Title
THE GRAETZ PROBLEM

Author
Newman, John.

Publication Date
1969

THE GRAETZ PROBLEM

John Newman

January 1969

LAWRENCE RADIATION LABORATORY
UNIVERSITY of CALIFORNIA BERKELEY

DISCLAIMER

Submitted to Journal of Heat Transfer

UCRL-18646
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract W-7405-eng-48

THE GRAETZ PROBLEM

John Newman

January 1969

The Graetz Problem

John Newman

Inorganic Materials Research Division,
Lawrence Radiation Laboratory, and
Department of Chemical Engineering,
University of California, Berkeley

January, 1969

Abstract

Previous work is reviewed, a new method is outlined for calculating the lower eigenvalues and coefficients for the Graetz series, and new asymptotic forms for large eigenvalues are presented. Good agreement can now be obtained between the Graetz series and the Lévêque series. The region where axial diffusion is important is treated at high Péclet numbers by the method of singular perturbations, thus providing a reasonably complete picture of the transfer process.

1. Introduction

A classical, nontrivial example of convective heat or mass transfer is the Graetz problem—transfer from the wall of a tube of radius R to a fluid in fully developed, laminar flow, with an insulating wall upstream of a point $z = 0$ and a wall maintained at a constant temperature or concentration downstream of the point $z = 0$. Phrased in terms of mass transfer, the partial differential equation is

$$v_z \frac{\partial c_i}{\partial z} = 2\langle v_z\rangle\left(1 - \frac{r^2}{R^2}\right)\frac{\partial c_i}{\partial z} = D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c_i}{\partial r}\right) + \frac{\partial^2 c_i}{\partial z^2}\right] , \qquad (1)$$

with boundary conditions which may be stated as follows:

$$\left.\begin{array}{c} c_i = c_o \text{ at } r = R,\ z > 0;\ \partial c_i/\partial r = 0 \text{ at } r = R,\ z < 0; \\ \partial c_i/\partial r = 0 \text{ at } r = 0 ; \\ c_i \to c_b \text{ as } z \to -\infty; \text{ and } c_i \to c_o \text{ as } z \to +\infty . \end{array}\right\} \qquad (2)$$

In terms of the dimensionless variables

$$\xi = \frac{r}{R} , \qquad \Theta = \frac{c_i - c_o}{c_b - c_o} , \qquad \zeta = \frac{zD}{2\langle v_z\rangle R^2} , \qquad (3)$$

the problem becomes

$$(1-\xi^2)\frac{\partial\Theta}{\partial\zeta} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right) + \frac{1}{Pe^2}\frac{\partial^2\Theta}{\partial\zeta^2} , \qquad (4)$$

where $Pe = 2R\langle v_z \rangle / D$ is the Péclet number, with the boundary conditions

$$\left.\begin{array}{c} \Theta = 0 \text{ at } \xi = 1,\ \zeta > 0;\ \partial\Theta/\partial\xi = 0 \text{ at } \xi = 1,\ \zeta < 0; \\ \partial\Theta/\partial\xi = 0 \text{ at } \xi = 0; \\ \Theta \rightarrow 1 \text{ as } \zeta \rightarrow -\infty; \text{ and } \Theta \rightarrow 0 \text{ as } \zeta \rightarrow +\infty \ . \end{array}\right\} \quad (5)$$

Graetz[6] treated this problem by the method of separation of variables under the condition that the last term in equation 4 could be neglected. This term, representing axial diffusion, will be small when the Péclet number is large, which holds for most practical applications. In this manner Graetz obtained a series solution involving functions of the radial distance, $R_k(\xi)$, which are defined by a Sturm-Liouville system where the separation parameter is restricted to discrete eigenvalues λ^2_k. Other workers have refined the Graetz solution. Results for eigenvalues and coefficients are summarized in table 1.

Asymptotic forms of the eigenvalues and coefficients have been worked out for large eigenvalues, notably by Lauwerier[10] and by Sellars, Tribus, and Klein.[24] Numerical solutions of the partial differential equation of the Graetz problem, with neglect of axial diffusion, have been given by Kays[9] and by Longwell.[15]

Near $z = 0$, the transfer rate becomes infinite, and many terms in the Graetz series are required for an accurate solution in this region. Here it is convenient to use a similarity solution developed by Lévêque[13] and extended by Newman.[17]

Table 1. Eigenvalues and coefficients for the Graetz solution. Additional results are given by Brinkman (1950), Brown (1960), Drew (1931), Lauwerier (1950), and Singh (1958).

Graetz (1883,1885)		Nusselt (1910)		Lee, Nelson, Cherry, Boelter (1939)	Jakob* (1949)		Schenk, Dumoré (1953)	
λ_k	M_k	λ_k	$\frac{1}{8}\lambda_k^2 M_k$	λ_k	$\frac{1}{8}\lambda_k^2 M_k$	M_k	λ_k	M_k
2.7043	0.81747	2.705	0.749	2.704	0.749	0.820	2.70437	0.819
6.50	0.0325	6.66	0.557	6.679	0.539	0.0972	6.6790	0.101
-	-	10.3	0.2515	10.673	0.179	0.0135	10.6733	0.032
-	-	-	-	14.671	-	-	14.6711	0.015

Abramowitz (1953)	Lipkis (1956)	present work		
λ_k	$\frac{1}{8}\lambda_k^2 M_k$	λ_k	$\frac{1}{8}\lambda_k^2 M_k$	M_k
2.7043644	0.74879	2.70436443	0.74877456	0.8190504
6.679032	0.54424	6.67903145	0.54382796	0.0975269
10.67338	0.46288	10.6733795	0.46286106	0.0325040
14.67108	0.41518	14.6710785	0.41541845	0.0154402
18.66987	0.38237	18.6698719	0.38291919	0.0087885

---

*Jakob took his values from Nusselt, but modified the values of M_k.

The axial diffusion term, neglected in the work already mentioned, becomes important at high Péclet numbers only in a small region near z = 0 and r = R, a region which becomes smaller as the Péclet number increases, as is typical of a singular-perturbation problem. With neglect of axial diffusion, the originally elliptic problem becomes parabolic. This region near z = 0 is the only place where the elliptic nature of the problem persists; thus, it is a small elliptic region embedded in an otherwise parabolic domain. Axial diffusion has been treated by Singh[26] and will be treated in section 7 as a singular-perturbation problem.

