Title
Using Mathematical Representations in the Classroom and the Effects on Student Learning

Author
Sprague, Savannah

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USING MATHEMATICAL REPRESENTATIONS IN THE CLASSROOM AND THE EFFECTS ON STUDENT LEARNING

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By

Savannah Layne Sprague

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APPROVED

Dr. Celeste Pilegard
Department of Graduate School of Education

Dr. Richard Cardullo, Howard H Hays Chair and Faculty Director, University Honors
Interim Vice Provost, Undergraduate Education

Abstract

Mathematical representations are models of mathematics concepts including both symbolic math, such as numbers and variables, and non-symbolic math, such as counting blocks or algebra tiles. In theory, mathematical representations supplement student learning in that they provide concrete ways for students to approach mathematics concepts and make connections between their experiences and symbolic math. Research has been done that shows a significant impact on student learning when using a variation of mathematical representations. The first part of the capstone project offers a summary analysis of research articles on mathematical representations and their effects on student learning. Based on the results of the research articles found, they are sorted into categories of effects in a discussion. The results of research articles provide evidence in support of using mathematical representations in mathematics classrooms to better student learning. As a second part of the capstone project, a lesson plan featuring different mathematical representations is provided as an example to use in a classroom.

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Finally, I would like to thank my parents for supporting me both financially and emotionally to pursue my dreams of becoming a math educator. I hope to one day raise a family just as they have raised myself and my siblings to be kind and compassionate individuals who contribute to the world in a positive way.

Table of Contents

Using Mathematical Representations in the Classroom and

the Effects on Student Learning

Remember learning algebra for the very first time? There were always these "unknowns" you had to find. Most of us were either incredibly lost and had no idea what was going, or it was so easy, we could barely fathom why others were struggling. Algebra is the very first time when learners transition from working with only concrete mathematics to making connections between concrete and abstract mathematics. The reason algebra can be very difficult for students is because it requires the ability of the learner to think in a very abstract, not of this world way, that is students must make connections between concrete numbers and abstract unknowns, which are types of mathematical representations.

Literature Review

Mathematical representations are models of mathematics concepts including both symbolic forms, such as variables and numerals, and non-symbolic forms such as algebra tiles and graphics. Representations can also be categorized as internal and external representations. Internal representations are abstract mathematical ideas or cognitive schemas developed by learner in their mind through mathematical experiences. For example, when asked to recall two times two, most can automatically answer four. External representations are physical and visual mathematics such as numerals, variables, graphs, and manipulatives (i.e., physical or virtual tools that learners can manipulate to represent mathematical concepts and operations), and they are used communicate math concepts to each other. As a learner progresses through mathematics, they rely less on concrete representations like pictures or numerals and begin to understand abstract math in forms of variables and formulas. Algebra is a monumental pivot point in a student's schooling that marks the transition from a concrete understanding of mathematics to an abstract understanding. Through this paper, I will examine the research and theory that relate to mathematics representations, how representations are used in the classroom, and the different expectations students and teachers have about mathematical representations.

School Standards

The National Council of Teachers of Mathematics (NCTM) is an organization that promotes an improved education campaign to reform mathematics classrooms across the United States. NCTM is responsible for the development and push towards the use of Common Core State Standards for Math (CCSSM), which sets nationwide goals for K-12 mathematics education. Not all fifty states in the U.S. have adopted the standards, but forty-five out fifty states have gone through the process of adopting and implementing CCSSM in public K-12 schools. The goal of NCTM and CCSSM is to prepare students for college and career-related mathematics. NCTM developed a set of eight mathematical teaching practices which they believe are necessary teaching skills to foster deep learning for students and one of the practices is to "use and connect mathematical representations." This means that effective mathematics teaching encourages students to make connections between different representations to deepen understanding of mathematics concepts and procedures to solve complex problems (NCTM 2014). "The abstractness of math, some only have access to constructing mathematical concepts through the use of representations of those ideas." (National Research Council 2001 p. 94)

Learning Theories

Jean Piaget (1896-1980) was a Swedish Psychologist that developed a widely accepted theory that children learn through developmental stages by trying to make sense of the world. Children go through four chronological stages of developmental learning: sensorimotor, pre-operational, concrete operational, and formal operational. Each stage is separated by distinct behaviors that children develop when learning (Byrnes 2008). Piaget's last development stage is formal operations stage, where children, from ages of twelve to adulthood, develop cognitive tools to think critically and abstractly about the world by making hypothesis and experiments. Piaget argues that children are not capable of thinking about abstract mathematical concepts such as algebra with unknown variables and inverse relationships, until the Formal Operations stage. Most students do not learn algebra in public schools until 8th and high school between the ages of twelve and fourteen, because of the research by Piaget and other researchers, that students are not developmentally ready to approach abstract math until a certain age.

