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General Relativistic Shock Waves that Induce Cosmic Acceleration

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General Relativistic Shock Waves that Induce Cosmic Acceleration

By

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DISSERTATION

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Abstract

This thesis concerns the construction and analysis of a new family of exact general relativistic shock waves. The construction resolves the open problem of determining the expanding waves created behind a shock-wave explosion into a static isothermal spacetime with an inverse square density and pressure profile. The construction involves matching two self-similar families of solutions to the perfect fluid Einstein field equations across a spherical shock surface. The matching is accomplished in Schwarzschild coordinates where the shock waves appear one derivative less regular than they actually are. Separately, both families contain singularities, but as matched shock-wave solutions, they are singularity free. There was no guarantee ahead of time that the matching of the two families could be achieved within the regions where both families are nonsingular. Indeed, for pure radiation equations of state, the matching occurs very near the singular point of the interior expanding wave, and this makes the analysis quite delicate, both numerically and formally. It is for this reason the construction is accompanied by a novel existence proof in the pure radiation case. The analysis is extended to demonstrate Lax stability in the pure radiation case and provide a criterion for stability in all other cases. These shock-wave solutions represent an intriguing new mechanism in General Relativity for exhibiting accelerations in perturbed Friedmann spacetimes, analogous to the accelerations modelled by the cosmological constant in the Standard Model of Cosmology. However, unlike in the Standard Model of Cosmology, these shock-wave solutions solve the Einstein field equations in the absence of a cosmological constant, opening up the question of whether a purely mathematical mechanism could account for the cosmic acceleration observed today, rather than dark energy.

Keywords: General Relativity, Shock Wave, Cosmology, Dark Energy

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I am humbled to be a part of the early era of human history for which our collective understanding of the large scale mechanics of the universe is so elegantly sophisticated and yet so far from complete. It is my admiration for the progress that has been made and my naive desire to confront the formidable outstanding problems of our time that motivates my interest in general relativistic shock wave theory and my choice to pursue a doctorate in Applied Mathematics. I seek to sustain this desire throughout the mounting competitiveness posed by a career in academia in the hope of reaching the limits of my capabilities in the pursuit of pushing the boundaries of mathematical creativity.

I would like to express my gratitude to all the staff and faculty for their commitment to providing the Department of Mathematics with both a rigorous graduate programme and a welcoming environment. In particular, I would like to thank Professors John Hunter, Andrew Waldron and Joseph Biello for not only writing academic references for me, sometimes at very short notice, but for their continued interest and encouragement throughout my research. I extend my thanks to Professor Craig Tracey for partially funding this research from his NSF grant *Integrable Structure of Interacting Particle Systems*. This funding helped extend my doctoral studies at UC Davis by two further years, providing invaluable time for the completion of this thesis. It goes without saying that I am highly thankful to my doctoral advisor, Professor Blake Temple, who has without question gone above and beyond in inspiring my research and being a beacon of encouragement at every opportunity. Blake has stood by me since the very beginning and I look forward to many more years of collaboration with him. Thank you Blake.

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CHAPTER 1

Introduction

The fields of General Relativity and Conservation Law Theory connect very naturally. This is probably not too surprising given the fact that one of the most well studied source terms of the Einstein field equations is that of a perfect fluid. The perfect fluid source term, which takes the form of a stress-energy-momentum tensor, is specifically crafted so that equating its divergence to zero yields the classical Euler equations in the limit of small fluid velocities and weak gravitational fields. It is within the study of the classical Euler equations, and its viscous sibling, the Navier-Stokes equations, that consideration of shock-wave formation is most extensive. It is only right then to apply such consideration to the generalised setting of curved spacetimes. This hosts two major advantages. The first is the accuracy gained in modelling self-gravitating fluids, especially compressible fluids, where the fluid pressure both influences and is influenced by the fluid mass. This advantage becomes most apparent when modelling fluids in strong gravitational fields. The second advantage is the extension of possible shock-wave solutions to classical conservation laws. In regard to the the classical Euler equations, in particular the shock tube problem, certain hydrodynamic variables must be balanced on each side of the shock surface to ensure stability of the shock, but this inflexibility in the hydrodynamic variables can be traded for an inflexibility in the geometry of spacetime when considered in the full generality of curved spacetimes.

This thesis focusses on the analysis of spherically symmetric self-similar perfect fluid spacetimes, an area that was first brought into focus by Cahill and Taub in 1971 [**3**]. In this analysis, solutions of the perfect fluid Einstein field equations are assumed to be spherically symmetric and self-similar in the variable $\xi = r/t$. These two assumptions reduce the Einstein field equations, a system of nonlinear partial differential equations, to a system of nonlinear ordinary differential equations in the single variable ξ. It is in this setting that Cahill and Taub establish criteria for the uniqueness of solutions, along with a method to form shock waves. The flat Friedmann-Lemaître-Robertson-Walker (FLRW) and Tolman-Oppenheimer-Volkoff (TOV) spacetimes are explicit solutions to these equations when admitting barotropic equations of state. The former spacetimes are central to the Standard Model of Cosmology and the latter are static models for the interior of a star. In fact, the TOV spacetimes form the unique family of static spherically symmetric perfect fluid spacetimes that are self-similar in ξ and play a central role in the construction of the general relativistic shock waves considered in this thesis.

The first Friedmann-static shock wave was constructed by Cahill and Taub by matching a pure radiation FLRW spacetime to a certain TOV spacetime across a spherical shock surface. Cahill and Taub claimed the existence of a two-parameter family of self-similar pure radiation spacetimes that could be matched to a TOV spacetime to form a shock wave in a subsequent paper that was not published and possibly never completed. Thus the construction of this two-parameter family of shock waves remains an open problem.

Friedmann-static shock waves were considered again by Smoller and Temple in 1994 [**17**], where they proved a number of theorems concerning the regularity of spherically symmetric shock waves. A year later, Smoller and Temple [**18**] generalised Cahill and Taub's shock wave to a one-parameter family of Friedmann-static shock waves, with the parameter corresponding to one of the equations of state either side of the shock. They also introduced a criteria for determining the Lax stability of these shock waves, that is, stability in the gas dynamical sense. Furthermore, Smoller and Temple in 2009 [**22**] derived a one-parameter family of exact self-similar perturbations of the FLRW spacetimes, opening up the possibility of forming new Friedmann-static shock waves from these perturbed spacetimes.

Unbeknown to Smoller and Temple, Carr and Yahil [**9**] were aware of the asymptotic form of these self-similar perturbations as early as 1990 and classified them as *asymptotically Friedmann* spacetimes. The complete classification of spherically symmetric self-similar in ξ solutions to the perfect fluid Einstein field equations was then completed by Carr and Coley in 2000 [**6**]. In addition to determining the number of free parameters present in each family of solutions, Carr and Coley provided a detailed discussion of the physical relevance of each of these families.

Friedmann-static shock waves model a general relativistic explosion within a static isothermal spacetime with an inverse square density and pressure profile. These static isothermal spheres may model the interior of a star, or possibly the early Universe, with the explosion then analogous to a supernova or big bang respectively. In either case, Friedmann-static shock waves offer the simplest dual-state model that incorporates conservation of mass-energy and momentum across the shock surface. The one-parameter family of asymptotically Friedmann spacetimes are of particular interest as they exhibit an accelerated expansion similar to the accelerated expansion found in the Standard Model of Cosmology, but solve the Einstein field equations in the absence of a cosmological constant. Temple conjectures that a Friedmann-static shock wave, constructed by matching an asymptotically Friedmann spacetime to a TOV spacetime, is a possible candidate for a cosmological model with an accelerated expansion but without a cosmological constant, and thus, without dark energy.

The objective of this thesis is the construction of these Friedmann-static shock waves, and in doing so, the determination of the expanding waves created behind a shock-wave explosion within a static isothermal sphere. These Friedmann-static shock waves form a two-parameter generalisation of the one-parameter family of Friedmann-static shock waves constructed by Smoller and Temple in 1995, noting that the equation of state parameter is included in this count. The additional parameter corresponds to the magnitude of perturbation, which in turn corresponds to the magnitude of accelerated expansion. For an interior pure radiation equation of state, these shock waves form a one-parameter subset of the two-parameter family of shock waves sought by Cahill and Taub, thus partially resolving their open problem.

Chapter 2 outlines the process of constructing a spherically symmetric shock-wave solution to the Einstein field equations and is based on Smoller and Temple's 1994 and 1995 papers. Unlike in classical shock-wave theory, the construction of a general relativistic shock wave requires the joining of two spacetime metrics and this requires finding a common set of coordinates for which the two metrics match Lipschitz continuously at the shock surface. If the two metrics can be matched, then consideration of the conservation of mass-energy and momentum across the shock surface can be made, which places a single constraint on the free parameters present in the two spacetimes. If there is sufficient parameter freedom then the resulting matched spacetime forms a shock-wave solution of the Einstein field equations. An analysis of the regularity and Lax stability of the shock wave can then be conducted. The first three sections of Chapter 2 outline this shock-wave construction process in full generality, with the remaining sections dedicated to an explicit example.

Chapter 3 considers spherically symmetric self-similar solutions of the perfect fluid Einstein field equations and closely follows the parts of Cahill and Taub's 1971 paper pertaining to barotropic equations of state. This chapter begins by deriving the system of ODE representing the spherically symmetric self-similar in ξ perfect fluid Einstein field equations in comoving coordinates. This system is used to establish the uniqueness of self-similar solutions, as well as to determine the compatibility of matching self-similar solutions with non-self-similar solutions across a spherical surface. The assumptions of spherical symmetry and self-similarity in ξ are shown to restrict the types of static solutions that can be found, as well as restricting the types of barotropic equations of state that can be modelled. A consideration of self-similar shock-wave solutions is then made, with the chapter concluding with the construction of an explicit self-similar shock-wave solution.

Chapter 4 analyses the asymptotic form of solutions to the system of ODE derived in Chapter 3 and closely follows Carr and Coley's 2000 paper. In this chapter, the method of solving the perfect fluid Einstein field equations by assuming spherical symmetry and self-similarity in the variable ξ is refined, making it easier to find solutions other than those with a pure radiation equation of state. This refinement is then used to provide a complete classification of the different asymptotic behaviours of spherically symmetric self-similar solutions at small, large and finite values of ξ.

Chapter 5 returns to Smoller and Temple's work by considering the asymptotically Friedmann spacetimes derived in their 2012 paper [**21**], which provides the details to their previous 2009 paper. Similar to Chapter 3, this chapter begins with the derivation of a system of ODE representing the spherically symmetric self-similar in ξ perfect fluid Einstein field equations. However, this is not the same system of ODE as derived in Chapter 3, since this new system is derived using Schwarzschild coordinates. There are advantages to both approaches, but the Schwarzschild coordinate approach is better suited to the construction of shock-wave solutions. Furthermore, unlike the ODE derived using comoving coordinates, the ODE derived using Schwarzschild coordinates are autonomous.

With this new system of ODE in place, the TOV, FLRW and asymptotically FLRW solutions are then derived in self-similar Schwarzschild coordinates. The chapter concludes by following Smoller and Temple's analysis of the asymptotically FLRW spacetimes, which exhibit an accelerated expansion similar to the accelerated expansion found in the Standard Model of Cosmology.

Chapter 6 brings together the methods considered in all previous chapters to construct a new family of Friedmann-static shock waves and resolve, or partially resolve, the open problems posed by Smoller and Temple and Cahill and Taub. This chapter begins with an alternative construction of the explicit one-parameter family of Friedmann-static shock waves originally derived by Smoller and Temple in 1995. This warm-up derivation introduces Lemma 6.1.2, which is central to the construction of the more general two-parameter family of Friedmann-static shock waves, noting that one of these parameters is an equation of state parameter. Since the asymptotically Friedmann spacetimes are not known explicitly, numerical approximations are used to construct the two-parameter family of Friedmann-static shock waves. This construction is followed by Lemma 6.2.1, which generalises the Lax characteristic conditions to an even broader family of general relativistic shock waves. An analysis of the Rankine-Hugoniot jump conditions then results in Theorem 6.2.3, which establishes the Lax stability of this broad family of shock waves. Chapter 6 concludes with Theorem 6.3.1, the main result of the thesis, which provides a rigorous proof of existence for the Friedmann-static pure radiation shock wave, that is, the unique Friedmann-static shock wave that models a perfect fluid with a pure radiation equation of state each side of the shock.

The conclusion introduces a conjecture regarding the existence of the full two-parameter family of Friedmann-static shock waves, along with a brief discussion of the possible future directions of this research.

CHAPTER 2

General Relativistic Shock Waves

This chapter summarises and extends the 1995 paper *Astrophysical Shock-Wave Solutions to Einstein's Equations* by Smoller and Temple [**18**]. The extension comes in two forms, the first of which is the inclusion of relevant exposition from the proceeding 1994 paper *Shock-Wave Solutions of the Einstein Equations: The Oppenheimer-Snyder Model of Gravitational Collapse Extended to the Case of Non-Zero Pressure* by Smoller and Temple [**17**]. The second form of extension comes from introducing new notation, all of the definitions, Proposition 2.1.4, Proposition 2.1.5 and Lemma 2.6.3, the latter of which provides an explicit example of the regularity results of Section 2.2.

2.1. Shock-Wave Construction

Consider first the Einstein field equations:

$$G = \kappa T \tag{2.1}$$

where G is the Einstein curvature tensor, T is the stress-energy-momentum tensor and κ is the constant:

$$\kappa = \frac{8\pi\mathcal{G}}{c^4}$$

Here, c is the speed of light and $\mathcal{G}$ is the gravitational constant. When modelling a perfect fluid, the stress-energy-momentum tensor takes the form:

$$T = \left(\rho + \frac{p}{c^2}\right) \boldsymbol{u} \otimes \boldsymbol{u} + pg \tag{2.2}$$

where g is the metric tensor, ρ is the fluid density, p is the fluid pressure and $\boldsymbol{u}$ is the fluid four-velocity. To simplify the algebraic calculations throughout this thesis, natural units will be used, with the exception of Chapter 3. In all chapters other than Chapter 3, this means that:

$$c = \mathcal{G} = 1$$

and distances, times, energies and masses are all measured in units of mass, with all speeds dimensionless and less than one. All spacetimes considered throughout this thesis will be solutions to the perfect fluid Einstein field equations, with the terms *solution* and *spacetime* being used interchangeably. Furthermore, all spacetimes will be spherically symmetric, so that the metric line element takes the general form:

$$ds^2 = -e^{2\varphi(t,r)}dt^2 + 2\mathscr{D}(t,r)dtdr + e^{2\psi(t,r)}dr^2 + \mathscr{R}^2(t,r)r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)$$

Since it is always possible to eliminate the $dtdr$ term with an appropriate change in the t and r coordinates, we do not lose any generality in restricting to the following diagonal form:

$$ds^2 = -e^{2\varphi(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + \mathscr{R}^2(t,r)r^2 d\Omega^2 \tag{2.3}$$

where $d\Omega^2$ is the standard metric on the two-sphere:

$$d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$$

Comoving coordinates are a common choice of coordinates in which to solve the perfect fluid Einstein field equations. The advantage of comoving coordinates is that the fluid four-velocity reduces to the form:

$$\boldsymbol{u} = (u^0, 0, 0, 0)$$

which for a diagonal metric means:

$$u^0 = \frac{1}{\sqrt{-g_{00}}} = \sqrt{-g^{00}}$$

The latter condition follows from the four-velocity normalisation requirement:

$$g(\boldsymbol{u}, \boldsymbol{u}) = -1 \tag{2.4}$$

Another common choice of coordinates are Schwarzschild coordinates. The advantage of using Schwarzschild coordinates is that the metric reduces to the form:

$$ds^2 = -B(t,r)dt^2 + \frac{1}{A(t,r)}dr^2 + r^2 d\Omega^2 \tag{2.5}$$

where A and B are strictly positive and the radial coordinate, r, corresponds to the radial distance in Minkowski spacetime. The use of B and A^{-1} as the respective coefficients of dt^2 and dr^2 follows the convention used by Smoller and Temple. In Schwarzschild coordinates the fluid four-velocity may be written without loss of generality as:

$$\boldsymbol{u} = (u^0, u^1, 0, 0)$$

and under the normalisation requirement (2.4), has only one independent component. This means that the fluid four-velocity can be fully specified through a single variable.

Definition 2.1.1. The *Schwarzschild coordinate velocity* is defined by:

$$v = \frac{1}{\sqrt{AB}} \frac{u^1}{u^0} \tag{2.6}$$

Together with A, B, ρ and p, the Schwarzschild coordinate velocity v is one of five unknown variables that completely specify a spherically symmetric perfect fluid solution. Similarly in comoving coordinates, these variables are the three metric components φ, ψ and $\mathscr{R}$ along with ρ and p. As there are only four independent components of the spherically symmetric perfect fluid Einstein field equations, an equation of state is required to close the system. Throughout this thesis, we will assume that solutions have a barotropic equation of state, that is, one of the form:

$$p = p(\rho) \tag{2.7}$$

The coordinate invariant nature of the spherically symmetric perfect fluid Einstein field equations means that any barotropic equation of state will close the system in any choice of coordinates. Both comoving and Schwarzschild coordinates reduce the complexity of the Einstein field equations by eliminating one of the variables to solve for. In the Schwarzschild case, this variable is $\mathscr{R}$, whereas in the comoving case this variable is $\boldsymbol{u}$ or v. In general, comoving coordinates are more useful for solving the perfect fluid Einstein field equations, whereas Schwarzschild coordinates make the process of matching metrics much simpler, as will be seen. In Section 3.6, an elegant method for transforming a general spherically symmetric metric into Schwarzschild form will be given.

Suppose that we have two spherically symmetric solutions to the perfect fluid Einstein field equations. Let us denote these solutions by the triples $(g, \rho, \boldsymbol{u})$ and $(\bar{g}, \bar{\rho}, \bar{\boldsymbol{v}})$ and assume also that these solutions have equations of state $p = p(\rho)$ and $\bar{p} = \bar{p}(\bar{\rho})$ respectively. Since we are assuming spherical symmetry, when specifying a set of coordinates (t, r, θ, ϕ), it is sufficient to only consider the coordinates (t, r). In this light, let metrics g and $\bar{g}$ be given in Schwarzschild coordinates (t, r) and $(\bar{t}, \bar{r})$ respectively as so:

$$
\begin{aligned}
ds^2 &= -B(t,r)dt^2 + \frac{1}{A(t,r)}dr^2 + r^2 d\Omega^2 \\
d\bar{s}^2 &= -\bar{B}(\bar{t},\bar{r})d\bar{t}^2 + \frac{1}{\bar{A}(\bar{t},\bar{r})}d\bar{r}^2 + \bar{r}^2 d\Omega^2
\end{aligned}
$$

where the coordinate variables (θ, ϕ) and $(\bar{\theta}, \bar{\phi})$ have been identified.

Definition 2.1.2. We say that two metrics can be *matched* on a spherical surface $\tilde{r} = \Phi(\tilde{t})$ if there exists a common set of coordinates $(\tilde{t}, \tilde{r})$ such that the coefficients of the metrics agree on this surface when written in these coordinates.

It is not required that the metrics be given in Schwarzschild coordinates in order to be matched, but it does provide a convenient set of coordinates from which the metrics can be compared. For metrics g and $\bar{g}$, we may simply take (t, r) as our common set of coordinates and ask which transformation of the form:

$$
\begin{aligned}
\bar{t} &= \bar{t}(t,r) \\
\bar{r} &= \bar{r}(t,r)
\end{aligned}
$$

is required in order to match these metrics. The reason Schwarzschild coordinates are so useful is because we automatically match the $d\Omega^2$ coefficients through the identification $\bar{r} = r$. This identification means that in order to avoid introducing $dtdr$ terms, the most general transformation that can be applied takes the form $\bar{t} = \bar{t}(t)$. Thus for two metrics given in Schwarzschild coordinates, the process of matching these metrics reduces to the existence of a spherical surface $r = \Phi(t)$ and a coordinate transformation $\bar{t} = \bar{t}(t)$ that satisfy the following algebraic-differential equations:

$$B(t, \Phi(t)) = \bar{B}(\bar{t}(t), \Phi(t))[\bar{t}'(t)]^2 \tag{2.8}$$

$$A(t, \Phi(t)) = \bar{A}(\bar{t}(t), \Phi(t)) \tag{2.9}$$

If these equations can be solved, then metrics g and $\bar{g}$ can be matched along the surface $r = \Phi(t)$. However, such a matching does not automatically imply that mass-energy and momentum are conserved across the surface. With this in mind, let us assume that there exists a set of coordinates (t, r) for which the metrics match on the spherical surface $r = \Phi(t)$ and define:

$$\Sigma = \{(t, r, \theta, \phi) : r = \Phi(t), t > 0\}$$

Note that even though this surface exists in four-dimensional spacetime, due to spherical symmetry we may refer to a point on this surface by its (t, r) coordinates only.

Definition 2.1.3. We say that the spacetime given by the matched metric $g \cup \bar{g}$, along with the associated hydrodynamic variables, forms a *shock-wave solution* of the perfect fluid Einstein field equations if the Rankine-Hugoniot jump conditions hold across the surface Σ. Furthermore, the spherical surface Σ is known as the *shock surface* or simply the *shock*.

As like in classical shock-wave theory, the Rankine-Hugoniot jump conditions express the weak form of the conservation of mass-energy and momentum across the shock-surface.

Proposition 2.1.4. *Let $\boldsymbol{p} \in \Sigma$ and U be a neighbourhood of $\boldsymbol{p}$, then the weak form of the conservation of mass-energy and momentum across $\Sigma \cap U$ is given by:*

$$\int_U T^{\mu\nu} \nabla_\nu \varphi \; d\boldsymbol{x} = 0 \;\forall\; \varphi \in C_c^\infty(U) \tag{2.10}$$

or equivalently:

$$\int_U G^{\mu\nu} \nabla_\nu \varphi \; d\boldsymbol{x} = 0 \;\forall\; \varphi \in C_c^\infty(U) \tag{2.11}$$

Proof. If each component of the stress-energy-momentum tensor T is differentiable in U, then the conservation of mass-energy and momentum in U is given by:

$$\nabla_\nu T^{\mu\nu} = 0$$

These conditions are equivalent to:

$$\int_U \varphi \nabla_\nu T^{\mu\nu} \, d\boldsymbol{x} = 0 \ \forall \ \varphi \in C_c^\infty(U)$$

and by using the identity:

$$\varphi \nabla_\nu T^{\mu\nu} = \nabla_\nu(\varphi T^{\mu\nu}) - T^{\mu\nu}\nabla_\nu \varphi$$

are then equivalent to:

$$\int_U \nabla_\nu(\varphi T^{\mu\nu}) \, d\boldsymbol{x} - \int_U T^{\mu\nu}\nabla_\nu \varphi \, d\boldsymbol{x} = 0 \ \forall \ \varphi \in C_c^\infty(U)$$

Now since φ is compactly supported within U, the divergence theorem implies:

$$\int_U \nabla_\nu(\varphi T^{\mu\nu}) \, d\boldsymbol{x} = \int_{\partial U} \varphi T^{\mu\nu} n_\nu \, d\boldsymbol{y} = 0 \ \forall \ \varphi \in C_c^\infty(U)$$

where $\boldsymbol{n}$ denotes the outward normal vector to Σ. Thus (2.10) yields the weak form of the conservation of mass-energy and momentum across $\Sigma \cap U$. Conditions (2.11) then follow from equation (2.1). □

The following proposition specifies the general relativistic form of the Rankine-Hugoniot jump conditions.

Proposition 2.1.5. *The Rankine-Hugoniot jump conditions are given by:*

$$[G^{\mu\nu}]n_\nu = 0$$

where:

$$[G^{\mu\nu}]n_\nu := G^{\mu\nu}(g)n_\nu - G^{\mu\nu}(\bar{g})n_\nu$$

Proof. Let $U = U_1 \cup U_2$ where $\partial U_1 \cap \partial U_2 = \Sigma \cap U$ and assume that g and $\bar{g}$ are sufficiently regular on their respective side of Σ, then:

$$\begin{aligned}
\int_U G^{\mu\nu}\nabla_\nu\varphi \; d\boldsymbol{x} &= \int_{U_1} G^{\mu\nu}(g)\nabla_\nu\varphi \; d\boldsymbol{x} + \int_{U_2} G^{\mu\nu}(\bar{g})\bar{\nabla}_\nu\varphi \; d\bar{\boldsymbol{x}} \\
&= \int_{U_1} \nabla_\nu(\varphi G^{\mu\nu}(g)) \; d\boldsymbol{x} - \int_{U_1} \varphi\nabla_\nu G^{\mu\nu}(g) \; d\boldsymbol{x} \\
&+ \int_{U_2} \bar{\nabla}_\nu(\varphi G^{\mu\nu}(\bar{g})) \; d\bar{\boldsymbol{x}} - \int_{U_2} \varphi\bar{\nabla}_\nu G^{\mu\nu}(\bar{g}) \; d\bar{\boldsymbol{x}} \\
&= \int_{\partial U_1} \varphi G^{\mu\nu}(g)n_\nu \; d\boldsymbol{y} - \int_{\partial U_2} \varphi G^{\mu\nu}(\bar{g})\bar{n}_\nu \; d\bar{\boldsymbol{y}} \\
&= \int_{\Sigma} \varphi G^{\mu\nu}(g)n_\nu \; d\boldsymbol{y} - \int_{\Sigma} \varphi G^{\mu\nu}(\bar{g})\bar{n}_\nu \; d\bar{\boldsymbol{y}} \\
&= \int_{\Sigma} \varphi[G^{\mu\nu}]n_\nu \; d\boldsymbol{y} \;\forall\; \varphi \in C_c^\infty(U)
\end{aligned}$$

Thus:

$$[G^{\mu\nu}]n_\nu = 0 \iff \int_U G^{\mu\nu}\nabla_\nu\varphi \; d\boldsymbol{x} = 0 \;\forall\; \varphi \in C_c^\infty(U)$$

□

2.2. Regularity

As like in the previous section, consider the solution triples $(g, \rho, \boldsymbol{u})$ and $(\bar{g}, \bar{\rho}, \bar{\boldsymbol{v}})$. Assume that these solutions can be matched Lipschitz continuously along a spherical surface Σ with a spacelike normal vector $\boldsymbol{n}$ to form the matched metric $g \cup \bar{g}$. Furthermore, let $g \cup \bar{g}$ satisfy the Rankine-Hugoniot jump condition across Σ so that $g \cup \bar{g}$ forms a shock-wave solution. For the rest of this section, the matched metric is to be referred to simply as the metric.

It is reasonable to be concerned with the regularity of such a solution, since a Lipschitz continuous shock wave has discontinuities in the first-order derivatives of the metric and delta function sources in the second-order derivatives. The Einstein tensor comprises second-order derivatives of the metric, so this too is expected to harbour delta function sources. On the other side of the Einstein field equations, the hydrodynamic variables ρ, p and $\boldsymbol{u}$, along with the metric, form the stress-energy-momentum tensor, and since the hydrodynamic variables are expected to be at worst discontinuous at the shock, so too is the stress-energy-momentum tensor. This is problematic, since the Einstein field equations cannot have different levels of regularity on the left and right hand sides of the equation. However, it turns out that even though delta function sources may appear in the second-order derivatives of the metric at the shock, with such being coordinate dependent, the Einstein tensor does not have any delta function sources, that is, the delta function sources cancel in the Einstein tensor. This result is summarised in the following theorem from [**18**].

THEOREM 2.2.1. *Let Σ denote a smooth, three-dimensional surface with a spacelike normal vector $\boldsymbol{n}$. Assume that the components of the metric are continuous on each side of Σ and Lipschitz continuous across Σ in some fixed coordinate system. Then the following statements are equivalent:*

(1) $[K] = 0$ *at each point of Σ, where K is the second fundamental form of the metric.*
(2) *The Riemann curvature and Einstein tensors, viewed as second-order operators on the metric components, produce no delta function sources on Σ.*
(3) *For each point $\boldsymbol{p} \in \Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighbourhood of $\boldsymbol{p}$ such that in the new coordinates, which can be taken to be the Gaussian normal coordinates for the surface, the metric components are $C^{1,1}$ functions of these coordinates.*
(4) *For each point $\boldsymbol{p} \in \Sigma$ there exists a coordinate frame that is locally Lorentzian at $\boldsymbol{p}$ and can be reached from the original coordinates by a $C^{1,1}$ coordinate transformation.*

Moreover, if any one of these statements hold, then the Rankine-Hugoniot jump conditions:

$$[G^{\mu\nu}]n_\nu = 0$$

hold at each point of Σ.

This theorem provides a criterion for the removal of the delta function sources and also a coordinate system for which the shock-wave solution can achieve optimal regularity, that is, when the metric has a Lipschitz continuous derivative at the shock. The following theorem, also from [**18**], provides convenient criteria for satisfying one of the equivalent statements of Theorem 2.2.1.

Theorem 2.2.2. *Assume the following:*

(1) *That g and $\bar{g}$ are two spherically symmetric metrics that match across a three-dimensional surface Σ to form the matched metric $g \cup \bar{g}$.*
(2) *The matched metric is Lipschitz continuous cross Σ.*
(3) *The normal $\boldsymbol{n}$ to Σ is non-null.*

Then the following are equivalent:

(1) $[G^{\mu\nu}]n_\nu = 0$
(2) $[G^{\mu\nu}]n_\mu n_\nu = 0$
(3) $[K] = 0$ *at each point of Σ, where K is the second fundamental form of the metric.*
(4) *The components of the matched metric in any Gaussian-normal coordinate system are $C^{1,1}$ functions of these coordinates across Σ.*

If the conditions of Theorem 2.2.2 are satisfied, then it is clear that the weak form of mass-energy and momentum conservation across the shock surface is equivalent to the single condition:

$$[T^{\mu\nu}]n_\mu n_\nu = 0$$

Thus the Rankine-Hugoniot jump conditions reduce to the single equivalent condition:

$$[G^{\mu\nu}]n_\mu n_\nu = 0$$

Therefore a shock-wave solution, which satisfies the Rankine-Hugoniot jump conditions by definition, only requires the metric to be continuous on each side of Σ and Lipschitz continuous across Σ to satisfy the equivalent statements of Theorems 2.2.1 and 2.2.2. The proofs of these theorems can be found in [**17**].

2.3. Lax Stability

The Lax stability of general relativistic shock waves will now be considered. This stability is determined in the gas dynamical sense, that is, a shock is considered Lax stable when characteristics in the same family as the shock impinge on the shock from both sides, see for example [**13**] and [**16**]. The conditions required for this, known as the Lax characteristic conditions, are derived in the same manner as done by [**18**]. Note that the Lax characteristic conditions lead to the time irreversibility of solutions, since characteristics impinge on the shock, entropy increases and information is lost. In classical gas dynamics, the density and pressure are always larger behind stable shock waves, which means spherically symmetric shock waves with a greater pressure and density on the interior are expected to expand.

Consider again the solution triples $(g, \rho, \hat{\boldsymbol{u}})$ and $(\bar{g}, \bar{\rho}, \hat{\boldsymbol{v}})$ with equations of state $p = p(\rho)$ and $\bar{p} = \bar{p}(\bar{\rho})$ respectively, and assume that these solutions form the shock-wave solution $g \cup \bar{g}$. As a spherical surface has an interior and exterior, let g represent the interior metric, which is given in comoving coordinates $(\hat{t}, \hat{r})$ as so:

$$d\hat{s}^2 = -e^{2\varphi}d\hat{t}^2 + e^{2\psi}d\hat{r}^2 + \mathscr{R}^2\hat{r}^2 d\Omega^2$$

Finally, let $\bar{g}$ represent the exterior metric, with the associated comoving coordinates denoted by $(\bar{t}, \bar{r})$. The objective is to determine the Lax characteristic conditions at the shock surface.

Lemma 2.3.1. *The shock speed relative to the interior fluid is given by:*

$$e^{\psi-\varphi}\dot{\Phi} \tag{2.12}$$

where $\hat{r} = \Phi(\hat{t})$ is the position of the shock in coordinates comoving with the interior fluid.

Proof. This proof largely follows an analogous proof provided by [**18**]. To begin, recall that the speed of a shock is a coordinate dependent quantity that can be interpreted in a special relativistic sense at a point $\boldsymbol{p}$ in coordinate systems for which:

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}_0^2 d\Omega^2 \tag{2.13}$$

where $\tilde{r}_0$ is the value of $\tilde{r}$ at $\boldsymbol{p}$. In a locally Minkowskian coordinate frame, a speed at $\boldsymbol{p}$ transforms according to the special relativistic velocity transformation law when a Lorentz transformation is performed. The shock speed at a point $\boldsymbol{p}$ on the shock in a locally Minkowskian frame that is comoving with the interior fluid will now be determined. To this end, let $\hat{r} = \Phi(\hat{t})$ be the position of the shock in $(\hat{t}, \hat{r})$ coordinates and let $(\tilde{t}, \tilde{r})$ coordinates correspond to a locally Minkowskian system at $\boldsymbol{p}$ obtained from $(\hat{t}, \hat{r})$ by a transformation of the form:

$$\tilde{t} = \tilde{t}(\hat{t})$$

$$\tilde{r} = \tilde{r}(\hat{r})$$

so that, in $(\tilde{t}, \tilde{r})$ coordinates:

$$d\tilde{s}^2 = -e^{2\varphi}\left(\frac{d\hat{t}}{d\tilde{t}}\right)^2 d\tilde{t}^2 + e^{2\psi}\left(\frac{d\hat{r}}{d\tilde{r}}\right)^2 d\tilde{r}^2 + \mathscr{R}^2\hat{r}^2 d\Omega^2$$

Choose $(\tilde{t}, \tilde{r})$ so that:

$$\frac{d\tilde{t}}{d\hat{t}} = e^{\varphi}$$
$$\frac{d\tilde{r}}{d\hat{r}} = e^{\psi}$$

Then in $(\tilde{t}, \tilde{r})$ coordinates at $\boldsymbol{p}$ the metric takes the form of (2.13). The $(\tilde{t}, \tilde{r})$ coordinates represent the class of locally Minkowskian coordinate frames that are fixed relative to the fluid particles of the interior spacetime at the point $\boldsymbol{p}$, that is, any two members of this class of coordinate frames differ only by higher order terms that do not affect the calculation of radial velocities at $\boldsymbol{p}$. Thus the speed $\dot{\tilde{r}}$ of a particle in $(\tilde{t}, \tilde{r})$ coordinates gives the value of the speed of the particle relative to the interior fluid in the special relativistic sense. If the speed of a particle in $(\hat{t}, \hat{r})$ coordinates is $\dot{\hat{r}}$, then its geometric speed relative to observers fixed with the interior fluid, and hence also fixed relative to the radial coordinate $\hat{r}$ of the metric g because the fluid is comoving, is equal to:

$$e^{\psi-\varphi}\dot{\hat{r}}$$

since:

$$\frac{d\hat{r}}{d\hat{t}} = \frac{d\hat{r}}{d\tilde{r}}\frac{d\tilde{t}}{d\hat{t}}\frac{d\tilde{r}}{d\tilde{t}} = e^{\varphi-\psi}\frac{d\tilde{r}}{d\tilde{t}} \tag{2.14}$$

Now considering the shock wave moves with speed $\dot{\Phi}$, therefore by (2.14) the speed of the shock relative to the interior fluid particles must be given by (2.12), which completes the proof. □

Let $\tilde{\lambda}^{+}_{Int}$ and $\tilde{\lambda}^{-}_{Int}$ denote the speeds of the interior characteristics in $(\tilde{t}, \tilde{r})$ coordinates. Since the characteristic speeds on the interior side of the shock equal the sound speeds in locally Minkowskian coordinates, we have:

$$\tilde{\lambda}^{\pm}_{Int} = \pm\sqrt{\frac{dp}{d\rho}}$$

The -,+ characteristics refer to the 1,2-characteristic families respectively. In the 1+1 dimensional theory of conservation laws, the Lax characteristic conditions state that the characteristic curves in the family of the shock impinge upon the shock from both sides. Since we are considering shocks that are outward moving with respect to $\hat{r}$ and $\bar{r}$, it follows that on the interior side, only the 2-characteristic can impinge on the shock, and thus the shock must be identified as a 2-shock. For more details on n-shocks, see [**16**]. Let $\tilde{\lambda}^{+}_{Ext}$ and $\tilde{\lambda}^{-}_{Ext}$ denote the speeds of the exterior characteristics in $(\tilde{t}, \tilde{r})$ coordinates. Since the shock has been identified as a 2-shock, the Lax characteristic conditions are given as the following inequalities:

$$\tilde{\lambda}^{+}_{Ext} < s < \tilde{\lambda}^{+}_{Int} \tag{2.15}$$

where s is the speed of the shock in $(\tilde{t}, \tilde{r})$ coordinates.

PROPOSITION 2.3.2. *For an expanding shock wave, the Lax characteristic conditions are given as the following inequalities:*

$$\frac{\tilde{w} + \sqrt{\frac{d\bar{p}}{d\bar{\rho}}}}{1 + \tilde{w}\sqrt{\frac{d\bar{p}}{d\bar{\rho}}}} < e^{\psi - \varphi}\dot{\Phi} < \sqrt{\frac{dp}{d\rho}} \tag{2.16}$$

where:

$$\tilde{w} = e^{\psi - \varphi}\frac{\partial \hat{r}}{\partial \bar{t}}\left(\frac{\partial \hat{t}}{\partial \bar{t}}\right)^{-1}$$

PROOF. This proof largely follows an analogous proof provided by [**18**]. Since the shock wave is expanding, it is a 2-shock, so the Lax characteristic conditions are given by (2.15), and by Lemma 2.3.1, s is given by (2.12). As we are working in $(\tilde{t}, \tilde{r})$ coordinates, $\tilde{\lambda}^{+}_{Int}$ is already known, so it remains to determine $\tilde{\lambda}^{+}_{Ext}$. Let $\hat{\boldsymbol{v}}$, $\bar{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}}$ denote the exterior fluid four-velocity given in interior comoving, exterior comoving and interior locally Minkowskian coordinates respectively. Since the aim is to compute the characteristic speed, which is a ratio of two vector components, a tangent vector of any length is sufficient. By writing $\hat{\boldsymbol{x}} = (\hat{t}, \hat{r})$ and $\bar{\boldsymbol{x}} = (\bar{t}, \bar{r})$, then:

$$\hat{v}^{\mu} = \frac{\partial \hat{x}^{\mu}}{\partial \bar{x}^{\nu}} \bar{v}^{\nu} = \frac{\partial \hat{x}^{\mu}}{\partial \bar{x}^{0}} \bar{v}^{0} = \frac{\partial \hat{x}^{\mu}}{\partial \bar{x}^{0}}$$

In light of this, the speed of the exterior fluid as measured in the interior coordinates $(\hat{t}, \hat{r})$ is given by:

$$\hat{w} = \frac{\hat{v}^{1}}{\hat{v}^{0}} = \frac{\partial \hat{x}^{1}}{\partial \bar{x}^{0}} \left(\frac{\partial \hat{x}^{0}}{\partial \bar{x}^{0}} \right)^{-1} = \frac{\partial \hat{r}}{\partial \bar{t}} \left(\frac{\partial \hat{t}}{\partial \bar{t}} \right)^{-1}$$

and so, by (2.14):

$$\tilde{w} = e^{\psi - \varphi} \hat{w}$$

This gives the exterior fluid speed in $(\tilde{t}, \tilde{r})$ coordinates, and since the sound speed in the exterior spacetime is given by:

$$\sqrt{\frac{d\bar{p}}{d\bar{\rho}}}$$

the relativistic addition of velocities formula yields:

$$\tilde{\lambda}^{+}_{Ext} = \frac{\tilde{w} + \sqrt{\frac{d\bar{p}}{d\bar{\rho}}}}{1 + \tilde{w}\sqrt{\frac{d\bar{p}}{d\bar{\rho}}}}$$

which completes the proof. □

2.4. FLRW Spacetimes

Now that the appropriate theory has been discussed regarding the construction of general relativistic shock waves, it is time to consider an explicit example. For this example, two well known spherically symmetric spacetimes will be matched to form a shock-wave solution of the perfect fluid Einstein field equations. This section will discuss the interior spacetime, which will be an FLRW spacetime. Note that in [**17**] and [**18**], the FLRW spacetimes are referred to as FRW spacetimes. These spatially homogeneous spacetimes have been studied in great detail and are the simplest examples of spacetimes that are expanding or contracting. The general form of an FLRW metric is given in comoving coordinates as so:

$$ds^2 = -dt^2 + R^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right) \tag{2.17}$$

where k takes the value of 0, 1 or -1 depending on whether the spacetime is flat, closed or open respectively. What makes an FLRW spacetime particularly simple is the fact that in comoving coordinates, the scale factor R and density ρ are both functions of time alone. This means that substituting (2.17) into the perfect fluid Einstein field equations yields a system of ODE in the single variable t, as so:

$$-\frac{3\ddot{R}}{R} - \frac{1}{2}\left(\frac{6R\ddot{R} + 6\dot{R}^2 + 6k}{R^2}\right)(-1) = 8\pi\mathcal{G}\rho$$

$$\frac{R\ddot{R} + 2\dot{R}^2 + 2k}{1-kr^2} - \frac{1}{2}\left(\frac{6R\ddot{R} + 6\dot{R}^2 + 6k}{R^2}\right)\left(\frac{R^2}{1-kr^2}\right) = \frac{8\pi\mathcal{G}pR^2}{1-kr^2}$$

and these equations can be simplified to:

$$3\ddot{R} = -4\pi\mathcal{G}(\rho + 3p)R \tag{2.18}$$

$$R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi\mathcal{G}(\rho - p)R^2 \tag{2.19}$$

Substituting equation (2.18) into (2.19) yields:

$$\dot{R}^2 + k = \frac{8\pi\mathcal{G}}{3}\rho R^2 \tag{2.20}$$

Proposition 2.4.1. *Local conservation of mass-energy and momentum implies:*

$$p = -\rho - \frac{R\dot{\rho}}{3\dot{R}} \tag{2.21}$$

Proof.

$$
\begin{aligned}
\nabla_\nu T^{\mu\nu} = 0 &\iff g^{\mu\nu}\nabla_\nu p + \nabla_\nu \left[(\rho+p)u^\mu u^\nu\right] = 0 \\
&\iff g^{\mu\nu}\partial_\nu p + \partial_\nu \left[(\rho+p)u^\mu u^\nu\right] + \Gamma^\mu{}_{\nu\sigma}(\rho+p)u^\sigma u^\nu + \Gamma^\sigma{}_{\nu\sigma}(\rho+p)u^\mu u^\nu = 0 \\
&\iff -\frac{d}{dt}p + \frac{d}{dt}(\rho+p) + 3\frac{\dot{R}}{R}(\rho+p) = 0 \\
&\iff p = -\rho - \frac{R\dot{\rho}}{3\dot{R}}
\end{aligned}
$$

□

Since the divergence-free nature of the stress-energy-momentum tensor follows from the Einstein field equations, equation (2.21) is implied by equations (2.18) and (2.19). With the introduction of a barotropic equation of state, that is (2.7), then (2.21) and (2.20) are two equations for the two remaining unknowns R and ρ. Note that it can be seen from (2.21) that $\dot{\rho}\dot{R} < 0$ and thus $(R(t), \rho(t))$ is a solution of (2.21) and (2.20) if and only if $(R(-t), \rho(-t))$ is, therefore every expanding solution has a corresponding contracting solution and vice versa. For all the shock waves with FLRW interiors constructed in this thesis, the only choice of k that will be considered is the $k = 0$ case, making the interior FLRW metric conformally flat. With $k = 0$, equation (2.20) can be rewritten as:

$$\dot{R} = \pm R\sqrt{\frac{8\pi\mathcal{G}\rho}{3}} \tag{2.22}$$

and substituting (2.22) into (2.21) yields:

$$p = -\rho \mp \frac{\dot{\rho}}{\sqrt{24\pi\mathcal{G}\rho}} \tag{2.23}$$

When $p = p(\rho)$ is assigned, equation (2.23) is independent of R and can thus be integrated explicitly to obtain:

$$t - t_0 = \mp \int_{\rho_0}^{\rho} \frac{1}{(p(\xi)+\xi)\sqrt{24\pi\mathcal{G}\xi}}\, d\xi \tag{2.24}$$

Formula (2.24) gives t as a function of ρ, this can be used along with (2.21) to obtain a closed form expression for R as a function of ρ. It follows from (2.23) that:

$$\dot{R} = \frac{d\rho}{dt}\frac{dR}{d\rho} = \mp(\rho + p)\sqrt{24\pi\mathcal{G}\rho}\frac{dR}{d\rho}$$

and combining this result with (2.21) yields:

$$\frac{1}{R}\frac{dR}{d\rho} = -\frac{1}{3(\rho + p)} \tag{2.25}$$

Finally, (2.25) can be solved explicitly to give:

$$R = R_0 \exp\left(-\int_{\rho_0}^{\rho} \frac{1}{3(p(\xi) + \xi)}\ d\xi\right) \tag{2.26}$$

Notice that the only assumptions required to construct an explicit solution of the FLRW type are that $k = 0$ and the equation of state is barotropic.

2.5. TOV Spacetimes

As like in the previous section, we now consider another well known spherically symmetric spacetime, with this spacetime being placed on the exterior of the shock surface. This spacetime is a TOV spacetime, which are characterised by the property of being static. Note that in [**17**] and [**18**], the TOV spacetimes are referred to as OT spacetimes. Everything in a spherically symmetric static scenario in General Relativity scales with the total mass of the system, so if we double the total mass of the system all lengths and time scales will also be doubled for example. The static nature of a TOV metric also means that it can be given in both Schwarzschild coordinates and comoving coordinates simultaneously. In such coordinates, a TOV metric takes the following form:

$$d\bar{s}^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{A(\bar{r})}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \tag{2.27}$$

As like for the FLRW spacetimes, the TOV spacetimes are simple in the respect that, in comoving coordinates, their density $\bar{\rho}$ and metric coefficients are functions of radius alone. Once again, substituting (2.27) into the perfect fluid Einstein field equations yields a system of ODE, but this time of the single variable r. Instead of substituting (2.27) into the Einstein field equations given by:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.28}$$

the metric will be substituted into the following alternative form of the Einstein field equations, given by:

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{2.29}$$

This alternative form is found by taking the trace of (2.28) so that R is equated to $-\kappa T$, which in turn is substituted back into (2.28) and rearranged to obtain (2.29). In this light, substituting (2.27) into (2.29) yields the following equations for A, B, ρ and p:

$$\frac{AB''}{2} + \frac{A'B'}{4} - \frac{A(B')^2}{4B} + \frac{AB'}{\bar{r}} = 4\pi\mathcal{G}(\bar{\rho} + 3\bar{p})B \tag{2.30}$$

$$-\frac{AB''}{2} - \frac{A'B'}{4} + \frac{A(B')^2}{4B} - \frac{A'B}{\bar{r}} = 4\pi\mathcal{G}(\bar{\rho} - \bar{p})B \tag{2.31}$$

$$\frac{B}{\bar{r}^2} - \frac{AB}{\bar{r}^2} - \frac{AB'}{2\bar{r}} - \frac{A'B}{2\bar{r}} = 4\pi\mathcal{G}(\bar{\rho} - \bar{p})B \tag{2.32}$$

Some of the following calculations are taken from [**23**]. To find an explicit expression for A, add (2.31) and two times (2.32) to (2.30) to obtain the following equation after simplification:

$$\frac{1}{\bar{r}^2} - \frac{A}{\bar{r}^2} - \frac{A'}{\bar{r}} = 8\pi\mathcal{G}\bar{\rho} \tag{2.33}$$

Requiring that $A(0)$ be finite, this boundary condition can be used to solve (2.33) to obtain:

$$A = 1 - \frac{2\mathcal{G}M}{\bar{r}} \tag{2.34}$$

where:

$$M' = 4\pi\bar{r}^2\bar{\rho} \tag{2.35}$$

PROPOSITION 2.5.1. *Local conservation of mass-energy and momentum implies:*

$$\frac{B'}{B} = -\frac{2\bar{p}'}{\bar{p} + \bar{\rho}} \tag{2.36}$$

PROOF.

$$
\begin{aligned}
\nabla_\nu T^{\mu\nu} = 0 &\iff g^{\mu\nu}\nabla_\nu \bar{p} + \nabla_\nu\left[(\bar{\rho}+\bar{p})u^\mu u^\nu\right] = 0 \\
&\iff g^{\mu\nu}\partial_\nu \bar{p} + \partial_\nu\left[(\bar{\rho}+\bar{p})u^\mu u^\nu\right] + \Gamma^\mu{}_{\nu\sigma}(\bar{\rho}+\bar{p})u^\sigma u^\nu + \Gamma^\sigma{}_{\nu\sigma}(\bar{\rho}+\bar{p})u^\mu u^\nu = 0 \\
&\iff A\bar{p}' + \frac{AB'}{2B}(\bar{\rho}+\bar{p}) = 0 \\
&\iff \frac{B'}{B} = -\frac{2\bar{p}'}{\bar{p}+\bar{\rho}}
\end{aligned}
$$

□

Finally, substituting (2.34) and (2.36) into (2.32) to eliminate A and B yields the following equation after non-trivial simplification:

$$
-\bar{r}^2\bar{p}' = \mathcal{G}M\bar{\rho}\left(1+\frac{\bar{p}}{\bar{\rho}}\right)\left(1+\frac{4\pi\bar{r}^3\bar{p}}{M}\right)\left(1-\frac{2\mathcal{G}M}{\bar{r}}\right)^{-1} \tag{2.37}
$$

Equation (2.37) is known as the *Oppenheimer-Volkov equation* and Weinberg remarks that it is the fundamental equation of Newtonian astrophysics, with the last three factors representing general relativistic corrections. As like in the FLRW case, it is again worth noting that equation (2.36) is not independent, but is implied by equations (2.30)-(2.32). Also note that if a barotropic equation of state is supplied, then (2.35) and (2.37) are two equations for the two unknowns M and $\bar{\rho}$. In reference to (2.35), M can be realised as the total mass-energy inside radius $\bar{r}$, that is:

$$
M(\bar{r}) = \int_0^{\bar{r}} 4\pi\xi^2\bar{\rho}(\xi)\; d\xi \tag{2.38}
$$

Note that the only assumption required to construct an explicit solution of the TOV type is that the equation of state be barotropic. Tolman was the first to notice that by assuming an equation of state of the form:

$$
\bar{p} = \bar{\sigma}\bar{\rho} \tag{2.39}
$$

for some constant $\bar{\sigma}$, and a density of the form:

$$
\bar{\rho} = \frac{\gamma}{\bar{r}^2} \tag{2.40}
$$

for some constant γ, that an explicit solution to (2.30)-(2.32) can be constructed. Since $\sqrt{\bar{\sigma}}$ is the speed of sound in the exterior fluid, we will require that $0 < \bar{\sigma} < 1$. Given (2.40), the total mass-energy within radius $\bar{r}$ will then be:

$$M = 4\pi\gamma\bar{r} \tag{2.41}$$

and substituting (2.39)-(2.41) into (2.37) yields:

$$\gamma = \frac{1}{2\pi\mathcal{G}}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right) \tag{2.42}$$

Since M has been specified, A is found through (2.34) to be:

$$A = 1 - 8\pi\mathcal{G}\gamma \tag{2.43}$$

To determine the TOV metric, it thus remains to solve for B. In this light, solving (2.36) is equivalent to solving:

$$\frac{1}{B}\frac{dB}{d\bar{\rho}}\frac{d\bar{\rho}}{d\bar{r}} = -\frac{2\bar{\sigma}}{(1+\bar{\sigma})\bar{\rho}}\frac{d\bar{\rho}}{d\bar{r}}$$

which simplifies to:

$$\frac{1}{B}\frac{dB}{d\bar{\rho}} = -\frac{2\bar{\sigma}}{(1+\bar{\sigma})\bar{\rho}}$$

and has the explicit solution:

$$B = B_0\left(\frac{\bar{\rho}}{\bar{\rho}_0}\right)^{-\frac{2\bar{\sigma}}{1+\bar{\sigma}}} = B_0\left(\frac{\bar{r}}{\bar{r}_0}\right)^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}} \tag{2.44}$$

By rescaling the time coordinate, $B_0 = 1$ at $\bar{r}_0 = 1$ can be set, in which case B is given by:

$$B = \bar{r}^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}} \tag{2.45}$$

The solutions derived in this section are singular at the spatial origin, since $\bar{p}$ and $\bar{\rho}$ become infinite at $\bar{r} = 0$ for all time. However, the shock wave constructed in the next section removes this singularity at $\bar{r} = 0$ for $t > 0$ by placing an expanding FLRW spacetime on the interior of the shock.