Schenk and Dumoré[20] have treated the effect of a nonzero transfer resistance of the tube wall, which serves to eliminate the infinite transfer rate at z = 0. Problems involving a catalytic reaction at the tube wall have been treated by Katz[8] and by Solomon and Hudson,[27] and transfer to non-Newtonian fluids has been treated by Schenk and Van Laar.[21]

It seems appropriate at this point in time to review the Graetz problem, which has been treated extensively, and to show how the various phases of the problem are interrelated. This will be done in the order in which they have been introduced. The author was motivated by a desire to try a new method for solving eigenvalue problems and to prepare for presentation in class the complete solution of a classical problem, including the effect of axial diffusion. Furthermore, in an earlier comparison of the Graetz series and the Lévêque series it became apparent that Lipkis's values of the coefficients could not be accurate to the number of significant figures given. (The precise results of Brown[4] were not discovered until the present work had been completed.)

New results are presented for the first five eigenvalues and coefficients (corroborating the results of Brown), for the asymptotic forms for large eigenvalues, and for the region of axial diffusion.

2. The Graetz series

With neglect of axial diffusion, the solution of equations 4 and 5 can be expressed as

$$\Theta = \sum_{k=1}^{\infty} C_k e^{-\lambda_k^2 \zeta} R_k(\xi) \quad , \tag{6}$$

where R_k and λ_k are eigenfunctions and eigenvalues of the Sturm-Liouville system

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dR}{d\xi}\right) + \lambda^2(1-\xi^2)R = 0 \quad , \tag{7}$$

$$R = 0 \text{ at } \xi = 1, \; dR/d\xi = 0 \text{ at } \xi = 0, \; R = 1 \text{ at } \xi = 0 \quad . \tag{8}$$

Furthermore, with neglect of axial diffusion, the solution for $\zeta < 0$ is simply $\Theta = 1$. Consequently, the Graetz series must satisfy the initial condition $\Theta = 1$ at $\zeta = 0$. Because they come from the Sturm-Liouville system 7 and 8, the eigenfunctions are orthogonal with respect to a weighting function of $\xi(1-\xi^2)$. Thus, the coefficients of equation 6 can be obtained by setting $\zeta = 0$, multiplying by $\xi(1-\xi^2)R_n(\xi)$, integrating from $\xi = 0$ to $\xi = 1$, and observing that only one term from the infinite series, that for k = n, is nonzero. The result is

$$C_n = \int_0^1 \xi(1-\xi^2)R_n(\xi)d\xi \bigg/ \int_0^1 \xi(1-\xi^2)R_n^2(\xi)d\xi \quad . \tag{9}$$

We should be interested in the total amount of material J transferred to the wall in a length L:

$$J = -\int_0^L D\frac{\partial c_i}{\partial r}\bigg|_{r=R} 2\pi R dz . \tag{10}$$

This can be expressed as

$$\Theta_m = 1 - \frac{J}{\pi R^2 (c_b - c_o) \langle v_z \rangle} = 1 - 2\frac{\overline{Nu}}{Pe}\frac{L}{R} = \sum_{k=1}^{\infty} M_k e^{-\lambda_k^2 \zeta} , \tag{11}$$

where $\overline{Nu}$ is the average Nusselt number based on the concentration difference at the inlet, $c_b - c_o$, and

$$M_k = 4C_k \int_0^1 \xi(1-\xi^2) R_k(\xi) d\xi \quad . \tag{12}$$

In equation 11, J is divided by the total amount of material which would be transferred in an infinite length of pipe. Equation 11 can also be regarded as expressing the dimensionless, "cup-mixing" concentration difference Θ_m between the wall and the fluid. The local Nusselt number based on the concentration difference at the inlet is readily obtained from equation 11:

$$Nu(\zeta) = \sum_{k=1}^{\infty} \frac{1}{2} \lambda_k^2 M_k e^{-\lambda_k^2 \zeta} \quad . \tag{13}$$

Finally, in later calculations it will be convenient to have available two additional formulas. If the solution R of the Sturm-Liouville system 7 and 8 is regarded as a function of the parameter λ as well as the variable ξ, where R satisfies the conditions at $\xi = 0$ but not necessarily the one at $\xi = 1$, then we obtain

$$\frac{dR}{d\xi}(\xi=1, \lambda=\lambda_k) = -\lambda_k^2 \int_0^1 \xi(1-\xi^2) R_k d\xi \quad , \tag{14}$$

and

$$2\lambda_k \int_0^1 \xi(1-\xi^2)R_k^2 d\xi = \frac{dR}{d\xi}\frac{dR}{d\lambda} \text{ evaluated at } \xi = 1 \text{ and } \lambda = \lambda_k \quad . \tag{15}$$

Consequently we can write

$$\frac{1}{8}\lambda_k^2 M_k = \frac{dR/d\xi}{\lambda dR/d\lambda} \text{ evaluated at } \xi = 1 \text{ and } \lambda = \lambda_k \quad . \tag{16}$$

3. Calculation of eigenvalues and eigenfunctions

Let us add to the Sturm-Liouville system 7 and 8 a second differential equation:

$$\frac{d\lambda^2}{d\xi} = 0 \quad . \tag{17}$$

Then if R and λ^2 are regarded as the unknowns, we have a nonlinear system of two coupled, ordinary differential equations with three boundary conditions at two values of ξ. It should now be possible to solve this problem by standard techniques.[16] For initial guesses we used

$$\left.\begin{aligned} R &= \cos[(k-\tfrac{1}{2})\pi\xi] \ , \\ \lambda^2 &= 2(k-\tfrac{1}{2})^2\pi^2 \quad , \quad k = 1, 2, 3, \ldots \quad . \end{aligned}\right\} \tag{18}$$

Here we know in advance that the solution is not unique, so we select an initial guess which is roughly similar to the Graetz functions. The calculation procedure converged for the first five eigenfunctions and eigenvalues, all that were tried. Calculations were carried out for 100, 200, 400, and 800 mesh intervals, and the results, extrapolated to an infinite number of intervals, are given in table 1. Figure 1 shows the first three eigenfunctions. In table 1, the eigenvalues of Abramowitz are seen to be quite accurate, while the coefficients of Lipkis are in some cases in error in the fourth significant figure. Our values are in good agreement with the more precise values of Brown.[4]

