JAKOB BERNOULLI'S THEORY OF INFERENCE

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Abstract. This review of Ars Conjectandi, written on the eve of its 300th anniversary, discusses an aspect of Bernoulli's magnum opus which hitherto has not received the attention it merits. Bernoulli envisioned a theory for the advancement of science based on the idea of pairing empirical evidence with the then-novel concept of probability. This theory of inference, which he termed "ars conjectandi", was intended to complement the predominant axiomatic-deductive method where the latter could not be applied successfully. In the 300 years since its publication, Bernoulli's idea went through ups and downs, but eventually ended up as the defining characteristic of statistical science and a cornerstone of modern science. This review discusses the historical context from which Bernoulli's idea was conceived, his sources of inspiration, and provides a detailed account of his theory of inference.

1. To the Reader

Jakob Bernoulli's *Ars Conjectandi* (1713) is widely recognized as a seminal text in the early history of probability theory. It is particularly noted for introducing the first probabilistic limit theorem: Bernoulli's limit theorem, or the law of large numbers. According to the theorem, the arithmetic mean can be made arbitrarily accurate through repeated observation; a result of immense importance to the theory of statistics.

Numerous reviews of Bernoulli's book have been written, including Pearson (1925), Hacking (1971), Shafer (1978), Schneider (1984), Sheynin (1986), Stigler (1986), Hald (1990), and Sylla (2006). In light of all these reviews, it would be rather natural to call into question the added value of writing another review; what more could possibly be said about this treatise? Well, the answer is that I feel that there is a fundamental aspect of *Ars Conjectandi* that has not yet been effectively discussed: Bernoulli's very motivation for writing the book, his theory of inference.

While Bernoulli's limit theorem is undoubtedly a vital result for the development of probability theory, the theorem occupies only eleven of *Ars Conjectandi*'s 305 pages, excluding the 35-page *Lettre à un Amy*. Furthermore, the limit theorem is arranged into what could almost be likened to an appendix. Yet reviews have hitherto focused almost solely on the limit theorem (Pearson (1925), Sheynin (1986), Stigler (1986), Hald (1990) and Sylla (2006)) and Bernoulli's use of the word probability (Hacking (1971), Shafer (1978), Schneider (1984) and Hald (1990)). Only Hald (1990) touches upon the focal point of the present review, but does not quite articulate the matter.

With *Ars Conjectandi*, Jakob Bernoulli developed a theory for the advancement of science through empirical evidence; a theory of inference. Central to the theory is that the concept of probability should be used as a device for transparent, reproducible and relatively objective valuation of empirical evidence. *Ars Conjectandi* is the first published text that proposes pairing empirical evidence with the concept of probability, an idea that presently constitutes the defining characteristic of statistical science and is a cornerstone of modern science.

On the eve of the 300th anniversary of its publication, I feel that discussing this aspect is a worthwhile contribution to the statistical community, and one that can enhance the understanding of the role and purpose of statistics in modern science.

2. Historical context

When Jakob Bernoulli (1654-1705) wrote *Ars Conjectandi*, probability theory was in its infancy and the philosophy of science was predominated by a school of thought, known as rationalism, according to which the use of empirical observation in science was not completely accepted. This section aims to summarize the aspects of the historical context up to and through Bernoulli's lifetime that are important to understanding his treatise. Section 6 discusses some of the developments from Bernoulli's passing through the twentieth century.

Circa 387 B.C., 2100 years before *Ars Conjectandi* was published, Greek mathematician Plato (c. 428-348 B.C.) returned to his native Athens with his mind set on realizing one of his long-held dreams: starting a mathematical society (Taylor, 1936). The society soon became referred to as Akademia, the site of the meetings was named after Greek mythological hero Akademos, and thus the first academy, and in a wider sense the scientific community, was founded.

Besides his mathematical work, Plato was active in the political arena and wrote extensively on constitutional issues, public policy, education, morality, et cetera. Scholars of Plato have identified a fundamental axiom that appears in all of his works (Ross, 1951), which has been named Plato's theory of forms. The axiom, in a condensed form, is formalized as follows.

Axiom (Plato's theory of forms)**.**

(i) Each and every phenomenon has an underlying true nature, a Platonic ideal form.
(ii) The senses are inherently unreliable and observations inevitably subject to flux.

