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ECONOMIC GROWTH AND GENERALIZED DEPRECIATION

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Abstract

The paper generalizes the familiar Solow one-sector model of economic growth by allowing for any depreciation schedule, not just exponential. Specifically, the asymptotic properties of the Solow model are preserved and there exists an average rate of depreciation which replaces the exponential one in the stationary state equation.

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Economic Growth and Generalized Depreciation[1]

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Introduction

As is widely recognized, the principal virtue of the assumption of exponential decay lies in its ability to capture the deterioration of the capital stock without a detailed knowledge of the entire history of investment. It has long been noted that this assumption generally does not conform to observation. An early study by Winfrey (1935) reported that the majority of physical assets have depreciation patterns other than exponential. Coen (1975) found that, for his sample, linear or one-hoss-shay depreciation were more common. Given these observations, it is natural to ask whether a departure from the assumption of exponential decay, will change the asymptotic behavior of the dynamic economy. Recently, Hakkio and Petersen (1988) pose this question with some examples of the Harrod-Domar model using linear and one-hoss-shay depreciation.

This note is intended to demonstrate the robustness of the exponential assumption in the neoclassical one sector model of economic growth - i.e., the Solow model. Specifically, we shall show that for virtually any pattern of depreciation, there exists an <u>average</u> depreciation rate that replaces the exponential one in the stationary state equation and, for the usual neoclassical version of the production function, the capital-labor ratio converges to a steady state.

Formal Structure

[1]The authors wish to acknowledge the most helpful comments of Professor Kenneth Wachter of the University of California, Berkeley and Professor Michael Rothschild of the University of California, San Diego.

The basic model follows Solow (1956): $Y(t) = F(K(t),L(t))$ is homogeneous of the first degree, $L(t)$ grows exponentially at the rate n, We shall depart in the specification of depreciation and suppose that there exists a function $\delta(u)$ such that for every unit of capital formed u years ago, the fraction $\delta(u)$ is lost in the current period.

Suppose that at some time zero the economy inherits a capital stock with a history $\{K(u) \mid u \in (-\infty,0]\}$. From that time onwards, new capital is formed with a gross savings rate of s out of $Y(t)$, i.e. $sF(K(t),L(t))$.

The current stock of capital may be written:

$$K(t) = \int_0^t sL(t-u)f[k(t-u)]\left[\int_u^\infty \delta(x)dx\right]du + \int_t^\infty K(t-u)\left[\int_u^\infty \delta(x)dx\right]du$$

where $k(t)$ denotes $K(t)/L(t)$. and so

$$k(t) = \int_0^t s\tau f[k(t-u]\varphi(u)du + \int_t^\infty \tau k(t-u)\varphi(u)du \qquad (1)$$

for

$$L(t-u) = L(t)e^{-nu}$$

$$\tau \equiv \int_0^\infty e^{-nu}\left[\int_u^\infty \delta(x)dx\right]du$$

$$\varphi(u) \equiv \frac{e^{-nu}\left[\int_u^\infty \delta(x)dx\right]}{\int_0^\infty e^{-nu}\left[\int_u^\infty \delta(x)dx\right]du}$$

Define $g(x) \equiv s\tau f[x]$. Then (1) may be rewritten:

$$k(t) = \int_0^t g[k(t-u)]\varphi(u)du + h(t) \qquad (2)$$

where g is a concave function in its argument and h(t) is the last expression from (1) representing the fading direct effects of inherited capital. Note in passing that $\lim_{t\to\infty} h(t) = 0$.

An equilibrium would require $k^* = g[k^*]$ or $k^* = \tau s f(k^*)$ as expected. Is the process stable?

The answer depends upon the shape of $f(\bullet)$. If $f(\bullet)$ has the traditional neoclassical shape and satisfies the Inada conditions,

$$\lim_{k\to 0} f(k) = \infty,\ \lim_{k\to\infty} f(k) = 0,\ f'(k) \geq 0,\ f''(k) \leq 0$$

then the system can be shown to converge to its unique equilibrium at k^*. In the Harrod-Domar case where $f(\bullet)$ is linear, then the system grows asymptotically where the warranted rate exceeds $1/\tau$ or collapses should it be less. Only in the separating case where the warranted rate exactly equals $1/\tau$ can there be a non-degenerate steady state.

The model is essentially identical to a cohort model of population growth where new births depend upon the relative size of each cohort and cohort fertility may depend upon cohort size.