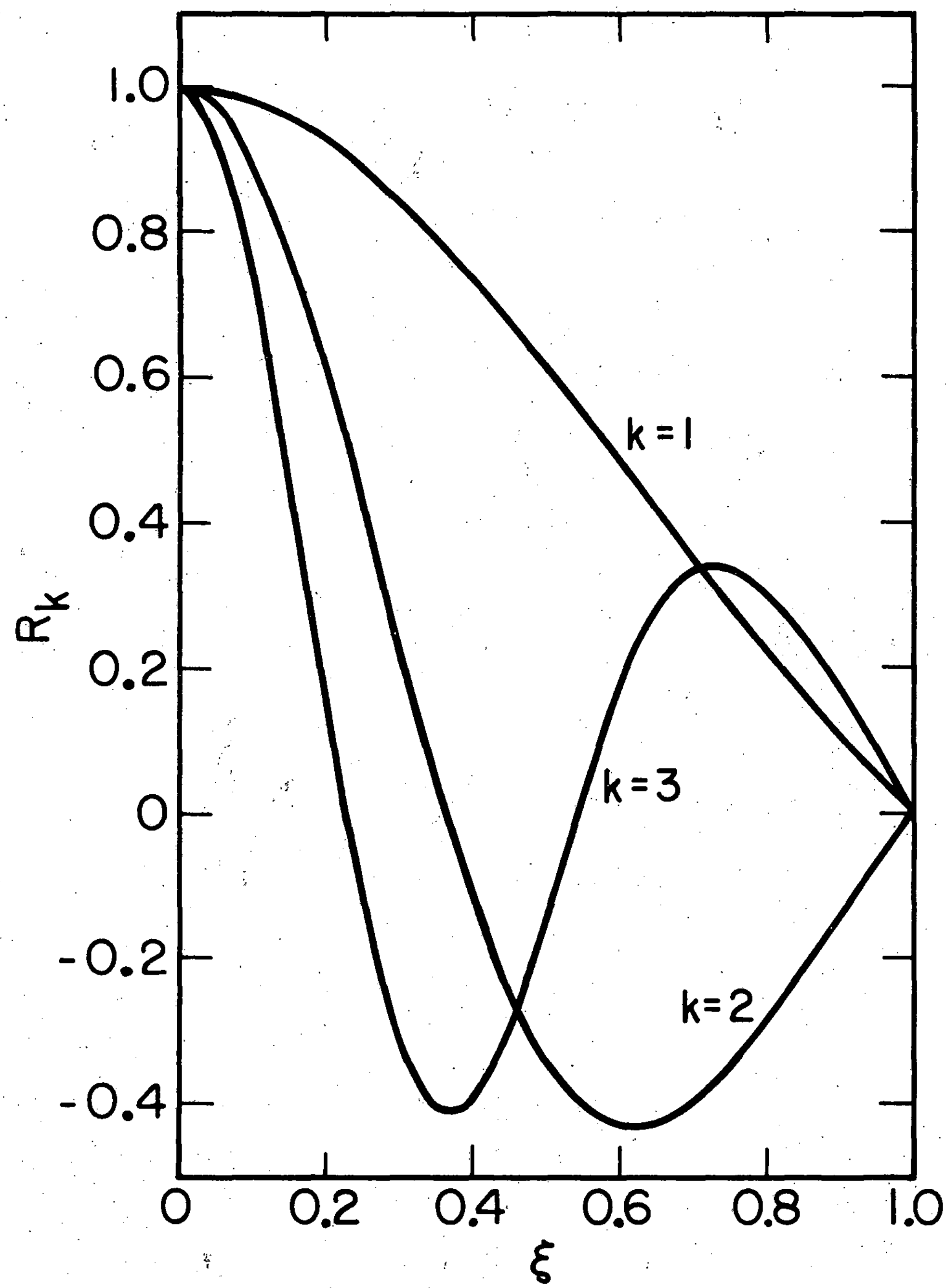


Figure 1. Graetz functions.

XBL6812-7376

4. Asymptotic forms for large eigenvalues

For large eigenvalues, the eigenvalues can be expressed as

$$\lambda = \lambda_o + S_1\lambda_o^{-4/3} + S_2\lambda_o^{-8/3} + S_3\lambda_o^{-10/3} + S_4\lambda_o^{-11/3} + O(\lambda_o^{-14/3}) , \qquad (19)$$

where

$$\left.\begin{aligned} &\lambda_o = 4k - 4/3, \; k = 1, 2, \ldots , \\ &S_1 = 0.159152288, \qquad S_3 = -0.224731440, \\ &S_2 = 0.0114856354, \qquad S_4 = -0.033772601 \; . \end{aligned}\right\} \qquad (20)$$

Lauwerier[10] gave values for λ_o, S_1, and S_2 (with an error in the sign of S_2), and Sellars, Tribus, and Klein[24] gave values for λ_o.

Lauwerier showed that the Graetz function $R(\xi,\lambda)$ can be expressed in terms of the confluent hypergeometric function and then obtained the asymptotic form

$$R(1,\lambda) = \frac{\sqrt{3}}{3\pi}\left(\frac{6}{\lambda}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)\left\{\cos\left[\left(\frac{\lambda}{4} - \frac{1}{6}\right)\pi\right]\sum_{p=0}\alpha_p\left(\frac{2}{\lambda}\right)^{2p}\right.$$

$$\left. + 3^{1/3}\,\frac{\Gamma(2/3)}{\Gamma(1/3)}\cos\left[\left(\frac{\lambda}{4} + \frac{1}{6}\right)\pi\right]\sum_{p=0}\beta_p\left(\frac{2}{\lambda}\right)^{2p+4/3}\right\} . \qquad (21)$$

By an extension of his method we can obtain (see also Lauwerier[11])

$$\frac{dR}{d\xi}\Big|_{\xi=1} = \frac{2\sqrt{3}}{\pi}\left(\frac{\lambda}{6}\right)^{1/3}\Gamma\left(\frac{2}{3}\right)\left\{\cos\left[\left(\frac{\lambda}{4}+\frac{1}{6}\right)\pi\right]\sum_{p=0}\gamma_p\left(\frac{2}{\lambda}\right)^{2p}\right.$$

$$\left. + \frac{1}{3^{1/3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}\cos\left[\left(\frac{\lambda}{4}-\frac{1}{6}\right)\pi\right]\sum_{p=0}\delta_p\left(\frac{2}{\lambda}\right)^{2p+2/3}\right\} \quad . \qquad (22)$$

The coefficients have the values

$$\begin{aligned}
\alpha_0 &= 1, & \gamma_0 &= 1,\\
\alpha_1 &= -0.01444444444, & \gamma_1 &= 0.03801587302,\\
\alpha_2 &= 0.009882467268, & \gamma_2 &= -0.01971517369,\\
\alpha_3 &= -0.02131664753, & \gamma_3 &= 0.03618529439,\\
\beta_0 &= -0.07857142857, & \delta_0 &= -0.3,\\
\beta_1 &= 0.02887167277, & \delta_1 &= 0.02189502164,\\
\beta_2 &= -0.04405535292, & \delta_2 &= -0.02140401420 \quad . \qquad (23)
\end{aligned}$$

These equations contain divergent, asymptotic series which cannot be summed to $p = \infty$. However, since only a finite number of coefficients are given, this causes no problem.

