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Author
Bander, Myron

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EQUIVALENCE OF LATTICE GAUGE AND SPIN THEORIES ☆

Myron BANDER
Department of Physics, University of California, Irvine, CA 92717, USA

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It is shown that a lattice gauge theory based on the group G is equivalent to a lattice spin theory invariant under a global group which is an infinite direct product of G's. A method of inducing a lattice gauge theory is presented.

Several authors have recently studied theories where Einstein gravity is obtained as an effective action after integrating out matter fields coupled to an external metric tensor; the Einstein–Hilbert lagrangian for pure gravity is not inserted explicitly but emerges after some of the matter fields are integrated out [1]. This procedure relies heavily on the use of dimensional analysis and the renormalizability of the theory. Namely, one may show that the only terms in the effective action that will survive in the large cutoff limit are the Einstein–Hilbert action and a cosmological constant.

It is tempting to try a similar exercise for gauge theories. One considers a set of fields coupled to an external gauge potential and then one integrates out the nongauge fields. By dimensional, renormalizability and invariance arguments one can convince oneself that in the limit of large cutoffs as well as large masses for the nongauge fields, the effective action will be of the Yang–Mills type (such a procedure has been previously used in an attempt to prove quark confinement [2]). However, as we shall show, the process of discarding the other terms, even if they appear to vanish in the appropriate limit, is not valid. This point is further elaborated on below.

In this paper we try to see whether a gauge theory can be induced on a lattice. We consider a theory of fields defined on sites of a lattice coupled to link gauge potentials and subsequently integrate out the nongauge fields, hoping to obtain an effective lattice gauge theory of the Wilson type. For any finite set of nongauge fields additional terms appear in the effective action; these terms contain products of link variables around contours of arbitrary lengths. However, in the limit where the number of these nongauge fields goes to infinity (with appropriately chosen coupling constants) one can recover the standard Wilson theory. *In the process, one shows that a gauge theory based on the group G is equivalent to a spin theory with the spins transforming under the group $G \times G \times G \times \ldots \times G$.* Neither the effect of this presentation on the previously discussed induced gravity theories, nor other applications is immediately obvious (at least not to the author).

We start with fields χ_r defined on sites of a D-dimensional euclidean lattice with lattice separation ϵ and transforming under an irreducible representation of some local group G. We shall refer to these fields as spins. Local gauge invariance is achieved by coupling these fields to the exponential of gauge potentials $U_{r,\epsilon}$ defined on the links of the lattice. These gauge fields are matrix representations of the gauge group. We shall consider only the fundamental representations of the gauge group. The spin fields likewise transform under this fundamental representation. At this stage we do not add a kinetic energy term for the gauge potentials. The euclidean partition function for a fixed background gauge potential is

$$Z[U] = \int [\mathrm{d}\chi_r \mathrm{d}\chi_r^\dagger] \exp\left((g^2 M)^{-1/4} \sum_{r,\epsilon} (\chi_{r+\epsilon}^\dagger U_{r,\epsilon} \chi_r + \text{h.c.})\right) . \quad (1)$$

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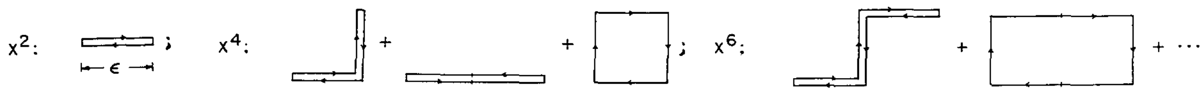

Fig. 1. Diagrammatic representation of terms contributing in various orders in x in the expansion of eq. (2). (The close double lines refer to backtracking on lattice links).

g will turn out to be a coupling constant, while M is a parameter which for now may be absorbed into g; its purpose will become apparent further on. The summation in the exponential is over all lattice sites and all positive link directions emanating from each site. The functional integration over χ is defined so as to insure

$$\int [d\chi_r d\chi_r^\dagger] = 1 , \quad \int [d\chi_r d\chi_r^\dagger] \chi_r^\dagger \chi_r = 1 . \qquad (2)$$

For nonlinear theories this may be achieved by a suitable normalization of the fields χ, while in the linear case by choosing a suitable mass term for this spin field.