Lev Vygotsky (1896-1934) was a Russian psychologist that contrasts Piaget's theory learning by suggesting that children develop and learn only through a social setting by interacting with peers and a teacher, instead of strict chronological development stages. One of his fundamental concepts of his theory is the zone of proximal development, which is the gap between what the learner can do without the help of an adult and what the learner can do with help from an adult. Students need the opportunity to work with others in an environment in order to have deeper learning.

Looking into the Classroom

According to NCTM (2014), teachers should model using multiple representations for students, while also giving students opportunities to practice using multiple representations. Teachers should focus on individual representations before using multiple representations. Teachers should also allow for time to be spent on discussing usefulness of different representations and making connections between representations by looking mathematical structures and patterns. Finally, teachers should evaluate students' abilities to use multiple representations when problem-solving.

In parallel, NCTM (2014) also states that students should use multiple forms of representations to make sense and understand mathematics in the classroom. Students should have the ability to make choices about which representation is best to use for problem situations. By using representations, students should also be able to justify their understanding and contextualize real-world problems. Lastly, students should be able to identify the advantages and limitations of representations when problem-solving.

When multiple mathematical representations are used every day in each lesson, the classroom builds a routine and expectation for both the teacher and the students to use multiple presentations when problem solving. Pape (2001) discusses how students' initial attempt at understanding mathematics in the informal, mental way, is only revised and refined through the process of sharing with others in an external way such as numbers, drawing or concrete tool when discussing with peer students and the teacher. As teachers make connections to help guide student understanding of different representations, students make their own connections and their internal representation of a mathematics concept changes and adapts. Pape (2001) describes this as a holistic approach to mathematics education where an individual develops their own understanding internally, and then improves their understanding externally through creative processes. In many ways, Pape's discussion and holistic approach is similar to Vygotsky's Social Learning theory, in which people learn socially by communicating and sharing ideas with others. Pape (2001) stresses that effective mathematics learning is when students practice every day by exploring, using and applying different representation to different problem situations. Students need to create and interact with internal and external representations, to discuss representations with other students and teacher, learn and understand various types of representations, and use representations as a tool to think critically and justify mathematical reasoning. As students become more familiar with different representations, they will become more flexible using different representations throughout problem solving.

Concrete to Abstract

One of the important roles multiple representations play in the mathematics classrooms is that students can use different representations to build their understanding of the most concrete mathematics concepts (e.g. numbers) to the most abstract mathematics concepts (e.g. algebra). Students build from concrete representations to abstract representations by changing their internal and external representations through social interactions with peers and the teacher. The teacher models for students to use multiple representations and guide students to make connections between different cognitive levels of representations. Once students can understand the similar structure and patterns between different representations, students can easily switch between looking at math with concrete and abstract perspectives.

Traditionally, mathematics is taught as "I do; we do; you do" where students copy the notes teachers have worked out solving a math problem, then both the students and teachers work together to solve a similar problem the same way during guided practice, and finally students attempt to solve more similar problems the same way during independent practice. Webb, Boswinkel, and Dekker (2008) assert that this "traditional" way of teaching is harmful to students, because students do not make necessary connections when learning and so do not process their learning into long-term memory. Consequently, teaching this way causes teachers to have to spend more time reteaching material to students. Webb et al. (2008) argues that if teachers taught by having students make connections between multiple representations, not only would students have both concrete and conceptual understanding of math concepts but will also have learned more meaningfully and can recall much more information later. Developed by

