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## On Rank of Block Hankel Matrix for 2-D Frequency Detection and Estimation

Howard Hua Yang and Yingbo Hua

**Abstract**—For detection and estimation of 2-D frequencies from a 2-D array of data using a subspace decomposition method, one needs to construct a block Hankel matrix. For reliable detection and estimation, the rank of the block Hankel matrix should be made equal to the number of 2-D frequencies inherent in the data in the absence of noise. In this work, we provide the conditions for achieving the desired rank.

### I. INTRODUCTION

It is well known that for detection and estimation of 1-D frequencies from a single sequence of data using a subspace decomposition method, one needs to form a Hankel matrix, (e.g., see [4]). The idea behind the Hankel matrix is also equivalent to the so-called moving-window-average in the context of array processing where multiple data sequences are available, (e.g., see [3]). Similarly, for detection and estimation of 2-D frequencies from a single 2-D array of data using a subspace decomposition method, one needs to form a block Hankel matrix where each block is a Hankel matrix. The block Hankel matrix is referred to as enhanced data matrix in [2]. An important property of the block Hankel matrix is that its rank can be equal to the number of 2-D frequencies inherent in the data in the absence of noise regardless of the distribution of the 2-D frequencies. This property clearly suggests that the number of the 2-D frequencies can be detected using the singular values of the block Hankel matrix. Furthermore, as shown in [2], if the rank of the block Hankel matrix is equal to the desired number (number of 2-D frequencies) in the absence of noise, the 2-D frequencies can be efficiently estimated from the principal subspace of the block Hankel matrix using a matrix pencil approach. A sufficient condition for the block Hankel matrix to achieve the desired rank was given in [2]. In this work, we provide a more complete and more rigorous analysis of the block Hankel matrix and show a number of possible (sufficient or/and necessary) conditions under which the block Hankel matrix has the desired rank.

In Section II, the block Hankel matrix is defined and decomposed in structure for later analysis. In Section III, the sufficient or/and necessary conditions for the rank property are discussed in detail. The main contribution of this paper is illustrated in Fig. 1.

### II. THE BLOCK HANKEL MATRIX

Consider a 2-D array of data defined by

$$x(m, n) = \sum_{i=1}^{N_s} a_i y_i^m z_i^n, \quad m = 0, 1, \dots, M-1, \\ n = 0, 1, \dots, N-1 \quad (1)$$

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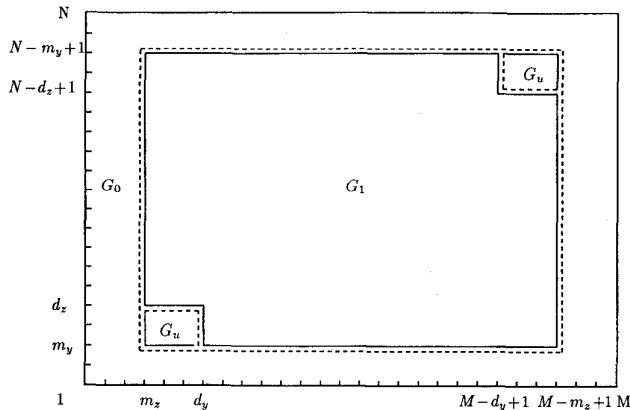


Fig. 1. Regions of window size parameters. For region  $G_1$ , the block Hankel matrix is of the desired rank (the number of 2-D sinusoids). For region  $G_0$ , the block Hankel matrix is of a rank less than the desired. Region  $G_u$  is the uncertain region where the block Hankel matrix may or may not have its rank equal to the desired.  $M$  and  $N$  define the size of the original data set.  $m_y$  and  $m_z$  denote the maximum multiplicities of poles in the first and second dimensions, respectively.  $d_y$  and  $d_z$  denote the numbers of distinct poles in the first and second dimensions, respectively. The sufficient condition given in [2] is a subset of  $G_1$ .

where  $a_i$  is the  $i$ th complex amplitude (none zero);  $(y_i, z_i) = (e^{j\omega_{1i}}, e^{j\omega_{2i}})$  defines the  $i$ th 2-D frequency  $(\omega_{1i}, \omega_{2i})$ ;  $N_s$  is the number of the 2-D frequencies. Naturally,  $\{(y_i, z_i)\}$  should be distinct. Let  $d_y$  and  $d_z$  be the numbers of distinct poles in  $\{y_i\}$  and  $\{z_i\}$ , respectively. Let  $m_y$  and  $m_z$  be the maximum multiplicity in  $\{y_i\}$  and the maximum multiplicity in  $\{z_i\}$ , respectively.

