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Author
Palamourdas, Konstantinos

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by

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Abstract of the Dissertation

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by

Konstantinos Palamourdas

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor Itay Neeman, Chair

It is well known by [1] that the Borel chromatic number of a graph generated by a Borel function is ω or at most 3. In this dissertation we will prove that the Borel chromatic number of a graph generated by n Borel functions that commute is ω or at most $2n+1$. On top of that, we will prove that the Borel chromatic number for graphs generated by 2 functions is ω or at most $2 \times 2 + 1 = 5$, while the Borel chromatic number for graphs generated by 3 functions is ω or at most 8.

The dissertation of Konstantinos Palamourdas is approved.

John Carriero

Donald Martin

Yiannis Moschovakis

Itay Neeman, Committee Chair

University of California, Los Angeles

2012

To my family

that supported me tirelessly

for so many years!

Table of Contents

List of Figures

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Vita

2003	High School degree from General Lyceum of Chalkis (Greece) with honors.
2006	Diploma in Mathematics from University of Athens (Greece) with honors, awards and scholarship.
2007	Awarded the Greek “Onassis-foundation” scholarship for academic excellence.
2010	MA in Mathematics from University of California Los Angeles (USA).
2011	Awarded the Sorgenfrey Teaching award from UCLA.
2012	PhD in mathematics from UCLA.

CHAPTER 1

Preliminaries

There has been a lot of work in descriptive set theory on actions on Borel spaces. Each such action can be viewed naturally as a directed graph induced by a Borel function. In [1], Kechris, Solecki and Todorcevic initiated the study of definable combinatorics of these graphs. More precisely, they defined the notions of Borel coloring and Borel chromatic number. The concept of Borel chromatic numbers is parallel to that of the usual chromatic numbers from graph theory. However, one can easily see that the two behave very differently, as there are examples of trees with infinite Borel chromatic numbers (see [1]).

In [1] they showed among other things that a Borel graph generated by one (Borel) function has a Borel chromatic number which is ω or at most 3. In this dissertation we will generalize the results to more than one functions.

In Chapter 2 we give a summary of these results. But first, we give all the relevant definitions and some background from [1]:

Definition 1.1. Let X be a set.

- A *(directed) graph* $\mathcal{G}$ on X is a binary relation $E \subseteq X \times X$ which is irreflexive (i.e. $(x, x) \notin E$). We write this as $\mathcal{G} = (X, E)$. Every $(x, y) \in E$ is called an *edge* of the graph $\mathcal{G}$ and every $x \in X$ is called a *vertex* or *node* of $\mathcal{G}$. If X is finite then $\mathcal{G}$ is a *finite graph*.

- A *forward path* in $\mathcal{G}$ is a sequence $(x_n)_{n \in k}$ where $k \in \omega + 1$, and $(x_i, x_{i+1}) \in E$ for all i s.t. $i + 1 \in k$. Since we do not deal with any other kinds of paths, we sometimes omit the word *"forward"*. If $k = \omega$ we say that the (forward) path is *infinite* or *unbounded*.

If $k \in \omega$, $x_0 = x$ and $x_{k-1} = y$ then we call this a *path from x to y*.

- A set $A \subseteq X$ is said to be *bounded* if it does not contain any infinite (forward) paths.

- A *cycle* in $\mathcal{G}$ is a sequence $(x_n)_{n \in k}$ where $k \in \omega$, each $x_n \in X$, $x_i \neq x_j$ for all $(i,j) \in (k \times k) \setminus \{(0, k-1)\}$, $x_0 = x_{k-1}$, and $(x_i, x_{i+1}) \in E$ for all $i+1 \in k$. A graph with no cycles is called *acyclic*

- A *connected component* of $\mathcal{G}$ is a set $A \subseteq X$ s.t. any two elements in A are connected via a path. If $\mathcal{G}$ has only one connected component we call it *connected*. An acyclic connected graph is called a *tree*. An acyclic (but not necessarily connected) graph is called a *forest*.

- A *successor* or *descendant* of a node $x \in X$ is every node $y \in X$ s.t. $(x,y) \in E$. The *out-degree* of x is the cardinality of the set of its successors.

- A *predecessor* or *ancestor* of a node $x \in X$ is every node $y \in X$ s.t. $(y,x) \in E$. The *in-degree* of x is the cardinality of the set of its predecessors.

- The *degree* of $x \in X$ is the cardinality of the sum of the sets of its successors and its predecessors.

- A finite graph $\mathcal{G} = (X, E)$ is called a *clique* if for every two $x, y \in X$ we either have $(x,y) \in E$ or $(y,x) \in E$. If in addition, X has exactly $k \in \omega$ elements then we call $\mathcal{G}$ a *k-clique*.

- Let I be an (index) set and $\mathcal{F}_\mathcal{I}$ be a family of functions $F_i : X \to X$ $(i \in I)$. We say that $\mathcal{G}$ is *generated* by $\mathcal{F}_\mathcal{I}$, and write $\mathcal{G} = \mathcal{G}_{\mathcal{F}_I}$, if $\mathcal{G} = (X, E)$ where $(x,y) \in E$ iff $x \neq y$ and $F_i(x) = y$ for some $i \in I$. Also, if $\mathcal{F}_\mathcal{I} = \{F_1, F_2, ..., F_n\}$ then we say that $\mathcal{G}$ is generated by $\{F_1, F_2, ..., F_n\}$ and we write $\mathcal{G} = \mathcal{G}_{F_1, F_2, ..., F_n}$.

- A *coloring* of $\mathcal{G}$ is a map: $c : X \to Y$, s.t. $(x,y) \in E \Rightarrow c(x) \neq c(y)$. If $|Y| = k$ then we call c a *k-coloring*. The smallest cardinal k for which $\mathcal{G}$ admits a k-coloring is called the *chromatic number* of $\mathcal{G}$. We write this as $\mathcal{X}(\mathcal{G}) = k$.

- A graph $\mathcal{G} = (X, E)$ on a standard Borel space X will be called *Borel* if the relation $E \subseteq X^2$ is Borel. Note that if I is countable and all functions F_i $(i \in I)$ are Borel then $\mathcal{G}_{\mathcal{F}_I}$ is Borel. In particular $\mathcal{G}_{F_1,F_2,...,F_n}$ is Borel for any $n \in \omega$ and any Borel functions $F_1, F_2, ..., F_n$.

Definition 1.2. Let $\mathcal{G} = (X, E)$ be a graph on a standard Borel space X. Let $n \in \omega + 1$. A *Borel n-coloring* of $\mathcal{G}$ is an n-coloring which is also Borel, meaning that its corresponding coloring map is a Borel function. Equivalently, the coloring may be viewed as a partition $X = \biguplus_{i \in I} A_i$ where $|I| = n$, every A_i is Borel, and if $x, y \in A_i$ then $(x, y) \notin E$ and $(y, x) \notin E$. We also define the *Borel chromatic number* of $\mathcal{G}$, denoted $\mathcal{X}_B(\mathcal{G})$, to be the smallest $n \in \omega+1$ s.t. $\mathcal{G}$ admits a Borel n-coloring. If such a coloring does not exist we say that: $\mathcal{X}_B(\mathcal{G}) > \omega$.

In what follows we will work with graphs $\mathcal{G}$ on a standard Borel space X that are generated by finitely many Borel functions $\{F_1, F_2, ..., F_n\}$ and thus they are Borel. By [1] we already know the following:

Fact 1.3. Let X be a standard Borel space, $n \in \omega$, and $F_i : X \to X$ be Borel functions for $i \in n$. Then for $\mathcal{G} = \mathcal{G}_{F_0,F_1,...,F_{n-1}}$ we have that $\mathcal{X}_B(\mathcal{G}) \leq \omega$.

Fact 1.4. There is a Borel space X and a Borel function $F : X \to X$ s.t. $\mathcal{X}_B(\mathcal{G}_F) = \omega$.

It is also a well known fact from graph theory that:

Fact 1.5. Let X be a finite set (with $|X| = k \in \omega$), $n \in \omega$, and $\mathcal{G}$ be a (finite) graph on X s.t. every node $x \in X$ has out-degree $\leq n$. Then $\mathcal{X}(\mathcal{G}) \leq 2n + 1$

Sketch of Proof. First, note that there is $x_0 \in X$ with degree $\leq 2n$. Otherwise, $\mathcal{G}$ would have more than $(2n)k/2 = nk$ edges. But this would imply that at least one of its k nodes has out-degree more than n. In the same manner, in the graph induced by $X \setminus \{x_0\}$ we can find a node $x_1 \in X \setminus \{x_0\}$ with degree $\leq 2n$ in the induced subgraph on $X \setminus \{x_0\}$. Recursively, we can enumerate $X = \{x_0, x_1, ..., x_{k-1}\}$ in such a way that for every $i \in k$, x_i has degree $\leq 2n$ in the graph induced by $\{x_i, x_{i+1}, ..., x_{k-1}\}$. Now, we can color $\mathcal{G}$ recursively as follows:

We start with the node x_{k-1} which we can color arbitrarily with any of the available $2n+1$ colors. Next, assuming that we have colored the nodes $x_{i+1},...,x_{k-1}$ we observe that x_i is connected to at most $2n$ vertices from the set $\{x_{i+1},...,x_{k-1}\}$ so by the pigeon-hole principle we can color it using one of the available $2n+1$ colors which is different than the colors of the nodes from $\{x_{i+1},...,x_{k-1}\}$ which are connected to x_i. □

Fact 1.6. Let $n \in \omega$. There is a standard Borel space X and $F_i : X \to X$ Borel functions (for $i \in n$) s.t. $\mathcal{X}_B(\mathcal{G}_{F_0,F_1,...,F_{n-1}}) \geq 2n+1$.

Proof. Let $X = 2n+1$. We define $F_i(x) = x+i \pmod{(2n+1)}$ for all $x \in X$ and $i \in n$. By the definition we observe that for $x, y \in X$ there is at most one $i \in n$ s.t. $F_i(x) = y$. Otherwise, if $i, j \in n$, $i \neq j$ and $F_i(x) = F_j(x) = y$ then $x+i = x+j \pmod{(2n+1)}$ which implies that $i = j \pmod{(2n+1)}$ and thus $i = j$ since $i, j < n$. Moreover, if $F_i(x) = y$ then there is no $j \in n$ s.t. $F_j(y) = x$. Otherwise, $x+i = y \pmod{(2n+1)}$ and thus $y+j+i = y \pmod{(2n+1)}$ which means that $j+i = 0 \pmod{(2n+1)}$ which is a contradiction since $j+i \leq 2n < 2n+1$. We conclude that $\mathcal{G} = \mathcal{G}_{F_0,F_1,...,F_{n-1}} = (X, E)$ has no double edges in the sense that if $x, y \in X$ with $(x, y) \in E$ then $(y, x) \notin E$ while there is a unique $F \in \{F_0, F_1, ..., F_{n-1}\}$ s.t. $F(x) = y$. Therefore, $\mathcal{G}$ has exactly $n*(2n+1) = \binom{2n+1}{2}$ (single) edges, which is the most a finite graph with $2n+1$ nodes and no double edges can have. We conclude that $\mathcal{G}$ is a clique and thus $\mathcal{X}(\mathcal{G}) \geq 2n+1$ which in turn means that $\mathcal{X}_B(\mathcal{G}) \geq 2n+1$. □

All the above lead naturally to the following question which was first asked in [1]:

Question 1.7. *Let X be a standard Borel space, $n \in \omega$ and $F_i : X \to X$ be Borel functions for $i \in n$. Is it true that $\mathcal{X}_B(\mathcal{G}_{F_0,F_1,...,F_{n-1}}) \in \{1, 2, ..., 2n+1\} \bigcup \{\omega\}$?*

By the results above, a positive answer would be *optimal*: By Facts 1.6 and 1.4, $2n+1$ and ω are both possible Borel chromatic numbers for graphs generated by n Borel functions.

Question 1.7 is the main topic of this dissertation. We will present several new results on various cases, that provide or approach positive answers. But first we present the results from [1] that provide a positive answer in case $n = 1$.

Theorem 1.8. *Let X be a standard Borel space and $F : X \to X$ be a Borel functions. Then $\mathcal{X}_B(\mathcal{G}_F) \in \{1, 2, 3, \omega\}$.*

Proof. We will give a proof slightly different than the one in [1]. This is because we want to be consistent with the ideas that we will use later on for proving the Least Available Subset Lemma 5.1. By Fact 1.3 $\mathcal{X}_B(\mathcal{G}) \leq \omega$. So, say that $\mathcal{X}_B(\mathcal{G}_F) = k \in \omega$ and $c : X \to k$ is the coloring function. Now, setting $A_i = c^{-1}(i)$ we have $X = \biguplus_{i \in k} A_i$ and each A_i is 1-colorable. We will re-partition X in a different way into the sets B and C (i.e. $X = B \uplus C$) using the A_i recursively as follows:

- We first set $B_0 = A_0$ and $C_0 = \emptyset$.
- Now (for $0 < i \leq k-1$) assuming that $A_0 \cup A_1 \cup ... \cup A_{i-1} \subseteq B_{i-1} \cup C_{i-1}$ we set $\bar{B}_i = \{x \in A_i \mid \text{the successor of } x \text{ (if it exists)} \notin B_{i-1}\}$. We also set $\bar{C}_i = A_i \setminus \bar{B}_i$. Next, we set $B_i = B_{i-1} \cup \bar{B}_i$ and $C_i = C_{i-1} \cup \bar{C}_i$.
- Finally, we set $B = B_{k-1}$ and $C = C_{k-1}$.

Clearly, we have that $B_i \subseteq B_{i+1}$ and $C_i \subseteq C_{i+1}$ for all $i < k-1$.

