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A continuation of supergravity solutions on warped spacetimes

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ABSTRACT OF THE DISSERTATION

A continuation of supergravity solutions on warped spacetimes

by

David Ross Corbino
Doctor of Philosophy in Physics
University of California, Los Angeles, 2021
Professor Eric D'Hoker, Chair

In this dissertation, we investigate the existence of solutions with sixteen supersymmetries to Type IIB supergravity on two sets of spacetimes that each contain an internal factor of two-dimensional Anti-de Sitter space (AdS_2).

The first case is $AdS_2 \times S^6$ warped over a Riemann surface Σ. We construct the general Ansatz for the bosonic supergravity fields and supersymmetry generators compatible with the $SO(2,1) \oplus SO(7)$ isometry algebra of the spacetime, which extends to the corresponding real form of the exceptional Lie superalgebra $F(4)$. We reduce the BPS equations to this Ansatz, obtain their general local solutions, and show that these local solutions solve the full Type IIB supergravity field equations and Bianchi identities. We contrast the $AdS_2 \times S^6$ solution with the closely related $AdS_6 \times S^2$ case and present the results for both in parallel.

In the second part of this work, we seek global half-BPS $AdS_2 \times S^6$ solutions corresponding to the near-horizon behavior of (p, q)-string junctions. The general local solution was obtained in terms of two holomorphic functions $\mathcal{A}_\pm$ on Σ, which are constrained by a set of positivity and regularity conditions. We identify the type of singularity in $\mathcal{A}_\pm$ needed at the boundary of Σ to match the solutions locally onto the classic (p, q)-string solution. We then construct and discuss solutions with multiple (p, q)-strings, however the existence of geodesically complete solutions remains unsettled.

The other case we consider is warped $AdS_2 \times S^5 \times S^1$. The existence of the Lie superalgebra $SU(1,1|4) \subset PSU(2,2|4)$, whose maximal bosonic subalgebra is $SO(2,1) \oplus SO(6) \oplus SO(2)$, motivates the search for half-BPS solutions with this same isometry that are asymptotic to $AdS_5 \times S^5$. We reduce the BPS equations to the Ansatz for the bosonic fields and supersymmetry generators compatible with these symmetries, then show that the only non-trivial solution is the maximally supersymmetric solution $AdS_5 \times S^5$. We argue that this implies that no solutions exist for fully back-reacted D7 probe or D7/D3 intersecting branes whose near-horizon limit is of the form $AdS_2 \times S^5 \times S^1$.

The dissertation of David Ross Corbino is approved.

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Chapter 2 is a version of [1], in collaboration with Eric D'Hoker and Christoph Uhlemann. Chapter 3 is a version of sections 2 and 3 of [2], in collaboration with Eric D'Hoker, Justin Kaidi, and Christoph Uhlemann. Chapter 4 is a version of [3].

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Publications

• D. Corbino, *Warped AdS_2 and $SU(1,1|4)$ symmetry in Type IIB*, *JHEP* **03** (2021) 060, [2004.12613].

• D. Corbino, E. D'Hoker, J. Kaidi and C. F. Uhlemann, *Global half-BPS $AdS_2 \times S^6$ solutions in Type IIB*, *JHEP* **03** (2019) 039, [1812.10206].

• D. Corbino, E. D'Hoker and C. F. Uhlemann, *$AdS_2 \times S^6$ versus $AdS_6 \times S^2$ in Type IIB supergravity*, *JHEP* **03** (2018) 120, [1712.04463].

CHAPTER 1

Introduction

Gauge/gravity duality has become a fundamental area of research in modern theoretical physics. The duality is a realization of the holographic principle, and posits an equivalence between a quantum field theory in d dimensions and a theory of gravity in $d+1$ dimensions. The best understood examples of gauge/gravity duality are given by the AdS/CFT correspondence [4, 5, 6] (for reviews, see [7, 8]), and involve conformal field theories (CFTs) with a large number of fields and their dual Anti-de Sitter (AdS) spacetimes with large degrees of supersymmetry. The original form of the AdS/CFT correspondence states that Type IIB superstring theory on the product spacetime $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory in four dimensions with gauge group $SU(N)$. Here, the string coupling g_s, Yang-Mills coupling g_{YM}, and radii L of both the AdS_5 and S^5 spaces are related as follows,

$$g_s = g_{YM}^2 \qquad\qquad L^4 = 4\pi g_s N(\alpha')^2 \tag{1.1}$$

where α' is the square of the Planck length. Defining the 't Hooft coupling $\lambda \equiv g_{YM}^2 N = g_s N$, the limit of $N \to \infty$ with λ fixed corresponds to weak coupling string perturbation theory, while for $\lambda \gg 1$ the string theory can be approximated by classical Type IIB supergravity.

A key feature within this correspondence is that of supersymmetry, the presence of which tightly constrains the set of possible supergravity solutions. The composition of supersymmetry transformations gives rise to additional bosonic symmetries, such as isometries, which further constrain the solutions. The supersymmetries and bosonic symmetries form a Lie superalgebra, under which the solution is invariant, with the isometries extending to the corresponding real form of the Lie superalgebra. Objects protected by supersymmetry play an important role in the correspondence, and among the simplest are those corresponding to so-called "half-BPS" solutions, which preserve sixteen of the maximal number of supersymmetries. Special cases involve half-BPS solutions to supergravity on pairs of spacetimes with internal factors related by "double analytic continuation", e.g. $AdS_p \times S^q$ and $AdS_q \times S^p$ for $p + q \leq 10$ in the case of Type IIB supergravity. One notable example where such pairs of half-BPS solutions to Type IIB supergravity have been constructed are for the following spacetimes warped over a two-dimensional Riemann surface Σ: $AdS_4 \times S^2 \times S^2 \times \Sigma$ [9] and $AdS_2 \times S^2 \times S^4 \times \Sigma$ [10]. These solutions provide the holographic duals to interface solutions and Wilson loops, respectively.

In this dissertation, we investigate the existence of half-BPS solutions to Type IIB supergravity on two sets of spacetimes that contain an internal (warped) AdS_2 factor, and which are the double-analytic counterpart to two previously studied cases. Experience with solutions to supergravity on spacetimes related through double analytic continuation therefore motivates the search for new solutions, and suggests that their solutions should be closely related mathematically, though their physical spacetime structure may be quite different. The study of AdS_2 holography provides further motivation for the search for these supergravity solutions. Holography on two-dimensional Anti-de Sitter spacetime is arguably less well understood than its higher-dimensional counterparts. This has lead to significant problems in the realization of the AdS_2/CFT_1 correspondence [11, 12, 13, 14], a common source of which is the disconnectedness of the AdS_2 boundary [15]. The technique of double analytic continuation is therefore a powerful guide in the construction of AdS_2 solutions. In addition to the solutions of [9] and [10] to Type IIB supergravity, a more recent example of AdS_2 solutions related by double analytic continuation are the $AdS_2 \times S^7$ solutions to massive Type IIA constructed in [16] from the $AdS_7 \times S^2$ solutions of [17].

In the first part, we consider half-BPS solutions to Type IIB supergravity on a spacetime of the form $AdS_2 \times S^6 \times \Sigma$ [1, 2]. By double analytic continuation, we expect that these solutions are related to those for a spacetime of the form $AdS_6 \times S^2 \times \Sigma$, which were obtained in [18, 19, 20, 21]. The motivation for that work was the construction of holographic duals to five-dimensional superconformal field theories (SCFTs). For $AdS_6 \times S^2$, the $SO(2,5) \oplus SO(3)$

isometry algebra of the spacetime manifold extends to invariance under the corresponding real form of the exceptional Lie superalgebra $F(4)$, which is the unique superconformal algebra in 5 spacetime dimensions. For $AdS_2 \times S^6$, the isometry algebra $SO(2,1) \oplus SO(7)$ now extends to a different real form of the exceptional Lie superalgebra $F(4)$, which is one of the superconformal algebras in 2 dimensions with 16 supercharges [22, 23]. The relation between these two problems is similar to the one encountered between gravity duals to Wilson loops [24] and interface solutions [25].

We begin by constructing the local form of half-BPS solutions to Type IIB supergravity for warped $AdS_2 \times S^6$ following the strategy used for $AdS_6 \times S^2$. We derive the reduced BPS equations for the general Ansatz dictated by $SO(2,1) \oplus SO(7)$ isometry for the fields of Type IIB supergravity. These equations are very closely related to the reduced BPS equations for the $AdS_6 \times S^2$ case, but differ by subtle and crucially important signs and factors of $i = \sqrt{-1}$. We provide a detailed comparison between the mathematical equations for both cases.

Using methods which are analogous to the ones developed to solve the reduced BPS equations for the $AdS_6 \times S^2$ case, we construct the general local solutions for the $AdS_2 \times S^6$ case in terms of two locally holomorphic functions $\mathcal{A}_\pm$ on the Riemann surface Σ. The differences between the $AdS_2 \times S^6$ and $AdS_6 \times S^2$ solutions are again subtle, but crucial, and to facilitate direct comparisons we discuss both cases in parallel. To solve the reduced BPS equations, we make use of the solution to the axion-dilaton Bianchi identities, but derive the Bianchi identity for the 3-form field strength from the BPS equations. To complete the discussion, we verify that the full set of Type IIB field equations are satisfied when the bosonic supergravity fields are given by the solutions to the BPS equations and axion-dilaton Bianchi identities. We show this for the $AdS_2 \times S^6$ and $AdS_6 \times S^2$ cases in parallel, and thus provide this check also for the solutions constructed in [26].

The solutions obtained for the supergravity fields satisfy the BPS and field equations, but they become physically viable only after certain reality, positivity and regularity conditions are enforced. We obtain the constraints on the functions $\mathcal{A}_\pm$ required by physical positivity and regularity conditions on the supergravity fields, and exhibit crucial differences between the $AdS_2 \times S^6$ and $AdS_6 \times S^2$ cases. We discuss the possibility of performing a "double analytic continuation" of the global $AdS_6 \times S^2$ solutions constructed in [19, 20] to the present case of $AdS_2 \times S^6$. Although such continuations are found to satisfy the field equations, they appear to be neither supersymmetric nor physically regular. Therefore, the construction of global $AdS_2 \times S^6$ solutions must be conducted independently of the $AdS_6 \times S^2$ case. With that objective in mind, we derive the explicit forms of the two-form and six-form potentials for the $AdS_2 \times S^6$ and the $AdS_6 \times S^2$ solutions.

We continue our investigation of this case in the second part, where we turn to the question of global half-BPS $AdS_2 \times S^6$ solutions. For $AdS_6 \times S^2$, globally regular and geodesically complete solutions sourced by the charges p, q of the complex three-form field strength of Type IIB were shown to provide full back-reacted geometries for the near-horizon region of general (p, q) five-brane webs [19, 20, 21]. Therefore, we shall investigate the existence of global $AdS_2 \times S^6$ solutions sourced by seven-form charges p, q, which are naturally associated with (p, q)-strings. We shall examine the emergence of (p, q)-string web solutions [27, 28, 29] in the near-horizon limit. Although the supergravity fields of the $AdS_2 \times S^6$ solutions differ from those of the $AdS_6 \times S^2$ solutions merely by certain sign reversals, these simple differences make the construction of globally regular $AdS_2 \times S^6$ solutions intricate and technically difficult. While we shall succeed in producing solutions with multiple (p, q)-strings in the near-horizon limit, the geodesic completeness of such solutions remains unsettled.

In the third part of this dissertation, we consider half-BPS solutions to Type IIB supergravity on a spacetime of the form $AdS_2 \times S^5 \times S^1 \times \Sigma$ [3]. In [23], a general correspondence was proposed between certain Lie superalgebras with 16 fermionic generators and half-BPS solutions to either Type IIB supergravity or M-theory. In the case of Type IIB, the semi-simple Lie superalgebras $\mathcal{H}$ are subalgebras of $PSU(2, 2|4)$, and the corresponding half-BPS solutions are invariant under $\mathcal{H}$ and locally asymptotic to the maximally supersymmetric solution $AdS_5 \times S^5$. It is shown that there exist a finite number of such subalgebras $\mathcal{H}$, and thus one obtains a classification of half-BPS solutions with the above asymptotics. Among these are the special classes of exact solutions previously found in [9] and [10], while those of [18, 19, 20, 21] and [1, 2] are absent since neither $F(4)$ nor any of its real forms are subalgebras of $PSU(2,2|4)$.

Half-BPS solutions related to D7 branes in Type IIB supergravity are of particular interest. The near-horizon limits of D7 probe or D7/D3 intersecting branes seem to support the existence of corresponding half-BPS solutions. However, D7 branes also produce flavor multiplets which ultimately break exact conformal invariance, and by the arguments of [30] and [31, 32] (which follows earlier work on D7 branes in [33, 34]) no fully back-reacted near-horizon limit solutions corresponding to D7 branes should exist. The classification of [23] reveals two cases, corresponding to the subalgebras $SU(1,1|4) \oplus SU(1,1)$ and $SU(2,2|4) \oplus SU(2)$, whose global symmetries and spacetime structure match those of the D7 probe or D7/D3 intersecting brane analysis. Each superalgebra contains a purely bosonic invariant subalgebra, respectively $SU(1,1)$ and $SU(2)$, which is not required by superconformal invariance. Additionally, these extra bosonic invariant subalgebras are incompatible with asymptotic $AdS_5 \times S^5$ behavior (see Section 5.4 of [23] and references therein). Their removal yields the respective cases $SU(1,1|4)$ and $SU(2,2|2)$, for which the superalgebra correspondence suggests the existence of half-BPS solutions. However, these cases no longer possess the symmetries necessary for fully back-reacted near-horizon D7 brane solutions.

Therefore, we consider half-BPS solutions with $SO(2,1) \oplus SO(6) \oplus SO(2)$ symmetry, corresponding to the maximal bosonic symmetry of the superalgebra $SU(1,1|4)$ and realized on a spacetime of the form $AdS_2 \times S^5 \times S^1$ warped over a Riemann surface Σ. Half-BPS solutions invariant under $SU(2,2|2)$ were investigated in [35], where it was shown that on either $AdS_5 \times S^2 \times S^1 \times \Sigma$ or $AdS_5 \times S^3 \times \Sigma$ the only non-trivial solution that exists is $AdS_5 \times S^5$. Employing the same strategy here, we prove that the only half-BPS solution invariant under $SO(2,1) \oplus SO(6) \oplus SO(2)$ is once again $AdS_5 \times S^5$. Thus, in the supergravity limit no fully back-reacted solutions of D7 branes can exist whose near-horizon limit match the symmetries and spacetime geometries of either case. Note that in contrast to the $SU(2,2|2)$ solutions, the bosonic invariant Lie subalgebra $SU(1,1)$ which is removed to obtain the $SU(1,1|4)$ case corresponds to a part of the isometry algebra for the original Anti-de Sitter space, and thus part of the conformal symmetry of the higher-dimensional dual CFT is broken. Additional arguments presented in Section 5.4 of [23] provide further evidence that the case of $SU(1,1|4) \oplus SU(1,1)$ cannot support half-BPS solutions with genuine asymptotic $AdS_5 \times S^5$ behavior, and so we do not consider such solutions here.

CHAPTER 2

$AdS_2 \times S^6$ versus $AdS_6 \times S^2$

2.1 $AdS_2 \times S^6 \times \Sigma$ Ansatz in Type IIB supergravity

In this section we begin by reviewing the salient features of Type IIB supergravity needed in this paper, and then obtain the $SO(2,1) \oplus SO(7)$-invariant Ansatz for the bosonic supergravity fields and the generator of supersymmetry transformations.

2.1.1 Type IIB supergravity review

The bosonic fields of Type IIB supergravity consist of the metric g_{MN}, the complex-valued axion-dilaton field B, a complex-valued two-form potential $C_{(2)}$ and a real-valued four-form field $C_{(4)}$. The field strengths of the potentials $C_{(2)}$ and $C_{(4)}$ are given as follows,

$$\begin{aligned} F_{(3)} &= dC_{(2)} \\ F_{(5)} &= dC_{(4)} + \frac{i}{16}(C_{(2)} \wedge \bar{F}_{(3)} - \bar{C}_{(2)} \wedge F_{(3)}) \end{aligned} \tag{2.1}$$

The field strength $F_{(5)}$ satisfies the well-known self-duality condition $F_{(5)} = *F_{(5)}$. Instead of the scalar field B and the 3-form $F_{(3)}$, the fields that actually enter the BPS equations are composite fields, namely the one-forms P, Q representing B, and the complex 3-form G representing $F_{(3)}$, given in terms of the fields defined above by the following relations,

$$\begin{aligned} P &= f^2\, dB \qquad\qquad f^2 = (1-|B|^2)^{-1} \\ Q &= f^2\, \mathrm{Im}\,(Bd\bar{B}) \\ G &= f(F_{(3)} - B\bar{F}_{(3)}) \end{aligned} \tag{2.2}$$

Under the $SU(1,1) \sim SL(2,\mathbb{R})$ global symmetry of Type IIB supergravity, the Einstein-frame metric g_{MN} and the four-form $C_{(4)}$ are invariant, while B and $C_{(2)}$ transform as,

$$
\begin{aligned}
B &\rightarrow (uB+v)/(\bar{v}B+\bar{u}) \\
C_{(2)} &\rightarrow uC_{(2)}+v\bar{C}_{(2)}
\end{aligned}
\tag{2.3}
$$

where $SU(1,1)$ is parametrized by $u, v \in \mathbb{C}$ with $|u|^2 - |v|^2 = 1$. The field B takes values in the coset $SU(1,1)/U(1)_q$ and Q plays the role of a composite $U(1)_q$ gauge field. The transformation laws for the composite fields are as follows [36],

$$
\begin{aligned}
P &\rightarrow e^{2i\theta}P & \theta &= \arg(v\bar{B}+u) \\
Q &\rightarrow Q+d\theta \\
G &\rightarrow e^{i\theta}G
\end{aligned}
\tag{2.4}
$$

Equivalently, one may formulate Type IIB supergravity directly in terms of g_{MN}, $F_{(5)}$, P, Q and G provided these fields are subject to the Bianchi identities [36, 37].

The fermionic fields are Weyl fermions with opposite 10-dimensional chirality, namely the dilatino λ satisfying $\Gamma_{11}\lambda = \lambda$ and the gravitino ψ_M satisfying $\Gamma_{11}\psi_M = -\psi_M$. The crucial information for the construction of supersymmetric solutions to Type IIB supergravity are the supersymmetry variations of the fermions, respectively $\delta\lambda$ and $\delta\psi_M$. The BPS equations are the conditions that the fermion fields and their variations vanish, and are given by,[1]

$$
\begin{aligned}
0 &= i(\Gamma \cdot P)\mathcal{B}^{-1}\varepsilon^* - \frac{i}{24}(\Gamma \cdot G)\varepsilon \\
0 &= (\nabla_M - \frac{i}{2}Q_M)\varepsilon + \frac{i}{480}(\Gamma \cdot F_{(5)})\Gamma_M\varepsilon - \frac{1}{96}(\Gamma_M(\Gamma \cdot G) + 2(\Gamma \cdot G)\Gamma_M)\mathcal{B}^{-1}\varepsilon^*
\end{aligned}
\tag{2.5}
$$

Here, ε is the generator of infinitesimal supersymmetry transformations. It transforms under the minus chirality Weyl spinor representation of $SO(1,9)$ and ∇_M is the covariant derivative

[1] The signature convention for the metric is $(- + \cdots +)$, the Dirac-Clifford algebra is defined by the relation $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} I_{32}$ and the charge conjugation matrix $\mathcal{B}$ is defined by the relations $\mathcal{B}^*\mathcal{B} = I$ and $\mathcal{B}\Gamma^M\mathcal{B}^{-1} = (\Gamma^M)^*$. Repeated indices are summed over, as usual, and complex conjugation is denoted by a *bar* for functions and by a *star* for spinors. We will also use the notation $\Gamma \cdot T \equiv \Gamma^{M_1 \cdots M_p} T_{M_1 \cdots M_p}$ for the contraction of an antisymmetric tensor field T of rank p with a Γ-matrix of the same rank.

acting on this representation. In sec. 2.5 we will show that the solutions with 16 supersymmetries to these BPS equations satisfy the field equations, and when formulated in terms of $P, Q, G, F_{(5)}$ also satisfy their Bianchi identities.

2.1.2 $SO(2,1) \oplus SO(7)$ invariant Ansatz for supergravity fields

We construct a general Ansatz for the bosonic fields of Type IIB supergravity invariant or covariant under the $SO(2,1) \oplus SO(7)$ symmetry algebra. The $SO(2,1)$ and $SO(7)$ parts are realized by a geometry which contains a factor AdS_2 as well as a factor S^6, so that the spacetime is given by,

$$AdS_2 \times S^6 \tag{2.6}$$

warped over a two-dimensional space Σ. To produce a geometry of Type IIB supergravity, Σ has to be orientable and carry a Riemannian metric, and is therefore a Riemann surface. The resulting $SO(2,1) \oplus SO(7)$-invariant Ansatz for the metric can be written as,

$$ds^2 = f_2^2 \, d\hat{s}^2_{AdS_2} + f_6^2 \, d\hat{s}^2_{S^6} + ds^2_\Sigma \tag{2.7}$$

where f_2, f_6, and ds^2_Σ are functions of Σ. We introduce an orthonormal frame,

$$\begin{aligned} e^m &= f_2 \, \hat{e}^m & \qquad m &= 0, 1 \\ e^i &= f_6 \, \hat{e}^i & \qquad i &= 2, 3, 4, 5, 6, 7 \\ e^a & & \qquad a &= 8, 9 \end{aligned} \tag{2.8}$$

where $\hat{e}^m$ and $\hat{e}^i$ respectively refer to orthonormal frames for the spaces AdS_2 and S^6 with unit radius and e^a is an orthonormal frame on Σ only. In particular, we have,

$$\begin{aligned} d\hat{s}^2_{AdS_2} &= \eta^{(2)}_{mn} \hat{e}^m \otimes \hat{e}^n & \qquad \eta^{(2)} &= \text{diag}(-+) \\ d\hat{s}^2_{S^6} &= \delta_{ij} \hat{e}^i \otimes \hat{e}^j \\ ds^2_\Sigma &= \delta_{ab} e^a \otimes e^b \end{aligned} \tag{2.9}$$

The requirement for $SO(2,1) \oplus SO(7)$-invariance restricts $F_{(5)} = 0$ as well as,

$$P = p_a e^a \qquad\qquad Q = q_a e^a \qquad\qquad G = g_a e^a \wedge e^0 \wedge e^1 \tag{2.10}$$

where the components p_a, q_a, and g_a are complex and depend on Σ only. We have thus parametrized the entire configuration in terms of functions that have non-trivial dependence only on Σ, and it will be convenient to set up the frame and coordinates on Σ more explicitly. We will use complex frame indices z, $\bar{z}$ with the following conventions,

$$\delta_{z\bar{z}} = 2 \qquad \delta^{z\bar{z}} = \frac{1}{2} \qquad e^z = \frac{1}{2}(e^8 + ie^9) \qquad e^{\bar{z}} = \frac{1}{2}(e^8 - ie^9) \tag{2.11}$$

We introduce local complex coordinates w, $\bar{w}$ such that the metric on Σ reads,

$$ds^2_\Sigma = 4\rho^2 |dw|^2 \tag{2.12}$$

and we have,

$$\begin{aligned} e^z &= \rho dw & D_z &= \rho^{-1}\partial_w & \hat{\omega}_z &= i\rho^{-2}\partial_w \rho \\ e^{\bar{z}} &= \rho d\bar{w} & D_{\bar{z}} &= \rho^{-1}\partial_{\bar{w}} & \hat{\omega}_{\bar{z}} &= -i\rho^{-2}\partial_{\bar{w}} \rho \end{aligned} \tag{2.13}$$

This completes the Ansatz for the bosonic fields.

2.1.3 $SO(2,1) \oplus SO(7)$ invariant Ansatz for susy generators

Next, we decompose the supersymmetry generator spinor ε in an $SO(2,1) \oplus SO(7)$-invariant basis of Killing spinors. The Killing spinor equations on AdS_2 and on S^6 were derived in Appendix B of [24] and are respectively given by,

$$\begin{aligned} \Big(\hat{\nabla}_m - \frac{1}{2}\eta_1 \gamma_m \otimes I_8\Big)\chi^{\eta_1,\eta_2}_\alpha &= 0 \\ \Big(\hat{\nabla}_i - \frac{i}{2}\eta_2 I_2 \otimes \gamma_i\Big)\chi^{\eta_1,\eta_2}_\alpha &= 0 \end{aligned} \tag{2.14}$$

where m and i are all *frame indices*. Note that $\hat{\nabla}_m$ and $\hat{\nabla}_i$ stand for the covariant spinor derivatives respectively on the spaces AdS_2 and S^6 with unit radius. The spinors $\chi^{\eta_1,\eta_2}_\alpha$ are 16-dimensional, and the parameters η_1 and η_2 can take the values $\pm$. For each value of (η_1, η_2), these equations admit solutions with a four-fold degeneracy, which is labelled by the index $\alpha = 1, 2, 3, 4$. The action of the chirality matrices is given by,

$$\begin{aligned} \big(\gamma_{(1)} \otimes I_8\big)\, \chi^{\eta_1,\eta_2}_\alpha &= \chi^{-\eta_1,\eta_2}_\alpha \\ \big(I_2 \otimes \gamma_{(2)}\big)\, \chi^{\eta_1,\eta_2}_\alpha &= \chi^{\eta_1,-\eta_2}_\alpha \end{aligned} \tag{2.15}$$

These equations can be understood as follows. Beginning with $\eta_1 = \eta_2 = +$, we pick a basis $\chi_\alpha^{+,+}$ for the four-dimensional vector space of spinors for fixed η_1, η_2 such that the action of $\gamma_{(1)}$ and $\gamma_{(2)}$ are diagonal. The basis for $\chi_\alpha^{\eta_1,\eta_2}$ can then simply be defined for the remaining three values of η_1, η_2 by the action of the chirality matrices above.

Using arguments similar to the ones used for the $AdS_6 \times S^2$ case, we relate the complex conjugate basis spinors to the original basis by,

$$\left(B_{(1)}^{-1} \otimes B_{(2)}^{-1}\right) (\chi_\alpha^{\eta_1,\eta_2})^* = \eta_2 \chi_\alpha^{\eta_1,\eta_2} \tag{2.16}$$

for all values of η_1, η_2, and α. Since this decomposition is now canonical in terms of the degeneracy index α, we will no longer indicate it explicitly. An arbitrary 32-component complex spinor ε may be decomposed onto the above Killing spinors as follows,

$$\varepsilon = \sum_{\eta_1,\eta_2=\pm} \chi^{\eta_1,\eta_2} \otimes \zeta_{\eta_1,\eta_2} \tag{2.17}$$

where ζ_{η_1,η_2} is a complex 2-component spinor for each η_1, η_2, and the four-fold degeneracy index is suppressed. As a supersymmetry generator in Type IIB, the spinor ε must be of definite chirality $\Gamma^{11}\varepsilon = -\varepsilon$, which imposes the following chirality requirements on ζ,

$$\gamma_{(3)} \zeta_{-\eta_1,-\eta_2} = -\zeta_{\eta_1,\eta_2} \tag{2.18}$$

Finally, the charge conjugate spinor is given by,

$$\mathcal{B}^{-1}\varepsilon^* = \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \star\zeta_{\eta_1,\eta_2} \qquad\qquad \star\zeta_{\eta_1,\eta_2} = -i\eta_2\sigma^2\zeta^*_{\eta_1,-\eta_2} \tag{2.19}$$

This completes the construction of the $SO(2,1) \oplus SO(7)$-invariant Ansatz.

2.2 Reducing the BPS equations

The residual supersymmetries, if any, of a configuration of purely bosonic Type IIB supergravity fields are governed by the BPS equations of (2.5). As we will discuss in more detail in sec. 2.5, any $SO(2,1) \oplus SO(7)$ invariant Ansatz for the supergravity fields and for the 16-component supersymmetry spinor as discussed in sec. 2.1.2 which satisfies the BPS equations will automatically solve the Bianchi and field equations, and thus automatically provides a half-BPS solution to Type IIB supergravity.

In this section, we reduce the BPS equations to the $AdS_2 \times S^6 \times \Sigma$ Ansatz, expose its residual symmetries, and solve those reduced equations which are purely algebraic in the supersymmetry spinor components. This will produce simple algebraic expressions for the metric factors f_2, f_6 in terms of the spinors. The remaining reduced BPS equations will be solved for the remaining bosonic fields in subsequent sections. The strategy employed here is the same as the one used in [26].

2.2.1 The reduced BPS equations

We use the τ matrix notation introduced originally in [38] in order to compactly express the action of the various γ matrices on the reduced supersymmetry generator ζ introduced in (2.17). Defining $\tau^{(ij)} = \tau^i \otimes \tau^j$ with $i, j = 0, 1, 2, 3$, we identify τ^0 with the identity matrix and τ^i for $i = 1, 2, 3$ with the standard Pauli matrices. The action of these matrices on ζ may be written in components as follows,

$$(\tau^{(ij)}\zeta)_{\eta_1,\eta_2} \equiv \sum_{\eta_1',\eta_2'} (\tau^i)_{\eta_1\eta_1'}(\tau^j)_{\eta_2\eta_2'}\zeta_{\eta_1'\eta_2'} \tag{2.20}$$

The reduced BPS equations may then be calculated using the decomposition of ε onto Killing spinors given in (2.17). The reduced dilatino equation is given by,

$$0 = -4p_a\gamma^a\sigma^2\zeta^* + g_a\tau^{(12)}\gamma^a\zeta \tag{2.21}$$

while the reduced gravitino equations take the following form,

$$\begin{aligned}
&(m) \qquad 0 = \frac{-i}{2f_2}\tau^{(21)}\zeta + \frac{D_a f_2}{2f_2}\gamma^a\zeta - \frac{3}{16}g_a\tau^{(12)}\gamma^a\sigma^2\zeta^* \\
&(i) \qquad 0 = \frac{1}{2f_6}\tau^{(02)}\zeta + \frac{D_a f_6}{2f_6}\gamma^a\zeta + \frac{1}{16}g_a\tau^{(12)}\gamma^a\sigma^2\zeta^* \\
&(a) \qquad 0 = \left(D_a + \frac{i}{2}\hat{\omega}_a\sigma^3 - \frac{i}{2}q_a\right)\zeta - \frac{3}{16}g_a\tau^{(12)}\sigma^2\zeta^* + \frac{1}{16}g_b\tau^{(12)}\gamma_a{}^b\sigma^2\zeta^*
\end{aligned} \tag{2.22}$$

The derivative D_a acts on functions of Σ only, and is defined with respect to the frame e^a of Σ, so that the total differential d_Σ on Σ takes the form $d_\Sigma = e^a D_a$, while the $U(1)$-connection with respect to frame indices is $\hat{\omega}_a$. The reduction is carried out in Appendix B.

2.2.2 Symmetries of the reduced BPS equations

The global $SU(1,1)$ symmetry of Type IIB supergravity, whose action on the bosonic fields was given in (2.3) and (2.4), survives the reduction to the $SO(2,1) \oplus SO(7)$ invariant Ansatz. It leaves the metric functions f_2, f_6, ρ invariant, transforms the axion-dilaton field B and the two-form $C_{(2)}$ as in (2.3), transforms the reduced supersymmetry spinor ζ by $\zeta \to e^{i\theta/2}\zeta$, and transforms the composite fields of (2.4) as follows,

$$U(1)_q : \; q_a \to q_a + D_a\theta \qquad p_a \to e^{2i\theta} p_a \qquad g_a \to g_a e^{i\theta} g_a \tag{2.23}$$

The reduced BPS equations are also invariant under the following discrete symmetries which act only on the reduced supersymmetry generator but not on the reduced supergravity fields,

$$\mathcal{I} : \; \zeta \to -\tau^{(11)}\sigma^3\zeta \qquad \mathcal{J} : \; \zeta \to \tau^{(32)}\zeta \tag{2.24}$$

Finally, charge conjugation $\mathcal{K}$ acts by,

$$\mathcal{K} : \; \zeta \to \tau^{(22)}\sigma^2\zeta^* \qquad q_a \to -q_a \qquad p_a \to \bar{p}_a \qquad g_a \to -\bar{g}_a \tag{2.25}$$

The chirality requirement of Type IIB restricts the spinor ζ to the subspace,

$$\mathcal{I}\zeta = -\tau^{(11)}\sigma^3\zeta = \zeta \tag{2.26}$$

The symmetries $\mathcal{I}, \mathcal{J}, \mathcal{K}$ commute with one another and may be diagonalized simultaneously. Both $\mathcal{I}$ and $\mathcal{J}$ commute with $U(1)_q$, but $\mathcal{K}$ does not commute with $U(1)_q$.

2.2.3 Restricting to a single subspace of $\mathcal{J}$

The eigenspace of $\mathcal{I}$ being already restricted by the chirality condition of (2.26), we now derive the restrictions to the eigenspaces of $\mathcal{J}$ and $\mathcal{K}$ which are implied by the reduced BPS equations, following the same procedure that was used for $AdS_6 \times S^2$ in [26]. For any value of g_a, we have the following quadratic relations in ζ,

$$g_a \zeta^t T \tau^{(12)} \sigma^2 \gamma^a \zeta = 0 \tag{2.27}$$

provided the matrix T belongs to the following set of $\tau^{(ij)}$-matrices,

$$T \in \mathcal{T} = \left\{ \tau^{(00)}, \tau^{(10)}, \tau^{(20)}, \tau^{(31)}, \tau^{(32)}, \tau^{(33)} \right\} \tag{2.28}$$

For these values of T, the combination $T\tau^{(12)}\sigma^2\gamma^a$ is anti-symmetric for $a = 1, 2$ and the relation (2.27) indeed holds automatically. The reduced dilatino equation implies,

$$\bar{p}_a \zeta^\dagger T \gamma^a \zeta = 0 \tag{2.29}$$

Making use also of the chirality condition (2.26), we obtain the further equation,

$$\bar{p}_a \zeta^\dagger T \tau^{(11)} \gamma^a \sigma^3 \zeta = 0 \tag{2.30}$$

When both T and $T\tau^{(11)}$ belong to $\mathcal{T}$, which is the case for only a single pair of matrices, namely $T = \tau^{(20)}$ or $T = \tau^{(31)}$, and assuming that p_a does not vanish identically, we may combine (2.29) and (2.30) to obtain the following relations,

$$\zeta^\dagger \tau^{(20)} \gamma^a \zeta = \zeta^\dagger \tau^{(31)} \gamma^a \zeta = 0 \tag{2.31}$$

which hold for $a = 1, 2$ and are equivalent to one another upon using the chirality condition.

Next, we analyze the gravitino equations. Multiplying equations (m) and (i) of (2.22) on the left by $\zeta^\dagger T \sigma^p$ for $p = 0, 3$, we obtain a cancellation of the last term when $T\tau^{(12)}$ is anti-symmetric (which is the same condition we had for the dilatino equation),

$$\begin{aligned} 0 &= -\frac{i}{2f_2} \zeta^\dagger T \tau^{(21)} \sigma^p \zeta + \frac{D_a f_2}{2f_2} \zeta^\dagger T \sigma^p \gamma^a \zeta \\ 0 &= \frac{1}{2f_6} \zeta^\dagger T \tau^{(02)} \sigma^p \zeta + \frac{D_a f_6}{2f_6} \zeta^\dagger T \sigma^p \gamma^a \zeta \end{aligned} \tag{2.32}$$

In view of (2.31), the second term will cancel when $T = \tau^{(20)}$ and $T = \tau^{(31)}$. so that we obtain the following relations from the remaining cancellation of the first term,

$$\begin{aligned} &\zeta^\dagger \tau^{(01)} \sigma^p \zeta = 0 \qquad\qquad p = 0, 3 \\ &\zeta^\dagger \tau^{(22)} \sigma^p \zeta = 0 \end{aligned} \tag{2.33}$$

and their chiral conjugates, which may be obtained by using the chirality condition.