Recalling (2) above, define $\Delta(t) \equiv k(t) - k^*$. Now

$$\Delta(t) = \int_0^t \left[g[k^* + \Delta(t-u)] - g[k^*] \right] \varphi(u)du + h(t) - g[k^*] \int_t^\infty \varphi(u)du \tag{3}$$

where $\Delta(t)$ has a long run solution at zero. We shall next argue that, under the Inada conditions, $\Delta = 0$ is stable and therefore $k(t)$ converges to k^*.

The argument below follows from Swick (1981) for the cohort model. Using standard perturbation analysis if all of the roots of

$$\frac{1}{g'(k^*)} = \int_0^\infty e^{-ru}\varphi(u)du \tag{4}$$

have negative real parts, then Δ is asymptotically stable. (An elaboration of a similar argument, examining the linear Taylor approximation for $g(\bullet)$ using Laplace transforms, is given in the appendix.) Now (4) has a largest real root which is negative since when r=0 the RHS is unity and less than the LHS (φ is normalized to integrate to one), and the RHS is decreasing in r (φ is non-negative). Then all of the complex roots must lie in the Left Hand Plane and the system is therefore locally stable at $\Delta = 0$ and $k = k^*$.

Global stability may be seen from bounding $f(\bullet)$ by a linear function tangent at $f(k^*)$ and finally noting that $k = 0$ is locally unstable as an equilibrium.

If the Inada conditions fail and the production function, though concave, exhibits $f'(k) > s\tau$ for all positive k, then the only solution to $k = g(k)$ is for $k = 0$ and the system is globally unstable and explodes. If $f'(k) < s\tau$ for all $k > 0$ then, again, the only solution to $k = g(k)$ is at $k = 0$ which is now a stable equilibrium.

Examples and Special Cases:

Supposing $n = 0$ for simplicity:

Common examples of δ functions along with their corresponding τ's are:

(a) T year linear depreciation: $\begin{cases} \delta(t) = (1\ /T) \text{ for } t \in [0,T] \\ \qquad 0 \qquad \text{for } t > T \\ \tau = (1/2)T \end{cases}$

(b) T year one-hoss shay: $\begin{cases} \delta(t) = 0 \text{ for all } t \neq T \\ \tau = T \end{cases}$

(c) Exponential Deprecation: $\begin{cases} \delta(t) = \delta e^{-\delta t} \\ \tau = (1/\delta) \end{cases}$

When $n \neq 0$, and depreciation is exponential , τ is simply $1/(n+\delta)$. The somewhat more complicated expressions for the other cases are straightforward.

Summary:

In conclusion, the extension of the Solow model to allow for a generalized pattern of depreciation does not appear to alter the fundamental stability or asymptotic properties of the analysis, though of course, the trajectory for the economy is modified.

Appendix

In order to explore the stability of k^*, let us take the linear terms of the Taylor expansion of $g(\bullet)$ about k^*:

$$g(k^*+\Delta) = g(k^*) + s\tau f'(k^*)\Delta$$

and substitute into (3) to obtain

$$\Delta(t) = \int_0^t s\tau f'(k^*)\Delta(t-u)\varphi(u)du + h(t) - k^*\int_t^\infty \varphi(u)du \tag{5}$$

Then the Laplace Transform of $\Delta(t)$ is defined by

$$\mathbb{L}(\Delta(t)) \equiv \int_0^\infty e^{-\lambda t}\Delta(t)dt$$

Denote $f'(k^*)$ by θ and, by applying the convolution theorem, the transform of (4) yields:

$$\mathbb{L}(\Delta(t)) = (\theta s\tau)\mathbb{L}(\Delta(t))\mathbb{L}(\varphi(t)) + \mathbb{L}(h(t)) - k^*\mathbb{L}\left[1-\int_0^t \varphi(u)du\right]$$

$$= (\theta s\tau)\mathbb{L}(\Delta(t))\mathbb{L}(\varphi(t)) + \mathbb{L}(h(t)) - k^*\left(\frac{1}{\lambda}\right)\left(1 - \mathbb{L}[\varphi(t)]\right)$$

Solving for $\mathbb{L}[\Delta]$ yields:

$$\mathbb{L}[\Delta(t)] = \frac{\mathbb{L}[h(t))-k^*\left(\frac{1}{\lambda}\right)\left(1 - \mathbb{L}[\varphi(t)]\right)}{[1-\theta s\tau\mathbb{L}[\varphi(t)]} \tag{6}$$

Define ρ to be the largest zero of $m(\lambda) = 1 - \theta s\tau\mathbb{L}[\varphi(t)]$.[2] For notational convenience, denote $\mathbb{L}[h(t)]$ as $H(\lambda)$ and $\mathbb{L}[\varphi(t)]$ as $d(\lambda)$.Observe $d(\lambda) < 1$ for $\lambda > 0$ and $d(0) = 1$.

