Title
Robust Cournot-Bertrand Equilibria on Power Networks

Authors
Munsing, Eric
Mather, Jonathan

Publication Date
2017-05-24

Peer reviewed

Robust Cournot-Bertrand Equilibria on Power Networks

Jonathan Mather[1], Eric Munsing[2]

***Abstract*— We present a convex model describing risk-averse strategies for electricity producers in congested electricity networks. Extending prior work on Cournot-Bertrand equilibria in Poolco-style spot markets with locational marginal pricing, we propose a formulation which integrates uncertainty through robust convex optimization. We find that producers uniformly benefit from robust strategies under small uncertainty intervals, and explore the impacts of congestion and network effects on strategic behavior. We demonstrate our results on a simple example and explore the impacts of robust strategies on social welfare, finding that these risk-averse strategies reduce welfare by restricting output and increasing prices. By formulating the equilibrium conditions as a convex optimization problem, we are able to scale our results to large networks and accommodate many sources of uncertainty.**

I. Introduction

Electricity is unique among commodities, having highly inelastic demand, very limited storage, network flows determined by Kirchoff's laws, and transmission constraints which can isolate consumers from low-cost suppliers. The deregulation of electricity generation in many regions has left the operation of the power grid in the hands of Independent System Operators (ISOs), who are tasked with collecting bids for supply and demand, clearing the market in such a way as to meet transmission and security constraints, and mitigating the use of market power.

However, the characteristics which make electricity unique also make energy markets prone to manipulation, and empirical studies have shown that these markets often operate as oligopolies in which participants affect outcomes by adjusting their bid curves to maximize profits [1],[2]. By modeling strategic equilibria in energy markets, researchers hope to measure social welfare impacts, design regulatory or technical changes which can promote more competitive markets, or identify noncompetitive behavior by comparing models with *ex-post* market outcomes.

A key decision for producers in energy markets is how to respond to uncertainty in supply, demand, and the actions of other producers. Risk-averse producers may hedge their production through long-term contracts, or submit their generation capacity as "must-take" (accepting any price). There have been a number of studies examining the impact of uncertainty on strategic equilibria in energy spot markets, particularly in two-settlement markets, for example [3], [4], [5]. However, these approaches can be intractable due to the computational burden required to model uncertainty on a large network using stochastic models [6], [7].

This work advances prior literature by showing how robust optimization can be used to integrate uncertainty into a convex model of strategic equilibrium in a single-settlement Poolco electricity market with network constraints, allowing us to scale our results to large networks while representing the most common form of spot market operation. The results will be of interest to system operators, regulators, and generators in the electricity market, as well as to economists and control theorists studying market design.

A. Relevant Literature

A number of game theoretic models of strategic competition have been used to model oligopoly behavior in electricity networks, and are reviewed in [6], [8], [9], [7]. We highlight three game theoretic models used to identify strategic equilibria in electricity markets: Cournot competition assumes that producers adjust their output to maximize profits [10], [11], [12], Bertrand competition assumes producers adjust prices [6], and Stackelberg leader-follower games assume that some firms may have greater decision-making power and serve as market leaders [13], [14]. Of these, Cournot models have gained particular attention for modeling electricity markets due to their mathematical and computational simplicity, as well as their ability to forecast market outcomes [15], [12], [8].

However, since electricity markets are coupled with complex engineered systems these game-theoretic models are not a panacea. Unlike most commodities, electricity markets are built on a transmission network with thousands of nodes [10], have temporal output constraints on generation equipment [16], and are typically structured as a series of sequential markets [17]. To address these issues, a parallel body of literature has sprung up in which engineering models are used to reflect the technical decisions faced by individual producers [18]. These models are often nonlinear, nonconvex, and computationally intractable for modeling the decisions of more than a single producer with a small fleet of generators.

Both game-theoretic and engineering models are challenged by the uncertainty inherent in electricity provisioning: demand is dependent on weather, generation plants may have unplanned outages, and increased penetration of renewable energy sources makes supply uncertain. A variety of techniques from stochastic optimization have been used to model the generators' decision process under these uncertainties

J. Mather is supported in part by the Lloyd S. Burr Memorial Endowment Fund. E. Munsing is supported in part by the National Science Foundation Graduate Research Fellowship under Grant No. 1106400. This research is partially supported by the National Science Foundation under Award NSF-CPS proposal 1646612.

[1]Mechanical Engineering, University of California at Berkeley jonathan.mather@berkeley.edu

[2]Energy Controls and Applications Laboratory, Civil and Environmental Engineering, University of California at Berkeley e.munsing@berkeley.edu

[19], [20]: historical energy prices can be used to construct Monte Carlo simulations [21], [22] or to fit parametric probability distribution models to sources of uncertainty [23].

These stochastic approaches can optimize expected profits, but struggle to deal with modeling uncertainty in the hundreds or thousands of nodes which characterize electricity grids. Robust optimization theory [24] and robust game theory [25] provide an alternative approach to integrating uncertainty, by seeking a solution which still performs well under a 'worst case' scenario. While this approach is anticipated to reduce the expected profits for the operators relative to their non-robust actions [26], it can be attractive for risk-averse players as it guarantees profits against uncertainty. These models are also mathematically appealing as they do not require any distributional assumptions on random variables and can preserve the convexity of the optimization problem, allowing the application of efficient solvers which can scale up to handle uncertainty across thousands of nodes.

Robust optimization has previously been applied to game theoretic problems, allowing the modeling of uncertainty in payoff matrices or competitors' strategies [27], [28]. However, robust optimization has only been applied to specific sub-problems in electricity market operation, e.g. the unit commitment problem of the system operator [29], [30], [31], [32], nonstrategic bidding as a price-taker [22], strategic equilibrium in a Stackelberg leader-follower game [33], and strategic equilibrium without congestion costs [34].

B. Novel contributions

We propose a convex formulation, computing the robust strategic equilibrium in an electricity network with congestion and demand uncertainty, and demonstrate it using a sample network.

The following contributions are unique to this work:

- Convex formulation of robust Cournot-Bertrand equilibrium in a single-settlement nodal Poolco electricity market
- Demonstration of the impact of congestion on robust strategic equilibria
- Demonstration of the impact of robust strategies on social welfare outcomes in electricity markets.