Next, we use the general result of [25] that the bilinear equation $\zeta^\dagger M\zeta = 0$ is solved by projecting ζ onto a subspace via a projection matrix Π that anti-commutes with M. Thus, we must find a projector Π with the following properties,

$$[\Pi, \tau^{(11)}\sigma^3] = \{\Pi\tau^{(01)}\sigma^p\} = \{\Pi, \tau^{(22)}\sigma^p\} = 0 \tag{2.34}$$

The solutions to these equations are $\tau^{(32)}$, $\tau^{(23)}$, $\tau^{(32)}\sigma^3$, and $\tau^{(23)}\sigma^3$. These four possibilities are pairwise equivalent under the chirality relation. The projector $\Pi = \tau^{(32)}$ precisely corresponds to the symmetry $\mathcal{J}$, so imposing a restriction on the spinor space by this operator is the only consistent restriction. Therefore, we will impose the restriction,

$$\tau^{(32)}\zeta = \nu\zeta \qquad\qquad \nu = \pm 1 \tag{2.35}$$

which solves all the above bilinear relations for either choice of ν, but not both.

We may solve the projection relations given in (2.26) and (2.35) in terms of two independent complex-valued one-component spinors α and β. Denoting the components of ζ by ζ_{abc}, where a, b label the τ-matrix basis, while c labels the chirality basis in which σ^3 is diagonal, and a, b, c take values $\pm$, we have,

$$\begin{aligned}
\bar\alpha &= \zeta_{+++} = -\zeta_{--+} = -i\nu\zeta_{+-+} = +i\nu\zeta_{-++} \\
\beta &= \zeta_{---} = +\zeta_{++-} = -i\nu\zeta_{-+-} = -i\nu\zeta_{+--}
\end{aligned} \tag{2.36}$$

2.2.4 The reduced BPS equations in component form

To decompose the reduced BPS equations in a basis of complex frame indices z, $\bar z$, we use (2.11) along with the following basis of γ-matrices compatible with a diagonal σ^3,

$$\gamma^z = \frac{1}{2}(\gamma^8 + i\gamma^9) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad\qquad \gamma^{\bar z} = \frac{1}{2}(\gamma^8 - i\gamma^9) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.37}$$

Using (2.36) the reduced dilatino equations become,

$$\begin{aligned}
4p_z\alpha + g_z\beta &= 0 \\
4p_{\bar z}\bar\beta + g_{\bar z}\bar\alpha &= 0
\end{aligned} \tag{2.38}$$

The gravitino equations which are purely algebraic in $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ are given by,

$$
\begin{aligned}
\frac{1}{2f_2}\bar{\alpha} + \frac{D_z f_2}{2f_2}\beta + \frac{3}{16}g_z\alpha = 0 \\
-\frac{1}{2f_2}\beta + \frac{D_{\bar{z}} f_2}{2f_2}\bar{\alpha} + \frac{3}{16}g_{\bar{z}}\bar{\beta} = 0 \\
\frac{\nu}{2f_6}\bar{\alpha} + \frac{D_z f_6}{2f_6}\beta - \frac{1}{16}g_z\alpha = 0 \\
\frac{\nu}{2f_6}\beta + \frac{D_{\bar{z}} f_6}{2f_6}\bar{\alpha} - \frac{1}{16}g_{\bar{z}}\bar{\beta} = 0
\end{aligned} \tag{2.39}
$$

while the gravitino equations which are differential in $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ are given by,

$$
\begin{aligned}
\left(D_z + \frac{i}{2}\hat{\omega}_z - \frac{i}{2}q_z\right)\bar{\alpha} + \frac{1}{4}g_z\bar{\beta} = 0 \\
\left(D_z - \frac{i}{2}\hat{\omega}_z - \frac{i}{2}q_z\right)\beta + \frac{1}{8}g_z\alpha = 0 \\
\left(D_{\bar{z}} + \frac{i}{2}\hat{\omega}_{\bar{z}} - \frac{i}{2}q_{\bar{z}}\right)\bar{\alpha} + \frac{1}{8}g_{\bar{z}}\bar{\beta} = 0 \\
\left(D_{\bar{z}} - \frac{i}{2}\hat{\omega}_{\bar{z}} - \frac{i}{2}q_{\bar{z}}\right)\beta + \frac{1}{4}g_{\bar{z}}\alpha = 0
\end{aligned} \tag{2.40}
$$

In addition, we have the complex conjugate equations to all of the equations above. Note that since G and P are complex-valued, we have in general $(g_z)^* \neq g_{\bar{z}}$ and $(p_z)^* \neq p_{\bar{z}}$.

2.2.5 Determining the radii f_2, f_6

One may solve for the radii starting from the algebraic gravitino equations of (2.39). Taking the linear combination of the first equation in (2.39) with coefficient $\bar{\beta}$ and the complex conjugate of the second equation α in (2.39) with coefficient $\bar{\alpha}$ on the one hand, and the third equation in (2.39) with coefficient $-\bar{\beta}$ and the complex conjugate of the fourth equation with coefficient $\bar{\alpha}$, we obtain the following equations,

$$
\begin{aligned}
\frac{D_z f_2}{2f_2}(\alpha\bar{\alpha} + \beta\bar{\beta}) &= -\frac{3}{16}g_z\alpha\bar{\beta} - \frac{3}{16}(g_{\bar{z}})^*\bar{\alpha}\beta \\
\frac{D_z f_6}{2f_6}(\alpha\bar{\alpha} - \beta\bar{\beta}) &= -\frac{1}{16}g_z\alpha\bar{\beta} + \frac{1}{16}(g_{\bar{z}})^*\bar{\alpha}\beta
\end{aligned} \tag{2.41}
$$

These combinations suggest that we should evaluate the covariant derivatives $D_z(\alpha\bar{\alpha} \pm \beta\bar{\beta})$ out of the differential equations (2.40) for α, β, $\bar{\alpha}$, $\bar{\beta}$, and we find,

$$
\begin{aligned}
D_z(\alpha\bar{\alpha} + \beta\bar{\beta}) &= -\frac{3}{8} g_z \alpha\bar{\beta} - \frac{3}{8} (g_{\bar{z}})^* \bar{\alpha}\beta \\
D_z(\alpha\bar{\alpha} - \beta\bar{\beta}) &= -\frac{1}{8} g_z \alpha\bar{\beta} + \frac{1}{8} (g_{\bar{z}})^* \bar{\alpha}\beta
\end{aligned} \tag{2.42}
$$

Eliminating all flux dependences between (2.41) and (2.42), we may integrate the resulting relations, to obtain the following expressions for the radii,

$$
\begin{aligned}
f_2 &= c_2(\alpha\bar{\alpha} + \beta\bar{\beta}) \\
f_6 &= c_6(\alpha\bar{\alpha} - \beta\bar{\beta})
\end{aligned} \tag{2.43}
$$

where c_2 and c_6 are integration constants.

2.2.6 Solving the remaining algebraic gravitino equations

To obtain the results of the previous subsection, we have taken only pairwise linear combinations of the algebraic gravitino equations. Here, we take the orthogonally conjugate pairwise linear combinations. To guarantee that the four resulting bilinear equations are equivalent to the original four algebraic gravitino equations, we must have that the determinant of the two linear combinations is $\alpha\bar{\alpha} + \beta\bar{\beta} \neq 0$. Therefore, we multiply the first equation by α and the second by $-\beta$, so that the terms in $D_z f_2$ and $D_z f_6$ cancel out, and we are left with,

$$
\begin{aligned}
\frac{1}{2c_2} + \frac{3}{16} g_z \alpha^2 - \frac{3}{16} (g_{\bar{z}})^* \beta^2 = 0 \\
\frac{\nu}{2c_6} - \frac{1}{16} g_z \alpha^2 + \frac{1}{16} (g_{\bar{z}})^* \beta^2 = 0
\end{aligned} \tag{2.44}
$$

The last equation may be simplified with the help of the first and yields,

$$
c_6 = -3\nu c_2 \tag{2.45}
$$

Recall that ν is allowed to take either value $\nu = \pm 1$, but not both.

2.2.7 Summary and comparison to $AdS_6 \times S^2$

In this subsection, we summarize the remaining reduced BPS equations for the $AdS_2 \times S^6$ case and present the result in parallel with the corresponding results for the remaining reduced BPS equations obtained for the case $AdS_6 \times S^2$ in [26]. To this end we introduce the quantities K and c to distinguish between the two cases as follows,

$$\begin{aligned} K &= i \qquad & c &= \nu c_6 \qquad & &\text{for } AdS_2 \times S^6 \\ K &= 1 \qquad & c &= c_6 \qquad & &\text{for } AdS_6 \times S^2 \end{aligned} \tag{2.46}$$

With the help of these quantities the remaining reduced BPS equations take on a remarkably unified form. The remaining reduced dilatino equations are,

$$\begin{aligned} -4iKp_z\alpha + g_z\beta &= 0 \\ 4p_{\bar z}\bar\beta - iKg_{\bar z}\bar\alpha &= 0 \end{aligned} \tag{2.47}$$

along with their complex conjugates. The radii in terms of the spinors α, β are given by,

$$\begin{aligned} f_2 &= -\frac{\nu}{3}c_6(\alpha\bar\alpha - K^2\beta\bar\beta) \\ f_6 &= c_6(\alpha\bar\alpha + K^2\beta\bar\beta) \end{aligned} \tag{2.48}$$

The remaining algebraic relation between the spinors and the fluxes is given by,

$$\frac{K}{2c} - \frac{i}{16}g_z\alpha^2 + \frac{i}{16}(g_{\bar z})^*\beta^2 = 0 \tag{2.49}$$

along with its complex conjugate. The remaining differential equations on the spinors are,

$$\begin{aligned} \left(D_z - \frac{i}{2}\hat\omega_z + \frac{i}{2}q_z\right)\alpha + \frac{i}{8K}(g_{\bar z})^*\beta &= 0 \\ \left(D_z - \frac{i}{2}\hat\omega_z - \frac{i}{2}q_z\right)\beta + \frac{i}{8K}g_z\alpha &= 0 \\ \left(D_z + \frac{i}{2}\hat\omega_z - \frac{i}{2}q_z\right)\bar\alpha - \frac{iK}{4}g_z\bar\beta &= 0 \\ \left(D_z + \frac{i}{2}\hat\omega_z + \frac{i}{2}q_z\right)\bar\beta - \frac{iK}{4}(g_{\bar z})^*\bar\alpha &= 0 \end{aligned} \tag{2.50}$$

along with their complex conjugates.

2.3 Local solutions to the BPS equations

The BPS equations have been reduced to the $AdS_2 \times S^6 \times \Sigma$ Ansatz and solved for the radii f_2 and f_6 in the previous section. In this section, the remaining equations, namely the dilatino BPS equations of (2.47), the remaining algebraic gravitino equation of (2.49) and the four differential equations (2.50), will be completely solved locally on Σ in terms of two locally holomorphic functions $\mathcal{A}_\pm$ on Σ. Thus we will obtain expressions for all bosonic supergravity fields that satisfy the BPS equations in terms of $\mathcal{A}_\pm$.

2.3.1 Eliminating the reduced flux fields

We start from the equations summarized in sec. 2.2.7 and keep K and c as defined in (2.46) for easier comparison with the $AdS_6 \times S^2$ case. We begin by eliminating the reduced flux fields g_z, $g_{\bar z}$ and their complex conjugates in favor of p_z, $p_{\bar z}$ and their complex conjugates using the dilatino BPS equations of (2.47). The algebraic relation (2.49) becomes,

$$p_z \frac{\alpha^3}{\beta} - (p_{\bar z})^* \frac{\beta^3}{\alpha} + \frac{2}{c} = 0 \tag{2.51}$$

The differential equations (2.50) take the following form,

$$\begin{aligned}
\left(D_z - \frac{i}{2}\hat\omega_z + \frac{i}{2}q_z\right)\alpha - \frac{1}{2}(p_{\bar z})^* \frac{\beta^2}{\alpha} &= 0 \\
\left(D_z - \frac{i}{2}\hat\omega_z - \frac{i}{2}q_z\right)\beta - \frac{1}{2}p_z \frac{\alpha^2}{\beta} &= 0 \\
\left(D_z + \frac{i}{2}\hat\omega_z - \frac{i}{2}q_z\right)\bar\alpha + K^2 p_z \frac{\alpha\bar\beta}{\beta} &= 0 \\
\left(D_z + \frac{i}{2}\hat\omega_z + \frac{i}{2}q_z\right)\bar\beta + K^2 (p_{\bar z})^* \frac{\bar\alpha\beta}{\alpha} &= 0
\end{aligned} \tag{2.52}$$

Equations (2.51) and (2.52) are the remaining relations to be solved. Their solution will give $\alpha, \beta, f_2, f_6, \rho, p_z$ and therefore B as well as the flux field $g_z, g_{\bar z}$ and their complex conjugates via the reduced dilatino equations.

2.3.2 Integrating the first pair of differential equations

Next, we use the expressions for $p_z, p_{\bar{z}}, q_z$ and their complex conjugates in terms of the axion-dilation field B via the relations (2.2) and (2.10) to solve the first two equations of (2.52) in terms of holomorphic functions. To do so, we multiply the first equation of (2.52) by α and the second equation of (2.52) by β, switch to conformally flat complex coordinates $(w, \bar{w})$ on Σ as introduced in (2.13), and use (2.2) and (2.10) to express p_z and q_z in terms of B,

$$\begin{aligned}\partial_w(\rho\alpha^2) &= -\frac{1}{2}f^2(B\partial_w\bar{B} - \bar{B}\partial_w B)\rho\alpha^2 + f^2(\partial_w\bar{B})\rho\beta^2 \\ \partial_w(\rho\beta^2) &= +\frac{1}{2}f^2(B\partial_w\bar{B} - \bar{B}\partial_w B)\rho\beta^2 + f^2(\partial_w B)\rho\alpha^2\end{aligned} \tag{2.53}$$

By taking suitable linear combinations we obtain the following equivalent equations,

$$\begin{aligned}\partial_w\left(\ln\{\rho(\alpha^2 - \bar{B}\beta^2)\} + \ln f\right) &= 0 \\ \partial_w\left(\ln\{\rho(B\alpha^2 - \beta^2)\} + \ln f\right) &= 0\end{aligned} \tag{2.54}$$

These equations are solved in terms of two independent holomorphic 1-forms $\kappa_\pm$, as follows,

$$\begin{aligned}\rho f\left(\alpha^2 - \bar{B}\beta^2\right) &= \bar{\kappa}_+ \\ \rho f\left(\beta^2 - B\alpha^2\right) &= \bar{\kappa}_-\end{aligned} \tag{2.55}$$

Inverting (2.55), we obtain the spinor components α, β, and their complex conjugates $\bar{\alpha}$, $\bar{\beta}$,

$$\begin{aligned}\rho\alpha^2 &= f(\bar{\kappa}_+ + \bar{B}\bar{\kappa}_-) \qquad & \rho\bar{\alpha}^2 &= f(\kappa_+ + B\kappa_-) \\ \rho\beta^2 &= f(B\bar{\kappa}_+ + \bar{\kappa}_-) \qquad & \rho\bar{\beta}^2 &= f(\bar{B}\kappa_+ + \kappa_-)\end{aligned} \tag{2.56}$$

The right side of all four equations involves only the holomorphic data $\kappa_\pm$ and the B-field and their complex conjugates. It remains to solve for the fields ρ and B.

2.3.3 Preparing the second pair of differential equations

Next, we express the third and fourth equation of (2.52) in terms of B, ρ and the local complex coordinates $(w, \bar{w})$, and obtain the following set of equations,

$$
\begin{aligned}
\left(\partial_w - \partial_w \ln \rho^2 - f^2 B \partial_w \bar{B}\right)(f\rho\bar{\alpha}^2) + 2K^2 f^2 (\partial_w B) f\rho \frac{\alpha\bar{\alpha}\bar{\beta}}{\beta} &= 0 \\
\left(\partial_w - \partial_w \ln \rho^2 - f^2 \bar{B} \partial_w B\right)(f\rho\bar{\beta}^2) + 2K^2 f^2 (\partial_w \bar{B}) f\rho \frac{\bar{\alpha}\beta\bar{\beta}}{\alpha} &= 0
\end{aligned}
\tag{2.57}
$$

The spinors α, β and their complex conjugates, as well as the derivatives of $f\rho\bar{\alpha}^2$ and $f\rho\bar{\beta}^2$ may be evaluated in terms of B, ρ, and $\kappa_\pm$ and their first derivatives using (2.52). After some simplifications, we obtain the following equivalent system of equations,

$$
\begin{aligned}
\partial_w \ln \rho^2 - f^2(\partial_w \bar{B}) \frac{\kappa_+ + B\kappa_-}{\bar{B}\kappa_+ + \kappa_-} - 2K^2 f^2 (\partial_w \bar{B}) e^{+i\vartheta} &= \frac{\bar{B}\partial_w \kappa_+ + \partial_w \kappa_-}{\bar{B}\kappa_+ + \kappa_-} \\
\partial_w \ln \rho^2 - f^2(\partial_w B) \frac{\bar{B}\kappa_+ + \kappa_-}{\kappa_+ + B\kappa_-} - 2K^2 f^2 (\partial_w B) e^{-i\vartheta} &= \frac{\partial_w \kappa_+ + B\partial_w \kappa_-}{\kappa_+ + B\kappa_-}
\end{aligned}
\tag{2.58}
$$

where we have used the following abbreviation for the phase angle ϑ,

$$
e^{i\vartheta} = \frac{\bar{\alpha}\beta}{\alpha\bar{\beta}} = \frac{(\kappa_+ + B\kappa_-)|B\bar{\kappa}_+ + \bar{\kappa}_-|}{|\bar{\kappa}_+ + \bar{B}\bar{\kappa}_-|(\bar{B}\kappa_+ + \kappa_-)}
\tag{2.59}
$$

The dependence of the algebraic relation (2.51) on α and β may also be eliminated using (2.56), while $p_z, p_{\bar{z}}$ may be expressed in terms of B, and we find the equivalent relation,

$$
f^3(\partial_w B) \frac{(\bar{\kappa}_+ + \bar{B}\bar{\kappa}_-)^{\frac{3}{2}}}{(B\bar{\kappa}_+ + \bar{\kappa}_-)^{\frac{1}{2}}} - f^3(\partial_w \bar{B}) \frac{(B\bar{\kappa}_+ + \bar{\kappa}_-)^{\frac{3}{2}}}{(\bar{\kappa}_+ + \bar{B}\bar{\kappa}_-)^{\frac{1}{2}}} + \frac{2\rho^2}{c} = 0
\tag{2.60}
$$

Equations (2.58) and (2.60) are supplemented by their complex conjugates.

Although on the face of it the remaining equations (2.58) and (2.60) depend on both $\kappa_\pm$ and their complex conjugates, the conformal invariance of these equations tells us that the dependence is actually only through the combination ρ^2/κ_- and the ratio,

$$
\lambda = \frac{\kappa_+}{\kappa_-}
\tag{2.61}
$$

In terms of these variables, the equations take the following form,

$$
\begin{aligned}
\partial_w \ln \frac{\rho^2}{\kappa_-} - f^2(\partial_w \bar{B}) \frac{\lambda + B}{\bar{B}\lambda + 1} - 2K^2 f^2 (\partial_w \bar{B}) e^{+i\vartheta} &= \frac{\bar{B}\partial_w \lambda}{\bar{B}\lambda + 1} \\
\partial_w \ln \frac{\rho^2}{\kappa_-} - f^2(\partial_w B) \frac{\bar{B}\lambda + 1}{\lambda + B} - 2K^2 f^2 (\partial_w B) e^{-i\vartheta} &= \frac{\partial_w \lambda}{\lambda + B}
\end{aligned}
\tag{2.62}
$$

where we now have,

$$e^{i\vartheta} = \frac{(\lambda + B)|B\bar{\lambda} + 1|}{|\bar{\lambda} + \bar{B}|(\bar{B}\lambda + 1)} \tag{2.63}$$

as well as the equation resulting from (2.60),

$$\frac{(\bar{\lambda} + \bar{B})^{\frac{3}{2}}}{(B\bar{\lambda} + 1)^{\frac{1}{2}}} f^3 \partial_w B - \frac{(B\bar{\lambda} + 1)^{\frac{3}{2}}}{(\bar{\lambda} + \bar{B})^{\frac{1}{2}}} f^3 \partial_w \bar{B} + \frac{2\rho^2}{c\,\bar{\kappa}_-} = 0 \tag{2.64}$$

In summary, we have prepared the remaining reduced BPS equations in the form of complex differential equations (2.62) and (2.64) along with their complex conjugate equations.

2.3.4 Decoupling by changing variables

In this subsection, we will perform two consecutive changes of variables to decouple the remaining equations.

2.3.4.1 First change of variables, from B to Z

A first change of variables replaces B by a complex field Z and is designed to parametrize the phase $e^{i\vartheta}$ in (2.63) without the square root required from its definition. We make the following rational change of variables to eliminate B in terms of a complex function Z,

$$Z^2 = \frac{\lambda + B}{B\bar{\lambda} + 1} \qquad\qquad B = \frac{Z^2 - \lambda}{1 - \bar{\lambda} Z^2} \tag{2.65}$$

which will allow us to express $e^{i\vartheta}$ and f^2 as rational functions of Z and its complex conjugate,

$$e^{i\vartheta} = \frac{Z}{\bar{Z}} \left(\frac{1 - \lambda \bar{Z}^2}{1 - \bar{\lambda} Z^2} \right) \qquad\qquad f^2 = \frac{(1 - \lambda \bar{Z}^2)(1 - \bar{\lambda} Z^2)}{(1 - |\lambda|^2)(1 - |Z|^4)} \tag{2.66}$$

The equations (2.62) now take the form,

$$\begin{aligned}
\partial_w \ln \frac{\rho^2}{\kappa_-} &= \frac{2Z^2\bar{Z} + 4K^2 Z}{1 - |Z|^4} \partial_w \bar{Z} + \frac{\bar{Z}^2 - \bar{\lambda} + 2K^2 Z\bar{Z}^3 - 2K^2 Z\bar{Z}\bar{\lambda}}{(1 - |\lambda|^2)(1 - |Z|^4)} \partial_w \lambda \\
\partial_w \ln \frac{\rho^2}{\kappa_-} &= \frac{2Z^{-1} + 4K^2 \bar{Z}}{1 - |Z|^4} \partial_w Z + \frac{Z^2\bar{Z}^2\bar{\lambda} - \bar{Z}^2 - 2K^2 \bar{Z} Z^{-1} + 2K^2 Z\bar{Z}\bar{\lambda}}{(1 - |\lambda|^2)(1 - |Z|^4)} \partial_w \lambda
\end{aligned} \tag{2.67}$$

Taking the difference of these two equations eliminates the dependence on ρ^2/κ_-,

$$(2 + 4K^2|Z|^2)\partial_w Z - Z^2(4K^2 + 2|Z|^2)\partial_w \bar{Z} \\ = \frac{2\bar{Z}(K^2 + |Z|^2 + K^2|Z|^4) - \bar{\lambda}Z(1 + 4K^2|Z|^2 + |Z|^4)}{1 - |\lambda|^2}\partial_w \lambda \tag{2.68}$$

while taking their sum gives,

$$\partial_w \ln \hat{\rho}^2 = \frac{1}{2}\partial_w \ln \frac{Z}{\bar{Z}} - K^2 \frac{\bar{Z}}{Z}\left(\frac{\partial_w \lambda}{1 - |\lambda|^2}\right) \tag{2.69}$$

where we have changed variables from ρ to $\hat{\rho}$ in the following way,

$$\hat{\rho}^2 = \frac{\rho^2}{c\,\kappa_- \bar{\kappa}_-}\frac{|1 - Z^2\bar{\lambda}|(1 - K^2|Z|^2)}{f|Z|(1 - |\lambda|^2)(1 + K^2|Z|^2)} \tag{2.70}$$

Finally, eliminating B in favor of Z in the algebraic flux equation (2.64) as well, we obtain,

$$(1 - |\lambda|^2)\partial_w \left(\frac{Z^2 + \bar{Z}^{-2}}{1 - |\lambda|^2}\right) - \frac{2\partial_w \lambda}{1 - |\lambda|^2} + 2\hat{\rho}^2 \kappa_- \frac{|Z|}{\bar{Z}^3}(1 + K^2|Z|^2)^2 = 0 \tag{2.71}$$

It remains to solve the system of equations (2.68), (2.69), and (2.71).

2.3.4.2 Second change of variables, from Z to R, ψ

A second change of variables is inspired by the form of equation (2.69), in which the norm of Z and its phase enter in distinct parts of the equation. We express the complex field Z in terms of two real variables, its absolute value R and phase ψ, as follows,

$$Z^2 = R\,e^{i\psi} \tag{2.72}$$

In terms of these variables (2.69) takes the form,

$$\partial_w \ln \hat{\rho}^2 - \frac{i}{2}\partial_w \psi + K^2 e^{-i\psi}\frac{\partial_w \lambda}{1 - |\lambda|^2} = 0 \tag{2.73}$$

while (2.68) becomes,

$$(1 - R^2)\frac{\partial_w R}{R} + (1 + 4K^2 R + R^2)\left(i\partial_w \psi + \frac{\bar{\lambda}\partial_w \lambda}{1 - |\lambda|^2}\right) - \frac{2K^2 e^{-i\psi}(1 + K^2 R + R^2)}{1 - |\lambda|^2}\partial_w \lambda = 0 \tag{2.74}$$

and (2.71) becomes,

$$(R^2-1)\frac{\partial_w R}{R}+(R^2+1)\left(i\partial_w\psi+\frac{\bar{\lambda}\partial_w\lambda}{1-|\lambda|^2}\right)-\frac{2R\partial_w\lambda}{1-|\lambda|^2}e^{-i\psi}+2\hat{\rho}^2\kappa_-e^{i\psi/2}(1+K^2R)^2=0 \quad (2.75)$$

The three equations (2.73), (2.74), and (2.75) are the basic starting point for the complete solution of the full system of reduced BPS equations.

2.3.4.3 Decoupling the equations for ψ and $\hat{\rho}^2$

Adding equations (2.74) and (2.75) cancels the terms proportional to $\partial_w R$, and concentrates the entire R-dependence of this sum in an overall multiplicative factor of $(1+K^2R)^2$. Omitting this factor, the sum becomes,

$$2i\partial_w\psi+\frac{2\bar{\lambda}\partial_w\lambda}{1-|\lambda|^2}-2K^2e^{-i\psi}\frac{\partial_w\lambda}{1-|\lambda|^2}+2\hat{\rho}^2\kappa_-e^{i\psi/2}=0 \quad (2.76)$$

Equations (2.73) and (2.76) involve only ψ and $\hat{\rho}^2$ but not R. Up to factors of K^2, this system is the same as the system in [26], and we will solve it with the same methods. Adding twice (2.73) to (2.76) eliminates the term proportional to $e^{-i\psi}$, and we obtain,

$$\partial_w\ln\hat{\rho}^2+\frac{i}{2}\partial_w\psi-\partial_w\ln(1-|\lambda|^2)+\hat{\rho}^2\kappa_-e^{i\psi/2}=0 \quad (2.77)$$

Clearly, this equation involves only the following specific complex combination of $\hat{\rho}^2$ and ψ,

$$K\xi=\hat{\rho}^{-2}e^{-i\psi/2} \quad (2.78)$$

where we have included a factor of K in the definition of ξ for later convenience. In terms of ξ we may express (2.77) as follows,

$$K\partial_w\left(\xi(1-|\lambda|^2)\right)=\kappa_-(1-|\lambda|^2)=\kappa_--\kappa_+\bar{\lambda} \quad (2.79)$$

where we have used the relation $\kappa_+=\lambda\kappa_-$. The integrable structure of the system of equations (2.68), (2.69), and (2.71) has therefore been exposed clearly with the help of this sequence of changes of variables. Indeed, equation (2.79) involves only the field ξ, which is the combination of $\hat{\rho}$ and ψ entering (2.78). Having obtained ξ, equation (2.77) may be solved for $\hat{\rho}$ and ψ. Finally, having $\hat{\rho}$ and ψ, equation (2.74) becomes an equation for R only, and we will see below that it can be solved as well.

2.3.5 Solving for ψ, $\hat{\rho}^2$, and R in terms of $\mathcal{A}_\pm$

Having decoupled the reduced BPS equations in the preceding subsection, we will solve the decoupled equations in the present section. We begin by solving (2.79) for ξ, then obtain ψ, $\hat{\rho}^2$, and R as described above. We introduce locally holomorphic functions $\mathcal{A}_\pm$ such that,

$$\kappa_\pm = K\partial_w \mathcal{A}_\pm \qquad\qquad \lambda = \frac{\partial_w \mathcal{A}_+}{\partial_w \mathcal{A}_-} \tag{2.80}$$

Given the one-forms $\kappa_\pm$, the functions $\mathcal{A}_\pm$ are unique up to an additive constant for each function.

With the conventions used to define ξ and $\mathcal{A}_\pm$, the equations governing ξ in terms of $\mathcal{A}_\pm$ are identical to those of the $AdS_6 \times S^2$ case, and we import their solution from [26],

$$\xi = \frac{\mathcal{L}}{1-|\lambda|^2} \qquad\qquad \mathcal{L} = \mathcal{A}_- - \bar{\mathcal{A}}_+ + \bar{\lambda}(\bar{\mathcal{A}}_- - \mathcal{A}_+) \tag{2.81}$$

Note that $\hat{\rho}$ and ψ are directly determined by ξ using equation (2.78).

To solve for R, we begin with equation (2.74) before using (2.73) to eliminate the term proportional to $e^{-i\psi}$. We then divide the resulting equation by R, and find,

$$\begin{aligned} 0 = \left(\frac{1}{R^2} - 1\right)\partial_w R + \left(R + \frac{1}{R} + 4K^2\right)\left(i\partial_w \psi - \partial_w \ln(1-|\lambda|^2)\right) \\ + \left(R + \frac{1}{R} + K^2\right)\left(2\partial_w \ln \hat{\rho}^2 - i\partial_w \psi\right) \end{aligned} \tag{2.82}$$

Changing variable from R to a new variable W, which we conveniently define by,

$$K^2 W = R + \frac{1}{R} \tag{2.83}$$

renders equation (2.82) linear in W with an inhomogeneous part,

$$\partial_w W - 2(W+1)\partial_w \ln \hat{\rho}^2 + (W+1)\partial_w \ln(1-\lambda\bar{\lambda}) = 3i\partial_w \psi - 3\partial_w \ln(1-\lambda\bar{\lambda}) \tag{2.84}$$

We note that this equation is now independent of K^2 and therefore coincides with the corresponding equation for the $AdS_6 \times S^2$ case, whose solution we import from [26],

$$W = 2 + \frac{6\kappa^2\,\mathcal{G}}{|\partial_w \mathcal{G}|^2} \tag{2.85}$$

where we have defined κ^2 and $\mathcal{G}$ through,

$$\begin{aligned} \kappa^2 &= -|\partial_w \mathcal{A}_+|^2 + |\mathcal{A}_-|^2 \\ \mathcal{G} &= |\mathcal{A}_+|^2 - |\mathcal{A}_-|^2 + \mathcal{B} + \bar{\mathcal{B}} \\ \partial_w \mathcal{B} &= \mathcal{A}_+ \partial_w \mathcal{A}_- - \mathcal{A}_- \partial_w \mathcal{A}_+ \end{aligned} \tag{2.86}$$

Since $\partial_w \mathcal{B}$ is a holomorphic 1-form, there exists a locally holomorphic function $\mathcal{B}$, defined up to the addition of an arbitrary complex constant. This completes the solution of the decoupled reduced BPS equations for the fields ψ, $\hat{\rho}$, and R.

2.4 Supergravity fields of the local solutions

The general local half-BPS solution to Type IIB supergravity with $SO(2,1) \oplus SO(7)$ symmetry can now be expressed in terms of the locally holomorphic functions $\mathcal{A}_\pm$ introduced above. Here we will translate from the local solution of the reduced BPS equations to the supergravity fields and discuss some of the immediate properties of the solutions.

For comparison we present the $AdS_6 \times S^2$ and $AdS_2 \times S^6$ cases in parallel. The five-form field strength and all fermion fields vanish. The remaining bosonic fields in both cases are distinguished merely by the parameter $\Lambda = K^2$. The metric Ansatz and Λ are given by,

$$\begin{aligned} ds^2 &= f_2^2 ds^2_{AdS_2} + f_6^2 ds^2_{S^6} + ds^2_\Sigma \qquad &&\text{for } AdS_2 \times S^6 \qquad &&\Lambda = -1 \\ ds^2 &= f_6^2 ds^2_{AdS_6} + f_2^2 ds^2_{S^2} + ds^2_\Sigma \qquad &&\text{for } AdS_6 \times S^2 \qquad &&\Lambda = +1 \end{aligned} \tag{2.87a}$$

The remaining fields in both cases are given by,

$$ds^2_\Sigma = 4\rho^2 dw d\bar{w} \qquad\qquad F_{(3)} = d\mathcal{C} \wedge \widehat{\mathrm{vol}}_2 \tag{2.87b}$$

where $\widehat{\mathrm{vol}}_2$ is the volume form on AdS_2 of unit radius for $\Lambda = -1$ and the volume form of S^2 with unit radius for the case $\Lambda = 1$. The metric functions f_2, f_6, ρ, and $\mathcal{C}$ and the dilaton-axion field B are all functions on Σ.

2.4.1 The metric functions

The metric functions f_2, f_6, and ρ are naturally expressed in terms of composite quantities κ^2 and $\mathcal{G}$ defined in (2.86), and the function R obtained by eliminating W between equations (2.83) and (2.85). The latter is given in terms of κ^2 and $\mathcal{G}$ by,

$$\Lambda R + \frac{1}{\Lambda R} = 2 + 6\frac{\kappa^2 \mathcal{G}}{|\partial_w \mathcal{G}|^2} \tag{2.88}$$

To obtain the explicit expressions for the metric functions, we begin by eliminating α and β from the combinations $(f_6 \pm \frac{3}{\nu} f_2)^2$ in favor of $\kappa_\pm$ and f using (2.48) and (2.56), and we find,

$$(f_6 + \frac{3}{\nu} f_2)^2 = \frac{4c^2 f^2}{\rho^2} |\kappa_-|^2 |B\bar{\lambda} + 1|^2 \qquad (f_6 - \frac{3}{\nu} f_2)^2 = \frac{4c^2 f^2}{\rho^2} |\kappa_-|^2 |\lambda + B|^2 \tag{2.89}$$

Changing variables from B to Z using (2.65) and (2.66), solving for f_2 and f_6, and expressing the result in terms of $|Z|^2 = R$, we obtain,

$$f_2^2 = \frac{c^2 \kappa^2 (1 - \Lambda R)}{9\rho^2 (1 + \Lambda R)} \qquad f_6^2 = \frac{c^2 \kappa^2 (1 + \Lambda R)}{\rho^2 (1 - \Lambda R)} \tag{2.90}$$

To calculate ρ^2 we express the result of (2.70) for ρ^2 in terms of $\hat{\rho}^2$, use (2.78) to obtain $\hat{\rho}^2$ in terms of ξ, and (2.81) to express ξ in terms of $\mathcal{L}$, which in turn is given by,

$$\hat{\rho}^4 = \frac{1}{\xi \bar{\xi}} = \frac{(1 - \lambda \bar{\lambda})^2}{\mathcal{L} \bar{\mathcal{L}}} \qquad \bar{\mathcal{L}} \partial_w \mathcal{A}_- = -\partial_w \mathcal{G} \tag{2.91}$$

Expressing the result in terms of R, $\mathcal{G}$ and κ^2, we find,

$$\rho^2 = \frac{c(\kappa^2)^{\frac{3}{2}}}{|\partial_w \mathcal{G}|} \frac{(R + \Lambda R^2)^{\frac{1}{2}}}{(1 - \Lambda R)^{\frac{3}{2}}} \tag{2.92}$$

Alternatively, after making use of (2.88), and eliminating ρ^2 from f_2^2 and f_6^2, we have,

$$f_2^2 = \frac{c}{9}\sqrt{6\Lambda \mathcal{G}} \left(\frac{1 - \Lambda R}{1 + \Lambda R} \right)^{\frac{3}{2}} \quad f_6^2 = c\sqrt{6\Lambda \mathcal{G}} \left(\frac{1 + \Lambda R}{1 - \Lambda R} \right)^{\frac{1}{2}} \quad \rho^2 = \frac{c\kappa^2}{\sqrt{6\Lambda \mathcal{G}}} \left(\frac{1 + \Lambda R}{1 - \Lambda R} \right)^{\frac{1}{2}} \tag{2.93}$$

Some care will be needed with the choice of the branch of the square root, which will be discussed in detail in section 2.6.