Flowing naturally from the axiom, Plato held the opinion that the task of science is to gain and build knowledge about the ideal forms, and that the senses, i.e. empirical observations, are inherently unsuitable for meaningful advancement of science (White, 1976). The logical consequence is that science must be advanced through use of the so-called axiomatic-deductive method.

An axiom is a formalized self-evident truth, and an axiomatic system is a set of noncontradictory axioms that work in conjunction. Within the system, the truth values of statements can be derived through logical deduction. Euclid's *Elements*, a mathematical treatise written circa 300 B.C., is a magnificent early example of the use of the axiomatic-deductive method. To this day, the axiomatic-deductive method and mathematics are so closely associated that the two terms are sometimes used interchangeably.

In the third century B.C., Archimedes (c. 287-212 B.C.) employed the axiomatic-deductive method for advancing the science of physics, see e.g. *On floating bodies* (c. 250 B.C.). In doing so, Archimedes also introduced the axiomatic-deductive method to a practical field beyond the abstractions of mathematics; a milestone in the history of science. The method has since been used in physics with great success, examples include Galilei (1638), Newton (1726), and Einstein (1905).

Attempts of applying the axiomatic-deductive method to other scientific disciplines, however, have met more challenge. In fields such as chemistry, biology and economics, predictions produced by axiomatic systems, no matter how carefully constructed, often do not represent experimental outcomes particularly well. Microeconomics, as a concrete example, represents an admirable attempt of creating an axiomatic system, based on self-evident truths such as utility maximizing agents, efficient markets and other quite reasonable axioms, but it has demonstrated limited success in predicting future outcomes. Even though theoretically infallible, the axiomatic-deductive method in practice bears the weakness that it often is difficult to apply; thus creating a need for an alternative scientific method.

Making inference from observations is something that comes natural to most of us humans, a fact which few would dispute. Throughout history, many people have tried creating methods that advance science through empirical evidence, for instance Plato's student Aristotle. However, early such methods demonstrated limited reproducibility and questionable objectivity, and through Bernoulli's lifetime they never gained wide acceptance. A particular difficulty is unquestionably the valuation of the evidence; even if all members of the scientific community agreed on what the empirical evidence is, they would still need to find agreement on its valuation.

It should be noted that the fact that a scientific work is strictly axiomatic-deductive does not necessarily imply that the author did not use empirical observation; the author could well have experimented but chosen not to mention it in writing. As an example, Archimedes does not mention any experiments in *On floating bodies*, although one may still imagine that Archimedes, at least to some limited extent, conducted experiments to properly convince himself that his axiomatic system was correctly constructed. In fact according to the Eureka legend, Archimedes formulated hypotheses while in a public bath, hinting at some interplay between logical deduction and experimentation.

Another example is *Two new sciences* (1638) by Galileo Galilei (1564-1642), which is an entirely axiomatic-deductive scientific work. However in a carefully constructed arrangement, Galilei incorporates a layman discussion in which the fictional laypersons provide detailed accounts of experiments that Galilei had conducted. Hence Galilei, through an intricate arrangement, gains the ability to include information about his experiments without tarnishing his scientific work. The fact that Galilei went to such lengths to distance his work from association with experiments provides circumstantial evidence that advancement of science through empirical observation at that time was not entirely accepted.

Throughout the seventeenth century, philosophers such as René Descartes (1596-1650), Baruch Spinoza (1632-1677) and Gottfried Leibniz (1646-1716) were influential advocates of Plato's view that science should be advanced through the axiomatic-deductive method. This school of thought, commonly referred to as *rationalism*, was predominant when the young Jakob Bernoulli studied philosophy and theology at the University of Basel. In his twenties, Bernoulli taught experimental physics for six years, and at age 32 he received a professorship in mathematics. With his set of competencies and interests, Bernoulli was in an ideal position, at the right moment in time, to envision and develop a new method for the advancement of science through empirical observation. In a letter to Leibniz (1703), Bernoulli expresses: "I scarcely think that anyone has thought more than I about these matters."

3. The elements of Bernoulli's method

Jakob Bernoulli never quite finished *Ars Conjectandi*; it was published posthumously in 1713. The reason for not publishing the manuscript was most certainly not a lack of interest; in correspondence with Leibniz (Bernoulli, 1703) Bernoulli expresses both enthusiasm and sense of purpose, but also mentions weak health and feeble writing.