The eigenvalues are to be determined by setting $R(1,\lambda)$ in equation 21 equal to zero. The first approximation λ_0 to the eigenvalues is obtained by setting $\cos[(\lambda_0/4 - 1/6)\pi]$ equal to zero, with the result given in equation 20. If we define a correction λ_1 to this first approximation by

$$\lambda = \lambda_0 + \lambda_1 \quad , \qquad (24)$$

then the eigenvalue equation becomes

$$\sin\left(\frac{\pi\lambda_1}{4}\right)\sum_{p=0}\alpha_p\left(\frac{2}{\lambda}\right)^{2p} = -\frac{3^{1/3}\Gamma(2/3)}{\Gamma(1/3)}\sin\left(\frac{\pi\lambda_1}{4}+\frac{\pi}{3}\right)\sum_{p=0}\beta_p\left(\frac{2}{\lambda}\right)^{2p+4/3} . \quad (25)$$

This can either be expanded to yield equation 19, or it can be solved directly to obtain the correction λ_1 to the eigenvalue. We favor the latter procedure since it allows full use of the coefficients 23 to be made, whereas equation 19 would have to be carried to terms of order λ_o^{-7} to obtain comparable accuracy. On the other hand, the equation

$$\lambda = \lambda_o + (S_1 - 0.183/\lambda_o^2)/\lambda_o^{4/3} \quad (26)$$

will do a fair job.

Eigenvalues have been calculated by solving equation 25. The first five deviate from those reported in table 1 by amounts decreasing rapidly from 0.15 percent for the first to one part in the tenth significant figure for the fifth. The next five eigenvalues are given in table 2.

Table 2. Eigenvalues and coefficients evaluated from the asymptotic forms.

k	λ_k	$\frac{1}{8}\lambda_k^2 M_k$
6	22.66914336	0.3586855666
7	26.66866200	0.3396221643
8	30.66832334	0.3240622113
9	34.66807382	0.3110140736
10	38.66788335	0.2998440377

The coefficients can be obtained by means of equation 16. Equation 21 is differentiated with respect to λ, and the result is put in the form

$$\frac{dR}{d\lambda}\Big|_{\xi=1} = (-1)^{k+1}\frac{\sqrt{3}}{6\pi}\left(\frac{6}{\lambda}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)\Bigg\{\frac{\pi}{2}\cos\left(\frac{\pi\lambda_1}{4}\right)\sum_{p=0}\alpha_p\left(\frac{2}{\lambda}\right)^{2p} - \sin\left(\frac{\pi\lambda_1}{4}\right)\sum_{p=1}2p\alpha_p\left(\frac{2}{\lambda}\right)^{2p+1}$$
$$+ 3^{1/3}\frac{\Gamma(2/3)}{\Gamma(1/3)}\left[\frac{\pi}{2}\cos\left(\frac{\pi\lambda_1}{4}+\frac{\pi}{3}\right)\sum_{p=0}\beta_p\left(\frac{2}{\lambda}\right)^{2p+4/3} - \sin\left(\frac{\pi\lambda_1}{4}+\frac{\pi}{3}\right)\sum_{p=0}\left(2p+\frac{4}{3}\right)\beta_p\left(\frac{2}{\lambda}\right)^{2p+7/3}\right]\Bigg\}. \tag{27}$$

Equation 22 becomes

$$\frac{dR}{d\xi}\Big|_{\xi=1} = (-1)^{k+1}\frac{2\sqrt{3}}{\pi}\left(\frac{\lambda}{6}\right)^{1/3}\Gamma\left(\frac{2}{3}\right)\Bigg\{\sin\left(\frac{\pi\lambda_1}{4}+\frac{\pi}{3}\right)\sum_{p=0}\gamma_p\left(\frac{2}{\lambda}\right)^{2p}$$
$$+ \frac{1}{3^{1/3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}\sin\left(\frac{\pi\lambda_1}{4}\right)\sum_{p=0}\delta_p\left(\frac{2}{\lambda}\right)^{2p+2/3}\Bigg\} . \tag{28}$$

Although we prefer to determine the coefficients from equation 16 by direct evaluation of equations 27 and 28, one can obtain the following asymptotic form:

$$\frac{1}{8}\lambda^2 M = \frac{C}{\lambda^{1/3}}\left\{1 + \frac{L_1}{\lambda_o^{4/3}} + \frac{L_2}{\lambda_o^{6/3}} + \frac{L_3}{\lambda_o^{7/3}} + \frac{L_4}{\lambda_o^{10/3}} + \frac{L_5}{\lambda_o^{11/3}} + O(\lambda_o^{-4})\right\} , \tag{29}$$

where

$$\left.\begin{array}{ll} C = 1.012787288, & L_3 = -0.21220305, \\ L_1 = 0.144335160, & L_4 = -0.187130142, \\ L_2 = 0.115555556, & L_5 = -0.0918850832 . \end{array}\right\} \tag{30}$$

Finally, the formula

$$\frac{1}{8}\lambda^2 M = (C/\lambda^{1/3})(1+0.2/\lambda^{4/3})/(1+0.051/\lambda^{4/3}) \tag{31}$$

gives good results.

The term $C/\lambda^{1/3}$ in equation 29 was obtained by Sellars, Tribus, and Klein,[24] though expressed as $C/\lambda_o^{1/3}$. They used a singular-perturbation expansion involving three regions—one near the center of the pipe, one near the wall, and one in between. We have extended their method to obtain the S_1 term in equation 19 and the L_1 term in equation 29. However, the method of Lauwerier yields the asymptotic forms with less effort.

The coefficients, evaluated from equations 16, 27, and 28, are given in table 2 for the sixth through tenth terms in the Graetz series. For the first five terms, the deviation from the values reported in table 1 varied from 0.21 percent to $6\cdot10^{-7}$ percent in a manner similar to that found for the eigenvalues. The eigenvalues and coefficients in table 2 are in excellent agreement with the results of Brown.[4]

5. The Lévêque series

Near z = 0, the transfer rate becomes infinite, and many terms in the Graetz series are required for accurate results. In such cases it is frequently possible to make approximations which permit a similarity solution valid near z = 0. Here the approximations follow from the fact that Θ differs from the inlet value only in a thin diffusion layer near the wall of the tube. This means that in this layer the velocity distribution can be approximated by

$$v_z = 4\langle v_z\rangle (1-r/R) \quad , \tag{32}$$

and the effect of the cylindrical geometry disappears in the sense that

$$\frac{1}{r}\frac{\partial\Theta}{\partial r} \ll \frac{\partial^2\Theta}{\partial r^2} \quad . \tag{33}$$

With these approximations, Lévêque[13] obtained the solution

$$\Theta = \frac{1}{\Gamma(4/3)}\int_0^{\eta} e^{-x^3}dx \quad , \tag{34}$$

in terms of the similarity variable

$$\eta = (1-\xi)(2/9\zeta)^{1/3} \quad . \tag{35}$$

By taking into account the approximations of Lévêque, Newman[17] extended this solution so that the local Nusselt number can be expressed as

$$Nu(\zeta) = 1.3565975\,\zeta^{-1/3} - 1.2 - 0.296919\,\zeta^{1/3} + O(\zeta^{2/3}) \quad . \tag{36}$$

6. Comparison of Graetz and Lévêque series

The local Nusselt number, proportional to the local transfer rate, is expressed by the Graetz series 13 or alternately by the Lévêque series 36. Figure 2 shows the local Nusselt number, divided by the first term of the Lévêque series so that this ratio approaches 1 as $\zeta \to 0$. The dashed lines indicate how well the Lévêque series approximates the exact solution. The dimensionless cup-mixing concentration difference (see equation 11) is also shown.