To show that ρ exists, just observe that since $\varphi(t)$ is non-negative,

[2]Note that $m(\lambda) = 0$ may be rewritten $\mathbb{L}[\varphi(t)] = 1/(\theta s\tau)$ which is identical to equation (4).

then $d(\lambda)$ is a decreasing function in λ so $m'(\lambda) \geq 0$. Since $\lim_{\lambda\to\infty} d(\lambda) = 0$ and $\lim_{\lambda\to -\infty} d(\lambda) = \infty$, and d is continuous, for some value of λ, d exactly equals one. Finally, $m(0) = 1-\theta s\tau$, and it is straightforward to show:

Lemma 1:

(i) $\theta s\tau \leq 1 \Rightarrow \rho \leq 0$.

(ii) $\theta s\tau > 1 \Rightarrow m(\lambda) = 1 - \theta s\tau d(\lambda)$ has unique, positive solution.

The critical relationship for stability, the sign of $(1-\theta s\tau)$ is quite readily interpretable: If the average duration of capital is τ, then each unit will result in the production of $(\tau\theta)$ additional units of output over its lifetime. Of this, the fraction s is saved resulting in replacement capital in the amount of $(s\tau\theta)$. If this will more than replace the original unit of capital then the system expands.

We may now present the main results in the following propositions:

Proposition 1: For $\theta s\tau <1$, $\Delta(t)$ approaches a limit of zero.

Proof:

Since $\rho \leq 0$ for $\theta s\tau < 1$, $\mathbb{L}(\Delta(t)$ is defined for all $\lambda > 0$. Applying the final value theorem of Laplace transforms, we have:

$$\lim_{t\to\infty} \Delta(t) = \lim_{\lambda\to 0_+} \lambda\mathbb{L}[\Delta(t)] = \lim_{\lambda\to 0_+} \frac{\lambda H(\lambda) - k^*\left[1 - d(\lambda)\right]}{1-\theta s\tau d(\lambda)}$$

Since the numerator goes to zero and the denominator to a positive number, Δ does to zero.

Proposition 2: When $\theta\tau > 1$, $\lim_{t\to\infty} k(t) \to \infty$ and the asymptotic growth rate of $k(t)$ is $\rho > 0$.

Proof: Now we have $\rho > 0$. As $\theta s\tau > 1$, $\mathbb{L}(k(t))$ is defined for $\lambda > \rho$.

For $\rho > 0$, $\mathbb{L}[e^{-\rho t}k(t)] < \infty$, and

$$\mathbb{L}[e^{-\rho t}\Delta(t)] = \frac{H(\lambda+\rho)-k^{*}\left[\frac{1}{\lambda+\rho}\right]\left[1 - d(\lambda+\rho)\right]}{[1-\theta s\tau d(\lambda+\rho)]}$$

$$\lim_{t\to\infty} e^{-\rho t}\Delta(t) = \lim_{\lambda\to 0_{+}} \frac{\lambda H(\lambda+\rho)-k^{*}\left[\frac{\lambda}{\lambda+\rho}\right]\left[1-d(\lambda+\rho)\right]}{[1-\theta s\tau d(\lambda+\rho)]}$$

By an application of L'Hospital's rule this becomes:

$$\frac{H(\rho)+k^{*}\left[\frac{1}{\rho}\right][1-d(\rho)]}{-\theta s\tau d'(\rho)}$$

which is finite under the conditions on the shape of $m(\lambda)$ and the hypothesis. So we have $\lim_{t\to\infty} \Delta(t) = \infty$ and $\Delta(t)$ growing at ρ in the limit.

Q.E.D.

Proposition 3: When $\theta s\tau = 1$, the capital stock converges to a steady state.

Proof: Once again, as in proposition 1,

$$\lim_{t\to\infty} \Delta(t) = \lim_{\lambda\to 0_{+}} \lambda\mathbb{L}[\Delta(t)] = \lim_{\lambda\to 0_{+}} \frac{\lambda H(\lambda)-k^{*}\left[1 - d(\lambda)\right]}{1-\theta s\tau d(\lambda)}$$

Now, as λ approaches zero and we apply L'Hospital's rule this limit becomes

$$\frac{H(0)+k^* d'(0)}{-\theta s \tau d'(0)} > 0$$

Q.E.D.

Thus the critical level of savings is given by $\theta s \tau = 1$. i.e. $s^* = 1/(\theta\tau)$, and when $s > s^*$, the economy demonstrates sustained growth at an asymptotic rate given by the solution of:

$$1 = \theta s \tau d(\rho)$$

and for $s < s^*$ the capital-labor ratio collapses to zero.

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