It follows from the definition that  $m_z \leq d_y$  and  $m_y \leq d_z$ .

The block Hankel matrix of  $x(m, n)$  is defined as

$$X_e = \begin{bmatrix} X_0 & X_1 & \cdots & X_{M-K} \\ X_1 & X_2 & \cdots & X_{M-K+1} \\ \cdots & \cdots & \cdots & \cdots \\ X_{K-1} & X_K & \cdots & X_{M-1} \end{bmatrix}$$

where for  $m = 0, 1, \dots, M-1$ ,

$$X_m = \begin{bmatrix} x(m, 0) & x(m, 1) & \cdots & x(m, N-L) \\ x(m, 1) & x(m, 2) & \cdots & x(m, N-L+1) \\ \cdots & \cdots & \cdots & \cdots \\ x(m, L-1) & x(m, L) & \cdots & x(m, N-1) \end{bmatrix}$$

in which  $K$  and  $L$  are called the window-size parameters.

Based on (1), the block Hankel matrix  $X_e$  can be decomposed as follows. We define

$$V(v, r) \triangleq \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_{N_s} \\ \cdots & \cdots & \cdots & \cdots \\ v_1^{r-1} & v_2^{r-1} & \cdots & v_{N_s}^{r-1} \end{pmatrix}$$

where  $v = (v_1, v_2, \dots, v_{N_s})^T$  is a vector. Then we can write (also shown in [2])

$$\begin{aligned} X_e &\triangleq X_e(K, L) \\ &= E_l A_d E_r \end{aligned} \quad (2)$$

where

$$\begin{aligned} E_l &\triangleq E_l(K, L) \\ &\triangleq \begin{bmatrix} V(z, L) \\ V(z, L)Y_d \\ \vdots \\ V(z, L)Y_d^{K-1} \end{bmatrix}_{KL \times N_s} \end{aligned}$$

$$\begin{aligned} z &\triangleq (z_1, \dots, z_{N_s})^T, \\ Y_d &\triangleq \text{diag}(y_1, \dots, y_{N_s}), \\ E_r &\triangleq E_r(K, L) \\ &= E_l^T (M - K + 1, N - L + 1) \end{aligned}$$

and

$$A_d \triangleq \text{diag}(a_1, \dots, a_{N_s}).$$

It is clear from (2) that  $\text{rank}(X_e) \leq N_s$ . What we need to address next is the conditions that should be satisfied by the window-size parameters  $K$  and  $L$  such that  $\text{rank}(X_e)$  is equal to the desired value  $N_s$ . The general results are shown in Fig. 1, and the detailed derivations are given in the next section.

### III. THE CONDITIONS ON THE WINDOW SIZE

Note that the matrix  $E_l$  takes a form identical to that of the following matrix:

$$\begin{aligned} O_k &\triangleq O_k(C, A) \\ &\triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \end{aligned} \quad (3)$$

where  $k \geq 1$ , and  $A$  and  $C$  are  $n \times n$  and  $m \times n$  matrices. When  $k = n$ ,  $O_n$  is known as the observability matrix of the pair  $(C, A)$ . Associated with the matrix  $O_k$ , the following two lemmas will be useful.

**Lemma 1:** Let  $n_1$  be the order of the minimum polynomial of  $A$ . Then for  $n' \geq n_1$ ,  $\text{rank}(O_{n'}) = n$  iff  $\text{rank}([{}^{\lambda I - A}_C]) = n$  for all eigenvalues  $\lambda$  of  $A$ .

*Proof:* This lemma is a generalization of the Theorem 9 in [1] (p. 240). Q.E.D.

**Lemma 2:** If  $\text{rank}([{}^{\lambda I - A}_C]) < n$  for some eigenvalue of  $A$ , then  $\text{rank}(O_k) < n$  for all  $k \geq 1$ .