During the development of this work, a comparison of the Graetz series and the Lévêque series served as a convenient check for errors and a measure of the significance of the various improvements in the Graetz series. This comparison was made with the quantity J, the total amount of material transferred to the wall in a given length. If ζ_o denotes the value of ζ where the error in the Graetz series is roughly comparable to the error in the three-term Lévêque series, then $\zeta_o = 0.006$ when the Graetz series is based on Abramowitz's eigenvalues, Lipkis's coefficients, and the asymptotic forms of Sellars, Tribus, and Klein. Refinement of the coefficients for the first five terms moves ζ_o down to 0.0017. Use of the modified asymptotic forms moves ζ_o down to about $2 \cdot 10^{-5}$, and refinement of the first five eigenvalues moves it slightly further, to about 10^{-5}. Here the error is about 10^{-4} percent. The use of Brown's results leads to some further improvement.

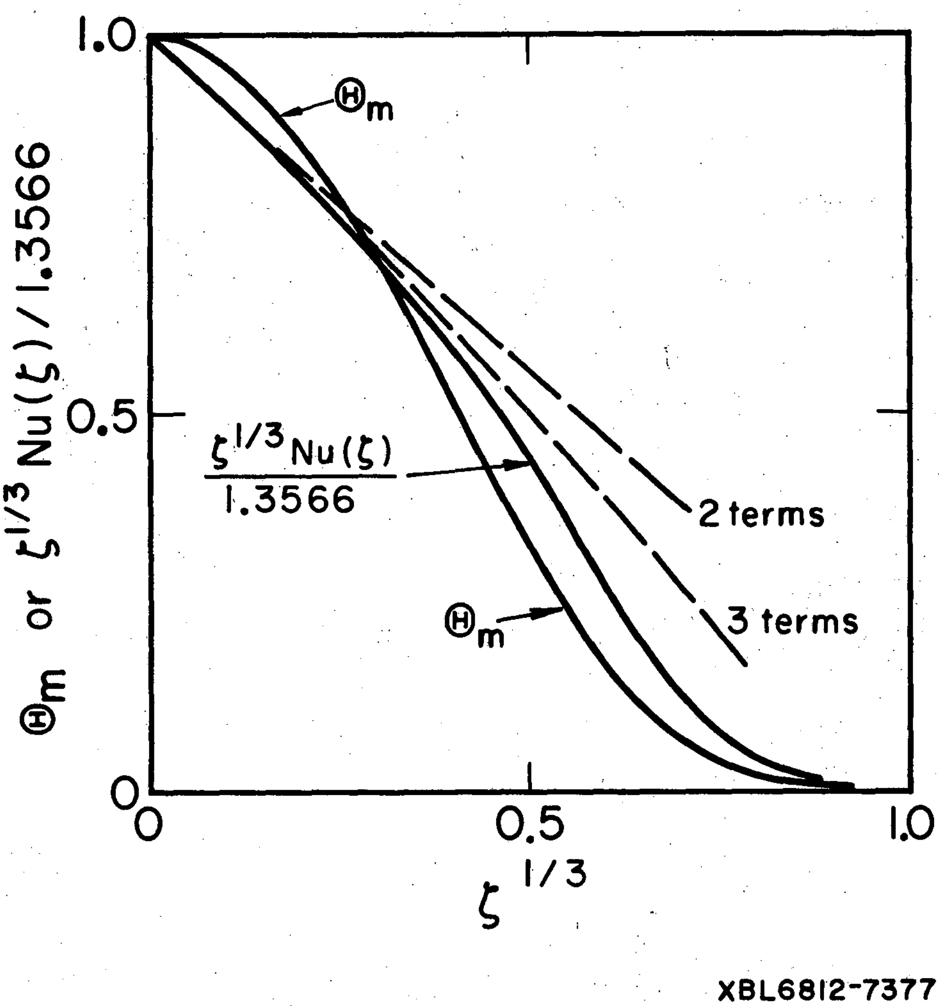


Figure 2. Dimensionless cup-mixing concentration difference Θ_m and the local Nusselt number (divided by Lévêque's solution). For comparison with the latter, the corresponding form of the Lévêque series is shown for 2 and 3 terms.

7. The region of axial diffusion

Graetz[6] included axial diffusion in his treatment of plug flow in a tube. Singh[26] did the same thing for a parabolic velocity profile in a tube. However, both of these authors took $\Theta = 1$ in the cross section at the beginning of the transfer section, without apparently recognizing the physically absurd consquences of this condition. The condition 2 or 5, on the other hand, represents a reasonable situation corresponding to an insulated wall upstream of the transfer section. If axial diffusion is important, then the fluid must become depleted, to some extent, upstream of the transfer section. It is only in the limit of an infinite Péclet number that axial diffusion can be neglected and the condition $\Theta = 1$ applied in the cross section $z = 0$. Furthermore, to have a surface with $\Theta = 1$ immediately adjoining a surface with $\Theta = 0$, here the tube wall for $z > 0$, will result in an infinite transfer rate near $z = 0$, a rate which cannot be integrated. In other words, with the boundary condition used by Graetz and Singh, the total amount of material transferred in any non-zero length z will be infinite.

Bodnarescu[2] treated axial diffusion with an upstream wall maintained at one temperature and a downstream wall at another temperature. Wilson[28] used the same boundary conditions with plug flow. As Drew[5] has pointed out, the total transfer will again be infinite. Schneider[22] treated the same situation as Wilson, but with the addition of a transfer resistance at the wall.*

In practical applications, with the possible exception of heat transfer in liquid metals, the Péclet number will be large, and axial diffusion will be important only in a small region near $z = 0$ and $r = R$. This region can be

---

*Half of Schneider's solutions <u>without</u> axial conduction are wrong.

treated by the method of singular perturbations. Stretched coördinates X and Y are introduced:

$$Y = (1-\xi)\sqrt{Pe} \text{ and } X = \zeta(Pe)^{3/2} = (z/R)\sqrt{Pe} \quad . \tag{37}$$

Substitution into equation 4 yields, in the limit $Pe \to \infty$, the appropriate differential equation for the region of axial diffusion

$$2Y\frac{\partial\Theta}{\partial X} = \frac{\partial^2\Theta}{\partial X^2} + \frac{\partial^2\Theta}{\partial Y^2} \quad . \tag{38}$$

The approximations introduced have the same meaning as in the case of the Lévêque solution—near the wall the effects of curvature of the wall can be neglected and the velocity profile can be approximated by equation 32—but the stretching of the coördinates makes it apparent that in this region axial diffusion is just as important as convection and radial diffusion.

