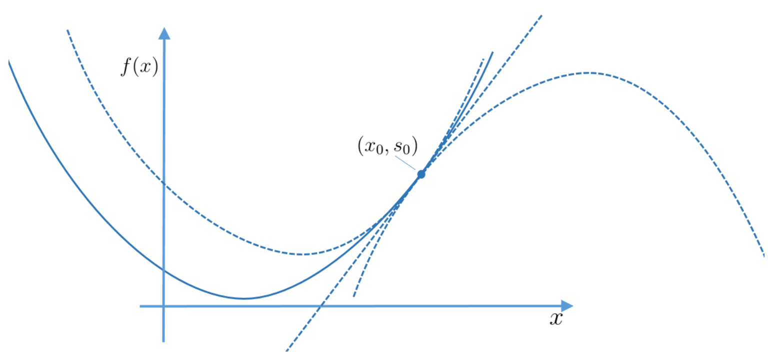

Figure 1.6: A supported-at-$(x_0, f(x_0))$ quadratic majorant to a function $f(\cdot)$, a supporting-at-$(x_0, f(x_0))$ affine minorant to a $f(\cdot)$, and a supporting-at-$(x_0, f(x_0))$ quadratic minorant to $f(\cdot)$, corresponding to simultaneous convexity and strong smoothness (at $(x_0, f(x_0))$).

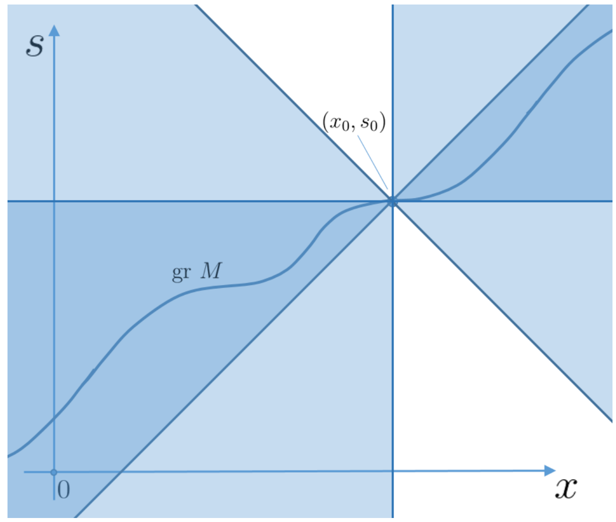

Figure 1.7: An operator that is both monotone and Lipschitz at (x_0, s_0).

one side of the bound that arises from Lipschitz-ness. See Figure 1.7.

Whereas convexity and strong smoothness lead to an affine supporting minorant and a quadratic (upward curving) supported majorant, when we describe a convex function as *strongly* convex, we are indicating the existence of a supporting quadratic (upward-curving) minorant, rather than the previous existence of a supporting affine minorant. See Figure 1.8.

Returning to the case for operators, we consider an operator that is monotone-and-inverse-Lipschitz-at-a-point. When we say an operator is both monotone and Lipschitz and some point, we again have bounds on the graph of the operator: now one of the relevant bounds comes from the “horizontal” bound that arises from monotonicity, the other comes from one side of the bound that arises from Lipschitz-ness. See Figure 1.9. In keeping with the pattern observed thus far, when we say that a monotone operator is strongly monotone, we are making a statement about a restriction on the graph of the operator corresponding to a “vertical” part of the quadrant bound coming from monotonicity and another “bottom-of-wedge” bound coming from a lower bound on the curvature.

When we consider a function that is both strongly smooth and strongly convex at a point, we have both a quadratic (upward curving) majorant supported at that point and a quadratic (upward curving) minorant supporting at that point; the corresponding

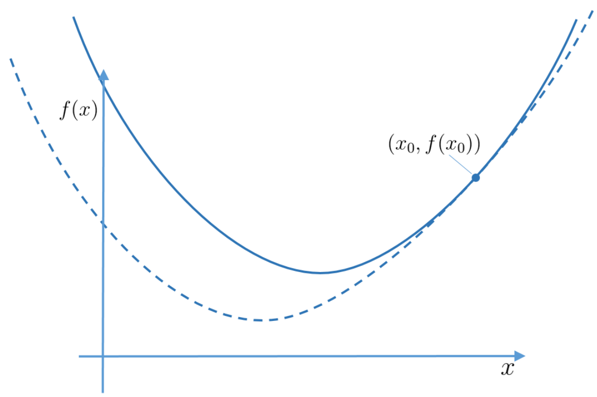

Figure 1.8: Strong convexity at $(x_0, f(x_0))$: there exists a supporting (upward curving) quadratic minorant to $f(\cdot)$ at $(x_0, f(x_0))$.

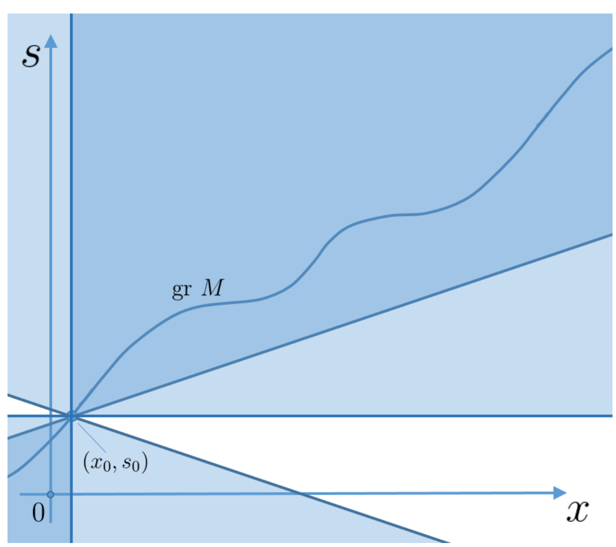

Figure 1.9: An operator that is both monotone and inverse Lipschitz at (x_0, s_0).

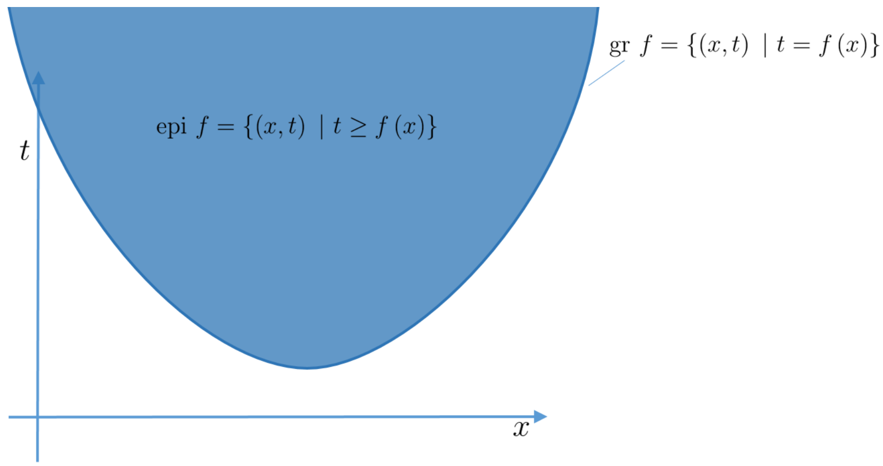

Figure 1.10: For a convex function, the epigraph (the set of argument, value pairs that are above the graph of the function) is a convex set.

statements for an monotone operator would be strong monotonicity and inverse strong monotonicity at the same point. As mentioned above, the most helpful way to think of these properties is in terms of "bounds" on the behavior of the function or the operator. This view of bounds on the behavior of the function of the operator arises particularly in convergence rate analyses.

We have been establishing relationships between the behavior of a function and the behavior of an operator associated with that function. It turns out that we have another path by which to consider the behavior of a function, since many statements about the behavior of a function can be equivalently stated in terms of the epigraph, a set that provides an equivalent representation of the function. See Figure 1.10.

When we have a foundation of viewpoints that provide us the ability to interpret any property of interest via the function, the epigraph of the function, or the related subdifferential operator we can move on to a specific idea that threads through each view: conjugacy. An initial view of conjugacy connects lines in the (primal argument, primal value) space of the (primal) function to points in the (dual argument, dual value) space of the associated (dual) conjugate function (as well as connecting dual lines to primal points). See Figure 1.11.

When we move to consider more general conjugacy between functions, we can base our picture in terms of precisely this sort of point-line duality: in the primal space,

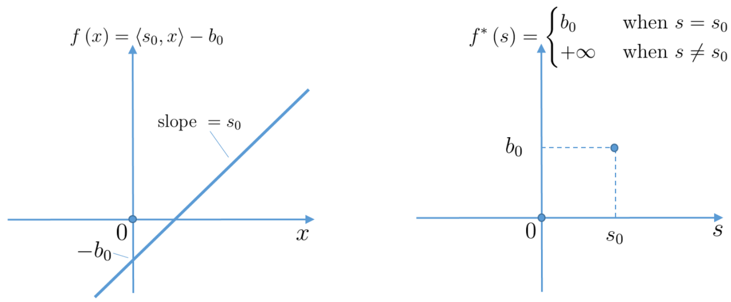

Figure 1.11: A line in the primal space corresponds (via conjugacy) to a point in the dual space. (After [BNO03]).

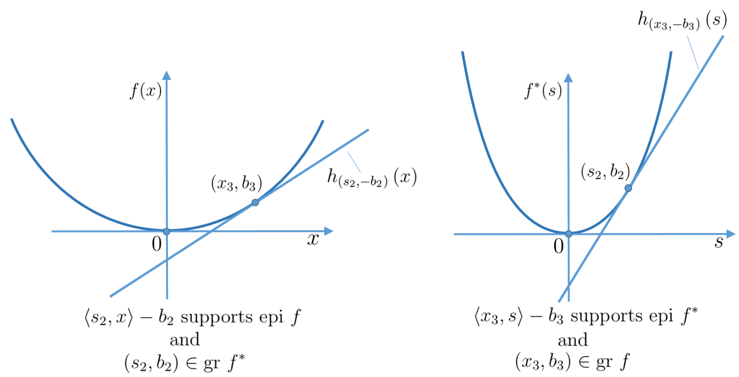

Figure 1.12: Primal lines correspond to dual points; primal points corresponds to dual lines. (After [RW04]).

the (slope,intercept) pair of any affine minorant of the primal (convex) function corresponds to a point in the epigraph of the dual function (see 1.12); the (slope,intercept) pair of a *supporting* affine minorant of the primal function corresponds to a point in the *graph* of the dual function. The same relationships hold when we consider affine minorants in the dual, and points in the epigraph in the primal.

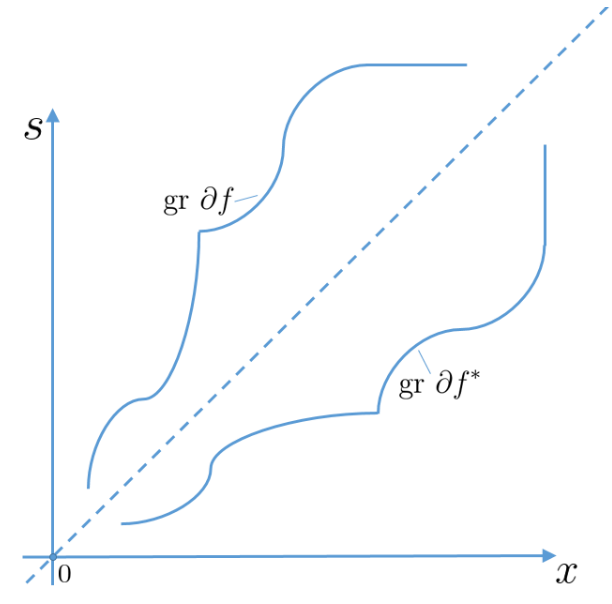

Figure 1.13: Illustrating $[(x,s) \in \partial f] \Longleftrightarrow [(s,x) \in \partial f^*]$. (After [HUL93a]).

We find another view of the connection between a function and its associated conjugate function when we shift our attention to the graph of the subdifferential. This view also connects to the fundamental Fenchel-Young inequality. The key to understanding this alternate perspective is that, when $f(\cdot)$ is a closed proper convex function, we have the symmetric relationship: $(x,s) \in \partial f$ if and only if $(s,x) \in \partial f^*$. See Figure 1.13.

This perspective emphasizes specifics of the association between a function and its conjugate. We can trace the relationship starting at the primal function, then consider the subdifferential of the primal function, then apply the symmetric relationship $(x,s) \in \partial f$ if and only if $(s,x) \in \partial f^*$ (to "flip" the primal subdifferential and thereby obtain the dual subdifferential), and then from the dual subdifferential "integrate" to arrive at the dual function (up to a constant shift). This process is illustrated in Figure 1.14 for the familiar case of the absolute value function.

For quadratics, the intermediate subdifferentials are lines; see Figure 1.15. In the special case $f(x) \stackrel{\text{set}}{=} \frac{1}{2}x^2$, it is immediately apparent that the primal and dual are identical.

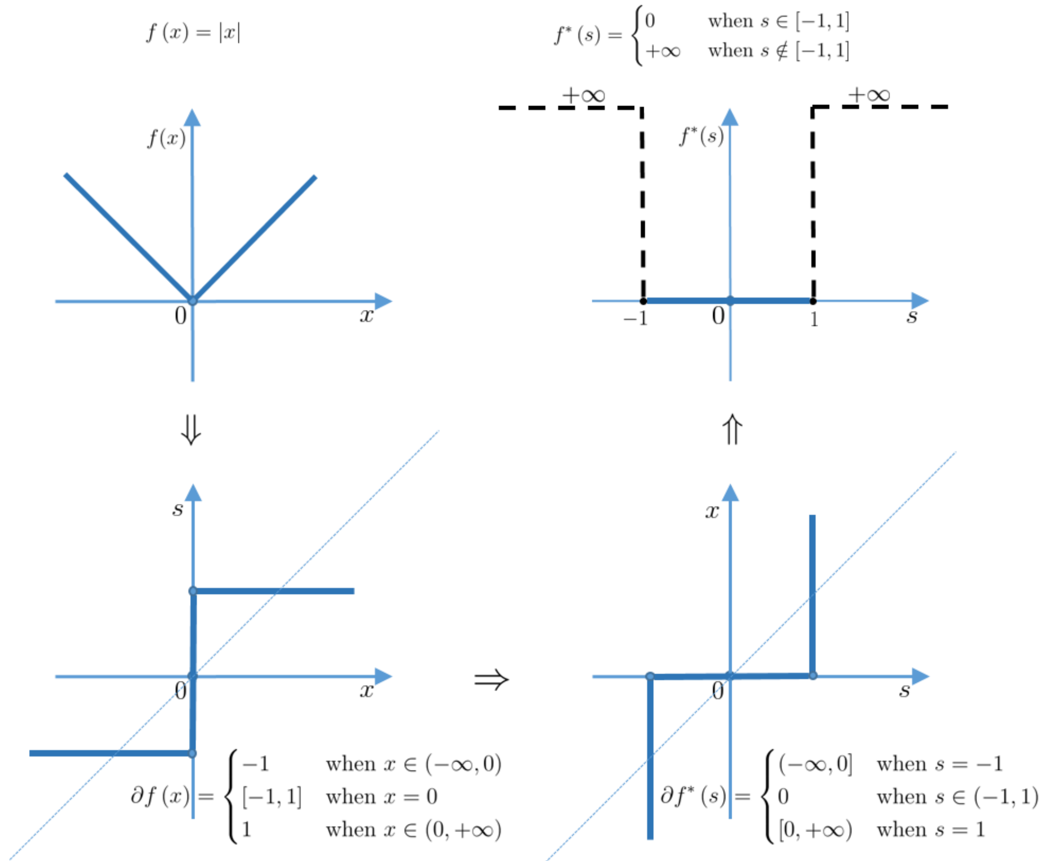

Figure 1.14: Function to conjugate via the subdifferential for $f(x) \overset{\text{set}}{=} |x|$. (After [Luc06]).

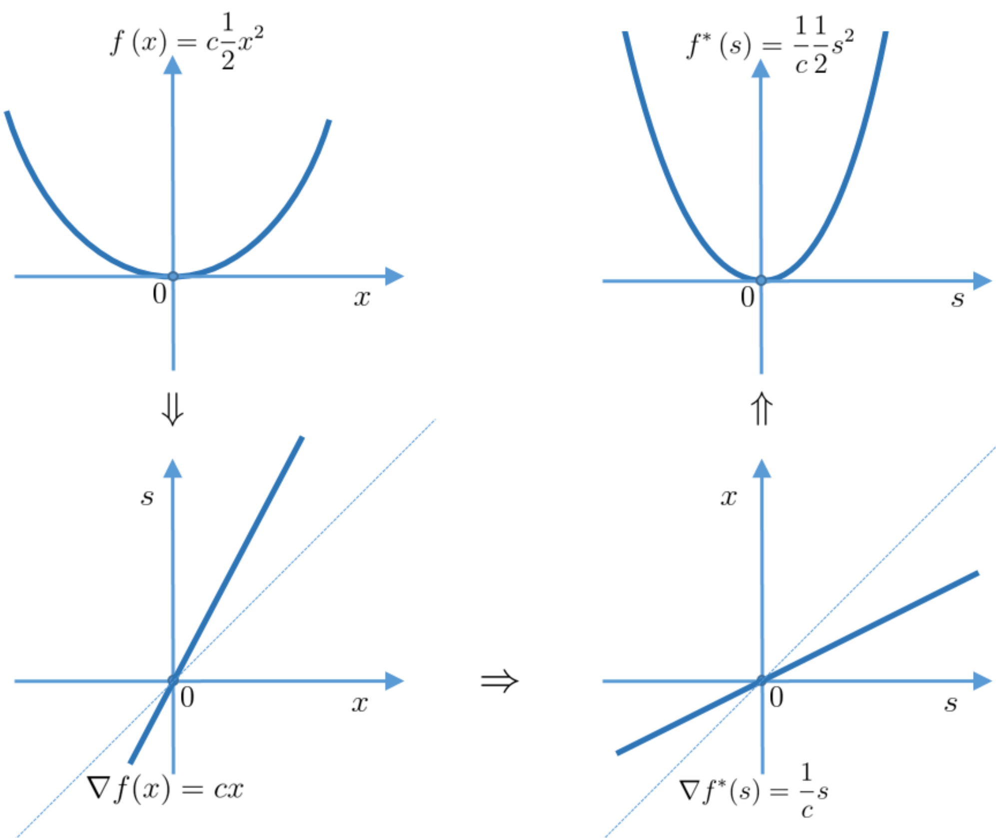

Figure 1.15: Function to conjugate via the subdifferential for $f(x) \overset{\text{set}}{=} c\frac{1}{2}x^2$. (After [Luc06]).

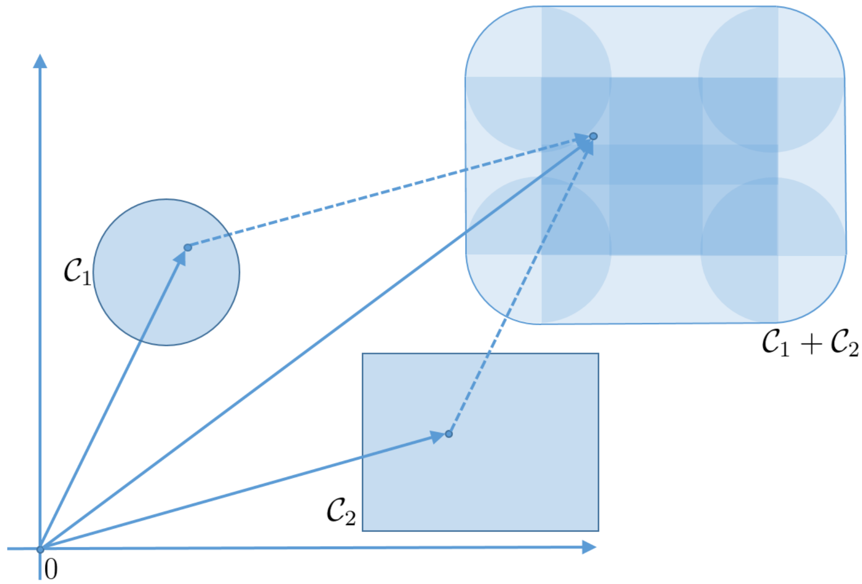

Figure 1.16: Minkowski sum of (convex) sets. (After [RW04]).

Returning to a point of view based on the epigraphs of the functions, we will also see that the recurring operation of "infimal convolution" can be visualized in terms of "set addition" of epigraphs. The process of addition of sets provides an immediate geometric interpretation to the standard definition of infimal convolution in terms of a specific optimization problem. See Figure 1.16.

The most prominent instance of infimal convolution involves some function of interest being "infimally convolved" with a scaled, squared norm; the result corresponds to the set addition of the epigraphs of the functions in question. The resulting function is called the Moreau envelope (of the function of interest). See Figure 1.17. We will later establish in detail a number of properties visible in this illustration, and provide an interpretation of these properties in terms of the associated subdifferential operators.

1.2.2 Assumptions and convergence rates from a standard perspective and a more general perspective

We highlight two aspects of the preceding illustrations: every time we make a statement about a convex function (or its corresponding conjugate function), we could

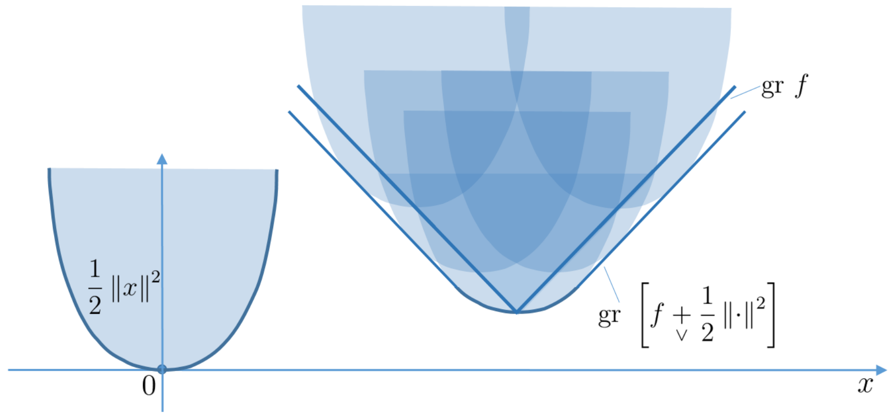

Figure 1.17: Smooth infimal convolution in terms of addition of epigraphs. (After [RW04]).

also make a statement about the graph of the subdifferential of the convex function (or the graph of the subdifferential of the conjugate function). This all will provide us with a visual means of considering the connection between properties of functions and corresponding properties of operators. Our goal will be to use these connections between function properties and operator properties to provide a unified analysis, novel in its generality, of the behavior (in terms of convergence and convergence rate) of a number of prominent methods for convex optimization.

The standard convergence-rate analyses of gradient descent for a convex objective function $f(\cdot)$ depend on what assumptions we make beyond convexity.

Specifically, when we assume that $f(\cdot)$ is convex and strongly smooth, a standard analysis of gradient descent yields rates on convergence in objective function suboptimality and in gradient norm, but not in argument distance to solution.

When we assume that $f(\cdot)$ is not only convex but also both strongly smooth and strongly convex, a standard analysis establishes that in this context gradient descent yields convergence in objective function suboptimality, gradient norm, and argument distance to solution, we also find that the convergence rate obtained is improved.

After seeing the arguments establishing these convergence rate results from the standard perspective, we move on to consider the operator theory perspective. In particular, we will see that making corresponding assumptions about iterative-update-operator-related properties recovers the standard arguments when considered for gradient descent.

We then move on to show that, as a consequence of our operator theoretic perspective, our argument immediately applies to a wide range of commonly-used convex optimization methods.

In each case, by establishing where the iterative update falls within our collection of relationships between "contractivity-type" and "monotonicity-type" properties, novel in comprehensiveness, we see what behavior the method is expected to display.

1.2.3 Overview of what follows

Now that we have a sense for the target areas of contribution, we provide an overview of the structure of the thesis.

The initial chapters provide necessary background covering concepts in convex analysis in such a way that the close connection to our later discussion of operator theoretic concepts can be easily seen. In Chapter 2 we discuss the state of machine learning optimization with particular reference to the papers for which there are connections to the operator theory perspective. In Chapters 3 through 10, we provide a discussion of basic facts from convex analysis. In Chapters 11 and 12, we provide a discussion of basic facts from optimization theory. In Chapter 13, we provide a discussion of basic facts from operator theory.

The remaining chapters include the primary contributions of this thesis. In Chapters 14, 15, and 16, we provide a discussion, novel in its comprehensiveness, of the fundamental relationships between "contractivity-type" properties of an iterative update operator and "monotonicity-type" properties of the associated displacement operator. In addition to providing (to the best of our knowledge) the most comprehensive discussion of the property categories and their relationships, we also identify an apparently completely novel collection of properties that we refer to as "displacement pseudocontractivity-type". We also use the relationships between the contractivity view and the corresponding monotonicity view to provide an alternative proof, novel in its perspective, of a result from [OY02]. We then move on in Chapter 17 to consider our framework for convergence rate analysis, novel in its scope of application. This analysis is based in the more general perspective made possible by considering the previously established relationships between "contractivity-type" properties of an iterative update operator and “monotonicity-type” properties of an associated displacement operator. Appendix A provides a discussion of the notation and the conventions used throughout the thesis.

Chapter 2

Related Literature

We first briefly cover some of the twists and turns of the terminology to refer to the properties, of contractivity-type and monotonicity-type, that are the focus of our discussion. We then move on to consider the appearances of some of these concepts in the optimization literature proper.

The terminology “pseudocontraction” and “strict pseudocontraction” appears to originate in [BP67]. The terminology strictly contractive, nonexpansive (unfortunately called “contractive”), strictly pseudocontractive, and pseudocontractive are used here; the latter two terms appear to be introduced here. We call particular attention to their characterization of strictly pseudocontractive via the expression

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + p\,\|Gx_{\#} - Gx_{\$}\|_2^2,$$

where $p \in (-\infty, 1)$. No special attention is given to the case in which $p \in (-\infty, 0)$ that we refer to as decreasing pseudocontractive. Subsequent usage fragments: some references, such as [VA95, VE09] pay attention only to $p \in (-\infty, 0)$ and call an operator that satisfies this property “pseudocontractive” (or, implicitly, strongly Fejer); no mention is made of the range $p \in (0, 1)$. On the other hand, much of the subsequent operator theory literature (for example [KCQ11, Zha11, OS09]) breaks with [BP67] in an opposite way: they use the term “strictly pseudocontractive” to refer not to $p \in (-\infty, 1)$ but only to $p \in (0, 1)$; these references make no reference to the case $p \in (-\infty, 0)$, instead lumping together this case as an undifferentiated subcase of nonexpansive.

On the topic of convergence rates, [BP67] make a comment that applies to much of the subsequent literature:

> There is a cost to the greater generality of our results which we must mention here explicitly. In addition to the greater complication of the approximation schemes, there is the fact that many (though not all) of the convergence proofs are not strictly constructive in the sense of giving explicit estimates for the error made at any given step of the approximation (a fact not uncommon in nonlinear numerical functional analysis).

The lack of convergence rate analysis persists in the subsequent literature. Most of the attention has been devoted to establishing strong or weak convergence of various classes of operators, under various iteration schemes, in more and more general spaces. This last leads to further fracturing, as in Banach space, there are two paths generally taken to modifying operator property characterization in the absence of an inner product; one path introduces a "semi-inner product" while the other proceeds with norm-based "analogous" definitions of all of the properties previously defined for the Hilbert space case. In the absence of a clear need for this generality in a specific machine learning application, we do not consider these more abstract settings, restricting our attention to $\mathbb{R}^n$.

In the apparent absence of common terminology for the case where $p \in (-\infty, 0)$, a number of authors refer to this class indirectly by describing with reference to nonexpansive operators: specifically, as "averaged nonexpansive" operators. This usage appears to originate in [BR77] and [BBR78]; a prominent recent usage nearer to optimization is [Com04] (we discuss the very recent [BNP14] below).

[BP67] serve as an early example of relating properties of "contractivity" type to properties of "monotonicity" type, although the subsequent literature appears to fail to follow this strong lead. In addition to their contractivity property discussion, [BP67] also use the terms monotone and strongly monotone; while they define the property that we call inverse strong monotonicity, they do not give this property a name. The term "inverse strong monotonicity" appears to originate with [Gol75] bu a variety of competing developed: for example co-coercivity, strong F-monotonicity, and the "Dunn property".

After [BP67] and [Dun76], most subsequent literature seems to have preferred to cite these works rather than to further explore the correspondences present in the operator classes. A recent exception is [BMW12], which provides the closest comprehensive discussion of operator properties to what we explore here, with the difference that the discussion there is couched with regard to firm nonexpansiveness throughout.

The most prominent contractivity-type property is strict contractivity. [Dun76] makes an observation about strict contractivity as a consequence of the combination of strong monotonicity and inverse strong monotonicity, although he does not name any of these properties. Another relevant result is [RW04, page 563], who in turn cites [Zar60]; closer to optimization, we have the result from [Nes04, page 66]). A more recent result that pursues this strict contractivity in the presence of strong monotonicity and inverse strong monotonicity (without citing any of this other work) is [AMS$^+$05]. This trail has been picked up very recently in [BNP14, BCPW13, DY14]; we discuss this further below.

We now move on to discuss some of the connections with the optimization community.

Decreasing pseudocontractive operators abound in machine learning optimization, but this fact is not explicitly mentioned (with the exception of some papers that call attention to the special case of "firm nonexpansiveness"). Most standard optimization texts (e.g., [Ber99, BNO03, NW06, BV04]) omit discussion of any of the operator properties/characterizations that form the basis of our discussion; namely operators that are characterized as being averaged (or averaged nonexpansive), or that satisfy the property of inverse strongly monotonicity (for a specific parameter range), or of decreasing pseudocontractivity. We now briefly consider those references that do mention one (or, in some cases, two) of these operator classes.

In [Eck89] the discussion covers (projected) gradient descent, the proximal-point method, forward-backward splitting, the alternating direction method of multipliers, and numerous other methods. The coverage does touch on the special case of 1-decreasing pseudocontractivity (under the name firm nonexpansiveness) and on the special case of 1-inverse strong monotonicity; however, these are specific cases of the more general concepts of ν-decreasing pseudocontractivity and σ-inverse strong monotonicity. Omitting these more general concepts (and the relationship between these concepts) significantly limits the discussion. One particularly notable limitation of a focus on firmly nonexpansive operators is that the class of firmly nonexpansive operators is not closed under composition.

[VA95] discusses both averaged operators and decreasing pseudocontractive operators (although these operators are called pseudocontractive), and also establishes the relationship between these classes; however, the general concept of inverse strong monotonicity is not present. In addition to the absence of the combined relationship between pseudocontractivity and inverse strong monotonicity, there is no unified convergence rate analysis of the type we discuss. On the other hand, [BC11] discusses averagedness and inverse strong monotonicity (called there co-coercivity). Both averagedness and inverse strong monotonicity are discussed extensively and the latter is the central property on which most of the subsequent analysis is based. However, pseudocontractivity is essentially absent and convergence rates are not discussed.

[GT96] discusses inverse strong monotonicity extensively in their discussion of gradient-type methods and in methods involving modified monotone mappings; however, the concepts of averagedness and decreasing pseudocontractivity are absent. The discussion in [Byr04] establishes that many iterative optimization methods have updates that are averaged; this work also includes the inverse strongly monotonicity properties satisfied by averaged nonexpansive operators. However, the concept of decreasing pseudocontractivity is absent. While the Krasnoselskii-Mann Theorem [Man53] is used to establish convergence of methods with averaged updates, there is no analysis of convergence rates.

[RW04] has a chapter on monotone mappings that covers maximality of monotone operators, observes that M monotone implies M^{-1} monotone, that M monotone implies λM monotone for any $\lambda \in \mathbb{R}_{++}$, and that the sum of monotone operators is monotone. It is one of the few references that names the operator $I - T$ (calling it the displacement operator associated with T) but this name is unfortunately used in exactly one sentence. Strong monotonicity and inverse strong monotonicity are mentioned but none of the relations to contractivity-type properties are mentioned, nor is the notion of averagedness.

[Kon07], in a chapter on the theory of variational inequality problems, includes the notion of monotonicity, strict monotonicity, and strong monotonicity as well as the relation of these properties to convexity, strict convexity, and strong convexity when considering a convex function; however, no mention is made of inverse strong monotonicity, nor the relationship between monotonicity-type properties and contractivity-type properties.

Of more recent papers discussing proximal methods, such as [CW05, CP11], the special case of firm nonexpansiveness appears, but not the more general cases of interest to us. The story is similar for other recent papers discussing the alternating direction method of multipliers, such as [BPC+11, PB13, HY12]. This, again, limits the utility and breadth of applicability of the results obtained.

In the machine learning community, as in the optimization community, the classes of operators that we consider pass largely unremarked. [LSS12] uses the proximal-point mapping and establishes the firm nonexpansiveness of some of the operations involved in their algorithm. The discussion in [DS09, SD09] squarely covers some of the methods that we touch on here; despite an apparent similarity in general outline, the analysis is pursued from a rather different perspective that is largely disjoint from our approach; in part this is due to a focus there on objective function value, in contrast to our focus on the sequences of iterates and displacements. One of the methods we discuss, the alternating direction method of multipliers, has recently been considered in stochastic and online variants [OHTG13, WB12], but our discussion here is restricted to the batch deterministic case.

The very recent paper [BNP14] should be regarded as establishing "regularity" conditions under which the iterative update operator (described there as averaged nonexpansive) does not simply satisfy decreasing pseudocontractivity but strict contractivity (and thus linear convergence). [BCPW13] provides an extremely close analysis of these ideas in the context of Douglas-Rachford splitting for the two subspace intersection problem. [DY14] applies these results to the more general settings (but still specialized relative to the breadth of possible applications) of Douglas-Rachford splitting (DRS), Peaceman-Rachford splitting (PRS), and alternating direction method of multipliers (ADMM) methods. That is, these papers consider what can be said in settings for which additional properties are assumed to hold.

Chapter 3

Sets and convexity

3.1 Introduction

In this chapter, we set notation and definitions useful in talking about sets and basic operations on sets. This material is standard; typical references include [BV04, Roc70, Ber09, HUL93a, HUL93b, Rus06]. While the material is standard, no single one of the references served to entirely cover the material in the fashion we sought as a foundation for our later discussion. The concepts, terminology, notation, and examples in this chapter provide the context in which to introduce later ideas.

3.2 Basic terminology and definitions

In the descriptions below, we often encounter parameterized expressions of the form

$$x[\theta] \stackrel{\text{set}}{=} (1-\theta)\, x_{\text{base}} + \theta x_{\text{tgt}}.$$

To see why we refer to x_{base} as the base vector, consider the case $\theta \stackrel{\text{set}}{=} 0$. This leads to

$$\begin{aligned} x[0] &= (1-0)\, x_{\text{base}} + 0 x_{\text{tgt}} \\ &= x_{\text{base}}. \end{aligned}$$

Thus, in some sense we view the parameterization as starting at x_{base}.

Similar consideration for the case $\theta \overset{\text{set}}{=} 1$ leads to us to the intuition behind our convention of referring to x_{tgt} as the target vector:

$$\begin{aligned} x[1] &= (1-1)\,x_{\text{base}} + 1x_{\text{tgt}} \\ &= x_{\text{tgt}}. \end{aligned}$$

Thus, in some sense we view the parameterization as heading from x_{base} toward x_{tgt}.

An alternate perspective would be to conceive of the expression in terms of a base vector x_{base} and a direction (or displacement) v_{dir}

$$x[t] \overset{\text{set}}{=} x_{\text{base}} + t v_{\text{dir}}.$$

To fit our previous expression in this base-plus-scaled-direction (or displacement) format, we would write

$$\begin{aligned} x[\theta] &= (1-\theta)\,x_{\text{base}} + \theta x_{\text{tgt}} \\ &= x_{\text{base}} + \theta\left(x_{\text{tgt}} - x_{\text{base}}\right). \end{aligned}$$

This last form will be of particular interest when we discuss the relationship between an operator T and its associated displacement operator $G \overset{\text{set}}{=} I - T$.

3.2.1 Affine combinations (lines), segments (line segments), and rays

We first introduce some basic geometric concepts involving two vectors in $\mathbb{R}^n$. Some rough patterns of usage are: we tend to use α for a parameter in the interval $[0,1]$; we tend to use t for a parameter in the range $(0,\infty)$; we often use θ in contexts where the range in question is the entire real line.

Definition 1 (Affine combinations)**.** The collection of all *affine combinations* of x_{base}, $x_{\text{tgt}} \in \mathbb{R}^n$ (sometimes described as the line through x_{base} and x_{tgt}) is a set in $\mathbb{R}^n$ denoted $\operatorname{aff}\left[x_{\text{base}}, x_{\text{tgt}}\right]$ and is defined via the expression

$$\operatorname{aff}\left[x_{\text{base}}, x_{\text{tgt}}\right] \overset{\text{set}}{=} \left\{x \in \mathbb{R}^n \mid x = (1-\theta)\,x_{\text{base}} + \theta x_{\text{tgt}} \text{ for some } \theta \in (-\infty, +\infty)\right\}.$$

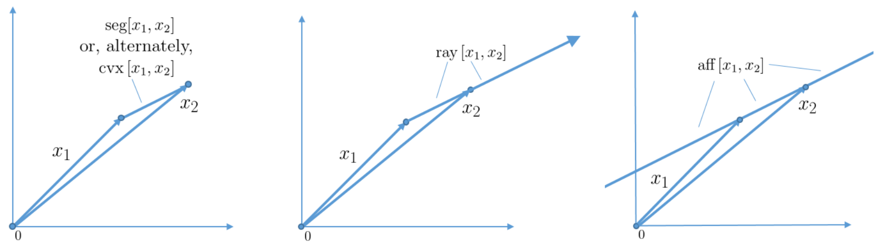

Figure 3.1: The line segment (or convex combination) of x_1, x_2; the ray from x_1 through x_2; the affine combinations of x_1, x_2.

We use the term "affine combinations of the vectors $x_{\text{base}}, x_{\text{tgt}} \in \mathbb{R}^n$" instead of "line through the vectors $x_{\text{base}}, x_{\text{tgt}} \in \mathbb{R}^n$" to avoid saying things like: "the line through x_{base} and x_{tgt} is not a linear set (subspace) because it does not pass through the origin".

Definition 2 (Segment)**.** The *segment* (or line-segment, or interval) between $x_{\text{base}}, x_{\text{tgt}} \in \mathbb{R}^n$ is a set in $\mathbb{R}^n$ denoted seg $\left[x_{\text{base}}, x_{\text{tgt}}\right]$ and is defined via the expression

$$\text{seg}\left[x_{\text{base}}, x_{\text{tgt}}\right] \overset{\text{set}}{=} \left\{x \in \mathbb{R}^n \mid x = (1-\alpha)\, x_{\text{base}} + \alpha x_{\text{tgt}} \text{ for some } \alpha \in [0,1]\right\}.$$

We will later connect the notion of segments between vectors to convex combinations of those vectors.

Definition 3 (Ray)**.** The *ray* emanating from $x_{\text{base}} \in \mathbb{R}^n$ and passing through $x_{\text{tgt}} \in \mathbb{R}^n$ is a set in $\mathbb{R}^n$ denoted ray $\left[x_{\text{base}}, x_{\text{tgt}}\right]$ and is defined via the expression

$$\text{ray}\left[x_{\text{base}}, x_{\text{tgt}}\right] \overset{\text{set}}{=} \left\{x \in \mathbb{R}^n \mid x = (1-t)\, x_{\text{base}} + t x_{\text{tgt}} \text{ for some } t \in (0, +\infty)\right\}.$$

As mentioned above, we have the following alternative perspective $(1-t)\, x_{\text{base}} + t x_{\text{tgt}} = x_{\text{base}} + t\left(x_{\text{tgt}} - x_{\text{base}}\right)$; this is arguably a more intuitive expression for a ray.

These concepts are illustrated in Figure 3.1.

3.2.2 Combinations

Consider a collection of K vectors $\{x_1, x_2, \ldots, x_K\}$, with each $x_k \in \mathbb{R}^n$. Consider also a collection of K corresponding "combination parameters" (alternately, a set of "multipliers") $\{\theta_1, \theta_2, \ldots, \theta_K\}$ with each $\theta_k \in \mathbb{R}$ for $k \in \{1, \ldots, K\}$.

Definition 4 ($\{\theta_k\}$-combination)**.** We describe a $\{\theta_k\}$-*combination* of the K vectors $\{x_1, x_2, \ldots, x_K\}$, as the vector expressible in the form $\sum_{k=1}^{K} \theta_k x_k = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_K x_K$.

Thus, a "combination" involves a collection of scaling parameters as well as addition. We consider several specific classes of combination parameter values, where each class is characterized by a property that holds for the combination parameters.

Definition 5 (Linear $\{\theta_k\}$-combination)**.** A *linear* $\{\theta_k\}$*-combination* of $\{x_1, x_2, \ldots, x_K\}$, denoted $\text{lin}\,[x_1, x_2, \ldots, x_K; \{\theta_k\}]$ is a $\{\theta_k\}$-combination of $\{x_1, x_2, \ldots, x_K\}$ for which the combination parameters satisfy $\theta_k \in \mathbb{R}$.

Note that there is always at least one choice of linear $\{\theta_k\}$-combination parameters for which the corresponding linear $\{\theta_k\}$-combination of $\{x_1, x_2, \ldots, x_K\}$, $\text{lin}[x_1, x_2, \ldots, x_K; \{\theta_k\}]$ is equal to 0: specifically, $\text{lin}\,[x_1, x_2, \ldots, x_K; \{0\}] = 0$.

Definition 6 (Affine $\{\theta_k\}$-combination)**.** An *affine* $\{\theta_k\}$*-combination* of $\{x_1, x_2, \ldots, x_K\}$, denoted $\text{aff}\,[x_1, x_2, \ldots, x_K; \{\theta_k\}]$ is a $\{\theta_k\}$-combination of $\{x_1, x_2, \ldots, x_K\}$ for which the combination parameters satisfy $\theta_k \in \mathbb{R}$ and $\sum_{k=1}^{K} \theta_k = 1$.

Unlike the case found in linear combinations, there need not be a choice of affine $\{\theta_k\}$-combination parameters for which the corresponding affine $\{\theta_k\}$-combination of $\{x_1, x_2, \ldots, x_K\}$, $\text{aff}\,[x_1, x_2, \ldots, x_K; \{\theta_k\}]$ is equal to 0. We also have the related notion of barycentric coordinates: consider a specific vector $x = \text{aff}\,[x_1, x_2, \ldots, x_K; \{\theta_k\}]$; the parameters $\{\theta_k\}$ in this affine $\{\theta_k\}$ combination of $\{x_1, x_2, \ldots, x_K\}$ are sometimes called the barycentric coordinates of x (with respect to $\{x_1, x_2, \ldots, x_K\}$) [Ewa96].

Definition 7 (Convex $\{\theta_k\}$-combination)**.** A *convex* $\{\theta_k\}$*-combination* of $\{x_1, x_2, \ldots, x_K\}$, denoted $\text{cvx}\,[x_1, x_2, \ldots, x_K; \{\theta_k\}]$ is a $\{\theta_k\}$-combination of $\{x_1, x_2, \ldots, x_K\}$ for which the combination parameters satisfy $\theta_k \in \mathbb{R}_+$ and $\sum_{k=1}^{K} \theta_k = 1$.

We will consider convex $\{\theta_k\}$-combinations most frequently. One typical interpretation of a convex combination is as a "mixture" or "weighted average" of the $\{x_1, x_2, \ldots, x_K\}$; in this interpretation, the kth combination parameter θ_k is sometimes described as the "fraction of x_k" in the weighted average.

Definition 8 (Convex conic $\{\theta_k\}$-combination)**.** A *convex conic* $\{\theta_k\}$*-combination* of $\{x_1, x_2, \ldots, x_K\}$, denoted ccone $[x_1, x_2, \ldots, x_K; \{\theta_k\}]$ is a $\{\theta_k\}$-combination of $\{x_1, x_2, \ldots, x_K\}$ for which the combination parameters satisfy $\theta_k \in \mathbb{R}_+$.

What we call a convex conic $\{\theta_k\}$-combination is often referred to as a conic $\{\theta_k\}$-combination; however, since the collection of all $\{\theta_k\}$-combinations of $\{x_1, x_2, \ldots, x_K\}$ for which the combination parameters satisfy $\theta_k \in \mathbb{R}_+$ always produces a convex cone, we prefer to use a naming convention that reflects this.

3.2.3 Set terminology

Definition 9 (Linear set or subspace)**.** We say that a set is a *linear set* (or subspace) when it contains every linear combination of two points in the sets. That is, for a linear set (subspace) $\mathcal{L}$, we have that $x_1, x_2 \in \mathcal{L}$ and $\theta_1, \theta_2 \in \mathbb{R}$ together imply that $\theta_1 x_1 + \theta_2 x_2 \in \mathcal{L}$.

Note that every linear set (subspace) necessarily contains the origin.

Definition 10 (Affine set or flat)**.** We say that a set is an *affine set* (alternately referred to as a linear manifold, linear variety, or a flat) when it contains every affine combination of two points in the set. That is, for an affine set $\mathcal{A}$, we have that $x_1, x_2 \in \mathcal{A}$ and $\theta_1, \theta_2 \in \mathbb{R}$ and $\theta_1 + \theta_2 = 1$ together imply that $\theta_1 x_1 + \theta_2 x_2 \in \mathcal{A}$.

Note that, unlike a linear set, an affine set need not contain the origin.

Definition 11 (Convex set)**.** We say that a set is a *convex set* when it contains every convex combination of two points in the set. That is, for a convex set $\mathcal{C}$, we have that x_1, $x_2 \in \mathcal{C}$ and $\theta_1, \theta_2 \in \mathbb{R}_+$ and $\theta_1 + \theta_2 = 1$ together imply that $\theta_1 x_1 + \theta_2 x_2 \in \mathcal{C}$.

Convex sets will be the focus of most of our attention.

Definition 12 (Nonnegatively homogeneous set or cone)**.** We say that a set is a *nonnegatively homogeneous set* (a conic set or cone) when it contains every nonnegative scaling of a point in the set. That is, for a nonnegatively homogeneous set $\mathcal{Q}$, we have that $x \in \mathcal{Q}$ and $\theta \in \mathbb{R}_+$ together imply that $\theta x \in \mathcal{Q}$. Note the absence of addition in this definition.

A nonnegatively homogeneous set is a set that is (membership) "homogeneous" with respect to nonnegative scaling.

Definition 13 (Convex conic set or convex cone)**.** We say that a set is a *convex conic set* when it contains every convex conic combination of two points in the set; that is, if it contains every convex combination of the nonnegative scalings of its elements. That is, for a convex conic set (convex cone) $\mathcal{K}$, we have that $x_1, x_2 \in \mathcal{K}$ and $\theta_1, \theta_2 \in \mathbb{R}_+$ together imply that $\theta_1 x_1 + \theta_2 x_2 \in \mathcal{K}$.

3.2.4 Hulls

From the starting point of an arbitrary set, for example $\mathcal{S} \in \mathbb{R}^n$, we can consider various types of combinations of all of the elements of the set. We refer to the resulting set as a "hull" of the set, with the specific kind of hull being determined by the type of combination used to generate the set. In particular, we consider the linear hull, the affine hull, the convex hull, and the convex conic hull. We first consider the notion of the linear hull (or span) of a finite set.

Definition 14 (Linear hull or span)**.** The *linear hull* (or span) of a finite set $\{x_1, x_2, \ldots, x_K\}$, each $x_k \in \mathbb{R}^n$, is the set in $\mathbb{R}^n$ denoted by $\operatorname{lin}[x_1, x_2, \ldots, x_K]$ and defined via the expression

$$\operatorname{lin}[x_1, x_2, \ldots, x_K] \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^n \mid x = \sum_{k=1}^{K} \theta_k x_k \text{ for some } \{\theta_k\} \text{ with } \theta_k \in \mathbb{R} \right\}.$$

In words, the linear hull of the set $\{x_1, x_2, \ldots, x_K\}$ is the set of all linear combinations of the elements of $\{x_1, x_2, \ldots, x_K\}$.

We also note that the linear hull of a set $\mathcal{S}$ is the intersection of all linear sets (subspaces) that contain $\mathcal{S}$.

Further, the linear hull of a set $\mathcal{S}$ is the smallest subspace that contains the set $\mathcal{S}$: any subspace (linear set) $\mathcal{L}$ containing $\mathcal{S}$ also contains $\operatorname{lin} \mathcal{S}$, the linear hull of $\mathcal{S}$. We may write this more explicitly as: if $\mathcal{L}$ is a subspace (linear set) and $\mathcal{S} \subseteq \mathcal{L}$, then $\operatorname{lin} \mathcal{S} \subseteq \mathcal{L}$.

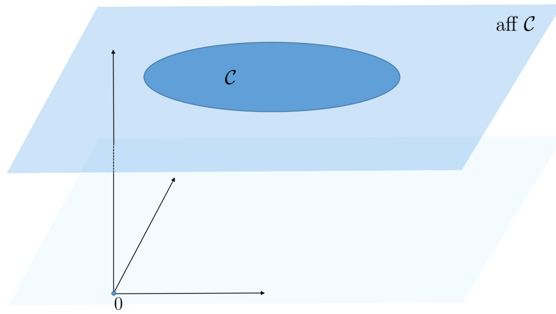

Figure 3.2: Affine hull of a set. (After [HUL93a]).

Definition 15 (Affine hull)**.** The *affine hull* of a finite set $\{x_1, x_2, \ldots, x_K\}$, each $x_k \in \mathbb{R}^n$, is the set in $\mathbb{R}^n$ denoted by aff $[x_1, x_2, \ldots, x_K]$ and defined via the expression

$$\text{aff}\,[x_1, x_2, \ldots, x_K] \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^n \mid x = \sum_{k=1}^{K} \theta_k x_k \text{ for some } \{\theta_k\} \text{ with } \theta_k \in \mathbb{R},\ \sum_{k=1}^{K} \theta_k = 1 \right\}.$$

In words, the affine hull of the set $\{x_1, x_2, \ldots, x_K\}$ is the set of all affine combinations of the elements of $\{x_1, x_2, \ldots, x_K\}$.

We also note that the affine hull of a set $\mathcal{S}$ is the intersection of all affine sets that contain $\mathcal{S}$.

Further, the affine hull of a set $\mathcal{S}$ is the smallest affine set that contains the set $\mathcal{S}$: any affine set $\mathcal{A}$ containing the set $\mathcal{S}$ also contains aff $\mathcal{S}$, the affine hull of $\mathcal{S}$. We may write this more explicitly as: if $\mathcal{A}$ is an affine set and $\mathcal{S} \subseteq \mathcal{A}$, then aff $\mathcal{S} \subseteq \mathcal{A}$. See Figure 3.2.

Definition 16 (Convex hull)**.** The *convex hull* of a finite set $\{x_1, x_2, \ldots, x_K\}$, each $x_k \in \mathbb{R}^n$, is the set in $\mathbb{R}^n$ denoted by cvx $[x_1, x_2, \ldots, x_K]$ and defined via the expression

$$\text{cvx}\,[x_1, x_2, \ldots, x_K] \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^n \mid x = \sum_{k=1}^{K} \theta_k x_k \text{ for some } \{\theta_k\} \text{ with } \theta_k \in \mathbb{R}_+,\ \sum_{k=1}^{K} \theta_k = 1 \right\}.$$

In words, the convex hull of the set $\{x_1, x_2, \ldots, x_K\}$ is the set of all convex combinations of the elements of $\{x_1, x_2, \ldots, x_K\}$.

We also note that the convex hull of a set $\mathcal{S}$ is the intersection of all convex sets that contain $\mathcal{S}$. Further, the convex hull of a set $\mathcal{S}$ is the smallest convex set that contains

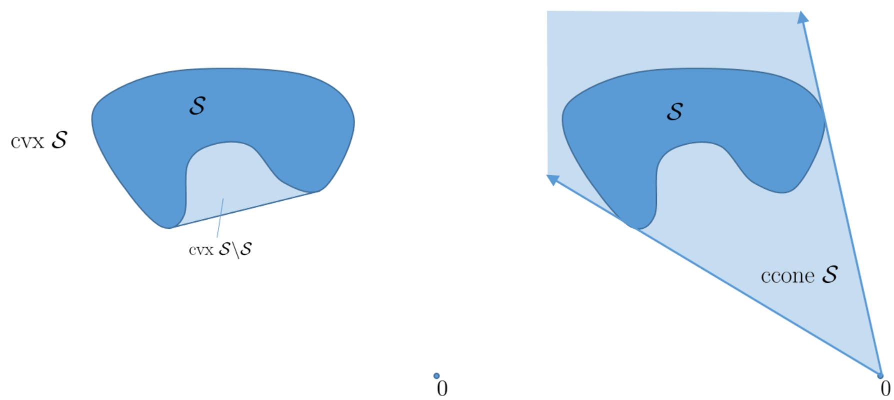

Figure 3.3: Convex hull and convex conic hull for a nonconvex set $\mathcal{S}$. (After [BV04]).

the set $\mathcal{S}$, in the following sense: any convex set $\mathcal{C}$ containing $\mathcal{S}$ also contains cvx $\mathcal{S}$, the convex hull of $\mathcal{S}$. We may write this more explicitly as: if $\mathcal{C}$ is a convex set and $\mathcal{S} \subseteq \mathcal{C}$, then cvx $\mathcal{S} \subseteq \mathcal{C}$. See Figure 3.3.

Definition 17 (Convex conic hull)**.** The *convex conic hull* (sometimes called the positive hull [Ewa96]) of a finite set $\{x_1, x_2, \ldots, x_K\}$, each $x_k \in \mathbb{R}^n$, is the set in $\mathbb{R}^n$ denoted by ccone $[x_1, x_2, \ldots, x_K]$ and defined via the expression

$$\text{ccone}\,[x_1, x_2, \ldots, x_K] \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^n \mid x = \sum_{k=1}^{K} \theta_k x_k \text{ for some } \{\theta_k\} \text{ with } \theta_k \in \mathbb{R}_+ \right\}.$$

In words, the convex conic hull of the set $\{x_1, x_2, \ldots, x_K\}$ is the set of all convex conic combinations of the elements of $\{x_1, x_2, \ldots, x_K\}$.

We also note that the convex conic hull of a set $\mathcal{S}$ is the intersection of all convex conic sets that contain $\mathcal{S}$.

Further, the convex conic hull of a set $\mathcal{S}$ is the smallest convex conic set (convex cone) that contains the set $\mathcal{S}$: any convex conic set (convex cone) $\mathcal{K}$ containing $\mathcal{S}$ also contains ccone $\mathcal{S}$, the convex conic hull of $\mathcal{S}$. We may write this more explicitly as: if $\mathcal{K}$ is a convex conic set (convex cone) and $\mathcal{S} \subseteq \mathcal{K}$, then ccone $\mathcal{S} \subseteq \mathcal{K}$. See Figure 3.3.

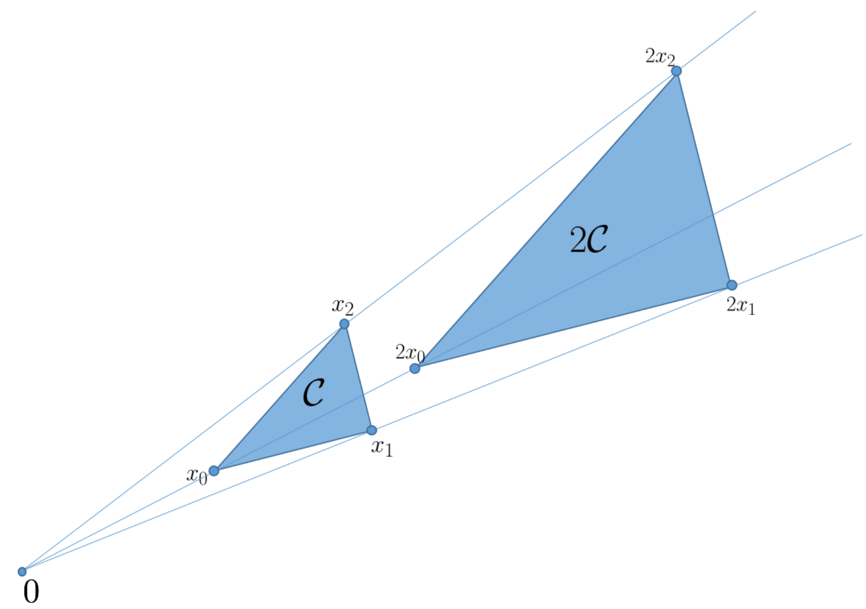

Figure 3.4: Scaling/dilation of a set.

3.3 Operations involving sets

3.3.1 Scaling of a set

Definition 18 (Scaling)**.** The λ*-scaling* of a set, say $\mathcal{X} \subseteq \mathbb{R}^n$, is denoted $\lambda\mathcal{X}$ and defined via the expression

$$\lambda\mathcal{X} \overset{\text{def}}{=} \{z \in \mathbb{R}^n \mid z = \lambda x \text{ where } x \in \mathcal{X} \text{ and } \lambda \in \mathbb{R}\}.$$

See Figure 3.4.

3.3.2 (Minkowski) sum of two sets

Another fundamental operation is "setwise addition" (also called Minkowski set addition, the Minkowski sum, or the vector sum) of (e.g., convex) sets. The Minkowski sum will be particularly relevant when we consider infimal convolution.

Definition 19 (Minkowski sum)**.** The *(Minkowski) sum* of two sets, say $\mathcal{X} \subseteq \mathbb{R}^n$ and

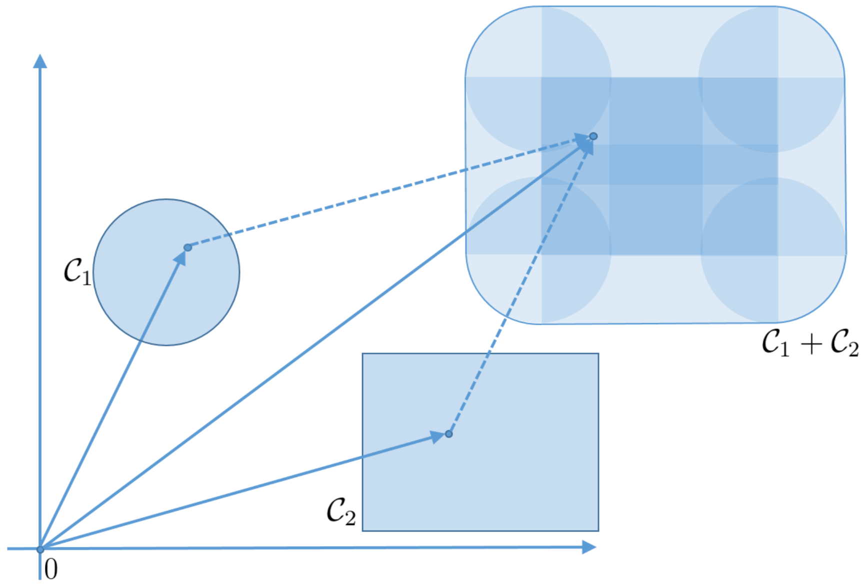

Figure 3.5: Minkowski sum of two sets. (After [RW04]).

$\mathcal{Y} \subseteq \mathbb{R}^n$, is denoted $\mathcal{X}+\mathcal{Y}$ and defined via the expression

$$\mathcal{X}+\mathcal{Y} \stackrel{\text{def}}{=} \{z \in \mathbb{R}^n \mid z = x+y \text{ where } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}.$$

See Figure 3.5.

3.4 Frequently relevant sets

3.4.1 Affine hyperplane

Definition 20. An *(affine) hyperplane* (or flat, or linear variety, or linear manifold) is a set denoted $\mathcal{H}_{s,b} \subseteq \mathbb{R}^n$, specified via $s \in \mathbb{R}^{n*} \setminus \{0\}$ and $b \in \mathbb{R}$, and defined via the expression

$$\mathcal{H}_{s,b} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle s, x \rangle = b\}.$$

The affine hyperplane $\mathcal{H}_{s,b}$ is the b-level set of the linear function $\langle s, \cdot \rangle$. We also introduce notation for the associated halfspaces and strict halfspaces as follows:

$$\begin{aligned}
\mathcal{H}_{s,b}^{>} &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle s,x \rangle > b\} \\
\mathcal{H}_{s,b}^{\geq} &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle s,x \rangle \geq b\} \\
\mathcal{H}_{s,b} &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle s,x \rangle = b\} \\
\mathcal{H}_{s,b}^{\leq} &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle s,x \rangle \leq b\} \\
\mathcal{H}_{s,b}^{<} &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle s,x \rangle < b\}.
\end{aligned}$$

Note that for the hyperplane, halfspaces and strict halfspaces above, we can alternately describe them as a level set, a (strict or nonstrict) sublevel set, or a (strict or nonstrict) superlevel set of a "reference" affine function.

3.4.2 Norm ball

A $\|\cdot\|_\diamond$-norm ball is a set in $\mathbb{R}^n$; here $\|\cdot\|_\diamond : \mathbb{R}^n \to \mathbb{R}$ denotes a generic norm. In general we specify a center $x_c \in \mathbb{R}^n$ and a radius $r \in \mathbb{R}_{++}$; the resulting norm ball is then denoted $\mathcal{B}_\diamond(x_c, r)$ and defined via the expression

$$\begin{aligned}
\mathcal{B}_\diamond(x_c, r) &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \|x - x_c\|_\diamond \leq r\} \\
&= \{x + ru \mid \|u\|_\diamond \leq 1\}.
\end{aligned}$$

When considering the specific case of the unit $\|\cdot\|_\diamond$-norm ball where $x_c \stackrel{\text{set}}{=} 0$ and $r \stackrel{\text{set}}{=} 1$ it is common practice to drop the explicit reference to x_c and r. That is, we have the following shorthand for the unit $\|\cdot\|_\diamond$-norm ball:

$$\mathcal{B}_\diamond \stackrel{\text{set}}{=} \mathcal{B}_\diamond(0,1).$$

3.4.3 Norm cones

The $\|\cdot\|_\diamond$-norm cone is the epigraph of the norm $\|\cdot\|_\diamond$; we will refer to the $\|\cdot\|_\diamond$-norm cone as $\mathcal{K}_\diamond \subseteq \mathbb{R}^n \times \mathbb{R}$. Explicitly, we have

$$\begin{aligned}
\mathcal{K}_\diamond &\stackrel{\text{set}}{=} \operatorname{epi} \|\cdot\|_\diamond \\
&= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \geq \|x\|_\diamond\}.
\end{aligned}$$

3.4.4 Polyhedron

Definition 21. A *polyhedron*, say $\mathcal{P} \subseteq \mathbb{R}^n$, is defined as the solution set of a finite number of linear equalities and inequalities:

$$\mathcal{P} \stackrel{\text{set}}{=} \left\{x \in \mathbb{R}^n \mid \langle s_i, x\rangle = b_i,\ i \in \{1,\dots,m\},\ \left\langle v_j, x\right\rangle \le b_j,\ j \in \{1,\dots,p\}\right\}.$$

In words: a polyhedron is the intersection of a finite number of hyperplanes and halfspaces. Particular examples of polyhedra include subspaces, hyperplanes, lines, any affine set, rays, line segments, and halfspaces.

3.4.5 Simplex, unit simplex, probability simplex

Simplexes are convex sets corresponding to specific instances of polyhedra.

Definition 22. A convex set is called a *K-simplex* if it is the convex hull of $K+1$ affinely independent vectors. For specificity, suppose that the $K+1$ vectors $\{x_0,\dots,x_K\}$ with $x_k \in \mathbb{R}^n$ are affinely independent; by this we mean that the K vectors $\{x_1 - x_0, \dots, x_K - x_0$ $\}$ are linearly independent.

Then the simplex of these $K+1$ affinely independent vectors $\{x_0,\dots,x_K\}$ is an alternate term for the convex hull of those vectors; explicitly

$$\begin{aligned}\text{splx } \{x_0,\dots,x_K\} &\stackrel{\text{set}}{=} \text{cvx } \{x_0,\dots,x_K\} \\ &= \left\{ x \in \mathbb{R}^n \mid x = \sum_{k=1}^{K} \theta_k x_k \text{ for some } \{\theta_k\} \text{ with } \theta_k \in \mathbb{R}_+,\ \sum_{k=1}^{K} \theta_k = 1 \right\}.\end{aligned}$$

The name K-simplex is a shorter form of K-dimensional simplex; this terminology is used because the affine dimension (that is, the dimension of the affine hull) of the convex hull of $K+1$ affinely independent points is K.

See Figure 3.6.

Definition 23. The *unit simplex* in $\mathbb{R}^n$ is the n-dimensional simplex constructed as the convex hull of the origin and $\{e_1, e_2, \dots, e_n\}$, the n unit vectors in $\mathbb{R}^n$: cvx $\{0, e_1, e_2, \dots,$ $e_n\}$. We note that we could alternately describe the unit simplex in $\mathbb{R}^n$ as the intersection of $n+1$ halfspaces:

$$\begin{aligned}\text{cvx } \{0, e_1, e_2, \dots, e_n\} &= \{x \in \mathbb{R}^n \mid x_i \ge 0,\ i \in \{1,\dots,n\} \text{ and } \langle 1, x\rangle \le 1\} \\ &= \{x \in \mathbb{R}^n \mid \langle e_i, x\rangle \ge 0,\ i \in \{1,\dots,n\} \text{ and } \langle 1, x\rangle \le 1\}.\end{aligned}$$

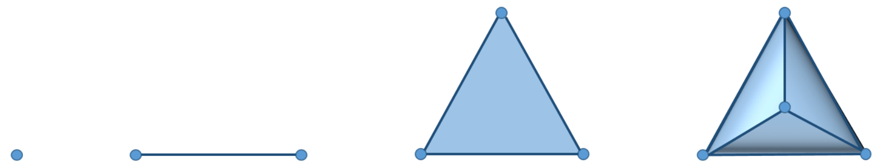

Figure 3.6: Simplexes.

Perhaps the most frequently seen simplex is the probability simplex.

Definition 24. The *probability simplex* in $\mathbb{R}^n$, denoted Δ_n, is the $(n-1)$-dimensional simplex constructed as the convex hull of $\{e_1, e_2, \dots, e_n\}$, the n unit vectors in $\mathbb{R}^n$: cvx $\{e_1, e_2, \dots, e_n\}$. We note that we could alternately describe the unit simplex in $\mathbb{R}^n$ as the intersection of n halfspaces and 1 hyperplane:

$$
\begin{aligned}
\Delta_n &\overset{\text{set}}{=} \text{cvx } \{e_1, e_2, \dots, e_n\} \\
&= \{x \in \mathbb{R}^n \mid x_i \geq 0,\ i \in \{1, \dots, n\} \text{ and } \langle 1, x \rangle = 1\} \\
&= \{x \in \mathbb{R}^n \mid \langle e_i, x \rangle \geq 0,\ i \in \{1, \dots, n\} \text{ and } \langle 1, x \rangle = 1\}.
\end{aligned}
$$

Chapter 4

Functions and convexity

4.1 Introduction

The material in this chapter is again standard. Typical references include [RW04, HUL93a, HUL93b, BV04, Ber09, Rus06]. We provide a coherent collection of the specific concepts that will assist in our subsequent discussions.

4.2 Extended-real-valued functions

When considering constrained minimization problems, it is often convenient to consider the notion of extended-real-valued functions.

A real-valued function is allowed to take on values in $\mathbb{R}$; an extended-real-valued function is allowed to take on values in $\mathbb{R} \cup \{+\infty\}$. In our discussions, this will allow us the possibility of alternate conceptions of constraints. In general, we write $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

Definition 25 (Effective domain)**.** The *effective domain* of an extended-real-valued function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is denoted dom $f(\cdot)$ and defined as the set of all arguments for which $f(\cdot)$ takes on finite value; that is

$$\text{dom } f(\cdot) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.$$

An important example of an extended-real-valued convex function is the indicator function of a convex set. We first define the notion of an indicator function for a

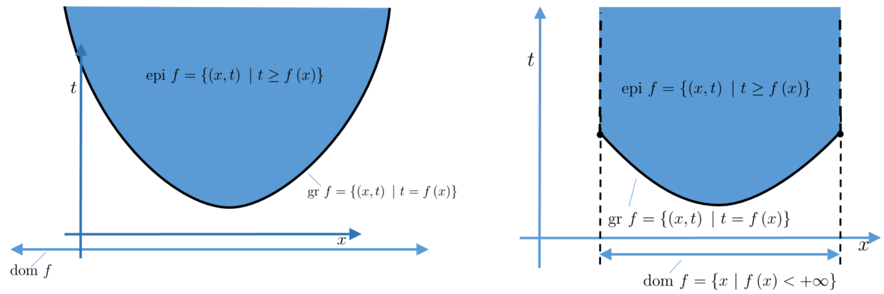

Figure 4.1: For a convex function, both the effective domain and the epigraph are convex sets.

general (not necessarily convex) set and then particularize to the case of the indicator function of a convex set.

Definition 26 (Indicator function)**.** Consider a set $\mathcal{S} \subseteq \mathbb{R}^n$. The *indicator function* of the set $\mathcal{S} \subseteq \mathbb{R}^n$ is denoted $I_{\mathcal{S}}[\cdot] : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and defined via the expression

$$I_{\mathcal{S}}[\cdot] \stackrel{\text{def}}{=} \begin{cases} 0 & \text{when } x \in \mathcal{S} \\ +\infty & \text{when } x \notin \mathcal{S}. \end{cases}$$

The effective domain of the indicator function of the set $\mathcal{S}$ is thus $\mathcal{S}$ itself. The use of indicator function of a set can provide us with an alternative means of expressing constrained optimization problems. We will also see the usefulness of extended-real-valued convex functions in our discussions of Legendre-Fenchel conjugacy and infimal convolution.