2.6. FLRW-TOV Shock Waves

In [**17**], a procedure for constructing a coordinate transformation $(\bar{t},\bar{r}) \to (t,r)$ is described such that the FLRW metric matches the TOV metric Lipschitz continuously along the shock surface. The details of this procedure will now be provided. Recall that the FLRW metric is given in comoving coordinates by (2.17) and the TOV metric is given in comoving Schwarzschild coordinates by (2.27). In order for the $d\Omega^2$ coefficients to be identified, we require:

$$\bar{r} = R(t)r$$

This identification provides a way of dynamically matching the radial coordinates on each side of the shock. The transformation $\bar{t} \to \bar{t}(t,r)$ is less simple to construct but an explicit construction will not be required, instead we will simply demonstrate the existence of this transformation. This forms an important step in proving that the matching is Lipschitz continuous, as we will see. In this light, applying the exterior derivative to $\bar{r}$ yields:

$$d\bar{r} = Rdr + \dot{R}rdt$$

so that:

$$dr = \frac{1}{R}d\bar{r} - \frac{\dot{R}}{R}rdt$$

and in particular:

$$dr^2 = \left(\frac{1}{R}d\bar{r} - \frac{\dot{R}r}{R}dt\right) \otimes \left(\frac{1}{R}d\bar{r} - \frac{\dot{R}r}{R}dt\right) = \frac{1}{R^2}d\bar{r}^2 + \frac{\dot{R}^2r^2}{R^2}dt^2 - 2\frac{\dot{R}r}{R^2}dtd\bar{r}$$

Thus the FLRW metric (2.17) is given in $(t,\bar{r})$ coordinates by:

$$ds^2 = -\left(1 - \frac{\dot{R}^2\bar{r}^2}{R^2 - k\bar{r}^2}\right)dt^2 + \frac{R^2}{R^2 - k\bar{r}^2}d\bar{r}^2 - \frac{2R\dot{R}\bar{r}}{R^2 - k\bar{r}^2}dtd\bar{r} + \bar{r}^2d\Omega^2$$

and this can be simplified with the use of (2.20) to become:

$$ds^2 = \frac{1}{R^2 - k\bar{r}^2}\left(-R^2\left[1 - \frac{8\pi\mathcal{G}}{3}\rho R^2r^2\right]dt^2 + R^2d\bar{r}^2 - 2R\dot{R}\bar{r}dtd\bar{r}\right) + \bar{r}^2d\Omega^2 \tag{2.46}$$

Now we need the transformation $t = t(\bar{t}, \bar{r})$ to eliminate the $dtd\bar{r}$ term in (2.46). Let us first consider such a transformation on a general metric of the form:

$$d\tilde{s}^2 = -C(t,\bar{r})dt^2 + D(t,\bar{r})d\bar{r}^2 + 2E(t,\bar{r})dtd\bar{r} \tag{2.47}$$

It is not difficult to show that if $\Psi = \Psi(t, \bar{r})$ is chosen to satisfy:

$$\frac{\partial}{\partial \bar{r}}(\Psi C) = -\frac{\partial}{\partial t}(\Psi E) \tag{2.48}$$

then:

$$d\bar{t} = \Psi\left(Cdt - Ed\bar{r}\right) \tag{2.49}$$

is an exact differential and the $(\bar{t}, \bar{r})$ line element for (2.47) becomes:

$$d\bar{s}^2 = -\frac{1}{\Psi^2 C}d\bar{t}^2 + \left(D + \frac{E^2}{C}\right)d\bar{r}^2 \tag{2.50}$$

Thus applying this general procedure to (2.46) yields:

$$C = R^2\left(1 - \frac{8\pi\mathcal{G}}{3}\rho\bar{r}^2\right) \tag{2.51}$$

$$D = R^2 \tag{2.52}$$

$$E = -R\dot{R}\bar{r} \tag{2.53}$$

By transforming the FLRW metric into $(\bar{t}, \bar{r})$ coordinates and equating the $d\bar{r}^2$ coefficients with the TOV metric, the shock surface Σ can be given implicitly by the equation:

$$\frac{1}{A} = \frac{1}{R^2 - k\bar{r}^2}\left[R^2 + \frac{R^2\dot{R}^2\bar{r}^2}{R^2\left(1 - \frac{8\pi\mathcal{G}}{3}\rho\bar{r}^2\right)}\right]$$

and we can simplify this using equation (2.20) to obtain:

$$M(\bar{r}(t)) = \frac{4\pi}{3}\rho(t)\bar{r}^3(t) \tag{2.54}$$

Equation (2.54) defines the radial coordinate $\bar{r}$ of the TOV metric as a function of the time coordinate t of the FLRW metric along the shock surface, that is, (2.54) is an equation parameterising the shock position $\bar{r}$, or equivalently r, in terms of t. Since there is no mention of p or $\bar{p}$ in (2.54), this equation holds when any barotropic equation of state $p = p(\rho)$ is assigned to the FLRW spacetime and any barotropic equation of state $\bar{p} = \bar{p}(\bar{\rho})$ is assigned to the TOV spacetime. Equation (2.54) is also a statement of the global conservation of mass-energy as the shock progresses. We will make the assumption $1 - kr^2 > 0$ for $k > 0$ in the calculations that follow and note that this assumption is equivalent to $R^2 - k\bar{r}^2 > 0$ for $k > 0$.

Returning to the construction of the transformation $t = t(\bar{t}, \bar{r})$, the next step is to determine the existence and regularity of Ψ, since this determines the existence and regularity of $\bar{t}$ through (2.49). To find Ψ, equation (2.48) must be solved subject to initial data on the shock surface, which according to [**17**] is determined through the requirement that the $d\bar{t}^2$ coefficient of the FLRW metric in $(\bar{t}, \bar{r})$ coordinates matches the $d\bar{t}^2$ coefficient of the TOV metric. Thus the initial data is given by:

$$B = \frac{1}{(R^2 - k\bar{r}^2)\Psi^2 C} \tag{2.55}$$

Now equation (2.48) can be rewritten in the form of a first-order linear partial differential equation as so:

$$E\partial_t\Psi + C\partial_{\bar{r}}\Psi = -(\partial_t E + \partial_{\bar{r}} C)\Psi \tag{2.56}$$

with the initial condition (2.55) rewritten as:

$$\Psi(t, \bar{r}(t)) = \left(B(t, \bar{r}(t))C(t, \bar{r}(t))[R^2(t) - k\bar{r}^2(t)]\right)^{-\frac{1}{2}} \tag{2.57}$$

and where all points $(t, \bar{r}(t))$ are constrained to lie on the shock surface given by (2.54). In solving a PDE initial value problem with initial data given on a surface, the solution in a neighbourhood of the surface can be determined by solving a set of ODE along the characteristic curves. If this surface contains a characteristic curve of the PDE then the initial data may not be freely imposed. This is because the value of any solution at a point further long the characteristic is determined by the initial data at that point as well as the initial condition for the corresponding ODE, the latter of which is specified on the surface.

If the shock surface given by (2.54) is characteristic with respect to (2.56) at some point $\boldsymbol{p}$ on the shock surface, then a solution cannot be guaranteed to exist locally about $\boldsymbol{p}$ for arbitrary compatible initial data. Thus we need to show that the shock surface (2.54) is non-characteristic with respect to the PDE (2.56). With this in mind, we have that the characteristic curves for (2.56) are parameterised by:

$$\frac{d\bar{r}}{dt} = \frac{C}{E}$$

and subject to the initial condition $\bar{r}(t_0) \in \Sigma$. The shock surface Σ is given by the following defining equation:

$$\Phi(t, \bar{r}) = M(\bar{r}) - \frac{4\pi}{3}\rho(t)\bar{r}^3 = 0$$

and this defining equation is non-characteristic with respect to (2.56) at a point $(t_0, \bar{r}_0)$ on the shock surface providing:

$$(E(t_0, \bar{r}_0), C(t_0, \bar{r}_0)) \cdot \nabla\Phi(t_0, \bar{r}_0) \neq 0 \tag{2.58}$$

To establish that the shock surface is non-characteristic with respect to (2.56), that is, establish (2.58), we will need the following proposition.

Proposition 2.6.1. *The density is larger on the interior of an expanding FLRW-TOV shock wave.*

Proof. Suppose for contradiction that $(t_0, \bar{r}_0)$ is a point on the shock surface Σ, defined by equation (2.54), and $\rho(t_0) \leq \bar{\rho}(\bar{r}_0)$. Then:

$$\frac{4\pi}{3}\rho(t_0)\bar{r}_0^3 \leq \frac{4\pi}{3}\bar{\rho}(\bar{r}_0)\bar{r}_0^3 < \int_0^{\bar{r}_0} 4\pi\bar{\rho}(\xi)\xi^2 \, d\xi = M(\bar{r}_0)$$

which is a contradiction, since the point $(t_0, \bar{r}_0)$ satisfies equation (2.54). Thus:

$$[\rho](t_0, \bar{r}_0) := \bar{\rho}(\bar{r}_0) - \rho(t_0) < 0$$

□

In such a case $\dot{\rho} < 0$, and since $\dot{\rho}\dot{R} < 0$, then $\dot{R} > 0$ as expected.

Proposition 2.6.2. *The shock surface Σ, defined by equation (2.54), is non-characteristic with respect to the PDE (2.56) providing:*

(1) *The shock surface is expanding.*
(2) *The shock surface remains within the region $1 - kr^2 > 0$ for $k > 0$.*
(3) *The shock surface remains within the Schwarzschild radius of (2.46).*

Proof. First note from (2.51) that C remains strictly positive when inside the Schwarzschild radius of (2.46), that is:

$$1 - \frac{8\pi\mathcal{G}}{3}\rho(t)\bar{r}^2 > 0$$

After taking the gradient of (2.54), condition (2.58) can be written as:

$$\frac{C(t_0,\bar{r}_0)}{E(t_0,\bar{r}_0)} \neq \frac{\dot{\rho}(t_0)\bar{r}_0}{3[\rho](t_0,\bar{r}_0)} \tag{2.59}$$

Letting $\bar{r} = \bar{r}(t)$ and taking the time derivative of (2.54) yields:

$$\dot{\bar{r}} = \frac{\dot{\rho}\bar{r}}{3[\rho]}$$

Thus condition (2.59) reduces, as expected, to:

$$\dot{\bar{r}} \neq \frac{C}{E}$$

Therefore, we see by comparing the sign on each side of (2.59) that this condition is satisfied for an expanding shock surface, since $\dot{\rho} < 0$, $\dot{R} > 0$ and, by Proposition 2.6.1, $[\rho] < 0$. □

Since the requirements of Proposition 2.6.2 are expected to hold for an expanding FLRW-TOV shock wave, thus making the shock surface non-characteristic, we are now in a position to determine the existence and regularity of Ψ.

Lemma 2.6.3. *Let Σ be non-characteristic with respect to (2.56), $E, C \in C^{2,1}(M)$ and $f, g, h \in C^{1,1}(Y)$ where M is a neighbourhood about a point on Σ and Y is an appropriate interval in $\mathbb{R}$. Then there exists a unique solution $\Psi \in C^{1,1}(M)$ to (2.56) with initial data (2.57).*

PROOF. The compatibility of the initial data will first be checked. The initial data (2.57) can be written in the form:

$$\Psi\left(f(y), g(y)\right) = h(y)$$

where $f(y) = y$, $h(y)$ is the right hand side of (2.57) in the variable y and $g(y)$ is given implicitly by:

$$M(g(y)) = \frac{4\pi}{3}\rho(y)g^3(y)$$

The initial data (2.57) is compatible with the PDE (2.56) if:

$$E\left(f(y), g(y)\right) g'(y) - C\left(f(y), g(y)\right) f'(y) \neq 0$$

which simplifies to the non-characteristic condition:

$$g' \neq \frac{C}{E}$$

Thus the initial data is compatible whenever the shock surface is non-characteristic. Now the characteristic equations of (2.56) for initial data (2.57) are given by the following system of first-order ODE:

$$\begin{aligned}
\frac{dt}{dx} &= E(t, \bar{r}) && t(0; y) = f(y) \\
\frac{d\bar{r}}{dx} &= C(t, \bar{r}) && \bar{r}(0; y) = g(y) \\
\frac{dz}{dx} &= -\left[\partial_t E(t, \bar{r}) + \partial_{\bar{r}} C(t, \bar{r})\right] z && z(0; y) = h(y)
\end{aligned}$$

of which the first two can be written as the system:

$$\begin{cases} \boldsymbol{v}'(x; y) &= \boldsymbol{F}(\boldsymbol{v}(x; y)) \\ \boldsymbol{v}(0; y) &= \boldsymbol{v}_0(y) \end{cases} \tag{2.60}$$

where:

$$\boldsymbol{v}(x;y) = (t(x;y), \bar{r}(x;y))$$

$$\boldsymbol{F}(\boldsymbol{v}) = (E(\boldsymbol{v}), C(\boldsymbol{v}))$$

$$\boldsymbol{v}_0(y) = (f(y), g(y))$$

By the Picard-Lindelöf theorem, if $\boldsymbol{F}$ is a Lipschitz continuous function of $\boldsymbol{v}$ on an open neighbourhood U of $\boldsymbol{v}_0$, then for some interval X containing 0 there exists a unique solution $\boldsymbol{v} \in C^{1,1}(X,U)$ of the initial value problem (2.60). If $E, C \in C^{0,1}(U)$ then $\boldsymbol{F} \in C^{0,1}(U)$, thus if $E, C \in C^{1,1}(U)$ and $f, g \in C^{1,1}(Y)$ for some appropriate interval Y in $\mathbb{R}$, then $t, \bar{r} \in C^{1,1}(X \times Y)$. If $\boldsymbol{v} \in C^{1,1}(X \times Y)$ and the initial data is compatible, then there exists a point $\boldsymbol{q} \in X \times Y$ such that $(t(\boldsymbol{q}), \bar{r}(\boldsymbol{q})) \in \Sigma$ and the Jacobian of $\boldsymbol{v}$ is invertible at $\boldsymbol{q}$. Thus by the inverse function theorem, there exists an open neighbourhood $N \subset X \times Y$ of $\boldsymbol{q}$ and an open neighbourhood $M \subset U$ of $\boldsymbol{v}(\boldsymbol{q})$ such that:

$$\boldsymbol{v}^{-1}(t, \bar{r}) = (x(t, \bar{r}), y(t, \bar{r})) \in C^{1,1}(M, N)$$

The final characteristic equation can then be solved to obtain:

$$z(x;y) = h(y) \exp\left(-\int_0^x \partial_t E(t(s;y), \bar{r}(s;y)) + \partial_{\bar{r}} C(t(s;y), \bar{r}(s;y))\; ds\right)$$

and therefore the unique solution to (2.56) with initial data (2.57) is given by:

$$\Psi(t, \bar{r}) = z(x(t, \bar{r}); y(t, \bar{r}))$$

Moreover, we see from our regularity assumptions that $\Psi \in C^{1,1}(M)$, which completes the proof. □

Corollary 2.6.1. *If:*

(1) *The shock surface Σ is expanding.*
(2) *The shock surface Σ remains within the region $1 - kr^2 > 0$ for $k > 0$.*
(3) *The shock surface Σ remains within the Schwarzschild radius of (2.46).*
(4) *The FLRW and TOV metrics are smooth on each side of Σ away from the origin.*

Then for each point $\boldsymbol{p} \in \Sigma \setminus \{\mathbf{0}\}$ there exists a $C^{1,1}$ coordinate transformation $(\bar{t}, \bar{r}) \to (t, r)$ defined in a neighbourhood of $\boldsymbol{p}$ such that the FLRW and TOV metrics match Lipschitz continuously across Σ and the FLRW metric components in $(\bar{t}, \bar{r})$ coordinates are $C^{1,1}$.

Proof. This result follows as a consequence of (2.49), Proposition 2.6.2 and Lemma 2.6.3. □

If the FLRW and TOV metrics are at least $C^{0,1}$ in their native coordinates then $C^{1,1}$ is the minimum regularity of a coordinate transformation between such coordinates that matches the metrics Lipschitz continuously across the shock surface. Since the function formed by the union of two adjacent Lipschitz continuous functions that agree on their intersection is a Lipschitz continuous function, then if the FLRW and TOV metrics are both Lipschitz continuous in barred coordinates and agree on the shock surface, then they match Lipschitz continuously on the shock surface.

Given that the metrics have been matched, we turn our attention to satisfying the Rankine-Hugoniot jump conditions, which ensure the conservation of mass-energy and momentum across the shock surface. Note that many of the proceeding points will follow [**17**] closely. Now if the conditions of Corollary 2.6.1 are met, then Corollary 2.6.1 implies that the conditions of Theorem 2.2.2 are also met, allowing the Rankine-Hugoniot jump conditions to be given as the following single condition:

$$[T^{\mu\nu}]n_\mu n_\nu = (\rho + p)n_0^2 - (\bar{\rho} + \bar{p})\frac{\bar{n}_0^2}{B} + (p - \bar{p})|\boldsymbol{n}|^2 = 0 \tag{2.61}$$

We will first determine the components n_μ and $\bar{n}_\mu$. Writing the shock surface as the scalar:

$$\Phi(t, r) = r - r(t) = 0$$

then:

$$d\Phi = n_\mu dx^\mu = -\dot{r}dt + dr$$

and so:

$$n_0 = -\dot{r}$$

$$n_1 = 1$$

Once again, writing the shock surface as the scalar:

$$\Phi(\bar{t},\bar{r}) = \frac{\bar{r}}{R(t(\bar{t},\bar{r}))} - r(t(\bar{t},\bar{r})) = 0$$

then:

$$d\Phi = \bar{n}_\mu d\bar{x}^\mu = \left(-\frac{\dot{R}\bar{r}}{R^2}\frac{\partial t}{\partial \bar{t}} - \dot{r}\frac{\partial t}{\partial \bar{t}}\right)d\bar{t} + \bar{n}_1 d\bar{r} = -\frac{\dot{\bar{r}}}{R}\frac{\partial t}{\partial \bar{t}}dt + \bar{n}_1 d\bar{r}$$

and so:

$$\bar{n}_0 = -\frac{\dot{\bar{r}}}{R}\frac{\partial t}{\partial \bar{t}}$$

Now noting that $t = t(\bar{t},\bar{r})$ means:

$$dt = \frac{\partial t}{\partial \bar{t}}d\bar{t} + \frac{\partial t}{\partial \bar{r}}d\bar{r}$$

and rearranging (2.49) gives:

$$\frac{\partial t}{\partial \bar{t}} = \frac{1}{\Psi C}$$

to then yield:

$$\bar{n}_0 = -\frac{\dot{\bar{r}}}{R\Psi C}$$

Because the coefficients of the FLRW and TOV metrics must agree on the shock surface, a number of relations can be found between these coefficients, but only on the shock surface. In this regard, on the shock surface defined by (2.54), we have:

$$C = R^2\left(1 - \frac{8\pi\mathcal{G}}{3}\rho\bar{r}^2\right) = R^2 A \tag{2.62}$$

and using (2.62) along with (2.55) yields:

$$\frac{1}{\Psi^2 C^2} = \frac{B}{A}(1 - kr^2)$$

and thus:

$$\bar{n}_0^2 = \frac{\dot{\bar{r}}^2 B}{R^2 A}(1 - kr^2)$$

Also note that:

$$|\boldsymbol{n}|^2 = -n_0^2 + \frac{1 - kr^2}{R^2} n_1^2 = -\dot{r}^2 + \frac{1 - kr^2}{R^2}$$

Now substituting the components n_μ and $\bar{n}_\mu$ into condition (2.61), we obtain:

$$[T^{\mu\nu}] n_\mu n_\nu = (\rho + \bar{p})\dot{r}^2 - (\bar{\rho} + \bar{p})\frac{\dot{\bar{r}}^2}{R^2 A}(1 - kr^2) + (p - \bar{p})\frac{1}{R^2}(1 - kr^2) = 0 \tag{2.63}$$

where $\dot{r}$ and $\dot{\bar{r}}$ denote the shock speeds in (t, r) and $(t, \bar{r})$ coordinates respectively. In [**17**], equation (2.21) is used to eliminate p from (2.63), thereby deriving an autonomous system of ODE in R and r as a function of t that determines the inner FLRW metric and shock position $r(t)$ in terms of the TOV metric. Thus for any assignment of equation of state $\bar{p} = \bar{p}(\bar{\rho})$ and initial condition for the TOV metric, this system of ODE determines the FLRW metric, $\rho(t)$ and $p(t)$ that match the given TOV metric Lipschitz continuously across the shock surface (2.54) such that (2.63) holds across the surface. However [**18**] proceeds differently to [**17**] by solving the aforementioned system of ODE by using an equivalent form of (2.63). This equivalent form will now be derived. To this end, differentiating (2.54) with respect to t yields:

$$\dot{\rho} = \frac{3\dot{\bar{r}}}{\bar{r}}(\bar{\rho} - \rho) \tag{2.64}$$

and solving for $\dot{\rho}$ in (2.21) gives:

$$\dot{\rho} = -\frac{3\dot{R}}{R}(\rho + p) \tag{2.65}$$

Combining (2.64) and (2.65) then yields:

$$\dot{\bar{r}} = \dot{R}r\left(\frac{\rho + p}{\rho - \bar{\rho}}\right) \tag{2.66}$$

Differentiating $\bar{r} = Rr$ with respect to t in (2.66) and solving for $\dot{r}$ gives:

$$\dot{r} = \frac{\dot{R}r}{R}\left(\frac{\bar{\rho}+p}{\rho-\bar{\rho}}\right) \tag{2.67}$$

and by substituting (2.66) and (2.67) into (2.63), we obtain the following equivalent form of (2.63) as so:

$$\left(\frac{1}{1-kr^2}\right)(\rho+\bar{p})(\bar{\rho}+p)^2 - \frac{1}{A}(\bar{\rho}+\bar{p})(\rho+p)^2 + \frac{1}{\dot{R}^2r^2}(p-\bar{p})(\rho-\bar{\rho})^2 = 0 \tag{2.68}$$

However, equation (2.68) is not yet the final form as an additional relation that holds only on the shock surface is required. Multiplying equation (2.20) by r^2, introducing M through equation (2.54) and then applying (2.34) yields this additional relation, which is given as:

$$\dot{R}^2r^2 = -A + (1-kr^2)$$

The final equivalent form of (2.63) is then given by:

$$(\rho+\bar{p})(\bar{\rho}+p)^2\left(1-\frac{A}{1-kr^2}\right) + (\bar{\rho}+\bar{p})(\rho+p)^2\left(1-\frac{1-kr^2}{A}\right) + (p-\bar{p})(\rho-\bar{\rho})^2 = 0 \tag{2.69}$$

What remains is to find FLRW and TOV solutions that satisfy the conservation constraint (2.69) at the shock interface. We begin this construction by assuming again that the equation of state for the TOV metric is given by $\bar{p} = \bar{\sigma}\bar{\rho}$, then a TOV solution is determined by (2.40)-(2.42) with explicit forms for A and B given by (2.43) and (2.45) respectively. We will also assume that $k = 0$ for the FLRW metric. Solving for ρ in (2.54) then using (2.41) specifies ρ on the shock surface in terms of $\bar{\rho}$ as so:

$$\rho = \frac{3M}{4\pi\bar{r}^3} = \frac{3\gamma}{\bar{r}^2} = 3\bar{\rho} \tag{2.70}$$

Given ρ, then p can be determined through the conservation condition (2.69). After substituting A into (2.69) the resulting equation is homogeneous of degree three in the ρ, $\bar{\rho}$, p and $\bar{p}$ variables. Since $\bar{p} = \bar{\sigma}\bar{\rho}$ and $\rho = 3\bar{\rho}$ on the shock surface, it is clear from homogeneity that (2.69) can be met if and only if $p = \sigma\rho$ for some constant σ. Substituting $p = \sigma\rho$ into (2.69) then yields the following constraint between σ and $\bar{\sigma}$:

$$\bar{\sigma} = H(\sigma) := \frac{1}{2}\sqrt{9\sigma^2 + 54\sigma + 49} - \frac{3}{2}\sigma - \frac{7}{2} \tag{2.71}$$

Alternatively, this constraint can be written as:

$$\sigma = \frac{\bar{\sigma}(7 + \bar{\sigma})}{3(1 - \bar{\sigma})}$$

This means imposing the Rankine-Hugoniot jump condition determines the interior equation of state when the exterior equation of state is specified and vice versa. But not only that, because $\bar{\rho}$ is a known explicit function of $\bar{r}$ and ρ is purely a function of t, then ρ is completely determined and in turn determines the full interior FLRW spacetime. Moreover, the constraint (2.71) guarantees that mass-energy and momentum conservation holds across the shock surface, so by Theorem 2.2.2 the equivalent statements of Theorem 2.2.1 also apply. Before deriving the full FLRW spacetime, it is worth noting that $H(0) = 0$ and as $\sigma \to 0$:

$$\bar{\sigma} = H(\sigma) \sim \frac{3}{7}\sigma + O(\sigma^2)$$

It is also easy to verify that for $0 \leq \sigma$, we have:

(1) $H'(\sigma) > 0$

(2) $\bar{\sigma} < 1$

(3) $\bar{\sigma} < \sigma$

as would be expected physically since $\rho = 3\bar{\rho} > \bar{\rho}$ at the shock surface. Some cases of note are:

$$\sigma = \frac{1}{3} \implies \bar{\sigma} = \sqrt{17} - 4 \approx 0.1231$$
$$\sigma = 1 \implies \bar{\sigma} = \frac{\sqrt{112}}{2} - 5 \approx 0.2915$$

Explicit formulas for $\rho(t)$ and $R(t)$ will now be obtained, along with the shock positions $r(t)$ and $\bar{r}(t) = r(t)R(t)$. In this light, substituting $p = \sigma\rho$ into (2.25) and the derivative of (2.24) yields:

$$\frac{1}{R}\frac{dR}{d\rho} = -\frac{1}{3(1+\sigma)\rho} \tag{2.72}$$

and:

$$\frac{d\rho}{dt} = \mp\sqrt{24\pi\mathcal{G}}(1+\sigma)\rho^{\frac{3}{2}} \tag{2.73}$$

respectively. Differentiating (2.70) yields:

$$\frac{d\rho}{d\bar{r}} = -\frac{2\rho^{\frac{3}{2}}}{\sqrt{3\gamma}} \tag{2.74}$$

and using (2.73), (2.74) and the chain rule gives:

$$\frac{d\bar{r}}{dt} = \pm\sqrt{18\pi\mathcal{G}\gamma}(1+\sigma) \tag{2.75}$$

Integrating (2.75) with respect to t then gives the following explicit formula for the shock position:

$$\bar{r}(t) = \pm\sqrt{18\pi\mathcal{G}\gamma}(1+\sigma)(t-t_0) + \bar{r}_0 \tag{2.76}$$

where $(t_0, \bar{r}_0)$ is the initial spacetime position of the shock wave. For a discussion of the physical relevance of R_0 and $(t_0, \bar{r}_0)$, see [**17**]. A positive sign in (2.76) corresponds to an expanding shock wave and a negative sign corresponds to a contracting one. Since (2.70) gives ρ in terms of $\bar{r}$, then (2.76) gives ρ in terms of t as so:

$$\rho(t) = \frac{3\gamma}{\bar{r}^2(t)} = \frac{3\gamma}{(\pm\sqrt{18\pi\mathcal{G}\gamma}(1+\sigma)(t-t_0)+\bar{r}_0)^2} \tag{2.77}$$

Now that ρ has been established, equation (2.72) can be solved to obtain R and such can be used to obtain the shock position r as well:

$$R(t) = R_0\left(\frac{\rho(t)}{\rho_0}\right)^{-\frac{1}{3(1+\sigma)}} = R_0\left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{\frac{2}{3(1+\sigma)}} \tag{2.78}$$

$$r(t) = \frac{\bar{r}(t)}{R(t)} = \frac{\bar{r}(t)}{R_0}\left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{-\frac{2}{3(1+\sigma)}} = \frac{\bar{r}_0}{R_0}\left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{\frac{1+3\sigma}{3(1+\sigma)}} \tag{2.79}$$

Differentiating (2.76) and (2.79) with respect to t gives the speed of the shock in $(t,\bar{r})$ and (t,r) coordinates respectively. Such speeds are found after some simplification to be:

$$\dot{\bar{r}} = 3(1+\sigma)\sqrt{\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}}$$
$$\dot{r} = \frac{1+3\sigma}{R}\sqrt{\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}} \tag{2.80}$$

Note that the solutions for (2.76)-(2.79) contain two arbitrary constants, R_0 and $(t_0,\bar{r}_0)$, which come from the initial value problems (2.22) and (2.23) respectively. From (2.77) it can be seen that running time backwards in the case of an expanding shock wave produces a singularity at time:

$$t_* = t_0 - \frac{\bar{r}_0}{\sqrt{18\pi\mathcal{G}\gamma}(1+\sigma)}$$

As $t \to t_*$ it is clear that $\bar{r}$, r and R tend to zero and ρ, $\bar{\rho}$, p and $\bar{p}$ tend to infinity. Taking this solution as a cosmological model, then $t = t_*$ represents an initial big bang singularity in which a shock wave emerges from $\bar{r} = 0$. We conclude this section with following theorem, which summarises its results.

THEOREM 2.6.1. *Assume the following:*

(1) *An equation of state of the form $p = \sigma\rho$ for the FLRW metric (2.17).*
(2) *An equation of state of the form $\bar{p} = \bar{\sigma}\bar{\rho}$ for the TOV metric (2.27).*
(3) *The constraint (2.71) holds.*
(4) $k = 0$.

Then:

(1) *The TOV solution is given by (2.40), (2.41), (2.43) and (2.44) where γ is given by (2.42).*
(2) *The FLRW solution is given by (2.77) and (2.78).*
(3) *The TOV metric matches the FLRW metric across the shock surface (2.76) such that the conservation of mass-energy and momentum holds across this surface.*
(4) *The coordinate identification $(t,r) \to (\bar{t},\bar{r})$ is given by $\bar{r} = Rr$ together with a $C^{1,1}$ function $\bar{t} = \bar{t}(t,r)$ that exists in a neighbourhood of the shock surface.*

Note that because the conditions of Theorem 2.2.2 are satisfied for this shock wave, all of the equivalent statements of Theorem 2.2.1 also hold. From this point onwards, the FLRW spacetime with curvature constant k and equation of state $p = \sigma\rho$ will be denoted by FLRW$(k, \sigma, 1)$ and the TOV spacetime with equation of state $\bar{p} = \bar{\sigma}\bar{\rho}$ will be denoted by TOV$(\bar{\sigma})$. The one in the third argument of FLRW$(k, \sigma, 1)$ denotes the fact that this spacetime has not been perturbed, the nature of which will be discussed in Chapter 5. We have shown that when $\bar{\sigma} = H(\sigma)$, can these two spacetimes be matched to form an FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock-wave solution of the perfect fluid Einstein field equations.

2.7. Properties

Now that we have constructed an FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock-wave solution of the perfect fluid Einstein field equations, what remains is an analysis of its properties, including an analysis of the Lax stability of the shock wave. Physically, such solutions model the general relativistic version of an explosion within a static, singular, isothermal sphere. The singular property stems from the fact that the TOV$(\bar{\sigma})$ density and pressure profiles are inverse square in $\bar{r}$, so ρ and p tend to infinity as $\bar{r} \to 0$. The isothermal property comes from the barotropic equation of state for both interior and exterior fluids, which in our case models the fluids with constant sound speeds and temperatures. The larger sound speed in the FLRW$(0, \sigma, 1)$ spacetime, compared to the smaller sound speed in the TOV$(\bar{\sigma})$ spacetime, can be interpreted as modelling an isothermal equation of state at a higher temperature. Smoller and Temple remark that this is consistent with the expected behaviour that shock waves should heat the fluid as they pass through it. They also remark that FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock-wave solutions are toy models for the scenario whereby the Big Bang begins with a shock-wave explosion. Such solutions can also be thought of as toy models for supernovae, which in the limit $\sigma \to 0$ recover the Newtonian case of low velocities and weak gravitational fields.

We will now determine the values of σ for which the Lax characteristic conditions hold, in addition to determining which values yield subluminal shock speeds, that is, shock speeds slower than the speed of light. An expanding FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock wave corresponds to taking the plus sign in (2.22) and the corresponding minus signs in equations (2.23) and (2.24). The objective of this section is to show that, in the case of an expanding FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock wave, there exist values $0 < \sigma_1 < \sigma_2 < 1$ such that the Lax characteristic conditions hold at the shock if and only if $0 < \sigma < \sigma_1$ and that the shock speed is subluminal if and only if $0 < \sigma < \sigma_2$.

COROLLARY 2.7.1 (Corollary of Lemma 2.3.1). *For $0 < \sigma < 1$, the shock speed relative to the FLRW fluid particles is given by:*

$$s = (1 + 3\sigma)\sqrt{\frac{\bar{\sigma}}{1 + 6\bar{\sigma} + \bar{\sigma}^2}} \tag{2.81}$$

PROOF. This result immediately follows from Lemma 2.3.1 and (2.80). $\square$

Note that from constraint (2.71), we can give s purely as a function of σ. The graph of $s(\sigma)$ is then given by Figure 2.1.

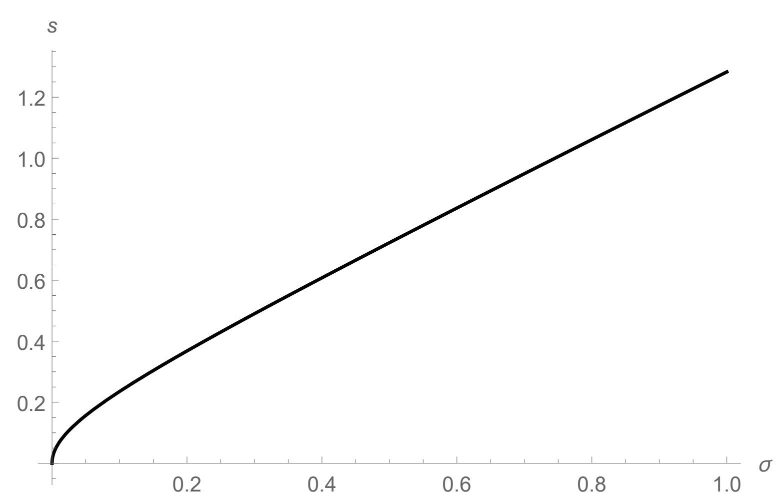

FIGURE 2.1. A plot of $s(\sigma)$.

Recall that through our choice of natural units, the speed of light is equal to unity.

THEOREM 2.7.1. *For $0 < \sigma < 1$, the shock speed is subluminal if and only if $\sigma < \sigma_2$, where:*

$$\sigma_2 = \frac{\sqrt{5}}{3} \approx 0.745$$

PROOF. This result follows from Lemma 2.3.1, equation (2.80) and direct numerical calculation since $1 - s(\sigma)$ is monotone for $0 < \sigma$ and negative for $\sigma > \sigma_2$. Alternatively, this calculation can be seen from Figure 2.2. $\square$

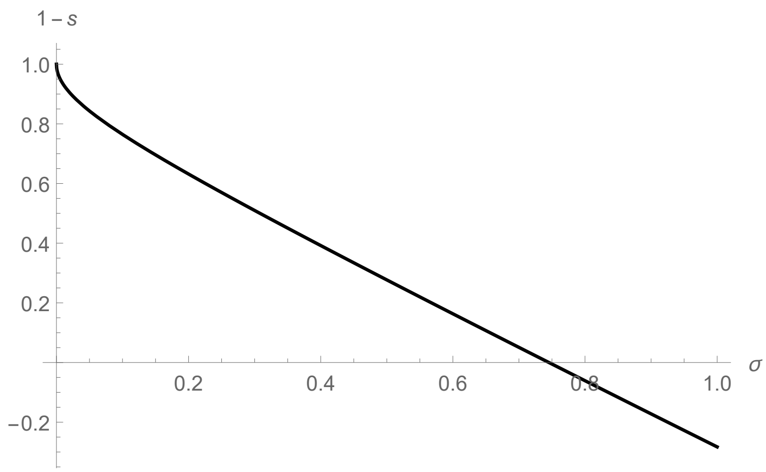

FIGURE 2.2. A plot of $1 - s(\sigma)$.

We now consider the values of σ for which the Lax characteristic conditions hold. We know that the Lax characteristic conditions are given by (2.15), which can be written as:

$$\tilde{\lambda}^{+}_{TOV} < s < \tilde{\lambda}^{+}_{FLRW}$$

where:

$$\tilde{\lambda}^{\pm}_{FLRW} = \pm\sqrt{\frac{dp}{d\rho}} = \pm\sqrt{\sigma}$$

LEMMA 2.7.1. *If* $0 < \sigma < 1$ *then the following inequalities hold:*

$$\tilde{\lambda}^{-}_{TOV} < \tilde{\lambda}^{+}_{TOV} < 0$$

Lemma 2.7.1 is key to the following theorems and its proof will be given towards the end of this section.

THEOREM 2.7.2. *For $0 < \sigma < 1$, the Lax characteristic conditions hold across the shock if and only if $\sigma < \sigma_1$, where:*

$$\sigma_1 = \frac{1+\sqrt{10}}{9} \approx 0.462$$

PROOF. By defining:

$$\Delta(\sigma) = \tilde{\lambda}^{+}_{FLRW}(\sigma) - s(\sigma)$$

then by (2.81), the right side inequality of the Lax characteristic conditions is equivalent to:

$$\Delta(\sigma) = \tilde{\lambda}^{+}_{FLRW}(\sigma) - s(\sigma) = \sqrt{\sigma} - (1+3\sigma)\sqrt{\frac{H(\sigma)}{1+6H(\sigma)+H^2(\sigma)}} > 0$$

The graph of $\Delta(\sigma)$, given by Figure 2.3, shows that $\Delta(\sigma)$ changes from positive to negative at the point $\sigma = \sigma_1$. Since $s(\sigma) > 0$ for $\sigma > 0$ then the left side inequality of the Lax characteristic conditions follows from Lemma 2.7.1. $\square$

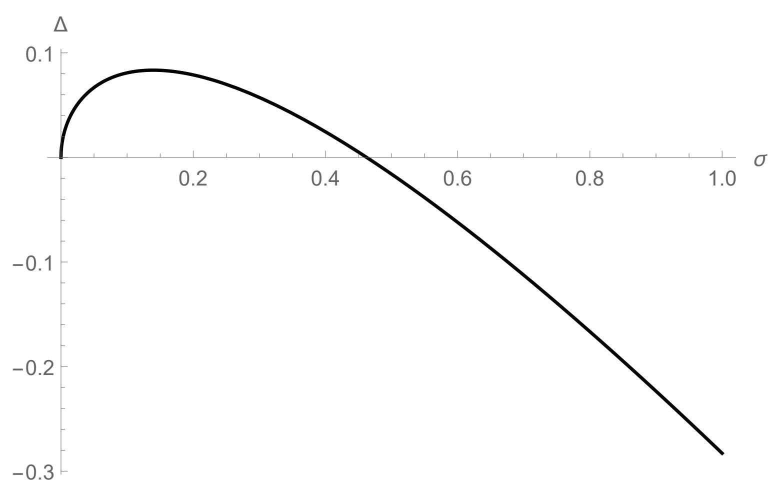

FIGURE 2.3. A plot of $\Delta(\sigma)$.

THEOREM 2.7.3. *If $\sigma_1 < \sigma < \sigma_2$, then the following inequalities hold:*

$$\tilde{\lambda}^{-}_{FLRW} < \tilde{\lambda}^{+}_{FLRW} < s \tag{2.82}$$

$$\tilde{\lambda}^{-}_{TOV} < \tilde{\lambda}^{+}_{TOV} < s \tag{2.83}$$

Proof. Inequalities (2.82) immediately follow from Theorem 2.7.2 and inequalities (2.83) from Lemma 2.7.1. $\square$

Note that for $\sigma_1 < \sigma < \sigma_2$, (2.82) and (2.83) describe a different type of shock wave in which the 1 and 2 characteristics both cross the shock. This is because the shock speed exceeds the characteristic speeds on both sides of the shock and occurs even though the sound speeds and shock speed all remain less than the speed of light. Put another way, Theorem 2.7.3 states that in general relativity a fluid sound speed of $\sqrt{\sigma} \approx \sqrt{0.744}$ is capable of driving the shock speed all the way to speed of light. It is also worth noting that:

(1) $\bar{\sigma}_1 = H(\sigma_1) \approx 0.161$
(2) $\bar{\sigma}_2 = H(\sigma_2) \approx 0.236$

Proof of Lemma 2.7.1. By Proposition 2.3.2 we have that:

$$\tilde{\lambda}^{+}_{TOV} = \frac{\tilde{w} + \sqrt{\frac{d\bar{p}}{d\bar{\rho}}}}{1 + \tilde{w}\sqrt{\frac{d\bar{p}}{d\bar{\rho}}}} = \frac{\tilde{w} + \sqrt{\bar{\sigma}}}{1 + \tilde{w}\sqrt{\bar{\sigma}}} \tag{2.84}$$

where:

$$\tilde{w} = e^{\psi - \varphi} w = \frac{1}{R}\frac{\partial r}{\partial \bar{t}}\left(\frac{\partial t}{\partial \bar{t}}\right)^{-1}$$

that is, the $(\hat{t}, \hat{r})$ coordinates are taken to be (t, r) coordinates. Now since:

$$\frac{\partial t}{\partial \bar{t}}(\bar{t}, \bar{r}) = \left(\frac{\partial \bar{t}}{\partial t}(t, \bar{r})\right)^{-1}$$

then:

$$w = \frac{\partial r}{\partial \bar{t}}(\bar{t}, \bar{r})\frac{\partial \bar{t}}{\partial t}(t, \bar{r}) = \frac{\partial r}{\partial t}(t, \bar{r})$$

and given that:

$$r(t, \bar{r}) = \frac{\bar{r}}{R(t)}$$

then in a neighbourhood of the shock surface we have:

$$w = \frac{\partial r}{\partial t}(t, \bar{r}) = \frac{\partial}{\partial t}\frac{\bar{r}(t)}{R(t)} = -\frac{\dot{R}(t)\bar{r}}{R^2(t)}$$

By (2.78):

$$\dot{R}(t) = \frac{2}{r(t)}\sqrt{\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}}$$

so:

$$w = -\frac{\dot{R}(t)\bar{r}}{R^2(t)} = -\frac{2}{R(t)}\sqrt{\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}}$$

and thus:

$$\tilde{w} = -2\sqrt{\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}} \tag{2.85}$$

Now by (2.85), (2.84) can be written as:

$$\tilde{\lambda}^+_{TOV} = -\frac{\sqrt{\bar{\sigma}}\left(2-\sqrt{1+6\bar{\sigma}+\bar{\sigma}^2}\right)}{\sqrt{1+6\bar{\sigma}+\bar{\sigma}^2}-2\bar{\sigma}}$$

and graphed by Figure 2.4. Because $\tilde{\lambda}^-_{TOV} < \tilde{\lambda}^+_{TOV}$ and we see that $\tilde{\lambda}^+_{TOV}(\sigma) < 0$ for $0 < \sigma < 1$, then the proof is complete. $\square$

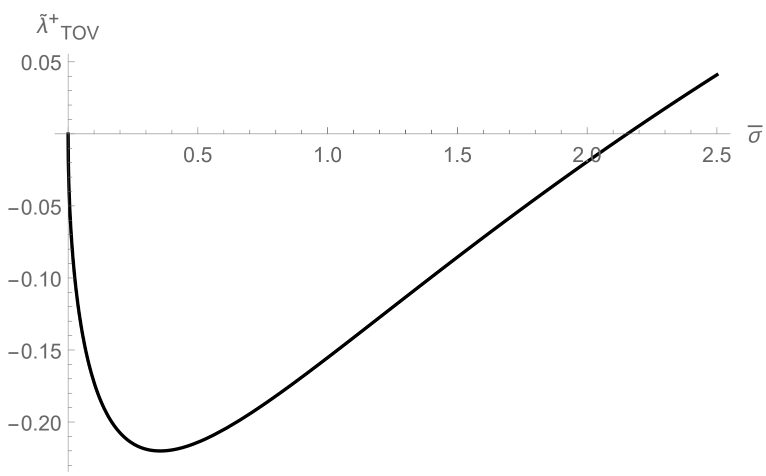

FIGURE 2.4. A plot of $\tilde{\lambda}^+_{TOV}(\bar{\sigma})$.

The fluid sound speed is constrained by the speed of light, however, it remains to say whether the shock speed is also constrained in the same way. We know that sound speeds of $\sqrt{\sigma} < \sqrt{\sigma_2}$ can drive shocks to the speed of light but the time reversal and stability properties of such shocks remain to be investigated. Smoller and Temple remark that for $0 < \sigma < \sigma_1$ many solutions will decay time-asymptotically to the same shock wave, thus in contrast to an FLRW$(0, \sigma, 1)$ solution, an FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock wave will not have a unique time reversal all the way back to the initial Big Bang singularity. Because the TOV$(\bar{\sigma})$ solution is singular at $\bar{r} = 0$, the solution is interpreted as being unstable since it requires an infinite pressure at $\bar{r} = 0$ to hold it up. An FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock wave removes this singularity at $\bar{r} = 0$ for strictly positive time and thus supplies the pressure required to stabilise a TOV$(\bar{\sigma})$ solution. As a final note, for $\sigma_1 < \sigma < \sigma_2$ the FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock-wave solutions represent a new type of fluid dynamical shock wave, that is, the shock speed exceeds all of the characteristic speeds both sides of the shock, since both the fast and slow characteristics cross the shock surface from the exterior side to the interior side.

CHAPTER 3

General Relativistic Self-Similar Waves

This chapter summarises, and in some parts extends, the majority of the 1971 paper *Spherically Symmetric Similarity Solutions of the Einstein Field Equations for a Perfect Fluid* by Cahill and Taub [**3**]. What is meant by the majority is that this summary only considers the barotropic equation of state results, rather than the results that apply to more general caloric equations of state. This restriction reduces the complexity of many results and arguably makes the paper both more readable and approachable. The last section on dust solutions has also been omitted. The extension comes in the form of short proofs to results that benefit from additional justification. One of the major advantages to including such a summary as part of this thesis, is the ability to unify the variables, notation and metric sign convention to match those of the other chapters, since the differences would otherwise be quite significant. Moreover, all of the results given in Cahill and Taub's paper have been put into proposition form, with definitions added for clarity.

3.1. Spherically Symmetric Einstein Field Equations

Chapter 2 considered generic spherically symmetric shock-wave solutions of the perfect fluid Einstein field equations and concluded with an explicit construction of an FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock wave. This chapter will also consider spherically symmetric solutions of the perfect fluid Einstein field equations, except we will now make the additional assumption of self-similarity. It will be seen later in this chapter and in particular in Chapter 6, that this additional assumption makes the shock-wave construction process much simpler. As like in Chapter 2, this chapter will consider the generic construction of spherically symmetric self-similar shock waves before providing a specific example.

We know that a self-similar variable is a combination of independent variables, which in the spherically symmetric case will be the time coordinate t and the radial coordinate r.