The integration over χ will result in an effective gauge invariant theory for the link variables U. Expanding (1) in powers of $x = (g^2M)^{-1/4}$ and using the linked cluster expansion, we obtain the first couple of terms in this effective action.

$$Z[U] = \exp\left[x^2 D \sum_r 1 + x^4 \left(2D^2 \sum_r 1 + \sum_P \prod_{\ell \in P} U_\ell \right) + x^6(...) + ... \right] . \qquad (3)$$

The symbol P indicates that we are summing over products of link variables around elementary plaquettes. The types of configurations contributing to the various powers of x are shown in fig. 1. In the continuum limit all these terms will contribute to an effective $\mathrm{tr} F_{\mu\nu} F_{\mu\nu}$ in the gauge action. At this stage we cannot limit the effective lattice lagrangian to just a single plaquette term of the Wilson action. If we could limit ourselves to diagrams involving only a finite number of links we could still recover an acceptable lattice gauge theory with an acceptable continuum limit. However, the situation is more serious. Closed lattice contours of arbitrary size will contribute terms to the lattice action which will give unacceptable results in the continuum limit. A curve whose length is L/ϵ lattice links will contribute

$$Z_C[U] = \exp\left(x^{L/4\epsilon} \,\mathrm{tr} \prod_{\ell \in C} U_\ell \right) , \qquad (4)$$

to the lattice action. The continuum limit is

$$Z_C[U] = \exp\left[\exp[(L/4\epsilon)\ln x] \,\mathrm{P}\exp\left(i \oint_C A \cdot d\ell \right) \right] . \qquad (5)$$

One might argue that for g sufficiently large and with ϵ tending towards zero this term will not contribute to the effective continuum action. On the contrary, such terms will contribute and result will be an effective theory that does not confine quarks as the vacuum expectation value of the Wilson loops will be proportional to the exponential of their circumference, rather than the exponential of the areas spanned by such loops. It is necessary to suppress the contribution of these large loops further; the best situation would result if we could suppress all contributions beyond the one-plaquette term. This may be achieved by raising eq. (3) to the Mth power and letting $M \to \infty$.

$$\lim_{M\to\infty} Z^M[U] = \exp\left[(M/g^2)^{1/2} D \sum_r 1 + \frac{1}{g^2} \left(2D^2 \sum_r 1 + \sum_P \prod_{\ell \in P} U_\ell \right) \right] . \qquad (6)$$

Aside from a constant term, i.e. independent of the link variable U, this is just the desired Wilson action. The procedure of taking the Mth power may be achieved by defining M spin fields, χ_r^α, and going through this procedure for each one. The result that we obtain is that for M spin fields χ_r^α coupled to a link variable U with coupling constant $(g^2M)^{1/4}$, in the limit $M\to\infty$ induces a Wilson lattice gauge theory of the standard type.

$$Z_W[U] = \exp\left(\frac{1}{g^2}\sum_P \prod_{\ell\in P} U_\ell\right)$$

$$= \lim_{M\to\infty} \int [d\chi_r^\alpha\, d\chi_r^{\dagger\alpha}] \exp\left(x \sum_{r,\epsilon,\alpha} (\chi_{r+\epsilon}^{\dagger\alpha} U_{r,\epsilon}\chi_r^\alpha + \text{h.c.})\right) \exp\left(-[(M/g^2)^{1/2}D + (2/g^2)D] \sum_r 1\right). \quad (7)$$

Rather than integrating out the spin fields and obtaining a field theory for the link variables U, we will integrate out the link variables and obtain a spin theory equivalent to the local gauge theory. We are interested in two quantities [cf. eq. (7)]

$$Z = \int [dU]\, Z_W[U], \qquad W[C] = \frac{1}{Z}\int [dU] Z_W[U] \prod_{\ell\in C} U_\ell. \quad (8,9)$$

C is an arbitrary closed curve on the lattice.

The integration over the link variables may be performed with the help of the following functions

$$I_0^G(\Omega) = \int dU \exp[\tfrac{1}{2}\,\text{tr}(U^\dagger\Omega + \Omega^\dagger U)], \qquad I_1^G(\Omega) = \int dU\, U \exp[\tfrac{1}{2}\,\text{tr}(U^\dagger\Omega + \Omega^\dagger U)]. \quad (10)$$

The superscript G refers to the gauge group of interest. (For G = SO(2) these functions reduce to the ordinary Bessel functions of imaginary argument.) We shall be interested in the limit of these functions, both for small and large Ω. (More precisely, we multiply Ω by a constant c and let $c\to\infty$.) In the first case we obtain

$$I_0^G(\Omega) \to 1 + (1/4R)\,\text{tr}\,\Omega^\dagger\Omega + \ldots, \qquad I_1^G(\Omega) \to (1/2R), \quad (11)$$

R is the dimension of the fundamental representation of the group G. In the second case

$$I_0^G(\Omega) \to \exp[(\text{tr}\,\Omega^\dagger\Omega)^{1/2}]/d_0(\Omega), \qquad I_1^G(Q) \to \exp[(\text{tr}\,Q^\dagger Q)^{1/2}]/d_1(G). \quad (12)$$

The d's are group dependent functions.