Claim 1.9. *The set B is bounded.*

Proof. Towards contradiction, assume that there is an unbounded path $P \subseteq B$. Since c is a finite coloring of $\mathcal{G}_F$, there is $x \in P$ s.t. $F(x) \neq x$, $F(x) \in P$ and $c(F(x)) \leq c(x)$. Since, $c(F(x)) \neq c(x)$ it follows that $c(F(x)) < c(x)$. Thus since we know that $F(x) \in B$ we must have that $F(x) \in B_{c(F(x))} \subseteq B_{c(x)-1}$. Therefore, during the $(c(x))$-th step of the recursion above $x \notin \bar{B}_{c(x)}$ (as the successor $F(x) \in B_{c(x)-1}$). We conclude that $x \in \bar{C}_{c(x)} \subseteq C$ which is of course a contradiction. $\square$

Claim 1.10. *The set B is 2-colorable.*

Proof. Since B is bounded then for every $x \in B$ there is a $k \in \omega$ s.t. $F^{k+1}(x) = F^k(x)$ or $F^k(x) \notin B$. If there is a $k \in \omega$ s.t. $F^{k+1}(x) = F^k(x)$ then set $e(x)$ equal to the minimum such k. Otherwise, set $e(x)$ to be $k-1$ where k is now the minimum integer s.t. $F^k(x) \notin B$. Note that if $F(x) \in B$ with $F(x) \neq x$ then $e(F(x)) = e(x) - 1$. Now define $d : B \to 2$ by setting $d(x) = e(x) mod 2$. Clearly, this produces a 2-coloring for B. □

Claim 1.11. *The set C is 1-colorable.*

Proof. Towards contradiction we assume that there is an $x \in C$ s.t. $F(x) \neq x$ and $F(x) \in C$. Suppose first that $c(x) > c(F(x))$. Then since $F(x) \in C$ and $C \cap B = \emptyset$, at stage $i = c(x)$ we have that $F(x) \notin B_{i-1}$. By construction this implies that $x \in B_i \subseteq B$ and thus $x \notin C$ which is a contradiction. Suppose next that $c(x) < c(F(x))$. If $c(x) = 0$ then $x \in B_0 \subseteq B$ and thus $x \notin C$ which is a contradiction. If $c(x) > 0$ then at stage $i = c(x)$ we have $F(x) \notin B_{i-1} \cup C_{i-1}$. By construction this implies that $x \in B_i \subseteq B$ and thus $x \notin C$ which is a contradiction. Therefore, $c(x) = c(F(x))$ which contradicts the definition of the coloring function. □

From all the above and the fact that $X = B \uplus C$ it is now obvious that $\mathcal{X}_B(\mathcal{G}_F) \leq 3$. □

Remark 1.12. Note that all $x \in X$ s.t. $F(x) = x$ and all $x \in B$ s.t. $F(x) \notin B$ are assigned the same color in the coloring we described above. All such x belong to B and are given the color $d(x) = 0$ in the proof of Claim 1.10. Moreover, the proof of Claim 1.10 does not use anything more than the fact that B is Borel and bounded. Therefore, we proved that each Borel and bounded set B for which each $x \in B$ has at most one successor is Borel 2-colorable, and all its elements with no successor in B are assigned the same color.

Finally, if we set $B^i = \{x \in B \mid d(x) = i\}$ then we have the following properties:

- $X = B^0 \uplus B^1 \uplus C$.
- The set $B^0 \uplus B^1 = B$ is bounded.
- Each B^i, C is Borel 1-colorable.
- Each $x \in B^1$ is followed by an element in B^0.

Based on the above theorem we can prove the following corollary that also appears in [1]:

Corollary 1.13. *Let X be a Borel space and $F_0, F_1, ..., F_{n-1}$ be Borel functions on X. Let $\mathcal{G} = \mathcal{G}_{F_0,F_1,...,F_{n-1}}$. If $\mathcal{X}_B(\mathcal{G}) < \omega$ then $\mathcal{X}_B(\mathcal{G}) \leq 3^n$.*

Proof. Since $\mathcal{X}_B(\mathcal{G}) < \omega$ then clearly $\mathcal{X}_B(\mathcal{G}_{F_i}) < \omega$ for each $i \in n$. Thus, from 1.8 we can define $c_i : X \to 3$ to be a Borel coloring of $\mathcal{G}_{F_i}$ for all $i \in n$. Then $c(x) = (c_0(x), c_1(x), ..., c_{n-1}(x))$ is easily a Borel coloring of $\mathcal{G}$. $\square$

Also, since the set C in theorem 1.8 is clearly Borel, we can give the following characterization which can be found in [2]:

Theorem 1.14. *(Finite Colorable Characterization for a Single Function) Let X be a Borel space, $f : X \to X$ be a Borel function with $f(x) \neq x$ for all $x \in X$, and let $\mathcal{G} = \mathcal{G}_f$ be the corresponding Borel graph generated by f. Then the following statements are equivalent:*

(*i*) $\chi_{\mathcal{B}}(\mathcal{G}) \leq 3$

(*ii*) $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$

(*iii*) *There is a Borel subset $A \subseteq X$ s.t. for each $x \in X$ there exists an $i \in \omega$ s.t. $f^i(x) \in A$ and $f^{i+1}(x) \notin A$.*

Proof. $(i) \to (ii)$: This is trivial.

$(ii) \to (iii)$: We take $A = C$ where C is the 1-colorable set described in the proof of Theorem 1.8. Now if $x \notin C$ then $x \in B$ (where again B is the 2-colorable set described in the proof of Theorem 1.8) and thus since $(\forall y) f(y) \neq y$, there should be an $i \in \omega$ s.t. $f^i(x) \notin B \Rightarrow f^i(x) \in C \Rightarrow f^i(x) \in A$. Moreover, since C is 1-colorable and $f^{i+1}(x) \neq f^i(x)$, we have that $f^{i+1}(x) \notin C = A$. Similarly if $x \in C$ then $f(x) \notin C$ and thus as before we can find a $i \in \omega$ s.t. $f^{i+1}(x) \in A$ while $f^{i+2}(x) \notin A$.

$(iii) \to (i)$: Let $A_0 \subseteq A$ be the set of all $x \in A$ s.t. $f(x) \notin A$. Clearly, A_0 is 1-colorable. Also, $X \setminus A_0$ is bounded. This is because if $x \in X \setminus A_0$ then by (iii) there is an $i \in \omega$ s.t. $f^i(x) \in A$ but $f^{i+1}(x) \notin A$. By the definition of A_0, this means that $f^i(x) \in A_0$ and thus $X \setminus A_0$ cannot contain unbounded paths. Thus $X \setminus A_0$ is bounded and therefore by the proof of Claim 1.10 it is 2-colorable. Since A_0 is 1-colorable it follows that $X = A_0 \uplus (X \setminus A_0)$ is 3-colorable. $\square$

CHAPTER 2

Summary of Results

In this section we will provide a list with all the new results proved in this dissertation. First of all, we will give a positive answer to Question 1.7 in the general case of n functions that commute with each other. More specifically:

Theorem 2.1. *Let X be a Borel space, $n \in \omega$, $F_1, F_2, ..., F_k : X \to X$ be Borel functions which commute with each other and $\mathcal{G} = \mathcal{G}_{F_1,F_2,...,F_k}$. Then either $\chi_{\mathcal{B}}(\mathcal{G}) \leq 2k+1$ or $\chi_{\mathcal{B}}(\mathcal{G}) = \omega$.*

Next, we will prove a key lemma that in its essence generalizes Theorem 1.8, and which will be essential for the proof of our major results in the next chapters. The idea of the lemma is that we can find non-trivial extensions of graphs generated by a single (Borel) function which have the property that they are Borel 3-colorable, provided that they are finitely Borel colorable. More formally:

Lemma 2.2. *(The simple 1-colorable Subset Lemma). Let X be a Borel space and $\mathcal{G}$ be a finitely (Borel) colorable Borel graph over X with the following two properties:*

- *Every $x \in X$ has at most two descendants in X.*
- *The set of splitting nodes $Y = \{x \in X | x \text{ has exactly } 2 \text{ descendants } \}$ is 1-colorable.*

Then $\mathcal{G}$ is Borel 3-colorable.

Then, we will use our new ideas we already used for the proof of Theorem 1.8 in order to produce a quadratic bound for the chromatic number of graphs generated by n Borel functions (for an arbitrary $n \in \omega$), which is already better than the exponential one given on Corollary 1.13. More precisely we will prove the following:

Lemma 2.3. ***(The Least Available Subset):*** *Let X be a Borel space, $k \in \omega$ and $F_i : X \to X$ be Borel functions for all $i \in k$. Let also $\mathcal{G} = \mathcal{G}_{F_0,F_1,\dots,F_{k-1}}$. Suppose finally that $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$. Then $\chi_{\mathcal{B}}(\mathcal{G}) \leq 1 + 2 + \dots + (k+1) = \frac{k^2}{2} + \frac{3k}{2} + 1 = \mathcal{O}(k^2)$.*

By combining the two Lemmas above, we will be able to give a positive answer to the main Question 1.7 for the case $n = 2$ when the functions do not necessarily commute with each other:

Theorem 2.4. ***(Non-commutative functions)*** *Let X be a Borel space and $F, G : X \to X$ be Borel functions. Let also $\mathcal{G} = \mathcal{G}_{F,G}$. Suppose finally that $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$. Then $\chi_{\mathcal{B}}(\mathcal{G}) \leq 5$.*

Then, by using a generalized version of the 1-colorable subset Lemma and by the Least Available Subset lemma, we will get a better than quadratic (but still not optimal) bound for the case $n = 3$ with arbitrary Borel functions:

Theorem 2.5. *Let X be a Borel space and $F, G, H : X \to X$ be Borel functions. Let also $\mathcal{G} = \mathcal{G}_{F,G,H}$. Suppose finally that $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$. Then $\chi_{\mathcal{B}}(\mathcal{G}) \leq 8$.*

Finally, we apply the above results and ideas to conclude bounds for Baire and μ-measurable chromatic numbers which are defined as follows:

Definition 2.6. Let X be a Polish space and $\mathcal{G}$ be a graph on X. Let also, $c : X \to \omega$ be a coloring of $\mathcal{G}$. Then:

- The Baire chromatic number of $\mathcal{G}$ ($\chi_{\mathcal{BP}}(\mathcal{G})$) is given by: $\chi_{\mathcal{BP}}(\mathcal{G}) = \min \{|c(X)|$ where c is a Baire measurable coloring of $\mathcal{G}\}$.
- The μ-measurable chromatic number of $\mathcal{G}$ ($\chi_{\mu}(\mathcal{G})$) is given by: $\chi_{\mu}(\mathcal{G}) = \min \{|c(X)|$ where c is a μ-measurable coloring of $\mathcal{G}\}$. (Here, μ is a probability measure on X).

For starters we give a different proof of the following theorem proved first in [2]:

Theorem 2.7. *Let X be a Polish space, μ be a probability measure on X, and $f : X \to X$ be a Borel function. Then $\chi_{\mathcal{BP}}(\mathcal{G}_f) \leq 3$ and $\chi_\mu(\mathcal{G}_f) \leq 3$.*

Then, we give two more results that use the bounds and ideas for the Borel chromatic number of graphs generated by n functions, that are described above:

Theorem 2.8. *Let X be a Polish space, μ be a probability measure on X, and $f_0, f_1 : X \to X$ be two Borel functions on X. Then $\chi_{\mathcal{BP}}(\mathcal{G}_{f_0,f_1}) \leq 5$ and $\chi_\mu(\mathcal{G}_{f_0,f_1}) \leq 5$.*

Theorem 2.9. *Let X be a Polish space, μ be a probability measure on X, $n \in \omega$ and $f_0, f_1, ..., f_{n-1} : X \to X$ be Borel functions on X. Then, we have that: $\chi_{\mathcal{BP}}(\mathcal{G}_{f_0,f_1,...,f_{n-1}}) < \omega$ and $\chi_\mu(\mathcal{G}_{f_0,f_1,...,f_{n-1}}) < \omega$.*

CHAPTER 3

Commuting Functions

The functions $F_0, F_1, ..., F_{n-1}$ on X are said to *commute* with each other if and only if $F_i(F_j(x)) = F_j(F_i(x))$ for all $x \in X$ and $i, j \in n$.

In the single function case (Theorem 1.8) the issue of commutativity does not come up, as a single function trivially commutes with itself. For $n > 1$, Question 1.7 splits naturally into two subquestions: One where the functions commute, and one where they do not. In this section we provide a general (positive) answer for all $n \in \omega$ in the case of commuting functions:

Theorem 3.1. *Let X be a Borel space, $k \in \omega$, $F_1, F_2, ..., F_k : X \to X$ be Borel functions which commute with each other and $\mathcal{G} = \mathcal{G}_{F_1,F_2,...,F_k}$. Then either $\chi_{\mathcal{B}}(\mathcal{G}) \leq 2k+1$ or $\chi_{\mathcal{B}}(\mathcal{G}) = \omega$.*

Proof. For the sake of simplicity we are going to prove the theorem for $k = 2$. The proof for $k > 2$ is very similar. Let $F, G : X \to X$ be Borel functions which commute with each other. By Fact 1.3 we have that $\mathcal{X}_B(\mathcal{G}) \leq \omega$. Suppose that $\mathcal{X}_B(\mathcal{G}) = n \in \omega$. Fix $c : X \to n$ to be a Borel coloring function for $\mathcal{G} = \mathcal{G}_{F,G}$.

We will construct a new coloring e, which uses only $2*2+1 = 5$ colors. Define $d : X \to n$ by $d(x) = c(F^n G^n(x))$. Due to commutativity of F and G it is easy to observe that d is a (Borel) n-coloring function. For example, if $x = F(y)$ then $d(x) = c(F^n G^n(x)) = c(F^n G^n(F(y))) = c(F(F^n G^n(y))) \neq c(F^n G^n(y)) = d(y)$. Similarly for $x = G(y)$. Now, it is enough to construct a Borel coloring function $e : X \to \{A, B, C, D, E\}$. Towards that we will first recursively define functions $e_i : X \to \{A, B, C, D, E\} \cup n$ for each $i \in n$ as follows:

[$i = 0$:] If $x \in d^{-1}(0)$ then set $e_0(x) = A$. Otherwise, set $e_0(x) = d(x)$.

[$0 < i < n$:] If $x \notin d^{-1}(i)$ then set $e_i(x) = e_{i-1}(x)$. Otherwise, we will prove that the set

$$\{A, B, C, D, E\} \setminus (\{e_{i-1}(F(x)), e_{i-1}(G(x))\} \cup \{e_{i-1}(y) | y \in F^{-1}(x) \cup G^{-1}(x)\})$$

is non-empty. Let $e_i(x)$ be the lexicographically least element of the above set.