DEFINITION 3.1.1. Solutions of conservation laws for which the self-similar variable takes the form:

$$\xi = \frac{r}{r_0}\left(\frac{t}{t_0}\right)^{-n}$$

for strictly positive constants t_0, r_0 and n, are called *similarity solutions* or *progressive waves.* For $n = 1$, such solutions are referred to as *similarity solutions of the first kind.*

The analysis of this chapter will be restricted to similarity solutions of the first kind and for which the perfect fluid is modelled by a barotropic equation of state of the form:

$$p = p(\rho) \tag{3.1}$$

where p is the fluid pressure and ρ is the fluid density. For convenience, the gravitational constant $\mathcal{G}$ and speed of light c will be scaled so that:

$$4\pi\mathcal{G} = c = 1 \tag{3.2}$$

As like in Chapter 2, the speed of sound in the fluid, $\sqrt{\sigma}$, is given by:

$$\sigma = \frac{dp}{d\rho} \tag{3.3}$$

Before constructing shock-wave solutions, it will be most convenient to use comoving coordinates for the spacetime metric, which will take the form:

$$ds^2 = -e^{2\varphi}dt^2 + e^{2\psi}dr^2 + \bar{r}^2 d\Omega^2 \tag{3.4}$$

where φ, ψ and $\bar{r}$ are functions of t and r. The perfect fluid Einstein field equations may be written in the form:

$$G^{\mu}_{\nu} = \kappa T^{\mu}_{\nu} = \kappa\left[(\rho + p)u^{\mu}u_{\nu} + p\delta^{\mu}_{\nu}\right] \tag{3.5}$$

where $\boldsymbol{u}$ is the fluid four-velocity and by our choice of scaling:

$$\kappa = \frac{8\pi\mathcal{G}}{c^4} = 2$$

Now substituting (3.4) into (3.5) yields four PDE:

$$e^{\varphi+\psi} + e^{\psi-\varphi}\left[\left(\frac{\partial\bar{r}}{\partial t}\right)^2 + 2\bar{r}\frac{\partial\bar{r}}{\partial t}\frac{\partial\psi}{\partial t}\right] - e^{\varphi-\psi}\left[2\bar{r}\frac{\partial^2\bar{r}}{\partial r^2} + \left(\frac{\partial\bar{r}}{\partial r}\right)^2 - 2\bar{r}\frac{\partial\bar{r}}{\partial r}\frac{\partial\psi}{\partial r}\right] = \kappa\rho\bar{r}^2 e^{\varphi+\psi} \tag{3.6}$$

$$e^{\varphi+\psi} + e^{\psi-\varphi}\left[2\bar{r}\frac{\partial^2\bar{r}}{\partial t^2} + \left(\frac{\partial\bar{r}}{\partial t}\right)^2 - 2\bar{r}\frac{\partial\bar{r}}{\partial t}\frac{\partial\varphi}{\partial t}\right] - e^{\varphi-\psi}\left[\left(\frac{\partial\bar{r}}{\partial r}\right)^2 + 2\bar{r}\frac{\partial\bar{r}}{\partial r}\frac{\partial\varphi}{\partial r}\right] = -\kappa p\bar{r}^2 e^{\varphi+\psi} \tag{3.7}$$

$$\begin{aligned} & e^{\psi-\varphi}\left[\frac{1}{\bar{r}}\frac{\partial^2\bar{r}}{\partial t^2} + \frac{\partial^2\psi}{\partial t^2} + \left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{\bar{r}}\frac{\partial\bar{r}}{\partial t}\frac{\partial\psi}{\partial t} - \frac{1}{\bar{r}}\frac{\partial\bar{r}}{\partial t}\frac{\partial\varphi}{\partial t} - \frac{\partial\psi}{\partial t}\frac{\partial\varphi}{\partial t}\right] - \\ & e^{\varphi-\psi}\left[\frac{1}{\bar{r}}\frac{\partial^2\bar{r}}{\partial r^2} + \frac{\partial^2\varphi}{\partial r^2} + \left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{1}{\bar{r}}\frac{\partial\bar{r}}{\partial r}\frac{\partial\varphi}{\partial r} - \frac{1}{\bar{r}}\frac{\partial\bar{r}}{\partial r}\frac{\partial\psi}{\partial r} - \frac{\partial\varphi}{\partial r}\frac{\partial\psi}{\partial r}\right] = -\kappa p e^{\varphi+\psi} \end{aligned} \tag{3.8}$$

$$\frac{\partial^2\bar{r}}{\partial t\partial r} - \frac{\partial\bar{r}}{\partial t}\frac{\partial\varphi}{\partial r} - \frac{\partial\bar{r}}{\partial r}\frac{\partial\psi}{\partial t} = 0 \tag{3.9}$$

the latter of which is equivalent to the statement that the coordinate system is comoving with the fluid, that is:

$$u^\mu = e^{-\varphi}\delta^\mu_0$$

The Bianchi identities give us two simplifications, namely:

$$\frac{\partial\varphi}{\partial r} = -\frac{1}{\rho+p}\frac{\partial p}{\partial r} \tag{3.10}$$

$$\frac{\partial\psi}{\partial t} = -\frac{1}{\rho+p}\frac{\partial\rho}{\partial t} - \frac{2}{\bar{r}}\frac{\partial\bar{r}}{\partial t} \tag{3.11}$$

We can integrate (3.10) and (3.11) to yield:

$$e^\varphi = \frac{g_0(t)\eta}{\rho+p} \tag{3.12}$$

$$e^\psi = \frac{f_0(r)}{\bar{r}^2\eta} \tag{3.13}$$

where f_0 and g_0 are arbitrary functions of their arguments and:

$$\frac{d\eta}{d\rho} = \frac{\eta}{\rho + p} \tag{3.14}$$

Now substituting (3.10) and (3.11) into (3.9) yields:

$$(\rho + p)\bar{r}^2 \frac{\partial^2 \bar{r}}{\partial t \partial r} + \bar{r}^2 \frac{\partial \bar{r}}{\partial t} \frac{\partial p}{\partial r} + \bar{r}^2 \frac{\partial \bar{r}}{\partial r} \frac{\partial \rho}{\partial t} + 2(\rho + p)\bar{r} \frac{\partial \bar{r}}{\partial r} \frac{\partial \bar{r}}{\partial t} = 0$$

and we may write this equation as:

$$\frac{\partial}{\partial t}\left(\rho \bar{r}^2 \frac{\partial \bar{r}}{\partial r}\right) + \frac{\partial}{\partial r}\left(p \bar{r}^2 \frac{\partial \bar{r}}{\partial t}\right) = 0$$

Hence there exists a function $m(t, r)$ such that:

$$\frac{\partial m}{\partial r} = \frac{\kappa}{2} \rho \bar{r}^2 \frac{\partial \bar{r}}{\partial r} \tag{3.15}$$

$$\frac{\partial m}{\partial t} = -\frac{\kappa}{2} p \bar{r}^2 \frac{\partial \bar{r}}{\partial t} \tag{3.16}$$

We next turn to (3.6) and (3.7), which may be written as:

$$\frac{\partial}{\partial r}\left[\bar{r} + \bar{r}e^{-2\varphi}\left(\frac{\partial \bar{r}}{\partial t}\right)^2 - \bar{r}e^{-2\psi}\left(\frac{\partial \bar{r}}{\partial r}\right)^2\right] = \kappa \rho \bar{r}^2 \frac{\partial \bar{r}}{\partial r} = 2\frac{\partial m}{\partial r} \tag{3.17}$$

$$\frac{\partial}{\partial t}\left[\bar{r} + \bar{r}e^{-2\varphi}\left(\frac{\partial \bar{r}}{\partial t}\right)^2 - \bar{r}e^{-2\psi}\left(\frac{\partial \bar{r}}{\partial r}\right)^2\right] = -\kappa p \bar{r}^2 \frac{\partial \bar{r}}{\partial t} = 2\frac{\partial m}{\partial t} \tag{3.18}$$

and equations (3.17) and (3.18) may be immediately integrated to give:

$$2m = \bar{r}\left[1 + e^{-2\varphi}\left(\frac{\partial \bar{r}}{\partial t}\right)^2 - e^{-2\psi}\left(\frac{\partial \bar{r}}{\partial r}\right)^2\right] \tag{3.19}$$

The function $m(t, r)$ satisfies (3.15) and (3.16) and hence is determined up to an additive constant of integration. Finally, note that (3.7) and (3.6) can be written as:

$$e^{-\varphi}\frac{\partial U}{\partial t} = e^{-\psi}\Gamma\frac{\partial \varphi}{\partial r} - \frac{\kappa}{2}p\bar{r} - \frac{m}{\bar{r}^2} \tag{3.20}$$

$$e^{-\psi}\frac{\partial \Gamma}{\partial r} = e^{-\varphi}U\frac{\partial \psi}{\partial t} - \frac{\kappa}{2}\rho\bar{r} + \frac{m}{\bar{r}^2} \tag{3.21}$$

respectively, where:

$$U = e^{-\varphi}\frac{\partial \bar{r}}{\partial t} \tag{3.22}$$

$$\Gamma = e^{-\psi}\frac{\partial \bar{r}}{\partial r} \tag{3.23}$$

From now onwards, we will apply our choice of scaling, that is, we will set $\kappa = 2$.

3.2. Compatible Equations of State

A spherically symmetric similarity solution admits a conformal Killing vector field Ξ satisfying:

$$\nabla_\nu \Xi_\mu + \nabla_\mu \Xi_\nu = 2g_{\mu\nu} \tag{3.24}$$

This means admitting the following conformal transformation:

$$\hat{g}_{\mu\nu} = g_{\sigma\tau}\frac{\partial x^\sigma}{\partial \hat{x}^\mu}\frac{\partial x^\tau}{\partial \hat{x}^\nu} = \frac{1}{\Omega^2}g_{\mu\nu}$$

where:

$$\hat{t} = \Omega t$$

$$\hat{r} = \Omega r$$

$$\hat{\theta} = \theta$$

$$\hat{\phi} = \phi$$

for some constant Ω. In the case that the stress-energy-momentum tensor models a perfect fluid, a consequence of (3.24) and the transformation properties of the Einstein tensor is that the fluid four-velocity $\boldsymbol{u}$ is conformally invariant, that is:

$$\Xi^\nu \nabla_\nu u^\mu - u^\nu \nabla_\nu \Xi^\mu = -u^\mu \tag{3.25}$$

Now with spherical symmetry we may write:

$$\Xi^\mu = \alpha\delta^\mu_0 + \beta\delta^\mu_1 \tag{3.26}$$

Equation (3.24) then becomes:

$$\alpha \frac{\partial \bar{r}}{\partial t} + \beta \frac{\partial \bar{r}}{\partial r} = \bar{r} \tag{3.27}$$

$$\alpha \frac{\partial \varphi}{\partial t} + \beta \frac{\partial \varphi}{\partial r} + \frac{\partial \alpha}{\partial t} = 1 \tag{3.28}$$

$$\alpha \frac{\partial \psi}{\partial t} + \beta \frac{\partial \psi}{\partial r} + \frac{\partial \beta}{\partial r} = 1 \tag{3.29}$$

$$e^{2\varphi} \frac{\partial \alpha}{\partial r} - e^{2\psi} \frac{\partial \beta}{\partial t} = 0 \tag{3.30}$$

Equation (3.25) reduces to two equations, one of these is (3.28), the other is:

$$\frac{\partial \beta}{\partial t} = 0 \tag{3.31}$$

It then follows from (3.30) that:

$$\frac{\partial \alpha}{\partial r} = 0 \tag{3.32}$$

If we define new independent variables $\tilde{t}$ and $\tilde{r}$ by the equations:

$$\frac{\partial \tilde{t}}{\partial t} = \frac{\tilde{t}}{\alpha}$$
$$\frac{\partial \tilde{r}}{\partial r} = \frac{\tilde{r}}{\beta}$$

and new dependent variables:

$$\tilde{\varphi} = \varphi + \log \alpha - \log \tilde{t}$$

$$\tilde{\psi} = \psi + \log \beta - \log \tilde{r}$$

then (3.27)-(3.29) become:

$$\begin{aligned} \tilde{t} \frac{\partial \bar{r}}{\partial \tilde{t}} + \tilde{r} \frac{\partial \bar{r}}{\partial \tilde{r}} &= \bar{r} \\ \tilde{t} \frac{\partial \varphi}{\partial \tilde{t}} + \tilde{r} \frac{\partial \varphi}{\partial \tilde{r}} &= 0 \\ \tilde{t} \frac{\partial \psi}{\partial \tilde{t}} + \tilde{r} \frac{\partial \psi}{\partial \tilde{r}} &= 0 \end{aligned} \tag{3.33}$$

and we may thus write:

$$\tilde{\varphi} = \tilde{\varphi}(\xi)$$

$$\tilde{\psi} = \tilde{\psi}(\xi) \tag{3.34}$$

$$\tilde{\bar{r}} = \tilde{r}\mathscr{R}(\xi)$$

where:

$$\xi = \frac{\tilde{r}}{\tilde{t}} \tag{3.35}$$

This means the metric can now be written as:

$$d\tilde{s}^2 = -e^{2\tilde{\varphi}}d\tilde{t}^2 + e^{2\tilde{\psi}}d\tilde{r}^2 + \mathscr{R}^2\tilde{r}^2 d\Omega^2$$

where $\tilde{t}$ and $\tilde{r}$ are also comoving coordinates. More importantly, we have shown there exists a set of comoving coordinates for which the components of the metric are self-similar. We will drop the tilde from here onwards and assume that φ, ψ and $\bar{r}$ are of the form given by (3.34). It follows from (3.34) and (3.15)-(3.19) that the variables m, p and ρ take the form:

$$m = r\mathscr{M}(\xi)$$

$$p = \frac{1}{r^2}\mathscr{P}(\xi) \tag{3.36}$$

$$\rho = \frac{1}{r^2}\mathscr{W}(\xi)$$

Furthermore, (3.22) and (3.23) then imply:

$$U = U(\xi)$$

$$\Gamma = \Gamma(\xi)$$

and we can introduce the the following quantities, which are functions of ξ alone:

$$
\begin{aligned}
M &= \frac{m}{\bar{r}} = \frac{\mathscr{M}}{\mathscr{R}} \\
P &= p\bar{r}^2 = \mathscr{P}\mathscr{R}^2 \\
W &= \rho\bar{r}^2 = \mathscr{W}\mathscr{R}^2
\end{aligned}
\tag{3.37}
$$

Note that the scale factor $\mathscr{R}$ is the self-similar analogue of R from Chapter 2. We conclude this section with a rather surprising proposition.

PROPOSITION 3.2.1. *The only barotropic equation of state consistent with a spherically symmetric similarity solution of the Einstein field equations is:*

$$p = \sigma\rho \tag{3.38}$$

for some constant σ.

PROOF. This result is an immediate consequence of (3.36) and balancing the variable r on either side of $p = p(\rho)$. □

3.3. Spherically Symmetric Self-Similar Einstein Field Equations

We now show that the perfect fluid Einstein field equations reduce to a system of ODE when the spacetime is assumed to be spherically symmetric and self-similar. The full equations will first be rewritten in terms of M, P and W, which will be considered as functions of ξ and r, rather than t and r. In this light, for any function $f = f(t, r)$, we may write:

$$
\begin{aligned}
\frac{\partial f}{\partial t}(t,r) &= -\frac{1}{t}\xi\frac{\partial f}{\partial \xi}(\xi,r) \\
\frac{\partial f}{\partial r}(t,r) &= \frac{1}{t}\frac{\partial f}{\partial \xi}(\xi,r) + \frac{\partial f}{\partial r}(\xi,r)
\end{aligned}
$$

and hence:

$$t\frac{\partial f}{\partial t}(t,r) = -\dot{f}$$
$$r\frac{\partial f}{\partial r}(t,r) = \dot{f} + \tilde{f}$$

where:

$$\dot{f} = \xi\frac{\partial f}{\partial \xi}(\xi,r)$$
$$\tilde{f} = r\frac{\partial f}{\partial r}(\xi,r)$$

Using this notation, we have that:

$$\frac{\partial \bar{r}}{\partial t}(t,r) = -\xi\dot{\mathscr{R}}(\xi,r) \tag{3.39}$$

$$\frac{\partial \bar{r}}{\partial r}(t,r) = \mathscr{R}(\xi,r) + \dot{\mathscr{R}}(\xi,r) + \tilde{\mathscr{R}}(\xi,r) \tag{3.40}$$

and so (3.19) may now be written as:

$$2M = 1 + \xi^2 e^{-2\varphi}\dot{\mathscr{R}}^2 - e^{-2\psi}(\mathscr{R} + \dot{\mathscr{R}} + \tilde{\mathscr{R}})^2 \tag{3.41}$$

Furthermore, (3.15) and (3.16) may now be written as:

$$\mathscr{R}\dot{M} + (P + M)\dot{\mathscr{R}} = 0 \tag{3.42}$$

$$M - W - (W + P)\frac{\dot{\mathscr{R}}}{\mathscr{R}} = (W - M)\frac{\tilde{\mathscr{R}}}{\mathscr{R}} - \tilde{M} \tag{3.43}$$

The integrability condition of (3.42) and (3.43) for determining $M(\xi,r)$ is equivalent to (3.9), and this condition takes the following equivalent form:

$$\ddot{\mathscr{R}} + \dot{\mathscr{R}} - (\mathscr{R} + \dot{\mathscr{R}} + \tilde{\mathscr{R}})\dot{\psi} - \dot{\mathscr{R}}\dot{\varphi} = \dot{\mathscr{R}}\tilde{\varphi} - \dot{\tilde{\mathscr{R}}} \tag{3.44}$$

We can also obtain the following equation by subtracting (3.7) from (3.6):

$$(3.45)\quad (\mathscr{R}+\tilde{\mathscr{R}})\dot{\varphi}-\xi^2 e^{2\psi-2\varphi}(\mathscr{R}+\tilde{\mathscr{R}})\dot{\psi}-e^{2\psi}\mathscr{R}^{-2}(W+P)=$$
$$\tilde{\tilde{\mathscr{R}}}+\dot{\tilde{\mathscr{R}}}+\tilde{\mathscr{R}}-\tilde{\psi}(\mathscr{R}+\dot{\mathscr{R}}+\tilde{\mathscr{R}})-(\mathscr{R}+\tilde{\mathscr{R}})\tilde{\varphi}-\xi^2 e^{2\psi-2\varphi}(\dot{\tilde{\mathscr{R}}}-\tilde{\varphi}\dot{\mathscr{R}})$$

Finally, equations (3.10) and (3.11) may be written as:

$$\dot{\varphi}+\frac{\dot{P}}{P+W}-\frac{2P}{P+W}\left(\frac{\mathscr{R}+\dot{\mathscr{R}}}{\mathscr{R}}\right)=\frac{1}{P+W}\left(2P\frac{\tilde{\mathscr{R}}}{\mathscr{R}}-\tilde{P}\right)-\tilde{\varphi}\tag{3.46}$$

$$\dot{\psi}+\frac{\dot{W}}{P+W}+\frac{2P}{P+W}\left(\frac{\dot{\mathscr{R}}}{\mathscr{R}}\right)=0\tag{3.47}$$

In light of the recent derivations, we know that if φ, ψ, $\mathscr{R}$, M, P and W are functions of ξ alone, as is the case for a similarity solution, then (3.41)-(3.46) reduce to the following system of ordinary differential equations:

$$1+\xi^2 e^{-2\varphi}\dot{\mathscr{R}}^2-e^{-2\psi}(\mathscr{R}+\dot{\mathscr{R}})^2=2M\tag{3.48}$$

$$\mathscr{R}\dot{M}+(P+M)\dot{\mathscr{R}}=0\tag{3.49}$$

$$W+(W+P)\frac{\dot{\mathscr{R}}}{\mathscr{R}}=M\tag{3.50}$$

$$\ddot{\mathscr{R}}+\dot{\mathscr{R}}-(\mathscr{R}+\dot{\mathscr{R}})\dot{\psi}-\dot{\mathscr{R}}\dot{\varphi}=0\tag{3.51}$$

$$e^{-2\psi}\dot{\varphi}-\xi^2 e^{-2\varphi}\dot{\psi}-(W+P)\frac{1}{\mathscr{R}^2}=0\tag{3.52}$$

$$\frac{2P}{W+P}\left(\frac{\mathscr{R}+\dot{\mathscr{R}}}{\mathscr{R}}\right)-\frac{\dot{P}}{W+P}+=\dot{\varphi}\tag{3.53}$$

3.4. Essential and Inessential Parameters

We now consider the number and type of parameters that spherically symmetric similarity solutions depend on.

PROPOSITION 3.4.1. *Solutions of (3.47)-(3.53) that are modelled by a barotropic equation of state depend on three parameters.*

Proof. We can eliminate M from (3.47)-(3.53) to yield six equations in the eleven dependent variables φ, $\dot{\varphi}$, ψ, $\dot{\psi}$, $\mathscr{R}$, $\dot{\mathscr{R}}$, $\ddot{\mathscr{R}}$, P, $\dot{P}$, W and $\dot{W}$. This leaves us with five integration constants to specify, although it turns out that we can reduce this to three. The assumption of a barotropic equation of state reduces the number of integration constants by one, since by Proposition 3.2.1 the barotropic equation of state must take the form:

$$P = \sigma W$$

for some constant σ, that is, the variables P and $\dot{P}$ are eliminated but the parameter σ is introduced. The other reduction in integration constants comes from combining equations (3.48) and (3.50), which yields:

$$F(\xi) := 1 + e^{-2\varphi}\xi^2\dot{\mathscr{R}}^2 - e^{2\psi}(\mathscr{R} + \dot{\mathscr{R}})^2 - 2\left(W + (W+P)\frac{\dot{\mathscr{R}}}{\mathscr{R}}\right) = 0$$

Since $F(\xi) \equiv 0$, then it must also be the case that $\dot{F}(\xi) \equiv 0$, and this introduces an additional constraint on the integration constants and parameter σ. Taking the dot derivative on any of the other equations would result in additional dependent variables, such as $\ddot{\varphi}$, so this would not yield any further constraints on the integration constants. As a result, solutions of (3.47)-(3.53) that are modelled by a barotropic equation of state are determined by two constants of integration in addition to the parameter σ. □

We will now discuss essential and inessential parameters.

Definition 3.4.2. The transformation:

$$t = \mathcal{T}_0 \hat{t} \tag{3.54}$$

$$r = \mathcal{R}_0 \hat{r} \tag{3.55}$$

where $\mathcal{T}_0$ and $\mathcal{R}_0$ are constants, will be referred to as a *scale transformation.*

Recall that the metric can be written as:

$$ds^2 = -e^{2\varphi}dt^2 + e^{2\psi}dr^2 + r^2\mathscr{R}^2 d\Omega^2$$

where φ, ψ and $\mathscr{R}$ are functions of ξ alone. Under a scale transformation the metric becomes:

$$d\hat{s}^2 = -e^{2\hat{\varphi}}d\hat{t}^2 + e^{2\hat{\psi}}d\hat{r}^2 + \hat{r}^2\hat{\mathscr{R}}^2 d\Omega^2$$

where $\hat{\varphi}$, $\hat{\psi}$ and $\hat{\mathscr{R}}$ are functions of the variable:

$$\hat{\xi} = \frac{\mathcal{T}_0}{\mathcal{R}_0}\xi \tag{3.56}$$

and defined by:

$$\begin{aligned} e^{\hat{\varphi}(\hat{\xi})} &= \mathcal{T}_0 e^{\varphi\left(\frac{\mathcal{R}_0}{\mathcal{T}_0}\hat{\xi}\right)} = \mathcal{T}_0 e^{\varphi(\xi)} \\ e^{\hat{\psi}(\hat{\xi})} &= \mathcal{R}_0 e^{\psi\left(\frac{\mathcal{R}_0}{\mathcal{T}_0}\hat{\xi}\right)} = \mathcal{R}_0 e^{\psi(\xi)} \\ \hat{\mathscr{R}}(\hat{\xi}) &= \mathcal{R}_0 \mathscr{R}\left(\frac{\mathcal{R}_0}{\mathcal{T}_0}\hat{\xi}\right) = \mathcal{R}_0\mathscr{R}(\xi) \end{aligned} \tag{3.57}$$

Definition 3.4.3. Quantities whose transformation law under a scale transformation involves the coefficients of this transformation explicitly, will be said to be *scale covariants*. Quantities for which their functional form and value are unaltered by a scale transformation will be referred to as *scale invariants*.

Quantities such as φ, ψ and $\mathscr{R}$ are scale covariants. The constants of integration which enter into the expression of two scale covariants may be transformed to any desired value by a scale transformation.

Definition 3.4.4. The constants of integration which enter into an expression of two scale covariants will be referred to as *inessential parameters*. Otherwise the parameters will be referred to as *essential parameters*.

The differential equations describing a similarity solution may be decomposed into two sets, one involving scale covariants and one involving scale invariants. Examples of scale invariants are:

$$V = \xi e^{\psi - \varphi} \tag{3.58}$$

$$I = V e^{-\psi}\mathscr{R} \tag{3.59}$$

since it may readily be verified as a consequence of (3.56) and (3.57) that:

$$\hat{V} = \hat{\xi} e^{\hat{\psi}-\hat{\varphi}} = \xi e^{\psi-\varphi} = V$$

$$\hat{I} = \hat{V} e^{-\hat{\psi}} \hat{\mathscr{R}} = V e^{-\psi} \mathscr{R} = I$$

If $f(\xi)$ is any scalar function, then:

$$\dot{f} = \xi \frac{df}{d\xi} = \hat{\xi} \frac{d\hat{f}}{d\hat{\xi}}$$

is a scale invariant. Do not confuse scalar functions with scalar valued functions representing tensor coefficients, such as φ, ψ or $\mathscr{R}$, which are defined as functions of the metric coefficients.

Proposition 3.4.5. *Solutions of (3.47)-(3.53) that are modelled by a barotropic equation of state depend on three essential parameters.*

Proof. We know from Proposition 3.4.1 that the number of parameters is three, so what remains to be shown is that these three parameters are all essential. We know from (3.57) that φ, ψ and $\mathscr{R}$ are all scale covariants, however, the quantities $\dot{\varphi}$, $\dot{\psi}$ and $\mathscr{R}^{-1}\dot{\mathscr{R}}$ are scale invariant, since:

$$\dot{\hat{\varphi}} e^{\hat{\varphi}} = \hat{\xi} \frac{d}{d\hat{\xi}} e^{\hat{\varphi}} = \xi \frac{d}{d\xi} (\mathcal{T}_0 e^{\varphi}) = \mathcal{T}_0 \dot{\varphi} e^{\varphi} = \dot{\varphi} e^{\hat{\varphi}}$$

$$\dot{\hat{\psi}} e^{\hat{\psi}} = \hat{\xi} \frac{d}{d\hat{\xi}} e^{\hat{\psi}} = \xi \frac{d}{d\xi} (\mathcal{R}_0 e^{\psi}) = \mathcal{R}_0 \dot{\psi} e^{\psi} = \dot{\psi} e^{\hat{\psi}}$$

$$\frac{\dot{\hat{\mathscr{R}}}}{\hat{\mathscr{R}}} = \hat{\xi} \frac{d\hat{\mathscr{R}}}{d\hat{\xi}} \frac{1}{\hat{\mathscr{R}}} = \xi \frac{d}{d\xi} (\mathcal{R}_0 \mathscr{R}) \frac{1}{\mathcal{R}_0 \mathscr{R}} = \frac{\dot{\mathscr{R}}}{\mathscr{R}}$$

Because these quantities are scale invariant, equations (3.48) and (3.50) tell us that M, P and W must also be scale invariants. We then see that equations (3.47)-(3.53) consist of only scale invariant quantities, that is, after minor manipulation. This means the three parameters are essential parameters by definition. □

The surface $\xi = \xi_0$ has the equation:

$$r - \xi_0 t = 0$$

and so its future oriented normal vector has components:

$$n_\mu = -\xi_0 \delta^0_\mu + \delta^1_\mu$$

In a coordinate frame comoving with the fluid, it can be checked that:

$$V = -\frac{u^\mu n_\mu}{\sqrt{(u^\mu n_\mu)^2 + n^\mu n_\mu}} \tag{3.60}$$

Physically, V represents the normal velocity of the moving sphere of radius $\xi_0 t$ relative to the fluid. Similarly, the scale invariant:

$$\tanh\omega = e^{\psi-\varphi}\frac{\partial \bar{r}}{\partial t}\left(\frac{\partial \bar{r}}{\partial r}\right)^{-1} = -\frac{V\dot{\mathscr{R}}}{\mathscr{R}+\dot{\mathscr{R}}} \tag{3.61}$$

represents the normal velocity of the fluid relative to the sphere $r = r(t)$ obtained by solving the equation:

$$\bar{r}(t, r) = \bar{r}_0$$

In this case, the normal has components:

$$n_\mu = \frac{\partial \bar{r}}{\partial t}\delta^0_\mu + \frac{\partial \bar{r}}{\partial r}\delta^1_\mu$$

Furthermore, it may be checked from equations (3.48) and (3.50) that:

$$\tanh\omega = \frac{V(W-M)}{M+P} \tag{3.62}$$

$$I^2 = \frac{(1-2M)(W+P)^2V^2}{(P+M)^2-(W-M)^2V^2} \tag{3.63}$$

Note that if the quantities M, P and V are given on a surface $\xi = \xi_0$, then the corresponding similarity solution is known. Consider an observer with the world-line:

$$r = \xi_0 t$$

$$\theta = \theta_0$$

$$\phi = \phi_0$$

with θ_0, ϕ_0 and ξ_0 constant. When $\xi_0 > 0$ the observer moves outward through the fluid toward higher values of r. When $\xi_0 < 0$ the observer moves inward. When $\xi_0 = 0$ the world line coincides with that of the fluid element at $r = 0$. On the other hand, consider now an observer with the world-line:

$$\bar{r}(t, r) = \bar{r}_0$$

$$\theta = \theta_0$$

$$\phi = \phi_0$$

with θ_0, ϕ_0 and ξ_0 constant. This observer has their velocity relative to the fluid determined by $\tanh\omega$. We know from (3.62) that when $V > 0$ and $W > M$, then $\tanh\omega > 0$, that is, even though the material is falling behind the first observer, when $t > 0$ it can be moving outward relative to the second observer.

3.5. Initial Data

Cahill and Taub remark that for a spherically symmetric similarity solution on the interior of an expanding or contracting shock wave, that the shock surface must be defined by the surface:

$$\xi = \xi_1$$

and the region in which the similarity solution holds must be given by:

$$\xi_0 \leq \xi \leq \xi_1$$

for some constants ξ_0 and ξ_1. By considering the initial data on such a surface, we can determine the existence and uniqueness of the solution in this region. In this light, for a barotropic equation of state, equations (3.12) and (3.13) can be written as:

$$e^{\varphi} = \frac{g_0(t)\eta}{\rho + p} \tag{3.64}$$

$$e^{\psi} = \frac{f_1(r)}{\mathscr{R}^2\eta} \tag{3.65}$$

respectively. By means of equation (3.14):

$$\dot{\varphi} = -\frac{\dot{g}_0}{g_0} - \sigma\frac{\dot{\eta}}{\eta} \tag{3.66}$$

$$\tilde{\varphi} = \frac{\dot{g}_0}{g_0} - \sigma\frac{\tilde{\eta}}{\eta} \tag{3.67}$$

$$\dot{\psi} = -\frac{\dot{\eta}}{\eta} - \frac{2\dot{\mathscr{R}}}{\mathscr{R}} \tag{3.68}$$

$$\tilde{\psi} = -\frac{\tilde{\eta}}{\eta} - \frac{2\tilde{\mathscr{R}}}{\mathscr{R}} - \frac{\tilde{f}_1}{f_1} \tag{3.69}$$

where:

$$\sigma = \frac{dp}{d\rho} = \sigma(\eta) \tag{3.70}$$

It can be checked that if (3.66)-(3.69) are substituted into (3.44)-(3.45), these equations and:

$$\dot{Y}^1 = Y^2 \tag{3.71}$$

$$\dot{Y}^3 = \tilde{Y}^2 \tag{3.72}$$

can be written in the form:

$$\dot{Y}^i = F^i(Y^j; \tilde{Y}^j; \xi, r) \tag{3.73}$$

for $i, j \in \{1, \ldots, 4\}$, where:

$$Y^1 = \mathscr{R}$$

$$Y^2 = \dot{\mathscr{R}}$$

$$Y^3 = \tilde{\mathscr{R}}$$

$$Y^4 = \eta$$

The Cauchy-Kowaleski theorem can be applied to this system of equations, and from it, we find that if:

$$(\mathscr{R} + \tilde{\mathscr{R}})(V^2 - \sigma) \neq 0 \tag{3.74}$$

on the initial surface $\xi = \xi_1$, then the equations have a unique and analytic solution in the neighbourhood of this surface. The quantities Y^i and $\tilde{Y}^i$ form the initial data of the problem. If they are constrained so that (3.43) holds, then a solution of the perfect fluid Einstein field equations is obtained, with these equations taking the form:

$$Z(Y^i; \tilde{Y}^i; \xi, r) = 0 \tag{3.75}$$

for $i \in \{1, \ldots, 4\}$. Since M, P and W can be given in terms of Y^i and $\tilde{Y}^i$, (3.42) follows from (3.43)-(3.45) and the definition of M. The surfaces defined by:

$$V^2 - \sigma = 0$$

are the hydrodynamical characteristic surfaces, that is, they describe the moving wavefronts of sound waves. These surfaces are referred to as *sonic surfaces* in Chapter 4. The surfaces defined by:

$$\mathscr{R} + \tilde{\mathscr{R}} = 0$$

satisfy:

$$1 - 2M = (1 - V^2)e^{-2\psi}\dot{\mathscr{R}}^2$$

Notice that if condition (3.74) is satisfied in the neighbourhood of an initial surface $\xi = \xi_1$, and if on this surface the $\tilde{Y}^i$ vanish, then a solution of the Einstein field equations must be a similarity solution. Because the solution is unique, then the only solution of the perfect fluid Einstein field equations which takes on constant values on a surface $\xi = \xi_1$ is a similarity solution. The variables being constant on the surface specifies the functions $f_1(r)$ and $g_0(t)$ in terms of the initial values of $\mathscr{R}$ and $\eta(\xi, r)$. For such a solution, (3.75) are of the form:

$$\dot{Y}^i = Z^i(Y^j; \xi) \tag{3.76}$$

Note that if p is not proportional to ρ, the solution determined by such data is not a similarity solution. We know this from Proposition 3.2.1, since other equations of state are not admissible.

3.6. Self-Similar Shock Waves

In this section we consider the conditions which relate the metric coefficients and hydrodynamic variables on each side of a spherical surface representing a shock wave. This surface will be assumed to separate spacetime into two regions, an interior region and an exterior region. In this light, the relation defining such a shock is given by:

$$r = \Phi(t) \tag{3.77}$$

If this shock is a surface $\xi = \xi_1$ in self-similar comoving coordinates, then we have:

$$\Phi(t) = \xi_1 t \tag{3.78}$$

where ξ_1 is a constant. We will now consider Schwarzschild coordinates, in particular, self-similar Schwarzschild coordinates, that is, those in which the metric (3.4) takes the form:

$$ds^2 = -e^{2\mu} d\bar{t}^2 + e^{2\nu} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \tag{3.79}$$

where variables μ and ν are functions of a single self-similar variable. Note that the following proposition does not require either the comoving or Schwarzschild coordinates to be self-similar.

Proposition 3.6.1. *The Schwarzschild coordinates $\bar{t}$ and $\bar{r}$ are related to the comoving coordinates t and r by the following equations:*

$$d\bar{t} = e^{-\mu}(e^{\varphi} \cosh \omega dt + e^{\psi} \sinh \omega dr) \tag{3.80}$$

$$d\bar{r} = e^{-\nu}(e^{\varphi} \sinh \omega dt + e^{\psi} \cosh \omega dr) \tag{3.81}$$

where:

$$\tanh \omega = e^{\psi-\varphi} \frac{\partial \bar{r}}{\partial t} \left(\frac{\partial \bar{r}}{\partial r} \right)^{-1} \tag{3.82}$$

$$e^{-2\nu} = e^{-2\psi} \left(\frac{\partial \bar{r}}{\partial r} \right)^2 - e^{-2\varphi} \left(\frac{\partial \bar{r}}{\partial t} \right)^2 = 1 - \frac{2m}{\bar{r}} \tag{3.83}$$

and μ is such that the right hand side of (3.80) is a perfect differential.

Proof. First note that:

$$
\begin{aligned}
ds^2 &= -e^{2\mu}d\bar{t}^2 + e^{2\nu}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \\
&= -(e^{\varphi}\cosh\omega dt + e^{\psi}\sinh\omega dr)^2 + (e^{\varphi}\sinh\omega dt + e^{\psi}\cosh\omega dr)^2 + \bar{r}^2 d\Omega^2 \\
&= -e^{2\varphi}dt^2 + e^{2\psi}dr^2 + \bar{r}^2 d\Omega^2
\end{aligned}
$$

and so it remains to show:

$$
\begin{aligned}
\frac{\partial\bar{t}}{\partial t} &= e^{\varphi-\mu}\cosh\omega \\
\frac{\partial\bar{t}}{\partial r} &= e^{\psi-\mu}\sinh\omega \\
\frac{\partial\bar{r}}{\partial t} &= e^{\varphi-\nu}\sinh\omega \\
\frac{\partial\bar{r}}{\partial r} &= e^{\psi-\nu}\cosh\omega
\end{aligned}
$$

The expression for $\tanh\omega$ comes from the fact that:

$$
d\bar{r} = \frac{\partial\bar{r}}{\partial t}dt + \frac{\partial\bar{r}}{\partial r}dr
$$

which yields:

$$
\begin{aligned}
\cosh^2\omega &= \left[1 - e^{2\psi-2\varphi}\left(\frac{\partial\bar{r}}{\partial t}\right)^2\left(\frac{\partial\bar{r}}{\partial r}\right)^{-2}\right]^{-1} = e^{2\nu-2\psi}\left(\frac{\partial\bar{r}}{\partial r}\right)^2 \\
\sinh^2\omega &= \left[1 - e^{2\psi-2\varphi}\left(\frac{\partial\bar{r}}{\partial t}\right)^2\left(\frac{\partial\bar{r}}{\partial r}\right)^{-2}\right]^{-1} - 1 = e^{2\nu-2\psi}\left(\frac{\partial\bar{r}}{\partial r}\right)^2 - 1 = e^{2\nu-2\varphi}\left(\frac{\partial\bar{r}}{\partial t}\right)^2
\end{aligned}
$$

and thus demonstrates the last two equalities. The first two equalities follow from the fact that $\bar{t}$ is indirectly specified by μ, which itself is determined by:

$$
\frac{\partial}{\partial r}(e^{\varphi-\mu}\cosh\omega) = \frac{\partial}{\partial t}(e^{\psi-\mu}\sinh\omega)
$$

□

In Schwarzschild coordinates the four-velocity is given by:

$$\bar{u}^\mu = u^\sigma \frac{\partial \bar{x}^\mu}{\partial x^\sigma} = e^{-\varphi} \frac{\partial \bar{x}^\mu}{\partial t}$$

that is, $\bar{u}^2 = \bar{u}^3 = 0$ and:

$$\bar{u}^0 = e^{-\mu} \cosh \omega$$

$$\bar{u}^1 = e^{-\nu} \sinh \omega$$

This means ω is related to the proper velocity associated with an element of the fluid when measured in Schwarzschild coordinates. In these coordinates, the surface described by (3.78) takes the form:

$$\bar{r} = \bar{\Phi}(\bar{t}) \tag{3.84}$$

and its unit normal has components $\bar{n}_2 = \bar{n}_3 = 0$ and:

$$\bar{n}_0 = -e^{\mu} \sinh \chi$$

$$\bar{n}_1 = e^{\nu} \cosh \chi$$

where:

$$\tanh \chi = e^{\nu - \mu} \frac{\partial \bar{\Phi}}{\partial \bar{t}} \tag{3.85}$$

This means χ is related to the proper velocity of the shock front in Schwarzschild coordinates. The quantities χ and ω together yield an interesting coordinate invariant:

$$\bar{u}^\mu \bar{n}_\mu = -\sinh(\chi - \omega) = u^\mu n_\mu \tag{3.86}$$

where n_μ is the unit normal to the shock surface in comoving coordinates. The normal components in comoving coordinates are given by $n_2 = n_3 = 0$ and:

$$n_0 = -e^{\varphi} \sinh(\chi - \omega)$$

$$n_1 = e^{\psi} \cosh(\chi - \omega)$$

with:

$$\tanh(\chi - \omega) = e^{\psi - \varphi} \frac{\partial \Phi}{\partial t} \tag{3.87}$$

The quantity $\chi - \omega$ thus measures the velocity of the shock surface relative to the fluid flow. Let us now introduce the notation:

$$f_{\pm} = \lim_{\epsilon \to 0} f(\bar{t}, \bar{\Phi}(\bar{t}) \pm \epsilon)$$

and:

$$[f] = f_{-} - f_{+}$$

where f is any function of $\bar{r}$ and $\bar{t}$. Similarly, in comoving coordinates we define:

$$f_{\pm} = \lim_{\epsilon \to 0} f(t, \Phi(t) \pm \epsilon)$$

Thus f_{-} and f_{+} are the values of the function f on the interior and exterior sides of the shock respectively, with $[f]$ denoting the jump in f across the shock. The values of f_{+} and f_{-} depend on the position of the shock and can be considered as functions of $\bar{t}$ and $\bar{r}$ or t and r.

Proposition 3.6.2. *It may be assumed without loss of generality that:*

$$\chi_{+} = \chi_{-} \tag{3.88}$$

Proof. The fundamental assumption is that a solution can be found that contains a shock, this means that the spacetimes both sides of the shock must agree on the position and thus the proper velocity of the shock. It then remains to show that this assumption yields (3.88) without loss of generality. Since the variable χ is determined by the velocity of the shock in Schwarzschild coordinates by (3.85), then let $(\bar{t}, \bar{r})$ and $(\tilde{t}, \tilde{r})$ represent the Schwarzschild coordinates of the interior and exterior spacetimes about the shock surface respectively. Because both coordinates are Schwarzschild coordinates then $\bar{r} = \tilde{r}$ and this implies $\bar{t}$ and $\tilde{t}$ can only differ by a temporal transformation of the form $\tilde{t} = \tilde{t}(\bar{t})$, thus:

$$\tanh \chi_- = e^{\nu-\mu}\frac{\partial\bar{\Phi}}{\partial\bar{t}} = \frac{\partial\tilde{r}}{\partial\bar{r}}\frac{\partial\bar{t}}{\partial\tilde{t}}e^{\tilde{\nu}-\tilde{\mu}}\frac{\partial\tilde{t}}{\partial\bar{t}}\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} = e^{\tilde{\nu}-\tilde{\mu}}\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} = \tanh \chi_+$$

and therefore (3.88) holds in any Schwarzschild coordinate system either side of the shock. □

PROPOSITION 3.6.3. *If the induced metric on the surface $\bar{r} = \bar{\Phi}(\bar{t})$ is continuous, then the following quantities are continuous across the shock, that is:*

$$\mu_+(\bar{t},\bar{r}) = \mu_-(\bar{t},\bar{r}) \tag{3.89}$$

$$\nu_+(\bar{t},\bar{r}) = \nu_-(\bar{t},\bar{r}) \tag{3.90}$$

$$\bar{r}_+(t,r) = \bar{r}_-(t,r) \tag{3.91}$$

$$\left(e^{2\varphi} - e^{2\psi}\left(\frac{\partial\Phi}{\partial t}\right)^2\right)_+ = \left(e^{2\varphi} - e^{2\psi}\left(\frac{\partial\Phi}{\partial t}\right)^2\right)_- \tag{3.92}$$

PROOF. The continuity of μ and ν follows from (3.85) and the continuity of the induced metric on the surface $\bar{r} = \bar{\Phi}(\bar{t})$. In the comoving coordinate system, (3.91) follows similarly, with (3.92) following from the continuity of ν, (3.77) and (3.83). □

Substituting (3.87) into (3.92) whilst noting the continuity of χ and the time derivative of Φ across the shock yields:

$$e^{-\varphi_+}\cosh(\chi-\omega_+) = e^{-\varphi_-}\cosh(\chi-\omega_-) \tag{3.93}$$

$$e^{-\psi_+}\sinh(\chi-\omega_+) = e^{-\psi_-}\sinh(\chi-\omega_-) \tag{3.94}$$

So in comoving coordinates the metric is discontinuous since φ and ψ are discontinuous, with the discontinuities constrained by (3.93) and (3.94). We will see that such discontinuities can also be expressed in terms of the discontinuity of the hydrodynamic variables.

PROPOSITION 3.6.4. *The function m is continuous across the shock.*

PROOF. This follows immediately from the continuity of $\bar{r}$ and ν and equation (3.83). However, this proposition may also be proved from (3.82), (3.87), (3.91), (3.93) and (3.94), as will now be shown. We have by differentiating (3.91) that:

$$\frac{\partial \bar{r}_+}{\partial r}\left(\frac{\partial \Phi}{\partial t}+\frac{\partial \bar{r}_+}{\partial t}\left(\frac{\partial \bar{r}_+}{\partial r}\right)^{-1}\right)=\frac{\partial \bar{r}_-}{\partial r}\left(\frac{\partial \Phi}{\partial t}+\frac{\partial \bar{r}_-}{\partial t}\left(\frac{\partial \bar{r}_-}{\partial r}\right)^{-1}\right)$$

and on using (3.82) and (3.87) we may write this as:

$$\left[\frac{\partial \bar{r}}{\partial r}e^{\varphi-\psi}\left(\tanh(\chi-\omega)+\tanh\omega\right)\right]=0$$

It then follows from (3.82), (3.91), (3.93) and (3.94) that:

$$\left[e^{-2\psi}\left(\frac{\partial \bar{r}}{\partial r}\right)^2-e^{-2\varphi}\left(\frac{\partial \bar{r}}{\partial t}\right)^2\right]=0$$

In view of the definition of m and the continuity of $\bar{r}$ it follows that m is continuous. □

The jump in the hydrodynamic variables is described by the Rankine-Hugoniot conditions, which we first introduced in Section 2.1. These conditions are given in Schwarzschild coordinates as so:

$$[(\rho+p)\bar{u}^\mu\bar{u}^\nu+pg^{\mu\nu}]\bar{n}_\nu=0 \tag{3.95}$$

Proposition 3.6.5. *By defining:*

$$\bar{u}=-\bar{u}^\mu\bar{n}_\mu=-u^\mu n_\mu=\sinh(\chi-\omega)$$

the Rankine-Hugoniot conditions may be written as:

$$\bar{u}_+^2(\rho_++p_+)+p_+=\bar{u}_-^2(\rho_-+p_-)+p_- \tag{3.96}$$

$$\bar{u}_+^2(\rho_++p_+)^2-\bar{u}_-^2(\rho_-+p_-)^2=(p_+-p_-)\left(\bar{u}_+^2(\rho_++p_+)+\bar{u}_-^2(\rho_-+p_-)\right) \tag{3.97}$$

Proof. Condition (3.96) directly follows from contracting the Rankine-Hugoniot conditions with $\bar{n}_\mu$, noting that $\bar{\boldsymbol{n}}$ is a normalised four-vector, so $g^{\mu\nu}\bar{n}_\mu\bar{n}_\nu=-1$. For (3.97), the Rankine-Hugoniot conditions will first need to be given in component form as so:

$$[\bar{u}(\rho+p)\cosh\omega-p\sinh\chi]=0$$

$$[\bar{u}(\rho+p)\sinh\omega-p\cosh\chi]=0$$

noting that the continuity of μ and ν means that they have been factored out of these equations. Subtracting the square of the second equation from the square of the first yields:

$$\begin{aligned}
0 &= [(\bar{u}(\rho+p)\cosh\omega - p\sinh\chi)^2 - (\bar{u}(\rho+p)\sinh\omega - p\cosh\chi)^2] \\
&= [\bar{u}^2(\rho+p)^2 + 2\bar{u}(\rho+p)p(\sinh\omega\cosh\chi - \cosh\omega\sinh\chi) - p^2] \\
&= [\bar{u}^2(\rho+p)^2 - 2\bar{u}(\rho+p)p\sinh(\chi-\omega) - p^2] \\
&= [\bar{u}^2(\rho+p)^2 - 2\bar{u}^2(\rho+p)p - p^2]
\end{aligned}$$

which can then be written as:

$$\bar{u}_+^2(\rho_+ + p_+)^2 - \bar{u}_-^2(\rho_- + p_-)^2 = 2\bar{u}_+^2(\rho_+ + p_+)p_+ + p_+^2 - 2\bar{u}_-^2(\rho_- + p_-)p_- - p_-^2$$

Finally, applying (3.96) yields:

$$\begin{aligned}
\bar{u}_+^2(\rho_+ + p_+)^2 - \bar{u}_-^2(\rho_- + p_-)^2 &= 2\bar{u}_+^2(\rho_+ + p_+)p_+ + p_+\left(\bar{u}_-^2(\rho_- + p_-) + p_- - \bar{u}_+^2(\rho_+ + p_+)\right) \\
&\quad - 2\bar{u}_-^2(\rho_- + p_-)p_- - p_-\left(\bar{u}_+^2(\rho_+ + p_+) + p_+ - \bar{u}_-^2(\rho_- + p_-)\right) \\
&= (p_+ - p_-)\left(\bar{u}_+^2(\rho_+ + p_+) + \bar{u}_-^2(\rho_- + p_-)\right)
\end{aligned}$$

□

Conditions (3.96) and (3.97) may be written as:

(3.98)
$$\bar{u}_-^2 = \frac{(p_- - p_+)(\rho_+ + p_-)}{(\rho_- + p_-)(\rho_- - \rho_+ + p_+ - p_-)}$$

(3.99)
$$\bar{u}_+^2 = \frac{(p_- - p_+)(\rho_- + p_+)}{(\rho_+ + p_+)(\rho_- - \rho_+ + p_+ - p_-)}$$

noting that:

(3.100)
$$\frac{\bar{u}_+^2}{\bar{u}_-^2} = \frac{(\rho_- + p_+)(\rho_- + p_-)}{(\rho_+ + p_-)(\rho_+ + p_+)}$$

It then follows that:

(3.101)
$$V^2 = \frac{\bar{u}^2}{1+\bar{u}^2} = \tanh^2(\chi - \omega) = e^{2\psi - 2\varphi}\left(\frac{\partial\Phi}{\partial t}\right)^2$$

and:

$$V_+^2 = \frac{(p_- - p_+)(\rho_- + p_+)}{(\rho_- - \rho_+)(\rho_+ + p_-)} \tag{3.102}$$

$$V_-^2 = \frac{(p_- - p_+)(\rho_+ + p_-)}{(\rho_- - \rho_+)(\rho_- + p_+)} \tag{3.103}$$

Hence:

$$\tanh(\omega_- - \omega_+) = \frac{V_+ - V_-}{1 - V_- V_+} = \sqrt{\frac{(p_- - p_+)(\rho_- - \rho_+)}{(\rho_+ + p_-)(\rho_- + p_+)}} = L \tag{3.104}$$

and:

$$\tanh \omega_- = \frac{L + \tanh \omega_+}{1 + L \tanh \omega_+} \tag{3.105}$$

Equations (3.93) and (3.94) may be written as:

$$e^{\varphi_- - \varphi_+} = \sqrt{\frac{(\rho_- + p_+)(\rho_+ + p_+)}{(\rho_+ + p_-)(\rho_- + p_-)}} \tag{3.106}$$

$$e^{\psi_- - \psi_+} = \sqrt{\frac{(\rho_+ + p_-)(\rho_+ + p_+)}{(\rho_- + p_+)(\rho_- + p_-)}} \tag{3.107}$$

respectively. The continuity of the function $m(t, r)$ across the shock surface implies that:

$$[m(t, \Phi(t))] = 0$$

and this means we have that:

$$\left[\frac{\partial m}{\partial r}\frac{\partial \Phi}{\partial t} + \frac{\partial m}{\partial t}\right] = 0$$

or through use of equations (3.15), (3.16) and (3.91):

$$\left[\rho \frac{\partial \bar{r}}{\partial r}\frac{\partial \Phi}{\partial t} - p \frac{\partial \bar{r}}{\partial t}\right] = 0 \tag{3.108}$$

It follows from the continuity of m and $\bar{r}$ across the shock surface that (3.102) and (3.103) can be written as:

$$V_-^2 = \frac{(P_- - P_+)(W_+ + P_-)}{(W_- - W_+)(W_- + P_+)} \tag{3.109}$$

$$V_+^2 = \frac{(P_- - P_+)(W_- + P_+)}{(W_- - W_+)(W_+ + P_-)} \tag{3.110}$$

respectively, in addition to:

$$M_- = M_+ \tag{3.111}$$

If the shock is a surface $\xi = \xi_1$ which separates two similarity solutions each described by a function $p(\rho)$, where $p_+(\rho)$ may differ from $p_-(\rho)$, the above equations relate the scale invariants M, P and V on each side of the shock. Since the values of these quantities on a surface $\xi = \xi_0$ determine the surface and a similarity solution, we see that a similarity solution on one side of a shock, and the position of the shock, that is, the value of ξ_1, determine the similarity solution on the other side of the shock consistent with the value of ξ_1. Equations (3.106) and (3.107) relate φ_- to φ_+ and ψ_- to ψ_+ across the shock when the same coordinates t and r are used on both sides. Suppose instead that the coordinates on one side are scaled according to (3.54) and (3.55) but those on the other side are left unchanged. We would then have:

$$t_- = t_+ = \mathcal{T}_0 \hat{t}_+$$

$$r_- = r_+ = \mathcal{R}_0 \hat{r}_+$$

In light of (3.57) it follows that:

$$\frac{t_-}{\hat{t}_+} e^{\varphi_- - \hat{\varphi}_+} = e^{\varphi_- - \varphi_+}$$

$$\frac{r_-}{\hat{r}_+} e^{\psi_- - \hat{\psi}_+} = e^{\psi_- - \psi_+}$$

Therefore, when $t_- \neq t_+$ and $r_- \neq r_+$, because different scales are used on each side of the shock, (3.106) and (3.107) may be written as:

$$\frac{t_-}{t_+}e^{\varphi_- - \varphi_+} = \sqrt{\frac{(W_- + P_+)(W_+ + P_+)}{(W_+ + P_-)(W_- + P_+)}} \tag{3.112}$$

$$\frac{r_-}{r_+}e^{\psi_- - \psi_+} = \sqrt{\frac{(W_+ + P_+)(W_+ + P_-)}{(W_- + P_-)(W_- + P_+)}} \tag{3.113}$$

3.7. Compatibility

We will now consider solutions compatible with similarity solutions. The generalised Rankine-Hugoniot conditions given by (3.98)-(3.105) and the conditions on the metric tensor given by (3.106) and (3.107) together with the continuity of the functions $m(t,r)$ and $\bar{r}(t,r)$, can be thought of as either determining the variables behind the shock in terms of those ahead of it or vice versa. Because $\bar{r}$ is continuous across the shock, it is possible to replace the variables p and ρ in (3.98)-(3.107) by P and W respectively. In this light, define:

$$\begin{aligned} \alpha &= \frac{P_+}{W_+} = \frac{p_+}{\rho_+} \\ \beta &= \frac{P_-}{W_+} = \frac{p_-}{\rho_+} \\ \gamma &= \frac{W_-}{P_-} = \frac{\rho_-}{p_-} \end{aligned} \tag{3.114}$$

Definition 3.7.1. If a solution ahead of a spherical shock is such that the variables so determined are the values that these variables take on in a similarity solution, then the solution ahead of the shock is said to be *compatible* with a similarity solution.