2.4.2 The axion-dilaton

The axion-dilaton field B is obtained using its expression of (2.65) in terms of Z, eliminating Z in favor of R and ψ using (2.72), and eliminating ψ in favor of ξ and $\mathcal{L}$ using (2.81),

$$B = \frac{Re^{i\psi} - \lambda}{1 - \bar{\lambda} Re^{i\psi}} = \frac{K^2 R\bar{\mathcal{L}} - \lambda \mathcal{L}}{\mathcal{L} - \bar{\lambda} K^2 R\bar{\mathcal{L}}} \tag{2.94}$$

Multiplying numerator and denominator by $|\kappa_-|^2$ and using (2.91) to eliminate $\mathcal{L}$ yields,

$$B = \frac{\partial_w \mathcal{A}_+ \partial_{\bar{w}} \mathcal{G} - \Lambda R\, \partial_{\bar{w}} \bar{\mathcal{A}}_- \partial_w \mathcal{G}}{\Lambda R\, \partial_{\bar{w}} \bar{\mathcal{A}}_+ \partial_w \mathcal{G} - \partial_w \mathcal{A}_- \partial_{\bar{w}} \mathcal{G}} \tag{2.95}$$

One verifies that this field automatically satisfies $|B| < 1$ provided $\kappa^2(1 - R^2) > 0$.

2.4.3 Two-form and six-form flux potentials

The evaluation of the two-form flux potential $C_{(2)}$ and of its magnetic dual six-form flux potential $C_{(6)}$ on the solutions to the BPS equations is considerably more involved than for the other supergravity fields. Here we shall summarize the result, and relay an account of the detailed calculations to Appendix C. As a byproduct, the calculations of the flux potentials will prove that the solutions to the BPS equations for 16 supersymmetries together with the Bianchi identities for the P, Q one-forms, imply the Bianchi identity and field equation for the three-form field G.

Consider the three-form $F_{(3)}$ and dual seven-form $F_{(7)}$ field strengths defined by,

$$\begin{aligned}
F_{(3)} &= f(G + B\bar{G}) \\
F_{(7)} &= \star f(G - B\bar{G}) + \frac{4i}{3}\left(2C_{(4)} \wedge F_{(3)} - F_{(5)} \wedge C_{(2)}\right)
\end{aligned} \tag{2.96}$$

where $\star$ denotes the Poincaré dual. It is a standard result that the Bianchi identity for the field $F_{(3)}$ is given by $dF_{(3)} = 0$. By inverting the relation between $F_{(3)}$ and G one deduces the well-known Bianchi identity for G, which takes the form,

$$dG - iQ \wedge G + P \wedge \bar{G} = 0 \tag{2.97}$$

where P and Q are given in terms of B by (2.2). The field equation for G is equivalent to the condition $dF_{(7)} = 0$. Here we shall be interested only in solutions to the field equations for which $F_{(5)} = C_{(2)} \wedge \bar{F}_{(3)} = 0$, so that the field equation for G reduces to,

$$\nabla^P \left(f(G_{MNP} - B\bar{G}_{MNP}) \right) = 0 \tag{2.98}$$

The closure conditions on the three-form $F_{(3)}$ and on the seven-form $F_{(7)}$ may be solved locally in terms of flux potentials $C_{(2)}$ and $C_{(6)}$ by,

$$\begin{aligned} dC_{(2)} &= f(G + B\bar{G}) \\ dC_{(6)} &= \star f(G - B\bar{G}) \end{aligned} \tag{2.99}$$

In view of the $SO(2,1) \oplus SO(7)$ isometry algebra of $AdS_2 \times S^6$ and the $SO(5,2) \oplus SO(3)$ isometry algebra of $AdS_6 \times S^2$, we have the following Ansatz for $C_{(2)}, C_{(6)}$ and G,

$$\begin{aligned} C_{(2)} &= \mathcal{C}\, \widehat{\text{vol}}_2 & \qquad G &= g_a e^a \wedge \text{vol}_2 \\ C_{(6)} &= \mathcal{M}\, \widehat{\text{vol}}_6 \end{aligned} \tag{2.100}$$

where $\widehat{\text{vol}}_2$ and $\widehat{\text{vol}}_6$ denote the volume forms of the maximally symmetric spaces of unit radius respectively of the two-dimensional and six-dimensional factors of the spacetime. Integrating these equations for our solutions results in the following flux potentials,

$$\begin{aligned} \mathcal{C} &= \frac{4ic}{9}\Lambda \left\{ \frac{\partial_{\bar{w}}\bar{\mathcal{A}}_- \partial_w \mathcal{G}}{\kappa^2} - 2\Lambda R \frac{(\partial_{\bar{w}}\mathcal{G}\partial_w \mathcal{A}_+ + \partial_w \mathcal{G}\partial_{\bar{w}}\bar{\mathcal{A}}_-)}{(1+\Lambda R)^2\kappa^2} - \bar{\mathcal{A}}_- - 2\mathcal{A}_+ \right\} \\ \mathcal{M} &= -24c^3\mathcal{G}\left(\frac{\partial_w \mathcal{G}\partial_{\bar{w}}\bar{\mathcal{A}}_-}{\kappa^2} + 3\bar{\mathcal{A}}_- + 2\mathcal{A}_+ \right) + 80c^3(\mathcal{W}_+ + \bar{\mathcal{W}}_-) \\ &\quad +40c^3(\mathcal{A}_+ + \bar{\mathcal{A}}_-)(|\mathcal{A}_+|^2 - |\mathcal{A}_-|^2) \end{aligned} \tag{2.101}$$

where $\mathcal{W}_\pm$ are locally holomorphic functions defined up to a constant by $\mathcal{A}_\pm \partial_w \mathcal{B} = \partial_w \mathcal{W}_\pm$.

2.4.4 $SU(1,1)$ transformations induced on the supergravity fields

The action of the global $SU(1,1)$ symmetry of Type IIB supergravity on the supergravity fields, as given in (2.3), is induced by an action of $SU(1,1) \otimes \mathbb{C}$ on $\mathcal{A}_\pm$, in parallel with the

$AdS_6 \times S^2$ case of [26],

$$\begin{aligned} \mathcal{A}_+ &\to \mathcal{A}'_+ = +u\mathcal{A}_+ - v\mathcal{A}_- + a \\ \mathcal{A}_- &\to \mathcal{A}'_- = -\bar{v}\mathcal{A}_+ + \bar{u}\mathcal{A}_- + \bar{a} \end{aligned} \tag{2.102}$$

where we have parametrized $SU(1,1)$ by $u, v \in \mathbb{C}$ with $|u|^2 - |v|^2 = 1$ and a is a complex constant. The transformation of $\mathcal{B}$ is given by,

$$\mathcal{B} \to \mathcal{B}' = \mathcal{B} + a\mathcal{A}'_- - \bar{a}\mathcal{A}'_+ \tag{2.103}$$

These transformations leave κ^2 and $\mathcal{G}$ and consequently also the metric functions invariant. The condition $F_{(5)} = 0$ is also left invariant. They transform B as given in (2.3), while $\mathcal{C}$ and $\mathcal{M}$ transform as follows,

$$\begin{aligned} \mathcal{C} &\to \mathcal{C}' = u\mathcal{C} + v\bar{\mathcal{C}} - \mathcal{C}_0 \\ \mathcal{M} &\to \mathcal{M}' = u\mathcal{M} - v\bar{\mathcal{M}} + \mathcal{M}_0 \end{aligned} \tag{2.104}$$

Consistently with the $SU(1,1)$ action of (2.3) and (2.4), $F_{(3)}$ and $F_{(7)}$ transform as follows,

$$\begin{aligned} F_{(3)} &\to uF_{(3)} + v\bar{F}_{(3)} \\ F_{(7)} &\to uF_{(7)} - v\bar{F}_{(7)} \end{aligned} \tag{2.105}$$

where the first transformation law follows from the second equation in (2.3) and $F_{(3)} = dC_{(2)}$.

2.5 Verifying the field equations

Whether the BPS equations for 16 residual supersymmetries imply the full set of Bianchi identities and field equations for the form fields $P, Q, G, F_{(5)}$, the spacetime metric, and the spin connection is, in general, an open problem. In our solution of the BPS equations, we have assumed the expression for the spin connection in terms of the metric, and we have assumed that the Bianchi identities for P, Q, given by,

$$\begin{aligned} 0 &= dP - 2iQ \wedge P \\ 0 &= dQ + iP \wedge \bar{P} \end{aligned} \tag{2.106}$$

have been solved in terms of the axion-dilaton field B by the first two equations in (2.2). The Bianchi identity for the $F_{(5)}$ field is trivially satisfied since in our solutions we have $F_{(5)} = 0$ as well as $G \wedge \bar{G} = 0$. But the Bianchi identity for the field G in (2.97) was not assumed from the outset and instead has been shown in subsection 2.4.3 to result from the solution to the BPS equations and the Bianchi identity for P, Q.

In this section we show that the field equations of Type IIB supergravity are obeyed for the general local solution obtained in section 2.4. We continue to treat the $AdS_2 \times S^6 \times \Sigma$ and $AdS_6 \times S^2 \times \Sigma$ cases in parallel, and establish the Type IIB field equations for both cases. In particular, we verify the field equations for the warped AdS_6 solutions obtained in [26], a result that was not completely obtained in that paper. The full Type IIB supergravity field equations for the bosonic fields are [36, 37],

$$\begin{aligned}
0 &= \nabla^M P_M - 2iQ^M P_M + \frac{1}{24} G_{MNP} G^{MNP} \\
0 &= \nabla^P G_{MNP} - iQ^P G_{MNP} - P^P \bar{G}_{MNP} + \frac{2}{3} i F_{(5)MNPQR} G^{PQR} \\
0 &= R_{MN} - P_M \bar{P}_N - \bar{P}_M P_N - \frac{1}{6}(F^2_{(5)})_{MN} \\
&\qquad - \frac{1}{8}(G_M{}^{PQ} \bar{G}_{NPQ} + \bar{G}_M{}^{PQ} G_{NPQ}) + \frac{1}{48} g_{MN} G^{PQR} \bar{G}_{PQR}
\end{aligned} \tag{2.107}$$

To show that these equations are satisfied, we start with Einstein's equations and then turn to the field equations for B and the three-form flux. We will need the components of the Ricci tensor, which are derived for general $AdS_p \times S^q \times \Sigma$ warped products in Appendix D. We will use the labels p and q for the dimensions of the AdS and S parts of the geometry throughout this section, as well as f_A and f_S for their respective radii. The general procedure for verifying the field equations is to reduce them to a form where they only involve quantities for which we have given explicit expressions in terms of the holomorphic data in sec. 2.4, and then verify them via a strategy that will be explained in sec. 2.5.4.

2.5.1 Einstein's equations

For easier reference we reproduce here the explicit expressions for the components of the Ricci tensor along Σ, with f_A, f_S, p and q defined in Appendix D,

$$\begin{aligned} R_{ww} &= -\frac{p}{f_A}\left[\partial_w - (\partial_w \ln\rho^2)\right]\partial_w f_A - \frac{q}{f_S}\left[\partial_w - (\partial_w \ln\rho^2)\right]\partial_w f_S \\ R_{w\bar{w}} &= -p\frac{\partial_w\partial_{\bar{w}} f_A}{f_A} - q\frac{\partial_w\partial_{\bar{w}} f_S}{f_S} - \partial_w\partial_{\bar{w}}\ln\rho^2 \end{aligned} \tag{2.108}$$

We will also use the explicit expansions of P and G,

$$P = p_z\rho dw + p_{\bar{z}}\rho d\bar{w} \qquad\qquad G = (g_z dw + g_{\bar{z}} d\bar{w})\rho f_2^2 \wedge \widehat{\text{vol}}_2 \tag{2.109}$$

where $\widehat{\text{vol}}_2$ is the canonical volume form on AdS_2 of unit radius for $AdS_2 \times S^6 \times \Sigma$ and on S^2 of unit radius for $AdS_6 \times S^2 \times \Sigma$.

2.5.1.1 Components along Σ

The ww component of Einstein's equations in (2.107) simplifies to,

$$0 = R_{ww} - 2P_w\bar{P}_w - \frac{1}{4}G_w{}^{PQ}\bar{G}_{wPQ} \tag{2.110}$$

Evaluating $G_w{}^{PQ}\bar{G}_{wPQ}$ amounts to contracting two volume forms on the two-dimensional space, for which the difference in signature between AdS_2 and S^2 is crucial. We thus find,

$$G_w{}^{PQ}\bar{G}_{wPQ} = 2\Lambda\rho^2 g_z(g_{\bar{z}})^* \tag{2.111}$$

Using the expression for the components of P in (2.47) and that $\bar{K}/K = \Lambda$, the ww components of Einstein's equations then become,

$$0 = R_{ww} - \frac{3\Lambda}{8}\rho^2 g_z(g_{\bar{z}})^* \tag{2.112}$$

With (C.2) this can be further evaluated to $\rho^2 g_z(g_{\bar{z}})^* = -16\Lambda f^4(\partial_w B)(\partial_w\bar{B})$, and the ww component of Einstein's equations consequently becomes,

$$0 = R_{ww} + 6f^4(\partial_w B)(\partial_w\bar{B}) \tag{2.113}$$

Note that $f^4 dBd\bar{B}$ is the Poincaré metric on the disc, and $SU(1,1)$ invariant.

Using the expansions of P and G in (2.109) and again that contractions of G produce overall factors as given in (2.111), the $w\bar{w}$ component of Einstein's equations becomes,

$$0 = R_{w\bar{w}} - \rho^2 \left[p_z(p_z)^* + p_{\bar{z}}(p_{\bar{z}})^*\right] - \frac{\Lambda}{4}\rho^2 \left[g_z(g_z)^* + g_{\bar{z}}(g_{\bar{z}})^*\right] + \frac{1}{48} g_{w\bar{w}} G^{PQR}\bar{G}_{PQR} \tag{2.114}$$

We can now use $g_{w\bar{w}} = 2\rho^2$ to evaluate,

$$G^{PQR}\bar{G}_{PQR} = 3\Lambda \left[g_z(g_z)^* + g_{\bar{z}}(g_{\bar{z}})^*\right] \tag{2.115}$$

Using also (2.47), the $w\bar{w}$ component of Einstein's equations consequently becomes,

$$0 = R_{w\bar{w}} - \frac{\rho^2}{16}\left[\left(2\Lambda + \frac{\beta\bar{\beta}}{\alpha\bar{\alpha}}\right)|g_z|^2 + \left(2\Lambda + \frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\right)|g_{\bar{z}}|^2\right] \tag{2.116}$$

Using (C.5) with (2.72) yields,

$$\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}} = R \tag{2.117}$$

With (C.2), this yields for the $w\bar{w}$ component of Einstein's equations,

$$0 = R_{w\bar{w}} - (2\Lambda R + 1)\, f^4 |\partial_w B|^2 - \left(\frac{2\Lambda}{R} + 1\right) f^4 |\partial_w \bar{B}|^2 \tag{2.118}$$

2.5.1.2 Components along $AdS_2 \times S^6$ and $AdS_6 \times S^2$

For the components of Einstein's equations in (2.107) along the six-dimensional space, AdS_6 for the case of $AdS_6 \times S^2$ and S^6 for the case of $AdS_2 \times S^6$, the only non-vanishing contributions are coming from the Ricci tensor and the last term. We thus find,

$$0 = R_{MN} + \frac{1}{48} g_{MN} G^{PQR}\bar{G}_{PQR} \qquad\qquad M,N \text{ along } AdS_6/S^6 \tag{2.119}$$

With (2.115), (C.2) and (2.117), we find,

$$\frac{1}{48} G^{PQR}\bar{G}_{PQR} = \Lambda \rho^{-2} f^4 \left[R|\partial_w B|^2 + R^{-1}|\partial_w \bar{B}|^2\right] \tag{2.120}$$

For the components along the six-dimensional spaces, AdS_6 and S^6, we thus have,

$$\begin{aligned} 0 &= R_{mn} + \eta_{mn}\rho^{-2} f^4 \left[R|\partial_w B|^2 + R^{-1}|\partial_w \bar{B}|^2\right] && \text{for } AdS_6 \\ 0 &= R_{ij} - \delta_{ij}\rho^{-2} f^4 \left[R|\partial_w B|^2 + R^{-1}|\partial_w \bar{B}|^2\right] && \text{for } S^6 \end{aligned} \tag{2.121}$$

The three-form field G has non-vanishing components along the two-dimensional space AdS_2/S^2, and the corresponding components of Einstein's equations therefore become,

$$0 = R_{MN} - \frac{1}{8}\left(G_M{}^{PQ}\bar{G}_{NPQ} + \bar{G}_M{}^{PQ}G_{NPQ}\right) + \frac{1}{48}g_{MN}G^{PQR}\bar{G}_{PQR} \tag{2.122}$$

The contraction of G in the last term has already been evaluated in (2.115). Explicitly evaluating the second term with M, N along AdS_2/S^2 produces a contribution similar to the last term, only with a different numerical coefficient. A factor Λ again originates from the difference in signature between AdS_2 and S^2. The components of Einstein's equations along AdS_2/S^2 become,

$$0 = R_{MN} - \frac{3\Lambda}{16}g_{MN}\left(|g_z|^2 + |g_{\bar{z}}|^2\right) \tag{2.123}$$

For the components along the two-dimensional spaces, AdS_2 and S^2, we thus have,

$$\begin{aligned} 0 &= R_{mn} + 3\eta_{mn}\rho^{-2}f^4\left(R|\partial_w B|^2 + R^{-1}|\partial_w \bar{B}|^2\right) && \text{for } AdS_2 \\ 0 &= R_{ij} - 3\delta_{ij}\rho^{-2}f^4\left(R|\partial_w B|^2 + R^{-1}|\partial_w \bar{B}|^2\right) && \text{for } S^2 \end{aligned} \tag{2.124}$$

2.5.2 Axion-dilaton field equations

We now turn to the axion-dilaton equation. We will perform the index contractions as contractions of spacetime indices instead of frame indices, without introducing new notation. The equation then reads,

$$0 = \partial^M P_M - g^{MN}\Gamma^R_{MN}P_R - 2iQ^M P_M + \frac{1}{24}G_{MNP}G^{MNP} \tag{2.125}$$

With the definitions of P and Q in (2.2), we find,

$$\partial^M P_M - 2iQ^M P_M = 2g^{w\bar{w}}\left(f^2\partial_w\partial_{\bar{w}}B + 2f^4\bar{B}(\partial_w B)\partial_{\bar{w}}B\right) \tag{2.126}$$

The connection term in the covariant derivative evaluates to,

$$g^{MN}\Gamma^R_{MN}P_R = -\frac{1}{2}g^{w\bar{w}}\left[P_w\partial_{\bar{w}}\ln(f_A^{2p}f_S^{2q}) + P_{\bar{w}}\partial_w\ln(f_A^{2p}f_S^{2q})\right] \tag{2.127}$$

This leaves only the term involving G to be evaluated. We find,

$$G_{MNQ}G^{MNQ} = 6\Lambda g_z g_{\bar{z}} = 96\Lambda\rho^{-2}\frac{\alpha\bar{\beta}}{\bar{\alpha}\beta}f^4(\partial_w B)(\partial_{\bar{w}}B) \tag{2.128}$$

where (C.2) was used to obtain the second equality. Using (2.117) and (2.56) shows,

$$\frac{\alpha\bar{\beta}}{\bar{\alpha}\beta} = R\frac{\bar{\beta}^2}{\bar{\alpha}^2} = R\frac{\bar{B}\kappa_+ + \kappa_-}{\kappa_+ + B\kappa_-} \tag{2.129}$$

The complete equation of motion, after dividing by $2g^{w\bar{w}}f^2$, becomes,

$$\begin{aligned} 0 &= \partial_w\partial_{\bar{w}}B + 2f^2\bar{B}(\partial_w B)\partial_{\bar{w}}B \\ &+ \frac{p}{4f_A^2}\left[(\partial_{\bar{w}}B)\partial_w f_A^2 + (\partial_w B)\partial_{\bar{w}} f_A^2\right] + \frac{q}{4f_S^2}\left[(\partial_{\bar{w}}B)\partial_w f_S^2 + (\partial_w B)\partial_{\bar{w}} f_S^2\right] \\ &+ 4\Lambda R\frac{\bar{B}\kappa_+ + \kappa_-}{\kappa_+ + B\kappa_-}f^2(\partial_w B)(\partial_{\bar{w}}B) \end{aligned} \tag{2.130}$$

2.5.3 The 3-form flux field equation

The field equation for the 3-form field G with vanishing $F_{(5)}$ reads,

$$0 = \nabla^P G_{MNP} - iQ^P G_{MNP} - P^P \bar{G}_{MNP} \tag{2.131}$$

We have already presented one proof that this field equations holds for our solution in section 5.3, by showing that the form $F_{(7)}$ is closed. Here, we provide a second proof, obtained by direct evaluation.

Analyzing (2.131), we see that the last two terms vanish unless M, N are both along AdS_2 for the $AdS_2 \times S^6$ case or correspondingly along S^2 for the $AdS_6 \times S^2$ case. The only non-trivial components of the entire equation are when M, N are either both on S^2/AdS_2, or one of them on S^2/AdS_2 and one on Σ. In the latter case, (2.131) reduces to an equation that is satisfied automatically due to metric compatibility of the connection on S^2/AdS_2. It therefore only remains to consider the case with both components on S^2/AdS_2. For notational convenience we will introduce coordinate indices μ, ν, which correspond to AdS_2 for the $AdS_2 \times S^6$ case and to S^2 for the $AdS_6 \times S^2$ case. We will also again perform index contractions as contractions of spacetime indices, without introducing additional notation.

When $M, N = \mu, \nu$ are both along S^2/AdS_2, the field equation reads,

$$\begin{aligned} 0 &= \partial^P G_{\mu\nu P} - iQ^P G_{\mu\nu P} - P^P \bar{G}_{\mu\nu P} \\ &\quad - g^{PQ}\Gamma^R_{PQ}G_{\mu\nu R} - g^{PQ}\left(\Gamma^R_{P\mu}G_{R\nu Q} + \Gamma^R_{P\nu}G_{\mu RQ}\right) \end{aligned} \tag{2.132}$$

Evaluating the connection terms in analogy with (2.127), the field equation becomes,

$$0 = \left(\partial_w - iQ_w + \frac{1}{2}\partial_w \ln \frac{f_A^{2p} f_S^{2q}}{f_2^8}\right) G_{\bar{w}\mu\nu} - P_w \bar{G}_{\bar{w}\mu\nu} + (w \leftrightarrow \bar{w}) \tag{2.133}$$

We now use the expansion (2.109) along with (C.2) to replace,

$$G_{w\mu\nu} = 4iK f_2^2 \frac{\alpha}{\beta} P_w \,\widehat{\mathrm{vol}}_{2\,\mu\nu} \qquad\qquad G_{\bar{w}\mu\nu} = -4iK\Lambda f_2^2 \frac{\bar{\beta}}{\bar{\alpha}} P_{\bar{w}} \,\widehat{\mathrm{vol}}_{2\,\mu\nu} \tag{2.134}$$

With (2.117), we then find,

$$\begin{aligned}
\frac{1}{4iK}\frac{\bar{\alpha}}{\bar{\beta}}\left(\partial_w G_{\bar{w}\mu\nu} + \partial_{\bar{w}} G_{w\mu\nu}\right) &= R\left(\partial_{\bar{w}} + \frac{1}{2}\partial_{\bar{w}} \ln \frac{\alpha^2}{\beta^2}\right) f_2^2 P_w \,\widehat{\mathrm{vol}}_{2\,\mu\nu} \\
&\quad - \Lambda\left(\partial_w + \frac{1}{2}\partial_w \ln \frac{\bar{\beta}^2}{\bar{\alpha}^2}\right) f_2^2 P_{\bar{w}} \,\widehat{\mathrm{vol}}_{2\,\mu\nu} \\
\frac{1}{4iK}\frac{\bar{\alpha}}{\bar{\beta}}\left(P_w \bar{G}_{\bar{w}\mu\nu} + P_{\bar{w}} G_{w\mu\nu}\right) &= f_2^2 \frac{\bar{\alpha}^2}{\bar{\beta}^2}\left(R^{-1}|P_{\bar{w}}|^2 - \Lambda |P_w|^2\right) \widehat{\mathrm{vol}}_{2\,\mu\nu}
\end{aligned} \tag{2.135}$$

The equation of motion, after dividing by $4i\Lambda K f_2^2 \bar{\beta}/\bar{\alpha}$ and separating off the volume form on the two-dimensional space, consequently becomes,

$$\begin{aligned}
0 &= \Lambda R\left[\partial_{\bar{w}} - iQ_{\bar{w}} + \frac{1}{2}\partial_{\bar{w}} \ln\left(\frac{f_A^{2p} f_S^{2q}}{f_2^4}\frac{\alpha^2}{\beta^2}\right)\right] P_w \\
&\quad - \left[\partial_w - iQ_w + \frac{1}{2}\partial_w \ln\left(\frac{f_A^{2p} f_S^{2q}}{f_2^4}\frac{\bar{\beta}^2}{\bar{\alpha}^2}\right)\right] P_{\bar{w}} - \frac{\bar{\alpha}^2}{\bar{\beta}^2}\left(\frac{|P_{\bar{w}}|^2}{\Lambda R} - |P_w|^2\right)
\end{aligned} \tag{2.136}$$

Evaluating the derivatives and using the components of Q as defined in (2.2) yields,

$$\begin{aligned}
0 &= (\Lambda R - 1)\left(f^2 \partial_w \partial_{\bar{w}} B + \frac{3}{2}\bar{B} P_w P_{\bar{w}}\right) + \left(\frac{1}{2}B\Lambda R + \frac{\bar{\alpha}^2}{\bar{\beta}^2}\right)\left(|P_w|^2 - \frac{|P_{\bar{w}}|^2}{\Lambda R}\right) \\
&\quad + \frac{1}{2}\Lambda R P_w \partial_{\bar{w}} \ln\left(\frac{f_A^{2p} f_S^{2q}}{f_2^4}\frac{\alpha^2}{\beta^2}\right) - \frac{1}{2}P_{\bar{w}} \partial_w \ln\left(\frac{f_A^{2p} f_S^{2q}}{f_2^4}\frac{\bar{\beta}^2}{\bar{\alpha}^2}\right)
\end{aligned} \tag{2.137}$$

With $P_w = f^2 \partial_w B$, $P_{\bar{w}} = f^2 \partial_{\bar{w}} B$ as well as,

$$\frac{\bar{\alpha}^2}{\bar{\beta}^2} = \frac{\partial_w \mathcal{A}_+ + B\partial_w \mathcal{A}_-}{\bar{B}\partial_w \mathcal{A}_+ + \partial_w \mathcal{A}_-} \tag{2.138}$$

2.5.4 Explicitly evaluating the equations

To summarize, the non-trivial components of Einstein's equations take the form given in (2.113) for the ww component, in (2.118) for the $w\bar{w}$ component, in (2.121) for the AdS_6/S^6

components, and in (2.124) for the AdS_2/S^2 components. The equation for the axion-dilaton scalar takes the form given in (2.130) and the non-trivial components of the equation for G are given in (2.137) with (2.138).

We will now describe the strategy to verify that these equations are satisfied. We use the explicit expressions for the metric functions in (2.93), for B in (2.95), and for the components of the Ricci tensor. This reduces the field equations to a set of equations involving only the holomorphic functions and their derivatives, as well as R, $\mathcal{G}$ and κ^2 along with their derivatives. We will avoid using the explicit definition for $\mathcal{G}$ or R, since $\mathcal{G}$ involves an integration that we have not performed for generic $\mathcal{A}_\pm$ while the definition of R involves a square root with a corresponding choice of branch that we do not wish to specify explicitly. The first step will be to make the expressions algebraic in R and $\mathcal{G}$, i.e. to eliminate all their derivatives. From the definitions for $\mathcal{G}$ in (2.86) and for R in (2.88), we straightforwardly derive,

$$\begin{aligned}
\partial_w R &= \frac{6\Lambda R^2}{R^2-1}\partial_w\left(\frac{\kappa^2\mathcal{G}}{|\partial_w\mathcal{G}|^2}\right) \\
\partial_w\mathcal{G} &= \left(\bar{\mathcal{A}}_+ - \mathcal{A}_-\right)\partial_w\mathcal{A}_+ + \left(\mathcal{A}_+ - \bar{\mathcal{A}}_-\right)\partial_w\mathcal{A}_-
\end{aligned} \tag{2.139}$$

The $\partial_{\bar{w}}$ derivatives of R and $\mathcal{G}$ are obtained by complex conjugation. Repeatedly using these relations to reduce the rank in derivatives acting on R and $\mathcal{G}$, we can eliminate all derivatives of $\mathcal{G}$ and R. We now use the definition of R in (2.88) to eliminate $\mathcal{G}$, by setting,

$$\mathcal{G} = \left(R + \frac{1}{R} - 2\Lambda\right)\frac{|\partial_w\mathcal{G}|^2}{6\Lambda\kappa^2} \tag{2.140}$$

Using also the explicit definition of κ^2, we have at this point reduced Einstein's equations to relations involving only the holomorphic functions and their differentials along with R. $\mathcal{G}$ and its derivatives as well as the derivatives of R are eliminated completely. Straightforward evaluation now shows that the equations are indeed satisfied for both, the AdS_6 and AdS_2 cases, with the corrsponding choices of Λ and K as well as of p and q for the dimensions of the AdS and S parts of the geometry. This shows that the local solution to the BPS equations presented in sec. 2.4 solves the field equations of Type IIB supergravity as well. We point out that, for the discussion of the BPS equations, c was assumed real and R constrained to be positive by its definition as absolute value of Z in (2.72), but that neither of these constraints appear to be necessary for the equations of motion to be satisfied.

We close this section with a comment on the sign of G. It is generally true that, for a solution to Type IIB supergravity, flipping the sign of G produces another solution, since G appears quadratically in the equations of motion. In general, supersymmetry is not preserved under this sign flip, since the BPS equations do depend on the sign of G. For our solutions, however, flipping the sign of G again produces a supersymmetric solution. This may be seen from the fact that a sign reversal in G corresponds to a special case of the $SU(1,1)$ transformations discussed in sec. 2.4.4. Choosing $u = -1$ and $v = 0$ indeed leaves all supergravity fields invariant except for the two-form potential $C_{(2)}$, on which it induces a sign flip that results in a sign reversal on G. The sign-flipped solution is therefore again in our class of supersymmetric solutions, although with a different form of the Killing spinors, which depend on the $\mathcal{A}_\pm$ directly.

2.6 Reality, positivity, and regularity conditions

The solutions obtained for the supergravity fields of the case $AdS_2 \times S^6$ in the previous section satisfy the BPS equations, but are physically viable solutions only after certain reality, positivity, and regularity conditions are enforced on the supergravity fields of the solutions. In this section we establish these conditions and uncover their implications on the functions $\mathcal{A}_\pm$. We set $\Lambda = -1$ throughout this section.

2.6.1 Reality and positivity conditions

For an acceptable solution with appropriate signature, the metric is real-valued and the functions $f_2^2, f_6^2\ \rho^2$ positive on Σ. There are no reality constraints on the fields B and $C_{(2)}$, but B is restricted by the condition of positive coupling constant, $|B| < 1$. We now extract the necessary and sufficient conditions on κ^2, $\mathcal{G}$, and R for these properties to hold.

Recalling that the functions κ^2 and $\mathcal{G}$, which were defined in (2.86), are real-valued by construction, and that R is real and non-negative by construction, (2.88) implies,

$$3\kappa^2\mathcal{G} < -2|\partial_w\mathcal{G}|^2 \tag{2.141}$$

In particular, κ^2 and $\mathcal{G}$ need to have opposite signs. Assuming positive ρ^2, the positivity of f_2^2 and f_6^2 in (2.90) furthermore requires,

$$(1-R)\,\kappa^2 > 0 \tag{2.142}$$

With this assumption ρ^2, given in (2.90), is real and can be made positive by appropriately choosing the sign of the constant c. To verify $|B| \leq 1$ we calculate f^2 using (2.66),

$$f^2 = \frac{1}{1-|B|^2} = 1 + \frac{|\lambda - Z^2|^2}{(1-|\lambda|^2)(1-|Z|^4)} \tag{2.143}$$

Since $\kappa^2(1-R) = |\kappa_-|^2(1-|\lambda|^2)(1-|Z|^2)$, (2.142) implies $f^2 \geq 1$. The reality and positivity conditions are therefore given by (2.141) and (2.142).

2.6.1.1 Inversion and complex conjugation

The space of allowed triplets $(\kappa^2, \mathcal{G}, R)$ naturally divides into two branches, according to whether the conditions (2.141) and (2.142) are realized for $R > 1$ or $R < 1$. We shall refer to these branches as $\mathfrak{B}_\pm$, defined by,

$$\begin{aligned}
\mathfrak{B}_+ &= \left\{\kappa^2 > 0, \qquad \mathcal{G} < 0, \qquad R < 1\right\} \\
\mathfrak{B}_- &= \left\{\kappa^2 < 0, \qquad \mathcal{G} > 0, \qquad R > 1\right\}
\end{aligned} \tag{2.144}$$

These two branches are mapped into one another by an involution, which is a combination of complex conjugation, reversal of the complex structure on Σ, and reversal of the indices $\pm$ on the functions $\mathcal{A}_\pm$, given by,

$$\mathcal{A}_\pm(w) \to \mathcal{A}'_\pm(w) = \bar{\mathcal{A}}_\mp(w) = \overline{\mathcal{A}_\mp(\bar{w})} \tag{2.145}$$

combined with $R \to R^{-1}$. This transformation leaves eq. (2.88) invariant and reverses the sign of κ^2 and $\mathcal{G}$. It leaves the metric functions f_2^2, f_6^2 and ρ^2 invariant and complex conjugates the fields B and $\mathcal{C}$,

$$\begin{aligned} B(w,\bar w) &\to B'(w,\bar w) = \bar B(w,\bar w) = \overline{B(\bar w, w)} \\ \mathcal{C}(w,\bar w) &\to \mathcal{C}'(w,\bar w) = \bar{\mathcal{C}}(w,\bar w) = \overline{\mathcal{C}(\bar w, w)} \end{aligned} \tag{2.146}$$

2.6.2 Global regularity and boundary conditions

By inspection of the metric functions f_2^2, f_6^2, and ρ^2 in (2.90), it is manifest that a supergravity solution considered on a compact subset U of Σ for which $(\kappa^2, \mathcal{G}, R)$ maps to a compact subset of either $\mathfrak{B}_+$ or $\mathfrak{B}_-$ (but not both) is *locally regular* in U. If the supergravity solution considered throughout a compact surface Σ is such that $(\kappa^2, \mathcal{G}, R)$ maps to a compact subset of either branches $\mathfrak{B}_+$ or $\mathfrak{B}_-$ then the supergravity solution is *globally regular* on Σ.