Claim 3.2. *Let $i \in n$. Then for any $j \leq i$ and any $x \in d^{-1}(j)$, $e_i(x) \in \{A, B, C, D, E\}$.*

Proof. Immediate from the definition of e_i. □

To facilitate the computations we also set $e_{-1} = d$. Also, for each $x \in X$ and $i \in n$ we define $P(x, i) : [-i, i]^2 \to n$ given by $P(x, i)(k, l) = c(F^{n+k}G^{n+l}(x))$.

Claim 3.3. *Let $0 < i \in n$. Then $P(F(x), i-1)$ can be determined uniformly from $P(x, i)$. In particular, if $x_1, x_2 \in X$ and $P(x_1, i) = P(x_2, i)$ then $P((F(x_1), i-1) = P((F(x_2), i-1)$. Similarly for G.*

Proof. Let $x_1, x_2 \in X$ and $P(x_1, i) = P(x_2, i)$. Let $k, l \in [-(i-1), i-1]$. Then:

$$P(F(x_1), i-1)(k, l) = c(F^{n+k+1}G^{n+l}(x_1)) \tag{3.1}$$

$$= P(x_1, i)(k+1, l) \tag{3.2}$$

$$P(F(x_2), i-1)(k, l) = c(F^{n+k+1}G^{n+l}(x_2)) \tag{3.3}$$

$$= P(x_2, i)(k+1, l) \tag{3.4}$$

By (3.2),(3.4) and the fact that $P(x_1, i) = P(x_2, i)$, we conclude that

$$P((F(x_1), i-1) = P((F(x_2), i-1)$$

. □

Claim 3.4. *Let $0 < i \in n$, and let $y \in F^{-1}(x)$. Then $P(y, i-1)$ can be determined uniformly from $P(x, i)$. In particular if $y_1, y_2 \in X$ and $P(F(y_1), i) = P(F(y_2, i)$ then $P(y_1, i-1) = P(y_2, i-1)$.*

Proof. Let $y_1, y_2 \in X$ and $P(F(y_1), i) = P(F(y_2, i)$. Let $k, l \in [-(i-1), i-1]$. Then:

$$P(y_1, i-1)(k, l) = c(F^{n+k}G^{n+l}(y_1)) \tag{3.5}$$

$$= P(F(y_1), i)(k-1, l) \tag{3.6}$$

$$P(y_2, i-1)(k, l) = c(F^{n+k}G^{n+l}(y_2)) \tag{3.7}$$

$$= P(F(y_2), i)(k-1, l) \tag{3.8}$$

By (3.6),(3.8) and the fact that $P(F(y_1), i) = P(F(y_2, i)$ we conclude that

$$P(y_1, i-1) = P(y_2, i-1)$$

. □

Claim 3.5. *Let $i \in n$ and $x \in X$. Then the value of $e_i(x)$ depends only on $P(x, i)$. Precisely this means that if $x_1, x_2 \in X$ are such that: $P(x_1, i) = P(x_2, i)$ then $e_i(x_1) = e_i(x_2)$.*

Proof. We will prove this using induction on $i \in n$. The base case is trivial since $e_0(x)$ depends only on $d(x) = c(F^nG^n(x)) = P(x, 0)(0, 0)$. Now, let $x_1, x_2 \in X$ and $0 < i < n$ with $P(x_1, i) = P(x_2, i)$. By the recursive construction above we know that both $e_i(x_j)$ ($j = 1, 2$) depend only on $d(x_j)$, $e_{i-1}(x_j)$, $e_{i-1}(F(x_j))$, $e_{i-1}(G(x_j))$ and all $e_{i-1}(y)$ for y s.t. $y \in F^{-1}(x_j) \cup G^{-1}(x_j)$.

Since $P(x_1, i) = P(x_2, i)$ we conclude that $P(x_1, 0) = P(x_2, 0)$ and $P(x_1, i-1) = P(x_2, i-1)$. Thus, $d(x_1) = d(x_2)$ and $e_{i-1}(x_1) = e_{i-1}(x_2)$. Also, since $P(x_1, i) = P(x_2, i)$ then by the claim above we get $P((F(x_1), i-1) = P((F(x_2), i-1)$. Therefore, by induction hypothesis we have $e_{i-1}(F(x_1)) = e_{i-1}(F(x_2))$. Similarly, $e_{i-1}(G(x_1)) = e_{i-1}(G(x_2))$. Also, let $y_1 \in F^{-1}(x_1)$ and $y_2 \in F^{-1}(x_2)$ then since $P(x_1, i) = P(x_2, i)$ and thus $P(F(y_1), i) = P(F(y_2, i)$, we can conclude from the claim above that $P(y_1, i-1) = P(y_2, i-1)$. Therefore, by induction hypothesis we have $e_{i-1}(y_1) = e_{i-1}(y_2)$.

Now using the last line of the first paragraph it follows that $e_i(x_1) = e_i(x_2)$. □

Claim 3.6. *Let $i \in n$ and $x \in d^{-1}(i)$. Then the following set is non-empty:*

$$Y = \{A, B, C, D, E\} \setminus (\{e_{i-1}(F(x)), e_{i-1}(G(x))\} \cup \{e_{i-1}(y) | y \in F^{-1}(x) \cup G^{-1}(x)\})$$

Proof. We will prove this using induction on $i \in n$. The base case is trivial. Now, we fix $0 < i < n$ and we further assume that all the e_{i-1} values are well-defined. If $y_1, y_2 \in X$ with $F(y_1) = F(y_2) = x$, then since we trivially have that $P(F(y_1), i) = P(F(y_2), i)$, we can conclude by a claim above that $P(y_1, i-1) = P(y_2, i-1)$. Therefore, again by a claim above we can deduce that $e_{i-1}(y_1) = e_{i-1}(y_2)$. Therefore, we conclude that $e_{i-1}(y)$ is the same for all $y \in F^{-1}(x)$. Similarly, $e_{i-1}(y)$ is the same for all $y \in G^{-1}(x)$. We conclude that the cardinality of $\{e_{i-1}(F(x)), e_{i-1}(G(x))\} \cup \{e_{i-1}(y) | y \in F^{-1}(x) \cup G^{-1}(x)\}$ is at most 4 and thus $Y \neq \emptyset$. $\square$

Claim 3.7. *Let $i \in n$. Then e_i is a Borel coloring.*

Proof. This is immediate by the above claim and the definition of e_i. $\square$

Claim 3.8. *e_{n-1} is a Borel coloring function from X to $\{A, B, C, D, E\}$*

Proof. This is immediate by the claims above. $\square$

We conclude the proof by setting $e = e_{n-1}$. This completes the proof for the case $k = 2$. The proof of the general case ($k > 2$) is exactly the same with the only difference that we define $d(x) = c(F_1^n F_2^n ... F_k^n(x))$ and that we have $2 * k$ restrictions for the values of e_i rather than just $2 * 2 = 4$. $\square$

From now on we will drop the assumption of commutativity and we will focus on graphs generated by not necessarily commuting functions. More specifically, in the following chapters we will give a proof for the case $n = 2$.

CHAPTER 4

The 1-colorable subset Lemma

The first step towards the proof of the case $n = 2$ is the following lemma which basically extends the graphs of theorem 1.8 while at the same time maintains the 3-coloring. But first, a few important definitions.

Definition 4.1. Let X be a set, $\mathcal{G} = (X, E)$ a graph on X and $A \subseteq X$. We say that A has *property* P_k ($k \in \omega$) if every $x \in A$ has at most k descendants in A. Also, if $k = 1$ we simply say that A has *property* P.

Fact 4.2. Let X be standard Borel space, $\mathcal{G} = (X, E)$ be a Borel graph on X and $A \subseteq X$ be a Borel set with propery P. If $\mathcal{G} \restriction A$ is finitely colorable then $\mathcal{X}_B(\mathcal{G} \restriction A) \leq 3$. In fact, there is a Borel coloring $c : A \to \{1, 2, 3\}$ with the property that if $x \in A$ has no descendants in A, then $c(x) = 1$. We call such x a *top element* of A.

Proof. We define $F : A \to A$ to be as follows: We set $F(x) = x$ if there is no $y \in A$ s.t. $(x, y) \in E \restriction A$. Otherwise, we set $F(x)$ to be the unique $y \in A$ s.t. $(x, y) \in E \restriction A$. F is clearly Borel and thus we can apply Theorem 1.8 on the graph $\mathcal{H} = (A, E \restriction A)$. Also, by Remark 1.12 all $x \in A$ with no descendants in A are given the same color. W.l.o.g. this color can be 1. $\square$

Lemma 4.3. *Let X be standard Borel and $\mathcal{G} = (X, E)$ be a Borel graph with a finite Borel coloring that has property P. Then we can partition $\mathcal{G}$ into two subgraphs $\mathcal{G}_1 = (X_1, E \restriction X_1)$ and $\mathcal{G}_2 = (X_2, E \restriction X_2)$ s.t. $\mathcal{G}_1$ is Borel 1-colorable and every connected component of $\mathcal{G}_2$ has at most 2 elements. On top of that:*

- *If $x \in X$ has no successors in $\mathcal{G}$ then $x \in X_2$.*

- *If K is a bounded connected component of $\mathcal{G}$, then $\mathcal{G}_2 \restriction K$ is 1-colorable.*

We call $\mathcal{G}_1$ the 1st part of $\mathcal{G}$ and $\mathcal{G}_2$ the 2nd part of $\mathcal{G}$.

Proof. For each $x \in X$ let K_x be the maximum connected component of $\mathcal{G}$ containing x. Now define $E = \{x \in X \mid K_x \text{ is bounded}\}$ and $U = \{x \in X \mid K_x \text{ is unbounded}\}$. Then by Remark 1.12 we can have $E = E^0 \uplus E^1$ and $U = B^0 \uplus B^1 \uplus C$ where E^0, E^1 are 1-colorable Borel subsets of X, E^0 contains all elements in E with no successor in X, and B^0, B^1, C are defined exactly as in Remark 1.12.

Let $X_1 = E^1 \uplus B^0$ and $X_2 = E^0 \uplus B^1 \uplus C$. Clearly $\mathcal{G}_1 = (X_1, E \restriction X_1)$ is 1-colorable. Also, by the definition of E and U, no element in E^0 is ever connected to any element in $B^1 \uplus C$, and at the same time no element in B^1 is followed by an element in C. Thus, using the additional fact that E^0, E^1, B^0, B^1 and C are all 1-colorable we can conclude that every connected component of $\mathcal{G}_2 = (X_2, E \restriction X_2)$ has at most 2 elements. Finally, again by Remark 1.12 and the fact that $E^0 \subseteq X_2$ we can deduce that $\mathcal{G}_2$ contains all $x \in X$ with no successors in $\mathcal{G}$. $\square$

Definition 4.4. Let X be an arbitrary set, $n \in \omega$, $c : X \to \{1, 2, ..., n\}$, and $r : X \to \mathcal{P}(\{1, 2, ..., n\})$. We say that c *is restricted by* r if $c(x) \notin r(x)$ for all $x \in X$. We also say that x *is restricted by* $i \in \{1, 2, ..., n\}$ if $i \in r(x)$.

Definition 4.5. Let X be any set, and $\mathcal{G} = (X, E)$ be a graph on X. Then the function $d_{\mathcal{G}} : X \to \mathcal{P}(X)$ defined by $d_{\mathcal{G}}(x) = \{y \in X \mid (x, y) \in E\}$, is called the *successor function* of $\mathcal{G}$. Also, for $A \subseteq X$ we define $d^A_{\mathcal{G}} : A \to \mathcal{P}(A)$ by $d^A_{\mathcal{G}}(x) = \{y \in A \mid (x, y) \in E\}$, to be the *successor function* of $\mathcal{G} \restriction A$.

Remark 4.6. If $\mathcal{G}$ is Borel then $d_{\mathcal{G}}$ is also Borel. Moreover, if $\mathcal{G}$ and $A \subseteq X$ are both Borel, then $d^A_{\mathcal{G}}$ is Borel too.

Lemma 4.7. *Let X be standard Borel, $\mathcal{G} = (X, E)$ also Borel, $n \in \omega$, and $r : X \to \mathcal{P}(\{1, ..., n\})$ be Borel as well. Also, let $A \subseteq X$ be Borel and bounded, with the additional* *property that* $(\forall x \in A)(|d^A_{\mathcal{G}}(x)| + |r(x)| \leq n)$. *Then there is a Borel coloring* $c : A \to \{1, ..., n+1\}$ *that is restricted by* r.

Proof. We start with a recursive definition of the function $rk : X \to \omega$ that assigns a rank to each element of A and a pseudo-rank to every element in $X \backslash A$:

Base Case: For all $x \notin A$ let $rk(x) = -1$

Recursive Step: Let $x \in A$ and assume that all $rk(y)$ for $y \in d^A_{\mathcal{G}}(x)$ have already been defined. Then we let $rk(x) = max\{rk(y) \mid y \in d^A_{\mathcal{G}}(x)\} + 1$.

Since, every node $x \in X$ of the graph $\mathcal{G} \restriction A$ has finite out-degree $\leq n$, then by Konig's lemma and the fact that A is bounded, we deduce that the function rk is well-defined and that $rk(x) \in \omega$ for all $x \in X$. Now, using this ranking function we can easily construct a Borel coloring function $c : A \to \{1, ..., n+1\}$ as follows:

Base Case: For all $x \in A$ s.t. $rk(x) = 0$ set $c(x)$ to be the least $m \in \{1, ..., n+1\} \setminus r(x)$.

Recursive Step: Let $x \in A$ s.t. $rk(x) = q \geq 1$. Then for all $y \in d^A_{\mathcal{G}}(x)$ we have $rk(y) \leq q - 1$ and hence $c(y)$ has already been defined. Set $c(x)$ to be the least $m \in \{1, ..., n+1\} \setminus (\{c(y) \mid y \in d^A_{\mathcal{G}}(x)\} \cup r(x))$. Such m exists since $|d^A_{\mathcal{G}}(x)| + |r(x)| \leq n$.

The function c is a Borel $(n+1)$-coloring for A. □

Remark 4.8. If $A \subseteq X$ has no restrictions, then by the above construction we conclude that every $x \in A$ with $c(x) = k$ is followed by a $y_m \in A$ with $c(y_m) = m$ for all $1 \leq m < k$.