Proposition 3.7.2. *If a spacetime behind a spherical shock is a similarity solution and the spacetime ahead of the shock is compatible with a similarity solution with its fluid characterised by an equation of state of the form $p_+ = p_+(\rho_+)$, then the spacetime ahead of the shock must also be a similarity solution.*

Proof. Since the variables P_-, W_-, φ_-, ψ_-, $\mathscr{R}_-$ and V_- are to describe a similarity solution, they are constant, that is, independent of r on the shock. Because we know that $p_+ = p_+(\rho_+)$, then equation (3.103) implies β and α are both constant, which is equivalent to both W_+ and P_+ being constant. From this, equations (3.102), (3.106) and (3.107) then imply that V_+, φ_+ and ψ_+ are constant. Because $\bar{r}_+ = \bar{r}_-(\xi_1, r) = r\mathscr{R}_-(\xi_1)$ then $\tilde{\mathscr{R}}_+$ vanishes. We also have that $\dot{\mathscr{R}}_+$ is constant from (3.82) and (3.104), so $\mathscr{R}_+$ is also constant. Hence if the region behind the shock is described by a similarity solution, then the $\tilde{Y}^i$ of (3.75) vanish. Therefore, the unique solution of equations (3.75) subject to the initial conditions derived from the jump relations is given by a similarity solution. Note that all of this is subject to the assumption that the shock satisfies (3.74). □

PROPOSITION 3.7.3. *If a spacetime behind a spherical shock is a similarity solution and the spacetime ahead of the shock is compatible with a similarity solution with one of the quantities: α, β, φ_+ or ψ_+ constant along the shock, then each of these quantities is independent of r.*

PROOF. This is an immediate consequence of (3.103), (3.106) and (3.107). □

3.8. Pure Radiation Solutions

In this section we consider spherically symmetric similarity perfect fluid solutions with a specific equation of state.

DEFINITION 3.8.1. If the source of the gravitational field is a perfect fluid with equation of state:

$$p = \frac{c^2}{3}\rho \tag{3.115}$$

then this state of matter is referred to as *pure radiation.*

A pure radiation equation of state is special for two reasons. The first is that the trace of the stress-energy-momentum tensor is null, that is:

$$T = g_{\mu\nu}T^{\mu\nu} = g_{\mu\nu}(\rho + p)u^\mu u^\nu + g_{\mu\nu}pg^{\mu\nu} = -(\rho + p) + 4p = 0$$

where the speed of light constant has again been set to unity. The second is that the Stefan-Boltzmann radiation law implies this equation state corresponds physically to a purely radiative state of matter, such as a photon gas. This equation of state also models the extreme relativistic limit of free particles. Since this equation of state is applicable to the Radiation Dominated Epoch after the Big Bang, it is reasonable to expect it to hold in the region behind a shock wave emanating from such an event. We will assume that such a medium constitutes the interior of a self-similar shock-wave solution, that is:

$$P_- = \frac{1}{3}W_- \tag{3.116}$$

This means that (3.102)-(3.107) become:

$$V_+^2 = \frac{(\beta-\alpha)(3\beta+\alpha)}{(3\beta-1)(1+\beta)} = \xi_1^2 e^{2\psi_+ - 2\varphi_+} \tag{3.117}$$

$$V_-^2 = \frac{(\beta-\alpha)(1+\beta)}{(3\beta-1)(3\beta+\alpha)} = \xi_1^2 e^{2\psi_- - 2\varphi_-} \tag{3.118}$$

$$L^2 = \frac{(\beta-\alpha)(3\beta-1)}{(1+\beta)(3\beta+\alpha)} \tag{3.119}$$

$$-\frac{\xi_1^2 e^{\psi_- - \varphi_+}\mathscr{R}_1'}{\mathscr{R}_1 + \xi_1 \mathscr{R}_1'} = \frac{L + \tanh\omega_+}{1 + L\tanh\omega_+} \tag{3.120}$$

$$e^{2\varphi_- - 2\varphi_+} = \frac{(3\beta+\alpha)(1+\alpha)}{4\beta(1+\beta)} \tag{3.121}$$

$$e^{2\psi_- - 2\psi_+} = \frac{(1+\beta)(1+\alpha)}{4\beta(3\beta+\alpha)} \tag{3.122}$$

respectively, where α and β are given by (3.114). Now from (3.115) and (3.118):

$$V_-^2 - \sigma_- = V_-^2 - \frac{1}{3} = -\frac{6(\beta-\beta_1)(\beta-\beta_2)}{(3\beta-1)(9\beta+\alpha)}$$

where:

$$2\beta_1 = 1 - \alpha + \sqrt{(1-\alpha)^2 - \frac{4}{3}\alpha}$$

$$2\beta_2 = 1 - \alpha - \sqrt{(1-\alpha)^2 - \frac{4}{3}\alpha} \leq 2\beta_1$$

Note that for $\alpha = \frac{1}{3}$ or $\alpha = 3$ then:

$$\beta_1 = \beta_2 = \frac{1-\alpha}{2}$$

It is reasonable to expect the outer fluid to be no hotter than the inner fluid, that is:

$$\alpha \leq \frac{1}{3}$$

which is equivalent to:

$$\rho_+ - 3p_+ \geq 0$$

and implies that both β_1 and β_2 are real. Now from (3.115) and (3.117) we have:

$$V_+^2 - 1 = \frac{(1+\alpha)(1-\alpha-2\beta)}{(3\beta-1)(1+\beta)} \leq 0$$

where the inequality comes from the fact that the speed of the shock relative to the material ahead of it is less than the speed of light in the fluid. This means that β must be restricted so that:

$$\frac{1-\alpha-2\beta}{3\beta-1} \leq 0$$

Note that:

$$\rho_- \geq \rho_+ \implies \beta \geq \frac{1}{3}$$

Now if we require:

$$\beta > \beta_1$$

then:

$$V_-^2 - \sigma_- < 0$$

$$V_+^2 - 1 < 0$$

and the perfect fluid Einstein field equations reduce to:

$$\ddot{\mathscr{R}} + \dot{\mathscr{R}} + \left(\frac{\dot{\mathscr{R}}}{2\mathscr{R}} - \frac{\dot{x}}{x}\right)\left(3\mathscr{R} + 4\dot{\mathscr{R}}\right) = 0 \tag{3.123}$$

$$\frac{1}{2}\mathscr{R}^4 + \frac{2\xi x^4 \dot{\mathscr{R}}}{\mathscr{R}} - \frac{\dot{x}}{x}\left(3\xi x^4 - \mathscr{R}^4\right) = 4x^2 C_0 \tag{3.124}$$

where:

$$e^{\varphi} = \xi^{\frac{1}{2}} x$$

$$e^{-\psi} = \frac{\mathscr{R}^2}{x^3} \tag{3.125}$$

$$W = \frac{3C_0\mathscr{R}^2}{x^4}$$

with C_0 being constant. Note that (3.125) is presented the way it is for comparison with an analogous set of relations that will be given in Chapter 4. Equations (3.123) and (3.124) have a first integral given by:

$$\frac{2C_0\mathscr{R}^2}{x^4}\left(3 + \frac{4\dot{\mathscr{R}}}{\mathscr{R}}\right) = 1 + \frac{\xi\dot{\mathscr{R}}^2}{x^2} - \frac{\mathscr{R}^4}{x^6}\left(\mathscr{R} + \dot{\mathscr{R}}\right)^2 \tag{3.126}$$

and this is obtained by equating (3.48) and (3.50). Consider now a solution to (3.123) and (3.124) given by:

$$x = x_0\xi^{-\frac{1}{2}}$$

$$\mathscr{R} = \mathscr{R}_0\xi^{-\frac{1}{2}}$$

where x_0 and $\mathscr{R}_0$ are related by:

$$x_0^2 = 2^3 C_0$$

$$\mathscr{R}_0^6 = 2^{11} C_0^3$$

The metric and density of this solution are then given by:

$$ds^2 = -x_0^2 dt^2 + \frac{\mathscr{R}_0^2}{\xi}\left(\frac{1}{4}dr^2 + r^2 d\Omega^2\right) \tag{3.127}$$

$$\rho = \frac{3C_0\xi^2}{x_0^4 r^2} \tag{3.128}$$

which under the scale transformation:

$$\hat{t} = x_0 t$$

$$\hat{r} = \mathscr{R}_0^2 x_0^{-1} r$$

becomes:

$$d\hat{s}^2 = -d\hat{t}^2 + \frac{1}{\hat{\xi}}\left(\frac{1}{4}d\hat{r}^2 + \hat{r}^2 d\Omega^2\right)$$
$$\rho = \frac{3\hat{\xi}^2}{8\hat{r}^2}$$

Alternatively, under the transformation:

$$\hat{t} = x_0 t$$

$$\hat{r} = \mathscr{R}_0 x_0^{-\frac{1}{2}} r^{\frac{1}{2}}$$

the metric and density take the form:

$$d\hat{s}^2 = -d\hat{t}^2 + R^2(\hat{t})(d\hat{r}^2 + \hat{r}^2 d\Omega^2)$$
$$\rho = \frac{3}{8\hat{t}^2}$$

where:

$$R(\hat{t}) = \hat{t}^{\frac{1}{2}}$$

This solution is the familiar FLRW$(0, \frac{1}{3}, 1)$ solution given in Chapter 2. This means that FLRW$(0, \frac{1}{3}, 1)$ is in fact a similarity solution. Note that the density is given differently than in Chapter 2 because we are assuming a scaling which implies $\kappa = 2$, whereas the scaling of all other chapters, including Chapter 2, implies $\kappa = 8\pi$. As a final note, and for use later, it follows from (3.48) that FLRW$(0, \frac{1}{3}, 1)$ satisfies the following relations:

$$P = M \tag{3.129}$$

$$V^2 = 2M \tag{3.130}$$

3.9. Static Solutions

In this section we consider spherically symmetric similarity perfect fluid solutions which are static.

PROPOSITION 3.9.1. *Spherically symmetric similarity solutions of the perfect fluid Einstein field equations are static if and only if the following conditions hold in comoving coordinates:*

$$\begin{aligned} \frac{\partial \bar{r}}{\partial t} &= 0 \\ \frac{\partial \rho}{\partial t} &= 0 \\ \frac{\partial p}{\partial t} &= 0 \end{aligned} \tag{3.131}$$

PROOF. Since these conditions apply to the metric:

$$ds^2 = -e^{2\varphi}dt^2 + e^{2\psi}dr^2 + \bar{r}^2 d\Omega^2$$

that is, a diagonal metric, then to prove this statement it is sufficient to show that there exists a temporal transformation for which φ, ψ and $\bar{r}$ are independent of t. We obtain time independence of $\bar{r}$ for free, and coupled with the time independence of p and ρ, (3.11) implies that we also immediately obtain time independence of ψ. We can also use (3.10) to obtain:

$$\varphi = \int -\frac{1}{\rho + p}\frac{\partial p}{\partial r}dr + g(t) = f_2(r) + g_1(t)$$

where f_2 and g_1 are arbitrary functions of their arguments. This means that:

$$e^{\varphi} = e^{f_2(r)+g_1(t)} = e^{f_2(r)}e^{g_1(t)}$$

and thus this metric component can be made time independent through the temporal transformation:

$$\hat{t} = \hat{t}(t)$$

$$\hat{r} = r$$

where:

$$\frac{\partial \hat{t}}{\partial t} = e^{g_1(t)}$$

□

PROPOSITION 3.9.2. *There is a single one-parameter family of static spherically symmetric similarity perfect fluid solutions.*

PROOF. For a static solution we have $\bar{r} = \bar{r}(r)$ and this implies:

$$\bar{r} = r\mathscr{R}(\xi) = r\mathscr{R}_1$$

that is, $\mathscr{R}$ must be constant. From (3.16) we obtain that $m = m(r)$ and thus (3.37) yields:

$$m = M_1\bar{r} = rM_1\mathscr{R}_1$$

for some constant M_1. Thus, from (3.19) we see that ψ must be constant and given by:

$$e^{2\psi} = \frac{\mathscr{R}_1^2}{1 - 2M_1}$$

In addition, (3.15) then yields:

$$\rho = \frac{M_1}{\mathscr{R}_1^2 r^2} \tag{3.132}$$

Since we already know from Proposition 3.2.1 that the most general barotropic equation of state for a spherically symmetric similarity solution takes the form $p = \sigma\rho$, for some constant σ, then we also obtain:

$$p = \frac{\sigma M_1}{\mathscr{R}_1^2 r^2}$$

Finally, from (3.10):

$$\frac{\partial \varphi}{\partial r} = -\frac{1}{\rho + p}\frac{\partial p}{\partial r} = \frac{2\sigma}{(1+\sigma)r}$$

and thus:

$$e^{\varphi} = g_1(t) r^{\frac{2\sigma}{1+\sigma}} \tag{3.133}$$

From this, (3.20) then specifies the constant M_1 as a function of σ, which is given by:

$$M_1(\sigma) = \frac{2\sigma}{1 + 6\sigma + \sigma^2}$$

The resulting static solution is then given as:

$$ds^2 = -C_1^2 \xi^{2n(\sigma)} dt^2 + \frac{\mathscr{R}_1^2}{1 - 2M_1(\sigma)} dr^2 + \mathscr{R}_1^2 r^2 d\Omega^2 \tag{3.134}$$

$$\rho = \frac{M_1(\sigma)}{\mathscr{R}_1^2 r^2} \tag{3.135}$$

where C_1 is an inessential parameter and:

$$n(\sigma) = \frac{2\sigma}{1 + \sigma}$$

$\square$

Since any purely temporal transformation, that is, a transformation of the form $\bar{t} = \bar{t}(t)$, will keep the fluid in a comoving coordinate frame, we can express the resulting static solution as:

$$ds^2 = -C_1^2 r^{2n(\sigma)} dt^2 + \frac{\mathscr{R}_1^2}{1 - 2M_1(\sigma)} dr^2 + \mathscr{R}_1^2 r^2 d\Omega^2$$

$$\rho = \frac{M_1(\sigma)}{\mathscr{R}_1^2 r^2}$$

Removing C_1 and $\mathscr{R}_1$ by a scale transformation yields a solution which is familiar to us from Chapter 2:

$$ds^2 = -r^{2n(\sigma)} dt^2 + \frac{1}{1 - 2M_1(\sigma)} dr^2 + r^2 d\Omega^2$$

$$\rho = \frac{M_1(\sigma)}{r^2}$$

That is, by requiring a spherically symmetric similarity solution to be static, we obtain the TOV solution. Since the only essential parameter in this solution is σ, the TOV solution is thus the unique spherically symmetric similarity solution that is static. To explicitly see TOV(σ) as a similarity solution, another purely temporal transformation puts TOV(σ) in self-similar comoving coordinates as so:

$$
\begin{aligned}
d\bar{s}^2 &= -\bar{\xi}^{2n(\sigma)}d\bar{t}^2 + \frac{1}{1-2M_1(\sigma)}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \\
\rho &= \frac{M_1(\sigma)}{\bar{r}^2}
\end{aligned}
$$

Now given that $M_1(\sigma)$ and $e^{2\psi}$ must be positive, then:

$$0 \leq 2M_1(\sigma) \leq 1$$

and σ will be real valued only when:

$$0 < 4M_1(\sigma) \leq 1 \tag{3.136}$$

Because σ represents the square of the sound speed in the fluid and the speed of light has been set to unity, it must be the case that:

$$\sigma \leq 1$$

Furthermore, if we further restrict σ so that:

$$\sigma \leq \frac{1}{3}$$

then we have:

$$0 \leq 14M_1(\sigma) \leq 3 \tag{3.137}$$

3.10. Shock-Wave Solutions

The FLRW metric (3.127) can be matched to a compatible TOV metric (3.134) for a particular value of M_1, or equivalently a particular value of σ. That is, we may construct a solution of the field equations such that for the region behind the shock:

$$0 \leq \xi \leq \xi_1$$

we have the FLRW metric and for the region ahead of the shock:

$$\xi_1 \leq \xi$$

we have the TOV metric. Such a construction was completed in Chapter 2, except this time we restrict to the pure radiation FLRW and have the advantage of using self-similar Schwarzschild coordinates. The shock-wave solution that will be constructed will be nonsingular everywhere away from the shock and the spatial and temporal origins. In this light, we must show that (3.117)-(3.118) and (3.121)-(3.122) can be satisfied. Recall that for the FLRW$(0, \frac{1}{3}, 1)$ we have:

$$P_- = M_- = \frac{1}{2}V_-^2 = \frac{1}{3}W_-$$

and that the continuity of M implies:

$$M_- = M_+ \equiv M_1(\sigma)$$

For TOV(σ) we have:

$$W_+ = M_1(\sigma)$$

$$P_+ = \sigma M_1(\sigma)$$

and when these equations are applied, (3.118) becomes an equation for M_1, that is:

$$2M_1 = \frac{M_1 - P_+(M_1)}{3M_1 + P_+(M_1)} = V_-^2 \tag{3.138}$$

where:

$$P_+(M_1) = \sigma(M_1)M_1$$

Equivalently, we have:

$$M_1(2M_1 - 1)(8M_1^2 + 6M_1 - 1) = 0 \tag{3.139}$$

The only value of M_1 that satisfies (3.139) and is consistent with inequalities (3.137) is:

$$M_1 = \frac{\sqrt{17} - 3}{8} \tag{3.140}$$

Hence:

$$V_-^2 = \frac{\sqrt{17} - 3}{4}$$

and we can use (3.117) to compute:

$$V_+^2 = \frac{(M_1 - P_+(M_1))(3M_1 + P_+(M_1))}{4M_1^2} \approx 0.685$$

since it follows from (3.140) that:

$$P_+ \approx 0.0173$$

If the scaling is set so that (3.128) and (3.135) hold, then from (3.129) we can compute the value of ξ_1 from:

$$2^{-\frac{7}{3}}\xi_1 = M_- = M_1 = \frac{\sqrt{17} - 3}{8}$$

The continuity of $\bar{r}$ requires that:

$$\mathscr{R}_1 = \frac{\mathscr{R}_0}{\sqrt{\xi_1}}$$

and (3.121) allows us to relate C_1 and x_0 as so:

$$\frac{x_0}{C_1\xi_1^{n(\sigma)}} = \sqrt{\frac{(3M_1 + P_+)(M_1 + P_+)}{8M_1^2}}$$

Equation (3.122) is satisfied as a consequence of (3.139) and the above equations, thus yielding a shock-wave solution. As a final note, equation (3.109) may be written as:

$$V_-^2 = \frac{(P_- - P_+(M_-))(M_- + P_-)}{(W_- - M_-)(W_- + P_+(M_-))} \tag{3.141}$$

and since V_- is determined as a function of P_- and M_- from the equations characterising a similarity solution, (3.141) imposes another condition on the similarity solutions that may be fitted to a static one. Cahill and Taub claim that there is a two-parameter family of pure radiation spacetimes that can be matched to TOV(σ) to form a shock-wave solution. Since the paper they claimed this result would appear in was never published, and possibly never completed, this claim remains an open problem. We will see in Chapter 6 that this open problem will be partially resolved.

CHAPTER 4

Classification of General Relativistic Self-Similar Waves

Having seen that the FLRW and TOV solutions are both self-similar from Chapter 3, it begs the question of whether other self-similar perfect fluid solutions can be found and matched to form shock-wave solutions. In this chapter we summarise the 2000 paper *Complete Classification of Spherically Symmetric Self-Similar Perfect Fluid Solutions* by Carr and Coley [**6**]. This summary closely follows Carr and Coley's paper, although not necessarily in the same order, with some parts rewritten, some parts omitted and some parts slightly extended, with definitions added for clarity. Moreover, the variables, notation and metric sign convention have been changed to match those of the other chapters. In Chapter 6, we will form shock-wave solutions from one of the families of solutions considered in this chapter, making the physical insights provided in this chapter all the more relevant.

4.1. Overview

The aim of this chapter will be to completely classify all solutions of the perfect fluid Einstein field equations with the following properties:

(1) Spherical symmetry.
(2) Self-similarity of the first kind ($\xi = r/t$).
(3) A barotropic equation of state ($p = \sigma\rho$ from Proposition 3.2.1).
(4) A positive sound velocity ($\sigma > 0$).
(5) A causal sound velocity ($|\sigma| \leq 1$).
(6) Positive mass ($\rho > 0$).
(7) Shock free.

Note however that a few solutions will be considered for which some of points four to six do not hold. Now for a given value of σ, the complete classification of solutions can be described with two additional parameters, those being β and γ. Carr and Coley use α to denote the equation of state parameter, however we will follow Smoller and Temple's convention by using σ. The solutions will be classified by their asymptotic behaviour at large and small distances from the origin. This corresponds to large and small values of $|\xi|$ in most cases, but since $\xi = r/t$, it may also correspond to finite ξ. The classification is based on [**5**], another paper by Carr and Coley which proves that all similarity solutions must be asymptotic to solutions which depend on either powers of ξ at large and small $|\xi|$ or powers of $\ln|\xi|$ at finite ξ. We will follow [**6**] in showing that there are only three similarity solutions which have an explicit power-law dependence on ξ, those being:

(1) The flat Friedmann solution, denoted by $\mathrm{FLRW}(0, \sigma, 1)$.
(2) A Kantowski-Sachs solution, denoted by $\mathrm{KS}(\sigma)$.
(3) A static solution, denoted by $\mathrm{TOV}(\sigma)$.

However, the Kantowski-Sachs solution is only physical for $-1 < \sigma < -\frac{1}{3}$, at least according to the definition given in the next section. At large values of $|\xi|$, we will show for each σ there is:

(1) A one-parameter family of asymptotically Friedmann solutions.
(2) A one-parameter family of asymptotically Kantowski-Sachs solutions.
(3) A two-parameter family of asymptotically quasi-static solutions.

At large values of r, and for each $\sigma > \frac{1}{5}$, we will show there is:

(1) A one-parameter family of solutions asymptotic to the Minkowski solution as $|\xi| \to \infty$.
(2) A two-parameter family of solutions asymptotic to the Minkowski solution at finite ξ.

Note that due to our assumptions, neither family of asymptotically Minkowski solutions contain the Minkowski solution itself. The asymptotic behaviours close to the origin depend on whether the solutions pass through the sonic surface.

(1) Solutions that are supersonic everywhere correspond to black holes or naked singularities. Their small $|\xi|$ behaviour is uniquely determined by their large $|\xi|$ behaviour. Not all of these solutions can be extended to $\xi = 0$ by passing through the sonic surface, the ones that cannot either encounter a shock or become unphysical.

(2) Solutions that enter the subsonic region by passing through the sonic surface may become discontinuous there. Solutions that do not become discontinuous or unphysical will then reach $\xi = 0$.

At small values of $|\xi|$, we will show for each σ there is:

(1) A one-parameter family of asymptotically Friedmann solutions.

(2) A one-parameter family of asymptotically Kantowski-Sachs solutions.

(3) No asymptotically static solutions besides TOV(σ).

The full family of solutions can then be found by combining the possible large and small distance behaviours. This chapter will also discuss the physical significance of the solutions given in it.

We know from Chapter 3 that the assumptions of spherical symmetry and self-similarity significantly reduce the complexity of the perfect fluid Einstein field equations, by reducing them to a system of ODE. We also know that self-similarity of the first kind corresponds geometrically to the existence of a conformal Killing vector. Carr and Coley remark that geometric self-similarity, a property of the metric, and physical self-similarity, a property of the fluid, coincide for a perfect fluid but that this need not be the case in general. Carr and Coley go on to state that the solutions classified in this chapter are good physical models for the long-time behaviour of explosions into homogeneous backgrounds, since initial fluctuations can become spherically symmetric and self-similar as time progresses, even in expanding backgrounds. Furthermore, the evolution of cosmic voids and gravitationally bound clouds collapsing from an initially static configuration may also be described by self-similar solutions at late times. According to Carr, he proposed the *similarity hypothesis*, which postulates that under certain circumstances, spherically symmetric solutions may naturally evolve to a self-similar form, even if they start out more complicated.

A complete classification of similarity dust solutions, that is, where $\sigma = 0$, has already been completed by Carr [**4**], but these will not be discussed in this chapter. Carr and Coley remark that unlike in the case of dust, similarity solutions with non-zero pressure tend to have a shock or pass through the sonic surface. Carr and Coley's classification only considers solutions which are regular at the sonic point, in the sense that they have a finite pressure gradient and can be continued beyond there. Some of these solutions will turn out to be unphysical, in the sense that they encounter either another irregular sonic point or a domain where the mass is negative.

As remarked previously, we will show that there are four possible behaviours at large distances from the origin, these are:

(1) Asymptotically Friedmann.
(2) Asymptotically Kantowski-Sachs.
(3) Asymptotically quasi-static.
(4) Asymptotically Minkowski.

with the last family being subdivided into two families, one of which corresponds to finite ξ. The possible behaviours at small distances are:

(1) Asymptotically Friedmann.
(2) Asymptotically Kantowski-Sachs.
(3) Exactly static, that is, TOV(σ).
(4) Singular, in the form of a black hole or naked singularity.

with the latter solutions being the ones that do not pass through a sonic point. If the solutions are required to be analytic at the sonic point, then they are determined uniquely by the large $|\xi|$ behaviour. However, if the solutions are only required to be C^1, then the small and large $|\xi|$ behaviours must be specified independently. The complete family of solutions is found by combining the four types of large distance behaviours with the four types of small-distance behaviours. Since Kantowski-Sachs type solutions can only link to each other, there are a total of ten different types of solution under these restrictions. A qualitative summary of each type of solution will now be given.

(1) Asymptotically Friedmann at large $|\xi|$:

  (a) For each σ there is a one-parameter family of solutions.

  (b) Solutions with $\xi > 0$ can be regarded as inhomogeneous big bang models which expand from an initial singularity at $\xi = \infty$ and then either expand indefinitely or recollapse to a black hole as ξ decreases.

  (c) The ever-expanding solutions can be interpreted as density fluctuations in a flat Friedmann model which grow at the same rate as the Universe and are asymptotically Friedmann for small $|\xi|$.

  (d) The transonic solutions can be either underdense or overdense relative to FLRW$(0, \sigma, 1)$.

  (e) There is a continuum of regular underdense solutions and these may be relevant to the existence of large-scale cosmic voids.

  (f) Regular overdense solutions may only occur in very narrow bands of σ and have the characteristic that they are all approximately static near the sonic point, although they depart from TOV(σ) and exhibit oscillations as they approach the origin.

(2) Asymptotically Kantowski-Sachs at large $|\xi|$:

  (a) For each σ there is a unique KS(σ) solution.

  (b) For each σ there exists a one-parameter family of solutions asymptotic to KS(σ) at both large and small values of $|\xi|$.

  (c) Solutions with $-\frac{1}{3} < \sigma < 1$ are unphysical because the mass is negative and are also tachyonic for $0 < \sigma < 1$.

  (d) Solutions with $-1 < \sigma < -\frac{1}{3}$ are not tachyonic and have positive mass.

  (e) Equations of state with negative values of σ violate the strong energy condition, although they could well arise in the early Universe due to inflation or particle production effects. Such solutions may be related to the growth of positive pressure bubbles formed at a phase transition in a negative pressure cosmological background.

(3) Asymptotically static at large $|\xi|$:

  (a) For each $\sigma > 0$ there is a unique TOV(σ) solution.

  (b) For each σ there exists a one-parameter family of solutions asymptotic to TOV(σ).

(c) There is a two-parameter family of solutions which are asymptotically quasi-static in the sense that they have an isothermal density profile at large values of $|\xi|$. Such solutions also exist in the dust case, although there is no exactly static dust solution.

(d) The two-parameter solutions may span both positive and negative values of ξ, whereas each solution of the other types is confined to either positive or negative ξ.

(e) The two-parameter solutions can be regarded as inhomogeneous big bang models in which the initial or final singularity occurs at a finite, rather than an infinite, value of ξ.

(f) Some of the two-parameter solutions expand or collapse monotonically, these necessarily pass through the sonic surface and may be attached to an asymptotically Friedmann solution in the subsonic domain. Others expand and then recollapse, these remain supersonic everywhere and contain two singularities at finite ξ, one of which may be naked.

(g) Asymptotically quasi-static solutions have been associated with the occurrence of naked singularities and the transonic ones are also associated with critical phenomena for $\sigma < 0.28$.

(4) Asymptotically Minkowski at large r:

(a) These solutions only exist for $\sigma > \frac{1}{5}$.

(b) There are two families of solutions.

(c) Members of the first family are described by one parameter and are asymptotically Minkowski as $|\xi| \to \infty$.

(d) Members of the second family are described by two parameters and are asymptotically Minkowski as ξ tends to some finite value, though this corresponds to an infinite physical distance unless $\sigma = 1$.

(e) As with the asymptotically Friedmann and asymptotically quasi-static solutions, these solutions may be either supersonic everywhere, in which case they contain a black hole or a naked singularity, or attached to $\xi = 0$ via a sonic point, in which case they are asymptotically Friedmann or exactly static at small $|\xi|$.

(f) The transonic solutions are associated with critical phenomena for $\sigma < 0.28$.

Note that the Kantowski Sachs solutions are the only ones for which negative values of σ will be considered. The next section will begin by providing a concise revision to the derivation of the spherically symmetric self-similar in ξ Einstein field equations, first completed by Cahill and Taub and summarised previously in Chapter 3. The subsequent sections will then address the derivations of each of the aforementioned families.

4.2. Revisiting the Spherically Symmetric Self-Similar Einstein Field Equations

Recall that a general spherically symmetric metric in comoving coordinates can be written as:

$$ds^2 = -e^{2\varphi}dt^2 + e^{2\psi}dr^2 + \bar{r}^2 d\Omega^2 \tag{4.1}$$

where φ, ψ and $\bar{r}$ are functions of t and r. We will return to using natural units, so:

$$c = \mathcal{G} = 1$$

and with this choice of scaling the perfect fluid Einstein field equations are given by:

$$G^{\mu\nu} = 8\pi\left[(\rho + p)u^\mu u^\nu + p g^{\mu\nu}\right] \tag{4.2}$$

where ρ, p and $\boldsymbol{u}$ denote the fluid density, pressure and velocity four-vector respectively. As derived in Chapter 3, a first integral of (4.2) is given by:

$$m(t,r) = \frac{1}{2}\bar{r}\left[1 + e^{-2\varphi}\left(\frac{\partial \bar{r}}{\partial t}\right)^2 - e^{-2\psi}\left(\frac{\partial \bar{r}}{\partial r}\right)^2\right] \tag{4.3}$$

where $m(t,r)$ can be interpreted as the mass within radius r at time t:

$$m(t,r) = 4\pi \int_0^r \rho \bar{r}^2 \frac{\partial \bar{r}}{\partial r'} dr' \tag{4.4}$$

Carr and Coley note that unless $p = 0$, the mass $m(t,r)$ decreases with increasing t because of the work done by the pressure. We can also write $m(t,r)$ as:

$$m(t,r) = 4\pi \int_0^t \rho \bar{r}^2 \frac{\partial \bar{r}}{\partial t'} dt' \tag{4.5}$$

and this will be useful for when there is no spatial origin, as will be the case for the Kantowski-Sachs solution. Equation (4.3) can be written as an equation for the energy per unit mass of the spherical shell with coordinate r, as so:

$$\mathcal{E} = \frac{1}{2}U^2 - \frac{m}{\bar{r}} \tag{4.6}$$

where:

$$U = e^{-\varphi}\frac{\partial \bar{r}}{\partial t}$$

We also know from Chapter 3 that a spherically symmetric similarity solution can be put into a form in which all quantities such as φ, ψ, $\mathcal{E}$ and:

$$\mathscr{R} = \frac{\bar{r}}{r} \tag{4.7}$$

$$M = \frac{m}{\bar{r}} \tag{4.8}$$

$$P = p\bar{r}^2 \tag{4.9}$$

$$W = \rho\bar{r}^2 \tag{4.10}$$

are functions only of the variable $\xi = r/t$. Changing the variables from t and r to ξ and r means that:

$$\frac{\partial}{\partial t} = -\frac{\xi^2}{r}\frac{d}{d\xi} \tag{4.11}$$

$$\frac{\partial}{\partial r} = \frac{\xi}{r}\frac{d}{d\xi} \tag{4.12}$$

and such reduces the perfect fluid Einstein field equations to a set of ODE in the variable ξ. One quantity of particular importance is the function:

$$V(\xi) = \xi e^{\psi-\varphi} \tag{4.13}$$

which represents the velocity of the surfaces of constant ξ relative to the fluid. These surfaces have the equation $r = \xi_0 t$, for some constant ξ_0, and therefore represent a family of spheres moving through the fluid. Recall from Section 3.4 that the spheres contract relative to the fluid for $\xi < 0$

and expand for $\xi > 0$. On the other hand, the velocity of the spheres of constant $\bar{r}$ relative to the fluid is given by:

$$V_R = -e^{\psi-\varphi}\frac{\partial\bar{r}}{\partial t}\left(\frac{\partial\bar{r}}{\partial r}\right)^{-1} \tag{4.14}$$

This quantity is positive if the fluid is collapsing and negative if it is expanding. Carr and Coley note that the value of ξ for which $|V(\xi)| = 1$ corresponds to a Cauchy horizon, such as a black hole or cosmological particle horizon, and the value of ξ for which $|V_R(\xi)| = 1$ corresponds to either a black hole or cosmological apparent horizon. We know from Proposition 2.1.4 that the only barotropic equation of state compatible with similarity solutions is one of the form:

$$p = \sigma\rho$$

Now the following analysis is guided by the work of Carr and Yahil [**9**]. In light of Proposition 2.1.4, it is convenient to use the function $x(\xi)$ defined by:

$$x(\xi) = \left(4\pi\rho r^2\right)^{-\frac{\sigma}{1+\sigma}} \tag{4.15}$$

This function is useful because the conservation equations:

$$\nabla_\nu T^{\mu\nu} = 0$$

can be integrated to give:

$$e^{\varphi} = \beta\xi^{\frac{2\sigma}{1+\sigma}}x \tag{4.16}$$

$$e^{-\psi} = \gamma x^{-\frac{1}{\sigma}}\mathscr{R}^2 \tag{4.17}$$

where β and γ are constants of integration. The remaining perfect fluid Einstein field equations reduce to the following system of ODE in the variables x and $\mathscr{R}$:

$$\ddot{\mathscr{R}} + \dot{\mathscr{R}} + \left(\frac{2}{1+\sigma}\frac{\dot{\mathscr{R}}}{\mathscr{R}} - \frac{1}{\sigma}\frac{\dot{x}}{x}\right)\left(\mathscr{R} + (1+\sigma)\dot{\mathscr{R}}\right) = 0 \tag{4.18}$$

$$\frac{2\sigma\gamma^2}{1+\sigma}\mathscr{R}^4 + \frac{2}{\beta^2}\xi^{\frac{2-2\sigma}{1+\sigma}}x^{\frac{2-2\sigma}{\sigma}}\frac{\dot{\mathscr{R}}}{\mathscr{R}} - \gamma^2\mathscr{R}^4\frac{\dot{x}}{x}\left(\frac{V^2}{\sigma} - 1\right) = (1+\sigma)x^{\frac{1-\sigma}{\sigma}} \tag{4.19}$$

$$x^{-\frac{1+\sigma}{\sigma}}\mathscr{R}^2\left(1 + (1+\sigma)\frac{\dot{\mathscr{R}}}{\mathscr{R}}\right) = M \tag{4.20}$$

$$\frac{1}{2} + \frac{1}{2\beta^2}\xi^{\frac{2-2\sigma}{1+\sigma}}x^{-2}\dot{\mathscr{R}}^2 - \frac{1}{2}\gamma^2 x^{-\frac{2}{\sigma}}\mathscr{R}^6\left(1 + \frac{\dot{\mathscr{R}}}{\mathscr{R}}\right)^2 = M \tag{4.21}$$

with the dot representing the operator:

$$\xi\frac{d}{d\xi}$$

In the variables x and $\mathscr{R}$, the velocity functions may be written as:

$$V = \frac{1}{\beta\gamma}\xi^{\frac{1-\sigma}{1+\sigma}}x^{\frac{1-\sigma}{\sigma}}\mathscr{R}^{-2} \tag{4.22}$$

$$V_R = \frac{V\dot{\mathscr{R}}}{\mathscr{R} + \dot{\mathscr{R}}} \tag{4.23}$$

and the energy function can be written as:

$$\mathcal{E} = \frac{1}{2}\gamma^2 x^{-\frac{2}{\sigma}}\mathscr{R}^6\left(1 + \frac{\dot{\mathscr{R}}}{\mathscr{R}}\right)^2 - \frac{1}{2} \tag{4.24}$$

noting that:

$$\mathcal{E} \geq -\frac{1}{2}$$

Given the rewriting of the energy and velocity functions, (4.21) can be written as:

$$M = \frac{1}{2} + \left(\mathcal{E} + \frac{1}{2}\right)\left(V_R^2 - 1\right) \tag{4.25}$$

noting that:

$$M = \frac{1}{2} \iff |V_R| = 1$$

Now it is possible to visualise the solutions of (4.18)-(4.21) by working in $(x, \mathscr{R}, \dot{\mathscr{R}})$ space. Indeed a similar approach inspired some of the new results in Chapter 6. For fixed σ and for any point in this space, (4.20) and (4.21) give the value of ξ, (4.19) then gives the value of $\dot{x}$, unless $|V| = \sqrt{\sigma}$, and (4.18) gives the value of $\ddot{\mathscr{R}}$. That is, equations (4.18)-(4.21) generate the vector field $(\dot{x}, \dot{\mathscr{R}}, \ddot{\mathscr{R}})$ and this specifies a trajectory at each point of the space. Each trajectory is parameterised by ξ and represents a solution. The space of solutions is thus a three-parameter space, given by parameters σ, β and γ.

Definition 4.2.1. The condition $V = \sqrt{\sigma}$ is known as the *sonic condition*, or in $(x, \mathscr{R}, \dot{\mathscr{R}})$ space, the *sonic surface*. Furthermore, the point on the trajectory of a solution in $(x, \mathscr{R}, \dot{\mathscr{R}})$ space that intersects the sonic surface will be known as a *sonic point*.

The sonic condition specifies a two-dimensional sonic surface because equations (4.20)-(4.22) allow ξ to be expressed in terms of x, $\mathscr{R}$ and $\dot{\mathscr{R}}$. Note that the same surface is generated from the condition $V = -\sqrt{\sigma}$. Equation (4.19) does not uniquely determine $\dot{x}$ where a trajectory intersects the sonic surface, so there can be a number of different solutions passing through the same point. However, solution trajectories intersect the sonic surface in a physically reasonable manner only if:

$$\frac{2\sigma\gamma^2}{1+\sigma}\mathscr{R}^4 + \frac{2}{\beta^2}\xi^{\frac{2-2\sigma}{1+\sigma}} x^{\frac{2-2\sigma}{\sigma}} \frac{\dot{\mathscr{R}}}{\mathscr{R}} = (1+\sigma) x^{\frac{1-\sigma}{\sigma}} \tag{4.26}$$

since otherwise the value of $\dot{x}$, and hence the density, pressure and velocity, will diverge there. Note that (4.26) follows directly from (4.19). Just like the sonic condition, (4.26) corresponds to another two-dimensional surface in $(x, \mathscr{R}, \dot{\mathscr{R}})$ space.

Definition 4.2.2. The line in $(x, \mathscr{R}, \dot{\mathscr{R}})$ space for which the (4.26) surface and sonic surface intersect is known as the *sonic line*. Furthermore, solution trajectories which hit the sonic surface on the sonic line are called *regular*.

Trajectories that do not hit the sonic surface on the sonic line will have to contain shock waves in order to be extended. Not all combinations of the parameters σ, β and γ will correspond to solutions that cross the sonic surface on the sonic line, some combinations will correspond to solutions that do not meet the sonic surface at all, this is how some families of solutions have less than two free parameters. From each point on the sonic line there will be regular trajectories with decreasing and increasing ξ. Any member of the first type can be joined to any member of the second type to extend the solution through the sonic surface. Physical solutions require a finite value of $\ddot{x}$ at the sonic surface, so the equations permit only two values of $\dot{x}$ at each point on the sonic line, corresponding to two values of $\dot{V}$, which we will denote by $\dot{V}_1$ and $\dot{V}_2$. There are various cases for the values of $\dot{V}_1$ and $\dot{V}_2$.

(1) If the values of $\dot{V}_1$ and $\dot{V}_2$ are complex, this corresponds to a focal point and such solutions will spiral around the sonic point and be unphysical.
(2) If the values of $\dot{V}_1$ and $\dot{V}_2$ are real, at least one of the values must be positive. If both values are positive, this corresponds to a nodal point and the smaller value is associated with a one-parameter family of solutions, while the larger value is associated with an isolated solution.
(3) If one of the values of $\dot{V}_1$ and $\dot{V}_2$ is negative, this corresponds to a saddle point and both values are associated with isolated solutions.

We are now in a position to provide the following working definition. Note however that this definition was not given in [**6**].

Definition 4.2.3. We say that a spherically symmetric similarity solution is *physical* if:

$$|\sigma| \leq 1$$

$$M \geq 0$$

and if passing through the sonic surface:

$$\dot{V}_1, \dot{V}_2 \in \mathbb{R}$$

In practice the assumption $|\sigma| < 1$ will be made. Carr and Coley state that there is a one-parameter family of regular solutions only in two intervals of $|\xi|$, with the intervals for positive ξ denoted by (ξ_1, ξ_2) and (ξ_3, ∞). They also state that there is a saddle point and a focal point in the intervals $(0, \xi_1)$ and (ξ_2, ξ_3) respectively. The values of ξ_1, ξ_2 and ξ_3 can be expressed in terms of σ but the expressions are complicated, so they were not given in [**6**], and consequently, will not be given here. If the FLRW$(0, \sigma, 1)$ and TOV(σ) sonic points are denoted by ξ_F and ξ_S repetitively, then we have that:

$$\sigma = \frac{1}{3} \implies \begin{cases} \xi_2 = \xi_S \\ \xi_3 = \xi_F \end{cases}$$

and in general:

$$\xi_2 < \xi_S < \xi_3 < \xi_F$$

where:

$$\xi_F = \left(\frac{3^{\frac{1}{2}} (2\sigma)^{\frac{1}{2}}}{1 + 3\sigma} \right)^{\frac{3+3\sigma}{1+3\sigma}} \tag{4.27}$$

$$\xi_S = \left(\frac{3^{\frac{5\sigma-1}{2\sigma-2}} (2\sigma)^{\frac{3}{2}}}{(1 + 3\sigma)\,(1 + 6\sigma + \sigma^2)} \right)^{\frac{1+\sigma}{1+3\sigma}} \tag{4.28}$$

It turns out that one-parameter solutions must hit the sonic line in the nodal intervals, with these solutions being physical only for certain bands of parameter values.

Definition 4.2.4. The region $|V| < \sqrt{\sigma}$ in $(x, \mathscr{R}, \dot{\mathscr{R}})$ space is known as the *subsonic region* and the region $|V| > \sqrt{\sigma}$ is known as the *supersonic region*. Furthermore, solution trajectories in the subsonic region are called *subsonic*, trajectories in the supersonic region are called *supersonic* and trajectories spanning both regions are called *transonic*.

On each side of a node, $\dot{V}$ may take either of its two possible values. The association of these values is directly tied to the regularity of the transonic solution.

(1) If the values are different, there will be a discontinuity in the pressure gradient, so the solution will only be C^0.

(2) If the values are the same, there may still be a discontinuity in the second derivative of V, in which case the solution will be C^1.

(3) If the values are the same, the isolated solution and at most a single member of the one-parameter family will be analytic.

Two-parameter solutions at large $|\xi|$ can also hit the sonic line in the saddle range and such solutions are analytic at the sonic point but may become unphysical in the subsonic region. Carr and Coley remark that in the case of a shock, V would be discontinuous.

4.3. Explicit Power-Law Solutions

Before considering asymptotic power-law solutions, we will first derive the explicit power-law solutions. We already know from Chapter 3 that FLRW$(0, \sigma, 1)$ and TOV(σ) are both power-law solutions, so we expect to find these, but we do not yet know if these are the only explicit power-law solutions that can be found. In this light, let us consider the power-law ansatz:

$$x = x_0 \xi^a \tag{4.29}$$

$$\mathscr{R} = \mathscr{R}_0 \xi^b \tag{4.30}$$

where x_0, $\mathscr{R}_0$, a and b are constants. Note that:

$$\frac{\dot{x}}{x} = a$$
$$\frac{\dot{\mathscr{R}}}{\mathscr{R}} = b$$

and that all four of these additional constants will depend at most on σ, β and γ, due to all solutions depending on at most three parameters. Now (4.18) implies:

$$a = \frac{b\sigma[3(b+1) + \sigma(3b+1)]}{(1+\sigma)[1+(1+\sigma)b]} \tag{4.31}$$

with (4.20) and (4.21) requiring the denominator to be non-zero. Substituting our ansatz into (4.19) yields an equation of the form:

$$A\xi^p + B\xi^q + C = 0 \tag{4.32}$$

where:

$$A = \frac{b[(\sigma-1)+(1+\sigma)(2\sigma-1)b]}{\beta^2(1+\sigma)[1+(1+\sigma)b]} x_0^{\frac{2-2\sigma}{\sigma}} \mathscr{R}_0^{-4} \tag{4.33}$$

$$B = -(1+\sigma)x_0^{\frac{1-\sigma}{\sigma}} \mathscr{R}_0^{-4} \tag{4.34}$$

$$C = \frac{\sigma\gamma^2(b+1)[2+3b(1+\sigma)]}{(1+\sigma)[1+(1+\sigma)b]} \tag{4.35}$$

$$p = 2a\left(\frac{1-\sigma}{\sigma}\right) - 4b + 2\left(\frac{1-\sigma}{1+\sigma}\right) \tag{4.36}$$

$$q = a\left(\frac{1-\sigma}{\sigma}\right) - 4b \tag{4.37}$$

With our practical assumption of $|\sigma| < 1$, B cannot be zero and this implies that there are three ways in which (4.32) holds to leading order as $\xi \to 0$ or $\xi \to \infty$. The three cases are summarised as so:

Case 1: $p = q$ and $A + B = 0$

Case 2: $q = 0$ and $B + C = 0$

Case 3: $p = 0$ and $A + C = 0$

Beginning with Case 1, we have:

$$p = q \implies a = -\frac{2\sigma}{1+\sigma} \tag{4.38}$$

$$A + B = 0 \implies x_0^{\frac{1-\sigma}{\sigma}} = \frac{1}{2}\beta^2\left(\frac{(1+\sigma)^2}{b(1+\sigma)+1}\right) \tag{4.39}$$

meaning that (4.31) requires:

$$b = -1 \text{ or } b = -\frac{2}{3(1+\sigma)} \tag{4.40}$$

with both values implying $C = 0$ from (4.33)-(4.35). Since (4.32) holds exactly in Case 1, there are no approximate solutions for $C \neq 0$. For the second choice of b in (4.40), equations (4.20) and (4.21) imply:

$$x_0^{\frac{\sigma-1}{\sigma}} = \frac{2}{3\beta^2(1+\sigma)^2} \tag{4.41}$$

$$\gamma^2 x_0^{-\frac{2}{\sigma}} \mathscr{R}_0^6 = \frac{9(1+\sigma)^2}{(1+3\sigma)^2} \tag{4.42}$$

and we can set $x_0 = \mathscr{R}_0 = 1$ if the r and t coordinates are scaled so that:

$$\beta = \frac{\sqrt{2}}{\sqrt{3}(1+\sigma)} \tag{4.43}$$

$$\gamma = \frac{3(1+\sigma)}{1+3\sigma} \tag{4.44}$$

This scaling yields:

$$x = \xi^{-\frac{2\sigma}{1+\sigma}} \tag{4.45}$$

$$\mathscr{R} = \xi^{-\frac{2}{3+3\sigma}} \tag{4.46}$$

and the resulting metric is:

$$ds^2 = -\beta^2 dt^2 + \gamma^{-2} \xi^{-\frac{4}{3+3\sigma}} dr^2 + \xi^{-\frac{4}{3+3\sigma}} r^2 d\Omega^2 \tag{4.47}$$

where β and γ are given in terms of σ as above, that is, this is an explicit one-parameter solution. Furthermore, the density, velocity and mass functions are given by:

$$\rho = \frac{\xi^2}{4\pi r^2} \tag{4.48}$$

$$V = \frac{1}{\sqrt{6}}(1+3\sigma)\xi^{\frac{1+3\sigma}{3+3\sigma}} \tag{4.49}$$

$$M = \frac{1}{3}\xi^{\frac{2+6\sigma}{3+3\sigma}} \tag{4.50}$$

We can put (4.47) in a more familiar form by making the coordinate transformation:

$$\hat{t} = \beta t \tag{4.51}$$

$$\hat{r} = \beta^{-\frac{2}{3+3\sigma}} r^{\frac{1+3\sigma}{3+3\sigma}} \tag{4.52}$$

and this yields:

$$d\hat{s}^2 = -d\hat{t}^2 + \hat{t}^{\frac{4}{3+3\sigma}} \left(d\hat{r}^2 + \hat{r}^2 d\Omega^2\right) \tag{4.53}$$

$$\rho = \frac{\beta^2}{4\pi t^2} \tag{4.54}$$

which we are familiar with as being FLRW$(0,\sigma,1)$. Solutions that are asymptotic to FLRW$(0,\sigma,1)$ will be discussed in the next section.