When the set in question is nonempty and convex, the corresponding indicator function is a proper convex extended-real-valued function; we introduce the (somewhat technical) concept of properness next. One may get a clearer notion of the significance of whether a convex function is proper or improper from the alternative description used by Aubin [Aub98]: he calls improper convex functions "trivial" and proper convex functions "nontrivial".

Definition 27 (Proper function)**.** We say that a convex extended-real-value function is *proper* when it is finite for at least one argument and never takes on the value $-\infty$.

The requirement that a proper convex extended-real-valued function be finite for at least one argument means that the effective domain of a proper extended-real-valued

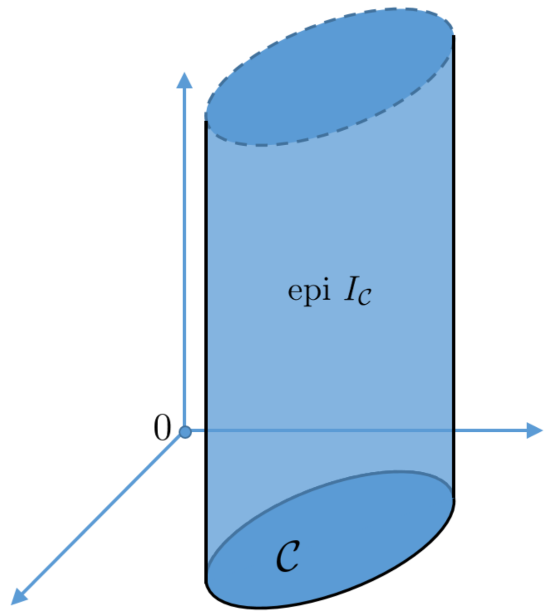

Figure 4.2: The epigraph of the indicator function of a convex set. (After [Luc06]).

convex function is non-empty. The requirement that a proper convex extended-real-valued function never take on the value $-\infty$ has the geometric meaning that the epigraph will not contain any vertical lines.

By restricting attention as much as possible to proper extended-real-valued convex functions we avoid technical problems that would otherwise arise. One might ask whether there are improper convex extended-real-valued functions of sufficient use that they might prompt us to put up with these technical problems; it turns out that the requirements of convexity imply that convex extended-real-valued functions that are improper can display only very limited behavior. If we further restrict attention to convex extended-real-valued functions for which lower semicontinuity holds (even if only at a single point), the behavior becomes even more limited. We now describe just what kinds of behavior are possible for a convex extended-real-valued function that is allowed to be improper; for this discussion only we consider convex functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

For any convex extended-real-valued function that takes on the value $-\infty$ anywhere, convexity implies that it must be the case that $f(x) = -\infty$ for every $x \in$ int dom

$f(\cdot)$. Further, an improper convex function satisfying lower semicontinuity can only take on infinite values: there must be a closed, convex set $\mathcal{D}$ such that $f(x) = -\infty$ for $x \in \mathcal{D}$ and $f(x) = +\infty$ for $x \notin \mathcal{D}$. Stated slightly differently: an extended-real-valued convex function that is lower semicontinuous at any point and that takes the value $-\infty$ anywhere cannot take on values other than $+\infty$ and $-\infty$. The preceding observations follow from the definition of convexity and from "extended-real-valued arithmetic": if $f(x_{\#}) = -\infty$ and $f(x_{\$}) < +\infty$, then at any intermediate point $x_\tau \overset{\text{set}}{=} (1-\tau)x_{\#} + \tau x_{\$}$, where $\tau \in (0,1)$, we must have $f(x_\tau) = -\infty$.

An example of an improper convex function taking on finite values in addition to infinite values is

$$f(x) = \begin{cases} -\infty & \text{when } x \in (0,+\infty) \\ 10 & \text{when } x = 0 \\ +\infty & \text{when } x \in (-\infty,0)\,; \end{cases}$$

note that this improper convex function taking on a finite value is not lower semicontinuous at any argument.

A similar example is

$$f(x) = \begin{cases} -\infty & \text{when } |x| < 1 \\ 2 & \text{when } |x| = 1 \\ +\infty & \text{when } |x| > 1. \end{cases}$$

While we will seek to avoid improper functions, they can arise even from standard operations that involve only proper convex extended-real-valued functions. For example, the conjugate of a proper convex function might be improper; further the infimal convolution of two proper convex extended-real-valued functions can be an improper convex function. For more detailed discussion of the preceding results, see [AT03], [BV10], and, especially, [RW04].

4.3 Sublevel sets, level sets, superlevel sets

Definition 28 (α-sublevel set)**.** The α-*sublevel set* (of arguments) of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a subset of $\mathbb{R}^n$ denoted $\mathrm{lvl}_{f(\cdot)\leq}(\alpha) \subseteq \mathbb{R}^n$ and defined via the expres-

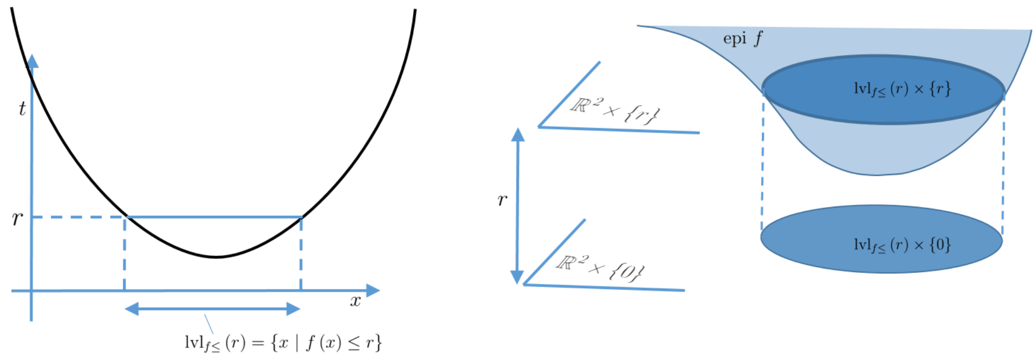

Figure 4.3: Illustrating the concept of sublevel set. (After [HUL93a]).

sion

$$\mathrm{lvl}_{f(\cdot)\leq}(\alpha) \stackrel{\text{set}}{=} \{x \in \mathrm{dom}\, f(\cdot) \mid f(x) \leq \alpha\}.$$

Discussions of α-sublevel sets frequently arise in the context of convex functions. Alternative names for α-sublevel sets of $f(\cdot)$ are "trenches" of $f(\cdot)$, "lower level sets" of $f(\cdot)$, "lower sections" of $f(\cdot)$, "wide sections" of $f(\cdot)$, or simply "sections" of $f(\cdot)$ [Aub98].

Definition 29 (α-level set)**.** The α-*level set* (of arguments) of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a subset of $\mathbb{R}^n$ denoted $\mathrm{lvl}_{f(\cdot)=}(\alpha) \subseteq \mathbb{R}^n$ and defined via the expression

$$\mathrm{lvl}_{f(\cdot)=}(\alpha) \stackrel{\text{set}}{=} \{x \in \mathrm{dom}\, f(\cdot) \mid f(x) = \alpha\}.$$

In convex analysis, the term "level set" is sometimes used for what we refer to as a sublevel set. We prefer to maintain the distinction, since, for example we would like to be able to describe a hyperplane as a level set of an affine (or, alternately, linear) function.

Definition 30 (α-superlevel set)**.** The α-*superlevel set* (of arguments) of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a subset of $\mathbb{R}^n$ denoted $\mathrm{lvl}_{f(\cdot)\geq}(\alpha) \subseteq \mathbb{R}^n$ and defined via the expression

$$\mathrm{lvl}_{f(\cdot)\geq}(\alpha) \stackrel{\text{set}}{=} \{x \in \mathrm{dom}\, f(\cdot) \mid f(x) \geq \alpha\}.$$

Discussions of α-superlevel sets frequently arise in the context of concave functions.

We note that the sublevel sets of a convex function are convex for any value of α; however, a function with all sublevel sets convex is not necessarily a convex function. As an example, consider $f(x) \stackrel{\text{set}}{=} \sqrt{|x|}$. Similarly, we note that the superlevel sets of a concave function are convex for any value of α; however, a function with all superlevel sets convex is not necessarily a concave function.

It is common practice to establish that a set is convex by expressing that set either as a sublevel set of a convex of a convex function or as a superlevel set of a concave function.

4.4 Graph, epigraph, hypograph

Definition 31 (Graph)**.** The *graph* of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a subset of $\mathbb{R}^n \times \mathbb{R}$ denoted by $\operatorname{gr} f(\cdot) \subset \mathbb{R}^n \times \mathbb{R}$ and defined via the expression

$$\begin{aligned} \operatorname{gr} f(\cdot) &\stackrel{\text{def}}{=} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \operatorname{dom} f(\cdot) \text{ and } t = f(x)\} \\ &= \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \operatorname{dom} f(\cdot)\}. \end{aligned}$$

We use the graph of a function as a reference for two other geometric notions associated with a function.

Definition 32 (Epigraph)**.** The *epigraph* of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is "everything that lies on or above the graph"; more precisely, a subset of $\mathbb{R}^n \times \mathbb{R}$ denoted $\operatorname{epi} f(\cdot) \subset \mathbb{R}^n \times \mathbb{R}$ and defined via the expression

$$\operatorname{epi} f(\cdot) \stackrel{\text{def}}{=} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \geq f(x) \text{ and } x \in \operatorname{dom} f(\cdot)\}.$$

We may describe the epigraph $\operatorname{epi} f(\cdot)$ as a collection of closed "half-lines" in $\mathbb{R}$ of the form $[f(x_\#), +\infty)$. The base (in $\mathbb{R}^n$) of each half-line is the corresponding input argument $x_\# \in \mathbb{R}^n$.

One link between convex sets and convex functions is via the notion of epigraph: A function $f(\cdot)$ is convex if and only if $\operatorname{epi} f(\cdot)$ is a convex set. See Figure 4.1.

Definition 33 (Strict epigraph)**.** The *strict epigraph* of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is "everything that lies strictly above the graph"; more precisely, a subset of $\mathbb{R}^n \times \mathbb{R}$ denoted s-epi $f(\cdot) \subset \mathbb{R}^n \times \mathbb{R}$ and defined via the expression

$$\text{s-epi } f(\cdot) \stackrel{\text{def}}{=} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t > f(x) \text{ and } x \in \text{dom } f(\cdot)\}.$$

We may describe the strict epigraph s-epi $f(\cdot)$ as a collection of open "half-lines" in $\mathbb{R}$ of the form $(f(x_\#), +\infty)$. The base (in $\mathbb{R}^n$) of each half-line is the corresponding input argument $x_\# \in \mathbb{R}^n$.

Definition 34 (Hypograph)**.** The *hypograph* of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is "everything that falls at or below the graph"; more precisely, a subset of $\mathbb{R}^n \times \mathbb{R}$ denoted hypo $f(\cdot) \subset \mathbb{R}^n \times \mathbb{R}$ and defined via the expression

$$\text{hypo } f(\cdot) \stackrel{\text{def}}{=} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \leq f(x) \text{ and } x \in \text{dom } f(\cdot)\}.$$

A function $f(\cdot)$ is concave if and only if hypo $f(\cdot)$ is a convex set.

Definition 35 (Strict hypograph)**.** The *strict hypograph* of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is "everything that falls strictly below the graph"; more precisely, a subset of $\mathbb{R}^n \times \mathbb{R}$ denoted s-hypo $f(\cdot) \subset \mathbb{R}^n \times \mathbb{R}$ and defined via the expression

$$\text{s-hypo } f(\cdot) \stackrel{\text{def}}{=} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t < f(x) \text{ and } x \in \text{dom } f(\cdot)\}.$$

Remarks Any function for which both the epigraph and the hypograph correspond to convex sets is affine.

Many results for convex functions can be established (or interpreted) geometrically considering the epigraph and applying results for convex sets.

For example, consider a slight re-expression of the first order condition for convexity of a continuously differentiable function $f(\cdot)$:

for each $x_\# \in \text{dom } f(\cdot)$, we can form a supporting affine minorant of $f(\cdot)$ at $x_\#$, denoted $l_{f(\cdot),x_\#}(x) \stackrel{\text{set}}{=} f(x_\#) + \langle \nabla f(x_\#), x - x_\# \rangle$ so that

$$f(x) \geq l_{f(\cdot),x_\#}(x) \text{ for all } x \in \mathbb{R}^n$$
$$f(x_\#) = l_{f(\cdot),x_\#}(x_\#).$$

We can interpret this basic inequality in terms of the epigraph epi $f(\cdot)$, as follows: $l_{f(\cdot),x_\#}(x) \overset{\text{set}}{=} f(x_\#) + \langle \nabla f(x_\#), x - x_\# \rangle$ is a supporting affine minorant to $f(\cdot)$ at $x_\#$. Thus, any $(x,t) \in \text{epi } f(\cdot)$ also satisfies $(x,t) \in \text{epi } l_{f(\cdot),x_\#}(\cdot)$. This in turn means

$$\begin{aligned}
&t \geq l_{f(\cdot),x_\#}(x) \\
&t \geq f(x_\#) + \langle \nabla f(x_\#), x - x_\# \rangle \\
&0 \geq f(x_\#) - t + \langle \nabla f(x_\#), x - x_\# \rangle \\
&0 \geq \left\langle \begin{pmatrix} \nabla f(x_\#) \\ 1 \end{pmatrix}, \begin{pmatrix} x - x_\# \\ f(x_\#) - t \end{pmatrix} \right\rangle.
\end{aligned}$$

Rearranging slightly,

$$\left\langle \begin{pmatrix} \nabla f(x_\#) \\ -1 \end{pmatrix}, \begin{pmatrix} x - x_\# \\ t - f(x_\#) \end{pmatrix} \right\rangle \leq 0$$

$$\left\langle \begin{pmatrix} \nabla f(x_\#) \\ -1 \end{pmatrix}, \begin{pmatrix} x \\ t \end{pmatrix} - \begin{pmatrix} x_\# \\ f(x_\#) \end{pmatrix} \right\rangle \leq 0.$$

This is precisely the characterization of a halfspace of the form $\langle u, y \rangle - b \leq 0$; this halfspace is in $\mathbb{R}^n \times \mathbb{R}$, with normal vector $\begin{pmatrix} \nabla f(x_\#) \\ -1 \end{pmatrix} \in \mathbb{R}^{n*} \times \mathbb{R}^*$ and bias value $\left\langle \begin{pmatrix} \nabla f(x_\#) \\ -1 \end{pmatrix}, \begin{pmatrix} x_\# \\ f(x_\#) \end{pmatrix} \right\rangle$.

This halfspace in fact supports the epigraph at $\begin{pmatrix} x \\ t \end{pmatrix} \overset{\text{set}}{=} \begin{pmatrix} x_\# \\ f(x_\#) \end{pmatrix}$, since

$$\left\langle \begin{pmatrix} \nabla f(x_\#) \\ -1 \end{pmatrix}, \begin{pmatrix} x_\# \\ f(x_\#) \end{pmatrix} - \begin{pmatrix} x_\# \\ f(x_\#) \end{pmatrix} \right\rangle = 0.$$

Alternately, we can say that 0-level set of the affine function $l_{f(\cdot),x_\#}(\cdot)$ is a hyperplane that supports epi $f(\cdot)$ at the point $\begin{pmatrix} x_\# \\ f(x_\#) \end{pmatrix}$. Another observation that is central to convex analysis is that the epigraph of the convex function $f(\cdot)$ coincides with the intersection of the epigraphs of the supporting-at-$x_\#$ affine minorants of $f(\cdot)$ over all possible arguments of support $x_\# \in \text{dom } f(\cdot)$; explicitly, epi $f(\cdot) = \bigcap_{x_\# \in \text{dom } f(\cdot)} \text{epi } l_{f(\cdot),x_\#}(\cdot)$. We define affine minorants in the next section.

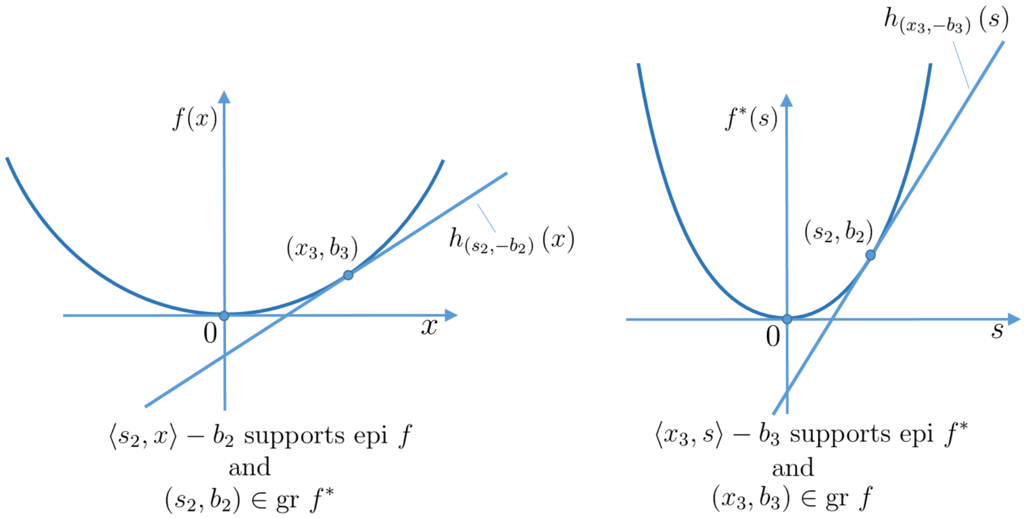

Figure 4.4: Primal lines correspond to dual points; primal points corresponds to dual lines. (After [RW04]).

4.5 Minorant, majorant, supporting minorant, supported majorant

Definition 36 (Minorization, minorant)**.** Consider two functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. We say that $g(\cdot)$ *minorizes* $f(\cdot)$ (or equivalently, that $f(\cdot)$ is minorized by $g(\cdot)$) if $g(\cdot)$ is pointwise less than or equal to $f(\cdot)$: that is

$$g(x) \leq f(x),$$

for each $x \in \mathbb{R}^n$.

When $g(\cdot)$ minorizes $f(\cdot)$ we alternately describe $g(\cdot)$ as a *minorant* of $f(\cdot)$.

A minorant of $f(\cdot)$ provides a pointwise (not-necessarily-strict) lower bound to $f(\cdot)$ everywhere in $\mathbb{R}^n$.

Definition 37 (Majorization, majorant)**.** Consider two functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. We say that $g(\cdot)$ *majorizes* $f(\cdot)$ (or equivalently, that $f(\cdot)$ is majorized by $g(\cdot)$) if $g(\cdot)$ is pointwise greater than or equal to $f(\cdot)$: that is

$$g(x) \geq f(x),$$

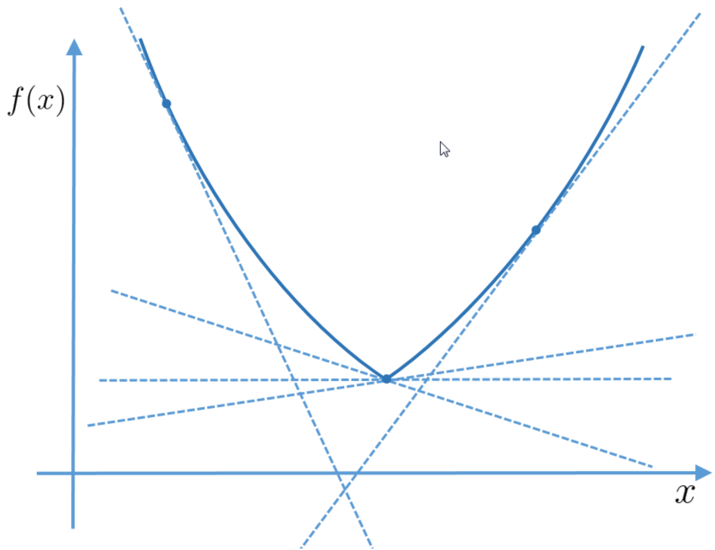

Figure 4.5: The supporting affine minorant characterization of convexity.

for each $x \in \mathbb{R}^n$.

When $g(\cdot)$ majorizes $f(\cdot)$ we alternately describe $g(\cdot)$ as a *majorant* of $f(\cdot)$.

A majorant of $f(\cdot)$ provides a pointwise (not-necessarily-strict) upper bound to $f(\cdot)$ everywhere in $\mathbb{R}^n$.

Definition 38 (Supporting minorant)**.** We say that $g(\cdot)$ is a *supporting minorant* of $f(\cdot)$ at $x_\#$ when $g(\cdot)$ is a minorant of $f(\cdot)$ that coincides with $f(\cdot)$ at $x_\#$; that is, when

$$g(x) \leq f(x) \text{ for all } x \in \mathbb{R}^n$$
$$\text{and}$$
$$g(x_\#) = f(x_\#).$$

We will later see that supporting minorants appear in the definitions of convexity and strong convexity.

Definition 39 (Supported majorant)**.** We say that $g(\cdot)$ is a *supported majorant* of $f(\cdot)$ at $x_\#$ when $g(\cdot)$ is a majorant of $f(\cdot)$ that coincides with $f(\cdot)$ at $x_\#$; that is, when

$$g(x) \geq f(x) \text{ for all } x \in \mathbb{R}^n$$
$$\text{and}$$
$$g(x_\#) = f(x_\#).$$

We will later see that supported majorants appear in the definition of "strong smoothness".

Remarks Note that if $g(\cdot)$ minorizes $f(\cdot)$, we have both that epi $g(\cdot) \supseteq$ epi $f(\cdot)$ and that $\mathrm{lvl}_{f(\cdot)\leq}(\alpha) \subseteq \mathrm{lvl}_{g(\cdot)\leq}(\alpha)$ for any level value $\alpha \in \mathbb{R}$.

Similarly, if $g(\cdot)$ majorizes $f(\cdot)$, we have both that epi $f(\cdot) \supseteq$ epi $g(\cdot)$ and that $\mathrm{lvl}_{g(\cdot)\leq}(\alpha) \subseteq \mathrm{lvl}_{f(\cdot)\leq}(\alpha)$ for any level value $\alpha \in \mathbb{R}$.

4.6 Affine, linear, and convex functions

Definition 40 (Affine function)**.** We say that a function $f(\cdot)$ is *affine* when, for any $\theta_1, \theta_2 \in \mathbb{R}$ satisfying that $\theta_1 + \theta_2 = 1$, it is the case that $f(\theta_1 x_1 + \theta_2 x_2) = \theta_1 f(x_1) + \theta_2 f(x_2)$.

The epigraph of an affine function is a half-space not necessarily containing the origin.

Definition 41 (Linear function)**.** We say that a function $f(\cdot)$ is *linear* when, for any $\theta_1, \theta_2 \in \mathbb{R}$, it is the case that $f(\theta_1 x_1 + \theta_2 x_2) = \theta_1 f(x_1) + \theta_2 f(x_2)$.

Note that this means that $f(0) = 0$ for any linear function.

The epigraph of a linear function is a half-space containing the origin.

Definition 42 (Subadditive function)**.** We say that a function $f(\cdot)$ is *subadditive* when $f(x_1 + y_1) \leq f(x_1) + f(x_2)$.

Any nonincreasing function is subadditive; any function of the form $f(x) \stackrel{\text{set}}{=} mx + b$ with $m, b \in \mathbb{R}_+$ is subadditive.

Definition 43 (Strictly positively homogeneous function)**.** We say that a function $f(\cdot)$ is *strictly positively homogeneous* when $f(\alpha x) = \alpha f(x)$ for any strictly positive scalar $\alpha \in \mathbb{R}_{++}$. The concept of a *nonnegatively homogeneous function* corresponds to the case $\alpha \in \mathbb{R}_+$.

Most of our attention will be devoted to convex functions.

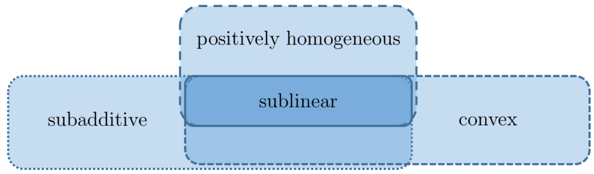

Figure 4.6: Relationships between convexity, sublinearity, positive homogeneity, and subadditivity. (After [HUL93a]).

Definition 44 (Convex function)**.** We say that a function $f(\cdot)$ is convex when, for any $\theta_1, \theta_2 \in \mathbb{R}_+$ satisfying $\theta_1 + \theta_2 = 1$, it is the case that $f(\theta_1 x_1 + \theta_2 x_2) \leq \theta_1 f(x_1) + \theta_2 f(x_2)$.

Geometrically, this means that the line segment between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is entirely contained in the epigraph of $f(\cdot)$; said slightly differently, that the chord connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies above the graph of $f(\cdot)$.

The epigraph of a convex function is a convex set.

Definition 45 (Sublinear function)**.** We say that a function $f(\cdot)$ is *sublinear* when it is both subadditive and strictly positively homogeneous.

The class of sublinear functions is alternately describable as any convex function that is also strictly positively homogeneous. The epigraph of any sublinear function is a convex cone.

The 0-sublevel set of any sublinear function is a cone.

Some authors refer to any function that is both subadditive and strictly positively homogeneous as a gauge function rather than a sublinear function; we reserve the term gauge function for another (admittedly related) usage defined later. We indicate the relationships between subadditivity, positive homogeneity, convexity, and sublinearity in Figure 4.6.

Remarks We further note that any affine function satisfies the convexity inequality as well as the corresponding inequality characterizing concavity; we may thus say that any affine function is both convex and concave. Conversely, any function that is both convex and concave is affine.

A function is convex if and only if that function is convex when restricted to any line that intersects dom $f(\cdot)$. That is, $f(\cdot)$ is convex if and only if $g(t) \overset{\text{set}}{=} f(x+tv)$ is convex on its domain dom $g(\cdot) = \{t \in \mathbb{R} \mid x+tv \in \text{dom } f(\cdot)\}$, for any $x \in \text{dom } f(\cdot)$ and all $v \in \mathbb{R}^n$. This means that we can check whether a function is convex by considering the convexity of its restriction to an arbitrary line.

Chapter 5

Strong smoothness and strong convexity

5.1 Introduction

In discussions of convex optimization, most analyses start from assumptions involving either Lipschitz continuity of the gradient of the function or a combination of Lipschitz continuity of the gradient of the function and strong convexity. As has long been known under various descriptions (for example, [Jam47], or more recently [KSST09]) there is a symmetry (via duality) between Lipschitz continuity of the gradient of a function and strong convexity of a function. Despite the symmetry in the relationships, it is apparent from our preceding statement that the usual descriptions are somewhat misaligned — one references the function via its gradient, the other directly refers to the function. Following the usage in [KSST09] we use the terms “strong smoothness” (specifically, of a convex function) and “strong convexity” when emphasizing properties of the function; when we later consider terminology for operators, the analogous terminology will be “inverse strong monotonicity” and “strong monotonicity”. The most prominent reference for this material is [Nes04].

5.2 Strong smoothness with parameter L

We say that the (at least once differentiable) convex function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is L-strongly smooth when it satisfies

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + L\frac{1}{2}\|y - x\|_2^2$$

for all $x, y \in \mathbb{R}^n$.

As most prominently demonstrated in [Nes04], this description can be expressed in several equivalent forms.

As L-Lipschitz continuity of the gradient of $f(\cdot)$:

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2 .$$

As secant-type inequalities:

$$[\alpha f(x) + (1-\alpha) f(y)] - [f(\alpha x + \{1-\alpha\} y)] \leq \alpha(1-\alpha) L\frac{1}{2}\|x - y\|_2^2$$

and

$$\alpha(1-\alpha)\frac{1}{L}\frac{1}{2}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq [\alpha f(x) + (1-\alpha) f(y)] - [f(\alpha x + \{1-\alpha\} y)] .$$

As expressions of inverse strong monotonicity:

$$\frac{1}{L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle$$

and

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq L\|x - y\|_2^2 .$$

As expressions related to the existence of a supported quadratic minorant:

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + L\frac{1}{2}\|y - x\|_2^2$$

and

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{L}\frac{1}{2}\|\nabla f(y) - \nabla f(x)\|_2^2 \leq f(y).$$

Finally, a re-expression of the immediately preceding expressions provides us with a statement involving bounds on the Bregman divergence using the function $f(\cdot)$, based at x and evaluated at y:

$$D_f(y;x) \leq L\frac{1}{2}\|y - x\|_2^2$$

and

$$\frac{1}{L}\frac{1}{2}\|\nabla f(y) - \nabla f(x)\|_2^2 \leq D_f(y;x).$$

Here $D_f(y;x)$ denotes the Bregman divergence using the function $f(\cdot)$, based at x and evaluated at y.

We collect these results in Table 5.1.

Nesterov uses the notation $\mathcal{F}_L^{1,1}(\mathbb{R}^n)$ to refer to the class of convex functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ for which the first derivative exists and for which L strong smoothness is satisfied.

5.3 Strong convexity with parameter μ

We say that the (say, at least once differentiable) convex function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is μ-strongly convex when it satisfies

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \mu\frac{1}{2}\|y - x\|_2^2$$

for all $x, y \in \mathbb{R}^n$.

Again, [Nes04] is the most prominent reference for the equivalent forms of this statement.

That is, the case in which $f(\cdot)$ is a function for which the first derivative exists, and for which $f(\cdot)$ is μ strongly convex.

Table 5.1: Strong smoothness equivalences.

	$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x-y\|_2$	
	$[\alpha f(x) + (1-\alpha) f(y)] - [f(\alpha x + \{1-\alpha\} y)] \leq \alpha(1-\alpha) L\frac{1}{2}\|x-y\|_2^2$	
	$\alpha(1-\alpha)\frac{1}{L}\frac{1}{2}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq [\alpha f(x) + (1-\alpha) f(y)] - [f(\alpha x + \{1-\alpha\} y)]$	
	$\frac{1}{L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq \langle \nabla f(x) - \nabla f(y), x-y\rangle$	
	$\langle \nabla f(x) - \nabla f(y), x-y\rangle \leq L\|x-y\|_2^2$	
	$f(y) \leq f(x) + \langle \nabla f(x), y-x\rangle + L\frac{1}{2}\|y-x\|_2^2$	
	$f(x) + \langle \nabla f(x), y-x\rangle + \frac{1}{L}\frac{1}{2}\|\nabla f(y) - \nabla f(x)\|_2^2 \leq f(y)$	
	$D_f(y;x) \leq L\frac{1}{2}\|y-x\|_2^2$	
	$\frac{1}{L}\frac{1}{2}\|\nabla f(y) - \nabla f(x)\|_2^2 \leq D_f(y;x)$	

As a condition similar in form to Lipschitz continuity:

$$\mu \left\|x-y\right\|_2 \leq \left\|\nabla f\left(x\right)-\nabla f\left(y\right)\right\|_2.$$

As secant-type inequalities:

$$\alpha\left(1-\alpha\right)\mu\frac{1}{2}\left\|x-y\right\|_2^2 \leq \left[\alpha f\left(x\right)+\left(1-\alpha\right)f\left(y\right)\right]-\left[f\left(\alpha x+\left\{1-\alpha\right\}y\right)\right]$$

and

$$\left[\alpha f\left(x\right)+\left(1-\alpha\right)f\left(y\right)\right]-\left[f\left(\alpha x+\left\{1-\alpha\right\}y\right)\right] \leq \alpha\left(1-\alpha\right)\frac{1}{\mu}\frac{1}{2}\left\|\nabla f\left(x\right)-\nabla f\left(y\right)\right\|_2^2.$$

As expressions of strong monotonicity:

$$\mu\left\|x-y\right\|_2^2 \leq \left\langle \nabla f\left(x\right)-\nabla f\left(y\right), x-y\right\rangle$$

and

$$\left\langle \nabla f\left(x\right)-\nabla f\left(y\right), x-y\right\rangle \leq \frac{1}{\mu}\left\|\nabla f\left(y\right)-\nabla f\left(x\right)\right\|_2^2.$$

As expressions related to the existence of a supporting quadratic minorant:

$$f\left(x\right)+\left\langle \nabla f\left(x\right), y-x\right\rangle+\mu\frac{1}{2}\left\|y-x\right\|_2^2 \leq f\left(y\right)$$

and

$$f\left(y\right) \leq f\left(x\right)+\left\langle \nabla f\left(x\right), y-x\right\rangle+\frac{1}{\mu}\frac{1}{2}\left\|\nabla f\left(y\right)-\nabla f\left(x\right)\right\|_2^2.$$

Again, a re-expression of the immediately preceding expressions provides us with a statement involving bounds on the Bregman divergence using the function $f(\cdot)$, based at x and evaluated at y:

$$\mu \frac{1}{2} \|y - x\|_2^2 \leq D_f(y;x)$$

and

$$D_f(y;x) \leq \frac{1}{\mu} \frac{1}{2} \|\nabla f(y) - \nabla f(x)\|_2^2 .$$

We collect these results in Table 5.2.

In Nesterov-style notation, $\mathcal{S}_\mu^1(\mathbb{R}^n)$ would refer to the class of convex functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ for which the first derivative exists and for which μ strong convexity is satisfied.

5.4 Both strong smoothness and strong convexity

The next result is central, for example, to the analysis in [Nes04]. Despite this centrality, it is less widely referenced than one might expect (although this seems to be changing).

Proposition 1. *When the convex function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a function for which the first derivative exists, and for which $f(\cdot)$ is both μ-strongly convex and L-strongly smooth, it is that case that*

$$\frac{1}{L+\mu} \|\nabla f(x) - \nabla f(y)\|_2^2 + \frac{L\mu}{L+\mu} \|x - y\|_2^2 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle$$

for all $x, y \in \mathbb{R}^n$.

For later reference, we also note that the expression above could be written

$$\frac{1}{\left(\frac{L+\mu}{2}\right)} \frac{1}{2} \|\nabla f(x) - \nabla f(y)\|_2^2 + \frac{1}{\left(\frac{\frac{1}{L}+\frac{1}{\mu}}{2}\right)} \frac{1}{2} \|x - y\|_2^2 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle .$$

Nesterov [Nes04] uses the notation $\mathcal{S}_{\mu,L}^{1,1}(\mathbb{R}^n)$ to refer to the class of convex functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ for which the first derivative exists and for which both L strong smoothness and μ strong convexity are satisfied.

Table 5.2: Strong convexity equivalences.

	$\mu \|x-y\|_2 \leq \|\nabla f(x) - \nabla f(y)\|_2$	
	$\alpha(1-\alpha)\mu\frac{1}{2}\|x-y\|_2^2 \leq [\alpha f(x) + (1-\alpha) f(y)] - [f(\alpha x + \{1-\alpha\} y)]$	
	$[\alpha f(x) + (1-\alpha) f(y)] - [f(\alpha x + \{1-\alpha\} y)] \leq \alpha(1-\alpha)\frac{1}{\mu}\frac{1}{2}\|\nabla f(x) - \nabla f(y)\|_2^2$	
	$\mu \|x-y\|_2^2 \leq \langle \nabla f(x) - \nabla f(y), x-y \rangle$	
	$\langle \nabla f(x) - \nabla f(y), x-y \rangle \leq \frac{1}{\mu}\|\nabla f(y) - \nabla f(x)\|_2^2$	
	$f(x) + \langle \nabla f(x), y-x \rangle + \mu\frac{1}{2}\|y-x\|_2^2 \leq f(y)$	
	$f(y) \leq f(x) + \langle \nabla f(x), y-x \rangle + \frac{1}{\mu}\frac{1}{2}\|\nabla f(y) - \nabla f(x)\|_2^2$	
	$\mu\frac{1}{2}\|y-x\|_2^2 \leq D_f(y;x)$	
	$D_f(y;x) \leq \frac{1}{\mu}\frac{1}{2}\|\nabla f(y) - \nabla f(x)\|_2^2$	

Chapter 6

Sets associated with other sets

6.1 Introduction

The material in this standard, although (as we comment below) surprisingly inconsistent in terminology and notation. Typical references include [Aub98, Ber09, BV04, HUL93a, HUL93b, RW04, Rus06, BSS06].

The title of this chapter is somewhat generic, so as to accurately encompass all of the material discussed. The majority of the discussion, however, is devoted to the much more specific topic of cones associated with various sets of interest. In the case of the cone generated by a set, or the polar cone associated with a set, or the dual cone associated with a set, the set in question is mentioned explicitly, but there is no explicit "reference point" mentioned (the origin is the "implicit" reference point in these cases). When instead considering feasible cones, tangent cones, or normal cones, the discussion requires an explicit statement of the set in question, as well as an explicit reference point (in the set).

The polar cone and dual cone provide a context for the relationship between corresponding tangent cones and normal cones. Normal cones most frequently (although implicitly) appear in our discussion in their relation to subdifferential sets.

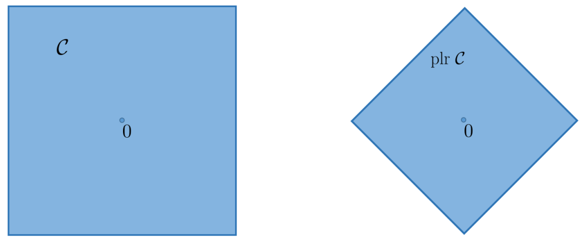

Figure 6.1: A convex set and its associated polar set.

6.1.1 Hulls: linear, affine, convex, convex conic

We have already seen the notion of hulls; many of the most fundamental and widely used results in convex analysis are stated in terms of hulls.

We consider further sets associated with sets below.

6.1.2 The polar set of a generic nonempty set

Definition 46 (Polar set)**.** Consider a generic set $\mathcal{S} \subseteq \mathbb{R}^n$. We associate with any such set another set in $\mathbb{R}^{n*}$ called the *polar* of the set $\mathcal{S}$, denoted plr $\mathcal{S}$ and defined via the expression

$$\operatorname{plr} \mathcal{S} \stackrel{\text{def}}{=} \{s \in \mathbb{R}^{n*} \mid \langle s, x \rangle \leq 1 \text{ for all } x \in \mathcal{S}\}.$$

The polar set of the empty set is ("vacuously") all of $\mathbb{R}^n$: explicitly, $\operatorname{plr} \emptyset = \mathbb{R}^n$.

We observe that whether or not the set $\mathcal{S}$ is convex, the associated polar set plr $\mathcal{S}$ is convex; this follows by noting that plr $\mathcal{S}$ is defined as an intersection of convex sets (namely, half-spaces). Moreover, it is always the case that $0 \in \operatorname{plr} \mathcal{S}$. See Figure 6.1.

The polar set of a generic set is relatively infrequently referenced in convex analysis ([Roc70, RW04, HUL93a] being notable exceptions); however, as we see below, the polar set associated with a specific convex cone arises very frequently and is known as the polar cone.

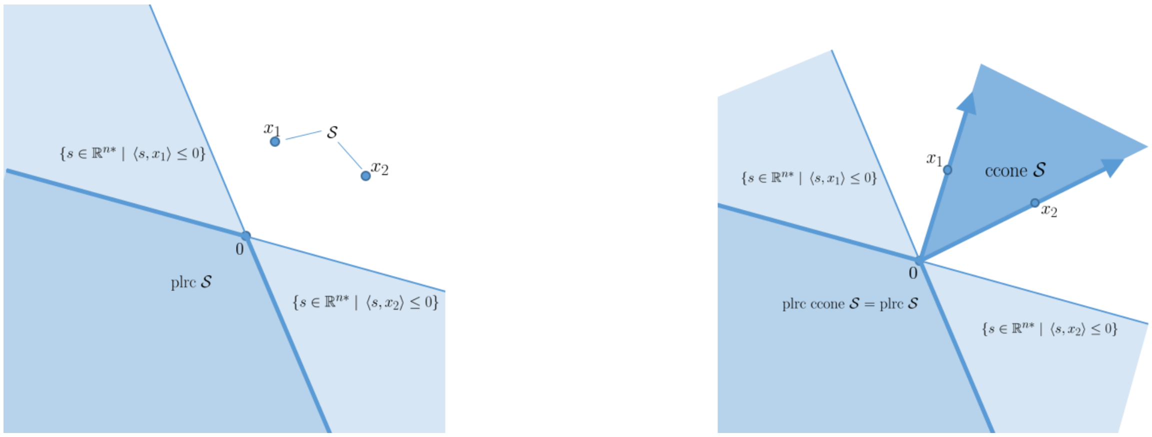

Figure 6.2: The polar cone of a set $S \overset{\text{set}}{=} \{x_1, x_2\}$ coincides with the polar cone of the convex conic hull of S : that is, plrc $\mathcal{S}$ = plrc ccone $\mathcal{S}$. (After [Ber09]).

6.2 Cones associated with any set

6.2.1 The polar cone of a generic set

Definition 47. Consider a nonempty generic set $\mathcal{S} \subseteq \mathbb{R}^n$. To any such set we associate a *polar cone* in $\mathbb{R}^{n*}$, denoted plrc $\mathcal{S}$ (or $\mathcal{S}^\circ$) and defined via the expression

$$\text{plrc}\,\mathcal{S} \overset{\text{def}}{=} \{s \in \mathbb{R}^{n*} \mid \langle s, x \rangle \le 0 \text{ for all } x \in \mathcal{S}\}.$$

We note that an alternative path to the polar cone of the set $\mathcal{S}$ is: begin with the nonempty set $\mathcal{S}$. Form ccone $\mathcal{S}$, the associated convex conic hull of $\mathcal{S}$. The polar set associated with ccone $\mathcal{S}$ coincides with the polar cone of the set $\mathcal{S}$: plr ccone $\mathcal{S}$ = plrc $\mathcal{S}$. See Figure 6.2.

The polar cone of the empty set is (“vacuously”) $\mathbb{R}^n$: explicitly, plrc $\emptyset = \mathbb{R}^n$.

For any arbitrary nonempty set $\mathcal{S}$, the polar cone associated with $\mathcal{S}$ is characterized as the intersection of closed convex sets — specifically, “homogeneous” halfspaces (that is, half-spaces including the origin as a point in the boundary). This implies that the polar cone of any arbitrary set $\mathcal{S}$ is a closed convex cone.

Other notation for the polar cone includes $\mathcal{S}^\perp$[Ber99], $\mathcal{S}^-$[BV10], $\mathcal{S}^\ominus$[Lue69], and (unfortunately) $\mathcal{S}^*$[RW04]. Other terms for the polar cone are the negative polar cone [BV10] or the negative conjugate cone [Lue69], the supplementary cone [KW79], and (unfortunately) the dual cone [Deu01].

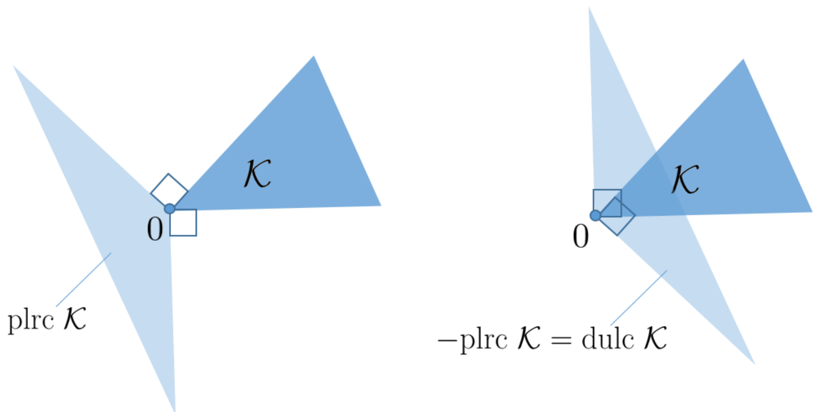

Figure 6.3: The polar cone, plrc $\mathcal{K}$, and the dual cone, dulc $\mathcal{K}$, of a convex cone $\mathcal{K}$.

6.2.2 The dual cone of a generic set

Definition 48. Consider a nonempty generic set $\mathcal{S} \subseteq \mathbb{R}^n$. To any such set we associate a *dual cone* in $\mathbb{R}^{n*}$, denoted dulc $\mathcal{S}$ (or $\mathcal{S}^*$) and defined via the expression

$$\operatorname{dulc} \mathcal{S} \stackrel{\text{def}}{=} \{s \in \mathbb{R}^{n*} \mid \langle s, x\rangle \geq 0 \text{ for all } x \in \mathcal{S}\}.$$

An alternative definition of the dual cone is as the negative of the polar cone: dulc $\mathcal{S} = -$plrc $\mathcal{S}$. See Figure

For any arbitrary nonempty set $\mathcal{S}$, the dual cone associated with $\mathcal{S}$ is characterized as the intersection of closed convex sets — specifically, "homogeneous" halfspaces. This implies that the dual cone of any arbitrary set $\mathcal{S}$ is a closed convex cone.

The dual cone of the empty set would ("vacuously") be $\mathbb{R}^{n*}$: explicitly, plrc $\emptyset = \mathbb{R}^{n*}$.

For a subspace $\mathcal{V}$, the associated dual cone dulc $\mathcal{V}$ and polar cone plrc $\mathcal{V}$ coincide with each other — they both correspond to the orthogonal subspace $\mathcal{V}^\perp$; explicitly, when $\mathcal{V}$ is a subspace dulc $\mathcal{V} =$ plrc $\mathcal{V} = \mathcal{V}^\perp$. For a vector to be orthogonal to a generic set $\mathcal{S}$, that vector must be a member of both the polar cone plrc $\mathcal{S}$ and the dual dulc $\mathcal{S}$: $s \in \mathcal{S}^\perp$ if and only if $s \in$ plrc $\mathcal{S} \cap$ dulc $\mathcal{S}$.

Other notation for the dual cone includes $\mathcal{S}^+$[BV10] and $\mathcal{S}^\oplus$[Lue69]. Other terms for the dual cone are the positive polar cone [BV10], the positive conjugate cone [Lue69], and the adjoint cone [Lev94].

Remark. There is a surprising lack of agreement on terminology and notation surrounding (polar and dual) cones. In [Ber99], the polar cone is denoted $\mathcal{S}^{\perp}$; the dual cone is not mentioned. In [BV10] the polar cone is called the negative polar cone and is denoted $\mathcal{S}^{-}$; the dual cone is called the positive polar cone and is denoted $\mathcal{S}^{+}$. Even though they do not use the term dual cone, they do use the term "self-dual" in describing a special class of cones. In [Deu01] the polar cone is called the dual cone (with a brief note indicating that negative polar cone is an alternative). In [Lev94] the polar cone is not mentioned; the dual cone is called the adjoint cone. In [Lue69] the polar cone is called the negative conjugate cone and is denoted $\mathcal{S}^{\ominus}$; the dual cone is called the positive conjugate cone and is denoted $\mathcal{S}^{\oplus}$; a brief remark notes that these cones are "related to" the polar cone and dual cone. In [BV04] the polar cone is not mentioned. In [RW04] the polar cone is denoted $\mathcal{S}^{*}$, even though the polar set is mentioned and denoted $\mathcal{S}^{\circ}$; the dual cone is not mentioned. In the apparently rare example of [Rus06] both the polar cone and the dual cone are mentioned.

6.3 Cones associated with a set and some point in that set

We now come to cones that are associated with a generic set and some point in that set. Denoting the generic set as $\mathcal{S}$ and the point as $x_{\#} \in \mathcal{S}$ we have

The cone of directions that are "feasible" with respect to the set $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$.

The cone of directions that are "tangent" with respect to the set $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$.

The cone of vectors that are "normal" with respect to the set $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$.

Just as one has the notion of a linear subspace, say $\mathcal{V}$, and a corresponding translated linear subspace, say $\mathcal{V}+x_{\#}$, we an analogous correspondence between a cone, say $\mathcal{Q}$, and a corresponding translated cone, say $\mathcal{Q}+x_{\#}$. Thus, rather than think of cones translated to $x_{\#} \in \mathcal{S}$, we may instead think of these cones as associated with the translated set $\mathcal{S}-x_{\#}$. See Figure 6.6. Since we restrict the usage of the term "cone" to the setting in which the origin is included, the notion that a "cone" is made up of directions is justified; we might then say that a translated cone is made up of translated directions. A discussion of “directions” as a class of objects distinct from a “vector starting at the origin” can be found in [RW04].

6.3.1 The cone of directions that are “feasible” with respect to the set $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$.

We will first consider the case of a generic set $\mathcal{S}$. We then will consider how the definition can be stated more simply when the set in question is a nonempty convex set $\mathcal{C}$.

First, we define the notion of a direction that is “feasible” (at the point $x_{\#} \in \mathcal{S}$) with respect to a generic set $\mathcal{S}$.

Definition 49. We say that a direction $d \in \mathbb{R}^n$ is a *feasible direction* with respect to the set $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$ if it is the case that $\operatorname{seg}[x_{\#}, x_{\#} + td] \subset \mathcal{S}$ for some strictly positive $t \in \mathbb{R}_{++}$.

Note that 0 is always a feasible direction from any $x_{\#} \in \mathcal{S}$.

We can now define the cone of feasible directions.

Definition 50. We call the collection of all directions that are feasible with respect to a generic set $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$ the *feasible cone* (with respect to $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$), denoted $\mathcal{F}_{\mathcal{S}}(x_{\#}) \subseteq \mathbb{R}^n$. Explicitly,

$$\mathcal{F}_{\mathcal{S}}(x_{\#}) \stackrel{\text{def}}{=} \{d \in \mathbb{R}^n \mid \operatorname{seg}[x_{\#}, x_{\#} + td] \subset \mathcal{S} \text{ for some } t \in \mathbb{R}_{++}\}.$$

We previously observed that 0 is always a feasible direction from any $x_{\#} \in \mathcal{S}$. If we begin with $d \in \mathcal{F}_{\mathcal{S}}(x_{\#})$, considering $d \stackrel{\text{set}}{=} \alpha d$ and $t \stackrel{\text{set}}{=} \frac{1}{\alpha} t$ for some $\alpha \in \mathbb{R}_{++}$, immediately establishes that the set of all feasible directions is a cone (although it need not be closed or convex). See Figure 6.4.

Roughly, we are saying that a direction d is feasible with respect to $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$ if we can travel from $x_{\#}$ in the direction d to some other point in $\mathcal{S}$ without ever leaving $\mathcal{S}$ along the way. This description emphasizes a method of “testing” a direction to see if it satisfies the requirement for being a feasible direction. An alternative approach would be to consider the point of interest $x_{\#} \in \mathcal{S}$, and then consider seg $[x_{\#}, x_{\%}]$ for some other $x_{\%} \in \mathcal{S}$; whenever seg $[x_{\#}, x_{\%}] \subset \mathcal{S}$, we conclude that the direction aligned with $x_{\%} - x_{\#}$ is feasible with respect to $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$ (and the direction aligned with $x_{\#} - x_{\%}$ is feasible with respect to $\mathcal{S}$ at the point $x_{\%} \in \mathcal{S}$).

This segment-based description of feasible directions indicates that the characterization of the cone of feasible directions can be more direct when the set in question is convex (since a convex set contains the entire line segment joining any two points in the set).

Proposition 2. *For a nonempty closed convex set $\mathcal{C}$, the feasible cone $\mathcal{F}_{\mathcal{C}}(x_{\#})$ with respect to $\mathcal{C}$ at the point $x_{\#} \in \mathcal{C}$ can be expressed as*

$$\mathcal{F}_{\mathcal{C}}(x_{\#}) = ccone(\mathcal{C} - x_{\#}).$$

When considering a convex set $\mathcal{C}$, the feasible cone $\mathcal{F}_{\mathcal{C}}(x_{\#})$ will always be a convex cone; however, it need not be closed. See Figure 6.5 for an example.

Other terminology for the cone of feasible directions includes the radial cone, the cone of interior directions, and the cone of attainable directions. It is sometimes indirectly referred to as "the cone generated by the set $\mathcal{C} - x_{\#}$".

6.3.2 The cone of directions that are "tangent" with respect to the set $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$.

For a nonconvex set $\mathcal{S}$, the notion of a direction being "tangent" with respect to the set $\mathcal{S}$ at the point $x_{\#} \in \mathcal{S}$ requires careful consideration. There are a number of possible alternative definitions, each of which is defined in terms of some limiting sequence (or limiting sequences). In the general case of nonconvex $\mathcal{S}$, these definitions can lead to significantly different sets; fortunately, when we consider a convex set, say $\mathcal{C}$, all of these definitions coincide. Not only do the various definitions of tangent direction coincide, it further turns out that the set of all directions that are "tangent" with respect to a convex set $\mathcal{C}$ at the point $x_{\#} \in \mathcal{C}$ has a notably direct characterization:

Definition 51. Consider a nonempty closed convex set $\mathcal{C} \subseteq \mathbb{R}^n$ and a point $x_{\#} \in \mathcal{C}$. We associate with any such set $\mathcal{C}$ and point $x_{\#} \in \mathcal{C}$ a set in $\mathbb{R}_n$ called the *tangent cone* to the

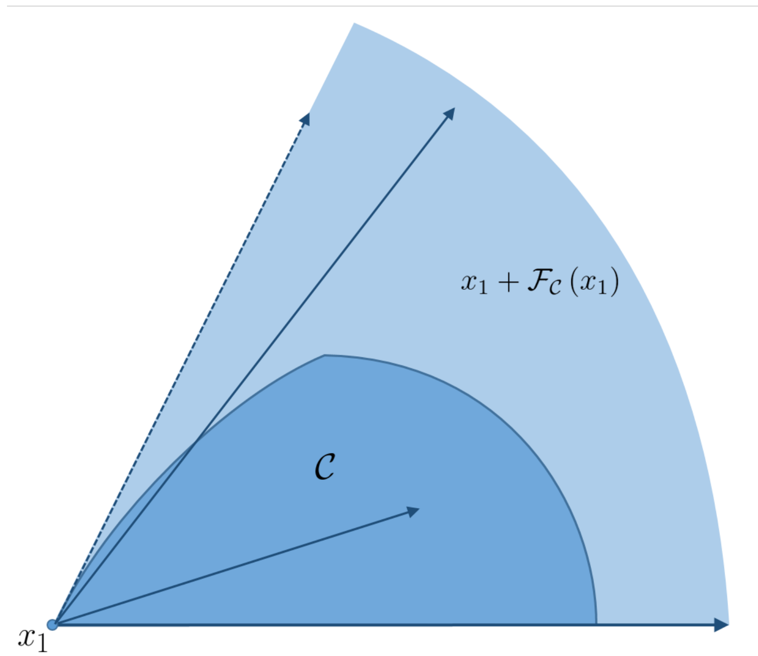

Figure 6.4: The cone $\mathcal{F}_{\mathcal{C}}(x_1)$ of directions that are feasible with respect to the convex set $\mathcal{C}$ at the point $x_1 \in \mathcal{C}$ (translated to the point x_1 at which it is calculated). Note that the "upper boundary ray" is not included as a feasible direction. (After [Rus06]).

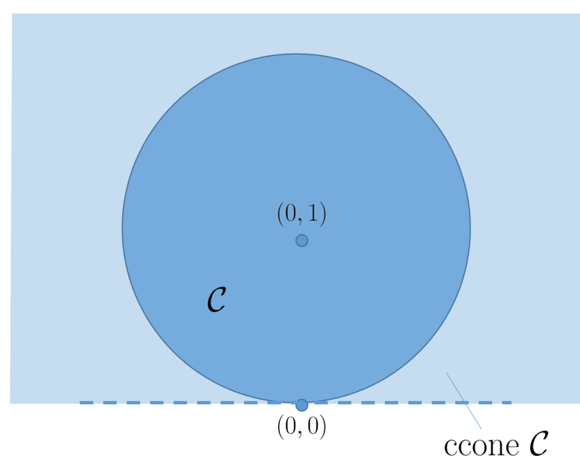

Figure 6.5: An example of a nonclosed feasible cone for a nonempty closed bounded convex set. (After [Ber09]).

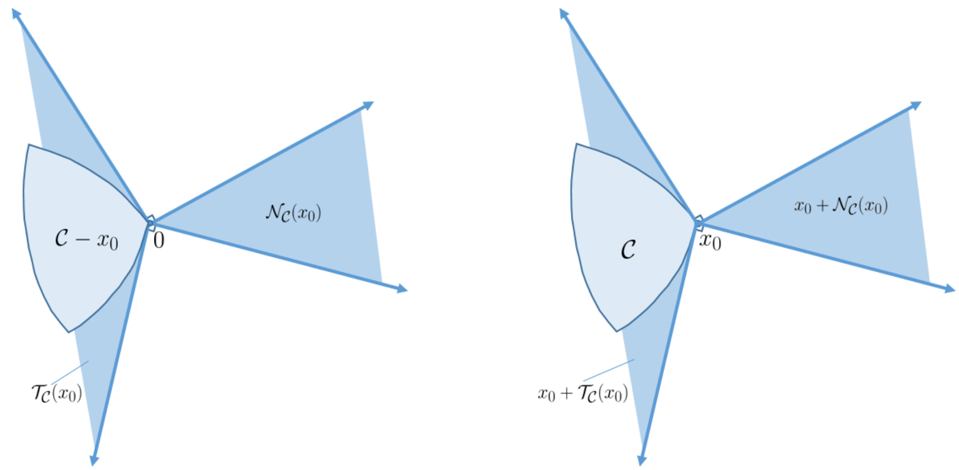

Figure 6.6: The tangent cone and normal cone to a convex set $\mathcal{C}$ at a point $x_0 \in \mathcal{C}$. We also emphasize the alternate means of illustrating tangent and normal cones: as cones or as "translated cones". (After [Aub98]).

convex set $\mathcal{C}$ at the point $x_\# \in \mathcal{C}$ (or the cone of directions that are "tangent" with respect to the convex set $\mathcal{C}$ at the point $x_\# \in \mathcal{C}$), denoted $\mathcal{T}_{\mathcal{C}}(x_\#) \subset \mathbb{R}^n$ and characterized via the expression

$$\begin{aligned} \mathcal{T}_{\mathcal{C}}(x_\#) &\stackrel{\text{def}}{=} \text{cl}\, \mathcal{F}_{\mathcal{C}}(x_\#) \\ &= \text{cl}\, \text{ccone}\,(\mathcal{C} - x_\#)\,. \end{aligned}$$

See Figure 6.6 for an illustration of a tangent cone. We have directly defined the cone of tangent directions above, rather than first defining the notion of tangent direction. Roughly, we may say that a direction d is a tangent direction to the convex set $\mathcal{C}$ at the point $x_\# \in \mathcal{C}$ if it can be expressed as the limit of a sequence of feasible directions (to $\mathcal{C}$ at $x_\# \in \mathcal{C}$). See Figure 6.7 for an example of the distinction between tangent cone and feasible cone.

The subtleties involved in defining tangent cones (and normal cones) to arbitrary sets are comprehensively discussed in [AF08]; specifically, they discuss the Clarke tangent cone (or circatangent cone), the contingent cone (or Bouligand tangent cone), the convex kernel of the contingent cone, the intermediate cone (or adjacent cone or derivable cone), and the paratingent cone. A less comprehensive but somewhat more accessible discussion can be found in [HUL93a]. When using a definition of tangent

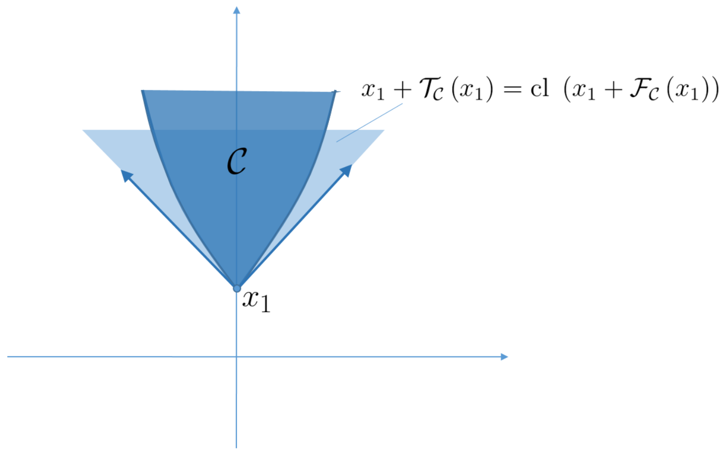

Figure 6.7: The tangent cone to $\mathcal{C}$ at $x_0 \in \mathcal{C}$ is the closure of the feasible cone to $\mathcal{C}$ at $x_0 \in \mathcal{C}$. (After [BNO03].

cone that yields convexity (either by virtue of its definition or because of the characteristics, e.g. regularity, of the set $\mathcal{S}$ at the point $x_\# \in \mathcal{S}$), an associated normal cone can be satisfactorily defined as the polar cone of the tangent cone. In less favorable situations, the definition of normal cone again requires additional care. By restricting our attention to nonempty convex sets, we need not address these complications. The terminology for the tangent cone of a nonempty convex set is fairly uniform; however, [SW70] introduce the notion of a "supporting cone" that (in specific settings) coincides with the tangent cone.

6.3.3 The cone of vectors that are "normal" with respect to the set $\mathcal{S}$ at the point $x_\# \in \mathcal{S}$.

In our discussion of tangent cones, we restricted our attention to the setting of nonempty convex sets; we continue this in our discussion of normal cones.

Definition 52. We say that a vector $s \in \mathbb{R}^{n*}$ is a *normal vector* to the convex set $\mathcal{C} \subseteq \mathbb{R}^n$ at the point $x_\# \in \mathcal{C}$ when

$$\langle s, x - x_\# \rangle \leq 0 \text{ for all } x \in \mathcal{C}.$$

In words, we can say that a vector is normal to the convex set $\mathcal{C}$ at the point $x_\#$ if it makes an obtuse angle with every direction that is feasible with respect to $\mathcal{C}$ at $x_\# \in \mathcal{C}$.

When $\mathcal{C}$ is smooth at $x_{\#} \in \mathcal{C}$, the feasible directions form a halfspace and the normal cone reduces to a single vector (orthogonal to the boundary of the feasible direction halfspace).

The normal cone is defined as the collection of all normal vectors:

Definition 53. We call the collection of all vectors that are normal to the convex set $\mathcal{C}$ at the point $x_{\#} \in \mathcal{C}$ the *normal cone* (to $\mathcal{C}$ at $x_{\#} \in \mathcal{C}$), denoted $\mathcal{N}_{\mathcal{C}}(x_{\#}) \subseteq \mathbb{R}^{n*}$. Explicitly,

$$\mathcal{N}_{\mathcal{S}}(x_{\#}) \stackrel{\text{def}}{=} \{s \in \mathbb{R}^{n*} \mid \langle s, x - x_{\#}\rangle \leq 0 \text{ for all } x \in \mathcal{C}\}.$$

We can interpret this definition in terms of a polar cone to the translated set $\mathcal{C} - x_{\#}$. In particular, note that $x \in \mathcal{C}$ corresponds to $x - x_{\#} \in \mathcal{C} - x_{\#}$; introducing the notation $d \stackrel{\text{set}}{=} x - x_{\#}$ and $\mathcal{C}_{\text{shft}} \stackrel{\text{set}}{=} \mathcal{C} - x_{\#}$, we observe that as x ranges over $\mathcal{C}$, the corresponding vector $d \stackrel{\text{set}}{=} x - x_{\#}$ ranges over $\mathcal{C}_{\text{shft}} \stackrel{\text{set}}{=} \mathcal{C} - x_{\#}$. In the language of this additional notation, we find

$$\begin{aligned} \langle s, x - x_{\#}\rangle \leq & 0 \text{ for all } x \in \mathcal{C} \\ & \text{becomes} \\ \langle s, d\rangle \leq & 0 \text{ for all } d \in \mathcal{C} - x_{\#}. \end{aligned}$$

Thus we find

$$\begin{aligned} \mathcal{N}_{\mathcal{S}}(x_{\#}) &\stackrel{\text{def}}{=} \{s \in \mathbb{R}^{n*} \mid \langle s, x - x_{\#}\rangle \leq 0 \text{ for all } x \in \mathcal{C}\} \\ &= \{s \in \mathbb{R}^{n*} \mid \langle s, d\rangle \leq 0 \text{ for all } d \in \mathcal{C} - x_{\#}\}. \end{aligned}$$

We now observe that the set $\{s \in \mathbb{R}^{n*} \mid \langle s, d\rangle \leq 0 \text{ for all } d \in \mathcal{C} - x_{\#}\}$ (of all vectors $s \in \mathbb{R}^{n*}$ satisfying the requirement for being a normal vector to $\mathcal{C}$ at $x_{\#} \in \mathcal{C}$) is precisely the definition of the polar cone $\operatorname{plrc}(\mathcal{C} - x_{\#})$ associated with the set $\mathcal{C} - x_{\#}$; explicitly we are recalling that

$$\operatorname{plrc}(\mathcal{C} - x_{\#}) \stackrel{\text{def}}{=} \{s \in \mathbb{R}^{n*} \mid \langle s, d\rangle \leq 0 \text{ for all } d \in \mathcal{C} - x_{\#}\}.$$

Thus, our explicit observation is

$$\mathcal{N}_{\mathcal{S}}(x_{\#}) = \operatorname{plrc}(\mathcal{C} - x_{\#}).$$

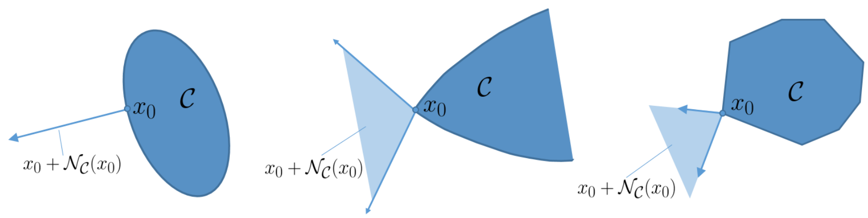

Figure 6.8: Normal cones for various convex sets. (After [Ber09]).

As a polar cone, the normal cone is a cone formed by the intersection of closed convex sets (closed halfspaces), and thus is itself a closed convex cone.

Finally, note that the polar cone of $\mathcal{C} - x_{\#}$ coincides with the polar cone of $\operatorname{ccone}(\mathcal{C} - x_{\#})$ and with the polar cone of $\operatorname{cl}\operatorname{ccone}(\mathcal{C} - x_{\#})$; since our definition of the tangent cone to $\mathcal{C}$ at $x_{\#} \in \mathcal{C}$ is $\mathcal{T}_{\mathcal{C}}(x_{\#}) \stackrel{\text{def}}{=} \operatorname{cl}\operatorname{ccone}(\mathcal{C} - x_{\#})$ this means that we have

$$\begin{aligned}
\mathcal{N}_{\mathcal{S}}(x_{\#}) &= \operatorname{plrc}(\mathcal{C} - x_{\#}) \\
&= \operatorname{plrc}(\operatorname{ccone}(\mathcal{C} - x_{\#})) \\
&= \operatorname{plrc}(\operatorname{cl}\operatorname{ccone}(\mathcal{C} - x_{\#})) \\
&= \operatorname{plrc}\mathcal{T}_{\mathcal{C}}(x_{\#}).
\end{aligned}$$

Since both the tangent cone $\mathcal{T}_{\mathcal{C}}(x_{\#})$ and the normal cone $\mathcal{N}_{\mathcal{C}}(x_{\#})$ are closed convex cones, they are thus cones that are mutually polar.

The normal cone is sometimes described as the cone of supporting functionals [Van84].

Chapter 7

Functions associated with sets

7.1 Introduction

We now encounter our (essentially) initial instance of a "primal" representation of an object (such as a set or a function) versus a "dual" representation of that object. In the present chapter, the objects in question are sets; the "primal" representation of the set is in terms of the indicator function of the set, while the "dual" representation of the set is in terms of the support function (or, more explicitly, supporting hyperplane function) of the set. When the set in question is the epigraph of a function, we will subsequently see how the "dual" representation of the epigraph of a function relates to the notion of (Fenchel) conjugacy; other connections lead to the subdifferential set and to the normal cone of the epigraph at a specified point.

This material is standard; typical references include [BV04, RW04, Rus06, Ber09, HUL93a, HUL93b].

7.2 Indicator function of an arbitrary set

We now introduce the notion of an indicator function of an arbitrary set. We may think of the indicator function of a set as a means of representing a set by a function. More specifically, the indicator function of a set can be described as a "primal" functional representation of the set. This is in contrast to the support function introduced

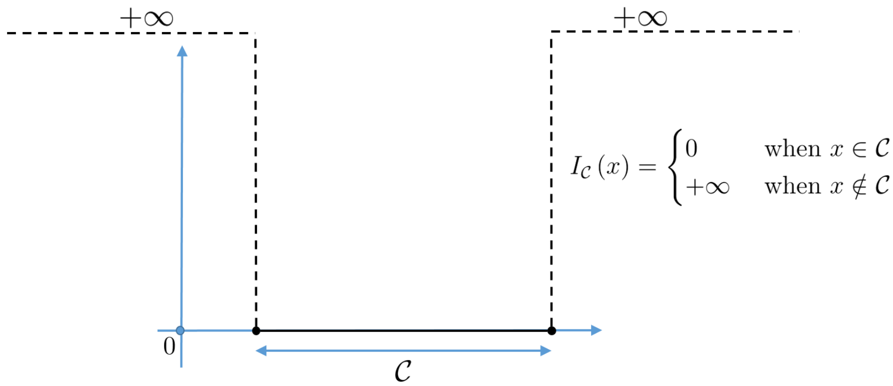

Figure 7.1: Indicator function of a convex set.

below, which we would analogously describe as a "dual" functional representation of the set. When the indicator function evaluates to the finite value 0, we know that the corresponding (primal) element argument is a member of the set. When the support function evaluates to a finite value, we know that the corresponding (dual) element argument is the "slope"/"normal vector" for some hyperplane that supports (the convex hull of) the set (and the specific finite value returned completes the description of that supporting hyperplane).

Definition 54 (Indicator function)**.** With any arbitrary nonempty set $\mathcal{S} \subseteq \mathbb{R}^n$ we associate an extended-real-valued function called the *indicator function* of the set $\mathcal{S}$. The indicator function of an arbitrary set $\mathcal{S}$ is denoted $I_{\mathcal{S}}(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and defined via the expression

$$I_{\mathcal{S}}(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{when } x \in \mathcal{S} \\ +\infty & \text{when } x \notin \mathcal{S}. \end{cases}$$

A typical indicator function is illustrated in Figure 7.1.