Lemma 4.9. *(The generalized 1-colorable Subset Lemma). Let* X *be a Borel space,* $r : X \to \mathcal{P}(\{1, 2, 3\})$ *also Borel, and* $\mathcal{G} = (X, E)$ *be a finitely (Borel) colorable Borel graph over* X *with the following properties:*

- *For every* $x \in X$ *we have that* $|d_{\mathcal{G}}(x)| + |r(x)| \leq 2$.
- *The (Borel) set of "splitting nodes"* $Y = \{x \in X \mid |d_{\mathcal{G}}(x)| + |r(x)| = 2\}$ *is 1-colorable.*

Then there is a Borel coloring $c : X \to \{1, 2, 3\}$ *on* $\mathcal{G}$ *which is restricted by* r.

Proof. In what follows, we will define c on different subsets of X that partition X. And every time we define c on a new subset $A \subseteq X$ we will make sure that $c(x) \neq r(x)$, while c maintains its coloring property, namely $c(x) \neq c(y)$ when $(x, y) \in E$ and $c(x)$, $c(y)$ are both defined. We will also call this procedure "coloring" the set $A \subseteq X$. We first define $Z = \{x \in X | x \text{ is followed by } y \in Y\}$. Note that $Z \cap Y = \emptyset$ by the assumption that Y is 1-colorable. Now we color X in steps:

Step 1: First we observe that since $X \setminus (Y \uplus Z)$ has no splitting nodes, it has property P. We can therefore color $X \setminus (Y \uplus Z)$ using 3 colors, namely 1,2,3. We will do that as follows: For the connected components of $X \setminus (Y \uplus Z)$ that have no elements with restrictions, we use Fact 4.2 to color our graph, while for all the other components, we use Lemma 4.7. That way, since the only elements in $X \setminus (Y \uplus Z)$ with restrictions are top elements which are not followed by an element in Z, we can arrange by 4.2, that no $z \in Z$ is preceded by a 2 or 3.

Step 2: Now we color all elements of Y which are not followed or restricted by 1, with the color 1. This does not violate the coloring property, as the only predecessors of a $y \in Y$ belong to Z which is yet to be colored.

Step 3: Finally let $W \subseteq X$ be the set of all nodes we have not assigned a color to yet. We can color W using the colors 2 and 3 without violating the property of the coloring function c. This is possible because:

(i) $W \subseteq Y \uplus Z$.

(ii) By the previous step, every $w \in W \cap Y$ is followed or restricted by at least one 1.

(iii) By its definition, $Z \subseteq W$ and every $z \in Z$ is followed by one element in Y, and it's not restricted by any $i \in \{1, 2, 3\}$.

(iv) We have already seen above that if $w \in W \cap Z \subseteq Z$ then w is not preceded by a 2 or 3. We also have that if $w \in W \cap Y \subseteq Y$ then w is preceded only by elements in Z which are yet to be colored. We conclude that W is not preceded by a 2 or 3.

(v) Since $W \subseteq Y \uplus Z$, then in every unbounded component K of W, every $w \in K \cap Y$ is followed exactly by a 1 and an element in $K \cap Z$, and every element in $w \in K \cap Z$ is followed exactly by an element in $K \cap Y$. Thus we can color this component by assigning the color 2 to every element in $K \cap Y$ and the color 3 to every element in $K \cap Z$.

(vi) Let K be a bounded component of W. By (ii) and (iii), the only element of K that can have a restriction other than a 1 is the top element. Hence, if we define $r'(x) = r(x) \setminus \{1\}$, we get that $|d_{\mathcal{G}}^{W}(x)| + |r'(x)| \leq 1$ for all x in bounded components of W. Now by Lemma 4.7 it is clear that we can color the bounded components of W using the colors 2 and 3 and without violating the coloring property.

This gives us a Borel 3-coloring $c : X \to \{1, 2, 3\}$ which is restricted by r. $\square$

Now we give as a corollary a weaker version of this lemma:

Corollary 4.10. *(The simple 1-colorable Subset Lemma). Let X be a Borel space and $\mathcal{G}$ be a finitely (Borel) colorable Borel graph over X with the following two properties:*

- *Every $x \in X$ has at most two descendants in X.*
- *The set of splitting nodes $Y = \{x \in X | x$ has exactly 2 descendants $\}$ is 1-colorable.*

Then $\mathcal{G}$ is Borel 3-colorable.

CHAPTER 5

The Least Available Subset Lemma

In this section we prove a Lemma that gives us a quadratic bound (even though not the optimal one) for the Borel chromatic number of a Borel graph generated by n Borel function. Note, that this is already a better bound than the obvious exponential bound (i.e. 3^n) we can derive from Corollary 1.13.

Lemma 5.1. ***(The Least Available Subset):*** *Let X be a Borel space, $k \in \omega$ and $F_i : X \to X$ be Borel functions for all $i \in k$. Let also $\mathcal{G} = \mathcal{G}_{F_0,F_1,\ldots,F_{k-1}}$. Suppose finally that $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$. Then, there is a Borel coloring c of $\mathcal{G}$ into pairs $\langle i, j\rangle$ s.t. $i, j \geq 0$, $i + j \leq k$ and if $c(x) = \langle i, j\rangle$ then x is followed by a $y_m \in X$ with $c(y_m) = \langle i, m\rangle$, and by a $z_l \in X$ with $c(z_l) = \langle l, j_l\rangle$ ($j_l \leq k - l$) for all $m < j$ and $l < i$. In particular, $\chi_{\mathcal{B}}(\mathcal{G}) \leq 1 + 2 + ... + (k + 1) = \frac{k^2}{2} + \frac{3k}{2} + 1 = \mathcal{O}(k^2)$.*

Proof. For clarity, we prove the lemma when $k = 2$. (The argument for the higher dimensions is a direct generalization to what follows and therefore can be easily deduced by the reader). Since $k = 2$ let $F_0 = F$ and $F_1 = G$. Also, since $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$ we can fix some $n \in \omega$ s.t. $\chi_{\mathcal{B}}(\mathcal{G}) \leq n$. Let $X = A_0 \uplus A_1 \uplus ... \uplus A_{n-1}$ be a n-Borel coloring of $\mathcal{G}$ over X. Using induction on $i \in n$ we will find a partition of X into three sets: B_0, B_1 and B_2. The inductive construction of the sets B_0, B_1 and B_2 goes as follows:

[$i = 0$:] In that case, we put all $x \in A_0$ into B_0. In other words $A_0 \subset B_0$.

[$i \to i+1$:] For that step we assume that $i+1 \in n$ and that $A_0 \uplus A_1 \uplus ... \uplus A_i \subseteq B_0 \uplus B_1 \uplus B_2$. Then, for every $x \in A_{i+1}$ we define $j(x) \in 3$ to be the minimal index $j \in 3$ s.t. both $F(x) \notin B_j$ and $G(x) \notin B_j$. Then we let $x \in B_{j(x)}$. In other words, we send x to the least "available" subset B_j. We note here that $j(x)$ is obviously well-defined, since $F(x)$ and $G(x)$ can belong to at most two different B_j's.

Claim 5.2. *The sets B_0, B_1 and B_2 are all bounded.*

Proof. Assume otherwise. Then for some $j \in 3$ we will have that B_j contains an infinite path $I = \{x_i \mid i \in \omega\} \subseteq B_j$. Since I is infinite and the coloring given by $A_0, \dots, A_{n-1}$ is finite, we can find $i, k, l \in \omega$ s.t. $x_i \in A_k$, $x_{i+1} \in A_l$ and $k > l$. So, in our inductive construction above, the element x_{i+1} gets priority over x_i. And thus, by the time we reach the k-th step of our inductive construction, the element x_{i+1} should already be in B_j. Then by construction, at stage k we put $x_i \in B_{j'}$ for some $j' \in 3$ s.t. $j' \neq j$. This is of course a contradiction since $x_i \in I \subseteq B_j$. We conclude that none of the B_j's contains an infinite path and therefore, they are all bounded. $\square$

Claim 5.3. *The set B_0 is Borel 3-colorable. Moreover, we can construct a Borel coloring $b_0 : B_0 \to \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle\}$ with the additional two properties:*

1. *If $x \in B_0$ is s.t. $b_0(x) = \langle 0,2\rangle$ then there are two descendants $y, z \in B_0$ of x s.t. $b_0(y) = \langle 0,0\rangle$ and $b_0(y) = \langle 0,1\rangle$.*
2. *If $x \in B_0$ is s.t. $b_0(x) = \langle 0,1\rangle$ then x has a descendant $y \in B_0$ s.t. $b_0(y) = \langle 0,0\rangle$.*

Proof. This is immediate by Lemma 4.7, Remark 4.8, and the fact that for all $x \in B_0$ we have $|d_{\mathcal{G}}^{B_0}(x)| \leq 2$ and no other restrictions. $\square$

Claim 5.4. *The set B_1 is (Borel) 2-colorable. Moreover, we can construct a Borel coloring $b_1 : B_1 \to \{\langle 1,0\rangle, \langle 1,1\rangle\}$ with the additional property that if $x \in B_1$ is s.t. $b_1(x) = \langle 1,1\rangle$ then x has a descendant $y \in B_1$ s.t. $b_1(y) = \langle 1,0\rangle$.*

Proof. First, we claim that by the definition of B_0 and B_1, every element $x \in B_1$ has at least one direct descendant in B_0. To see this, let's assume that $x \in A_k$ for some $k \in n$. Then in the k-th step of our inductive construction we will look at the direct descendants of x. But if none of them are in B_0 then we would have had that $x \in B_0$ which is a contradiction.

Since each $x \in B_1$ has at most two descendants overall, and one of them belongs to B_0, we have $|d_{\mathcal{G}}^{B_1}(x)| \leq 1$. Thus we can use Lemma 4.7 and Remark 4.8 conclude the claim. $\square$

Claim 5.5. *The set B_2 is (Borel) 1-colorable.*

Proof. Assume otherwise. Then there should be two elements $x, y \in B_2$ s.t. y is a successor of x. But $x \in B_2$ iff x has successors in both B_0 and B_1. Since x has two successors, it cannot then have a successor $y \in B_2$. $\square$

By the above claim we can construct a Borel coloring $b_2 : B_2 \to \{\langle 2, 0\rangle\}$.

Now, by all the claims above and the fact that $X = B_0 \uplus B_1 \uplus B_2$, we can construct a Borel function $c : X \to \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 1,0\rangle, \langle 1,1\rangle, \langle 2,0\rangle\}$ by $c(x) = b_i(x)$ for $x \in B_i$. Clearly, c is a coloring function. Moreover, by the properties of all b_i, if $c(x) = \langle i, j\rangle$ then x is followed by a $y_m \in X$ with $c(y_m) = \langle i, m\rangle$, and by a $z_l \in X$ with $c(z_l) = \langle l, j_l\rangle$ ($j_l \leq k - l$) for all $m < j$ and $l < i$. This completes the proof of the lemma for $k = 2$.

Remark 5.6. By all the above, it is now fairly obvious that in the general case of k-functions we will have that $X = B_0 \uplus ... \uplus B_k$ where each B_i will be bounded and have property P_{k-i}, and each element $x \in B_i$ will have successors in each of the sets $B_0, \ldots, B_{i-1}$.

The last observation together with Remark 4.8 are enough to complete the proof of the lemma in the general case of $k > 2$. $\square$

We now give a helpful definition and some remarks in order to clarify some of the proof ideas of the previous lemma:

Definition 5.7. ***(White-Blue-Red elements)***

- We call every element $x \in B_0$ a White element.
- We call every element $x \in B_1$ a Blue element.
- We call every element $x \in B_2$ a Red element.

White:	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 0,2\rangle$
Blue:	$\langle 1,0\rangle$	$\langle 1,1\rangle$	
Red:	$\langle 2,0\rangle$		

Figure 5.1: Coloring partition for 2 functions

Notational Abuse: In what follows, we will call an element x of X by the number which is assigned to it by the corresponding b_j coloring function. For example: If $x \in B_1$ and $b_1(x) = \langle 1,1\rangle$ then we will say that this x is a $\langle 1,1\rangle$, if $y \in B_2$ then we will say that this y is a $\langle 2,0\rangle$ etc... Figure 5.1 summarizes the coloring partition of Lemma 5.1 to White/Blue/Red sections, as well as the coloring within each component of the partition. All the above leads to the following Remark:

Remark 5.8. For the coloring c of Lemma 5.1 the following properties hold:

1. Every $\langle 2,0\rangle$ is followed by exactly one White and one Blue element.
2. Every $\langle 1,1\rangle$ is followed by a White element and a $\langle 1,0\rangle$.
3. Every $\langle 1,0\rangle$ is followed by at least one White element.
4. Every $\langle 0,2\rangle$ is followed by exactly a $\langle 0,0\rangle$ and a $\langle 0,1\rangle$.
5. Every $\langle 0,1\rangle$ is followed by at least one $\langle 0,0\rangle$.
6. A $\langle 0,0\rangle$ gives us inconclusive info, in the sense that it could be followed by any two non-$\langle 0,0\rangle$ elements.

We will use the above observations to obtain an optimal coloring for $n = 2$ in the following chapter.

CHAPTER 6

The 2 functions case (no restrictions)

Theorem 6.1. ***(Non-commutative functions)*** *Let X be a Borel space and $F, G : X \to X$ be Borel functions. Let also $\mathcal{G} = \mathcal{G}_{F,G}$. Suppose finally that $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$. Then $\chi_{\mathcal{B}}(\mathcal{G}) \leq 5$.*

Proof. By the L.A.S. lemma 5.1, our space X can be partitioned into three sets, the *White* set W, the *Blue* set B, and the *Red* set R, each of which is bounded and has respectively, property P_2, property P_1 and property P_0. Moreover, every (Blue) element in B is followed by at least one (White) element in W and every (Red) element in R is followed by exactly one (White) element in W and one (Blue) element in B. To sum up:

1. $X = W \uplus B \uplus R$
2. All sets W, B, R are bounded
3. W has property P_2 and thus since it's also bounded it is Borel 3-colorable.
4. B has property P_1 and thus since it's also bounded it is Borel 2-colorable.
5. R has property P_0 and thus it's Borel 1-colorable.
6. Every Blue element is followed by at least one White element.
7. Every Red element is followed by exactly one Blue and one White element.