For boundary conditions we have

1. $\Theta = 0$ at $Y = 0$, $X > 0$.
2. $\partial\Theta/\partial Y = 0$ at $Y = 0$, $X < 0$.
3. $\Theta \to 1$ as $X \to -\infty$ or as $Y \to \infty$.
4. As $X \to \infty$, the term $\partial^2\Theta/\partial X^2$ should become negligible compared to the other terms in equation 38.

The last of these conditions is somewhat unusual. It expresses the fact that this elliptic region where axial diffusion is important is embedded in an otherwise parabolic domain. From this last condition and equation 38 one can derive the asymptotic form

$$\Theta \to \frac{1}{\Gamma(4/3)}\int_0^{\eta} e^{-x^3}\,dx \text{ as } X \to \infty \quad , \tag{39}$$

where

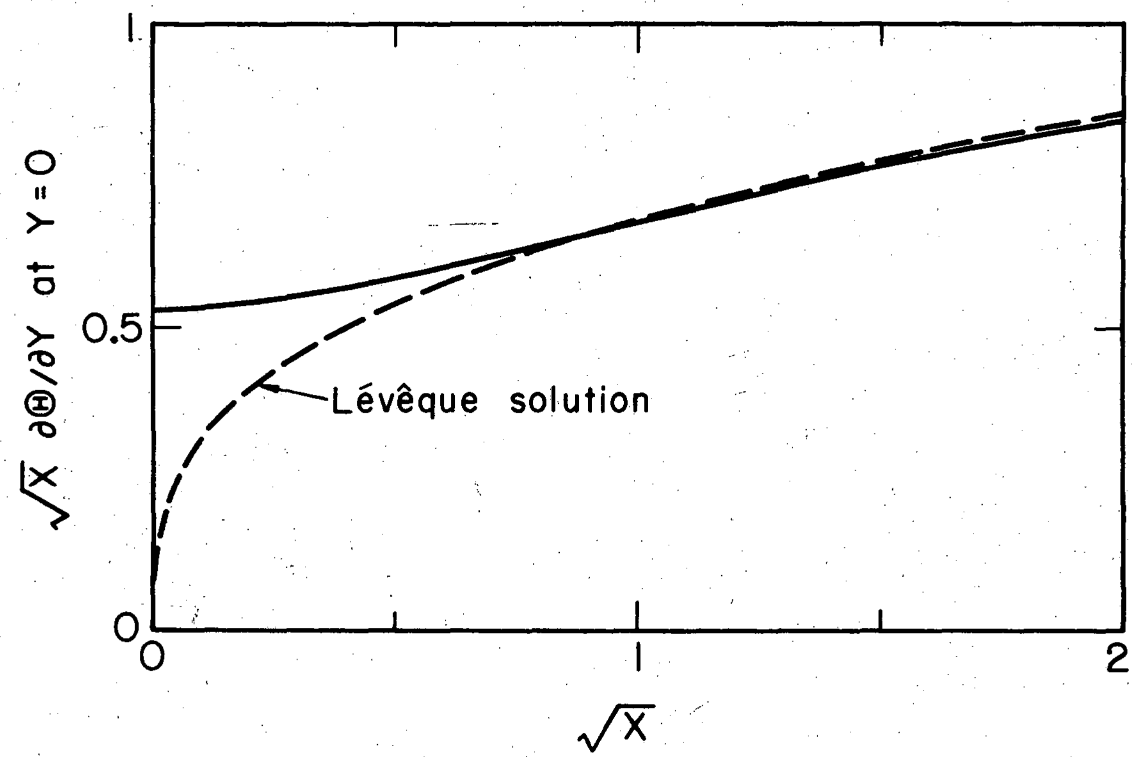
$$\eta = Y(2/9X)^{1/3} \quad . \tag{40}$$

In other words, Θ approaches the Lévêque solution far downstream in the region of axial diffusion. However, since effects are transmitted downstream in the parabolic regions, a better phrasing would be to say that Θ in the Lévêque or Graetz regions should approach the form given by equation 39 as $\zeta \to 0$. In other words, the development leading to equation 39 is the proper justification for the validity of the Lévêque solution.

Here we have formulated a proper singular-perturbation problem, now independent of the explicit appearance of the Péclet number. The situation and mathematical treatment are very similar to those encountered in the calculation of rates of transfer to the rear of bluff objects in a free stream in slow, laminar flow.[25,18]

The problem posed here was solved numerically by successive over-relaxation in parabolic coördinates, coördinates selected so that the derivatives of Θ would be finite at the origin. In each direction, 120 mesh intervals were used. Difficulties were encountered away from the wall where convection becomes dominant. These were circumvented by using an over-relaxation factor which decreases as convection becomes more important and by setting $\Theta = 1$ when convection becomes dominant.

Figure 3 presents the dimensionless transfer rate in the region of axial diffusion. As $X \to 0$, $\partial\Theta/\partial Y$ at $Y = 0$ approaches $0.52945/\sqrt{X}$. Thus, if axial diffusion is taken into account, the transfer rate becomes infinite like $1/z^{1/2}$ rather than $1/z^{1/3}$, as predicted by the Lévêque solution. As $X \to \infty$,



XBL6812-7378

Figure 3. The normal derivative at the wall in the region of axial diffusion, with the Lévêque solution for comparison.

the transfer rate approaches that given by the Lévêque solution. Figure 4 shows the dimensionless concentration distribution along the upstream, insulating wall and in the cross section at z = 0, the beginning of the transfer section, and gives a quantitative indication of the extent of the region of axial diffusion.