We shall now return to the expressions of interest, namely eqs. (8) and (9). From now on we shall omit writing explicitly the constant factor appearing on the right hand side of eq. (7) as it will cancel out in the expressions for all expectation values. The partition function or vacuum to vacuum amplitude becomes

$$Z = \lim_{M\to\infty} \int [d\chi_r^\alpha\, d\chi_r^{\dagger\alpha}] \prod_{r,\epsilon} I_0^G(2\chi_r^\alpha \chi_{r+\epsilon}^{\dagger\alpha}), \quad (13)$$

and the expectation value of the Wilson loop acquires the form

$$W[C] = \frac{1}{Z}\lim_{M\to\infty} \int [d\chi_r^\alpha\, d\chi_r^{\dagger\alpha}] \prod_{r,\epsilon} I_G^0(2\chi_r^\alpha\chi_{r+\epsilon}^{\dagger\alpha}) \prod_{\substack{r\in C\\(r+\epsilon)\in C}} \frac{I_1(2\Sigma_\alpha \chi_r^\alpha \chi_{r+\epsilon}^{\dagger\alpha})}{I_0(2\Sigma_\alpha \chi_r^\alpha \chi_{r+\epsilon}^{\dagger\alpha})}. \quad (14)$$

These equations provide the equivalence between a lattice gauge theory based on the group G and a lattice spin theory based on the group G × G × G × ... × G. For strong coupling, confinement holds only if the limit $M\to\infty$ is taken before the Wilson loop is made large. For any finite M, the expectation value of a large Wilson loop is proportional to the exponential of its circumference.

Gauge invariance is assured by the fact that the function $I_0^G(\Omega)$ is invariant under the transformation $\Omega\to V\Omega W$, with V and W belonging to the group G. A convenient gauge fixing may be obtained by rotating all the χ_r^α for some fixed α into a standard form. Let us say we rotate χ_r^M to the form

$$\chi_r^M = (1, 0, 0, \ldots, 0)^T. \quad (15)$$

(Note, that in going from a single spin field χ_r to M fields χ_r^α we may mix linear and nonlinear spin representations and thus can always choose χ_r^M such that $\chi_r^{\dagger M}\chi_r^M = 1$.) After this gauge fixing we are left with only a global invariance under the group G.

Having fixed a definite gauge it is possible to go to a continuum limit. This is achieved by approximating the I^{G} functions as indicated in eq. (12) and expanding the product of two nearby spin fields as a sum of product of fields and the product of fields and their derivatives. The result obtained is the same if had we integrated out the gauge potentials in a theory given by the lagrangian

$$L = \left(\frac{1}{g^2 M}\right)^{1/4} \sum_\alpha (\partial_\mu \chi^{\dagger\alpha} + \mathrm{i}\chi^{\dagger\alpha} A_\mu)(\partial_\mu \chi^\alpha - \mathrm{i}A_\mu \chi^\alpha) . \tag{16}$$

The above holds for the nonlinear spin theory with $\chi_r^{\dagger\alpha}\chi_r^\alpha = 1$; a similar expression may be obtained for a linear spin model. In the above, we assume that a gauge has been chosen so that χ_r^M is fixed throughout space. In the absence of gauge fixing the continuum limit does not exist due to the presence of singularities for various field configurations.

One might be tempted to simplify the theory by using the approximation of eq. (11) in order to obtain a gauge invariant spin theory whose lattice partition function is

$$Z = [\mathrm{d}\chi_r^\alpha \, \mathrm{d}\chi_r^{\dagger\alpha}] \exp\left(\sum_{r,\epsilon} \sum_{\alpha,\beta} \frac{1}{R} (\chi_r^{\dagger\alpha} \chi_r^\beta \chi_{r+\epsilon}^{\dagger\beta} \chi_{r+\epsilon}^\alpha) \right) . \tag{17}$$

Unfortunately, this model does not exist; it is equivalent to the gauge theories considered by Weingarten [3]. In order to obtain a sensible theory one would have to go to the next order in the approximation of eq. (11) resulting in a lagrangian involving eighth order polynomials in the fields.

Instead of using Bose spin fields one could achieve the same induced gauge theory using M lattice Fermi fields.

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