Moreover, using the colors 1,2,3 in this order for the White colors $\langle 0,0\rangle$, $\langle 0,1\rangle$, $\langle 0,2\rangle$ (of Lemma 5.1), 4,5 for the Blue colors $\langle 1,0\rangle$, $\langle 1,1\rangle$, and 6 for the Red color $\langle 2,0\rangle$, we have by Remark 5.8 that:

- Every 2 and 3 is followed by at least one 1
- Every 5 is followed by at least a 4

Now we define $D \subseteq X$ to be the set of all 2's, 3's, 5's and 6's. By the above and since every element in D has trivially at most two descendants, we have that every 2,3 and 5 has at most one descendant in D and thus only the 1-colorable subset of 6's can have two descendants in D. We conclude that the set D satisfies the assumptions of the 1-colorable subset Lemma 4.10 and thus it should be 3-colorable. Now, clearly, $X = D \uplus (X \setminus D)$. D is Borel 3-colorable by the above, and $X \setminus D$ is trivially Borel 2-colorable. Therefore, $\mathcal{G}$ will be Borel 5-colorable as desired. □

Using all the above, we can give the following characterization of all the finitely Borel-colorable graphs generated by two Borel functions:

Theorem 6.2. *Let X be a Borel space, $F, G : X \to X$ be arbitrary Borel functions and $\mathcal{G} = \mathcal{G}_{F,G}$, the corresponding Borel graph generated by them. Then the following statements are equivalent:*

i. $\chi_{\mathcal{B}}(\mathcal{G}) \leq 5$

ii. $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$

iii. There are 3 Borel Subsets A, B and C s.t.

  - $X = A \uplus B \uplus C$
  - *All A, B and C are bounded*
  - *If $x \in B$ then $F(x) \in A$ or $G(x) \in A$*
  - *If $x \in C$ then ($F(x) \in A$, $G(x) \in B$) or ($F(x) \in B$, $G(x) \in A$).*

Proof. It's enough to prove that $i \to ii$, $ii \to iii$ and $iii \to i$.

[$i \to ii$:] This is trivial.

[$ii \to iii$:] This implication follows from the construction in the proof of the L.A.S. Lemma 5.1 and Claim 5.2.

[$iii \to i$:] This is just by the proof of the theorem 6.1. I.e. the set A will be our White set, B will be our Blue set and C will be our Red set.

$\square$

CHAPTER 7

The 3 functions case (no restrictions)

In this chapter, we explore the $n = 3$ case:

Theorem 7.1. *Let X be a Borel space and $F, G, H : X \to X$ be Borel functions. Let also $\mathcal{G} = \mathcal{G}_{F,G,H}$. Suppose finally that $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$. Then $\chi_{\mathcal{B}}(\mathcal{G}) \leq 8$.*

Proof. According to the L.A.S. lemma 5.1 we can color the graph $\mathcal{G}$ in 10 colors and in such a way that we can get the following properties (see Figure 7.1):

- All elements colored by 1,2,3 or 4 are further labelled 'white'
- All elements colored by 5,6 or 7 are further labelled 'blue'
- All elements colored by 8 or 9 are further labelled 'red'
- All elements colored by 10 are further labelled 'black'
- All 2's are followed by at least a 1
- All 3's are followed by at least a 2 and a 1
- All 4's are followed by at least a 3, a 2 and a 1

White:	1	2	3	4
Blue:	5	6	7	
Red:	8	9		
Black:	10			

Figure 7.1: Coloring partition for 3 functions

- All 5's are followed by at least a white element
- All 6's are followed by at least a 5 and a white element
- All 7's are followed by at least a 6, a 5 and a white element
- All 8's are followed by at least a blue and a white element
- All 9's are followed by at least an 8, a blue and a white element
- All 10's are followed by at least a red, a blue and a white element

Now let's define $Y \subseteq X$ to be the set of all 2's, 3's, 4's, 6's, 7's, 9's and 10's. It's enough to prove that $\mathcal{G} \upharpoonright Y$ is 5-colorable. We will color $\mathcal{G} \upharpoonright Y$ using the colors A, B, C, D and E. Also for notational simplicity, when we talk about *coloring on* Z we will really mean *coloring on* $\mathcal{G} \upharpoonright Z$. We will color Y in 5 colors using the following steps:

Step 1: By the listed properties above, except for the 10's, every element in Y has at most 2 successors in Y.

Step 2: We set $Y_0 \subseteq Y$ to be the set of all $y \in Y$ which has a path that ends with a 10. I.e. all $y \in Y$ s.t. there exists an $n \in \omega$ and a sequence $L_i \in \{F, G, H\}$ (for all $i \in n$) s.t. $L_{n-1}(...(L_1(L_0(y)))...)$ is a 10. Note also that all 10's are trivially in Y_0.

Step 3: We color the set $Y \setminus Y_0$ in 5-colors (A, B, C, D, E). This is simply by theorem 6.1, the fact that Y is finitely Borel colorable and the fact that every element in $Y \setminus Y_0$ has at most two successors in $Y \setminus Y_0$.

Step 4: We set $Z_0 \subseteq Y_0$ to be the set of all 10's.

Step 5: For $y \in Y_0 \setminus Z_0$ we define $dist(y)$ to be the minimal $n \in \omega$ s.t. there are $L_i \in \{F, G, H\}$ (for all $i \in n$) s.t. $L_{n-1}(...(L_1(L_0(y)))...)$ is a 10.

Step 6: We set $Z_1 = \{y \in Y_0 \setminus Z_0 | dist(y) \text{ is odd }\}$, $Z_2 = \{y \in Y_0 \setminus Z_0 | dist(y) \text{ is even }\}$.

Step 7: The set Z_1 has property P. This is simply because every element $y \in Z_1$ is followed by at least one element in $Z_0 \cup Z_2$ (and one element in $X \setminus Y$). Similarly the set Z_2 also has property P. This is simply because every element $y \in Z_2$ is followed by at least one element in Z_1 (and one element in $X \setminus Y$).

Step 8: We set $W_1 \subseteq Z_1$ to be the set of all elements in Z_1 that have two successors in $M = Y_0 \cup C \cup D \cup E$, namely, either in Y_0, or in $Y \setminus Y_0$ and colored with C, D or E. (In addition there is a third successor outside Y.) W_1 has property P since it's contained in Z_1. By lemma 4.3, applied to the graph $\mathcal{G} \restriction W_1$, let V_1 be the 1st part of W_1. By this construction we clearly have that the set of all elements in $Z_1 \setminus V_1$ with exactly two successors in $M \setminus V_1$ is 1-colorable. This is because by lemma 4.3 if $y \in Z_1 \setminus V_1$ has this property then:

- y has to be an element of the 2nd part of W_1.
- Every successor of an element in the 2nd part of W_1 which is also in the 2nd part of W_1, has a successor on the 1st part of W_1.

Also every element in V_1 is immediately followed by exactly one element in $Z_1 \setminus V_1$, because if not then by lemma 4.3 such element would be the first of a bounded component which means it would belong in the 2nd part of W_1 and thus in $W_1 \setminus V_1$. As a corollary, every element in V_1 is followed by at most one element in $Z_0 \uplus Z_2$.

Step 9: We set $W_2 \subseteq Z_2$ to be the set of all elements in Z_2 that have two successors in $M \setminus V_1$. By lemma 4.3 let V_2 be the 2nd part of W_2. By this construction we clearly have that every element in $Z_2 \setminus V_2$ has at most one successor in $M \setminus (V_1 \uplus V_2)$. This is again because of lemma 4.3 and the fact that if $y \in W_2 \setminus V_2$ then it has to be in the 1st part of W_2 and thus it should be followed by an element in V_2.

Step 10: Now we set $V_0 \subseteq Z_0$ to be all the 10's which are followed by at most one element in $V_1 \cup V_2 \cup A \cup B$ (this means that the 10 is followed by at most one element which is in $V_1 \cup V_2$ or it's in $Y \setminus Y_0$ and it's colored by A or B).

Step 11: By all the above, the set $V = V_0 \uplus V_1 \uplus V_2$ has property P. On top of that, by construction of V_2, an element in V_2 is never followed by an element in $V_0 \uplus V_1$ and thus the only unbounded components of V contain elements only from the 1-colorable sets V_0 and V_1. We conclude that the restriction of V to its unbounded components is 2-colorable. Since we also have by Lemma 4.7 that every bounded component with property P is Borel 2-colorable, we conclude that V is 2-colorable. Moreover, we can color the entire set V using exactly the colors A and B without violating the coloring property with respect to nodes that have already been colored, namely nodes in $Y \setminus Y_0$. This is simply because: (a) all predecessors of nodes in V belong to Y_0, (b) no element in $V_1 \uplus V_2$ is followed by an element in $(Y \setminus Y_0) \cap (A \cup B)$, and (c) if $y \in V_0$ is followed by an $(A \cup B)$-element in $Y \setminus Y_0$ then the connected component K_y of V that contains y is bounded and has y as its first/top element. The latter implies that by Lemma 4.7 we can treat the $(A \cup B)$-successor of y as a restriction, when coloring the bounded components. Moreover, Lemma 4.7 gives a two coloring into $\{A, B\}$ that respects these restrictions.

Step 12: The remaining graph on $U = Y_0 \setminus V$ is 3-colorable and we can color it using C, D and E without violating the coloring property. This is because $U \subseteq Z_0 \uplus Z_1 \uplus Z_2$ and all elements in $(Z_0 \uplus Z_2) \setminus V$ have at most one successor in $U \cup (C \cup D \cup E)$. To verify this, we observe that any $y \in Z_0 \setminus V$ is followed by at least two elements in $A \cup B$ (this includes elements of V that were colored using A and B in step 11) and any $y \in Z_2 \setminus V$ is followed by at least one element in V_2 and one element in $X \setminus Y$. So, if $U_0 \subseteq U$ is the set of all elements in U that have two successors in $U \cup (C \cup D \cup E)$ then $U_0 \subseteq Z_1 \setminus V$. But we have already shown above that such a set should be 1-colorable. Define $r : U \to \mathcal{P}(\{C, D, E\})$ by "$a \in r(y)$" iff "there is a successor of y which is colored by $a \in \{C, D, E\}$". Then, by the generalized 1-colorable subset lemma 4.9 we can find a Borel coloring $c : U \to \{C, D, E\}$ on U which is restricted by r and thus it does not violate the coloring property.

All the above shows that we can color Y in 5-colors, i.e. A, B, C, D, E and thus we can color X in 8-colors, namely $A, B, C, D, E, 1, 5, 8$. Therefore, $\chi_{\mathcal{B}}(\mathcal{G}) \leq 8$ $\square$

Using all the above, we can give the following characterization:

Theorem 7.2. *Let X be a Borel space, $F_0, F_1, F_2 : X \to X$ be arbitrary Borel functions and $\mathcal{G} = \mathcal{G}_{F_0,F_1,F_2}$, the corresponding Borel graph generated by them. Then the following statements are equivalent:*

i. $\chi_{\mathcal{B}}(\mathcal{G}) \leq 8$

ii. $\chi_{\mathcal{B}}(\mathcal{G}) < \omega$

iii. *There are 4 Borel Subsets A, B, C and D s.t.*

- $X = A \uplus B \uplus C \uplus D$
- *All A, B, C and D are bounded*
- *If $x \in B$ then $F_0(x) \in A$ or $F_1(x) \in A$ or $F_2(x) \in A$*
- *If $x \in C$ then there are $i, j \in 3$ with $i \neq j$ s.t. $F_i(x) \in A$ and $F_j(x) \in B$*
- *If $x \in D$ then there are $i, j, k \in 3$ with $i \neq j, i \neq k, j \neq k$ s.t. $F_i(x) \in A$, $F_j(x) \in B$ and $F_k(x) \in C$.*

Proof. $i \to ii$: This is trivial.

$ii \to iii$: This implication is an immediate result of the L.A.S. Lemma 5.1.

$iii \to i$: This is just by the proof of the theorem 7.1. I.e. the set A will be our White set, B will be our Blue set, C will be our Red set and D will be our Black set. □

CHAPTER 8

Other chromatic numbers

We recall the following definitions from Chapter 2:

Definition 8.1. Let X be a Polish space and $\mathcal{G}$ be a graph on X. Then:

- The Baire chromatic number of $\mathcal{G}$ ($\chi_{\mathcal{BP}}(\mathcal{G})$) is given by: $\chi_{\mathcal{BP}}(\mathcal{G}) = \min\ \{|c(X)|$ where c is a Baire measurable coloring of $\mathcal{G}\}$.
- The μ-measurable chromatic number of $\mathcal{G}$ ($\chi_\mu(\mathcal{G})$) is given by: $\chi_\mu(\mathcal{G}) = \min\ \{|c(X)|$ where c is a μ-measurable coloring of $\mathcal{G}\}$. (Here, μ is a probability measure on X).

Definition 8.2. A set $C \subseteq X$ of a graph (X, E) is called upward invariant if $x \in C$ implies $y \in C$ for all successors y of x. The same set $C \subseteq X$ is called downward invariant if $x \in C$ implies $y \in C$ for all predecessors y of x.

Proposition 8.3. *If $C \subseteq X$ is comeager, C is upward invariant, $\mathcal{G} = \mathcal{G}_{f_0, f_1, \ldots, f_n}$ is generated by n Borel functions and $\chi_{\mathcal{B}}(\mathcal{G} \restriction C) \leq k$, then $\chi_{\mathcal{BP}}(\mathcal{G}) \leq max\{2n+1, k\}$. (Similarly for χ_μ, if $C \subseteq X$ has μ-measure 1).*

Proof. If such a C exists, then we will first color $\mathcal{G} \restriction C$ in a Borel — and thus also in a Baire — way, using k colors. Then we will color the remaining graph on $X \setminus C$ using at most $2n+1$ colors without violating the coloring property. This is possible since we could do this for any finite subset of $\mathcal{G} \restriction (X \setminus C)$. Therefore, by applying compactness (AC) we can get a $2n+1$ coloring on $\mathcal{G} \restriction (X \setminus C)$ which may not be Borel but it will be Baire, since the set $X \setminus C$ is meager. $\square$

Ben Miller showed in [2] that in the case of graphs $\mathcal{G}$ generated by one Borel function, there is always a both upward and downward invariant co-meager Borel set $C \subseteq X$ s.t. $\chi_{\mathcal{B}}(\mathcal{G} \upharpoonright C)$ is finite (and similarly with μ-measure 1 set C). From this, Theorem 1.14 and the last remark, it follows that $\chi_{\mathcal{BP}}(\mathcal{G}) \leq 3$ and $\chi_{\mu}(\mathcal{G}) \leq 3$.