To our knowledge this is the first attempt to characterize robust Cournot-Bertrand equilibria in electricity markets on transmission networks, extending the work of [35], [36], [37] to incorporate uncertainty and risk-averse producers.

C. Outline

In Section II we present the ISO and producer problems as Cournot-Bertrand competition on electricity networks. We formulate the resulting equilibrium as a monotone linear complementarity problem (LCP), which can be solved as a convex QP. We apply the results of [38], [39] to develop a robust LCP and formulate the robust counterpart of the corresponding convex QP. In Section III we present results for a simple example problem. The impact on producer profits, consumer surplus, and net social benefit is discussed, and we conclude in Section IV.

II. Problem Formulation

This formulation builds on the work of [35] and [37], obtaining the equilibrium as the solution of a linear complementarity problem, for which a convex robust counterpart is developed. Background on modeling energy markets with complementarity problems is provided in [6] and background on robust optimization theory can be found in [24].

A. Network Modeling

The power network is modeled by a connected undirected graph $\mathcal{G}(\mathcal{N},\mathcal{E})$, where $\mathcal{N} := \{1,\ldots,n\}$ is the set of n nodes, and $\mathcal{E} := \{(i,j) : i,j \in \mathcal{N}\}$ is the set of m edges (transmission lines), and $(i,j) \in \mathcal{E}$ means that there is a line connecting buses i and j. Throughout this paper we will assume the standard linear DC power flow model, where lines are lossless, and power flows on the network are governed by a shift-factor (PTDF) matrix $H \in \mathbb{R}^{2m\times n}$ which linearly maps the vector of nodal net injections $r \in \mathbb{R}^n$ to the vector of bidirectional line flows. We denote $T \in \mathbb{R}_+^{2m}$ as the vector of line capacities. Given that nodal injections must sum to zero across the network, we can write the set of feasible power injections as the polytope $\mathcal{R} \subset \mathbb{R}^n$.

$$\mathcal{R} := \left\{ r \in \mathbb{R}^n \mid Hr \leq T,\, \mathbf{1}^\top r = 0 \right\} \tag{1}$$

We assume there are $|\mathcal{F}|$ firms, owning generation units at nodes $i \in \mathcal{N}_f \subset \mathcal{N}$, $f \in \mathcal{F}$. Each firm f makes production quantity decisions for its generators ($\{q_i\}_{i\in\mathcal{N}_f}$), where $0 \leq q_i \leq \overline{q}_i$. For simplicity, we assume that there is at most one generation unit per node.[1] Generation costs $C(q_i)$ are assumed to be convex and quadratic.

At each node, we assume a (steeply) decreasing affine inverse demand function $P(x_i)$, where x_i is the quantity demanded at node i. This is a common assumption for relatively inelastic electricity markets [1], and can represent a linearization of a more complicated inverse demand function. It is worth noting that in general x_i is endogenous, and is calculated as $x_i = r_i + q_i$.

B. The ISO Problem

The ISO controls the import (export) $r_i > 0$ ($r_i < 0$) at each node $i \in \mathcal{N}$ and sets the corresponding locational marginal prices (LMPs). These quantities must satisfy the network feasibility constraints, determined by the set $\mathcal{R}$. The ISO's objective is to maximize social welfare, taken as the aggregated area under the nodal inverse demand functions $P_i(\cdot)$, less the sum of all generation costs $C_i(\cdot)$. Mathematically, the ISO solves the following problem, parametric on the firms' production decisions ($\{q_i\}_{i\in\mathcal{N}}$):

$$\begin{aligned} \underset{r_i}{\text{maximize}} \quad & \sum_{i\in\mathcal{N}} \left(\int_0^{r_i+q_i} P_i(\tau_i) d\tau_i - C_i(q_i) \right) \\ \text{subject to} \quad & 0 = \mathbf{1}^\top r, \ : \ \gamma \\ & 0 \leq T - Hr, \ : \ \mu \end{aligned} \tag{2}$$

[1] This can be achieved in practice by introducing dummy nodes into the network.

As in [37], we have excluded the nonnegativity constraints $r_i + q_i \geq 0$, $i \in \mathcal{N}$, by implicitly assuming an interior solution with respect to these constraints. The KKT conditions are as follows

$$\begin{aligned} 0 &= P_i(q_i + r_i) - \gamma - \psi_i, \quad i \in \mathcal{N} \\ 0 &= \psi - H^\top \mu \\ 0 &= \mathbf{1}^\top r \\ 0 &\leq \mu \perp T - Hr \geq 0 \end{aligned} \tag{3}$$

The first KKT condition implies that

$$q_i + r_i = (P_i)^{-1}(\gamma + \psi_i), \quad i \in \mathcal{N} \tag{4}$$

And consequently,

$$\sum_{i \in \mathcal{N}} q_i = \sum_{i \in \mathcal{N}} (P_i)^{-1}(\gamma + \psi_i) \tag{5}$$

This equation represents the aggregate demand function in the network relating the total consumption quantity to the reference node price γ and the nodal price premiums $\{\psi_i\}_{i \in \mathcal{N}}$, which determine the relative value of LMPs. We denote the LMP vector λ as

$$\lambda = \gamma \mathbf{1} + \psi \tag{6}$$

To prevent arbitrage between nodes i and j, the corresponding congestion charge must be $\psi_j - \psi_i$.

C. The Firm's Problem

We assume that generation firms do not anticipate the impact of their production on the congestion prices set by the ISO. We model this 'bounded rationality' as a game where the ISO and generation firms move simultaneously. Similar to [37] we use a mixed Cournot-Bertrand model, where the ISO behaves a la Bertrand, setting locational price differences, while the generation firms are Cournot players with respect to each other (*i.e.* set quantities), but treat the ISO as a price setter. The reasons for choosing the ISO as a Bertrand player are well discussed in [37].