If Σ has a non-empty boundary, $\partial\Sigma$, additional regularity conditions have to be satisfied on $\partial\Sigma$. We assume geodesic completeness of the spacetime manifold allowed for a supergravity solution, so that the boundary of spacetime is at infinite geodesic distance (modulo issues of the Minkowski signature of the AdS_2-factor). The only way we know how to realize this when Σ has a boundary is by closing off the sphere S^6, namely $f_6^2 \to 0$, while keeping f_2^2 finite. In view of the expression obtained from (2.90) for the ratio,

$$\frac{f_6^2}{f_2^2} = 9\frac{(1-R)^2}{(1+R)^2} \tag{2.147}$$

this corresponds to the boundary condition $R = 1$. As $R \to 1$, factors of $1 - R$ in the expressions for the metric functions vanish at a number of places, and having a regular limit therefore imposes additional constraints: From the expression for f_2^2 in (2.93), we see that finiteness of the AdS_2 radius needs $\mathcal{G} = \mathcal{O}\left((1-R)^3\right)$ as the boundary is approached. Similarly, from the finiteness of ρ^2 we then conclude that $\kappa^2 = \mathcal{O}(1-R)$. The boundary $\partial\Sigma$ is therefore mapped to the common boundary of the two branches $\mathfrak{B}_\pm$, namely $\kappa^2 = \mathcal{G} = 0$ and $R = 1$. Lastly, in view of (2.88) we also have the constraint that $\partial_w \mathcal{G} = \mathcal{O}((1-R)^2)$.

2.6.3 Implications of regularity and boundary conditions

In this subsection we will discuss some immediate implications of the global regularity conditions for the structure of the solutions.

2.6.3.1 No smooth solutions for compact Σ without boundary

Assuming that $\mathcal{G}$ is smooth, there are no globally regular solutions on a compact surface Σ without boundary. The argument is parallel to the one already given for the case $AdS_6 \times S^2$. It is based on the following differential equation,

$$\partial_w \partial_{\bar{w}} \mathcal{G} = -\kappa^2 \tag{2.148}$$

which readily follows from the definitions of κ^2 and $\mathcal{G}$ in (2.86). If $\mathcal{G}$ is smooth, then on a compact surface without boundary, the integral of the left side over Σ must vanish. But the sign of κ^2 is constant throughout Σ so the integral of the right side cannot vanish, which is in contradiction to our assumptions. Hence such globally regular solutions cannot exist. We are thus left with two options: either Σ has a non-empty boundary, or Σ is compact without boundary and the functions $\mathcal{A}_\pm$ have singularities in Σ.

2.6.3.2 No smooth solutions for compact Σ with boundary

We show that for smooth $\mathcal{G}$ and an arbitrary Riemann surface Σ with non-empty boundary $\partial\Sigma$, the conditions $\mathcal{G}|_{\partial\Sigma} = 0$ and $\text{sgn}(\kappa^2) = -\text{sgn}(\mathcal{G})$ can not be satisfied simultaneously. We start from (2.148) and solve this equation along with the boundary condition $\mathcal{G}|_{\partial\Sigma} = 0$ to obtain the following integral equation,

$$\mathcal{G}(w) = H(w) + \frac{1}{\pi} \int_\Sigma d^2 z \; G(w,z) \kappa^2(z) \tag{2.149}$$

Here, $G(w,z)$ is the scalar Green function on Σ, which is symmetric $G(z,w) = G(w,z)$ and vanishes on the boundary $\partial\Sigma$,

$$\begin{aligned} \partial_w \partial_{\bar{w}} G(w,z) &= -\pi \delta(w,z) \\ G(w,z)|_{w \in \partial\Sigma} &= 0 \end{aligned} \tag{2.150}$$

As shown in detail in sec. 2.3 of [20], for any two points w, z in the interior of Σ, the function $G(w,z)$ is strictly positive. $H(w)$ is a harmonic function. Since $\mathcal{G}(w)$ vanishes on the boundary by assumption, and by construction the Green function $G(w,z)$ vanishes for $w \in \partial\Sigma$, then H itself must also vanish on $\partial\Sigma$, and we have

$$\begin{aligned} &\partial_w \partial_{\bar{w}} H(w) = 0 \\ &H(w)\big|_{w\in\partial\Sigma} = 0 \end{aligned} \tag{2.151}$$

By the min-max principle for harmonic functions, $H(w)$ takes its minimum and maximum values on the boundary of Σ. Since $H(w) = 0$ for $w \in \partial\Sigma$, this implies that $H(w) = 0$ both on $\partial\Sigma$ and in the interior of Σ. But this is incompatible with $\mathrm{sgn}(\kappa^2) = -\mathrm{sgn}(\mathcal{G})$, due to the positivity of the Green function $G(w,z)$, which implies that the integral term in (2.149) is strictly positive for the branch $\mathfrak{B}_+$ with $\kappa^2 > 0$ and strictly negative for the branch $\mathfrak{B}_-$ with $\kappa^2 < 0$. Thus, no regular supergravity solutions exist when $\mathcal{G}$ vanishes on $\partial\Sigma$.

2.7 Double analytic continuation

In this section we study the relation between $AdS_2 \times S^6$ and $AdS_6 \times S^2$ via double analytic continuation of the spacetime manifold metrics in more detail, and discuss the implications from the perspective of the solutions to the BPS equations. At the level of the geometry, one may perform an analytic continuation from $AdS_6 \times S^2$ to $AdS_2 \times S^6$ via[2]

$$AdS_6 \to -S^6 \qquad\qquad S^2 \to -AdS_2 \tag{2.152}$$

and these continuations can be extended straightforwardly to the remaining bosonic supergravity fields. This does not produce a ten-dimensional spacetime of the desired signature, since the eight-dimensional symmetric space has changed signature from mostly plus to mostly minus while the metric on Σ remains positive definite, but it does formally produce

[2]The signs can be understood as follows. Starting from AdS in mostly plus signature, one obtains Euclidean hyperbolic space by a standard Wick rotation in Poincaré coordinates. From Euclidean hyperbolic space, for which we can take global coordinates such that $ds^2 = dr^2 + \sinh^2 r \, d\Omega^2$, we can then get to a sphere by setting $r = i\theta$. The resulting metric is $ds^2 = -(d\theta^2 + \sin^2\theta \, d\Omega^2)$, i.e. a sphere with negative signature.

a solution to the equations of motion. We may therefore wonder whether we can recover the analytic continuation of the globally regular $AdS_6 \times S^2 \times \Sigma$ solutions constructed in [19, 20] as a special case of the $AdS_2 \times S^6 \times \Sigma$ solutions presented in sec. 2.4. In the remainder of this section we will show that this is indeed the case, but that the result is neither regular (even leaving aside the signature issue) nor supersymmetric.

The construction in [19, 20] started from Σ a disc, realized as the upper half plane. The holomorphic functions $\mathcal{A}_\pm$ were given by,

$$\mathcal{A}_\pm = \mathcal{A}_\pm^0 + \sum_{\ell=1}^{L} Z_\pm^\ell \ln(w - p_\ell) \qquad\qquad Z_+^\ell = \sigma \prod_{n=1}^{L-2}(p_\ell - s_n) \prod_{k\neq\ell}^{L} \frac{1}{p_\ell - p_k} \tag{2.153}$$

where w is a complex coordinate on the upper half plane, s_n a collection of points inside the upper half plane and p_ℓ a set of poles of the differentials $\partial_w \mathcal{A}_\pm$ on the boundary of the upper half plane. With a suitable choice of the integration constant implicit in $\mathcal{G}$, this produced,

$$\kappa^2, \mathcal{G} > 0 \;\text{ on } \operatorname{int}(\Sigma) \qquad\qquad \kappa^2 = \mathcal{G} = 0 \;\text{ on } \partial\Sigma \tag{2.154}$$

We can assume the same choice of holomorphic data as input for the $AdS_2 \times S^6 \times \Sigma$ solutions. The expressions for κ^2 and $\mathcal{G}$ in terms of the locally holomorphic functions are the same for $AdS_2 \times S^6 \times \Sigma$ and $AdS_6 \times S^2 \times \Sigma$, such that we realize (2.154) in both cases. Eq. (2.88) then implies $\Lambda R > 0$ in the interior of the upper half plane and $\Lambda R \to 1$ on the boundary, and we choose the branch $0 < \Lambda R \leq 1$. This implies that R is positive for $AdS_6 \times S^2 \times \Sigma$ and negative for $AdS_2 \times S^6 \times \Sigma$. We note that negative R was not acceptable for solving the BPS equations, where R was positive by construction. But, as noted at the end of sec. 2.5, neither R nor the constant c are constrained by the equations of motion. So at the level of the equations of motion these $AdS_2 \times S^6$ configurations are acceptable, and we have,

$$(\Lambda R)_{AdS_2\times S^6\times\Sigma} = (\Lambda R)_{AdS_6\times S^2\times\Sigma} \tag{2.155}$$

The expressions for the supergravity fields in (2.93), (2.95) and (2.101) depend on R only through this combination ΛR. The form of the axion-dilaton B in (2.95) is, in fact, exactly the same for $AdS_2 \times S^6 \times \Sigma$ and $AdS_6 \times S^2 \times \Sigma$. The metric functions in (2.93) are real provided that c is chosen real for $AdS_6 \times S^2 \times \Sigma$ and imaginary for $AdS_2 \times S^6 \times \Sigma$, to compensate for the phase in $\sqrt{\Lambda \mathcal{G}}$. They only differ between $AdS_2 \times S^6 \times \Sigma$ and $AdS_6 \times S^2 \times \Sigma$ through their signs: while f_2^2, f_6^2 and ρ^2 all have the same sign for $AdS_6 \times S^2 \times \Sigma$, the sign of f_2^2 and f_6^2 is opposite to that of ρ^2 for $AdS_2 \times S^6 \times \Sigma$. This is precisely as expected for solutions connected by the analytic continuation in (2.152). The gauge potential $\mathcal{C}$ in (2.101) differs only by an overall factor of i between $AdS_2 \times S^6 \times \Sigma$ and $AdS_6 \times S^2 \times \Sigma$. This produces the expected behavior under a Wick rotation for the three-form field strength, where one of the components along S^2 becomes timelike and picks up a factor of i. We have thus recovered the analytic continuation of the global $AdS_6 \times S^2 \times \Sigma$ solutions to $AdS_2 \times S^6 \times \Sigma$ via (2.152), which is simply realized by the same choice of locally holomorphic functions.

This naive analytic continuation does, however, not lead to physically regular solutions. Aside from the inappropriate signs for the metric functions, it is still the two-dimensional space that collapses on the boundary of Σ. This was the desired behavior for the $AdS_6 \times S^2 \times \Sigma$ case, with the collapsing S^2 smoothly closing off spacetime. But it is not desirable for the $AdS_2 \times S^6 \times \Sigma$ solutions to have the AdS_2 cap off on $\partial\Sigma$. Moreover, the solutions are not supersymmetric, since we do not recover them from the BPS equations where $R \geq 0$ was required by construction. The loss of supersymmetry under Wick rotation may be understood from the change in the Clifford algebra due to the changed signature in the two- and six-dimensional spaces.

2.8 Discussion

We have constructed the general local form of solutions to Type IIB supergravity that are invariant under $SO(2,1) \oplus SO(7)$ and sixteen supersymmetries. The geometry takes the form $AdS_2 \times S^6$ warped over a two-dimensional Riemann surface Σ, and the local form of the solutions is strikingly similar to the $AdS_6 \times S^2$ case considered in [26]. The entire solution is summarized, in parallel with the $AdS_6 \times S^2$ case, in sec. 2.4, and we have verified for both cases that the solution to the BPS equations also satisfies the field equations of

Type IIB supergravity in sec. 2.5. The differences between the local solutions for $AdS_2 \times S^6$ and $AdS_6 \times S^2$ are subtle, and encoded entirely in sign flips at various places.

To obtain physically acceptable solutions additional positivity and regularity conditions have to be imposed on the general local form of the solutions. We have presented a preliminary analysis of these conditions for $AdS_2 \times S^6$ in sec. 2.6. Subtle but crucial differences between the solutions for $AdS_2 \times S^6$ and $AdS_6 \times S^2$ appear to render ineffective the strategy followed in [20] to obtain global solutions for $AdS_6 \times S^2$. An analytic continuation of the physically regular $AdS_6 \times S^2$ solutions to $AdS_2 \times S^6$, discussed in sec. 2.7, gives rise to field configurations which solve the field equations, but are neither regular nor supersymmetric. The construction of physically regular $AdS_2 \times S^6$ solutions is the subject of the next chapter.

Finally, we comment on the superalgebra structure of these AdS_2 solutions. While the five-dimensional superconformal algebra is unique and corresponds to a specific real form of $F(4)$, there exist several superconformal algebras for AdS_2 [22, 23]. They are $SU(1,1|4), OSp(8|2,\mathbb{R})$, and $OSp(4^*|4)$, with maximal bosonic subalgebras respectively realized by $AdS_2 \times S^5 \times S^1 \times \Sigma$, $AdS_2 \times S^7 \times L$, and $AdS_2 \times S^2 \times S^4 \times \Sigma$, where Σ is a Riemann surface and L is a one-dimensional line. The first case is the subject of Chapter 4; the last case was solved already in [24].

CHAPTER 3

Global half-BPS $AdS_2 \times S^6$ solutions

3.1 Local solution and regularity conditions

For convenience, we once again summarize the local form of Type IIB supergravity solutions with 16 supersymmetries and spacetime of the form $AdS_2 \times S^6$ warped over a Riemann surface Σ, and discuss the conditions for physical positivity and regularity of the supergravity fields. The local solutions are invariant under the real form of the exceptional Lie superalgebra $F(4)$ which has maximal bosonic subalgebra $SO(1,2) \oplus SO(7)$.

3.1.1 Supergravity fields

Invariance under $SO(1,2) \oplus SO(7)$ dictates the general form of the supergravity fields of the solutions. All Fermi fields vanish and the spacetime metric takes the form,

$$ds^2 = f_2^2 ds^2_{AdS_2} + f_6^2 ds^2_{S^6} + 4\rho^2 |dw|^2 \tag{3.1}$$

The five-form field strength vanishes $F_{(5)} = 0$ and the three-form field strength $F_{(3)}$ and its Poincaré dual $F_{(7)}$ are given by,

$$\begin{aligned} F_{(3)} &= dC_{(2)} & C_{(2)} &= \mathcal{C} \ \mathrm{vol}_{AdS_2} \\ F_{(7)} &= dC_{(6)} & C_{(6)} &= \mathcal{M} \ \mathrm{vol}_{S^6} \end{aligned} \tag{3.2}$$

Throughout, w is a local complex coordinate on Σ while f_2, f_6, and ρ are real-valued functions on Σ. The fields $\mathcal{C}$, $\mathcal{M}$, and the axion-dilaton $B = (1 + i\tau)/(1 - i\tau)$ are complex-valued functions on Σ. The line elements $ds^2_{AdS_2}$, $ds^2_{S^6}$, and the volume forms vol_{AdS_2}, vol_{S^6} are for maximally symmetric AdS_2 and S^6 with unit radius.

The solutions are parametrized by two locally holomorphic functions $\mathcal{A}_\pm$ and expressed conveniently in terms of the composite quantities κ^2, $\mathcal{G}$, and T given in terms of $\mathcal{A}_\pm$ by,

$$\kappa^2 = -|\partial_w \mathcal{A}_+|^2 + |\partial_w \mathcal{A}_-|^2 \qquad \partial_w \mathcal{B} = \mathcal{A}_+ \partial_w \mathcal{A}_- - \mathcal{A}_- \partial_w \mathcal{A}_+$$
$$\mathcal{G} = |\mathcal{A}_+|^2 - |\mathcal{A}_-|^2 + \mathcal{B} + \bar{\mathcal{B}} \qquad T = \frac{1-R}{1+R} = \left(1 + \frac{2|\partial_w \mathcal{G}|^2}{3\kappa^2 \mathcal{G}}\right)^{1/2} \tag{3.3}$$

where $\kappa^2 = -\partial_w \partial_{\bar{w}} \mathcal{G}$. By construction, the functions κ^2, $\mathcal{G}$, and R are real. Furthermore, R is non-negative so that T is real and satisfies $T \in [-1, 1]$. In terms of these composites, the metric functions are given by,

$$f_2^2 = \frac{1}{9}\left(\frac{-6\mathcal{G}}{T^3}\right)^{1/2} \qquad f_6^2 = (-6\mathcal{G}T)^{1/2} \qquad \rho^2 = \kappa^2 \left(\frac{T}{-6\mathcal{G}}\right)^{1/2} \tag{3.4}$$

The functions $\mathcal{C}$ and $\mathcal{M}$ parametrizing the two- and six-form potentials are given by,

$$\mathcal{C} = -\frac{2i}{3}\left(\frac{U}{3T^2} - \bar{\mathcal{A}}_- - \mathcal{A}_+\right)$$
$$\mathcal{M} = 80(\mathcal{W}_+ + \bar{\mathcal{W}}_-) - 12\mathcal{G}U + 20(\mathcal{A}_+ + \bar{\mathcal{A}}_-)\left(2|\mathcal{A}_+|^2 - 2|\mathcal{A}_-|^2 - 3\mathcal{G}\right) \tag{3.5}$$

where U and $\mathcal{W}_\pm$ are defined by,

$$\kappa^2 U = \overline{\partial_w \mathcal{G}} \partial_w \mathcal{A}_+ + \partial_w \mathcal{G} \overline{\partial_w \mathcal{A}_-} \qquad \partial_w \mathcal{W}_\pm = \mathcal{A}_\pm \partial_w \mathcal{B} \tag{3.6}$$

The axion-dilaton scalar field is given,

$$B = -\frac{\partial_w \mathcal{A}_+ \partial_{\bar{w}} \mathcal{G} + R\, \partial_{\bar{w}} \bar{\mathcal{A}}_- \partial_w \mathcal{G}}{R\, \partial_{\bar{w}} \bar{\mathcal{A}}_+ \partial_w \mathcal{G} + \partial_w \mathcal{A}_- \partial_{\bar{w}} \mathcal{G}} \tag{3.7}$$

The global $SU(1,1)$ symmetry transformations of the Type IIB supergravity fields are induced by the following transformations of $\mathcal{A}_\pm$ under the group $SU(1,1) \otimes \mathbb{C}$,

$$\mathcal{A}_+ \quad \rightarrow \quad \mathcal{A}'_+ = +u\mathcal{A}_+ - v\mathcal{A}_- + a$$
$$\mathcal{A}_- \quad \rightarrow \quad \mathcal{A}'_- = -\bar{v}\mathcal{A}_+ + \bar{u}\mathcal{A}_- + \bar{a} \tag{3.8}$$

where $SU(1,1)$ is parametrized by $u, v \in \mathbb{C}$ with $|u|^2 - |v|^2 = 1$ and the complex shift parameter a has the effect of producing gauge transformations in $\mathcal{C}$ and $\mathcal{M}$ only.

3.1.2 Positivity and regularity conditions

Minkowski signature of the ten-dimensional spacetime metric imposes the positivity conditions $f_2^2, f_6^2, \rho^2 > 0$ which require $\kappa^2 > 0$ and $\mathcal{G}T < 0$, assuming that all square roots of positive real arguments in (3.4) are taken to be positive. Without loss of generality, we may choose the branch $T > 0$ for the square root in (3.3), so that $0 < R < 1$. As a result, the positivity conditions become,

$$\kappa^2 > 0 \qquad\qquad \mathcal{G} < 0 \qquad\qquad 0 < R < 1 \tag{3.9}$$

Regularity of the supergravity fields of the solutions in the interior of Σ requires that the inequalities of (3.9) be obeyed strictly. If Σ has a non-empty boundary $\partial\Sigma$, then geodesic completeness of the solutions requires that the six-sphere shrinks to zero size $f_6 \to 0$ at the boundary, while the radius of AdS_2 remains finite. Since we have $f_6^2/f_2^2 = 9T^2$ this means that $T \to 0$ and $R \to 1$ as the boundary is being approached. Regularity of the solution at the boundary then requires the following behavior as $r \equiv 1 - R \to 0$,

$$\kappa^2 = \mathcal{O}(r) \qquad\qquad \mathcal{G} = \mathcal{O}(r^3) \qquad\qquad \partial_w\mathcal{G} = \mathcal{O}(r^2) \tag{3.10}$$

The explicit expression for R in terms of $\kappa^2\mathcal{G}$ and $\partial_w\mathcal{G}$ in (3.3) furthermore requires,

$$\frac{\kappa^2\mathcal{G}}{|\partial_w\mathcal{G}|^2} \to -\frac{2}{3} \tag{3.11}$$

Note that the boundary condition $\partial_w\mathcal{G} = 0$ on $\partial\Sigma$ is stronger than the corresponding condition for the AdS_6 case, where $(\partial_w + \partial_{\bar{w}})\mathcal{G} = 0$ was sufficient [18].

3.1.3 Realizing the regularity conditions at the boundary $\partial\Sigma$

The boundary conditions discussed in sec. 3.1.2 can be realized naturally by imposing a conjugation condition on the holomorphic functions $\mathcal{A}_\pm$ on $\partial\Sigma$. We shall take $\partial\Sigma$ to consist of only a single connected boundary component, though the construction may be easily generalized to the case when $\partial\Sigma$ has several components. We may map the boundary $\partial\Sigma$ to the real line by a Schwarz-Christoffel transformation, which is piecewise conformal. Let

$w, \bar{w}$ be local complex coordinates in terms of which a boundary segment is given by $w = \bar{w}$. The conjugation condition is then given by,

$$\overline{\mathcal{A}_\pm(\bar{w})} = \mathcal{A}_\mp(w) \tag{3.12}$$

This condition readily implies $\kappa^2 = 0$ on $\partial\Sigma$ and, noting that we have,

$$\partial_w \mathcal{G}(w,\bar{w}) = \Big(\overline{\mathcal{A}_+(w)} - \mathcal{A}_-(w)\Big)\,\partial_w \mathcal{A}_+(w) + \Big(\mathcal{A}_+(w) - \overline{\mathcal{A}_-(w)}\Big)\,\partial_w \mathcal{A}_-(w) \tag{3.13}$$

it also implies $\partial_w \mathcal{G} = 0$ on $\partial\Sigma$. Consequently, $\mathcal{G}$ is constant along each boundary segment and can be made to vanish on any one single segment by fixing the integration constant implicit in the definitions of $\mathcal{B}$ and $\mathcal{G}$. The behaviors near the boundary in (3.10) are implied by the relations between $\mathcal{G}$, $\partial_w\mathcal{G}$, and κ^2 via differentiation, which in turn imply (3.11).

We conclude this section by drawing a comparison between the boundary conditions for the $AdS_2 \times S^6$ case studied here and the boundary conditions for the $AdS_6 \times S^2$ case studied in [20]. The conjugation relation between the differentials resulting from (3.12) differs from the analogous condition for the differentials in the $AdS_6 \times S^2$ solutions of [20] only by a sign. More importantly, it was sufficient in [20] to implement a conjugation condition on the differentials $\partial_w \mathcal{A}_\pm$ to ensure $(\partial_w + \partial_{\bar{w}})\mathcal{G}|_{\partial\Sigma} = 0$, whereas here we impose the conjugation relation on the functions $\mathcal{A}_\pm$ themselves in order to implement the stronger condition $\partial_w \mathcal{G}|_{\partial\Sigma} = 0$.

Furthermore, the conjugation condition of (3.12) is incompatible with the presence of logarithmic branch cuts in $\mathcal{A}_\pm$ starting at branch points on the boundary $\partial\Sigma$ and with branch cuts along the boundary. Suppose that we have a branch point at $w = 0$,

$$\mathcal{A}_\pm(w) = \mathcal{A}_\pm^{(0)}(w) + \mathcal{A}_\pm^{(1)}(w)\ln w \qquad\qquad \overline{\mathcal{A}_\pm^{(i)}(\bar{w})} = \mathcal{A}_\mp^{(i)}(w) \tag{3.14}$$

where $\mathcal{A}_\pm^{(0)}(w)$ and $\mathcal{A}_\pm^{(1)}(w)$ are regular and single-valued in a neighborhood of $w = 0$. Upon encircling $w = 0$ counterclockwise, $\mathcal{A}_\pm \to \mathcal{A}_\pm + i\pi \mathcal{A}_\pm^{(1)}$. This is compatible with (3.12) and the assumed conjugation properties of $\mathcal{A}_\pm^{(i)}$ only if $\mathcal{A}_\pm^{(1)}$ is zero as a function. Hence such branch cuts are ruled out, contrary to the case of $AdS_6 \times S^2$ where they were crucial ingredients in the construction of the global solutions.

3.2 Towards string junction solutions

In this section we determine the behavior needed for the functions $\mathcal{A}_\pm$ to source the seven-form charges associated with (p,q)-strings. We shall show that, in addition to reproducing the charges, $\mathcal{A}_\pm$ with this behavior correctly reproduces the metric, axion-dilaton, and two-form fields of the near-horizon limit of the classic $(p,q)-string solution$, provided we carry out a certain coordinate inversion to be explained below. Though we will be able to write down the functions $\mathcal{A}_\pm$ producing (p,q)-string charges at multiple points on $\partial\Sigma$, the question of whether these supergravity solutions are actually geodesically complete for some choices of the parameters remains unsettled.

3.2.1 Realizing the charge and the S^7 of the (p,q)-string solution

To realize a (p,q)-string charge in an $AdS_2 \times S^6$ supergravity solution, we begin by determining the behavior of the functions $\mathcal{A}_\pm$ near a point $b \in \partial\Sigma$ where a (p,q)-string charge resides. A first ingredient is that the supergravity fields should be regular in a neighborhood of the point b with b itself removed, and the seven-form should support (p,q) charge. A second ingredient is the fact that the classic (p,q)-string solution exhibits a round S^7 in the directions transverse to the string. Noting that the metric function f_6^2 vanishes on $\partial\Sigma$, we conclude that the S^7 is realized by a fibration of S^6 over a curve in Σ which begins and ends on $\partial\Sigma$. The angular dependence required to realize this fibration smoothly will constrain the functions $\mathcal{A}_\pm$.

Consider a point $b \in \partial\Sigma$ and local complex coordinates $w, \bar{w}$ which vanish at this point. Regularity and single-valuedness of the supergravity fields f_2^2, f_6^2, ρ^2, and B near $w = 0$ require $\mathcal{A}_\pm$ to be single-valued in a neighborhood of $w = 0$, just as was the case for $AdS_6 \times S^2$ solutions. The extra condition that the factor $d\mathcal{C}$ in $F_{(3)}$ be residue-free at $w = 0$ ensures the absence of five-brane charges and excludes logarithmic branch cuts emanating from $w = 0$. Thus, we shall assume that $\mathcal{A}_\pm$ has a Laurent expansion in w at $w = 0$. While $\mathcal{A}_\pm$ and $\mathcal{B} + \bar{\mathcal{B}}$ are single-valued near $w = 0$, this set-up still allows the factor $d\mathcal{M}$ of $F_{(7)}$ to have a non-zero residue and thus to carry a non-zero (p,q)-string charge.

Next, we determine the order of the pole in $\mathcal{A}_\pm$ by requiring a smooth S^6 slicing of S^7. We shall assume that $\mathcal{A}_\pm$ has a pole at $w=0$ of order at most $p-1$,

$$\mathcal{A}_\pm(w) = \frac{\alpha_\pm}{w^{p-1}} + \frac{\beta_\pm}{w^{p-2}} + \frac{\gamma_\pm}{w^{p-3}} + \cdots \tag{3.15}$$

The coefficients are constrained by (3.12), so that $\bar{\alpha}_\pm = \alpha_\mp$ and likewise for $\beta_\pm$, $\gamma_\pm$, which forces the orders of the poles in $\mathcal{A}_\pm$ to coincide with one another. Whether a smooth 7-cycle is formed around the pole at $w=0$ can be inferred from the ratio f_6^2/ρ^2. In terms of polar coordinates $w = re^{i\theta}$ near the pole, the metric (3.1) may be written as,

$$ds^2 = f_2^2 ds^2_{AdS_2} + 4\rho^2 \left(dr^2 + r^2 d\theta^2 + \frac{f_6^2}{4\rho^2} ds^2_{S^6} \right) \tag{3.16}$$

A smooth cycle is formed if f_6^2/ρ^2 is positive for $\theta \in (0,\pi)$ and approaches zero quadratically as $\theta \to 0$ and $\theta \to \pi$. For $p=2$ no smooth 7-cycle is formed. For $p \geq 3$ we find,

$$\frac{f_6^2}{4\rho^2} = -\frac{3\mathcal{G}}{2\kappa^2} = 3r^2 \frac{(2p-3)\sin\theta - \sin(2p\theta - 3\theta)}{2(p-1)(p-2)(2p-3)\sin\theta} + \mathcal{O}(r^3) \tag{3.17}$$

For $p=3$ we find $f_6^2/4\rho^2 \approx r^2 \sin^2\theta$, giving rise to a round S^7 from S^6 and the polar part of the metric on Σ. For integer $p>3$, a smooth 7-cycle is formed which is not a round S^7. We conclude that the poles in $\mathcal{A}_\pm$ must be double, with $p=3$.

3.2.2 Supergravity fields near a double pole in $\mathcal{A}_\pm$

To obtain the supergravity fields near a double pole in $\mathcal{A}_\pm$, we need the Laurent expansions of these functions to order w^3,

$$\mathcal{A}_\pm = \frac{\alpha_\pm}{w^2} + \frac{\beta_\pm}{w} + \gamma_\pm + \delta_\pm w + \epsilon_\pm w^2 + \chi_\pm w^3 + \mathcal{O}(w^4) \tag{3.18}$$

along with the conjugation condition implied by (3.12) so that $\bar{\alpha}_\pm = \alpha_\mp$, etc. The first regularity condition is that $\kappa^2 > 0$ in the interior of Σ. The leading behavior of κ^2 is obtained from (3.18),

$$\kappa^2 = \frac{2\zeta_{\alpha\beta}\,\mathrm{Im}\,w}{|w|^6} + \mathcal{O}(|w|^{-3}) \qquad\qquad \zeta_{\alpha\beta} = 2i(\alpha_+\beta_- - \alpha_-\beta_+) \tag{3.19}$$

The conjugation conditions imply that $\zeta_{\alpha\beta}$ is real, and positivity of κ^2 in the upper half-plane requires $\zeta_{\alpha\beta} > 0$. In addition, the function $\mathcal{G}$, and hence $\mathcal{B}+\bar{\mathcal{B}}$, must be single-valued. Computing $\mathcal{B}$ in terms of (3.18), we find,

$$\mathcal{B} = \frac{i\zeta_{\alpha\beta}}{6w^3} + \frac{i\zeta_{\alpha\gamma}}{2w^2} + \frac{i(3\zeta_{\alpha\delta}+\zeta_{\beta\gamma})}{2w} - 2i(2\zeta_{\alpha\epsilon}+\zeta_{\beta\delta})\ln w + \mathcal{O}(|w|) \tag{3.20}$$

with $\zeta_{\alpha\gamma}$ etc. defined in analogy with $\zeta_{\alpha\beta}$. Single-valuedness of $\mathcal{B}+\bar{\mathcal{B}}$ requires the purely imaginary coefficient of $\ln w$ to vanish,

$$2\zeta_{\alpha\epsilon}+\zeta_{\beta\delta} = 0 \tag{3.21}$$

The functions $\mathcal{G}$ and T, in terms of which the metric functions f_2^2, f_6^2, ρ^2 are given by (3.4), take the following form near $w=0$,

$$\mathcal{G} \approx -\frac{4\zeta_{\alpha\beta}(\mathrm{Im}\, w)^3}{3|w|^6} \qquad T \approx 4\xi |w|^2 (\mathrm{Im}\, w)^2 \qquad -\xi = \frac{2\zeta_{\alpha\chi}+\zeta_{\beta\epsilon}}{\zeta_{\alpha\beta}} + \frac{\zeta_{\alpha\delta}^2}{\zeta_{\alpha\beta}^2} \tag{3.22}$$

The condition $\zeta_{\alpha\beta} > 0$, which already guaranteed $\kappa^2 > 0$, is seen to also guarantee that $\mathcal{G} < 0$, as is indeed required by the regularity of the supergravity solution. In addition, we impose the requirement $\xi > 0$ to render T positive.

With these conditions fulfilled, the behavior of the functions f_2^2, f_6^2, ρ^2, and $\mathcal{C}$ near the pole is given as follows in terms of polar coordinates $w = re^{i\theta}$ near $w=0$,

$$\rho^2 \approx \frac{\zeta_{\alpha\beta}^{1/2}\xi^{1/4}}{r^{5/2}} \qquad f_6^2 \approx 4r^2\sin^2\theta\rho^2 \qquad f_2^2 \approx \frac{\zeta_{\alpha\beta}^{1/2}}{9\xi^{3/4}r^{9/2}} \qquad \mathcal{C} \approx \frac{-2i\alpha_+}{9\xi r^6} \tag{3.23}$$

The complex axion-dilaton field τ, for $\alpha_+ \neq \alpha_- = \bar{\alpha}_+$, is given by,

$$\mathrm{Re}(\tau) \approx \frac{\mathrm{Re}(\alpha_+)}{\mathrm{Im}(\alpha_+)} \qquad \mathrm{Im}(\tau) \approx \frac{\xi^{1/2}\zeta_{\alpha\beta}}{4\mathrm{Im}(\alpha_+)^2} r^3 \tag{3.24}$$

Finally, the presence of string charge at the pole may be verified by examining the potential $\mathcal{M}$ for the seven-form field strength $F_{(7)}$. By inspection of (3.5) we see that all terms in $\mathcal{M}$ are single-valued by construction, except for the contributions from the locally holomorphic functions $\mathcal{W}_\pm$, whose behavior near $w=0$ is given as follows,

$$\mathcal{W}_\pm(w) = \mathcal{W}_\pm^s(w) - \frac{3}{2}i\Big((3\zeta_{\alpha\chi}+\zeta_{\beta\epsilon})\beta_\pm + \zeta_{\alpha\delta}\delta_\pm - \zeta_{\alpha\beta}\chi_\pm\Big)\ln w \tag{3.25}$$

where $\mathcal{W}_\pm^s$ denotes the single-valued part. Therefore, as w encircles the pole at $w = 0$ counterclockwise in Σ by a 180° degree arc, the potential $\mathcal{M}$ shifts as follows,

$$\mathcal{M} \to \mathcal{M} + 240\pi\Big((3\zeta_{\alpha\chi} + \zeta_{\beta\epsilon})\beta_+ + \zeta_{\alpha\delta}\delta_+ - \zeta_{\alpha\beta}\chi_+\Big) \tag{3.26}$$

The shift in $\mathcal{M}$ gives the integral of the seven-form field strength over the S^7, producing a formula for the p, q string charges in terms of the coefficients of the Laurent series,

$$p + iq = \int_{S^7} F_{(7)} = 80\pi^5\Big((3\zeta_{\alpha\chi} + \zeta_{\beta\epsilon})\beta_+ + \zeta_{\alpha\delta}\delta_+ - \zeta_{\alpha\beta}\chi_+\Big) \tag{3.27}$$

where we have used $\text{vol}(S^7) = \pi^4/3$. Note that the dependence of the string charges p, q on the coefficients of the Laurent series is trilinear, in contrast with the $AdS_6 \times S^2$ case where the five-brane charges had linear dependence.