The epigraph of the indicator function consists of all half-lines starting at $(x,0) \in \mathcal{S} \times \{0\}$ and extending upward to $+\infty$. Explicitly, epi $I_{\mathcal{S}}(\cdot) = \mathcal{S} \times [0,+\infty)$. See Figure 7.2.

The indicator function of a nonempty set is a proper extended-real-valued function.

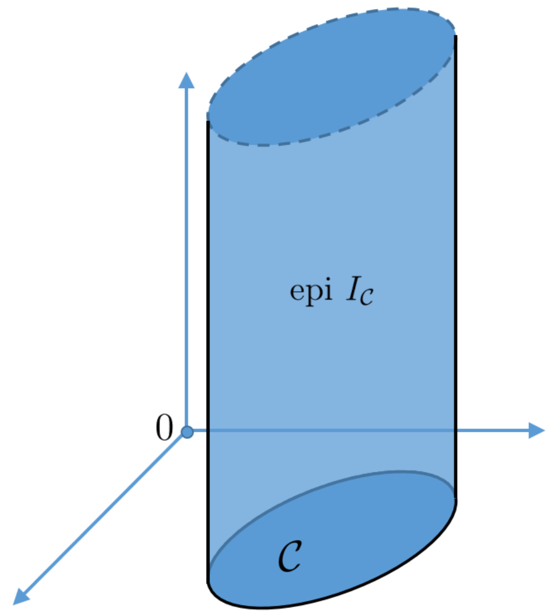

Figure 7.2: Epigraph of the indicator function of a convex set. (After [Luc06]).

The indicator function of a nonempty convex set is a proper convex extended-real-valued function.

The indicator function of a nonempty closed convex set is a closed proper convex extended-real-valued function.

We will later devote particular attention to the indicator function of a closed convex function and to the indicator function of a unit norm ball.

Alternative notations for the indicator function of a set $\mathcal{S}$ include $\delta_{\mathcal{S}}(\cdot)$ [Ber09], $\iota_{\mathcal{S}}(\cdot)$ [BC11], $\psi_{\mathcal{S}}(\cdot)$ [Aub98], and $\chi_{\mathcal{S}}(\cdot)$ [Sin06].

An alternative term for the indicator function of a set is the "characteristic function" of that set [Aub98].

7.3 Support function of an arbitrary set

We now come to the notion of a support function that we may associate with any arbitrary set; a more fully descriptive name for the support function might be the "supporting hyperplane function" (an even more fully descriptive name would be "a function by which we can specify a particular 'supporting' member of the family of hyperplanes with (shared) normal vector provided as the input argument").

As we discussed above, the support function associated with an arbitrary set is a means of representing a set (or more specifically, the closed convex hull of that set) in the form of a function. We have also mentioned that the support function is a "dual"-type function representation: we provide a dual vector s and the support function returns the scalar specifying a hyperplane with "normal" s that supports the set; if the set extends without bound in the direction corresponding to s, the support function returns $+\infty$.

In contrast to the "primal"-type function representation provided by the indicator function, we may view definition of the support function as containing "built-in" convexification of the set; one manifestation of this convexification is that the support function associated with a set coincides with the support function of the convex hull of the set, and also coincides with the support function associated with the closure of the convex hull of the set.

Definition 55 (Support function)**.** With any arbitrary nonempty set $\mathcal{S} \subseteq \mathbb{R}^n$ we associate an extended-real-valued function called the *support function* of the set $\mathcal{S}$. The support function of an arbitrary set $\mathcal{S}$ is denoted $\sigma_{\mathcal{S}}(\cdot) : \mathbb{R}^{n*} \to \mathbb{R} \cup \{+\infty\}$ and defined via the expression

$$\sigma_{\mathcal{S}}(s) \stackrel{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \ \{\langle s, x\rangle \mid x \in \mathcal{S}\}.$$

When $0 \in \mathcal{K}$, it will be the case that $\sigma_{\mathcal{K}}(\cdot) \geq 0$ for all arguments.

Alternative notations for the support function of a set $\mathcal{S}$ include $I^*_{\mathcal{S}}(\cdot)$ [Roc70], $h_{\mathcal{S}}(\cdot)$ [Lue69], $H_{\mathcal{S}}(\cdot)$ [Bar92] and $s_{\mathcal{S}}(\cdot)$.

7.4 Distance function associated with a convex set (and a generic norm)

As its name suggests, the distance function associated with a convex set measures how far the provided argument is from the convex set, as measured by some yardstick function (the 2-norm is the most typical yardstick function, but we will state the definition for a generic norm).

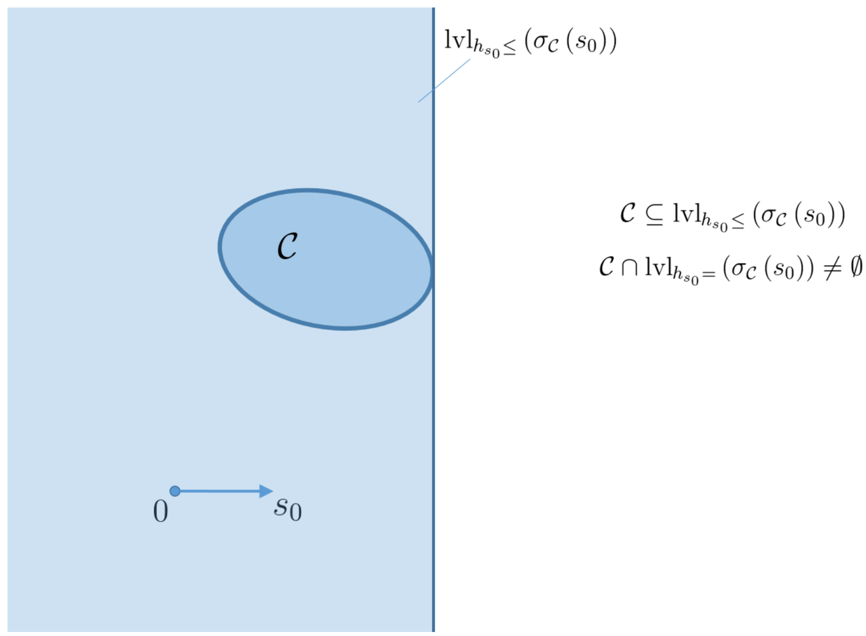

Figure 7.3: One evaluation (for the argument $s_0 \in \mathbb{R}^*$) of the support function $\sigma_{\mathcal{C}}(\cdot)$ of the nonempty closed convex set $\mathcal{C}$; note the corresponding halfspace containing $\mathcal{C}$.

Definition 56. Consider a nonempty convex set $\mathcal{C} \subset \mathbb{R}^n$. The $\|\cdot\|_\diamond$-norm *distance function* (using a generic norm $\|\cdot\|_\diamond : \mathbb{R}^n \to \mathbb{R}$ to measure distance to the set $\mathcal{C}$) is a real-valued function denoted $\mathrm{dist}_{\mathcal{C},\diamond}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and defined via the expression

$$\mathrm{dist}_{\mathcal{C},\diamond}(x_\#) \stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}} \left\{\|x_\# - x\|_\diamond \mid x \in \mathcal{C}\right\}.$$

7.5 Gauge function associated with a convex set containing the origin

Whereas a distance function begins with convex set and a yardstick (e.g. the 2-norm) and then uses that yardstick function to measure distance with respect to that set, the gauge function takes some convex set (which, for reasons discussed below, we require to contain the origin) and uses that set to define a yardstick. That is, we use the specified set as a means of "gauging" distance from the origin.

Definition 57. Consider a closed nonempty convex set $\mathcal{C} \subset \mathbb{R}^n$ containing the origin; that is, $0 \in \mathcal{C}$. The *gauge function* associated with any such set $\mathcal{C}$ is an extended-real-valued function denoted $\gamma_{\mathcal{C}}(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and defined via the expression

$$\gamma_{\mathcal{C}}(x) \stackrel{\text{def}}{=} \underset{\lambda \in \mathbb{R}_{++}}{\text{infimum}} \; \{\lambda \mid x \in \lambda \mathcal{C}\}.$$

In words: when called with the argument $x \in \mathbb{R}^n$, the gauge function associated with $\mathcal{C}$ returns the (infimum of the) amount of scaling needed so that x is contained in the scaled version of $\mathcal{C}$.

Gauge functions can be seen as a generalization of the concept of norm, in the sense that we can express any norm as a gauge function: using $\mathcal{B}_{\diamond} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \|x\|_{\diamond} \leq 1\}$ to denote the unit ball of the $\|\cdot\|_{\diamond}$-norm, we see that the $\|\cdot\|_{\diamond}$-norm can be expressed as the gauge function of the unit ball $\mathcal{B}_{\diamond}$: explicitly, $\|\cdot\|_{\diamond} = \gamma_{\mathcal{B}_{\diamond}}(\cdot)$.

For any closed nonempty convex set $\mathcal{C}$ the associated gauge function $\gamma_{\mathcal{C}}(\cdot)$ is convex. More specifically, $\gamma_{\mathcal{C}}(\cdot)$ is sublinear (nonnegatively homogeneous and subadditive).

Whenever $x \in \mathcal{C}$, the corresponding value of the gauge function does not exceed 1: $\gamma_{\mathcal{C}}(x) \leq 1$ if and only if $x \in \mathcal{C}$.

When the set $\mathcal{C}$ is absorbing [1] we can say more:

For $\mathcal{C}$ absorbing, whenever $x \in \operatorname{int} \mathcal{C}$, the corresponding value of the gauge function is strictly less than 1: $\gamma_{\mathcal{C}}(x) < 1$ if and only if $x \in \operatorname{int} \mathcal{C}$.

We also note that when $\mathcal{C}$ is absorbing, the associated gauge function $\gamma_{\mathcal{C}}(\cdot)$ is finite for all arguments.

Alternative terms for the gauge function include the Minkowski gauge, the Minkowski functional [Lue69], the Minkowski distance function [Lay82], the calibration function [IT68], and the distance function [Ben66].

Alternative notations for the gauge function include $p_{\mathcal{C}}(\cdot)$, $\rho_{\mathcal{C}}(\cdot)$, $m_{\mathcal{C}}(\cdot)$, and $\mu_{\mathcal{C}}(\cdot)$.

[1] $\mathcal{C}$ absorbing means that the entire space can be expressed as $\mathcal{X} = \bigcup_{\lambda \in \mathbb{R}_+} \lambda \mathcal{C}$

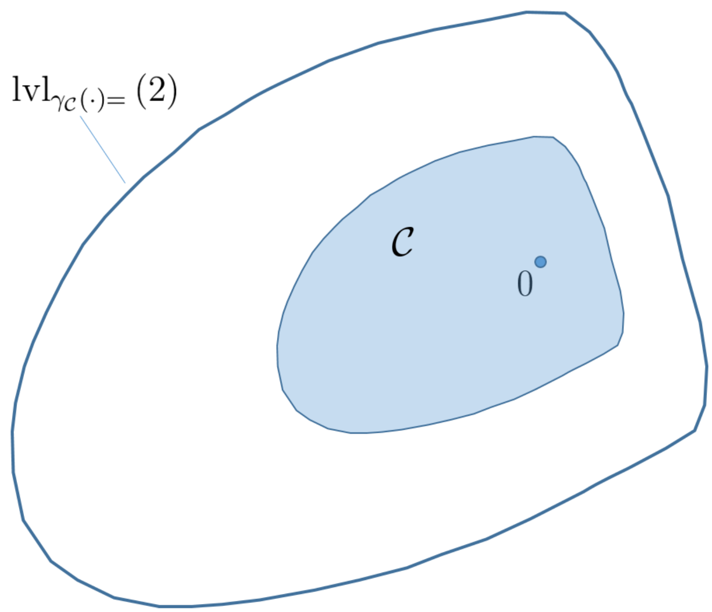

Figure 7.4: The 2-level set of the gauge function $\gamma_{\mathcal{C}}(\cdot)$ associated with the convex set $\mathcal{C}$ (containing the origin). (After [Lue69]).

Chapter 8

Conjugacy in convex analysis

8.1 Introduction

Rockafellar [Roc70] makes the following very relevant observation:

> There are two ways of viewing a classical curve or surface like a conic: the "primal" view, as the locus of points or the "dual" view, as an envelope of "tangents" (each tangent has a corresponding normal).
>
> When we consider the strictly convex case there is a one-to-one relationship between points and "tangents"; nonsmoothness will lead us to point-to-set mappings relating points and "tangents".

In Har-Peled [Har11], this relationship appears as duality between points and lines. In Vasin and Eremin [VE09], this relationship appears as duality between forms of representation of a (convex) set. In portions of the engineering literature, the correspondence between points (primal) and tangents (dual) is referred to as the slope transform (or, sometimes, as the maximum transform). In computer vision, this slope transform is also referred to as the Hough line transform. In classical mechanics, this transform corresponds to the Legendre transform (and the related notion of the fiber derivative) linking the Lagrangian mechanics perspective and the Hamiltonian mechanics perspective. In thermodynamics, this transform shifts the representation of the energy in terms of an "extensive" variable to a representation of the energy in terms of the (conjugate) "intensive" variable.

We have previously seen an instance of this duality in our two "complementary" methods for describing a set by means of some associated function: the "primal"-type representation using the indicator function and the “dual”-type representation using the support(ing hyperplane) function. We now consider “primal”-type (indicator function) and “dual”-type (support function) representations of a very specific type of set: sets corresponding to the epigraph of some function. When we consider our primal and dual representations of the epigraph of a function we will find a wide range results referred to collectively under the umbrella of “conjugacy”. We will initially establish results explicitly stated for the epigraph, but we will soon move to directly state our results in terms of the functions being considered. Finally, we have one additional aspect to watch for: just as we observed when we discussed sets in general, we will find that the “dual”-type representations here have convexity as part of their construction while “primal”-type representations are only convex when considered for convex functions.

This material is standard; typical references include [BV04, Rus06, Ber09]and especially [RW04, HUL93a, HUL93b].

8.2 Conjugate function definition via the support function of the epigraph of a function

First, consider an extended-real-valued function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$; recall the definition of the epigraph epi $f(\cdot) \subset \mathbb{R}^n \times \mathbb{R}$ of any such function:

$$\text{epi } f(\cdot) \stackrel{\text{def}}{=} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \geq f(x)\}.$$

Next, we state (in a slightly more convenient form) the definition of the support function $\sigma_{\mathcal{S}}(\cdot) : \mathbb{R}^{n*} \times \mathbb{R}^* \to \mathbb{R} \cup \{+\infty\}$ associated with an arbitrary set $\mathcal{S} \subset \mathbb{R}^n \times \mathbb{R}$:

$$\sigma_{\mathcal{S}}\left(\begin{bmatrix} v \\ u \end{bmatrix}\right) \stackrel{\text{def}}{=} \underset{\begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}}{\text{supremum}} \left\{ \left\langle \begin{bmatrix} v \\ u \end{bmatrix}, \begin{bmatrix} x \\ t \end{bmatrix} \right\rangle \mid \begin{bmatrix} x \\ t \end{bmatrix} \in \mathcal{S} \right\}.$$

In order to connect our support function characterization to the traditional definition of conjugacy, we must make an additional observation: specifically, recall that the support function is nonnegatively homogeneous; explicitly, $\sigma_{\mathcal{S}}\left(\alpha \begin{bmatrix} v_\# \\ u_\# \end{bmatrix}\right)$ is equal to

$\alpha\sigma_{\mathcal{S}}\left(\begin{bmatrix} v_\# \\ u_\# \end{bmatrix}\right)$ for any nonnegative scaling $\alpha \in \mathbb{R}_+$. For our current purposes, the significance of this statement lies in observation that, so long as $\alpha \in \mathbb{R}_{++}$ the hyperplane with normal vector $\begin{bmatrix} v \\ u \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} v_\# \\ u_\# \end{bmatrix}$ and scalar term $\sigma_{\mathcal{S}}\left(\begin{bmatrix} v_\# \\ u_\# \end{bmatrix}\right)$ exactly coincides with the hyperplane with normal vector $\begin{bmatrix} v \\ u \end{bmatrix} \overset{\text{set}}{=} \alpha \begin{bmatrix} v_\# \\ u_\# \end{bmatrix}$ and scalar term $\alpha\sigma_{\mathcal{S}}\left(\begin{bmatrix} v_\# \\ u_\# \end{bmatrix}\right)$; explicitly, we are stating the equality of the following two sets:

$$\left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R} \mid \left\langle \begin{bmatrix} v_\# \\ u_\# \end{bmatrix}, \begin{bmatrix} x \\ t \end{bmatrix} \right\rangle = \sigma_{\mathcal{S}}\left(\begin{bmatrix} v_\# \\ u_\# \end{bmatrix}\right) \right\}$$

is equal to

$$\left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R} \mid \left\langle \alpha \begin{bmatrix} v_\# \\ u_\# \end{bmatrix}, \begin{bmatrix} x \\ t \end{bmatrix} \right\rangle = \alpha\sigma_{\mathcal{S}}\left(\begin{bmatrix} v_\# \\ u_\# \end{bmatrix}\right) \right\}.$$

So long as the normal vector in question does not correspond to a "vertical" hyperplane, the scalar portion, the $u \in \mathbb{R}$ in $\begin{bmatrix} v \\ u \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}$, will be nonzero; thus, for any nonvertical hyperplane we can choose to consider the specific scaled normal vector for which the scalar portion equals -1. With this scaling argument in mind, we consider the "scaled" support function $\sigma_{\text{epi } f(\cdot)}(\cdot) \ : \ \mathbb{R}^{n*} \times \mathbb{R}^* \to \mathbb{R} \cup \{+\infty\}$ of the epigraph of the extended-real-valued function $f(\cdot)$ to be

$$\begin{aligned}
\sigma_{\operatorname{epi} f(\cdot)}\left(\begin{bmatrix} s \\ -1 \end{bmatrix}\right) &\stackrel{\text{def}}{=} \underset{\begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}}{\text{supremum}} \left\{ \left\langle \begin{bmatrix} s \\ -1 \end{bmatrix}, \begin{bmatrix} x \\ t \end{bmatrix} \right\rangle \mid \begin{bmatrix} x \\ t \end{bmatrix} \in \operatorname{epi} f(\cdot) \right\} \\
&= \underset{\begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}}{\text{supremum}} \left\{ \left\langle \begin{bmatrix} s \\ -1 \end{bmatrix}, \begin{bmatrix} x \\ t \end{bmatrix} \right\rangle \mid t \geq f(x) \right\} \\
&= \underset{\begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}}{\text{supremum}} \left\{ \langle s, x \rangle - t \mid t \geq f(x) \right\} \\
&= \underset{\begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}}{\text{supremum}} \left\{ \langle s, x \rangle - t \mid -t \leq -f(x) \right\} \\
&= \underset{x \in \mathbb{R}^n.}{\text{supremum}} \left\{ \langle s, x \rangle - f(x) \right\}.
\end{aligned}$$

This last expression is the standard definition for value of the conjugate function evaluated for argument $s \in \mathbb{R}^n$.

We will next consider an argument emphasizing supporting affine minorants that turns out to coincide (although in different language) with the support function argument described above.

8.3 Conjugate function definition via supporting affine minorants

From the locus-of-points versus envelope-of-tangents view described in the introduction, our process of establishing a connection between a "primal" locus-of-points view of a function and a "dual" envelope-of-tangents view of the same function could be expected to begin from either a point or a slope. When the function in question is smooth, starting from some point in the graph of the function is a natural approach (pursued via the derivative). However, when we consider nonsmooth functions, the standard

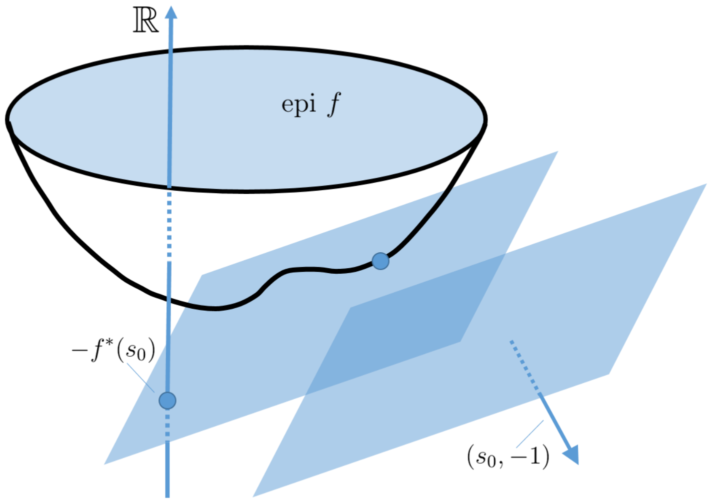

Figure 8.1: Relating the conjugate function of $f(\cdot)$ to the support function of the epigraph of $f(\cdot)$. (After [HUL93b]).

definition of the derivative breaks down and we are instead motivated to consider an approach that begins by considering a possible "slope" that could be possessed by some affine minorant to the function.

More specifically, our process begins by considering a possible slope s and then asking: does there exist any affine minorant (with slope $s \in \mathbb{R}^{n*}$) to the function $f(\cdot)$? For the absolute value function, the answer would be "no" if we asked whether there was an affine minorant with slope $s \overset{\text{set}}{=} 2$, (or any slope in the range $(-\infty, 1) \cup (1, +\infty)$) but "yes" if we asked whether there was an affine minorant with slope $s \overset{\text{set}}{=} 1$ (or any slope in the range $[-1, 1]$).

If there *does* exist some slope-s affine minorant to $f(\cdot)$, we proceed by asking "what is the maximum value of the associated bias term b such that the affine function $\ell_{s,b}(x) = \langle s, x \rangle + b$ is still an affine minorant"? Suppose that $b_{\max}$ is that maximum bias. Then the affine minorant $\ell_{s,b_{\max}}(\cdot)$ will support epi $f(\cdot)$ at one or more points (otherwise we could increase the bias until the minorant did support the epigraph). Call some such point of support $(x_@, f(x_@))$. If $f(\cdot)$ is differentiable at $x_@$, $f(\cdot)$ will have slope s at $x_@$. We illustrate supporting affine minorants in Figure 8.2.

This process is encapsulated in the following questions:

1. For which "slope" vectors $s \in \mathbb{R}^{n*}$ do affine minorants of the form $\ell_{s,b}(x) =$

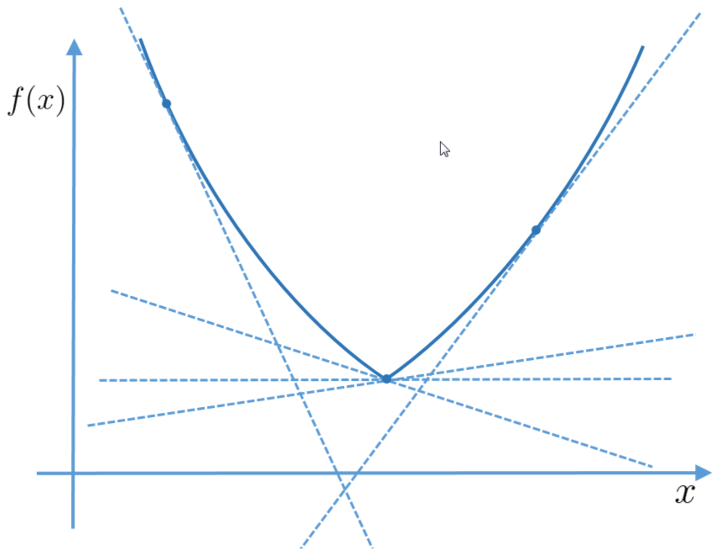

Figure 8.2: Illustrating the supporting affine minorant characterization of convexity.

$\langle s,x\rangle + b$ exist?

2. Out of all slope-s affine minorants to $f(\cdot)$, is there finite supremum of the associated bias values b?

3. Is it possible to represent $f(\cdot)$ as a pointwise supremum of all possible affine minorants, that is, over all possible "slope" vectors $s \in \mathbb{R}^{n*}$?

This last point is precisely the "dual" envelope-of-tangents perspective.

We will see how the process considered above leads us to the notion of a conjugate function. We will begin with the standard definition and then see how this definition corresponds to the questions we have asked above.

Definition 58. *Conjugate function.* Consider a proper extended-real-valued function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Associated with such any such function we have the conjugate function, denoted $f^*(\cdot) : \mathbb{R}^{n*} \to \mathbb{R} \cup \{+\infty\}$ and defined via the expression

$$f^*(s) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \{\langle s,x\rangle - f(x)\}.$$

At the moment, the connection between the process described above and the definition of the conjugate function may be somewhat unclear. However, we can see how our the the process of determining a supporting affine minorant leads to this definition as follows:

The problem we are motivated by is

$$\begin{aligned}&\underset{b\in\mathbb{R}}{\text{supremize}}\ b\\&\text{subject to }\ \langle s,x\rangle+b\leq f(x)\ \text{ for all } x\in\mathbb{R}^n.\end{aligned}$$

Or, to more explicitly emphasize the “affine minorant” aspect of our process

$$\begin{aligned}&\underset{b\in\mathbb{R}}{\text{supremize}}\ b\\&\text{subject to }\ \left\{\ell_{s,b}(\cdot)\overset{\text{set}}{=}\langle s,\cdot\rangle+b\right\}\ \text{minorize } f(\cdot)\,.\end{aligned}$$

If no affine minorant with slope s exists for the function $f(\cdot)$, then the supremal value $b^\star$ will equal$-\infty$; this possible outcome is in fact the reason that we have written “supremize” rather than “maximize”.

If an affine minorant with slope s for the function $f(\cdot)$ does exist, then the supremal value $b^\star$ will be finite; we then define the value of the conjugate function $f^*(\cdot)$ at argument s as $f^*(s)\overset{\text{set}}{=}-b^\star$.

To get from the motivating problem to the standard definition of the conjugate function, we reformulate the problem with variable $a\overset{\text{set}}{=}-b$.

$$\begin{aligned}&\underset{a\in\mathbb{R}}{\text{supremize}}\ -a\\&\text{subject to }\ \langle s,x\rangle-a\leq f(x)\ \text{ for all } x\in\mathbb{R}^n,\end{aligned}$$

which leads to

$$\begin{aligned}&\underset{a\in\mathbb{R}}{\text{infimize}}\ a\\&\text{subject to }\ \langle s,x\rangle-a\leq f(x)\ \text{ for all } x\in\mathbb{R}^n,\end{aligned}$$

and thence to

$$\begin{aligned}&\underset{a\in\mathbb{R}}{\text{infimize}}\ a\\&\text{subject to }\ \langle s,x\rangle-f(x)\leq a \text{ for all } x\in\mathbb{R}^n.\end{aligned}$$

Note that in the alternative formulation immediately above, if no affine minorant with slope s exists for the function $f(\cdot)$, then the infimal value $a^{\star}$will equal $+\infty$; this means that our definition of conjugate will lead to a potentially extended-real-valued function.

This last formulation can now be connected to the conventional definition of the conjugate. In order for a to be feasible, we must satisfy $\langle s,x\rangle - f(x) \leq a$ for all $x \in \mathbb{R}^n$; this condition comes directly from the requirement that $\ell_{s,-a}(\cdot) \stackrel{\text{set}}{=} \langle s,\cdot\rangle - a$ minorize $f(\cdot)$. An alternative means of finding the smallest a that is still feasible, we can instead ask "What is the largest value of $\langle s,x\rangle - f(x)$ over all $x \in \mathbb{R}^n$? That value will coincide with the smallest feasible value for a." This is precisely the conventional definition of the value of the conjugate $f^*(\cdot)$ for the argument s:

$$f^*(s) \stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - f(x)\}\,.$$

This process is somewhat roundabout, but does lead to a conjugate function that is a convex function in the "slope" argument $s \in \mathbb{R}^{n*}$. To see that this is the case, observe that $f^*(\cdot)$ is defined as a pointwise supremum of convex functions — more precisely, as a pointwise supremum of the affine-in-s functions $\langle s,x\rangle - f(x)$. Moreover, the argument establishes that the conjugate function $f^*(\cdot)$ is a convex function of s whether or not $f(\cdot)$ is a convex function of x, since $\langle s,x\rangle - f(x)$ is affine in s whether or not $f(\cdot)$ is a convex function in x. This is an instance of the "convexity by construction" that we observed for "dual"-type representations.

In Figure (8.3) we can see dom $f^*(\cdot)$, the set of slopes $s \in \mathbb{R}^{n*}$ for which $f^*(s)$ is finite. Note that dom $f^*(\cdot)$ is determined by the behavior of $f(\cdot)$ "at infinity" (or "at the boundary of dom $f(\cdot)$", for the case of "barrier" functions as used in interior point methods).

Informally, dom $f^*(\cdot)$ is the answer to the question "which slopes occur for $f(\cdot)$?" — that is, dom $f^*(\cdot)$ consists of the slopes $s \in \mathbb{R}^{n*}$ for which the supremum $\langle s,x\rangle - f(x)$ is finite. Another restatement would be dom $f^*(\cdot)$ consists of the slopes $s \in \mathbb{R}^{n*}$ for which the difference $\langle s,x\rangle - f(x)$ is bounded above over all $x \in \mathbb{R}^n$. Slopes that "don't occur" lead to $f^*(s) = +\infty$; for slopes that "do occur", consider $x_@ \in \text{dom}\, f$ and any associated subgradient $s_@ \in \partial f(x_@)$. The corresponding affine minorant $f(x_@) + \langle s_@, x - x_@\rangle$ supports epi $f(\cdot)$ at $(x_@, f(x_@))$, leading to the observation that $f^*(s_@) =$

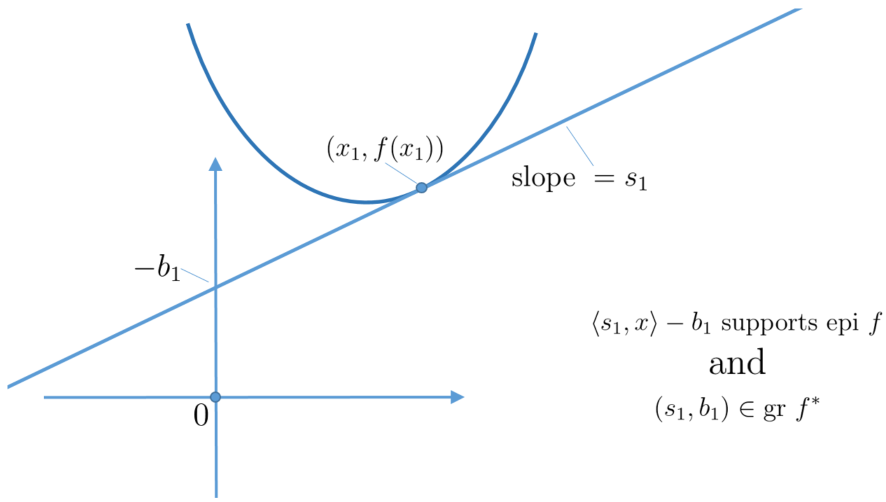

Figure 8.3: One evaluation of a conjugate function.

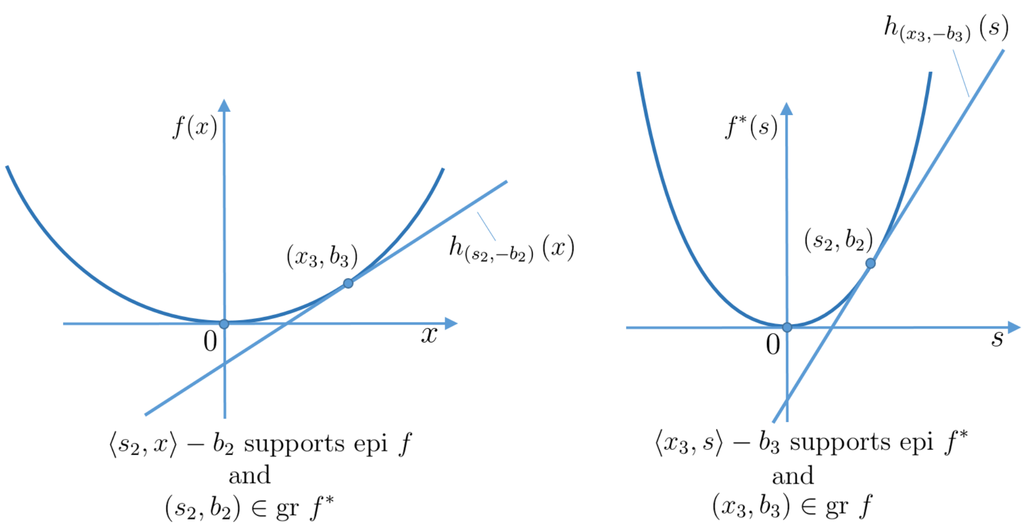

Figure 8.4: Correspondence between a function and its conjugate by repeated point-line conjugacy. (After [RW04]).

$\langle s_@, x_@\rangle - f(x_@)$ for the slope $s_@$. This follows because $s_@ \in \partial f(x_@)$ tells us that there exists a slope-$s_@$ affine minorant for $f(\cdot)$. Thus, we can go from the supremum form $f^*(s_@) \stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \{\langle s_@, x\rangle - f(x)\}$ to the maximum form $f^*(s_@) \stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{maximum}}$ $\{\langle s_@, x\rangle - f(x)\}$. Continuing, we observe that $x_@ = \underset{x\in\mathbb{R}^n}{\text{argmax}} \{\langle s_@, x\rangle - f(x)\}$ so that we can plug in this attaining argument to find $f^*(s_@) \stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{maximum}} \{\langle s_@, x\rangle - f(x)\} = \langle s_@, x_@\rangle - f(x_@)$ as previously claimed.

Remark. For finite, coercive, convex, twice continuously differentiable functions $f(\cdot)$ with Hessian everywhere positive definite, the conjugacy operation defined above coincides with the "classical" Legendre transform [RW04]; however, the conjugacy operation above remains valid even in settings where the requirements for the classical Legendre transform definition do not hold. The conjugacy operation is called the Young transform in [IT68]. The conjugacy operation (although with a focus on concavity instead of convexity) is called the maximum transform in [BK61, BK62]. A closely related idea in mathematical morphology is the slope transform [DV94, Mar95]. The conjugate function is called the polar function in [Mor67, ET99]; other common descriptions for the conjugate function include the Fenchel conjugate, the Legendre-Fenchel conjugate, and the dual function. In some references, e.g. [BL06, Sin06], a (closely-related) definition is given for a concave conjugate function (differing essentially in replacing supremum by infimum); in such settings the definition above is distinguished by being called the convex conjugate function. Luenberger [Lue69] uses the term conjugate concave function instead of concave conjugate function.

In some application-oriented references the conjugate of $f(\cdot)$ is denoted by $f^\star(\cdot)$ instead of $f^*(\cdot)$; we prefer to keep a superscript "$*$" as an indication of conjugacy/duality and to use a superscript "$\star$" as an indication of argument optimality in optimization problems.

8.4 Basic Properties of the Conjugate

8.4.1 Fenchel-Young inequality

Note that the definition $f^*(s) \stackrel{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \{\langle s, x \rangle - f(x)\}$ immediately yields the *Fenchel-Young inequality*

$$f(x) + f^*(s) \geq \langle s, x \rangle,$$

for all $x \in \mathbb{R}^n$, $s \in \mathbb{R}^{n*}$.

This is because

$$f^*(s) \stackrel{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \{\langle s, x \rangle - f(x)\},$$

which by the definition of supremum means that

$$f^*(s) \geq \langle s, x \rangle - f(x) \text{ for all } x \in \mathbb{R}^n,\ s \in \mathbb{R}^{n*}.$$

We then get the Fenchel-Young inequality by adding $f(x)$ to both sides:

$$f^*(s) + f(x) \geq \langle s, x \rangle \text{ for all } x \in \mathbb{R}^n,\ s \in \mathbb{R}^{n*}.$$

We illustrate examples of strict inequality in Figure 8.5; the case of equality is found in Figure 8.6.

8.4.2 The case of equality in the Fenchel-Young inequality

We first recall the definitions of subgradient and subdifferential.

Definition 59. For a closed proper convex extended-real-valued function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ we say that $s \in \mathbb{R}^{n*}$ is a *subgradient* of $f(\cdot)$ at the point $x_\# \in \mathbb{R}^n$ when

$$f(x) \geq f(x_\#) + \langle s, x - x_\# \rangle \text{ for all } x \in \mathbb{R}^n.$$

The subdifferential is the collection of all subgradients:

Definition 60. For a closed proper convex extended-real-valued function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ the collection of all subgradients of $f(\cdot)$ at the point $x_\# \in \mathbb{R}^n$ is a set in

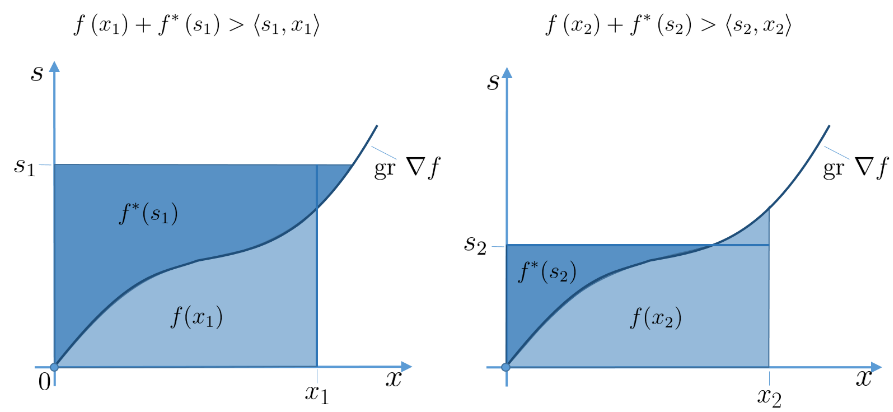

Figure 8.5: Connecting a function, its subdifferential, the subdifferential of the conjugate, the conjugate, and the Fenchel-Young inequality: examples of strict inequality. (After [AN09]).

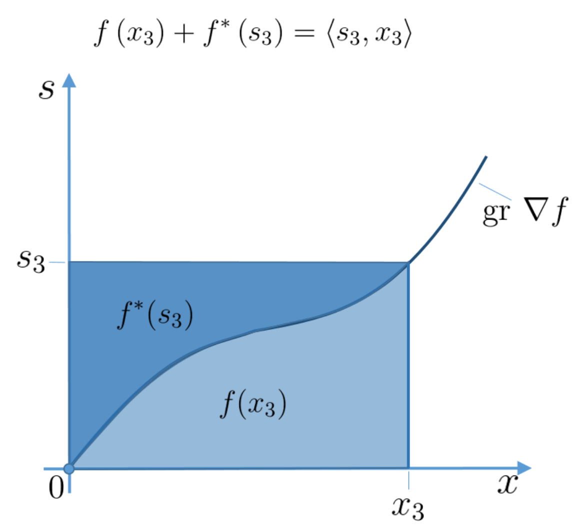

Figure 8.6: Connecting a function, its subdifferential, the subdifferential of the conjugate, the conjugate, and the Fenchel-Young inequality: the case of equality.

$\mathbb{R}^{n*}$ called the *subdifferential* of $f(\cdot)$ at $x_\#$, denoted $\partial f(x_\#) \subset \mathbb{R}^{n*}$ and defined via the expression

$$\partial f(x_\#) \stackrel{\text{def}}{=} \{s \in \mathbb{R}^{n*} \mid f(x) \geq f(x_\#) + \langle s, x - x_\# \rangle \text{ for all } x \in \mathbb{R}^n\}.$$

The language remains the same when we consider the conjugate function:

Definition 61. For a closed proper convex extended-real-valued function $f^*(\cdot) : \mathbb{R}^{n*} \to \mathbb{R} \cup \{+\infty\}$ we say that $x \in \mathbb{R}^n$ is a *subgradient* of $f^*(\cdot)$ at the point $s_\$ \in \mathbb{R}^{n*}$ when

$$f^*(s) \geq f^*(s_\$) + \langle s - s_\$, x \rangle \text{ for all } s \in \mathbb{R}^{n*}.$$

and

Definition 62. For a closed proper convex extended-real-valued function $f^*(\cdot) : \mathbb{R}^{n*} \to \mathbb{R} \cup \{+\infty\}$ the collection of all subgradients of $f^*(\cdot)$ at the point $s_\$ \in \mathbb{R}^{n*}$ is a set in $\mathbb{R}^n$ called the *subdifferential* of $f^*(\cdot)$ at s^*, denoted $\partial f^*(s^*) \subset \mathbb{R}^n$ and defined via the expression

$$\partial f^*(s_\$) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid f^*(s) \geq f^*(s_\$) + \langle s - s_\$, x \rangle \text{ for all } s \in \mathbb{R}^{n*}\}.$$

For a closed proper convex function extended-real-valued function $f(\cdot)$, the case of equality in the Fenchel-Young inequality corresponds to a characterization of membership in the subdifferential:

Proposition 3 (Subdifferentials and conjugacy)**.** *Consider a closed proper convex extended real valued function* $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. *For any such a function we have the following equivalences:*

$$[s \in \partial f(x)] \Longleftrightarrow [f(x) + f^*(s) = \langle s, x \rangle] \Longleftrightarrow [x \in \partial f^*(s)].$$

Proof. We first demonstrate that $[s \in \partial f(x)] \Longrightarrow [f(x) + f^*(s) = \langle s, x \rangle]$. Suppose that $s_\$ \in \mathbb{R}^{n*}$ and $x_\# \in \mathbb{R}^n$ are such that $s_\$ \in \partial f(x_\#)$. From the definition of $s_\$ \in \partial f(x_\#)$ we have

$$f(x) \geq f(x_\#) + \langle s_\$, x - x_\# \rangle \text{ for all } x \in \mathbb{R}^n.$$

We observe that

$$
\begin{aligned}
f(x) &\geq f(x_{\#}) + \langle s_{\$}, x - x_{\#}\rangle \text{ for all } x \in \mathbb{R}^n \\
f(x) &\geq f(x_{\#}) + \langle s_{\$}, x\rangle - \langle s_{\$}, x_{\#}\rangle \text{ for all } x \in \mathbb{R}^n \\
\langle s_{\$}, x_{\#}\rangle - f(x_{\#}) &\geq \langle s_{\$}, x\rangle - f(x) \text{ for all } x \in \mathbb{R}^n.
\end{aligned}
$$

The preceding statement $\langle s_{\$}, x_{\#}\rangle - f(x_{\#}) \geq \langle s_{\$}, x\rangle - f(x)$ for all $x \in \mathbb{R}^n$ means that $\langle s_{\$}, x_{\#}\rangle - f(x_{\#})$ is an upper bound for $\langle s_{\$}, x\rangle - f(x)$ over $x \in \mathbb{R}^n$; we can see that $\langle s_{\$}, x_{\#}\rangle - f(x_{\#})$ is in fact the least upper bound because $\langle s_{\$}, x\rangle - f(x)$ attains the upper bound for the choice $x \overset{\text{set}}{=} x_{\#}$. Our conclusion that $\langle s_{\$}, x_{\#}\rangle - f(x_{\#})$ is the least upper bound for $\langle s_{\$}, x\rangle - f(x)$ over $x \in \mathbb{R}^n$ can be explicitly written

$$
\langle s_{\$}, x_{\#}\rangle - f(x_{\#}) = \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{\langle s_{\$}, x\rangle - f(x)\right\}.
$$

Recognizing that $\underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{\langle s_{\$}, x\rangle - f(x)\right\}$ is the definition of $f^*(s_{\$})$, we have thus established that

$$
[s_{\$} \in \partial f(x_{\#})] \Longrightarrow [\langle s_{\$}, x_{\#}\rangle - f(x_{\#}) = f^*(s_{\$})]
$$

which is only a slight re-expression of our desired goal.

We now demonstrate that $[f(x) + f^*(s) = \langle s, x\rangle] \Longrightarrow [s \in \partial f(x)]$. Suppose that $s_{\$} \in \mathbb{R}^{n*}$ and $x_{\#} \in \mathbb{R}^n$ are such that $f(x_{\#}) + f^*(s_{\$}) = \langle s_{\$}, x_{\#}\rangle$; that is, such that $\langle s_{\$}, x_{\#}\rangle - f(x_{\#}) = f^*(s_{\$})$. Recalling again that $f^*(s_{\$})$ is defined as $\underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{\langle s_{\$}, x\rangle - f(x)\right\}$ we have

$$
\begin{aligned}
f^*(s_{\$}) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{\langle s_{\$}, x\rangle - f(x)\right\} \\
f^*(s_{\$}) &\geq \langle s_{\$}, x\rangle - f(x) \text{ for all } x \in \mathbb{R}^n.
\end{aligned}
$$

We have assumed that $s_{\$} \in \mathbb{R}^{n*}$ and $x_{\#} \in \mathbb{R}^n$ are such that $\langle s_{\$}, x_{\#}\rangle - f(x_{\#}) = f^*(s_{\$})$ and thus we can continue from the preceding inequality to find

$$
\begin{aligned}
f^*(s_{\$}) &\geq \langle s_{\$}, x\rangle - f(x) \text{ for all } x \in \mathbb{R}^n \\
\langle s_{\$}, x_{\#}\rangle - f(x_{\#}) = f^*(s_{\$}) &\geq \langle s_{\$}, x\rangle - f(x) \text{ for all } x \in \mathbb{R}^n \\
\langle s_{\$}, x_{\#}\rangle - f(x_{\#}) &\geq \langle s_{\$}, x\rangle - f(x) \text{ for all } x \in \mathbb{R}^n \\
f(x) &\geq f(x_{\#}) + \langle s_{\$}, x - x_{\#}\rangle \text{ for all } x \in \mathbb{R}^n.
\end{aligned}
$$

We recognize the final inequality as the definition of $s_\$ \in \partial f(x_\#)$. Thus, we have established that $[\langle s_\$, x_\# \rangle - f(x_\#) = f^*(s_\$)] \Longrightarrow [s_\$ \in \partial f(x_\#)]$. In summary, we have shown the equivalence $[s_\$ \in \partial f(x_\#)] \Longleftrightarrow [f(x_\#) + f^*(s_\$) = \langle s_\$, x_\# \rangle]$.

We next show that, by virtue of our assumption that $f(\cdot)$ is a *closed* proper convex extended-real-valued function, we can establish an additional equivalence, for the final claimed result $[s_\$ \in \partial f(x_\#)] \Longleftrightarrow [f(x_\#) + f^*(s_\$) = \langle s_\$, x_\# \rangle] \Longleftrightarrow [x_\# \in \partial f^*(s_\$)]$.

Introduce the notation $g(\cdot) \overset{\text{set}}{=} f^*(\cdot)$. Reasoning identical to that above demonstrates that $[x_\# \in \partial g(s_\$)] \Longleftrightarrow [g^*(x_\#) + g(s_\$) = \langle s_\$, x_\# \rangle]$. From our assumption that $f(\cdot)$ is a *closed* proper convex extended-real-valued function we have that $f(\cdot) = f^{**}(\cdot)$; expressed in the notation $g(\cdot) \overset{\text{set}}{=} f^*(\cdot)$ this means $f(\cdot) = f^{**}(\cdot) = g^*(\cdot)$. The fact that $g^*(\cdot) = f(\cdot)$ means that we have

$$[x_\# \in \partial g(s_\$)] \Longleftrightarrow [g^*(x_\#) + g(s_\$) = \langle s_\$, x_\# \rangle]$$
$$[x_\# \in \partial f^*(s_\$)] \Longleftrightarrow [f^{**}(x_\#) + f^*(s_\$) = \langle s_\$, x_\# \rangle]$$
$$[x_\# \in \partial f^*(s_\$)] \Longleftrightarrow [f(x_\#) + f^*(s_\$) = \langle s_\$, x_\# \rangle].$$

Combined with the previous equivalence, we have established the desired result: $[s_\$ \in \partial f(x_\#)] \Longleftrightarrow [f(x_\#) + f^*(s_\$) = \langle s_\$, x_\# \rangle] \Longleftrightarrow [x_\# \in \partial f^*(s_\$)]$. □

The implication of

$$[f(x_\#) + f^*(s_\$) = \langle s_\$, x_\# \rangle] \Longleftrightarrow [s_\$ \in \partial f(x_\#)]$$

is that the pairs $(x, s) \in \mathbb{R}^n \times \mathbb{R}^{n*}$ for which we have equality in the Fenchel-Young inequality correspond precisely to gr $\partial f(\cdot)$, the graph of the subdifferential of $f(\cdot)$.

The implication of

$$[(x, s) \in \partial f] \Longleftrightarrow [(s, x) \in \partial f^*]$$

is that we can form gr ∂f^*, the graph of the subdifferential of the conjugate function $f^*(\cdot)$, from gr ∂f, the graph of the subdifferential of the primal function $f(\cdot)$; more specifically, every pair $(x, s) \in \partial f$ corresponds to a pair $(s, x) \in \partial f^*$.

We illustrate the result

$$[(x, s) \in \partial f] \Longleftrightarrow [(s, x) \in \partial f^*]$$

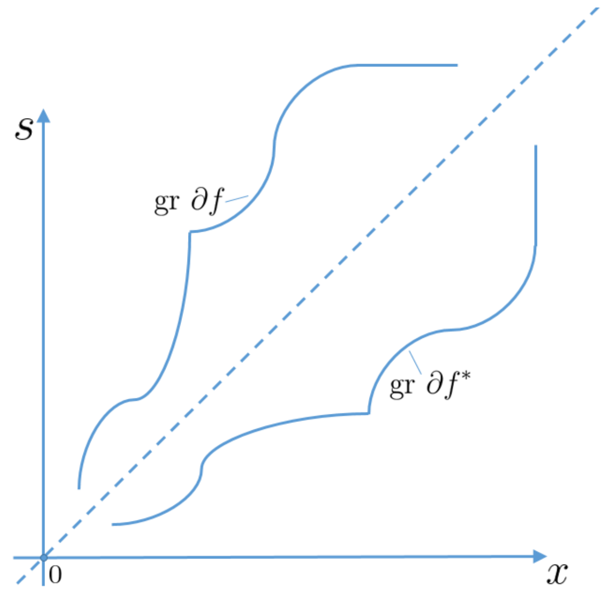

Figure 8.7: Illustrating $[(x,s) \in \partial f] \Longleftrightarrow [(s,x) \in \partial f^*]$. (After [HUL93a]).

in Figure 8.7.

The symmetry present in the expression $[(x,s) \in \partial f] \Longleftrightarrow [(s,x) \in \partial f^*]$ is sometimes written as $\partial f^* = [\partial f]^{-1}$. This "inverse" notation emphasizes that we go from ∂f to ∂f^* by exchanging x and s; examples of this "inversion"/"argument flipping" process can be seen in the relationship between a function $f(\cdot)$ and the corresponding conjugate function $f^*(\cdot)$ illustrated in Figures 8.8 and 8.9. This is also present in the earlier illustration of the Fenchel-Young inequality 8.5.

8.4.3 Basic transformation results for the conjugate

We collect for later use a number of results describing how simple operations applied to a function of interest are reflected in the conjugate of that function [HUL93b].

Proposition 4 (Basic results on the conjugate)**.** *The functions $f(\cdot)$ and the $f_j(\cdot)$ appearing below are assumed to be closed proper convex extended-real-valued functions*

(1) The conjugate of the function $g(x) \overset{set}{=} f(x) + \alpha$ is $g^*(s) = f^*(s) - \alpha$.

(2) For any $\alpha \in \mathbb{R}_{++}$, the conjugate of the function $g(x) \overset{set}{=} \alpha f(x)$ is $g^*(s) = \alpha f^*\left(\frac{1}{\alpha}s\right)$.

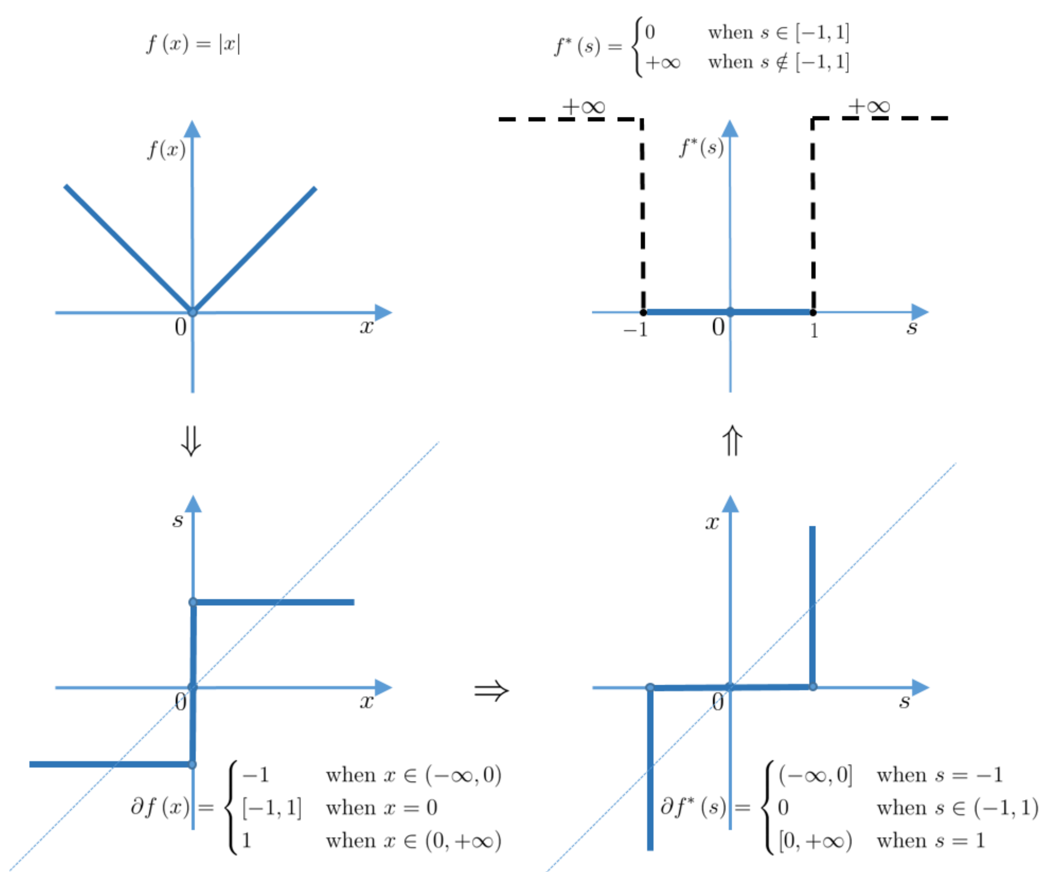

Figure 8.8: Function to subdifferential to subdifferential of conjugate to conjugate: $f(x) \overset{\text{set}}{=} |x|$. (After [Luc06]).

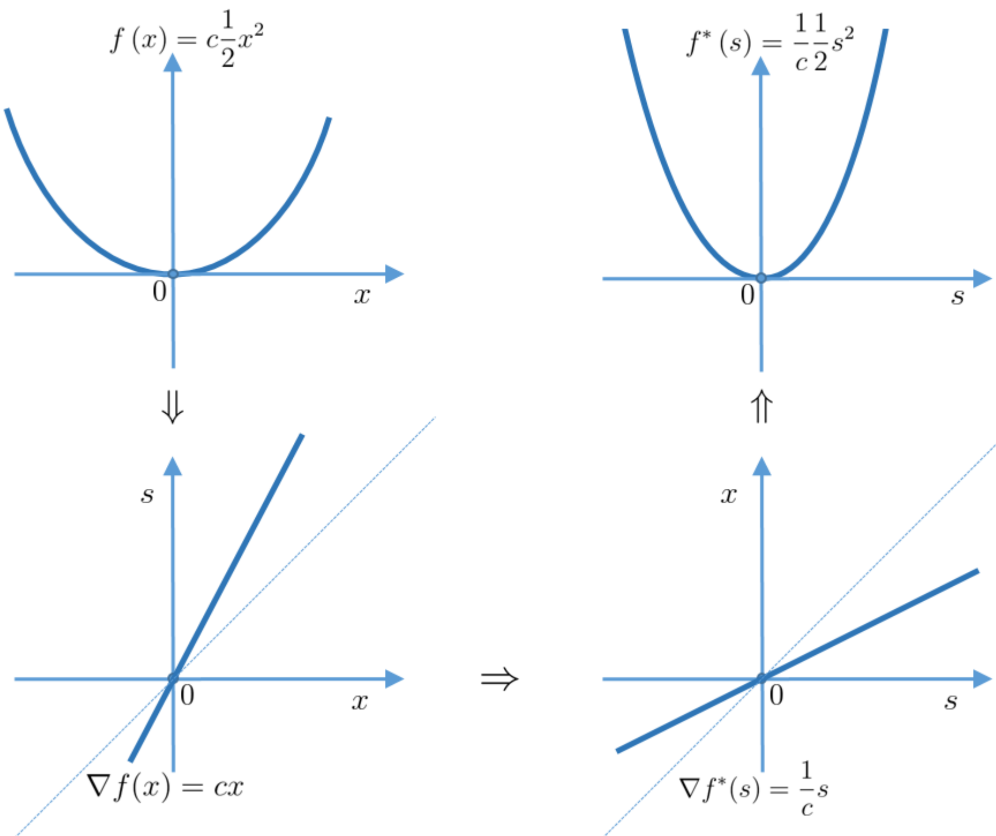

Figure 8.9: Function to subdifferential to subdifferential of conjugate to conjugate: $f(x) \stackrel{\text{set}}{=} c\frac{1}{2}x^2$. (After [Luc06]).

(3) For any nonzero scalar $\alpha \neq 0$, *the conjugate of the function* $g(x) \overset{set}{=} f(\alpha x)$ *is* $g^*(s) = f^*\left(\frac{1}{\alpha}s\right)$.

(4) A generalization of the above result is: if A is an invertible linear operator, $(f \circ A)^*(\cdot) = f^* \circ \left(A^{-1}\right)^*(\cdot)$.

(5) The conjugate of the function $g(x) \overset{set}{=} f(x - x_0)$ *is* $g^*(s) = f^*(s) + \langle s, x_0 \rangle$.

(6) The conjugate of the function $g(x) \overset{set}{=} f(x) + \langle s_0, x \rangle$ *is* $g^*(s) = f^*(s - s_0)$.

(7) If $f_1(\cdot) \leq f_2(\cdot)$ *then* $f_1^*(\cdot) \geq f_2^*(\cdot)$.

(8) "Convexity" of the conjugation: If $\operatorname{dom} f_1(\cdot) \cap \operatorname{dom} f_2(\cdot) \neq \emptyset$ *and if* $\alpha \in (0,1)$, *we have*

$$[(1-\alpha) f_1(\cdot) + \alpha f_2(\cdot)]^*(\cdot) \leq (1-\alpha) f_1^*(\cdot) + \alpha f_2^*(\cdot)$$

(9) The Legendre-Fenchel transform preserves decomposition: Suppose that $x \in \mathbb{R}^n := \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$, *and "decomposable" objective function*

$$f(x) \overset{set}{=} \sum_{j=1}^{m} f_j\left(x_j\right).$$

If we assume that $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ *has the scalar product of a product-space, then the conjugate is*

$$f^*(s_1, \ldots, s_m) = \sum_{j=1}^{m} f_j^*\left(s_j\right).$$

Chapter 9

Convex analysis results: preliminaries

9.1 Introduction

In this chapter we direct attention to some specific examples of support functions in contexts that occur frequently. This material is standard, but in many cases is relegated to exercises only.

9.2 Support function examples

9.2.1 Support function of the empty set

For the empty set, $\mathcal{S} \stackrel{\text{set}}{=} \emptyset$, the associated support function $\sigma_{\mathcal{S}}(s)$ is the (improper) function identically equal to $-\infty$:

$$
\begin{aligned}
\sigma_{\mathcal{S}}(s) &\stackrel{\text{def}}{=} \underset{x\in\mathcal{S}}{\text{supremum}}\ \langle s,x\rangle\,.\\
&= \underset{x\in\emptyset}{\text{supremum}}\ \langle s,x\rangle\\
\sigma_{\emptyset}(s) &= -\infty.
\end{aligned}
$$

9.2.2 Support function of $\mathbb{R}^n$

The support function of $\mathbb{R}^n$ is the indicator function of the singleton set $\{0\}$:

$$\sigma_{\mathbb{R}^n}(\cdot) = I_{\{0\}}(\cdot).$$

9.2.3 Support function of a singleton set

For a singleton set $\mathcal{S} \overset{\text{set}}{=} \{x_\#\}$, the associated support function $\sigma_{\{x_\#\}}(s)$ coincides with "the scalar product of s with $x_\#$":

$$\begin{aligned} \sigma_{\mathcal{S}}(s) &\overset{\text{def}}{=} \underset{x\in\mathcal{S}}{\text{supremum}}\ \langle s,x\rangle. \\ &= \underset{x\in\{x_\#\}}{\text{supremum}}\ \langle s,x\rangle \\ \sigma_{\{x\}}(s) &= \langle s,x_\#\rangle. \end{aligned}$$

This simplest example of this is $\sigma_{\{0\}}(\cdot) = 0$.

We will see later how this provides a perspective on the way in which the gradient is the "singleton case" of the subdifferential.

9.2.4 Support function of a linear subspace

For an arbitrary linear subspace $\mathcal{S} \overset{\text{set}}{=} \mathcal{V}$ the associated support function $\sigma_{\mathcal{V}}(s)$ is $I_{\mathcal{V}^\perp}(\cdot)$ (the indicator function of the orthogonal subspace $\mathcal{V}^\perp$ associated with the subspace cone $\mathcal{V}$):

$$\begin{aligned} \sigma_{\mathcal{S}}(s) &\overset{\text{def}}{=} \underset{x\in\mathcal{S}}{\text{supremum}}\ \langle s,x\rangle. \\ &= \underset{x\in\mathcal{V}}{\text{supremum}}\ \langle s,x\rangle \\ &= \begin{cases} +\infty & \text{when } s \notin \mathcal{V}^\perp \\ 0 & \text{when } s \in \mathcal{V}^\perp \end{cases} \\ \sigma_{\mathcal{V}}(s) &= I_{\mathcal{V}^\perp}(\cdot). \end{aligned}$$

9.2.5 Support function of an arbitrary cone

For an arbitrary cone $\mathcal{S} \overset{\text{set}}{=} \mathcal{Q}$ the associated support function $\sigma_{\mathcal{Q}}(s)$ is $I_{\text{plrc }\mathcal{Q}}(\cdot)$ (the indicator function of the polar cone plrc $\mathcal{Q}$ associated with the cone $\mathcal{Q}$):

$$\begin{aligned}
\sigma_{\mathcal{S}}(s) &\overset{\text{def}}{=} \underset{x\in\mathcal{S}}{\text{supremum}}\ \langle s,x\rangle . \\
&= \underset{x\in\mathcal{Q}}{\text{supremum}}\ \langle s,x\rangle \\
&= \begin{cases} +\infty & \text{when } s \notin \text{plrc } \mathcal{K} \\ 0 & \text{when } s \in \text{plrc } \mathcal{K} \end{cases} \\
\sigma_{\mathcal{Q}}(s) &= I_{\text{plrc }\mathcal{Q}}[\cdot] .
\end{aligned}$$

To see this more clearly, let us recall the definition of the polar cone plrc $\mathcal{Q}$ associated with the cone $\mathcal{Q}$. Specifically

- $s \in \text{plrc } \mathcal{Q}$ means $\langle s,x\rangle \le 0$ for all $x \in \mathcal{Q}$. Since $0 \in \text{cl } \mathcal{Q}$ for any cone, we conclude that $s \in \text{plrc } \mathcal{Q}$ implies that $\underset{x\in\mathcal{Q}}{\text{supremum}}\ \langle s,x\rangle = 0$.
- $s \notin \text{plrc } \mathcal{Q}$ means there exists at least one argument, say $x_> \in \mathcal{Q}$, for which $\langle s,x_>\rangle > 0$; for specificity, introduce the notation $b_> \overset{\text{set}}{=} \langle s,x_>\rangle > 0$. Since a cone is a nonnegatively homogeneous set, we have that $x_> \in \mathcal{Q}$ means that $\lambda x_> \in \mathcal{Q}$ for any nonnegative scalar $\lambda \in \mathbb{R}_+$. Thus, we observe that since $\underset{x\in\mathcal{K}}{\text{supremum}}\ \langle s,x\rangle \ge \underset{\lambda\in\mathbb{R}_+}{\text{supremum}}\ \langle s,\lambda x_>\rangle = \underset{\lambda\in\mathbb{R}_+}{\text{supremum}}\ \lambda b_> = +\infty$, we are led to the conclusion that $s \notin \text{plrc } \mathcal{Q}$ implies that $\underset{x\in\mathcal{K}}{\text{supremum}}\ \langle s,x\rangle = +\infty$.

9.2.6 Support function of the polar set of a convex set that contains the origin

Consider a nonempty convex set $\mathcal{C} \subset \mathbb{R}^n$ containing the origin: $0 \in \mathcal{C}$. Denote the polar set to $\mathcal{C}$ as plr $\mathcal{C}$.

The support function of the polar set plr $\mathcal{C}$ coincides with the gauge function $\gamma_{\mathcal{C}}(\cdot)$ of the original set:

$$\sigma_{\text{plr }\mathcal{C}}(x) = \gamma_{\mathcal{C}}(x) .$$

For more details, see [Lue69, page 142], [Roc70], [RW04], or [HUL93a].

9.2.7 Support function of a unit norm ball

For a norm ball $\mathcal{S} \stackrel{\text{set}}{=} \mathcal{B}_\diamond$ the associated support function $\sigma_{\mathcal{B}_\diamond}(\cdot)$ is $\|\cdot\|_*$ (the dual norm to $\|\cdot\|_\diamond$):

$$\begin{aligned}\sigma_{\mathcal{S}}(s) &\stackrel{\text{def}}{=} \underset{x\in\mathcal{S}}{\text{supremum}}\ \langle s,x\rangle .\\ &= \underset{x\in\mathcal{B}_\diamond}{\text{supremum}}\ \langle s,x\rangle\\ &= \underset{x\in\mathcal{X}}{\text{supremum}}\ \{\langle s,x\rangle \mid \|x\|_\diamond \le 1\}\\ \sigma_{\mathcal{B}_\diamond}(s) &= \|s\|_* .\end{aligned}$$

9.2.8 Support function of the epigraph of an arbitrary function

We have already seen a relationship between (a slightly restricted version of) the support function of the epigraph of an arbitrary function and the conjugate $f^*(\cdot)$ of that function:

$$f^*(s) = \sigma_{\text{epi } f}(s,-1) = \underset{x,t\in\mathbb{R}^n\times\mathbb{R}}{\text{supremum}} \left\{ \left\langle \begin{matrix} s \\ -1 \end{matrix}, \begin{matrix} x \\ t \end{matrix} \right\rangle \mid (x,t)\in \text{epi } f(\cdot) \right\}.$$

A more complete description of the support function of the epigraph of an arbitrary function is

$$\sigma_{\text{epi } f}(s,-\alpha) = \begin{cases} \alpha f^*\left(\frac{1}{\alpha}s\right) & \text{when } \alpha\in\mathbb{R}_{++}\\ \sigma_{\text{epi } f}(s,0) = \sigma_{\text{dom } f}(s) & \text{when } \alpha = 0\\ +\infty & \text{when } \alpha\in\mathbb{R}_{--}. \end{cases}$$

See [HUL93b] for more details.

Chapter 10

Convex analysis results on Fenchel conjugates

10.1 Introduction

In this section we direct attention to convex analysis results for several specific functions that occur frequently in applications. Specifically, we consider Fenchel conjugacy results for a wide range of basic functions that occur in common practice. This material is standard, but is usually relegated to exercises. Typical references include [Rus06, BV04, RW04, Ber09, HUL93a, HUL93b].

10.2 Fenchel conjugates

10.2.1 Conjugate of a linear function

Consider the linear function $f(\cdot) \overset{\text{set}}{=} \langle s_{\#}, \cdot \rangle \,:\, \mathbb{R}^n \to \mathbb{R}$. From the definition of the Fenchel conjugate, we have

$$\begin{aligned} f^*(s) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - f(x)\} \\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - [\langle s_{\#},x\rangle]\} \\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s - s_{\#},x\rangle\} \\ &= \begin{cases} 0 & \text{when } s = s_{\#} \\ +\infty & \text{when } s \neq s_{\#} \end{cases} \\ &= I_{\{s_{\#}\}}(s). \end{aligned}$$

10.2.2 Conjugate of an affine function

Consider the affine function $f(\cdot) \overset{\text{set}}{=} \langle s_{\#}, \cdot \rangle - b_{\$} \,:\, \mathbb{R}^n \to \mathbb{R}$. From the definition of the Fenchel conjugate, we have

$$\begin{aligned} f^*(s) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - f(x)\} \\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - [\langle s_{\#},x\rangle - b_{\$}]\} \\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s - s_{\#},x\rangle + b_{\$}\} \\ &= \begin{cases} b_{\$} & \text{when } s = s_{\#} \\ +\infty & \text{when } s \neq s_{\#}. \end{cases} \end{aligned}$$

See Figure 10.1.

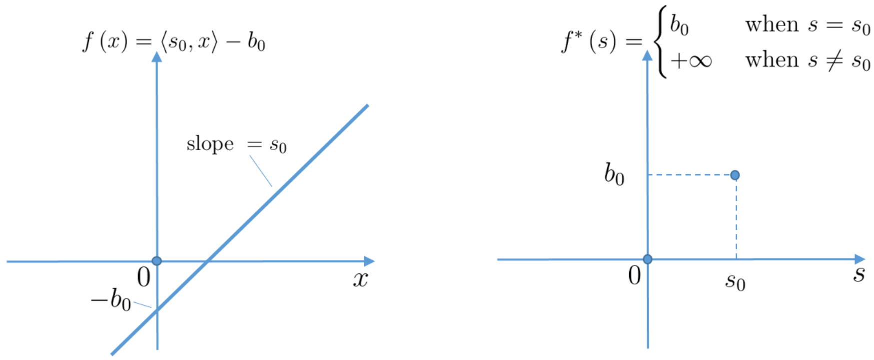

Figure 10.1: Conjugate function of an affine function. (After [Ber09]).