Each firm chooses its production quantities to maximize profits with respect to the residual demand defined implicitly by (5). In this formulation, the reference bus price γ is determined implicitly by the aggregate production decisions of all the generation firms, just as in a regular Cournot game. However, these production decisions and the implied reference node price also depend on the nodal premiums $\{\psi_i\}$ set by the ISO. The resulting problem solved by each firm $f \in \mathcal{F}$ is

$$\begin{aligned} \underset{q_i : i \in \mathcal{N}_f, \gamma}{\text{maximize}} \quad & \sum_{i \in \mathcal{N}_f} (\gamma + \psi_i) q_i - C_i(q_i) \\ \text{subject to} \quad & 0 \leq q_i \leq \overline{q}_i, \ : \ \nu_i^-, \nu_i^+, \quad i \in \mathcal{N}_f, \\ & \sum_{i \in \mathcal{N}} q_i = \sum_{i \in \mathcal{N}} (P_i)^{-1}(\gamma + \psi_i), \ : \ \beta_f \end{aligned} \tag{7}$$

The KKT conditions are as follows

$$\begin{aligned} 0 &= \gamma + \psi_i - \frac{\partial C_i(q_i)}{\partial q_i} + v_i^- - v_i^+ - \beta_f, \quad i \in \mathcal{N}_f \\ 0 &= \sum_{i \in \mathcal{N}_f} q_i + \beta_f \sum_{i \in \mathcal{N}} \frac{\partial (P_i)^{-1}(\gamma + \psi_i)}{\partial \gamma} \\ 0 &= \sum_{i \in \mathcal{N}} (P_i)^{-1}(\gamma + \psi_i) - \sum_{i \in \mathcal{N}} q_i \\ 0 &\leq \nu_i^- \perp q_i \geq 0, \quad i \in \mathcal{N}_f \\ 0 &\leq \nu_i^+ \perp \overline{q}_i - q_i \geq 0, \quad i \in \mathcal{N}_f \end{aligned} \tag{8}$$

We only consider a single market (e.g. spot market) and do not consider optimization across different energy markets (e.g. forward markets or ancillary services), however we will show that it is possible to represent uncertainty with respect to the outcomes of different energy markets.

These assumptions are consistent with other literature [35], [17] and with the approaches used by most ISOs for scheduling hour-ahead and real-time markets, where the computational benefits of the (convex) lossless DC power flow model are important.

D. Equilibrium Conditions of the Deterministic Game

Aggregating the KKT conditions for the firms' and the ISO's programs yields the equilibrium conditions, which in general form a mixed nonlinear complementarity problem. It becomes a mixed LCP when both the nodal demand functions and the marginal cost functions are linear, as is assumed henceforth.

Let the inverse demand functions and the cost functions be, respectively

$$P_i(x_i) = a_i - b_i x_i, \quad i \in \mathcal{N} \tag{9}$$

$$C_i(q_i) = d_i q_i + \frac{1}{2} s_i q_i^2, \quad i \in \mathcal{N} \tag{10}$$

where $a_i, b_i, d_i, s_i \geq 0$. We denote $a = \text{vec}(a_i)$, $B = \text{diag}(b_i)$, $d = \text{vec}(d_i)$, $S = \text{diag}(s_i)$.

We denote $L \in \mathbb{R}^{|\mathcal{N}| \times |\mathcal{F}|}$ as the firm-node assignment matrix, where $L_{ij} = 1$ if node i is owned by firm j, and $L_{ij} = 0$ otherwise. We also denote $\beta \in \mathbb{R}^{|\mathcal{F}|}$, where β_i is the dual variable associated with firm i. Also denoting $\sum_{i \in \mathcal{N}} \frac{1}{b_i} = \mathbf{1}^\top B^{-1} \mathbf{1} = c$, the equilibrium conditions are then

$$0 = \gamma \mathbf{1} + H^\top \mu - d - Sq + v^- - v^+ - L\beta \tag{11}$$

$$0 = L^\top q - \beta c \tag{12}$$

$$0 = \gamma + \frac{\mathbf{1}^\top q}{c} - \frac{\mathbf{1}^\top B^{-1} H^\top \mu}{c} - \frac{\mathbf{1}^\top B^{-1} a}{c} \tag{13}$$

$$0 \leq \nu^- \perp q \geq 0 \tag{14}$$

$$0 \leq \nu^+ \perp \overline{q} - q \geq 0 \tag{15}$$

$$0 = \mathbf{1}^\top r \tag{16}$$

$$0 = a - B(q + r) - \gamma \mathbf{1} - H^\top \mu \tag{17}$$

$$0 \leq \mu \perp T - Hr \geq 0 \tag{18}$$

Here, (11)-(15) are the aggregated KKT conditions for the firms' problems, and (16)-(18) are the aggregated KKT conditions for the ISO's problem. Under the assumption of linear demand functions and quadratic convex cost functions, the firms' and the ISO's programs are strictly concave-maximization problems, so (11)-(18) are also sufficient. Note that (13) can be excluded from the preceding market equilibrium conditions because it is implied by (16) and (17). This set of equations constitutes a mixed linear complementarity program (mLCP).

We wish to turn these set of conditions into a compact LCP. This derivation closely follows that in [37]. We first write out equations (16) and (17) as follows

$$\begin{bmatrix} a \\ 0 \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} q - \begin{bmatrix} B & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix} \begin{bmatrix} r \\ \gamma \end{bmatrix} - \begin{bmatrix} H^\top \\ 0 \end{bmatrix} \mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

Rearranging and solving for γ and r yields

$$r = Qa - QBq - QH^\top \mu \tag{20}$$

$$\gamma = \frac{\mathbf{1}^\top B^{-1}}{c} a - \frac{\mathbf{1}^\top}{c} q - \frac{\mathbf{1}^\top B^{-1}}{c} H^\top \mu \tag{21}$$

where, denoting $\mathbf{E} = \mathbf{1}\mathbf{1}^\top$

$$Q = B^{-1} - \frac{B^{-1}\mathbf{E}B^{-1}}{\mathbf{1}^\top B^{-1} \mathbf{1}} \tag{22}$$

We note that

$$QB = I - \frac{B^{-1}\mathbf{E}}{c}, \quad BQ = I - \frac{\mathbf{E}B^{-1}}{c} \tag{23}$$

We now consider equations (11) and (12). We have that $\beta = \frac{L^\top q}{c}$, and that $LL^\top = Y$, where $Y_{ij} = 1$ if either $i = j$, or if $i \neq j$ but node i and j are owned by the same firm, $Y_{ij} = 0$ otherwise. Using substitution we rewrite equation (11) as

$$\begin{aligned} 0 = \mathbf{1}\Big(\frac{\mathbf{1}^\top B^{-1}}{c} a - \frac{\mathbf{1}^\top}{c} q - \frac{\mathbf{1}^\top B^{-1}}{c} H^\top \mu\Big) + H^\top \mu \\ -d - Sq + \nu^- - \nu^+ - \frac{Yq}{c} \end{aligned} \tag{24}$$