3.2.3 Satisfying the regularity conditions near a double pole

The various positivity and regularity conditions derived in the preceding subsection may be satisfied simultaneously. To see this, we use the $SU(1,1)$ symmetry of supergravity to rotate $\alpha_\pm$ to be real, and furthermore scale it to 1 without loss of generality. The conditions then reduce to the following relations

$$\zeta_{\alpha\beta} = 4\text{Im}(\beta_+) > 0, \qquad \text{Im}(\epsilon_+) = -\frac{1}{8}\zeta_{\beta\delta}, \qquad \xi = -\frac{(\text{Im}\,\delta_+)^2}{(\text{Im}\,\beta_+)^2} - \frac{8\text{Im}\,\chi_+ + \zeta_{\beta\epsilon}}{4\text{Im}\,\beta_+} > 0 \tag{3.28}$$

For given $\text{Re}(\beta_+)$, $\text{Im}(\beta_+) > 0$, δ_+, and ϵ_+, we may always choose $\text{Im}(-\chi_+)$ large enough to satisfy the remaining condition $\xi > 0$. The expression for the charge p is unenlightening, but the charge q takes the simple form $q = -320\pi^5\xi(\text{Im}\,\beta_+)^2$ and must be negative.

We conclude this subsection with a remark on a no-go result derived in sec. 2.6.3.2. Assuming certain regularity conditions on κ^2 and $\mathcal{G}$, it was argued that $\kappa^2 > 0$ and $\mathcal{G} < 0$ cannot both be realized for compact Σ with boundary. This argument was based on an integral representation for $\mathcal{G}$, obtained by solving the differential relation $\kappa^2 = -\partial_w\partial_{\bar{w}}\mathcal{G}$

$$\mathcal{G}(w) = H(w) + \frac{1}{\pi}\int_\Sigma d^2z\; G(w,z)\kappa^2(z) \tag{3.29}$$

with harmonic H and G the Green's function on Σ. The functions κ^2 and $\mathcal{G}$ obtained here circumvent this no-go result, because they are too singular at the pole to allow for the integral representation (3.29). Indeed, the singularity in κ^2 is not integrable against the Green function, as may be seen from the form of κ^2 and $\mathcal{G}$ given in (3.19) and (3.22). This shows that the assumptions entering the argument of sec. 2.6.3.2 do not hold here.

3.2.4 Matching with the classic (p, q)-string solutions

The classic (p, q)-string solutions of Type IIB supergravity constructed in [39] are labeled by a pair of integers (q_1, q_2) which characterize the charges. The metric, two-form, and axion-dilaton $\tau = \chi + ie^{-\phi}$ are given by,

$$
\begin{aligned}
ds^2 &= A_q^{-3/4} ds^2_{\mathbb{R}^{1,1}} + A_q^{1/4}(dy^2 + y^2 ds^2_{S^7}) & \tau &= \frac{q_1\chi_0 - q_2|\tau_0|^2 + iq_1 e^{-\phi_0} A_q^{1/2}}{q_1 - q_2\chi_0 + iq_2 e^{-\phi_0} A_q^{1/2}} \\
B_{01}^{(i)} &= \left(\mathcal{M}_0^{-1}\right)_{ij} q_j \Delta_q^{-1/2} A_q^{-1} & A_q &= 1 + \frac{\alpha_q}{y^6}
\end{aligned}
\tag{3.30}
$$

The asymptotic values of the axion-dilaton are given by $\tau_0 = \chi_0 + ie^{-\phi_0}$, and we have,

$$
\alpha_q = \Delta_q^{1/2} Q \qquad \Delta_q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}^t \mathcal{M}_0^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \qquad \mathcal{M} = e^{\phi} \begin{pmatrix} |\tau|^2 & \chi \\ \chi & 1 \end{pmatrix}
\tag{3.31}
$$

As $y \to \infty$ we recover flat spacetime $\mathbb{R}^{1,9}$. The near-horizon limit corresponds to $y^6 \ll \alpha_q$, so that the first term in A_q may be neglected in this limit and we have simply $A_q(y) \to \alpha_q/y^6$. The supergravity fields take the following form,

$$
\begin{aligned}
ds^2 &= \frac{y^{9/2}}{\alpha_q^{3/4}} ds^2_{\mathbb{R}^{1,1}} + \frac{\alpha_q^{1/4}}{y^{3/2}}(dy^2 + y^2 ds^2_{S^7}) & \tau &= \frac{q_1\chi_0 - q_2|\tau_0|^2 + iq_1 e^{-\phi_0}\sqrt{\alpha_q}/y^3}{q_1 - q_2\chi_0 + iq_2 e^{-\phi_0}\sqrt{\alpha_q}/y^3} \\
B_{01}^{(i)} &= \left(\mathcal{M}_0^{-1}\right)_{ij} q_j \Delta_q^{-1/2} \frac{y^6}{\alpha_q}
\end{aligned}
\tag{3.32}
$$

In this limit, $y \to 0$ corresponds to the location of the string, but this is a strong coupling limit since the dilaton blows up there. The limit $y \to \infty$ corresponds to the other end of the throat which is also a strong coupling limit. Clearly, identifying the coordinate r of (3.23) with y does not lead to a match between the supergravity fields of the $AdS_2 \times S^6$ solutions and the supergravity fields of the classic (p, q)-string solution to Type IIB. However, if we perform a coordinate inversion on y in the string solution by setting,

$$y = L/r \tag{3.33}$$

then the supergravity fields of the string solution in terms of r are given by,

$$\begin{aligned} ds^2 &= \frac{L^{9/2}}{\alpha_q^{3/4} r^{9/2}} ds^2_{\mathbb{R}^{1,1}} + \frac{L^{1/2}\alpha_q^{1/4}}{r^{5/2}}(dr^2 + r^2 ds^2_{S^7}) \qquad \tau = \frac{q_1\chi_0 - q_2|\tau_0|^2 + iq_1 e^{-\phi_0}\sqrt{\alpha_q} r^3/L^3}{q_1 - q_2\chi_0 + iq_2 e^{-\phi_0}\sqrt{\alpha_q} r^3/L^3} \\ B_{01}^{(i)} &= \left(\mathcal{M}_0^{-1}\right)_{ij} q_j \Delta_q^{-1/2} \frac{L^6}{\alpha_q r^6} \end{aligned} \tag{3.34}$$

which perfectly match with the $AdS_2 \times S^6$ solution provided we identify the parameters,

$$L^3 = \frac{\zeta_{\alpha\beta}}{3} \qquad\qquad \alpha_q = \xi(3\zeta_{\alpha\beta}^2)^{2/3} \tag{3.35}$$

and a corresponding identification for the flux field. Note that the worldvolume for the $AdS_2 \times S^6$ solution is AdS_2, whereas for the classic string solution it is $\mathbb{R}^{1,1}$. The inversion in the identification (3.33) may play a role in the physical interpretation of potential global solutions.

3.2.5 Multiple (p, q) charge solutions on the upper half-plane

In the previous subsection, we have shown that a double pole in the functions $\mathcal{A}_\pm$ on the boundary $\partial\Sigma$ produces supergravity fields which may be identified locally, i.e. in a finite neighborhood of the pole, with the supergravity fields of the classic (p, q)-string solution. Here we shall extend this construction to the case of multiple double poles which are all located on the boundary $\partial\Sigma$. For simplicity, we shall consider the case where Σ has the topology of the upper half-plane, for which the boundary is the real line. Hence we shall consider functions $\mathcal{A}_\pm$ with N double poles, located at points $p_l \in \mathbb{R}$ for $l = 1, \cdots, N$.

To make further progress, we shall assume that $\mathcal{A}_\pm$ are rational functions of w and that $w = \infty$ is a regular point (which may always be achieved by conformal mapping). The functions may therefore be decomposed into partial fractions in w as follows,

$$\mathcal{A}_\pm = \mathcal{A}_\pm^{(0)} + \sum_{l=1}^{N} \left(\frac{Y_\pm^l}{(w - p_l)^2} + \frac{Z_\pm^l}{w - p_l} \right) \tag{3.36}$$

where $Y_-^l = \bar{Y}_+^l$, $Z_-^l = \bar{Z}_+^l$, and $\mathcal{A}_\pm(0)$ are complex parameters which are independent of w. This Ansatz implements the reflection condition (3.12), as a result of which κ^2 and $\mathcal{G}$ vanish on $\partial\Sigma = \mathbb{R}$. It remains to enforce the positivity requirement $\kappa^2 > 0$ everywhere in the interior of the upper half-plane, which in particular requires that $\partial\mathcal{A}_-$ has no zeros in the upper half-plane. We also need the condition that the function $\mathcal{B} + \bar{\mathcal{B}}$ be single-valued.

An alternative formulation starts from the differentials $\partial\mathcal{A}_\pm$, which have a triple pole at each $w = p_l$. We may easily enforce the conditions that the zeros of $\partial\mathcal{A}_+$ and $\partial\mathcal{A}_-$ all be located in the upper and lower half-planes, respectively, by the following parametrization (analogous to the parametrization used for the AdS_6 case in [20]),

$$\partial_w \mathcal{A}_\pm = P_\pm(w) \prod_{l=1}^{N} \frac{1}{(w-p_l)^3}, \quad P_+(w) = \prod_{n=1}^{3N-2} (w - s_n), \quad P_-(w) = \prod_{n=1}^{3N-2} (w - \bar{s}_n) \tag{3.37}$$

with $\mathrm{Im}(s_n) > 0$. In order to integrate to single-valued functions $\mathcal{A}_\pm$ and $\mathcal{B} + \bar{\mathcal{B}}$, the differentials $\partial\mathcal{A}_\pm$ must have vanishing residues at p_l, while the imaginary part of the residue of the differential $\partial_w\mathcal{B}$ must also vanish,

$$\mathrm{Res}(\partial_w \mathcal{A}_\pm)\Big|_{w=p_l} = 0 \qquad\qquad \mathrm{Res}(\partial_w \mathcal{B})\Big|_{w=p_l} \in \mathbb{R} \tag{3.38}$$

The counting of parameters shows that, for a given arrangement of poles p_l, there are $3N-2$ complex zeros, subject to $3N-3$ real residue conditions. Thus parameter counting allows for the existence of large families of solutions. While it is clear that the positivity and regularity conditions are satisfied in the neighborhood of each pole, and that the supergravity fields match onto a classical (p,q)-string solution in the near-horizon limit, it is unclear how to ensure regularity throughout the upper half-plane. The solutions found numerically thus far have all been geodesically incomplete, and this includes the cases with one and two charges. The situation will be discussed explicitly for the case of three charges in the next subsection.

3.2.6 Three charges

In this final subsection, we analyze the case of three double poles in $\mathcal{A}_\pm$, and thus three (p,q)-string charges. In order to conveniently exploit as much symmetry of the configuration as possible, we shall work on the unit disc with complex coordinates $z, \bar{z}$ rather than on the upper half-plane with complex coordinates $w, \bar{w}$. The conjugation condition (3.12) on the disc becomes $\overline{\mathcal{A}_\pm(1/\bar{z})} = \mathcal{A}_\mp(z)$, and we may exploit $SU(1,1)$ symmetry of the unit disc to map the positions of the poles to $1, \varepsilon, \varepsilon^2$ where ε is a non-trivial cube root of unity. The differentials and polynomials of (3.37) then take the form,

$$\partial_z \mathcal{A}_\pm(z) = \frac{P_\pm(z)}{(z^3-1)^3}, \qquad P_+(z) = \sum_{k=0}^{7} c_k z^k, \qquad P_-(z) = \sum_{k=0}^{7} \bar{c}_{7-k} z^k \tag{3.39}$$

The vanishing of the residues of $\partial_z \mathcal{A}_\pm$ at the poles gives two complex linearly independent relations between the coefficients c_k,

$$c_3 = 5c_0 + 2c_6 \qquad\qquad c_4 = 5c_7 + 2c_1 \tag{3.40}$$

while the vanishing of the imaginary part of the residues of $\partial_z \mathcal{B}$ gives two independent real relations, which may be combined into one complex relation between the coefficients c_k,

$$\begin{aligned} 0 = {} & 27|c_7|^2 + 9|c_6|^2 - 2|c_5|^2 + 2|c_2|^2 - 9|c_1|^2 - 27|c_0|^2 \\ & - 18\bar{c}_7 c_6 + 21\bar{c}_2 c_7 - 9\bar{c}_1 c_7 - 36\bar{c}_7 c_1 - 3\bar{c}_6 c_2 \\ & + 9\bar{c}_0 c_6 + 36\bar{c}_6 c_0 + 3\bar{c}_5 c_1 - 21\bar{c}_0 c_5 + 18\bar{c}_1 c_0 \end{aligned} \tag{3.41}$$

where we have eliminated c_3, c_4 using (3.40). Global regularity and geodesic completeness requires furthermore that we have $\kappa^2 > 0$, $\mathcal{G} < 0$, and T real. The condition $\kappa^2 > 0$ in the interior of the disc requires that all the zeros of $P_+(z)$ be in the interior of the disc, which implies that all the zeros of $P_-(z)$ will be outside the disc. We have not been able to solve this condition in any general form, nor numerically for any particular choice of parameters c_k. However, we have also not been able to show convincingly that no solutions exist. The cases with 4 poles have also been explored, but the complexity of the conditions required is then even more involved. In the absence of these results, we are left only with solutions with (p,q)-string charges which are not geodesically complete.

3.3 Discussion

We have constructed an Ansatz for global Type IIB supergravity solutions with 16 supersymmetries on a spacetime of the from $AdS^2 \times S^6$ warped over the unit disc or equivalently the upper half-plane, which may allow for an identification with string junctions. These solutions circumvent the no-go results of sec. 2.6.3.2, and naturally implement the boundary conditions on $\partial\Sigma$ which impose stronger constraints than in the $AdS_6 \times S^2$ case. The remaining conditions for regularity and geodesic completeness were reduced to algebraic constraints on the parameters of the Ansatz, whose complete solution remains an open problem. In analogy with the relation of AdS_6 solutions to M5-brane curves [40], one may expect the data $(\Sigma, \mathcal{A}_\pm)$ for solutions corresponding to string junctions to define the holomorphic curve wrapped by the M2-brane in the M-theory uplift of the string junctions [41, 42, 43, 44].

Finally, we briefly summarize sec. 5 of [2], in which the T-duals of the D0-F1-D8 system in massive Type IIA supergravity [16] is studied. Such solutions take the form $AdS_2 \times S^7$ warped over an interval. It is shown that T-dualizing along the S^1 fiber in the fibration over $\mathbb{CP}^3$ yields a configuration that could naturally arise from Type IIB solutions of the form $AdS_2 \times \mathbb{CP}^3$ warped over a Riemann surface Σ, where the appropriate superalgebra would be $SU(1,1|4)$. However, $\mathbb{CP}^3$ is not maximally symmetric and $AdS_2 \times \mathbb{CP}^3$ can only support 12 supersymmetries instead of the desired number of 16 (see e.g. [45]). Another option for T-duality is the S^1 in the $S^5 \times S^1$ slicing of S^7. While this does not produce an S^6 in the T-dual geometry, such solutions would also realize $SU(1,1|4)$ symmetry and so we consider the case of $AdS_2 \times S^5 \times S^1 \times \Sigma$ in the next chapter.

CHAPTER 4

Warped AdS_2 and $SU(1,1|4)$ symmetry

4.1 $AdS_2 \times S^5 \times S^1 \times \Sigma$ Ansatz in Type IIB supergravity

In this section, we again review key aspects of Type IIB supergravity, then obtain the Ansatz for bosonic supergravity fields and susy generators with $SO(2,1) \oplus SO(6) \oplus SO(2)$ symmetry.

4.1.1 Type IIB supergravity review

The bosonic fields of Type IIB supergravity consist of the metric g_{MN}, the complex-valued axion-dilaton field B, a complex-valued two-form potential $C_{(2)}$ and a real-valued four-form field $C_{(4)}$. The field strengths of the potentials $C_{(2)}$ and $C_{(4)}$ are given as follows,

$$F_{(3)} = dC_{(2)} \qquad\qquad F_{(5)} = dC_{(4)} + \frac{i}{16}(C_{(2)} \wedge \bar{F}_{(3)} - \bar{C}_{(2)} \wedge F_{(3)}) \tag{4.1}$$

The field strength $F_{(5)}$ satisfies the well-known self-duality condition $F_{(5)} = *F_{(5)}$. Instead of the scalar field B and the 3-form $F_{(3)}$, the fields that actually enter the BPS equations are composite fields, namely the one-forms P, Q representing B, and the complex 3-form G representing $F_{(3)}$, given in terms of the fields defined above by the following relations,

$$\begin{aligned} P &= f_B^2\, dB & \qquad f_B^2 &= (1 - |B|^2)^{-1} \\ Q &= f_B^2\, \mathrm{Im}\,(B\, d\bar{B}) \\ G &= f_B(F_{(3)} - B\bar{F}_{(3)}) \end{aligned} \tag{4.2}$$

Under the $SU(1,1) \sim SL(2,\mathbb{R})$ global symmetry of Type IIB supergravity, the Einstein-frame metric g_{MN} and the four-form $C_{(4)}$ are invariant, while B and $C_{(2)}$ transform as,

$$B \to \frac{uB + v}{\bar{v}B + \bar{u}} \qquad\qquad C_{(2)} \to uC_{(2)} + v\bar{C}_{(2)} \tag{4.3}$$

where $SU(1,1)$ is parametrized by $u, v \in \mathbb{C}$ with $|u|^2 - |v|^2 = 1$. The field B takes values in the coset $SU(1,1)/U(1)_q$ and Q acts as a composite $U(1)_q$ gauge field. The $SU(1,1)$ symmetry induces the following transformations on the composite fields [36],

$$\begin{aligned} &P \to e^{2i\theta} P && \theta = \arg(v\bar{B} + u) \\ &Q \to Q + d\theta && \\ &G \to e^{i\theta} G && \end{aligned} \tag{4.4}$$

Equivalently, one may formulate Type IIB supergravity directly in terms of g_{MN}, $F_{(5)}$, P, Q and G provided these fields are subject to the Bianchi identities [36, 37],

$$0 = dP - 2iQ \wedge P \tag{4.5}$$

$$0 = dQ + iP \wedge \bar{P} \tag{4.6}$$

$$0 = dG - iQ \wedge G + P \wedge \bar{G} \tag{4.7}$$

$$0 = dF_{(5)} - \frac{i}{8} G \wedge \bar{G} \tag{4.8}$$

The fermion fields of Type IIB supergravity are the dilatino λ and the gravitino ψ_M. The conditions that these fields and their variations $\delta\lambda$, $\delta\psi_M$ vanish yield the BPS equations, [1]

$$\begin{aligned} 0 &= i(\Gamma \cdot P)\mathcal{B}^{-1}\varepsilon^* - \frac{i}{24}(\Gamma \cdot G)\varepsilon \\ 0 &= (\nabla_M - \frac{i}{2}Q_M)\varepsilon + \frac{i}{480}(\Gamma \cdot F_{(5)})\Gamma_M \varepsilon - \frac{1}{96}(\Gamma_M(\Gamma \cdot G) + 2(\Gamma \cdot G)\Gamma_M)\mathcal{B}^{-1}\varepsilon^* \end{aligned} \tag{4.9}$$

where ε is the supersymmetry generator transforming under the minus chirality Weyl spinor representation of $SO(1,9)$ and ∇_M is the covariant derivative acting on this representation.

4.1.2 $SO(2,1) \oplus SO(6) \oplus SO(2)$-invariant Ansatz for supergravity fields

We construct a general Ansatz for the bosonic fields of Type IIB supergravity consistent with the $SO(2,1) \oplus SO(6) \oplus SO(2)$ symmetry algebra. A natural realization is a spacetime

[1] Repeated indices are summed over, and complex conjugation is denoted by a *bar* for functions and by a *star* for spinors. We use the notation $\Gamma \cdot T \equiv \Gamma^{A_1 \cdots A_p} T_{A_1 \cdots A_p}$ for the contraction of an antisymmetric tensor field T of rank p with a Γ-matrix of the same rank. The matrices Γ^A and $\mathcal{B}$ are defined in Appendix A.

geometry of the form $AdS_2 \times S^5 \times S^1$ warped over a two-dimensional Riemann surface Σ. The $SO(2,1) \oplus SO(6) \oplus SO(2)$-invariant Ansatz for the metric is then of the following form,

$$ds^2 = f_2^2\, d\hat{s}^2_{AdS_2} + f_5^2\, d\hat{s}^2_{S^5} + f_1^2\, d\hat{s}^2_{S^1} + ds^2_\Sigma \tag{4.10}$$

where the radii f_2, f_5, f_1 and ds^2_Σ are functions of Σ. We define an orthonormal frame,

$$\begin{aligned} e^m &= f_2\, \hat{e}^m && m = 0,1 \\ e^i &= f_5\, \hat{e}^i && i = 2,3,4,5,6 \\ e^a &= \rho\, \hat{e}^a && a = 7,8 \\ e^9 &= f_1\, \hat{e}^9 && \end{aligned} \tag{4.11}$$

where $\hat{e}^m$, $\hat{e}^i$, and $\hat{e}^9$ respectively refer to orthonormal frames for the spaces AdS_2, S^5, and S^1 with unit radius. Here, e^a is an orthonormal frame on Σ only, so that we have,

$$\begin{aligned} d\hat{s}^2_{AdS_2} &= \eta^{(2)}_{mn}\, \hat{e}^m \otimes \hat{e}^n && d\hat{s}^2_{S^5} = \delta_{ij}\, \hat{e}^i \otimes \hat{e}^j \\ ds^2_\Sigma &= \delta_{ab}\, e^a \otimes e^b && d\hat{s}^2_{S^1} = \hat{e}^9 \otimes \hat{e}^9 \end{aligned} \tag{4.12}$$

The axion-dilaton field B is a function of Σ only, so the 1-forms P and Q can be written as,

$$P = p_a e^a \qquad\qquad Q = q_a e^a \tag{4.13}$$

where the components p_a, q_a are complex and depend on Σ only. Finally, the complex 3-form G and self-dual 5-form field strength $F_{(5)} = *F_{(5)}$ are given as follows,

$$G = ig_{\bar{a}} e^{01\bar{a}} + h e^{789} \qquad\qquad F_{(5)} = f\left(e^{01789} + e^{23456}\right) \tag{4.14}$$

where the indices $\bar{a}$ run over the values $7,8,9$. The coefficients are constrained by $SO(2,1) \oplus SO(6) \oplus SO(2)$ invariance, so that both the real-valued functions f, q_a and complex-valued functions p_a, h, g_a, g_9 depend only on Σ. This completes the Ansatz for the bosonic fields.

4.1.3 $SO(2,1) \oplus SO(6) \oplus SO(2)$-invariant Ansatz for susy generators

We decompose the supersymmetry generator ε onto the Killing spinors of the various components of $AdS_2 \times S^5 \times S^1$. The Killing spinor equations on AdS_2 and on S^5 were derived in the appendices of [10] and [46], and are given (respectively) by,

$$\left(\hat{\nabla}_m - \frac{1}{2}\eta_1\gamma_m \otimes I_4\right)\chi^{\eta_1,\eta_2} = 0 \qquad \left(\hat{\nabla}_i - \frac{i}{2}\eta_2 I_4 \otimes \gamma_i\right)\chi^{\eta_1,\eta_2} = 0 \tag{4.15}$$

Here, $\hat{\nabla}_m$ and $\hat{\nabla}_i$ are the covariant spinor derivatives on the respective spaces, and integrability requires that $\eta_1^2 = \eta_2^2 = 1$. The action of the chirality matrices is given by,

$$\left(\gamma_{(1)} \otimes I_8\right)\chi^{\eta_1,\eta_2} = \chi^{-\eta_1,\eta_2} \qquad \left(I_2 \otimes \gamma_{(2)}\right)\chi^{\eta_1,\eta_2} = \chi^{\eta_1,+\eta_2} \tag{4.16}$$

while under charge conjugation we have,

$$\chi^{\eta_1,\eta_2} \to (\chi^c)^{\eta_1,\eta_2} = \left(B_{(1)} \otimes B_{(2)}\right)^{-1}(\chi^{\eta_1,\eta_2})^* \propto \chi^{-\eta_1,-\eta_2} \tag{4.17}$$

The components are found by first choosing $(\chi^c)^{++} \equiv \chi^{--}$, then using the chirality matrix $\gamma_{(1)}$ and charge conjugation matrices $B_{(1)}, B_{(2)}$ to obtain the following relations for all η_1, η_2:

$$(\chi^c)^{\eta_1,\eta_2} = \eta_2\chi^{-\eta_1,-\eta_2} \tag{4.18}$$

Killing spinors χ^{η_3} on S^1 are single functions for each value of η_3 which solve the equation,

$$\left(\hat{\nabla}_9 - \frac{i}{2}\eta_3\right)\chi^{\eta_3} = 0 \tag{4.19}$$

As explained in [35], we may set $(\chi^{\eta_3})^* = \chi^{-\eta_3}$, with the values $\eta_3 = \pm 1$ corresponding to a double-valued representation for the spinors. The most general 32-component complex spinor ε that can be decomposed onto the Killing spinors of $AdS_2 \times S^5 \times S^1$, and which is consistent with the 10-dimensional chirality condition $\Gamma^{11}\varepsilon = -\varepsilon$, is of the following form,

$$\varepsilon = \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} \otimes \phi \tag{4.20}$$

where we have defined the constant spinor,

$$\phi \equiv e^{-i\pi/4}\begin{pmatrix}1\\0\end{pmatrix} + e^{i\pi/4}\begin{pmatrix}0\\1\end{pmatrix} \tag{4.21}$$

Finally, the charge conjugate spinor is given by,

$$\mathcal{B}^{-1}\varepsilon^* = \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \star\zeta_{\eta_1,\eta_2,\eta_3} \otimes \phi \qquad \star\zeta_{\eta_1,\eta_2,\eta_3} = -i\eta_2\sigma^2\zeta^*_{-\eta_1,-\eta_2,-\eta_3} \tag{4.22}$$

This completes the construction of the $SO(2,1) \oplus SO(6) \oplus SO(2)$-invariant Ansatz.

4.2 Reducing the BPS equations

For purely bosonic Type IIB supergravity fields, half-BPS configurations are those for which the BPS equations yield 16 independent solutions. In this section, we reduce the BPS equations to the Ansatz, employing the same strategy and methods as those used in [35].

4.2.1 The reduced BPS equations

As before, we use the τ matrix notation introduced originally in [38] to compactly express the action of the various γ matrices on ζ. Defining $\tau^{(ijk)} = \tau^i \otimes \tau^j \otimes \tau^k$ with $i, j, k = 0, 1, 2, 3$, τ^0 the identity matrix and τ^i for $i = 1, 2, 3$ the standard Pauli matrices, we can write,

$$(\tau^{(ijk)}\zeta)_{\eta_1,\eta_2,\eta_3} \equiv \sum_{\eta'_1,\eta'_2,\eta'_3} (\tau^i)_{\eta_1\eta'_1}(\tau^j)_{\eta_2\eta'_2}(\tau^k)_{\eta_3\eta'_3}\zeta_{\eta'_1,\eta'_2,\eta'_3} \tag{4.23}$$

The reduction of the BPS equations (4.9) using the decomposition of ε (4.20) onto the Killing spinors (4.15) is discussed in Appendix B. The reduced dilatino BPS equation is given by,

$$(d) \qquad 0 = 4p_a\gamma^a\sigma^2\zeta^* + ig_{\bar{a}}\tau^{(021)}\gamma^{\bar{a}}\zeta - ih\tau^{(121)}\zeta \tag{4.24}$$

while the various components of the reduced gravitino BPS equations are as follows,

$$\begin{aligned}
(m) \qquad 0 &= \frac{1}{2f_2}\tau^{(300)}\zeta + \frac{D_a f_2}{2f_2}\tau^{(100)}\gamma^a\zeta + \frac{1}{2}f\zeta \\
&\quad + \frac{1}{16}\left(3ig_{\bar{a}}\tau^{(121)}\gamma^{\bar{a}}\sigma^2\zeta^* + ih\tau^{(021)}\sigma^2\zeta^*\right) \\
(i) \qquad 0 &= \frac{1}{2f_5}\tau^{(030)}\zeta + \frac{D_a f_5}{2f_5}\tau^{(100)}\gamma^a\zeta - \frac{1}{2}f\zeta \\
&\quad + \frac{1}{16}\left(-ig_{\bar{a}}\tau^{(121)}\gamma^{\bar{a}}\sigma^2\zeta^* + ih\tau^{(021)}\sigma^2\zeta^*\right) \\
(a) \qquad 0 &= \left(D_a + \frac{i}{2}\hat{\omega}_a\sigma^3\right)\zeta - \frac{i}{2}q_a\zeta + \frac{1}{2}f\tau^{(100)}\gamma_a\zeta \\
&\quad + \frac{1}{16}\left(3ig_a\tau^{(021)}\sigma^2\zeta^* - ig_{\bar{b}}\tau^{(021)}\gamma_a{}^{\bar{b}}\sigma^2\zeta^* - 3ih\tau^{(121)}\gamma_a\sigma^2\zeta^*\right) \\
(9) \qquad 0 &= \frac{i}{2f_1}\tau^{(103)}\sigma^3\zeta + \frac{D_a f_1}{2f_1}\tau^{(100)}\gamma^a\zeta + \frac{1}{2}f\zeta \\
&\quad + \frac{1}{16}\left(3ig_9\tau^{(121)}\sigma^3\sigma^2\zeta^* - ig_a\tau^{(121)}\gamma^a\sigma^2\zeta^* - 3ih\tau^{(021)}\sigma^2\zeta^*\right)
\end{aligned} \tag{4.25}$$

The derivative D_a is defined with respect to the frame e^a of Σ, so that the total differential d_Σ takes the form $d_\Sigma = e^a D_a$, while the $U(1)$-connection with respect to frame indices is $\hat{\omega}_a$.

4.2.2 Symmetries of the reduced BPS equations

The global $SU(1,1)$ symmetry of Type IIB, whose action on the bosonic fields was given in (4.3) and (4.4), survives the reduction to the $SO(2,1) \oplus SO(6) \oplus SO(2)$-invariant Ansatz. Upon reduction to the Ansatz, the $U(1)_q$ gauge transformations of (4.4) are now given by,

$$\begin{aligned} \zeta &\to e^{i\theta/2}\zeta & g_a &\to e^{i\theta} g_a \\ q_a &\to q_a + D_a\theta & g_9 &\to e^{i\theta} g_9 \\ p_a &\to e^{2i\theta} p_a & h &\to e^{i\theta} h \end{aligned} \tag{4.26}$$

In addition to the continuous symmetries, there are linear discrete symmetries which leave the reduced supergravity fields unchanged and act on the supersymmetry generator as follows,

$$\zeta \to \zeta' = S\zeta \qquad\qquad S \in \mathcal{S}_0 \equiv \left\{ I, \tau^{(033)}, i\tau^{(030)}, i\tau^{(003)} \right\} \tag{4.27}$$

Finally, composing complex conjugation with an arbitrary $U(1)_q$ transformation, we have,

$$\begin{aligned} \zeta &\to \mathcal{K}\zeta = e^{i\theta}\tau^{(033)}\sigma^1\zeta^* & g_a &\to \mathcal{K}g_a = e^{2i\theta} g_a^* \\ p_a &\to \mathcal{K}p_a = e^{4i\theta} p_a^* & g_9 &\to \mathcal{K}g_9 = -e^{2i\theta} g_9^* \\ q_a &\to \mathcal{K}q_a = -q_a + 2D_a\theta & h &\to \mathcal{K}h = e^{2i\theta} h^* \end{aligned} \tag{4.28}$$

while the pure discrete complex conjugation corresponds to the special case where $\theta = 0$.