Now for the first choice of b in (4.40), then (4.39) requires:

$$\beta^2 = -\frac{2\sigma}{(1+\sigma)^2} x_0^{\frac{1-\sigma}{\sigma}} \tag{4.55}$$

with equations (4.20) and (4.21) implying:

$$x_0^{-\frac{1+\sigma}{\sigma}} \mathscr{R}_0^2 = \frac{2\sigma}{(1+\sigma)^2 - 4\sigma^2} \tag{4.56}$$

This shows that $x_0 = \mathscr{R}_0 = 1$ cannot be set in this case. However, if β and γ are set by (4.43) and (4.44) for $\sigma < 0$ and i times those for $\sigma > 0$, then we get the same scaling as used for FLRW$(0,\sigma,1)$. From this, (4.55) and (4.56) give:

$$x_0 = (3|\sigma|)^{-\frac{\sigma}{1-\sigma}} \tag{4.57}$$

$$\mathscr{R}_0^2 = \frac{2\sigma(3|\sigma|)^{-\frac{1+\sigma}{1-\sigma}}}{(1+3\sigma)(1-\sigma)} \tag{4.58}$$

Irrespective of this scaling, the power-law ansatz yields:

$$x = x_0 \xi^{-\frac{2\sigma}{1+\sigma}} \tag{4.59}$$

$$\mathscr{R} = \mathscr{R}_0 \xi^{-1} \tag{4.60}$$

and the resulting metric is:

$$ds^2 = -\beta^2 x_0^2 dt^2 + \gamma^{-2} x_0^{\frac{2}{\sigma}} \mathscr{R}_0^{-4} \xi^{\frac{4\sigma}{1+\sigma}} dr^2 + \mathscr{R}_0^2 \xi^{-2} r^2 d\Omega^2 \tag{4.61}$$

The t coordinate is spacelike and the r coordinate is timelike for $\sigma > 0$ because of the i factors in β and γ. This is not the case for $-\frac{1}{3} < \sigma < 0$, however from (4.57) and (4.58) the circumferential coordinate is timelike as $\mathscr{R}_0$ is imaginary in this case. The metric (4.61) can be put in a more standard form by making the coordinate transformation:

$$\tilde{t} = \beta x_0 t \tag{4.62}$$

$$\tilde{r} = \gamma^{-1} (\beta x_0)^{\frac{2\sigma}{1+\sigma}} x_0^{\frac{1}{\sigma}} \mathscr{R}_0^{-2} r^{\frac{1+3\sigma}{1+\sigma}} \tag{4.63}$$

which yields:

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{t}^{-\frac{4\sigma}{1+\sigma}} d\tilde{r}^2 + \beta^{-2} x_0^{-2} \mathscr{R}_0^2 \tilde{t}^2 d\Omega^2 \tag{4.64}$$

and where it is worth noting that:

$$\beta^{-2} x_0^{-2} \mathscr{R}_0^2 = -\frac{(1+\sigma)^2}{(1+\sigma)^2 - 4\sigma^2}$$

This explicit spherically symmetric similarity solution is one that we have not encountered yet, but for those that are familiar with such a solution, they will know that it is the self-similar Kantowski-Sachs solution, which we will denote the by KS(σ). The density, velocity and mass functions of KS(σ) are given by:

$$\rho = \frac{(3|\sigma|)^{\frac{1+\sigma}{1-\sigma}} \xi^2}{4\pi r^2} \tag{4.65}$$

$$V = -\frac{(1-\sigma)(1+3\sigma)^2 (3|\sigma|)^{\frac{2\sigma}{1-\sigma}}}{\sqrt{24}\sigma} \xi^{\frac{1+3\sigma}{1+\sigma}} \tag{4.66}$$

$$M = -\frac{2\sigma^2}{(1-\sigma)(1+3\sigma)} \tag{4.67}$$

and we will define:

$$M_{KS}(\sigma) = \frac{2\sigma^2}{(\sigma-1)(1+3\sigma)}$$

The function V is negative for $0 < \sigma < 1$, corresponding to tachyonic solutions, while M is negative for $-\frac{1}{3} < \sigma < 1$, corresponding to negative mass solutions. By our definition, only solutions with $-1 < \sigma < -\frac{1}{3}$ are physical. Carr and Coley note that (4.4) does not apply for $-1 < \sigma < -\frac{1}{3}$ as there is no well-defined origin. Moreover, everything is on a shell, since (4.59) and (4.60) imply that $\bar{r}$ is independent of r. Carr and Coley go on to state that the value of m must instead be interpreted as the mass of the whole Universe at time t, as indicated by (4.5). Furthermore, this solution corresponds to the special case $\mathcal{E} = -\frac{1}{2}$, as V_R diverges and $M \neq \frac{1}{2}$. Solutions that are asymptotic to KS(σ) will be discussed in Section 4.5.

For Case 2, (4.31), (4.36) and (4.37) imply that the only solution for $\sigma > 0$ and $V > 0$ is given by:

$$a = b = 0 \tag{4.68}$$

that is, x and $\mathscr{R}$ must be constant. Carr and Coley note that the condition $q = 0$ allows for another value of b other than the value given by (4.68) but this implies $C < 0$ for $\sigma > 0$, so the condition $B + C = 0$ cannot be satisfied. Condition (4.68) implies $A = 0$ and so (4.32) is satisfied identically, hence there are no approximate solutions with $A \neq 0$. In this light, and for $\sigma > 0$, $B + C = 0$ means:

$$\mathscr{R}_0^2 = \frac{1+3\sigma}{\sqrt{18\sigma}} x_0^{\frac{1-\sigma}{2\sigma}} \tag{4.69}$$

and the resulting metric is given by:

$$ds^2 = -\beta^2 x_0^2 \xi^{\frac{4\sigma}{1+\sigma}} dt^2 + \gamma^{-2} x_0^{\frac{2}{\sigma}} \mathscr{R}_0^{-4} dr^2 + \mathscr{R}_0^2 r^2 d\Omega^2 \tag{4.70}$$

Furthermore, the density, velocity and mass functions of this solution are given by:

$$\rho = \frac{1}{4\pi r^2} x_0^{-\frac{1+\sigma}{\sigma}} \tag{4.71}$$

$$V = \sqrt{3\sigma} x_0^{\frac{1-\sigma}{2\sigma}} \xi^{\frac{1-\sigma}{1+\sigma}} \tag{4.72}$$

$$M = \frac{2\sigma}{1 + 6\sigma + \sigma^2} \tag{4.73}$$

We can put (4.70) in a more familiar form by making the coordinate transformation:

$$\bar{t} = \frac{1+\sigma}{1-\sigma}\beta x_0 \mathscr{R}_0^{-\frac{2\sigma}{1+\sigma}} t^{\frac{1-\sigma}{1+\sigma}} \tag{4.74}$$

$$\bar{r} = \mathscr{R}_0 r \tag{4.75}$$

which yields:

$$d\bar{s}^2 = -\bar{r}^{\frac{4\sigma}{1+\sigma}} d\bar{t}^2 + \gamma^{-2} x_0^{\frac{2}{\sigma}} \mathscr{R}_0^{-6} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \tag{4.76}$$

$$\rho = \frac{\mathscr{R}_0^2}{4\pi r^2} x_0^{-\frac{1+\sigma}{\sigma}} \tag{4.77}$$

That is, this solution is the familiar TOV(σ) metric. We can again set β and γ by (4.43) and (4.44) so that we have the same scaling as used for FLRW$(0,\sigma,1)$. In such a case, (4.20), (4.21) and (4.69) imply that x_0 and $\mathscr{R}_0$ are given by:

$$x_0 = \left(\frac{(1+3\sigma)\left(1+6\sigma+\sigma^2\right)}{3(2\sigma)^{\frac{3}{2}}}\right)^{\frac{2\sigma}{1+3\sigma}} \tag{4.78}$$

$$\mathscr{R}_0 = \left(\frac{(1+3\sigma)^{1+\sigma}\left(1+6\sigma+\sigma^2\right)^{\frac{1-\sigma}{2}}}{3^{1+\sigma}(2\sigma)}\right)^{\frac{1}{1+3\sigma}} \tag{4.79}$$

Thus there is only one TOV solution for each equation of state parameter σ, as expected. Note that we also have $V_R = 0$ from (4.23), also as expected. Carr and Coley remark that there are no static solutions in the dust case, essentially because the $\sigma^{-1}\dot{x}$ term in (4.18) cannot be null for $\sigma = 0$. Solutions asymptotic to TOV(σ) will be discussed in Section 4.6. Before considering the asymptotic solutions of Case 2, the following proposition highlights an interesting connection between the TOV(σ) and KS(σ) solutions.

Proposition 4.3.1. *By interchanging the t and r coordinates in the TOV(σ) metric (4.70) and changing the equation of state parameter to:*

$$\sigma' = -\frac{\sigma}{1+2\sigma} \tag{4.80}$$

then we obtain the KS(σ') metric (4.61).

PROOF. The proof is a direct consequence of the results derived in this section. $\square$

Note that by (4.80) if $0 < \sigma < 1$, then $-\frac{1}{3} < \sigma' < 0$ and $\sigma = \sigma'$ only for $\sigma = 0$ and $\sigma = -1$. However, KS(σ') is only physical for $-1 < \sigma' < -\frac{1}{3}$, which by (4.80) corresponds to $|\sigma| > 1$ and so does not give physical TOV(σ) solutions. Carr and Coley remark that the mass of both the TOV(σ) and KS(σ) solutions tends to zero as σ does, even though both solutions do not exist in the limit $\sigma = 0$.

At the start of Case 2, we assumed $V > 0$. If instead we assume $V < 0$, with β^2 and γ^2 changing their sign as was seen for KS(σ), then another solution can be obtain in the limits $\xi \to \infty$. This asymptotic solution satisfies:

$$a = \frac{4\sigma\left(1 + 6\sigma + \sigma^2\right)}{(1 + 7\sigma)\left(1 - \sigma^2\right)} \tag{4.81}$$

$$b = -\frac{\left(1 + 6\sigma + \sigma^2\right)}{(1 + 7\sigma)(1 + \sigma)} \tag{4.82}$$

for:

$$\mathscr{R}_0^2 = x_0^{\frac{1-\sigma}{2\sigma}} \left(\frac{(1 + 7\sigma)(1 - \sigma)}{18\sigma}\right)^{\frac{1}{2}} \tag{4.83}$$

and the mass and velocity functions satisfy:

$$V \sim \xi^{-\frac{(1+3\sigma)^2}{(1+7\sigma)(1+\sigma)}} \tag{4.84}$$

$$M \sim \xi^{\frac{(2+6\sigma)(1+6\sigma+\sigma^2)}{(1+7\sigma)(1-\sigma^2)}} \tag{4.85}$$

These asymptotic solutions will be revisited in Section 4.5.

In Case 3, $p = 0$ implies:

$$b = \frac{1}{2}\left(\frac{1 - \sigma}{\sigma}\right) a + \frac{1}{2}\left(\frac{1 - \sigma}{1 + \sigma}\right) \tag{4.86}$$

and (4.22) implies V tend to the finite value:

$$V_* = \beta^{-1}\gamma^{-1}x_0^{\frac{1-\sigma}{\sigma}}\mathscr{R}_0^{-2} \tag{4.87}$$

The condition $A + C = 0$ now leads to:

$$a = \frac{V_*^2(1-\sigma) + 2\sigma}{(V_*^2 - 1)\,(1+\sigma)} \tag{4.88}$$

$$b = \frac{(1-\sigma)\left(V_*^2 + \sigma\right)}{2\sigma(1+\sigma)\,(V_*^2 - 1)} \tag{4.89}$$

with (4.36) and (4.37) yielding:

$$q = \frac{(1-\sigma)V_*^2}{\sigma\,(1 - V_*^2)} \tag{4.90}$$

Since $B \neq 0$ from (4.34), an exact solution to (4.32) cannot be obtained in Case 3. In this light, condition (4.90) is only a consistent solution of (4.32) for large ξ if $V_*^2 > 1$ and for small ξ if $V_*^2 < 1$. Equation (4.20) gives negative values of M, and hence unphysical solutions, unless $V_*^2 < \sigma$ and this last condition will also turn out to be inconsistent, so focus will be on the $V_*^2 > 1$ case. Equation (4.20) yields:

$$M \sim \frac{V^2 - \sigma}{V^2 - 1} \xi^{-\frac{V_*^2(1-\sigma)+1+3\sigma}{(V_*^2-1)(1+\sigma)}} \tag{4.91}$$

and this tends to zero as $\xi \to \infty$. Carr and Coley include the coefficient of M to demonstrate that the mass is negative for $\sigma < V_*^2 < 1$. Equation (4.21) implies:

$$M \sim \frac{1}{2} + \left[b^2(V_*^2 - 1) - 2b - 1\right] \xi^{-\frac{V_*^2(1-\sigma)-\sigma(1+3\sigma)}{(V_*^2-1)\sigma(1+\sigma)}} \tag{4.92}$$

and if the exponent of ξ is positive, $M \to 0$ as $\xi \to \infty$ only if the term in square brackets is zero and this means:

$$b = \frac{1}{V_* - 1}$$

Conditions (4.88) and (4.89) give a quadratic equation for V_*:

$$(1-\sigma)V_*^2 - 2\sigma(1+\sigma)V_* - \sigma(1+3\sigma) = 0 \tag{4.93}$$

with the real positive solution given by:

$$V_* = \frac{\sigma(1+\sigma) + \sqrt{\sigma\,(1 + 3\sigma - \sigma^2 + \sigma^3)}}{1-\sigma} \tag{4.94}$$

Continuing to follow [**6**] closely, we note that (4.93) implies that the exponent of ξ in (4.92) is positive, matching our assumption. In addition, V_* decreases from ∞ to $\sqrt{\sigma}$ as σ decreases from 1 to 0, which prevents $V_* < \sqrt{\sigma}$, so there are no subsonic solutions of this type as $\xi \to 0$. The value of V_* given by (4.94) exceeds one, as required, only for $\sigma > \frac{1}{5}$, so these solutions do not exist in the dust case. Putting this all together, we have that the metric takes following asymptotic form:

$$ds^2 \sim -\xi^{\frac{2V_*^2}{V_*^2-1}} dt^2 + \xi^{\frac{2}{V_*^2-1}} dr^2 + \xi^{\frac{2}{V_*-1}} r^2 d\Omega^2 \tag{4.95}$$

and this can be put in Minkowski form by a change of coordinates. Because (4.87) and (4.94) impose a constraint on x_0 and $\mathscr{R}_0$, these solutions depend on a single parameter.

4.4. Asymptotically Friedmann Solutions

In this section we follow the work of Carr and Yahil [**9**] by considering solutions which are asymptotically FLRW$(0, \sigma, 1)$ for large and small values of ξ. We will refer to these solutions as being *asymptotically Friedmann* and they will play an important role in Chapters 5 and 6. To start, we introduce functions $X(\xi)$ and $Y(\xi)$ defined by:

$$x = \xi^{-\frac{2\sigma}{1+\sigma}} e^X \tag{4.96}$$

$$\mathscr{R} = \xi^{-\frac{2}{3+3\sigma}} e^Y \tag{4.97}$$

Then FLRW$(0, \sigma, 1)$ is given by:

$$X \equiv Y \equiv 0$$

It will be assumed that $\xi > 0$, otherwise ξ will need to be replaced with $|\xi|$ in what follows. The analysis is trivially extended to the $\xi < 0$ case since r is always taken to be positive so the $\xi < 0$ solutions are just the time-reverse of the $\xi > 0$ ones, making the solutions symmetric in ξ. We will first consider solutions that are asymptotically Friedmann as $\xi \to \infty$, that is, as $r \to \infty$ for fixed t or as $t \to 0$ for fixed r. By substituting (4.96) and (4.97) into (4.18)-(4.21), the system of ODE in x and $\mathscr{R}$ now become ODE in X and Y. We can find the leading order solutions by linearising these ODE in the limit $\xi \to \infty$. In this light, equations (4.18) and (4.19) give us:

$$\ddot{Y} = \frac{1}{3\sigma}\dot{X} - \frac{1+3\sigma}{1+\sigma}\dot{Y} \tag{4.98}$$

$$\dot{Y} = \frac{1}{2\sigma}\dot{X} - \frac{\sigma-1}{3\sigma(1+\sigma)}X \tag{4.99}$$

Differentiating (4.99) and eliminating $\dot{Y}$ and $\ddot{Y}$ gives us a single second-order ODE for X as so:

$$\ddot{X} + \frac{9\sigma-1}{3+3\sigma}\dot{X} + \frac{2(1+3\sigma)(1-\sigma)}{3(1+\sigma)^2}X = 0 \tag{4.100}$$

Equation (4.100) has two solutions and these are given by:

$$X \sim \xi^{-\frac{2+6\sigma}{3+3\sigma}} \tag{4.101}$$

$$X \sim \xi^{\frac{1-\sigma}{1+\sigma}} \tag{4.102}$$

Solution (4.102) can be discarded as X would diverge to infinity as $\xi \to \infty$ for any $\sigma < 1$. Thus we take the first solution and this yields:

$$X = -\frac{\sigma(1+3\sigma)}{1+\sigma}C_0\xi^{-\frac{2+6\sigma}{3+3\sigma}} \tag{4.103}$$

$$Y = Y_\infty - C_0\xi^{-\frac{2+6\sigma}{3+3\sigma}} \tag{4.104}$$

where:

$$Y_\infty = \lim_{\xi\to\infty} Y(\xi)$$

and C_0 are integration constants. Note that in [**6**] the constant k is used in place of C_0, however this may be slightly misleading as it could imply that the solution is asymptotic to FLRW$(k, \sigma, 1)$, which it is not. Equations (4.20) and (4.21) constrain C_0 and Y_∞ through:

$$C_0 = \frac{3(1+\sigma)(e^{-2Y_\infty} - e^{4Y_\infty})}{2(1+3\sigma)(5+3\sigma)} \tag{4.105}$$

thus making these asymptotic solutions a one-parameter family for each fixed σ. From (4.24):

$$\mathcal{E} = E + O\left(\xi^{-\frac{4+12\sigma}{3+3\sigma}}\right) \tag{4.106}$$

where the asymptotic energy E is given by:

$$E = \frac{1}{2}\left(e^{6Y_\infty} - 1\right)$$

It is clear that FLRW$(0, \sigma, 1)$ corresponds to the value $Y_\infty = 0$. From (4.105), (4.103) and (4.15), we can interpret Y_∞ as being a perturbation of the density, with $Y_\infty < 0$ corresponding to an overdense perturbation and $Y_\infty > 0$ corresponding to an underdense perturbation. Carr and Coley note that there is a one-parameter continuum of regular underdense solutions but the overdense solutions only lie in discrete bands. Through a numerical analysis, the behaviour of these solutions can be summarised as so:

(1) If Y_∞ lies in the interval $(-\infty, Y_\infty^{\text{crit}})$ for some $Y_\infty^{\text{crit}} < 0$, then as ξ decreases, V reaches a minimum value $V_{\text{min}} > \sqrt{\sigma}$ before rising again to infinity. The value of ξ for which $V = V_{\text{min}}$ corresponds to a singularity. These solutions are thus supersonic and contain black holes which grow as fast as the Universe.

(2) If Y_∞ lies in the subinterval $(Y_\infty^*, Y_\infty^{\text{crit}})$, for some $Y_\infty^* < Y_\infty^{\text{crit}}$, then $V_{\text{min}} < 1$ and there is an event and particle horizon.

(3) If Y_∞ lies in the subinterval $(-\infty, Y_\infty^*)$, then $V_{\text{min}} > 1$ and the whole universe is inside a black hole.

(4) If Y_∞ lies in the interval $(Y_\infty^{\text{crit}}, \infty)$, then the solutions hit the sonic surface. The solutions will be regular only if they hit the sonic line.

We will now consider solutions that are asymptotically Friedmann as $\xi \to 0$, that is, as $t \to \infty$ for fixed r or as $r \to 0$ for fixed t. Goliath, Nilsson and Uggla [**10**] describe solutions that are asymptotically Friedmann in the limit $\xi \to 0$ as solutions that have a *regular centre*. This description is consistent with the regularity assumption made by Smoller and Temple [**21**], as we will see in Chapter 5. Since we need $\mathscr{R} \to \infty$ and $V \to 0$ in the limit $t \to \infty$, then X and Y need to be finite at $\xi = 0$, so:

$$\dot{X}(0) = \dot{Y}(0) = 0 \tag{4.107}$$

Carr and Coley note that $m/r = M\mathscr{R}$ must be finite in the limit $r \to 0$, so we must have $M(0) = 0$. This distinguishes these solutions from the static case where $m/r \to \infty$. Now from equation (4.21):

$$X(0) = 3\sigma Y(0) \tag{4.108}$$

so as like in the $\xi \to \infty$ case, these solutions form a one-parameter family. Also similar to the $\xi \to \infty$ case, the constant $X(0)$, which we will denote by the parameter X_0, determines whether the asymptotic solution is underdense or overdense compared to FLRW$(0, \sigma, 1)$. We see this from (4.15), (4.96) and (4.97), as this gives:

$$X_0 = \frac{\sigma}{1+\sigma} \ln\left(\frac{\rho_F(0)}{\rho(0)}\right) \tag{4.109}$$

where ρ_F denotes the density of FLRW$(0, \sigma, 1)$. So as like for Y_∞, we have that $X_0 > 0$ and $X_0 < 0$ correspond to underdense and overdense perturbations respectively. We now discuss Carr and Coley's numerical analysis of the $\xi \to 0$ solutions, noting that this will be of particular relevance to a similar analysis conducted in Chapter 6.

(1) For some range of values of X_0, the solutions must hit the sonic surface since the solution with $X_0 = 0$ does, with regular solutions hitting the surface on the sonic line. The point of intersection will be denoted by ξ_{AFS}.

(2) As X_0 decreases from positive values to some value $X_0^{\text{crit}} < 0$, we have that ξ_{AFS} decreases continuously to ξ_1. Solutions with $\xi_{AFS} > \xi_3$ are regular at the sonic point, whereas those with $\xi_1 < \xi_{AFS} < \xi_3$ are not.

(3) As X_0 decreases below X_0^{crit}, the $V(\xi)$ curves develop an inflection and ξ_{AFS} now increases to the value ξ_2. This is not a continuous increase, as not every value of ξ_{AFS} between ξ_1 and ξ_2 is attained. There is a small band of values within (ξ_1, ξ_2) for which the solutions are C^1.

(4) As X_0 decreases further, ξ_{AFS} increases and decreases within (ξ_1, ξ_2) and the $V(\xi)$ curves increasingly oscillate. These solutions can be grouped according to the number of oscillations, with each group containing a small band of solutions that are C^1, with one of these being analytic.

Carr and Coley remark that the oscillating groups also arise in the Newtonian case, as investigated by Whitworth and Summers [**24**]. They also mention that the non-analytic solutions may all be unstable to what is termed the *kink instability* and form shocks. They refer the reader to Amos and Pirin [**15**] for further details. It is also interesting to note that all overdense solutions are nearly static close to the sonic point, with this approximation worsening closer to the origin. However, the greater the number of oscillations, the better the approximation is closer to the origin.

Now if we denote the ξ value of intersection with the sonic surface in the $\xi \to \infty$ case by ξ_{AFL}, then we can construct sound waves, that is, transonic solutions, providing $\xi_1 < \xi_{AFS} = \xi_{AFL} < \xi_2$ or $\xi_{AFS} = \xi_{AFL} > \xi_3$. We would also expect the value of $\dot{X}$, which corresponds to the density and velocity gradient, to be continuous at the sonic point. Carr and Coley interpret such solutions as density perturbations that grow at the same rate as the Universe. Note that for $\xi_{AFL} = \xi_S$, there exists a supersonic asymptotically Friedmann solution that can be attached to TOV(σ) on the sonic surface for each σ. The same can be done for a subsonic asymptotically Friedmann solution.

Carr and Coley remark that attention originally focussed on solutions containing black holes because there was interest in whether black holes could grow at the same rate as the particle horizon. Carr and Hawking [**7**] have shown that such solutions exist in the case of pure radiation and dust but only if the Universe is asymptotically rather than exactly Friedmann, that is, there is no solution that can be formed by attaching a black hole to FLRW$(0, \sigma, 1)$ by a sound wave. Carr and Coley go on to state that this has the important implication that black holes formed through purely local processes cannot grow as fast as the Universe. Carr in his PhD Thesis, and Bicknell and Henriksen in [**1**], extended the aforementioned result to a general $0 < \sigma < 1$, while Lin et al. [**14**], and Bicknell and Henriksen [**2**], considered the case of a stiff perfect fluid, that is, one for which $\sigma = 1$.

4.5. Asymptotically Kantowski-Sachs Solutions

In this section we follow the work of Carr and Koutras [**8**] by considering solutions which are asymptotically KS(σ) for large and small values of ξ. We will refer to these solutions as being *asymptotically Kantowski-Sachs* and proceed similarly to Section 4.4 by introducing functions $X(\xi)$

and $Y(\xi)$ defined by:

$$x = x_0 \xi^{-\frac{2\sigma}{1+\sigma}} e^X \tag{4.110}$$

$$\mathscr{R} = \mathscr{R}_0 \xi^{-1} e^Y \tag{4.111}$$

where x_0 and $\mathscr{R}_0$ are given by (4.57) and (4.58). As like in Section 4.4, we will also assume $\xi > 0$. We begin by substituting (4.110) and (4.111) into equations (4.18)-(4.21) to form ODE in the variables X and Y. Unlike in Section 4.4, we find the leading order solutions by linearising these ODE in the limit $|V| \to \infty$. We will then do the same in the limit $|V| \to 0$. These solutions will be referred to as the supersonic and subsonic solutions respectively. In this light, the linearisation of equations (4.18) and (4.19) in the limit $|V| \to \infty$ are given by:

$$\ddot{Y} = -\dot{X} + \frac{1+3\sigma}{1+\sigma}\dot{Y} \tag{4.112}$$

$$\dot{Y} = \frac{1}{2\sigma}\dot{X} + \frac{1-\sigma}{1+\sigma}X \tag{4.113}$$

Differentiating (4.113) and eliminating $\dot{Y}$ and $\ddot{Y}$ gives us a single second-order ODE for X as so:

$$\ddot{X} + \frac{\sigma-1}{\sigma+1}\dot{X} - \frac{2\sigma(1+3\sigma)(1-\sigma)}{(1+\sigma)^2} = 0 \tag{4.114}$$

Equation (4.114) has two solutions and these are given by:

$$X \propto \xi^{-p_+}$$

$$X \propto \xi^{-p_-}$$

where:

$$p_\pm = \frac{-1+\sigma \pm \sqrt{(1-\sigma)\,(1+7\sigma+24\sigma^2)}}{2(1+\sigma)} \tag{4.115}$$

Thus the general solution takes the form:

$$X = X_\infty \xi^{-p_\pm} \tag{4.116}$$

$$Y = X_\infty \left(\frac{1}{2\sigma} - \left(\frac{1-\sigma}{1+\sigma}\right)\frac{1}{p_\pm}\right)\xi^{-p_\pm} \tag{4.117}$$

where X_∞ is a constant of integration. Equations (4.20) and (4.21) fix the integration constant of Y in (4.117) to be zero, thus these asymptotically Kantowski-Sachs solutions are at most a one-parameter family. The sign of p is determined by σ as the following cases demonstrate.

(1) For $-1 < \sigma < -\frac{1}{3}$, KS($\sigma$) has $|V| \to \infty$ as $\xi \to 0$, so the negative root p_- must be chosen.

(2) For $-\frac{1}{3} < \sigma < 0$, KS($\sigma$) has $|V| \to \infty$ as $\xi \to \infty$ but both p_+ and p_- are negative, so there is no solution as $\xi \to \infty$.

(3) For $0 < \sigma < 1$, KS(σ) has $|V| \to \infty$ as $\xi \to \infty$, so the positive root p_+ must be chosen.

By Proposition 4.3.1, we have that the KS(σ) solutions with $-\frac{1}{3} < \sigma < 0$ correspond to static solutions with $0 < \sigma < 1$ if t and r (and hence ξ and ξ^{-1}) are interchanged. This corresponds to the fact that there are no asymptotically static solutions as $\xi \to 0$ for $0 < \sigma < 1$, as we will see in the next section.

We now consider the limit $|V| \to 0$, which corresponds to $\xi \to 0$ for $0 < \sigma < 1$. In this limit, (4.20), (4.21) and the condition $\dot{X}(0) = \dot{Y}(0) = 0$ imply that X and Y tend to constants which are related by:

$$e^{2Y_0} = \frac{1}{2}\left[M_{KS}e^{-\frac{X_0(1+\sigma)}{\sigma}} - \left(M_{KS} - \frac{1}{2}\right)e^{-2X_0}\right]^{-1} \tag{4.118}$$

where M_{KS} is defined by (4.67). Both the supersonic and subsonic solutions only depend on a single parameter, thus the asymptotically Kantowski-Sachs solutions are a one-parameter family. As like for the asymptotically Friedmann parameters, X_0 and X_∞ can be interpreted as density perturbations. Carr and Coley remark that it is possible to show that there are only isolated solutions at a sonic point when $0 < \sigma < 1$, so any asymptotically Kantowski-Sachs solution which hits the sonic surface is unlikely to be regular there. As like for the supersonic solutions, the behaviour of the subsonic solutions depends on the value of σ, as the following cases consider.

(1) For $-1 < \sigma < -\frac{1}{3}$, we have $V \to 0$ as $\xi \to \infty$, so as before, there is a one-parameter family of solutions.

(2) For $-\frac{1}{3} < \sigma < 0$, there is a two-parameter family of solutions as $\xi \to 0$ related to the two-parameter family of asymptotically quasi-static solutions with $0 < \sigma < 1$ as $\xi \to \infty$.

(3) For $0 < \sigma < 1$, we will consider this case for the remainder of the section.

We now follow [**6**] closely in describing the behaviour of the supersonic solutions with $V < -\sqrt{\sigma}$. In this case the underdense solutions have X_∞ positive. As ξ decreases from infinity, all solutions cross $V = -1$ with a smaller value of ξ than KS(σ). These solutions reach a maximum between $V = -\sqrt{\sigma}$ and $V = -1$ and so do not hit the sonic point. As ξ continues to decreases they hit the $V = -1$ surface again, all with $\dot{V} = 1$ and the same value of ξ, with M and ρ tending to zero. Carr and Coley remark that this behaviour is analogous to that which arises for the solutions which are asymptotically Minkowski at finite ξ, as we will see in Section 4.7. The overdense supersonic solutions have X_∞ negative and, as ξ decreases, all hit the sonic line with a larger value of ξ than KS(σ). As X_∞ decreases, the point at which these solutions hit the sonic line tends to infinity. All the supersonic solutions have $M < 0$ and as $\xi \to \infty$, both X and Y tend to 0, meaning that V tends to the exact KS(σ) form.

We now consider the behaviour of the subsonic solutions with $-\sqrt{\sigma} < V < 0$. For the overdense solutions, X_0 is negative and none of the solutions hit the sonic surface. The solutions reach a minimum as ξ decreases and then asymptotically approach $V = 0$. Carr and Coley remark that an interesting feature of these solutions is that M, which is negative at the origin, goes through zero and becomes positive as ξ increases. As for the underdense solutions, they have X_0 positive and hit the sonic line with a smaller value of ξ than KS(σ).

4.6. Asymptotically Quasi-Static Solutions

In this section we consider solutions which are *asymptotically static* and *asymptotically quasi-static*. We will proceed similarly to Sections 4.4 and 4.5 by introducing functions $X(\xi)$ and $Y(\xi)$ defined by:

$$x = x_0 e^X \tag{4.119}$$

$$\mathscr{R} = \mathscr{R}_0 e^Y \tag{4.120}$$

where x_0 and $\mathscr{R}_0$ are given by (4.78) and (4.79). We will begin by assuming $\xi > 0$, but for a full description of the solutions, we will need to consider the $\xi < 0$ case as well. As like in the previous two sections, we begin by substituting (4.110) and (4.111) into equations (4.18)-(4.21) to form ODE in the variables X and Y. Equations (4.18) and (4.19) become:

$$\ddot{Y} + 3\dot{Y}^2 - \frac{1}{\sigma}\dot{X} + \frac{\sigma+3}{\sigma+1}\dot{Y} - \frac{1+\sigma}{\sigma}\dot{X}\dot{Y} = 0 \tag{4.121}$$

$$\left(\dot{Y} - \frac{1}{2\sigma}\dot{X}\right)V^2 + \frac{1}{2}\dot{X} - \frac{\sigma}{1+\sigma}\left(e^{-4Y+\frac{1-\sigma}{\sigma}X} - 1\right) = 0 \tag{4.122}$$

As like in Section 4.5, to find the leading order solution of (4.121) and (4.122) as $V \to \infty$, that is, as $\xi \to \infty$, we linearise these equation in this limit to yield:

$$\ddot{Y} = \frac{1}{\sigma}\dot{X} - \frac{\sigma+3}{\sigma+1}\dot{Y} \tag{4.123}$$

$$\dot{Y} = \frac{1}{2\sigma}\dot{X} \tag{4.124}$$

By eliminating $\dot{X}$ from (4.123) we obtain a single second-order ODE for Y given by:

$$\ddot{Y} + \frac{1-\sigma}{1+\sigma}\dot{Y} = 0 \tag{4.125}$$

and we can solve this equation to obtain:

$$X = X_\infty + C_1\xi^{-\frac{1-\sigma}{1+\sigma}} \tag{4.126}$$

$$Y = Y_\infty + \frac{C_1}{2\sigma}\xi^{-\frac{1-\sigma}{1+\sigma}} \tag{4.127}$$

where X_∞, Y_∞ and C_1 are constants of integration. Equations (4.20) and (4.21) place a single constraint on these constants, which is given as so:

$$C_1 \propto \left(e^{2Y_\infty - \frac{1+\sigma}{\sigma}X_\infty} - \frac{1+6\sigma+\sigma^2}{4\sigma} + \frac{(1+\sigma)^2}{4\sigma}e^{6Y_\infty - \frac{2}{\sigma}X_\infty}\right)^{\frac{1}{2}} e^{X_\infty - Y_\infty} \tag{4.128}$$

where the proportional symbol indicates that we have omitted a coefficient which depends only on σ. Notice that (4.128) implies $C_1 = 0$ when X_∞ and Y_∞ are, which corresponds to TOV(σ). Thus these asymptotically solutions form a two-parameter family. As like for asymptotically Kantowski-Sachs solutions, the parameter X_∞ measures the asymptotic density perturbation relative to TOV(σ), with $X_\infty > 0$ corresponding to underdense solutions and $X_\infty < 0$ corresponding to overdense solutions. The second parameter Y_∞ specifies the asymptotic value of the scale factor relative to its value in TOV(σ). Because relation (4.128) contains a square root with two possible signs, there will be two solutions for a given value of Y_∞. However, for large ξ, (4.23), (4.126) and (4.127) imply:

$$V_R \approx -\frac{(1-\sigma)C_1}{2\sigma(1+\sigma)} \tag{4.129}$$

and since TOV(σ) satisfies $V_R = 0$, then this distinguishes $C_1 = 0$ as the one-parameter subset that most closely approximates TOV(σ).

DEFINITION 4.6.1. The one-parameter subset of solutions corresponding to $C_1 = 0$ define the family of *asymptotically static* solutions, whereas the general two-parameter family with $C_1 \neq 0$ define the family of *asymptotically quasi-static* solutions.

From (4.119) and (4.126) we see that asymptotically quasi-static solutions exhibit an isothermal density profile at large ξ, that is, ρr^2 is constant at large ξ. Carr and Coley remark that the behaviour of asymptotically quasi-static solutions at large ξ is similar to the behaviour exhibited in the dust case, where the solutions are also described by two parameters. The first parameter in the dust case is related to the asymptotic energy E, which we will now specify in terms of X_∞ and Y_∞. For large ξ, (4.23), (4.126) and (4.127) imply:

$$\mathcal{E} = \frac{(1+\sigma)^2}{2\left(1+6\sigma+\sigma^2\right)} e^{6Y_\infty - \frac{2}{\sigma}X_\infty} \left(1 - \frac{C_1(3-\sigma)}{\sigma(1+\sigma)}\xi^{-\frac{1-\sigma}{1+\sigma}}\right) - \frac{1}{2} \tag{4.130}$$

and so we can deduce that:

$$E = \frac{(1+\sigma)^2}{2\left(1+6\sigma+\sigma^2\right)} e^{6Y_\infty - \frac{2}{\sigma}X_\infty} - \frac{1}{2} \tag{4.131}$$

This relation provides an explicit correspondence between one of the parameters in the dust case and the parameters of asymptotic quasi-static solutions. The second parameter in the dust case corresponds to the value of ξ associated with a big bang or big crunch singularity, that is, $D = |\xi|^{-1}$, where the singularity occurs at ξ. Except for TOV(σ), where $D = \infty$, Carr and Coley note that D can only be determined numerically when there is pressure, and so this parameter cannot be given explicitly in terms of X_∞ and Y_∞. However, the second parameter in the dust case can be associated with the asymptotic value of V_R, given in terms of C_1 by (4.129), although this implicit relationship is complicated.

We now turn to deriving solutions asymptotic to TOV(σ) as $\xi \to 0$. This corresponds to solutions for which $V \to 0$ and X and Y are finite at $\xi = 0$, that is, $\dot{X}(0) = \dot{Y}(0) = 0$. Now equation (4.23) implies that these solutions satisfy $V_R = 0$ and this in turn implies $\dot{X} = \dot{Y} = 0$. However, coupling this with (4.122) gives:

$$4Y_0 = \frac{1-\sigma}{\sigma} X_0 \tag{4.132}$$

with equations (4.20) and (4.21) then implying $X_0 = Y_0 = 0$, which corresponds to the explicit TOV(σ) solution. This means there are no solutions asymptotic to TOV(σ) as $\xi \to 0$ except the TOV(σ) solution itself. Carr and Coley note that if solutions are instead considered for which $V \to 0$ and $\dot{X}(0)$ and $\dot{Y}(0)$ are finite and non-zero, then (4.122) gives:

$$\dot{X} = -\frac{2\sigma}{1+\sigma} \tag{4.133}$$

and substituting (4.133) into (4.121) yields:

$$(3+3\sigma)\dot{Y}^2 + (5+3\sigma)\dot{Y} + 2 = 0 \tag{4.134}$$

The solutions of this quadratic equation are given by:

$$\dot{Y} = -\frac{2}{3+3\sigma} \tag{4.135}$$

$$\dot{Y} = -1 \tag{4.136}$$

however, these solutions correspond to asymptotic solutions that we have already considered in Sections 4.4 and 4.5.

We will now describe the physical nature of asymptotically static and quasi-static solutions based on their parameter values D and E. Note that the asymptotically static solutions can be described in terms of the single parameter E_{sym}. As briefly mentioned at the beginning of this section, the behaviour of these asymptotic solutions necessarily span both positive and negative ξ. All solutions that are not time reversed correspond to cosmological models which start off expanding from a big bang singularity at $\xi = -D^{-1}$, then tend to the asymptotically quasi-static form as $\xi \to -\infty$ and then cross over to $\xi = \infty$.

(1) If $E < E_{\text{crit}}(D)$ for some $E_{\text{crit}}(D)$, then these solutions recollapse to another singularity.
(2) If $E_*(D) < E < E_{\text{crit}}(D)$ for some $E_*(D)$, then there is an event and particle horizon.
(3) If $E > E_{\text{crit}}(D)$, then these solutions expand forever and hit the sonic surface.

The remaining solutions are the time reverse of these and all collapse to a final big crunch singularity singularity at $\xi = D^{-1}$.

(1) If $E > E_+(D)$ for some $E_+(D) > E_{\text{crit}}(D)$ then this singularity is a black hole.
(2) If $E < E_+(D)$ then this singularity is naked.

These solutions may start off either expanding from a white hole or collapsing from infinity. For asymptotically static solutions, the second singularity is given by $\xi = D^{-1}$ and the time reversed and unreversed solutions coincide. Carr and Coley remark that the $\xi < 0$ solutions can be obtained from the $\xi > 0$ solutions by reflection, that is, the time reversed solutions give complete information about the unreversed solutions and vice versa. However, it is interesting to note that both sets of solutions are needed to follow the trajectory of a solution that passes between these sets. Because of the square root in relation (4.128), there are two trajectories for each asymptotic value of $\mathscr{R}$, which results in two values of its derivative. In order for a solution to pass from $\xi = -\infty$ to $\xi = \infty$ the derivative must be preserved.

We will now describe the evolution of $V(\xi)$ for the asymptotically static and quasi-static solutions. We begin with the ever collapsing solutions, for which the evolution of $V(\xi)$ is described by the following points.

(1) These solutions start with $V(0) = 0$.
(2) As ξ decreases, $V(\xi)$ passes through the sonic surface $V = -\sqrt{\sigma}$.
(3) As ξ continues to decrease, $V(\xi)$ passes through the Cauchy horizon $V = -1$.

(4) As $\xi \to -\infty$, $V(\xi)$ tends to the quasi-static form at $\xi = -\infty$.

(5) The solutions then jump to $\xi = \infty$ and enter the $\xi > 0$ regime.

(6) As ξ decreases again, $V(\xi)$ reaches a minimum, denoted by $V_{\rm min}$.

(7) If $E > E_+(D)$ then $V_{\rm min} > 1$.

(8) If $E < E_+(D)$ then $V_{\rm min} < 1$.

(9) If $V_{\rm min} < \sqrt{\sigma}$ then the solutions would need to have a second sonic point, however, it is unlikely that such solutions would be regular at this second point, so these solutions are not expected to exist.

(10) As $\xi \to D^{-1}$, $V(\xi) \to \infty$ as it encounters the big crunch singularity.

The evolution of $V(\xi)$ for the ever-expanding solutions is then just the time reverse of its evolution for the ever-collapsing ones. We now consider the evolution of $V(\xi)$ for the expanding-recollapsing solutions, which arise if $E < E_{\rm crit}$. There are two solutions of this kind, the first of which is described by the following points.

(1) These solutions start with $V(\xi)$ at $-\infty$ when $\xi = -D^{-1}$.

(2) As ξ increases, $V(\xi)$ reaches a maximum, denoted by $V_{\rm max}$.

(3) If $E > E_+(D)$ then $V_{\rm max} < -1$.

(4) If $E < E_+(D)$ then $V_{\rm max} > -1$.

(5) As $\xi \to \infty$, $V(\xi)$tends to the quasi-static form at $\xi = \infty$.

(6) The solutions then jump to $\xi = -\infty$ and enter the $\xi < 0$ regime.

(7) As ξ increases again, $V(\xi)$ reaches a minimum, denoted by $V_{\rm min}$.

(8) If $E > E_*(D)$ then $V_{\rm min} < 1$.

(9) If $E < E_*(D)$ then $V_{\rm min} > 1$.

(10) As $\xi \to \xi_s$, where ξ_s denotes the singularity, then $V(\xi) \to \infty$.

For reasons indicated previously, it is likely that the maximum is less than $-\sqrt{\sigma}$ and the minimum is more than $\sqrt{\sigma}$, in which case these solutions have no sonic points. The second kind of expanding-recollapsing solution is the time reverse of the first and starts from $\xi = -\xi_s$ and ends at $\xi = D^{-1}$. In such a case there is a naked singularity at $\xi = D^{-1}$ if $E < E_+(D)$.

4.7. Asymptotically Minkowski Solutions

In this section we consider solutions which are *asymptotically Minkowski.* In the previous sections the asymptotic solutions are derived by perturbing an explicitly known self-similar perfect fluid solution. However, even though we know the Minkowski solution explicitly, it is not a perfect fluid solution. Instead, the asymptotically Minkowski solutions are found by considering a logarithmic power-law expansion. If we recall from Section 4.1, a logarithmic power-law expansion is one of the two possible expansions that completely classify all spherically symmetric self-similar perfect fluid solutions of the first kind. So in this light, and by analogy with (4.29) and (4.30), solutions will be sought with the following form:

$$x = x_0 |L|^a \tag{4.137}$$

$$\mathscr{R} = \mathscr{R}_0 |L|^b \tag{4.138}$$

where $\xi \to \xi_*$ and:

$$L = \ln\left(\frac{\xi}{\xi_*}\right)$$

The modulus signs are put in (4.137) and (4.138) as L may be negative and have fractional exponents. Since the circumferential coordinate is given by $\bar{r} = \mathscr{R} r$, then $\xi = \xi_*$ corresponds to an infinite distance from the origin for $b < 0$ and corresponds to zero distance for $b > 0$. Now substituting (4.137) and (4.138) into equation (4.18) leads to:

$$b\left[3b - 1 - \left(\frac{1+\sigma}{\sigma}\right)a\right] + \left[\left(\frac{3+\sigma}{1+\sigma}\right)b - \frac{a}{\sigma}\right]L = 0 \tag{4.139}$$

with the first term being zero for:

$$b = \frac{1}{3} + \left(\frac{1+\sigma}{3\sigma}\right)a \tag{4.140}$$

Carr and Coley remark that the second term in (4.139) cannot be zero, meaning that there cannot be any explicit solutions for our ansatz. There are now two possible cases, which are based on the asymptotic value of V, these are:

(1) $V \to V_*$ as $\xi \to \xi_*$.

(2) $V \to \infty$ as $\xi \to \xi_*$.

For Case 1, equation (4.19) can be written as:

$$aL^{-1} = \frac{2\sigma b V^2}{V^2 - \sigma} L^{-1} + \frac{2\sigma^2}{(1+\sigma)\left(V_*^2 - \sigma\right)} - \frac{\sigma(1+\sigma)\beta^2 V_*^2}{V_*^2 - \sigma} \xi_*^{-\frac{2-2\sigma}{1+\sigma}} \left(x_0 |L|^a\right)^{\frac{\sigma-1}{\sigma}} \tag{4.141}$$

where the first term on the right hand side does not substitute V for V_* because $(V^2 - V_*^2)L^{-1}$ may become constant in the limit $\xi \to \xi_*$. The only way to obtain a solution from (4.141) is to set:

$$a = \left(\frac{2\sigma V_*^2}{V_*^2 - \sigma}\right) b \tag{4.142}$$

and since (4.22) requires:

$$a = \left(\frac{2\sigma}{1-\sigma}\right) b \tag{4.143}$$

then this implies $V_*^2 = 1$. In addition, (4.140) and (4.143) now form two equations for a and b. Solving these equations and substituting a and b back into our ansatz thus yields:

$$x \approx x_0 |L|^{\frac{2\sigma}{1-5\sigma}} \tag{4.144}$$

$$\mathscr{R} \approx \mathscr{R}_0 |L|^{\frac{1-\sigma}{1-5\sigma}} \tag{4.145}$$

The requirement that $V_* = 1$ imposes a constraint on the constants $x_0, \mathscr{R}_0$ and ξ_* by virtue of equation (4.22). This means these solutions form a two-parameter family. With $V_* = 1$ then equation (4.20) implies:

$$M \sim \frac{1-\sigma}{5\sigma - 1} |L|^{\frac{1-\sigma}{5\sigma-1}} \tag{4.146}$$

The coefficient is included to show that the mass is negative when $\sigma < \frac{1}{5}$. Thus only solutions which have $\sigma > \frac{1}{5}$ are physical. Carr and Coley remark that ξ is required to tend to ξ_* from below in order to keep the mass positive. Now equation (4.21) can be written as:

$$M = \frac{1}{2} + \frac{1}{2}\gamma^2 x^{-\frac{2}{\sigma}} \mathscr{R}^6 \left[\left(\frac{\dot{\mathscr{R}}}{\mathscr{R}}\right)^2 \left(V^2 - 1\right) - \frac{2\dot{\mathscr{R}}}{\mathscr{R}} - 1\right] \tag{4.147}$$

and since:

$$x^{-\frac{2}{\sigma}}\mathscr{R}^6 \sim |L|^{\frac{6\sigma-2}{5\sigma-1}}$$

goes to infinity for $\sigma < \frac{1}{3}$ and zero for $\sigma > \frac{1}{3}$, then we need the term in square brackets in (4.147) to go to zero and infinity in these cases respectively. The second term on the right hand side of (4.147) can be rewritten as:

$$\sim |L|^{\frac{\sigma-1}{5\sigma-1}} \left[\frac{\dot{\mathscr{R}}}{\mathscr{R}}\left(V^2-1\right) - 2 - \frac{\mathscr{R}}{\dot{\mathscr{R}}}\right] \tag{4.148}$$

and since the exponent of $|L|$ is negative, then the term in square brackets is required to go to zero and scale as:

$$\sim |L|^{\frac{1-\sigma}{5\sigma-1}}$$

As a result, we need:

$$\frac{\dot{\mathscr{R}}}{\mathscr{R}}\left(V^2-1\right) \to 2 \tag{4.149}$$

Using the approximation:

$$\frac{V^2}{V^2-\sigma} \approx \frac{1}{1-\sigma}\left[1 - \frac{\sigma}{1-\sigma}\left(V^2-1\right)\right] \tag{4.150}$$

equations (4.22) and (4.141) then tell us that:

$$\frac{\dot{V}}{V} \to \frac{1-5\sigma}{1-\sigma} < 0 \tag{4.151}$$

Because condition (4.149) depends on σ we will need to introduce higher order terms, that is, we consider:

$$x \approx x_0|L|^a\left(1 + C_A|L|^k + C_C L\right) \tag{4.152}$$

$$\mathscr{R} \approx \mathscr{R}_0|L|^b\left(1 + C_B|L|^k + C_D L\right) \tag{4.153}$$

where a and b are given by (4.144) and (4.145), and k, C_A, C_B, C_C and C_D are constants.

In this light, (4.146) and (4.147) imply:

$$O\left(|L|^{\frac{1-\sigma}{5\sigma-1}}\right) = 1 + O\left(|L|^{\frac{\sigma-1}{5\sigma-1}}\right)\left[\frac{\dot{\mathscr{R}}}{\mathscr{R}}\left(V^2-1\right) - 2 - \frac{\mathscr{R}}{\dot{\mathscr{R}}}\right] \tag{4.154}$$

and matching the exponents of $|L|$ means:

$$k = \frac{1-\sigma}{5\sigma-1}$$

Now:

(1) For $\sigma > \frac{1}{3}$, we have $k < 1$ and so the dominant term in (4.152) and (4.153) is L^k.

(2) For $\sigma < \frac{1}{3}$, we have $k > 1$ and so the dominant term in (4.152) and (4.153) is L.