10.2.3 Conjugate of absolute value

Consider the function $f(\cdot) \overset{\text{set}}{=} |\cdot| \, : \, \mathbb{R}^n \to \mathbb{R}$. From the definition of the Fenchel conjugate, we have

$$\begin{aligned} f^*(s_\$) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \, \{\langle s_\$, x\rangle - f(x)\} \\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \, \{\langle s_\$, x\rangle - |x|\} \\ &= -\underset{x\in\mathbb{R}^n}{\text{infimum}} \, \{|x| - \langle s_\$, x\rangle\} \\ &= \begin{cases} 0 & \text{when } s \in [-1,1] \\ +\infty & \text{when } s \notin [-1,1]. \end{cases} \end{aligned}$$

See Figure 10.2.

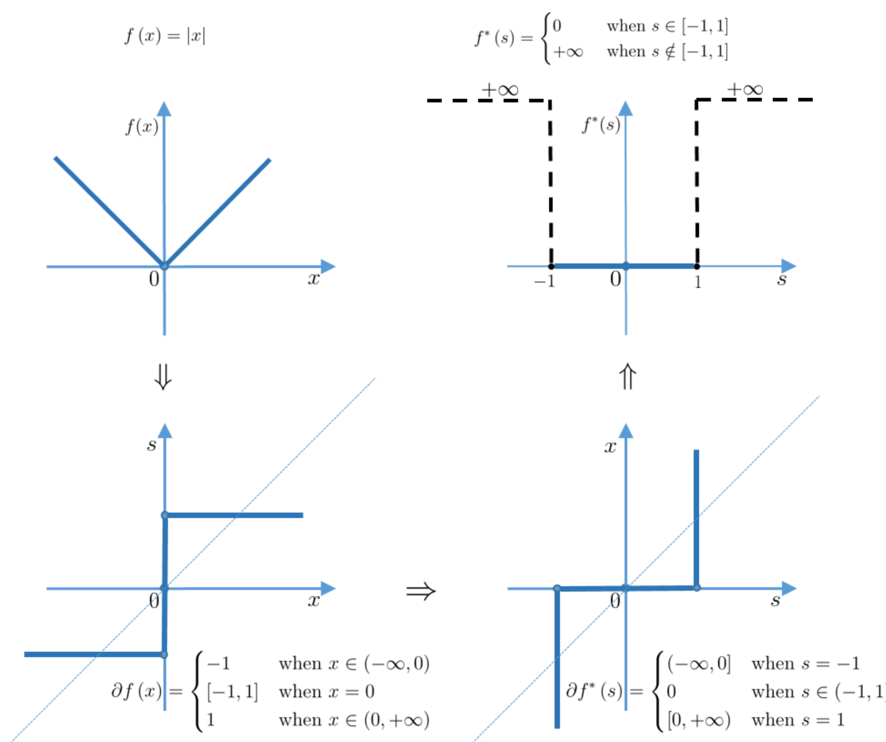

Figure 10.2: Conjugate function of an affine function. (After [Luc06]).

10.2.4 Conjugate of $c \cdot \frac{1}{2}x^2$ for strictly positive c

Consider the function $f(\cdot) \overset{\text{set}}{=} c\frac{1}{2}x^2 \,:\, \mathbb{R}^n \to \mathbb{R}$, with $c \in \mathbb{R}_{++}$. From the definition of the Fenchel conjugate, we have

$$
\begin{aligned}
f^*(s_\$) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \,\{s_\$ x - f(x)\} \\
&= \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{ s_\$ x - c \cdot \frac{1}{2}x^2 \right\} \\
&= -\underset{x\in\mathbb{R}^n}{\text{infimum}} \left\{ c \cdot \frac{1}{2}x^2 - s_\$ x \right\} \\
&= -\underset{x\in\mathbb{R}^n}{\text{minimum}} \left\{ c \cdot \frac{1}{2}x^2 - s_\$ x \right\} \\
&= -c \cdot \frac{1}{2}\left(\frac{1}{c}s_\$\right)^2 + s_\$ \cdot \frac{1}{c}s_\$ \\
&= \frac{1}{c} \cdot \frac{1}{2}s_\$^2.
\end{aligned}
$$

See Figure 10.3.

10.2.5 Conjugate of the indicator function of a generic set

Consider a generic set $\mathcal{S} \subseteq \mathbb{R}^n$ with associated indicator function $I_{\mathcal{S}}(\cdot) \,:\, \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The corresponding Fenchel conjugate $I_{\mathcal{S}}^*(\cdot) \,:\, \mathbb{R}^{n*} \to \mathbb{R} \cup \{+\infty\}$ coincides with $\sigma_{\mathcal{S}}(\cdot) \,:\, \mathbb{R}^{n*} \to \mathbb{R} \cup \{+\infty\}$, the support function of the generic set $\mathcal{S}$:

$$
I_{\mathcal{S}}^*(s) = \sigma_{\mathcal{S}}(s).
$$

Consider a generic set $\mathcal{S} \subseteq \mathbb{R}^n$. Recall that the indicator function of a set $\mathcal{S}$ is defined as

$$
I_{\mathcal{S}}(x) \overset{\text{def}}{=} \begin{cases} 0 & \text{when } x \in \mathcal{S} \\ +\infty & \text{when } x \notin \mathcal{S}. \end{cases}
$$

The definition of the Fenchel conjugate of a function $f(\cdot) \,:\, \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is

$$
f^*(s) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \,\{\langle s, x\rangle - f(x)\}.
$$

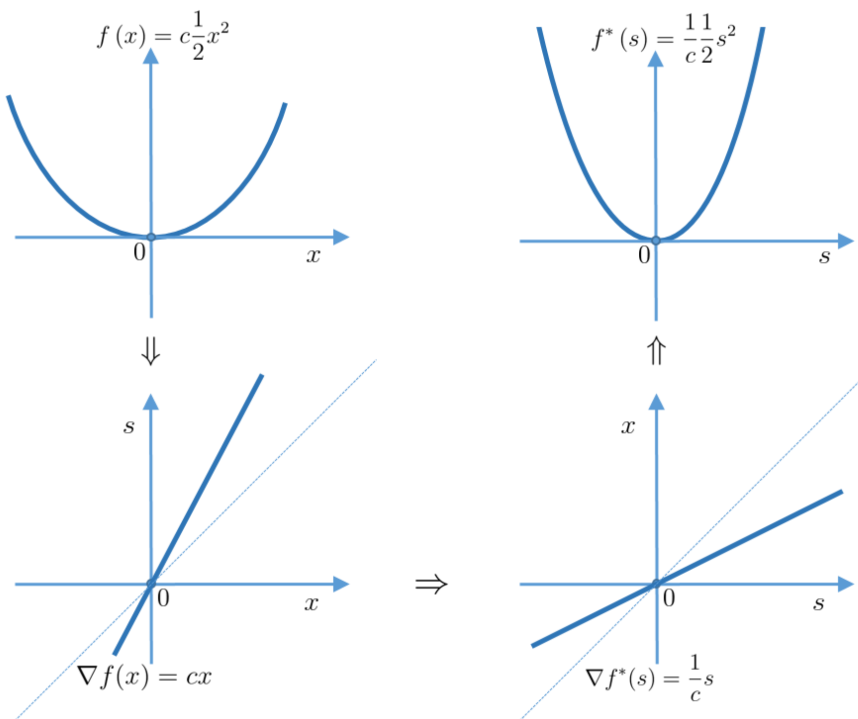

Figure 10.3: Conjugate function for $f(x) \overset{\text{set}}{=} c\frac{1}{2}x^2$.

For the specific case of $f(\cdot) \overset{\text{set}}{=} I_{\mathcal{S}}(\cdot)$ we have

$$\begin{aligned} I_{\mathcal{S}}^{*}(s) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \ \{\langle s,x\rangle - I_{\mathcal{S}}(x)\} \\ &= \underset{x\in\mathcal{S}}{\text{supremum}} \ \langle s,x\rangle \\ &\overset{\text{def}}{=} \sigma_{\mathcal{S}}(s). \end{aligned}$$

10.2.6 Conjugate of the indicator function of a closed convex set

The correspondence between closed convex sets and their corresponding support functions is “imbedded” within Fenchel conjugacy:

$$\text{when } \mathcal{C} \text{ is a closed convex set we have} \quad \begin{aligned} I_{\mathcal{C}}(\cdot) &\underset{*}{\longleftrightarrow} \sigma_{\mathcal{C}}(\cdot) \\ I_{\mathcal{C}}^{*}(\cdot) &= \sigma_{\mathcal{C}}(\cdot) \\ \sigma_{\mathcal{C}}^{*}(\cdot) &= I_{\mathcal{C}}(\cdot). \end{aligned}$$

Further, the following are all equivalent

$$\begin{aligned} s_{\$} &\in \mathcal{N}_{\mathcal{C}}(x_{\#}) \\ x_{\#} &\in \partial\sigma_{\mathcal{C}}(s_{\$}) \\ x_{\#} &\in \mathcal{C} \text{ and } \sigma_{\mathcal{C}}(s_{\$}) = \langle s_{\#}, x_{\#}\rangle \\ I_{\mathcal{C}}(x_{\#}) + I_{\mathcal{C}}^{*}(s_{\$}) &= \langle s_{\$}, x_{\#}\rangle \\ I_{\mathcal{C}}(x_{\#}) + \sigma_{\mathcal{C}}(s_{\$}) &= \langle s_{\$}, x_{\#}\rangle. \end{aligned}$$

See [RW04] for further discussion.

10.2.7 Conjugate of a positively homogeneous function

For a nonnegatively homogeneous function $h(\cdot) : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ we have

$$h^{*}(s) = \begin{cases} 0 & \text{when } \langle s,x\rangle \le h(x) \text{ for all } x\in\mathbb{R}^n \\ +\infty & \text{when } \langle s,x\rangle > h(x) \text{ for some } x\in\mathbb{R}^n \end{cases}.$$

Said slightly differently: for any nonnegatively homogeneous function (that is, for any function whose epigraph is a cone), say $h(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the corresponding Fenchel conjugate function is the indicator function of the set denoted $\mathcal{U}$ and given by $\{s \in \mathbb{R}^{n*} \, | s \in \mathbb{R}^{n*} \mid \langle s,x\rangle \leq h(x) \text{ for all } x \in \mathbb{R}^n\}$.

Consider a nonnegatively homogeneous function $h(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

We will show that

$$\begin{aligned} h^*(s) &= \begin{cases} 0 & \text{when } \langle s,x\rangle \leq h(x) \text{ for all } x \in \mathbb{R}^n \\ +\infty & \text{when } \langle s,x\rangle > h(x) \text{ for some } x \in \mathbb{R}^n \end{cases} \\ &= \begin{cases} 0 & \text{when } s \in \mathcal{U} \\ +\infty & \text{when } s \notin \mathcal{U} \end{cases} \\ &= I_{\mathcal{U}}(s), \end{aligned}$$

where $\mathcal{U} \overset{\text{set}}{=} \{s \in \mathbb{R}^{n*} \mid \langle s,x\rangle \leq h(x) \text{ for all } x \in \mathbb{R}^n\}$.

The definition of the Fenchel conjugate of a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is

$$f^*(s) \overset{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - f(x)\}.$$

For the specific case of $f(\cdot) \overset{\text{set}}{=} h(\cdot)$ we have

$$h^*(s) \overset{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - h(x)\}.$$

Suppose that $s \in \mathcal{U}$.

In this case we have $\langle s,x\rangle \leq h(x)$ for all $x \in \mathbb{R}^n$, which implies that $\langle s,x\rangle - h(x) \leq 0$ for all $x \in \mathbb{R}^n$. Since $\langle s,x\rangle - h(x) \leq 0$ for all $x \in \mathbb{R}^n$ and we can attain 0 with the choice $x \overset{\text{set}}{=} 0$, since we then have$\langle s,0\rangle - h(0) = 0 - 0 = 0$, we conclude that

$$\text{when } s \in \mathcal{U} \text{ we have } h^*(s) \overset{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - h(x)\} = 0.$$

Suppose that $s \notin \mathcal{U}$.

This means that there exists some $x \in \mathbb{R}^n$, say $x_>$, such that $\langle s,x_>\rangle > h(x_>)$. This in turn implies that $\langle s,x_>\rangle - h(x_>) > 0$; for convenience, let us introduce the notation $b_> \overset{\text{set}}{=} \langle s,x_>\rangle - h(x_>) > 0$. We now recall that, because the function $h(\cdot)$ was assumed to be nonnegatively homogeneous, we have $h(\lambda x_>) = \lambda h(x_>)$ whenever $\lambda \in \mathbb{R}_+$. This in turn tells us that $\langle s, \lambda x_{>} \rangle - h(\lambda x_{>}) = \lambda \langle s, x_{>} \rangle - \lambda h(x_{>}) = \lambda b_{>}$ whenever $\lambda \in \mathbb{R}_{+}$. We continue by observing that

$$\text{when } s \notin \mathcal{U}$$

we have, recalling that $h^*(s) \stackrel{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \{\langle s, x \rangle - h(x)\}$,

$$h^*(s) \geq \underset{\lambda \in \mathbb{R}_{+}}{\text{supremum}} \{\langle s, \lambda x_{>} \rangle - h(\lambda x_{>})\} = \underset{\lambda \in \mathbb{R}_{+}}{\text{supremum}}\, \lambda b = +\infty.$$

We can summarize this as

$$\text{when } s \notin \mathcal{U} \text{ we have } h^*(s) \stackrel{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \{\langle s, x \rangle - h(x)\} = +\infty.$$

Thus we have the claimed result:

$$\begin{aligned} h^*(s) &= \begin{cases} 0 & \text{when } \langle s, x \rangle \leq h(x) \text{ for all } x \in \mathbb{R}^n \\ +\infty & \text{when } \langle s, x \rangle > h(x) \text{ for some } x \in \mathbb{R}^n \end{cases} \\ &= \begin{cases} 0 & \text{when } s \in \mathcal{U} \\ +\infty & \text{when } s \notin \mathcal{U} \end{cases} \\ &= I_{\mathcal{U}}(s). \end{aligned}$$

10.2.8 Conjugate of the indicator function of a generic cone

For a generic cone $\mathcal{Q} \subseteq \mathbb{R}^n$, we have $I^*_{\mathcal{Q}}(\cdot) = I_{\text{plrc}\mathcal{Q}}(\cdot) = \sigma_{\mathcal{Q}}(\cdot)$.

That is, for a cone $\mathcal{Q} \subseteq \mathbb{R}^n$ the conjugate of the indicator function $I_{\mathcal{Q}}(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of the corresponding polar cone.

10.2.9 Conjugate of the indicator function of a convex cone

For a convex cone $\mathcal{K} \subseteq \mathbb{R}^n$, we have $I^*_{\mathcal{K}}(\cdot) = I_{\text{plrc}\,\mathcal{K}}(\cdot) = \sigma_{\mathcal{K}}(\cdot)$.

We will show that the conjugate of the indicator function of a convex cone $\mathcal{K}$ is the indicator function of the polar cone plrc $\mathcal{K}$; explicitly,

$$I^*_{\mathcal{K}}(\cdot) = I_{\text{plrc}\,\mathcal{K}}(\cdot).$$

Consider a convex cone $\mathcal{K} \subseteq \mathbb{R}^n$ and its indicator function

$$I_{\mathcal{K}}(\cdot) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in \mathcal{K} \\ +\infty & \text{if } x \notin \mathcal{K}. \end{cases}$$

Let us calculate the conjugate of this indicator function, $I^*_{\mathcal{K}}(\cdot)$. We have

$$\begin{aligned} I^*_{\mathcal{K}}(s) &\stackrel{\text{def}}{=} \underset{x \in \mathcal{K}}{\text{supremum}} \ \{\langle s,x \rangle - I_{\mathcal{K}}(x)\} \\ &= \underset{x \in \mathcal{K}}{\text{supremum}} \ \langle s,x \rangle \\ &= \sigma_{\mathcal{K}}(s). \end{aligned}$$

We claim that

$$\begin{aligned} \underset{x \in \mathcal{K}}{\text{supremum}} \ \langle s,x \rangle &= I_{\text{plrc}\,\mathcal{K}}(s) \\ &= \begin{cases} 0 & \text{when } s \in \text{plrc}\,\mathcal{K} \\ +\infty & \text{when } s \notin \text{plrc}\,\mathcal{K} \end{cases} \\ &= \begin{cases} 0 & \text{when } \langle s,x \rangle \leq 0 \text{ for all } x \in \mathcal{K} \\ +\infty & \text{when } \langle s,x \rangle > 0 \text{ for some } x \in \mathcal{K}. \end{cases} \end{aligned}$$

Let us consider the two cases:

Case 1: suppose that $s \in \text{plrc}\,\mathcal{K}$.

When $s \in \text{plrc}\,\mathcal{K}$ we have that $\langle s,x \rangle \leq 0$ for all $x \in \mathcal{K}$. Moreover, for the choice $x \stackrel{\text{set}}{=} 0$ we have $\langle s,0 \rangle = 0$. From these observations we conclude that

$$\text{when } s \in \text{plrc}\,\mathcal{K}$$

it is the case that

$$\underset{x \in \mathcal{K}}{\text{supremum}} \ \langle s,x \rangle = 0.$$

Case 2: suppose that $s \notin \text{plrc}\,\mathcal{K}$.

This means that there exists some $x \in \mathbb{R}^n$, say $x_>$, such that $\langle s,x_> \rangle > 0$. For convenience, let us introduce the notation $b_> \stackrel{\text{set}}{=} \langle s,x_> \rangle > 0$. We now observe that $\langle s,\lambda x_> \rangle = \lambda \langle s,x_> \rangle = \lambda b_> > 0$ whenever $\lambda \in \mathbb{R}_{++}$. We continue by observing that

$$\text{when } s \notin \text{plrc}\,\mathcal{K}$$

we have

$$\underset{x\in\mathcal{K}}{\text{supremum}}\ \langle s,x\rangle \geq \underset{\lambda\in\mathbb{R}_{++}}{\text{supremum}}\ \langle s,\lambda x_{>}\rangle = \underset{\lambda\in\mathbb{R}_{+}}{\text{supremum}}\ \lambda b_{>} = +\infty.$$

We can summarize this as

$$\text{when } s\notin \text{plrc}\ \mathcal{K} \text{ we have } \underset{x\in\mathcal{K}}{\text{supremum}}\ \langle s,x\rangle = +\infty.$$

Thus we have the claimed result:

$$\begin{aligned} I_{\mathcal{K}}^{*}(s) &= \begin{cases} 0 & \text{when } \langle s,x\rangle \leq 0 \text{ for all } x\in\mathbb{R}^{n} \\ +\infty & \text{when } \langle s,x\rangle > 0 \text{ for some } x\in\mathbb{R}^{n} \end{cases} \\ &= \begin{cases} 0 & \text{when } s\in \text{plrc}\ \mathcal{K} \\ +\infty & \text{when } s\notin \text{plrc}\ \mathcal{K} \end{cases} \\ &= I_{\text{plrc}\ \mathcal{K}}(s). \end{aligned}$$

That is, we have that $I_{\mathcal{K}}^{*}(\cdot) = I_{\text{plrc}\ \mathcal{K}}(\cdot)$.

10.2.10 Conjugate of the indicator function of a generic norm unit ball

We will show that the conjugate function, $I_{\mathcal{B}_{\diamond}}^{*}(\cdot)\ :\ \mathbb{R}^{n*}\to\mathbb{R}$, of the indicator function of a generic $\|\cdot\|_{\diamond}$-norm unit ball is $\|\cdot\|_{*}\ :\ \mathbb{R}^{n*}\to\mathbb{R}$, the dual norm associated with that generic norm:

$$I_{\mathcal{B}_{\diamond}}^{*}(s) = \|s\|_{*}.$$

Consider a generic norm in $\mathbb{R}^{n}$, say $\|\cdot\|_{\diamond}\ :\ \mathbb{R}^{n}\to\mathbb{R}\cup\{+\infty\}$. Consider $\mathcal{B}_{\diamond}\subset\mathbb{R}^{n}$, the unit $\|\cdot\|_{\diamond}$-norm ball in $\mathbb{R}^{n}$:

$$\mathcal{B}_{\diamond} \overset{\text{def}}{=} \{x\in\mathbb{R}^{n}\ |\ \|x\|_{\diamond}\leq 1\}.$$

Recall that $I_{\mathcal{B}_{\diamond}}(\cdot)\ :\ \mathbb{R}^{n}$, the indicator function of the generic norm unit ball, is defined as

$$I_{\mathcal{B}_{\diamond}}(x) \overset{\text{def}}{=} \begin{cases} 0 & \text{when } \|x\|_{\diamond}\leq 1 \\ +\infty & \text{when } \|x\|_{\diamond} > 1. \end{cases}$$

The conjugate function to the indicator function of the generic norm unit ball, $I^*_{\mathcal{B}_\diamond}(\cdot) : \mathbb{R}^{n*} \to \mathbb{R}$, has the form

$$\begin{aligned} I^*_{\mathcal{B}_\diamond}(s) &\stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - I_{\mathcal{B}_\diamond}(x)\} \\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle \mid x \in \mathcal{B}_\diamond\} \\ &= \sigma_{\mathcal{B}_\diamond}(s) \\ &= \underset{x\in\mathcal{B}_\diamond}{\text{supremum}} \; \langle s,x\rangle \\ &= \underset{\|x\|_\diamond \le 1}{\text{supremum}} \; \langle s,x\rangle . \end{aligned}$$

However, we note that this last expression is precisely the definition of the dual norm $\|s\|_*$. Thus, we have

$$I^*_{\mathcal{B}_\diamond}(s) = \underset{x\in\mathcal{B}_\diamond}{\text{supremum}} \; \langle s,x\rangle = \|s\|_* .$$

so

$$\begin{aligned} I^*_{\mathcal{B}_\diamond}(s) &= \|s\|_* \\ &= \sigma_{\mathcal{B}_\diamond}(s) . \end{aligned}$$

10.2.11 Conjugate of the support function of a generic set

Summary: $\sigma^*_{\mathcal{S}}(s) = I^{**}_{\mathcal{S}}(s) = \text{cl cvx } I_{\mathcal{S}}(s) = I_{\text{cl cvx } \mathcal{S}}(s)$.

Consider a generic set $\mathcal{S} \subseteq \mathbb{R}^n$. Recall that the support function of the set $\mathcal{S}$ is denoted $\sigma_{\mathcal{S}}(s) : \mathbb{R}^{n*} \to \mathbb{R} \cup \{+\infty\}$ and is defined via the expression

$$\sigma_{\mathcal{S}}(s) \stackrel{\text{def}}{=} \underset{x\in\mathcal{S}}{\text{supremum}} \; \langle s,x\rangle .$$

Slight reformulation of the expression above yields

$$\begin{aligned} \sigma_{\mathcal{S}}(s) &\stackrel{\text{def}}{=} \underset{x\in\mathcal{S}}{\text{supremum}} \; \langle s,x\rangle \\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle \mid x \in \mathcal{S}\} \\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \; \{\langle s,x\rangle - I_{\mathcal{S}}(x)\} \\ &= I^*_{\mathcal{S}}(s) . \end{aligned}$$

Thus, this demonstrates that the support function $\sigma_{\mathcal{S}}(\cdot)$ of the set $\mathcal{S}$ coincides with the conjugate function, $I_{\mathcal{S}}^{*}(\cdot)$, of the indicator function of the set $\mathcal{S}$. From this observation that

$$\sigma_{\mathcal{S}}(\cdot) = I_{\mathcal{S}}^{*}(\cdot),$$

we further note that the conjugate $\sigma_{\mathcal{S}}^{*}(\cdot)$ of the support function of the set $\mathcal{S}$ is equal to the biconjugate $I_{\mathcal{S}}^{**}(\cdot) : \mathbb{R}^{n} \to \mathbb{R} \cup \{+\infty\}$ of the indicator function of the set $\mathcal{S}$:

$$\sigma_{\mathcal{S}}^{*}(\cdot) = I_{\mathcal{S}}^{**}(\cdot).$$

By the Fenchel-Moreau Theorem (or the Biconjugate Function Theorem), we observe that

$$\begin{aligned} \text{epi } I_{\mathcal{S}}^{**}(\cdot) &= \text{cl cvx}\left(\text{epi } I_{\mathcal{S}}(\cdot)\right) \\ &= \text{epi } I_{\text{cl cvx } \mathcal{S}}(\cdot). \end{aligned}$$

Combining this with our previous result, we have

$$\begin{aligned} \sigma_{\mathcal{S}}^{*}(\cdot) &= I_{\mathcal{S}}^{**}(\cdot) \\ &= I_{\text{cl cvx } \mathcal{S}}(\cdot). \end{aligned}$$

That is, if the set $\mathcal{S}$ is not assumed to be convex and closed, the biconjugate $I_{\mathcal{S}}^{**}(\cdot)$ of the indicator function of the (neither convex nor closed) set $\mathcal{S}$ is $I_{\text{cl cvx } \mathcal{S}}(\cdot)$, the indicator function of the closed convex hull of the set $\mathcal{S}$; explicitly, $I_{\mathcal{S}}^{**}(\cdot) = I_{\text{cl cvx } \mathcal{S}}(\cdot)$.

10.2.12 Conjugate of the support function of a closed convex set

If, on the other hand, the set $\mathcal{S}$ is convex and closed, say $\mathcal{S} \overset{\text{set}}{=} \mathcal{C}$, we have the standard statement that $\mathcal{C} = \text{cl cvx } \mathcal{C}$; from this statement and the immediately preceding result we conclude that $\sigma_{\mathcal{C}}(\cdot)$, the support function of the closed convex set $\mathcal{C}$, is "mutually conjugate" to $I_{\mathcal{C}}(\cdot)$, the indicator function of that same closed convex set $\mathcal{C}$; explicitly,

$$\text{for a closed convex set } \mathcal{C}$$

$$\sigma_{\mathcal{C}}^{*}(\cdot) = I_{\mathcal{C}}(\cdot)$$

$$\text{and}$$

$$I_{\mathcal{C}}^{*}(\cdot) = \sigma_{\mathcal{C}}(\cdot).$$

10.2.13 Conjugate of the support function of a norm unit ball

We note in light of the earlier example (on the conjugate of support function of a closed convex set) that a generic norm $\|\cdot\|_\diamond$ can be seen as the support function of the dual norm unit ball $\mathcal{B}_*$.

Explicitly, we consider the set $\mathcal{B}_\diamond \overset{\text{def}}{=} \{x \in \mathbb{R}^n \mid \|x\|_\diamond \leq 1\}$. Since any norm is a closed convex function, and since sublevel sets of closed proper convex functions are closed convex sets, we conclude that the unit norm ball is a closed convex set. Proceeding from this observation, we have "mutual conjugacy" between support function and indicator function:

$$\begin{aligned} &\text{since } \mathcal{B}_\diamond \text{ is closed and convex} \\ &\sigma^*_{\mathcal{B}_\diamond}(\cdot) = I^{**}_{\mathcal{B}_\diamond}(\cdot) = I_{\mathcal{B}_\diamond}(\cdot) \\ &\quad \text{and} \\ &I^*_{\mathcal{B}_\diamond}(\cdot) = \sigma_{\mathcal{B}_\diamond}(\cdot). \end{aligned}$$

We have an analogous result for the case of the dual norm unit ball $\mathcal{B}_* \overset{\text{def}}{=} \{s \in \mathbb{R}^{n*} \mid \|s\|_* \leq 1\}$:

$$\begin{aligned} &\text{since } \mathcal{B}_* \text{ is closed and convex} \\ &\sigma^*_{\mathcal{B}_*}(\cdot) = I_{\mathcal{B}_*}(\cdot) \\ &\quad \text{and} \\ &I^*_{\mathcal{B}_*}(\cdot) = \sigma_{\mathcal{B}_*}(\cdot). \end{aligned}$$

We can say a little more, since we have also previously seen that the conjugate, $I^*_{\mathcal{B}_\diamond}(\cdot)$, of the indicator function of the generic norm unit ball corresponds to the dual norm: $I^*_{\mathcal{B}_\diamond}(\cdot) = \|\cdot\|_*$. This observation is demonstrates that the dual norm $\|\cdot\|_*$ is the support function of the (primal) $\|\cdot\|_\diamond$-norm unit ball $\mathcal{B}_\diamond$; explicitly

$$\|\cdot\|_* = I^*_{\mathcal{B}_\diamond}(\cdot) = \sigma_{\mathcal{B}_\diamond}(\cdot).$$

Likewise, we see later that $[\|\cdot\|_\diamond]^*(\cdot) = I_{\mathcal{B}_*}(\cdot)$, from which we can conjugate both sides to find $\|\cdot\|_\diamond = I^*_{\mathcal{B}_*}(\cdot)$. Together with the result that $I^*_{\mathcal{B}_*}(\cdot) = \sigma_{\mathcal{B}_*}(\cdot)$, this tells us that the primal norm $\|\cdot\|_\diamond$ is the support function of the (dual) $\|\cdot\|_*$-norm unit ball; explicitly

$$\|\cdot\|_\diamond = I^*_{\mathcal{B}_*}(\cdot) = \sigma_{\mathcal{B}_*}(\cdot).$$

10.2.14 Conjugate of the support function of a convex cone

The support function of any cone (convex or otherwise) is the indicator function of the polar cone.

Thus, the conjugate of the support function of any cone is the conjugate of the indicator function of the polar cone, and thus (from the earlier result $I^*_{\mathcal{K}}(\cdot) = I_{\text{plrc}\,\mathcal{K}}(\cdot) = \sigma_{\mathcal{K}}(\cdot)$) we have

$$\begin{aligned} I^*_{\text{plrc}\,\mathcal{K}}(\cdot) &= I_{\text{plrc plrc}\,\mathcal{K}}(\cdot) \\ &= I_{\text{cl cvx}\,\mathcal{K}}(\cdot) \\ &= \sigma_{\text{plrc}\,\mathcal{K}}(\cdot)\,. \end{aligned}$$

10.2.15 Conjugate of a generic norm

We next show that the conjugate function, $[\|\cdot\|_\diamond]^*(\cdot) \ : \ \mathbb{R}^{n*} \to \mathbb{R}$, of a generic norm coincides with the indicator function $I_{\mathcal{B}_*}(\cdot)$ of the dual norm unit ball: $[\|\cdot\|_\diamond]^*(\cdot) = I_{\mathcal{B}_*}(\cdot)$.

Consider a generic norm $\|\cdot\|_\diamond \ : \ \mathbb{R}^n \to \mathbb{R}$. The conjugate function to the $\|\cdot\|_\diamond$-norm is defined via the expression

$$[\|\cdot\|_\diamond]^*(s) \stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{supremum}}\{\langle s,x\rangle - \|x\|_\diamond\}\,.$$

We consider two cases:

on the one hand we will consider any $s \in \mathbb{R}^{n*}$ for which it is the case that $\langle s,x\rangle - \|x\|_\diamond \leq 0$ for all $x \in \mathbb{R}^n$. We claim that this is the case for any s in the dual norm unit ball:

if $s \in \mathcal{B}_*$ it will be the case that $\langle s,x\rangle - \|x\|_\diamond \leq 0$ for all $x \in \mathbb{R}^n$.

on the other hand we will consider any $s \in \mathbb{R}^{n*}$ for which it is the case that there exists some $x \in \mathbb{R}^n$ for which $\langle s,x\rangle - \|x\|_\diamond > 0$. We claim that this is the case for any s not in the dual norm unit ball: if $s \notin \mathcal{B}_*$ it will be the case that there exists some $x \in \mathbb{R}^n$ for which $\langle s,x\rangle - \|x\|_\diamond > 0$.

We first establish that $s \notin \mathcal{B}_*$ implies that there exists some $x \in \mathbb{R}^n$ for which $\langle s,x\rangle - \|x\|_\diamond > 0$; we then show that this in turn implies that $[\|\cdot\|_\diamond]^*(s) = \underset{x\in\mathbb{R}^n}{\text{supremum}}\{\langle s,x\rangle - \|x\|_\diamond\} = +\infty$.

By definition, $s \notin \mathcal{B}_*$ means that $\|s\|_* > 1$. For specificity, let us introduce the reference notation $\|s\|_* \overset{\text{set}}{=} \alpha_> > 1$.

We claim that, as a particular example, the choice of $x \overset{\text{set}}{=} \frac{2}{\alpha_>^2} s$ satisfies the inequality being considered $\langle s,x \rangle - \|x\|_\diamond > 0$.

To see why, note that for the current case we have $\|s\|_* \overset{\text{set}}{=} \alpha_> > 1$ and $x \overset{\text{set}}{=} \frac{2}{\alpha_>^2} s$; this leads to $\langle s,x \rangle = \left\langle s, \frac{2}{\alpha_>^2} s \right\rangle = \frac{2}{\alpha_>^2} \langle s,s \rangle = \frac{2}{\alpha_>^2} \alpha_>^2 = 2$. On the other hand, we have $\|x\|_\diamond = \left\| \frac{2}{\alpha_>^2} s \right\|_\diamond = \frac{2}{\alpha_>^2} \|s\|_\diamond = \frac{2}{\alpha_>^2} \alpha_> = \frac{2}{\alpha_>} < 2$; this last inequality follows from the implication that $s \notin \mathcal{B}_*$ means $\|s\|_* \overset{\text{set}}{=} \alpha_> > 1$.

Taken together we have seen that when $\|s\|_* \overset{\text{set}}{=} \alpha_> > 1$ the choice $x \overset{\text{set}}{=} \frac{2}{\alpha_>^2} s$ leads to $\langle s,x \rangle = 2$ and to $\|x\|_\diamond = \frac{2}{\alpha_>} < 2$. Thus there does indeed exist an $x \in \mathbb{R}^n$ such that

$$\langle s,x \rangle - \|x\|_\diamond > 0.$$

Having established (by construction) that $s \notin \mathcal{B}_*$ implies the existence of an $x \in \mathbb{R}^n$ for which $\langle s,x \rangle - \|x\|_\diamond > 0$, we will now show that for any such $s \notin \mathcal{B}_*$ we have the consequence $[\|\cdot\|_\diamond]^* (s) = \underset{x \in \mathbb{R}^n}{\text{supremum}} \{\langle s,x \rangle - \|x\|_\diamond\} = +\infty$.

Returning to our argument: We have shown by construction that $s \notin \mathcal{B}_*$ implies the existence of an $x \in \mathbb{R}^n$ for which $\langle s,x \rangle - \|x\|_\diamond > 0$. We introduce the notation $x_>$ for a particular such x; that is, $x_>$ denotes an element of $\mathbb{R}^n$ for which $\langle s,x_> \rangle - \|x_>\|_\diamond > 0$. Our previous construction is one example of such an element of $\mathbb{R}^n$.

For specificity, we introduce the reference notation $\langle s,x_> \rangle - \|x_>\|_\diamond \overset{\text{set}}{=} b_> > 0$. We now observe that

$$[\|\cdot\|_\diamond]^* (s) = \underset{x \in \mathbb{R}^n}{\text{supremum}} \{\langle s,x \rangle - \|x\|_\diamond\} \geq \underset{\lambda \in \mathbb{R}_{++}}{\text{supremum}} \{\langle s, \lambda x_> \rangle - \|\lambda x_>\|_\diamond\},$$

and that

$$\begin{aligned}
\underset{\lambda \in \mathbb{R}_{++}}{\text{supremum}} \{\langle s, \lambda x_> \rangle - \|\lambda x_>\|_\diamond\} &= \underset{\lambda \in \mathbb{R}_{++}}{\text{supremum}} \{\lambda \left(\langle s,x_> \rangle - \|x_>\|_\diamond\right)\} \\
&= \underset{\lambda \in \mathbb{R}_{++}}{\text{supremum}} \{\lambda b_>\} \\
&= +\infty.
\end{aligned}$$

Thus, we have demonstrated that $s \notin \mathcal{B}_*$ implies $[\|\cdot\|_\diamond]^* (s) = +\infty$.

We now consider the case $s \in \mathcal{B}_*$.

We first establish that $s \in \mathcal{B}_*$ implies that $\langle s, x\rangle - \|x\|_\diamond \leq 0$ for all $x \in \mathbb{R}^n$; we then show that this in turn implies that $[\|\cdot\|_\diamond]^* (s) = \underset{x\in\mathbb{R}^n}{\text{supremum}} \{\langle s, x\rangle - \|x\|_\diamond\} = 0$.

By definition, $s \in \mathcal{B}_*$ means that $\|s\|_* \leq 1$. We observe that this means

$$\|s\|_* \leq 1$$
$$\|s\|_* \|x\|_\diamond \leq \|x\|_\diamond \text{ for all } x \in \mathbb{R}^n.$$

The (generic norm) Cauchy-Schwarz inequality tells us $\langle s, x\rangle \leq \|s\|_* \|x\|_\diamond$ for all $s \in \mathbb{R}^{n*}$ and $x \in \mathbb{R}^n$; in particular it holds for any $s \in \mathcal{B}_*$. Taken together, we have for any $s \in \mathcal{B}_*$

$$\langle s, x\rangle \leq \|s\|_* \|x\|_\diamond \leq \|x\|_\diamond \text{ for all } x \in \mathbb{R}^n$$
$$\langle s, x\rangle \leq \|x\|_\diamond \text{ for all } x \in \mathbb{R}^n$$
$$\langle s, x\rangle - \|x\|_\diamond \leq 0 \text{ for all } x \in \mathbb{R}^n.$$

We now observe that we have seen that $s \in \mathcal{B}_*$ implies $\langle s, x\rangle - \|x\|_\diamond \leq 0$ for all $x \in \mathbb{R}^n$; we also note that (no matter what value s takes on), the choice

$x \overset{\text{set}}{=} 0$ leads to $\langle s, 0\rangle - \|0\|_\diamond = 0$.

Since $s \in \mathcal{B}_*$ implies that $\langle s, x\rangle - \|x\|_\diamond \leq 0$ for all $x \in \mathbb{R}^n$ and since $\langle s, x\rangle - \|x\|_\diamond = 0$ for $x \overset{\text{set}}{=} 0$, we thus have that $s \in \mathcal{B}_*$ implies

$$[\|\cdot\|_\diamond]^* (s) = \underset{x\in\mathbb{R}^n}{\text{supremum}} \{\langle s, x\rangle - \|x\|_\diamond\} = 0.$$

Collecting our results from the $s \notin \mathcal{B}_*$ and $s \in \mathcal{B}_*$ cases together, we have

$$[\|\cdot\|_\diamond]^* (s) = \underset{x\in\mathbb{R}^n}{\text{supremum}} \{\langle s, x\rangle - \|x\|_\diamond\} = \begin{cases} 0 & \text{when } s \in \mathcal{B}_* \\ +\infty & \text{when } s \notin \mathcal{B}_*. \end{cases}$$

This is exactly the indicator function of the dual norm unit ball, so:

$$[\|\cdot\|_\diamond]^* (\cdot) = I_{\mathcal{B}_*} (\cdot) .$$

Remark: Norms and dual norm unit balls are mutually conjugate. We have seen that

$$I^*_{\mathcal{B}_\diamond} (\cdot) = \|\cdot\|_* ,$$

and that

$$[\|\cdot\|_\diamond]^* (\cdot) = I_{\mathcal{B}_*} (\cdot) .$$

10.2.16 Conjugate of (one-half) generic norm squared

Suppose that our original function is

$$f(x) \overset{\text{set}}{=} \frac{1}{2} \|x\|_\diamond^2 ,$$

where $\|\cdot\|_\diamond : \mathbb{R}^n \to \mathbb{R}$ is a generic norm. We demonstrate that the conjugate function to $f(x) \overset{\text{set}}{=} \frac{1}{2} \|x\|_\diamond^2$ is $f^*(s) = \frac{1}{2} \|s\|_*^2$.

We will proceed by showing first that it is the case that $\frac{1}{2} \|s_\$\|_*^2$ is an upper bound for $\left\{ \langle s_\$, x \rangle - \frac{1}{2} \|x\|_\diamond^2 \mid x \in \mathbb{R}^n \right\}$ and then showing that it is the case that we actually have attainment, so that $\frac{1}{2} \|s_\$\|_*^2$ is in fact the least upper bound: $\frac{1}{2} \|s_\$\|_*^2 = f^*(s_\$) \overset{\text{set}}{=} \underset{x \in \mathbb{R}^n}{\text{supremum}} \left\{ \langle s_\$, x \rangle - \frac{1}{2} \|x\|_\diamond^2 \right\}$.

We first show that $\frac{1}{2} \|s_\$\|_*^2$ is an upper bound for $\left\{ \langle s_\$, x \rangle - \frac{1}{2} \|x\|_\diamond^2 \mid x \in \mathbb{R}^n \right\}$. We recall that the dual norm is defined as

$$\|s\|_* \overset{\text{def}}{=} \underset{x \in \mathcal{X}}{\text{supremum}} \left\{ \langle s, x \rangle \mid \|x\|_\diamond \leq 1 \right\} .$$

From the definition of the dual norm, we note in particular that

$$\langle s, x \rangle \leq \|s\|_* \|x\|_\diamond \text{ for all } s \in \mathbb{R}^{n*} \text{ and } x \in \mathbb{R}^n.$$

From this, we see that for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned} \langle s, x \rangle &\leq \|s\|_* \|x\|_\diamond \\ \langle s, x \rangle - \|s\|_* \|x\|_\diamond &\leq 0 \\ 2 \langle s, x \rangle - 2 \|s\|_* \|x\|_\diamond &\leq 0 \end{aligned}$$

We also note that

$$0 \leq \left(\|x\|_\diamond - \|s\|_* \right)^2 .$$

Combining the respective last lines, we find that for all $x \in \mathbb{R}^n$ and all $s \in \mathbb{R}^{n*}$ it is the case that

$$
\begin{aligned}
2\langle s,x\rangle - 2\|s\|_* \|x\|_\diamond &\le 0 \le (\|x\|_\diamond - \|s\|_*)^2 \\
2\langle s,x\rangle - 2\|s\|_* \|x\|_\diamond &\le (\|x\|_\diamond - \|s\|_*)^2 \\
2\langle s,x\rangle - 2\|s\|_* \|x\|_\diamond &\le \|x\|_\diamond^2 - 2\|x\|_\diamond \|s\|_* + \|s\|_*^2 \\
2\langle s,x\rangle &\le \|x\|_\diamond^2 + \|s\|_*^2 \\
\langle s,x\rangle &\le \frac{1}{2}\|x\|_\diamond^2 + \frac{1}{2}\|s\|_*^2 \\
\langle s,x\rangle - \frac{1}{2}\|x\|_\diamond^2 &\le \frac{1}{2}\|s\|_*^2 \text{ for all } x \in \mathbb{R}^n,\ s \in \mathbb{R}^{n*}.
\end{aligned}
$$

From the last line above, we have that $\frac{1}{2}\|s_\$\|_*^2$ is an upper bound on $\{\langle s_\$,x\rangle - \frac{1}{2}\|x\|_\diamond^2 \mid x \in \mathbb{R}^n\}$ for any choice $s_\$ \in \mathbb{R}^{n*}$. We next show that there exists an $x \in \mathbb{R}^n$ for which we have attainment, thus demonstrating that $\frac{1}{2}\|s_\$\|_*^2$ is the least upper bound of $\{\langle s_\$,x\rangle - \frac{1}{2}\|x\|_\diamond^2 \mid x \in \mathbb{R}^n\}$; in other words, that $\frac{1}{2}\|s_\$\|_*^2 = f^*(s_\$)$.

We need to demonstrate that for any $s_\$ \in \mathbb{R}^{n*}$, there is a corresponding choice of x, say $x \stackrel{\text{set}}{=} \tilde{x} \in \mathbb{R}^n$ for which we have attainment: $\langle s_\$,\tilde{x}\rangle - \frac{1}{2}\|\tilde{x}\|_\diamond^2 = \frac{1}{2}\|s_\$\|_*^2$.

We proceed by construction.

Case 1: $s = 0$. In the case where $s_\$ = 0$, we note that the choice $x \stackrel{\text{set}}{=} 0$ achieves equality since then both sides evaluate to 0.

Case 2: $s \neq 0$. We recall that the dual norm is defined as

$$
\|s\|_* \stackrel{\text{def}}{=} \underset{x \in \mathcal{X}}{\text{supremum}} \{\langle s,x\rangle \mid \|x\|_\diamond \le 1\}.
$$

A convex function attains its supremum on a closed nonempty convex set at an argument in the boundary of the set. Introduce the notation $x_{s_\$}^+$for $\underset{x\in\mathbb{R}^n}{\text{argmax}} \{\langle s_\$,x\rangle \mid \|x\|_\diamond \le 1\}$; this optimizing argument satisfies $\|s_\$\|_* = \left\langle s_\$, x_{s_\$}^+\right\rangle$ and $\left\|x_{s_\$}^+\right\|_\diamond = 1$, so that we also recognize that $s_\$$ and $x_{s_\$}^+$ attain equality in the (generic norm) Cauchy-Schwarz inequality: $\|s_\$\|_* \left\|x_{s_\$}^+\right\|_\diamond = \left\langle s_\$, x_{s_\$}^+\right\rangle$.

For such an equality achieving $x_{s_\$}^+ \in \mathbb{R}^n$, construct a new (scaled) value $\tilde{x} \stackrel{\text{set}}{=}$

$\frac{\|s_\$\|_*}{\left\|x^+_{s_\$}\right\|_\diamond} x^+_{s_\$}$. We note that for this scaled $\tilde{x}$, it is the case that

$$\begin{aligned}
\|\tilde{x}\|_\diamond &= \left\| \frac{\|s_\$\|_*}{\left\|x^+_{s_\$}\right\|_\diamond} x^+_{s_\$} \right\|_\diamond \\
&= \frac{\|s_\$\|_*}{\left\|x^+_{s_\$}\right\|_\diamond} \left\|x^+_{s_\$}\right\|_\diamond \\
&= \|s_\$\|_* .
\end{aligned}$$

Further, since we began with $x^+_{s_\$}$ satisfying

$$\left\langle s_\$, x^+_{s_\$} \right\rangle = \|s_\$\|_* \left\|x^+_{s_\$}\right\|_\diamond ,$$

we examine what the result of inner product with $s_\$$ is for the scaled $\tilde{x}$:

$$\begin{aligned}
\langle s_\$, \tilde{x} \rangle &= \left\langle s_\$, \frac{\|s_\$\|_*}{\left\|x^+_{s_\$}\right\|_\diamond} x^+_{s_\$} \right\rangle \\
&= \frac{\|s_\$\|_*}{\left\|x^+_{s_\$}\right\|_\diamond} \left\langle s_\$, x^+_{s_\$} \right\rangle \\
\text{using } \left\langle s_\$, x^+_{s_\$} \right\rangle &= \|s_\$\|_* \left\|x^+_{s_\$}\right\|_\diamond \\
&= \frac{\|s_\$\|_*}{\left\|x^+_{s_\$}\right\|_\diamond} \left(\|s_\$\|_* \left\|x^+_{s_\$}\right\|_\diamond \right) \\
&= \|s_\$\|_*^2 .
\end{aligned}$$

Since we have established that

$$\langle s_\$, \tilde{x} \rangle = \|s_\$\|_*^2 ,$$

if we subtract $\frac{1}{2} \|\tilde{x}\|_\diamond^2$ from both sides, we get

$$\begin{gathered}
\langle s_\$, \tilde{x} \rangle - \frac{1}{2} \|\tilde{x}\|_\diamond^2 = \|s_\$\|_*^2 - \frac{1}{2} \|\tilde{x}\|_\diamond^2 \\
\text{recalling that } \|\tilde{x}\|_\diamond = \|s_\$\|_*^2 , \\
\text{so that } \frac{1}{2} \|\tilde{x}\|_\diamond = \frac{1}{2} \|s_\$\|_*^2 , \\
\text{we substitute on the right hand side to find} \\
\langle s_\$, \tilde{x} \rangle - \frac{1}{2} \|\tilde{x}\|_\diamond^2 = \|s_\$\|_*^2 - \frac{1}{2} \|s_\$\|_*^2 \\
\langle s_\$, \tilde{x} \rangle - \frac{1}{2} \|\tilde{x}\|_\diamond^2 = \frac{1}{2} \|s_\$\|_*^2 .
\end{gathered}$$

Thus, the choice $x \stackrel{\text{set}}{=} \tilde{x}$ yields attainment.

Since $\frac{1}{2}\left\|s_\$\right\|_*^2$ is an upper bound on $\{\langle s_\$, x\rangle - \frac{1}{2}\left\|x\right\|_\diamond^2 \mid x \in \mathbb{R}^n\}$ and since for each $s_\$ \in \mathbb{R}^n$ there exists a corresponding choice of x argument for which that upper bound is attained , we conclude that

$$\frac{1}{2}\left\|s_\$\right\|_*^2 = \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{\langle s_\$, x\rangle - \frac{1}{2}\left\|x\right\|_\diamond^2\right\} \stackrel{\text{def}}{=} f^*\left(s_\$\right).$$

10.2.17 Conjugate of the generic $\|\cdot\|_\diamond$-norm measured distance from a point $x_\#$ to a closed convex set $\mathcal{C}$

Consider a closed nonempty convex set $\mathcal{C} \subset \mathbb{R}^n$. Consider also a generic norm, say $\|\cdot\|_\diamond : \mathbb{R}^n \to \mathbb{R}$. The distance to such a closed nonempty convex set $\mathcal{C}$, as measured by the generic $\|\cdot\|_\diamond$-norm will be denoted $\text{dist}_{\mathcal{C},\|\cdot\|_\diamond}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and defined via the expression

$$\text{dist}_{\mathcal{C},\|\cdot\|_\diamond}(x_\#) \stackrel{\text{def}}{=} \underset{x\in\mathcal{C}}{\text{minimum}} \left\|x_\# - x\right\|_\diamond.$$

Introduce the temporary notation $f(\cdot) \stackrel{\text{set}}{=} \text{dist}_{\mathcal{C},\|\cdot\|_\diamond}(\cdot)$, we claim that the corresponding conjugate function $f^*(\cdot) = \text{dist}^*_{\mathcal{C},\|\cdot\|_\diamond}(\cdot) : \mathbb{R}^{n*} \to \mathbb{R}\cup\{+\infty\}$ has the form

$$f^*(s) = \sigma_{\mathcal{C}}(s) + I_{\mathcal{B}_*}(s),$$

where $\sigma_{\mathcal{C}}(\cdot) : \mathbb{R}^{n*} \to \mathbb{R}\cup\{+\infty\}$ is the support function of the closed nonempty convex set $\mathcal{C}$ and $I_{\mathcal{B}_*}(\cdot) : \mathbb{R}^{n*} \to \mathbb{R}\cup\{+\infty\}$ is the indicator function of the dual norm unit ball $\mathcal{B}_* \stackrel{\text{set}}{=} \{s \in \mathbb{R}^{n*} \mid \|s\|_* \leq 1\}$.

We will later use the immediately preceding result to calculate the subdifferential of this function $f(\cdot) \stackrel{\text{set}}{=} \text{dist}_{\mathcal{C},\|\cdot\|_\diamond}(\cdot)$.

As an initial step, we recall that the support function of any set, say $\mathcal{S} \subseteq \mathbb{R}^n$, is denoted $\sigma_{\mathcal{S}}(\cdot) : \mathbb{R}^{n*} \to \mathbb{R}\cup\{+\infty\}$ and defined via the expression

$$\sigma_{\mathcal{S}}(s) \stackrel{\text{def}}{=} \underset{x\in\mathcal{S}}{\text{supremum}} \langle s, x\rangle.$$

Also, the explicit statement of the indicator function of the dual norm unit ball is

$$I_{\mathcal{B}_*}(s) \stackrel{\text{def}}{=} \begin{cases} 0 & s \in \mathcal{B}_* \\ +\infty & s \notin \mathcal{B}_* \end{cases}$$

$$= \begin{cases} 0 & \|s\|_* \leq 1 \\ +\infty & \|s\|_* > 1. \end{cases}$$

This means that we can more explicitly state our claim about the conjugate as

$$\begin{aligned} f^*(s) &= \sigma_{\mathcal{C}}[s] + I_{\mathcal{B}_*}[s] \\ &= \begin{cases} \underset{x \in \mathcal{C}}{\text{supremum}} \ \langle s, x \rangle & \text{when } s \in \mathcal{B}_* \\ +\infty & \text{when } s \notin \mathcal{B}_* \end{cases} \\ &= \begin{cases} \underset{x \in \mathcal{C}}{\text{supremum}} \ \langle s, x \rangle & \text{when } \|s\|_* \leq 1 \\ +\infty & \text{when } \|s\|_* > 1 \end{cases} \end{aligned}$$

Continuing, we observe that the distance function is an instance of infimal convolution.

In particular, we have

$$\begin{aligned} \text{dist}_{\mathcal{C},\|\cdot\|_\diamond}(x_\#) &\stackrel{\text{def}}{=} \underset{x \in \mathcal{C}}{\text{minimum}} \|x_\# - x\|_\diamond \\ &= \underset{x \in \mathbb{R}^n}{\text{minimum}} \{I_{\mathcal{C}}(x) + \|x_\# - x\|_\diamond\} \\ &= \left[I_{\mathcal{C}}(\cdot) \underset{\vee}{+} \|\cdot\|_\diamond \right](x_\#). \end{aligned}$$

We have also seen that the conjugate of the infimal convolution is the sum of the conjugates: in the current case, this means

$$\begin{aligned} \text{dist}^*{}_{\mathcal{C},\|\cdot\|_\diamond}(s_\$) &= \left[I_{\mathcal{C}}(\cdot) \underset{\vee}{+} \|\cdot\|_\diamond \right]^*(s_\$) \\ &= I_{\mathcal{C}}^*(s_\$) + [\|\cdot\|_\diamond]^*(s_\$). \end{aligned}$$

Further, we have seen that the conjugate of a generic norm $\|\cdot\|_\diamond$ coincides with indicator function of the unit dual norm unit ball: $[\|\cdot\|_\diamond]^*(\cdot) = I_{\mathcal{B}_*}(\cdot)$.

Finally, we know that $I_{\mathcal{C}}^*(\cdot) = \sigma_{\mathcal{C}}(\cdot)$.

Combined together, this gives us

$$\begin{aligned}\mathrm{dist}^*{}_{\mathcal{C},\|\cdot\|_\diamond}(s_\$) &= I_{\mathcal{C}}^*(s_\$) + [\|\cdot\|_\diamond]^*(s_\$) \\ &= \sigma_{\mathcal{C}}(s_\$) + I_{\mathcal{B}_*}(s_\$).\end{aligned}$$

10.2.18 Conjugate of (one-half) squared generic norm distance function of a point from a set

Consider a closed nonempty convex set $\mathcal{C} \subset \mathbb{R}^n$. Consider also a generic norm, say $\|\cdot\|_\diamond : \mathbb{R}^n \to \mathbb{R}$. One-half the squared distance to such a closed nonempty convex set $\mathcal{C}$, as measured by the generic $\|\cdot\|_\diamond$-norm will be denoted $\frac{1}{2}\mathrm{dist}^2_{\mathcal{C},\|\cdot\|_\diamond}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and defined via the expression

$$\frac{1}{2}\mathrm{dist}^2_{\mathcal{C},\|\cdot\|_\diamond}(x_\#) \overset{\mathrm{def}}{=} \underset{x\in\mathcal{C}}{\mathrm{minimum}} \frac{1}{2}\|x_\# - x\|_\diamond^2.$$

Introducing the temporary notation $f(\cdot) \overset{\mathrm{set}}{=} \frac{1}{2}\mathrm{dist}^2_{\mathcal{C},\|\cdot\|_\diamond}(\cdot)$, we claim that the corresponding conjugate function $f^*(\cdot) = \left[\frac{1}{2}\mathrm{dist}^2_{\mathcal{C},\|\cdot\|_\diamond}\right]^*(\cdot) : \mathbb{R}^{n*} \to \mathbb{R}$ has the form

$$f^*(s) = \sigma_{\mathcal{C}}(s) + \frac{1}{2}\|s\|_*^2.$$

To establish our claim, we first observe that one-half the square of the distance function is an instance of infimal convolution.

In particular, we have

$$\begin{aligned}\frac{1}{2}\mathrm{dist}^2{}_{\mathcal{C},\|\cdot\|_\diamond}(x_\#) &\overset{\mathrm{def}}{=} \underset{x\in\mathcal{C}}{\mathrm{minimum}} \frac{1}{2}\|x_\# - x\|_\diamond^2 \\ &= \underset{x\in\mathbb{R}^n}{\mathrm{minimum}} \left\{ I_{\mathcal{C}}(x) + \frac{1}{2}\|x_\# - x\|_\diamond^2 \right\} \\ &= \left[I_{\mathcal{C}}(\cdot) \underset{\vee}{+} \frac{1}{2}\|\cdot\|_\diamond^2 \right](x_\#).\end{aligned}$$

We have also seen that the conjugate of the infimal convolution is the sum of the conjugates: in the current case, this means

$$\begin{aligned}\left[\frac{1}{2}\mathrm{dist}^2_{\mathcal{C},\|\cdot\|_\diamond}\right]^*(s_\$) &= \left[I_{\mathcal{C}}(\cdot) \underset{\vee}{+} \frac{1}{2}\|\cdot\|_\diamond^2 \right]^*(s_\$) \\ &= I_{\mathcal{C}}^*(s_\$) + \left[\frac{1}{2}\|\cdot\|_\diamond^2\right]^*(s_\$).\end{aligned}$$

Further, we have seen that the conjugate of $\frac{1}{2}\left\|\cdot\right\|_{\diamond}^{2}$ for a generic norm $\left\|\cdot\right\|_{\diamond}$ coincides with $\frac{1}{2}\left\|\cdot\right\|_{*}^{2}$.

Finally, we know that $I_{\mathcal{C}}^{*}\left(\cdot\right)=\sigma_{\mathcal{C}}\left(\cdot\right)$.

Combined together, this gives us

$$\begin{aligned}\left[\frac{1}{2}\mathrm{dist}_{\mathcal{C},\left\|\cdot\right\|_{\diamond}}^{2}\right]^{*}\left(s_{\$}\right) &= I_{\mathcal{C}}^{*}\left(s_{\$}\right)+\left[\frac{1}{2}\left\|\cdot\right\|_{\diamond}^{2}\right]^{*}\left(s_{\$}\right)\\ &= \sigma_{\mathcal{C}}\left(s_{\$}\right)+\frac{1}{2}\left\|\cdot\right\|_{*}^{2}.\end{aligned}$$

Chapter 11

Optimization theory

11.1 Introduction

For simplicity, we only consider problems with equality constraints; in this limited setting we establish basic results that can subsequently be related to results in more general contexts. On one hand, we see below that when we determine an optimizing argument to the Lagrangian problem for some specified λ value, we also determine a subgradient of (the negative of) the Lagrange dual function $g(\cdot)$. On the other hand, we establish that the primal optimal value function $p^\star(\cdot)$ is conjugate to (the negative of) the Lagrange dual function$g(\cdot)$ (with a negative applied to the argument). These results, when considered in the appropriate contexts, provide the path by which to reach results in distributed optimization and/or the use of inexact subproblem solutions.

While these results are well-established, standard textbook coverage tends to be (relatively) much less extensive than one might expect. Our discussion here follows (albeit with some modifications) the unusually comprehensive coverage in [HUL93b]. Some of the results we discuss below are related to areas known by a wide range of names: for example, parametric programming, envelope theorems, data perturbation, postoptimal analysis, marginal analysis, perturbation analysis, or sensitivity analysis.

11.2 Equality constraints only

11.2.1 Problem: Primal optimization problem

We write

$$\begin{aligned}&\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f_0(x)\\&\text{subject to } h_i(x)=0 \text{ for } i\in\{1,\dots,p\}.\end{aligned}$$

Alternately

$$\begin{aligned}&\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f_0(x)\\&\text{subject to } h(x)=0.\end{aligned}$$

Here the use of "h" is intended to suggest "hyperplane"; this would be an accurate name when the equality constraints are required to be affine, as would be needed to ensure that the optimization problem involved a convex constraint set.

Primal optimal value function $p^\star(\cdot)$

Introduce $p^\star(\cdot)\ :\ \mathbb{R}^p\to\mathbb{R}$.

$$\begin{aligned}p^\star(b)&\overset{\text{def}}{=}\underset{x\in\mathbb{R}^n}{\text{minimum}}\ \{f_0(x)\mid h_i(x)=b_i \text{ for } i\in\{1,\dots,p\}\}\\&=\underset{x\in\mathbb{R}^n}{\text{minimum}}\ \{f_0(x)\mid h(x)=b\}.\end{aligned}$$

Note that the optimal value of the unmodified primal optimization problem is expressible in this notation as

$$p^\star(0)=\underset{x\in\mathbb{R}^n}{\text{minimum}}\ \{f_0(x)\mid h(x)=0\}.$$

Other names for the primal optimal value function include the "primal function", the "(infimal, extremal, primal) value function", the "marginal function", and the "perturbation function".

Primal optimizing argument and primal optimizing argument set

For each of the family of optimization problems

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f_0(x)$$
$$\text{subject to } h(x) = b$$

we introduce the notation $x^\star(b)$ to refer generically refer to an optimizing primal argument for the corresponding primal optimization problem. The optimizing argument point-to-point mapping takes input from the constraint space $\mathbb{R}^p$ and returns an element of the argument space $\mathbb{R}^n$ that attains the minimum of the corresponding optimization problem. That is, $x^\star(\cdot) : \mathbb{R}^p \to \mathbb{R}^n$ and is defined via

$$x^\star(b) \overset{\text{def}}{\in} \underset{x\in\mathbb{R}^n}{\text{Argmin}} \{f_0(x) \mid h(x) = b\}.$$

The optimizing argument point-to-set mapping takes an input from the constraint space $\mathbb{R}^p$ and returns all elements of the argument space $\mathbb{R}^n$ that satisfy the optimality conditions for the corresponding optimization problem. That is, $\mathcal{X}^\star(\cdot) : \mathbb{R}^p \to 2^{\mathbb{R}^n}$ and is defined via

$$\mathcal{X}^\star(b) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{Argmin}} \{f_0(x) \mid h(x) = b\}.$$

11.2.2 Lagrangian function $L(\cdot,\cdot)$

$L(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^{p*} \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}.$

$$L(x,\lambda) \overset{\text{def}}{=} f_0(x) + \langle \lambda, h(x) \rangle.$$

11.2.3 The Lagrange dual function $g(\cdot)$

$g(\cdot) : \mathbb{R}^{p*} \to \mathbb{R}$

$$\begin{aligned} g(\lambda) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}}\ L(x,\lambda) \\ &= \underset{x\in\mathbb{R}^n}{\text{infimum}} \{f_0(x) + \langle \lambda, h(x) \rangle\}. \end{aligned}$$

We note that the Lagrange dual function $g(\cdot) : \mathbb{R}^{p*} \to \mathbb{R}$ is concave whether or not the original primal optimization problem is convex. To see this, we will observe that the negative of the Lagrange dual function is convex. Explicitly, we have

$$\begin{aligned}
[-g](\lambda) &\stackrel{\text{def}}{=} -\underset{x\in\mathbb{R}^n}{\text{infimum}}\, L(x,\lambda) \\
&= -\underset{x\in\mathbb{R}^n}{\text{infimum}}\ \{f_0(x) + \langle\lambda, h(x)\rangle\} \\
&= \underset{x\in\mathbb{R}^n}{\text{supremum}}\ -\{f_0(x) + \langle\lambda, h(x)\rangle\} \\
&= \underset{x\in\mathbb{R}^n}{\text{supremum}}\ \{-f_0(x) + \langle\lambda, -h(x)\rangle\} \\
&= \underset{x\in\mathbb{R}^n}{\text{supremum}}\ \{\langle\lambda, -h(x)\rangle - f_0(x)\}.
\end{aligned}$$

Considering the last expression, we note that $\{-f_0(x) + \langle\lambda, -h(x)\rangle\}$ is affine in λ for any choice of $x \in \mathbb{R}^n$; this is immediate from the observation that $-f_0(x)$ is constant with respect to λ and $\langle\lambda, -h(x)\rangle$ is linear in λ. Thus, $[-g](\cdot)$ is a pointwise-in-λ supremum-indexed-by-x of affine-in-λ functions, and so we conclude that $[-g](\cdot)$ is convex in λ (and therefore that $g(\cdot)$ is concave in λ).

11.2.4 Problem: Evaluate the Lagrange dual function for some $\lambda_\$$

For the original optimization problem stated above, we consider the notion of a Lagrangian $L(\cdot, \lambda_\$)$ optimization-over-x problem for the specific λ-variable value $\lambda \stackrel{\text{set}}{=} \lambda_\$ \in \mathbb{R}^{p*}$. Specifically, this problem is

$$\begin{aligned}
&\underset{x\in\mathbb{R}^n}{\text{infimize}}\, L(x,\lambda_\$) \\
&= \underset{x\in\mathbb{R}^n}{\text{infimize}}\ \{f_0(x) + \langle\lambda_\$, h(x)\rangle\}.
\end{aligned}$$

This problem is sometimes referred to as the “Lagrangian relaxation” of the Lagrange primal problem above; the term “relaxation” is used here to refer to the fact that the Lagrange primal problem has the equality constraint $h(x) = 0$ whereas the Lagrangian $L(\cdot, \lambda_\$)$ problem has no such equality constraint — the constraint has been “relaxed”. In particular, where we previously incurred a penalty of $+\infty$ whenever $h(x) \neq 0$ (“hard constraint”), we now incur an additional (penalty) of the form $\langle\lambda_\$, h(x)\rangle$ when

$h(x) \neq 0$. We may describe $\lambda_{\$}$ as the price-per-unit-of-violation that is incurred whenever $h(x) \neq 0$.

11.2.5 Problem: Optimize the Lagrange dual function over all $\lambda \in \mathbb{R}^{p*}$

For the original optimization problem stated above, the corresponding Lagrange dual optimize-$g(\cdot)$-over-λ problem is

$$\underset{\lambda \in \mathbb{R}^{p*}}{\text{supremize}}\; g(\lambda).$$

The optimal objective value of this Lagrange dual problem will be denoted $d^\star$, defined as

$$d^\star \stackrel{\text{def}}{=} \underset{\lambda \in \mathbb{R}^{m*}}{\text{supremum}}\; g(\lambda).$$

In our current circumstance the Lagrange dual problem does not have any feasibility constraint requirements on the value of the λ-variable, so we will not presently introduce the notion of the Lagrange dual problem optimal value function — there are no constraints to shift. We note that because the Lagrange dual function is always a concave function (whether or not the Lagrange primal problem is a convex problem), the Lagrange dual problem is always a concave optimization problem (whether or not the Lagrange primal problem is a convex problem).

Any λ-argument that attains the optimal Lagrange dual problem objective value $d^\star$ will be denoted as $\lambda^\star$.

$$\lambda^\star \stackrel{\text{def}}{\in} \underset{\lambda \in \mathbb{R}^{m*}}{\text{Argmax}}\; g(\lambda).$$

The set of all such optimal-objective-value-attaining λ-arguments will be denoted $\Lambda^\star$, defined as

$$\Lambda^\star \stackrel{\text{def}}{=} \underset{\lambda \in \mathbb{R}^{m*}}{\text{Argmax}}\; g(\lambda),$$

or equivalently as

$$\Lambda^\star \stackrel{\text{def}}{=} \{\lambda \in \mathbb{R}^{m*} \mid g(\lambda) = d^\star\}.$$

11.2.6 Relations involving one or more of the three problems above

We now have three problems:

(b-perturbed) primal problem:

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \{f_0(x)\mid h(x)=b\}.$$

$\lambda_\$$-Lagrangian problem: Evaluate the Lagrange dual function $g(\cdot)$ at some dual argument $\lambda_\$\in\mathbb{R}^{p*}$:

$$\underset{x\in\mathbb{R}^n}{\text{infimize}}\, L(x,\lambda_\$).$$

Lagrange dual problem: Supremize the Lagrange dual function $g(\cdot)$ over all $\lambda\in\mathbb{R}^{p*}$:

$$\underset{\lambda\in\mathbb{R}^{p*}}{\text{supremize}}\, g(\lambda).$$

In order to evaluate the Lagrange dual function $g(\cdot)$ we determine an infimum of the Lagrangian $L(\cdot,\lambda_\$)$ problem

In order to evaluate the Lagrange dual function $g(\cdot)$ at $\lambda\overset{\text{set}}{=}\lambda_\$$ we determine the infimum of the Lagrangian $L(\cdot,\lambda_\$)$ problem. This was precisely the definition of the Lagrange dual function

$$\begin{aligned}g(\lambda_\$)&\overset{\text{def}}{=}\underset{x\in\mathbb{R}^n}{\text{infimum}}\, L(x,\lambda_\$)\\&=\underset{x\in\mathbb{R}^n}{\text{infimum}}\ \{f_0(x)+\langle\lambda_\$,h(x)\rangle\}.\end{aligned}$$

Whenever the infimum in the evaluation of $g(\cdot)$ is attained, we have also determined a solution to a perturbed version of the primal problem

In order to evaluate the Lagrange dual function $g(\cdot)$ at an argument $\lambda_\$\in\mathbb{R}^{p*}$, we find an infimum:

$$\begin{aligned}g(\lambda_\$)&\overset{\text{def}}{=}\underset{x\in\mathbb{R}^n}{\text{infimum}}\, L(x,\lambda_\$)\\&=\underset{x\in\mathbb{R}^n}{\text{infimum}}\ \{f_0(x)+\langle\lambda_\$,h(x)\rangle\}.\end{aligned}$$

When there exists some x argument, say $x^+_{\lambda_\$}$, for which this infimum is attained, it is in fact the case that $x^+_{\lambda_\$}$ is also an optimizing argument for a perturbed version of the primal problem. Explicitly, we are saying that

$$\left[x^+_{\lambda_\$} \in \operatorname*{Argmin}_{x\in\mathbb{R}^n} L(x,\lambda_\$)\right] \Longrightarrow \left[x^+_{\lambda_\$} \in \operatorname*{Argmin}_{x\in\mathbb{R}^n} \left\{f_0(x) \mid h(x) = b_{\lambda_\$}\right\}\right],$$

where $b_{\lambda_\$} \overset{\text{set}}{=} h\left(x^+_{\lambda_\$}\right)$.