Collecting terms, using (23), and solving for ν^- we get

$$\nu^- = (BQ - I)a + d + (S + \frac{Y}{c} + \frac{\mathbf{E}}{c})q - BQH^\top \mu + \nu^+ \tag{25}$$

We denote $N = (S + \frac{Y}{c} + \frac{\mathbf{E}}{c})$, where $N \in \mathbb{S}_+$ is positive semi-definite, and has the following properties

$$N_{ij} = \begin{cases} \frac{2}{c} + s_i, & \text{if } i = j, \\ \frac{2}{c}, & \text{if } i \neq j, \text{ and the units at nodes } i \text{ and } j \\ & \text{belong to the same firm,} \\ \frac{1}{c}, & \text{otherwise} \end{cases} \tag{26}$$

We can now write out the following LCP

$$w = \begin{bmatrix} \overline{q} - q \\ \nu^- \\ T - Hr \end{bmatrix}, \qquad z = \begin{bmatrix} \nu^+ \\ q \\ \mu \end{bmatrix},$$

$$t = \begin{bmatrix} \overline{q} \\ (BQ - I)a + d \\ T - HQa \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -I & 0 \\ I & N & -BQH^\top \\ 0 & HQB & HQH^\top \end{bmatrix}$$

where $w = t + Mz$, $w \geq 0$, $z \geq 0$, $w^\top z = 0$. We notice that M is positive semidefinite but not symmetric. Since it is square we can write M as the sum of a symmetric matrix P and a skew symmetric matrix K, such that $M = P + K$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & HQH^\top \end{bmatrix} + \begin{bmatrix} 0 & -I & 0 \\ I & 0 & -BQH^\top \\ 0 & HQB & 0 \end{bmatrix} \tag{27}$$

Due to the fact that $z^\top M z = \frac{1}{2} z^\top (M + M^\top) z = z^\top P z$, we can solve the LCP by solving the following convex QP

$$\begin{aligned} \underset{z \geq 0}{\text{minimize}} \quad & h(z) = z^\top P z + t^\top z \\ \text{subject to} \quad & Mz + t \geq 0 \end{aligned} \tag{28}$$

with any solution z^* solving the LCP(M, t), iff $h(z) = 0$ [6].

E. Formulating a Robust Counterpart

We wish to identify strategies for the producers that are robust to uncertainty, an increasingly prevalent feature of modern power systems. Three potential sources of uncertainty for a generator are: the parameters of the inverse demand function, the quantity of zero marginal cost renewable generation in the network, and the volume of forward contracts signed by other generation firms. All of these sources can be represented as aggregate uncertainty in the residual demand curve faced by a producers, however we will see that this formulation can additionally capture more general sources of uncertainty.

We seek a robust equilibrium where producers maximize their profits, robust to demand uncertainty, while assuming that other producers are adopting strategies robust to demand uncertainty. A robust optimization problem takes the form

$$\min_{x \in X} \max_{u \in \mathcal{U}} f(x; u) \tag{29}$$

which determines the best possible action $x^\star$ under a worst case realization of uncertainty $u \in \mathcal{U}$. As seen in (28), the equilibrium solution of the deterministic problem is an LCP which can be formulated as a convex QP. We obtain a robust equilibrium solution by considering a robust LCP, and formulating the robust counterpart to the equivalent convex QP.

A nominal LCP(M, t) has the form

$$0 \leq z \perp Mz + t \geq 0 \tag{30}$$

The function $h(z) = z^\top (Mz + t)$ is known as the residual of LCP(M, t), with $h(z) = 0$ iff z solves LCP(M, t). Applying the results of [38] we define an uncertain LCP(u) as

$$0 \leq z \perp M(u)z + t(u) \geq 0 \tag{31}$$

where $M(u), t(u)$ are parametric on the realization of a random variable $u \in \mathcal{U}$. A robust solution to the LCP seeks to find a feasible solution z^* which minimizes the residual function $h(z; u)$ under a worst case uncertainty realization u^*. This takes the form

$$\begin{aligned} \min_{z\geq 0}\max_{u\in\mathcal{U}} \quad & z^\top (M(u)z + t(u)) \\ \text{subject to} \quad & \min_{u\in\mathcal{U}} M(u)_i z + t_i(u) \geq 0,\ \forall i \end{aligned} \tag{32}$$

In [39], the authors show that this problem is tractable for affine uncertainty sets of the form

$$\begin{aligned} t(u) &= t_0 + \sum_{l=1}^{L} u_l t_l \\ M(u) &= M_0 + \sum_{k=1}^{K} u_k M_k,\ M_0 \succeq 0,\ M_k \succeq 0,\ \forall k \\ u &\in \mathcal{U} \subseteq \mathbb{R}^{L+K} \end{aligned} \tag{33}$$

where L and K are general scalars defining the affine uncertainty set, and $\mathcal{U}$ can take any of the following forms[2]

$$\begin{aligned} \mathcal{U}_1 = \{u\ :\ \|u\|_1 \leq 1\},\ \mathcal{U}_2 = \{u\ :\ \|u\|_2 \leq 1\} \\ \mathcal{U}_\infty = \{u\ :\ \|u\|_\infty \leq 1\} \end{aligned} \tag{34}$$

While the formulation is general, for exposition we restrict our attention to uncertainty in $t(u)$, such that $M(u) = M$, $\forall u \in \mathcal{U}$. If $\mathcal{U} = \mathcal{U}_\infty$, then (32) takes the form

$$\begin{aligned} \min_{z\geq 0} \quad & z^\top (Mz + t_0) + \sum_{l=1}^{L} \|z^\top t_l\|_1 \\ \text{subject to} \quad & M_i z + t_0 - \sum_{l=1}^{L} \|(t_l)_i\|_1 \geq 0,\ \forall i \end{aligned} \tag{35}$$

As stated previously we consider uncertainty in the residual demand function, which we treat as interval uncertainty in the intercept of the inverse demand functions at each node. We consider functions of the form

$$\begin{aligned} P_i(x_i;\zeta_i) &= a_i(\zeta_i) - b_i x_i,\ i \in \mathcal{N} \\ a_i(\zeta_i) &= a_0 + \zeta_i a_{il},\ \|\zeta_i\|_\infty \leq 1 \end{aligned} \tag{36}$$

Where $a_{il} \geq 0$ is a common belief among producers regarding the bounds of the uncertainty interval at node i. We implicitly assume that all firms have the same belief regarding the uncertainty in the inverse demand function at each node. This assumption is required for the robust LCP model we have used here. Incorporating different beliefs regarding uncertainty to capture the presence of firms with different risk preferences should be possible, although will require a much closer treatment of the individual robust optimization that each firm faces.