In either case, substituting (4.152) and (4.153) into (4.154) enables us to determine C_A, C_B, C_C and C_D without placing a constraint on x_0, $\mathscr{R}_0$ and ξ_*. The resulting metric is then given by:

$$ds^2 \sim |L|^{\frac{4\sigma}{1-5\sigma}}\left(-dt^2 + dr^2 + |L|^{\frac{6\sigma-2}{5\sigma-1}} r^2 d\Omega^2\right) \tag{4.155}$$

We now consider Case 2, that is, the case in which $V \to \infty$ as $\xi \to \xi_*$. In this limit, equation (4.19) becomes:

$$aL^{-1} = 2\sigma b L^{-1} - \sigma(1+\sigma)\beta^2 \xi_*^{-\frac{2-2\sigma}{1+\sigma}} \left(x_0|L|^a\right)^{\frac{\sigma-1}{\sigma}} \tag{4.156}$$

In order to satisfy this equation, we will need all terms to scale as L^{-1}, which requires:

$$a = \frac{\sigma}{1-\sigma}$$

Relation (4.140) then gives:

$$b = \frac{2}{3-3\sigma}$$

and thus:

$$x \approx x_0|L|^{\frac{\sigma}{1-\sigma}} \tag{4.157}$$

$$\mathscr{R} \approx \mathscr{R}_0|L|^{\frac{2}{3-3\sigma}} \tag{4.158}$$

Now from equation (4.20) we have:

$$M \sim |L|^{-\frac{2}{3-3\sigma}} \tag{4.159}$$

and Carr and Coley note that in order for the mass to be positive, ξ must approach ξ_* from above. Since (4.156) gives the same relation between C_A, C_B and ξ_* as (4.20) and (4.21), then these solutions also depend on two parameters. Thus the asymptotically Minkowski solutions form a two-parameter family and the resulting metric in the $V \to \infty$ case is given by:

$$ds^2 \sim -|L|^{\frac{2\sigma}{1-\sigma}} dt^2 + |L|^{-\frac{2}{3-3\sigma}} dr^2 + |L|^{\frac{4}{3-3\sigma}} r^2 d\Omega^2 \tag{4.160}$$

We now provide physical insight into the asymptotically Minkowski solutions. Beginning with the $V \to V_*$ case, Carr and Coley note that the scalar curvature tends to zero as $\xi \to \xi_*$ providing $\frac{1}{5} < \sigma < 1$, that is, the physical solutions are flat on the surface $\xi = \xi_*$. This surface is null because $V_* = 1$. It turns out that although r tends to a finite value for finite t, this is just a coordinate anomaly because (4.144) and (4.145) tell us that the physical distance still diverges. We also note that (4.144) and (4.145) imply that the scale factor does not diverge at ξ_* in the limit $\sigma = 1$, although the density still tends to zero.

In the $V \to \infty$ case, the scale factor goes to zero and the density goes to infinity at ξ_*, which corresponds to a singularity at the physical origin. Carr and Coley note that this is a Schwarzschild-type singularity in the sense that $g_{00} \to 0$ and $g_{11} \to \infty$ as $\xi \to \xi_*$.

In both the $V \to V_*$ and $V \to \infty$ cases we cannot describe the solutions in terms of the parameter E, since the energy function diverges, that is, (4.24), (4.88), (4.89), (4.144) and (4.145) imply:

$$\mathcal{E} \sim \left|\ln\left|\frac{\xi}{\xi_*}\right|\right|^{\frac{4\sigma}{1-5\sigma}} \tag{4.161}$$

$$\mathcal{E} \sim \xi^{\frac{2V_*}{V_*^2-1}} \tag{4.162}$$

in the two cases. Note that double modulus signs are used in (4.161) since ξ is now permitted to be negative.

Carr and Coley remark that in both cases, there are solutions which collapse monotonically to a central singularity and solutions which collapse and then bounce into an expansion phase, with the latter hitting the sonic surface. As like for the other asymptotic solutions, the $\xi < 0$ solutions are simply the time reverse of the $\xi > 0$ solutions. Although asymptotically Minkowski solutions represent a large fraction of the complete solution space, Carr and Coley state at the time of writing that many of their features remain unexplored.

CHAPTER 5

General Relativistic Self-Similar Waves that Induce Cosmic Acceleration

This chapter summarises and extends the 2012 paper *General Relativistic Self-Similar Waves that Induce an Anomalous Acceleration into the Standard Model of Cosmology* by Smoller and Temple [**21**]. One of the extensions is to the derivation of the autonomous spherically symmetric self-similar in ξ Einstein field equations, which have now been derived for general σ. These equations, which form an autonomous system of nonlinear ODE, are central to the construction of the general relativistic shock waves considered in Chapter 6. As for the other extensions, Sections 5.2 and 5.3 have been added and corrections have been supplied to the Taylor expansions of the asymptotically Friedmann spacetimes in Section 5.4.

5.1. Autonomous Spherically Symmetric Self-Similar Einstein Field Equations

Before deriving the autonomous spherically symmetric self-similar in ξ perfect fluid Einstein field equations, let us revisit the flat FLRW spacetime, which represents our current Standard Model of Cosmology. In particular, we will consider this spacetime during the Radiation Dominated Epoch of the Universe, that is, when the spacetime is modelled with a pure radiation equation of state. We know from Chapter 2 that the flat FLRW metric takes the following form in comoving coordinates:

$$ds^2 = -dt^2 + R^2(t)\left(dr^2 + r^2 d\Omega^2\right) \tag{5.1}$$

Here, $R(t)$ is the cosmological scale factor and the variable $\bar{r} = R(t)r$ measures the arc length distance at fixed time t. We note that for spherically symmetric spacetimes, the coordinate system is comoving if the radial coordinate is constant along particle paths. Now substituting the metric (5.1) into the Einstein field equations (2.1) with a perfect fluid source (2.2) yields the following system of ODE:

$$H = \frac{\kappa}{3}\rho \tag{5.2}$$

$$\dot{\rho} = -3(\rho + p)H \tag{5.3}$$

where H is the Hubble constant, defined by:

$$H = \frac{\dot{R}}{R} \tag{5.4}$$

Recall from Section 3.8, that during the Radiation Dominated Epoch, the Stefan-Boltzmann radiation law implies the pure radiation equation of state:

$$p = \frac{c^2}{3}\rho \tag{5.5}$$

Explicit solutions of (5.2) and (5.3) are given in the following theorem from [**21**], which is a corollary of Theorem 2 from [**22**].

THEOREM 5.1.1. *Let (5.1) solve the perfect fluid Einstein field equations with a pure radiation equation of state and assume an expanding universe, that is,* $\dot{R} > 0$. *Then the solution of (5.2) and (5.3) satisfying* $R(0) = 0$ *and* $R(1) = 1$ *is given by:*

$$\rho(t) = \frac{3}{4\kappa t^2} \tag{5.6}$$

$$R(t) = t^{\frac{1}{2}} \tag{5.7}$$

Note that (5.7) implies the Hubble constant is given by:

$$H(t) = \frac{1}{2t} \tag{5.8}$$

and that H and $\bar{r}$ are scale independent relative to the scaling law:

$$r \to \alpha r$$

$$t \to \alpha t$$

for some positive constant α. The following theorem, also from [**21**], is a specific application of Proposition 3.6.1 from Section 3.6. However, Smoller and Temple were not aware of Cahill and Taub's work at the time.

THEOREM 5.1.2. *Assume $p = \frac{1}{3}\rho$ and $R(t) = \sqrt{t}$, then the FLRW metric (5.1) under the change of coordinates:*

$$\bar{t} = \Psi_0 \left(1 + \left[\frac{R(t)r}{2t}\right]^2\right) t \tag{5.9}$$

$$\bar{r} = R(t)r \tag{5.10}$$

transforms to the Schwarzschild coordinate metric:

$$d\bar{s}^2 = -B(\xi)d\bar{t}^2 + \frac{1}{A(\xi)}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \tag{5.11}$$

where:

$$\xi = \frac{\bar{r}}{\bar{t}} \tag{5.12}$$

and:

$$A = 1 - v^2 \tag{5.13}$$

$$B = \frac{1}{\Psi_0^2(1 - v^2)} \tag{5.14}$$

$$v = \frac{1}{\sqrt{AB}} \frac{\bar{u}^1}{\bar{u}^0} \tag{5.15}$$

Furthermore, the Schwarzschild coordinate velocity satisfies:

$$v = \frac{\zeta}{2} \tag{5.16}$$

$$\Psi_0 \xi = \frac{2v}{1 + v^2} \tag{5.17}$$

where Ψ_0 is an inessential parameter and:

$$\zeta = \frac{\bar{r}}{t} \tag{5.18}$$

Recall that $\bar{u}^0$ and $\bar{u}^1$ denote the $(\bar{t}, \bar{r})$ components of the normalised four-velocity $\boldsymbol{u}$ in Schwarzschild coordinates. Smoller and Temple remark that the constant Ψ_0 is included to later account for the time rescaling freedom in (5.11). The Jacobian and inverse Jacobian corresponding to the transformation (5.9) and (5.10) are given by:

$$J \equiv \frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}} = \begin{pmatrix} \Psi_0 & \Psi_0 \sqrt{t} \frac{\zeta}{2} \\ \frac{\zeta}{2} & \sqrt{t} \end{pmatrix} \tag{5.19}$$

$$J^{-1} \equiv \frac{\partial \boldsymbol{x}}{\partial \bar{\boldsymbol{x}}} = \frac{1}{|J|} \begin{pmatrix} \sqrt{t} & -\Psi_0 \sqrt{t} \frac{\zeta}{2} \\ -\frac{\zeta}{2} & \Psi_0 \end{pmatrix} \tag{5.20}$$

where:

$$|J| = \Psi_0 \sqrt{t} \left(1 - \frac{\zeta^2}{4} \right) \tag{5.21}$$

For metrics taking the Schwarzschild form (5.11), we will now show how the Einstein field equations reduce to a system of three ODE. This is in the same spirit as Cahill and Taub, except the equations will be derived using Schwarzschild coordinates. The use of Schwarzschild coordinates yields significant advantages when it comes to forming shock-wave solutions. Furthermore, these ODE will turn out to be autonomous, paving the way to a new phase space analysis that will be conducted in Chapter 6. In this light, by putting the Schwarzschild metric ansatz (5.11) into the Einstein field equations, the following four PDE are obtained:

$$-\bar{r} \frac{1}{A} \frac{\partial A}{\partial \bar{r}} + \frac{1-A}{A} = \kappa \frac{B}{A} \bar{r}^2 T^{00} \tag{5.22}$$

$$\bar{r} \frac{1}{A} \frac{\partial A}{\partial \bar{t}} = \kappa \frac{B}{A} \bar{r}^2 T^{01} \tag{5.23}$$

$$\bar{r} \frac{1}{B} \frac{\partial B}{\partial \bar{r}} - \frac{1-A}{A} = \kappa \frac{1}{A^2} \bar{r}^2 T^{11} \tag{5.24}$$

$$-\frac{\partial^2}{\partial \bar{t}^2} \left(\frac{1}{A} \right) + \frac{\partial^2}{\partial \bar{r}^2} B - \Theta = 2\kappa \frac{B}{A} \bar{r}^2 T^{22} \tag{5.25}$$

where:

$$\Theta = \frac{1}{2A^2 B} \frac{\partial B}{\partial \bar{t}} \frac{\partial A}{\partial \bar{t}} - \frac{1}{2A^3} \left(\frac{\partial A}{\partial \bar{t}} \right)^2 - \frac{1}{\bar{r}} \frac{\partial B}{\partial \bar{r}} - \frac{1}{\bar{r}} \frac{B}{A} \frac{\partial A}{\partial \bar{r}} + \frac{1}{2B} \left(\frac{\partial B}{\partial \bar{r}} \right)^2 - \frac{1}{2A} \frac{\partial B}{\partial \bar{r}} \frac{\partial A}{\partial \bar{r}}$$

The main purpose of this section is to prove the following theorem, which is a modification of Theorem 4 from [**21**]. The original theorem is given for a pure radiation equation of state, whereas this modified theorem generalises the result to equations of state of the form $p = \sigma\rho$. Theorem 5 of [**21**] demonstrates that FLRW$(0, \frac{1}{3}, 1)$ is a particular solution of equations (5.27)-(5.29), however this theorem is replaced by Proposition 5.3.2 and the proceeding remark given in Section 5.3.

THEOREM 5.1.3. *Let:*

$$G = \frac{\xi}{\sqrt{AB}} \tag{5.26}$$

and assume that $A(\xi)$, $G(\xi)$ and $v(\xi)$ solve the following ODE:

$$\xi\frac{dA}{d\xi} = -\frac{(3+3\sigma)(1-A)v}{\{\cdot\}_S} \tag{5.27}$$

$$\xi\frac{dG}{d\xi} = -G\left[\left(\frac{1-A}{A}\right)\frac{(3+3\sigma)[(1+v^2)G-2v]}{2\{\cdot\}_S} - 1\right] \tag{5.28}$$

$$\xi\frac{dv}{d\xi} = -\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\left[3\sigma\{\cdot\}_S + \left(\frac{1-A}{A}\right)\frac{(3+3\sigma)^2\{\cdot\}_N}{4\{\cdot\}_S}\right] \tag{5.29}$$

where:

$$\{\cdot\}_S = 3(G-v) - 3\sigma v(1-Gv) \tag{5.30}$$

$$\{\cdot\}_N = -3(G-v)^2 + 3\sigma v^2(1-Gv)^2 \tag{5.31}$$

$$\{\cdot\}_D = \frac{3}{4}(3+3\sigma)\left[(G-v)^2 - \sigma(1-Gv)^2\right] \tag{5.32}$$

and the density is given by the constraint:

$$\kappa\rho\bar{r}^2 = \frac{3(1-v^2)(1-A)G}{\{\cdot\}_S} \tag{5.33}$$

Then the metric:

$$d\bar{s}^2 = -B(\xi)d\bar{t}^2 + \frac{1}{A(\xi)}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \tag{5.34}$$

with Schwarzschild coordinate velocity v and equation of state $p = \sigma\rho$ solves the perfect fluid Einstein field equations.

Note that under the change of variable:

$$\xi = e^s \tag{5.35}$$

the equations become explicitly autonomous, since:

$$\xi \frac{d}{d\xi} = \frac{d}{ds} \tag{5.36}$$

The autonomous nature of these equations distinguish them from the self-similar ODE derived by Cahill and Taub. It is also worth noting that the variable G is the Schwarzschild coordinate analogue of the variable V introduced by Cahill and Taub and used throughout Chapters 3 and 4.

Proof of Theorem 5.1.3. In a previous paper by Groah and Temple [**12**], it was shown for smooth solutions that (5.22)-(5.25) are equivalent to (5.22)-(5.24) together with $\nabla_\mu T^{\mu 1} = 0$, with the latter equation taking the following Schwarzschild coordinate form:

$$\begin{aligned} &\frac{\partial}{\partial \bar{t}} T_M^{01} + \frac{\partial}{\partial \bar{r}} \left(\sqrt{AB} T_M^{11} \right) \\ &\quad = -\frac{1}{2}\sqrt{AB} \left(\frac{4}{\bar{r}} T_M^{11} + \frac{1}{\bar{r}} \frac{1-A}{A} (T_M^{00} - T_M^{11}) + 2\kappa \bar{r} \frac{1}{A} (T_M^{00} T_M^{11} - (T_M^{01})^2 - 4\bar{r} T_M^{22}) \right) \end{aligned} \tag{5.37}$$

where the Minkowski stresses $T_M^{\mu\nu}$ are defined by:

$$\begin{aligned} T_M^{00} &= BT^{00} = (\rho + p)\frac{1}{1-v^2} - p \\ T_M^{01} &= \sqrt{\frac{B}{A}} T^{01} = (\rho + p)\frac{v}{1-v^2} \\ T_M^{11} &= \frac{1}{A} T^{11} = (\rho + p)\frac{v^2}{1-v^2} + p \end{aligned} \tag{5.38}$$

and:

$$T_M^{22} = T^{22} = \sigma \rho \bar{r}^2 \tag{5.39}$$

In this light, we need to show equations (5.22)-(5.24) and (5.37) close and reduce to equations (5.27)-(5.29) with constraint (5.33). Smoller and Temple's strategy is to show that when $A(\xi)$, $G(\xi)$ and $v(\xi)$ are substituted into (5.22)-(5.24) and (5.37), all terms not depending on ξ can be written in the form $\bar{r}^2 T_M^{\mu\nu}$, which are be shown to be of the form $\rho\bar{r}^2$ multiplied by functions of the velocity. We will follow this strategy and start by substituting $A(\xi)$, $G(\xi)$ and $v(\xi)$ into (5.22)-(5.24). By defining:

$$S^{\mu\nu} = \kappa\bar{r}^2 T_M^{\mu\nu} \tag{5.40}$$

we can write (5.22)-(5.24) as:

$$\xi\frac{dA}{d\xi} = 1 - A - S^{00} \tag{5.41}$$

$$\xi\frac{dA}{d\xi} = -\frac{1}{G}S^{01} \tag{5.42}$$

$$\xi\frac{1}{B}\frac{dB}{d\xi} = \frac{1}{A}(1 - A + S^{11}) \tag{5.43}$$

where:

$$S^{00} = \kappa\rho\bar{r}^2\frac{1+\sigma v^2}{1-v^2} = \kappa\left(\frac{\rho\bar{r}^2}{3(1-v^2)}\right)(3+3\sigma v^2) \tag{5.44}$$

$$S^{01} = \kappa\rho\bar{r}^2\frac{1+\sigma}{1-v^2}v = \kappa\left(\frac{\rho\bar{r}^2}{3(1-v^2)}\right)(3+3\sigma)v \tag{5.45}$$

$$S^{11} = \kappa\rho\bar{r}^2\frac{\sigma+v^2}{1-v^2} = \kappa\left(\frac{\rho\bar{r}^2}{3(1-v^2)}\right)(3\sigma+3v^2) \tag{5.46}$$

In addition, for $\mu, \nu \in \{0, 1\}$ we define:

$$S^{\mu\nu} = \kappa w V^{\mu\nu} \tag{5.47}$$

$$S^{22} = \frac{1}{\bar{r}^2}\kappa w V^{22} \tag{5.48}$$

where:

$$w = \frac{\rho\bar{r}^2}{3(1-v^2)} \tag{5.49}$$

and:

$$V^{00} = 3 + 3\sigma v^2 \tag{5.50}$$

$$V^{01} = (3 + 3\sigma)v \tag{5.51}$$

$$V^{11} = 3\sigma + 3v^2 \tag{5.52}$$

$$V^{22} = 3\sigma(1 - v^2) \tag{5.53}$$

Now equating (5.41) and (5.42) implies:

$$G(1 - A) - GS^{00} = -S^{01} \tag{5.54}$$

and substituting (5.44) and (5.45) into (5.54) then yields the constraint (5.33). Furthermore, by applying (5.49) to (5.33) we obtain the equivalent constraint:

$$\kappa w = \frac{(1 - A)G}{\{\cdot\}_S} \tag{5.55}$$

Using (5.55), κw can be eliminated from (5.47), and thus from (5.41)-(5.43). In this light, substituting (5.55) into (5.47) yields:

$$S^{\mu\nu} = \frac{(1 - A)G}{\{\cdot\}_S} V^{\mu\nu} \tag{5.56}$$

and so (5.44)-(5.46) can be written independent of w and $\rho\bar{r}^2$ as so:

$$S^{00} = \frac{(1 - A)G}{\{\cdot\}_S}(3 + 3\sigma v^2) \tag{5.57}$$

$$S^{01} = \frac{(1 - A)G}{\{\cdot\}_S}(3 + 3\sigma)v \tag{5.58}$$

$$S^{11} = \frac{(1 - A)G}{\{\cdot\}_S}(3\sigma + 3v^2) \tag{5.59}$$

By substituting (5.57) into (5.41) we then obtain (5.27). Now differentiating (5.26) we find that:

$$\xi\frac{dG}{d\xi} = G\left[1 - \frac{1}{2}\left(\frac{\xi}{A}\frac{dA}{d\xi} + \frac{\xi}{B}\frac{dB}{d\xi}\right)\right]$$

and using (5.41) and (5.43) we have:

$$\xi \frac{dG}{d\xi} = G\left[1 - \left(\frac{1-A}{A} + \frac{S^{11} - S^{00}}{2A}\right)\right] \tag{5.60}$$

By substituting (5.57) and (5.59) into (5.60) we then obtain (5.28). So far we have shown that if A, G and v are functions of ξ, then (5.22)-(5.24) are equivalent to (5.27) and (5.28) together with the constraint (5.33). It remains to show that when $A(\xi)$, $G(\xi)$ and $v(\xi)$ are substituted into (5.37), the relations (5.57)-(5.59) can be used to obtain (5.29). To start, we multiply (5.37) by $\bar{r}^3$ and use (5.40) to get:

$$0 = \bar{r}\frac{\partial}{\partial \bar{t}} S^{01} + \bar{r}\frac{\partial}{\partial \bar{r}}\left(\sqrt{AB}S^{11}\right) - 2\sqrt{AB}S^{11}$$
$$- \frac{1}{2}\sqrt{AB}\left(4S^{11} + \frac{1-A}{A}(S^{00} - S^{11}) + \frac{2\kappa}{A}\left[S^{00}S^{11} - (S^{01})^2 - 4\bar{r}^2 S^{22}\right]\right)$$

Using (5.41) and (5.43) to eliminate S^{01} and S^{11} in the two terms quadratic in $S^{\mu\nu}$, we then obtain:

$$0 = -\xi^2 \frac{d}{d\xi} S^{01} + \xi \frac{d}{d\xi}\left(\sqrt{AB}S^{11}\right) - 2\sqrt{AB}S^{11}$$
$$- \frac{1}{2}\sqrt{AB}\left(\frac{2\xi^2}{A\sqrt{AB}}\frac{dA}{d\xi}S^{01} + \left(4 - \frac{\xi}{A}\frac{dA}{d\xi}\right)S^{11} + \frac{\xi}{B}\frac{dB}{d\xi}S^{00} - 4\bar{r}^2 S^{22}\right)$$

Evaluating the derivative in the second term and cancelling terms of the form:

$$\pm\frac{1}{2}\xi^2 \frac{E}{A}\frac{dA}{d\xi}S^{11}$$

leads to the following expression after multiplying by ξ^{-1}:

$$0 = -\xi\frac{d}{d\xi}S^{01} + \xi E\frac{d}{d\xi}S^{11} + \frac{\xi}{A}\frac{d}{d\xi}S^{01} + \frac{1}{2}\xi\frac{E}{B}\frac{dB}{d\xi}(S^{00} + S^{11}) - 2E\bar{r}^2 S^{22} \tag{5.61}$$

where:

$$E = \frac{1}{G} = \frac{\sqrt{AB}}{\xi} \tag{5.62}$$

From (5.47), the derivatives of S^{01} and S^{11} in (5.61) can be given in terms of the derivatives of v and w, to yield:

(5.63)
$$0 = (EV^{11} - V^{01})\frac{\xi}{w}\frac{dw}{d\xi} + (6Ev - 3 - 3\sigma)\xi\frac{dv}{d\xi} + \frac{\xi}{A}\frac{dA}{d\xi}V^{01} + \frac{1}{2}\xi\frac{E}{B}\frac{dB}{d\xi}(V^{11} + V^{00}) - 2EV^{22}$$

Before proceeding, we note that:

$$\kappa w = \frac{1 - A}{V^{00} - EV^{01}} \tag{5.64}$$

follows from (5.33) and we will show that:

$$\frac{1}{w}\frac{dw}{d\xi} = \frac{(3 + 3\sigma)E - 6\sigma v}{V^{00} - EV^{01}}\frac{dv}{d\xi} + \frac{(3 + 3\sigma)v}{2\xi^2 E(V^{00} - EV^{01})}\frac{d}{d\xi}(AB) \tag{5.65}$$

In this light, differentiating (5.64) yields:

$$\frac{1}{w}\frac{dw}{d\xi} = \frac{D}{1 - A}\frac{dw}{d\xi} = -\frac{1}{1 - A}\frac{dA}{d\xi} - \frac{1}{D}\frac{dD}{d\xi} \tag{5.66}$$

where:

$$D = V^{00} - EV^{01} \tag{5.67}$$

By (5.42), (5.64) and (5.67), we have:

$$-\frac{1}{1 - A}\frac{dA}{d\xi} = \frac{(3 + 3\sigma)vE}{\xi D} \tag{5.68}$$

and by (5.62), (5.64) and (5.67), we have:

$$\frac{1}{D}\frac{dD}{d\xi} = \frac{6\sigma v - (3 + 3\sigma)E}{D}\frac{dv}{d\xi} + \frac{(3 + 3\sigma)vE}{\xi D} - \frac{(3 + 3\sigma)v}{2\xi^2 ED}\frac{d}{d\xi}(AB) \tag{5.69}$$

Then by substituting (5.68) and (5.69) in (5.66) we thus obtain (5.65). Now that we have (5.65), we can insert this expression into (5.63), use (5.48) and solve the resulting equation for the derivative of v to get:

$$(5.70)\quad 8\xi\frac{dv}{d\xi}\{\cdot\}_D^* = \frac{\xi}{A}\frac{dA}{d\xi}\left\{(EV^{11}-V^{01})EV^{01}+2DV^{01}\right\}_A$$
$$+\,\xi\frac{E}{B}\frac{dB}{d\xi}\left\{(EV^{11}-V^{01})V^{01}+D(V^{00}+V^{11})\right\}_B - 4EDV^{22}$$

where:

$$(5.71)\qquad 2\{\cdot\}_D^* = -(EV^{11}-V^{01})(2E-V)+(2-3Ev)D$$

We can then use (5.41)-(5.43) to replace the derivatives of A and B to obtain:

$$(5.72)\quad 2\xi\frac{dv}{d\xi}\{\cdot\}_D^*\frac{4AD}{(1-A)E} = -(V^{01})^2\left\{(EV^{11}-V^{01})E+2D\right\}_A$$
$$+\,(D+V^{11})\left\{(EV^{11}-V^{01})V^{01}+D(V^{00}+V^{11})\right\}_B - \frac{4AD^2}{1-A}V^{22}$$

Given that V^{00}, V^{01} and V^{11} only depend on v, then we can see that the $\{\cdot\}_A$, $\{\cdot\}_B$ and $\{\cdot\}_D^*$ terms in (5.72) are all quadratic polynomials in E with polynomials in v as coefficients. Such coefficients will now be found. Beginning with $\{\cdot\}_D^*$, we have by (5.71) that:

$$2\{\cdot\}_D^* = \left[\frac{1}{2}(3+3\sigma)V^{00}-3\sigma V^{01}v\right]+\left[3\sigma V^{11}v-3V^{00}v\right]E+\left[3V^{01}v-\frac{1}{2}(3+3\sigma)V^{11}\right]E^2$$

and then by (5.50)-(5.52) we obtain:

$$(5.73)\qquad 2\{\cdot\}_D^* = \frac{1}{2}(3+3\sigma)(3-3\sigma v^2)+3(3\sigma^2 v-3v)E+\frac{1}{2}(3+3\sigma)(3v^2-3\sigma)E^2 = 2E^2\{\cdot\}_D$$

where $\{\cdot\}_D$ is given by (5.32). For $\{\cdot\}_A$, we apply (5.67) to see that:

$$-(V^{01})^2\{\cdot\}_A = -(V^{01})^2\left\{(EV^{11}-V^{01})E+2V^{00}-2EV^{01}\right\}_A$$

and then by (5.50)-(5.52) we obtain:

$$(5.74)\qquad -(V^{01})^2\{\cdot\}_A = -(3+3\sigma)^2v^2\left\{(3\sigma+3v^2)E^2-3(3+3\sigma)vE+2(3+3\sigma v^2)\right\}_A$$

Similarly, for $\{\cdot\}_B$ we have that:

$$
\begin{aligned}
(D+V^{11})\{\cdot\}_B &= (D+V^{11})\left\{(EV^{11}-V^{01})V^{01}+D(V^{00}+V^{11})\right\}_B \\
&= (V^{00}+V^{11}-EV^{01})\left\{V^{00}(V^{00}+V^{11})-(V^{01})^2-V^{00}V^{01}E\right\}_B
\end{aligned}
$$

and again by (5.50)-(5.52) we obtain:

$$
\begin{aligned}
(5.75)\quad (D+V^{11})\{\cdot\}_B &= (3+3\sigma)^2(1+v^2)(3+3\sigma v^4) \\
&\quad -2(3+3\sigma)^2 v\left[3+\frac{1}{2}(3+3\sigma)v^2+3\sigma v^4\right]E+(3+3\sigma)^2(3+3\sigma v^2)v^2E^2
\end{aligned}
$$

Now by adding (5.74) and (5.75) we obtain the following after some simplification:

$$
-(V^{01})^2\{\cdot\}_A+(D+V^{11})\{\cdot\}_B=\{\cdot\}_0+\{\cdot\}_1E+\{\cdot\}_2E^2 \tag{5.76}
$$

where:

$$
\{\cdot\}_0=(3+3\sigma)^2(1-v^2)(3-3\sigma v^4) \tag{5.77}
$$

$$
\{\cdot\}_1=-2(3+3\sigma)^2v(1-v^2)(3-3\sigma v^2) \tag{5.78}
$$

$$
\{\cdot\}_2=(3-3\sigma)(3+3\sigma)^2v^2(1-v^2) \tag{5.79}
$$

We can then use (5.76) to see that:

$$
-(V^{01})^2\{\cdot\}_A+(D+V^{11})\{\cdot\}_B=-(3+3\sigma)^2(1-v^2)E^2\{\cdot\}_N \tag{5.80}
$$

where $\{\cdot\}_N$ is given by (5.31). Finally, inserting (5.80) into (5.72) and using (5.73) and (5.53) gives the following equation, which equivalent to (5.37):

$$
2\xi\frac{dv}{d\xi}\frac{ADE}{1-A}\{\cdot\}_D=-\frac{1}{4}(3+3\sigma)^2(1-v^2)E^2\{\cdot\}_N-3\sigma(1-v^2)\frac{AD^2}{1-A} \tag{5.81}
$$

By dividing (5.81) by:

$$
\frac{ADE}{1-A}\{\cdot\}_D
$$

and using:

$$D = E\left[(3+3\sigma v^2)G - (3+3\sigma)v\right]$$

we see that equation (5.81) is equivalent to (5.29), which completes the proof. $\square$

Just as with the ODE derived in comoving coordinates in Chapter 3, we see that the temporal scaling $\bar{t} \to \Psi_0 \bar{t}$ preserves solutions of (5.22)-(5.24) and the constraint (5.37). The following theorem, also from [**21**], confirms Ψ_0 is indeed an inessential parameter.

THEOREM 5.1.4. *The replacement $\bar{t} \to \Psi_0 \bar{t}$ takes:*

$$A(\xi) \to A\left(\frac{\xi}{\Psi_0}\right)$$
$$G(\xi) \to G\left(\frac{\xi}{\Psi_0}\right)$$
$$v(\xi) \to v\left(\frac{\xi}{\Psi_0}\right)$$

and this scaling preserves solutions of (5.22)-(5.24) and (5.37). Moreover, this is the only such scaling law in the sense that any two solutions of (5.22)-(5.24) and (5.37) that are not related by this scaling will describe distinct spacetimes.

PROOF. Following Smoller and Temple's proof, we need to show that the only coordinate transformation that preserves the Schwarzschild coordinate form (5.34) is the time scaling $\bar{t} \to \alpha \bar{t}$ for constant α. The problem reduces to demonstrating that the only coordinate transformation of the form $(\bar{t}, \bar{r}) \to (\tilde{t}, \tilde{r})$ taking a metric:

$$d\bar{s}^2 = -B(\xi)d\bar{t}^2 + \frac{1}{A(\xi)}d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

to a metric:

$$d\tilde{s}^2 = -\tilde{B}(\tilde{\xi})d\tilde{t}^2 + \frac{1}{\tilde{A}(\tilde{\xi})}d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$$

where $\tilde{\xi} = \tilde{r}/\tilde{t}$, is the time scaling:

$$\tilde{r} = \bar{r}$$

$$\tilde{t} = \alpha \bar{t}$$

As remarked in Section 2.1, because both metrics are in Schwarzschild form then they must share the same radial variable. So to avoid introducing $dtdr$ terms into the metric, it must be the case that:

$$\tilde{t} = \Lambda(\bar{t})$$

for some function Λ. This means the transformation must meet the condition:

$$\tilde{A}\left(\frac{\bar{r}}{\Lambda(\bar{t})}\right) = A\left(\frac{\bar{r}}{\bar{t}}\right)$$

Differentiating with respect to $\bar{r}$ then yields:

$$\frac{\Lambda(\bar{t})}{\bar{t}} = \frac{\tilde{A}'}{A'}$$

Since the left hand side of the above is independent of $\bar{r}$, and the right hand side is not, we must have that both sides are constant, implying:

$$\Lambda(\bar{t}) = \alpha \bar{t}$$

for some positive constant α, as claimed. □

We conclude this section with the statement of Theorem 8 from [**21**]. This theorem establishes the important fact that only the $k = 0$ FLRW$(k, \sigma, 1)$ spacetimes are solutions of equations (5.27)-(5.29), that is, the parameter k is not one of the two parameters, for each fixed σ, specifying solutions to (5.27)-(5.29).

THEOREM 5.1.5. *Spacetime metrics defined by solutions of (5.27)-(5.29) are distinct form the $k \neq 0$ FLRW spacetimes.*

5.2. Revisiting the TOV Spacetimes

We know from Chapter 2 that the TOV spacetimes are the family of spherically symmetric static spacetimes. We then found in Chapter 3, that the self-similar subset of these spacetimes with a barotropic equation of state, form the unique family of spherically symmetric self-similar static spacetimes, which we denote by TOV(σ). In the context of equations (5.27)-(5.29), the TOV(σ) spacetimes are distinguished by having a Schwarzschild coordinate velocity that is identically zero.

Proposition 5.2.1. *Spherically symmetric self-similar perfect fluid spacetimes are static if and only if* $v \equiv 0$.

Proof. Since it is known that the family of static spherically symmetric self-similar perfect fluid spacetimes are unique, it is sufficient to demonstrate that solutions with zero Schwarzschild coordinate velocity are static. In this light, substituting $v \equiv 0$ into equation (5.27) implies that $A \equiv A_0$ for some constant A_0. Furthermore, substituting $v \equiv 0$ into equation (5.29) requires:

$$9\sigma - \frac{1}{4}(3+3\sigma)^2\left(\frac{1-A_0}{A_0}\right) = 0$$

to ensure $v' \equiv 0$. This means A_0 can be given as a function of σ as so:

$$A_0(\sigma) = 1 - 2M(\sigma)$$

where:

$$M(\sigma) = \frac{2\sigma}{1+6\sigma+\sigma^2}$$

Now substituting both $v \equiv 0$ and $A \equiv A_0$ into equation (5.28) and solving for G yields:

$$G(\xi) = C_1 \xi^{\frac{1-\sigma}{1+\sigma}}$$

for some positive constant C_1. Putting these results together gives us the following metric in self-similar Schwarzschild coordinates:

$$d\bar{s}^2 = -C_2 \xi^{\frac{4\sigma}{1+\sigma}} d\bar{t}^2 + \frac{1}{1-2M(\sigma)} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

for some positive constant C_2, with the density given by:

$$\rho = \frac{2M(\sigma)}{\kappa \bar{r}^2}$$

Note that because $v \equiv 0$, this coordinate frame is also comoving with the fluid. Finally, making the temporal transformation:

$$\begin{aligned}\tilde{t} &= \frac{1+\sigma}{1-\sigma}\bar{t}^{\frac{1-\sigma}{1+\sigma}} \\ \tilde{r} &= \bar{r}\end{aligned}$$

puts the metric in the following explicitly static form:

$$d\tilde{s}^2 = -C_2\tilde{r}^{\frac{4\sigma}{1+\sigma}}d\tilde{t}^2 + \frac{1}{1-2M(\sigma)}d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$$

noting that the density also remains static. $\square$

The TOV(σ) spacetimes are a remarkably convenient and simple set of solutions to the perfect fluid Einstein field equations, mostly because they can be placed in a coordinate system which is comoving, explicitly self-similar and in Schwarzschild form simultaneously.

Proposition 5.2.2. *The one-parameter family of TOV spacetimes, denoted by TOV(σ), are given in self-similar comoving Schwarzschild coordinates as:*

$$\begin{aligned}d\bar{s}^2 &= -\alpha^2\xi^{\frac{4\sigma}{1+\sigma}}d\bar{t}^2 + \frac{1}{1-2M(\sigma)}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \\ \rho &= \frac{2M(\sigma)}{\kappa\bar{r}^2} \\ p &= \sigma\rho\end{aligned}$$

where α is an inessential parameter and:

$$M(\sigma) = \frac{2\sigma}{1+6\sigma+\sigma^2}$$

Proof. This follows from the proof of Proposition 5.2.1. $\square$

5.3. Revisiting the FLRW Spacetimes

We know from Chapter 2 that the flat FLRW spacetimes are the family of spatially homogeneous spherically symmetric spacetimes. We also know from Chapter 4 that the self-similar subset of FLRW spacetimes which solve the perfect fluid Einstein field equations with a barotropic equation of state take the following form in self-similar comoving coordinates:

$$\begin{aligned} d\hat{s}^2 &= -e^{2\varphi}d\hat{t}^2 + e^{2\psi}d\hat{r}^2 + \mathscr{R}^2\hat{r}^2 d\Omega^2 \\ \rho &= \frac{2\hat{\xi}^2}{\kappa\hat{r}^2} \\ p &= \sigma\rho \end{aligned}$$

where:

$$\begin{aligned} e^{2\varphi} &= \beta^2 \\ e^{2\psi} &= \gamma^{-2}\hat{\xi}^{-\frac{4}{3+3\sigma}} \\ \mathscr{R}^2 &= \hat{\xi}^{-\frac{4}{3+3\sigma}} \end{aligned}$$

and:

$$\begin{aligned} \beta &= \frac{\sqrt{6}}{3+3\sigma} \\ \gamma &= \frac{3+3\sigma}{1+3\sigma} \end{aligned}$$

Note that the density is independent of r and that the metric can also be put into an explicitly spatially homogeneous form through the purely radial transformation:

$$\begin{aligned} t &= \hat{t} \\ r &= \hat{r}^{\frac{1+3\sigma}{3+3\sigma}} \end{aligned}$$

to yield:

$$ds^2 = -\beta^2 dt^2 + t^{\frac{4}{3+3\sigma}}\left(dr^2 + r^2 d\Omega^2\right)$$

Proposition 5.3.1. *The one-parameter family of self-similar perfect fluid FLRW spacetimes with barotropic equations of state, denoted by FLRW$(0, \sigma, 1)$, are given in self-similar Schwarzschild coordinates as:*

$$\begin{aligned}
d\bar{s}^2 &= -\Psi_0^{-2}\left[1+\frac{1}{3}(1+3\sigma)\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-\frac{1-3\sigma}{1+3\sigma}}\left[1-\frac{2}{3}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-1}d\bar{t}^2+\left[1-\frac{2}{3}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-1}d\bar{r}^2+\bar{r}^2d\Omega^2\\
v &= \frac{2}{\sqrt{6}}\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}\\
\rho &= \frac{3v^2}{\kappa\bar{r}^2}\\
p &= \sigma\rho
\end{aligned}$$

where Ψ_0 is an inessential parameter and:

$$\xi = \frac{1}{\sqrt{6}}\Psi_0^{-1}(3+3\sigma)\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}\left[1+\frac{1}{3}(1+3\sigma)\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-\frac{3+3\sigma}{2+6\sigma}} \tag{5.82}$$

Recall that the zero in the first argument of FLRW$(0, \sigma, 1)$ corresponds to the flat subset of FLRW spacetimes, that is, those with $k = 0$ in reduced circumference Schwarzschild coordinates. The one in the third argument corresponds to the lack of perturbation, which will be defined in the next section.

Proof of Proposition 5.3.1. To change FLRW$(0, \sigma, 1)$ from self-similar comoving coordinates to self-similar Schwarzschild coordinates, we can use the transformation given by Proposition 3.6.1 from Section 3.6:

$$d\bar{t} = e^{-\mu}\left(e^{\varphi}\cosh\omega\ d\hat{t} + e^{\psi}\sinh\omega\ d\hat{r}\right) \tag{5.83}$$

$$d\bar{r} = e^{-\nu}\left(e^{\varphi}\sinh\omega\ d\hat{t} + e^{\psi}\cosh\omega\ d\hat{r}\right) \tag{5.84}$$

where:

$$\tanh\omega = e^{\psi-\varphi}\frac{\partial_{\hat{t}}(\mathscr{R}\hat{r})}{\partial_{\hat{r}}(\mathscr{R}\hat{r})} \tag{5.85}$$

$$e^{-2\nu} = e^{-2\psi}[\partial_{\hat{r}}(\mathscr{R}\hat{r})]^2 - e^{-2\varphi}[\partial_{\hat{t}}(\mathscr{R}\hat{r})]^2 \tag{5.86}$$

and μ is such that $d\bar{t}$ is a perfect differential. Relations (5.85) and (5.86) come from setting $\bar{r} = \mathscr{R}\hat{r}$. The resulting self-similar Schwarzschild form of the metric is then given by:

$$d\bar{s}^2 = -e^{2\mu}d\bar{t}^2 + e^{2\nu}d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

To begin, the $\tanh\omega$ term is computed as so:

$$\begin{aligned}
\tanh\omega &= e^{\psi-\varphi}\frac{\partial_{\hat{t}}(\mathscr{R}\hat{r})}{\partial_{\hat{r}}(\mathscr{R}\hat{r})} \\
&= \beta^{-1}\gamma^{-1}\hat{\xi}^{-\frac{2}{3+3\sigma}}\frac{-\hat{\xi}^2\partial_{\hat{r}}\mathscr{R}}{\mathscr{R}+\hat{r}\partial_{\hat{r}}\mathscr{R}} \\
&= \beta^{-1}\gamma^{-1}\hat{\xi}^{-\frac{2}{3+3\sigma}}\frac{-\hat{\xi}^2(-\frac{2}{3+3\sigma})\hat{\xi}^{-\frac{2}{3+3\sigma}-1}}{\hat{\xi}^{-\frac{2}{3+3\sigma}}+\hat{\xi}(-\frac{2}{3+3\sigma})\hat{\xi}^{-\frac{2}{3+3\sigma}-1}} \\
&= 2(1+3\sigma)^{-1}\beta^{-1}\gamma^{-1}\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}
\end{aligned}$$

and this yields:

$$\begin{aligned}
\cosh\omega &= (1-\tanh^2\omega)^{-\frac{1}{2}} = \left[1-4(1+3\sigma)^{-2}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-\frac{1}{2}} \\
\sinh\omega &= \tanh\omega(1-\tanh^2\omega)^{-\frac{1}{2}} = 2(1+3\sigma)^{-1}\beta^{-1}\gamma^{-1}\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}\left[1-4(1+3\sigma)^{-2}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-\frac{1}{2}}
\end{aligned}$$

The $e^{-2\nu}$ term is computed similarly:

$$\begin{aligned}
e^{-2\nu} &= e^{-2\psi}[\partial_{\hat{r}}(\mathscr{R}\hat{r})]^2 - e^{-2\varphi}[\partial_{\hat{t}}(\mathscr{R}\hat{r})]^2 \\
&= \gamma^2\hat{\xi}^{\frac{4}{3+3\sigma}}\left[\hat{\xi}^{-\frac{2}{3+3\sigma}}+\hat{\xi}\left(-\frac{2}{3+3\sigma}\right)\hat{\xi}^{-\frac{2}{3+3\sigma}-1}\right]^2 - \beta^{-2}\left[-\hat{\xi}^2\left(-\frac{2}{3+3\sigma}\right)\hat{\xi}^{-\frac{2}{3+3\sigma}-1}\right]^2 \\
&= (1+3\sigma)^2(3+3\sigma)^{-2}\gamma^2\left[1-4(1+3\sigma)^{-2}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]
\end{aligned}$$

and this results in:

$$\begin{aligned}
d\bar{t} &= \beta e^{-\mu}\left[1-4(1+3\sigma)^{-2}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-\frac{1}{2}}\left[d\hat{t}+2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{-1}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}d\hat{r}\right] \\
d\bar{r} &= 2(3+3\sigma)^{-1}\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}\left[d\hat{t}+\frac{1}{2}(1+3\sigma)\hat{\xi}^{-1}d\hat{r}\right]
\end{aligned}$$

Now given that μ is such that the right hand side of (5.83) is a perfect differential, then:

$$\frac{\partial}{\partial \hat{r}} e^{-\eta} = \frac{\partial}{\partial \hat{t}} \left[2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{-1}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}} e^{-\eta}\right]$$

where:

$$e^{-\eta} = e^{-\mu}\left[1 - 4(1+3\sigma)^{-2}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-\frac{1}{2}}$$

The equation for η can be solved as so:

$$\begin{aligned}
&\frac{\partial}{\partial \hat{r}} e^{-\eta} = \frac{\partial}{\partial \hat{t}} \left[2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{-1}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}} e^{-\eta}\right] \\
\iff\ &\frac{1}{\hat{t}}\frac{d}{d\hat{\xi}} e^{-\eta} = -\frac{\hat{r}}{\hat{t}^2}\frac{d}{d\hat{\xi}} \left[2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{-1}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}} e^{-\eta}\right] \\
&\iff \eta' = 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}e^{\eta}\left[-\hat{\xi}^{-1}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\eta' e^{-\eta} + (3\sigma-1)(3+3\sigma)^{-1}\hat{\xi}^{-\frac{4}{3+3\sigma}} e^{-\eta}\right] \\
&\iff \eta' = 2(1+3\sigma)^{-1}(3\sigma-1)(3+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{-1}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\left[1 + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-1} \\
&\iff \eta = (3\sigma-1)(2+6\sigma)^{-1}\log\left[1 + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right] + C_3
\end{aligned}$$

where C_3 is a constant. This then yields:

$$e^{-\eta} = \Psi_0\left[1 + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{\frac{1-3\sigma}{2+6\sigma}}$$

for some positive constant Ψ_0, thus:

$$d\bar{t} = \Psi_0\beta\left[1 + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{\frac{1-3\sigma}{2+6\sigma}}\left[d\hat{t} + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{-1}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}} d\hat{r}\right]$$

Because the transformation is taking the metric from one self-similar form to another, let:

$$\bar{t} = \mathscr{T}(\hat{\xi})\hat{t}$$

so that:

$$\begin{aligned}
\frac{\partial \bar{t}}{\partial \hat{t}} &= \mathscr{T}(\hat{\xi}) - \hat{\xi}\mathscr{T}'(\hat{\xi}) = \Psi_0\beta\left[1 + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{\frac{1-3\sigma}{2+6\sigma}} \\
\frac{\partial \bar{t}}{\partial \hat{r}} &= \mathscr{T}'(\hat{\xi}) = 2\Psi_0(1+3\sigma)^{-1}\beta^{-1}\gamma^{-2}\hat{\xi}^{-1}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\left[1 + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{\frac{1-3\sigma}{2+6\sigma}}
\end{aligned}$$

Solving these equations yields the same function for $\mathscr{T}(\hat{\xi})$ only when the integration constant is zero, thus:

$$\mathscr{T}(\hat{\xi}) = \Psi_0 \beta \left[1 + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{\frac{3+3\sigma}{2+6\sigma}}$$

Now the fluid four-velocity $\boldsymbol{u}$ is given in self-similar comoving coordinates as:

$$\boldsymbol{u} = (\hat{u}^0, \hat{u}^1, \hat{u}^2, \hat{u}^3) = (e^{-\mu}, 0, 0, 0) = (\beta^{-1}, 0, 0, 0)$$

and in self-similar Schwarzschild coordinates as:

$$\boldsymbol{u} = (\bar{u}^0, \bar{u}^1, \bar{u}^2, \bar{u}^3) = \left(\hat{u}^0 \frac{\partial \bar{t}}{\partial \hat{t}}, \hat{u}^0 \frac{\partial \bar{r}}{\partial \hat{t}}, 0, 0\right) = \left(\beta^{-1} \frac{\partial \bar{t}}{\partial \hat{t}}, \beta^{-1} \frac{\partial \bar{r}}{\partial \hat{t}}, 0, 0\right)$$
$$= \left(\Psi_0 \left[1 + 2(1+3\sigma)^{-1}\beta^{-2}\gamma^{-2}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{\frac{1-3\sigma}{2+6\sigma}}, 2(3+3\sigma)^{-1}\beta^{-1}\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}, 0, 0\right)$$

Therefore, by Definition 2.1.1:

$$v = e^{\nu-\mu}\frac{\bar{u}^1}{\bar{u}^0} = 2(3+3\sigma)^{-1}\beta^{-1}\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}$$

Finally, by substituting in β and γ and noting that:

$$\xi = \frac{\bar{r}}{\bar{t}} = \frac{\mathscr{R}(\hat{\xi})\hat{r}}{\mathscr{T}(\hat{\xi})\hat{t}} = \frac{\mathscr{R}(\hat{\xi})}{\mathscr{T}(\hat{\xi})}\hat{\xi}$$

the rest follows. □

Spacetimes that solve equations (5.27)-(5.29) can be denoted by the triple (A, G, v), which specifies the metric through A and G, the fluid four-velocity through v and the density through the constraint (5.33). This notation will be used frequently in Chapter 6.

PROPOSITION 5.3.2. *FLRW*$(0, \sigma, 1)$ *is given implicitly by:*

(5.87) $$A = 1 - v^2$$

(5.88) $$G = \frac{1}{2}(3+3\sigma)v\left(1 + \frac{1}{2}(1+3\sigma)v^2\right)^{-1}$$

(5.89) $$v = \frac{2}{\sqrt{6}}\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}$$

PROOF. First note that (5.89) is immediately obtained from Proposition 5.3.1. Then by definition:

$$A = e^{-2\nu} = 1 - \frac{2}{3}\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}$$
$$G = \xi e^{\nu-\mu} = \frac{1}{\sqrt{6}}(3+3\sigma)\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}\left[1 + \frac{1}{3}(1+3\sigma)\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-1}$$

which yields (5.87) and (5.88). □

To check that (5.87)-(5.89) satisfies equations (5.27)-(5.29), it is recommended to first show the following:

$$\xi\frac{d}{d\xi} = \xi\frac{d\hat{\xi}}{d\xi}\frac{d}{d\hat{\xi}} = \frac{(3+3\sigma)^2}{2+6\sigma}\frac{v}{AG}\hat{\xi}\frac{d}{d\hat{\xi}}$$

and secondly show:

$$\hat{\xi}\frac{dA}{d\hat{\xi}} = -\frac{2+6\sigma}{3+3\sigma}v^2$$
$$\hat{\xi}\frac{dG}{d\hat{\xi}} = \frac{2+6\sigma}{(3+3\sigma)^2}\left(1 - \frac{1}{2}(1+3\sigma)v^2\right)\frac{G^2}{v}$$
$$\hat{\xi}\frac{dv}{d\hat{\xi}} = \frac{1+3\sigma}{3+3\sigma}v$$

Then it is not difficult to confirm that (5.87)-(5.89) solves equations (5.27)-(5.29).