In this chapter we generalize the results to graphs generated by more functions. More precisely, for any $n \in \omega$ and any Borel functions $f_0, f_1, \ldots, f_n$ there is an upward invariant co-meager $C \subseteq X$ s.t. $\chi_{\mathcal{B}}(\mathcal{G}_{f_0,\ldots,f_n} \upharpoonright C)$ is finite. (Similarly for μ-measure 1 set C.)

Historical Remark: After we mentioned our generalization to Ben Miller, he found a simple variant of his theorem in [2] saying that for any Borel functions $f_0, f_1, \ldots, f_{n-1}$ and for any $i < n$ there is a comeager $C \subseteq X$ which is upward invariant under all functions $f_0, f_1, \ldots, f_{n-1}$ and s.t. $\chi_{\mathcal{B}}(\mathcal{G}_{f_i} \upharpoonright C) \leq 3$ (similarly for μ-measure 1 sets). Applying this to each $i < n$ and intersecting the sets C, one gets an upward invariant co-meager (or μ-measure 1) $C^* \subseteq X$ s.t. $\chi_{\mathcal{B}}(\mathcal{G}_{f_0,\ldots,f_n} \upharpoonright C^*) \leq 3^n < \omega$. We give our full original proof here, in case its other ideas become useful for other work.

We start by giving a different proof of the theorem first proved in [2] while keeping some of his details and ideas intact.

Theorem 8.4. *Let X be a Polish space, μ be a probability measure on X, and $f : X \to X$ be a Borel function. Then there is a co-meager upward and downward invariant set C s.t. $\chi_{\mathcal{B}}(\mathcal{G}_f \upharpoonright C) \leq 3$, and similarly for μ-measure 1 set C. In particular, $\chi_{\mathcal{BP}}(\mathcal{G}_f) \leq 3$ and $\chi_{\mu}(\mathcal{G}_f) \leq 3$.*

Proof. We know that $\chi_{\mathcal{B}}(\mathcal{G}_f) \leq \omega$, so we can fix an ω Borel coloring D_i for $\mathcal{G}_f$ $(i \in \omega)$. Now, for each $n \in \omega$ we let $F_n = D_0 \uplus D_1 \uplus \ldots \uplus D_n$. Clearly, $F_n \subseteq F_{n+1}$ and $\bigcup F_n = X$. Finally, we let $A_n = \{x \in X | \forall i \in \omega \exists j > i \text{ s.t } f^j(x) \in F_n\}$.

Lemma 8.5. *The Borel chromatic number of $\mathcal{G}_f \upharpoonright A_n$ is at most 3.*

Proof. Fix $n \in \omega$. By looking at $A_n \cap D_0$, $A_n \cap D_1$, ..., $A_n \cap D_n$, we can find a 1-colorable set $Y_n \subseteq A_n$ s.t. if $x \in A_n$ then there exists an $m \in \omega$ s.t. $f^m(x) \in Y_n$. Clearly, the set $A_n \setminus Y_n$

is bounded by Y_n and thus it is Borel 2-colorable. We conclude that the entire $\mathcal{G}_f \restriction A_n$ is Borel 3-colorable. $\square$

A close inspection to the above construction shows that we can do it in such a way that:

1. $Y_n \subseteq Y_{n+1}$ (for all $n \in \omega$)
2. $\bigcup_{n\in\omega} Y_n$ is 1-colorable
3. On the n-th step, we recursively preserve the colors we assigned during the previous steps. That is, on step n we only assign colors to the elements of A_n that haven't been colored during the steps 0,1,...,$n-1$, while we keep the previously assigned colors for the rest of the elements of A_n.
4. In the end the entire set $\biguplus_{n\in\omega} A_n$ is Borel 3-colorable.

Since A_n is both upward and downward invariant, the lemma above allows us to assume without loss of generality that $A_n = \emptyset$ for all $n \in \omega$. This means that for each $x \in X$ and $n \in \omega$ there exists some $i \in \omega$ s.t. $f^j(x) \notin F_n$ for all $j \geq i+1$. Now, for each $\alpha \in 2^{\leq\omega}$, we define $C_\alpha = \bigcup_{\alpha(n)=1} D_n$.

Lemma 8.6. *There is a comeager both upward and downward invariant Borel set $C \subseteq X$ such that $\chi_{\mathcal{B}}(\mathcal{G}_f \restriction C) \leq 3$.*

Proof. For all $x \in X$, $k \in \omega$ and $s \in 2^{<\omega}$, there exist $t \supseteq s$ and $i \geq k$ s.t. $f^i(x) \in C_t$ and $f^{i+1}(x) \in F_{|t|} \setminus C_t$. To see this, let $n = |s|$. Since $A_n = \emptyset$ there is some $i \geq k$ s.t. $f^i(x) \notin F_n$ and $f^{i+1}(x) \notin F_n$. So, we can find $m, l > n$ ($m \neq l$) that satisfy: $f^i(x) \in D_m$ and $f^{i+1}(x) \in D_l$. We now expand s to $t \in 2^{<\omega}$ in such a way that $t(m) = 1$, $t(l) = 0$, $t \restriction |s| = s$ and $|t| = max\{m, l\} + 1$. The latter gives us that $f^i(x) \in C_t$ and $f^{i+1}(x) \in F_{|t|} \setminus C_t$ as desired. We now have that:

$$\forall x \in X \forall^* \alpha \in 2^\omega \forall k \in \omega \exists i \in \omega (f^{i+k}(x) \in C_\alpha \text{ and } f^{i+k+1} \notin C_\alpha)$$

where the part "$\forall^* \alpha \in 2^\omega \phi(\alpha)$" indicates that the set $\{\alpha \in 2^\omega | \phi(\alpha)\}$ is comeager. Now the Kuratowski-Ulam Theorem gives us that for comeagerly many $\alpha \in 2^\omega$, the set:

$$C^\alpha = \{x \in X | \forall k \in \omega \exists i \in \omega (f^{i+k}(x) \in C_\alpha \text{ and } f^{i+1+k}(x) \notin C_\alpha)\} \text{ is comeager}$$

Thus, in order to complete the lemma it's enough to fix any α_0 with that property. By the Finite Colorable Characterization for a Single Function (Theorem 1.14), it is very easy to see that $\chi_{\mathcal{B}}(\mathcal{G}_f \restriction C^{\alpha_0}) \leq 3$. Moreover, a straightforward computations shows that C^{α_0} is also both upward and downward invariant. The proof of the lemma is now complete. □

Lemma 8.7. *There is a μ-conull both upward and downward invariant Borel set $C \subseteq X$ such that $\chi_{\mathcal{B}}(\mathcal{G}_f \restriction C) \leq 3$.*

Proof. First we observe that for each $\epsilon > 0$, $p \in \omega$, and $n \in \omega$ there exists an $m > n$ s.t. $\mu(\{x \in X | \exists i \in \omega \text{ s.t. } f^{p+i}(x) \in F_m \setminus F_n\}) \geq 1 - \epsilon/2$. To see this we consider $i_n(x)$ to be the least $i \in \omega$ s.t. $f^j(x) \notin F_n$ for all $j \geq i$ (This is well defined since $A_n = \emptyset$). Now, we define $B_k = \{x \in X | f^{i_n(x)}(x) \in D_k\}$. Clearly, $\bigcup_{k>n} B_k = X$ since $f^{i_n(x)}(x) \notin F_n$. Thus, we can find $m > n$ big enough s.t. $\mu(\bigcup_{n<k\leq m} B_k) \geq 1 - \epsilon/2$. Thus, $\mu(\{x \in X | \exists i \in \omega$ s.t. $f^{p+i}(x) \in F_m \setminus F_n\}) \geq 1 - \epsilon/2$ as desired.

Now, an argument like the one above, allows us to also find an $l > m$ s.t.:

$\mu(\{x \in X | \exists i \in \omega \text{ s.t. } f^{p+i}(x) \in F_m \setminus F_n \text{ and } f^{p+i+1}(x) \in F_l \setminus F_m\}) \geq 1 - \epsilon$.

Hence, for each $n \in \omega$ and each $s \in 2^n$, we can find $t \in 2^{l+1}$ such that:

$\mu(\{x \in X | \exists i \in \omega \text{ s.t. } f^{p+i}(x) \in C_t \text{ and } f^{p+i+1}(x) \in F_{|t|} \setminus C_t\}) \geq 1 - \epsilon$. (We just set $t \restriction n = s$, $t(k) = 1$ for all $n \leq k \leq m$ and $t(k) = 0$ for all $m < k \leq l$). Now we can recursively construct an $\alpha \in 2^\omega$ s.t.:

$\{x \in X | \forall p \in \omega \exists i \in \omega \text{ s.t. } f^{p+i}(x) \in C_\alpha \text{ and } f^{p+i+1}(x) \notin C_\alpha\}$ is μ-conull. To see this, we observe that we can find strictly increasing sequence of $t_k \in 2^{<\omega}$ ($t_k \subsetneq t_{k+1}$) s.t. $\mu(H_k) \geq 1 - 1/2^k$ for all $k \in \omega$, where:

$H_k = \{x \in X | \exists i \in \omega \text{ s.t. } f^{p_k+i}(x) \in C_{t_k} \setminus F_{|t_{k-1}|} \text{ and } f^{p_k+i+1}(x) \in F_{|t_k|} \setminus C_{t_k}\}$, while $\{p_k | k \in \omega\}$ lists all natural numbers each repeated infinitely many times. Then, we let

$T_j = \bigcap_{k>j} H_j$. Set $T = \bigcup_j T_j$. Clearly, $\mu(T_j) \geq 1 - 1/2^j$ and $\mu(T) = 1$. Thus, $\alpha = \bigcup_k t_k$ is as desired. This is simply because by the definition of H_k, for each $x \in T_j$ we have that $\forall p \in \omega \exists i \in \omega$ s.t. $f^{p+i}(x) \in C_\alpha$ and $f^{p+i+1}(x) \notin C_\alpha$.

Then let $C = \{x \in X | \forall p \in \omega \exists i \in \omega \text{ s.t. } f^{p+i}(x) \in C_\alpha \text{ and } f^{p+i+1}(x) \notin C_\alpha\}$. Clearly, C is a μ-conull both upward and downward invariant borel subset of X. Also, by the Finite Colorable Characterization for a Single Function (Theorem 1.14), it is very easy to see that $\chi_{\mathcal{B}}(\mathcal{G}_f \upharpoonright C) \leq 3$, as desired. □

□

Theorem 8.8. *Let X be a Polish space, μ be a probability measure on X, and $f_0, f_1 : X \to X$ be two Borel functions on X. Then there is an upward invariant and co-meager set C s.t. $\chi_{\mathcal{B}}(\mathcal{G}_{f_0,f_1} \upharpoonright C) \leq 5$, and similarly for μ-measure 1 set C. In particular, $\chi_{\mathcal{BP}}(\mathcal{G}_{f_0,f_1}) \leq 5$ and $\chi_\mu(\mathcal{G}_{f_0,f_1}) \leq 5$.*

Proof. Let X be any Polish space and $f_0, f_1 : X \to X$ be any two Borel functions on X. For the first part it is enough to show that there is a comeager $C \subseteq X$ s.t. $\chi_{\mathcal{B}}(\mathcal{G}_{f_0,f_1} \upharpoonright C) < \omega$. Our plan is to partition X into sets E, E_0, E_1, E_2, M_{α_0}, and B, in such a way that, on an upward invariant comeager set (and similarly for a co-null set):

1. E, E_0, E_1, E_2, and M_{α_0} all have property P.

2. B is bounded.

Then B can be Borel 3-colored using Lemma 4.7, while E, E_0, E_1, E_2, and M_{α_0} can be Borel 3-colored on a comeager (co-null) set using Theorem 8.4. The entire graph can then be Borel $3 \times 6 = 18$ colored on a comeager (co-nulll) set.

We know that $\chi_{\mathcal{B}}(\mathcal{G}_{f_0,f_1}) \leq \omega$, so we can fix an ω Borel coloring $\biguplus_{i\in\omega} D_i$ for $\mathcal{G}_{f_0,f_1}$. Now, for each $n \in \omega$ we let $F_n = D_0 \uplus D_1 \uplus ... \uplus D_n$. Clearly, $F_n \subseteq F_{n+1}$ and $\bigcup F_n = X$. We now give the following very useful notational definition:

Definition 8.9. Let $x \in X$, $n \in \omega$ and $s \in 2^n$. Then we will use the following notation:

$$f^s(x) = f_{s(n-1)}...f_{s(1)}f_{s(0)}(x) \text{ (Clearly,} f^{\emptyset}(x) = x)$$

We now let $A_n = \{x \in X | \exists^\infty s \in 2^{<\omega} \text{ with } f^s(x) \in F_n\}$, where $\exists^\infty s \in 2^{<\omega}$ is an abbreviation for "there exist infinitely many distinct $s \in 2^{<\omega}$". We set, $A = \bigcup A_n$. We also define the following sets: $E^n = (A_n \setminus \bigcup_{j<n} A_j) \cap D_n$ for every $n \in \omega$. We also set $E = \bigcup_{n\in\omega} E^n$.

Claim 8.10. *The set E has property P.*

Proof. Let $x \in E^n$ for some $n \in \omega$. We will prove that either $f_0(x) \notin E$ or $f_1(x) \notin E$. This is enough to prove the claim.

$[n = 0]$*:* In this case, $x \in D_0$ and either $f_0(x) \in A_0$ or $f_1(x) \in A_0$ (by definition of A_0). If $f_0(x) \in A_0$ then $f_0(x) \notin E^0$ (since $f_0(x) \notin D_0$), while $f_0(x) \notin E^j$ for all $j > 0$ since $E^j \cap A_0 = \emptyset$ for all $j > 0$. We conclude that $f_0(x) \notin E$. If $f_1(x) \in A_0$ then we work similarly.

$[n > 0]$*:* In this case, $x \in D_n$ and either $f_0(x) \in A_n$ or $f_1(x) \in A_n$ (by definition of A_n). If $f_0(x) \in A_n$ then $f_0(x) \notin E^n$ (since $f_0(x) \notin D_n$) while $f_0(x) \notin E^j$ for all $j > n$ since $E^j \cap A_n = \emptyset$ for all $j > n$. Also, $f_0(x) \notin E^j$ for all $j < n$. This is simply because if there was some $m < n$ s.t. $f_0(x) \in E^m$ then that would imply that $f_0(x) \in A_m$ and thus $x \in A_m$ which is impossible since $x \in E^n$ and thus $x \notin A_j$ for all $j < n$. We conclude that $f_0(x) \notin E$. If $f_1(x) \in A_0$ then we work similarly.