From $x^+_{\lambda_\$} \in \operatorname*{Argmin}_{x\in\mathbb{R}^n} L(x,\lambda_\$)$ we find

$$\begin{aligned} x^+_{\lambda_\$} &\in \operatorname*{Argmin}_{x\in\mathbb{R}^n} L(x,\lambda_\$) \\ L\left(x^+_{\lambda_\$},\lambda_\$\right) &\le L(x,\lambda_\$) \text{ for all } x\in\mathbb{R}^n \\ f_0\left(x^+_{\lambda_\$}\right) + \left\langle \lambda_\$, h\left(x^+_{\lambda_\$}\right)\right\rangle &\le f_0(x) + \langle \lambda_\$, h(x)\rangle \text{ for all } x\in\mathbb{R}^n \\ f_0\left(x^+_{\lambda_\$}\right) &\le f_0(x) \text{ for all } x \in \left\{x\in\mathbb{R}^n \mid h(x) = b_{\lambda_\$}\right\}, \end{aligned}$$

recalling that $b_{\lambda_\$} \overset{\text{set}}{=} h\left(x^+_{\lambda_\$}\right)$, so that every x in the constraint set also yields $h(x) = b_{\lambda_\$} = h\left(x^+_{\lambda_\$}\right)$.

We recognize the last expression as an alternative method of stating $x^+_{\lambda_\$} \in \operatorname*{Argmin}_{x\in\mathbb{R}^n} \left\{f_0(x) \mid h(x) = b_{\lambda_\$}\right\}$.

This result is sometimes referred to [Las70, HUL93b] as Everett's Theorem [Eve63].

In order to determine a subgradient of $[-g](\cdot)$ we determine an optimizing argument of the Lagrangian $L(\cdot,\lambda_\$)$ problem

Consider the Lagrange dual function evaluation problem: with $\lambda_\$$ fixed, optimize $L(\cdot,\lambda_\$)$ over x.

Suppose that this infimal value is attained by some $x\in\mathbb{R}^n$,say

$$\begin{aligned} x^+_{\lambda_\$} &\overset{\text{set}}{\in} \operatorname*{Argmin}_{x\in\mathbb{R}^n} L(x,\lambda_\$) \\ &\in \operatorname*{Argmin}_{x\in\mathbb{R}^n} \left\{f_0(x) + \langle \lambda_\$, h(x)\rangle\right\}. \end{aligned}$$

We denote the set of all such Lagrangian $L(\cdot,\lambda_\$)$ infimal-value attaining x-arguments as $\mathcal{X}^+_{\lambda_\$}$, with definition

$$\mathcal{X}^+_{\lambda_\$} \stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, L(x,\lambda_\$)\,.$$

We now show how such an optimal argument $x^+_{\lambda_\$}$ can be used to determine an element of $\partial\,[-g]\,(\lambda_\$)$, the subdifferential set of the (convex by construction) function $[-g]\,(\cdot)$ at the argument $\lambda_\$$.

Note that the optimal argument $x^+_{\lambda_\$}$ satisfies $g\,(\lambda_\$) = L\left(x^+_{\lambda_\$},\lambda_\$\right)$ which is in turn equal to $\left\{f_0\left(x^+_{\lambda_\$}\right) + \left\langle \lambda_\$, h\left(x^+_{\lambda_\$}\right)\right\rangle\right\}$.

Keeping in mind our notation $x^+_{\lambda_\$}$ and $\mathcal{X}^+_{\lambda_\$}$, we observe that $-h\left(x^+_{\lambda_\$}\right)$ is an element of $\partial\,[-g]\,(\lambda_\$)$, the subdifferential set of the (convex by construction) function $[-g]\,(\cdot)$ at the argument $\lambda_\$$. Explicitly,

$$-h\left(x^+_{\lambda_\$}\right) \in \partial\,[-g]\,(\lambda_\$)\,.$$

To see this, first recall that $g\,(\lambda) \stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}}\ \{f_0\,(x) + \langle\lambda, h\,(x)\rangle\}$. From this, we have

$$\begin{aligned}
g\,(\lambda) &\stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}}\ \{f_0\,(x) + \langle\lambda, h\,(x)\rangle\} \\
g\,(\lambda) &\leq f_0\,(x) + \langle\lambda, h\,(x)\rangle \ \text{ for all } x\in\mathbb{R}^n \\
g\,(\lambda) &\leq f_0\left(x^+_{\lambda_\$}\right) + \left\langle\lambda, h\left(x^+_{\lambda_\$}\right)\right\rangle \ \text{ since } x^+_{\lambda_\$}\in\mathbb{R}^n \\
[-g]\,(\lambda) &\geq -f_0\left(x^+_{\lambda_\$}\right) - \left\langle\lambda, h\left(x^+_{\lambda_\$}\right)\right\rangle \ \text{ since } x^+_{\lambda_\$}\in\mathbb{R}^n.
\end{aligned}$$

We next seek to relate the expression above to $g\,(\lambda_\$) = L\left(x^+_{\lambda_\$},\lambda_\$\right)$ which we have seen is equal to $f_0\left(x^+_{\lambda_\$}\right) + \left\langle\lambda_\$, h\left(x^+_{\lambda_\$}\right)\right\rangle$; more precisely, to $[-g]\,(\lambda_\$) = -f_0\left(x^+_{\lambda_\$}\right) -$

$\left\langle \lambda_{\$}, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle$. We do this by adding and subtracting $\left\langle \lambda_{\$}, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle$, to find

$$\begin{aligned}
[-g](\lambda) &\geq -f_0\left(x^{+}_{\lambda_{\$}}\right) - \left\langle \lambda, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle \text{ since } x^{+}_{\lambda_{\$}} \in \mathbb{R}^n. \\
&= -f_0\left(x^{+}_{\lambda_{\$}}\right) + \left[\left\langle \lambda_{\$} - \lambda_{\$}, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle\right] - \left\langle \lambda, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle \\
&= -f_0\left(x^{+}_{\lambda_{\$}}\right) + \left\langle \lambda_{\$}, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle - \left\langle \lambda_{\$}, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle - \left\langle \lambda, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle \\
&= -f_0\left(x^{+}_{\lambda_{\$}}\right) - \left\langle \lambda_{\$}, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle + \left\langle \lambda_{\$} - \lambda, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle \\
&= [-g](\lambda_{\$}) + \left\langle \lambda_{\$} - \lambda, h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle \\
&= [-g](\lambda_{\$}) + \left\langle \lambda - \lambda_{\$}, -h\left(x^{+}_{\lambda_{\$}}\right)\right\rangle.
\end{aligned}$$

Since this last expression holds for any $\lambda \in \mathbb{R}^{p*}$ and any $\lambda_{\$} \in \mathbb{R}^{p*}$, we have established $-h\left(x^{+}_{\lambda_{\$}}\right) \in \partial[-g](\lambda_{\$})$.

Lagrange duality relationships between the Lagrange dual function $g(\cdot)$ and the primal objective function $f_0(\cdot)$ (evaluated in the feasible set): "weak duality"

First, consider any dual λ-argument, say $\lambda_{\$} \in \mathbb{R}^{p*}$.

Now consider any feasible primal argument; say $\tilde{x}$; explicitly $\tilde{x}$ is an element of the set denoted $\tilde{\mathcal{X}}$ and characterized as $\{x \in \mathbb{R}^n \mid h(x) = 0\}$.

For any such (dual, feasible primal) pair $(\lambda_{\$}, \tilde{x}) \in \mathbb{R}^{p*} \times \tilde{\mathcal{X}}$ pair, the following relationship always holds:

$$g(\lambda_{\$}) \leq f_0(\tilde{x}).$$

As one specific case of the above expression: if we consider an optimal (Lagrange dual problem) λ-argument $\lambda \overset{\text{set}}{=} \lambda^{\star}$ and an optimal (and so feasible for primal problem) x-argument $x \overset{\text{set}}{=} x^{\star}$, we have [1]

$$g(\lambda^{\star}) \leq f_0(x^{\star}),$$

[1] One somewhat abstract way to describe settings where we have "strong duality" $d^{\star} = p^{\star}(0)$ is: we have strong duality when the optimal value function $p^{\star}(\cdot)$ is lower semicontinuous at 0; when we consider a problem with inequality constraints only, the satisfaction of Slater's condition ensures that the optimal value function $p^{\star}(\cdot)$ is continuous at 0.

which we may alternately write using our previously introduced optimal value notation as

$$d^\star \le p^\star(0).$$

The conjugacy of $p^\star(\cdot)$ and $[-g](-\cdot)$.

It would be pleasing if the primal problem optimal value function $p^\star(\cdot) : \mathbb{R}^p \to \mathbb{R}\cup\{+\infty\}$ and the Lagrange dual function $g(\cdot) : \mathbb{R}^{p*} \to \mathbb{R}\cup\{-\infty\}$ were related by conjugacy. This turns out not to be the case (in some sense because of sign conventions chosen elsewhere): for one thing, the Lagrange dual function $g(\cdot)$ is defined to be concave while our definition of conjugacy was chosen to yield convex functions; on the other hand, we defined the Lagrangian function as $L(x,\lambda) \overset{\text{def}}{=} f_0(x) + \langle\lambda, h(x)\rangle$ rather than $f_0(x) - \langle\lambda, h(x)\rangle$. This convention prevents $[-g](\cdot)$ (the negative of the Lagrange dual function, without an additional negative being applied to the input argument) from being the conjugate of the primal optimal value function $p^\star(\cdot)$.

The actual relationship is that $[-g](-\cdot)$ is the conjugate of $p^\star(\cdot)$. Explicitly: if dom $[-g](\cdot) \neq \emptyset$, we have

$$[p^\star(\cdot)]^*(\lambda_\$) = [-g](-\lambda_\$) \text{ for any } \lambda_\$ \in \mathbb{R}^{p*}.$$

We can describe the operations by which (convex) function $[-g](-\cdot)$ comes from the (concave) Lagrange dual function $g(\cdot)$ as follows: given an input argument, start by "mirroring" the argument; pass this mirrored argument to the concave Lagrange dual function $g(\cdot)$; return the negative of the resulting value (returning the negative means that we deal with convexity instead of concavity).

To see why this conjugacy relationship holds, observe

$$g(\lambda) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}} \{f_0(x) + \langle\lambda, h(x)\rangle\}$$

$$g(\lambda) = -\underset{x\in\mathbb{R}^n}{\text{supremum}} \{-f_0(x) - \langle\lambda, h(x)\rangle\}.$$

Shifting the minus to the left hand side yields

$$[-g](\lambda) = \underset{x\in\mathbb{R}^n}{\text{supremum}} \{-f_0(x) - \langle\lambda, h(x)\rangle\}.$$

Considering the mirrored argument, we get

$$
\begin{aligned}
[-g](-\lambda) &= \underset{x\in\mathbb{R}^n}{\text{supremum}}\ \{-f_0(x) - \langle -\lambda, h(x)\rangle\} \\
&= \underset{x\in\mathbb{R}^n}{\text{supremum}}\ \{-f_0(x) + \langle \lambda, h(x)\rangle\} \\
&= \underset{x\in\mathbb{R}^n}{\text{supremum}}\ \{\langle \lambda, h(x)\rangle - f_0(x)\}
\end{aligned}
$$

We next introduce an “empty” supremum over the constraint variable $b \in \mathbb{R}^p$

$$
[-g](-\lambda) = \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\underset{x\in\mathbb{R}^n}{\text{supremum}}\ \{\langle \lambda, h(x)\rangle - f_0(x)\}\right]
$$

We next introduce a constraint specified in terms of the constraint variable $b \in \mathbb{R}^p$

$$
\begin{aligned}
[-g](-\lambda) &= \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\underset{x\in\mathbb{R}^n}{\text{supremum}}\ \{\langle \lambda, b\rangle - f_0(x)\ \mid h(x) = b\}\right] \\
&= \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\langle \lambda, b\rangle + \underset{x\in\mathbb{R}^n}{\text{supremum}}\ \{-f_0(x)\ \mid h(x) = b\}\right] \\
&= \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\langle \lambda, b\rangle - \underset{x\in\mathbb{R}^n}{\text{infimum}}\ \{f_0(x)\ \mid h(x) = b\}\right] \\
&= \underset{b\in\mathbb{R}^p}{\text{supremum}}\ [\langle \lambda, b\rangle - p^\star(b)] \\
&= [p^\star]^*(\lambda).
\end{aligned}
$$

When we determine an optimizing argument of the Lagrangian $L(\cdot, \lambda_\$)$ problem, we determine a subgradient of $[p^\star]^{**}(\cdot)$

We first note that $-h\left(x^+_{\lambda_\$}\right) \in \partial[-g](\lambda_\$)$ implies that $h\left(x^+_{\lambda_\$}\right) \in \partial[-g](-\lambda_\$)$ [2]. We have previously seen that $[(s,x) \in \partial f^*(\cdot)] \Longleftrightarrow [(x,s) \in \partial f^{**}(\cdot)]$. In the present

---

[2]Introduce the temporary notation $q(\lambda) \overset{\text{set}}{=} [-g](-\lambda)$.

We note that b is a subgradient of $[-g](\cdot)$ at $\lambda_\$$ if and only if $-b$ is a subgradient of $q(\cdot) \overset{\text{set}}{=} [-g](-\cdot)$ at $-\lambda_\$$:

$$
\begin{aligned}
[-g](\lambda) &\geq [-g](\lambda_\$) + \langle b, \lambda - \lambda_\$\rangle \\
q(-\lambda) &\geq q(-\lambda_\$) + \langle b, \lambda - \lambda_\$\rangle \\
q(-\lambda) &\geq q(-\lambda_\$) + \langle -b, -(\lambda - \lambda_\$)\rangle \\
q(-\lambda) &\geq q(-\lambda_\$) + \langle -b, (-\lambda) - (-\lambda_\$)\rangle.
\end{aligned}
$$

situation, this becomes

$$
\begin{aligned}
[(s,x) \in \partial f^{*}(\cdot)] &\Longleftrightarrow [(x,s) \in \partial f^{**}(\cdot)] \\
[(s,x) \in \partial [-g](-\cdot)] &\Longleftrightarrow \left[(x,s) \in \partial [p^{\star}]^{**}(\cdot)\right] \\
\left[\left(\lambda_{\$}, h\left(x^{+}_{\lambda_{\$}}\right)\right) \in \partial [-g](-\cdot)\right] &\Longleftrightarrow \left[\left(h\left(x^{+}_{\lambda_{\$}}\right), \lambda_{\$}\right) \in \partial [p^{\star}]^{**}(\cdot)\right].
\end{aligned}
$$

In particular, this means that anytime we have attainment in the evaluation of the Lagrange dual function we have a supporting affine minorant to $[p^{\star}]^{**}(\cdot) = \mathrm{cl\ cvx}\ p^{\star}(\cdot)$; the crossing point of this affine minorant with the "function evaluation axis"/"vertical axis of the epigraph" (which is precisely the evaluated value $g(\lambda_{\$})$) provides a lower bound on $p^{\star}(0)$ (the value of $p^{\star}(\cdot)$ at this axis).

Chapter 12

Optimization theory with modified functions

12.1 Introduction

In the previous section, we considered the "standard" Lagrange duality approach; in this section, we consider a "$\rho\frac{1}{2}\|\cdot\|_2^2$-modified" Lagrange duality approach. We will see that (under some conditions) the $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange duality approach will lead to a $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange dual function that will be smooth (whether or not the unmodified Lagrange dual function is smooth). Note that we explicitly mention the specific function ($\rho\frac{1}{2}\|\cdot\|_2^2$) by which we modify the Lagrange duality approach because we wish emphasize that there exist many possible modification functions that can be applied to the "standard" Lagrange duality approach.

We again only consider problems with equality constraints; these results can of course be considered in more general contexts. In analogy with our previous discussion, we establish results that provide the path by which to reach results in distributed optimization and/or the use of inexact subproblem solutions. The results in this chapter specifically extend to settings in which modification (or "augmentation") is used in place of standard, unmodified approaches.

These results are again well-established, but with standard textbook coverage that tends to be (relatively) much less extensive than one might expect. Our discussion here again follows (with some alterations) the unusually comprehensive coverage in [HUL93b].

12.2 Equality constraints only

We recall the equality constrained optimization problem previously discussed before introducing the modifications that will be the focus of the present chapter.

12.2.1 Problem: Primal optimization problem

We write

$$\begin{aligned}&\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f_0(x)\\&\text{subject to } h_i(x)=0 \text{ for } i\in\{1,\dots,p\}.\end{aligned}$$

Alternately

$$\begin{aligned}&\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f_0(x)\\&\text{subject to } h(x)=0.\end{aligned}$$

Here the use of “h” is intended to suggest “hyperplane”; this would be an accurate name when the equality constraints are required to be affine, as would be needed to ensure that the optimization problem involved a convex constraint set.

12.2.2 Introducing modified functions

In our previous discussion of Lagrange duality, we introduced the following notation and terminology:

- primal objective function $f_0(\cdot)\ :\ \mathbb{R}^n\to\mathbb{R}$
- primal optimal value function $p^\star(\cdot)\ :\ \mathbb{R}^p\to\mathbb{R}$
- Lagrangian function $L(\cdot,\cdot)\ :\ \mathbb{R}^n\times\mathbb{R}^{p*}\to\mathbb{R}$

- Lagrange dual function $g(\cdot)\ :\ \mathbb{R}^{p*}\to\mathbb{R}$.

We will now consider "modified" versions of each of these functions. More precisely, we will consider $\rho\frac{1}{2}\|\cdot\|_2^2$-modified versions (involving a nonnegative scalar $\rho\in\mathbb{R}_+$) of each of these functions:

- $\rho\frac{1}{2}\|\cdot\|_2^2$-modified primal objective function $f_\rho(\cdot)\ :\ \mathbb{R}^n\to\mathbb{R}$
- $\rho\frac{1}{2}\|\cdot\|_2^2$-modified primal optimal value function $p_\rho^\star(\cdot)\ :\ \mathbb{R}^p\to\mathbb{R}$
- $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrangian function $L_\rho(\cdot,\cdot)\ :\ \mathbb{R}^n\times\mathbb{R}^{p*}\to\mathbb{R}$
- $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange dual function $g_\rho(\cdot)\ :\ \mathbb{R}^{p*}\to\mathbb{R}$.

$\rho\frac{1}{2}\|\cdot\|_2^2$-modified primal objective function $f_\rho(\cdot)\ :\ \mathbb{R}^n\to\mathbb{R}$

The $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange primal problem objective function $f_\rho(\cdot)$ for our $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange primal problem (discussed below) is denoted $f_\rho(\cdot)\ :\ \mathbb{R}^n\to\mathbb{R}$, and is defined as

$$f_\rho(x)\overset{\text{set}}{=}f_0(x)+\frac{1}{2}\|h(x)\|_2^2.$$

$\rho\frac{1}{2}\|\cdot\|_2^2$-modified primal optimal value function $p_\rho^\star(\cdot)\ :\ \mathbb{R}^p\to\mathbb{R}$

The $\rho\frac{1}{2}\|\cdot\|_2^2$-modified primal problem optimal value function $p_\rho^\star(\cdot)$ associated with our $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange primal problem (discussed below) is denoted $p_\rho^\star(\cdot)$: $\mathbb{R}^p\to\mathbb{R}$ and is defined as

$$p_\rho^\star(b)\overset{\text{set}}{=}\underset{x\in\mathbb{R}^n}{\text{infimum}}\ \left\{f_\rho(x)\ \mid h(x)=b\right\}.$$

We note that the modified primal optimal value function can alternately be described as $p_\rho^\star(b)=p^\star(b)+\rho\frac{1}{2}\|b\|_2^2$.

To see this, note that

$$
\begin{aligned}
p_{\rho}^{\star}(b) &\overset{\text{set}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}} \;\left\{ f_{\rho}(x) \;\middle|\; h(x) = b \right\} \\
&= \underset{x\in\mathbb{R}^n}{\text{infimum}} \;\left\{ f_0(x) + \rho\frac{1}{2}\|h(x)\|_2^2 \;\middle|\; h(x) = b \right\} \\
&= \underset{x\in\mathbb{R}^n}{\text{infimum}} \;\left\{ f_0(x) + \rho\frac{1}{2}\|b\|_2^2 \;\middle|\; h(x) = b \right\} \\
&= \underset{x\in\mathbb{R}^n}{\text{infimum}} \;\left\{ f_0(x) \;\middle|\; h(x) = b \right\} + \rho\frac{1}{2}\|b\|_2^2 \\
&= p^{\star}(b) + \rho\frac{1}{2}\|b\|_2^2 .
\end{aligned}
$$

Note the presence of $\rho\frac{1}{2}\|b\|_2^2$ rather than $\rho\frac{1}{2}\|h(x)\|_2^2$, reflecting the fact that the optimal value function requires the constraints to be satisfied.

$\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrangian function $L_{\rho}(\cdot,\cdot) \,:\, \mathbb{R}^n \times \mathbb{R}^{p*} \to \mathbb{R}$

The $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrangian function $L_{\rho}(\cdot,\cdot)$ associated with our $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange primal problem (discussed below) is denoted $L_{\rho}(\cdot,\cdot) \,:\, \mathbb{R}^n \times \mathbb{R}^{p*} \to \mathbb{R}$. The definition of the $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrangian function is

$$
\begin{aligned}
L_{\rho}(x,\lambda) &\overset{\text{set}}{=} f_{\rho}(u) + \langle \lambda, h(x) \rangle \\
&= f_0(u) + \rho\frac{1}{2}\|h(x)\|_2^2 + \langle \lambda, h(x) \rangle .
\end{aligned}
$$

We note that the modified Lagrangian function can alternately be described as $L_{\rho}(x,\lambda) = L(x,\lambda) + \rho\frac{1}{2}\|h(x)\|_2^2$, since

$$
\begin{aligned}
L_{\rho}(x,\lambda) &\overset{\text{def}}{=} f_{\rho}(x) + \langle \lambda, h(x) \rangle \\
&= f_0(x) + \rho\frac{1}{2}\|h(x)\|_2^2 + \langle \lambda, h(x) \rangle \\
&= f_0(x) + \langle \lambda, h(x) \rangle + \rho\frac{1}{2}\|h(x)\|_2^2 \\
&= L(x,\lambda) + \rho\frac{1}{2}\|h(x)\|_2^2 .
\end{aligned}
$$

$\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange dual function $g_{\rho}(\cdot) \,:\, \mathbb{R}^{p*} \to \mathbb{R}$.

The$\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange dual function $g_{\rho}(\cdot)$ associated with our $\rho\frac{1}{2}\|\cdot\|_2^2$-modified Lagrange primal problem (discussed below) is denoted $g_{\rho}(\cdot) \,:\, \mathbb{R}^{m*} \to \mathbb{R}$ and defined via the expression

$$
\begin{aligned}
g_\rho(\lambda) &\overset{\text{set}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}}\, L_\rho(x,\lambda) \\
&= \underset{x\in\mathbb{R}^n}{\text{infimum}} \left\{ L(x,\lambda) + \rho\frac{1}{2}\|h(x)\|_2^2 \right\} \\
&= \underset{x\in\mathbb{R}^n}{\text{infimum}} \left\{ f_0(x) + \langle\lambda, h(x)\rangle + \rho\frac{1}{2}\|h(x)\|_2^2 \right\}.
\end{aligned}
$$

12.2.3 Introducing modified problems

In our previous discussion of Lagrange duality, we introduced the following problems:

- primal optimization problem:
  - Determine $p^\star(0) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}} \{f_0(x) \mid h(x) = 0\}$
- primal "perturbed" optimization problem:
  - Determine $p^\star(b) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}} \{f_0(x) \mid h(x) = b\}$
- evaluate the Lagrange dual function at $\lambda_\$$:
  - Determine $g(\lambda_\$) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}}\, L(x,\lambda_\$)$
- dual optimization problem:
  - Determine $d^\star \overset{\text{set}}{=} \underset{\lambda\in\mathbb{R}^{p*}}{\text{supremum}}\, g(\lambda)$.

We will now consider "modified" versions of each of these functions. More precisely, we will consider $\rho\frac{1}{2}\|\cdot\|_2^2$-modified versions of each of these functions.

- $\rho\frac{1}{2}\|\cdot\|_2^2$-modified primal optimization problem:
  - Determine $p_\rho^\star(0) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}} \left\{f_\rho(x) \mid h(x) = 0\right\}$
- $\rho\frac{1}{2}\|\cdot\|_2^2$-modified primal "perturbed" optimization problem:

– Determine $p_{\rho}^{\star}(b) \overset{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{infimum}} \left\{ f_{\rho}(x) \mid h(x) = b \right\}$

- evaluate the $\rho \frac{1}{2} \|\cdot\|_2^2$-modified Lagrange dual function at $\lambda_{\$}$:

  – Determine $g_{\rho}(\lambda_{\$}) \overset{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{infimum}}\, L_{\rho}(x, \lambda_{\$})$

- $\rho \frac{1}{2} \|\cdot\|_2^2$-modified dual optimization problem:

  – Determine $d_{\rho}^{\star} \overset{\text{set}}{=} \underset{\lambda \in \mathbb{R}^{p*}}{\text{supremum}}\, g_{\rho}(\lambda)$.

12.2.4 Observations on the modified problems

The primal optimization problem coincides with the modified primal optimization problem

We first note that $\rho \frac{1}{2} \|\cdot\|_2^2$-modified primal optimization problem is in fact a completely equivalent restatement of the original primal optimization problem; explicitly, $p_{\rho}^{\star}(0) = p^{\star}(0)$ for any $\rho \in \mathbb{R}_+$ (in fact, for the equality constrained case, this holds for any $\rho \in \mathbb{R}$). This is immediate from the definition of the $\rho \frac{1}{2} \|\cdot\|_2^2$-modified primal optimization problem:

$$
\begin{aligned}
p_{\rho}^{\star}(0) &\overset{\text{def}}{=} \underset{x \in \mathbb{R}^n}{\text{infimum}} \left\{ f_{\rho}(x) \mid h(x) = 0 \right\} \\
&= \underset{x \in \mathbb{R}^n}{\text{infimum}} \left\{ f_0(x) + \rho \frac{1}{2} \|h(x)\|_2^2 \mid h(x) = 0 \right\} \\
&= \underset{x \in \mathbb{R}^n}{\text{infimum}} \left\{ f_0(x) + \rho \frac{1}{2} \|0\|_2^2 \mid h(x) = 0 \right\} \\
&= \underset{x \in \mathbb{R}^n}{\text{infimum}} \left\{ f_0(x) \mid h(x) = 0 \right\} \\
&= p^{\star}(0).
\end{aligned}
$$

The conjugate of the modified primal optimal function

We previously saw that the conjugate of the primal optimal value function $p^{\star}(\cdot) : \mathbb{R}^p \to \mathbb{R}$ was $[-g](-\cdot) : \mathbb{R}^{p*} \to \mathbb{R}$. In the modified setting, all of the previous arguments follow immediately, so that the result is now that the conjugate of the modified primal optimal value function $p_\rho^\star(\cdot)$ is $\left[-g_\rho\right](-\cdot)$; explicitly

$$\left[p_\rho^\star\right]^*(\cdot) = \left[-g_\rho\right](-\cdot).$$

For specificity, the steps to establish this are

$$g_\rho(\lambda) \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}} \left\{f_\rho(x) + \langle\lambda, h(x)\rangle\right\}$$
$$g_\rho(\lambda) = -\underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{-f_\rho(x) - \langle\lambda, h(x)\rangle\right\}.$$

Shifting the minus to the left hand side yields

$$\left[-g_\rho\right](\lambda) = \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{-f_\rho(x) - \langle\lambda, h(x)\rangle\right\}.$$

Considering the mirrored argument, we get

$$\begin{aligned}\left[-g_\rho\right](-\lambda) &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{-f_\rho(x) - \langle-\lambda, h(x)\rangle\right\}\\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{-f_\rho(x) + \langle\lambda, h(x)\rangle\right\}\\ &= \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{\langle\lambda, h(x)\rangle - f_\rho(x)\right\}\end{aligned}$$

We next introduce an "empty" supremum over the constraint variable $b \in \mathbb{R}^p$

$$\left[-g_\rho\right](-\lambda) = \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{\langle\lambda, h(x)\rangle - f_\rho(x)\right\}\right]$$

We next introduce a constraint specified in terms of the constraint variable $b \in \mathbb{R}^p$

$$\begin{aligned}\left[-g_\rho\right](-\lambda) &= \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{\langle\lambda, b\rangle - f_\rho(x) \mid h(x) = b\right\}\right]\\ &= \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\langle\lambda, b\rangle + \underset{x\in\mathbb{R}^n}{\text{supremum}} \left\{-f_\rho(x) \mid h(x) = b\right\}\right]\\ &= \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\langle\lambda, b\rangle - \underset{x\in\mathbb{R}^n}{\text{infimum}} \left\{f_\rho(x) \mid h(x) = b\right\}\right]\\ &= \underset{b\in\mathbb{R}^p}{\text{supremum}} \left[\langle\lambda, b\rangle - p_\rho^\star(b)\right]\\ &= \left[p_\rho^\star\right]^*(\lambda).\end{aligned}$$

Interpreting $[-g](-\cdot)$ in light of infimal convolution

We just established that $\left[p^\star_\rho\right]^*(\lambda) = \left[-g_\rho\right](-\lambda)$. We have previously seen that $p^\star_\rho(b) = p^\star(b) + \rho\frac{1}{2}\|b\|_2^2$. Further, we know that the conjugate a sum is (the closure of) the infimal convolution of the individual conjugates [1]: $(f_1+f_2)^*(\cdot) = f_1^*(\cdot) \underset{\vee}{+} f_2^*(\cdot)$.

Taken together, we have

$$\begin{aligned}
\left[p^\star_\rho\right]^*(\lambda) &= \left[-g_\rho\right](-\lambda) \\
\left[p^\star(b) + \rho\frac{1}{2}\|b\|_2^2\right]^*(\lambda) &= \left[-g_\rho\right](-\lambda) \\
\left[[p^\star]^*(\cdot) \underset{\vee}{+} \frac{1}{\rho}\frac{1}{2}\|\cdot\|_2^2\right](\lambda) &= \left[-g_\rho\right](-\lambda) \\
\left[[-g](-\cdot) \underset{\vee}{+} \frac{1}{\rho}\frac{1}{2}\|\cdot\|_2^2\right](\lambda) &= \left[-g_\rho\right](-\lambda),
\end{aligned}$$

so that the modified function $\left[-g_\rho\right](-\cdot)$ is the Moreau envelope of the previous unmodified function $[-g](-\cdot)$. In particular, we observe that the modified function $\left[-g_\rho\right](-\cdot)$ will be smooth whether or not the unmodified function $[-g](-\cdot)$ was.

In order to evaluate the modified Lagrange dual function $g_\rho(\cdot)$ we determine an infimum of the Lagrangian $L_\rho(\cdot,\lambda_\$)$ problem

In order to evaluate the modified Lagrange dual function $g_\rho(\cdot)$ at $\lambda \overset{\text{set}}{=} \lambda_\$$ we determine the infimum of the modified Lagrangian $L_\rho(\cdot,\lambda_\$)$ problem. This was precisely the definition of the modified Lagrange dual function

$$\begin{aligned}
g_\rho(\lambda_\$) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}}\, L_\rho(x,\lambda_\$) \\
&= \underset{x\in\mathbb{R}^n}{\text{infimum}} \left\{f_\rho(x) + \langle\lambda_\$, h(x)\rangle\right\}.
\end{aligned}$$

[1] The statement here is for closed proper convex extended-real-valued functions $f_1(\cdot), f_2(\cdot) : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ such that relint dom $f_1(\cdot) \cap$ relint dom $f_2(\cdot) \neq \emptyset$. [HUL93b]. If we have instead simply dom $f_1(\cdot) \cap$ dom $f_2(\cdot) \neq \emptyset$ we must take the closure of the infimal convolution of the conjugates. With the earlier relint dom intersection "qualification condition", the infimal convolution is "exact" and the closure operation is superfluous (because the infimal convolution of the conjugates is already a closed function).

Whenever the infimum in the evaluation of $g_\rho(\cdot)$ is attained, we have also determined a solution to a perturbed version of the modified primal problem

In order to evaluate the modified Lagrange dual function $g_\rho(\cdot)$ at an argument $\lambda_\$ \in \mathbb{R}^{p*}$, we find an infimum:

$$\begin{aligned} g_\rho(\lambda_\$) &\stackrel{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}}\, L_\rho(x,\lambda_\$) \\ &= \underset{x\in\mathbb{R}^n}{\text{infimum}}\, \left\{ f_\rho(x) + \langle \lambda_\$, h(x)\rangle \right\}. \end{aligned}$$

When there exists some x argument, say $x^+_{\rho,\lambda_\$}$, for which this infimum is attained, it is in fact the case that $x^+_{\rho,\lambda_\$}$ is also an optimizing argument for a perturbed version of the modified primal problem. Explicitly, we are saying that

$$\left[x^+_{\rho,\lambda_\$} \in \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, L_\rho(x,\lambda_\$) \right] \Longrightarrow \left[x^+_{\rho,\lambda_\$} \in \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, \left\{ f_\rho(x) \mid h(x) = b_{\rho,\lambda_\$} \right\} \right],$$

where $b_{\rho,\lambda_\$} \stackrel{\text{set}}{=} h\left(x^+_{\rho,\lambda_\$}\right)$.

From $x^+_{\rho,\lambda_\$} \in \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, L_\rho(x,\lambda_\$)$ we find

$$\begin{aligned} x^+_{\rho,\lambda_\$} &\in \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, L_\rho(x,\lambda_\$) \\ L\left(x^+_{\rho,\lambda_\$}, \lambda_\$\right) &\le L(x,\lambda_\$) \text{ for all } x \in \mathbb{R}^n \\ f_\rho\left(x^+_{\rho,\lambda_\$}\right) + \left\langle \lambda_\$, h\left(x^+_{\rho,\lambda_\$}\right)\right\rangle &\le f_\rho(x) + \langle \lambda_\$, h(x)\rangle \text{ for all } x \in \mathbb{R}^n \\ f_\rho\left(x^+_{\rho,\lambda_\$}\right) &\le f_\rho(x) \text{ for all } x \in \left\{ x \in \mathbb{R}^n \mid h(x) = b_{\rho,\lambda_\$} \right\}, \end{aligned}$$

recalling that $b_{\rho,\lambda_\$} \stackrel{\text{set}}{=} h\left(x^+_{\rho,\lambda_\$}\right)$, so that every x in the constraint set also yields $h(x) = b_{\rho,\lambda_\$} = h\left(x^+_{\rho,\lambda_\$}\right)$.

We recognize the last expression as an alternative method of stating $x^+_{\rho,\lambda_\$} \in \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, \left\{ f_\rho(x) \mid h(x) = b_{\rho,\lambda_\$} \right\}$.

This is Everett's Theorem for the modified primal problem.

In order to determine a subgradient of the modified $[-g_\rho](\cdot)$ we determine an optimizing argument of the modified Lagrangian $L_\rho(\cdot,\lambda_\$)$ problem

Consider the modified Lagrange dual function evaluation problem: with $\lambda_\$$ fixed, optimize $L_\rho(\cdot,\lambda_\$)$ over x.

Suppose that this infimal value is attained by some $x \in \mathbb{R}^n$,say

$$\begin{aligned} x^+_{\rho,\lambda_\$} &\overset{\text{set}}{\in} \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, L_\rho\left(x,\lambda_\$\right) \\ &\in \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, \left\{f_\rho\left(x\right)+\left\langle\lambda_\$,h\left(x\right)\right\rangle\right\}. \end{aligned}$$

We denote the set of all such modified Lagrangian-$L\left(\cdot,\lambda_\$\right)$-infimal-value-attaining x-arguments as $\mathcal{X}^+_{\rho,\lambda_\$}$, with definition

$$\mathcal{X}^+_{\rho,\lambda_\$} \overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{Argmin}}\, L\left(x,\lambda_\$\right).$$

We now show how such an optimal argument $x^+_{\rho,\lambda_\$}$ can be used to determine an element of $\partial\left[-g_\rho\right]\left(\lambda_\$\right)$, the subdifferential set of the (convex by construction) modified function $\left[-g_\rho\right]\left(\cdot\right)$ at the argument $\lambda_\$$.

Note that the optimal argument $x^+_{\rho,\lambda_\$}$ satisfies $g_\rho\left(\lambda_\$\right)=L_\rho\left(x^+_{\rho,\lambda_\$},\lambda_\$\right)$ which is in turn equal to $\{f_\rho\left(x^+_{\rho,\lambda_\$}\right)+\left\langle\lambda_\$,h\left(x^+_{\rho,\lambda_\$}\right)\right\rangle\}$.

Keeping in mind our notation $x^+_{\rho,\lambda_\$}$ and $\mathcal{X}^+_{\rho,\lambda_\$}$, we observe that $-h\left(x^+_{\rho,\lambda_\$}\right)$ is an element of $\partial\left[-g_\rho\right]\left(\lambda_\$\right)$, the subdifferential set of the (convex by construction) modified function $\left[-g_\rho\right]\left(\cdot\right)$ at the argument $\lambda_\$$. Explicitly,

$$-h\left(x^+_{\rho,\lambda_\$}\right)\in\partial\left[-g_\rho\right]\left(\lambda_\$\right).$$

To see this, first recall that $g_\rho\left(\lambda\right)$ is defined as $\underset{x\in\mathbb{R}^n}{\text{infimum}}\,\left\{f_\rho\left(x\right)+\left\langle\lambda,h\left(x\right)\right\rangle\right\}$. From this, we have

$$\begin{aligned} g_\rho\left(\lambda\right) &\overset{\text{def}}{=} \underset{x\in\mathbb{R}^n}{\text{infimum}}\,\left\{f_\rho\left(x\right)+\left\langle\lambda,h\left(x\right)\right\rangle\right\} \\ g_\rho\left(\lambda\right) &\le f_\rho\left(x\right)+\left\langle\lambda,h\left(x\right)\right\rangle \text{ for all } x\in\mathbb{R}^n \\ g_\rho\left(\lambda\right) &\le f_0\left(x^+_{\rho,\lambda_\$}\right)+\left\langle\lambda,h\left(x^+_{\rho,\lambda_\$}\right)\right\rangle \text{ since } x^+_{\rho,\lambda_\$}\in\mathbb{R}^n \\ \left[-g_\rho\right]\left(\lambda\right) &\ge -f_0\left(x^+_{\rho,\lambda_\$}\right)-\left\langle\lambda,h\left(x^+_{\rho,\lambda_\$}\right)\right\rangle \text{ since } x^+_{\rho,\lambda_\$}\in\mathbb{R}^n. \end{aligned}$$

We next seek to relate the expression above to $g_\rho\left(\lambda_\$\right)=L_\rho\left(x^+_{\rho,\lambda_\$},\lambda_\$\right)=f_\rho\left(x^+_{\rho,\lambda_\$}\right)+\left\langle\lambda_\$,h\left(x^+_{\rho,\lambda_\$}\right)\right\rangle$; more precisely, to $\left[-g_\rho\right]\left(\lambda_\$\right)=-f_\rho\left(x^+_{\rho,\lambda_\$}\right)-\left\langle\lambda_\$,h\left(x^+_{\rho,\lambda_\$}\right)\right\rangle$. We do this by adding and subtracting $\left\langle \lambda_{\$}, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle$, to find

$$\begin{aligned}
\left[-g_{\rho}\right](\lambda) &\geq -f_{\rho}\left(x^{+}_{\rho,\lambda_{\$}}\right) - \left\langle \lambda, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle \text{ since } x^{+}_{\rho,\lambda_{\$}} \in \mathbb{R}^{n}. \\
&= -f_{\rho}\left(x^{+}_{\rho,\lambda_{\$}}\right) + \left[\left\langle \lambda_{\$} - \lambda_{\$}, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle\right] - \left\langle \lambda, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle \\
&= -f_{\rho}\left(x^{+}_{\rho,\lambda_{\$}}\right) + \left\langle \lambda_{\$}, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle - \left\langle \lambda_{\$}, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle - \left\langle \lambda, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle \\
&= -f_{\rho}\left(x^{+}_{\rho,\lambda_{\$}}\right) - \left\langle \lambda_{\$}, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle + \left\langle \lambda_{\$} - \lambda, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle \\
&= \left[-g_{\rho}\right](\lambda_{\$}) + \left\langle \lambda_{\$} - \lambda, h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle \\
&= \left[-g_{\rho}\right](\lambda_{\$}) + \left\langle \lambda - \lambda_{\$}, -h\left(x^{+}_{\rho,\lambda_{\$}}\right)\right\rangle.
\end{aligned}$$

Since this last expression holds for any $\lambda \in \mathbb{R}^{p*}$ and any $\lambda_{\$} \in \mathbb{R}^{p*}$, we have established $-h\left(x^{+}_{\rho,\lambda_{\$}}\right) \in \partial\left[-g\right](\lambda_{\$})$.

Lagrange duality relationships between the modified Lagrange dual function $g(\cdot)$ and the modified primal objective function $f_{\rho}(\cdot)$ (evaluated in the feasible set): "weak duality"

First, consider any dual λ-argument, say $\lambda_{\$} \in \mathbb{R}^{p*}$.

Now consider any feasible primal argument, say $\tilde{x}$; explicitly $\tilde{x}$ is an element of the set denoted $\tilde{\mathcal{X}}$ and characterized as $\{x \in \mathbb{R}^{n} \mid h(x) = 0\}$.

For any such (dual, feasible primal) pair $(\lambda_{\$}, \tilde{x}) \in \mathbb{R}^{p*} \times \tilde{\mathcal{X}}$ pair, the following relationship always holds:

$$g_{\rho}(\lambda_{\$}) \leq f_{\rho}(\tilde{x}) = f_{0}(\tilde{x}).$$

As one specific case of the above expression: if we consider an optimal (modified Lagrange dual problem) λ-argument $\lambda \overset{\text{set}}{=} \lambda^{\star}_{\rho}$ and an optimal (and so feasible for modified primal problem) x-argument $x \overset{\text{set}}{=} x^{\star} \in \tilde{\mathcal{X}}$, we have

$$g_{\rho}\left(\lambda^{\star}_{\rho}\right) \leq f_{\rho}(x^{\star}) = f_{0}(x^{\star}),$$

which we may alternately write using our previously introduced optimal value notation as

$$d^{\star}_{\rho} \leq p^{\star}_{\rho}(0) = p^{\star}(0).$$

Chapter 13

Operator theory basics

13.1 Introduction

We first establish informal conventions for use of the terms “function”, “operator”, and “mapping”.

We refer to objects that look like $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ as functions.

We refer to objects that look like $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ or $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n*}$ as operators.

We refer to objects that look like $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ or $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^{m*}$ as mappings.

We refer to objects that look like $f(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}}$ as multifunctions, where $2^{\mathbb{R}}$ denotes the power set of $\mathbb{R}$ (the set made up of all sets of elements in $\mathbb{R}$) .

We refer to objects that look like $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ or $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^{n*}}$ as point-to-set operators.

We refer to objects that look like $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ or $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^{m*}}$ as point-to-set mappings.

The conventions above are informal in the sense that there appears to be no consensus in the literature on what should distinguish the term “operator” from the term “mapping”. In general usage, when we use the term “mapping” the comment will also apply to operators. Other descriptions of point-to-set operators are include multivalued operators and set-valued operators.

In general, we will use upper case letters to denote operators and mappings and lower case letters to denote functions.

Instead of the power-set-based notation $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$, an alternative notation sometimes used to indicate a point-to-set mapping is $T(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. We prefer $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$, since the power set notation $2^{\mathbb{R}^m}$ (as a reference to "the set of all sets in $\mathbb{R}^m$") is arguably a more explicit statement of the setting in which the mapping takes on values (that is, the mapping takes in some element of $\mathbb{R}^n$ and returns a set of elements that are each in $\mathbb{R}^m$). An alternative notation for the power set $2^{\mathbb{R}^m}$ is $\mathcal{P}(\mathbb{R}^m)$.

As an example operator, consider the gradient descent iterative update from x^k to x^{k+1}:

$$x^{k+1} \stackrel{\text{set}}{=} x^k - \alpha \nabla f\left(x^k\right).$$

To interpret this update as the result of applying an operator, we introduce the notation $T(x) \stackrel{\text{set}}{=} x - \alpha \nabla f(x)$.

13.2 Definitions

Definition 63 (Point-to-set mapping)**.** A *point-to-set mapping* $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is characterized by the subset of the product space $\mathbb{R}^n \times \mathbb{R}^m$ made up of all pairs of (input, output) elements $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ that occur. The "set" in point-to-set indicates that a single input element x may be associated (via the mapping) with an entire set of output elements; we denote the set of output elements from $\mathbb{R}^m$ associated (via the mapping $T(\cdot)$) with an input element x by $T(x) \subseteq \mathbb{R}^m$.

A point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ as introduced above associates with each x a subset of $\mathbb{R}^m$; in particular, note that the empty set is a subset of $\mathbb{R}^m$ and so we may have $F(x) = \emptyset$. We also introduce the convention that set addition between any set and the empty set yields the empty set: explicitly, $\mathcal{S} + \emptyset = \emptyset$ for any set $\mathcal{S} \subseteq \mathbb{R}^n$. (This is equivalent to the addition rule for extended-real-valued functions by which the sum of any real value with infinity is equal to infinity).

Definition 64 (Effective domain)**.** We refer to the collection of all input arguments to a point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ that do not map to the empty set as the *effective* *domain* of the mapping $T(\cdot)$. This is a subset in $\mathbb{R}^n$ denoted dom $T(\cdot) \subseteq \mathbb{R}^n$ and defined via the expression

$$\text{dom } T(\cdot) \overset{\text{def}}{=} \{x \in \mathbb{R}^n \mid T(x) \neq \emptyset\}.$$

The effective domain dom $T(\cdot) \subseteq \mathbb{R}^n$ can be visualized as the "projection" of gr $T(\cdot)$ onto the input space $\mathbb{R}^n$. An alternate name for the effective domain of a point-to-set mapping is "the set of definition".

We also have a notion of properness for point-to-set mappings.

Definition 65 (Properness)**.** We say that a point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is *proper* (or nontrivial) when it has non-empty effective domain, explicitly

$$\text{dom } T(\cdot) \neq \emptyset.$$

That is, we refer to a point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ as proper when there exists at least one $x_\# \in \mathbb{R}^n$ for which $T(x_\#) \neq \emptyset$.

Definition 66 (Graph)**.** The *graph* (or graphical representation) of a point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is a subset of $\mathbb{R}^n \times \mathbb{R}^m$ denoted gr $T(\cdot) \subset \mathbb{R}^n \times \mathbb{R}^m$ and defined via the expression

$$\text{gr } T(\cdot) \overset{\text{def}}{=} \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in T(x)\}.$$

In particular, the expression $(x,y) \in \text{gr } T(\cdot)$ is equivalent to the expression $y \in T(x)$. In some settings, the "gr" is omitted and the expression $(x,y) \in T(\cdot)$ is used as a shorthand for $(x,y) \in \text{gr } T(\cdot)$. An alternative path to point-to-set operators views them directly as subsets of the (input,output) space $\mathbb{R}^n \times \mathbb{R}^m$; these subsets are described as "relations" (of $\mathbb{R}^n$ and $\mathbb{R}^m$) or "correspondences" (of $\mathbb{R}^n$ and $\mathbb{R}^m$). Our approach to this subset view is by means of the graph gr $T(\cdot) \subset \mathbb{R}^n \times \mathbb{R}^m$.

Definition 67 (Image of a point)**.** We refer to the set $T(x) \subseteq \mathbb{R}^m$ as the *image of the point* $x \in \mathbb{R}^n$ with respect to the point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$.

We can generalize this notion to an entire set.

Definition 68 (Image of a set)**.** The *image of a set*, say $\mathcal{S} \subseteq \mathbb{R}^n$ of the input space $\mathbb{R}^n$ with respect to a point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$is the subset of the output space $\mathbb{R}^m$ described as the union of all image sets for input arguments from $\mathcal{S}$. The image of $\mathcal{S}$ with respect to $T(\cdot)$ is denoted $T(\mathcal{S}) \subseteq \mathbb{R}^m$ and defined via the expression

$$T(\mathcal{S}) \stackrel{\text{def}}{=} \bigcup_{x \in \mathcal{S}} T(x).$$

We can also consider the image of the entire input space with respect to $T(\cdot)$; this is sometimes referred to as the "image of the point-to-set-operator".

Definition 69 (Image of input space)**.** The *image of the input space* with respect to the point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$(or the range of $T(\cdot)$ or set of values of $T(\cdot)$) is the subset of the output space $\mathbb{R}^m$ described as the union of all image sets for input arguments from $\mathbb{R}^n$. The image of $\mathbb{R}^n$ with respect to $T(\cdot)$ is denoted image $T(\cdot) \subseteq \mathbb{R}^m$ (or more explicitly $T(\mathbb{R}^n)$) and defined via the expression

$$\text{image } T(\cdot) \stackrel{\text{def}}{=} T(\mathbb{R}^n) = \bigcup_{x \in \mathbb{R}^n} T(x) = \bigcup_{x \in \text{dom } T(\cdot)} T(x).$$

The image $T(\mathbb{R}^n) \subseteq \mathbb{R}^m$ of the input space with respect to the point-to-set operator $T(\cdot)$ can be visualized as the "projection" of gr $T(\cdot)$ onto the output space $\mathbb{R}^m$.

We can state the effective domain and the image of $T(\cdot)$ is a more matched way by referencing the graph gr $T(\cdot)$:

$$\begin{aligned} \text{dom } T(\cdot) &= \{x \in \mathbb{R}^n \mid \text{there exists some } y \in \mathbb{R}^m \text{ with } (x,y) \in \text{gr } T(\cdot)\} \\ \text{image } T(\cdot) &= \{y \in \mathbb{R}^m \mid \text{there exists some } x \in \mathbb{R}^n \text{ with } (x,y) \in \text{gr } T(\cdot)\}. \end{aligned}$$

The graph gr $T(\cdot)$ is also helpful in describing the inverse of a point-to-set mapping.

Definition 70 (Inverse mapping)**.** The *inverse mapping* associated with a point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is the (potentially) point-to-set mapping denoted $T^{-1}(\cdot) : \mathbb{R}^m \to 2^{\mathbb{R}^n}$ and defined (via its graph) through the expression

$$\text{gr } T^{-1}(\cdot) \stackrel{\text{def}}{=} \{(y,x) \in \mathbb{R}^m \times \mathbb{R}^n \mid (x,y) \in \text{gr } T(\cdot)\}.$$

A more direct statement might be:

$$(y,x) \in \text{gr } T^{-1}(\cdot) \Longleftrightarrow (x,y) \in \text{gr } T(\cdot)$$

or more completely

$$(y,x) \in \text{gr } T^{-1}(\cdot) \Longleftrightarrow x \in T^{-1}(y) \Longleftrightarrow y \in T(x) \Longleftrightarrow (x,y) \in \text{gr } T(\cdot).$$

From the statement immediately above and our matched expressions from dom $T(\cdot)$ and image $T(\cdot)$ we observe expressions for dom $T^{-1}(\cdot)$ and image $T^{-1}(\cdot)$:

$$\text{image } T^{-1}(\cdot) = \text{dom } T(\cdot) = \{x \in \mathbb{R}^n \mid \text{there exists some } y \in \mathbb{R}^m \text{ with } (x,y) \in \text{gr } T(\cdot)\}$$
$$\text{dom } T^{-1}(\cdot) = \text{image } T(\cdot) = \{y \in \mathbb{R}^m \mid \text{there exists some } x \in \mathbb{R}^n \text{ with } (x,y) \in \text{gr } T(\cdot)\}.$$

Definition 71 (Constant mapping)**.** We say that a point-to-set mapping $T(\cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^{m*}}$ is a *constant* point-to-set mapping if there exists some subset of the output space $\mathbb{R}^m$, say $\mathcal{S} \subseteq \mathbb{R}^m$ such that $\mathcal{S}$ is the image set for every element of the input space. Explicitly,

$$T(x) \stackrel{\text{set}}{=} \mathcal{S} \text{ for each } x \in \mathbb{R}^m.$$

Since our goal is to use operator theoretic ideas to analyze iterative methods of optimization, we need to introduce some additional terms. Note that we are now considering operators (mappings of the form $\mathbb{R}^n \to \mathbb{R}^n$ or $\mathbb{R}^n \to \mathbb{R}^{n*}$) rather than mappings.

The identity operator is as one would expect:

Definition 72 (Identity operator)**.** The identity operator $I(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ returns whatever its input is; we define it (via its graph) through the expression

$$\text{gr } I(\cdot) \stackrel{\text{def}}{=} \{(x,x) \in \mathbb{R}^n \times \mathbb{R}^n \mid \text{for all } x \in \mathbb{R}^n\}.$$

Note that the identity operator is always single-valued.

Optimization methods typically iterate until some argument remains unchanged by the update process; these are the fixed points of the update.

Definition 73 (Fixed point)**.** We say that a point $x \in \mathbb{R}^n$ is a *fixed point* of the operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ when it remains unchanged after being acted on by $T(\cdot)$; explicitly

$$T(x) = x.$$

To indicate that a particular argument is a fixed point, we may use the notation $x^{\star}, x^{+}$, or x_{fix}.

The collection of all fixed points of $T(\cdot)$ is the fixed point set of $T(\cdot)$.

Definition 74 (Fixed point set)**.** Associated with any operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is the collection of all fixed points of $T(\cdot)$, a set in $\mathbb{R}^n$ denoted Fix $T(\cdot) \subseteq \mathbb{R}^n$ and defined via the expression

$$\text{Fix } T(\cdot) \overset{\text{def}}{=} \{x \in \mathbb{R}^n \mid T(x) = x\}.$$

Note that an operator need not have any fixed points; consider the choice $T(x) \overset{\text{set}}{=} x + b$.

An idea that (in some sense) runs parallel to the notion of a fixed point of an operator is that of a zero of an operator. As we will discuss later, the "types" of operator for which we are interested in zeros are not the same as the "types" of operators for which we are interested in fixed points (although there will generally be a close correspondence). To emphasize this distinction, we will shift from our previous use of $T(\cdot)$ to denote a generic operator to a use of $M(\cdot)$.

Definition 75 (Zero)**.** We say that a point $x \in \mathbb{R}^n$ is a *zero* of the operator $M(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n*}$ when it is mapped to zero under $M(\cdot)$; explicitly

$$M(x) = 0.$$

The collection of all zeros of $M(\cdot)$ is the zero set of $M(\cdot)$.

Definition 76 (Zero set)**.** Associated with any operator $M(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n*}$ is the collection of all zeros of $T(\cdot)$, a set in $\mathbb{R}^n$ denoted Zeros $T(\cdot) \subseteq \mathbb{R}^n$ and defined via the expression

$$\text{Zeros } M(\cdot) \overset{\text{def}}{=} \{x \in \mathbb{R}^n \mid M(x) = 0\}.$$

We next consider the notion of a displacement operator associated with some operator.

Definition 77. For any point-to-set operator $T(\cdot)\ :\ \mathbb{R}^n \to 2^{\mathbb{R}^n}$, we have an associated *displacement* operator, $G(\cdot)$, characterized via the expression

$$G(\cdot) \overset{\text{set}}{=} [I - T](\cdot).$$

The displacement operator is called the complement operator by [Byr08]; this follows from the observation that G is the additive complement of T, since $G + T = I$. Many references do not provide a name for this operator at all, yet it shows up repeatedly and in important ways. Note that the fixed point set of an operator T coincides with the zeros of the associated displacement operator $G \overset{\text{set}}{=} I - T$; explicitly, $\text{Fix}\ T(\cdot) = \text{Zeros}\ G(\cdot)$.

Definition 78. For any point-to-set operator $T(\cdot)\ :\ \mathbb{R}^n \to 2^{\mathbb{R}^n}$, we have an associated *reflection* operator, $R(\cdot)$, characterized via the expression

$$R(\cdot) \overset{\text{set}}{=} 2T(\cdot) - I(\cdot).$$

The reflection operator of a point-to-set operator will be useful in discussing "over- or under-relaxation" in the context of our iterative updates.

Definition 79. For any point-to-set operator $M(\cdot)\ :\ \mathbb{R}^n \to 2^{\mathbb{R}^{n*}}$, we have an associated operator called the *λ-Yosida regularization* (or approximation) of M, parameterized by a strictly positive scalar $\lambda \in \mathbb{R}_{++}$, typically denoted by $M_\lambda(\cdot)\ :\ \mathbb{R}^n \to \mathbb{R}^n$ and defined via the expression

$$M_\lambda(\cdot) \overset{\text{def}}{=} \left[\lambda I + M^{-1}\right]^{-1}(\cdot).$$

See Figure 13.1.

The standard reference for this material is [Ber63]; a reference notable for its valuable comments is [AF08]. Other references include [KK95] and [ALM13].

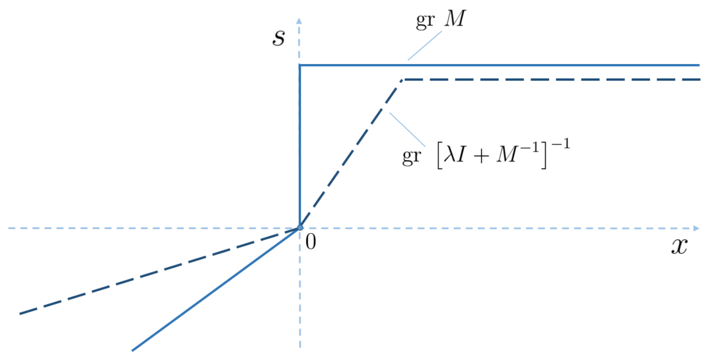

Figure 13.1: λ-Yosida regularization of a monotone operator M. (After [RW04]).

13.3 Relationships

13.3.1 Polarization identity

Consider a generic operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and its associated displacement operator $G(\cdot) \overset{\text{set}}{=} I(\cdot) - T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$.

We have

$$\langle Tx - Ty, x - y \rangle = \frac{1}{2} \|Tx - Ty\|_2^2 + \frac{1}{2} \|x - y\|_2^2 - \frac{1}{2} \|Gx - Gy\|_2^2 .$$

This is an expression of the "polarization identity" [AST10]; we also may recognize this as a "vector form" of the law of cosines:

$$\begin{aligned}
\|u - v\|_2^2 &= \|u\|_2^2 + \|v\|_2^2 - 2\langle u, v \rangle \\
\|[Tx - Ty] - [x - y]\|_2^2 &= \|Tx - Ty\|_2^2 + \|x - y\|_2^2 - 2\langle Tx - Ty, x - y \rangle \\
\|[x - y] - [Tx - Ty]\|_2^2 &= \|Tx - Ty\|_2^2 + \|x - y\|_2^2 - 2\langle Tx - Ty, x - y \rangle \\
\|Gx - Gy\|_2^2 &= \|Tx - Ty\|_2^2 + \|x - y\|_2^2 - 2\langle Tx - Ty, x - y \rangle \\
\langle Tx - Ty, x - y \rangle &= \frac{1}{2} \|Tx - Ty\|_2^2 + \frac{1}{2} \|x - y\|_2^2 - \frac{1}{2} \|Gx - Gy\|_2^2 .
\end{aligned}$$

Note that we could equivalently express the immediately preceding result as

$$\langle Gx - Gy, x - y \rangle = \frac{1}{2} \|Gx - Gy\|_2^2 + \frac{1}{2} \|x - y\|_2^2 - \frac{1}{2} \|Tx - Ty\|_2^2 .$$

Two additional forms will sometimes be useful:

$$\|Tx - Ty\|_2^2 = \|x - y\|_2^2 - \|Gx - Gy\|_2^2 - 2\langle Tx - Ty, x - y\rangle$$
$$\|Gx - Gy\|_2^2 = \|x - y\|_2^2 - \|Tx - Ty\|_2^2 - 2\langle Tx - Ty, x - y\rangle .$$

13.3.2 Another identity

Consider a generic operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and its associated displacement operator $G(\cdot) \stackrel{\text{set}}{=} I(\cdot) - T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$.

We have

$$\langle Tx - Ty, x - y\rangle - \|Tx - Ty\|_2^2 = \langle Gx - Gy, x - y\rangle - \|Gx - Gy\|_2^2 .$$

See [Byr08] for further discussion.

Chapter 14

Contractivity-type properties and monotonicity-type properties

14.1 Contractivity-type properties

When one represents the iterative update of some optimization method as an operator, questions about the convergence of the optimization method can instead be rephrased as questions about properties of the operator.

In this section, we consider categorization of operator properties in the form of inequalities relating $\|Tx_\# - Tx_\$\|_2^2$ (the squared distance between vectors after applying the operator) to two other quantities: $\|x_\# - x_\$\|_2^2$ (the squared distance between vectors before applying the operator) and $\|Gx_\# - Gx_\$\|_2^2$ (the squared distance after applying the associated displacement operator to the vectors).

All of the main contractivity-type property categories that we consider can be described in terms of parameter value ranges associated with a single inequality:

$$\|Tx_\# - Tx_\$\|_2^2 \leq a\,\|x_\# - x_\$\|_2^2 + b\,\|Gx_\# - Gx_\$\|_2^2\,.$$

For most categories, this inequality is required to hold for all $x_\#, x_\$ \in \mathbb{R}^n$; in some situations we may qualify the discussion to require the arguments (and/or their associated displacement mappings) to be distinct.

We consider parameter ranges $a \in (-\infty, 1]$ and $b \in (-\infty, 1]$; we discuss more specific terminology depending on specific subranges.

As we discuss below (and collect in Table 14.1), different values of the parameters a and b above correspond to different operator classes:

- when T satisfies the inequality above for $a \overset{\text{set}}{=} 1$, $b \overset{\text{set}}{=} 1$, we say that $T(\cdot)$ is *pseudocontractive*
- when T satisfies the inequality above for $a \overset{\text{set}}{=} 1$, $b \in (0,1)$, we say that $T(\cdot)$ is *strictly pseudocontractive*
- when T satisfies the inequality above for $a \in (0,1)$, $b \overset{\text{set}}{=} 1$, we say that $T(\cdot)$ is *displacement strictly pseudocontractive*
- when T satisfies the inequality above for $a \overset{\text{set}}{=} 1$, $b \overset{\text{set}}{=} 0$, we say that $T(\cdot)$ is *nonexpansive*
- when T satisfies the inequality above for $a \overset{\text{set}}{=} 0$, $b \overset{\text{set}}{=} 1$, we say that $T(\cdot)$ is *displacement nonexpansive*
- when T satisfies the inequality above for $a \overset{\text{set}}{=} 1$, $b \in (-\infty,0)$, we say that $T(\cdot)$ is *decreasing pseudocontractive*
- when T satisfies the inequality above for $a \in (-\infty,0)\,, b \overset{\text{set}}{=} 1$, we say that $T(\cdot)$ is *displacement decreasing pseudocontractive*
- when T satisfies the inequality above for $a \overset{\text{set}}{=} 1$, $b \overset{\text{set}}{=} -1$, we say that $T(\cdot)$ is *firmly nonexpansive*
- when T satisfies the inequality above for $a \overset{\text{set}}{=} -1$, $b \overset{\text{set}}{=} 1$, we say that $T(\cdot)$ is *displacement firmly nonexpansive*
- when T satisfies the inequality above for $a \in [0,1)$, $b \overset{\text{set}}{=} 0$, we say that $T(\cdot)$ is *strictly contractive*
- when T satisfies the inequality above for $a \overset{\text{set}}{=} 0$, $b \in [0,1)$, we say that $T(\cdot)$ is *displacement strictly contractive*.

The terminology above merits some additional discussion, because it differs from the existing literature in two important respects. First, the existing literature appears to make no reference to any of the properties that we describe with the prefatory adjective "displacement"; nonetheless, as we show when we consider the correspondence between contractivity-type properties and monotonicity-type properties, any discussion of contractivity-type properties is incomplete without these displacement-operator-centric categories. Second, the existing terminology is frequently fragmented and contradictory. For example, our category of "decreasing pseudocontractivity" is in some places simply referred to as "pseudocontractivity" (see [Byr08] or [Var09]); these references omit any discussion of the categories that we refer to as "strictly pseudocontractive" or "pseudocontractive". On the other hand, references that refer to the categories that we refer to as "pseudocontractive" and "strictly pseudocontractive" often omit any separate discussion of the category that we refer to as "decreasing pseudocontractive" (see [BP67]). Sometimes the category that we refer to as decreasing pseudocontractive is instead referred to as "averaged nonexpansive" (see [BC11] or [Com04]); we feel that this term breaks the connection that exists among the various varieties that we refer to in common as "pseudocontractive" and so will not use this term.

Another example of contradictory use of the term "pseudocontractive" occurs in [BT89], where the term "pseudocontractive" is used to refer to what we later describe as "strictly contractive with respect to the fixed point set Fix T". We will call further attention to this notational collision when we subsequently consider contractivity-type conditions with respect to the fixed point set Fix T. A related notion applied to a slightly different context is introduced in [CZ97] under the name "targeted contraction".

Finally, we mention one additional thread of terminology as exemplified by [VE09]. What we call below "nonexpansive with respect to the nonempty fixed point set $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$" [VE09] refer to either as $\mathcal{F}$-Fejer or $\mathcal{F}$-quasi-nonexpansive. The category that we call below "strictly contractive with respect to the nonempty fixed point set $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$" corresponds (essentially) to what [VE09] refer to as $\mathcal{F}$-Fejer or strictly $\mathcal{F}$-quasi-nonexpansive; this category also essentially corresponds what [BT89] refer to as pseudocontractive (with respect to the nonempty fixed point set Fix T) and to what [Byr08] refers to as paracontractive (with respect to the nonempty fixed point set Fix T). What we refer to as "pseudocontractive with respect to the nonempty fixed point set $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$" essentially corresponds to what [VE09] refer to as strongly $\mathcal{F}$-Fejer or $\mathcal{F}$-

pseudocontractive.

We formally define these property classes below and display the classes in Table 14.1.

The geometric picture for these property classes is that each class of the properties above corresponds to the iterate being contained in a ball of a radius reflecting the class. We separately describe firm nonexpansiveness (because this term is widely used in the literature) but we emphasize that firm nonexpansiveness is a specific instance of decreasing pseudocontractivity.

Definition 80 (Strictly contractive with parameter c)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *c-strictly contractive* when there exists a constant $c \in [0,1)$ such that

$$\|Tx_{\#} - Tx_{\$}\|_2 \leq c\,\|x_{\#} - x_{\$}\|_2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

We denote the class of all strictly contractive operators by $\mathcal{S}_{\text{sc}}$. When we wish to include specific reference to the strict contractivity parameter c, we write $\mathcal{S}_{\text{sc}}(c)$ to denote the class of all operators that satisfy the c-strict contractivity condition.

Definition 81 (Displacement strictly contractive with parameter d)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *d-displacement strictly contractive* when there exists a constant $d \in [0,1)$ such that

$$\|Tx_{\#} - Tx_{\$}\|_2 \leq d\,\|Gx_{\#} - Gx_{\$}\|_2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

We denote the class of all displacement strictly contractive operators by $\mathcal{S}_{\text{dsc}}$. When we wish to include specific reference to the displacement strict contractivity parameter d, we write $\mathcal{S}_{\text{dsc}}(d)$ to denote the class of all operators that satisfy the d-displacement strict contractivity condition.

Definition 82 (Firmly nonexpansive)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *firmly nonexpansive* when

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 - \|Gx_{\#} - Gx_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

Table 14.1: Contractivity-type properties

description	notation	characterizing inequality	parameter range
strictly contractive	$\mathcal{S}_{\text{sc}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq c^2 \|x_\# - x_\$\|_2^2$	$c \in [0,1)$
displacement strictly contractive	$\mathcal{S}_{\text{dsc}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq d^2 \|Gx_\# - Gx_\$\|_2^2$	$d \in [0,1)$
firmly nonexpansive	$\mathcal{S}_{\text{fne}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \|Gx_\# - Gx_\$\|_2^2$	
displacement firmly nonexpansive	$\mathcal{S}_{\text{dfne}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 - \|x_\# - x_\$\|_2^2$	
decreasing pseudocontractive	$\mathcal{S}_{\text{dpc}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \nu \|Gx_\# - Gx_\$\|_2^2$	$\nu \in (0,+\infty)$
displacement decreasing pseudocontractive	$\mathcal{S}_{\text{ddpc}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 - \kappa \|x_\# - x_\$\|_2^2$	$\kappa \in (0,+\infty,)$
nonexpansive	$\mathcal{S}_{\text{ne}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2$	
displacement nonexpansive	$\mathcal{S}_{\text{dne}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2$	
strictly pseudocontractive	$\mathcal{S}_{\text{spc}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 + p \|Gx_\# - Gx_\$\|_2^2$	$p \in (0,1)$
displacement strictly pseudocontractive	$\mathcal{S}_{\text{dspc}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 + q \|x_\# - x_\$\|_2^2$	$q \in (0,1)$
pseudocontractive	$\mathcal{S}_{\text{pc}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2$	

We denote the class of all firmly nonexpansive operators by $\mathcal{S}_{\text{fne}}$.

Definition 83 (Displacement firmly nonexpansive)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *displacement firmly nonexpansive* when

$$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 - \|x_\# - x_\$\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all displacement firmly nonexpansive operators by $\mathcal{S}_{\text{dfne}}$.

Definition 84 (Decreasing pseudocontractive with parameter ν)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *ν-decreasing pseudocontractive*, for some $\nu \in (0, +\infty)$, when

$$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \nu \|Gx_\# - Gx_\$\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all decreasing pseudocontractive operators by $\mathcal{S}_{\text{dpc}}$. When we wish to include specific reference to the decreasing pseudocontractivity parameter ν, we write $\mathcal{S}_{\text{dpc}}(\nu)$ to denote the class of all operators that satisfy the ν-decreasing pseudocontractivity condition.