Ars Conjectandi contains remarkably few references. The only cited reference in the first part is Huygens' *De ratiociniis in ludo aleæ* (1657), and the only reference in part three, except for the numerous citations of Huygens, is a text on the card game Basset (Sauveur, 1679), which Bernoulli uses for the construction of a problem. The fourth and last part of *Ars Conjectandi*, the part of greatest interest to this review, cites only one reference: Arnauld & Nicole's *La logique, ou l'art de penser* (1662).

De ratiociniis is a thirteen page text, largely recreational in nature, on the mathematically correct valuation of games, lottery tickets and the like. Written by Dutch astronomer, physicist and mathematician Christiaan Huygens (1629-1695), it was the first published text on probability theory (Hald, 1990). Its main result is both short and simple:

> If the number of chances to gain a is p, and the number of chances to gain b is q, then, assuming all chances occur equally easily, the mathematically correct value of the game is $(ap + bq)/(p + q)$.

Huygens' expression is noted as the first definition of the expected value of a random variable, but more immediately it is a weighted mean. Hence Huygens gives a formula for the mathematically correct weighting of different possibilities. The influence this text had on Bernoulli's thinking is indisputable; it was reprinted in its entirety, with annotations, as the first part of *Ars Conjectandi*.

La logique is a historically influential and widely read textbook on logical deduction. It was for example used by the Universities of Cambridge and Oxford well into the nineteenth century (Buroker, 1996). Antoine Arnauld (1612-1694) and Pierre Nicole (1625-1695) were French philosophers, mathematicians, and Catholic theologians, and their book discusses axiomatic-deductive theory: constructing axiomatic systems, proving propositions, deriving conclusions from premises, et cetera. *La logique* endorses the two parts of Plato's theory of forms and is influenced by Descartes' rationalism. In his early twenties, Bernoulli had travelled to Paris and studied the works of Descartes' and his followers (Hald, 1990), presumably including Arnauld & Nicole.

The fourth and last part of *La logique* concludes the discussion on scientific methodology with a summary of the material into eight general rules. The summary is a natural conclusion of the book, but Arnauld & Nicole state that, "before ending, it will be good to discuss another kind of knowledge". The remaining twenty pages of the 355-page book, which are arranged into what could be likened to an appendix, discuss faith and are theological in nature, making the pages distinct from the rest of the book. Interestingly, it is these twenty pages that Bernoulli expressly cites.

A focus of the final twenty pages of *La logique* is reconciling the use of the axiomatic-deductive method with belief in miracles, i.e. events declared by the Vatican to result from divine interventions. The evidence surrounding putative miracles often consists exclusively of witness accounts, and therefore, according to strict rationalism, the evidence typically cannot yield complete certainty as to whether or not a putative miracle really is the result of a divine intervention. However, Arnauld & Nicole argue that even though one cannot reach complete certainty, one can sometimes reach *moral certainty*. Chapter twelve reads:

> Human faith is in itself subject to error because all humans are liars, according to Scripture, and it can happen that people who assure us that something is true may themselves be mistaken. As we have already indicated, however, some things we know only by human faith, which we ought to consider as certain and as indubitable as if we had mathematical demonstrations of them. [*An example of such an indubitable certainty, discussed in a later chapter, is that St. Peter visited Rome.*] [...]
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> It is true that it is often fairly difficult to mark precisely when human faith has attained this certainty and when it has not. [...] We can, however, mark certain limits that must be reached in order to attain this human certainty, [...]

Bernoulli seized upon the idea of a continuum of certainty and a limit for moral certainty which, if reached, ought to make the authors and all readers, i.e. mankind, consider the statement in question as certain as if it had been mathematically demonstrated. Arnauld & Nicole had no suggestion as to how this degree of certainty should be determined, but Bernoulli had a splendid idea: the degree of certainty can be determined through application of Huygens' formula.

Idea (Bernoulli (1713))**.** *The concept of probability can be used for the valuation of empirical evidence, and hence pairing the two yields a method for the advancement of science through empirical evidence.*

As a result, Bernoulli had identified an entirely new use of Huygens' formula: pairing it with empirical observations to produce a theory of the correct valuation of empirical evidence. And thus Bernoulli's idea was formed; the defining characteristic of statistical science and a cornerstone of modern science.

Bernoulli defined ars conjectandi, his theory of inference, as the theory of estimating, as exactly as possible, our degree of certainty in things. He added: "in this alone lies all the wisdom of the philosopher".