The proof is now complete. We conclude that E has in fact property P. □

Claim 8.11. *Let $x, y \in X$, $s \in 2^{<\omega}$ and $y = f^s(x)$. If $y \in A$ then $x \in A$ as well.*

Proof. Since $y \in A$ then we can fix the minimum $n \in \omega$ s.t. $y \in A_n$. But since $y = f^s(x)$ then it is immediate that $x \in A_n$ by the definition of A_n. □

As a corollary of the claim above we get that the set $X \setminus A$ is upward invariant.

Claim 8.12. *For all $x \in A$ there exists some $s \in 2^{<\omega}$ s.t. $f^s(x) \in E$.*

Proof. Let $x \in A$. Then we can find an $n \in \omega$ such that $x \in A_n \setminus \bigcup_{j<n} A_j$. By definition of A_n this means that: $\exists^\infty s \in 2^{<\omega}$ s.t. $f^s(x) \in D_n \setminus \bigcup_{j<n} A_j$. However, the latter is equivalent to: $\exists^\infty s \in 2^{<\omega}$ s.t. $f^s(x) \in E_n$. $\square$

We now define recursively on A the following rank function: $rk : A \to \omega$, which is well defined by the previous claim.

- If $x \in E \subseteq A$ then $rk(x) = 0$.
- If $x \in A \setminus E$ then $rk(x) = min\{|s| \text{ s.t. } f^s(x) \in E\}$

The above, allows us to partition $A \setminus E = \biguplus_{i \geq 1} X_i$ where $x \in X_i$ iff $rk(x) = i$. Next we will recursively re-partition $A \setminus E$ into the sets $E_0 \uplus E_1 \uplus E_2$.

$(i = 1)$ In that case, we set $X_1 \subseteq E_0$.

$(i > 1)$ For this one, we recursively assume that $X_1 \uplus \cdots \uplus X_{i-1} \subseteq E_0 \uplus E_1 \uplus E_2$. For each $x \in X_i$ we let $x \in E_j$ where $j \in 3$ is the minimum index such that $f_0(x) \notin E_j$ and $f_1 \notin E_j$.

This construction clearly gives us: $A \setminus E = E_0 \uplus E_1 \uplus E_2$ from which we can clearly get: $A = E \uplus E_0 \uplus E_1 \uplus E_2$.

Claim 8.13. *The sets E_0, E_1 and E_2, they all have property P.*

Proof. It is enough to prove that for each $j \in 3$ and $x \in E_j$ we can find $i \in 2$ s.t. $f_i(x) \notin E_j$. Since $E_0 \uplus E_1 \uplus E_2 = E \setminus A$, we can prove this using induction on the rank of $x \in E \setminus A$:

$(rk(x) = 1)$ Then, x is followed by an element in E.

$(rk(x) > 1)$ By definition of the rk function, we know that there is an $i \in 2$ s.t.: $f_i(x) \in X_{rk(x)-1} \subseteq E_0 \uplus E_1 \uplus E_2$. However, by the recursive definition of the sets E_0, E_1, E_2 above, we know that $f_i(x) \notin E_j$ as desired. The proof of the lemma is now complete. $\square$

Remark 8.14. We can alternatively partition $A = E \uplus E^0 \uplus E^1$ in such a way that both E^0 and E^1 have property P.

Proof. As before we recursively define on A the following function: $rk : A \to \omega$:

- If $x \in E \subseteq A$ then $rk(x) = 0$.
- If $x \in A \setminus E$ then $rk(x) = min\{|s| \text{ s.t. } f^s(x) \in E\}$

Now we set $E^0 = \{x \in A | rk(x) \text{ is even but not zero }\}$ and $E^1 = \{x \in A | rk(x) \text{ is odd }\}$. Then it's clear that every element in E^0 is followed by an element in E^1 and every element in E^1 is followed by an element in $E \cup E^0$. $\square$

We now concentrate on $X \setminus A$. For convenience, let us simply work on X while assuming that $A = \emptyset$. We will later return to the general situation of $A \neq \emptyset$. As in theorem 8.4 we set $C_\alpha = \bigcup_{\alpha(n)=1} D_n$ for all $\alpha \in 2^{\leq \omega}$. We now prove the following useful lemma:

Lemma 8.15. *For each $x \in X$ and $s \in 2^{<\omega}$ there exist some $r \in \omega$ and $t \in 2^{<\omega}$ (with $t \supseteq s$) that satisfy the following:*

$$\forall h_0 \in 2^r \exists h_1 \subseteq h_0 : f^{h_1}(x) \in C_t \textit{ but } f^{h_1 \frown <i>}(x) \in F_{|t|} \setminus C_t \textit{ (some } i \in 2)(*)$$

Proof. Let $x \in X$, $n \in \omega$ and $s \in 2^n$. Let $B_n = \{y \in X | y = x \text{ or } y = f^u(x) \in F_n \text{ for some } u \in 2^{<\omega}\}$. Clearly, B_n is finite (since $A = \emptyset$). Now, we find $m > n$ big enough s.t.: if $y \in F_n$ or $y = x$ then both $f_0(y) \in F_m$ and $f_1(y) \in F_m$. We can do this because B_n is finite. Now, we let $B_m = \{y \in X | y = x \text{ or } y = f^u(x) \in F_m \text{ for some } u \in 2^{<\omega}\}$. Again, B_m is finite. Now we let $B_m^0 \subseteq B_m$ to be the set of all $y \in B_m$ s.t. either $f_0(y) \notin B_m$ or $f_1(y) \notin B_m$. Clearly by the definition of m, $B_m^0 \cap (F_n \cup \{x\}) = \emptyset$. Finally, we find $l > m$ big enough s.t.: if $y \in B_m^0$ then both $f_0(y) \in F_l$ and $f_1(y) \in F_l$.

Now, let $U = \{u \in 2^{<\omega} | f^u(x) \in B_m\}$. Since $A = \emptyset$, U is finite. Set $r \in \omega$ to be $r = max\{|u| : u \in U\}$. We also set $t \in 2^{l+1}$ to be $t \upharpoonright n = s$, $t(n) = t(n+1) = ... = t(m) = 1$ and $t(m+1) = t(m+2) = ... = t(l) = 0$. We will now prove that $(*)$ holds for these r and t. Towards that, let $h_0 \in 2^r$. Then, there must be some $k \leq r$ s.t. $f^{h_0 \upharpoonright k}(x) \in B_m^0$.

If not, then $f^{h_0 \upharpoonright k}(x) \in B_m \setminus B_m^0$ for all $k \leq r$. In particular, $f^{h_0 \upharpoonright r}(x) = f^{h_0}(x) \in B_m \setminus B_m^0$ which means that both $f^{h_0 \frown <0>}(x) \in B_m$ and $f^{h_0 \frown <1>}(x) \in B_m$, which contradicts the maximality of r, as both $|h_0 \frown (0)| = r + 1$ and $|h_0 \frown (1)| = r + 1$.

We set $h_1 = h_0 \upharpoonright k \subseteq h_0$. Then $f^{h_1}(x) \in F_m \setminus F_n$ but $f^{h_1 \frown <i>}(x) \in F_l \setminus F_m$ for some $i \in 2$. Thus, $f^{h_1}(x) \in C_t$ but $f^{h_1 \frown <i>}(x) \in F_{|t|} \setminus C_t$ for some $i \in 2$. We conclude that the $(*)$ and thus the lemma holds. $\square$

Hence, $\forall x \in X \forall^* \alpha \in 2^\omega \exists r \in \omega$ s.t. $\forall h_0 \in 2^r \exists h_1 \subseteq h_0$ s.t. $[f^{h_1}(x) \in C_\alpha$ and $f^{h_1 \frown <i>} \notin C_\alpha$ (some $i \in 2)]$, where the part "$\forall^* \alpha \in 2^\omega \phi(\alpha)$" indicates that the set $\{\alpha \in 2^\omega | \phi(\alpha)\}$ is comeager.

We now return to the situation where $A \neq \emptyset$. Using the fact that $X \setminus A$ is upward invariant, the work above gives us the following statement:

$\forall x \forall^* \alpha$ if $x \in X \setminus A$ then $\exists r \forall h_0 \in 2^r \exists h_1 \subseteq h_0$ s.t. $f^{h_1}(x) \in C_\alpha$ but $f^{h_1 \frown <i>}(x) \notin C_\alpha$ for some $i \in 2$.

For each such α we can define $M_\alpha = \{x \in C_\alpha | f_0(x) \notin C_\alpha \text{ or } f_1(x) \notin C_\alpha\}$. That way, we make sure that M_α has property P. Moreover:

$\forall x \forall^* \alpha$ if $x \in X \setminus A$ then $\exists r \forall h_0 \in 2^r \exists h_1 \subseteq h_0$ s.t. $f^{h_1}(x) \in M_\alpha$

For simplicity, we set $Q(x, \alpha) = \exists r \forall h_0 \in 2^r \exists h_1 \subseteq h_0$ s.t. $f^{h_1}(x) \in M_\alpha$.

Therefore, so far we have:

$\forall x \forall^* \alpha$ if $x \in X \setminus A$ then $Q(x, \alpha)$.

Now using the proof of theorem 8.4 and the fact that E has property P we can get $\bar{A}_E \subseteq E$ s.t. $\chi_{\mathcal{B}}(\mathcal{G} \upharpoonright \bar{A}_E) \leq 3$ and on $E \setminus \bar{A}_E$ we have:

$\forall x \forall^* \beta$ if $x \in E \setminus \bar{A}_E$ then $\exists s \in 2^{<\omega}$ s.t. $f^{s \upharpoonright n}(x) \in E$ for all $n \leq lh(s)$ and $[(f^{s \frown <i>}(x) \notin E$ for all $i \in 2)$ or $(f^s(x) \in C_\beta$ but $f^{s \frown <i>}(x) \in E \setminus C_\beta$ for some $i \in 2)]$

Of course, the same statements are true for all the sets E_0, E_1, E_2 and M_α (for the comeagerly many α's) since they all have property P.

Set $R(x, \beta, Y) = \exists s \in 2^{<\omega}$ s.t. $f^{s \upharpoonright n}(x) \in Y$ for all $n \leq lh(s)$ and $[(f^{s \frown <i>}(x) \notin Y$ for all $i \in 2)$ or $(f^s(x) \in C_\beta$ but $f^{s \frown <i>}(x) \in Y \setminus C_\beta$ for some $i \in 2)]$.

We then have by the above that for each $Y \in \{E, E_0, E_1, E_2, M_\alpha\}$ there are sets $\bar{A}_Y \subseteq Y$ s.t. $\chi_{\mathcal{B}}(\mathcal{G} \upharpoonright \bar{A}_Y) \leq 3$ and:

$(\forall x)(\forall^* \alpha)(\forall^* \beta)(x \in Y \setminus \bar{A}_Y \to R(x, \beta, Y))$

Let:

$$T(x, \alpha, \beta) = \left(\bigwedge_{Y \in \{E, E_0, E_1, E_2, M_\alpha\}} \big(x \in Y \setminus \bar{A}_Y \to R(x, \beta, Y)\big) \right) \& \, (x \in X \setminus A \to Q(x, \alpha)).$$

Then, $(\forall x)(\forall^* \alpha)(\forall^* \beta) T(x, \alpha, \beta)$. Also, replacing x by $f^t(x)$, where $t \in 2^{<\omega}$, we get equivalently that:

$(\forall x)(\forall t \in 2^{<\omega})(\forall^* \alpha)(\forall^* \beta) T(f^t(x), \alpha, \beta)$.

Now, it's finally time to "switch quantifiers" by using the Kuratowski-Ulam theorem:

For comeagerly many α's and β's the following set is comeager:

$C^{\alpha, \beta} = \{x \in X | \forall t \in 2^{<\omega} T(f^t(x), \alpha, \beta)\}$.

Finally, all we have to do is fix some α_0 and β_0 such that the set $C = C^{\alpha_0, \beta_0}$ is comeager. Clearly, C is upward invariant. We check that it is also Borel finitely colorable. Since the set C is partitioned into $\left(\biguplus_{Y \in \{E, E_0, E_1, E_2, M_{\alpha_0}\}} (Y \cap C) \right) \uplus (C \setminus (A \cup M_{\alpha_0}))$, it is enough to check that the graph $\mathcal{G}$ is Borel 3-colorable on each of the pieces. Then the entire graph will be Borel 18 colorable in C. But, each $Y \cap C$ is Borel 3-colorable as in the proof of Theorem 8.4, using the property that $(\forall x \in (Y \setminus \bar{A}_Y) \cap C) R(x, \beta, Y)$ and the fact that $\bar{A}_Y$ (and thus $\bar{A}_Y \cap C$) is upward and downward invariant inside Y as well as Borel 3-colorable. At the same time, the set $C \setminus (A \cup M_{\alpha_0})$ is bounded by $A \cup M_{\alpha_0}$ and thus it's also Borel 3-colorable by Lemma 4.7. Combining these colorings we get that $\chi_{\mathcal{B}}(\mathcal{G} \restriction C) \leq 18$ and hence by Theorem 6.1, $\chi_{\mathcal{B}}(\mathcal{G} \restriction C) \leq 5$.