From a practical point of view, axial diffusion is not important for large Péclet numbers, although its proper mathematical treatment is of some interest. The Lévêque solution could be extended[17] to higher order terms in ζ because those corrections did not involve axial diffusion but only curvature effects and the parabolic nature of the velocity profile. The correction of the Lévêque solution for the effects of axial diffusion would require as a basis a uniformly valid approximation such as that presented here. In fact, the entire Lévêque solution is included here in the solution for the region of axial diffusion, and corrections to the Lévêque solution solely for the effects of axial diffusion are merely higher order terms in the asymptotic solution of equation 38 for large values of X. Thus

$$\frac{\partial\Theta}{\partial Y}\Big|_{Y=0} \rightarrow \frac{(2/9X)^{1/3}}{\Gamma(4/3)}\left(1 + \frac{0.1548}{X^{4/3}} + O(X^{-8/3})\right) \quad \text{as } X \rightarrow \infty \tag{41}$$

or

$$Nu(\zeta) \rightarrow \frac{2}{\Gamma(4/3)}\left(\frac{2}{9\zeta}\right)^{1/3}\left(1 + \frac{0.1548}{\zeta^{4/3}Pe^{2}} + O(Pe^{-4})\right) \quad \text{as } Pe \rightarrow \infty \text{ and } \zeta \rightarrow 0 \,. \tag{42}$$

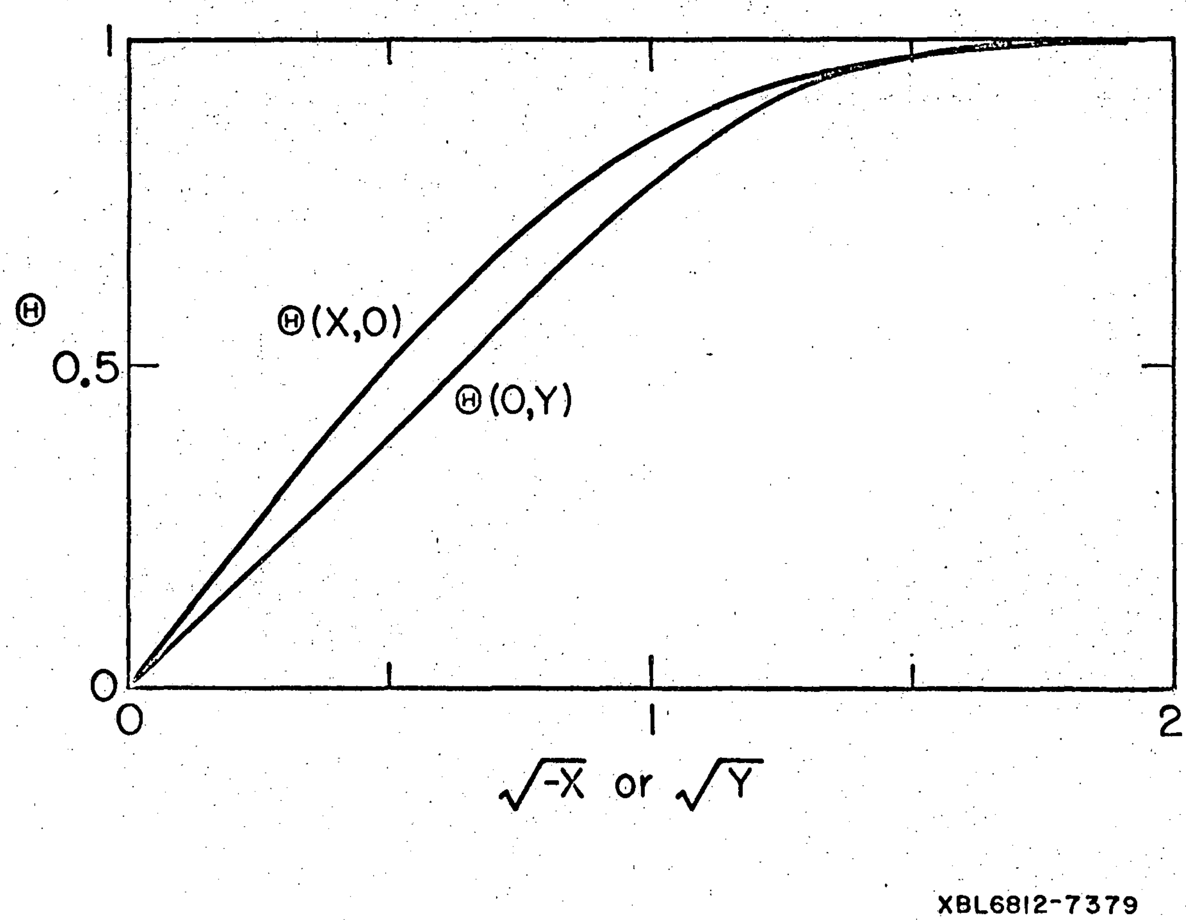


Figure 4. Dimensionless concentration distribution $\Theta(X,0)$ along the upstream wall and $\Theta(0,Y)$ in the cross section at $z = 0$.

It might be asked whether any useful information about the rôle of axial diffusion can be obtained from the solutions of Graetz and Singh or whether their method can be extended to yield useful results. The conclusion of Singh that the effect of axial diffusion becomes negligible for large Péclet numbers is certainly valid, but where axial diffusion is important, at low Péclet numbers, one cannot eliminate with any confidence the effect of the non-integrable singularity in the transfer rate at z = 0. The method of separation of variables used by Graetz and Singh will not be useful if it becomes necessary to consider the upstream, insulated pipe. An artificial way to avoid the infinite total transfer while retaining the method of separation of variables would be to specify instead the flux at the cross section at z = 0, say by the condition

$$(1-\xi^2)(\Theta-1) = (1/Pe^2)\partial\Theta/\partial\zeta \text{ at } \zeta = 0. \qquad (43)$$

Another artificial way would be to specify the concentration at $\zeta = 0$ not by $\Theta = 1$ but perhaps by means of figure 4 with some adjustment so that the total flux corresponds to the convective flow far upstream. For an idea of the difficulties involved, it would be pertinent to consult the work of Bodnarescu,[2] Schneider,[22] Singh,[26] and Wilson.[28]

8. Conclusions

If the eigenvalue is regarded as an unknown variable, a Sturm-Liouville system can be solved numerically by standard techniques with quadratic convergence characteristics. In the present case this involved only about four hours of programming, key punching, and debugging, which is regarded as a minimal amount of personnel effort. The results confirm the more precise values of Brown.[4]

Numerical results for the lower eigenvalues are complemented by asymptotic forms for large eigenvalues, developed along the lines of Lauwerier.[10,11] With the new results, the Graetz series yields accurate results even for fairly small values of the axial distance. Thus, good agreement can now be obtained between the Graetz series and the extended solution of Lévêque, a situation which did not prevail in 1955.[23]

It should, perhaps, be emphasized that asymptotic forms should not replace accurately calculated values of the lower eigenvalues and coefficients. The first eigenvalue calculated even by Graetz is more accurate than the asymptotic form of Sellars, Tribus, and Klein.

For high Péclet numbers, axial diffusion is important only in a small region near $z = 0$ and $r = R$ and of spatial dimensions on the order of $R/Pe^{1/2}$. For an insulating wall upstream and a constant wall concentration downstream, the transfer rate becomes infinite near $z = 0$ like $1/z^{1/2}$. The Lévêque solution, which neglects axial diffusion, predicts a behavior like $1/z^{1/3}$.