4. Bernoulli's theory of inference

In the fourth and last part of *Ars Conjectandi*, Bernoulli lays out the details of his theory of inference. Like Arnauld & Nicole, Bernoulli endorses Plato's theory of forms through a theological argument. Every statement is in and of itself either inherently true or inherently false; however we, mankind, cannot obtain this ideal form through the senses. Bernoulli defines *certainty* as our, mankind's, subjective level of knowledge concerning a statement's truth value. *Complete certainty* corresponds to perfect knowledge that the statement's truth value is true, and can be obtained through the axiomatic-deductive method or, presumably, Scripture. *Complete impossibility* corresponds to perfect knowledge that the statement's truth value is false, and is thus the antipode of complete certainty.

Bernoulli continues by postulating that complete certainty is represented by the number 1, and that *degree of certainty* relates to complete certainty as the part of a whole, i.e. a number in the interval $[0, 1]$. The degree of certainty 0 represents complete impossibility, and the degree of certainty 0.5, consequently, represents utter uncertainty. A statement is *morally certain* if the degree of certainty is such that the difference relative to complete certainty cannot be perceived. Bernoulli provides the numbers 0.99 and 0.999 as examples of limits that can be used.

Complete certainty and complete impossibility correspond to the statement's numeric truth value, 1 if the statement is true and 0 if it is false. The numeric truth value has the symmetry property that if a statement has numeric truth value x, then its negation has numeric truth value $1 - x$. Bernoulli does not explicitly discuss whether this symmetry property holds for degrees of certainty, but alludes to this by stating that if the limit for moral certainty is 0.999 then the limit for moral impossibility is 0.001.

Bernoulli defines *probability* as degree of certainty, and uses the two terms interchangeably. For the purpose of this review, use of the word probability is slightly problematic because the word has meanings to the modern scientist that it did not have in Bernoulli's lifetime, when probability theory was in its infancy. For instance, the so-called frequency and Bayesian interpretations, as well as Kolmogorov's construction, did not exist during Bernoulli's lifetime; the word probable had a pre-numerate sense, as discussed by Hacking (1971). The best understanding of Bernoulli's use of the word is likely to accept his wording precisely as written: that probability is defined as a synonym for degree of certainty, which has its own definition.

As a historical remark, Locke (1700) defines probability as incomplete proof, which is a distinct and now defunct sense of the word. As an exemplification: A proof shows a statement to be true, while a probability induces the mind to judge the statement to be true. This sense of the word reflects an older English tradition and has no relationship with probability theory (cf. Uzgalis, 2012).

In a pair of axioms, Bernoulli postulates the role of his theory of inference relative to the then-predominant axiomatic-deductive method. By the first axiom, there is no place for inference if complete certainty can be reached, and by the last axiom, if complete certainty cannot be reached then necessity and use ordain that a morally certain statement be taken as if it were completely certain. The last part echoes Arnauld & Nicole's claim that some things that attain a particular limit of certainty should be considered as certain as if there existed mathematical demonstrations of them.

Hacking (1971) and Hald (1990) claim that Bernoulli intended *Ars Conjectandi* as an extension of *La logique*. There are many obvious similarities between the two works: *La logique* is named *Ars Cogitandi* in Latin, and the two works are similarly arranged into four parts. Moreover, *La logique* ends with a brief, unfinished discussion on a continuum of certainty and the concept of moral certainty, which constitute the main foci of *Ars Conjectandi*. Bernoulli's theory of inference complements Arnauld & Nicole's theory of deduction; *The Theory of Inference* is in fact a passably correct English translation of *Ars Conjectandi*.

As to the practical application of his theory, Bernoulli proposes two ways of estimating the probability of a statement: one that uses Huygens' formula, and one that uses the arithmetic mean. If all the parts and pieces that are relevant to the statement have known degrees of certainty, i.e. probabilities, then they can be correctly weighted through use of the expected value operator. The probabilities that go in to the equation could be known either through symmetry or because the parts are deemed completely certain or utterly uncertain. Bernoulli exemplifies the weighting of probabilities through a legal trial situation in which the court weights different arguments that variously favor or disfavor the defendant.

If a probability cannot be estimated through Huygens' formula, the expected value operator, then Bernoulli proposes a method of empirical estimation through repeated experiments and the use of the arithmetic mean. This unassuming proposal constitutes one of the most important contributions to the evolution of modern science. It is in arguing the merits of this proposed method that Bernoulli states and proves his famous limit theorem, which Poisson (1837) termed the law of large numbers.