Since a only appears in t in the deterministic LCP, it is straightforward to translate uncertainty in a to uncertainty in t, with the resulting robust LCP having the form of (35). This is a convex optimization problem, and thus a robust equilibrium solution exists but may not be unique. The problem can be solved using standard convex optimization solvers.

[2]For the case when $\mathcal{U} = \mathcal{U}_2$, the M_0, M_k are restricted to be symmetric matrices.

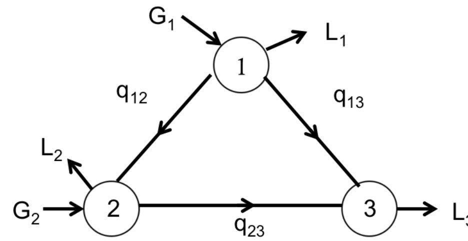

Fig. 1. 3 Node Network

III. Example and Results

We demonstrate our model on an example network, and compare outcomes with conventional (non-robust) Nash-Cournot equilibrium. Networks with up to 300 buses were simulated and were all found to demonstrate qualitatively similar behavior, thus for expositional clarity the simple 3-node network shown in Fig. 1 is used here, similar to the network modeled in [40].

A. Example Network

The three buses $i = 1, 2, 3$ have customers with inverse demand functions $P_i(x_i) = 40 - 0.08q_i$, $i = 1, 2$, and $P_3(x_3) = 35 - 0.05q_3$ \$/MWh (node 3 has greater demand elasticity). Each pair of buses is connected by a single transmission line, and all three lines have equal impedance. The market has two firms $f = 1, 2$ each with a single generator in its fleet; Firm 1's generator is sited at $i = 1$, while Firm 2's generator is at $i = 2$. Both generators have a maximum capacity of $q_i = 1000$MW. Each generator has a constant marginal cost: d_1 = \$15/MWh for firm 1, and d_2 = \$20/MWh for firm 2. We simulate a congested scenario by imposing a 20MW constraint on the line between nodes 1 and 2, and a 35MW constraint on the line between nodes 1 and 3.

In the examples that follow, we assume that $a_{il} = a_l$, $\forall i$. That is the uncertainty at each node is independent but lies in the same interval. The uncertainty a_l is swept over a range of \$0-15/MWh.

For comparison, the non-robust quantity decisions (i.e. classic Nash-Cournot equilibrium) are also modeled. For each scenario, the non-robust quantity is intercepted with the realized demand curve to produce the price at each node. This simulates the scenario where uncertainty is present, but firms behave as if there is no uncertainty. The system is modeled within Matlab using CVX and the Gurobi solver.

B. Results and Discussion

The profits for the two generation firms are shown in Fig. 2 for the case when the transmission lines are unconstrained and there is no congestion in the network. We plot the results for the range of potential uncertainty realizations as a shaded region. These shaded regions are bounded by the profits achieved at the maximum and minimum limits of the uncertainty interval, plotted as thick lines. The results for the nominal value of the uncertainty are plotted as a third solid line through each shaded region. By construction, the

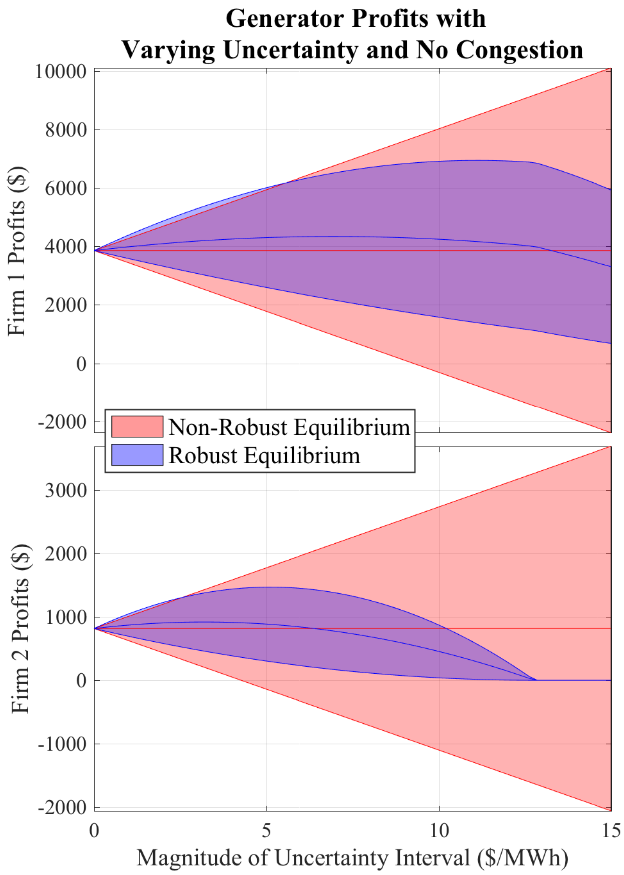

Fig. 2. Profits of both generation firms under a range of uncertainty intervals. Both robust and non-robust strategies are shown, and the three lines indicate high, expected, and low realizations of demand. The robust equilibrium increases prices for low uncertainty intervals, and limits exposure to downside risk.