Freudenthal Institute for Teachers in Netherlands, the “Iceberg Model” is used by teachers in the U.S. as a visual example of how to teach mathematics (Webb et al., 2008). The goal of instruction is that teachers want students to learn and understand a formal and abstract math concept, which is the tip of the iceberg and is visible above water. However, most of the iceberg lays below the water and is not visible which represents informal and concrete understandings of math concepts. First students should think of their most informal and concrete representations of math, such as life examples, and then students start to make connections between their informal representations and preformal representations, which are more formal representations such as numbers and diagrams. Informal and preformal representations should be most of the representations that students use to understand math concepts at first. Once the students have spent time making connections and practicing with informal and preformal representations, students should be introduced to the formal representation such as a formula or rule. Now with experience of using informal and preformal representations students can make sense of and build connections between representations so that the learning is meaningful and is not forgotten. Thus, teachers that follow the “iceberg model” will have to reteach less and spend more time with students having students make connections between informal and preformal representations of mathematics concepts, and then students can easily access and learn formal representations.

Stylianou (2010) compares middle school students’ use of representations and experts’ use of representations. Experts have an abstract understanding of mathematics and use multiple representations of math concepts to explore their understanding, record their thinking, and evaluate their argument. Experts primarily use multiple representations to either to show their depth of understanding or to reduce the abstraction of a mathematical concept. Similarly, Stylianou (2010) also found that middle school students use representations as tools in learning to explore their understanding and evaluate their argument. The difference between students and experts is that students are limited in evaluating their mathematical arguments with representations. Stylianou (2010) further argues that students need social interactions with their peers and the teacher to develop and improve their use of representations while problem solving. The teacher plays a key role in facilitating the discussion between students of the meanings behind multiple representations used in problem-solving. In conclusion, students need to first understand the role of each representation in problem-solving and then move towards using multiple representations when problem-solving.

Schultz and Waters (2000) analyzed which representations students use to solve systems of linear equations. Students can use all different types of representations from the most concrete (manipulatives and tables) to the most abstract (symbolic algebra and matrices), and each representation has advantages and disadvantages when solving systems of equations. While concrete representations allow for all students at all ability levels to have access to solving the problem by making the mathematics concept more tangible, abstract representations allow for students to build a higher level of understanding. Concluding their article, Schultz and Waters (2008) suggest that by using multiple representations and understanding the connections, students build procedural fluency, conceptual understanding, and problem-solving skills.

Cognitive Load Theory

Cognitive load theory argues that students have limited working memory capacity available to process information. They can therefore become overwhelmed when introduced to more than one mathematical representation. Their working memory is overloaded with too many representations and so therefore they do not encode the concept. Moreno and Mayer (1999) make three predictions of student learning when using representations in the classroom. Multiple representations theory argues that students introduced to multiple representations enriches their learning experiences and promotes higher-level of understanding. Single representation theory, supported by evidence from Cognitive Load Theory, argues that students who focus on one representation will have a deep understanding of a concept. The last prediction is a combination of the multiple representation theory and cognitive load theory and suggests that high-achieving students in multiple representations group will outperform in high-achieving students single representations group, but that low-achieving students will score the same in multiple and single representation groups. In the experiment, high-achieving students in the multiple representation group did perform higher than scores for high-achieving students in control group, but low-achieving students in both groups performed at the same level (Moreno & Mayer 1999). Their conclusion supports their prediction that multiple representations can overload students, especially students who struggle with math, and does not help students gain better understanding.

Conversely, Maccini and Hughes (2000) work with students with learning disabilities and multiple representations to find that students with disabilities can benefit from exposure to multiple representations. Their findings support the counter-argument to Cognitive Load theory, that students who struggle with math can improve their understanding with the use of multiple representations (Maccini and Hughes 2000). For students with learning disabilities, it is sometimes difficult for them to succeed in algebra, for they struggle with algebraic math skills including basic skills and terminology, problem representation, problem solution and self-monitoring. In their case study, Maccini and Hughes (2000) had students learn and represent addition, subtraction, multiplication and division problems with multiple representations. The multiple representation strategy STAR helped students by acting as a cue for students to process, represent and solve problems. After practicing with STAR, students' representations became more accurate and thorough, and thus more accurate set-up, plans and solutions to algebraic equations followed. Students scored higher on near-transfer than far-transfer exam, but results show that students with learning disabilities can be taught to represent and solve integer word problems.

Therefore, while evidence from Moreno and Mayer (1999) cautions teachers from using multiple representations that overload students' cognitive capacities, Maccini and Hughes (2000) provide support that students of all skill levels in math can use multiple representations to further their understanding. Teachers should be careful when introducing multiple representations by taking the time to help students develop multiple representations and make connections between them in order to not overload students' cognitive loads.