Definition 85 (Displacement decreasing pseudocontractive with parameter κ)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *κ-displacement decreasing pseudocontractive*, for some $\kappa \in (0, +\infty)$, when

$$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 - \kappa \|x_\# - x_\$\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all displacement decreasing pseudocontractive operators by $\mathcal{S}_{\text{ddpc}}$. When we wish to include specific reference to the displacement decreasing pseudocontractivity parameter κ, we write $\mathcal{S}_{\text{ddpc}}(\kappa)$ to denote the class of all operators that satisfy the κ-decreasing pseudocontractivity condition.

Definition 86 (Nonexpansive)**.** We say that operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *nonexpansive* when

$$\|Tx_\# - Tx_\$\|_2 \leq \|x_\# - x_\$\|_2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$.

We denote the class of all nonexpansive operators by $\mathcal{S}_{\text{ne}}$.

Definition 87 (Displacement nonexpansive)**.** We say that operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *displacement nonexpansive* when

$$\|Tx_\# - Tx_\$\|_2 \le \|Gx_\# - Gx_\$\|_2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$.

We denote the class of all displacement nonexpansive operators by $\mathcal{S}_{\text{dne}}$.

Definition 88 (Lipschitz with parameter L)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *L-Lipschitz*, for some nonnegative $L \in \mathbb{R}_+$, when

$$\|Tx_\# - Tx_\$\|_2 \le L\,\|x_\# - x_\$\|_2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$.

We denote the class of all Lipschitz operators by $\mathcal{S}_{\text{Lip}}$. When we wish to include specific reference to the Lipschitz parameter L, we write $\mathcal{S}_{\text{Lip}}(L)$ to denote the class of all operators that satisfy the L-Lipschitz condition.

Definition 89 (Displacement Lipschitz with parameter Λ)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *Λ-displacement Lipschitz*, for some nonnegative $\Lambda \in \mathbb{R}_+$, when

$$\|Tx_\# - Tx_\$\|_2 \le \Lambda\,\|x_\# - x_\$\|_2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$.

We denote the class of all displacement Lipschitz operators by $\mathcal{S}_{\text{dLip}}$. When we wish to include specific reference to the displacement Lipschitz parameter Λ, we write $\mathcal{S}_{\text{dLip}}(\Lambda)$ to denote the class of all operators that satisfy the Λ-displacement Lipschitz condition.

Definition 90 (Strictly pseudocontractive with parameter p)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *p-strictly pseudocontractive*, for some $p \in (0,1)$ when

$$\|Tx_\# - Tx_\$\|_2^2 \le \|x_\# - x_\$\|_2^2 + p\,\|Gx_\# - Gx_\$\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all strictly pseudocontractive operators by $\mathcal{S}_{\text{spc}}$. When we wish to include specific reference to the strict pseudocontractivity parameter p, we write $\mathcal{S}_{\text{spc}}(p)$ to denote the class of all operators that satisfy the p-strict pseudocontractivity condition.

Definition 91 (Displacement strictly pseudocontractive with parameter q)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *q-displacement strictly pseudocontractive*, for some $q \in (0,1)$ when

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + q\,\|Gx_{\#} - Gx_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all displacement strictly pseudocontractive operators by $\mathcal{S}_{\text{dspc}}$. When we wish to include specific reference to the displacement strict pseudocontractivity parameter q, we write $\mathcal{S}_{\text{dspc}}(q)$ to denote the class of all operators that satisfy the q-displacement strict pseudocontractivity condition.

Definition 92 (Pseudocontractive)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *pseudocontractive* when

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + \|Gx_{\#} - Gx_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all pseudocontractive operators by $\mathcal{S}_{\text{pc}}$.

We note the following inclusion relationships: $\mathcal{S}_{\text{sc}} \subset \mathcal{S}_{\text{dpc}} \subset \mathcal{S}_{\text{ne}} \subset \mathcal{S}_{\text{spc}} \subset \mathcal{S}_{\text{pc}}$. We do not explicitly mention the class of firmly nonexpansive operators because we regard this class as a special case of the class of decreasing pseudocontractive operators. The analogous inclusion relationships for the displacement operator classifications are $\mathcal{S}_{\text{dsc}} \subset \mathcal{S}_{\text{ddpc}} \subset \mathcal{S}_{\text{dne}} \subset \mathcal{S}_{\text{dspc}} \subset \mathcal{S}_{\text{pc}}$. Note that the standard definition of pseudocontractivity coincides with what we might have called displacement pseudocontractive; this motivates the use of the term pseudocontractive in either perspective.

14.2 Monotonicity-type properties

In this section, we consider categorization of operator properties in the form of inequalities relating $\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle$, the scalar product between the difference of applications of the operator G and the difference of initial argument values, to two other quantities: $\|x_{\#} - x_{\$}\|_2^2$, the squared distance between vectors before applying the operator and $\|Gx_{\#} - Gx_{\$}\|_2^2$, the squared distance after applying the operator G to the vectors. We deliberately state our monotonicity-type properties in terms of an operator referred to as G because we will see later that contractivity type properties of an operator, say T, correspond to monotonicity-type properties of the associated displacement operator $G \stackrel{\text{set}}{=} I - T$; thus, since we have referred to contractivity-type properties of an operator denoted T above, it will be appropriately suggestive to consider monotonicity-type properties of an operator denoted G.

With the exception of strict monotonicity, all of the categories that we consider can be described in terms of parameter value ranges associated with a single inequality:

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \sigma \|x_{\#} - x_{\$}\|_2^2 + \rho \|Gx_{\#} - Gx_{\$}\|_2^2 .$$

For most categories, this inequality is required to hold for all $x_{\#}, x_{\$} \in \mathbb{R}^n$; in some situations we may qualify the discussion to require the arguments (and/or their associated displacement mappings) to be distinct.

As we discuss below (and display in Table 14.2), different values of the parameters σ and ρ above correspond to different operator classes:

- when G satisfies the inequality above for some $\sigma \in \mathbb{R}_{++}$ and some $\rho \in \mathbb{R}_{++}$, we say that $G(\cdot)$ is combined strongly monotone
- when G satisfies the inequality above for some $\sigma \in \mathbb{R}_{++}$ and $\rho \stackrel{\text{set}}{=} 0$ we say that $G(\cdot)$ is strongly monotone
- when G satisfies the inequality above for some $\sigma \stackrel{\text{set}}{=} 0$ and some $\rho \in \mathbb{R}_{++}$, we say that $G(\cdot)$ is inverse strongly monotone
- when G satisfies the inequality above for $\sigma \stackrel{\text{set}}{=} 0$ and $\rho \stackrel{\text{set}}{=} 0$, we say that $G(\cdot)$ is monotone.

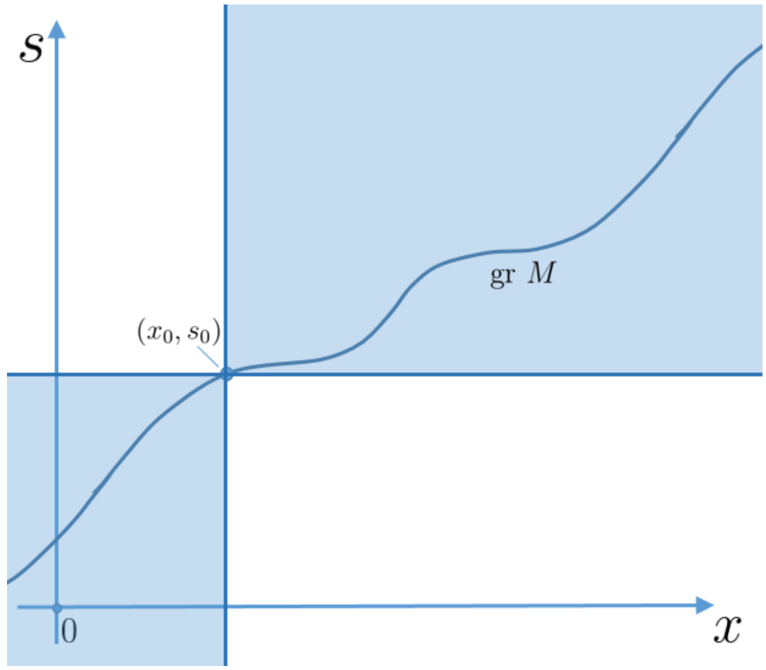

Figure 14.1: Monotone operator M (illustrated with respect to the point $(x_0, s_0) \in \text{gr } M$). (After[AN09]).

We will see later how the parameter ranges considered here for monotonicity-type properties of the displacement operator $G \overset{\text{set}}{=} I - T$ correspond to various categories of contractivity type properties of the operator T.

The monotonicity-type property that we refer to as “inverse strong monotonicity” is sometimes referred to as “co-coercivity” (see [Com09]), the “Dunn property”, or “strong F monotonicity”. We prefer our terminology as it more fully emphasizes the symmetry that exists between the categories under discussion and more explicitly emphasizes the close connection to the important application area of convex optimization. We also note that the category that we refer to as “combined strong monotonicity” does not seem to appear in the existing literature; however, the discussion in Nesterov [Nes04] provides great impetus to introduce such an explicit reference.

We formally define these property classes below and display the classes in Table 14.2. We also recall the previous illustrations of some of these properties; see Figures 14.1 and 14.2.

Definition 93 (Combined strongly monotone with parameters σ, ρ)**.** We say that an op-

Table 14.2: Monotonicity-type properties

description	notation	characterizing inequality	parameter range
combined strongly monotone	$\mathcal{M}_{\mathrm{csm}}$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \lVert x_{\#} - x_{\$} \rVert_2^2 + \rho \lVert Gx_{\#} - Gx_{\$} \rVert_2^2$	$\sigma \in \mathbb{R}_{++}, \rho \in \mathbb{R}_{++}$
strongly monotone	$\mathcal{M}_{\mathrm{sm}}$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \lVert x_{\#} - x_{\$} \rVert_2^2$	$\sigma \in \mathbb{R}_{++}, \rho = 0$
inverse strongly monotone	$\mathcal{M}_{\mathrm{ism}}$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \rho \lVert Gx_{\#} - Gx_{\$} \rVert_2^2$	$\sigma = 0, \rho \in \mathbb{R}_{++}$
strictly monotone	$\mathcal{M}_{\mathrm{ms}}$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle > 0$	
monotone	$\mathcal{M}_{\mathrm{m}}$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq 0$	$\sigma = 0, \rho = 0$

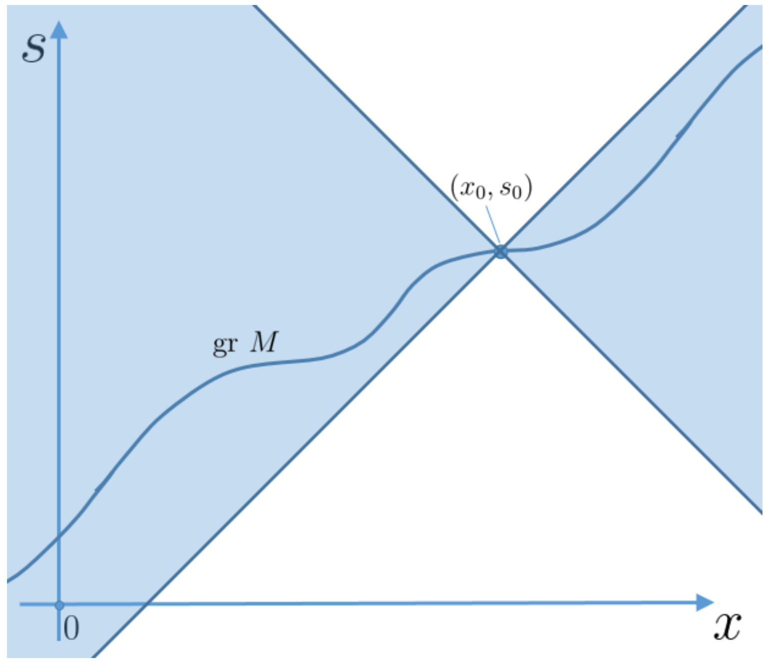

Figure 14.2: Lipschitz operator at the point $(x_0, s_0) \in \text{gr } M$. (After [AN09]).

erator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is (σ, ρ)-*combined strongly monotone* for some strictly positive $\sigma \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}_{++}$when

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \|x_{\#} - x_{\$}\|_2^2 + \rho \|Gx_{\#} - Gx_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

We denote the class of combined strongly monotone operators by $\mathcal{M}_{\text{csm}}$. When we wish to include specific reference to the parameters σ and ρ, we write $\mathcal{M}_{\text{csm}}(\sigma, \rho)$ to denote the class of all operators that satisfy the (σ, ρ)-combined strong monotonicity condition.

Definition 94 (Strongly monotone with parameter σ)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is σ-*strongly monotone* for some strictly positive $\sigma \in \mathbb{R}_{++}$ when

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \|x_{\#} - x_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

We denote the class of strongly monotone operators by $\mathcal{M}_{\text{sm}}$. When we wish to include specific reference to the parameter σ, we write $\mathcal{M}_{\text{sm}}(\sigma)$ to denote the class of all operators that satisfy the σ-strong monotonicity condition.

Definition 95 (Inverse strongly monotone with parameter ρ)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is ρ-*inverse strongly monotone* for some strictly positive $\rho \in \mathbb{R}_{++}$ when

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \rho \, \|Gx_{\#} - Gx_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

We denote the class of inverse strongly monotone operators by $\mathcal{M}_{\text{ism}}$. When we wish to include specific reference to the parameter ρ, we write $\mathcal{M}_{\text{ism}}(\rho)$ to denote the class of all operators that satisfy the ρ-inverse strong monotonicity condition.

Definition 96 (Separately strongly monotone with parameters σ, ρ)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is (σ, ρ)-*separately strongly monotone* for some strictly positive $\sigma \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}_{++}$when

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \, \|x_{\#} - x_{\$}\|_2^2$$

and

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \rho \, \|Gx_{\#} - Gx_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

Note that the category of separate strong monotonicity has a slightly asymmetric relationship to the category of combined strong monotonicity: the separate assumptions of separate strong monotonicity can be averaged to yield a range of combined strong monotonicity results, but the respective parameter values do not directly correspond to σ or ρ. On the other hand, an assumption of combined strong monotonicity immediately implies separate strong monotonicity with the same individual parameter values (simply drop the term that is not being considered). We include this category in our discussion because analysis is frequently based on separate strong monotonicity assumptions rather than a combined strong monotonicity assumption.

We denote the class of separately strongly monotone operators by $\mathcal{M}_{\text{ssm}}$. When we wish to include specific reference to the parameters σ and ρ, we write $\mathcal{M}_{\text{ssm}}(\sigma, \rho)$ to denote the class of all operators that satisfy the (σ, ρ)-separate strong monotonicity condition.

Definition 97 (Strictly monotone)**.** We say that an operator $G(\cdot)\,:\,\mathbb{R}^n \to \mathbb{R}^n$ is strictly monotone when

$$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle > 0$$

for all $x_\#, x_\$ \in \mathbb{R}^n$ such that $x_\# \neq x_\$$ and $Gx_\# \neq Gx_\$$.

We denote the class of strictly monotone operators by $\mathcal{M}_{\mathrm{ms}}$.

Definition 98 (Monotone)**.** We say that an operator $G\,:\,\mathbb{R}^n \to \mathbb{R}^n$ is *monotone* (or, more specifically, monotone nondecreasing) when

$$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq 0$$

for all $x_\#, x_\$ \in \mathbb{R}^n$.

We denote the class of monotone operators by $\mathcal{M}_{\mathrm{m}}$.

We note the following inclusion relationships: $\mathcal{M}_{\mathrm{csm}} \subset \mathcal{M}_{\mathrm{ism}} \subset \mathcal{M}_{\mathrm{m}}$ and $\mathcal{M}_{\mathrm{csm}} \subset \mathcal{M}_{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{m}}$. We discussed the placement of separate strong monotonicity above; in the preceding inclusion statement, separate strong monotonicity would essentially overlap with combined strong monotonicity.

14.3 Contractivity-type properties with respect to a fixed point

The definitions above require the contractivity-type inequalities to be satisfied for all possible pairs $x_\#, x_\$ \in \mathbb{R}^n$. We can also consider versions of these properties in which we require the inequalities to hold in a more specialized way: we specify a single fixed point, say $x_{\mathrm{fix}} \in \mathrm{Fix}\, T(\cdot)$, and require the inequalities to be satisfied with respect to that fixed point as the other argument ranges over all of $\mathbb{R}^n$. This is precisely analogous to the setting in convex optimization where we might require a Lipschitz condition to hold at the optimal argument. This correspondence to the case of restricting attention to a Lipschitz condition at the optimal argument highlights the reason we state contractivity-type with respect to a fixed point, rather than with respect to a generic

Table 14.3: Contractivity-type properties with respect to a fixed point. The subscript "p" is intended to suggest "point".

description	notation	characterizing inequality	parameter range
strictly contractive wrt x_{fix}	$\mathcal{S}_{\text{sc,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq c^2 \\|x_\# - x_{\text{fix}}\\|_2^2$	$c \in [0,1)$
displacement strictly contractive wrt x_{fix}	$\mathcal{S}_{\text{dsc,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq d^2 \\|Gx_\#\\|_2^2$	$d \in [0,1)$
firmly nonexpansive wrt x_{fix}	$\mathcal{S}_{\text{fne,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|x_\# - x_{\text{fix}}\\|_2^2 - \\|Gx_\#\\|_2^2$	
displacement firmly nonexpansive wrt x_{fix}	$\mathcal{S}_{\text{dfne,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|Gx_\#\\|_2^2 - \\|x_\# - x_{\text{fix}}\\|_2^2$	
decreasing pseudocontractive wrt x_{fix}	$\mathcal{S}_{\text{dpc,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|x_\# - x_{\text{fix}}\\|_2^2 - \nu \\|Gx_\#\\|_2^2$	$\nu \in (0,+\infty)$
displacement decreasing pseudocontractive wrt x_{fix}	$\mathcal{S}_{\text{ddpc,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|Gx_\#\\|_2^2 - \kappa \\|x_\# - x_{\text{fix}}\\|_2^2$	$\kappa \in (0,+\infty,)$
nonexpansive wrt x_{fix}	$\mathcal{S}_{\text{ne,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|x_\# - x_{\text{fix}}\\|_2^2$	
displacement nonexpansive wrt x_{fix}	$\mathcal{S}_{\text{dne,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|Gx_\#\\|_2^2$	
strictly pseudocontractive wrt x_{fix}	$\mathcal{S}_{\text{spc,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|x_\# - x_{\text{fix}}\\|_2^2 + p \\|Gx_\#\\|_2^2$	$p \in (0,1)$
displacement strictly pseudocontractive wrt x_{fix}	$\mathcal{S}_{\text{dspc,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|Gx_\#\\|_2^2 + q \\|x_\# - x_{\text{fix}}\\|_2^2$	$q \in (0,1)$
pseudocontractive wrt x_{fix}	$\mathcal{S}_{\text{pc,p}}$	$\\|Tx_\# - x_{\text{fix}}\\|_2^2 \leq \\|x_\# - x_{\text{fix}}\\|_2^2 + \\|Gx_\#\\|_2^2$	

point (although the statement in for generic points is implicit in the previous statement that considers all possible pairs of points).

The corresponding definitions closely match those we have previously considered; we formally define these property classes below and display the classes in Table 14.3. To more closely see the correspondence between properties that hold for all pairs of arguments and properties that hold with respect to a fixed point $x_{\text{fix}} \in \text{Fix } T(\cdot)$, note that $x_{\text{fix}} \in \text{Fix } T(\cdot)$ implies $T(x_{\text{fix}}) = x_{\text{fix}}$ and $G(x_{\text{fix}}) = x_{\text{fix}} - T(x_{\text{fix}}) = 0$. Again, we separately describe firm nonexpansiveness (because of the wide use of this term in the literature) but we recognize that firm nonexpansiveness is a specific instance of decreasing pseudocontractivity.

Definition 99 (Strictly contractive (with parameter c) with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *c-strictly contractive with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix } T(\cdot)$, when there exists a constant $c \in [0,1)$ such that

$$\|Tx_{\#} - x_{\text{fix}}\|_2 \leq c \|x_{\#} - x_{\text{fix}}\|_2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$.

We denote the class of all operators strictly contractive with respect to a fixed point by $\mathcal{S}_{\text{sc,p}}$. When we wish to include specific reference to the strict contractivity parameter c, we write $\mathcal{S}_{\text{sc,p}}(c)$ to denote the class of all operators that satisfy the c-strict contractivity condition with respect to a fixed point.

Definition 100 (Displacement strictly contractive (with parameter d) with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *d-displacement strictly contractive with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix } T(\cdot)$, when there exists a constant $d \in [0,1)$ such that

$$\|Tx_{\#} - x_{\text{fix}}\|_2 \leq d \|Gx_{\#}\|_2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$.

We denote the class of all operators that are displacement strictly contractive with respect to a fixed point by $\mathcal{S}_{\text{dsc,p}}$. When we wish to include specific reference to the displacement strict contractivity parameter d, we write $\mathcal{S}_{\text{dsc,p}}(d)$ to denote the class of all operators that satisfy the d-displacement strict contractivity condition with respect to a fixed point.

Definition 101 (Firmly nonexpansive with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *firmly nonexpansive with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix } T(\cdot)$, when

$$\|Tx_{\#} - x_{\text{fix}}\|_2^2 \leq \|x_{\#} - x_{\text{fix}}\|_2^2 - \|Gx_{\#}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are firmly nonexpansive with respect to a fixed point by $\mathcal{S}_{\text{fne,p}}$.

Definition 102 (Displacement firmly nonexpansive with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *displacement firmly nonexpansive with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix } T(\cdot)$, when

$$\|Tx_{\#} - x_{\text{fix}}\|_2^2 \leq \|Gx_{\#}\|_2^2 - \|x_{\#} - x_{\text{fix}}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are displacement firmly nonexpansive with respect to a fixed point by $\mathcal{S}_{\text{dfne,p}}$.

Definition 103 (Decreasing pseudocontractive (with parameter ν) with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *ν-decreasing pseudocontractive with respect to a fixed point*, $x_{\text{fix}} \in \text{Fix } T(\cdot)$, for some $\nu \in (0, +\infty)$, when

$$\|Tx_{\#} - x_{\text{fix}}\|_2^2 \leq \|x_{\#} - x_{\text{fix}}\|_2^2 - \nu \|Gx_{\#}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators decreasing pseudocontractive with respect to a fixed point by $\mathcal{S}_{\text{dpc,p}}$. When we wish to include specific reference to the decreasing pseudocontractivity parameter ν, we write $\mathcal{S}_{\text{dpc,p}}(\nu)$ to denote the class of all operators that satisfy the ν-decreasing pseudocontractivity condition with respect to a fixed point.

Definition 104 (Displacement decreasing pseudocontractive (with parameter κ) with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *κ-displacement decreasing pseudocontractive with respect to a fixed point*, $x_{\text{fix}} \in \text{Fix}\, T(\cdot)$, for some $\kappa \in (0, +\infty)$, when

$$\|Tx_{\#} - x_{\text{fix}}\|_2^2 \leq \|Gx_{\#}\|_2^2 - \kappa \|x_{\#} - x_{\text{fix}}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix}\, T(\cdot)$, where $G \stackrel{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are displacement decreasing pseudocontractive with respect to a fixed point by $\mathcal{S}_{\text{ddpc,p}}$. When we wish to include specific reference to the displacement decreasing pseudocontractivity parameter κ, we write $\mathcal{S}_{\text{ddpc,p}}(\kappa)$ to denote the class of all operators that satisfy the κ-displacement decreasing pseudocontractivity condition with respect to a fixed point.

Definition 105 (Nonexpansive with respect to a fixed point)**.** We say that operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *nonexpansive with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix}\, T(\cdot)$, when

$$\|Tx_{\#} - x_{\text{fix}}\|_2 \leq \|x_{\#} - x_{\text{fix}}\|_2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix}\, T(\cdot)$.

We denote the class of all operators nonexpansive with respect to a fixed point by $\mathcal{S}_{\text{ne,p}}$.

Definition 106 (Displacement nonexpansive with respect to a fixed point)**.** We say that operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *displacement nonexpansive with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix}\, T(\cdot)$, when

$$\|Tx_{\#} - x_{\text{fix}}\|_2 \leq \|Gx_{\#}\|_2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix}\, T(\cdot)$.

We denote the class of all operators that are displacement nonexpansive with respect to a fixed point by $\mathcal{S}_{\text{dne,p}}$.

Definition 107 (Strictly pseudocontractive (with parameter p) with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *p-strictly pseudocontractive with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix } T(\cdot)$, for some $p \in (0,1)$ when

$$\|Tx_{\#} - x_{\text{fix}}\|_2^2 \leq \|x_{\#} - x_{\text{fix}}\|_2^2 + p\,\|Gx_{\#}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are strictly pseudocontractive with respect to a fixed point by $\mathcal{S}_{\text{spc,p}}$. When we wish to include specific reference to the strict pseudocontractivity parameter p, we write $\mathcal{S}_{\text{spc,p}}(p)$ to denote the class of all operators that satisfy the p-strict pseudocontractivity condition with respect to a fixed point.

Definition 108 (Displacement strictly pseudocontractive (with parameter q) with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *q-displacement strictly pseudocontractive with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix } T(\cdot)$, for some $q \in (0,1)$ when

$$\|Tx_{\#} - x_{\text{fix}}\|_2^2 \leq \|Gx_{\#}\|_2^2 + q\,\|x_{\#} - x_{\text{fix}}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are displacement strictly pseudocontractive with respect to a fixed point by $\mathcal{S}_{\text{dspc,p}}$. When we wish to include specific reference to the displacement strict pseudocontractivity parameter q, we write $\mathcal{S}_{\text{dspc,p}}(q)$ to denote the class of all operators that satisfy the q-strict pseudocontractivity condition with respect to a fixed point.

Definition 109 (Pseudocontractive with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *pseudocontractive with respect to a fixed point,* say $x_{\text{fix}} \in \text{Fix } T(\cdot)$, when

$$\|Tx_{\#} - x_{\text{fix}}\|_2^2 \leq \|x_{\#} - x_{\text{fix}}\|_2^2 + \|Gx_{\#}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are pseudocontractive with respect to a fixed point by $\mathcal{S}_{\text{pc,p}}$.

Definition 110 (Lipschitz (with parameter L) with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *L-Lipschitz with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix}\, T(\cdot)$, for some nonnegative $L \in \mathbb{R}_+$, when

$$\|Tx_\# - x_{\text{fix}}\|_2 \leq L \|x_\# - x_{\text{fix}}\|_2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix}\, T(\cdot)$.

We denote the class of all operators that are Lipschitz with respect to a fixed point by $\mathcal{S}_{\text{Lip,p}}$. When we wish to include specific reference to the Lipschitz parameter L, we write $\mathcal{S}_{\text{Lip,p}}(L)$ to denote the class of all operators that satisfy the L-Lipschitz condition with respect to a fixed point.

Definition 111 (Displacement Lipschitz (with parameter Λ) with respect to a fixed point)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is Λ-*displacement Lipschitz with respect to a fixed point*, say $x_{\text{fix}} \in \text{Fix}\, T(\cdot)$, for some nonnegative $\Lambda \in \mathbb{R}_+$, when

$$\|Tx_\# - x_{\text{fix}}\|_2 \leq \Lambda \|Gx_\#\|_2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix}\, T(\cdot)$.

We denote the class of all operators that are displacement Lipschitz with respect to a fixed point by $\mathcal{S}_{\text{dLip,p}}$. When we wish to include specific reference to the displacement Lipschitz parameter Λ, we write $\mathcal{S}_{\text{dLip,p}}(\Lambda)$ to denote the class of all operators that satisfy the Λ-displacement Lipschitz condition with respect to a fixed point.

We note the following inclusion relationships: $\mathcal{S}_{\text{sc,p}} \subset \mathcal{S}_{\text{dpc,p}} \subset \mathcal{S}_{\text{ne,p}} \subset \mathcal{S}_{\text{spc,p}} \subset \mathcal{S}_{\text{pc,p}}$. Also: $\mathcal{S}_{\text{dsc,p}} \subset \mathcal{S}_{\text{ddpc,p}} \subset \mathcal{S}_{\text{dne,p}} \subset \mathcal{S}_{\text{dspc,p}} \subset \mathcal{S}_{\text{pc,p}}$.

14.4 Monotonicity-type properties with respect to a zero point of the operator

When stated with respect to a point, contractivity properties are naturally connected to elements of the fixed point set of the operator in question. For monotonicity-type properties stated with respect to a point, the natural point to consider is an element of the set of zeros of the operator. This statement follows immediately from the relationship between T and G, since $G = I - T$ means that $x \in \text{Fix}\ T$ if and only if $x \in \text{Zeros}\ G$.

As before, with the exception of strict monotonicity, all of the categories that we consider can be described in terms of parameter value ranges associated with a single inequality:

$$\langle Gx_{\#}, x_{\#} - x_{\text{zer}} \rangle \geq \sigma \left\| x_{\#} - x_{\text{zer}} \right\|_2^2 + \rho \left\| Gx_{\#} \right\|_2^2 \text{ for all } x_{\#} \in \mathbb{R}^n \backslash \text{Zeros}\ G.$$

We formally define these property classes below and display the classes in Table 14.4.

Definition 112 (Combined strongly monotone (with parameters σ, ρ) with respect to a zero point)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is (σ, ρ)-*combined strongly monotone with respect to a zero point*, say $x_{\text{zer}} \in \text{Zeros}\ G$, for some strictly positive $\sigma \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}_{++}$when

$$\langle Gx_{\#}, x_{\#} - x_{\text{zer}} \rangle \geq \sigma \left\| x_{\#} - x_{\text{zer}} \right\|_2^2 + \rho \left\| Gx_{\#} \right\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Zeros}\ G$.

We denote the class of operators that are combined strongly monotone with respect to a zero point by $\mathcal{M}_{\text{csm,p}}$. When we wish to include specific reference to the parameters σ and ρ, we write $\mathcal{M}_{\text{csm,p}}(\sigma, \rho)$ to denote the class of all operators that satisfy the (σ, ρ)-combined strong monotonicity condition with respect to a zero point.

Definition 113 (Strongly monotone (with parameter σ) with respect to a zero point)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is σ-*strongly monotone with respect to a zero point*, say $x_{\text{zer}} \in \text{Zeros}\ G$, for some strictly positive $\sigma \in \mathbb{R}_{++}$ when

$$\langle Gx_{\#}, x_{\#} - x_{\text{zer}} \rangle \geq \sigma \left\| x_{\#} - x_{\text{zer}} \right\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Zeros}\ G$.

We denote the class of operators that are strongly monotone with respect to a zero point by $\mathcal{M}_{\text{sm,p}}$. When we wish to include specific reference to the parameter σ, we write $\mathcal{M}_{\text{sm,p}}(\sigma)$ to denote the class of all operators that satisfy the σ-strong monotonicity condition with respect to a zero point.

Table 14.4: Monotonicity-type properties with respect to a zero point of G. The subscript "p" is intended to suggest "point".

description	notation	characterizing inequality	parameter range
combined strongly monotone wrt x_{zer}	$\mathcal{M}_{\text{csm,p}}$	$\langle Gx_{\#}, x_{\#} - x_{\text{fix}} \rangle \geq \sigma \left\Vert x_{\#} - x_{\text{fix}} \right\Vert_2^2 + \rho \left\Vert Gx_{\#} \right\Vert_2^2$	$\sigma \in \mathbb{R}_{++}, \rho \in \mathbb{R}_{++}$
strongly monotone wrt x_{zer}	$\mathcal{M}_{\text{sm,p}}$	$\langle Gx_{\#}, x_{\#} - x_{\text{fix}} \rangle \geq \sigma \left\Vert x_{\#} - x_{\text{fix}} \right\Vert_2^2$	$\sigma \in \mathbb{R}_{++}, \rho = 0$
inverse strongly monotone wrt x_{zer}	$\mathcal{M}_{\text{ism,p}}$	$\langle Gx_{\#}, x_{\#} - x_{\text{fix}} \rangle \geq \rho \left\Vert Gx_{\#} \right\Vert_2^2$	$\sigma = 0, \rho \in \mathbb{R}_{++}$
strictly monotone wrt x_{zer}	$\mathcal{M}_{\text{ms,p}}$	$\langle Gx_{\#}, x_{\#} - x_{\text{fix}} \rangle > 0$	
monotone wrt x_{zer}	$\mathcal{M}_{\text{m,p}}$	$\langle Gx_{\#}, x_{\#} - x_{\text{fix}} \rangle \geq 0$	$\sigma = 0, \rho = 0$

Definition 114 (Inverse strongly monotone (with parameter ρ) with respect to a zero point)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is ρ-*inverse strongly monotone with respect to a zero point*, say $x_{\text{zer}} \in \text{Zeros } G$, for some strictly positive $\rho \in \mathbb{R}_{++}$ when

$$\langle Gx_{\#}, x_{\#} - x_{\text{zer}} \rangle \geq \rho \left\| Gx_{\#} \right\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Zeros } G$.

We denote the class of operators that are inverse strongly monotone with respect to a zero point by $\mathcal{M}_{\text{ism,p}}$. When we wish to include specific reference to the parameter ρ, we write $\mathcal{M}_{\text{ism,p}}(\rho)$ to denote the class of all operators that satisfy the ρ-inverse strong monotonicity condition with respect to a zero point.

Definition 115 (Strictly monotone with respect to a zero point)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *strictly monotone with respect to a zero point*, say $x_{\text{zer}} \in \text{Zeros } G$, when

$$\langle Gx_{\#}, x_{\#} - x_{\text{zer}} \rangle > 0$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Zeros } G$.

We denote the class of operators that are strictly monotone with respect to a zero point by $\mathcal{M}_{\text{ms,p}}$.

Definition 116 (Monotone with respect to a zero point)**.**

We say that an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is *monotone with respect to a zero point*, say $x_{\text{zer}} \in \text{Zeros } G$, when

$$\langle Gx_{\#}, x_{\#} - x_{\text{zer}} \rangle \geq 0$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Zeros } G$.

We denote the class of operators that are monotone with respect to a zero point by $\mathcal{M}_{\text{m,p}}$.

We note the following inclusion relationships: $\mathcal{M}_{\text{csm,p}} \subset \mathcal{M}_{\text{ism,p}} \subset \mathcal{M}_{\text{m,p}}$ and $\mathcal{M}_{\text{csm,p}} \subset \mathcal{M}_{\text{sm,p}} \subset \mathcal{M}_{\text{m,p}}$.

14.5 Contractivity-type properties with respect to a fixed point set

We have seen statements of contractivity-type properties with respect to a single fixed point; we can also consider versions of these properties stated relative to a fixed point set. In particular, we state this version with respect to a projection onto the fixed point set and so requires the inequalities to be satisfied with respect to the appropriate "closest fixed point" as the other argument ranges over all of $\mathbb{R}^n$. For convenience, we will introduce the notation $\mathcal{F} \overset{\text{set}}{=} \text{Fix } T$.

The corresponding definitions closely match those we have previously considered; we formally define these property classes below and display the classes in Table 14.5. Again, note that $x_{\text{fix}} \in \text{Fix } T(\cdot)$ implies $T(x_{\text{fix}}) = x_{\text{fix}}$ and $G(x_{\text{fix}}) = x_{\text{fix}} - T(x_{\text{fix}}) = 0$.

The literature again includes some alternative terminology for the contractivity-type property categories listed above. Byrne [Byr08] includes the somewhat nonspecific category of "paracontractions" (implicitly with respect to a fixed point set). An operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is said to be paracontractive (with respect to Fix T) when $\|Tx_\# - x_{\text{fix}}\|_2 < \|x_\# - x_{\text{fix}}\|_2$ for every $x_\# \in \mathbb{R}^n \backslash \text{Fix } T$.

As mentioned above, another example of contradictory use of the term "pseudocontractive" occurs in [BT89], where the term "pseudocontractive" is used to refer to what we describe below as "strictly contractive with respect to the fixed point set Fix T". A related notion applied to a slightly different context is introduced in [CZ97] under the name "targeted contraction".

Further, we again mention the additional thread of alternative terminology exemplified by [VE09]. What we call below "nonexpansive with respect to the nonempty fixed point set $\mathcal{F} \overset{\text{set}}{=} \text{Fix } T$" essentially corresponds to what [VE09] refer to either as $\mathcal{F}$-Fejer or $\mathcal{F}$-quasi-nonexpansive. The category that we call below "strictly contractive with respect to the nonempty fixed point set $\mathcal{F} \overset{\text{set}}{=} \text{Fix } T$" essentially corresponds to what [VE09] refer to as $\mathcal{F}$-Fejer or strictly $\mathcal{F}$-quasi-nonexpansive; this category also essentially corresponds what [BT89] refer to as pseudocontractive (with respect to the nonempty fixed point set Fix T) and to what [Byr08] refers to as paracontractive (with

Table 14.5: Contractivity-type properties with respect to a fixed point. The subscript "s" is intended to suggest "set".

description	notation	characterizing inequality	parameter range
strictly contractive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{sc,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le c^2 \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2$	$c \in [0,1)$
displacement strictly contractive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{dsc,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le d^2 \|Gx_\#\|_2^2$	$d \in [0,1)$
firmly nonexpansive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{fne,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 - \|Gx_\#\|_2^2$	
displacement firmly nonexpansive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{dfne,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|Gx_\#\|_2^2 - \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2$	
decreasing pseudocontractive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{dpc,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 - \nu \|Gx_\#\|_2^2$	$\nu \in (0,+\infty)$
displacement decreasing pseudocontractive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{ddpc,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|Gx_\#\|_2^2 - \kappa \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2$	$\kappa \in (0,+\infty)$
nonexpansive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{ne,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2$	
displacement nonexpansive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{dne,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|Gx_\#\|_2^2$	
strictly pseudocontractive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{spc,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 + p \|Gx_\#\|_2^2$	$p \in (0,1)$
displacement strictly pseudocontractive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{dspc,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|Gx_\#\|_2^2 + q \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2$	$q \in (0,1)$
pseudocontractive wrt $\mathcal{F} \overset{\text{set}}{=} \operatorname{Fix} T$	$\mathcal{S}_{\text{pc,s}}$	$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \le \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 + \|Gx_\#\|_2^2$	

respect to the nonempty fixed point set Fix T). What we refer to as "pseudocontractive with respect to the nonempty fixed point set $\mathcal{F} \overset{\text{set}}{=} \text{Fix } T$" essentially corresponds to what [VE09] refer to as strongly $\mathcal{F}$-Fejer or $\mathcal{F}$-pseudocontractive. Throughout each of these examples, the distinction is that our use of the projection onto the fixed point set as the reference point, rather than having the reference point range over any element of the fixed point set. We adopt this to convention to more closely align our usage with the usage of such concepts as "restricted strong convexity" that has recently gained prominence in the machine learning optimization literature.

Definition 117 (Strictly contractive (with parameter c) with respect to Fix T)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *c-strictly contractive with respect to its fixed point set* Fix $T(\cdot)$ when there exists a constant $c \in [0,1)$ such that

$$\|Tx_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2 \leq c\,\|x_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$.

We denote the class of all operators that are strictly contractive with respect to their fixed point set by $\mathcal{S}_{\text{sc,s}}$. When we wish to include specific reference to the strict contractivity parameter c, we write $\mathcal{S}_{\text{sc,s}}(c)$ to denote the class of all operators that satisfy the c-strict contractivity condition with respect to their fixed point set.

Definition 118 (Displacement strictly contractive (with parameter d) with respect to Fix T)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *d-displacement strictly contractive with respect to its fixed point set* Fix $T(\cdot)$ when there exists a constant $d \in [0,1)$ such that

$$\|Tx_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2 \leq d\,\|Gx_{\#}\|_2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$.

We denote the class of all operators that are displacement strictly contractive with respect to their fixed point set by $\mathcal{S}_{\text{dsc,s}}$. When we wish to include specific reference to the displacement strict contractivity parameter d, we write $\mathcal{S}_{\text{dsc,s}}(c)$ to denote the class of all operators that satisfy the d-displacement strict contractivity condition with respect to their fixed point set.

Definition 119 (Firmly nonexpansive with respect to Fix T)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *firmly nonexpansive with respect to its fixed point set* Fix $T(\cdot)$ when

$$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \leq \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 - \|Gx_\#\|_2^2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \stackrel{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are firmly nonexpansive with respect to their fixed point set by $\mathcal{S}_{\text{fne,s}}$.

Definition 120 (Displacement firmly nonexpansive with respect to Fix T)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *displacement firmly nonexpansive with respect to its fixed point set* Fix $T(\cdot)$ when

$$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \leq \|Gx_\#\|_2^2 - \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \stackrel{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are displacement firmly nonexpansive with respect to their fixed point set by $\mathcal{S}_{\text{dfne,s}}$.

Definition 121 (Decreasing pseudocontractive (with parameter ν) with respect to Fix T)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is ν*-decreasing pseudocontractive with respect to its fixed point set* Fix $T(\cdot)$, for some $\nu \in (0, +\infty)$, when

$$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \leq \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 - \nu \|Gx_\#\|_2^2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \stackrel{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are decreasing pseudocontractive with respect to their fixed point set by $\mathcal{S}_{\text{dpc,s}}$. When we wish to include specific reference to the decreasing pseudocontractivity parameter ν, we write $\mathcal{S}_{\text{dpc,s}}(\nu)$ to denote the class of all operators that satisfy the ν-decreasing pseudocontractivity condition with respect to their fixed point set.

Definition 122 (Displacement decreasing pseudocontractive (with parameter κ) with respect to Fix T)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *κ-displacement decreasing pseudocontractive with respect to its fixed point set* Fix $T(\cdot)$, for some $\kappa \in (0, +\infty)$, when

$$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2 \leq \|Gx_\#\|_2^2 - \kappa \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2^2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are displacement decreasing pseudocontractive with respect to their fixed point set by $\mathcal{S}_{\text{ddpc,s}}$. When we wish to include specific reference to the displacement decreasing pseudocontractivity parameter κ, we write $\mathcal{S}_{\text{ddpc,s}}(\kappa)$ to denote the class of all operators that satisfy the κ-displacement decreasing pseudocontractivity condition with respect to their fixed point set.

Definition 123 (Nonexpansive with respect to Fix T)**.** We say that operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *nonexpansive with respect to its fixed point set* Fix $T(\cdot)$ when

$$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2 \leq \|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$.

We denote the class of all operators that are nonexpansive with respect to their fixed point set by $\mathcal{S}_{\text{ne,s}}$.

Definition 124 (Displacement nonexpansive with respect to Fix T)**.** We say that operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *displacement nonexpansive with respect to its fixed point set* Fix $T(\cdot)$ when

$$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2 \leq \|Gx_\#\|_2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$.

We denote the class of all operators that are displacement nonexpansive with respect to their fixed point set by $\mathcal{S}_{\text{dne,s}}$.

Definition 125 (Strictly pseudocontractive (with parameter p) with respect to Fix T)**.** We say that an operator $T(\cdot)\ :\ \mathbb{R}^n \to \mathbb{R}^n$ is *p-strictly pseudocontractive with respect to its fixed point set* Fix $T(\cdot)$, for some $p \in (0,1)$ when

$$\|Tx_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2^2 \leq \|x_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2^2 + p\,\|Gx_{\#}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are strictly pseudocontractive with respect to their fixed point set by $\mathcal{S}_{\text{spc,s}}$. When we wish to include specific reference to the strict pseudocontractivity parameter p, we write $\mathcal{S}_{\text{spc,s}}(p)$ to denote the class of all operators that satisfy the p-strict pseudocontractivity condition with respect to their fixed point set.

Definition 126 (Displacement strictly pseudocontractive (with parameter q) with respect to Fix T)**.** We say that an operator $T(\cdot)\ :\ \mathbb{R}^n \to \mathbb{R}^n$ is *q-displacement strictly pseudocontractive with respect to its fixed point set* Fix $T(\cdot)$, for some $q \in (0,1)$ when

$$\|Tx_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2^2 \leq \|Gx_{\#}\|_2^2 + q\,\|x_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are displacement strictly pseudocontractive with respect to their fixed point set by $\mathcal{S}_{\text{dspc,s}}$. When we wish to include specific reference to the displacement strict pseudocontractivity parameter q, we write $\mathcal{S}_{\text{dspc,s}}(q)$ to denote the class of all operators that satisfy the q-displacement strict pseudocontractivity condition with respect to their fixed point set.

Definition 127 (Pseudocontractive with respect to Fix T)**.** We say that an operator $T(\cdot)\ :\ \mathbb{R}^n \to \mathbb{R}^n$ is *pseudocontractive with respect to its fixed point set* Fix $T(\cdot)$, when

$$\|Tx_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2^2 \leq \|x_{\#} - \Pi_{\mathcal{F}}(x_{\#})\|_2^2 + \|Gx_{\#}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

We denote the class of all operators that are pseudocontractive with respect to their fixed point set by $\mathcal{S}_{\text{pc,s}}$.

Definition 128 (Lipschitz (with parameter L) with respect to Fix T)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *L-Lipschitz with respect to its fixed point set* Fix $T(\cdot)$, for some nonnegative $L \in \mathbb{R}_+$, when

$$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2 \leq L\|x_\# - \Pi_{\mathcal{F}}(x_\#)\|_2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$.

We denote the class of all operators that are Lipschitz with respect to their fixed point set by $\mathcal{S}_{\text{Lip,s}}$. When we wish to include specific reference to the Lipschitz parameter L, we write $\mathcal{S}_{\text{Lip,s}}(L)$ to denote the class of all operators that satisfy the L-Lipschitz condition with respect to their fixed point set.

Definition 129 (Displacement Lipschitz (with parameter Λ) with respect to Fix T)**.** We say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is Λ-*displacement Lipschitz with respect to its fixed point set* Fix $T(\cdot)$, for some nonnegative $\Lambda \in \mathbb{R}_+$, when

$$\|Tx_\# - \Pi_{\mathcal{F}}(x_\#)\|_2 \leq \Lambda\|Gx_\#\|_2$$

for all $x_\# \in \mathbb{R}^n \backslash \text{Fix } T(\cdot)$.

We denote the class of all operators that are displacement Lipschitz with respect to their fixed point set by $\mathcal{S}_{\text{dLip,s}}$. When we wish to include specific reference to the displacement Lipschitz parameter Λ, we write $\mathcal{S}_{\text{dLip,s}}(\Lambda)$ to denote the class of all operators that satisfy the Λ-displacement Lipschitz condition with respect to their fixed point set.

We note the following inclusion relationships: $\mathcal{S}_{\text{dsc,s}} \subset \mathcal{S}_{\text{ddpc,s}} \subset \mathcal{S}_{\text{dne,s}} \subset \mathcal{S}_{\text{dspc,s}} \subset \mathcal{S}_{\text{dpc,s}}$.

14.6 Monotonicity-type properties with respect to a zero point set

When stated with respect to a point, contractivity properties are naturally connected to elements of the fixed point set of the operator in question. For monotonicity-type properties stated with respect to a point, the natural point to consider is an element of the set of zero points of the operator. This statement follows immediately from the relationship between T and G, since $G = I - T$ means that $x \in \operatorname{Fix} T$ if and only if $x \in \operatorname{Zeros} G$. In like fashion, for the analog of contractivity-type properties with respect to the fixed point set we are led to monotonicity-type properties with respect to a zero point set. For convenience, we introduce the notation $\mathcal{Z} \overset{\text{set}}{=} \operatorname{Zeros} G$.

As before, with the exception of strict monotonicity, all of the categories that we consider can be described in terms of parameter value ranges associated with a single inequality:

$$\langle Gx_{\#}, x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq \sigma \left\|x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\right\|_2^2 + \rho \left\|Gx_{\#}\right\|_2^2 \text{ for all } x_{\#} \in \mathbb{R}^n \backslash \operatorname{Zeros} G.$$

We formally define these property classes below and display the classes in Table 14.6.

Definition 130 (Combined strongly monotone (with parameters σ, ρ) with respect to Zeros G)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is (σ, ρ)-*combined strongly monotone with respect to its zero point set* Zeros G, for some strictly positive $\sigma \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}_{++}$when

$$\langle Gx_{\#}, x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq \sigma \left\|x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\right\|_2^2 + \rho \left\|Gx_{\#}\right\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \operatorname{Zeros} G$.

We denote the class of operators that are combined strongly monotone with respect to their zero point set by $\mathcal{M}_{\text{csm,s}}$. When we wish to include specific reference to the parameters σ and ρ, we write $\mathcal{M}_{\text{csm,s}}(\sigma, \rho)$ to denote the class of all operators that satisfy the (σ, ρ)-combined strong monotonicity condition with respect to their zero point set.

Definition 131 (Strongly monotone (with parameter σ) with respect to Zeros G)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is σ-*strongly monotone with respect to its zero point set* Zeros G, for some strictly positive $\sigma \in \mathbb{R}_{++}$ when

$$\langle Gx_{\#}, x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq \sigma \left\|x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\right\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \operatorname{Zeros} G$.

Table 14.6: Monotonicity-type properties with respect to a zero point of G. The subscript "s" is intended to suggest "set".

description	notation	characterizing inequality	parameter range
combined strongly monotone wrt $\mathcal{Z} \overset{\text{set}}{=} \text{Zeros}\, G$	$\mathcal{M}_{\text{csm,s}}$	$\langle Gx_\#, x_\# - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq \sigma \|x_\# - \Pi_{\mathcal{Z}}(x_{\text{zer}})\|_2^2 + \rho \|Gx_\#\|_2^2$	$\sigma \in \mathbb{R}_{++}, \rho \in \mathbb{R}_{++}$
strongly monotone wrt $\mathcal{Z} \overset{\text{set}}{=} \text{Zeros}\, G$	$\mathcal{M}_{\text{sm,s}}$	$\langle Gx_\#, x_\# - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq \sigma \|x_\# - \Pi_{\mathcal{Z}}(x_{\text{zer}})\|_2^2$	$\sigma \in \mathbb{R}_{++}, \rho = 0$
inverse strongly monotone wrt $\mathcal{Z} \overset{\text{set}}{=} \text{Zeros}\, G$	$\mathcal{M}_{\text{ism,s}}$	$\langle Gx_\#, x_\# - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq \rho \|Gx_\#\|_2^2$	$\sigma = 0, \rho \in \mathbb{R}_{++}$
strictly monotone wrt $\mathcal{Z} \overset{\text{set}}{=} \text{Zeros}\, G$	$\mathcal{M}_{\text{ms,s}}$	$\langle Gx_\#, x_\# - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle > 0$	
monotone wrt $\mathcal{Z} \overset{\text{set}}{=} \text{Zeros}\, G$	$\mathcal{M}_{\text{m,s}}$	$\langle Gx_\#, x_\# - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq 0$	$\sigma = 0, \rho = 0$

We denote the class of operators that are strongly monotone with respect to their zero point set by $\mathcal{M}_{\text{sm,s}}$. When we wish to include specific reference to the parameter σ, we write $\mathcal{M}_{\text{sm,s}}(\sigma)$ to denote the class of all operators that satisfy the σ-strong monotonicity condition with respect to their zero point set.

Definition 132 (Inverse strongly monotone (with parameter ρ) with respect to Zeros G)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is ρ*-inverse strongly monotone with respect to its zero point set* Zeros G, for some strictly positive $\rho \in \mathbb{R}_{++}$ when

$$\langle Gx_{\#}, x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq \rho \|Gx_{\#}\|_2^2$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Zeros } G$.

We denote the class of operators that are inverse strongly monotone with respect to their zero point set by $\mathcal{M}_{\text{ism,s}}$. When we wish to include specific reference to the parameter ρ, we write $\mathcal{M}_{\text{ism,s}}(\rho)$ to denote the class of all operators that satisfy the ρ-inverse strong monotonicity condition with respect to their zero point set.

Definition 133 (Strictly monotone with respect to Zeros G)**.** We say that an operator $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is *strictly monotone with respect to its zero point set* Zeros G when

$$\langle Gx_{\#}, x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle > 0$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Zeros } G$.

We denote the class of operators that are strictly monotone with respect to their zero point set by $\mathcal{M}_{\text{ms,s}}$.

Definition 134 (Monotone with respect to Zeros G)**.**

We say that an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is *monotone with respect to its zero point set* Zeros G when

$$\langle Gx_{\#}, x_{\#} - \Pi_{\mathcal{Z}}(x_{\text{zer}})\rangle \geq 0$$

for all $x_{\#} \in \mathbb{R}^n \backslash \text{Zeros } G$.

We denote the class of operators that are monotone with respect to their zero point set by $\mathcal{M}_{\text{m,s}}$.

We note the following inclusion relationships: $\mathcal{M}_{\text{csm,s}} \subset \mathcal{M}_{\text{ism,s}} \subset \mathcal{M}_{\text{m,s}}$ and $\mathcal{M}_{\text{csm,s}} \subset \mathcal{M}_{\text{sm,s}} \subset \mathcal{M}_{\text{m,s}}$.

Chapter 15

Relationships between "contractivity" properties of T and "monotonicity" properties of $G = I - T$

15.1 Introduction

We will see here that there is a close relationship between the properties of an operator T and the properties of the associated displacement operator $G = I - T$; specifically, we will see that contractivity-type properties of T correspond to monotonicity-type properties of the associated displacement operator $G = I - T$ (and vice versa).

These relationships all hinge on one central identity:

$$\|Tx_{\#} - Tx_{\$}\|_2^2 = \|x_{\#} - x_{\$}\|_2^2 + \|Gx_{\#} - Gx_{\$}\|_2^2 - 2\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle$$

or the equivalent re-expression

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle = \frac{1}{2}\|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2}\|Gx_{\#} - Gx_{\$}\|_2^2 - \frac{1}{2}\|Tx_{\#} - Tx_{\$}\|_2^2.$$

As previously noted, this identity can be described as (a vector form of) the law of cosines; in other contexts, it is referred to as "polarization identity".

15.2 Starting from contractivity-type properties

When starting from contractivity-type properties, the most convenient form of the central identity is

$$\|Tx_{\#}-Tx_{\$}\|_2^2=\|x_{\#}-x_{\$}\|_2^2+\|Gx_{\#}-Gx_{\$}\|_2^2-2\left\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\right\rangle.$$

Proposition 5 (Strict contractivity and combined strong monotonicity)**.** *When T is c-strictly contractive, $G=I-T$ is $\left(\frac{1-c^2}{2},1\right)$-combined strongly monotone.*

Proof. Recall that we say an operator $T(\cdot)\ :\ \mathbb{R}^n\to\mathbb{R}^n$ is c-strictly contractive when there exists a constant $c\in[0,1)$ such that

$$\|Tx_{\#}-Tx_{\$}\|_2\leq c\cdot\|x_{\#}-x_{\$}\|_2$$

for all $x_{\#},x_{\$}\in\mathbb{R}^n$.

From T being c-strictly contractive we have

$$\|Tx_{\#}-Tx_{\$}\|_2\leq c\cdot\|x_{\#}-x_{\$}\|_2;$$

combined with the central identity

$$\|Tx_{\#}-Tx_{\$}\|_2^2=\|x_{\#}-x_{\$}\|_2^2+\|Gx_{\#}-Gx_{\$}\|_2^2-2\left\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\right\rangle,$$

we get

$$\|x_{\#}-x_{\$}\|_2^2+\|Gx_{\#}-Gx_{\$}\|_2^2-2\left\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\right\rangle\leq c^2\cdot\|x_{\#}-x_{\$}\|_2^2$$

from which

$$\begin{aligned}
-2\left\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\right\rangle &\leq c^2\cdot\|x_{\#}-x_{\$}\|_2^2-\|x_{\#}-x_{\$}\|_2^2-\|Gx_{\#}-Gx_{\$}\|_2^2\\
2\left\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\right\rangle &\geq -c^2\cdot\|x_{\#}-x_{\$}\|_2^2+\|x_{\#}-x_{\$}\|_2^2+\|Gx_{\#}-Gx_{\$}\|_2^2\\
\left\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\right\rangle &\geq \left(\frac{1-c^2}{2}\right)\cdot\|x_{\#}-x_{\$}\|_2^2+\frac{1}{2}\|Gx_{\#}-Gx_{\$}\|_2^2.
\end{aligned}$$

This establishes that when T is c-strictly contractive, $G=I-T$ is $\left(\frac{1-c^2}{2},\frac{1}{2}\right)$-combined strongly monotone (and thus $\left(\frac{1-c^2}{2}\right)$-strongly monotone, $\frac{1}{2}$-inverse strongly monotone, and monotone). □

Proposition 6 (Displacement strict contractivity and combined strong monotonicity)**.** *When T is d-displacement strictly contractive, $G = I - T$ is $\left(1, \frac{1-d^2}{2}\right)$-combined strongly monotone.*

Proof. Recall that we say an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is d-displacement strictly contractive when there exists a constant $d \in [0,1)$ such that

$$\|Tx_\# - Tx_\$\|_2 \leq d \cdot \|Gx_\# - Gx_\$\|_2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$.

From T being d-displacement strictly contractive we have

$$\|Tx_\# - Tx_\$\|_2^2 \leq d^2 \cdot \|Gx_\# - Gx_\$\|_2^2;$$

combined with the central identity

$$\|Tx_\# - Tx_\$\|_2^2 = \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle,$$

we get

$$\|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq d^2 \cdot \|Gx_\# - Gx_\$\|_2^2$$

from which

$$\begin{aligned}
-2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle &\leq d^2 \cdot \|Gx_\# - Gx_\$\|_2^2 - \|x_\# - x_\$\|_2^2 - \|Gx_\# - Gx_\$\|_2^2 \\
2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle &\geq -d^2 \cdot \|Gx_\# - Gx_\$\|_2^2 + \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 \\
\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle &\geq \left(\frac{1-d^2}{2}\right) \cdot \|Gx_\# - Gx_\$\|_2^2 + \frac{1}{2}\|x_\# - x_\$\|_2^2.
\end{aligned}$$

This establishes that when T is d-displacement strictly contractive, $G = I - T$ is $\left(\frac{1}{2}, \frac{1-d^2}{2}\right)$-combined strongly monotone (and thus $\frac{1}{2}$-strongly monotone, $\left(\frac{1-d^2}{2}\right)$-inverse strongly monotone, and monotone). □

Proposition 7 (Firm nonexpansiveness and 1-inverse strong monotonicity)**.** *When T is firmly nonexpansive, $G = I - T$ is 1-inverse strongly monotone.*

Proof. Recall that we say an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is firmly nonexpansive when

$$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \|Gx_\# - Gx_\$\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

From T firmly nonexpansive

$$\|Tx_\# - Tx_\$\|_2 \leq \|x_\# - x_\$\|_2^2 - \|Gx_\# - Gx_\$\|_2^2,$$

and the central identity

$$\|Tx_\# - Tx_\$\|_2^2 = \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle,$$

we get

$$\|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq \|x_\# - x_\$\|_2^2 - \|Gx_\# - Gx_\$\|_2^2$$

from which

$$-2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq -2\|Gx_\# - Gx_\$\|_2^2$$
$$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \|Gx_\# - Gx_\$\|_2^2.$$

This establishes that when T is firmly nonexpansive, $G = I - T$ is 1-inverse strongly monotone (and thus monotone). □

Proposition 8 (Displacement firm nonexpansiveness and 1-strong monotonicity)**.** *When T is displacement firmly nonexpansive, $G = I - T$ is 1-strongly monotone.*

Proof. Recall that we say an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is displacement firmly nonexpansive when

$$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 - \|x_\# - x_\$\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

From T displacement firmly nonexpansive

$$\|Tx_\# - Tx_\$\|_2 \leq \|Gx_\# - Gx_\$\|_2^2 - \|x_\# - x_\$\|_2^2,$$

and the central identity

$$\|Tx_\# - Tx_\$\|_2^2 = \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle,$$

we get

$$\|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq \|Gx_\# - Gx_\$\|_2^2 - \|x_\# - x_\$\|_2^2$$

from which

$$-2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq -2\|x_\# - x_\$\|_2^2$$
$$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \|x_\# - x_\$\|_2^2.$$

This establishes that when T is displacement firmly nonexpansive, $G = I - T$ is 1-strongly monotone. □

Proposition 9 (Decreasing pseudocontractivity and inverse strong monotonicity)**.** *When T is ν-decreasing pseudocontractive, $G = I - T$ is $\left(\frac{1+\nu}{2}\right)$-inverse strongly monotone.*

Proof. Recall that we say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is ν-decreasing pseudocontractive, for some $\nu \in (0,1)$, when

$$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \nu\|Gx_\# - Gx_\$\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \stackrel{\text{set}}{=} I - T$ is the displacement operator associated with T.

From T ν-decreasing pseudocontractive

$$\|Tx_\# - Tx_\$\|_2 \leq \|x_\# - x_\$\|_2^2 - \nu\|Gx_\# - Gx_\$\|_2^2,$$

and the central identity

$$\|Tx_\# - Tx_\$\|_2^2 = \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle,$$

we get

$$\|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq \|x_\# - x_\$\|_2^2 - \nu\|Gx_\# - Gx_\$\|_2^2$$

from which

$$-2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq -\nu\|Gx_\# - Gx_\$\|_2^2 - \|Gx_\# - Gx_\$\|_2^2$$
$$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \left(\frac{1+\nu}{2}\right)\|Gx_\# - Gx_\$\|_2^2.$$

This establishes that when T is ν-decreasing pseudocontractive, $G = I - T$ is $\left(\frac{1+\nu}{2}\right)$-inverse strongly monotone. □

Proposition 10 (Displacement decreasing pseudocontractivity and strong monotonicity)**.** *When T is μ-displacement decreasing pseudocontractive, $G = I - T$ is $\left(\frac{1+\mu}{2}\right)$-strongly monotone.*

Proof. Recall that we say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is μ-displacement decreasing pseudocontractive, for some $\mu \in (0,1)$, when

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|Gx_{\#} - Gx_{\$}\|_2^2 - \mu \|x_{\#} - x_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

From T μ-displacement decreasing pseudocontractive

$$\|Tx_{\#} - Tx_{\$}\|_2 \leq \|Gx_{\#} - Gx_{\$}\|_2^2 - \mu \|x_{\#} - x_{\$}\|_2^2,$$

and the central identity

$$\|Tx_{\#} - Tx_{\$}\|_2^2 = \|x_{\#} - x_{\$}\|_2^2 + \|Gx_{\#} - Gx_{\$}\|_2^2 - 2\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle,$$

we get

$$\|x_{\#} - x_{\$}\|_2^2 + \|Gx_{\#} - Gx_{\$}\|_2^2 - 2\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \leq \|Gx_{\#} - Gx_{\$}\|_2^2 - \mu \|x_{\#} - x_{\$}\|_2^2$$

from which

$$\begin{aligned} -2\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle &\leq -\mu \|x_{\#} - x_{\$}\|_2^2 - \|x_{\#} - x_{\$}\|_2^2 \\ \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle &\geq \left(\frac{1+\mu}{2}\right) \|x_{\#} - x_{\$}\|_2^2. \end{aligned}$$

This establishes that when T is μ-displacement decreasing pseudocontractive, $G = I - T$ is $\left(\frac{1+\mu}{2}\right)$-strongly monotone. □

Proposition 11 (Nonexpansiveness and $\frac{1}{2}$-inverse strong monotonicity)**.** *When T is nonexpansive, $G = I - T$ is $\frac{1}{2}$-inverse strongly monotone.*

Proof. Recall that we say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive when

$$\|Tx_{\#} - Tx_{\$}\|_2 \leq \|x_{\#} - x_{\$}\|_2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

From T nonexpansive

$$\|Tx_{\#}-Tx_{\$}\|_2^2 \leq \|x_{\#}-x_{\$}\|_2^2,$$

and the central identity

$$\|Tx_{\#}-Tx_{\$}\|_2^2 = \|x_{\#}-x_{\$}\|_2^2 + \|Gx_{\#}-Gx_{\$}\|_2^2 - 2\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle,$$

we get

$$\begin{aligned}
\|x_{\#}-x_{\$}\|_2^2 + \|Gx_{\#}-Gx_{\$}\|_2^2 - 2\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle &\leq \|x_{\#}-x_{\$}\|_2^2 \\
-2\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle &\leq -\|Gx_{\#}-Gx_{\$}\|_2^2 \\
2\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle &\geq \|Gx_{\#}-Gx_{\$}\|_2^2 \\
\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle &\geq \frac{1}{2}\|Gx_{\#}-Gx_{\$}\|_2^2.
\end{aligned}$$

This establishes that when T is nonexpansive, $G=I-T$ is $\frac{1}{2}$-inverse strongly monotone.

□

Proposition 12 (Displacement nonexpansiveness and $\frac{1}{2}$-strong monotonicity)**.** *When T is displacement nonexpansive, $G=I-T$ is $\frac{1}{2}$-strongly monotone.*

Proof. Recall that we say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is displacement nonexpansive when

$$\|Tx_{\#}-Tx_{\$}\|_2 \leq \|Gx_{\#}-Gx_{\$}\|_2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

From T displacement nonexpansive

$$\|Tx_{\#}-Tx_{\$}\|_2^2 \leq \|Gx_{\#}-Gx_{\$}\|_2^2,$$

and the central identity

$$\|Tx_{\#}-Tx_{\$}\|_2^2 = \|x_{\#}-x_{\$}\|_2^2 + \|Gx_{\#}-Gx_{\$}\|_2^2 - 2\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle,$$

we get

$$\begin{aligned}
\|x_{\#}-x_{\$}\|_2^2 + \|Gx_{\#}-Gx_{\$}\|_2^2 - 2\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle &\leq \|Gx_{\#}-Gx_{\$}\|_2^2 \\
-2\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle &\leq -\|x_{\#}-x_{\$}\|_2^2 \\
2\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle &\geq \|x_{\#}-x_{\$}\|_2^2 \\
\left\langle Gx_{\#}-Gx_{\$}, x_{\#}-x_{\$}\right\rangle &\geq \frac{1}{2}\|x_{\#}-x_{\$}\|_2^2.
\end{aligned}$$

This establishes that when T is displacement nonexpansive, $G = I - T$ is $\frac{1}{2}$-strongly monotone. $\square$

Proposition 13 (Strict pseudocontractivity and inverse strong monotonicity)**.** *When T is p-strictly pseudocontractive, $G = I - T$ is $\left(\frac{1-p}{2}\right)$-inverse strongly monotone.*

Proof. Recall that we say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is p-strictly pseudocontractive, for some $p \in (0,1)$ when

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + p\,\|Gx_{\#} - Gx_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

From T p-strictly pseudocontractive

$$\|Tx_{\#} - Tx_{\$}\|_2 \leq \|x_{\#} - x_{\$}\|_2^2 + p\,\|Gx_{\#} - Gx_{\$}\|_2^2,$$

and the central identity

$$\|Tx_{\#} - Tx_{\$}\|_2^2 = \|x_{\#} - x_{\$}\|_2^2 + \|Gx_{\#} - Gx_{\$}\|_2^2 - 2\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle,$$

we get

$$\|x_{\#} - x_{\$}\|_2^2 + \|Gx_{\#} - Gx_{\$}\|_2^2 - 2\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \leq \|x_{\#} - x_{\$}\|_2^2 + p\,\|Gx_{\#} - Gx_{\$}\|_2^2,$$

from which

$$\begin{aligned} -2\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle &\leq p\,\|Gx_{\#} - Gx_{\$}\|_2^2 - \|Gx_{\#} - Gx_{\$}\|_2^2 \\ \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle &\geq \left(\frac{1-p}{2}\right)\|Gx_{\#} - Gx_{\$}\|_2^2. \end{aligned}$$

This establishes that when T is p-strictly pseudocontractive, $G = I - T$ is $\left(\frac{1-p}{2}\right)$-inverse strongly monotone. $\square$

Proposition 14 (Displacement strict pseudocontractivity and strong monotonicity)**.** *When T is q-displacement strictly pseudocontractive, $G = I - T$ is $\left(\frac{1-q}{2}\right)$-strongly monotone.*

Proof. Recall that we say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is q-displacement strictly pseudocontractive, for some $q \in (0,1)$ when

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|Gx_{\#} - Gx_{\$}\|_2^2 + q\,\|x_{\#} - x_{\$}\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

From T q-displacement strictly pseudocontractive

$$\|Tx_\# - Tx_\$\|_2 \leq \|Gx_\# - Gx_\$\|_2^2 + q\,\|x_\# - x_\$\|_2^2,$$

and the central identity

$$\|Tx_\# - Tx_\$\|_2^2 = \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\,\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle\,,$$

we get

$$\|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\,\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq \|Gx_\# - Gx_\$\|_2^2 + q\,\|x_\# - x_\$\|_2^2,$$

from which

$$\begin{aligned} -2\,\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle &\leq q\,\|x_\# - x_\$\|_2^2 - \|x_\# - x_\$\|_2^2 \\ \langle Gx_\# - Gx_\$, x_\# - x_\$\rangle &\geq \left(\frac{1-q}{2}\right)\|x_\# - x_\$\|_2^2. \end{aligned}$$

This establishes that when T is q-displacement strictly pseudocontractive, $G = I - T$ is $\left(\frac{1-q}{2}\right)$-strongly monotone. □

Proposition 15 (Pseudocontractivity and monotonicity)**.** *When T is pseudocontractive (equivalent to displacement pseudocontractive), $G = I - T$ is monotone.*

Proof. Recall that we say that an operator $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is pseudocontractive when

$$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$, where $G \overset{\text{set}}{=} I - T$ is the displacement operator associated with T.

From T pseudocontractive

$$\|Tx_\# - Tx_\$\|_2 \leq \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2,$$

and the central identity

$$\|Tx_\# - Tx_\$\|_2^2 = \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\,\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle\,,$$

we get

$$\|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2 - 2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2,$$
$$-2\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \leq 0$$
$$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq 0.$$

This establishes that when T is pseudocontractive (equivalent to displacement pseudocontractive), $G = I - T$ is monotone. □

Comparison of implications of contractivity conditions To get a clearer comparison, we re-write the implications together and in matching form. See Tables 15.1 and 15.2.