COROLLARY 5.3.1. *$FLRW(0, \frac{1}{3}, 1)$ is given in self-similar Schwarzschild coordinates as:*

$$d\bar{s}^2 = \frac{1}{1-v^2}\left(-\Psi_0^{-2}d\bar{t}^2 + d\bar{r}^2\right) + \bar{r}^2 d\Omega^2$$
$$v = \frac{1-\sqrt{1-\Psi_0^2\xi^2}}{\Psi_0\xi}$$
$$\rho = \frac{3v^2}{\kappa\bar{r}^2}$$
$$p = \sigma\rho$$

where Ψ_0 is an inessential parameter.

PROOF. From Proposition 5.3.1 in the case $\sigma = \frac{1}{3}$, relation (5.82) can be inverted to yield v. The metric then follows from using this inversion and some algebraic manipulation. $\square$

COROLLARY 5.3.2. *FLRW*$(0, \frac{1}{3}, 1)$ *is given implicitly by:*

$$A = 1 - v^2 \tag{5.90}$$

$$G = \frac{2v}{1+v^2} \tag{5.91}$$

$$G = \Psi_0 \xi \tag{5.92}$$

PROOF. Relations (5.90) and (5.91) follow immediately from Proposition 5.3.2 and relation (5.92) follows from (5.26) and the identity $B = \Psi_0^{-2} A^{-1}$ from Corollary 5.3.1. $\square$

5.4. Self-Similar Perturbations of FLRW Spacetimes

Let (A, G, v) denote a solution of equations (5.27)-(5.29). Since these equations are autonomous, solutions can be represented by non-intersecting trajectories in (A, G, v) space. The FLRW$(0, \sigma, 1)$ spacetimes solve equations (5.27)-(5.29) and constraint (5.33) with the trajectories emanating from the point:

$$(A, G, v) = (1, 0, 0)$$

The nature of equations (5.27)-(5.29) suggests that to analyse this point, it is helpful to rewrite these equations as functions of v, A and H, where H is defined as the ratio:

$$H = \frac{G}{v}$$

This was completed by Smoller and Temple [**21**] for $\sigma = \frac{1}{3}$, however it is not difficult to reproduce these equations for general σ, especially when working from equations (5.27)-(5.29). Recalling (5.35) and (5.36), equations (5.27)-(5.33) are given in autonomous form as functions of v, A and H as so:

$$\frac{dv}{ds} = -v\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\left[3\sigma\{\cdot\}_S^* + \left(\frac{1-A}{A}\right)\frac{(3+3\sigma)^2\{\cdot\}_N^*}{4\{\cdot\}_S^*}\right] \tag{5.93}$$

$$\frac{dA}{ds} = -\frac{(3+3\sigma)(1-A)}{\{\cdot\}_S^*} \tag{5.94}$$

$$\frac{dH}{ds} = -H\left[\left(\frac{1-A}{A}\right)\frac{(3+3\sigma)[(1+v^2)H-2]}{2\{\cdot\}_S^*} - 1\right] - \frac{H}{v}\frac{dv}{ds} \tag{5.95}$$

with:

$$\kappa\rho\bar{r}^2 = \frac{3(1-v^2)(1-A)H}{\{\cdot\}_S^*} \tag{5.96}$$

and where:

$$\{\cdot\}_S^* = -(3+3\sigma) + (3+3\sigma v^2)H$$

$$\{\cdot\}_N^* = -(3-3\sigma) + 2(3-3\sigma v^2)H - (3-3\sigma v^4)H^2$$

$$\{\cdot\}_D = -\frac{1}{4}(3+3\sigma)(3\sigma-3v^2) - \frac{3}{2}(3-3\sigma^2)Hv^2 + \frac{1}{4}(3+3\sigma)(3-3\sigma v^2)H^2v^2$$

In variables v, A and H, the point of interest is given by:

$$(v, A, H) = \left(0, 1, \frac{1}{2}(3+3\sigma)\right)$$

and it is not difficult to check that this is a fixed point of the system of equations (5.93)-(5.95). Following Smoller and Temple, a linear analysis of this fixed point can be achieved by first representing equations (5.93)-(5.95) as:

$$v' = F_1(v, A, H)$$

$$A' = F_2(v, A, H)$$

$$H' = F_3(v, A, H)$$

and then denoting these equations by:

$$\boldsymbol{U}' = \boldsymbol{F}(\boldsymbol{U})$$

where:

$$\boldsymbol{U} = (v, A, H)^T$$
$$\boldsymbol{F} = (F_1(\boldsymbol{U}), F_2(\boldsymbol{U}), F_3(\boldsymbol{U}))^T$$

Next, the Jacobian of $\boldsymbol{F}$ at the fixed point is calculated. Note that when Smoller and Temple first calculated this Jacobian there was a small error, so a brief new derivation will be produced. To begin, denote the fixed point by $\boldsymbol{U}_0$ and note that:

$$dF_2(\boldsymbol{U}_0) = \left(\frac{\partial F_2}{\partial v}, \frac{\partial F_2}{\partial A}, \frac{\partial F_2}{\partial H}\right)\Bigg|_{\boldsymbol{U}_0} = (0, 2, 0)$$

Neglecting terms second order in v and second order in terms that vanish at $\boldsymbol{U}_0$ on the right hand side of (5.93) gives:

$$dF_1(\boldsymbol{U}_0) = d\left[-v\left(-\frac{2}{3\sigma(3+3\sigma)}\right)[9\sigma H - 3\sigma(3+3\sigma)]\right]_{\boldsymbol{U}_0} = (1, 0, 0)$$

and similarly for (5.95) gives:

$$\begin{aligned}
dF_3(\boldsymbol{U}_0) &= d\left[H - H\left(\frac{1-A}{A}\right)\frac{(3+3\sigma)(H-2)}{2[3H-(3+3\sigma)]}\right]_{\boldsymbol{U}_0} \\
&\quad + d\left[H\left(-\frac{2}{3\sigma(3+3\sigma)}\right)\left[9\sigma H - 3\sigma(3+3\sigma) + \left(\frac{1-A}{A}\right)\frac{(3+3\sigma)^2[(3\sigma-3)+6H-3H^2]}{4[3H-(3+3\sigma)]}\right]\right]_{\boldsymbol{U}_0} \\
&= d\left[3H - \frac{6H^2}{3+3\sigma} + \left(\frac{1-A}{A}\right)\frac{(3+3\sigma)(1-H)H}{6\sigma}\right]_{\boldsymbol{U}_0} \\
&= \left(0, \left(-\frac{1}{A^2}\right)\frac{(3+3\sigma)(1-H)H}{6\sigma}, 3 - \frac{12H}{3+3\sigma}\right)\Bigg|_{\boldsymbol{U}_0} \\
&= \left(0, -\frac{(1+3\sigma)(3+3\sigma)^2}{24\sigma}, -3\right)
\end{aligned}$$

Thus the Jacobian is given by:

$$d\boldsymbol{F}(\boldsymbol{U}_0) = \begin{pmatrix} dF_1(\boldsymbol{U}_0) \\ dF_2(\boldsymbol{U}_0) \\ dF_3(\boldsymbol{U}_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & N(\sigma) & -3 \end{pmatrix}$$

where:

$$N(\sigma) = -\frac{(1+3\sigma)(3+3\sigma)^2}{24\sigma}$$

This means $\boldsymbol{U}_0$ is a hyperbolic rest point of the system of equations (5.93)-(5.95) with eigenvalues:

$$\lambda_1 = 1$$

$$\lambda_2 = 2$$

$$\lambda_3 = -3$$

Therefore the solutions:

$$\boldsymbol{U}(s) = \boldsymbol{U}_0 + \boldsymbol{V}(s)$$

where $\boldsymbol{V}(s)$ solves the linearised equations:

$$\boldsymbol{V}' = d\boldsymbol{F}(\boldsymbol{U}_0) \cdot \boldsymbol{V}$$

lie in the two-dimensional unstable manifold $\mathcal{M}_0$ of $\boldsymbol{U}_0$, given by:

$$\mathcal{M}_0 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}(3+3\sigma) \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^s + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2s} \right\}$$

In particular:

$$\boldsymbol{U}(s) = \begin{pmatrix} C_4 e^s \\ 1 + C_5 e^{2s} \\ \frac{1}{2}(3+3\sigma) \end{pmatrix}$$

for arbitrary constants C_4 and C_5. In the variable ξ, the solutions are given by:

$$A_1(\xi) = 1 + C_5\xi^2 \tag{5.97}$$

$$G_1(\xi) = \frac{1}{2}(3+3\sigma)C_4\xi \tag{5.98}$$

$$v_1(\xi) = c\xi \tag{5.99}$$

with the subscript denoting the fact that (A_1, G_1, v_1) represents a solution to the linearised version of equations (5.27)-(5.29). Now for small ξ, functions A, G and v of FLRW$(0, \sigma, 1)$ are given to leading order as:

$$A \approx 1 - \frac{4}{(3+3\sigma)^2}\Psi_0^2\xi^2 \tag{5.100}$$

$$G \approx \Psi_0\xi \tag{5.101}$$

$$v \approx \frac{2}{3+3\sigma}\Psi_0\xi \tag{5.102}$$

Comparing (5.97)-(5.99) to (5.100)-(5.102) suggests setting C_4 and C_5, without loss of generality, as so:

$$C_4 = \frac{2}{3+3\sigma}\Psi_0$$
$$C_5 = -\frac{4}{(3+3\sigma)^2}\Psi_0^2 a^2$$

where a is an essential parameter. Including σ, we have that (5.97)-(5.99) is a two-parameter family of solutions originating from the fixed point $\boldsymbol{U}_0$ with the leading order approximations of FLRW$(0, \sigma, 1)$ as a one-parameter subset. The FLRW$(0, \sigma, 1)$ spacetimes correspond to $a = 1$ and any other value of a represents a self-similar perturbation from FLRW$(0, \sigma, 1)$. From this point onwards the value of Ψ_0 will be fixed as:

$$\Psi_0 = \frac{1}{4}(3+3\sigma) \tag{5.103}$$

so as to simplify calculations and match the notation and proceeding definition used by Smoller and Temple in [**21**].

Definition 5.4.1. The *asymptotically Friedmann* spacetimes, denoted by FLRW$(0, \sigma, a)$, are defined as the two-parameter family of solutions to (5.27)-(5.33) with the following leading order form as $\xi \to 0$:

$$\begin{aligned} A(\xi) &\approx 1 - \frac{1}{4}a^2\xi^2 \\ G(\xi) &\approx \frac{1}{4}(3+3\sigma)\xi \\ v(\xi) &\approx \frac{1}{2}\xi \end{aligned}$$

Furthermore, the parameter a is referred to as the *acceleration parameter.*

The asymptotic form of the FLRW$(0, \sigma, a)$ spacetimes were first found by [**6**], although this asymptotic form was given in comoving coordinates. The FLRW$(0, \sigma, a)$ spacetimes are exact solutions of Einstein's field equations, even though they are not known explicitly. Despite this, we can still give a leading order approximation of the FLRW$(0, \sigma, a)$ solutions local to the centre of expansion.

Proposition 5.4.2. *The FLRW$(0, \sigma, a)$ spacetimes are given in self-similar Schwarzschild coordinates to leading order as $\xi \to 0$ as so:*

$$\begin{aligned} d\bar{s}^2 &\approx -\frac{16}{(3+3\sigma)^2}\left(1+\frac{1}{4}a^2\xi^2\right)d\bar{t}^2 + \left(1+\frac{1}{4}a^2\xi^2\right)d\bar{r}^2 + \bar{r}^2 d\Omega^2 \\ v &\approx \frac{1}{2}\xi \\ \rho &\approx \frac{3a^2\xi^2}{4\kappa\bar{r}^2} \\ p &= \sigma\rho \end{aligned} \tag{5.104}$$

Proof. This follows from Proposition 5.3.1 and Definition 5.4.1 by noting that:

$$B = \frac{\xi^2}{AG^2}$$

□

In the pure radiation case, that is, for $\sigma = \frac{1}{3}$, Smoller and Temple [**21**] summarise an extension of the leading order expansion in their Theorem 10, which is paraphrased as follows.

THEOREM 5.4.1. *Let (v, A, H) denote an FLRW$(0, \frac{1}{3}, a)$ spacetime and let (v_1, A_1, H_1), with $G_1 = H_1 v_1$, denote FLRW$(0, \frac{1}{3}, 1)$. Then the following estimates hold:*

$$v(\xi) = v_1(\xi) + \frac{1-a^2}{20}\xi^3 + |a-1| O\left(\xi^5\right) \tag{5.105}$$

$$A(\xi) = 1 - a^2 v_1(\xi) + \frac{3a^2(a^2-1)}{40}\xi^4 + |a-1| O\left(\xi^6\right) \tag{5.106}$$

$$H(\xi) = H_1(\xi) + \frac{a^2-1}{5}\xi^2 + |a-1| O\left(\xi^4\right) \tag{5.107}$$

$$= \frac{\xi}{v(\xi)} + |a-1| O\left(\xi^4\right) \tag{5.108}$$

$$G(\xi) = \xi - \frac{6-7a^2+a^4}{100}\xi^5 + |a-1| O\left(\xi^7\right) \tag{5.109}$$

$$A(\xi)B(\xi) = 1 - \frac{6-7a^2+a^4}{50}\xi^4 + |a-1| O\left(\xi^6\right) \tag{5.110}$$

The proof of this theorem will not be given as such expansions can be found without much difficulty using modern symbolic manipulation software. Note that (5.110) implies:

$$\sqrt{A(\xi)B(\xi)} = 1 - \frac{6-7a^2+a^4}{100}\xi^4 + |a-1| O\left(\xi^6\right) \tag{5.111}$$

The factor $\sqrt{AB}$ is what converts the velocity v, measured relative to the speed of light, over to coordinate velocity:

$$\frac{d\bar{r}}{d\bar{t}} = \frac{\bar{u}^1}{\bar{u}^0} = \sqrt{AB}v$$

Smoller and Temple remark that expansion (5.111) implies that for small ξ and $|a-1|$, it could be difficult to measure the dilation of time between the $a \neq 1$ FLRW$(0, \sigma, a)$ spacetimes and the Standard Model.

COROLLARY 5.4.1. *When $a = 1$, the density $\rho = \rho_1$ of FLRW$(0, \sigma, a)$ is given exactly by:*

$$\kappa \rho_1 \bar{r}^2 = \Phi_1(\xi) \tag{5.112}$$

$$\Phi_1(\xi) = \frac{3(1+v_1^2)v_1^2}{1-v_1^2} \tag{5.113}$$

and when $a \neq 1$ the density satisfies the following expansion:

$$\kappa\rho\bar{r}^2 = \Phi(\xi) \tag{5.114}$$

$$\Phi(\xi) = \Phi_1(\xi) + \frac{3}{4}(a^2-1)\xi^2 - \frac{3}{8}(2-3a^2+a^4)\xi^4 + |a-1|O\left(\xi^6\right) \tag{5.115}$$

We can gain a slightly more intuitive comparison between the FLRW$(0, \frac{1}{3}, a)$ spacetimes and the Standard Model in the Radiation Dominated Epoch, that is, to FLRW$(0, \frac{1}{3}, 1)$, by considering an extension of the coordinate transformation (5.9)-(5.10) to $a \neq 1$. That is, if we make the transformation:

$$\bar{t} = \left(1 + \frac{a^2}{4}\zeta^2\right)t \tag{5.116}$$

$$\bar{r} = t^{\frac{a}{2}} r \tag{5.117}$$

then we find the metric (5.104) can be given by:

$$ds^2 = -dt^2 + a(1-a)\zeta dt d\bar{r} + t^a\left(dr^2 + r^2 d\Omega^2\right) \tag{5.118}$$

This metric is close to the FLRW$(0, \frac{1}{3}, 1)$ metric in comoving coordinates, that is, metric (5.1) with $R(t)$ given by (5.7). We see that the difference is a correction to the scale factor, that is, $R_a(t) = t^{\frac{a}{2}}$ instead of $R(t) = t^{\frac{1}{2}}$, and an additional $dtd\bar{r}$ term. Smoller and Temple remark that the constant time slices $t = t_0$ in (5.118) are all the flat space $\mathbb{R}^3$, as they are for FLRW$(0, \frac{1}{3}, 1)$. Furthermore, the $\bar{r} = \bar{r}_0$ slices agree with the FLRW$(0, \frac{1}{3}, 1)$ metric when modified with scale factor $R_a(t)$. It follows that the $t = t_0$ surfaces given by (5.116) and (5.117) define a foliation of spacetime into flat three-dimensional spacelike slices, thus when $a \neq 1$, (5.118) exhibits many of the flat space properties characteristic of the FLRW$(0, \frac{1}{3}, 1)$ spacetime. As a cosmological model, the closer the acceleration parameter is to one, the more spatially homogeneous the associated universe is, with homogeneity increasing closer to centre of expansion. Despite this, any FLRW$(0, \sigma, a)$ spacetime with $a \neq 1$ is still inhomogeneous and thus violates the Cosmological Principle. The acceleration parameter is also responsible for the rate of acceleration of the spacetime, as the next section will discuss. Observational data suggests that the cosmic acceleration in the Radiation Dominated Epoch was small, which corresponds to an acceleration parameter only slightly larger than one.

Thus an FLRW$(0, \frac{1}{3}, a)$ universe with a similar acceleration parameter will appear homogeneous close to the centre of expansion and exhibit a small accelerated expansion similar to the cosmic acceleration found in the Standard Model of Cosmology during the Radiation Dominated Epoch.

5.5. Cosmic Acceleration

Even though the metric (5.118) is similar to the comoving metric (5.1) with (5.7), it is not comoving with the velocity v of (5.105), even at the leading order, when $a \neq 1$. However, because v only depends on a at third order in ξ, we can use the inverse of the original $a = 1$ transformation, given by (5.9) and (5.10), to put (5.104) in approximate comoving coordinates. The advantage of using the inverse of the $a = 1$ transformation over the inverse of the $a \neq 1$ transformation is that we can compare the Hubble constant and redshift vs luminosity relations for FLRW$(0, \frac{1}{3}, a)$ to the Hubble constant and redshift vs luminosity relations for FLRW$(0, \frac{1}{3}, 1)$ in the same coordinate system. In this light, we will let (t, r) denote the comoving coordinates of FLRW$(0, \frac{1}{3}, 1)$ and define $(\bar{t}, \bar{r})$ by (5.9) and (5.10), or (5.116) and (5.117) with $a = 1$. The coordinates (t, r) will thus be our approximate comoving coordinates for FLRW$(0, \frac{1}{3}, a)$. Now from (5.16) and (5.17) we have as $\xi \to 0$:

$$\zeta = \xi + O(\xi^3) \tag{5.119}$$

and this will be helpful in understanding the following theorem, which gives us the approximate comoving form of (5.104). Note that this theorem, along with the rest of the results in this section, all originate from [**21**] unless stated otherwise.

Theorem 5.5.1. *The inverse of the coordinate transformation (5.9) and (5.10) maps (5.104) to (t, r) coordinates as:*

$$ds^2 = F_a^2(\zeta)\left(-dt^2 + tdr^2\right) + tr^2 d\Omega^2 \tag{5.120}$$

where:

$$F_a^2(\zeta) = \frac{1 - \frac{1}{4}\zeta^2}{1 - \frac{a^2}{4}\zeta^2} = 1 + \frac{a^2 - 1}{4}\zeta^2 + |a - 1| O(\zeta^4) \tag{5.121}$$

and the Schwarzschild coordinate velocity v, given by (5.105), maps to the approximate comoving velocity:

$$w = -\frac{a^2-1}{20}\zeta^3 + |a-1|O(\zeta^4) \tag{5.122}$$

Furthermore, we have by (5.105) and (5.119) that:

$$w = v - v_1 + |a-1|O(\zeta^4) \tag{5.123}$$

Now Smoller and Temple remark that the variable $\zeta = \bar{r}/t$ is a natural dimensionless perturbation parameter that has a physical interpretation in (t, r) coordinates because, assuming $c = 1$, ζ ranges from 0 to 1 as $\bar{r}$ ranges from zero to the horizon distance in FLRW$(0, \frac{1}{3}, 1)$. According to Weinberg [**23**], this is approximately the Hubble distance c/H, a measure of the furthest we can see from $r = 0$ at time t after the Big Bang, that is:

$$\zeta \approx \frac{\text{Distance}}{\text{Hubble Length}} \tag{5.124}$$

Thus according to Smoller and Temple in [**20**], an expansion in ζ is an expansion in the fractional distance to the Hubble length. We can use this to compare the expansion of FLRW$(0, \frac{1}{3}, a)$ to the expansion of FLRW$(0, \frac{1}{3}, 1)$, but first we need the following definition.

Definition 5.5.1. The *Hubble constant at parameter value a* is defined by:

$$H_a(t,\zeta) = \frac{1}{R_a}\frac{\partial}{\partial t}R_a$$

where:

$$R_a(t,\zeta) = F_a(\zeta)\sqrt{t}$$

is the square root of the coefficient of dr^2 in (5.120).

In this light, we see that:

$$H_a(t,\zeta) = \frac{1}{2t}\left(1 - \frac{3}{8}(a^2-1)\zeta^2 + |a-1|O(\zeta^4)\right) \tag{5.125}$$

and thus the fractional change in the Hubble constant due to the acceleration parameter is given by:

$$\frac{H_a - H}{H} = \frac{3}{8}(a^2-1)\zeta^2 + |a-1|O(\zeta^4)$$

We now calculate the $a \neq 1$ corrections to the redshift vs luminosity relation of FLRW$(0, \frac{1}{3}, 1)$ up to third order in ζ. Smoller and Temple remark that this is a purely theoretical relation, since the Universe was not transparent during the Radiation Dominated Epoch. Our derivation of the redshift vs luminosity relation for (5.120) will follow Grøn-Hervik [**11**], page 289. In this light, the redshift vs luminosity relation for FLRW$(0, \frac{1}{3}, 1)$, is given by:

$$d_l = 2ct_0 z \tag{5.126}$$

and we will now generalise this result to (5.120). To begin, assume that radiation of frequency ν_e is emitted from a source moving at velocity w relative to a comoving observer at (t_e, r_e) within a spacetime described by metric (5.120). Then assume that this radiation is observed at $(t_0, 0)$ with frequency ν_0. Furthermore, let $\bar{\nu}_e$ denote the intermediate frequency of the emitted radiation as measured by the comoving observer fixed at position (t_e, r_e). Now for FLRW$(0, \frac{1}{3}, 1)$, $w = 0$ so $\nu_e = \bar{\nu}_e$, but this is not the case for FLRW$(0, \frac{1}{3}, 1)$, since (5.122) tells us $w \neq 0$ and thus $\nu_e \neq \bar{\nu}_e$. To account for this, first let λ_e denote the wavelength of the radiation emitted at (t_e, r_e) and λ_0 the wavelength observed at $(t_0, 0)$, that is, at $\zeta = 0$. Then define:

$$L := \text{ Absolute Luminosity } = \frac{\text{Energy Emitted by Source}}{\text{Time}} \tag{5.127}$$

$$l := \text{ Apparent Luminosity } = \frac{\text{Power Received}}{\text{Area}} \tag{5.128}$$

and let:

$$d_l := \text{ Luminosity Distance } = \left(\frac{L}{4\pi l}\right)^{\frac{1}{2}} \tag{5.129}$$

$$z := \text{ Redshift Factor } = \frac{\lambda_0 - \lambda_e}{\lambda_e} \tag{5.130}$$

Using the fact that (5.120) is a diagonal metric in comoving coordinates and that the circumferential coordinate does not depend on a, Smoller and Temple remark that the arguments given in [**11**], Section 11.8, can be modified to give the following theorem, which extends the results of [**22**].

THEOREM 5.5.2. *The luminosity distance d_l, as measured by an observer positioned at the radial centre of the spacetime described by metric (5.104) with velocity profile (5.105), is given by the exact formula:*

$$d_l = ct_0\zeta\sqrt{\frac{1+w}{1-w}} \tag{5.131}$$

where the approximate comoving velocity w satisfies:

$$w = -\frac{a^2-1}{20}\zeta^3 + |a-1|O(\zeta^4) \tag{5.132}$$

and the self similar variable ζ satisfies:

$$\zeta = 2z + (a^2-1)z^2 + \frac{(a^2-1)(5a^2+4)}{5}z^3 + |a-1|O(z^4) \tag{5.133}$$

Furthermore, putting (5.132) and (5.133) into (5.131) gives:

$$d_l = 2ct_0\left(z + \frac{a^2-1}{2}z^2 + \frac{(a^2-1)(5a^2+4)}{10}z^3 + |a-1|O(z^4)\right) \tag{5.134}$$

We see that for $a = 1$, (5.134) reduces to (5.126), as expected. Moreover, we see that w does not affect the redshift vs luminosity relation until the fourth order in ζ. Smoller and Temple remark that (5.134) gives the leading order quadratic and cubic corrections to the redshift vs luminosity relation when $a \neq 1$, thereby improving the quadratic estimate (6.5) of [**22**]. Furthermore, since the term (a^2-1) appears in front of the leading order correction in (5.134), it follows by continuous dependence of solutions on their parameters, that the leading order part of any anomalous correction to the redshift vs luminosity relation of the Standard Model, observed at a time after the Radiation Dominated Epoch, can be accounted for by suitable adjustment of the parameter a. Smoller and Temple also note that when $a > 1$, the leading order correction in (5.134) implies a blue-shift of the radiation relative to the Standard Model.

We now give the statements of Proposition 5.5.2 and Lemmas 5.5.3 and 5.5.4. As like with Theorems 5.5.1 and 5.5.2, the proofs will not be given, but can instead be found in [**21**]. In this light, let P denote the power of radiation received at the mirror of a reflecting telescope of area $\mathcal{A}$, positioned at the coordinate centre transverse to the radial direction, with the radiation being emitted at a distant source moving at velocity w at (t_e, r_e) and received at $(t_0, 0)$. From [**11**], page 289, we have:

$$P = \frac{\Delta \text{Energy}}{\Delta \tau_0} = L \cdot f_{\mathcal{A}} \cdot \frac{\nu_0}{\nu_e} \cdot \frac{\Delta \tau_e}{\Delta \tau_0} \tag{5.135}$$

where:

$$L = \frac{\Delta \text{Energy}}{\Delta \tau_e}$$

is the absolute luminosity, the energy per time emitted by the source (5.128). Now the ratio of the frequencies, given by:

$$\frac{\nu_0}{\nu_e} = \frac{1}{1+z}$$

accounts for the energy lost from the red-shifting at the source, whereas the ratio of proper times, given by:

$$\frac{\Delta \tau_e}{\Delta \tau_0} = \frac{1}{1+z}$$

accounts for the proper time change from the receiver to the source. If we define $f_{\mathcal{A}}$ to be the fraction of the emitted radiation received at the mirror $\mathcal{A}$, then for FLRW$(0, \frac{1}{3}, 1)$, equation (11.116), page 289 of [**11**] tells us that:

$$f_{\mathcal{A}} = \frac{\mathcal{A}}{4\pi t_0 r_e^2} \tag{5.136}$$

The following proposition then provides a correction for FLRW$(0, \frac{1}{3}, a)$.

Proposition 5.5.2. *The value of $f_{\mathcal{A}}$ for the family of FLRW$(0, \frac{1}{3}, a)$ spacetimes is given by:*

$$f_{\mathcal{A}} = \frac{\mathcal{A}}{4\pi t_0 r_e^2} C_a \tag{5.137}$$

where C_a has the exact expression:

$$C_a = \frac{1}{F_a^2(\zeta)} \frac{(1+w)^2}{1-w^2} = 1 - \frac{a^2-1}{4}\zeta^2 + |a-1|O(\zeta^3) \tag{5.138}$$

Proposition 5.5.2 solves what Smoller and Temple term the *mirror problem*, that is, it gives the ratio C_a of an area $\mathcal{A}$ of light received from a distant source at a mirror positioned at the origin of FLRW$(0, \frac{1}{3}, a)$, to the corresponding area when the mirror is positioned at the origin of FLRW$(0, \frac{1}{3}, 1)$, in the limit $\mathcal{A} \to 0$. The limit states that the mirror is small in comparison to the distance to the source. We conclude this section with the following two lemmas.

Lemma 5.5.3. *Assume that radiation of frequency ν_e is emitted by a source moving at velocity w at (t_e, r_e) and observed at frequency ν_0 at $(t_0, 0)$ within a spacetime described by metric (5.120). Then r_e is related to t_0 by:*

$$r_e = \frac{\zeta}{1+\frac{\zeta}{2}}\sqrt{t_0} \tag{5.139}$$

where, for this section:

$$\zeta = \frac{r_e}{\sqrt{t_e}}$$

$$d_l = \left(\frac{L}{4\pi l}\right)^{\frac{1}{2}} = \frac{t_0(1+z)\zeta}{\left(1+\frac{\zeta}{2}\right)\sqrt{C_a}} \tag{5.140}$$

$$1+z = \left(1+\frac{\zeta}{2}\right)\frac{1}{F_a(\zeta)}\sqrt{\frac{1+w}{1-w}} \tag{5.141}$$

Lemma 5.5.4. *The following relation holds between the frequency ν_e emitted by a source moving at velocity w at (t_e, r_e) and the frequency $\bar{\nu}_e$ as measured in the comoving frame at (t_e, r_e):*

$$\frac{\nu_e}{\bar{\nu}_e} = \sqrt{\frac{1+w}{1-w}} \tag{5.142}$$

CHAPTER 6

General Relativistic Shock Waves that Induce Cosmic Acceleration

This chapter combines the techniques employed by Smoller and Temple, Cahill and Taub and Carr and Coley to construct a new family of general relativistic shock waves with asymptotically Friedmann interiors. All results in this chapter are either new or offer simpler alternatives to proving some of the previously stated results. As will be seen, some of the previously stated theorems from Chapter 2 fall out almost effortlessly from a few lemmas given in the first two sections. This chapter introduces two new theorems, the first of which addresses the Lax stability of all subluminal general relativistic shock waves with static exteriors under the assumptions of spherical symmetry and self-similarity of the first kind. The second theorem formalises the existence of a general relativistic shock wave with an asymptotically Friedmann interior and a pure radiation equation of state each side of the shock. The latter theorem introduces a novel dynamical systems method not previously considered in this field of study.

6.1. Friedmann-Static Shock Waves

The objective of this section is the construction of the family of FLRW$(0, \sigma, a)$-TOV$(\bar{\sigma})$ shock waves. As like in Chapter 5, we will use the triple (A, G, v) to denote a solution of the spherically symmetric self-similar perfect fluid Einstein field equations (5.27)-(5.29).

Definition 6.1.1. A shock-wave solution with an FLRW$(0, \sigma, a)$ spacetime on the interior and a TOV$(\bar{\sigma})$ spacetime on the exterior will be referred to as a *Friedmann-static shock wave* and denoted by FLRW$(0, \sigma, a)$-TOV$(\bar{\sigma})$.

Since all Friedmann-static shock waves share a TOV($\bar{\sigma}$) exterior, the following lemma will be of great utility in their construction.

LEMMA 6.1.2. *Let* (A, G, v) *denote a spherically symmetric similarity solution to the perfect fluid Einstein field equations with equation of state* $p = \sigma\rho$*. If there exists a* $\xi_0 > 0$ *such that:*

$$A(\xi_0) = 1 - 2M(\bar{\sigma}) \tag{6.1}$$

then (A, G, v) *can be matched to TOV(*$\bar{\sigma}$*) on the surface* $\xi = \xi_0$ *and the Rankine-Hugoniot jump condition is given by:*

$$\frac{[\sigma + v^2(\xi_0)]G(\xi_0) - (1+\sigma)G^2(\xi_0)v(\xi_0)}{[1 + \sigma v^2(\xi_0)]G(\xi_0) - (1+\sigma)v(\xi_0)} = \bar{\sigma} \tag{6.2}$$

PROOF. Let the metric of the (A, G, v) solution in self-similar Schwarzschild coordinates be given by:

$$ds^2 = -B(\xi)dt^2 + \frac{1}{A(\xi)}dr^2 + r^2 d\Omega^2$$

and recall by Proposition 5.2.2 that TOV($\bar{\sigma}$) is given in self-similar comoving Schwarzschild coordinates as:

$$\begin{aligned}
d\bar{s}^2 &= -\bar{\xi}^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}} d\bar{t}^2 + \frac{1}{1 - 2M(\bar{\sigma})} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \\
\bar{\rho} &= \frac{M(\bar{\sigma})}{4\pi\bar{r}^2} \\
\bar{p} &= \bar{\sigma}\bar{\rho}
\end{aligned}$$

where the inessential parameter has been set to one and:

$$M(\bar{\sigma}) = \frac{2\bar{\sigma}}{1 + 6\bar{\sigma} + \bar{\sigma}^2}$$

Because both metrics are specified in Schwarzschild coordinates, the $d\Omega^2$ components automatically match under the identification $\bar{r} = r$. Matching the dr^2 components implies that the shock surface is defined by $\xi = \xi_0$, with the constant ξ_0 given implicitly by (6.1). This also implies that B is constant on the surface. A temporal rescaling of the form:

$$\bar{t} = \alpha t$$

implies:

$$\alpha\bar{\xi} = \xi = \xi_0$$

and matches the dt^2 coefficients providing α satisfies:

$$\alpha^2 \bar{\xi}_0^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}} = B(\xi_0) \tag{6.3}$$

With the matching in place, recall from Section 2.1 that the Rankine-Hugoniot jump condition is equivalent to:

$$[T^{\mu\nu}]n_\mu n_\nu = T^{\mu\nu}(g, \rho, p, \boldsymbol{u}_{FLRW})n_\mu n_\nu - T^{\mu\nu}(g, \bar{\rho}, \bar{p}, \boldsymbol{u}_{TOV})n_\mu n_\nu = 0$$

where $\boldsymbol{n}$ is the outward normal to the shock surface. Using this and $p = \sigma\rho$ we obtain:

$$(1+\sigma)\rho u^\mu_{FLRW} u^\nu_{FLRW} n_\mu n_\nu + \sigma\rho|\boldsymbol{n}|^2 - (1+\bar{\sigma})\bar{\rho} u^\mu_{TOV} u^\nu_{TOV} n_\mu n_\nu - \bar{\sigma}\bar{\rho}|\boldsymbol{n}|^2 = 0$$

Now since the surface is defined by $\xi = \xi_0$, which is equivalent to:

$$r - \xi_0 t = 0$$

then the components of the normal satisfy:

$$n_\mu dx^\mu = d(r - \xi_0 t) = -\xi_0 dt + dr$$

and so:

$$n_0 = -\xi_0$$

$$n_1 = 1$$

Noting that the metric components are identified on the surface, the following identities are obtained:

$$
\begin{aligned}
|\boldsymbol{n}|^2 &= A(\xi_0) - \xi_0^2 B^{-1}(\xi_0) \\
u^0_{FLRW} &= [1 - v^2(\xi_0)]^{-\frac{1}{2}} B^{-\frac{1}{2}}(\xi_0) \\
u^1_{FLRW} &= v(\xi_0)[1 - v^2(\xi_0)]^{-\frac{1}{2}} A^{\frac{1}{2}}(\xi_0) \\
u^\mu_{FLRW} u^\nu_{FLRW} n_\mu n_\nu &= [1 - v^2(\xi_0)]^{-1} \left[v(\xi_0) A^{\frac{1}{2}}(\xi_0) - \xi_0 B^{-\frac{1}{2}}(\xi_0) \right]^2 \\
u^\mu_{TOV} u^\nu_{TOV} n_\mu n_\nu &= \xi_0^2 B^{-1}(\xi_0)
\end{aligned}
$$

Applying these identities puts the Rankine-Hugoniot jump condition in the following form:

$$
\begin{aligned}
0 = {} & (1+\sigma)[1 - v^2(\xi_0)]^{-1} \left[v(\xi_0) A^{\frac{1}{2}}(\xi_0) - \xi_0 B^{-\frac{1}{2}}(\xi_0) \right]^2 \rho \\
& - (1+\bar{\sigma}) \xi_0^2 B^{-1}(\xi_0) \bar{\rho} \\
& + \left[A(\xi_0) - \xi_0^2 B^{-1}(\xi_0) \right] (\sigma\rho - \bar{\sigma}\bar{\rho})
\end{aligned}
$$

Dividing by $A(\xi_0)$ and substituting $B(\xi_0)$ for $G(\xi_0)$ then yields:

$$
\begin{aligned}
0 = {} & (1+\sigma)[1 - v^2(\xi_0)]^{-1} [v(\xi_0) - G(\xi_0)]^2 \rho \\
& - (1+\bar{\sigma}) G^2(\xi_0) \bar{\rho} \\
& + [1 - G^2(\xi_0)](\sigma\rho - \bar{\sigma}\bar{\rho})
\end{aligned}
$$

Finally, applying (5.33) and (6.1) gives (6.2), which completes the proof. □

As FLRW$(0, \sigma, 1)$ is known explicitly, it is possible to construct an explicit FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock wave. Such a construction is shown in Chapter 3 in the pure radiation case and in full generality in Chapter 2. However, the result can instead be derived directly from Lemma 6.1.2.

Theorem 6.1.1. *For each $0 < \sigma < 1$, FLRW$(0, \sigma, 1)$ can be matched to TOV$(\bar{\sigma})$ to form a general relativistic shock wave providing:*

$$
\bar{\sigma} = H(\sigma)
$$

where:

$$H(\sigma) = \frac{1}{2}\sqrt{9\sigma^2 + 54\sigma + 49} - \frac{3}{2}\sigma - \frac{7}{2} \tag{6.4}$$

PROOF. The matching follows similarly to the matching completed in the proof of Lemma 6.1.2, but with (6.1) and (6.3) replaced with:

$$1 - \frac{2}{3}\hat{\xi}_0^{\frac{2+6\sigma}{3+3\sigma}} = 1 - 2M(\bar{\sigma})$$

and:

$$\alpha^2 \xi_0^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}} = \frac{16}{(3+3\sigma)^2}\left[1 + \frac{1}{3}(1+3\sigma)\hat{\xi}_0^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-\frac{2-6\sigma}{2+6\sigma}}\left[1 - \frac{2}{3}\hat{\xi}_0^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-1}$$

respectively, where the inessential parameter is set by (5.103) and $\hat{\xi}$ is defined implicitly by:

$$\xi = \frac{4}{\sqrt{6}}\hat{\xi}^{\frac{1+3\sigma}{3+3\sigma}}\left[1 + \frac{1}{3}(1+3\sigma)\hat{\xi}^{\frac{2+6\sigma}{3+3\sigma}}\right]^{-\frac{3+3\sigma}{2+6\sigma}}$$

Note that this matching is Lipschitz continuous, as any $0 < \sigma < 1$ and $0 < \bar{\sigma} < 1$ imply that the components of the interior and exterior metrics are continuous in a neighbourhood of the surface when given in (t, r) coordinates. Thus it remains to show that the condition $\bar{\sigma} = H(\sigma)$ is equivalent to the Rankine-Hugoniot jump condition, which we know by Lemma 6.1.2 is given by:

$$\frac{[\sigma + v^2(\xi_0)]G(\xi_0) - (1+\sigma)G^2(\xi_0)v(\xi_0)}{[1 + \sigma v^2(\xi_0)]G(\xi_0) - (1+\sigma)v(\xi_0)} = \bar{\sigma}$$

By Proposition 5.3.2, $G(\xi_0)$ can be substituted for $v(\xi_0)$, which in turn can be substituted for $A(\xi_0)$ to yield:

$$\frac{(3\sigma + 3[1 - A(\xi_0)])(2 + (1+3\sigma)[1 - A(\xi_0)]) - (3+3\sigma)^2[1 - A(\xi_0)]}{A(\xi_0)(2 + (1+3\sigma)[1 - A(\xi_0)])} = \bar{\sigma}$$

Finally, substituting $A(\xi_0)$ for $1 - 2M(\bar{\sigma})$ yields:

$$\sigma = \frac{\bar{\sigma}(7 + \bar{\sigma})}{3(1 - \bar{\sigma})}$$

which is equivalent to $\bar{\sigma} = H(\sigma)$. $\square$

Definition 6.1.3. The *Rankine-Hugoniot curve*, denoted by:

$$v = \Gamma_{RH}(G; \sigma, \bar{\sigma})$$

is the curve in (A, G, v) space generated by constraints (6.1) and (6.2).

We are now in a position to extend the family of FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock waves to the family of FLRW$(0, \sigma, a)$-TOV$(\bar{\sigma})$ shock waves. Even though the FLRW$(0, \sigma, a)$ spacetimes are exact solutions, they are not known explicitly away from $\xi = 0$, so these solutions need to be approximated numerically. One way of describing FLRW$(0, \sigma, a)$ solutions is to numerically generate their trajectories in (A, G, v) space, such as in Figure 6.1.

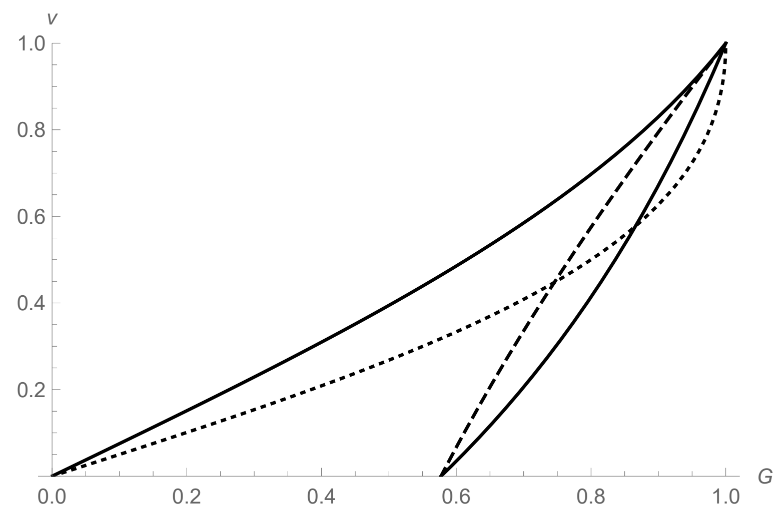

Figure 6.1. This figure is a side view of (A, G, v) space and depicts the most important features. The left and right unbroken curves represent the surfaces $\{\cdot\}_S = 0$ and $\{\cdot\}_D = 0$ respectively. These surfaces have no dependence on A and so remain the same in any constant A plane. The Rankine-Hugoniot curve is represented by the dashed curve and lives in the plane $A = 1 - 2M(\bar{\sigma})$. The dotted curve represents the explicitly known FLRW$(0, \frac{1}{3}, 1)$ trajectory.

The FLRW$(0, \frac{1}{3}, 1)$ trajectory obeys the implicit relationship given by Corollary 5.3.2, that is, as ξ increases from zero, G increases linearly with ξ, v increases according to (5.91) and A decreases according to (5.90). General FLRW$(0, \sigma, a)$ trajectories are similar to the FLRW$(0, \frac{1}{3}, 1)$ trajectory for small ξ but differ as ξ increases. One characteristic that remains similar for larger ξ is the near linear dependence of G on ξ. Note that because equations (5.27)-(5.29) are autonomous, all trajectories, the Rankine-Hugoniot curve and surfaces $\{\cdot\}_S = 0$ and $\{\cdot\}_D = 0$ are all independent of ξ. Thus it is often easier to think of G as the independent variable and consider the trajectory as a function of G.

The TOV($\bar{\sigma}$) trajectories are simple to represent in (A, G, v) solution space as they are the lines defined by $A = 1 - 2M(\bar{\sigma})$ and $v = 0$. Now because:

$$\min_{0 \leq \bar{\sigma} \leq 1} \{1 - 2M(\bar{\sigma})\} = \frac{1}{2} \tag{6.5}$$

the TOV($\bar{\sigma}$) trajectories span the surface:

$$\frac{1}{2} < A < 1$$
$$v = 0$$

The reason considering solutions in (A, G, v) space is so useful, is the immediate implication that any trajectory that crosses the $A = 1 - 2M(\bar{\sigma})$ plane Lipschitz continuously can be matched to the TOV($\bar{\sigma}$) solution. Furthermore, if the solution trajectory crosses the $A = 1 - 2M(\bar{\sigma})$ plane and intersects the Rankin-Hugoniot curve, which lies in this plane, then the solution can be matched to the TOV($\bar{\sigma}$) solution to form a general relativistic shock wave. In the case of FLRW$(0, \sigma, a)$ trajectories, changing the parameters σ and a changes the trajectory, so certain combinations of σ and a result in an intersection with the Rankine-Hugoniot curve, and thus the formation of an FLRW$(0, \sigma, a)$-TOV($\bar{\sigma}$) shock wave. We already know from Theorem 6.1.1 that for $a = 1$ the relationship between σ and $\bar{\sigma}$ obeys $\bar{\sigma} = H(\sigma)$. For $a \neq 1$, trajectories can be generated numerically and the parameters a, σ and $\bar{\sigma}$ can be adjusted to achieve the intersection. Since the intersection imposes a single constraint on the parameters a, σ and $\bar{\sigma}$, we conclude that the family of FLRW$(0, \sigma, a)$-TOV($\bar{\sigma}$) shock waves is a one-parameter family for each σ. Fixing $\sigma = \frac{1}{3}$, the resulting family partially answers Cahill and Taub's claim given in Section 3.10 by determining a subset of the self-similar pure radiation spacetimes that can be matched to TOV($\bar{\sigma}$) to form a general relativistic shock wave.

A physically important Friedmann-static shock wave is the one for which the equation of state on each side of the shock models pure radiation, since these shock waves may have been present during the Radiation Dominated Epoch. As demonstrated in Figure 6.2 for $\sigma = \bar{\sigma} = \frac{1}{3}$, the value of a can be varied in order to achieve an intersection and thus form the Friedmann-static pure radiation shock wave.

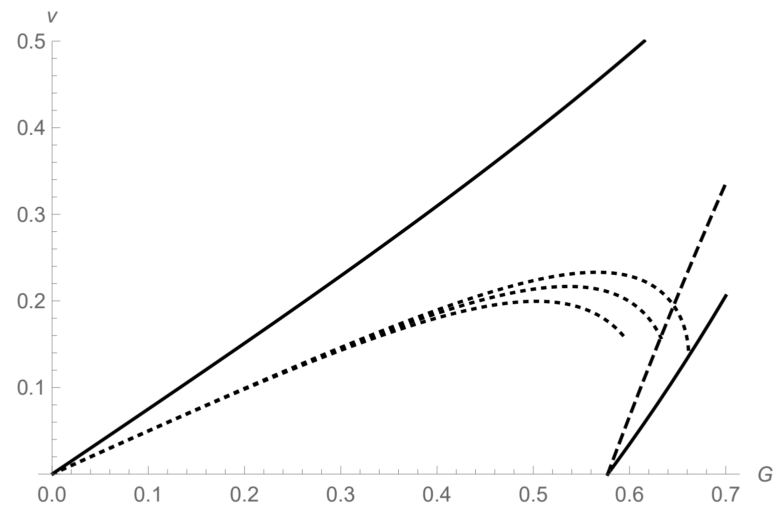

FIGURE 6.2. This figure depicts the same features as Figure 6.1, except the FLRW$(0, \frac{1}{3}, 1)$ trajectory is replaced by three FLRW$(0, \frac{1}{3}, a)$ trajectories with varying values of a. Unlike in Figure 6.1, the trajectories given in this figure are terminated once they reach the $A = 1 - 2M(\bar{\sigma})$ plane.

In Figure 6.2, the leftmost trajectory overshoots the curve and rightmost trajectory undershoots it. The leftmost, centre and the rightmost trajectories are generated for:

$$a = 2.8$$

$$a = 2.58$$

$$a = 2.4$$

respectively. Therefore, the value of the acceleration parameter for the Friedmann-static pure radiation shock wave is approximated by:

$$a \approx 2.58$$

with the corresponding point of intersection approximated by:

$$\xi_0 \approx 0.706$$

We know from Chapter 5 that the FLRW$(0,\sigma,a)$ spacetimes can exhibit an accelerated expansion similar to the accelerated expansion found in the Standard Model of Cosmology when $a \approx 1$. It is conjectured by Temple that the accelerated expansion observed today is not the result of dark energy, but instead from being within a vast primordial shock wave with an FLRW$(0,\sigma,a)$ interior. By vast, we mean a shock wave with a shock surface that lies beyond the Hubble radius, that is, not presently observable. What makes this proposal particularly interesting is that the magnitude of acceleration, parameterised by a, is determined purely mathematically by the equation of state parameter each side of the shock, assuming a TOV$(\bar{\sigma})$ exterior. However, with $a \approx 2.58$, the Friedmann-static pure radiation shock wave exhibits a cosmic acceleration many orders of magnitude larger than what is observed today, and observational data suggests that cosmic acceleration has only increased since the Radiation Dominated Epoch. Furthermore, the Friedmann-static pure radiation shock surface lies within the Hubble radius. Each of these properties rule out the Friedmann-static pure radiation shock wave as a cosmological model, but does not rule out a shock-wave cosmological model consisting of an interior FLRW$(0,\sigma,a)$ spacetime matched to a non-TOV$(\bar{\sigma})$ exterior. For $a = 1$, one such shock wave was constructed with the shock surface lying beyond the Hubble radius by Smoller and Temple in [**19**]. Further details of this construction were then provided in [**20**]. This was completed in 2003, before Smoller and Temple had derived their family of asymptotically Friedmann spacetimes. It remains an open problem to construct shock waves with $a \neq 1$ FLRW$(0,\sigma,a)$ interiors for which the resulting shock surface lies beyond the Hubble radius.