4.2.3 Further reduction and chiral form of the BPS equations

We now derive the restrictions to one of the linear discrete symmetries which are implied by the reduced BPS equations, following the same procedure that was used for [35]. From (4.27), we see that only $\tau^{(033)} \in \mathcal{S}_0$ commutes with the BPS differential operator and admits real eigenvalues. Therefore, we may diagonalize this symmetry simultaneously with the BPS operator and analyze separately the restriction of the BPS equations to the two eigenspaces,

$$\zeta \to \tau^{(033)}\zeta = \nu\zeta \qquad\qquad \nu = \pm 1 \tag{4.29}$$

The non-zero components of ζ are then redefined in terms of a new ζ-spinor with two indices,

$$\nu = +1 \begin{cases} \zeta_{++} & \equiv \zeta_{+++} \\ \zeta_{+-} & \equiv \zeta_{+--} \\ \zeta_{-+} & \equiv \zeta_{-++} \\ \zeta_{--} & \equiv \zeta_{---} \end{cases} \qquad\qquad \nu = -1 \begin{cases} \zeta_{++} & \equiv \zeta_{++-} \\ \zeta_{+-} & \equiv \zeta_{+-+} \\ \zeta_{-+} & \equiv \zeta_{-+-} \\ \zeta_{--} & \equiv \zeta_{--+} \end{cases} \tag{4.30}$$

The remaining elements $i\tau^{(030)}, i\tau^{(003)} \in \mathcal{S}_0$ (4.27) map between identical ν, and along with the complex conjugations symmetries (4.28) reduce under (4.30) to the following transformations,

$$i\tau^{(030)}\zeta \to i\tau^{(03)}\zeta \qquad\qquad i\tau^{(003)}\zeta \to i\nu\tau^{(03)}\zeta \qquad\qquad \mathcal{K}\zeta \to \nu e^{i\theta}\sigma^1\zeta^* \tag{4.31}$$

We then decompose the spinors ζ_{η_1,η_2} in terms of complex frame basis $e^a = (e^z, e^{\bar{z}})$ on Σ, with a metric $\delta_{z\bar{z}} = \delta_{\bar{z}z} = 2$ and Clifford algebra generators $\gamma^a = (\gamma^z, \gamma^{\bar{z}})$ defined as follows,

$$e^z = \frac{1}{2}\left(e^7 + ie^8\right) \qquad e^{\bar{z}} = \frac{1}{2}\left(e^7 - ie^8\right) \qquad \gamma^z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \gamma^{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.32}$$

Similar relations hold for p_a, q_a, g_a, e.g. $p_z = p_7 - ip_8$ and $p_{\bar{z}} = p_7 + ip_8$. In this same 2-dimensional spinor basis, we decompose the two-index spinor ζ into the chirality components,

$$\zeta_{\eta_1,\eta_2} = \begin{pmatrix} \tau^{(02)}\xi^*_{\eta_1,\eta_2} \\ \psi_{\eta_1,\eta_2} \end{pmatrix} \tag{4.33}$$

where $\xi^*_{\eta_1,\eta_2}$, ψ_{η_1,η_2} are 1-component spinors. In this basis, the reduced dilatino equation is,

$$\begin{aligned} &(d_1) \qquad\qquad 0 = 4ip_z\xi - ig_z\psi - ig_9\tau^{(02)}\xi^* + ih\tau^{(12)}\xi^* \\ &(d_2) \qquad\qquad 0 = 4ip^*_{\bar{z}}\psi - ig^*_{\bar{z}}\xi - ig^*_9\tau^{(02)}\psi^* - ih\tau^{(12)}\psi^* \end{aligned} \tag{4.34}$$

The components of the reduced gravitino equation along AdS_2, S^5, S^1 are given by,

$$
\begin{aligned}
&(m_1) \quad 0 = \frac{-i}{2f_2}\tau^{(22)}\xi^* + \frac{D_z f_2}{2f_2}\psi + \frac{1}{2}f\tau^{(12)}\xi^* + \frac{1}{16}\left(3g_z\xi + 3g_9\tau^{(02)}\psi^* + h\tau^{(12)}\psi^*\right) \\
&(m_2) \quad 0 = \frac{i}{2f_2}\tau^{(22)}\psi^* + \frac{D_z f_2}{2f_2}\xi - \frac{1}{2}f\tau^{(12)}\psi^* + \frac{1}{16}\left(3g_{\bar z}^*\psi + 3g_9^*\tau^{(02)}\xi^* - h^*\tau^{(12)}\xi^*\right) \\
&(i_1) \quad 0 = \frac{-i}{2f_5}\tau^{(11)}\xi^* + \frac{D_z f_5}{2f_5}\psi - \frac{1}{2}f\tau^{(12)}\xi^* + \frac{1}{16}\left(-g_z\xi - g_9\tau^{(02)}\psi^* + h\tau^{(12)}\psi^*\right) \\
&(i_2) \quad 0 = \frac{-i}{2f_5}\tau^{(11)}\psi^* + \frac{D_z f_5}{2f_5}\xi + \frac{1}{2}f\tau^{(12)}\psi^* + \frac{1}{16}\left(-g_{\bar z}^*\psi - g_9^*\tau^{(02)}\xi^* - h^*\tau^{(12)}\xi^*\right) \\
&(9_1) \quad 0 = \frac{\nu}{2f_1}\tau^{(01)}\xi^* + \frac{D_z f_1}{2f_1}\psi + \frac{1}{2}f\tau^{(12)}\xi^* + \frac{1}{16}\left(-g_z\xi + 3g_9\tau^{(02)}\psi^* - 3h\tau^{(12)}\psi^*\right) \\
&(9_2) \quad 0 = \frac{\nu}{2f_1}\tau^{(01)}\psi^* + \frac{D_z f_1}{2f_1}\xi - \frac{1}{2}f\tau^{(12)}\psi^* + \frac{1}{16}\left(-g_{\bar z}^*\psi + 3g_9^*\tau^{(02)}\xi^* + 3h^*\tau^{(12)}\xi^*\right)
\end{aligned}
\tag{4.35}
$$

together with the components along Σ,

$$
\begin{aligned}
&(+_1) \qquad 0 = \left(D_{\bar z} - \frac{i}{2}\hat\omega_{\bar z} + \frac{i}{2}q_{\bar z}\right)\xi + \frac{1}{4}g_z^*\psi \\
&(+_2) \qquad 0 = \left(D_z - \frac{i}{2}\hat\omega_z - \frac{i}{2}q_z\right)\psi + f\tau^{(12)}\xi^* + \frac{1}{8}\left(g_z\xi - g_9\tau^{(02)}\psi^* - 3h\tau^{(12)}\psi^*\right) \\
&(-_1) \qquad 0 = \left(D_z - \frac{i}{2}\hat\omega_z + \frac{i}{2}q_z\right)\xi - f\tau^{(12)}\psi^* + \frac{1}{8}\left(g_{\bar z}^*\psi - g_9^*\tau^{(02)}\xi^* + 3h^*\tau^{(12)}\xi^*\right) \\
&(-_2) \qquad 0 = \left(D_{\bar z} - \frac{i}{2}\hat\omega_{\bar z} - \frac{i}{2}q_{\bar z}\right)\psi + \frac{1}{4}g_{\bar z}\xi
\end{aligned}
\tag{4.36}
$$

where $\hat\omega_z = i(\partial_w\rho)/\rho^2$. The action of the complex conjugation symmetry (4.31) is given by,

$$
\xi \to \xi' = -\nu e^{-i\theta}\tau^{(02)}\psi \qquad\qquad \psi \to \psi' = -\nu e^{+i\theta}\tau^{(02)}\xi \tag{4.37}
$$

with the transformations on the bosonic fields (4.28) translated as follows in the chiral basis,

$$
\begin{aligned}
&p_z \to p'_z = e^{4i\theta}(p_{\bar z})^* \qquad && g_z \to g'_z = e^{2i\theta}(g_{\bar z})^* \qquad && h \to h' = e^{2i\theta}h^* \\
&q_z \to q'_z = -q_z + 2D_z\theta && g_9 \to g'_9 = -e^{2i\theta}g_9^* &&
\end{aligned}
\tag{4.38}
$$

Finally, we note that shifting the metric factor $f_1 \to \nu f_1$ removes all explicit dependence on ν from the reduced BPS equations, which is irrelevant since the supergravity fields only ever depend on the square f_1^2. Thus, for every solution to the reduced BPS equations with $\nu = +1$, there exists another solution with $\nu = -1$ so that a systematic doubling of the total number of spinor solutions is produced. Together with the counting of components for the basis of Killing spinors in (4.20), this implies that any solution with $\nu = +1$ produces 16 linearly independent solutions to the BPS equations, thereby generating a half-BPS solution.

4.3 Metric factors in terms of spinor bilinears

In this section, we use the gravitino BPS equations to solve for the metric factors f_1, f_2, f_5. We find that their solutions may be related to bilinears of the spinors ψ, ξ. The reality properties of the metric factors impose the conditions that the spinor bilinears be real and invariant under $U(1)_q$ transformations. The only combinations that satisfy these requirements are those of the form $\psi^\dagger \tau^{(\alpha\beta)} \psi$, $\xi^\dagger \tau^{(\alpha\beta)} \xi$. We seek relations that hold for *generic* values of the supergravity fields $f_1, f_2, f_5, f, g_z, g_{\bar{z}}, h$ and g_9. Following the same procedure that was used for [35], we will use combinations of the differential equations $(\pm)$ in (4.36),

$$\begin{aligned}
D_z \left(\psi^\dagger \tau^{(\alpha\beta)} \psi\right) =& - f \psi^\dagger \tau^{(\alpha\beta)} \tau^{(12)} \xi^* - \frac{1}{4} g_{\bar{z}}^* \xi^\dagger \tau^{(\alpha\beta)} \psi \\
&+ \frac{1}{8} \left(-g_z \psi^\dagger \tau^{(\alpha\beta)} \xi + g_9 \psi^\dagger \tau^{(\alpha\beta)} \tau^{(02)} \psi^* + 3h \psi^\dagger \tau^{(\alpha\beta)} \tau^{(12)} \psi^*\right) \\
D_z \left(\xi^\dagger \tau^{(\alpha\beta)} \xi\right) =& + f \xi^\dagger \tau^{(\alpha\beta)} \tau^{(12)} \psi^* - \frac{1}{4} g_z \psi^\dagger \tau^{(\alpha\beta)} \xi \\
&+ \frac{1}{8} \left(-g_{\bar{z}}^* \xi^\dagger \tau^{(\alpha\beta)} \psi + g_9^* \xi^\dagger \tau^{(\alpha\beta)} \tau^{(02)} \xi^* - 3h^* \xi^\dagger \tau^{(\alpha\beta)} \tau^{(12)} \xi^*\right)
\end{aligned} \tag{4.39}$$

and of the algebraic gravitino BPS equations (4.35) to find relations of the following type,

$$D_z \left(r_1 \psi^\dagger \tau^{(\alpha\beta)} \psi + r_2 \xi^\dagger \tau^{(\alpha\beta)} \xi\right) + \frac{D_z f_i}{f_i} \left(r_3 \psi^\dagger \tau^{(\alpha\beta)} \psi + r_4 \xi^\dagger \tau^{(\alpha\beta)} \xi\right) = 0 \tag{4.40}$$

where $i = 1, 2, 5$, and the coefficients r_1, r_2, r_3, r_4 may depend on i and α, β, but not on Σ.

4.3.1 AdS_2 metric factor

Left-multiplying equation (m_1) by $\psi^\dagger$ and the complex conjugate of equation (m_2) by $\xi^\dagger$, then term by term cancellation imposes the following requirements for generic fields,

$$
\begin{aligned}
(f_2) \qquad & 0 = r_3\tau^{(\alpha\beta)}\tau^{(22)} - r_4\tau^{(22)}\left(\tau^{(\alpha\beta)}\right)^t \\
(f) \qquad & 0 = (r_1+r_3)\tau^{(\alpha\beta)}\tau^{(12)} + (r_2+r_4)\tau^{(12)}\left(\tau^{(\alpha\beta)}\right)^t \\
(g^*_{\bar{z}}) \qquad & 0 = 2r_1 + r_2 + 3r_4 \\
(g_z) \qquad & 0 = 2r_2 + r_1 + 3r_3 \\
(g_9) \qquad & 0 = (r_1 - 3r_3)g_9\psi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\psi^* = (r_2 - 3r_4)g_9^*\xi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\xi^* \\
(h) \qquad & 0 = (3r_1 - r_3)h\psi^\dagger\tau^{(\alpha\beta)}\tau^{(12)}\psi^* = -(3r_2 - r_4)h^*\xi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\xi^*
\end{aligned}
\tag{4.41}
$$

If $r_3 = 0$, then (f_2) implies $r_4 = 0$, and (g_z) and $(g^*_{\bar{z}})$ imply $r_1 = r_2 = 0$. Therefore, for non-trivial solutions we have $r_3 \neq 0$, and without loss of generality we set $r_3 = 1$. Then (f_2) implies that $|r_4| = 1$, so that (g_z) and $(g^*_{\bar{z}})$ reduce to $r_1 = -2r_2 - 3$ and $r_4 = r_2 + 2$, with the condition that $|r_2 + 1| = 1$. For $r_4 = \pm 1$, (f_2) and (f) yield two sets of solutions,

$$
\begin{aligned}
(r_1, r_2, r_3, r_4) = (-1, -1, +1, +1) \qquad & \tau^{(\alpha\beta)} \in \left\{\tau^{(00)}, \tau^{(11)}, \tau^{(12)}, \tau^{(13)}, \tau^{(21)}, \right. \\
& \qquad \left. \tau^{(22)}, \tau^{(23)}, \tau^{(31)}, \tau^{(32)}, \tau^{(33)}\right\} \\
(r_1, r_2, r_3, r_4) = (+3, -3, +1, -1) \qquad & \tau^{(\alpha\beta)} \in \left\{\tau^{(10)}, \tau^{(20)}\right\}
\end{aligned}
\tag{4.42}
$$

4.3.2 S^5 metric factor

Left-multiplying equation (i_1) by $\psi^\dagger$ and the complex conjugate of equation (i_2) by $\xi^\dagger$, then term by term cancellation imposes the following requirements for generic fields,

$$
\begin{aligned}
&(f_5) && 0 = r_3\tau^{(\alpha\beta)}\tau^{(11)} + r_4\tau^{(11)}\left(\tau^{(\alpha\beta)}\right)^t \\
&(f) && 0 = (r_3 - r_1)\tau^{(\alpha\beta)}\tau^{(12)} + (r_4 - r_2)\tau^{(12)}\left(\tau^{(\alpha\beta)}\right)^t \\
&(g^*_{\bar{z}}) && 0 = 2r_1 + r_2 - r_4 \\
&(g_z) && 0 = 2r_2 + r_1 - r_3 \\
&(g_9) && 0 = (r_1 + r_3)g_9\psi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\psi^* = (r_2 + r_4)g_9^*\xi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\xi^* \\
&(h) && 0 = (3r_1 - r_3)h\psi^\dagger\tau^{(\alpha\beta)}\tau^{(12)}\psi^* = -(3r_2 - r_4)h^*\xi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\xi^*
\end{aligned} \tag{4.43}
$$

If $r_3 = 0$, then (f_5) implies $r_4 = 0$, and (g_z) and $(g^*_{\bar{z}})$ imply $r_1 = r_2 = 0$. Therefore, for non-trivial solutions we have $r_3 \neq 0$, and without loss of generality we set $r_3 = 1$. Then (f_5) implies that $|r_4| = 1$, so that (g_z) and $(g^*_{\bar{z}})$ reduce to $r_1 = -2r_2 + 1$ and $r_4 = -3r_2 + 2$, with the condition that $|-3r_2 + 2| = 1$. For $r_4 = \pm 1$, (f_5) and (f) yield two sets of solutions,

$$
\begin{aligned}
&(r_1, r_2, r_3, r_4) = (-1, +1, +1, -1) && \tau^{(\alpha\beta)} \in \left\{\tau^{(00)}, \tau^{(10)}, \tau^{(20)}, \tau^{(33)}\right\} \\
&(r_1, r_2, r_3, r_4) = (1/3, 1/3, +1, +1) && \tau^{(\alpha\beta)} \in \left\{\tau^{(03)}, \tau^{(13)}, \tau^{(23)}, \tau^{(30)}\right\}
\end{aligned} \tag{4.44}
$$

4.3.3 S^1 metric factor

Left-multiplying equation (9_1) by $\psi^\dagger$ and the complex conjugate of equation (9_2) by $\xi^\dagger$, then term by term cancellation imposes the following requirements for generic fields,

$$
\begin{aligned}
&(f_1) && 0 = r_3\tau^{(\alpha\beta)}\tau^{(01)} + r_4\tau^{(01)}\left(\tau^{(\alpha\beta)}\right)^t \\
&(f) && 0 = (r_1 + r_3)\tau^{(\alpha\beta)}\tau^{(12)} + (r_2 + r_4)\tau^{(12)}\left(\tau^{(\alpha\beta)}\right)^t \\
&(g^*_{\bar{z}}) && 0 = 2r_1 + r_2 - r_4 \\
&(g_z) && 0 = 2r_2 + r_1 - r_3 \\
&(g_9) && 0 = (r_1 - 3r_3)g_9\psi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\psi^* = (r_2 - 3r_4)g_9^*\xi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\xi^* \\
&(h) && 0 = (r_1 + r_3)h\psi^\dagger\tau^{(\alpha\beta)}\tau^{(12)}\psi^* = -(r_2 + r_4)h^*\xi^\dagger\tau^{(\alpha\beta)}\tau^{(02)}\xi^*
\end{aligned} \tag{4.45}
$$

If $r_3 = 0$, then (f_1) implies $r_4 = 0$, and (g_z) and $(g^*_{\bar{z}})$ imply $r_1 = r_2 = 0$. Therefore, for non-trivial solutions we have $r_3 \neq 0$, and without loss of generality we set $r_3 = 1$. Then (f_1) implies that $|r_4| = 1$, so that (g_z) and $(g^*_{\bar{z}})$ reduce to $r_1 = -2r_2 + 1$ and $r_4 = -3r_2 + 2$, with the condition that $|-3r_2 + 2| = 1$. For $r_4 = \pm 1$, (f_1) and (f) yield two sets of solutions,

$$\begin{aligned}
(r_1, r_2, r_3, r_4) &= (-1, +1, +1, -1) \qquad & \tau^{(\alpha\beta)} &\in \big\{\tau^{(00)}, \tau^{(01)}, \tau^{(02)}, \tau^{(10)}, \tau^{(11)}, \\
& & &\qquad \tau^{(12)}, \tau^{(23)}, \tau^{(30)}, \tau^{(31)}, \tau^{(32)}\big\} \\
(r_1, r_2, r_3, r_4) &= (1/3, 1/3, +1, +1) \qquad & \tau^{(\alpha\beta)} &\in \big\{\tau^{(03)}, \tau^{(13)}, \tau^{(21)}, \tau^{(22)}\big\}
\end{aligned} \tag{4.46}$$

4.3.4 Summary of expressions

Imposing the (g_9) and (h) conditions in each case, then in terms of the Hermitian forms,

$$H_\pm^{(\alpha\beta)} \equiv \psi^\dagger \tau^{(\alpha\beta)} \psi \pm \xi^\dagger \tau^{(\alpha\beta)} \xi \tag{4.47}$$

we have the following generic relations, valid for arbitrary values of all the supergravity fields,

$$\begin{aligned}
f_2^{-1} H_+^{(00)} &= C_2^{(00)} & & \\
f_2^{1/3} H_-^{(\alpha\beta)} &= C_2^{(\alpha\beta)} & \tau^{(\alpha\beta)} &\in \big\{\tau^{(10)}, \tau^{(20)}\big\} \\
& & & \\
f_5^{-1} H_-^{(\alpha\beta)} &= C_5^{(\alpha\beta)} & \tau^{(\alpha\beta)} &\in \big\{\tau^{(00)}, \tau^{(10)}, \tau^{(20)}, \tau^{(33)}\big\} \\
f_5^{3} H_+^{(\alpha\beta)} &= C_5^{(\alpha\beta)} & \tau^{(\alpha\beta)} &\in \big\{\tau^{(23)}, \tau^{(30)}\big\} \\
& & & \\
f_1^{-1} H_-^{(\alpha\beta)} &= C_1^{(\alpha\beta)} & \tau^{(\alpha\beta)} &\in \big\{\tau^{(00)}, \tau^{(10)}, \tau^{(23)}, \tau^{(30)}\big\}
\end{aligned} \tag{4.48}$$

Since both f_2 and $\psi^\dagger\psi + \xi^\dagger\xi$ must be positive, we rescale ξ and ψ by a real constant, so that,

$$H_+^{(00)} = f_2 \tag{4.49}$$

4.4 Vanishing Hermitian forms

We can use the reality properties of various combinations of the BPS equations to show that certain Hermitian forms vanish automatically. We consider the following Hermitian forms,

$$\begin{aligned} H_{\pm}^{(\alpha\beta)} &\equiv \psi^\dagger \tau^{(\alpha\beta)} \psi \pm \xi^\dagger \tau^{(\alpha\beta)} \xi \\ H_{g\pm}^{(\alpha\beta)} &\equiv g_9 \psi^\dagger \tau^{(\alpha\beta)} \xi \pm g_9^* \xi^\dagger \tau^{(\alpha\beta)} \psi \\ H_{h\pm}^{(\alpha\beta)} &\equiv h \psi^\dagger \tau^{(\alpha\beta)} \xi \pm h^* \xi^\dagger \tau^{(\alpha\beta)} \psi \end{aligned} \tag{4.50}$$

where $H_{\pm}^{(\alpha\beta)}$, $H_{g+}^{(\alpha\beta)}$, $H_{h+}^{(\alpha\beta)}$ are real, while $H_{g-}^{(\alpha\beta)}$, $H_{h-}^{(\alpha\beta)}$ are purely imaginary. In the following sections, we consider three particular combinations. Then in section 4.4.4, separating out the real and imaginary parts yields the full sets of vanishing and non-trivial Hermitian relations.

4.4.1 First set of Hermitian relations

We consider the linear combination $(m) + 2(i) + (9)$ of the BPS equations (4.35). Note that all the terms containing f, g_z, $g_{\bar{z}}$, h, h^* are cancelled. Multiplying the first equation by $\xi^t \tau^{(\alpha\beta)}$, the second by $-\psi^t \tau^{(\alpha\beta)t}$, then adding them and taking the transpose, we obtain,

$$\begin{aligned} 0 = {}& \psi^\dagger \left(-\frac{i}{f_2} \tau^{(22)} + \frac{2i}{f_5} \tau^{(11)} - \frac{\nu}{f_1} \tau^{(01)} \right) \tau^{(\alpha\beta)} \psi - \frac{1}{2} g_9 \psi^\dagger \tau^{(02)} \tau^{(\alpha\beta)^t} \xi \\ & + \xi^\dagger \left(-\frac{i}{f_2} \tau^{(22)} - \frac{2i}{f_5} \tau^{(11)} + \frac{\nu}{f_1} \tau^{(01)} \right) \tau^{(\alpha\beta)^t} \xi + \frac{1}{2} g_9^* \xi^\dagger \tau^{(02)} \tau^{(\alpha\beta)} \psi \end{aligned} \tag{4.51}$$

4.4.2 Second set of Hermitian relations

We eliminate the $D_z f_i$, g_z, $g_{\bar{z}}^*$ terms in each set of equations (m), (i), or (9). We calculate only the (m) and (i) equations, since the relations for the (9) equation can be obtained from a linear combination of the (m) and (i) equations, together with the first set of relations.

For each pair of the f_2 and f_5 equations in the BPS equations (4.35), we multiply the first by $\xi^t \tau^{(\alpha\beta)}$ and the second by $\psi^t \tau^{(\alpha\beta)}$. The g_z, $g_{\bar{z}}^*$ terms then vanish automatically if

$\tau^{(\alpha\beta)^t} = -\tau^{(\alpha\beta)}$. Adding both to cancel the $D_z f_i$ terms, then taking the transpose, we have,

$$
\begin{aligned}
(m): \quad 0 &= \psi^\dagger \left(\frac{i}{f_2}\tau^{(22)} + f\tau^{(12)} \right) \tau^{(\alpha\beta)}\psi - \frac{3}{8} g_9 \psi^\dagger \tau^{(02)}\tau^{(\alpha\beta)}\xi - \frac{1}{8} h \psi^\dagger \tau^{(12)}\tau^{(\alpha\beta)}\xi \\
&+ \xi^\dagger \left(-\frac{i}{f_2}\tau^{(22)} - f\tau^{(12)} \right) \tau^{(\alpha\beta)}\xi - \frac{3}{8} g_9^* \xi^\dagger \tau^{(02)}\tau^{(\alpha\beta)}\psi + \frac{1}{8} h^* \xi^\dagger \tau^{(12)}\tau^{(\alpha\beta)}\psi \\
(i): \quad 0 &= \psi^\dagger \left(-\frac{i}{f_5}\tau^{(11)} - f\tau^{(12)} \right) \tau^{(\alpha\beta)}\psi + \frac{1}{8} g_9 \psi^\dagger \tau^{(02)}\tau^{(\alpha\beta)}\xi - \frac{1}{8} h \psi^\dagger \tau^{(12)}\tau^{(\alpha\beta)}\xi \\
&+ \xi^\dagger \left(-\frac{i}{f_5}\tau^{(11)} + f\tau^{(12)} \right) \tau^{(\alpha\beta)}\xi + \frac{1}{8} g_9^* \xi^\dagger \tau^{(02)}\tau^{(\alpha\beta)}\psi + \frac{1}{8} h^* \xi^\dagger \tau^{(12)}\tau^{(\alpha\beta)}\psi \quad (4.52)
\end{aligned}
$$

4.4.3 Third set of Hermitian relations

Finally, we consider the combination $(i) - (9)$. We multiply the first equation by $\xi^t \tau^{(\alpha\beta)}$ and the second by $\psi^t \tau^{(\alpha\beta)}$, with $\tau^{(\alpha\beta)^t} = +\tau^{(\alpha\beta)}$. Taking the difference and then the transpose,

$$
\begin{aligned}
0 &= \psi^\dagger \left(\frac{i}{f_5}\tau^{(11)} + \frac{\nu}{f_1}\tau^{(01)} + 2f\tau^{(12)} \right) \tau^{(\alpha\beta)}\psi + \psi^\dagger \left(\frac{1}{2} g_9 \tau^{(02)} - \frac{1}{2} h \tau^{(12)} \right) \tau^{(\alpha\beta)}\xi \\
&+ \xi^\dagger \left(-\frac{i}{f_5}\tau^{(11)} - \frac{\nu}{f_1}\tau^{(01)} + 2f\tau^{(12)} \right) \tau^{(\alpha\beta)}\xi + \xi^\dagger \left(-\frac{1}{2} g_9^* \tau^{(02)} - \frac{1}{2} h^* \tau^{(12)} \right) \tau^{(\alpha\beta)}\psi \quad (4.53)
\end{aligned}
$$

4.4.4 Summary of all Hermitian relations

The full set of vanishing Hermitian relations is given by,

$$
\begin{aligned}
&H_+^{\alpha\beta} = 0 \qquad (\alpha\beta) \in \{(03), (11), (12), (13), (23), (33)\} \\
&H_-^{\alpha\beta} = 0 \qquad (\alpha\beta) \in \{(00), (01), (02), (10), (20), (21), (22), (30), (31), (32)\} \\
\\
&H_{g+}^{\alpha\beta} = 0 \qquad (\alpha\beta) \in \{(00), (10), (23), (30)\} \\
&H_{g-}^{\alpha\beta} = 0 \qquad (\alpha\beta) \in \{(03), (11), (12), (13), (31), (32), (33)\} \\
\\
&H_{h+}^{\alpha\beta} = 0 \qquad (\alpha\beta) \in \{(03), (11), (12), (13), (23)\} \\
&H_{h-}^{\alpha\beta} = 0 \qquad (\alpha\beta) \in \{(00), (10), (20), (33)\} \qquad (4.54)
\end{aligned}
$$

The remaining non-trivial Hermitian relations are as follows. We have the first set,

$$\begin{aligned}
&(00) && \frac{1}{f_2}H_+^{(22)} - \frac{2}{f_5}H_-^{(11)} - \frac{i}{2}H_{g-}^{(02)} = 0 \\
&(03) && \frac{1}{f_2}H_+^{(21)} + \frac{2}{f_5}H_-^{(12)} - i\frac{1}{2}H_{g-}^{(01)} = 0 \\
&(10) && \frac{1}{f_2}H_+^{(32)} + \frac{\nu}{f_1}H_-^{(11)} = 0 \\
&(13) && \frac{1}{f_2}H_+^{(31)} - \frac{\nu}{f_1}H_-^{(12)} = 0 \\
&(20) && \frac{2}{f_5}H_+^{(31)} + \frac{\nu}{f_1}H_+^{(21)} - \frac{1}{2}H_{g+}^{(22)} = 0 \\
&(21) && \frac{1}{f_2}H_-^{(03)} + \frac{2}{f_5}H_+^{(30)} + \frac{\nu}{f_1}H_+^{(20)} = 0 \\
&(22) && \frac{1}{f_2}H_+^{(00)} + \frac{2}{f_5}H_-^{(33)} + \frac{\nu}{f_1}H_-^{(23)} - \frac{i}{2}H_{g-}^{(20)} = 0 \\
&(23) && \frac{2}{f_5}H_+^{(32)} + \frac{\nu}{f_1}H_+^{(22)} + \frac{1}{2}H_{g+}^{(21)} = 0
\end{aligned} \tag{4.55}$$

the second set,

$$\begin{aligned}
&(20) && 3H_{g+}^{(22)} + iH_{h-}^{(32)} = 0 && \frac{1}{f_5}H_+^{(31)} + \frac{1}{8}H_{g+}^{(22)} - i\frac{1}{8}H_{h-}^{(32)} = 0 \\
&(21) && \frac{1}{f_2}H_-^{(03)} + fH_-^{(33)} = 0 && \frac{1}{f_5}H_+^{(30)} - fH_-^{(33)} = 0 \\
&(23) && 3H_{g+}^{(21)} + iH_{h-}^{(31)} = 0 && \frac{1}{f_5}H_+^{(32)} - \frac{1}{8}H_{g+}^{(21)} + \frac{i}{8}H_{h-}^{(31)} = 0
\end{aligned} \tag{4.56}$$

and finally the third set,

$$\begin{aligned}
&(00) && \frac{1}{f_5}H_-^{(11)} - i\frac{1}{2}H_{g-}^{(02)} = 0 \\
&(03) && \frac{1}{f_5}H_-^{(12)} + i\frac{1}{2}H_{g-}^{(01)} = 0 \\
&(10) && \frac{\nu}{f_1}H_-^{(11)} + 2fH_+^{(02)} - \frac{1}{2}H_{h+}^{(02)} = 0 \\
&(13) && \frac{\nu}{f_1}H_-^{(12)} - 2fH_+^{(01)} + \frac{1}{2}H_{h+}^{(01)} = 0 \\
&(22) && \frac{1}{f_5}H_-^{(33)} - \frac{\nu}{f_1}H_-^{(23)} - 2fH_+^{(30)} + \frac{i}{2}H_{g-}^{(20)} + \frac{1}{2}H_{h+}^{(30)} = 0 \\
&(30) && 2fH_+^{(22)} - \frac{1}{2}H_{h+}^{(22)} = 0 \\
&(33) && 2fH_+^{(21)} - \frac{1}{2}H_{h+}^{(21)} = 0
\end{aligned} \tag{4.57}$$

4.4.5 Implications for the metric factors

Together with (4.48), the above relations imply the vanishing of the following constants,

$$\begin{aligned} 0 &= C_2^{(10)} = C_2^{(20)} \\ 0 &= C_5^{(00)} = C_5^{(10)} = C_5^{(20)} = C_5^{(23)} \\ 0 &= C_1^{(00)} = C_1^{(10)} = C_1^{(30)} \end{aligned} \tag{4.58}$$

which leaves the following non-vanishing Hermitian forms,

$$\begin{aligned} f_2^{-1} H_+^{(00)} &= 1 \\ f_5^{-1} H_-^{(33)} &= C_5^{(33)} \\ f_5^3 H_+^{(30)} &= C_5^{(30)} \\ f_1^{-1} H_-^{(23)} &= C_1^{(23)} \end{aligned} \tag{4.59}$$

where we have used the normalization $C_2^{(00)} = 1$.

4.5 General solutions to the reduced BPS equations

In this section, we use the vanishing Hermitian forms to solve the reduced BPS equations. We follow the same procedure and reach the same conclusion as in [35], namely that the only solution to the reduced BPS equations is the maximally supersymmetric solution $AdS_5 \times S^5$.

4.5.1 Solving the Hermitian relations $H_\pm^{(\alpha\beta)} = 0$

Grouping the vanishing Hermitian relations $H_\pm^{(\alpha\beta)} = 0$ from (4.54) into four sets, we obtain the following relations between the spinor components for $\eta_1 = \pm$ and $\eta_2 = \pm$ independently,

$$\begin{aligned} 0 = H_-^{(00)} = H_-^{(30)} = H_+^{(03)} = H_+^{(33)} \quad &\Longrightarrow \quad \xi^*_{\eta_1,\eta_2}\xi_{\eta_1,\eta_2} - \psi^*_{\eta_1,-\eta_2}\psi_{\eta_1,-\eta_2} = 0 \\ 0 = H_-^{(10)} = H_-^{(20)} = H_+^{(13)} = H_+^{(23)} \quad &\Longrightarrow \quad \xi^*_{\eta_1,\eta_2}\xi_{-\eta_1,\eta_2} - \psi^*_{\eta_1,-\eta_2}\psi_{-\eta_1,-\eta_2} = 0 \\ 0 = H_-^{(01)} = H_-^{(31)} = H_-^{(02)} = H_-^{(32)} \quad &\Longrightarrow \quad \xi^*_{\eta_1,\eta_2}\xi_{\eta_1,-\eta_2} - \psi^*_{\eta_1,\eta_2}\psi_{\eta_1,-\eta_2} = 0 \\ 0 = H_+^{(11)} = H_-^{(21)} = H_+^{(12)} = H_-^{(22)} \quad &\Longrightarrow \quad \xi^*_{\eta_1,\eta_2}\xi_{-\eta_1,-\eta_2} + \psi^*_{-\eta_1,\eta_2}\psi_{\eta_1,-\eta_2} = 0 \end{aligned} \tag{4.60}$$

When the ψ_{η_1,η_2} are all generic and non-vanishing, the solutions to (4.60) are of the form,

$$\begin{aligned}
\psi_{++} &= r_{++}e^{i\Lambda+i\Phi} & \xi_{++} &= e^{i\theta_1}\psi_{+-} = r_{+-}e^{i\Lambda'+i\Phi} \\
\psi_{+-} &= r_{+-}e^{i\Lambda-i\Phi} & \xi_{+-} &= e^{i\theta_2}\psi_{++} = r_{++}e^{i\Lambda'-i\Phi} \\
\psi_{-+} &= r_{-+}e^{i\Lambda+i\Phi+i\pi/2} & \xi_{-+} &= e^{i\theta_1}\psi_{--} = r_{--}e^{i\Lambda'+i\Phi+i\pi/2} \\
\psi_{--} &= r_{--}e^{i\Lambda-i\Phi+i\pi/2} & \xi_{--} &= e^{i\theta_2}\psi_{-+} = r_{-+}e^{i\Lambda'-i\Phi+i\pi/2}
\end{aligned} \tag{4.61}$$

parametrized in terms of 4 real functions r_{η_1,η_2} plus the angles $\theta_1 = 2\Phi+2\Phi'$, $\theta_2 = -2\Phi+2\Phi'$, $\Lambda' = 2\Phi' + \Lambda$, and Λ arbitrary. The case where one component $\psi_{\eta_1,\eta_2} = 0$ can be viewed as the limit in which $r_{\eta_1,\eta_2} = 0$. The only exception is when $\psi_{\eta_1,\eta_2} = \psi_{-\eta_1,\eta_2} = 0$ and $\psi_{\eta_1,-\eta_2} = \psi_{-\eta_1,-\eta_2} \neq 0$. We consider the case $\psi_{+-} = \psi_{--} = 0$ and $\psi_{++}, \psi_{-+} \neq 0$, which may be parametrized by four real fuctions $r_{++} \equiv r_1, r_{-+} \equiv r_3, \Lambda_1, \Lambda_3$, plus an angle θ, as follows,

$$\begin{aligned}
\psi_{++} &= r_1 e^{i\Lambda_1} & \xi_{+-} &= e^{i\theta}\psi_{++} = r_1 e^{i(\Lambda_1+\theta)} \\
\psi_{-+} &= r_3 e^{i\Lambda_3} & \xi_{--} &= e^{i\theta}\psi_{-+} = r_3 e^{i(\Lambda_3+\theta)}
\end{aligned} \tag{4.62}$$

The solutions (4.61) - which we will refer to as the "first type" of solutions - reproduce all the relations $H_\pm^{(\alpha\beta)} = 0$ in (4.54), as well as two additional relations not listed in section 4.4.4,

$$H_+^{(10)} = 0 \qquad\qquad H_-^{(13)} = 0 \tag{4.63}$$

The solutions (4.62) - which we will refer to as the "second type" of solutions - reproduce all the relations $H_\pm^{(\alpha\beta)} = 0$ in (4.54), as well as the following additional vanishing conditions,

$$0 = H_+^{(01)} = H_+^{(02)} = H_+^{(21)} = H_+^{(22)} = H_+^{(31)} = H_+^{(32)} = H_-^{(11)} = H_-^{(12)} \tag{4.64}$$

4.5.2 Solving the Hermitian relations $H_{g\pm}^{(\alpha\beta)}$, $H_{h\pm}^{(\alpha\beta)}$

Next, we use the solutions of the previous section to obtain conditions from the remaining vanishing Hermitian forms. It is then straightforward but tedious to show that the only non-trivial solutions are those with $g_9 = 0$. The calculation parallels the one in [35], so we will summarize the results while highlighting any notable differences. We define the quantities,

$$r = \begin{pmatrix} r_{++} \\ r_{+-} \\ r_{-+} \\ r_{--} \end{pmatrix} \equiv \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} \qquad \begin{Bmatrix} \mathcal{G}_\pm \\ \\ \mathcal{H}_\pm \end{Bmatrix} \equiv \begin{Bmatrix} g_9 e^{i(\Lambda'-\Lambda)} \pm g_9^* e^{-i(\Lambda'-\Lambda)} \\ \\ h e^{i(\Lambda'-\Lambda)} \pm h^* e^{-i(\Lambda'-\Lambda)} \end{Bmatrix} \tag{4.65}$$

4.5.2.1 First type of solution

For the first type of solutions (4.61), the following Hermitian forms vanish automatically,

$$H_{g\pm}^{(\alpha\beta)} = H_{h\pm}^{(\alpha\beta)} = 0 \qquad (\alpha\beta) \in \{(03), (10), (11), (12), (23), (33)\} \tag{4.66}$$

The remaining vanishing Hermitian forms (4.54) yield two sets of conditions for $H_{g\pm}^{(\alpha\beta)} = 0$:

$$\mathcal{G}_+ r^t \tau^{(\gamma\delta)} r = 0 \qquad (\gamma\delta) \in \{(01), (22), (31)\} \tag{4.67}$$

$$\mathcal{M}_g \begin{pmatrix} r^t \tau^{(30)} r \\ r^t \tau^{(33)} r \end{pmatrix} \equiv \begin{pmatrix} \cos(2\Phi)\, \mathcal{G}_- & -i \sin(2\Phi)\, \mathcal{G}_+ \\ -\sin(2\Phi)\, \mathcal{G}_- & -i\cos(2\Phi)\, \mathcal{G}_+ \end{pmatrix} \begin{pmatrix} r^t \tau^{(30)} r \\ r^t \tau^{(33)} r \end{pmatrix} = 0 \tag{4.68}$$

From the relations (4.54), we also have a set of conditions for the Hermitian forms $H_{h\pm}^{(\alpha\beta)} = 0$:

$$\mathcal{H}_- r^t \tau^{(\gamma\delta)} r = 0 \qquad (\gamma\delta) \in \{(01), (11), (22)\} \tag{4.69}$$

From (4.68) we have $\det \mathcal{M}_g = -i\mathcal{G}_+\mathcal{G}_-$. For trivial solutions corresponding to $\mathcal{G}_+\mathcal{G}_- \neq 0$, the conditions (4.67) and (4.68), together with the (22) relations in (4.55) and (4.57), imply $H_+^{(00)} = 0$ and thus all $r_{\eta_1,\eta_2} = 0$. Non-trivial solutions correspond to $\mathcal{G}_+\mathcal{G}_- = 0$, and one can show that for either choice $\mathcal{G}_\pm = 0$ and $\mathcal{G}_\mp \neq 0$, the conditions (4.67) and (4.68) plus the non-trivial relations from section 4.4.4, imply that all $r_{\eta_1,\eta_2} = 0$. For example, if $\mathcal{G}_+ = 0$ and $\mathcal{G}_- \neq 0$ then (4.67) is automatically satisfied, while (4.68) and the second (21) relation in (4.56) imply,

$$r^t \tau^{(30)} r = 0: \qquad H_-^{(33)} = \frac{1}{f f_5} H_+^{(30)} = 0 \qquad \Longrightarrow \qquad r^t \tau^{(33)} r = 0 \tag{4.70}$$

The Hermitian forms that vanish under $r^t\tau^{(30)}r = r^t\tau^{(33)}r = 0$ cause a number of relations in section 4.4.4 to become trivial, which in turn produce conditions that can only be satisfied if all $r_{\eta_1,\eta_2} = 0$. The only remaining possibility is $g_9 = 0$, which yields extra vanishing forms:

$$
\begin{aligned}
0 &= H_+^{(21)} = H_+^{(22)} = H_+^{(31)} = H_+^{(32)} = H_-^{(11)} = H_-^{(12)} \\
0 &= H_{h+}^{(21)} = H_{h+}^{(22)} = H_{h-}^{(31)} = H_{h-}^{(32)}
\end{aligned}
\tag{4.71}
$$

The top line of (4.71) plus the original vanishing Hermitian forms imply the conditions,

$$
r_1 r_4 = r_2 r_3 = 0 \qquad\qquad r_1 r_2 - r_3 r_4 = 0 \tag{4.72}
$$

Without loss of generality, we choose $r_4 = 0$, so that either $r_1 = r_3 = 0$ or $r_2 = 0$, and examine the dilatino equation (4.34). If $r_1, r_3 \neq 0$ and $r_2 = r_4 = 0$, then we must have,

$$
p_z = p_{\bar{z}} = 0 \qquad\qquad |h|^2 - |g_z|^2 = 0 \qquad\qquad |h|^2 - |g_{\bar{z}}|^2 = 0 \tag{4.73}
$$

for non-vanishing spinor solutions. But in order to have non-trivial solutions while satisfying both the original conditions (4.69) and the bottom line of (4.71), we must set $h = 0$ so that,

$$
p_z = p_{\bar{z}} = g_z = g_{\bar{z}} = h = 0 \tag{4.74}
$$

On the other hand, if we take $r_1 = r_3 = r_4 = 0$ and $r_3 \neq 0$, then this result is automatic.