*Proof:* Let  $\lambda$  be the eigenvalue of  $A$  such that  $\text{rank}([{}^{\lambda I - A}_C]) < n$ . Then there is a vector  $x \neq 0$  such that  $Ax = \lambda x$  and  $Cx = 0$ . So  $CA^k x = 0$  for all  $k \geq 1$ . Hence,  $O_k x = 0$  for all  $k \geq 1$ . Therefore,  $\text{rank}(O_k) < n$  for all  $k \geq 1$ . Q.E.D.

Now we are ready to provide the following.

**Theorem 1:**

- a) If  $K \geq d_y$ , then  $\text{rank}[E_l(K, L)] = N_s$  iff  $L \geq m_y$ ;  
if  $L \geq d_z$ , then  $\text{rank}[E_l(K, L)] = N_s$  iff  $K \geq m_z$ .
- b) If  $K \leq M - d_y + 1$ , then  $\text{rank}[E_r(K, L)] = N_s$  iff  $L \leq N - m_y + 1$ ;  
if  $L \leq N - d_z + 1$ , then  $\text{rank}[E_r(K, L)] = N_s$  iff  $K \leq M - m_z + 1$ .
- c)  $\text{rank}[X_e(K, L)] = N_s$  if the conditions in a) and b) are met.

*Proof:* First, we consider  $E_l(K, L)$ . Let  $(C, A) = [V(z, L), Y_d]$ . Since  $A$  is a diagonal matrix, all eigenvalues of  $A$  are the elements on the diagonal. Let  $\lambda = y_i$  be an element in  $\{y_k\}$  with a multiplicity  $m$ , then we can transform, by a simple column permutation as follows:

$$[{}^{\lambda I - A}_C] \text{ into } \begin{bmatrix} D_1 & 0 \\ 0 & 0 \\ V_1 & V_2 \end{bmatrix} \quad (4)$$

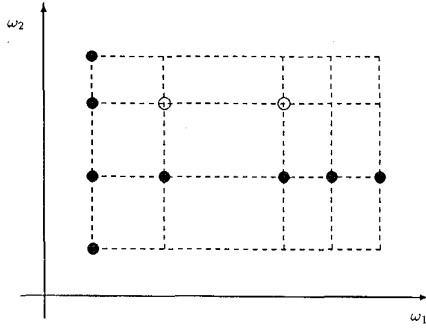


Fig. 2. Constellation of 2-D frequencies that satisfy the condition of Corollary 1.

and, hence, we have

$$\begin{aligned} \text{rank} \left( \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \right) &= \text{rank} \left( \begin{bmatrix} D_1 & 0 \\ 0 & 0 \\ V_1 & V_2 \end{bmatrix} \right) \\ &= \text{rank}(D_1) + \text{rank}(V_2) \end{aligned}$$

where  $D_1 = \text{diag}(y_1 - y'_1, \dots, y_i - y'_{N_s-m})$ ,  $V_2 = V[(z'_1, \dots, z'_m)^T, L]$ , none of  $\{y'_1, \dots, y'_{N_s-m}\} \subset \{y_1, \dots, y_{N_s}\}$  is equal to  $y_i$ , and the poles in  $\{z'_1, \dots, z'_m\} \subset \{z_1, \dots, z_{N_s}\}$  are distinct. So the rank of  $D_1$  is  $N_s - m$ . The rank of  $V_2$  is  $m$  iff  $L \geq m$ . Hence,  $\text{rank} \left( \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \right) = N_s$  for  $\lambda = y_i$  iff  $L \geq m$ . Therefore,  $\text{rank} \left( \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \right) = N_s$  for all  $\lambda \in \{y_i\}$  iff  $L \geq m_y$ . Since  $A$  has  $d_y$  distinct diagonal elements, the order of the minimum polynomial of  $A$  is equal to  $d_y$ . So, by Lemma 1, for  $K \geq d_y$ ,  $\text{rank}[E_l(K, L)] = N_s$  iff  $L \geq m_y$ .

We can symmetrically get the other half conclusion of (a) by using the fact that there exists a permutation matrix  $P$  [2] such that

$$\begin{aligned} PE_l &= \begin{bmatrix} V(y, K) \\ V(y, K)Z_d \\ \vdots \\ V(y, K)Z_d^{L-1} \end{bmatrix} \\ &= O_L[V(y, K), Z_d] \end{aligned} \quad (5)$$

where  $y \hat{=} (y_1, \dots, y_{N_s})^T$  and  $Z_d = \text{diag}(z_1, \dots, z_{N_s})$ .