It is felt that an insulating wall upstream of the transfer section represents a more relaistic boundary condition than either a specified concentration in the cross section $z = 0$ or a specified upstream wall concentration, even if a nonzero resistance at the wall is considered in the last case.

Acknowledgment

This work was supported by the United States Atomic Energy Commission.

Nomenclature

c_b - upstream concentration.

c_i - concentration of species i.

c_o - constant concentration at the downstream wall.

C - constant in asymptotic form for M_k.

C_k - coefficients in Graetz series for Θ.

D - diffusion coefficient (cm^2/sec).

J - amount transferred in a given length of tube.

L_i - coefficients in asymptotic form for M_k.

M_k - coefficients in Graetz series for Θ_m.

Nu - Nusselt number.

Pe - Péclet number.

r - radial distance from axis of tube (cm).

R - radius of tube (cm).

R_k - radial functions in Graetz series for Θ.

S_i - coefficients in asymptotic form for λ_k.

v_z - fluid velocity in axial direction (cm/sec).

$\langle v_z \rangle$ - average value of v_z (cm/sec).

X - stretched axial distance in elliptic region.

Y - stretched distance from the wall in the elliptic region.

z - axial distance from the beginning of the transfer section (cm).

$\alpha, \beta, \gamma, \delta$ - coefficients in asymptotic form for R_k and $dR/d\xi$.

Γ - gamma function.

ζ - dimensionless axial distance.

η - Lévêque's similarity variable.

Θ - dimensionless concentration.

Θ_m - dimensionless cup-mixing concentration.

λ_k - eigenvalues.

λ_o - first asymptotic approximation to the eigenvalues.

ξ - dimensionless radial distance.

References

1. Milton Abramowitz. "On the Solution of the Differential Equation Occurring in the Problem of Heat Convection in Laminar Flow through a Tube." Journal of Mathematics and Physics, 32, 184-187 (1953).

2. Musat Vasile Bodnarescu. "Beitrag zur Theorie des Wärmeübergangs in laminarer Strömung." VDI-Forschungsheft 450, 21B, 19-27 (1955).

3. H. C. Brinkman. "Heat Effects in Capillary Flow I." Applied Scientific Research, A2, 120-124 (1950).

4. George Martin Brown. "Heat or Mass Transfer in a Fluid in Laminar Flow in a Circular or Flat Conduit." A.I.Ch.E. Journal, 6, 179-183 (1960).

5. Thomas Bradford Drew. "Mathematical Attacks on Forced Convection Problems: a Review." Transactions of the American Institute of Chemical Engineers, 26, 26-79 (1931).

6. L. Graetz. "Ueber die Wärmeleitungsfähigkeit von Flüssigkeiten." Annalen der Physik und Chemie, 18, 79-94 (1883); 25, 337-357 (1885).

7. Max Jakob. Heat Transfer. Volume I. New York: John Wiley & Sons, Inc., 1949, pp. 451-464.

8. S. Katz. "Chemical reactions catalysed on a tube wall." Chemical Engineering Science, 10, 202-211 (1959).

9. W. M. Kays. "Numerical Solutions for Laminar-Flow Heat Transfer in Circular Tubes." Transactions of the American Society of Mechanical Engineers, 77, 1265-1272 (1955).

10. H. A. Lauwerier. "The Use of Confluent Hypergeometric Functions in Mathematical Physics and the Solution of an Eigenvalue Problem." Applied Scientific Research, A2, 184-204 (1950).

11. H. A. Lauwerier. "Poiseuille Functions." Applied Scientific Research, A3, 58-72 (1951).

12. Allyn Lee, W. O. Nelson, V. H. Cherry, and L. M. K. Boelter. "Pressure Drop and Velocity Distribution for Incompressible Viscous Non-Isothermal Flow in the Steady State through a Pipe." Proceedings of the Fifth International Congress for Applied Mechanics. New York: John Wiley & Sons, Inc., 1939, pp. 571-577.

13. M. A. Lévêque. "Les Lois de la Transmission de Chaleur par Convection." Annales des Mines, Memoires, ser. 12, 13, 201-299, 305-362, 381-415 (1928).

14. R. P. Lipkis. "Discussion." Transactions of the American Society of Mechanical Engineers, 78, 447-448 (1956).

15. P. A. Longwell. "Graphical Solution of Turbulent-flow Diffusion Equations." A.I.Ch.E. Journal, 3, 353-360 (1957).

16. John Newman. "Numerical Solution of Coupled, Ordinary Differential Equations." Industrial and Engineering Chemistry Fundamentals, 7, 514-517 (1968).

17. John Newman. "Extension of the Lévêque Solution." Journal of Heat Transfer, to be published. (UCRL-17600, June, 1967)

18. John Newman. "Mass Transfer to the Rear of a Cylinder at High Schmidt Numbers." Industrial and Engineering Chemistry Fundamentals, to be published. (UCRL-17599, June, 1967)

19. Wilhelm Nusselt. "Die Abhängigkeit der Wärmeübergangszahl von der Rohrlänge." Zeitschrift des Vereines deutscher Ingenieure, 54, 1154-1158 (1910).

20. J. Schenk and J. M. Dumoré. "Heat Transfer in Laminar Flow through Cylindrical Tubes." Applied Scientific Research, A4, 39-51 (1953).

21. J. Schenk and J. Van Laar. "Heat Transfer in Non-Newtonian Laminar Flow in Tubes." Applied Scientific Research, A7, 449-462 (1958).

22. P. J. Schneider. "Effect of Axial Fluid Conduction on Heat Transfer in the Entrance Regions of Parallel Plates and Tubes." Transactions of the American Society of Mechanical Engineers, 79, 765-773 (1957).

23. John Sellars, John Klein, and Myron Tribus. "Discussion." Transactions of the American Society of Mechanical Engineers, 77, 1273-1274 (1955).

24. J. R. Sellars, Myron Tribus, and J. S. Klein. "Heat Transfer to Laminar Flow in a Round Tube or Flat Conduit—The Graetz Problem Extended." Transactions of the American Society of Mechanical Engineers, 78, 441-447 (1956).

25. Ping Huei Sih and John Newman. "Mass Transfer to the Rear of a Sphere in Stokes Flow." International Journal of Heat and Mass Transfer, 10, 1749-1756 (1967).

26. S. N. Singh. "Heat Transfer by Laminar Flow in a Cylindrical Tube." Applied Scientific Research, A7, 325-340 (1958).

27. R. L. Solomon and J. L. Hudson. "Heterogeneous and Homogeneous Reactions in a Tubular Reactor." A.I.Ch.E. Journal, 13, 545-550 (1967).

28. Harold A. Wilson. "On Convection of Heat." Proceedings of the Cambridge Philosophical Society, 12, 406-423 (1904).

TECHNICAL INFORMATION DIVISION
LAWRENCE RADIATION LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720