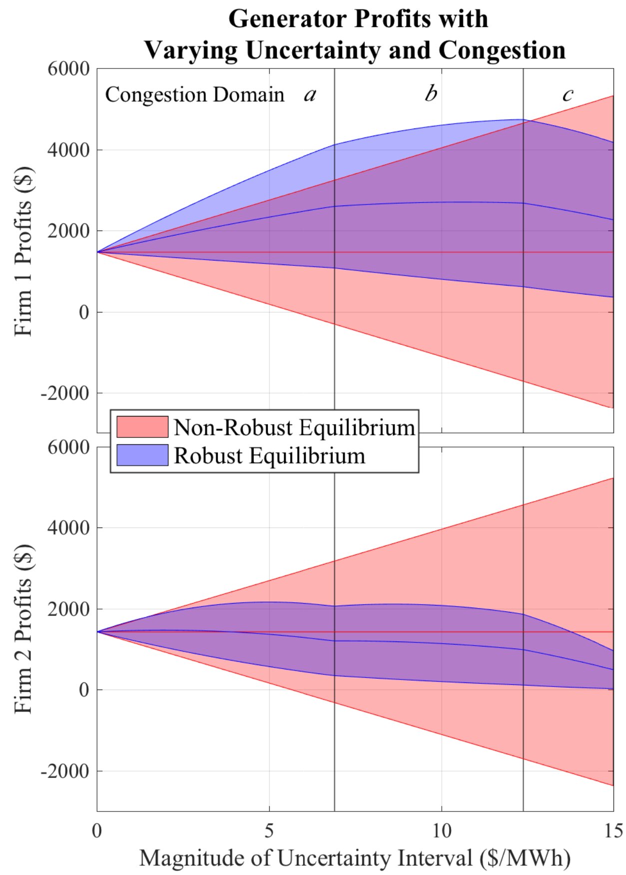

Fig. 3. Profits of each generating firm when congestion is present on the network. Note that the shape of the curves changes over distinct domains, dictated by the congestion on the network: in domain *a* line 1-3 is congested; in domain *b* lines 1-3 and 1-2 are congested, and in domain *c* only line 1-2 is congested.

uncertainty interval spans all potential realizations of demand and thus all possible market outcomes.

Compared with classic Nash-Cournot equilibrium, we see that the primary goal of the robust optimization is met: the robust strategy always results in higher profits for the worst case realization of demand. For small ranges of uncertainty, we see that the firms actually make *greater* profits at the robust equilibrium than at the non-robust equilibrium, regardless of the level of demand. This can be explained by observing that each firm restricts its output in order to protect itself from low prices, contracting the net supply curve and driving up prices.

Eventually the reduction in demand due to the higher prices offsets the initial gain in profits, and we see that the Nash-Cournot equilibrium results in higher profits in nominal and high demand scenarios. At low demand the robust scenario still guarantees that the generators will not incur a loss, whereas a Nash-Cournot equilibrium can actually result in a net loss for generators as realized prices fall below the marginal cost of generation.

It is important to emphasize that these results assume that all firms follow the same robust optimization behavior, *i.e.* that they have the same belief about uncertainty and the same sensitivity to risk. However, for a less risk-sensitive firm, there is an incentive to increase production in order to increase expected profits. The final equilibrium would be dependent on the firms' risk acceptance, with greater risk aversion driving firms towards the robust optimization, and lower risk sensitivity driving them towards Cournot optimization. This is explored in greater detail in [25].

C. Network Effects

As firms restrict their production in response to uncertainty, network flows change and can shift the congestion patterns on the network. We can divide the uncertainty range into distinct domains with unique congestion patterns, highlighted in Figure 3. Within each domain the residual demand curves for each generator stay constant, and equilibrium follows the principles outlined above for the uncongested case. However as uncertainty increases and the congestion pattern changes, there is a discontinuous shift in the residual demand curve each firm faces, seen as a change of curvature in Figure 3.

In our example, comparing Figures 2 and 3 shows that congestion reduces the profits of Firm 1, but also makes a robust strategy more attractive for low and moderate

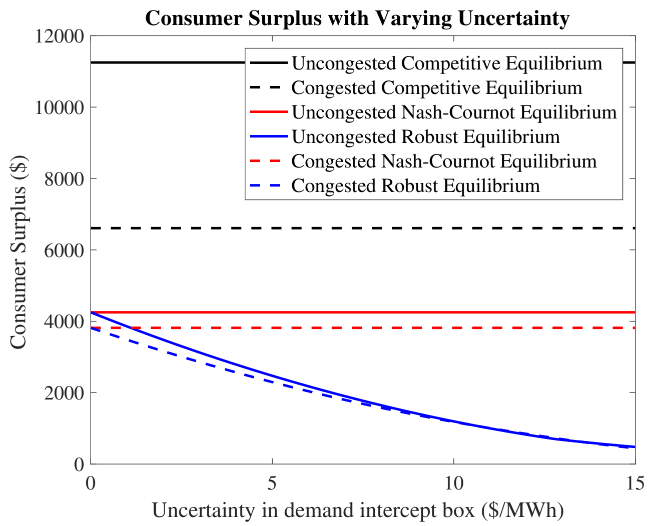

Fig. 4. Consumer surplus under a number of equilibrium models: perfect competition, Nash-Cournot equilibrium, and Nash-Cournot robust equilibrium. When producers restrict production to be robust to uncertainty, consumers are clearly impacted. Congestion further reduces consumer surplus by introducing congestion charges.

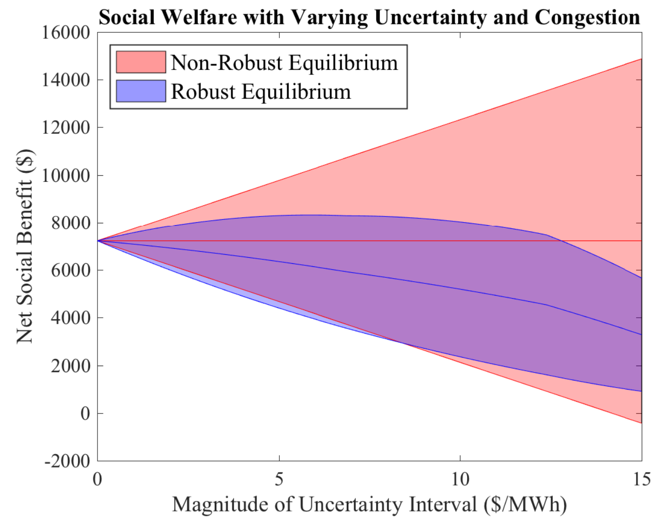

Fig. 5. Net Social Benefit (sum of consumer surplus, producer profits, and merchandising surplus) under both robust and non-robust equilibrium.

uncertainty. Firm 2 benefits from congestion rents, but sees profits more threatened by uncertainty in domain a. As uncertainty increases and the line between nodes 1 and 2 becomes congested in domain b, Firm 2 sees greater benefit from uncertainty and the profits of Firm 1 are eroded. This effect is repeated more dramatically as uncertainty increases into domain c. These effects are dependent on the exact network structure, and were found to be particularly complex for larger networks.