Part (b) follows readily from part (a) since  $E_r = E_l^T(M - K + 1, N - L + 1)$ .

From (2), we know that  $\text{rank}(X_e) = N_s$  iff  $\text{rank}(E_l) = \text{rank}(E_r) = N_s$ . Combining (a) and (b), we have (c). Q.E.D.

**Theorem 2:**

- 1)  $\text{rank}[E_l(K, L)] < N_s$  if  $L < m_y$  or  $K < m_z$ .
- 2)  $\text{rank}[E_r(K, L)] < N_s$  if  $L > N - m_y + 1$  or  $K > M - m_z + 1$ .
- 3)  $\text{rank}[X_e(K, L)] < N_s$  if  $L < m_y$  or  $K < m_z$  or  $L > N - m_y + 1$  or  $K > M - m_z + 1$ .

*Proof:* Following the first paragraph of the proof for Theorem 1, one can show that if  $\lambda = y_i$  that has the maximum multiplicity  $m_y$ , then  $\text{rank}(V_2) < m_y$  if  $L < m_y$  and hence  $\text{rank} \left( \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \right) < N_s$  if  $L < m_y$ . Therefore, by Lemma 2,  $\text{rank}[E_l(K, L)] < N_s$  if  $L < m_y$ . Other proofs can be done similarly. Q.E.D.

By Theorems 1 and 2, the whole window size set  $G = \{(K, L): 1 \leq K \leq M, 1 \leq L \leq N\}$  is divided into three sets  $G = G_0 \cup G_1 \cup G_u$  (see Fig. 1). For the sets  $G_0$  and  $G_1$ , we have definite answers: i.e.,  $\text{rank}[X_e(K, L)] < N_s$  when  $(K, L) \in G_0$ , and  $\text{rank}[X_e(K, L)] = N_s$  when  $(K, L) \in G_1$ . But for  $G_u$ , the

answer is uncertain depending on the distribution of the signal poles. In other words,  $\text{rank}[X_e(K, L)]$  may be less than or equal to  $N_s$  for  $(K, L) \in G_u$ . It should be noted that the sufficient condition, given in [2], i.e.,  $N_s \leq K \leq M - N_s + 1$  and  $N_s \leq L \leq N - N_s + 1$ , is a subset of  $G_1$ .

When  $m_y = d_z$  and  $m_z = d_y$ , the set  $G_u$  is empty. From Theorems 1 and 2, we have the following sufficient and necessary constraint on the window size under which the rank of the enhanced data matrix is equal to the desired number.

*Corollary 1:* If  $m_y = d_z$  and  $m_z = d_y$ , then

$$\begin{aligned} \text{rank}[X_e(K, L)] = N_s \text{ iff } & d_y \leq K \leq M - d_y + 1, \\ & d_z \leq L \leq N - d_z + 1. \end{aligned}$$

Note that the condition of Corollary 1 is satisfied when all 2-D frequencies are scattered on a rectangular grid (not necessarily uniform) and at least one straight line in each dimension on the grid is fully occupied by 2-D frequencies, which is illustrated in Fig. 2.

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## High-Speed Systolic Ladder Structures for Multidimensional Recursive Digital Filters

Xiaojian Liu and Leonard T. Bruton

**Abstract**—We propose a multilevel approach for designing high-speed systolic ladder structures for multidimensional (MD) recursive digital filters. Based on appropriately selected 1-D filter structures for each filter dimension (or level), a large variety of MD systolic filter structures may be derived. In particular, we introduce a new 1-D filter structure that proves the most suitable structure in terms of a systolic ladder implementation, because it leads to MD ladder filter structures possessing such important properties as the shortest critical path (for filters without order augmentation), the canonic number of high-level storage registers (e.g., line and frame registers of images), and local interconnectivity.

## I. INTRODUCTION

High-speed multidimensional (MD) digital filtering is very useful for real-time video signal processing such as video image coding, bandwidth compression, sampling rate conversion and the enhancement of television signals. In this contribution, we are concerned

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