Teacher Motivation and Interest

Moyer (2002) examines teachers' interest and motivation in using multiple representations in the mathematics classroom. The teachers in the study first went to a professional development class over the summer that gave them examples of different representations to use in class. The teachers were then asked to record how often they used multiple representations throughout the school year and why. Looking at the results, teachers thought that representations were for "fun" and should only be used if there was free time, because representations were non-essential to lessons. Teachers did not use manipulatives for the purpose of helping students make connections between concrete and abstract representations of math to build to higher levels of thinking. Moyer (2002) remarks that teachers face tremendous amounts of pressure from standardized testing, and so teachers are more focused on teaching students to prepare for and succeed on the test, rather than spending time on meaningful learning.

Moyer (2002) further argues that if the factor of pressure from standardized testing is removed, teachers would see more value in using multiple representations to deepen student learning.

Moyer-Packenham, Salkind, and Bolyard (2008) analyzed which roles representations played in the context of a teacher's lesson. The teachers first went to a professional development class where they learned about different virtual manipulatives and were given example lessons using virtual manipulatives. The teachers were then asked plan lessons using virtual manipulatives and submit a reflection of their teaching and student learning along with the lesson they planned. The authors describe five distinct ways teachers use virtual manipulatives in a lesson: (1) to investigate a mathematical concept; (2) for skill solidification to build procedural fluency; (3) to introduce a mathematical concept; (4) to model understanding of a mathematical concept; and (5) to extend understanding of a mathematical concept by looking at patterns and structure. Most teachers in the case study used virtual manipulatives in their lessons for investigation and skill solidification, while very few teachers used virtual manipulatives as an introduction and extension to their lessons. In conclusion, the case study found that the two contributing factors that influenced which virtual manipulative teachers used and how teachers used virtual manipulatives in the lesson were the teachers' familiarity with using the manipulatives and their personal beliefs in "mathematical, cognitive and pedagogical" justification of manipulatives. (Moyer-Pockenham et al. 2008)

Student Motivation and Interest

The two factors that impact students' perceptions of mathematical representations are their previous knowledge and the amount of experience that have using multiple representations. The more students are exposed to different representations and understand the advantages and limits of each representation, the more students are comfortable with using multiple representations in a problem and use representations to justify their reasoning. Ozgun-Koca (1998) looks at what representations students in remedial college math class and asked the students why they chose that representation. Many of the students picked an equation as the representation they were most comfortable with, compared to a table or a graph. However, Ozgun-Koca (1998) points out that during this decade, mathematics curriculum focused on equations as the primary representation for problem solving, so students were obviously uncomfortable with using graphs or tables because they lacked experience in using these representations to solve problems. In the future, there is a need for mathematics curriculum to focus on using multiple representations for problem solving.

Preston and Garner (2003) analyzed the use of multiple representations as a problem-solving strategy and a communication tool in middle school mathematics lessons. In the study, students were given a multi-step problem and asked to solve the problem using any mathematics they knew. Students were given the opportunity to choose their own representation and as a class, most students chose a different representation to use. Due to the context of the problem, some representations were more effective when problem solving than others, but still students used the strategy they were most comfortable with. A unique finding in the study was that although student may have chosen one representation to solve the problem on their own, some students chose a different representation to explain their understanding of the problem to another student.

In summary, students use the representation that they are most comfortable with, and thus the more time a student spends using a representation to solve problems, the more comfortable they are in using that representation. When teachers want to help their students use multiple representations, they need to factor the amount of time that is needed for students learn a representation and make connections to others. Another key point is that in specific scenarios, students can make a sophisticated choice of which representation works best when communicating their understanding to their peers, which is not always the same representation they used when working through their own internal representation of the problem.