6.2. Lax Stability of Shock Waves with Static Exteriors

LEMMA 6.2.1. *Let (A,G,v) denote a spherically symmetric similarity solution to the perfect fluid Einstein field equations with equation of state $p = \sigma\rho$. If there exists a $\xi_0 > 0$ such that (A,G,v) can be matched to TOV$(\bar{\sigma})$ to form a shock-wave solution, then the Lax characteristic conditions* *are given by:*

$$\frac{\sqrt{\bar\sigma}-v(\xi_0)}{1-\sqrt{\bar\sigma}v(\xi_0)} < \frac{G(\xi_0)-v(\xi_0)}{1-G(\xi_0)v(\xi_0)} < \sqrt{\sigma} \tag{6.6}$$

Proof. As a reverse to the coordinate transformation introduced in Proposition 3.6.1, we begin by transforming a general solution given in self-similar Schwarzchild coordinates, to a solution given in self-similar comoving coordinates. Noting that B and $\boldsymbol{u}$ are given implicitly by the triple (A, G, v), we can write this solution in self-similar Schwarzschild coordinates as so:

$$\begin{aligned} ds^2 &= -B(\xi)dt^2 + \frac{1}{A(\xi)}dr^2 + r^2 d\Omega^2 \\ \boldsymbol{u} &= (u^0, u^1, 0, 0) \end{aligned}$$

where p and ρ are determined by $p = \sigma\rho$ and (5.33) respectively. In self-similar comoving coordinates, the solution can be written as:

$$\begin{aligned} d\hat{s}^2 &= -e^{2\varphi}d\hat{t}^2 + e^{2\psi}d\hat{r}^2 + \mathscr{R}^2\hat{r}^2 d\Omega^2 \\ \hat{\boldsymbol{u}} &= (\hat{u}^0, 0, 0, 0) \end{aligned}$$

To eliminate the radial component of the four-velocity, the transformation from Schwarzschild to comoving coordinates must satisfy:

$$\hat{u}^1 = u^0\frac{\partial\hat{r}}{\partial t} + u^1\frac{\partial\hat{r}}{\partial r} = 0$$

which is equivalent to:

$$\frac{\partial\hat{r}}{\partial t} = -\frac{\xi v}{G}\frac{\partial\hat{r}}{\partial r} \tag{6.7}$$

Now given that:

$$\begin{aligned} d\hat{t} &= \frac{\partial\hat{t}}{\partial t}dt + \frac{\partial\hat{t}}{\partial r}dr \\ d\hat{r} &= \frac{\partial\hat{r}}{\partial t}dt + \frac{\partial\hat{r}}{\partial r}dr \end{aligned}$$

then:

$$dt = \left(\frac{\partial \hat{t}}{\partial t}\frac{\partial \hat{r}}{\partial r} - \frac{\partial \hat{t}}{\partial r}\frac{\partial \hat{r}}{\partial t}\right)^{-1}\left(\frac{\partial \hat{r}}{\partial r}d\hat{t} - \frac{\partial \hat{t}}{\partial r}d\hat{r}\right)$$

$$dr = \left(\frac{\partial \hat{t}}{\partial t}\frac{\partial \hat{r}}{\partial r} - \frac{\partial \hat{t}}{\partial r}\frac{\partial \hat{r}}{\partial t}\right)^{-1}\left(-\frac{\partial \hat{r}}{\partial t}d\hat{t} + \frac{\partial \hat{t}}{\partial t}d\hat{r}\right)$$

Thus to keep the metric diagonal, the following condition is also needed:

$$B\frac{\partial \hat{r}}{\partial r}\frac{\partial \hat{t}}{\partial r} - \frac{1}{A}\frac{\partial \hat{r}}{\partial t}\frac{\partial \hat{t}}{\partial t} = 0$$

which by (6.7) is equivalent to:

$$\frac{\partial \hat{t}}{\partial r} = -\frac{Gv}{\xi}\frac{\partial \hat{t}}{\partial t} \tag{6.8}$$

The most general transformation that preserves self-similarity takes the form:

$$\hat{t} = \mathcal{T}(\xi)t$$

$$\hat{r} = \mathcal{R}(\xi)r$$

and conditions (6.7) and (6.8) determine the functions $\mathcal{T}(\xi)$ and $\mathcal{R}(\xi)$. In self-similar Schwarzschild coordinates the shock speed is given by $\xi = \xi_0$, so in self-similar comoving coordinates the shock speed is given by $\hat{\xi} = \hat{\xi}_0$, where:

$$\hat{\xi} = \frac{\hat{r}}{\hat{t}} = \frac{\mathcal{R}(\xi)r}{\mathcal{T}(\xi)t} = \frac{\mathcal{R}(\xi)}{\mathcal{T}(\xi)}\xi$$

Thus by Lemma 2.3.1 the shock speed is given in interior locally Minkowskian coordinates by:

$$e^{\psi-\varphi}\hat{\xi}_0$$

By Proposition 2.3.2, it remains to determine $e^{\psi-\varphi}$ and $\tilde{w}$. In this light:

$$\begin{aligned} e^{2\varphi} &= \frac{1}{A}\left(\frac{\partial \hat{r}}{\partial t}\right)^2 - \frac{\xi^2}{AG^2}\left(\frac{\partial \hat{r}}{\partial r}\right)^2 \\ &= \frac{\xi^2(1-v^2)}{AG^2}\left(\frac{\partial \hat{r}}{\partial r}\right)^2 \end{aligned}$$

and:

$$\begin{aligned} e^{2\psi} &= \frac{1}{A}\left(\frac{\partial \hat{t}}{\partial t}\right)^2 - \frac{\xi^2}{AG^2}\left(\frac{\partial \hat{t}}{\partial r}\right)^2 \\ &= \frac{1-v^2}{A}\left(\frac{\partial \hat{t}}{\partial t}\right)^2 \end{aligned}$$

Now:

$$\begin{aligned} \frac{\partial \hat{r}}{\partial t} &= -\xi^2 \mathcal{R}'(\xi) \\ \frac{\partial \hat{r}}{\partial r} &= \mathcal{R}(\xi) + \xi \mathcal{R}'(\xi) \end{aligned}$$

so (6.7) yields:

$$\begin{aligned} -\xi^2 \mathcal{R}' &= -\frac{\xi v}{G}(\mathcal{R} + \xi \mathcal{R}') \\ \iff \xi \mathcal{R}' &= \frac{v}{G-v}\mathcal{R} \end{aligned}$$

Similarly:

$$\begin{aligned} \frac{\partial \hat{t}}{\partial t} &= \mathcal{T}(\xi) - \xi \mathcal{T}'(\xi) \\ \frac{\partial \hat{t}}{\partial r} &= \mathcal{T}'(\xi) \end{aligned}$$

to which (6.8) yields:

$$\begin{aligned} \mathcal{T}' &= -\frac{Gv}{\xi}(\mathcal{T} - \xi \mathcal{T}') \\ \iff \xi \mathcal{T}' &= -\frac{Gv}{1-Gv}\mathcal{T} \end{aligned}$$

Therefore the shock speed is given in interior locally Minkowskian coordinates by:

$$\begin{aligned} e^{\psi-\varphi}\hat{\xi}_0 &= G(\xi_0)\frac{\partial \hat{t}}{\partial t}\left(\frac{\partial \hat{r}}{\partial r}\right)^{-1}\frac{\mathcal{R}(\xi_0)}{\mathcal{T}(\xi_0)} \\ &= \frac{G(\xi_0) - v(\xi_0)}{1 - G(\xi_0)v(\xi_0)} \end{aligned}$$

By Proposition 5.2.2, TOV($\bar{\sigma}$) is comoving in Schwarzschild coordinates, and given that TOV($\bar{\sigma}$) is matched to (A, G, v) in (t, r) coordinates, then the $(\bar{t}, \bar{r})$ coordinates of Proposition 2.3.2 are identified with (t, r), so:

$$\begin{aligned}
\tilde{w} &= e^{\psi-\varphi}\frac{\partial \hat{r}}{\partial \bar{t}}\left(\frac{\partial \hat{t}}{\partial \bar{t}}\right)^{-1} \\
&= e^{\psi-\varphi}\frac{\partial \hat{r}}{\partial t}\left(\frac{\partial \hat{t}}{\partial t}\right)^{-1} \\
&= \frac{G(\xi_0)}{\xi_0}\frac{\partial \hat{r}}{\partial t}\left(\frac{\partial \hat{r}}{\partial r}\right)^{-1} \\
&= -v(\xi_0)
\end{aligned}$$

Finally, substituting $e^{\psi-\varphi}\hat{\xi}_0$, $\tilde{w}$ and the equations of state into (2.16) yields (6.6). $\square$

The following theorem was proved in Chapter 2, but can instead be obtained directly from Lemma 6.2.1. In Smoller and Temple's original proof, the value of σ_1 is approximated, however it is now possible to obtain an exact value.

THEOREM 6.2.1. *The FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock-wave solutions satisfy the Lax characteristic conditions for:*

$$0 < \sigma < \sigma_1$$

where:

$$\sigma_1 = \frac{1+\sqrt{10}}{9} \approx 0.462$$

PROOF. By Lemma 6.1.2 and Proposition 5.3.2 we know that FLRW$(0, \sigma, 1)$ satisfies (6.2) and:

$$G(\xi_0) = \frac{1}{2}(3+3\sigma)v(\xi_0)\left(1+\frac{1}{2}(1+3\sigma)v^2(\xi_0)\right)^{-1} \tag{6.9}$$

at the point of intersection with the shock surface. Solving (6.2) and (6.9) for $G(\xi_0)$ and $v(\xi_0)$ yields:

$$G(\xi_0) = \frac{1}{2}(3+\bar{\sigma})v(\xi_0) \tag{6.10}$$

$$v(\xi_0) = \sqrt{\frac{2(3\sigma-\bar{\sigma})}{(1+3\sigma)(3+\bar{\sigma})}} \tag{6.11}$$

Thus using (6.4), (6.10) and (6.11), the left hand inequality of (6.6) is found to be satisfied for $0<\sigma<1$ and the right hand inequality is found to be satisfied for for $0<\sigma<\sigma_1$. □

Lemma 6.2.2. *Let (A, G, v) denote a spherically symmetric similarity solution to the perfect fluid Einstein field equations with equation of state $p = \sigma\rho$. If there exists a $\xi_0 > 0$ such that (A, G, v) can be matched to $TOV(\bar{\sigma})$ to form a shock-wave solution, then the shock speed is subluminal if:*

$$G(\xi_0) < 1 \tag{6.12}$$

and in such a case the Lax characteristic conditions reduce to:

$$G(\xi_0) > \sqrt{\bar{\sigma}} \tag{6.13}$$

$$\{\cdot\}_D(\xi_0) < 0 \tag{6.14}$$

Proof. By Lemma 6.2.1, the shock speed is subluminal if:

$$\frac{G(\xi_0)-v(\xi_0)}{1-G(\xi_0)v(\xi_0)} < 1$$

which for $0<v<1$ is equivalent to (6.12). For $G<1$ it is then not difficult to check that the left hand inequality of (6.6) is equivalent to (6.13). Thus it remains to demonstrate that the right hand inequality is equivalent to (6.14). In this light, we have:

$$\begin{aligned}\{\cdot\}_D &= \frac{3}{4}(3+3\sigma)\left[(G-v)^2-\sigma(1-Gv)^2\right]\\ &= \frac{3}{4}(3+3\sigma)\left[G-v+\sqrt{\sigma}(1-Gv)\right]\left[G-v-\sqrt{\sigma}(1-Gv)\right]\end{aligned}$$

and for $0 < v < G < 1$ we see that $\{\cdot\}_D = 0$ is equivalent to:

$$\frac{G - v}{1 - Gv} = \sqrt{\sigma}$$

which completes the proof. $\square$

The following theorem, also proved in Chapter 2, demonstrates that even though FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock waves can be constructed mathematically, their physical applicability is limited for $\sigma > \sigma_2$.

THEOREM 6.2.2. *The FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock-wave solutions have subluminal shock speeds for:*

$$0 < \sigma < \sigma_2$$

where:

$$\sigma_2 = \frac{\sqrt{5}}{3} \approx 0.745$$

PROOF. This follows directly from Lemma 6.2.2 and relations (6.4), (6.10) and (6.11). $\square$

The following definition is consistent with the sonic surface definition given in Chapter 4, that is, the surfaces have the same physical interpretation.

DEFINITION 6.2.3. The *singular surface* and *sonic surface* are defined in (A, G, v) space by $\{\cdot\}_S = 0$ and $\{\cdot\}_D = 0$ respectively. Moreover, the *subsonic region* and *supersonic region* are defined by $\{\cdot\}_D < 0$ and $\{\cdot\}_D > 0$ respectively.

As a consequence of Lemma 6.2.2, the sonic surface serves as a convenient indicator for the Lax stability of a general relativistic shock wave with a TOV$(\bar{\sigma})$ exterior. This is particularly useful for numerical approximations, since if the intersection with the Rankine-Hugoniot curve is in the subsonic region and to the right of the $G = \sqrt{\bar{\sigma}}$ plane, the resulting shock wave is stable in the Lax sense. For $\sigma \neq \bar{\sigma}$, condition (6.13) is not automatically satisfied, since the Rankine-Hugoniot jump condition curve does not intersect the $v = 0$ plane at $G = \sqrt{\sigma}$, as Figure 6.3 demonstrates.

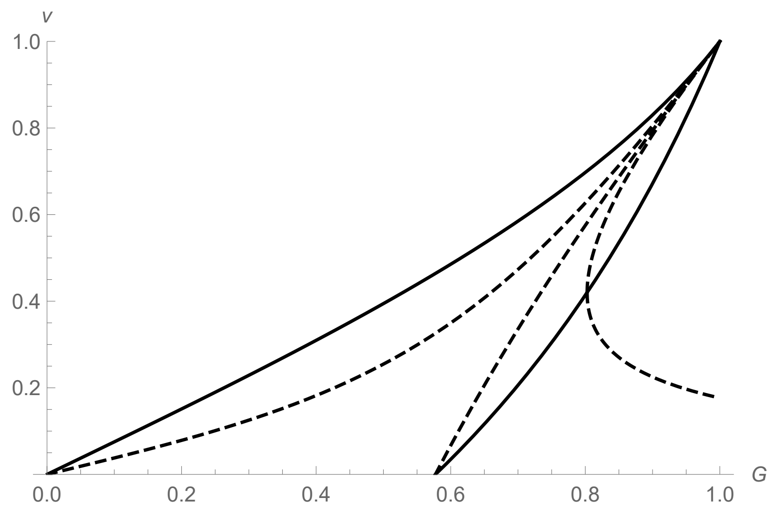

FIGURE 6.3. This figure depicts the singular and sonic surfaces as unbroken curves and three Rankine-Hugoniot curves by dashed curves, all for $\sigma = \frac{1}{3}$.

In Figure 6.3, the leftmost, centre and rightmost dashed curves correspond to:

$$\sigma < \bar{\sigma} = \frac{2}{3}$$
$$\sigma = \bar{\sigma} = \frac{1}{3}$$
$$\sigma > \bar{\sigma} = \frac{1}{6}$$

respectively. In the $\sigma < \bar{\sigma}$ case, the Rankine-Hugoniot curve always touches the singular surface at $(G, v) = (0, 0)$ and $(G, v) = (1, 1)$. In the $\sigma = \bar{\sigma}$ case, the Rankine-Hugoniot curve always touches the sonic surface at $(G, v) = (\sqrt{\sigma}, 0)$ and $(G, v) = (1, 1)$. In the $\sigma > \bar{\sigma}$ case, the Rankine-Hugoniot curve also touches the sonic surface at $(G, v) = (1, 1)$ and intersects it at:

$$G = \frac{\sqrt{\sigma}(1 + \bar{\sigma}) + \sqrt{(\sigma - \bar{\sigma})(1 - \sigma\bar{\sigma})}}{1 + \sigma}$$
$$v = \sqrt{\frac{\sigma - \bar{\sigma}}{1 - \sigma\bar{\sigma}}}$$

Thus for a Friedmann-static shock wave to be unstable in the Lax sense, the solution trajectory must either hit the Rankine-Hugoniot curve before the $G = \sqrt{\bar{\sigma}}$ plane or after passing through the sonic surface. Since conditions (6.13) and (6.14) are always satisfied for $\sigma = \bar{\sigma}$, then the Friedmann-static shock waves for which $\sigma = \bar{\sigma}$ are always stable in the Lax sense, as the following theorem summarises.

THEOREM 6.2.3. *Let (A, G, v) denote a spherically symmetric similarity solution to the perfect fluid Einstein field equations with equation of state $p = \sigma\rho$. If there exists a $\xi_0 > 0$ such that (A, G, v) can be matched to TOV($\bar{\sigma}$) to form a shock-wave solution with a subluminal shock speed, then the Lax characteristic conditions are satisfied if:*

(1) $\sigma = \bar{\sigma}$ *or*
(2) $\sigma < \bar{\sigma}$ *and* $G(\xi_0) > \sqrt{\bar{\sigma}}$ *or*
(3) $\sigma > \bar{\sigma}$ *and* $\{\cdot\}_D(\xi_0) < 0$.

PROOF. This is an immediate consequence of Lemma 6.2.2 and the discussion proceeding Definition 6.2.3. □

6.3. Existence of Friedmann-Static Pure Radiation Shock Waves

Previously in this chapter the Friedmann-static pure radiation shock wave was constructed numerically. This section provides a rigorous proof of this construction. We know from Proposition 5.3.2 that FLRW$(0, \sigma, 1)$ solutions have a certain structure that allow us to determine if and when the solution trajectory crosses the singular or sonic surfaces. However, even though FLRW$(0, \sigma, a)$ solutions can be expected to behave similar to FLRW$(0, \sigma, 1)$ solutions for $a \approx 1$, there is no guarantee that they remain similar as ξ increases, or for larger perturbations of a. We know from Figure 6.2 that the FLRW$(0, \frac{1}{3}, 2.4)$ trajectory differs significantly from the FLRW$(0, \frac{1}{3}, 1)$ trajectory, in particular, the FLRW$(0, \frac{1}{3}, 2.4)$ trajectory encounters a singularity in equation (5.29) by hitting the sonic surface. The following lemma helps to predict the behaviour of FLRW$(0, \sigma, a)$ trajectories.

LEMMA 6.3.1. *Let $0 < \sigma < 1$ and $a > 0$. Then so long as FLRW$(0, \sigma, a)$ satisfies:*

$$A > 1 - 2M(\sigma)$$

$$\{\cdot\}_D < 0$$

it also satisfies:

$$A' < 0 \tag{6.15}$$

$$G' > 0 \tag{6.16}$$

$$v > 0 \tag{6.17}$$

$$\{\cdot\}_S > 0 \tag{6.18}$$

PROOF. Note from Figure 6.1 that $\{\cdot\}_D < 0$ implies the trajectory remains to the left of the sonic surface and $\{\cdot\}_S > 0$ implies that the trajectory remains below the singular surface. The monotonicity of A and G implies that the trajectory advances to the right whilst simultaneously approaching the $A = 1 - 2M(\sigma)$ surface. Now because the FLRW$(0, \sigma, a)$ trajectory begins by satisfying inequalities (6.15)-(6.18), it is sufficient to show that each one of the four inequalities is implied by the other three. In this light, assume $v > 0$ and $\{\cdot\}_S > 0$, then equation (5.27) and the initial conditions $A = 1$ and $A' < 0$ imply inequality (6.15). For inequality (6.16), assume $v > 0$ and $\{\cdot\}_S > 0$ and note that $\{\cdot\}_S > 0$ and $\{\cdot\}_D < 0$ imply $v < 1$. Given these constraints, equation (5.28) implies:

$$\begin{aligned}
\xi \frac{dG}{d\xi} &= -G \left[\left(\frac{1-A}{A} \right) \frac{(3+3\sigma)[(1+v^2)G - 2v]}{2\{\cdot\}_S} - 1 \right] \\
&= G \left[1 - \left(\frac{1-A}{A} \right) \frac{(3+3\sigma)(1+v^2)G - (6+6\sigma)v}{(6+6\sigma v^2)G - (6+6\sigma)v} \right] \\
&> G \left[1 - \left(\frac{1-A}{A} \right) \right] \\
&> 0
\end{aligned}$$

with the last line following from (6.5) and the initial condition $G' > 0$. Now it is sufficient to demonstrate inequality (6.17) in the interval $0 < G < \sqrt{\sigma}$, since the sonic surface intersects the $v = 0$ plane at $G = \sqrt{\sigma}$ and we are assuming that the trajectory stays off the sonic surface. In this light, assume $A' < 0$, $G' > 0$ and $\{\cdot\}_S > 0$ and note that $A' < 0$ implies $A < 1$ and $G' > 0$ implies $G > 0$. By equation (5.29), the sign of v' on the plane $v = 0$ in the region bounded by $1 - 2M(\sigma) < A < 1$ and $0 < G < \sqrt{\sigma}$ is strictly positive, since:

$$
\begin{aligned}
\xi \frac{dv}{d\xi} &= -\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\left[3\sigma\{\cdot\}_S + \left(\frac{1-A}{A}\right)\frac{(3+3\sigma)^2\{\cdot\}_N}{4\{\cdot\}_S}\right] \\
&= \left(\frac{2G}{3(3+3\sigma)(\sigma - G^2)}\right)\left[9\sigma - \frac{(3+3\sigma)^2}{4}\left(\frac{1-A}{A}\right)\right] \\
&> 0
\end{aligned}
$$

Thus any trajectory that begins above the $v = 0$ plane remains above the plane. Therefore the initial condition $v' > 0$ then implies inequality (6.17). Note that this result still holds when $1 - 2M(\bar{\sigma}) < A < 1$ for $0 < \bar{\sigma} \leq \sigma < 1$, since:

$$
\begin{aligned}
9\sigma - \frac{(3+3\sigma)^2}{4}\left(\frac{1-A}{A}\right) &> 9\sigma - \frac{(3+3\sigma)^2}{4}\left(\frac{2M(\bar{\sigma})}{1-2M(\bar{\sigma})}\right) \\
&= 9\sigma - \frac{9\bar{\sigma}(3+3\sigma)^2}{(3+3\bar{\sigma})^2} \\
&= (3+3\sigma)^2\left(\frac{9\sigma}{(3+3\sigma)^2} - \frac{9\bar{\sigma}}{(3+3\bar{\sigma})^2}\right) \\
&\geq 0
\end{aligned}
$$

Finally, inequality (6.18) is demonstrated in a similar manner to inequality (6.17) by showing that trajectories stay away from the surface $\{\cdot\}_S = mv$ for some $0 < m < \frac{3}{2}$. The upper bound for m ensures FLRW$(0, \sigma, a)$ trajectories initially satisfy inequality (6.18). Now assume inequalities (6.15)-(6.17) and note that the inequalities additionally imply $A < 1$ and $G > 0$. Since $\{\cdot\}_S = mv$ is equivalent to:

$$
G = \frac{(3+3\sigma+m)v}{3+3\sigma v^2} \tag{6.19}
$$

then by equation (5.28) and (6.19), we have:

$$
\begin{aligned}
q_A(v;\sigma,m) &= \xi\frac{d}{d\xi}\left(G-\frac{(3+3\sigma+m)v}{3+3\sigma v^2}\right)\Bigg|_{\{\cdot\}_S=mv} \\
&= \left(\xi\frac{dG}{d\xi}-\frac{(3+3\sigma+m)(3-3\sigma v^2)}{(3+3\sigma v^2)^2}\xi\frac{dv}{d\xi}\right)\Bigg|_{\{\cdot\}_S=mv} \\
&= \frac{(3+3\sigma+m)v}{3+3\sigma v^2}\left[1-\left(\frac{1-A}{A}\right)\frac{(3+3\sigma)[(3+3\sigma+m)(1+v^2)-2(3+3\sigma v^2)]}{2m(3+3\sigma v^2)}\right] \\
&+ \frac{(3+3\sigma+m)(3-3\sigma v^2)}{(3+3\sigma v^2)^2}\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\left[3\sigma mv+\left(\frac{1-A}{A}\right)\frac{(3+3\sigma)^2\{\cdot\}_N}{4mv}\right] \\
&= \frac{(3+3\sigma+m)v}{(3+3\sigma v^2)^2}\left[3+3\sigma v^2+\frac{3\sigma m(1-v^2)(3-3\sigma v^2)}{2\{\cdot\}_D}+\left(\frac{1-A}{A}\right)(\{\cdot\}_A+\{\cdot\}_B+\{\cdot\}_C)\right]
\end{aligned}
$$

where:

$$
\begin{aligned}
\{\cdot\}_A &= \frac{(3+3\sigma)(3-3\sigma-m)(1-v^2)}{2m} \\
\{\cdot\}_B &= \frac{(3+3\sigma)^2(1-v^2)(3-3\sigma v^2)\{\cdot\}_N}{8mv^2\{\cdot\}_D} \\
\{\cdot\}_C &= -(3+3\sigma)v^2
\end{aligned}
$$

The objective for this part is to find an m such that $q_A(v;\sigma,m)>0$ for all $0<\sigma<1$ and $0<v<v_*$ for arbitrary $v_*<1$. Note that it is always possible to choose an m small enough to ensure $v_*<v_I(\sigma,m)$, where $v_I(\sigma,m)$ is the intersection of surfaces (6.19) and $\{\cdot\}_D=0$, since:

$$
\lim_{m\to 0} v_I(\sigma,m)=1
$$

Now even though it can be shown that $\{\cdot\}_A+\{\cdot\}_B+\{\cdot\}_C>0$ for a certain interval of v, it is easier to show $\{\cdot\}_A+\{\cdot\}_B>0$ for the whole interval $0<v<v_I$. This the case since:

$$
\begin{aligned}
\{\cdot\}_A+\{\cdot\}_B &= \frac{(3+3\sigma)(1-v^2)}{2m}\left[3-3\sigma-m+\frac{(3+3\sigma)(3-3\sigma v^2)\{\cdot\}_N}{4v^2\{\cdot\}_D}\right] \\
&= \frac{(3+3\sigma)(1-v^2)}{8m(-\{\cdot\}_D)v^2}\left[4(3-3\sigma-m)(-\{\cdot\}_D)v^2-(3+3\sigma)(3-3\sigma v^2)\{\cdot\}_N\right] \\
&= \frac{3\sigma(3+3\sigma)^2(1-v^2)(3-3v^2+n)}{8(-\{\cdot\}_D)(3+3\sigma v^2)^2}\left[3-3v^2+\sigma(9+n)v^2-\sigma(9+n\sigma)v^4\right] \\
&> 0
\end{aligned}
$$

where $m = n\sigma$ for some $0 < n < \frac{3}{2}$. With $\{\cdot\}_A + \{\cdot\}_B > 0$ and $\{\cdot\}_C < 0$, then for $\frac{1}{2} < A < 1$ we have:

$$\left(\frac{1-A}{A}\right)(\{\cdot\}_A + \{\cdot\}_B + \{\cdot\}_C) > \left(\frac{1-A}{A}\right)\{\cdot\}_C > \{\cdot\}_C$$

Thus for any $0 < \sigma < 1$ and $0 < v < v_*$:

$$\begin{aligned}
\lim_{m\to 0} q_A(v;\sigma,m) &= \lim_{n\to 0} q_A(v;\sigma,n\sigma) \\
&> \lim_{n\to 0} \frac{(3+3\sigma+n\sigma)v}{(3+3\sigma v^2)^2}\left[3+3\sigma v^2 + \frac{3n\sigma^2(1-v^2)(3-3\sigma v^2)}{2\{\cdot\}_D} + \{\cdot\}_C\right] \\
&= \lim_{n\to 0} \frac{(3+3\sigma+n\sigma)(1-v^2)v}{(3+3\sigma v^2)^2}\left[3 - \frac{2n\sigma(3+3\sigma)^{-1}(3-3\sigma v^2)(3+3\sigma v^2)^2}{(3-3v^2-n\sigma v^2)^2 - \sigma v^2(3-3v^2+n)^2}\right] \\
&> 0
\end{aligned}$$

Therefore, for any interval $0 < v < v_*$ with $v_* < 1$, there exists an $0 < n < \frac{3}{2}$ such that the surface $\{\cdot\}_S = n\sigma v$ cannot be crossed. Now assume for contradiction that a trajectory crosses the $\{\cdot\}_S = 0$ surface. Because $v_I(\sigma, 0) = 1$ and we assume $\{\cdot\}_D < 0$, the trajectory cannot cross the surface $\{\cdot\}_S = 0$ at $v = 1$, so it must intersect at some point $0 < v_{**} < 1$. Given that FLRW$(0,\sigma,a)$ satisfies $\{\cdot\}_S > n\sigma v$ initially for any $0 < n < \frac{3}{2}$ and we can pick a v_* such that $v_{**} < v_* < 1$, we know that the surface $\{\cdot\}_S = n\sigma v$ cannot be crossed in the interval $0 < v < v_*$, which is a contradiction. Thus under our assumptions, FLRW$(0,\sigma,a)$ satisfies inequality (6.18) and completes the proof. $\square$

With Lemma 6.3.1 in place, we are now in a position to prove the main result.

THEOREM 6.3.1. *There exists an $a > 1$ such that FLRW$(0, \frac{1}{3}, a)$ can be matched to TOV$(\frac{1}{3})$ to form a pure radiation general relativistic shock wave that satisfies the Lax characteristic conditions.*

PROOF. By Lemma 6.1.2 and Definition 6.1.3, for FLRW$(0,\frac{1}{3},a)$ to match with TOV$(\frac{1}{3})$ to form a general relativistic shock wave, then FLRW$(0,\frac{1}{3},a)$ must satisfy:

$$A(\xi_0) = \frac{4}{7} \tag{6.20}$$

$$v(\xi_0) = \Gamma_{RH}\left(G(\xi_0); \frac{1}{3}, \frac{1}{3}\right) \tag{6.21}$$

for some positive constant ξ_0. We know from Theorem 6.1.1 that FLRW$(0, \frac{1}{3}, 1)$ cannot form a general relativistic shock wave with TOV$(\frac{1}{3})$, since $\sigma = \bar{\sigma} = \frac{1}{3}$ is not a solution of $\bar{\sigma} = H(\sigma)$. Instead, when the FLRW$(0, \frac{1}{3}, 1)$ trajectory hits the $A = \frac{4}{7}$ plane, then:

$$v(\xi_0) < \Gamma_{RH}\left(G(\xi_0); \frac{1}{3}, \frac{1}{3}\right)$$

That is, the FLRW$(0, \frac{1}{3}, 1)$ trajectory passes under the Rankine-Hugoniot curve. Note that the explicitly known FLRW$(0, \frac{1}{3}, 1)$ solution is able to cross the sonic surface without becoming singular due to the cancellation of $\{\cdot\}_D$ in equation (5.29) when on the sonic surface. General FLRW$(0, \sigma, a)$ solutions typically become singular at the sonic point, that is, the point of intersection with the sonic surface. Now suppose that there exists a $b > 1$ such that the FLRW$(0, \frac{1}{3}, b)$ trajectory hits the plane $A = \frac{4}{7}$ with:

$$v(\xi_0) > \Gamma_{RH}\left(G(\xi_0); \frac{1}{3}, \frac{1}{3}\right) \tag{6.22}$$

then providing the transition of the FLRW$(0, \frac{1}{3}, 1)$ trajectory to the FLRW$(0, \frac{1}{3}, b)$ trajectory crosses the Rankine-Hugoniot curve, there exists an $1 < a < b$ such that (6.20) and (6.21) are satisfied. An example of this process is demonstrated numerically in Figure 6.2. Lemma 6.3.1 establishes the fact that if the FLRW$(0, \frac{1}{3}, a)$ trajectory remains in the subsonic region, then it must eventually hit the $A = \frac{4}{7}$ plane. The continuous dependence of FLRW$(0, \frac{1}{3}, a)$ on the parameter a means that there is a continuous transition from FLRW$(0, \frac{1}{3}, 1)$ to FLRW$(0, \frac{1}{3}, b)$, at least up until the trajectory hits the $A = \frac{4}{7}$ plane or hits the sonic surface. This continuous transition, along with Lemma 6.3.1, guarantees the crossing of the Rankine-Hugoniot curve in the $\sigma = \bar{\sigma} = \frac{1}{3}$ case, since the transition from hitting the sonic surface to hitting the $A = \frac{4}{7}$ plane occurs on the intersection of the sonic surface with the $A = \frac{4}{7}$ plane, which lies under the Rankine-Hugoniot curve. Thus it is sufficient to rigorously demonstrate the existence of an FLRW$(0, \frac{1}{3}, b)$ solution that satisfies (6.20) and (6.22). We know from Figure 6.2 that a numerical approximation of the FLRW$(0, \frac{1}{3}, \frac{14}{5})$ trajectory passes above the Rankine-Hugoniot curve, so existence is considered for $b = \frac{14}{5}$. Because the FLRW$(0, \frac{1}{3}, a)$ trajectories originate from the fixed point of an unstable manifold, the vector field generated by the system of equations (5.27)-(5.29) points toward the FLRW$(0, \frac{1}{3}, a)$ trajectories when moving away from the fixed point. This fact allows for the construction of a trapping region around the trajectory and this is how the FLRW$(0, \frac{1}{3}, \frac{14}{5})$ trajectory is shown to overshoot the Rankine-Hugoniot curve. Since Lemma 6.3.1 establishes the monotonicity of G as a function of ξ, A and v can be considered as functions of G, with equations (5.27)-(5.29) becoming:

$$\frac{dA}{dG} = -\left(\xi\frac{dG}{d\xi}\right)^{-1}\frac{(3+3\sigma)(1-A)v}{\{\cdot\}_S} \tag{6.23}$$

$$\xi\frac{dG}{d\xi} = -G\left[\left(\frac{1-A}{A}\right)\frac{(3+3\sigma)[(1+v^2)G-2v]}{2\{\cdot\}_S} - 1\right]$$

$$\frac{dv}{dG} = -\left(\xi\frac{dG}{d\xi}\right)^{-1}\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\left[3\sigma\{\cdot\}_S + \left(\frac{1-A}{A}\right)\frac{(3+3\sigma)^2\{\cdot\}_N}{4\{\cdot\}_S}\right] \tag{6.24}$$

In this sense, the trajectory of FLRW$(0, \frac{1}{3}, \frac{14}{5})$ can be represented as $(A(G), v(G))$, with G parameterising the progress of the trajectory towards the $A = \frac{4}{7}$ plane. This step provides a considerable simplification, since the trapping region now only needs to contain A and v. The next step is to construct a trapping region using the Taylor polynomials of A and v about $G = 0$. In this light, define:

$$P_{2N+1}(G) = \sum_{n=0}^{N}\frac{A^{(2n)}(0)}{(2n)!}G^{2n}$$

$$Q_{2N+1}(G) = \sum_{n=0}^{N}\frac{v^{(2n+1)}(0)}{(2n+1)!}G^{2n+1}$$

noting that A and v have even and odd expansions respectively. Furthermore, define:

$$A_M(G) = P_{2N-1}(G) + M_A G^{2N}$$

$$A_m(G) = P_{2N-1}(G) + m_A G^{2N}$$

$$v_M(G) = Q_{2N-1}(G) + M_v G^{2N+1}$$

$$v_m(G) = Q_{2N-1}(G) + m_v G^{2N+1}$$

where M_A, m_A, M_v and m_v are chosen so that:

$$m_A < \frac{A^{(2N)}(0)}{(2N)!} < M_A$$

$$m_v < \frac{v^{(2N+1)}(0)}{(2N+1)!} < M_v$$

The functions A_M and A_m are used to bound A from above and below respectively, with v_M and v_m providing analogous bounds for v. The objective is to show:

$$v_m(G_0) > \Gamma_{RH}\left(G_0; \frac{1}{3}, \frac{1}{3}\right) \tag{6.25}$$

where G_0 is found implicitly through:

$$A_M(G_0) = \frac{4}{7} \tag{6.26}$$

This is so the lowest point of the Taylor trapping region of v remains above the Rankine-Hugoniot curve for the most conservative value of G, which is given by the intersection of the highest point of the Taylor trapping region of A with the $A = \frac{4}{7}$ plane. For large enough N, it is possible to find values for M_A, m_A, M_v and m_v such that (6.25) and (6.26) are satisfied and inequalities:

$$A_m(G) < A < A_M(G) \tag{6.27}$$

$$v_m(G) < v < v_M(G) \tag{6.28}$$

hold for $0 < G < G_0$. This can be done through extensive trial and error, using a numerical approximation of A and v as a guide. Note that the Taylor expansions of A and v converge quicker for smaller values of a, but larger values of a allow for (6.25) to be more easily satisfied, this is why $a = \frac{14}{5}$ is chosen, as it provides a good compromise. In this light, and using a numerical approximation of A and v as a guide, it is found that $N = 16$ and the following values satisfy (6.25)-(6.28):

$$M_A = (1 + 2^{-7})\frac{A^{(32)}(0)}{(32)!}$$
$$m_A = 2^{-1}\frac{A^{(32)}(0)}{(32)!}$$
$$M_v = 2^{-1}\frac{v^{(33)}(0)}{(33)!}$$
$$m_v = 2^{5}\frac{v^{(33)}(0)}{(33)!}$$

noting that M_v and m_v are chosen in the knowledge that $v^{(33)}(0)$ is negative.

With M_A, m_A, M_v and m_v specified, the Taylor polynomials of A and v can be computed and A_M, A_m, v_M and v_m become known explicitly. The graphs of these bounding functions are given in Figures 6.4 and 6.5.

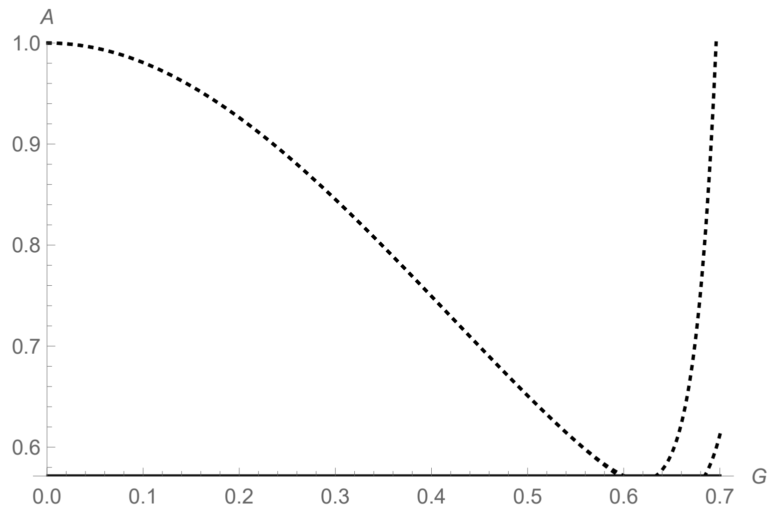

FIGURE 6.4. This figure depicts $A_M(G)$ and $A_m(G)$ by the top and bottom dotted curves respectively. Note that these curves are almost indistinguishable until they cross the $A = \frac{4}{7}$ plane, which is given by the unbroken line at the bottom.

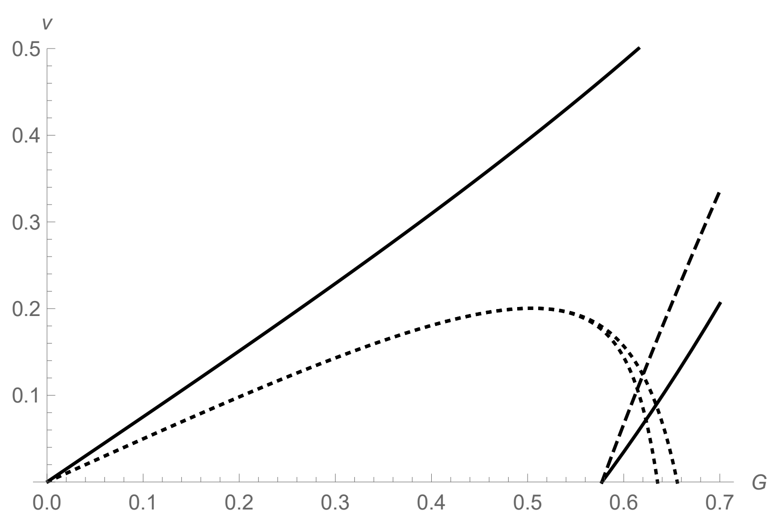

FIGURE 6.5. This figure depicts $v_M(G)$ and $v_m(G)$ by the top and bottom dotted curves respectively. The Rankine-Hugoniot curve is given by the dashed curve and the singular and sonic surfaces are given as unbroken curves.

Even at 33rd order, Figure 6.5 shows that v_M and v_m noticeably diverge after passing the Rankine-Hugoniot curve. This is due to the trajectory approaching the sonic surface, where the solution is likely to become singular, resulting in a slower convergence of the Taylor polynomials. With A_M known explicitly, equation (6.26) can be solved, at least approximately, to yield:

$$G_0 \approx 0.601$$

and this results in inequality (6.25) being satisfied, since v_m is also known explicitly. The final, and most difficult step, is to show that inequalities (6.27) and (6.28) hold in the interval $0 < G < G_0$. To do this, the structure of equations (6.23) and (6.24) can be exploited, that is, it is possible to show:

$$\frac{\partial}{\partial v}\frac{dA}{dG} < 0 \tag{6.29}$$

$$\frac{\partial}{\partial A}\frac{dv}{dG} > 0 \tag{6.30}$$

within the region given by (6.27) and (6.28). Starting with (6.29), we have:

$$\begin{aligned}
\frac{\partial}{\partial v}\frac{dA}{dG} &= -\frac{4(1-A)v}{\{\cdot\}_S}\frac{\partial}{\partial v}\left(\xi\frac{dG}{d\xi}\right)^{-1} - \left(\xi\frac{dG}{d\xi}\right)^{-1}\frac{\partial}{\partial v}\frac{4(1-A)v}{\{\cdot\}_S} \\
&= -\frac{4(1-A)G^2}{\{\cdot\}_S^3}\left(\xi\frac{dG}{d\xi}\right)^{-2}\left[4v^2(3-v^2)\left(\frac{1-A}{A}\right) + (3-v^2)\{\cdot\}_S - 2(1-v^2)\{\cdot\}_S\left(\frac{1-A}{A}\right)\right] \\
&< 0
\end{aligned}$$

which holds in the more general region described by $\frac{2}{5} < A < 1$, $v > 0$, $\{\cdot\}_S > 0$ and $\{\cdot\}_D < 0$. For (6.30) we have:

$$\begin{aligned}
\frac{\partial}{\partial A}\frac{dv}{dG} &= -\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\left[\{\cdot\}_S + 4\left(\frac{1-A}{A}\right)\frac{\{\cdot\}_N}{\{\cdot\}_S}\right]\frac{\partial}{\partial A}\left(\xi\frac{dG}{d\xi}\right)^{-1} \\
&\quad - \left(\xi\frac{dG}{d\xi}\right)^{-1}\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\frac{\partial}{\partial A}\left[\{\cdot\}_S + 4\left(\frac{1-A}{A}\right)\frac{\{\cdot\}_N}{\{\cdot\}_S}\right] \\
&= \frac{G}{A^2\{\cdot\}_S}\left(\xi\frac{dG}{d\xi}\right)^{-2}\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\left[(2(1+v^2)G - 4v)\{\cdot\}_S + 4\{\cdot\}_N\right] \\
&> 0
\end{aligned}$$

which holds in the region described by $v > 0$, $\{\cdot\}_D < 0$ and:

$$(2(1+v^2)G - 4v)\{\cdot\}_S + 4\{\cdot\}_N < 0$$

This region is slightly smaller than the region described by $v > 0$, $\{\cdot\}_D < 0$ and $\{\cdot\}_S > 0$, but includes the region given by (6.27) and (6.28) nonetheless. Now by construction, we know that (6.27) and (6.28) are satisfied in the interval $0 < G < G_\epsilon$ for some small $G_\epsilon > 0$, so to demonstrate (6.27) and (6.28) in the interval $0 < G < G_0$, it is sufficient to demonstrate:

$$\frac{d}{dG}(A_M - A)|_{A=A_M} \geq 0 \tag{6.31}$$

$$\frac{d}{dG}(A - A_m)|_{A=A_m} \geq 0 \tag{6.32}$$

$$\frac{d}{dG}(v_M - v)|_{v=v_M} \geq 0 \tag{6.33}$$

$$\frac{d}{dG}(v - v_m)|_{v=v_m} \geq 0 \tag{6.34}$$

in the interval $G_\epsilon \leq G < G_0$. Note that the left hand sides of (6.31)-(6.34) are functions of A, v and G, so (6.29) can be used to determine the most conservative value of v in (6.31) and (6.32), and (6.30) can be used to determine the most conservative value of A in (6.33) and (6.34). In particular, the most conservative choice out of v_M and v_m for (6.31) is v_m and the most conservative choice for (6.32) is v_M. Likewise, the the most conservative choice out of A_M and A_m for (6.33) is A_M and the most conservative choice for (6.34) is A_m. This can be interpreted as remaining within the right wall of the trapping region implies remaining below the ceiling, and remaining below the ceiling implies remaining within the left wall and so on. Such an interpretation can be summarised as so:

$$\begin{array}{ccc} A < A_M & \Rightarrow & v < v_M \\ \Uparrow & & \Downarrow \\ v > v_m & \Leftarrow & A > A_m \end{array}$$

Now using these conservative choices, the left hand sides of (6.31)-(6.34) become explicitly known functions of G and thus the interval for which they remain positive can be calculated, at least approximately.

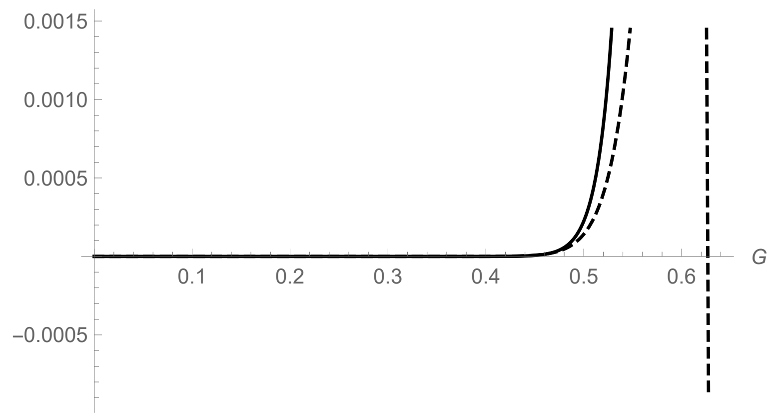

FIGURE 6.6. This figure depicts $\frac{d}{dG}(A_M - A)|_{A=A_M}$ and $\frac{d}{dG}(A - A_m)|_{A=A_m}$ as unbroken and dashed curves respectively.

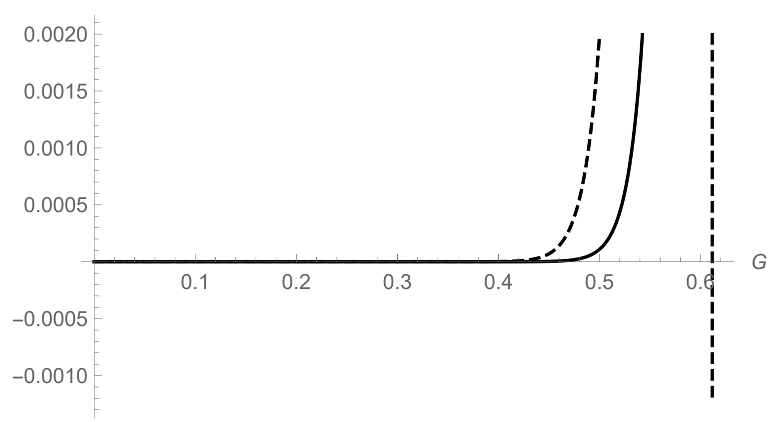

FIGURE 6.7. This figure depicts $\frac{d}{dG}(v_M - v)|_{v=v_M}$ and $\frac{d}{dG}(v - v_m)|_{v=v_m}$ as unbroken and dashed curves respectively.

From Figures 6.6 and 6.7, the intervals for which (6.31)-(6.34) hold are given by:

$$0 < G < G_1$$

$$0 < G < G_2$$

$$0 < G < G_3$$

$$0 < G < G_4$$

respectively, where:

$$G_1 > G_I$$

$$G_2 \approx 0.627$$

$$G_3 > G_I$$

$$G_4 \approx 0.612$$

and G_I is the value of G for which v_m intersects the sonic surface. Since (6.25) has already been established, then $G_0 < G_I$ and thus:

$$G_0 < \min\{G_1, G_2, G_3, G_4\}$$

Therefore (6.27) and (6.28) hold in the interval $0 < G < G_0$ and since the Lax characteristic conditions follow by Theorem 6.2.3, the proof is complete. $\square$

CHAPTER 7

Concluding Remarks

Throughout the preceding five chapters, the necessary machinery was introduced, and in some places developed, to enable the construction of a new family of exact general relativistic shock waves that induce a cosmic acceleration. This construction partially resolves an open problem posed by Cahill and Taub in 1970 and fully resolves Smoller and Temple's open problem of determining the expanding waves created behind a shock-wave explosion into a static isothermal sphere. We saw in Chapter 2 that this family of shock waves are one derivative less regular in Schwarzschild coordinates than they actually are and that any delta function sources are cancelled within the Einstein tensor. In Chapter 3 we found that the interior and exterior spacetimes are uniquely determined by their initial data on the shock surface and that spherical symmetry and self-similarity of the first kind restrict the types of barotropic equations of state that can be modelled. Chapter 4 introduced us to the different families of spacetimes that we could consider for additional self-similar shock-wave models and provided a physical insight into the interior of the new family of shock waves. Chapter 5 built the machinery required for a phase space analysis and demonstrated the cosmic acceleration property inherent in asymptotically Friedmann spacetimes. Chapter 6 brought all this machinery together and established the existence and Lax stability of this new family of general relativistic shock waves, and in doing so, provided a mechanism for exhibiting an accelerated expansion whilst removing the central singularity from the exterior static isothermal sphere.

Given that formal existence has been demonstrated in the pure radiation case, the obvious follow-up question is whether it is possible to formally demonstrate the existence of the full two-parameter family. For a certain range of values of σ and $\bar{\sigma}$, there is no reason to suspect otherwise.

Conjecture 7.0.1. *For $0 < \bar{\sigma} \leq \sigma \leq \frac{1}{3}$, there exists an $a > 0$ such that $FLRW(0, \sigma, a)$ can be matched to $TOV(\bar{\sigma})$ to form a general relativistic shock wave.*

The resolution of this conjecture is one avenue of future research. The continuous dependence of the solution trajectories on the parameters means that formal existence is all but guaranteed for $\bar{\sigma}, \sigma \approx \frac{1}{3}$ and $\bar{\sigma} \approx H(\sigma)$. Moreover, the existence proof in the pure radiation case is readily modified to demonstrate the formal existence for any fixed pair $0 < \bar{\sigma} \leq \sigma \leq \frac{1}{3}$. The difficulty arises when generalising the proof from fixed parameter values to two-dimensional parameter spaces, since conservative estimates need to be satisfied for all values of σ and $\bar{\sigma}$ in such spaces. It is likely to be possible to construct such a proof by patching together many subproofs demonstrating existence in small two-dimensional parameter spaces, although this method may be rather tedious.

Another avenue of future research is in regard to the possible cosmological applications of Friedmann-static shock waves. It is shown in Chapter 6 that the Friedmann-static pure radiation shock wave yields an acceleration parameter value of $a \approx 2.58$. For reference, the acceleration parameter that would be expected in the Radiation Dominated Epoch, according to Smoller and Temple, would likely satisfy $a \approx 1$. In addition, Smoller and Temple demonstrate in [**20**] that Friedmann-static shock waves have shock fronts that would already be observable, as the shock surface would lie within our current Hubble radius. Just one of these implications rules out Friedmann-static shock waves as cosmological models in the Radiation Dominated Epoch, but there remains an interesting modification to these shock waves that keeps Temple's conjecture open.

Smoller and Temple demonstrate in [**20**] that it is possible to construct a shock wave, with a shock surface beyond the Hubble radius, by modelling the entire Universe as a finite mass explosion within the Schwarzschild radius of a time-reversed black hole. Their shock wave consisted of an FLRW spacetime on the interior and a modified TOV spacetime on the exterior. This modification was not known explicitly but incorporated the swapping of the temporal and radial variables in the metric to account for being within the Schwarzschild radius of a black hole.

The possibility remains to construct modified Friedmann-static shock waves with shock surfaces beyond the Hubble radius and determine the resulting rate of expansion. If the predicted rate of expansion lies within current estimates, then such a model offers a mathematically independent derivation for the cosmic acceleration observed today without dark energy.

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