Bernoulli's theory of inference has many merits. By using the concept of probability as a device, and Huygens' formula for the mathematically correct weighting of different possibilities, the valuation of empirical evidence is made transparent, reproducible and relatively objective. Further, Bernoulli's theory does not challenge or seek to overturn notions of the axiomatic-deductive method, but builds on them. By Bernoulli's last axiom, the theory should only be employed if the axiomatic-deductive method cannot be applied successfully, and hence the two methods cannot conflict. In fact, Bernoulli's theory is itself axiomatic-deductive in nature.

According to the second part of Plato's theory of forms, the senses are inherently unreliable and observations inevitably subject to flux. The conclusion that Plato and many others came to was that empirical evidence cannot yield meaningful knowledge about the ideal forms of phenomena. But through Bernoulli's limit theorem, an observation can be made arbitrarily accurate through repeated observation and use of the arithmetic mean, regardless of how inaccurate the individual observations are. Remarkably, in the last paragraph of *Ars Conjectandi*, Bernoulli reaches out to Plato, discussing how he would have reacted had he been aware of this result.

5. The originality question

The arithmetic mean is a composition of simple arithmetic operations which has been employed since ancient times, for instance for division of inheritance. However, using the arithmetic mean as a statistical method to obtain an estimate more accurate than a single observation is a more recent practice. This section aims to investigate whether Bernoulli's proposal of the method is an original contribution of *Ars Conjectandi*. Because of insufficiently detailed historical sources, the originality question may never be definitively answered, alas moral certainty may never be reached, but analysis of existing sources is interesting nonetheless.

Galilei (1638) measured the time of descent of a hard bronze ball through experiments repeated a full hundred times, but did not use the arithmetic mean. Galilei, one of the foremost scientists of his day, only considered the difference between the greatest and the smallest observations, which he noted was less than a tenth of a pulse-beat. He did not in any way combine the observations to obtain a more accurate estimate of the time of descent. Galilei was a leading astronomer, having improved the telescope and made several empirical discoveries, such as the moons of Jupiter, and should reasonably have been acquainted with the most recent advances in empirical observation, including the method of the arithmetic mean had it existed.

Simpson (1755) argues that it is advantageous to combine observations through the arithmetic mean rather than relying on a single observation taken with due care. Simpson's text implies that the method existed in 1755 but had not yet, as Simpson puts it, been so generally received. Hence the literature indicates that the arithmetic mean as a statistical method first appeared sometime between years 1638 and 1755.

Interestingly, Simpson (1755) contains a perceived counter-argument, which facilitates understanding of why use of the arithmetic mean at the time was not instinctively favored. An astronomer who measures angles on the celestial sphere may develop, through experience or intuition, a sense of the accuracy of the individual measurements. If conditions at one particular instance, atmospheric, technical or other, are such that the astronomer deems the observation to be highly accurate, then it could possibly seem counter-intuitive to contaminate the accuracy of that observation by pooling it with less accurate observations.

In the nineteenth century, by contrast, it became well accepted that through use of the arithmetic mean the observational errors, whether small or large, on average cancel. Therefore the arithmetic mean yields an estimate that is more accurate than any one observation (see, e.g., Gauss, 1809). However, understanding this result presupposes probability theory, the first published text being Huygens (1657), and the law of large numbers, first version published in *Ars Conjectandi* (1713).

As to Bernoulli's own writings, in private correspondence with Leibniz (Bernoulli, 1703, 1704), he repeatedly refers to estimation through the arithmetic mean as his method. But in *Ars Conjectandi*, Bernoulli emphatically denies that using the arithmetic mean is his proposal. Besides the denial being out of place, there are a number of reasons to question the validity of Bernoulli's denial.

In denying originality, Bernoulli uses a markedly different language relative to the rest of *Ars Conjectandi*, claiming: "everyone consistently does the same thing in daily practice. [...] even the most foolish person". Simpson (1755) attests that everyone at the time did not use the arithmetic mean in daily practice, including highly intelligent people. Secondly, Bernoulli claims that Arnauld & Nicole (1662) contains a similar recommendation, but this claim is demonstrably false; Arnauld & Nicole (1662) contains no mention of the arithmetic mean, repeated observation, combination of observations, or any similar notion.

