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Classification of ribbon categories with the fusion rules of $SO(N)$

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by

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The dissertation of Daniel R. Copeland is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

To Alex,
for everything.

Table of Contents

List of Notation and Abbreviations

$[n]_q$	the quantum number $\frac{q^n-q^{-n}}{q-q^{-1}}$
$\mathbb{Z}_N$	the group of integers mod N
$\mathbb{C}^\times$	non-zero complex numbers
$\mathcal{C} \boxtimes \mathcal{D}$	Deligne product
$\mathrm{Gr}^+(\mathcal{C})$	Grothendieck semigroup/semiring
$\mathrm{Gr}(\mathcal{C})$	Grothendieck ring
$U(\mathcal{C})$	universal grading group
$H^3(G)$	third cohomology group of G with values in $\mathbb{C}^\times$
$H^3_{\mathrm{a}b}(G)$	third abelian cohomology of G
$\dim_{\mathcal{C}}(X)$	categorical dimension of an object X
$c_{X,Y}$	braiding map $X \otimes Y \to Y \otimes X$
θ_X	twist on the object X in a ribbon category
$\Gamma, \Gamma(\mathcal{C}), \Gamma(G)$	sets of irreducible isotypes
W	Weyl group of $SO(N)$
P	weight lattice for $SO(N)$
$\mathcal{V}(q, r)$	orthogonal type category
Rep $SO(N)_q$	quantum group category for $SO(N)$
$\mathcal{B}(k)$	kth level of the Bratteli diagram
J_k	k-th Jucys-Murphy element
Δ_k^2	k-th full twist
AB_2	affine braid group on two strands

List of Figures

List of Tables

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ABSTRACT OF THE DISSERTATION

Classification of Ribbon Categories with the Fusion Rules of $SO(N)$

by

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Professor Hans Wenzl, Chair

We classify ribbon categories with the tensor product rules of the finite-dimensional complex representations of $SO(N)$, for $N \geq 5$ and $N = 3$. The strategy is to study representations of the braid group which appear in $\mathrm{End}(X^{\otimes k})$, where X corresponds to the defining representation. The fusion rules serve to define path bases for these algebras, and we prove that the matrix representations of the braid elements are uniquely determined by the eigenvalues of a braid operator on $X \otimes X$. We use this to show that the equivalence class of a category with $SO(N)$ fusion rules depends only on one of the eigenvalues of the braid operator. The classification applies both to generic $SO(N)$ tensor product rules and to certain fusion rings having only finitely many simple objects.

1

Introduction

This thesis is concerned with the classification of ribbon categories whose tensor product (a.k.a. fusion) rules resemble those of the special orthogonal group $SO(N, \mathbb{C})$. We consider both the actual tensor product rules for finite dimensional $\mathbb{C}$-representations of $SO(N)$, involving infinitely many simple objects, and certain related fusion rings spanned by finitely many simple elements, which come from the representation theory of $SO(N)$ in a more complicated way. These ribbon categories appear in several different contexts, notably Drinfel'd-Jimbo quantum groups, Turaev-Wenzl style skein theory, and representations of affine Lie alegbras. Such categories are perhaps best known as algebraic ingredients for producing low dimensional TQFT's and the resulting quantum invariants of 3-manifolds [Wal] [Tur16], e.g. the Reshetikhin-Turaev [RT91] and Turaev-Viro invariants [TV92]. However, this thesis is entirely algebraic. Our main result is that if a ribbon category has certain prescribed fusion rules associated to $SO(N)$, then it must "come from" a quantum group – meaning it can be obtained from a known quantum group category through a standard construction. Without referring to quantum groups we can roughly state our result as the following.

Theorem 1.0.1. *Categories with $SO(N)$ type fusion rules are determined (after a standard normalization) by the eigenvalues of the braid operator $c_{X,X}$ acting on $X \otimes X$ where X is a simple object corresponding to the N-dimensional defining representation of $SO(N)$.*

More precisely, the fusion rules dictate that $X \otimes X$ splits into 3 irreducible subobjects, so that the braid element $c_{X,X}$ has 3 eigenvalues. Part of the theorem is that these eigenvalues can be written $(q, -q^{-1}, q^{-(N-1)})$ where q is some non-zero complex number. Hence (up to the "standard normalization") these categories depend only on their fusion rules and the parameter q. Similar results were previously obtained for $SL(N)$ by Kazhdan and Wenzl [KW93] and for $O(N)$ and $Sp(N)$ by Tuba and Wenzl [TW05].