D. Welfare Effects

The impact of the robust strategy on consumers is less nuanced. The consumer surplus is calculated as the area above the market clearing price and below the demand curve, representing the surplus value which consumers would have been willing to pay for electricity [41]:

$$CS = \frac{1}{2}(a - \lambda)^\top x \tag{37}$$

The total consumer surplus for the market is shown in Figure 4 for competitive, Nash-Cournot, and robust equilibria. As we assume that producers offer a fixed quantity of power into the market, the consumer surplus is invariant to the realized inverse demand function for a given uncertainty interval. When firms restrict their output to be robust to low realizations of demand, prices rise above competitive levels, demand decreases, and consumer surplus drops below the Cournot oligopoly level.

The total efficiency of the market can be measured by its net social benefit: the sum of consumer surplus, producer profits, and merchandising surplus[3] [41]. Building on (37), this can be written as

$$NSB = CS + (\lambda^\top q - C(q)) + \mu^\top T \tag{38}$$

Since robust behavior restricts supply below Nash-Cournot equilibrium levels, the net social benefit decreases monotonically [41] as shown in Figure 5.

IV. Conclusion

Electricity markets are particularly susceptible to non-competitive behavior, making it important to understand strategic equilibria in order to inform better market design and policies. The complicated structure of electricity networks, and the many sources of uncertainty in supply and demand, also make it important to have scalable tools for studying the impact of uncertainty on energy markets.

We extend a model of strategic equilibria in electricity networks to include robustness to uncertain demand, reflecting the behavior of risk-averse generation firms. The robust optimization model remains convex, allowing it to be scaled to large power networks. The model is not intended to describe the optimal bidding strategies or bid curves of individual producers, however it provides an efficient way of simulating the impact of uncertainty on market outcomes.

Whereas robust optimization in competitive markets may reduce profits for producers, we see that robustness with small uncertainty intervals *uniformly increases* profits for the generating firms relative to Nash-Cournot equilibrium. The impact of the robust equilibrium on consumers is uniformly negative, as firms restrict their output leading to an increase in prices, similar to that which would be seen under collusive behavior. Thus the ”price of robustness” is seen in a reduction of the net social benefit of the market.

Congestion affects different firms unevenly, as it simultaneously creates congestion rents and increases market power for some generators. We show that uncertainty can affect congestion patterns in robust equilibrium, with the exact

[3]This is the rent collected by the system operator in the presence of congestion, and can be shown to be equal to $\mu^\top T$.

relationship between generator profits and congestion being dependent on network topology.

These results can be applied to reflect uncertainty in supply due to intermittent renewable electricity generation, by modeling the uncertainty in net load (demand less must-take renewables). The results can also represent uncertainty in the forward contracts signed by other firms, which will contract the residual supply curve in the spot market. By incorporating robustness into the strategic equilibrium, we offer additional insight that producers, utilities, and regulators can use to understand outcomes in real markets.

REFERENCES

[1] S. Borenstein, J. Bushnell, and F. Wolak, "Measuring Market Inefficiencies in Californias Restructured Wholesale Electricity Market," *Power*, 2000.

[2] A. Hortaçsu and S. L. Puller, "Understanding strategic bidding in multi-unit auctions: A case study of the Texas electricity spot market," *RAND Journal of Economics*, vol. 39, no. 1, pp. 86–114, 2008.

[3] A. Kannan, U. V. Shanbhag, and H. M. Kim, "Addressing supply-side risk in uncertain power markets: stochastic Nash models, scalable algorithms and error analysis," *Optimization Methods and Software*, vol. 28, no. 5, pp. 1095–1138, 2013.

[4] A. Kannan, U. V. Shanbhag, and H. M. Kim, "Strategic behavior in power markets under uncertainty," *Energy Systems*, vol. 2, no. 2, pp. 115–141, 2011.

[5] D. Zhang, H. Xu, and Y. Wu, "A two stage stochastic equilibrium model for electricity markets with two way contracts," *Mathematical Methods of Operations Research*, vol. 71, no. 1, pp. 1–45, 2010.

[6] S. a. Gabriel, A. J. Conejo, D. Fuller, B. Hobbs, and C. Ruiz, *Complementarity Modeling in Energy Markets*. Spring International Series in Operations Research & Management Science, 2010.

[7] W. Tang and R. Jain, "Game-theoretic analysis of the nodal pricing mechanism for electricity markets," *Proceedings of the IEEE Conference on Decision and Control*, pp. 562–567, 2013.

[8] C. J. Day, B. F. Hobbs, and J. S. Pang, "Oligopolistic competition in power networks: A conjectured supply function approach," *IEEE Transactions on Power Systems*, vol. 17, no. 3, pp. 597–607, 2002.

[9] M. Ventosa, Á. . Baíllo, A. Ramos, and M. Rivier, "Electricity market modeling trends," *Energy Policy*, vol. 33, no. 7, pp. 897–913, 2005.

[10] B. F. Hobbs, C. Metzler, and J. S. Pang, "Strategic gaming analysis for electric power networks: An MPEC approach," *IEEE Transactions on Power Systems*, vol. 15, no. 2, pp. 638–645, 2000.

[11] J. Barquín and M. Vázquez, "Cournot equilibrium in power networks," *Working Paper, Submitted to IEEE Transactions on Power Systems*, no. April, pp. 1–8, 2005.

[12] K. Neuhoff, J. Barquin, M. G. Boots, A. Ehrenmann, B. F. Hobbs, F. A. M. Rijkers, and M. Vazquez, "Network-constrained Cournot models of liberalized electricity markets: the devil is in the details," *Energy Economics*, vol. 27, no. 3, pp. 495–525, 2005.