Lastly and most profoundly, Bernoulli argues the merits of the method by stating and proving his limit theorem; the theoretical result proving with impeccable rigor that the arithmetic mean can reach any level of accuracy through repeated observation. Bernoulli's limit theorem is generally accepted as the first result showing that the more observations that are taken, the more accurate an estimate becomes (Hald, 1990). A result vital to the development of probability theory and statistics, Bernoulli expressed that he valued this result more than if he had solved the ancient open problem of squaring the circle (Bernoulli, 1688).

A likely explanation for Bernoulli's denial of originality is that he suspected that empirical determination of probabilities through the arithmetic mean would be perceived as a weakness of his work, one that would invoke spontaneous criticism. From the rationalistic point of view, empirical estimation of a statement's truth value could well been seen as a rather absurd proposal. By eschewing ownership of the method, Bernoulli could preemptively fend off some criticism, since methods, practices and results are typically subject to less criticism if they are perceived as commonplace rather than original contributions.

When all are taken together: private correspondence, Bernoulli's limit theorem, his out-of-place denial, and the historical context; then existing sources point towards originality.

The investigation concludes that the arithmetic mean, used as a method to obtain a more accurate estimate than a single observation, is an original contribution of *Ars Conjectandi* and should be credited to Bernoulli. Even if a reader would dismiss this analysis, according to scientific principles a result should be attributed to the earliest published text in which it appeared, and as of now the earliest known such text is *Ars Conjectandi*.

6. Epilogue

In eulogies of Jakob Bernoulli, Bernard le Bovier de Fontanelle and Joseph Saurin discussed his work on probability, his limit theorem, and not least that Bernoulli applied the concept of probability for valuation of evidence, which they claimed held the greatest originality and promise (Sylla, 2006). According to Hald (1990), Bernoulli's successors recognized the importance of his limit theorem, and successful work was continued by de Moivre (1718) and others, but Bernoulli's theory of inference was seemingly soon forgotten.

Although his theory of inference was forgotten, Bernoulli's idea of pairing empirical observations with the concept of probability persevered through the years. Jakob's nephew Daniel Bernoulli (1778) discussed an objection first raised by Leibniz; that if there are infinitely many outcomes, each with probability zero, then estimating probabilities of outcomes is not fruitful. Daniel Bernoulli's solution consists of instituting a statistical criterion, i.e. a convention by which certain values are considered relatively probable and others relatively improbable, that utilizes probability density. Gauss (1809) applied this density criterion for determining the most probable Kepler orbit given observations of a heavenly body, which under a normal distribution assumption yielded the method of least squares. During the nineteenth century, the method of least squares gained widespread use in astronomy (see, e.g., Airy, 1875), and as a consequence Bernoulli's idea started to gain popularity.

In the late nineteenth century, Bernoulli's theory of inference was reinvented by a fellow philosopher, physicist and mathematician: Karl Pearson. In *The Grammar of Science* (1892), Pearson used Bernoulli's idea in his verification postulate: that verification through means of empirical evidence is the demonstration of overwhelming probability. The concept of a Pearson-verified statement is practically identical to that of a morally certain statement, and the resurrection of Bernoulli's theory of inference constitutes a remarkable historical development. Using a different statistical criterion, the distance criterion, Pearson (1900) constructed the statistical hypothesis test.

In the year 1911, the method of least squares and Pearson's hypothesis test were collected in George Udny Yule's *An Introduction to the Theory of Statistics*. The fact that Yule published these methods, from astronomy and biology, under the name *Statistics* greatly affected the scientific discipline in the United Kingdom (Hill, 1984), and gave the word statistics its modern meaning. During the twentieth century, aided by the electronic computer, the adoption of statistics grew enormously. At present it is the de facto standard methodology in nearly every scientific discipline, save mathematics and theoretical physics where the axiomatic-deductive method is used successfully.

Jakob Bernoulli envisioned an alternative scientific method, based on pairing empirical evidence with the concept of probability, that complements the axiomatic-deductive method where the latter cannot be applied successfully. Today, 300 years after the publication of *Ars Conjectandi*, Bernoulli's vision is reality. Other than Plato's founding of the original academy, it is difficult to conceive of a greater contribution to the scientific community than inventing the principal scientific method that is used daily by nearly all of its members.

Acknowledgements

This work was supported by the Swedish Council for Working Life and Social Research, project 2010-1406. The author is grateful to Andrew Bray and Rick Paik Schoenberg for valuable comments and suggestions for improvements.

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