4.5.2.2 Second type of solution

For the second type of solutions (4.62), the following Hermitian forms vanish automatically,

$$
H_{g\pm}^{(\alpha\beta)} = H_{h\pm}^{(\alpha\beta)} = 0 \qquad\qquad (\alpha\beta) \in \{(00), (03), (10), (13), (20), (23), (30), (33)\} \tag{4.75}
$$

and the extra forms $H_{h+}^{(30)} = H_{g-}^{(20)} = 0$ modify the (22) relations in (4.55) and (4.57) as follows,

$$
\frac{1}{f_2} H_+^{(00)} + \frac{2}{f_5} H_-^{(33)} + \frac{\nu}{f_1} H_-^{(23)} = 0 \qquad\qquad \frac{1}{f_5} H_-^{(33)} - \frac{\nu}{f_1} H_-^{(23)} - 2f H_+^{(30)} = 0 \tag{4.76}
$$

From the remaining cases of $H_{g\pm}^{(\alpha\beta)} = 0$ and $H_{h\pm}^{(\alpha\beta)} = 0$ in (4.54), we obtain the conditions,

$$
\begin{aligned}
\left[e^{i(\Lambda_1 - \Lambda_3)} + e^{-i(\Lambda_1 - \Lambda_3)}\right] \left(e^{i\theta} g_9 \pm e^{-i\theta} g_9^*\right) r_1 r_3 &= 0 \\
\left(e^{i\theta} g_9 \pm e^{-i\theta} g_9^*\right) \left(r_1^2 - r_3^2\right) &= 0
\end{aligned}
\tag{4.77}
$$

$$\left[e^{i(\Lambda_1-\Lambda_3)}+e^{-i(\Lambda_1-\Lambda_3)}\right]\left(e^{i\theta}h\pm e^{-i\theta}h^*\right)r_1r_3=0 \tag{4.78}$$

For non-trivial solutions with $g_9\neq 0$, we must have $\mathrm{Re}\left[e^{i(\Lambda_1-\Lambda_3)}\right]=0$ and $r_1^2=r_3^2$. Under this choice, the (22) relations of (4.55) and (4.57) reduce to $H_+^{(00)}=0$ and thus all $r_{\eta_1,\eta_2}=0$. So we again must have $g_9=0$, which then yields the additional vanishing Hermitian forms,

$$0=H_{h+}^{(21)}=H_{h+}^{(22)}=H_{h-}^{(31)}=H_{h-}^{(32)} \tag{4.79}$$

Examining the dilatino equation (4.34), we find the same constraints as (4.73) on the supergravity fields. The extra forms in (4.76) together with (4.79) impose the following conditions:

$$\Lambda_{\eta_1}\Theta_{\eta_2}r_1r_3=0 \qquad\qquad \Theta_{\eta_3}\left(r_1^2-r_3^2\right)=0 \tag{4.80}$$

where the $\eta_i=\pm$ for $i=1,2,3$ independently, and we have defined the quantities,

$$\Lambda_{\eta_1}=e^{i(\Lambda_1-\Lambda_3)}+\eta_1e^{-i(\Lambda_1-\Lambda_3)} \qquad\qquad \Theta_{\eta_2}=e^{i\theta}h+\eta_2e^{-i\theta}h^* \tag{4.81}$$

An analysis similar to the one used for the first type of solution again yields the result (4.74).

4.5.3 Vanishing G implies the $AdS_5\times S^5$ solution

When $G=0$, we have $g_z=g_{\bar z}=g_9=h=0$. For half-BPS solutions, ψ and ξ cannot both vanish, and the reduced dilatino equation (4.36) implies $p_z=p_{\bar z}=0$. By the Bianchi identities (4.5), $P=0$ implies $dQ=0$, and we use the $U(1)_q$ gauge symmetry to set $Q=0$. Therefore, the requirements (4.74) can be obtained directly by imposing the vanishing of G.

4.5.3.1 Using the discrete symmetries

The generators $\tau^{(033)},\tau^{(030)}$ (4.27) and $\mathcal{K}$ (4.28) may be simultaneously diagonalized as follows,

$$\tau^{(033)}\zeta=\nu\zeta \qquad\qquad \tau^{(030)}\zeta=\gamma\zeta \qquad\qquad \mathcal{K}\zeta=\mu\zeta \tag{4.82}$$

where ν, γ, μ take on the values ± 1 independently. The $\tau^{(033)}$ projection was used to obtain the chiral form of the reduced BPS equation. We define the projections of $\tau^{(030)}$ and $\mathcal{K}$ as,

$$\begin{aligned} &\tau^{(030)}: && \tau^{(03)}\psi = \gamma\psi && \tau^{(03)}\xi = -\gamma\xi \\ &\mathcal{K}: && \xi = \tau^{(02)}\psi && \mu \equiv -\nu e^{i\theta} \end{aligned} \tag{4.83}$$

using the $U(1)_q$ gauge symmetry to fix the sign of the $\mathcal{K}$ projection. For $\gamma = +1$, we take,

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \psi_{++} \\ \psi_{-+} \end{pmatrix} \qquad \xi = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} \xi_{+-} \\ \xi_{--} \end{pmatrix} = -i \begin{pmatrix} \psi_{++} \\ \psi_{-+} \end{pmatrix} \tag{4.84}$$

to be two-component spinors with η_2 fixed. The remaining reduced BPS equations are then,

$$\begin{aligned} &(m) && \pm\frac{1}{f_2}\psi_\mp^* + \frac{D_z f_2}{f_2}\psi_\pm - f\psi_\mp^* = 0 \\ &(i) && -\frac{1}{f_5}\psi_\mp^* + \frac{D_z f_5}{f_5}\psi_\pm + f\psi_\mp^* = 0 \\ &(9) && -\frac{i\nu}{f_1}\psi_\pm^* + \frac{D_z f_1}{f_1}\psi_\pm - f\psi_\mp^* = 0 \\ &(-) && \left(D_{\bar z} - \frac{i}{2}\hat\omega_{\bar z}\right)\psi_\pm = 0 \\ &(+) && \left(D_z - \frac{i}{2}\hat\omega_z\right)\psi_\pm - f\psi_\mp^* = 0 \end{aligned} \tag{4.85}$$

4.5.3.2 Generic solutions when $G = 0$

Using $\hat\omega_z = i(\partial_z\rho)/\rho^2$ and $D_z = \rho^{-1}\partial_z$, the solution to the $(-)$ equation of (4.85) is given by,

$$\psi_+ = \sqrt{\rho}\alpha \qquad \psi_- = \sqrt{\rho}\beta \qquad \partial_{\bar z}\alpha = \partial_{\bar z}\beta = 0 \tag{4.86}$$

Employing the same strategy as [35], the $(\pm)$ equations are used in combination with the (m), (i), (9) equations of (4.85) to obtains solutions in terms of $\psi_\pm$ for the metric factors,

$$f_2 = |\psi_+|^2 + |\psi_-|^2, \qquad f_5 = |\psi_+|^2 - |\psi_-|^2, \qquad f_1 = c_1\left(\psi_+^*\psi_- - \psi_-^*\psi_+\right) \tag{4.87}$$

where $D_z f_5 = 0$ and we set $f_5 \equiv 1$. We can rewrite the $(\pm)$ equations (4.85) involving f_5 as,

$$\left|\frac{D_z f_5}{f_5}\right|^2 \psi_+\psi_-^* - \left(\frac{1}{f_5} - f\right)^2 \psi_+\psi_-^* = 0 \tag{4.88}$$

If $\psi_+\psi_-^* \neq 0$, then $D_z f_5 = 0$ implies that $ff_5 = 1$. If either $\psi_+ = 0$ or $\psi_-^* = 0$, the (i) equations of (4.85) also imply that $ff_5 = 1$. The $(+)$ equations of (4.85) then reduce to,

$$\beta\partial_z\alpha - \alpha\partial_z\beta + 1 = 0 \tag{4.89}$$

The solution is given, in terms of an arbitrary holomorphic function $A(z)$, by the expressions,

$$\alpha(z) = \frac{1}{\sqrt{\partial_z A(z)}} \qquad\qquad \beta(z) = \frac{A(z)}{\sqrt{\partial_z A(z)}} \tag{4.90}$$

4.5.3.3 Solution of $AdS_5 \times S^5$

Choosing $A(z) = -e^{-2z}$, with $c_1 = i$ so that f_1 is real, the 10-dimensional metric becomes,

$$ds^2 = (\coth x_7)^2\, ds^2_{AdS_2} + ds^2_{S^5} + \frac{dx_7^2 + dx_8^2}{(\sinh x_7)^2} + \frac{\sin^2 x_8 dx_9^2}{(\sinh x_7)^2} \tag{4.91}$$

where $z = (x_7 + ix_8)/2$. Performing the following transformation on the x_7 coordinate,

$$e^{x_7} = \tanh\left(\frac{\theta}{2}\right) \tag{4.92}$$

we recover the $AdS_5 \times S^5$ metric in the standard form,

$$\begin{aligned} ds^2 &= \cosh^2\theta\, ds^2_{AdS_2} + ds^2_{S^5} + \sinh^2\theta\left(dx_7^2 + dx_8^2\right) + \sinh^2\theta\sin^2 x_8 dx_9^2 \\ &= \cosh^2\theta\, ds^2_{AdS_2} + ds^2_{S^5} + d\theta^2 + \sinh^2\theta dx_8^2 + \sinh^2\theta\sin^2 x_8 dx_9^2 \\ &= \left[d\theta^2 + \cosh^2\theta\, ds^2_{AdS_2} + \sinh^2\theta\, ds^2_{S^2}\right] + ds^2_{S^5} \end{aligned} \tag{4.93}$$

The solution to the spinor ζ is characterized by the three projections,

$$\sigma^1\zeta^* = -\zeta \qquad\qquad \tau^{(033)}\zeta = \nu\zeta \qquad\qquad \tau^{(030)}\zeta = \gamma\zeta \tag{4.94}$$

With 8 independent Killing spinors χ in (4.20) and 4 independent solutions to ζ of the form,

$$\nu = \eta_2\eta_3 \qquad\qquad \gamma = \eta_2 : \qquad\qquad \zeta_{\pm,\eta_2,\eta_3} = \begin{pmatrix} \zeta_\pm \\ -\bar{\zeta}_\pm \end{pmatrix} \tag{4.95}$$

we indeed recover 32 supersymmetries for the maximally supersymmetric solution $AdS_5 \times S^5$.

4.6 Discussion

We have proven that for a spacetime of the form $AdS_2 \times S^5 \times S^1$ warped over a Riemann surface Σ, the only solution with at least 16 supersymmetries is just the maximally supersymmetric solution $AdS_5 \times S^5$. As we discussed, this then implies that no supergravity solutions exist for fully back-reacted D7 probe or D7/D3 intersecting branes whose near-horizon limit has the same spacetime structure with corresponding $SO(2,1) \oplus SO(6) \oplus SO(2)$ symmetry. Thus the $SU(1,1|4)$-invariant AdS_2 solutions are *rigid* in exactly the same sense as the two $SU(2,2|2)$-symmetric cases considered in [35], while the case of $SU(1,1|4) \oplus SU(1,1)$ is left for consideration in a future work.

Finally, we observe that for both the present case as well as the two cases in [35], one of the internal factors of the corresponding maximally supersymmetric solution is present in the warped spacetime of the half-BPS solutions, either AdS_5 or S^5 for Type IIB supergravity. One can show (as was done in Sec. 2.3 of [35]) that in each case the Bianchi identity for $F_{(5)}$ yields the same constraint on the corresponding metric factor and field strength as was obtained by solving the BPS equations. Namely, the condition that the product $f f_5^5$ is constant, which in each case could only be satisfied if both f and f_5 are constant, and in turn leaves only the $AdS_5 \times S^5$ solution. An open question is whether such rigidity extends to any half-BPS solution that has a spacetime factor in common with the corresponding maximally supersymmetric solution, for example warped $AdS_2 \times S^7$ solutions to M-theory.

APPENDIX A

Clifford algebra basis adapted to the Ansatz

A.1 $AdS_2 \times S^6 \times \Sigma$

The signature of the spacetime metric is chosen to be $(- + \cdots +)$. The Dirac-Clifford algebra is defined by $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} I_{32}$, where A, B are the 10-dimensional frame indices. We construct a basis for the Clifford algebra that is well-adapted to $AdS_2 \times S^6 \times \Sigma$ Ansatz, with the frame labeled as in (2.8),

$$\begin{aligned}
\Gamma^m &= \gamma^m \otimes I_8 \otimes I_2 && m = 0, 1 \\
\Gamma^i &= \gamma_{(1)} \otimes \gamma^i \otimes I_2 && i = 2, 3, 4, 5, 6, 7 \\
\Gamma^a &= \gamma_{(1)} \otimes \gamma_{(2)} \otimes \gamma^a && a = 8, 9
\end{aligned} \tag{A.1}$$

where a convenient basis for the lower dimensional Dirac-Clifford algebra is as follows,

$$\begin{aligned}
\gamma^0 &= i\sigma^2 & \gamma^2 &= \sigma^1 \otimes I_2 \otimes I_2 & & \\
\gamma^1 &= \sigma^1, & \gamma^3 &= \sigma^2 \otimes I_2 \otimes I_2 & & \\
& & \gamma^4 &= \sigma^3 \otimes \sigma^1 \otimes I_2 & & \\
& & \gamma^5 &= \sigma^3 \otimes \sigma^2 \otimes I_2 & & \\
& & \gamma^6 &= \sigma^3 \otimes \sigma^3 \otimes \sigma^1 & \gamma^8 &= \sigma^1 \\
& & \gamma^7 &= \sigma^3 \otimes \sigma^3 \otimes \sigma^2, & \gamma^9 &= \sigma^2
\end{aligned} \tag{A.2}$$

We will also need the chirality matrices on the various components of $AdS_2 \times S^6 \times \Sigma$,

$$
\begin{aligned}
\Gamma^{01} &= \gamma_{(1)} \otimes I_8 \otimes I_2 && \gamma_{(1)} = \sigma^3 \\
\Gamma^{234567} &= -iI_2 \otimes \gamma_{(2)} \otimes I_2 && \gamma_{(2)} = \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \\
\Gamma^{89} &= iI_2 \otimes I_8 \otimes \gamma_{(3)} && \gamma_{(3)} = \sigma^3
\end{aligned} \tag{A.3}
$$

The 10-dimensional chirality matrix in this basis is then given by,

$$
\Gamma^{11} = \Gamma^{0123456789} = \gamma_{(1)} \otimes \gamma_{(2)} \otimes \gamma_{(3)} = \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \tag{A.4}
$$

The complex conjugation matrices in each component are defined as follows,

$$
\begin{aligned}
(\gamma^m)^* &= +B_{(1)}\gamma^m B_{(1)}^{-1} && \left(B_{(1)}\right)^* B_{(1)} = +I_2 && B_{(1)} = I_2 \\
(\gamma^i)^* &= -B_{(2)}\gamma^i B_{(2)}^{-1} && \left(B_{(2)}\right)^* B_{(2)} = +I_8 && B_{(2)} = \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \\
(\gamma^a)^* &= -B_{(3)}\gamma^a B_{(3)}^{-1} && \left(B_{(3)}\right)^* B_{(3)} = -I_2 && B_{(3)} = \sigma^2
\end{aligned} \tag{A.5}
$$

where in the last column we have also listed the form of these matrices in our particular basis. The 10-dimensional complex conjugation matrix $\mathcal{B}$ satisfies,

$$
(\Gamma^M)^* = \mathcal{B}\Gamma^M \mathcal{B}^{-1} \qquad \mathcal{B}^*\mathcal{B} = I_{32} \qquad [\mathcal{B}, \Gamma^{11}] = 0 \tag{A.6}
$$

and in this basis is given by,

$$
\mathcal{B} = -iB_{(1)} \otimes \left(B_{(2)}\gamma_{(2)}\right) \otimes B_{(3)} = I_2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \tag{A.7}
$$

A.2 $AdS_2 \times S^5 \times S^1 \times \Sigma$

The Dirac-Clifford algebra is defined by $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} I_{32}$, where A, B are 10-dimensional frame indices and $\eta^{AB} = \mathrm{diag}(- + \cdots +)$. We choose a basis for the Clifford algebra which is well-adapted to the $AdS_2 \times S^5 \times \Sigma \times S^1$ Ansatz, with the frame labeled as in (4.11),

$$
\begin{aligned}
\Gamma^m &= \gamma^m \otimes I_4 \otimes I_2 \otimes \sigma^1 && m = 0, 1 \\
\Gamma^i &= I_2 \otimes \gamma^i \otimes I_2 \otimes \sigma^3 && i = 2, 3, 4, 5, 6 \\
\Gamma^a &= \sigma^3 \otimes I_4 \otimes \gamma^a \otimes \sigma^1 && a = 7, 8 \\
\Gamma^9 &= \sigma^3 \otimes I_4 \otimes \gamma^9 \otimes \sigma^1
\end{aligned} \tag{A.8}
$$

where the lower dimensional Dirac-Clifford algebra is defined as follows,

$$
\begin{aligned}
&\gamma^0 = i\sigma^1 && \gamma^2 = \sigma^1 \otimes I_2 && \\
&\gamma^1 = \sigma^2 && \gamma^3 = \sigma^2 \otimes I_2 && \\
& && \gamma^4 = \sigma^3 \otimes \sigma^1 && \gamma^7 = \sigma^1 \\
& && \gamma^5 = \sigma^3 \otimes \sigma^2 && \gamma^8 = \sigma^2 \\
& && \gamma^6 = \sigma^3 \otimes \sigma^3 && \gamma^9 = \sigma^3
\end{aligned}
\tag{A.9}
$$

The chirality matrices on the various components of $AdS_2 \times S^5 \times \Sigma \times S^1$ are given by,

$$
\begin{aligned}
\Gamma^{01} &= -\gamma_{(1)} \otimes I_4 \otimes I_2 \otimes I_2 & \gamma_{(1)} &= -\gamma^0\gamma^1 = \sigma^3 \\
\Gamma^{23456} &= -I_2 \otimes \gamma_{(2)} \otimes I_2 \otimes \sigma^3 & \gamma_{(2)} &= -\gamma^2 \cdots \gamma^6 = I_4 \\
\Gamma^{78} &= iI_2 \otimes I_4 \otimes \gamma_{(3)} \otimes I_2 & \gamma_{(3)} &= -i\gamma^7\gamma^8 = \sigma^3
\end{aligned}
\tag{A.10}
$$

which yields the following 10-dimensional chirality matrix,

$$
\Gamma^{11} = \Gamma^{0123456789} = -I_2 \otimes I_4 \otimes I_2 \otimes \sigma^2 \tag{A.11}
$$

The complex conjugation matrices in each component are defined by,

$$
\begin{aligned}
(\gamma^m)^* &= -B_{(1)}\gamma^m B_{(1)}^{-1} & \left(B_{(1)}\right)^* B_{(1)} &= +I_2 & B_{(1)} &= I_2 \\
(\gamma^i)^* &= +B_{(2)}\gamma^i B_{(2)}^{-1} & \left(B_{(2)}\right)^* B_{(2)} &= -I_4 & B_{(2)} &= \sigma^1 \otimes \sigma^2 \\
(\gamma^a)^* &= -B_{(3)}\gamma^a B_{(3)}^{-1} & \left(B_{(3)}\right)^* B_{(3)} &= -I_2 & B_{(3)} &= \sigma^2 \\
(\gamma^9)^* &= +B_{(4)}\gamma^9 B_{(4)}^{-1} & \left(B_{(4)}\right)^* B_{(4)} &= +I_2 & B_{(4)} &= I_2
\end{aligned}
\tag{A.12}
$$

where in the last column we have also listed the form of these matrices in our particular basis. The 10-dimensional complex conjugation matrix $\mathcal{B}$ satisfies,

$$
(\Gamma^M)^* = \mathcal{B}\Gamma^M\mathcal{B}^{-1} \qquad \mathcal{B}^*\mathcal{B} = I_{32} \qquad \{\mathcal{B}, \Gamma^{11}\} = 0 \tag{A.13}
$$

and in this basis has the following form,

$$
\mathcal{B} = I_2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^3 \tag{A.14}
$$

APPENDIX B

Derivation of the BPS equations

B.1 $AdS_2 \times S^6 \times \Sigma$

In reducing the BPS equations, we will use the following decompositions of ε and $\mathcal{B}^{-1}\varepsilon^*$,

$$\varepsilon = \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \zeta_{\eta_1,\eta_2} \qquad \mathcal{B}^{-1}\varepsilon^* = \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \star\zeta_{\eta_1,\eta_2} \tag{B.1}$$

where we use the abbreviations,

$$\star\zeta_{\eta_1,\eta_2} = -i\sigma^2\eta_2\zeta^*_{\eta_1,-\eta_2} \qquad \star\zeta = \tau^{(02)} \otimes \sigma^2\zeta^* \tag{B.2}$$

in τ-matrix notation. We will also need the chirality relations,

$$\sigma^3\zeta_{\eta_1,\eta_2} = -\zeta_{-\eta_1,-\eta_2} \qquad \tau^{(11)} \otimes \sigma^3\zeta = -\zeta \tag{B.3}$$

B.1.1 The dilatino equation

Reduced to the Ansatz, and using the above decomposition, the dilatino equation is,

$$\begin{aligned} 0 &= iP_A\Gamma^A\mathcal{B}^{-1}\varepsilon^* - \frac{i}{24}\Gamma \cdot G\varepsilon \\ &= ip_a\Gamma^a \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \star\zeta_{\eta_1,\eta_2} - \frac{i}{4}g_a\Gamma^a \sum_{\eta_1,\eta_2} \chi^{-\eta_1,\eta_2} \otimes \zeta_{\eta_1,\eta_2} \end{aligned} \tag{B.4}$$

where we have used the following simplifications for the inner products,

$$\begin{aligned} P_A\Gamma^A &= p_a\Gamma^a \\ \Gamma \cdot G &= 3!\, g_a\Gamma^{01a} = 6g_a\Gamma^a\gamma_{(1)} \otimes I_8 \otimes I_2 \end{aligned} \tag{B.5}$$

Using the expression for $\star\zeta$ and reversing the sign of η_2, we extract an equation satisfied by the ζ-spinors, and recover the reduced dilatino BPS equation announced in (2.21).

B.1.2 The gravitino equation

The gravitino equation is,

$$
\begin{aligned}
0 &= (d+\omega)\varepsilon - \frac{i}{2}Q\varepsilon + \mathfrak{g}\mathcal{B}^{-1}\varepsilon^* \\
\omega &= \frac{1}{4}\omega_{AB}\Gamma^{AB} \\
\mathfrak{g} &= -\frac{1}{96}e_A\left(\Gamma^A(\Gamma\cdot G) + 2(\Gamma\cdot G)\Gamma^A\right)
\end{aligned}
\tag{B.6}
$$

where A, B are the 10-dimensional frame indices.

B.1.2.1 The calculation of $(d+\omega)\varepsilon$

The spin connection components for a generic spacetime of the form $AdS_p \times S^q \times \Sigma$ are worked out in Appendix D. Here we quote results for the case of $p = 2$ and $q = 6$, and in particular we reproduce equation (D.5),

$$
\begin{aligned}
\omega^m{}_n &= \hat{\omega}^m{}_n & \qquad \omega^m{}_a &= e^m D_a \ln f_2 \\
\omega^i{}_j &= \hat{\omega}^i{}_j & \qquad \omega^i{}_a &= e^i D_a \ln f_6
\end{aligned}
\tag{B.7}
$$

which gives the relevant spin connection components, along with $\omega^a{}_b$ whose explicit form we will not need. The hats refer to the canonical connections on AdS_2 and S^6, respectively. We replace the covariant derivative of the spinor along the AdS_2 and S^6 directions by the corresponding group action, $SO(2,1)$ and $SO(7)$, as defined in (2.14). The additional term that appears in going from ∇ to $\hat{\nabla}$ is due to the warp factors in the ten-dimensional metric. The covariant derivatives along AdS_2 and S^6, respectively, are given by,

$$
\begin{aligned}
&(m) \qquad & \nabla_m\varepsilon &= \left(\frac{1}{f_2}\hat{\nabla}_m + \frac{D_a f_2}{2f_2}\Gamma_m\Gamma^a\right)\varepsilon \\
&(i) \qquad & \nabla_i\varepsilon &= \left(\frac{1}{f_6}\hat{\nabla}_i + \frac{D_a f_6}{2f_6}\Gamma_i\Gamma^a\right)\varepsilon
\end{aligned}
\tag{B.8}
$$

as well as $\nabla_a \varepsilon$ along Σ. Using the Killing spinor equations (2.14) to eliminate the hatted covariant derivatives, as well as the equation $\Gamma^a = \gamma_{(1)} \otimes \gamma_{(2)} \otimes \gamma^a$, we have,

$$(m) \qquad \nabla_m \varepsilon = \Gamma_m \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \left(\frac{1}{2f_2} \eta_1 \zeta_{\eta_1,\eta_2} + \frac{D_a f_2}{2f_2} \gamma^a \zeta_{-\eta_1,-\eta_2} \right)$$
$$(i) \qquad \nabla_i \varepsilon = \Gamma_i \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \left(\frac{i}{2f_6} \eta_2 \zeta_{-\eta_1,\eta_2} + \frac{D_a f_6}{2f_6} \gamma^a \zeta_{-\eta_1,-\eta_2} \right) \tag{B.9}$$

As we will show, each term in the gravitino equation contains $\Gamma_A \chi^{\eta_1,\eta_2}$, which we argue are linearly independent. Therefore, we will require that the coefficients vanish independently along the various directions of AdS_2, S^6, and Σ.

B.1.2.2 The calculation of $\mathfrak{g}\mathcal{B}^{-1}\varepsilon^*$

The relevant expression is as follows,

$$\mathfrak{g}\mathcal{B}^{-1}\varepsilon^* = -\frac{3!}{96} e_B g_a (\Gamma^B \Gamma^{01a} + 2\Gamma^{01a}\Gamma^B)\mathcal{B}^{-1}\varepsilon^* \tag{B.10}$$

We make use of the following identities,

$$\Gamma^m \Gamma^{01b} + 2\Gamma^{01b}\Gamma^m = 3\Gamma^m \Gamma^{01b}$$
$$\Gamma^i \Gamma^{01b} + 2\Gamma^{01b}\Gamma^i = -\Gamma^i \Gamma^{01b}$$
$$\Gamma^a \Gamma^{01b} + 2\Gamma^{01b}\Gamma^a = \Gamma^{01}(3\delta^{ab} - \Gamma^{ab}) = (\gamma_1 \otimes I_8 \otimes I_2)(3\delta^{ab} I_2 - \gamma^{ab}) \tag{B.11}$$

where $\gamma^{ab} \equiv i\varepsilon^{ab}\sigma^3$ and $\varepsilon^{89} = +1$. Projecting along the various directions, we obtain,

$$(m) \qquad \Gamma_m \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \left(-\frac{3}{16} g_a \gamma^a \star \zeta_{\eta_1,-\eta_2} \right)$$
$$(i) \qquad \Gamma_i \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \left(\frac{1}{16} g_a \gamma^a \star \zeta_{\eta_1,-\eta_2} \right)$$
$$(a) \qquad \sum_{\eta_1,\eta_2} \chi^{\eta_1,\eta_2} \otimes \left(-\frac{3}{16} g_a \star \zeta_{-\eta_1,\eta_2} + \frac{1}{16} g_b \gamma^{ab} \star \zeta_{-\eta_1,\eta_2} \right) \tag{B.12}$$

B.1.2.3 The complete gravitino BPS equation

We now assemble the reduced gravitino equations. Requiring the coefficients of $\Gamma_A \chi^{\eta_1,\eta_2}$ to vanish independently, then rewriting the relations using the τ-matrix notation, we have,

$$\begin{aligned}
&(m) \qquad 0 = \frac{1}{2f_2}\tau^{(30)}\zeta + \frac{D_a f_2}{2f_2}\tau^{(11)}\gamma^a\zeta - \frac{3}{16}g_a\tau^{(01)}\gamma^a \star\zeta \\
&(i) \qquad 0 = \frac{i}{2f_6}\tau^{(13)}\zeta + \frac{D_a f_6}{2f_6}\tau^{(11)}\gamma^a\zeta + \frac{1}{16}g_a\tau^{(01)}\gamma^a \star\zeta \\
&(a) \qquad 0 = \left(D_a + \frac{i}{2}\hat{\omega}_a\sigma^3 - \frac{i}{2}q_a\right)\zeta - \frac{3}{16}g_a\tau^{(10)}\star\zeta + \frac{1}{16}g_b\tau^{(10)}\gamma^{ab}\star\zeta
\end{aligned} \tag{B.13}$$

where $\hat{\omega}_a = (\hat{\omega}_{89})_a$ is the spin connection along Σ, and we have used the fact $\Gamma^{89} = i\sigma^3$. Eliminating the star using the definition (B.15), then multiplying the (m) and (i) equations by $\tau^{(11)}$, we recover the system of reduced gravitino BPS equations announced in (2.22).

B.2 $AdS_2 \times S^5 \times S^1 \times \Sigma$

In reducing the BPS equations, we will use the following decompositions of ε and $\mathcal{B}^{-1}\varepsilon^*$,

$$\varepsilon = \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} \otimes \phi \qquad \mathcal{B}^{-1}\varepsilon^* = \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \star\zeta_{\eta_1,\eta_2,\eta_3} \otimes \phi \tag{B.14}$$

where we have used the abbreviations,

$$\star\zeta_{\eta_1,\eta_2,\eta_3} = -i\eta_2\sigma^2\zeta^*_{-\eta_1,-\eta_2,-\eta_3} \qquad\qquad \star\zeta = \tau^{(121)}\sigma^2\zeta^* \tag{B.15}$$

in τ-matrix notation. The strategy employed here is the same as the one used in Appendix B of [35], therefore we will summarize the derivation while highlighting certain key details.