Conclusion

Mathematical representations play an important role in mathematics instruction and how students develop an understanding of our world through a mathematical lens. Based on the research reviewed here, students should be exposed to many different kinds of representations, from concrete and visual forms to abstract forms. Mathematical representations also allow for students at all levels of understanding have an entry point when solving a problem, as long as the teacher encourages students brainstorm their own representation of a problem and then bring all the students together to make connections between the different representations of a problem. The teacher should be both introducing many different forms of representations and spending time with each individual representation to give students the depth of understanding. Traditionally, we teach mathematics one way and occasionally two ways, but very rarely do we teachers show “multiple” solution strategies for one problem. From reading through the research-based case studies, common themes are that teachers should show students multiple representations but give students the time to work with and understand both the usefulness and the limits of a representation, and finally give students the ultimate choice to choose the representation that they think makes the most sense for them to solve a problem. Teachers should be cautious of overwhelming students when using multiple representations in the classroom and keep in mind Cognitive Load Theory by building on students’ previous knowledge of representations and making connections between representations. How teachers plan their instruction has a tremendous effect on student learning outcomes, and by implementing multiple representations into their lessons, teachers will positively affect student learners.

Original Lesson Plan: Solving One-Step Equations

As a second part to my Capstone project, I took what I learned about representations from the research and created a lesson plan to implement in a math classroom. According to the Common Core State Standards, 7th grade students are expected to be introduced to solving one-step and two-step equations and inequalities and have a mastery before the end of the year. This is an important transition for students in their K-12 education, because this is when the transition from concrete mathematics to abstract mathematics starts and continues to build on to higher-level abstract math in the following years. First, I will explain why I chose the lesson plan based on the research, and then I will provide a description and example of the lesson.

Commentary

In my lesson plan (see Table 1), I want to use as many different representations for solving equations, starting from the most concrete representations and building up to the most abstract as evidence-based research by Webb et al. (2008). I want to give students the time with each representation by itself, before guiding students to make connections between concrete and abstract representations as evidence-based research by Webb et al. (2008). I want students to also build both internal and external representations through communicating with their peers as evidence-based research by Pape and Tchoshanov (2001). Lastly, I want students to be able to make the choice of which representation is best for them personally to use when solving one-step equations as evidence-based research by Preston and Garner (2000) and Schultz and Waters (2000).

On the first day, students will first look at number sentences with no variables (e.g. 2 + 2 = 4) and use the algebra tiles to represent the problem. The algebra tiles are one type of representation called a manipulative, and students can use the algebra tiles with their hands to model number sentences. The number sentence is a type of symbolic representation that students use to write math. The algebra tiles represent the symbolic math, and, in the lesson, I want to encourage students to make connections between the two representations. Next, students will also look at diagrams of balances and discuss with each other of how to find unknowns on one side of the balance. The balance diagrams serve as a visual representation for students to be able see and understand the concept of balancing the equation. The number sentences, algebra tiles and balance diagrams are concrete representations that many students are familiar with and have previous experience or knowledge of. By spending the first reviewing and encouraging students to recall their most concrete representations of numbers and operations, then I can move towards building more abstract representations such as equations with unknown variables while building on the connections between representations. On the second day, students will look at symbolic forms of equations with unknown variables and use the algebra tiles to solve the equations. Instead manipulating the algebra tiles, students will be drawing diagrams of the algebra tiles to solve the equations. Students will become more comfortable using the algebra tiles and drawing diagrams to represent the algebra tiles by practicing problems. I want students to continue building on their understanding of inverse operations by making connections between the algebra tiles and the drawings. On the third day, students will solve symbolic equations with inverse operations and make connections between algebra tiles and symbolic operations. The symbolic equations represent the most abstract form of the mathematical concept. I want students to be able to see the patterns between solving equations with algebra tiles and solving equations symbolically using inverse operations. On the fourth day, students use their tools for solving one-step equations and apply them to solving real-life problem. Students will be able to use multiple representations while problem-solving and have the opportunity to choose the representation that they think is best for the problem situation. On the last day, I want the students to be able take what they learned from the week and be able fluidly switch using different representations and justify their understanding of problems using the representations. Students will look at pictures of different foods and the amount of food that equals the recommended two-thousand calorie nutrition limit. Students will then use different representations to model what foods and the amount of food they would eat in a day to reach the two-thousand calorie nutrition limit, and then pick one representation they think best models their solution to the problem. At the end of the lesson plan, students should have ability to use multiple representations to solve one-step equations, to identify advantages and limitations of representations when solving one-step equations, and to have a deep understanding of solving for unknown variables.

Table 1

Lesson Plan Materials

Day 1: Transition from Concrete to Abstract Day 1

<u>Description:</u> First introduce algebra tiles and what each tile represents. Students will use algebra tiles to verbally solve one step equations while looking at a balance.