We note in particular a correspondence between some contractivity-type properties considered for T and inverse strong monotonicity on $G = I - T$. This provides us with connections to, on one hand, the behavior of gradient-type methods when the only assumption is a Lipschitz-type condition on the gradient; on the other hand, to a crucial remark made by Nemirovskii and Yudin [NY83] in their motivation for mirror-descent methods: "it is important to go over to the dual space".

15.3 Starting from monotonicity-type properties

When starting from contractivity-type properties, the most convenient form of the central identity is

$$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle = \frac{1}{2}\|x_\# - x_\$\|_2^2 + \frac{1}{2}\|Gx_\# - Gx_\$\|_2^2 - \frac{1}{2}\|Tx_\# - Tx_\$\|_2^2.$$

Proposition 16 (Implication of combined strong monotonicity.)**.** *When G is (σ, ρ)-combined strongly monotone, T satisfies an inequality similar in form to pseudocontractivity with constants* $(1 - 2\sigma, 1 - 2\rho)$.

Proof. Recall that we say an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is (σ, ρ)-combined strongly monotone for some strictly positive $\sigma \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}_{++}$when

$$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \sigma\|x_\# - x_\$\|_2^2 + \rho\|Gx_\# - Gx_\$\|_2^2$$

Table 15.1: Contractivity conditions on T, implications for $G = I - T$.

	Contractivity condition on T		Monotonicity condition on $G = I - T$	
$\mathcal{S}_{\text{sc}}(c)$	$\|Tx_\# - Tx_\$\|_2^2 \leq c^2 \|x_\# - x_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \left(\frac{1-c^2}{2}\right) \cdot \|x_\# - x_\$\|_2^2 + \frac{1}{2}\|Gx_\# - Gx_\$\|_2^2$	$\mathcal{M}_{\text{csm}}\left(\frac{1-c^2}{2}, \frac{1}{2}\right)$
$\mathcal{S}_{\text{dsc}}(d)$	$\|Tx_\# - Tx_\$\|_2^2 \leq + d^2 \|Gx_\# - Gx_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \frac{1}{2}\|x_\# - x_\$\|_2^2 + \left(\frac{1-d^2}{2}\right)\|Gx_\# - Gx_\$\|_2^2$	$\mathcal{M}_{\text{csm}}\left(\frac{1}{2}, \frac{1-d^2}{2}\right)$
$\mathcal{S}_{\text{fne}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \|Gx_\# - Gx_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \|Gx_\# - Gx_\$\|_2^2$	$\mathcal{M}_{\text{ism}}(1)$
$\mathcal{S}_{\text{dfne}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 - \|x_\# - x_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \|x_\# - x_\$\|_2^2$	$\mathcal{M}_{\text{sm}}(1)$
$\mathcal{S}_{\text{dpc}}(\nu)$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \nu\|Gx_\# - Gx_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \left(\frac{1+\nu}{2}\right)\|Gx_\# - Gx_\$\|_2^2$	$\mathcal{M}_{\text{ism}}\left(\frac{1+\nu}{2}\right)$
$\mathcal{S}_{\text{ddpc}}(\mu)$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 - \mu\|x_\# - x_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \left(\frac{1+\mu}{2}\right)\|x_\# - x_\$\|_2^2$	$\mathcal{M}_{\text{sm}}\left(\frac{1+\mu}{2}\right)$
$\mathcal{S}_{\text{ne}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \frac{1}{2}\|Gx_\# - Gx_\$\|_2^2$	$\mathcal{M}_{\text{ism}}\left(\frac{1}{2}\right)$
$\mathcal{S}_{\text{dne}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \frac{1}{2}\|x_\# - x_\$\|_2^2$	$\mathcal{M}_{\text{sm}}\left(\frac{1}{2}\right)$
$\mathcal{S}_{\text{spc}}(p)$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 + p\|Gx_\# - Gx_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \left(\frac{1-p}{2}\right)\|Gx_\# - Gx_\$\|_2^2$	$\mathcal{M}_{\text{ism}}\left(\frac{1-p}{2}\right)$
$\mathcal{S}_{\text{dspc}}(q)$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|Gx_\# - Gx_\$\|_2^2 + q\|x_\# - x_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \left(\frac{1-q}{2}\right)\|x_\# - x_\$\|_2^2$	$\mathcal{M}_{\text{sm}}\left(\frac{1-q}{2}\right)$
$\mathcal{S}_{\text{psc}}$	$\|Tx_\# - Tx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq 0$	$\mathcal{M}_{\text{m}}$

Table 15.2: Combined language for conditions on T, implications for $G = I - T$. The subscript "gc" in $\mathcal{S}_{\text{gc}}(a,b)$ stands for "generalized contractivity-type".

$\mathcal{S}_{\text{gc}}(a,b)$	Contractivity condition on T		Monotonicity condition on $G = I - T$	$\mathcal{M}_{\text{csm}}(\sigma,\rho)$
$\left(c^2, \; 0\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le c^2 \lVert x_\# - x_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \left(\frac{1-c^2}{2}\right) \cdot \lVert x_\# - x_\$\rVert_2^2 + \frac{1}{2}\lVert Gx_\# - Gx_\$\rVert_2^2$	$\left(\frac{1-c^2}{2}, \; \frac{1}{2}\right)$
$\left(0, \; d^2\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le \quad d^2 \lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \frac{1}{2}\lVert x_\# - x_\$\rVert_2^2 + \left(\frac{1-d^2}{2}\right)\lVert Gx_\# - Gx_\$\rVert_2^2$	$\left(\frac{1}{2}, \; \frac{1-d^2}{2}\right)$
$\left(1, \; -1\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le \; 1\lVert x_\# - x_\$\rVert_2^2 - 1\lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \quad 1\lVert Gx_\# - Gx_\$\rVert_2^2$	$\left(0, \; 1\right)$
$\left(-1, \; 1\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le -1\lVert x_\# - x_\$\rVert_2^2 + 1\lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge 1\lVert x_\# - x_\$\rVert_2^2$	$\left(1, \; 0\right)$
$\left(1, \; -\nu\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le \; 1\lVert x_\# - x_\$\rVert_2^2 - \nu\lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \quad \left(\frac{1+\nu}{2}\right)\lVert Gx_\# - Gx_\$\rVert_2^2$	$\left(0, \; \frac{1+\nu}{2}\right)$
$\left(-\mu, \; 1\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le -\mu\lVert x_\# - x_\$\rVert_2^2 + 1\lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \left(\frac{1+\mu}{2}\right)\lVert x_\# - x_\$\rVert_2^2$	$\left(\frac{1+\mu}{2}, \; 0\right)$
$\left(1, \; 0\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le \; 1\lVert x_\# - x_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \quad \frac{1}{2}\lVert Gx_\# - Gx_\$\rVert_2^2$	$\left(0, \; \frac{1}{2}\right)$
$\left(0, \; 1\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le \quad 1\lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \frac{1}{2}\lVert x_\# - x_\$\rVert_2^2$	$\left(\frac{1}{2}, \; 0\right)$
$\left(1, \; p\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le \; 1\lVert x_\# - x_\$\rVert_2^2 + p\lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \quad \left(\frac{1-p}{2}\right)\lVert Gx_\# - Gx_\$\rVert_2^2$	$\left(0, \; \frac{1-p}{2}\right)$
$\left(q, \; 1\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le \; q\lVert x_\# - x_\$\rVert_2^2 + \lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \left(\frac{1-q}{2}\right)\lVert x_\# - x_\$\rVert_2^2$	$\left(\frac{1-q}{2}, \; 0\right)$
$\left(1, \; 1\right)$	$\lVert Tx_\# - Tx_\$\rVert_2^2 \le \; 1\lVert x_\# - x_\$\rVert_2^2 + \lVert Gx_\# - Gx_\$\rVert_2^2$	$\Longrightarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge 0$	$\left(0, \; 0\right)$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

From G being (σ, ρ)-combined strongly monotone

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2 + \rho \left\| Gx_{\#} - Gx_{\$} \right\|_2^2$$

and the central identity

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle = \frac{1}{2} \left\| x_{\#} - x_{\$} \right\|_2^2 + \frac{1}{2} \left\| Gx_{\#} - Gx_{\$} \right\|_2^2 - \frac{1}{2} \left\| Tx_{\#} - Tx_{\$} \right\|_2^2,$$

we get

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2 + \rho \left\| Gx_{\#} - Gx_{\$} \right\|_2^2$$

and so

$$\frac{1}{2} \left\| x_{\#} - x_{\$} \right\|_2^2 + \frac{1}{2} \left\| Gx_{\#} - Gx_{\$} \right\|_2^2 - \frac{1}{2} \left\| Tx_{\#} - Tx_{\$} \right\|_2^2 \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2 + \rho \left\| Gx_{\#} - Gx_{\$} \right\|_2^2$$
$$\left(\frac{1}{2} - \sigma \right) \left\| x_{\#} - x_{\$} \right\|_2^2 + \left(\frac{1}{2} - \rho \right) \left\| Gx_{\#} - Gx_{\$} \right\|_2^2 \geq \frac{1}{2} \left\| Tx_{\#} - Tx_{\$} \right\|_2^2$$

so that

$$\left\| Tx_{\#} - Tx_{\$} \right\|_2^2 \leq (1 - 2\sigma) \left\| x_{\#} - x_{\$} \right\|_2^2 + (1 - 2\rho) \left\| Gx_{\#} - Gx_{\$} \right\|_2^2.$$

This establishes that $G = I - T$ being (σ, ρ)-combined strongly monotone implies that T satisfies an inequality similar in form to pseudocontractivity with constants $(1 - 2\sigma, 1 - 2\rho)$. □

Proposition 17 (Implication of simultaneous application of separate strong monotonicity.)**.** *When $G : \mathbb{R}^n \to \mathbb{R}^n$ is (σ, ρ)-separately strongly monotone for some(ρ, σ) satisfying $\rho \in \left(\frac{1}{2}, \infty \right)$ and $\left[1 - (2\rho - 1)\, \sigma^2 \right] \in (0, 1)$, the operator T satisfies something like an improved strict contractivity result.*

Proof. Recall that we say an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is (σ, ρ)-separately strongly monotone for some strictly positive $\sigma \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}_{++}$when both of the following expressions hold:

$$\frac{1}{\sigma} \left\| Gx_{\#} - Gx_{\$} \right\|_2^2 \geq \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2$$
$$\frac{1}{\rho} \left\| x_{\#} - x_{\$} \right\|_2^2 \geq \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \rho \left\| Gx_{\#} - Gx_{\$} \right\|_2^2$$

for all $x_\#, x_\$ \in \mathbb{R}^n$.

Consider an operator $S \overset{\text{set}}{=} G - \sigma I$; with respect to this operator, observe that $G = S + \sigma I$ and note that

$$\frac{1}{\rho}\|x_\# - x_\$\|_2^2 \geq \langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \geq \sigma \|x_\# - x_\$\|_2^2$$
$$\frac{1}{\rho}\|x_\# - x_\$\|_2^2 \geq \langle [S + \sigma I]\, x_\# - [S + \sigma I]\, x_\$, x_\# - x_\$\rangle \geq \sigma \|x_\# - x_\$\|_2^2$$

so that

$$\frac{1}{\rho}\|x_\# - x_\$\|_2^2 \geq \langle [Sx_\# + \sigma x_\#] + [Sx_\$ + \sigma x_\$]\,, x_\# - x_\$\rangle \geq \sigma \|x_\# - x_\$\|_2^2$$
$$\frac{1}{\rho}\|x_\# - x_\$\|_2^2 \geq \langle [Sx_\# - Sx_\$] + [\sigma x_\# - \sigma x_\$]\,, x_\# - x_\$\rangle \geq \sigma \|x_\# - x_\$\|_2^2$$

from which

$$\left(\frac{1}{\rho} - \sigma\right)\|x_\# - x_\$\|_2^2 \geq \langle Sx_\# - Sx_\$, x_\# - x_\$\rangle \geq 0.$$

This is equivalent to

$$\langle Sx_\# - Sx_\$, x_\# - x_\$\rangle \geq \frac{1}{\left(\frac{1}{\rho} - \sigma\right)}\|Sx_\# - Sx_\$\|_2^2$$
$$\langle [G - \sigma I]\, x_\# - [G - \sigma I]\, x_\$, x_\# - x_\$\rangle \geq \frac{1}{\left(\frac{1}{\rho} - \sigma\right)}\|[G - \sigma I]\, x_\# - [G - \sigma I]\, x_\$\|_2^2$$
$$\langle [Gx_\# - \sigma x_\#] - [Gx_\$ - \sigma x_\$]\,, x_\# - x_\$\rangle \geq \frac{1}{\left(\frac{1}{\rho} - \sigma\right)}\|[Gx_\# - \sigma x_\#] - [Gx_\$ - \sigma x_\$]\|_2^2$$
$$\langle [Gx_\# - Gx_\$] - [\sigma x_\# - \sigma x_\$]\,, x_\# - x_\$\rangle \geq \frac{1}{\left(\frac{1}{\rho} - \sigma\right)}\|[Gx_\# - Gx_\$] - [\sigma x_\# - \sigma x_\$]\|_2^2.$$

Continuing, we have

$$\left(\frac{1}{\rho} - \sigma\right)\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle - \sigma\left(\frac{1}{\rho} - \sigma\right)\|x_\# - x_\$\|_2^2$$
$$\geq \|[Gx_\# - Gx_\$] - [\sigma x_\# - \sigma x_\$]\|_2^2$$

and

$$\left(\frac{1}{\rho}-\sigma\right)\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle-\sigma\left(\frac{1}{\rho}-\sigma\right)\|x_{\#}-x_{\$}\|_2^2$$
$$\geq\left\{\|Gx_{\#}-Gx_{\$}\|_2^2+\|\sigma x_{\#}-\sigma x_{\$}\|_2^2-2\sigma\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle\right\}$$

and

$$\left(\frac{1}{\rho}-\sigma+2\sigma\right)\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle-\sigma\left(\frac{1}{\rho}-\sigma\right)\|x_{\#}-x_{\$}\|_2^2$$
$$\geq\|Gx_{\#}-Gx_{\$}\|_2^2+\sigma^2\|x_{\#}-x_{\$}\|_2^2.$$

Further, we have

$$\left(\frac{1}{\rho}+\sigma\right)\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle$$
$$\geq\|Gx_{\#}-Gx_{\$}\|_2^2+\left[\sigma\left(\frac{1}{\rho}-\sigma\right)+\sigma^2\right]\|x_{\#}-x_{\$}\|_2^2$$

and

$$\left(\frac{1}{\rho}+\sigma\right)\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle$$
$$\geq\|Gx_{\#}-Gx_{\$}\|_2^2+\left[\sigma\frac{1}{\rho}-\sigma^2+\sigma^2\right]\|x_{\#}-x_{\$}\|_2^2$$

and

$$\left(\frac{1}{\rho}+\sigma\right)\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle$$
$$\geq\|Gx_{\#}-Gx_{\$}\|_2^2+\sigma\frac{1}{\rho}\|x_{\#}-x_{\$}\|_2^2$$

and so

$$\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle\geq\frac{1}{\left(\frac{1}{\rho}+\sigma\right)}\|Gx_{\#}-Gx_{\$}\|_2^2+\frac{\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\|x_{\#}-x_{\$}\|_2^2.$$

This together with the central identity

$$\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle=\frac{1}{2}\|x_{\#}-x_{\$}\|_2^2+\frac{1}{2}\|Gx_{\#}-Gx_{\$}\|_2^2-\frac{1}{2}\|Tx_{\#}-Tx_{\$}\|_2^2,$$

yields

$$\frac{1}{2}\|x_{\#}-x_{\$}\|_2^2+\frac{1}{2}\|Gx_{\#}-Gx_{\$}\|_2^2-\frac{1}{2}\|Tx_{\#}-Tx_{\$}\|_2^2$$
$$\geq \frac{1}{\left(\frac{1}{\rho}+\sigma\right)}\|Gx_{\#}-Gx_{\$}\|_2^2+\frac{\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\|x_{\#}-x_{\$}\|_2^2$$

from which

$$\frac{1}{2}\|x_{\#}-x_{\$}\|_2^2+\frac{1}{2}\|Gx_{\#}-Gx_{\$}\|_2^2$$
$$\geq \frac{1}{\left(\frac{1}{\rho}+\sigma\right)}\|Gx_{\#}-Gx_{\$}\|_2^2+\frac{\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\|x_{\#}-x_{\$}\|_2^2+\frac{1}{2}\|Tx_{\#}-Tx_{\$}\|_2^2$$

and

$$\|x_{\#}-x_{\$}\|_2^2+\|Gx_{\#}-Gx_{\$}\|_2^2$$
$$\geq \frac{2}{\left(\frac{1}{\rho}+\sigma\right)}\|Gx_{\#}-Gx_{\$}\|_2^2+\frac{2\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\|x_{\#}-x_{\$}\|_2^2+\|Tx_{\#}-Tx_{\$}\|_2^2$$

so that

$$\|Tx_{\#}-Tx_{\$}\|_2^2 \leq \left[1-\frac{2\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|x_{\#}-x_{\$}\|_2^2+\left[1-\frac{2}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|Gx_{\#}-Gx_{\$}\|_2^2$$
$$\|Tx_{\#}-Tx_{\$}\|_2^2 \leq \left[1-\frac{2\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|x_{\#}-x_{\$}\|_2^2-\left[\frac{2}{\left(\frac{1}{\rho}+\sigma\right)}-1\right]\|Gx_{\#}-Gx_{\$}\|_2^2$$

□

We have not discussed step factors in iterative methods; if step factors are considered, it is typical to restrict attention to the setting in which the expression $-\left[\frac{2}{\left(\frac{1}{\rho}+\sigma\right)}-1\right]$ multiplying the term $\|Gx_{\#}-Gx_{\$}\|_2^2$ is nonpositive. In this setting, one approach is to drop the $\|Gx_{\#}-Gx_{\$}\|_2^2$ term and only consider the relationship between $\|Tx_{\#}-Tx_{\$}\|_2^2$ and $\|x_{\#}-x_{\$}\|_2^2$; this relationship turns out to correspond to strict contractivity.

This result corresponds to a more general perspective on a result used in [Nes04].

Proposition 18 (Strong monotonicity and either displacement decreasing pseudocontractivity or displacement strict pseudocontractivity.)**.** *When $G = I - T$ is σ-strongly monotone the operator T is either $(1-2\sigma)$-displacement decreasing pseudocontractive or $(1-2\sigma)$-displacement strictly pseudocontractive; which of these depends the specific value of σ, since that determines whether $1-2\sigma$ is, for example, negative or positive).*

Proof. Recall that we say an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is σ-strongly monotone for some strictly positive $\sigma \in \mathbb{R}_{++}$ when

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

From $G = I - T$ being σ-strongly monotone

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2$$

and the central identity

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle = \frac{1}{2} \left\| x_{\#} - x_{\$} \right\|_2^2 + \frac{1}{2} \left\| Gx_{\#} - Gx_{\$} \right\|_2^2 - \frac{1}{2} \left\| Tx_{\#} - Tx_{\$} \right\|_2^2,$$

we get

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2$$

from which

$$\frac{1}{2} \left\| x_{\#} - x_{\$} \right\|_2^2 + \frac{1}{2} \left\| Gx_{\#} - Gx_{\$} \right\|_2^2 - \frac{1}{2} \left\| Tx_{\#} - Tx_{\$} \right\|_2^2 \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2$$

$$\left(\frac{1}{2} - \sigma \right) \left\| x_{\#} - x_{\$} \right\|_2^2 + \frac{1}{2} \left\| Gx_{\#} - Gx_{\$} \right\|_2^2 \geq \frac{1}{2} \left\| Tx_{\#} - Tx_{\$} \right\|_2^2$$

so that

$$\left\| Tx_{\#} - Tx_{\$} \right\|_2^2 \leq (1 - 2\sigma) \left\| x_{\#} - x_{\$} \right\|_2^2 + \left\| Gx_{\#} - Gx_{\$} \right\|_2^2 .$$

This establishes that $G = I - T$ being σ-strongly monotone implies that T is either $(1-2\sigma)$-displacement decreasing pseudocontractive or $(1-2\sigma)$-displacement strictly pseudocontractive (depending the specific value of σ, since that determines whether $1-2\sigma$ is, for example, negative or positive). □

Proposition 19 (Inverse strong monotonicity and either decreasing pseudocontractivity or strict pseudocontractivity.)**.** *When $G = I - T$ is ρ-strongly monotone the operator T is either $(1-2\rho)$-displacement decreasing pseudocontractive or $(1-2\rho)$-displacement strictly pseudocontractive; which of these T is depends the specific value of ρ, since that determines whether $1-2\rho$ is, for example, negative or positive).*

Proof. Recall that we say an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is ρ-inverse strongly monotone for some strictly positive $\rho \in \mathbb{R}_{++}$ when

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \rho \, \|Gx_{\#} - Gx_{\$}\|_2^2$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

From $G = I - T$ being ρ-inverse strongly monotone

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \rho \, \|Gx_{\#} - Gx_{\$}\|_2^2$$

and the central identity

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle = \frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2} \|Gx_{\#} - Gx_{\$}\|_2^2 - \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2 ,$$

we get

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \rho \, \|Gx_{\#} - Gx_{\$}\|_2^2$$

from which

$$\begin{aligned} \frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2} \|Gx_{\#} - Gx_{\$}\|_2^2 - \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2 &\geq \rho \, \|Gx_{\#} - Gx_{\$}\|_2^2 \\ \frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \left(\frac{1}{2} - \rho\right) \|Gx_{\#} - Gx_{\$}\|_2^2 &\geq \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2 \end{aligned}$$

so that

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + (1-2\rho) \, \|Gx_{\#} - Gx_{\$}\|_2^2 .$$

This establishes that $G = I - T$ being ρ-inverse strongly monotone implies that T is either $(1-2\rho)$-decreasing pseudocontractive or $(1-2\rho)$-strictly pseudocontractive (depending the specific value of ρ, since that determines whether $1-2\rho$ is, for example, negative or positive). □

Proposition 20 (Monotonicity and pseudocontractivity.). *When $G = I - T$ is monotone, T is pseudocontractive.*

Proof. Recall that we say an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is monotone when

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq 0$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$.

From $G = I - T$ being monotone

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq 0$$

and the central identity

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle = \frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2} \|Gx_{\#} - Gx_{\$}\|_2^2 - \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2,$$

we get

$$\begin{aligned} \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle &\geq 0 \\ \frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2} \|Gx_{\#} - Gx_{\$}\|_2^2 - \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2 &\geq 0 \end{aligned}$$

from which

$$\frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2} \|Gx_{\#} - Gx_{\$}\|_2^2 \geq \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2$$

so that

$$\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + \|Gx_{\#} - Gx_{\$}\|_2^2.$$

This establishes that $G = I - T$ being monotone implies that T is pseudocontractive. □

The relationship between convexity and monotonicity and between monotonicity of $G = I - T$ and pseudocontractivity of T provides a strong impetus to consider pseudocontractivity rather than only paying attention to decreasing pseudocontractivity (or "averaged nonexpansiveness").

Table 15.3: Monotonicity conditions on $G = I - T$, implications for T.

	Contractivity condition on T		Monotonicity condition on $G = I - T$	
$\mathcal{S}_{\text{sc}}\left(c^2\right)$	$\|Tx_\# - Tx_\$\|_2^2 \le c^2 \|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \left(\frac{1-c^2}{2}\right) \cdot \|x_\# - x_\$\|_2^2 + \frac{1}{2}\|Gx_\# - Gx_\$\|_2^2$	$\mathcal{M}_{\text{csm}}\left(\frac{1-c^2}{2}, \frac{1}{2}\right)$
$\mathcal{S}_{\text{dsc}}\left(d^2\right)$	$\|Tx_\# - Tx_\$\|_2^2 \le d^2 \|Gx_\# - Gx_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \frac{1}{2}\|x_\# - x_\$\|_2^2 + \left(\frac{1-d^2}{2}\right)\|Gx_\# - Gx_\$\|_2^2$	$\mathcal{M}_{\text{csm}}\left(\frac{1}{2}, \frac{1-d^2}{2}\right)$
$\mathcal{S}_{\text{fne}}$	$\|Tx_\# - Tx_\$\|_2^2 \le \|x_\# - x_\$\|_2^2 - \|Gx_\# - Gx_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \rho\|Gx_\# - Gx_\$\|_2^2$ with $\rho = 1$	$\mathcal{M}_{\text{ism}}(\rho = 1)$
$\mathcal{S}_{\text{dfne}}$	$\|Tx_\# - Tx_\$\|_2^2 \le \|Gx_\# - Gx_\$\|_2^2 - \|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \sigma\|x_\# - x_\$\|_2^2$ with $\sigma = 1$	$\mathcal{M}_{\text{sm}}(\sigma = 1)$
$\mathcal{S}_{\text{dpc}}(2\rho - 1)$	$\|Tx_\# - Tx_\$\|_2^2 \le \|x_\# - x_\$\|_2^2 - (2\rho - 1)\|Gx_\# - Gx_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \rho\|Gx_\# - Gx_\$\|_2^2$ with $\rho \in \left(\frac{1}{2}, +\infty\right)$	$\mathcal{M}_{\text{ism}}\left(\rho \in \left(\frac{1}{2}, +\infty\right)\right)$
$\mathcal{S}_{\text{ddpc}}(2\sigma - 1)$	$\|Tx_\# - Tx_\$\|_2^2 \le \|Gx_\# - Gx_\$\|_2^2 - (2\sigma - 1)\|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \sigma\|x_\# - x_\$\|_2^2$ with $\sigma \in \left(\frac{1}{2}, +\infty\right)$	$\mathcal{M}_{\text{sm}}\left(\sigma \in \left(\frac{1}{2}, +\infty\right)\right)$
$\mathcal{S}_{\text{ne}}$	$\|Tx_\# - Tx_\$\|_2^2 \le \|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \rho\|Gx_\# - Gx_\$\|_2^2$ with $\rho = \frac{1}{2}$	$\mathcal{M}_{\text{ism}}\left(\rho = \frac{1}{2}\right)$
$\mathcal{S}_{\text{dne}}$	$\|Tx_\# - Tx_\$\|_2^2 \le \|Gx_\# - Gx_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \sigma\|x_\# - x_\$\|_2^2$ with $\sigma = \frac{1}{2}$	$\mathcal{M}_{\text{sm}}\left(\sigma = \frac{1}{2}\right)$
$\mathcal{S}_{\text{spc}}(1 - 2\rho)$	$\|Tx_\# - Tx_\$\|_2^2 \le \|x_\# - x_\$\|_2^2 + (1 - 2\rho)\|Gx_\# - Gx_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \rho\|Gx_\# - Gx_\$\|_2^2$ with $\rho \in \left(0, \frac{1}{2}\right)$	$\mathcal{M}_{\text{ism}}\left(\rho \in \left(0, \frac{1}{2}\right)\right)$
$\mathcal{S}_{\text{dspc}}(1 - 2\sigma)$	$\|Tx_\# - Tx_\$\|_2^2 \le \|Gx_\# - Gx_\$\|_2^2 + (1 - 2\sigma)\|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge \sigma\|x_\# - x_\$\|_2^2$ with $\sigma \in \left(0, \frac{1}{2}\right)$	$\mathcal{M}_{\text{sm}}\left(\sigma \in \left(0, \frac{1}{2}\right)\right)$
$\mathcal{S}_{\text{pc}}$	$\|Tx_\# - Tx_\$\|_2^2 \le \|x_\# - x_\$\|_2^2 + \|Gx_\# - Gx_\$\|_2^2$	$\Longleftarrow$	$\langle Gx_\# - Gx_\$, x_\# - x_\$\rangle \ge 0$	$\mathcal{M}_{\text{m}}$

Table 15.4: Combined language for conditions on $G = I - T$, implications for T. The subscript "gc" in $\mathcal{S}_{\text{gc}}(a,b)$ stands for "generalized contractivity-type".

$\mathcal{S}_{\text{gc}}(a,b)$	Contractivity condition on T		Monotonicity condition on $G = I - T$	$\mathcal{M}_{\text{csm}}(\sigma,\rho)$
$c^2, 0$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq c^2 \lVert x_{\#} - x_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \left(\frac{1-c^2}{2}\right) \cdot \lVert x_{\#} - x_{\$}\rVert_2^2 + \frac{1}{2}\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\frac{1-c^2}{2}, \frac{1}{2}$
$0, d^2$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq d^2 \lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \frac{1}{2}\lVert x_{\#} - x_{\$}\rVert_2^2 + \left(\frac{1-d^2}{2}\right)\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\frac{1}{2}, \frac{1-d^2}{2}$
$1, -1$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq 1\lVert x_{\#} - x_{\$}\rVert_2^2 - 1\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \rho\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$ with $\rho = 1$	$0, 1$
$-1, 1$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq -1\lVert x_{\#} - x_{\$}\rVert_2^2 + 1\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \sigma\lVert x_{\#} - x_{\$}\rVert_2^2$ with $\sigma = 1$	$1, 0$
$1, -(2\rho - 1) \in (-\infty, 0)$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq 1\lVert x_{\#} - x_{\$}\rVert_2^2 - (2\rho - 1)\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \rho\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$ with $\rho \in \left(\frac{1}{2}, +\infty\right)$	$0, \rho \in \left(\frac{1}{2}, +\infty\right)$
$-(2\sigma - 1) \in (-\infty, 0), 1$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq -(2\sigma - 1)\lVert x_{\#} - x_{\$}\rVert_2^2 + 1\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \sigma\lVert x_{\#} - x_{\$}\rVert_2^2$ with $\sigma \in \left(\frac{1}{2}, +\infty\right)$	$\sigma \in \left(\frac{1}{2}, +\infty\right), 0$
$1, 0$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq 1\lVert x_{\#} - x_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \rho\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$ with $\rho = \frac{1}{2}$	$0, \frac{1}{2}$
$0, 1$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq 1\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \sigma\lVert x_{\#} - x_{\$}\rVert_2^2$ with $\sigma = \frac{1}{2}$	$\frac{1}{2}, 0$
$1, (1 - 2\rho) \in (0, 1)$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq 1\lVert x_{\#} - x_{\$}\rVert_2^2 + (1 - 2\rho)\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \rho\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$ with $\rho \in \left(0, \frac{1}{2}\right)$	$0, \rho \in \left(0, \frac{1}{2}\right)$
$(1 - 2\sigma) \in (0, 1), 1$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq (1 - 2\sigma)\lVert x_{\#} - x_{\$}\rVert_2^2 + 1\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq \sigma\lVert x_{\#} - x_{\$}\rVert_2^2$ with $\sigma \in \left(0, \frac{1}{2}\right)$	$\sigma \in \left(0, \frac{1}{2}\right), 0$
$1, 1$	$\lVert Tx_{\#} - Tx_{\$}\rVert_2^2 \leq 1\lVert x_{\#} - x_{\$}\rVert_2^2 + 1\lVert Gx_{\#} - Gx_{\$}\rVert_2^2$	$\Longleftarrow$	$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \geq 0$	$0, 0$

Comparison of implications of monotonicity conditions To get a clearer comparison, we re-write the implications together and in matching form.

Proposition 21 (Strict monotonicity and strict pseudocontractivity.)**.** *When $G = I - T$ is strictly monotone, T is strictly pseudocontractive (without specifying a parameter value).*

Proof. Recall that we say an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ is strictly monotone when

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle > 0$$

for all $x_{\#}, x_{\$} \in \mathbb{R}^n$ such that $x_{\#} \neq x_{\$}$ and $Gx_{\#} \neq Gx_{\$}$. ☐

From $G = I - T$ being strictly monotone

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle > 0$$

and the central identity

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle = \frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2} \|Gx_{\#} - Gx_{\$}\|_2^2 - \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2,$$

we get

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle > 0$$

from which

$$\frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2} \|Gx_{\#} - Gx_{\$}\|_2^2 - \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2 > 0$$

so that

$$\begin{aligned} \frac{1}{2} \|x_{\#} - x_{\$}\|_2^2 + \frac{1}{2} \|Gx_{\#} - Gx_{\$}\|_2^2 &> \frac{1}{2} \|Tx_{\#} - Tx_{\$}\|_2^2 \\ \|Tx_{\#} - Tx_{\$}\|_2^2 &< \|x_{\#} - x_{\$}\|_2^2 + \|Gx_{\#} - Gx_{\$}\|_2^2. \end{aligned}$$

This establishes that $G = I - T$ being strictly monotone implies that T is "strictly pseudocontractive" (without specifying a parameter value).

Chapter 16

Operator scaling, transformation, combination, and composition

16.1 Introduction

Now that we have considered the most prominent collections of operator properties, we move on to consider how those properties are affected by various processes performed on those operators. In particular, we will consider scaling, "rays", combination, and composition.

16.2 Monotonicity properties under scaling

Here we consider scaling by a strictly positive scalar $\gamma \in \mathbb{R}_{++}$; in particular, we consider monotonicity properties under scaling by examining $F \overset{\text{set}}{=} \gamma G$.

16.2.1 Scaling an operator G that is strongly monotone

When G is σ-strongly monotone, $F \overset{\text{set}}{=} \gamma G$ is $\gamma\sigma$ strongly monotone:

$$\begin{aligned}
\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle &\geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2 \\
\gamma \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle &\geq \gamma\sigma \left\| x_{\#} - x_{\$} \right\|_2^2 \\
\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$} \rangle &\geq \gamma\sigma \left\| x_{\#} - x_{\$} \right\|_2^2 .
\end{aligned}$$

We interpret this in the light of the previously-established implications from σ-strong monotonicity of $G = I - T$ to contractivity properties of T. We first recall that

- when $\sigma \in \left(0, \frac{1}{2}\right)$, we have $T \in \mathcal{S}_{\text{dspc}}(1-2\sigma)$;
  - as σ increases from 0 to $\frac{1}{2}$, $1-2\sigma$ decreases from 1 to 0
  - $\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|Gx_{\#} - Gx_{\$}\|_2^2 + (1-2\sigma)\|x_{\#} - x_{\$}\|_2^2$
- when $\sigma = \frac{1}{2}$, we have $T \in \mathcal{S}_{\text{dspc}}(0) = \mathcal{S}_{\text{dne}}$;
  - with σ at $\frac{1}{2}$, $1-2\sigma$ is 0
  - $\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|Gx_{\#} - Gx_{\$}\|_2^2$
- when $\sigma \in \left(\frac{1}{2}, +\infty\right)$, we have $T \in \mathcal{S}_{\text{dspc}}(1-2\sigma) = \mathcal{S}_{\text{ddpc}}(2\sigma-1)$;
  - as σ increases from $\frac{1}{2}$ to $+\infty$, $1-2\sigma$ decreases from 0 to $-\infty$
  - as σ increases from $\frac{1}{2}$ to $+\infty$, $2\sigma-1$ increases from 0 to $+\infty$
  - $\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|Gx_{\#} - Gx_{\$}\|_2^2 + (1-2\sigma)\|x_{\#} - x_{\$}\|_2^2$
  - $\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|Gx_{\#} - Gx_{\$}\|_2^2 - (2\sigma-1)\|x_{\#} - x_{\$}\|_2^2$

When we move to consider the scaling $F \overset{\text{set}}{=} \gamma G$ with G σ-strongly monotone, we thus have the following implications for the contractivity properties of $S = I - F = I - \gamma G$

- when $\gamma\sigma \in \left(0, \frac{1}{2}\right)$, we have $S \in \mathcal{S}_{\text{dspc}}(1-2\gamma\sigma)$;
  - as $\gamma\sigma$ increases from 0 to $\frac{1}{2}$, $1-2\gamma\sigma$ decreases from 1 to 0
  - $\|Sx_{\#} - Sx_{\$}\|_2^2 \leq \|Fx_{\#} - Fx_{\$}\|_2^2 + (1-2\gamma\sigma)\|x_{\#} - x_{\$}\|_2^2$
- when $\gamma\sigma = \frac{1}{2}$, we have $S \in \mathcal{S}_{\text{dspc}}(0) = \mathcal{S}_{\text{dne}}$;
  - with $\gamma\sigma$ at $\frac{1}{2}$, $1-2\gamma\sigma$ is 0
  - $\|Sx_{\#} - Sx_{\$}\|_2^2 \leq \|Fx_{\#} - Fx_{\$}\|_2^2$
- when $\gamma\sigma \in \left(\frac{1}{2}, +\infty\right)$, we have $S \in \mathcal{S}_{\text{dspc}}(1-2\gamma\sigma) = \mathcal{S}_{\text{ddpc}}(2\gamma\sigma-1)$;

- as $\gamma\sigma$ increases from $\frac{1}{2}$ to $+\infty$, $1-2\gamma\sigma$ decreases from 0 to $-\infty$
- as $\gamma\sigma$ increases from $\frac{1}{2}$ to $+\infty$, $2\gamma\sigma-1$ increases from 0 to $+\infty$
- $\|Sx_{\#}-Sx_{\$}\|_2^2 \leq \|Fx_{\#}-Fx_{\$}\|_2^2+(1-2\gamma\sigma)\|x_{\#}-x_{\$}\|_2^2$
- $\|Sx_{\#}-Sx_{\$}\|_2^2 \leq \|Fx_{\#}-Fx_{\$}\|_2^2-(2\gamma\sigma-1)\|x_{\#}-x_{\$}\|_2^2$

We may interpret the observations above as follows: if we know that G is strongly monotone for some value σ, scaling by γ gives us the ability to shift $F \overset{\text{set}}{=} \gamma G$ into whatever parameter range we desire. Presumably one would usually choose γ so that we have $\gamma\sigma \in \left(\frac{1}{2},+\infty\right)$, so as to ensure that $I-\gamma G$ is displacement decreasing pseudocontractive.

16.2.2 Scaling an operator G that is inverse strongly monotone

When G is ρ-inverse strongly monotone, $F \overset{\text{set}}{=} \gamma G$ is $\frac{1}{\gamma}\rho$-inverse strongly monotone.

$$
\begin{aligned}
\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle &\geq \rho\,\|Gx_{\#}-Gx_{\$}\|_2^2 \\
\gamma\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle &\geq \gamma\rho\,\|Gx_{\#}-Gx_{\$}\|_2^2 \\
\gamma\langle Gx_{\#}-Gx_{\$},x_{\#}-x_{\$}\rangle &\geq \frac{1}{\gamma^2}\gamma^2\cdot\gamma\rho\,\|Gx_{\#}-Gx_{\$}\|_2^2 \\
\langle \gamma Gx_{\#}-\gamma Gx_{\$},x_{\#}-x_{\$}\rangle &\geq \frac{1}{\gamma^2}\cdot\gamma\rho\,\|\gamma Gx_{\#}-\gamma Gx_{\$}\|_2^2 \\
\langle Fx_{\#}-Fx_{\$},x_{\#}-x_{\$}\rangle &\geq \frac{1}{\gamma}\rho\,\|Fx_{\#}-Fx_{\$}\|_2^2.
\end{aligned}
$$

We interpret this in the light of the previously-established implications from ρ-inverse strong monotonicity of $G=I-T$ to contractivity properties of T. We first recall that

- when $\rho \in \left(0,\frac{1}{2}\right)$, we have $T \in \mathcal{S}_{\text{spc}}(1-2\rho)$;
  - as ρ increases from 0 to $\frac{1}{2}$, $1-2\rho$ decreases from 1 to 0
  - $\|Tx_{\#}-Tx_{\$}\|_2^2 \leq \|x_{\#}-x_{\$}\|_2^2+(1-2\rho)\|Gx_{\#}-Gx_{\$}\|_2^2$
- when $\rho=\frac{1}{2}$, we have $T \in \mathcal{S}_{\text{spc}}(0)=\mathcal{S}_{\text{ne}}$;
  - with ρ at $\frac{1}{2}$, $1-2\rho$ is 0

  - $\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2$
- when $\rho \in \left(\frac{1}{2}, +\infty\right)$, we have $T \in \mathcal{S}_{\text{spc}}(1-2\rho) = \mathcal{S}_{\text{dpc}}(2\rho - 1)$;
  - as ρ increases from $\frac{1}{2}$ to $+\infty$, $1-2\rho$ decreases from 0 to $-\infty$
  - as ρ increases from $\frac{1}{2}$ to $+\infty$, $2\rho - 1$ increases from 0 to $+\infty$
  - $\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + (1-2\rho)\|Gx_{\#} - Gx_{\$}\|_2^2$
  - $\|Tx_{\#} - Tx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + (2\rho - 1)\|Gx_{\#} - Gx_{\$}\|_2^2$

When we move to consider the scaling $F \overset{\text{set}}{=} \gamma G$ with G ρ strongly monotone, we thus have the following implications for the contractivity properties of $S = I - F = I - \gamma G$

- when $\frac{1}{\gamma}\rho \in \left(0, \frac{1}{2}\right)$, we have $S \in \mathcal{S}_{\text{spc}}\left(1 - 2\frac{1}{\gamma}\rho\right)$;
  - as $\frac{1}{\gamma}\rho$ increases from 0 to $\frac{1}{2}$, $1 - 2\frac{1}{\gamma}\rho$ decreases from 1 to 0
  - $\|Sx_{\#} - Sx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + \left(1 - 2\frac{1}{\gamma}\rho\right)\|Fx_{\#} - Fx_{\$}\|_2^2$
- when $\frac{1}{\gamma}\rho = \frac{1}{2}$, we have $S \in \mathcal{S}_{\text{spc}}(0) = \mathcal{S}_{\text{ne}}$;
  - with $\frac{1}{\gamma}\rho$ at $\frac{1}{2}$, $1 - 2\frac{1}{\gamma}\rho$ is 0
  - $\|Sx_{\#} - Sx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2$
- when $\frac{1}{\gamma}\rho \in \left(\frac{1}{2}, +\infty\right)$, we have $S \in \mathcal{S}_{\text{spc}}\left(1 - 2\frac{1}{\gamma}\rho\right) = \mathcal{S}_{\text{dpc}}\left(2\frac{1}{\gamma}\rho - 1\right)$;
  - as $\frac{1}{\gamma}\rho$ increases from $\frac{1}{2}$ to $+\infty$, $1 - 2\frac{1}{\gamma}\rho$ decreases from 0 to $-\infty$
  - as $\frac{1}{\gamma}\rho$ increases from $\frac{1}{2}$ to $+\infty$, $2\frac{1}{\gamma}\rho - 1$ increases from 0 to $+\infty$
  - $\|Sx_{\#} - Sx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 + \left(1 - 2\frac{1}{\gamma}\rho\right)\|Fx_{\#} - Fx_{\$}\|_2^2$
  - $\|Sx_{\#} - Sx_{\$}\|_2^2 \leq \|x_{\#} - x_{\$}\|_2^2 - \left(2\frac{1}{\gamma}\rho - 1\right)\|Fx_{\#} - Fx_{\$}\|_2^2$

We may interpret the observations above as follows: if we know that G is inverse strongly monotone for some value ρ, scaling by γ gives us the ability to shift $F \overset{\text{set}}{=} \gamma G$ into whatever parameter range we desire. Presumably one would usually choose γ so that we have $\frac{1}{\gamma}\rho \in \left(\frac{1}{2}, +\infty\right)$, so as to ensure that $S \overset{\text{set}}{=} I - F = I - \gamma G$ is decreasing pseudocontractive.

Note that while we want both $\frac{1}{\gamma}\rho \in \left(\frac{1}{2}, +\infty\right)$ and $\gamma\sigma \in \left(\frac{1}{2}, +\infty\right)$, the effect of the scaling parameter γ is opposite between the two situations.

16.2.3 Scaling an operator G that is both strongly monotone and inverse strongly monotone

This immediately tells us that when G is both σ-strongly monotone and ρ-inverse strongly monotone, $F \overset{\text{set}}{=} \gamma G$ is both $\gamma \cdot \sigma$ strongly monotone and $\frac{1}{\gamma}\rho$-inverse strongly monotone; this in turn tells us

$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2$$
$$\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle \geq \rho \left\| Gx_{\#} - Gx_{\$} \right\|_2^2$$

imply

$$\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$} \rangle \geq \gamma\sigma \left\| x_{\#} - x_{\$} \right\|_2^2$$
$$\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$} \rangle \geq \frac{1}{\gamma}\rho \left\| Fx_{\#} - Fx_{\$} \right\|_2^2$$

and further,

$$\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$} \rangle \geq \frac{(\gamma\sigma)\frac{1}{\left(\frac{1}{\gamma}\rho\right)}}{\left(\frac{1}{\left(\frac{1}{\gamma}\rho\right)} + (\gamma\sigma)\right)} \left\| x_{\#} - x_{\$} \right\|_2^2 + \frac{1}{\left(\frac{1}{\left(\frac{1}{\gamma}\rho\right)} + (\gamma\sigma)\right)} \left\| Fx_{\#} - Fx_{\$} \right\|_2^2$$
$$\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$} \rangle \geq \gamma\frac{\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho} + \sigma\right)} \left\| x_{\#} - x_{\$} \right\|_2^2 + \frac{1}{\gamma}\frac{1}{\left(\frac{1}{\rho} + \sigma\right)} \left\| Fx_{\#} - Fx_{\$} \right\|_2^2.$$

We can further observe that $F \overset{\text{set}}{=} \gamma G$ being both $\gamma\sigma$ strongly monotone and $\frac{1}{\gamma}\rho$-inverse strongly monotone implies that $S \overset{\text{set}}{=} I - F = I - \gamma G$ will satisfy the contractivity condition

$$\begin{aligned}
&\left\| Sx_{\#} - Sx_{\$} \right\|_2^2 \\
&\leq \left[1 - \frac{2(\gamma\sigma)\frac{1}{\left(\frac{1}{\gamma}\rho\right)}}{\left(\frac{1}{\left(\frac{1}{\gamma}\rho\right)} + (\gamma\sigma)\right)} \right] \left\| x_{\#} - x_{\$} \right\|_2^2 + \left[1 - \frac{2}{\left(\frac{1}{\left(\frac{1}{\gamma}\rho\right)} + (\gamma\sigma)\right)} \right] \left\| Fx_{\#} - Fx_{\$} \right\|_2^2
\end{aligned}$$

from which

$$\|Sx_{\#}-Sx_{\$}\|_2^2 \leq \left[1-\gamma\frac{2\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|x_{\#}-x_{\$}\|_2^2+\left[1-\frac{1}{\gamma}\frac{2}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|Fx_{\#}-Fx_{\$}\|_2^2$$

$$\|Sx_{\#}-Sx_{\$}\|_2^2 \leq \left[1-\gamma\frac{2\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|x_{\#}-x_{\$}\|_2^2+\left[1-\frac{1}{\gamma}\frac{2}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|\gamma Gx_{\#}-\gamma Gx_{\$}\|_2^2$$

$$\|Sx_{\#}-Sx_{\$}\|_2^2 \leq \left[1-\gamma\frac{2\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|x_{\#}-x_{\$}\|_2^2+\left[1-\frac{1}{\gamma}\frac{2}{\left(\frac{1}{\rho}+\sigma\right)}\right]\gamma^2\|Gx_{\#}-Gx_{\$}\|_2^2$$

$$\|Sx_{\#}-Sx_{\$}\|_2^2 \leq \left[1-\gamma\frac{2\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|x_{\#}-x_{\$}\|_2^2+\gamma\left[\gamma-\frac{2}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|Gx_{\#}-Gx_{\$}\|_2^2$$

so that

$$\|Sx_{\#}-Sx_{\$}\|_2^2 \leq \left[1-\gamma\frac{2\sigma\frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\right]\|x_{\#}-x_{\$}\|_2^2-\gamma\left[\frac{2}{\left(\frac{1}{\rho}+\sigma\right)}-\gamma\right]\|Gx_{\#}-Gx_{\$}\|_2^2.$$

We collect the relationships between scaled operator monotonicity-type conditions and the corresponding contractivity conditions in Table 16.1.

16.2.4 Ray from the identity through an operator T

We can use a perspective from [BP67] to provide some additional interpretations of the correspondences in Table 16.1.

We will now consider operators along the ray starting at I passing through T.

We denote the set of all such operators as $\text{Iray}(T)$ with definition

$$\text{Iray}(T) \overset{\text{set}}{=} \{W \mid W = I + t(T-I) \text{ for some } t \in \mathbb{R}_{++}\}.$$

We note that we can view an element of $\text{Iray}(T)$ from several perspectives:

$$W = I + t(T-I)$$
$$W = (1-t)I + tT$$
$$W = I - t(I-T) = I - tG.$$

Each of these expressions emphasizes a different viewpoint:

Table 16.1: Monotonicity conditions on $F \overset{\text{set}}{=} \gamma G$, implications for $S \overset{\text{set}}{=} I - F = I - \gamma G$.

Contractivity condition for $S = I - \gamma G$		Monotonicity condition for $F = \gamma G$
$\|Sx_\# - Sx_\$\|_2^2 \leq \left[1 - \gamma \frac{2\sigma \frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)}\right] \|x_\# - x_\$\|_2^2 + \left[1 - \frac{1}{\gamma} \frac{2}{\left(\frac{1}{\rho}+\sigma\right)}\right] \|Fx_\# - Fx_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \gamma \frac{\sigma \frac{1}{\rho}}{\left(\frac{1}{\rho}+\sigma\right)} \|x_\# - x_\$\|_2^2 + \frac{1}{\gamma} \frac{1}{\left(\frac{1}{\rho}+\sigma\right)} \|Fx_\# - Fx_\$\|_2^2$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \|Fx_\# - Fx_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \frac{1}{\gamma}\rho \|Fx_\# - Fx_\$\|_2^2$ with $\frac{1}{\gamma}\rho = 1$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|Fx_\# - Fx_\$\|_2^2 - \|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \gamma\sigma \|x_\# - x_\$\|_2^2$ with $\gamma\sigma = 1$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 - \left(2\frac{1}{\gamma}\rho - 1\right) \|Fx_\# - Fx_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \frac{1}{\gamma}\rho \|Fx_\# - Fx_\$\|_2^2$ with $\frac{1}{\gamma}\rho \in \left(\frac{1}{2}, +\infty\right)$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|Fx_\# - Fx_\$\|_2^2 - (2\gamma\sigma - 1) \|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \gamma\sigma \|x_\# - x_\$\|_2^2$ with $\gamma\sigma \in \left(\frac{1}{2}, +\infty\right)$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \frac{1}{\gamma}\rho \|Fx_\# - Fx_\$\|_2^2$ with $\frac{1}{\gamma}\rho = \frac{1}{2}$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|Fx_\# - Fx_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \gamma\sigma \|x_\# - x_\$\|_2^2$ with $\gamma\sigma = \frac{1}{2}$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 + \left(1 - 2\frac{1}{\gamma}\rho\right) \|Fx_\# - Fx_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \frac{1}{\gamma}\rho \|Fx_\# - Fx_\$\|_2^2$ with $\frac{1}{\gamma}\rho \in \left(0, \frac{1}{2}\right)$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|Fx_\# - Fx_\$\|_2^2 + (1 - 2\gamma\sigma) \|x_\# - x_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq \gamma\sigma \|x_\# - x_\$\|_2^2$ with $\gamma\sigma \in \left(0, \frac{1}{2}\right)$
$\|Sx_\# - Sx_\$\|_2^2 \leq \|x_\# - x_\$\|_2^2 + \|Fx_\# - Fx_\$\|_2^2$	$\Longleftarrow$	$\langle Fx_\# - Fx_\$, x_\# - x_\$ \rangle \geq 0$

The form $W = I + t\,(T - I)$ emphasizes that W is along the “ray of operators” starting at I and passing through T.

The form $W = (1-t)I + tT$ more closely resembles the standard expression used in convex combinations.

Finally, the form $W = I - t\,(I - T)$ directs attention to an interpretation as the displacement operator $I - tG$ associated with tG, which is in turn a t-scaled version of the displacement operator $G = I - T$.

Continuing, we note that an immediate consequence of $W = I + t\,(T - I)$ is $\text{Fix}\, W = \text{Fix}\, T$. We may describe this connection as follows:

$$\begin{aligned}
\text{Fix}\, W &= \text{Fix}\,(I - tG) \\
&= \text{Zeros}\,(tG) \\
&= \text{Zeros}\,(G) \\
&= \text{Fix}\,(I - G) \\
&= \text{Fix}\, T.
\end{aligned}$$

We can also “reverse” our perspective (from viewing W as constructed from T to viewing T as constructed from W) via the equivalence of the expressions

$$\begin{aligned}
W &= I + t\,(T - I) \\
T &= I + \frac{1}{t}\,(W - I)\,.
\end{aligned}$$

A less explicit statement of the immediately preceding equivalence is $W \in \text{Iray}\,(T) \iff T \in \text{Iray}\,(W)\,.$

Passing from T through $G = I - T$ and tG to $W = I - t\,(I - T)$

We can now state three additional results, one neglecting parameter values and two tracking parameter values.

Pseudocontractivity and monotonicity First, consider a pseudocontractive operator T. Every $W \in \text{Iray}\,(T)$ is also pseudocontractive.

Explicitly, we are saying: when W is of the form $I+t(T-I)$ we have

$$\|Tx-Ty\|_2^2 \leq \|x-y\|_2^2 + \|[I-T]x-[I-T]y\|_2^2$$
$$\text{implies}$$
$$\|Wx-Wy\|_2^2 \leq \|x-y\|_2^2 + \|[I-W]x-[I-W]y\|_2^2.$$

This result provides an example of the convenience of using the correspondence between contractivity properties of an operator and monotonicity properties of the associated displacement operator, since we can pass from a contractivity property of T to a monotonicity property of $G=I-T$, to a monotonicity property of tG, to a contractivity property of $W=I-tG=I-tT$.

More specifically, when T is pseudocontractive, $G=I-T$ is monotone. When $G=I-T$ is monotone, so too is $tG=t(I-T)$ for any $t \in \mathbb{R}_{++}$. From tG monotone, we observe that $W=I-tG=I-t(I-T)$ is pseudocontractive.

Thus, we pass from the contractivity properties of T to the contractivity properties of $W=I-tG=I-t(I-T)$ by means of the monotonicity properties of $G=I-T$ and tG.

p-strict pseudocontractivity and inverse strong monotonicity Consider a p-strictly pseudocontractive operator T; every $W \in \operatorname{Iray}(T)$ is also strictly pseudocontractive, with constant $k \overset{\text{set}}{=} 1-\frac{1}{t}(1-p)$.

Explicitly, we are saying that

$$\|Tx-Ty\|_2^2 \leq \|x-y\|_2^2 + p\,\|[I-T]x-[I-T]y\|_2^2$$
$$\text{implies}$$
$$\|Wx-Wy\|_2^2 \leq \|x-y\|_2^2 + k\,\|[I-W]x-[I-W]y\|_2^2,$$

where $k \overset{\text{set}}{=} 1-\frac{1}{t}(1-p)$. We note that this relationship mirrors the previously observed relationship between T and $W=I-tG=I-tT$.

This result provides another example of the convenience of using the correspondence between contractivity properties of an operator and monotonicity properties of the associated displacement operator, since we can again pass from a contractivity property of T to a monotonicity property of $G=I-T$, to a monotonicity property of tG, to a contractivity property of $W = I - tG = I - tT$; in contrast to the previous discussion, in which no specific parameter values were tracked, we will now be very specific about the parameter values involved.

Specifically, we have seen that when T is p-strictly pseudocontractive, $G = I - T$ is $\left(\frac{1-p}{2}\right)$-inverse strongly monotone[1]. Further, our results on scaling and monotonicity properties tell us that when $G = I - T$ is $\left(\frac{1-p}{2}\right)$-inverse strongly monotone, the scaled version $tG = t\left(I - T\right) = I - W$ is $\frac{1}{t}\left(\frac{1-p}{2}\right)$-inverse strongly monotone. Finally, when $tG = t\left(I - T\right) = I - W$ is $\frac{1}{t}\left(\frac{1-p}{2}\right)$-inverse strongly monotone we have that W is $k = \left(1 - 2\left(\frac{1}{t}\left(\frac{1-p}{2}\right)\right)\right) = 1 - \frac{1}{t}\left(1 - p\right)$ strictly pseudocontractive. Just as we had several views of the relationship between T and $W = I + t\left(T - I\right)$, namely,

$$W = I + t\left(T - I\right)$$
$$W = (1 - t)I + tT$$
$$W = I - t\left(I - T\right) = I - tG,$$

we can have several views of the pseudocontractivity parameter k associated with W as we alter t (recalling that p is strictly less than 1):

$$k = 1 + \frac{1}{t}\left(p - 1\right)$$
$$k = \left(1 - \frac{1}{t}\right) + \frac{1}{t}p$$
$$k = 1 - \frac{1}{t}\left(1 - p\right).$$

We would argue that the third form is clearest: because p is strictly less than 1, we know that $1 - p$ is strictly greater than 0; in the limit of large t scaling, we have a k near 1 (and a corresponding tG that is near to being only monotone); a t scaling of 1 leads to k coinciding with p (and a corresponding tG that is $\left(\frac{1-p}{2}\right)$-inverse strongly monotone, observing that $\left(\frac{1-p}{2}\right) < \frac{1}{2}$); a t scaling of $(1 - p)$ will yield a k of 0 (and a corresponding tG that is $\frac{1}{2}$-inverse strongly monotone); a t scaling smaller than $(1 - p)$

[1] The case $p \stackrel{\text{set}}{=} 1$ would correspond to 0-inverse strongly monotone (that is, to simply being monotone); decreasing p from 1 to 0 would correspond to $\left(\frac{1-p}{2}\right)$ increasing from 0 to $\frac{1}{2}$.

will yield a k that is negative (and a corresponding tG that is inverse strongly monotone with parameter in the range $\left(\frac{1}{2},+\infty\right)$).

q-displacement strict pseudocontractivity and strong monotonicity We saw a relationship between strict pseudocontractivity and inverse strong monotonicity in the previous section; we now consider an analogous relationship between displacement strict pseudocontractivity and strong monotonicity.

Consider a q-displacement strictly pseudocontractive operator T; every $W \in \mathrm{Iray}(T)$ is also displacement strictly pseudocontractive, with constant $h \overset{\text{set}}{=} 1-\frac{1}{t}(1-q)$.

Explicitly, we are saying that

$$\|Tx-Ty\|_2^2 \leq \|x-y\|_2^2 + q\,\|[I-T]\,x-[I-T]\,y\|_2^2$$
$$\text{implies}$$
$$\|Wx-Wy\|_2^2 \leq \|x-y\|_2^2 + h\,\|[I-W]\,x-[I-W]\,y\|_2^2,$$

where $h \overset{\text{set}}{=} 1-\frac{1}{t}(1-q)$. We note that this relationship again mirrors the previously observed relationship between T and $W = I-tG = I-tT$.

This result provides a further example of the convenience of using the correspondence between contractivity properties of an operator and monotonicity properties of the associated displacement operator, since we can again pass from a contractivity property of T to a monotonicity property of $G = I-T$, to a monotonicity property of tG, to a contractivity property of $W = I-tG = I-t(I-T)$.

Of particular relevance to the current discussion, we have seen that when T is q-displacement strictly pseudocontractive, $G = I-T$ is $\left(\frac{1-q}{2}\right)$-strongly monotone[2]. Further, our results on scaling and monotonicity properties tell us that when $G = I-T$ is $\left(\frac{1-q}{2}\right)$-strongly monotone, the scaled version $tG = t(I-T) = I-W$ is $\frac{1}{t}\left(\frac{1-q}{2}\right)$-strongly monotone. Finally, when $tG = t(I-T) = I-W$ is $\frac{1}{t}\left(\frac{1-q}{2}\right)$-strongly monotone we have that W is $h = \left(1-2\left(\frac{1}{t}\left(\frac{1-q}{2}\right)\right)\right) = 1-\frac{1}{t}(1-q)$ displacement strictly pseudocontractive. Again recalling the several views of the relationship between T and $W = I+t(T-I)$, namely,

[2]The case $q \overset{\text{set}}{=} 1$ would correspond to 0-strongly monotone (that is, to simply being monotone); decreasing q from 1 to 0 would correspond to $\left(\frac{1-q}{2}\right)$ increasing from 0 to $\frac{1}{2}$-strongly monotone.

$$W = I + t(T - I)$$
$$W = (1-t)I + tT$$
$$W = I - t(I - T) = I - tG,$$

we again have several views of the displacement pseudocontractivity parameter h associated with W as we alter t (recalling that p is strictly less than 1):

$$h = 1 + \frac{1}{t}(q-1)$$
$$h = \left(1 - \frac{1}{t}\right) + \frac{1}{t}q$$
$$h = 1 - \frac{1}{t}(1-q).$$

Again, we feel that the third form is clearest: because q is strictly less than 1, we know that $1-q$ is strictly greater than 0; in the limit of large t scaling, we have an h near 1 (and a corresponding tG that is near to being only monotone); a t scaling of 1 leads to h coinciding with q (and a corresponding tG that is $\left(\frac{1-q}{2}\right)$-strongly monotone, observing that $\left(\frac{1-q}{2}\right) < \frac{1}{2}$); a t scaling of $(1-q)$ will yield an h of 0 (and a corresponding tG that is $\frac{1}{2}$-strongly monotone); a t scaling smaller than $(1-q)$ will yield an h that is negative (and a corresponding tG that is strongly monotone with parameter in the range $\left(\frac{1}{2}, +\infty\right)$).

Strictly pseudocontractive operators and decreasing pseudocontractive operators We saw above that for a p-strictly pseudocontractive operator T, every $W \in \text{Iray}(T)$ is also strictly pseudocontractive, with constant $k \overset{\text{set}}{=} 1 - \frac{1}{t}(1-p)$. In particular, we note that if we start with T a p-strictly pseudocontractive operator with $p \in (0,1)$, forming $W = I + t(T - I)$ with any t smaller than $(1-p)$ will yield a negative k value for W, so that W will be decreasing pseudocontractive.

Displacement strictly pseudocontractive operators and displacement decreasing pseudocontractive operators We saw above that for a q-strictly pseudocontractive operator T, every $W \in \text{Iray}(T)$ is also displacement strictly pseudocontractive, with constant $h \overset{\text{set}}{=} 1 - \frac{1}{t}(1-q)$. In particular, we note that if we start with T a q-strictly pseudocontractive operator with $q \in (0,1)$, forming $W = I + t(T - I)$ with any t smaller than $(1-q)$ will yield a negative h value for W, so that W will be displacement decreasing pseudocontractive.

Remark. Our discussion has now equipped us with a perspective on the category of "averaged nonexpansive" operators. If we restricted our attention to initial operators T that were exactly nonexpansive (rather than considering the broader category of strictly pseudocontractive operators) the reasoning used above would tell us that any t between 0 and 1 would yield a decreasing pseudocontractive operator. This path is a common one in some portions of the literature; however, this starting point serves to obscure that we could arrive at a decreasing pseudocontractive operator from any strictly pseudocontractive operator, and not simply from a nonexpansive operator.

16.3 Operator addition with the identity

Here we consider $F \overset{\text{set}}{=} G + \gamma I$ and $D \overset{\text{set}}{=} G - \gamma I$, where G satisfies various properties and where we have a strictly positive scale parameter $\gamma \in \mathbb{R}_{++}$.

When G is σ-strongly monotone, $F \overset{\text{set}}{=} G + \gamma I$ is $\sigma + \gamma$ strongly monotone.

$$\begin{aligned}
\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle &\geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2 \\
\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle + \gamma \left\| x_{\#} - x_{\$} \right\|_2^2 &\geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2 + \gamma \left\| x_{\#} - x_{\$} \right\|_2^2 \\
\langle [G + \gamma I] x_{\#} - [G + \gamma I] x_{\$}, x_{\#} - x_{\$} \rangle &\geq (\sigma + \gamma) \left\| x_{\#} - x_{\$} \right\|_2^2 \\
\langle F x_{\#} - F x_{\$}, x_{\#} - x_{\$} \rangle &\geq (\sigma + \gamma) \left\| x_{\#} - x_{\$} \right\|_2^2 .
\end{aligned}$$

When G is σ-strongly monotone, $D \overset{\text{set}}{=} G - \gamma I$ is $\sigma - \gamma$ strongly monotone; suppose that $\sigma > \gamma$.

$$\begin{aligned}
\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle &\geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2 \\
\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$} \rangle - \gamma \left\| x_{\#} - x_{\$} \right\|_2^2 &\geq \sigma \left\| x_{\#} - x_{\$} \right\|_2^2 - \gamma \left\| x_{\#} - x_{\$} \right\|_2^2 \\
\langle [G - \gamma I] x_{\#} - [G - \gamma I] x_{\$}, x_{\#} - x_{\$} \rangle &\geq (\sigma - \gamma) \left\| x_{\#} - x_{\$} \right\|_2^2 \\
\langle Dx_{\#} - Dx_{\$}, x_{\#} - x_{\$} \rangle &\geq (\sigma - \gamma) \left\| x_{\#} - x_{\$} \right\|_2^2 .
\end{aligned}$$

When G is ρ-inverse strongly monotone, $F \stackrel{\text{set}}{=} G + \gamma I$ is $\frac{1}{\left(\frac{1}{\rho}+\gamma\right)}$-inverse strongly monotone.

$$\begin{aligned}
\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle &\geq \rho \left\|Gx_{\#} - Gx_{\$}\right\|_2^2 \\
\frac{1}{\rho}\left\|x_{\#} - x_{\$}\right\|_2^2 &\geq \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \\
\frac{1}{\rho}\left\|x_{\#} - x_{\$}\right\|_2^2 + \gamma\left\|x_{\#} - x_{\$}\right\|_2^2 &\geq \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle + \gamma\left\|x_{\#} - x_{\$}\right\|_2^2 \\
\frac{1}{\rho}\left\|x_{\#} - x_{\$}\right\|_2^2 + \gamma\left\|x_{\#} - x_{\$}\right\|_2^2 &\geq \langle [G+\gamma I]\,x_{\#} - [G+\gamma I]\,x_{\$}, x_{\#} - x_{\$}\rangle \\
\left(\frac{1}{\rho} + \gamma\right)\left\|x_{\#} - x_{\$}\right\|_2^2 &\geq \langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$}\rangle \\
\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$}\rangle &\geq \frac{1}{\left(\frac{1}{\rho}+\gamma\right)}\left\|Fx_{\#} - Fx_{\$}\right\|_2^2.
\end{aligned}$$

When G is ρ-inverse strongly monotone, $D \stackrel{\text{set}}{=} G - \gamma I$ is $\frac{1}{\left(\frac{1}{\rho}-\gamma\right)}$ strongly monotone; suppose that $\frac{1}{\rho} > \gamma$.

$$\begin{aligned}
\langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle &\geq \rho \left\|Gx_{\#} - Gx_{\$}\right\|_2^2 \\
\frac{1}{\rho}\left\|x_{\#} - x_{\$}\right\|_2^2 &\geq \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle \\
\frac{1}{\rho}\left\|x_{\#} - x_{\$}\right\|_2^2 - \gamma\left\|x_{\#} - x_{\$}\right\|_2^2 &\geq \langle Gx_{\#} - Gx_{\$}, x_{\#} - x_{\$}\rangle - \gamma\left\|x_{\#} - x_{\$}\right\|_2^2 \\
\frac{1}{\rho}\left\|x_{\#} - x_{\$}\right\|_2^2 - \gamma\left\|x_{\#} - x_{\$}\right\|_2^2 &\geq \langle [G-\gamma I]\,x_{\#} - [G-\gamma I]\,x_{\$}, x_{\#} - x_{\$}\rangle \\
\left(\frac{1}{\rho} - \gamma\right)\left\|x_{\#} - x_{\$}\right\|_2^2 &\geq \langle Dx_{\#} - Dx_{\$}, x_{\#} - x_{\$}\rangle \\
\langle Dx_{\#} - Dx_{\$}, x_{\#} - x_{\$}\rangle &\geq \frac{1}{\left(\frac{1}{\rho}-\gamma\right)}\left\|Dx_{\#} - Dx_{\$}\right\|_2^2.
\end{aligned}$$

When G is both σ-strongly monotone and ρ-inverse strongly monotone, we have seen that $F \stackrel{\text{set}}{=} G + \gamma I$ is $\sigma + \gamma$ strongly monotone and $\frac{1}{\left(\frac{1}{\rho}+\gamma\right)}$-inverse strongly monotone. This corresponds to the combined expression

$$\begin{aligned}
&\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$}\rangle \\
\geq &\frac{1}{\left(\left(\frac{1}{\rho}+\gamma\right) + (\sigma+\gamma)\right)}\left\|Fx_{\#} - Fx_{\$}\right\|_2^2 + \frac{(\sigma+\gamma)\left(\frac{1}{\rho}+\gamma\right)}{\left(\left(\frac{1}{\rho}+\gamma\right) + (\sigma+\gamma)\right)}\left\|x_{\#} - x_{\$}\right\|_2^2
\end{aligned}$$

or

$$\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$} \rangle$$
$$\geq \frac{1}{\left(\frac{1}{\rho} + \sigma + 2\gamma\right)} \|Fx_{\#} - Fx_{\$}\|_2^2 + \frac{(\sigma+\gamma)\left(\frac{1}{\rho}+\gamma\right)}{\left(\frac{1}{\rho} + \sigma + 2\gamma\right)} \|x_{\#} - x_{\$}\|_2^2.$$

Similarly, when G is both σ-strongly monotone and ρ-inverse strongly monotone, we have seen that $D \stackrel{\text{set}}{=} G - \gamma I$ is $\sigma - \gamma$ strongly monotone and $\frac{1}{\left(\frac{1}{\rho}-\gamma\right)}$-inverse strongly monotone.