B.2.1 The dilatino equation

Using the explicit form of the supergravity fields and Γ-matrices, the dilatino equation is,

$$0 = \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \left[\eta_2 p_a\gamma^a\sigma^2\zeta^*_{\eta_1,-\eta_2,-\eta_3} - \frac{1}{4}g_{\bar{a}}\gamma^{\bar{a}}\zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{4}h\zeta_{-\eta_1,\eta_2,\eta_3}\right] \otimes \phi^* \tag{B.16}$$

The linear independence of the $\chi^{\eta_1,\eta_2}\chi^{\eta_3}$ implies that each coefficient in the square brackets must vanish separately. Rewriting the result in terms of τ-matrix notation, we recover (4.24):

$$0 = 4p_a\gamma^a\sigma^2\zeta^* + ig_{\bar{a}}\tau^{(021)}\gamma^{\bar{a}}\zeta - ih\tau^{(121)}\zeta \tag{B.17}$$

B.2.2 The gravitino equation

The components of the covariant derivative $\nabla_M\varepsilon$ along AdS_2, S^5, and S^1 are given by,

$$\begin{aligned}
&(m) \qquad && \nabla_m\varepsilon = \left(\frac{1}{f_2}\hat{\nabla}_m + \frac{D_af_2}{2f_2}\Gamma_m\Gamma^a\right)\varepsilon \\
&(i) \qquad && \nabla_i\varepsilon = \left(\frac{1}{f_5}\hat{\nabla}_i + \frac{D_af_5}{2f_5}\Gamma_i\Gamma^a\right)\varepsilon \\
&(9) \qquad && \nabla_9\varepsilon = \left(\frac{1}{f_1}\hat{\nabla}_9 + \frac{D_af_1}{2f_1}\Gamma_9\Gamma^a\right)\varepsilon
\end{aligned} \tag{B.18}$$

as well as $\nabla_a\varepsilon$ along Σ. The hats refer to the canonical connections on AdS_2, S^5, and S^1, respectively, and the additional term that appears in going from ∇ to $\hat{\nabla}$ is due to the warp factors in the ten-dimensional metric, with $D_af = \rho^{-1}\partial_af$. Using the Killing spinor equations (4.15) to eliminate the hatted covariant derivatives, we have,

$$\begin{aligned}
&(m) \qquad \nabla_m\varepsilon = \Gamma_m\sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2}\chi^{\eta_3}\otimes\left(\frac{\eta_1}{2f_2}\zeta_{\eta_1,\eta_2,\eta_3} + \frac{D_af_2}{2f_2}\gamma^a\zeta_{-\eta_1,\eta_2,\eta_3}\right)\otimes\phi^* \\
&(i) \qquad \nabla_i\varepsilon = \Gamma_i\sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2}\chi^{\eta_3}\otimes\left(\frac{\eta_2}{2f_5}\zeta_{\eta_1,\eta_2,\eta_3} + \frac{D_af_5}{2f_5}\gamma^a\zeta_{-\eta_1,\eta_2,\eta_3}\right)\otimes\phi^* \\
&(9) \qquad \nabla_9\varepsilon = \Gamma_9\sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2}\chi^{\eta_3}\otimes\left(\frac{i\eta_3}{2f_1}\gamma^9\zeta_{-\eta_1,\eta_2,\eta_3} + \frac{D_af_1}{2f_1}\gamma^a\zeta_{-\eta_1,-\eta_2,\eta_3}\right)\otimes\phi^*
\end{aligned} \tag{B.19}$$

For the additional terms involving ε, we project along the various directions and obtain,

$$\begin{aligned}
&(m) \qquad && \Gamma_m\sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2}\chi^{\eta_3}\otimes\left(\frac{1}{2}f\zeta_{\eta_1,\eta_2,\eta_3}\right)\otimes\phi^* \\
&(i) \qquad && \Gamma_i\sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2}\chi^{\eta_3}\otimes\left(-\frac{1}{2}f\zeta_{\eta_1,\eta_2,\eta_3}\right)\otimes\phi^* \\
&(a) \qquad && \sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2}\chi^{\eta_3}\otimes\left(-\frac{i}{2}q_a\zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}f\gamma^a\zeta_{-\eta_1,\eta_2,\eta_3}\right)\otimes\phi \\
&(9) \qquad && \Gamma_9\sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2}\chi^{\eta_3}\otimes\left(\frac{1}{2}f\zeta_{\eta_1,\eta_2,\eta_3}\right)\otimes\phi^*
\end{aligned} \tag{B.20}$$

while for the terms involving ε^* we have,

$$
\begin{aligned}
&(m) \qquad \Gamma_m \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \frac{1}{16}\left(3ig_{\bar{a}}\gamma^{\bar{a}} \star \zeta_{\eta_1,\eta_2,\eta_3} + ih \star \zeta_{-\eta_1,\eta_2,\eta_3}\right) \otimes \phi^* \\
&(i) \qquad \Gamma_i \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \frac{1}{16}\left(-ig_{\bar{a}}\gamma^{\bar{a}} \star \zeta_{\eta_1,\eta_2,\eta_3} + ih \star \zeta_{-\eta_1,\eta_2,\eta_3}\right) \otimes \phi^* \\
&(a) \qquad \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \frac{1}{16}\left[\left(3ig^a - ig_{\bar{b}}\gamma^{a\bar{b}}\right) \star \zeta_{-\eta_1,\eta_2,\eta_3} - 3ih\gamma^a \star \zeta_{\eta_1,\eta_2,\eta_3}\right] \otimes \phi \\
&(9) \qquad \Gamma_9 \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2}\chi^{\eta_3} \otimes \frac{1}{16}\left[\left(3ig^9\sigma^3 - ig_a\gamma^a\right) \star \zeta_{\eta_1,\eta_2,\eta_3} - 3ih \star \zeta_{-\eta_1,\eta_2,\eta_3}\right] \otimes \phi^*
\end{aligned}
\tag{B.21}
$$

We observe that each term in the gravitino equation contains $\Gamma_A\chi^{\eta_1,\eta_2}\chi^{\eta_3}$ or $\chi^{\eta_1,\eta_2}\chi^{\eta_3}$, which we argue are linearly independent. Requiring the coefficients to vanish independently, then rewriting the relations using the τ-matrix notation and eliminating the star using the definition (B.15), we recover the system of reduced gravitino BPS equations announced in (4.25).

APPENDIX C

Calculation of the $AdS_2 \times S^6 \times \Sigma$ flux potentials

In this appendix, we present the calculations of the reduced flux potentials $\mathcal{C}$ and $\mathcal{M}$.

C.1 Calculation of the flux potential $\mathcal{C}$

Expressing the field strength G in terms of g_a, the equations for the derivatives of the potential $\mathcal{C}$ are given by,

$$\begin{aligned}\partial_w \mathcal{C} &= \rho f_2^2 f (g_z + B \bar{g}_z) \\ \partial_{\bar{w}} \mathcal{C} &= \rho f_2^2 f (g_{\bar{z}} + B \bar{g}_{\bar{z}})\end{aligned} \tag{C.1}$$

Converting G into P and then into derivatives of B using (2.2) and (2.47) yields,

$$\rho g_z = 4iK \frac{\alpha}{\beta} f^2 \partial_w B \qquad\qquad \rho g_{\bar{z}} = -4iK^3 \frac{\bar{\beta}}{\bar{\alpha}} f^2 \partial_{\bar{w}} B \tag{C.2}$$

along with $\bar{g}_z = (g_{\bar{z}})^*$ and $\bar{g}_{\bar{z}} = (g_z)^*$, and we obtain the following expressions,

$$\begin{aligned}\partial_w \mathcal{C} &= 4iK f_2^2 f^3 \left(\frac{\alpha}{\beta} \partial_w B + B \frac{\beta}{\alpha} \partial_w \bar{B} \right) \\ \partial_w \bar{\mathcal{C}} &= 4iK f_2^2 f^3 \left(\frac{\beta}{\alpha} \partial_w \bar{B} + \bar{B} \frac{\alpha}{\beta} \partial_w B \right)\end{aligned} \tag{C.3}$$

We will now apply the same changes of variables used to solve the BPS equations. In the derivation of $\mathcal{C}$ as well as $\mathcal{M}$, it will be useful to have the derivatives of B and $\bar{B}$ expressed in terms of the new variables, and these derivatives are given by,

$$\begin{aligned}\partial_w B &= \frac{1 - |\lambda|^2}{(1 - \bar{\lambda} Z^2)^2} \partial_w Z^2 - \frac{\partial_w \lambda}{1 - \bar{\lambda} Z^2} \\ \partial_w \bar{B} &= \frac{1 - |\lambda|^2}{(1 - \lambda \bar{Z}^2)^2} \partial_w \bar{Z}^2 + \frac{\bar{Z}^2 (\bar{Z}^2 - \bar{\lambda}) \partial_w \lambda}{(1 - \lambda \bar{Z}^2)^2}\end{aligned} \tag{C.4}$$

C.1.1 Expressing variables in terms of holomorphic functions

Recall that we have,

$$\frac{\alpha}{\beta} = \left(\frac{\bar\lambda + \bar B}{\bar\lambda B + 1}\right)^{\frac{1}{2}} = \bar Z \left(\frac{1 - \bar\lambda Z^2}{1 - \lambda \bar Z^2}\right)^{\frac{1}{2}}, \qquad \left|\frac{\alpha}{\beta}\right| = |Z| \tag{C.5}$$

as well as the expressions for B in (2.65) and f^2 in (2.66). Using these, along with the rearrangement formula following from (2.78),

$$\frac{K}{\hat\rho^2 \bar Z |Z|} = \frac{\bar\xi}{|Z|^2} \tag{C.6}$$

we write (C.3) as,

$$\begin{aligned}
\partial_w \mathcal{C} &= \frac{4ic}{9}\frac{(1-K^2|Z|^2)}{(1+K^2|Z|^2)^3}\frac{\bar\xi}{|Z|^2}\left(\partial_w |Z|^4 - \lambda(\partial_w \bar Z^2 + \bar Z^4 \partial_w Z^2) - \bar Z^2(1-|Z|^4)\partial_w \lambda\right) \\
\partial_w \bar{\mathcal{C}} &= \frac{4ic}{9}\frac{(1-K^2|Z|^2)}{(1+K^2|Z|^2)^3}\frac{\bar\xi}{|Z|^2}\left(-\bar\lambda \partial_w |Z|^4 + \partial_w \bar Z^2 + \bar Z^4 \partial_w Z^2\right)
\end{aligned} \tag{C.7}$$

Next, we define the combination $\bar{\mathcal{C}} + \bar\lambda \mathcal{C}$ by its derivative with respect to w as,

$$\partial_w(\bar{\mathcal{C}} + \bar\lambda \mathcal{C}) = \frac{4iK^2 c}{9}\mathcal{P}_w \tag{C.8}$$

where $\mathcal{P}_w$ is given by,

$$\mathcal{P}_w = \xi \frac{(1-K^2|Z|^2)}{(1+K^2|Z|^2)^3}\left((1-|\lambda|^2)(\partial_w \ln \bar Z^2 + \bar Z^2 \partial_w Z^2) + (1-|Z|^4)\partial_w(1-|\lambda|^2)\right) \tag{C.9}$$

Using $\xi(1-|\lambda|^2) = \mathcal{L}$ to eliminate ξ and then changing variables via $Z^2 = Re^{i\psi}$, we find,

$$\mathcal{P}_w = \mathcal{L}\frac{(1-K^2 R)}{(1+K^2 R)^3}\left\{\left(R + \frac{1}{R}\right)\partial_w R - (1-R^2)\left(i\partial_w \psi - \partial_w \ln(1-|\lambda|^2)\right)\right\} \tag{C.10}$$

The combinations of R and $\partial_w W$ can be expressed in terms of W defined in (2.83) and its derivative $\partial_w W$. The latter is given by equation (2.84), from which we eliminate $\hat\rho$ in favor of ξ, as well as eliminate ξ in favor of $\mathcal{L}$ and λ, to obtain,

$$\partial_w W = -(W+1)\partial_w \ln \mathcal{L}\bar{\mathcal{L}} + (W-2)\partial_w \ln(1-|\lambda|^2) + 3i\partial_w \psi \tag{C.11}$$

Using the relation $e^{i\psi} = K^2 \bar{\mathcal{L}}/\mathcal{L}$ to express the derivative $i\partial_w \psi$ in terms of $\mathcal{L}$, then separating the dependence of $\mathcal{L}$ and $\bar{\mathcal{L}}$, we find,

$$\mathcal{P}_w = \frac{2\mathcal{L}}{(W+2)^2}[(W^2 + 2W - 2)\partial_w \ln \mathcal{L} + (W-2)\{\partial_w \ln(1-|\lambda|^2) - \partial_w \ln \bar{\mathcal{L}}\}] \tag{C.12}$$

The last term inside the square brackets can be replaced using the differential equation for W in (C.11), which upon expressing ψ in terms of $\mathcal{L}$ takes the form,

$$(W-2)\{\partial_w \ln(1-|\lambda|^2) - \partial_w \ln \bar{\mathcal{L}}\} = \partial_w W + (W+4)\partial_w \ln \mathcal{L} \tag{C.13}$$

Eliminating this term between (C.11) and (C.12) and integrating the result, we find,

$$\mathcal{P}_w = \partial_w \left(\frac{2\mathcal{L}(W+1)}{W+2} \right) \tag{C.14}$$

Therefore, we have, with some holomorphic function $\mathcal{K}_1$,

$$\bar{\mathcal{C}} + \bar{\lambda}\mathcal{C} = \frac{4iK^2c}{9} \left(\frac{2\mathcal{L}(W+1)}{W+2} + \bar{\mathcal{K}}_1 \right) \tag{C.15}$$

Proceeding analogously for $\bar{\mathcal{C}}$, and using the equations for $\partial_w \xi$ and $\partial_w \bar{\xi}$, we find,

$$\bar{\mathcal{C}} = \frac{4iK^2c}{9} \left(-2\frac{\xi + \bar{\lambda}\bar{\xi}}{W+2} + \frac{\mathcal{L}}{1-|\lambda|^2} + \mathcal{A}_- + \bar{\mathcal{K}}_2 \right) \tag{C.16}$$

for some holomorphic function $\mathcal{K}_2$. To determine $\mathcal{K}_2$, we equate the two different expressions for $\bar{\mathcal{C}} + \bar{\lambda}\mathcal{C}$, eliminate the dependence on W and separate holomorphic and anti-holomorphic dependences to obtain,

$$\mathcal{C} = \frac{4iK^2c}{9} \left(+2\frac{\bar{\xi} + \lambda\xi}{W+2} - \frac{\bar{\mathcal{L}}}{1-|\lambda|^2} - (\bar{\mathcal{A}}_- + 2\mathcal{A}_+ + \mathcal{K}_0) \right) \tag{C.17}$$

where $\mathcal{K}_0$ is an arbitrary complex constant. Eliminating ξ and $\mathcal{L}$ leads to (2.101).

C.2 Calculation of the reduced flux potential $\mathcal{M}$

To evaluate $\mathcal{M}$, we proceed in analogy with the calculation of $\mathcal{C}$. Reducing $G_{(3)}$ to P and then expressing this quantity in terms of derivatives of B using (2.2) and (2.47) yields,

$$\begin{aligned} \partial_w \mathcal{M} &= 4K f_6^6 f^3 \left(-\frac{\alpha}{\beta}\partial_w B + B\frac{\beta}{\alpha}\partial_w \bar{B} \right) \\ \partial_w \bar{\mathcal{M}} &= 4K f_6^6 f^3 \left(-\frac{\beta}{\alpha}\partial_w \bar{B} + \bar{B}\frac{\alpha}{\beta}\partial_w B \right) \end{aligned} \tag{C.18}$$

Using (C.5) and (C.4) to express α/β and the derivatives of B and $\bar{B}$ in terms of Z, λ and $\hat{\rho}^2$, as well as the following expression,

$$f_6^6 f^3 = \frac{c^3}{\hat{\rho}^6} \frac{|1-\bar{\lambda}Z^2|^3}{|Z|^3} \tag{C.19}$$

we find, after some simplifications,

$$\begin{aligned}
\partial_w \mathcal{M} &= 4K\frac{c^3\bar{Z}}{\hat{\rho}^6|Z|^3}\Big(-(1-|\lambda|^2)(1-\lambda\bar{Z}^2)\partial_w Z^2 + (1-|\lambda|^2)(Z^2-\lambda)\partial_w \ln \bar{Z}^2 \\
&\qquad + |1-\lambda\bar{Z}^2|^2\partial_w\lambda + |Z^2-\lambda|^2\partial_w\lambda\Big) \\
\partial_w \bar{\mathcal{M}} &= 4K\frac{c^3\bar{Z}}{\hat{\rho}^6|Z|^3}\Big(-(1-|\lambda|^2)(1-\bar{\lambda}Z^2)\partial_w \ln \bar{Z}^2 + (1-|\lambda|^2)(\bar{Z}^2-\bar{\lambda})\partial_w Z^2 \\
&\qquad -2(\bar{Z}^2-\bar{\lambda})(1-\bar{\lambda}Z^2)\partial_w\lambda\Big)
\end{aligned} \tag{C.20}$$

Significant cancellations occur when combining these results into the following form,

$$\partial_w(\bar{\mathcal{M}}-\bar{\lambda}\mathcal{M}) = 4K\frac{c^3\bar{Z}^3}{\hat{\rho}^6|Z|^3}(1-|\lambda|^2)^3\left(\frac{\partial_w(Z^2+\bar{Z}^{-2})}{1-|\lambda|^2} + \frac{\bar{\lambda}(Z^2+\bar{Z}^{-2})-2}{(1-|\lambda|^2)^2}\partial_w\lambda\right) \tag{C.21}$$

Combining the relations,

$$K\xi = \frac{e^{-i\psi/2}}{\hat{\rho}^2} \qquad\qquad \frac{\bar{Z}}{|Z|} = e^{-i\psi/2} \qquad\qquad \xi(1-|\lambda|^2) = \mathcal{L} \tag{C.22}$$

we find,

$$\frac{\bar{Z}^3}{\hat{\rho}^6|Z|^3}(1-|\lambda|^2)^3 = K^3\mathcal{L}^3 \tag{C.23}$$

Recognizing a total derivative in the expression for $\partial_w(\bar{\mathcal{M}}-\bar{\lambda}\mathcal{M})$, we may rearrange the result as follows,

$$\partial_w(\bar{\mathcal{M}}-\bar{\lambda}\mathcal{M}) = 4c^3\mathcal{L}^3\,\partial_w\left(\frac{Z^2+\bar{Z}^{-2}-2\bar{\lambda}^{-1}}{1-|\lambda|^2}\right) \tag{C.24}$$

Extracting a total derivative again significantly simplifies the expression, and we have,

$$\partial_w(\bar{\mathcal{M}}-\bar{\lambda}\mathcal{M}) = 12c^3\mathcal{S}_w + \partial_w\left[4c^3\mathcal{L}^3\left(\frac{Z^2+\bar{Z}^{-2}-2\bar{\lambda}^{-1}}{1-|\lambda|^2}\right)\right] \tag{C.25}$$

where $\mathcal{S}_w$ is given by,

$$\mathcal{S}_w = -\left(\frac{Z^2+\bar{Z}^{-2}-2\bar{\lambda}^{-1}}{1-|\lambda|^2}\right)\mathcal{L}^2\partial_w\mathcal{L} \tag{C.26}$$

Using the relation $\partial_w\mathcal{L} = (1-|\lambda|^2)\partial_w\mathcal{A}_-$, we find a remarkable cancellation of the factor $(1-|\lambda|^2)$, and recover an expression which, after converting Z and $\bar{Z}$ to R and ψ using (2.72), is given as follows,

$$\mathcal{S}_w = -\mathcal{L}^2 e^{i\psi}\left(R+\frac{1}{R}\right)\partial_w\mathcal{A}_- + \frac{2\mathcal{L}^2}{\bar{\lambda}}\partial_w\mathcal{A}_- \tag{C.27}$$

Next, we make use of the identity $\mathcal{L}e^{i\psi} = \Lambda\bar{\mathcal{L}}$, and then use $\partial_w \mathcal{G} = -\bar{\mathcal{L}}\partial_w \mathcal{A}_-$ to obtain the following expression,

$$\mathcal{S}_w = -\left(\Lambda R + \frac{1}{\Lambda R}\right)\frac{|\partial_w \mathcal{G}|^2}{\partial_{\bar{w}}\bar{\mathcal{A}}_-} + \frac{2\mathcal{L}^2}{\bar{\lambda}}\partial_w \mathcal{A}_- \tag{C.28}$$

Using the definition of R as well as $\kappa^2 = -\partial_w \partial_{\bar{w}} \mathcal{G}$, this can be rewritten as follows,

$$\mathcal{S}_w = \partial_w \left(6\frac{\mathcal{G}\partial_{\bar{w}}\mathcal{G}}{\partial_{\bar{w}}\bar{\mathcal{A}}_-}\right) + 8\mathcal{L}\partial_w \mathcal{G} + \frac{2\mathcal{L}^2}{\bar{\lambda}}\partial_w \mathcal{A}_- \tag{C.29}$$

The first term already is in integrated form, and the last two terms are of degree at most two in $\mathcal{A}_+$ and $\mathcal{A}_-$ and of degree at most one in $\partial_w \mathcal{A}_+$ and $\partial_w \mathcal{A}_+$, and they may be integrated in their holomorphic dependence on w.

C.2.1 Integrating for $\bar{\mathcal{M}} - \bar{\lambda}\mathcal{M}$

We introduce two locally holomorphic functions, $\mathcal{W}_\pm$ such that

$$\partial_w \mathcal{W}_\pm = \mathcal{A}_\pm \partial_w \mathcal{B} \tag{C.30}$$

One may then verify by straightforward evaluation that the second term in (C.29) can be integrated as follows

$$\mathcal{L}\partial_w \mathcal{G} = -\partial_w \left[\left(\frac{1}{2}\mathcal{G} + \mathcal{B}\right)(\bar{\mathcal{A}}_+ - \bar{\lambda}\bar{\mathcal{A}}_-) - \frac{1}{2}\left(|\mathcal{A}_+|^2 - |\mathcal{A}_-|^2\right)\mathcal{L} + \bar{\lambda}\mathcal{W}_+ - \mathcal{W}_-\right] \tag{C.31}$$

For the remaining term we use repeatedly the relation $\partial_w \mathcal{L} = \partial_w \mathcal{A}_- - \bar{\lambda}\partial_w \mathcal{A}_+$ and find

$$\begin{aligned}\mathcal{L}^2 \partial_w \mathcal{A}_- = \partial_w \Big[& \mathcal{L}^2 \mathcal{A}_- + (\bar{\lambda}\mathcal{A}_+ - \mathcal{A}_-)\mathcal{A}_- \left(\mathcal{L} + \frac{\bar{\lambda}\mathcal{A}_+ - \mathcal{A}_-}{3}\right) \\ & + \bar{\lambda}\left(\bar{\mathcal{A}}_+ - \bar{\lambda}\bar{\mathcal{A}}_-\right)\mathcal{B} + \frac{2}{3}\bar{\lambda}^2 \mathcal{W}_+ - \frac{2}{3}\bar{\lambda}\mathcal{W}_-\Big]\end{aligned} \tag{C.32}$$

Combining (C.31) and (C.32) with the expression for S_w in (C.29) shows that we have thus integrated S_w. Coming back to the expression for $\partial_w(\bar{\mathcal{M}} - \bar{\lambda}\mathcal{M})$ in (C.25), we can thus integrate for $\bar{\mathcal{M}} - \bar{\lambda}\mathcal{M}$. After a number of simplifications and using that $\bar{\lambda}\bar{\mathcal{L}} - \mathcal{L} = (1 - |\lambda|^2)(\bar{\mathcal{A}}_+ - \mathcal{A}_-)$, we find

$$\begin{aligned}\frac{\bar{\mathcal{M}} - \bar{\lambda}\mathcal{M}}{4c^3} = & \frac{2\mathcal{L}^2}{\bar{\lambda}}(\bar{\mathcal{A}}_+ - \mathcal{A}_-) - 12\left(\mathcal{G} - |\mathcal{A}_+|^2 + |\mathcal{A}_-|^2\right)\mathcal{L} + \frac{2\mathcal{A}_-}{\bar{\lambda}}(\bar{\lambda}\mathcal{A}_+ - \mathcal{A}_-)^2 \\ & - 6\left(\bar{\mathcal{A}}_+ - \bar{\lambda}\bar{\mathcal{A}}_-\right)\left(3\mathcal{B} + 2\mathcal{G} + \frac{\mathcal{L}\mathcal{A}_-}{\bar{\lambda}}\right) - 20(\bar{\lambda}\mathcal{W}_+ - \mathcal{W}_-) + \bar{\mathcal{V}}_1\end{aligned} \tag{C.33}$$

where $\mathcal{V}_1$ is a so far arbitrary locally holomorphic function. The terms of degree -1 in $\bar{\lambda}$ combine to a purely anti-holomorphic function and can be absorbed into $\mathcal{V}_1$.

C.2.2 Integrating $\bar{\mathcal{M}}$

To fix the so far unspecified locally holomorphic function, we go back to the expression for $\partial_w \bar{\mathcal{M}}$ in (C.20). Using (C.23), and combining all the derivative terms, it can be rewritten as

$$\partial_w \bar{\mathcal{M}} = -4c^3 \mathcal{L}^3 \bar{\lambda}^{-1} \partial_w \left(\frac{(1-\bar{\lambda} Z^2)(1-\bar{\lambda}\bar{Z}^{-2})}{(1-|\lambda|^2)^2} \right) \tag{C.34}$$

To evaluate the terms in the derivative further, we perform the variable changes to R and ψ and then to $\mathcal{L}$, which yields

$$\begin{aligned} (1-\bar{\lambda} Z^2)(1-\bar{\lambda}\bar{Z}^{-2}) &= 1 - \frac{\bar{\lambda}\bar{\mathcal{L}}}{\mathcal{L}}\left(\Lambda R + \frac{1}{\Lambda R}\right) + \frac{\bar{\lambda}^2 \bar{\mathcal{L}}^2}{\mathcal{L}^2} \\ &= \mathcal{L}^{-2}\left(\mathcal{L} - \bar{\lambda}\bar{\mathcal{L}}\right)^2 - 6\bar{\lambda}\frac{\bar{\mathcal{L}}}{\mathcal{L}}\frac{\kappa^2 \mathcal{G}}{|\partial_w \mathcal{G}|^2} \end{aligned} \tag{C.35}$$

Using that $\bar{\lambda}\bar{\mathcal{L}} - \mathcal{L} = (1-|\lambda|^2)(\bar{\mathcal{A}}_+ - \mathcal{A}_-)$ and $\partial_w \mathcal{G} = -\bar{\mathcal{L}}\partial_w \mathcal{A}_-$, yields

$$\frac{(1-\bar{\lambda} Z^2)(1-\bar{\lambda}\bar{Z}^{-2})}{1-|\lambda|^2} = \mathcal{L}^{-2}(\bar{\mathcal{A}}_+ - \mathcal{A}_-)^2 - \frac{6\bar{\lambda}\mathcal{G}\mathcal{L}^{-2}}{1-|\lambda|^2} \tag{C.36}$$

We thus have

$$\partial_w \bar{\mathcal{M}} = -4c^3 \mathcal{L}^3 \partial_w \left(\bar{\lambda}^{-1}\mathcal{L}^{-2}(\bar{\mathcal{A}}_+ - \mathcal{A}_-)^2 - \frac{6\mathcal{G}\mathcal{L}^{-2}}{1-|\lambda|^2} \right) \tag{C.37}$$

Extracting a total derivative and using again that $\partial_w \mathcal{L} = (1-|\lambda|^2)\partial_w \mathcal{A}_-$, this expression can be further evaluated. Extracting total derivatives iteratively eventually yields

$$\begin{aligned} \frac{\partial_w \bar{\mathcal{M}}}{4c^3} &= \partial_w \left(\frac{6\mathcal{L}\mathcal{G}}{1-|\lambda|^2} + (\mathcal{A}_+ - \bar{\mathcal{A}}_-)(\bar{\mathcal{A}}_+ - \mathcal{A}_-)^2 - 18\mathcal{A}_-\mathcal{G} \right) \\ &\quad + 18\mathcal{A}_-\partial_w \mathcal{G} - 3(\bar{\mathcal{A}}_+ - \mathcal{A}_-)^2 \partial_w \mathcal{A}_+ \end{aligned} \tag{C.38}$$

The terms in the second line are again of degree at most two in $\mathcal{A}_\pm$ and at most one in $\partial_w \mathcal{A}_-$, and can be integrated straightforwardly. We find

$$\begin{aligned} 18\mathcal{A}_-\partial_w \mathcal{G} - 3(\bar{\mathcal{A}}_+ - \mathcal{A}_-)^2 \partial_w \mathcal{A}_+ = \partial_w \Big[& 12|\mathcal{A}_+|^2 \mathcal{A}_- - 12\bar{\mathcal{A}}_+ \mathcal{B} - 9|\mathcal{A}_-|^2 \mathcal{A}_- \\ & - 3|\mathcal{A}_+|^2 \bar{\mathcal{A}}_+ + 20\mathcal{W}_- - \mathcal{A}_+ \mathcal{A}_-^2 \Big] \end{aligned} \tag{C.39}$$

This completes the integration and we conclude that

$$\bar{\mathcal{M}} = 4c^3\Big[\frac{6\mathcal{L}\mathcal{G}}{1-|\lambda|^2} + (\mathcal{A}_+ - \bar{\mathcal{A}}_-)(\bar{\mathcal{A}}_+ - \mathcal{A}_-)^2 - 18\mathcal{A}_-\mathcal{G} + 12|\mathcal{A}_+|^2\mathcal{A}_- \\ - 12\bar{\mathcal{A}}_+\mathcal{B} - 9|\mathcal{A}_-|^2\mathcal{A}_- - 3|\mathcal{A}_+|^2\bar{\mathcal{A}}_+ + 20\mathcal{W}_- - \mathcal{A}_+\mathcal{A}_-^2 + \bar{\mathcal{V}}_2\Big] \tag{C.40}$$

with a locally holomorphic function $\mathcal{V}_2$. To fix $\mathcal{V}_1$ and $\mathcal{V}_2$, we equate the two expressions for $\mathcal{M}$ in (C.40) and (C.33) and isolate the holomorphic and anti-holomorphic dependences. This yields

$$\mathcal{V}_2 = 20\mathcal{W}_+ - 12\mathcal{A}_+\mathcal{B} + \mathcal{A}_-\mathcal{A}_+^2 \tag{C.41}$$

The final form of $\mathcal{M}$ then becomes

$$\mathcal{M} = 8c^3\Big[3\mathcal{G}\left(\frac{\partial_w\mathcal{G}}{\bar{\lambda}\partial_w\mathcal{A}_+ - \partial_w\mathcal{A}_-} - 3\bar{\mathcal{A}}_- - 2\mathcal{A}_+\right) + 10(\mathcal{W}_+ + \bar{\mathcal{W}}_-) \\ + 5(\mathcal{A}_+ + \bar{\mathcal{A}}_-)(|\mathcal{A}_+|^2 - |\mathcal{A}_-|^2)\Big] \tag{C.42}$$

where $\mathcal{W}_\pm$ are defined up to complex constants by (C.30).

APPENDIX D

Curvature for $AdS_p \times S^q \times \Sigma$ warped products

To compute the curvature for the $AdS_6 \times S^2$ and $AdS_2 \times S^6$ cases in parallel, we generalize the setting to $AdS_p \times S^q$ warped over a Riemann surface Σ, with metric

$$ds^2 = 4\rho^2 |dw|^2 + f_A^2 \, d\hat{s}^2_{AdS_p} + f_S^2 \, d\hat{s}^2_{S^q} \tag{D.1}$$

The functions ρ^2, f_A^2, f_S^2 depend only on Σ and $d\hat{s}^2_{AdS_p}$, $d\hat{s}^2_{S^q}$ are respectively the $SO(2,p-1)$ and $SO(q+1)$-invariant metric of unit radius on AdS_p and S^q.

With $\hat{e}^m$ and $\hat{e}^i$ denoting the orthonormal frames for the unit radius AdS_p and S^q, respectively, we make the following choice for the orthonormal frame,

$$\begin{aligned} e^m &= f_A \, \hat{e}^m & m &= 1, \cdots, p \\ e^i &= f_S \, \hat{e}^i & i &= p+1, \cdots, p+q \end{aligned} \tag{D.2}$$

combined with e^z, $e^{\bar{z}}$ for Σ as defined in (2.13). We collectively denote the frame indices by A, B. The frame metrics are given as in (2.11) for Σ and $\eta_{mn} = \text{diag}\,[- + \cdots +]$, $\delta_{ij} = \text{diag}\,[+ \cdots +]$. Denoting the connection 1-form by $\omega^A{}_B$, the torsion equations are,

$$de^A + \omega^A{}_B \wedge e^B = 0 \tag{D.3}$$

The connection forms of the symmetric spaces, denoted by a hat in analogy to the notation for the frame, are defined by the analogous vanishing torsion conditions. The connection components on Σ are given by

$$\omega^z{}_z = dw \, \partial_w \ln \rho - d\bar{w} \, \partial_{\bar{w}} \ln \rho \tag{D.4}$$

We use the notation of (2.13) for the frame-covariant derivative when no connection is needed, and analogous notation when a connection is needed, e.g. $D_z v^a = \rho^{-1}(\partial_w v^a + \omega_w{}^a{}_b v^b)$. The remaining components are then,

$$
\begin{aligned}
\omega^m{}_n &= \hat\omega^m{}_n & \omega^m{}_a &= e^m D_a \ln f_A \\
\omega^i{}_j &= \hat\omega^i{}_j & \omega^i{}_a &= e^i D_a \ln f_S
\end{aligned}
\tag{D.5}
$$

and $\omega^m{}_i = 0$. The components of the Riemann tensor, defined via

$$
\frac{1}{2} R^A{}_{BCD} e^C \wedge e^D = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \tag{D.6}
$$

are then found as

$$
\begin{aligned}
R^m{}_{nAB} &= -\frac{1+|D_a f_A|^2}{f_A^2}\, \delta^m{}_{[A}\, \eta_{B]n} & R^i{}_{jAB} &= \frac{1-|D_a f_S|^2}{f_S^2}\, \delta^i{}_{[A}\, \delta_{B]j} \\
R^a{}_{bAB} &= R^{(2)}\, \delta^a{}_{[A}\, \delta_{B]b} & R^m{}_{iAB} &= -\frac{D_a f_A\, D^a f_S}{f_A f_S}\, \delta^m{}_{[A}\, \delta_{B]i} \\
R^m{}_{aAB} &= -\frac{D_b D_a f_A}{f_A}\, \delta^b{}_{[A}\, \delta_{B]}{}^m & R^i{}_{aAB} &= -\frac{D_b D_a f_S}{f_S} \delta^b{}_{[A}\, \delta_{B]}{}^i
\end{aligned}
\tag{D.7}
$$

where we use the notation $|D_a \ln f|^2 \equiv (D^a \ln f)(D_a \ln f)$ and square brackets denote anti-symmetrization of the enclosed indices, e.g. $\delta^m{}_{[A}\, \eta_{B]n} = \delta^m{}_A\, \eta_{Bn} - \delta^m{}_B\, \eta_{An}$. Furthermore,

$$
R^{(2)} = -\frac{1}{\rho^2} \partial_w \partial_{\bar w} \ln \rho \tag{D.8}
$$

where the normalization is such that $R^{(2)} = -1$ when $\rho^2 = y^{-2}$ corresponds to the Poincaré metric on the upper half space. The components of the Ricci tensor (in frame index convention) are defined by $R_{AB} = R^C{}_{ACB}$, and the non-vanishing ones are given by,

$$
\begin{aligned}
R_{mn} &= \eta_{mn} \left(-\frac{p-1}{f_A^2} - (p-1)\frac{|D_a f_A|^2}{f_A^2} - q\frac{D^a f_A D_a f_S}{f_A f_S} - \frac{D^a D_a f_A}{f_A} \right) \\
R_{ij} &= \delta_{ij} \left(+\frac{q-1}{f_S^2} - (q-1)\frac{|D_a f_S|^2}{f_S^2} - p\frac{D^a f_A D_a f_S}{f_A f_S} - \frac{D^a D_a f_S}{f_S} \right) \\
R_{ab} &= -p\frac{D_b D_a f_A}{f_A} - q\frac{D_b D_a f_S}{f_S} + R^{(2)} \delta_{ab}
\end{aligned}
\tag{D.9}
$$

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