<u>Example:</u>

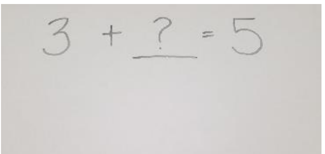

Figure 1

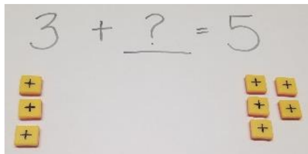

Figure 2

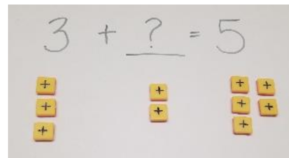

Figure 3

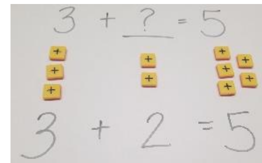

Figure 4

Day 2: Transition from Concrete to Abstract Day 2

<u>Description:</u> Given symbolic equations, students will solve one-step equations by drawing pictures of algebra tiles.

<u>Example:</u>

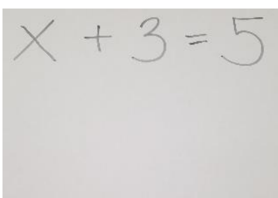

Figure 5

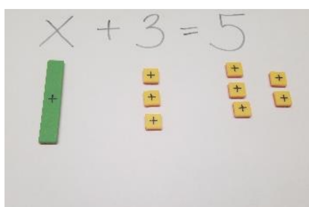

Figure 6

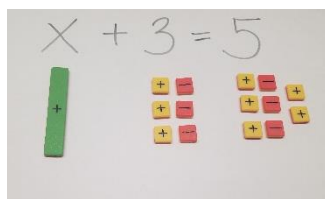

Figure 7

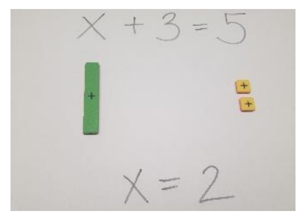

Figure 8

Day 3: Abstract Representation

<u>Description:</u> Students will solve symbolic equations with inverse operations by making connections between algebra tiles and symbolic operations.

<u>Example:</u>

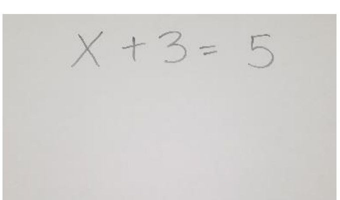

Figure 9

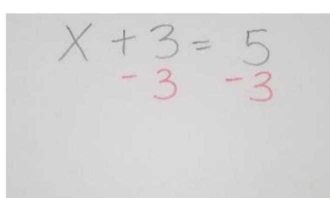

Figure 10

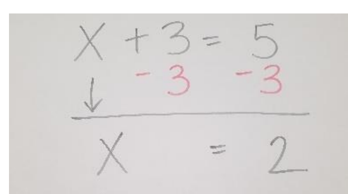

Figure 11

Day 4: Application

Description: In Kaplinsky's task, students will use different representations to solve one-step equations and also pick one representation to focus on.

Example:

Figure 12

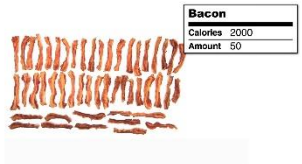

Figure 13

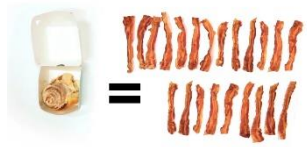

Figure 14

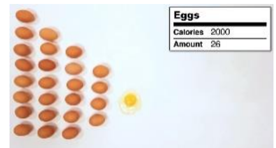

Figure 15

Students would answer the following questions:

1. How many calories is one _____________?
2. How many ___________ could you eat if you only wanted to eat exactly ________ calories? (one-step equation)
3. How many ___________ could you eat if you only wanted to eat at most ________ calories? (one-step inequality)
4. How many more _________ could you eat if you have already eaten ______ of them and want to only eat exactly _______ calories? (two-step equation)
5. How many more _________ could you eat if you have already eaten ______ of them and want to eat at most _______ calories? (two-step inequality)

Source: Robert Kaplinsky's "What does 2000 calories look like?"

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