This corresponds to the combined expression

$$\langle Dx_{\#} - Dx_{\$}, x_{\#} - x_{\$} \rangle$$
$$\geq \frac{1}{\left(\left(\frac{1}{\rho} - \gamma\right) + (\sigma - \gamma)\right)} \|Dx_{\#} - Dx_{\$}\|_2^2 + \frac{(\sigma-\gamma)\left(\frac{1}{\rho}-\gamma\right)}{\left(\left(\frac{1}{\rho} - \gamma\right) + (\sigma - \gamma)\right)} \|x_{\#} - x_{\$}\|_2^2$$

or

$$\langle Dx_{\#} - Dx_{\$}, x_{\#} - x_{\$} \rangle$$
$$\geq \frac{1}{\left(\frac{1}{\rho} + \sigma - 2\gamma\right)} \|Dx_{\#} - Dx_{\$}\|_2^2 + \frac{(\sigma-\gamma)\left(\frac{1}{\rho}-\gamma\right)}{\left(\frac{1}{\rho} + \sigma - 2\gamma\right)} \|x_{\#} - x_{\$}\|_2^2.$$

16.4 Operator addition

When G_1 is σ_1-strongly monotone, and G_2 is σ_2-strongly monotone the sum $F \stackrel{\text{set}}{=} G_1 + G_2$ is $(\sigma_1 + \sigma_2)$-strongly monotone.

$$\begin{aligned}
\langle G_1x_{\#} - G_1x_{\$}, x_{\#} - x_{\$} \rangle &\geq \sigma_1 \|x_{\#} - x_{\$}\|_2^2 \\
\langle G_2x_{\#} - G_2x_{\$}, x_{\#} - x_{\$} \rangle &\geq \sigma_2 \|x_{\#} - x_{\$}\|_2^2 \\
\langle G_1x_{\#} - G_1x_{\$}, x_{\#} - x_{\$} \rangle + \langle G_2x_{\#} - G_2x_{\$}, x_{\#} - x_{\$} \rangle &\geq \sigma_1 \|x_{\#} - x_{\$}\|_2^2 + \sigma_2 \|x_{\#} - x_{\$}\|_2^2 \\
\langle [G_1 + G_2]\, x_{\#} - [G_1 + G_2]\, x_{\$}, x_{\#} - x_{\$} \rangle &\geq (\sigma_1 + \sigma_2) \|x_{\#} - x_{\$}\|_2^2 \\
\langle Fx_{\#} - Fx_{\$}, x_{\#} - x_{\$} \rangle &\geq (\sigma_1 + \sigma_2) \|x_{\#} - x_{\$}\|_2^2.
\end{aligned}$$

When G_1 is ρ_1-inverse strongly monotone, and G_2 is ρ_2-inverse strongly monotone the sum $F \overset{\text{set}}{=} G_1 + G_2$ is $\left(\frac{1}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}\right)$-inverse strongly monotone[3].

$$\begin{aligned}
\langle G_1 x_\# - G_1 x_\$, x_\# - x_\$ \rangle &\geq \rho_1 \left\| G_1 x_\# - G_1 x_\$ \right\|_2^2 \\
\frac{1}{\rho_1} \left\| x_\# - x_\$ \right\|_2^2 &\geq \langle G_1 x_\# - G_1 x_\$, x_\# - x_\$ \rangle \\
\langle G_2 x_\# - G_2 x_\$, x_\# - x_\$ \rangle &\geq \rho_2 \left\| G_2 x_\# - G_2 x_\$ \right\|_2^2 \\
\frac{1}{\rho_2} \left\| x_\# - x_\$ \right\|_2^2 &\geq \langle G_2 x_\# - G_2 x_\$, x_\# - x_\$ \rangle \\
\frac{1}{\rho_1} \left\| x_\# - x_\$ \right\|_2^2 + \frac{1}{\rho_2} \left\| x_\# - x_\$ \right\|_2^2 &\geq \langle G_1 x_\# - G_1 x_\$, x_\# - x_\$ \rangle + \langle G_2 x_\# - G_2 x_\$, x_\# - x_\$ \rangle \\
\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \left\| x_\# - x_\$ \right\|_2^2 &\geq \langle [G_1 + G_2] x_\# - [G_1 + G_2] x_\$, x_\# - x_\$ \rangle \\
\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \left\| x_\# - x_\$ \right\|_2^2 &\geq \langle F x_\# - F x_\$, x_\# - x_\$ \rangle \\
\langle F x_\# - F x_\$, x_\# - x_\$ \rangle &\geq \frac{1}{\frac{1}{\rho_1} + \frac{1}{\rho_2}} \left\| F x_\# - F x_\$ \right\|_2^2 .
\end{aligned}$$

When G_1 is σ_1-strongly monotone and ρ_1-inverse strongly monotone, and G_2 is σ_2-strongly monotone and ρ_2-inverse strongly monotone, the sum $F \overset{\text{set}}{=} G_1 + G_2$ is both $(\sigma_1 + \sigma_2)$-strongly monotone and $\left(\frac{1}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}\right)$-inverse strongly monotone. Thus we have

$$\begin{aligned}
&\langle F x_\# - F x_\$, x_\# - x_\$ \rangle \\
&\geq \frac{1}{\left(\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right] + [\sigma_1 + \sigma_2]\right)} \left\| F x_\# - F x_\$ \right\|_2^2 + \frac{[\sigma_1 + \sigma_2]\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right]}{\left(\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right] + [\sigma_1 + \sigma_2]\right)} \left\| x_\# - x_\$ \right\|_2^2 .
\end{aligned}$$

16.5 Operator affine combination

We now show that the displacement operator of an affine combination is the affine combination of the displacement operators.

The "line of operators" passing through $T_\#$ and $T_\$$ can be written

$$T_{\#\$}[t] \overset{\text{set}}{=} \operatorname{aff}[T_\#, T_\$; (1-t), t] = T_\# + t (T_\$ - T_\#) = (1-t) T_\# + t T_\$.$$

[3] Compare to "parallel addition" of matrices [AD69], to "resistors in parallel", and to the harmonic mean.

The displacement operator, say $G_{\#\$}[t] \overset{\text{set}}{=} I - T_{\#\$}[t]$, associated with $T_{\#\$}[t]$ then has the form

$$\begin{aligned} G_{\#\$}[t] &\overset{\text{set}}{=} I - T_{\#\$}[t] \\ &= I - [(1-t)\,T_{\#} + tT_{\$}] \\ &= [(1-t)\,I + tI] - [(1-t)\,T_{\#} + tT_{\$}] \\ &= (1-t)\,[I - T_{\#}] + t\,[I - T_{\$}] \\ &= (1-t)\,G_{\#} + tG_{\$} \\ &= \operatorname{aff}[G_{\#}, G_{\$}; (1-t), t]\,. \end{aligned}$$

We have not developed the machinery to properly describe situations where $t \notin [0,1]$; in particular, this would lead to a negative scaling for either $G_{\#}$ or $G_{\$}$ (and thus to a change in the sign of the characterizing inequality). This is a matter for further investigation.

16.6 Operator convex combination

We do have the machinery to more fully investigate the situations in which $t \in (0,1)$.

We will first consider a convex combination of operators considered from a monotonicity perspective before moving on to consider a convex combination of operators considered from a contractivity perspective.

16.6.1 Operator convex combination: monotonicity perspective

Convex combination of inverse strongly monotone operators

Our reasoning for the convex combination of inverse strongly monotone operators is as follows: we have seen that when $G_{\#} \overset{\text{set}}{=} I - T_{\#}$ is $\rho_{\#}$-inverse strongly monotone, the strictly positively scaled version $\alpha_{\#}G_{\#}$ is $\frac{1}{\alpha_{\#}}\rho_{\#}$-inverse strongly monotone; likewise when $G_{\$} \overset{\text{set}}{=} I - T_{\$}$ is $\rho_{\$}$-inverse strongly monotone, the nonnegatively scaled version $\alpha_{\$}G_{\$}$ is $\frac{1}{\alpha_{\$}}\rho_{\$}$-inverse strongly monotone.

We also have seen that when G_1 is ρ_1-inverse strongly monotone, and G_2 is ρ_2-inverse strongly monotone, the sum $F \stackrel{\text{set}}{=} G_1 + G_2$ is $\left(\frac{1}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}\right)$-inverse strongly monotone.

Thus, the sum of a $\rho_1 \stackrel{\text{set}}{=} \frac{1}{\alpha_\#}\rho_\#$-inverse strongly monotone operator and a $\rho_2 \stackrel{\text{set}}{=} \frac{1}{\alpha_\$}\rho_\$$-inverse strongly monotone operator is an operator that is inverse strongly monotone with parameter

$$\left(\frac{1}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}\right) = \frac{1}{\frac{1}{\left(\frac{1}{\alpha_\#}\rho_\#\right)} + \frac{1}{\left(\frac{1}{\alpha_\$}\rho_\$\right)}} = \frac{1}{\alpha_\# \frac{1}{\rho_\#} + \alpha_\$ \frac{1}{\rho_\$}}.$$

Note that we can view this as a “weighted harmonic mean” of the inverse strong monotonicity parameters.

Convex combination of strongly monotone operators

Our reasoning for the convex combination of strongly monotone operators is as follows: we have seen that when $G_\# \stackrel{\text{set}}{=} I - T_\#$ is $\sigma_\#$-strongly monotone, the strictly positively scaled version $\alpha_\# G_\#$ is $\alpha_\#\sigma_\#$-strongly monotone; likewise when $G_\$ \stackrel{\text{set}}{=} I - T_\$$ is $\sigma_\$$-strongly monotone, the strictly positively scaled version $\alpha_\$ G_\$$ is $\alpha_\$\sigma_\$$-strongly monotone.

We also have seen that when G_1 is σ_1-strongly monotone, and G_2 is σ_2-strongly monotone, the sum $F \stackrel{\text{set}}{=} G_1 + G_2$ is $(\sigma_1 + \sigma_2)$-strongly monotone.

Thus, the sum of a $\sigma_1 \stackrel{\text{set}}{=} \alpha_\#\sigma_\#$-strongly monotone operator and a $\rho_2 \stackrel{\text{set}}{=} \alpha_\$\sigma_\$$-strongly monotone operator is an operator that is strongly monotone with parameter

$$(\sigma_1 + \sigma_2) = \alpha_\#\sigma_\# + \alpha_\$\sigma_\$.$$

Note that we can view this as a “weighted mean” of the strong monotonicity parameters.

16.6.2 Operator convex combination: contractivity perspective

Convex combination of pseudocontractive operators

We will now move on to consider a convex combination of pseudocontractive operators; we will do this by leveraging our previous result from the convex combination of inverse strongly monotone operators. Specifically, consider $T_{\#}$ $p_{\#}$-strictly pseudocontractive and $T_{\$}$ $p_{\$}$-strictly pseudocontractive. We have seen that the corresponding $G_{\#} \overset{\text{set}}{=} I - T_{\#}$ is $\frac{1-p_{\#}}{2}$ inverse strongly monotone and $G_{\$} \overset{\text{set}}{=} I - T_{\$}$ is $\frac{1-p_{\$}}{2}$ inverse strongly monotone.

The "line segment of operators" passing through $T_{\#}$ and $T_{\$}$ (that is, the convex combinations of $T_{\#}$ and $T_{\$}$) can be written

$$T_{\#\$}[\alpha] \overset{\text{set}}{=} \text{cvx}[T_{\#}, T_{\$}; (1-\alpha), \alpha] = T_{\#} + \alpha (T_{\$} - T_{\#}) = (1-\alpha) T_{\#} + \alpha T_{\$},$$

where $\alpha \in (0,1)$.

In order to make clearer some subsequent results, we will re-express this in terms that do not "privilege" the convex combination parameter associated with $T_{\$}$: instead of considering $(1-\alpha)$ and α we will consider convex combination parameters $\alpha_{\#}$ and $\alpha_{\$}$ that are required to satisfy $\alpha_{\#}, \alpha_{\$} \in \mathbb{R}_{+}$ and $\alpha_{\#} + \alpha_{\$} = 1$. This leads us to

$$T_{\#\$}[\alpha_{\#}, \alpha_{\$}] \overset{\text{set}}{=} \text{cvx}[T_{\#}, T_{\$}; \alpha_{\#}, \alpha_{\$}] = \alpha_{\#} T_{\#} + \alpha_{\$} T_{\$},$$

where $\alpha_{\#}, \alpha_{\$} \in \mathbb{R}_{+}$ and $\alpha_{\#} + \alpha_{\$} = 1$.

The displacement operator, say $G_{\#\$}[\alpha_{\#}, \alpha_{\$}] \overset{\text{set}}{=} I - T_{\#\$}[\alpha_{\#}, \alpha_{\$}]$, associated with $T_{\#\$}[\alpha]$ then has the form

$$\begin{aligned}
G_{\#\$}[\alpha_{\#}, \alpha_{\$}] &\overset{\text{set}}{=} I - T_{\#\$}[\alpha_{\#}, \alpha_{\$}] \\
&= I - [\alpha_{\#} T_{\#} + \alpha_{\$} T_{\$}] \\
&= [\alpha_{\#} I + \alpha_{\$} I] - [\alpha_{\#} T_{\#} + \alpha_{\$} T_{\$}] \\
&= \alpha_{\#}[I - T_{\#}] + \alpha_{\$}[I - T_{\$}] \\
&= \alpha_{\#} G_{\#} + \alpha_{\$} G_{\$} \\
&= \text{cvx}[G_{\#}, G_{\$}; \alpha_{\#}, \alpha_{\$}].
\end{aligned}$$

We have seen that when $G_{\#} \overset{\text{set}}{=} I - T_{\#}$ is $\frac{1-p_{\#}}{2}$-inverse strongly monotone, the nonnegatively scaled version $\alpha_{\#} G_{\#}$ is $\frac{1}{\alpha_{\#}} \frac{1-p_{\#}}{2}$-inverse strongly monotone; likewise when $G_{\$} \overset{\text{set}}{=}$

$I - T_\$$ is $\frac{1-p_\$}{2}$-inverse strongly monotone, the nonnegatively scaled version $\alpha_\$ G_\$$ is $\frac{1}{\alpha_\$}\frac{1-p_\$}{2}$-inverse strongly monotone.

We have seen that when G_1 is ρ_1-inverse strongly monotone, and G_2 is ρ_2-inverse strongly monotone, the sum $F \overset{\text{set}}{=} G_1 + G_2$ is $\left(\frac{1}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}\right)$-inverse strongly monotone.

Thus, the sum of a $\frac{1}{\alpha_\#}\frac{1-p_\#}{2}$-inverse strongly monotone operator and a $\frac{1}{\alpha_\$}\frac{1-p_\$}{2}$-inverse strongly monotone operator is an operator that is inverse strongly monotone with parameter

$$\begin{aligned}\left(\frac{1}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}\right) &= \frac{1}{\frac{1}{\left(\frac{1}{\alpha_\#}\frac{1-p_\#}{2}\right)} + \frac{1}{\left(\frac{1}{\alpha_\$}\frac{1-p_\$}{2}\right)}} \\ &= \frac{1}{\alpha_\# \frac{1}{\left(\frac{1-p_\#}{2}\right)} + \alpha_\$ \frac{1}{\left(\frac{1-p_\$}{2}\right)}}.\end{aligned}$$

We can use our previous results to see that the corresponding contractivity-type operator will be of pseudocontractive type with parameter value

$$\begin{aligned}1 - 2\frac{1}{\alpha_\# \frac{1}{\left(\frac{1-p_\#}{2}\right)} + \alpha_\$ \frac{1}{\left(\frac{1-p_\$}{2}\right)}} &= 1 - 2\frac{1}{\alpha_\# \frac{2}{1-p_\#} + \alpha_\$ \frac{2}{1-p_\$}} \\ &= 1 - \frac{1}{\alpha_\# \frac{1}{1-p_\#} + \alpha_\$ \frac{1}{1-p_\$}}.\end{aligned}$$

We note that this appears to provide a novel alternative proof of a result that includes result (a) in Theorem 3 of [OY02] as a special case (when we restrict our attention only to decreasing pseudocontractive operators). This result in turn is a strict improvement of the result in part 2) of Theorem 1.8 of [VE09].

Convex combination of displacement pseudocontractive operators

We will now move on to consider a convex combination of displacement pseudocontractive operators; we will do this by leveraging our previous result from the convex combination of strongly monotone operators. Specifically, consider $T_\#$ $q_\#$-displacement strictly pseudocontractive and $T_\$$ $q_\$$-displacement strictly pseudocontractive. We have seen that the corresponding $G_\# \overset{\text{set}}{=} I - T_\#$ is $\frac{1-q_\#}{2}$-strongly monotone and $G_\$ \overset{\text{set}}{=} I - T_\$$ is $\frac{1-q_\$}{2}$-strongly monotone.

The "line segment of operators" passing through $T_\#$ and $T_\$$ (that is, the convex combinations of $T_\#$ and $T_\$$) can be written

$$T_{\#\$}[\alpha] \overset{\text{set}}{=} \operatorname{cvx}[T_\#, T_\$; (1-\alpha), \alpha] = T_\# + \alpha(T_\$ - T_\#) = (1-\alpha)T_\# + \alpha T_\$,$$

where $\alpha \in (0,1)$.

In order to make clearer some subsequent results, we will re-express this in terms that do not "privilege" the convex combination parameter associated with $T_\$$: instead of considering $(1-\alpha)$ and α we will consider convex combination parameters $\alpha_\#$ and $\alpha_\$$ that are required to satisfy $\alpha_\#, \alpha_\$ \in \mathbb{R}_+$ and $\alpha_\# + \alpha_\$ = 1$. This leads us to

$$T_{\#\$}[\alpha_\#, \alpha_\$] \overset{\text{set}}{=} \operatorname{cvx}[T_\#, T_\$; \alpha_\#, \alpha_\$] = \alpha_\# T_\# + \alpha_\$ T_\$,$$

where $\alpha_\#, \alpha_\$ \in \mathbb{R}_+$ and $\alpha_\# + \alpha_\$ = 1$.

The displacement operator, say $G_{\#\$}[\alpha_\#, \alpha_\$] \overset{\text{set}}{=} I - T_{\#\$}[\alpha_\#, \alpha_\$]$, associated with $T_{\#\$}[\alpha]$ then has the form

$$\begin{aligned} G_{\#\$}[\alpha_\#, \alpha_\$] &\overset{\text{set}}{=} I - T_{\#\$}[\alpha_\#, \alpha_\$] \\ &= I - [\alpha_\# T_\# + \alpha_\$ T_\$] \\ &= [\alpha_\# I + \alpha_\$ I] - [\alpha_\# T_\# + \alpha_\$ T_\$] \\ &= \alpha_\# [I - T_\#] + \alpha_\$ [I - T_\$] \\ &= \alpha_\# G_\# + \alpha_\$ G_\$ \\ &= \operatorname{cvx}[G_\#, G_\$; \alpha_\#, \alpha_\$]. \end{aligned}$$

We have seen that when $G_\# \overset{\text{set}}{=} I - T_\#$ is $\frac{1-q_\#}{2}$-strongly monotone, the strictly positively scaled version $\alpha_\# G_\#$ is $\alpha_\# \frac{1-q_\#}{2}$-strongly monotone; likewise when $G_\$ \overset{\text{set}}{=} I - T_\$$ is $\frac{1-q_\$}{2}$-strongly monotone, the strictly positively scaled version $\alpha_\$ G_\$$ is $\alpha_\$ \frac{1-q_\$}{2}$-strongly monotone.

We have seen that when G_1 is σ_1-strongly monotone, and G_2 is σ_2-strongly monotone, the sum $F \overset{\text{set}}{=} G_1 + G_2$ is $(\sigma_1 + \sigma_2)$-inverse strongly monotone.

Thus, the sum of an $\alpha_\# \frac{1-q_\#}{2}$-strongly monotone operator and an $\alpha_\$ \frac{1-q_\$}{2}$-strongly monotone operator is an operator that is strongly monotone with parameter

$$(\sigma_1 + \sigma_2) = \alpha_\# \frac{1-q_\#}{2} + \alpha_\$ \frac{1-q_\$}{2}.$$

We can use our previous results to see that the corresponding contractivity-type operator will be displacement pseudocontractive with parameter value

$$1-2\left[\alpha_{\#}\frac{1-q_{\#}}{2}+\alpha_{\$}\frac{1-q_{\$}}{2}\right]=1-\left[\alpha_{\#}\left(1-q_{\#}\right)+\alpha_{\$}\left(1-q_{\$}\right)\right]$$
$$=\alpha_{\#}q_{\#}+\alpha_{\$}q_{\$}.$$

16.7 Operator Composition

We cite the following results on composition of decreasing pseudocontractive operators; one result is simple but loose, the other is more complicated but tight.

Theorem 1.8 from [VE09] tells us:

Theorem 1. *Theorem 1.8 from [VE09]Consider a collection of m operators* $\{T_1,\ldots,T_m\}$. *Suppose that each operator* $T_i(\cdot)\,:\,\mathcal{H}\to\mathcal{H}$ *satisfies* p_i*-decreasing pseudocontractivity with respect to its (nonempty) fixed point set* $\mathcal{M}_i\overset{set}{=}Fix\ T_i$. *Further, suppose that the intersection of all of the fixed point sets is nonempty:* $\mathcal{M}\overset{set}{=}\bigcap_{i\in\{1,\ldots,m}\mathcal{M}_i\neq\emptyset$. *In such a setting, the composition operator* $T_{comp}\,:\,\mathcal{H}\to\mathcal{H}$ *characterized via the expression*

$$T_{comp}\overset{set}{=}T_m\circ T_{m-1}\circ\cdots\circ T_1$$

satisfies p_{comp}*-decreasing pseudocontractivity (with respect to the intersection set* $\mathcal{M}$*) when* $p_{comp}\overset{set}{=}\frac{1}{2^{m-1}}minimum\{p_1,\ldots,p_m\}$.

Note that the result above is stated in terms of decreasing pseudocontractivity parameters that are defined by convention to be nonnegative.

The more complicated (tight) result comes from [OY02]

Theorem 2. *Theorem 3 part (b) from [OY02] Consider an* α_1*-averaged operator* $T_1\,:\,\mathcal{H}\to\mathcal{H}$ *and an* α_2*-averaged operator* $T_2\,:\,\mathcal{H}\to\mathcal{H}$*, where* $\alpha_1\in[0,1)$ *and* $\alpha_1\in[0,1)$. *In such a setting, we have* T_1T_2 α_{comp}*-averaged, with*

$$\alpha_{comp}\overset{set}{=}\frac{\alpha_1+\alpha_2-2\alpha_1\alpha_2}{1-\alpha_1\alpha_2}.$$

Note that $\alpha_{comp}\in[0,1)$.

A broader perspective on a result in Aubin [Aub98]

Consider a closed bounded convex set $\mathcal{C}$ in a Hilbert space $\mathcal{H}$.

Consider a nonexpansive operator $N(\cdot) : \mathcal{C} \to \mathcal{C}$; explicitly, this means that

$$\|T(x_{\#}) - T(x_{\$})\|_2 \leq \|x_{\#} - x_{\$}\|_2 \text{ for all } x, y \in \mathcal{C}.$$

Consider some specific (but otherwise arbitrary) element, say x_{c}, of the set $\mathcal{C}$.

We previously considered operators, denoted $T_{I,t}$, along the ray from I to T, of the form

$$\begin{aligned} T_{I,t} &\overset{\text{set}}{=} I + t(T - I) \\ &= (1-t)I + tT \\ &= I - t(I - T). \end{aligned}$$

We saw previously that for any strictly pseudocontractive T, there exists a range of (sufficiently small) t values within which $I - t(I - T)$ is decreasing pseudocontractive [[from $\mathcal{C}$ to $\mathcal{C}$]].

We now consider operators, denoted $T_{x_{\mathrm{C}},t}$, along the ray from (the "constant operator") x_{c} to T

$$\begin{aligned} T_{x_{\mathrm{C}},t} &\overset{\text{set}}{=} x_{\mathrm{c}} + t(T - I) \\ &= (1-t)x_{\mathrm{c}} + tT \\ &= x_{\mathrm{c}} - t(I - T). \end{aligned}$$

We will see that for any p-strictly pseudocontractive T, there exists a range of (sufficiently small) t values within which $x_{\mathrm{c}} - t(I - T)$ is a strict contraction from $\mathcal{C}$ to $\mathcal{C}$.

Consider

$$\begin{aligned} \|T_{x_{\mathrm{C}},t}(x_{\#}) - T_{x_{\mathrm{C}},t}(x_{\$})\|_2^2 &= \|[(1-t)x_{\mathrm{c}} + tTx_{\#}] - [(1-t)x_{\mathrm{c}} + tTx_{\$}]\|_2^2 \\ &= \|[(1-t)x_{\mathrm{c}} + tTx_{\#}] - [(1-t)x_{\mathrm{c}} + tTx_{\$}]\|_2^2 \\ &= \|tTx_{\#} - tTx_{\$}\|_2^2 \\ &= t^2\|Tx_{\#} - Tx_{\$}\|_2^2 \\ &= t^2\|Tx_{\#} - Tx_{\$}\|_2^2 \leq t^2\left(\|x_{\#} - x_{\$}\|_2^2 + p\|Gx_{\#} - Gx_{\$}\|_2^2\right) \end{aligned}$$

$$\|T_{x_{\mathrm{C}},t}(x_{\#}) - T_{x_{\mathrm{C}},t}(x_{\$})\|_2^2 \leq t^2\|x_{\#} - x_{\$}\|_2^2 + t^2 p\|Gx_{\#} - Gx_{\$}\|_2^2.$$

The standard result involving this (for example, in [Aub98] or [VE09]) considers the case in which T is nonexpansive, for which $p = 0$ and the expression above indicates that $T_{x_C,t}$ is a t strict contraction (when $t \in (0,1)$).

When T is in fact decreasing pseudocontractive (corresponding to a negative value for p), this expression indicates that the update is “a little bit more than just contractive” (again, when $t \in (0,1)$).

Chapter 17

Decreasing pseudocontractivity in methods for convex optimization

17.1 Introduction

Optimization methods are traditionally viewed and analyzed largely separately from one another. In this Chapter, we highlight a common feature, namely *decreasing pseudocontractivity*, of the iterative updates of many methods, and we show how recognizing this commonality can lead to a unified analysis that can contribute to a deeper understanding of a variety of prominent convex optimization methods.

We summarize this Chapter as follows: we first establish the (essentially unremarked) prevalence of decreasing pseudocontractive updates in many convex optimization methods; we then prove a result (novel in its scope of application), Lemma 26, establishing a bound on the decrease of the norm of the displacement operator associated with any decreasing pseudocontractive operator; we move on to use the Lemma 26 result on monotone norm decrease to prove a result (novel in its scope of application), Theorem 5, establishing that the error criterion $\left\|x^N - Tx^N\right\|_2^2$ is $o\left(\frac{1}{N}\right)$ for any method involving decreasing pseudocontractive updates (where N is the number of iterations and T is the iteration operator).

We continue with an extension of the initial result on the error criterion, indicating a setting where that error criterion bound can be used to establish to a bound on the objective function suboptimality. We then indicate a setting in which we can expect to have updates that are strictly contractive rather than just decreasing pseudocontractive.

17.1.1 Preliminaries

We restrict our attention to the finite-dimensional case and to operators having nonempty fixed point sets. When convenient, we switch freely between operator notation and relation notation; for example, we might state the operator inclusion $y \in T(x)$ equivalently in relation form as $(x,y) \in T$.

17.2 Classes of operators

We begin with a definition.

Definition 135. An operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is called α-*averaged* (or α-averaged nonexpansive) when it can be written in the form

$$T = (1-\alpha)I + \alpha N,$$

where N is a nonexpansive operator and $\alpha \in (0,1)$.

We note that we can write the characterization of α-averagedness in the equivalent forms

$$\begin{aligned} T &= (1-\alpha)I + \alpha N \\ &= I + \alpha (N - I) \\ &= I - \alpha (I - N). \end{aligned}$$

We thus recognize an averaged operator as "an operator that lies along the line segment (of operators) from the identity to some nonexpansive operator". This is therefore a restriction of a ray of strictly pseudocontractive operators, so that we may in particular refer to the collection of α-averaged operators as a truncation of the cone of strictly pseudocontractive operators. We note that this is the region of the cone of strictly pseudocontractive operators that contains all of the decreasing pseudocontractive operators. This observation does not appear to be made anywhere in the contemporary literature, and yet clearly more completely describes the situation. The description "averaged" does in some sense locate the "relative position" of these operators but fails to concretely refer to either the contractivity-type or the monotonicity-type properties that we have extensively discussed elsewhere.

We remark in passing that the use of the term averaged as defined above corresponds to the generalization from the special case in which the convex combination parameter $\alpha \overset{\text{set}}{=} \frac{1}{2}$.

17.2.1 Decreasing pseudocontractive updates: Prevalence and common convergence rate analysis

We discuss the details in later sections after necessary theoretical developments; for now we state the two main themes of this Chapter.

Proposition (Details in Section 17.3)**.** *The following prominent convex optimization methods have decreasing pseudocontractive updates:*

(1) Orthogonal projection onto a nonempty closed convex set is 1 *decreasing pseudocontractive.*

(2) Over- or -under relaxed orthogonal projection methods are $\left(2\frac{1}{\omega}-1\right)$ *decreasing pseudocontractive, where* $\omega \in (0,2)$ *is the relaxation parameter.*

(3) For a convex function $f(\cdot)$ *with an L Lipschitz gradient, the gradient descent iterative update operator* $I-\gamma\nabla f$ *is* $\left(\frac{2}{\gamma L}-1\right)$ *decreasing pseudocontractive for any step factor* $\gamma \in \left(0,\frac{2}{L}\right)$.

(4) The proximal-point mapping, $prox_{f,\lambda}(\cdot)$*, associated with a convex function* $f(\cdot)$ *is* 1 *decreasing pseudocontractive for any* $\lambda > 0$.

(5) The ω*-relaxed proximal-point mapping,* $prox_{\omega}(\cdot)$*, where* $\omega \in (0,2)$*, associated with a convex function* $f(\cdot)$ *is* $\left(2\frac{1}{\omega}-1\right)$ *decreasing pseudocontractive for any* $\lambda > 0$.

(6) Projected gradient descent involving a convex function $f(\cdot)$ *with an L Lipschitz gradient is decreasing pseudocontractive with parameter* $\left(1-\frac{\gamma L}{2}\right)$*, where* $\gamma \in \left(0,\frac{2}{L}\right)$ *is the gradient descent step factor.*

(7) Over- or -under-relaxed projected gradient descent is decreasing pseudocontractive.

(8) Forward-backward splitting in which the smooth convex function $f(\cdot)$ has an L Lipschitz gradient is decreasing pseudocontractive with parameter $\left(1-\frac{\gamma L}{2}\right)$, where $\gamma \in \left(0, \frac{2}{L}\right)$ is the gradient descent step factor.

(9) The most basic version of the alternating direction method of multipliers update is 1 *decreasing pseudocontractive.*

In addition to highlighting the common role played by decreasing pseudocontractive updates in all of the methods above, we also present a proof, novel in its scope of application, establishing that the error criterion $\left\|x^N - Tx^N\right\|_2^2$ is $o\left(\frac{1}{N}\right)$ for any method with a decreasing pseudocontractive update.

Theorem (Details in Section 17.4)**.** *The error criterion $\left\|x^N - Tx^N\right\|^2$ is $o(\frac{1}{N})$ for any method with a decreasing pseudocontractive iteration operator.*

Having stated our primary results, we now begin introducing the background concepts that we will use in establishing these results.

17.2.2 Relationships

We note that the following statements are equivalent:

- T is $\left(\frac{1}{\alpha}-1\right)$-decreasing pseudocontractive
- T is α-averaged
- $I-T$ is $\frac{1}{\alpha}\frac{1}{2}$-inverse strongly monotone

These relationships correspond to restricted versions of the broader relationships established in our previous discussions of operators. Since $\alpha \in (0,1)$, we observe that $\left(\frac{1}{\alpha}-1\right) \in (0,+\infty)$ and $\frac{1}{2\alpha} \in \left(\frac{1}{2},+\infty\right)$.

The relationship between decreasing pseudocontractive and averaged can be found in [VA95]. The inverse strong monotonicity property satisfied by an averaged operator is discussed in [Byr04]. We also note an equivalence in a special case: when T satisfies 1-decreasing pseudocontractivity, T also satisfies 1-inverse strong monotonicity; see Corollary 1. As stated, this observation somewhat muddies our usual practice of carefully distinguishing between contractivity-type properties of T and monotonicity-type properties of $I-T$; this is an issue that deserves fuller exploration subsequently.

Over- or under-relaxation involving a firmly nonexpansive operator

Finally, we note that if we start with a firmly nonexpansive (or 1-decreasing pseudocontractive in the language we have been using) operator F, the operator $T = I + \omega(F - I) = I - \omega(I - F)$ is decreasing pseudocontractive for any $\omega \in (0,2)$. We can see this by using previously established relationships: when F is 1-decreasing pseudocontractive, $G = I - F$ is 1-inverse strongly monotone; ωG is then $\frac{1}{\omega}$-inverse strongly monotone and $I - \omega G = I - \omega(I - F)$ is $\left(2\frac{1}{\omega} - 1\right)$-decreasing pseudocontractive. When $\omega \overset{\text{set}}{=} 2$ we get $I - \omega(I - F) = 2F - I$ is 0-decreasing pseudocontractive; more succinctly, nonexpansive. This argument thus recovers as a special case the standard result that F firmly nonexpansive implies that $2F - I$ is nonexpansive; the standard (shorter but much more specialized) proof is Theorem 12.1 of [GK90].

We summarize this as: when F is firmly nonexpansive, its associated reflection[1] operator $R \overset{\text{set}}{=} 2F - I$ is nonexpansive.

We again call attention to the fact that the "relaxation" argument considered above corresponds to a special case of the more general operator scaling relationships that we have discussed previously. More specifically, we have seen how the decreasing pseudocontractivity parameter for an operator T relates to the decreasing pseudocontractivity parameter of $I - t(I - T) = (1 - t)I + tT$; in particular, we have seen how to do this no matter what decreasing pseudocontractivity parameter T possesses. We have not seen this explicit observation elsewhere in the optimization or operator theory literature.

Although relaxation is a special case of our more general results, we state the result on relaxation as a Proposition for future reference anyway:

Proposition 22. *For a firmly nonexpansive operator F, the ω-relaxed version of the operator, denoted $F_\omega \overset{set}{=} (1 - \omega)I + \omega F = I + \omega(F - I) = I - \omega(I - F)$, is $\left(2\frac{1}{\omega} - 1\right)$-decreasing pseudocontractive for $\omega \in (0,2)$.*

The significance of the result above is that iterations involving under- or over-relaxation of a decreasing pseudocontractive update operator are also covered by our re-

[1] The use of the term reflection for this operator is most immediately clear when we consider the case of projection onto a hyperplane H. Denoting projection onto the hyperplane H by P_H and denoting reflection across the hyperplane by R_H, we observe that $P_H = \frac{1}{2}I + \frac{1}{2}R_H$. The usage in the case above is a natural generalization.

sults. Over-relaxation, in which $\omega \in (1,2)$, is a commonly-used technique to accelerate convergence of iterative methods in numerical linear algebra [Var09]; the observation that over-relaxation of a firmly nonexpansive operator yields a decreasing pseudocontractive update thus connects our coverage to these approaches.

17.2.3 Closure properties

The convex combination of nonexpansive operators is a nonexpansive operator. The composition of nonexpansive operators is a nonexpansive operator. The convex combination of decreasing pseudocontractive operators is a decreasing pseudocontractive operator. The composition of decreasing pseudocontractive operators is a decreasing pseudocontractive operator. These results can be found, for example, in [VA95] or [Byr08]; however, we can in fact be even more specific:

Theorem 3 ([OY02] Theorem 3)**.** *When T_1 is α_1-averaged, where $\alpha_1 \in [0,1)$, and T_2 is α_2-averaged, where $\alpha_2 \in [0,1)$, we have*

(a) for $t \in [0,1]$, the t-convex combination $T_{cc} = (1-t)T_1 + tT_2$ is α_{cc}-averaged, where $\alpha_{cc} \overset{set}{=} (1-t)\,\alpha_1 + t\alpha_2$,

(b) the composition $T_{co} = T_1 T_2$ is α_{co}-averaged, where $\alpha_{co} \overset{set}{=} \frac{\alpha_1+\alpha_2-2\alpha_1\alpha_2}{1-\alpha_1\alpha_2}$.

Using the relationship between averaged and decreasing pseudocontractive, we note that the convex combination T_{cc} is ν_{cc}-decreasing pseudocontractive, where $\nu_{\mathrm{cc}} = \frac{1}{\alpha_{\mathrm{cc}}} - 1$; similarly, the composition T_{co} is ν_{co}-decreasing pseudocontractive, where $\nu_{\mathrm{co}} = \frac{1-\alpha_1-\alpha_2+\alpha_1\alpha_2}{\alpha_1+\alpha_2-2\alpha_1\alpha_2}$.

Note that while this result establishes that both T_1T_2 and T_2T_1 are decreasing pseudocontractive, the respective fixed point sets need not coincide[2].

The significance of these closure properties is twofold: first, closure under composition and convex combination means that new algorithms can very naturally be constructed by using previous algorithms as building blocks; second, the resulting constructed algorithms will maintain the desirable properties that the individual building

[2]For example, consider the case of successive orthogonal projection onto two closed, convex, nonempty sets $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A} \cap \mathcal{B} = \emptyset$. In this situation, $P_{\mathcal{A}}P_{\mathcal{B}}$ and $P_{\mathcal{B}}P_{\mathcal{A}}$ each have nonempty fixed point sets, but Fix $P_{\mathcal{A}}P_{\mathcal{B}}$ is the set of points in $\mathcal{A}$ that are at the minimal distance to $\mathcal{B}$, whereas Fix $P_{\mathcal{B}}P_{\mathcal{A}}$ is the set of points in $\mathcal{B}$ that are at the minimal distance to $\mathcal{A}$.

block updates possessed (and using Theorem 3 we can track the associated decreasing pseudocontractivity parameter values from the simple methods to the more complex combined methods).

17.3 Methods with decreasing pseudocontractive updates

In this section we demonstrate that decreasing pseudocontractive updates are widespread in many popular convex optimization methods.

17.3.1 Projection

Most discussions of projection methods only establish that projection is nonexpansive; the following known result demonstrates that projection is in fact 1-decreasing pseudocontractive (i.e., firmly nonexpansive).

Proposition 23. *Orthogonal projection $P_{\mathcal{C}}$ onto a nonempty closed convex set $\mathcal{C} \subset \mathbb{R}^n$ is* 1*-decreasing pseudocontractive. For convenience we show the equivalent result that projection is* 1*-inverse strongly monotone; that is,*

$$\langle P_{\mathcal{C}}x - P_{\mathcal{C}}y, x-y \rangle \geq \|P_{\mathcal{C}}x - P_{\mathcal{C}}y\|^2,$$

for all $x, y \in \mathbb{R}^n$.

Proof. The variational characterization of projection is: z is the projection of x onto $\mathcal{C}$ if and only if $\langle c - z, x - z \rangle \leq 0$ for every $c \in \mathcal{C}$ (see, e.g., [Rus06]); we will express this as

$$\langle c - P_{\mathcal{C}}x, x - P_{\mathcal{C}}x \rangle \leq 0,$$

for every $c \in \mathcal{C}$. From the preceding characterization, we have, for any $x, y \in \mathbb{R}^n$

$$\langle P_{\mathcal{C}}x - P_{\mathcal{C}}y, y - P_{\mathcal{C}}y \rangle \leq 0$$
$$\langle P_{\mathcal{C}}y - P_{\mathcal{C}}x, x - P_{\mathcal{C}}x \rangle \leq 0,$$

yielding the immediate conclusion

$$\|P_{\mathcal{C}}x - P_{\mathcal{C}}y\|^2 \leq \langle P_{\mathcal{C}}x - P_{\mathcal{C}}y, x-y \rangle,$$

for any $x, y \in \mathbb{R}^n$. The result immediately above establishes that $P_{\mathcal{C}}$ is 1-inverse strongly monotone, so that Corollary 1 implies that $P_{\mathcal{C}}$ is 1-decreasing pseudocontractive. □

17.3.2 Over- or under-relaxed projection

Using the notation P for orthogonal projection onto the nonempty closed convex set $\mathcal{C}$, we introduce the ω-relaxed projection operator $P_\omega = (1-\omega)I + \omega P = I - \omega(I-P)$, where $\omega \in (0,2)$. Since we just established that P is 1-decreasing pseudocontractive, our previous reasoning for 22 tells us that P_ω is $\left(2\frac{1}{\omega} - 1\right)$-decreasing pseudocontractive.

We note in passing that reasoning similar to the reasoning in the previous two subsections can also be used to establish that cyclic subgradient projection (CSP) methods (in both their relaxed and unrelaxed forms) [CZ97] involve decreasing pseudocontractive updates.

17.3.3 Gradient descent

Typical discussions of gradient descent do not make any reference to operator theory, or, more specifically, to decreasing pseudocontractivity; the known result below establishes conditions under which the gradient descent iterative update is decreasing pseudocontractive. The explicit form of the gradient descent iterative update is

$$x^{k+1} = [I - \gamma \nabla f]\left(x^k\right).$$

Proposition 24. *For a convex function f with an L-Lipschitz gradient, the gradient descent iterative update operator $I - \gamma\nabla f$ is $\left(2\frac{1}{\gamma L} - 1\right)$-decreasing pseudocontractive for any step factor $\gamma \in \left(0, \frac{2}{L}\right)$.*

Proof. We have assumed that ∇f is L-Lipschitz, which implies that $\frac{1}{L}\nabla f$ is nonexpansive. Theorem 6.9 from Chapter 1 of [GT96] establishes that $\frac{1}{L}\nabla f$ nonexpansive implies that $\frac{1}{L}\nabla f$ is 1-decreasing pseudocontractive.

Corollary 1 establishes that $\frac{1}{L}\nabla f$ being 1-decreasing pseudocontractive is equivalent to $\frac{1}{L}\nabla f$ being 1-inverse strongly monotone. Lemma 3 establishes that $\frac{1}{L}\nabla f$ being 1-inverse strongly monotone implies that $\gamma\nabla f$ is $\frac{1}{\gamma L}$-inverse strongly monotone, since $\gamma L > 0$. We have seen that $\gamma\nabla f$ being $\frac{1}{\gamma L}$-inverse strongly monotone corresponds to $I - \gamma\nabla f$ being $\left(1 - 2\frac{1}{\gamma L}\right)$-strictly pseudocontractive. For decreasing pseudocontractivity, we want the value $1 - 2\frac{1}{\gamma L}$ to be strictly negative, which in turn means that we need $\gamma \in \left(0, \frac{2}{L}\right)$. □

17.3.4 Proximal-point method

As is the case with projection, most discussions of proximal-point methods only establish that the proximal-point mapping is nonexpansive; the following known result demonstrates that the proximal-point mapping is more specifically 1-decreasing pseudocontractive (i.e., firmly nonexpansive). We note that the proximal-point mapping associated with $f(\cdot)$ can be written in either of the two equivalent forms

$$\begin{aligned}
\operatorname{prox}_{f,\lambda}(x_\#) &= \operatorname*{argmin}_{x\in\mathbb{R}^n}\left\{f(x)+\frac{1}{\lambda}\frac{1}{2}\|x-x_\#\|^2\right\}\\
\operatorname{prox}_{f,\lambda}(x_\#) &= [I+\lambda\partial f]^{-1}(x_\#),
\end{aligned}$$

where $\lambda > 0$; see, e.g., [Roc70]. The operator $[I+\lambda\partial f]^{-1}$ is called the λ-resolvent of the subdifferential operator $\partial f(\cdot)$.

Proposition 25. *For a convex function f, the associated proximal-point update is* 1-*decreasing pseudocontractive, for any $\lambda > 0$.*

Proof. Consider $x^+ \overset{\text{set}}{=} \operatorname{prox}_{f,\lambda}(x)$ and $y^+ \overset{\text{set}}{=} \operatorname{prox}_{f,\lambda}(y)$. Lemma 2 tells us that

$$\begin{aligned}
\left(x^+, x-x^+\right) &\in \lambda\partial f\\
\left(y^+, y-y^+\right) &\in \lambda\partial f.
\end{aligned}$$

The subdifferential mapping of a convex function is monotone [Roc70]; thus $\lambda\partial f$, where $\lambda > 0$, is also monotone. From this we have

$$\begin{aligned}
\left\langle x^+-y^+, \left(x-x^+\right)-\left(y-y^+\right)\right\rangle &\geq 0\\
\left\langle x^+-y^+, (x-y)-\left(x^+-y^+\right)\right\rangle &\geq 0\\
\left\langle x^+-y^+, x-y\right\rangle &\geq \left\|x^+-y^+\right\|_2^2.
\end{aligned}$$

This establishes that $\operatorname{prox}_{f,\lambda}(\cdot)$ is 1-inverse strongly monotone; applying Corollary 1 then establishes that $\operatorname{prox}_{f,\lambda}(\cdot)$ is 1-decreasing pseudocontractive for any $\lambda > 0$. □

17.3.5 Relaxed proximal-point method

Lightening our notation and for the moment taking $\operatorname{prox}(\cdot)$ to denote the λ-proximal-point mapping associated with the convex function $f(\cdot)$, we introduce the ω-relaxed proximal-point iteration operator $\operatorname{prox}_\omega \overset{\text{set}}{=} (1-\omega)I+\omega\operatorname{prox} = I-\omega(I-\operatorname{prox})$, where $\omega \in (0,2)$. Since we just established that $\text{prox}(\cdot)$ is 1-decreasing pseudocontractive, our previous reasoning for 22 tells us that $\text{prox}_{\omega}(\cdot)$ is $\left(2\frac{1}{\omega}-1\right)$-decreasing pseudocontractive.

17.3.6 Projected gradient descent

The iterative update for projected gradient descent involving a convex function $f(\cdot)$ with an L-Lipschitz gradient and a nonempty closed convex constraint set $\mathcal{C}$ corresponds to the composition $P_{\mathcal{C}} \circ [I-\gamma\nabla f]$. We have established that the projection $P_{\mathcal{C}}$ is 1-decreasing pseudocontractive and that the gradient descent update $I-\gamma\nabla f$ is $(\frac{2}{\gamma L}-1)$-decreasing pseudocontractive, when $\gamma \in \left(0,\frac{2}{L}\right)$. Since the composition of decreasing pseudocontractive updates is decreasing pseudocontractive [VA95], we have thus established that projected gradient descent is decreasing pseudocontractive. We also note that under- or over-relaxation of the projection portion of the update still yields decreasing pseudocontractivity.

We can use Theorem 3b to be more specific. Basic projected gradient descent is the composition of the gradient descent step, which is $\frac{\gamma L}{2}$-averaged, and the projection step, which is $\frac{1}{2}$-averaged; the composition thus is $\left(1-\frac{\gamma L}{2}\right)$-decreasing pseudocontractive. Under- or over-relaxed projected gradient descent has the $\frac{1}{2}$-averaged projection replaced with ω-relaxed projection, which we have seen is $\left(2\frac{1}{\omega}-1\right)$-decreasing pseudocontractive.

17.3.7 Forward-backward splitting

Consider the optimization problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\; f(x)+g(x),$$

where $f(\cdot)$ and $g(\cdot)$ are each convex and $f(\cdot)$ has an L-Lipschitz gradient. The explicit statement of the forward-backward splitting method is

$$x^{k+1} \overset{\text{set}}{=} \text{prox}_{g,\lambda} \circ [I-\gamma\nabla f]\left(x^k\right).$$

When used as a gradient descent step followed by a proximal step, so long as the gradient descent step factor is $\gamma \in \left(0, \frac{2}{L}\right)$, we have the composition of decreasing pseudocontractive operators, which is in turn decreasing pseudocontractive.

We can again use Theorem 3b to be more specific. Basic forward-backward splitting is the composition of the gradient descent step, which is $\frac{\gamma L}{2}$-averaged, and the proximal step, which is $\frac{1}{2}$-averaged; the composition thus is $\left(1 - \frac{\gamma L}{2}\right)$-decreasing pseudocontractive. When we use under- or over-relaxation of the proximal step in forward-backward splitting, the $\frac{1}{2}$-averaged proximal step is replaced with an ω-relaxed generalized proximal step, which we have seen is $\left(2\frac{1}{\omega} - 1\right)$-decreasing pseudocontractive.

17.3.8 Alternating direction method of multipliers

Consider the optimization problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\ f(x) + g(x),$$

where f and g are each convex. Introducing the notation $R_f = 2\text{prox}_{f,\lambda} - I$ and $R_g = 2\text{prox}_{g,\lambda} - I$, we can use an observation from [Boy11] to express the most basic version of the alternating direction method of multipliers (ADMM) update in terms of operators as

$$x^{k+1} \overset{\text{set}}{=} \left[\frac{1}{2}I + \frac{1}{2}R_g R_f\right]\left(x^k\right).$$

Proposition 25 establishes that $\text{prox}_{f,\lambda}(\cdot)$ and $\text{prox}_{g,\lambda}(\cdot)$ are each 1-decreasing pseudocontractive, for any $\lambda > 0$. Proposition 27 then establishes that the respective reflection operators R_f and R_g are each nonexpansive. The composition of nonexpansive operators is nonexpansive, so $R_g R_f$ is nonexpansive. We thus observe that the most basic version of the ADMM update is $\frac{1}{2}$-averaged, and thus is 1-decreasing pseudocontractive. We also immediately observe that the more general convex-combination ADMM-like update

$$x^{k+1} \overset{\text{set}}{=} \left[(1-\alpha)I + \alpha R_g R_f\right]\left(x^k\right),$$

where $\alpha \in (0,1)$, will be $(\frac{1}{\alpha} - 1)$-decreasing pseudocontractive.

17.4 Convergence rate of methods with ν-decreasing pseudocontractive updates

The worst-case convergence rate is well-known for each of the methods that we discuss in Section 17.3. The standard approach generally requires separate arguments to establish this rate for each method; in contrast, by leveraging the fact that the updates in each of these methods are decreasing pseudocontractive (and making use of the relationships between decreasing pseudocontractivity, averagedness, and inverse strong monotonicity) we are able to use one argument for all of these methods.

Consider a method for which the iteration operator T is ν-decreasing pseudocontractive. We will establish

Proposition 26. *For a ν-decreasing pseudocontractive iteration operator T, the norm of the displacement operator $G = I - T$ satisfies the expression*

$$\left\|Gx^{k+1}\right\|_2^2 \leq \left\|Gx^k\right\|_2^2 - \nu \left\|Gx^k - Gx^{k+1}\right\|_2^2$$

for $k \in \{0,1,\ldots\}$.

Proof. From the relationship between decreasing pseudocontractivity and inverse strong monotonicity, we note that when T is ν-decreasing pseudocontractive, we have that $G = I - T$ is $\frac{1+\nu}{2}$-inverse strongly monotone. That is, we have

$$\langle Gx - Gy, x - y\rangle \geq \left(\frac{1+\nu}{2}\right) \|Gx - Gy\|^2,$$

for all $x, y \in \mathbb{R}^n$. Thus,

$$\begin{aligned}
\left\langle Gx^k - Gx^{k+1}, x^k - x^{k+1}\right\rangle &\geq \frac{1+\nu}{2}\left\|Gx^k - Gx^{k+1}\right\|_2^2 \\
\left\langle Gx^k - Gx^{k+1}, Gx^k\right\rangle &\geq \frac{1+\nu}{2}\left\|Gx^k - Gx^{k+1}\right\|_2^2 \\
\left\|Gx^k\right\|_2^2 - \left\langle Gx^{k+1}, Gx^k\right\rangle &\geq \frac{1+\nu}{2}\left\|Gx^k - Gx^{k+1}\right\|_2^2,
\end{aligned}$$

which yields

$$\begin{aligned}
2\left\|Gx^k\right\|_2^2 - 2\left\langle Gx^{k+1}, Gx^k\right\rangle &\geq (1+\nu)\left\|Gx^k - Gx^{k+1}\right\|_2^2 \\
2\left\|Gx^k\right\|_2^2 - \left(\left\|Gx^{k+1}\right\|_2^2 + \left\|Gx^k\right\|_2^2 - \left\|Gx^{k+1} - Gx^k\right\|_2^2\right) &\geq (1+\nu)\left\|Gx^k - Gx^{k+1}\right\|_2^2 \\
\left\|Gx^k\right\|_2^2 - \left\|Gx^{k+1}\right\|_2^2 + \left\|Gx^{k+1} - Gx^k\right\|_2^2 &\geq (1+\nu)\left\|Gx^k - Gx^{k+1}\right\|_2^2 \\
\left\|Gx^k\right\|_2^2 - \left\|Gx^{k+1}\right\|_2^2 &\geq \nu\left\|Gx^k - Gx^{k+1}\right\|_2^2
\end{aligned}$$

so that

$$\left\|Gx^{k+1}\right\|^2 \leq \left\|Gx^k\right\|_2^2 - \nu\left\|Gx^k - Gx^{k+1}\right\|_2^2.$$

□

Using Lemma 26, we can now show that

Theorem 4. *Any method with a ν-decreasing pseudocontractive iteration operator has error criterion satisfying $\left\|x^N - Tx^N\right\|_2^2 \leq \frac{1}{(N+1)\nu}\left\|x^0 - x^\star\right\|_2^2$ at iteration N.*

Proof. An alternate expression of ν-decreasing pseudocontractivity with one argument, $x^\star$, coming from the fixed point set is

$$\nu\left\|Gx^k\right\|_2^2 \leq \left\|x^k - x^\star\right\|_2^2 - \left\|x^{k+1} - x^\star\right\|_2^2.$$

Summing over all the iterations from 0 to N yields

$$\sum_{k=0}^{N}\left\|Gx^k\right\|_2^2 \leq \frac{1}{\nu}\left\|x^0 - x^\star\right\|_2^2.$$

Lemma 26 tells us that $\left\|Gx^{k+1}\right\|_2^2 \leq \left\|Gx^k\right\|_2^2$ for each k, so we conclude that

$$\left\|Gx^N\right\|_2^2 \leq \frac{1}{(N+1)\nu}\left\|x^0 - x^\star\right\|_2^2.$$

□

We can establish a stronger result by leveraging a useful technical Lemma from [Don13].

Lemma 1 ([Don13]). *Consider three sequences of strictly positive numbers* $\{\alpha_k\}, \{\beta_k\}$, $\{\gamma_k\}$. *When these sequences are such that* $\{\beta_k\}$ *is unsummable,* $\{\gamma_k\}$ *is nonincreasing, and the relationship* $\alpha_{k+1}^2 \leq \alpha_k^2 - \beta_k \gamma_k$ *holds for each* $k \in \{0, 1, \ldots\}$, *then it will also be the case that there exists another sequence* $\{\varepsilon_k\}$ *such that*

$\varepsilon_k \in (\alpha_k, \alpha_0)$,
$\lim_{k \to +\infty} \varepsilon_k = \lim_{k \to +\infty} \alpha_k$,
$\gamma_k \cdot \left[\sum_{i=0}^{k} \beta_i\right] \leq 2 \cdot \alpha_0 \cdot \varepsilon_k$.

We now use this Lemma to establish

Theorem 5. *For any method with a* ν*-decreasing pseudocontractive iteration operator, the error criterion* $\left\|x^N - Tx^N\right\|^2$ *is* $o\left(\frac{1}{N}\right)$.

Proof. We will apply Lemma 1 with $\alpha_k = \left\|x^k - x^\star\right\|$, $\beta_k = \nu$, and $\gamma_k = \left\|Gx^k\right\|^2$. An infinite sequence of constants is clearly unsummable and Lemma 26 establishes that the sequence $\left\{\left\|Gx^k\right\|_2^2\right\}$ of squared norms of displacement mappings is nonincreasing.

Moreover, the definition of ν-decreasing pseudocontractivity when one argument, $x^\star$, comes from the fixed point set provides us with the necessary relationship

$$\left\|x^{k+1} - x^\star\right\|^2 \leq \left\|x^k - x^\star\right\|^2 - \nu \left\|Gx^k\right\|^2$$

for each $k \in \{0, 1, \ldots\}$.

Thus, Lemma 1 tells us that there exists a sequence $\{\varepsilon_k\}$ such that
$\varepsilon_k \in \left(\left\|x^k - x^\star\right\|, \left\|x^0 - x^\star\right\|\right)$,
$\lim_{k \to +\infty} \varepsilon_k = \lim_{k \to +\infty} \left\|x^k - x^\star\right\|_2$,
$\left\|Gx^k\right\|_2^2 \cdot \left[\sum_{i=0}^{k} \nu\right] \leq 2 \cdot \left\|x^0 - x^\star\right\|_2 \cdot \varepsilon_k$.

Taking limsup on both sides of the preceding inequality and noting that $\lim_{k \to +\infty} \varepsilon_k$ exists and is equal to $\lim_{k \to +\infty} \left\|x^k - x^\star\right\|_2$ we have

$$\limsup_{k \to \infty} \left\{\left\|Gx^k\right\|_2^2 \left[\sum_{i=0}^{k} \nu\right]\right\} \leq \lim_{k \to \infty} \left\{2 \left\|x^0 - x^\star\right\|_2 \varepsilon_k\right\}$$

$$\limsup_{k \to \infty} \left\{\left\|Gx^k\right\|_2^2 \left[\nu (k+1)\right]\right\} \leq 2 \left\|x^0 - x^\star\right\|_2 \lim_{k \to \infty} \left\|x^k - x^\star\right\|_2$$

$$\limsup_{k\to\infty}\left\{\left\|Gx^k\right\|_2^2[\nu(k+1)]\right\}\leq 0,$$

where we have used $\lim_{k\to+\infty}\left\|x^k-x^\star\right\|_2=0$ (for our finite dimensional setting, this follows because, e.g., the Krasnoselskii-Mann Theorem [Man53] establishes weak convergence of methods with averaged updates [and thus with decreasing pseudocontractive updates] and weak convergence coincides with strong convergence in finite dimension).

The final inequality above establishes the desired result: in terms of iteration N, we observe that $\left\|Gx^N\right\|_2^2$ is $o\left(\frac{1}{N}\right)$. □

17.5 An additional result on objective function suboptimality

The result above establishes $o\left(\frac{1}{N}\right)$ convergence for $\left\|Gx^N\right\|_2^2$, the squared norm of displacement operator. When we are considering gradient descent for a closed proper convex L-strongly smooth function $f(\cdot)$, a standard argument demonstrates that we can also bound the objective function suboptimality. Specifically, from convexity of $f(\cdot)$ we have

$$f(x)\geq f(x_\#)+\langle\nabla f(x_\#),x-x_\#\rangle\text{ for any }x,x_\#\in\mathbb{R}^n.$$

In particular, consider this expression for the choices $x\overset{\text{set}}{=}x^\star$ and $x_\#\overset{\text{set}}{=}x^N$, yielding

$$\begin{aligned}f(x^\star)&\geq f\left(x^N\right)+\left\langle\nabla f\left(x^N\right),x^\star-x^N\right\rangle\\ f\left(x^N\right)-f(x^\star)&\leq\left\langle\nabla f\left(x^N\right),x^N-x^\star\right\rangle\\ f\left(x^N\right)-f(x^\star)&\leq\left\|\nabla f\left(x^N\right)\right\|_2\left\|x^N-x^\star\right\|_2\leq\left\|\nabla f\left(x^N\right)\right\|_2\left\|x^0-x^\star\right\|_2,\end{aligned}$$

where $\left\|x^N-x^\star\right\|_2\leq\left\|x^0-x^\star\right\|_2$ holds because the updates are decreasing pseudocontractive (and thus nonexpansive).

Our bound on the norm of the displacement operator thus implies a bound on the objective function suboptimality. A standard example indicates that, in the absence of additional assumptions, small objective function suboptimality need not imply small norm of distance to an optimal argument. Consider two intersecting hyperplanes $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$; suppose that our objective function is the sum of one-half the squared distance to each hyperplane. That is, $f(x_{\#}) \overset{\text{set}}{=} \frac{1}{2}\text{dist}^2_{\mathcal{H}_{\mathcal{A}}}(x_{\#}) + \frac{1}{2}\text{dist}^2_{\mathcal{H}_{\mathcal{B}}}(x_{\#})$. The respective gradients for each term of the objective are $\left[I - \Pi_{\mathcal{H}_{\mathcal{A}}}\right](x_{\#})$ and $\left[I - \Pi_{\mathcal{H}_{\mathcal{B}}}\right](x_{\#})$. We observe the following: we can have an argument, say $x_{@}$, for which objective value $f(x_{@})$ is arbitrarily small and the associated sum of squared norms of gradients is arbitrarily small while still being arbitrarily far from the intersection of the hyperplanes — in particular, this is what can happen when the angle between the hyperplanes is allowed to be arbitrarily close to 0. In such a setting, it is possible to be very close to $\mathcal{H}_{\mathcal{A}}$ and very close to $\mathcal{H}_{\mathcal{B}}$ while still being very far from their intersection $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathcal{B}}$.

17.6 Strictly contractive updates

We now briefly touch on settings in which the updates will be strictly contractive. The most familiar example occurs when we have an objective function that is both strongly convex and strongly smooth (either everywhere, or with respect to a minimizing argument, or with respect to an entire set of minimizing arguments); more generally, we have seen that if the displacement operator associated with the iterative update operator is both strongly monotone and inverse strongly monotone, the corresponding iterative update operator will be strictly contractive.

Another example is mentioned by [RW04, page 563]: if one starts with an iterative update operator for which the associated displacement operator G is only known to be strongly monotone, the iterative update operator $I - \tau Y_G$ associated with the (scaled) Yosida 1-regularization of G, $Y_G \overset{\text{set}}{=} \left[I + G^{-1}\right]^{-1}$ is strictly contractive when $\tau \in (0,2)$. We may intuitively say that this holds because we know that the Yosida regularization of a (maximal) monotone operator will be inverse strongly monotone; since we started with an operator that was strongly monotone, we thus end with an operator that is both strongly monotone and inverse strongly monotone (and so the corresponding iterative update operator will be a strict contraction).

One further example of a strictly contractive iterative update comes when we consider a convex combination between a "constant operator" (that is, an operator that returns the same vector output for every possible input) and a nonexpansive operator [Aub98, page 314]. This example differs from the previous two because here the contractive iterative update will not (except in unusual situations) have the same fixed point(s) as the initial nonexpansive operator. In this setting, introducing a sequence of iterations with an "appropriately chosen" sequence of convex combination parameters (tending in the limit to a 0 parameter associated with the constant) recovers the element of the fixed point set of the nonexpansive operator that is nearest to the constant with which the convex combinations have been formed.

17.7 Necessary results

The following known results are necessary for establishing results elsewhere in the chapter.

Lemma 2. *For a convex function $f(\cdot)$, the proximal-point $x^+ \overset{set}{=} prox_{f,\lambda}(x)$ satisfies the inclusion*

$$x - x^+ \in \lambda \partial f\left(x^+\right),$$

where $\lambda > 0$.

Proof. The resolvent statement of the proximal-point mapping is

$$\text{prox}_{f,\lambda}(x) = [I + \lambda \partial f]^{-1}(x).$$

From $x^+ \overset{\text{set}}{=} \text{prox}_{f,\lambda}(x)$, we thus have

$$\begin{aligned} \left(x, x^+\right) &\in [I + \lambda \partial f]^{-1} \\ \left(x^+, x\right) &\in I + \lambda \partial f \\ x - x^+ &\in \lambda \partial f\left(x^+\right), \end{aligned}$$

where we have freely shifted our view from relation to operator as convenient. □

Lemma 3. *T being σ-inverse strongly monotone implies that, for $\gamma > 0$, the scaled operator γT is $\frac{\sigma}{\gamma}$-inverse strongly monotone.*

Proof. Immediate from the definition of inverse strongly monotone. □

Proposition 27. *[GK90], Theorem 12.1 For a firmly nonexpansive operator F, the associated reflection operator $R \overset{set}{=} 2F - I$ is nonexpansive.*

Proof. We have

$$\begin{aligned} \|Rx - Ry\|^2 &= 4\|Fx - Fy\|^2 + \|x - y\|^2 - 4\langle Fx - Fy, x - y\rangle \\ &\leq \|x - y\|^2, \end{aligned}$$

for all $x, y \in \mathbb{R}^n$, where we have used the fact that F firmly nonexpansive implies that F is 1-inverse strongly monotone; that is, that $\langle Fx - Fy, x - y\rangle \geq \|Fx - Fy\|^2$, for all $x, y \in \mathbb{R}^n$. □

Proposition 28. *An operator F is* 1*-inverse strongly monotone if and only if the displacement operator $G = I - F$ is* 1*-inverse strongly monotone.*

Proof. The relevant expressions for either direction are

$$\begin{aligned} \|Gx - Gy\|^2 &\leq \langle Gx - Gy, x - y\rangle \\ \|(x - y) - (Fx - Fy)\|^2 &\leq \langle x - y, x - y\rangle - \langle Fx - Fy, x - y\rangle \\ \|Fx - Fy\|^2 &\leq \langle x - y, Fx - Fy\rangle . \end{aligned}$$

□

Proposition 29. *An operator F is* 1*-decreasing pseudocontractive if and only if the displacement operator $G = I - F$ is* 1*-decreasing pseudocontractive.*

Proof. Immediate from the definition of 1-decreasing pseudocontractive. □

Proposition 30. *An operator F is* 1*-decreasing pseudocontractive if and only if the displacement mapping $G = I - F$ is* 1*-inverse strongly monotone.*

Proof. The relevant expressions for either direction are

$$\begin{aligned} &\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|Gx - Gy\|^2 \\ &\|Tx - Ty\|^2 + \|[x - y] - [Tx - Ty]\|^2 \leq \|x - y\|^2 \\ &\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle . \end{aligned}$$

□

Corollary 1. *An operator F is* 1*-decreasing pseudocontractive if and only if F is* 1*-inverse strongly monotone.*

Proof. Proposition 30 shows that F is 1-decreasing pseudocontractive if and only if $G = I - F$ is 1-inverse strongly monotone. Proposition 28 shows that $G = I - F$ is 1-inverse strongly monotone if and only if F is 1-inverse strongly monotone. Together these establish the result. □

Chapter 17, in part, is currently being prepared for submission for publication of the material. Gallagher, Patrick; Tu, Zhuowen. The dissertation author was the primary author of this material.

Appendix A

Notation and conventions

- As much as possible, do not rely on "context makes clear"
- When considering multiple arguments coming from the same space/set use subscripted symbols to distinguish arguments rather than expanding to other letters or using subscripted numbers
  - $x_{\#}, x_{\$}, x_{\%}, x_{@} \in \mathcal{X}$ instead of $x, y, z, w \in \mathcal{X}$ or $x_0, x_1, x_2, x_3 \in \mathcal{X}$
  - $x_{\#}, x_{\$}, x_{\%}, x_{@} \in \mathcal{X}$ immediately indicates that all of the arguments "play the same role" use of $x, y, z, w \in \mathcal{X}$ requires looking elsewhere
  - $x_0, x_1, x_2, x_3 \in \mathcal{X}$ could indicate that all of the arguments "play the same role", but in the context of machine learning applications it is arguably more natural to use numeric indexes to refer to coordinates/position within a data set
- Calligraphic fonts are used to denote sets or spaces
- Capital letters are used to denote operators/mappings
- A superscript "$*$" Denotes dual or conjugate
  - We explicitly distinguish between primal arguments and dual arguments at all times
    - For: getting the "units" right; as in examples from economics or physics

    - Against: Hilbert space isomorphism

- A superscript "$\star$" denotes optimality
- A superscript "$+$" denotes subproblem/auxiliary problem optimality; "we solve a subproblem for some specific parameter setting and get this as the minimizing argument for the subproblem"
- Dependence on some other argument/parameter will be indicated via subscript or $[\cdot]$
- We seek to avoid adjacency of $(\cdot)$ or of $\{\cdot\}$ or of $[\cdot]$
- We use "$\stackrel{\text{set}}{=}$" set equal to (for temporary assignment or informal convention)
- We use "$\stackrel{\text{def}}{=}$" defined equal to (for formal definition)
- We use "$\leftarrow$" updated to be (in the context of iterative update of a parameter)
- Some variable naming conventions
- x generic primal argument
- s generic dual argument (s for "slope")
- e_i ith unit vector: the ith coordinate is equal to 1, every other coordinate is equal to 0
- $\mathcal{S}$ generic set
- $\mathcal{L}$ subspace/vector space
- $\mathcal{V}$ subspace/vector space
- $\mathcal{A}$ affine subspace/affine set
- $\mathcal{C}$ convex set
- $\mathcal{K}$ cone
- $\mathcal{X}$ a space; typically either a Hilbert space or a Banach space

- $\mathcal{X}^*$ the dual space associated with the space $\mathcal{X}$
- Generic norm $\|\cdot\|_\diamond$ and associated dual norm $\|\cdot\|_*$
- M monotone operator
- F firmly nonexpansive operator
- N nonexpansive operator
- A averaged operator
- T generic operator; any special characteristics to be described at the time
- S generic operator; any special characteristics to be described at the time
- "mapping" when going from some element of one space to some element of not-necessarily-the-same space
  - for example $\mathcal{X} \to \mathcal{Y}$ or $\mathbb{R}^n \to \mathbb{R}^m$ or $\mathbb{R}^n \to \mathbb{R}^{m*}$
- "operator" when going from some element of the space to another element of the same number of dimensions
  - for example $\mathcal{X} \to \mathcal{X}$ or $\mathbb{R}^n \to \mathbb{R}^n$ or $\mathbb{R}^n \to \mathbb{R}^{n*}$
- "function" when going from some space to $\mathbb{R}$
  - for example $\mathcal{X} \to \mathbb{R}$ or $\mathbb{R}^n \to \mathbb{R}$
- whenever possible we prefer to use abbreviations of the descriptions in place of additional symbology
  - for example,
    * the closure of a set $\mathcal{S}$ will be denoted cl $\mathcal{S}$ rather than $\overline{\mathcal{S}}$
    * the boundary of a set $\mathcal{S}$ will be denoted bd $\mathcal{S}$ rather than $\partial\mathcal{S}$
    * the interior of a set $\mathcal{S}$ will be denoted int $\mathcal{S}$ rather than $\mathcal{S}^0$ or $\mathring{\mathcal{S}}$

- “Argmax” refers to the set of all maximum-attaining arguments; “Argmin” is analogous
- “argmax” refers to a single arbitrary element of the set of all maximum-attaining arguments; “argmin” is analogous

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