Next we turn to the μ-measurable chromatic number. We start with the following useful lemma:

Lemma 8.16. *For each $\epsilon > 0$, $u \in 2^{<\omega}$ and $s \in 2^{<\omega}$ there exists some $t \in 2^{<\omega}$ (with $t \supseteq s$) s.t.: $\mu(\{x \in X | f^u(x) \in X \setminus A \to [\exists r \in \omega \forall h_0 \in 2^r \exists h_1 \subseteq h_0 : f^{u \frown h_1}(x) \in C_t, f^{u \frown h_1 \frown <i>}(x) \in F_{|t|} \setminus C_t$ (some $i \in 2$)$]\}) \geq 1 - \epsilon$.*

Proof. Towards that, we let $\epsilon > 0$, $u \in 2^{<\omega}$, $n \in \omega$ and $s \in 2^n$. Then we can find $m > n$ such that:

$\mu(\{x \in X | f^u(x) \in X \setminus A \to [\exists r \in \omega \forall h_0 \in 2^r \exists h_1 \subseteq h_0 \colon f^{u \frown h_1}(x) \in F_m \setminus F_n]\} \geq 1 - \epsilon/2.$

To see this, let $r_n(x)$ to be the least $r \in \omega$ s.t. $f^{u \frown h}(x) \notin F_n$ for all $h \in 2^{<\omega}$ with $|h| \geq r$. Such r exists since $f^u(x) \notin A$. We also set $B_k = \{x \in X | f^{u \frown h}(x) \in F_k$ for all $h \in 2^{\leq r_n(x)}\}$. Clearly, $\bigcup_{k>n} B_k = X$ and thus we can find $m > n$ big enough s.t. $\mu(\bigcup_{n<k\leq m} B_k) \geq 1 - \epsilon/2$. In the same manner, we can find $l > m$ such that:

$\mu(\{x \in X | f^u(x) \in X \setminus A \to [\exists r \in \omega \forall h_0 \in 2^r \exists h_1 \subseteq h_0 \colon f^{u \frown h_1}(x) \in F_m \setminus F_n, f^{u \frown h_1 \frown <i>}(x) \in F_l \setminus F_m$ (some $i \in 2$)$]\ \} \geq 1 - \epsilon$. Now, we define $t \in 2^{l+1}$ to be such that: $t \restriction n = s$, $t(k) = 1$ for all $n \leq k \leq m$ and $t(k) = 0$ for all $m < k \leq l$. Clearly, the t in question satisfies the properties of the lemma. □

Now we can recursively construct an $\alpha \in 2^\omega$ s.t.:

The set $C = \{x \in X | (\forall u \in 2^{<\omega})[f^u(x) \in X \setminus A \to \exists r \in \omega \forall h_0 \in 2^r \exists h_1 \subseteq h_0 \colon f^{u \frown h_1}(x) \in C_\alpha$ but $f^{u \frown h_1 \frown <i>}(x) \notin C_\alpha$ (some $i \in 2$)$]\ \}$ is μ-conull.

To see this, we observe that we can find strictly increasing sequence of $t_k \in 2^{<\omega}$ ($t_k \subsetneq t_{k+1}$) s.t. $\mu(H_k) \geq 1 - 1/2^k$ for all $k \in \omega$, where:

$H_k = \{x \in X | f^u(x) \in X \setminus A \to [\exists r \in \omega \forall h_0 \in 2^r \exists h_1 \subseteq h_0 \colon f^{u_k \frown h_1}(x) \in C_{t_k} \setminus F_{|t_{k-1}|}$ but $f^{u_k \frown h_1 \frown <i>}(x) \in F_{|t_k|} \setminus C_{t_k}$ (some $i \in 2$)$]\ \}$, while $\{u_k | k \in \omega\}$ lists all finite sequences $u \in 2^{<\omega}$ each repeated infinitely many times, and $F_{|t_{-1}|}$ is defined to be the empty set. Then, we let $T_j = \bigcap_{k>j} H_j$ and $T = \bigcup_j T_j$. Clearly, $\mu(T_j) \geq 1 - 1/2^j$ and $\mu(T) = 1$. Thus, $\alpha = \bigcup_k t_k$ is as desired. This is simply because by the definition of H_k, for each $x \in T_j$ we have that $(\forall u \in 2^{<\omega})[f^u(x) \in X \setminus A \to \exists r \in \omega$ s.t. $\forall h_0 \in 2^r \exists h_1 \subseteq h_0 \colon f^{u \frown h_1}(x) \in C_\alpha$ but $f^{u \frown h_1 \frown <i>}(x) \notin C_\alpha]$.

Next, we let $D_\alpha \subseteq C_\alpha$ to be the set of all $x \in C_\alpha$ s.t. either $f_0(x) \notin C_\alpha$ or $f_1(x) \notin C_\alpha$. It is then easy to see that:

$C = \{x \in X | (\forall u \in 2^{<\omega})[f^u(x) \in X \setminus A \to \exists r \in \omega \forall h_0 \in 2^r \exists h_1 \subseteq h_0 \colon f^{u \frown h_1}(x) \in D_\alpha]\}$ is μ-conull.

For notational simplicity, we set $E = E_3$ and $D_\alpha = E_4$. Thus, $A \cup D_\alpha = E_0 \uplus E_1 \uplus$

$E_2 \uplus E_3 \uplus E_4$. Note that all E_j's have property P. Using this fact, we can define a function $g : \bigcup_{j\in 5} E_j \rightharpoonup \bigcup_{j\in 5} E_j$ s.t. $g(x) = f_i(x)$ if both $x, f_i(x) \in E_j$ (for some $i \in 2$ and $j \in 5$), or else $g(x)\uparrow$ (meaning that $g(x)$ is undefined). For each $Y \in \{E_0, E_1, E_2, E_3, E_4\}$ let $\bar{A}_Y$ be the union of the sets A_n defined at the start of proof of Theorem 8.4, for the graph $\mathcal{G} \restriction Y$. Also, for $s, t, u \in 2^{<\omega}$ and $Y \in \{E_0, E_1, E_2, E_3, E_4\}$ we define:

$H^u_Y(s,t) = \{x \in X | y = f^u(x) \in Y \setminus \bar{A}_Y \to [(\exists k \in \omega \text{ s.t. } g^k(y)\uparrow) \text{ or } (\exists k \in \omega \text{ s.t. } g^k(y) \in C_t \setminus F_{|s|} \text{ but } g^{k+1}(y) \in F_{|t|} \setminus C_t)]\}$.

Lemma 8.17. *Let $\epsilon > 0$, $s, u \in 2^{<\omega}$ and $Y \in \{E_0, E_1, E_2, E_3, E_4\}$. Then we can find $t \in 2^{<\omega}$ s.t. $s \subseteq t$ and $\mu(H^u_Y(s,t)) \geq 1 - \epsilon$.*

Proof. Let $\epsilon > 0$, $n \in \omega$, $s \in 2^n$, $u \in 2^{<\omega}$, and $Y \in \{E_0, E_1, E_2, E_3, E_4\}$. Clearly, by the definition of $Y \setminus \bar{A}_Y$, when $g^j(x)$ is defined for all $j \in \omega$, and $x \notin \bar{A}_Y$, we can also define $i_n(x)$ to be the least $i \in \omega$ s.t. $g^j(x) \notin F_n$ for all $j \geq i$. Now, define $B_k = \{x \in X | y = f^u(x) \in Y \setminus \bar{A}_Y \to [(\exists k \in \omega \text{ s.t. } g^k(y)\uparrow) \text{ or } (g^{i_n(y)}(y) \in D_k)]\}$. Clearly, $\bigcup_{k>n} B_k = X$. Thus we can find $m > n$ s.t. $\mu(\bigcup_{n<k\leq m} B_k) \geq 1 - \epsilon/2$. Therefore:

$\mu(\{x \in X | y = f^u(x) \in Y \setminus \bar{A}_Y \to [(\exists k \in \omega \text{ s.t. } g^k(y)\uparrow) \text{ or } (\exists k \in \omega \text{ s.t. } g^k(y) \in F_m \setminus F_n)]\}) \geq 1 - \epsilon/2$. Similarly, we can find an $l > m$ s.t.:

$\mu(\{x \in X | y = f^u(x) \in Y \setminus \bar{A}_Y \to [(\exists k \in \omega \text{ s.t. } g^k(y)\uparrow) \text{ or } (\exists k \in \omega \text{ s.t. } g^k(y) \in F_m \setminus F_n \text{ but } g^{k+1}(y) \in F_l \setminus F_m)]\}) \geq 1 - \epsilon$.

Finally, we define $t \in 2^{l+1}$ to be such that: $t \restriction n = s$, $t(k) = 1$ for all $n \leq k \leq m$ and $t(k) = 0$ for all $m < k \leq l$. Clearly, for that t we have that $\mu(H^u_Y(s,t)) \geq 1 - \epsilon$, which proves the lemma. □

Now, let $d_k = (E_{j_k}, u_k)$ be an enumeration of all pairs (E_j, u) with $j \in 5$ and $u \in 2^{<\omega}$, where each pair appears infinitely many times in the enumeration. Next, starting with $t_0 = \emptyset$ we recursively construct a sequence $\{t_i\}_{i\in\omega}$ of elements in $2^{<\omega}$ s.t. $\mu(H_i) > 1 - 1/2^i$ where $H_i = H^{u_i}_{E_{j_i}}(t_i, t_{i+1})$. We also set $H = \bigcup_j \bigcap_{i>j} H_i$. Clearly $\mu(H) = 1$. We also let $\beta = \bigcup_i t_i$. By all the above, it should also be clear that:

$H = \{x \in X | (\forall u \in 2^\omega)[y = f^u(x) \in \biguplus_{j\in 5} \left(E_j \setminus \bar{A}_{E_j}\right) \to [(\exists k \in \omega \text{ s.t. } g^k(y) \uparrow) \text{ or } (\exists k \in \omega$ s.t. $g^k(y) \in C_\beta$ but $g^{k+1}(y) \notin C_\beta)]]\}$ which is μ-conull.

Let $C' = C^{\alpha,\beta} = \{x \in X | \forall u \in 2^{<\omega} T(f^u(x), \alpha, \beta)\}$, where T is the predicate defined in the proof for the comeager case above. Clearly $C' = H \cap C$ and thus C' is μ-conull. Moreover, C' is easily upward invariant.

We also check that C' is Borel finitely colorable: Since the set C' is partitioned into $\left(\biguplus_{j\in 5}(E_j \cap C')\right) \uplus \left(C' \setminus \left(\biguplus_{j\in 5} E_j\right)\right)$, it is enough to check that the graph $\mathcal{G}$ is Borel 3-colorable on each of the pieces. Then the entire graph will be Borel 18 colorable in C'. But, each $E_j \cap C'$ is Borel 3-colorable exactly as in the proof of Theorem 8.4, using the property that $(\forall x \in (E_j \setminus \bar{A}_{E_j}) \cap C') R(x, \beta, E_j)$, where R is the predicate defined in the proof for the comeager case above, and the fact that $\bar{A}_{E_j}$ (and thus $\bar{A}_{E_j} \cap C'$) is upward and downward invariant inside E_j as well as Borel 3-colorable. At the same time, the set $C' \setminus (\biguplus_{j\in 5} E_j)$ is bounded by $\biguplus_{j\in 5} E_j$ and thus it's also Borel 3-colorable by Lemma 4.7. Combining these colorings we get that $\chi_{\mathcal{B}}(\mathcal{G} \restriction C') \leq 18$ and hence by Theorem 6.1, $\chi_{\mathcal{B}}(\mathcal{G} \restriction C') \leq 5$. □

The method used in the proof of the above theorem can be used to obtain the following theorem as a corollary:

Theorem 8.18. *Let X be a Polish space, μ be a probability measure on X, and $\mathcal{G} = (X, E)$ be a graph generated by countably many Borel functions. Then let $E_0, \ldots, E_{n-1}$ be disjoint subsets of X with property P_k. Then $\biguplus_{j\in n} E_j$ is Borel finitely colorable on an upward invariant comeager subset of X. Similarly for μ measure 1 set. In particular, if $f_0, f_1, ..., f_{k-1} : X \to X$ are Borel functions on X, then there is an upward invariant and comeager set C s.t. $\chi_{\mathcal{B}}(\mathcal{G}_{f_0,f_1,...,f_{k-1}} \restriction C) < \omega$, and similarly for μ-measure 1 set C. It follows, we have that: $\chi_{\mathcal{BP}}(\mathcal{G}_{f_0,f_1,...,f_{k-1}}) < \omega$ and $\chi_\mu(\mathcal{G}_{f_0,f_1,...,f_{k-1}}) < \omega$.*

Proof. The proof of Theorem 8.8 shows that given any graph $\mathcal{G}$ generated by $k+1$ Borel functions, we can find an upward invariant comeager (μ-conull) set $C \subseteq X$ s.t. C can be partitioned into sets $E_0, E_1, \ldots, E_{k+3}$ and $C \setminus (\biguplus_{j\in k+4} E_j)$ where all E_j have property P_k

and $C \setminus (\biguplus_{j\in k+4} E_j)$ is bounded by $\biguplus_{j\in k+4} E_j$. By induction, there is an upward invariant and comeager (μ-conull) $C' \subseteq C$ s.t. $C' \cap E_j$ is Borel finitely colorable. Moreover, $C' \setminus (\biguplus_{j\in k+4} E_j)$ is also Borel finitely colorable by Lemma 4.7. We conclude that the graph $\mathcal{G}$ is Borel finitely colorable on C'.

Now let $E_0, \dots, E_{n-1}$ be disjoint sets with property P_{k+1}. By the proof of Theorem 8.8 the following set is upward invariant and comeager:

$$C = C^\alpha = \{x \in X | (\forall u \in \omega^{<\omega}) \bigwedge_{i\in n} (f^u(x) \in E_i \setminus A_{E_i} \to Q(f^u(x), \alpha))\}$$

Now, again by the proof of Theorem 8.8, we also get that inside $E_i \cap C$ we can find some disjoint sets $E_i^0, \dots, E_i^{k+3}$ s.t. each E_i^j has property P_k while $(E_i \cap C) \setminus \left(\biguplus_{j\in k+4} E_i^j\right)$ is bounded by $\biguplus_{j\in k+4} E_i^j$.

Now, let: $E = \biguplus_{i\in n} \biguplus_{j\in k+4} E_i^j$. By induction, we get that E is Borel finitely colorable on a upward invariant comeager set $C' \subseteq C$. Moreover, since $(E_i \cap C') \setminus \left(\biguplus_{j\in k+4} E_i^j\right)$ is bounded by $\biguplus_{j\in k+4} E_i^j$, then by Lemma 4.7 we have that: $\biguplus_{i\in n} E_i$ is Borel finitely colorable on C'.

For the μ measurable chromatic number we work similarly. $\square$

References

[1] A. S. Kechris, S. Solecki, and S. Todorcevic. Borel chromatic numbers. *Adv. Math.*, 141(1):1–44, 1999.

[2] Benjamin D. Miller. Measurable chromatic numbers. *J. Symbolic Logic*, 73(4):1139–1157, 2008.