[13] K. H. Lee, "Strategy equilibrium in Stackelberg model with transmission congestion in electricity market," *Journal of Electrical Engineering and Technology*, vol. 9, no. 1, pp. 90–97, 2014.

[14] J. Yao, S. S. Oren, and B. F. Hobbs, "Hybrid Bertrand-Cournot Models of Electricity Markets with Multiple Strategic Subnetworks and Common Knowledge Constraints," *Restructured Electric Power Systems: Analysis of Electricity Markets with Equilibrium Models*, pp. 167–192, 2010.

[15] O. L. Mangasarian and H. Stone, "Two-Person Nonzero-Sum Games and Quadratic Programming," *Journal of Mathematical Analysis and Applications*, vol. 9, no. 3, pp. 348–355, 1964.

[16] F. Wen and a. K. David, "Optimal bidding strategies for competitive generators and large consumers," *International Journal of Electrical Power and Energy System*, vol. 23, no. 1, pp. 37–43, 2001.

[17] J. Yao, S. S. Oren, and I. Adler, "Two-settlement electricity markets with price caps and Cournot generation firms," *European Journal of Operational Research*, vol. 181, no. 3, pp. 1279–1296, 2007.

[18] R. H. Kwon and D. Frances, "Optimization-Based Bidding in Day-Ahead Electricity Auction Markets: A Review of Models for Power Producers," in *Handbook of Networks in Power Systems I*, 2012.

[19] A. J. Conejo, M. Carrion, and J. M. Morales, *Decision Making Under Uncertainty in Electricity Markets*. New York, NY: Springer Science+Business Media, 2010.

[20] R. Rajagopal, E. Bitar, P. Varaiya, and F. Wu, "Risk-limiting dispatch for integrating renewable power," *International Journal of Electrical Power and Energy Systems*, vol. 44, no. 1, pp. 615–628, 2013.

[21] L. Wang, M. Mazumdar, M. D. Bailey, and J. Valenzuela, "Oligopoly models for market price of electricity under demand uncertainty and unit reliability," *European Journal of Operational Research*, vol. 181, no. 3, pp. 1309–1321, 2007.

[22] M. Dicorato, G. Forte, M. Trovato, and E. Caruso, "Risk-Constrained Profit Maximization in Day-Ahead Electricity Market," *IEEE Transactions on Power Systems on Power Systems*, vol. 24, no. 3, pp. 1107–1114, 2009.

[23] A. Baillo, M. Ventosa, M. Rivier, and A. Ramos, "Optimal offering strategies for generation companies operating in electricity spot markets," *IEEE Transactions on Power Systems*, vol. 19, no. 2, pp. 745–753, 2004.

[24] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski, *Robust Optimization*. Princeton University Press, 2009.

[25] D. Bertsimas and M. Sim, "The Price of robustness," *Operations Research*, vol. 52, pp. 35–53, 2004.

[26] D. Bertsimas, D. B. D. Brown, and C. Caramanis, "Theory and Applications of Robust Optimization," *Operations Research*, p. 50, 2010.

[27] M. Aghassi and D. Bertsimas, "Robust game theory," *Mathematical Programming*, vol. 107, no. 1-2, pp. 231–273, 2006.

[28] R. Nishimura, S. Hayashi, and M. Fukushima, "Robust nash equilibria in N-person non-cooperative games: Uniqueness and reformulation," *Pacific Journal of Optimization*, vol. 5, no. 2, pp. 237–259, 2009.

[29] A. Street, F. Oliveira, and J. M. Arroyo, "Contingency-Constrained Unit Commitment With Security Criterion: A Robust Optimization Approach," *IEEE Transactions on Power Systems*, vol. 26, no. 3, pp. 1581–1590, 2011.

[30] R. Jiang, J. Wang, and Y. Guan, "Robust unit commitment with wind power and pumped storage hydro," *IEEE Transactions on Power Systems*, vol. 27, no. 2, pp. 800–810, 2012.

[31] D. Bertsimas, E. Litvinov, X. A. Sun, J. Zhao, and T. Zheng, "Adaptive Robust Optimization for the Security Constrained Unit Commitment Problem," *IEEE Transactions on Power Systems*, vol. 28, pp. 1–8, 2013.

[32] T. Summers, J. Warrington, M. Morari, and J. Lygeros, "Stochastic optimal power flow based on convex approximations of chance constraints," *Proceedings - 2014 Power Systems Computation Conference, PSCC 2014*, vol. 72, pp. 116–125, 2014.

[33] L. Baringo and A. J. Conejo, "Offering strategy via robust optimization," *IEEE Transactions on Power Systems*, vol. 26, no. 3, pp. 1418–1425, 2011.

[34] H. Haghighat, "Strategic offering under uncertainty in power markets," *International Journal of Electrical Power and Energy Systems*, vol. 63, pp. 1070–1077, 2014.

[35] C. Metzler, B. F. Hobbs, and J.-S. Pang, "Nash-Cournot Equilibria in Power Markets on a Linearized DC Network with Arbitrage: Formulations and Properties," *Networks and Spatial Economics*, vol. 3, pp. 123–150, 2003.

[36] B. F. Hobbs, E. Bartholomew, Y. Chen, G. Drayton, and W. Lise, "Improved transmission representations in oligopolistic market models," *2006 IEEE PES Power Systems Conference and Exposition*, pp. 81–86, 2006.

[37] J. Yao, I. Adler, and S. S. Oren, "Modeling and Computing Two-Settlement Oligopolistic Equilibrium in a Congested Electricity Network," *Operations Research*, vol. 56, no. 1, pp. 34–47, 2008.

[38] Y. Xie and U. V. Shanbhag, "On robust solutions to uncertain monotone linear complementarity problems (LCPs) and their variants," *53rd IEEE Conference on Decision and Control*, pp. 2834–2839, 2014.

[39] Y. Xie and U. V. Shanbhag, "On robust solutions to uncertain monotone linear complementarity problems and their variants," *Siam Journal of Optimization*, 2017.

[40] B. F. Hobbs, "Linear complementarity models of nash-Cournot competition in bilateral and POOLCO power markets," *IEEE Transactions on Power Systems*, vol. 16, no. 2, pp. 194–202, 2001.

[41] W. F. Samuelson and S. G. Marks, *Managerial Economics*. John Wiley & Sons, Inc., 7 ed., November 2011.