Ribbon categories are beautiful algebraic structures that are fundamental to a branch of mathematics called quantum algebra.[1] This broad field has many facets, interrelating representation theory, low dimensional topology, Hopf algebras and noncommutative geometry, subfactor theory, higher category theory, quantum field theory, and quantum computation. To get a feeling for the type of objects in this field the reader is reminded of a central gadget, the *Jones polynomial*. This is a knot invariant with a simple combinatorial definition discovered by Jones [Jon87] while studying subfactors. Jones realized the polynomial as a Markov trace on the Iwahori-Hecke algebra of type A – a well known q-deformation of the group algebra of the symmetric group. Later, Witten [Wit89] observed a connection between the Jones polynomial and topological quantum field theories (TQFTs) built out of $SU(2)$ Chern-Simons theory at certain roots of unity. It was later realized that these structures could be mathematically described by a quantum group – in this case a certain q-deformation of the classical group $SL(2)$. Such quantum groups had been previously introduced and studied by Drinfel'd [Dri86] and Jimbo [Jim86], thanks to a connection with solvable models in statistical mechanics. In fact, Jimbo realized that the Hecke algebras are in duality with the quantum groups of type $SL(N)$ in exactly the same way that the symmetric group and $SL(N)$ are in duality via classical Schur-Weyl duality [Wey66]. [2] The ribbon category connected to all

[1] Or *quantum topology* if you prefer.

[2] In a fascinating development from this millenium, Freedman, Larsen and Wang [FLW02] showed that simulating certain of the theories considered by Witten was a "BQP-complete" problem (both universally difficult and doable by a quantum computer). This was later elaborated by Aharanov, Jones and Landau [AJL09] to show that computing an approximation for the Jones polynomial of certain links

of these objects is constructed from the representation category of the Drinfel'd-Jimbo quantum group $U_q(\mathfrak{sl}_2)$, which is a Hopf algebra by definition. In this thesis we are concerned with the "special orthogonal" version of these objects. For instance, the Jones polynomial is replaced by the Kauffman polynomial [Kau90] and the Hecke algebra by the Birman-Murakami-Wenzl algebra [3] [BW89], [Wen90].

Someone who has seen invariant theory, quantum computing, or even classical circuit diagrams, will be familiar with the use of planar diagrams to represent various operations. Tensor categories provide a rigorous framework in which to interpret such diagrams (e.g. consisting of boxes and wires) as morphisms between tensor products of objects. Ribbon categories are a further specialization which allow us to interpret braids and tangles (knotted strands of wires[4]) as morphisms in a way that only depends on the topology of the tangle. We will frequently use the graphical calculus of tensor categories.

We hope to have persuaded the reader that ribbon categories are interesting objects worthy of investigation. To understand ribbon categories one might start with examples. Almost all known ribbon categories come from finite groups or Lie groups through various constructions, which are sometimes complicated but seem to be well understood by specialists. Searching for so called *exotic* ribbon categories that aren't already coming from groups (at least in a known way) is an exciting and difficult challenge. Relatively few are known, such as the Haagerup category and its relatives [BPMS12] and Izumi's examples [Izu01]. If one does not know where to look for new examples then they may refine the search using some invariant. Probably the most fundamental invariant is the *Grothendieck ring* of the category, which encodes how tensor products decompose into simple objects. Having an invariant, one would like to know to what degree it differentiates ribbon categories. We want to answer questions such as: how many distinct

is also BQP-complete.

[3] The BMW algebras are actually "full orthogonal" or "symplectic" versions of the Hecke algebra, depending on 2 parameters. Part of the challenge of this thesis is the absence of an analogue for the Hecke algebra in the special orthogonal situation.

[4] Technically our wires must be framed, oriented links or "ribbons", see [Tur16].

ribbon categories have a particular Grothendieck ring? What information besides the fusion rules (if any) is needed to specify a ribbon category, up to equivalence? This is the *categorification problem* for fusion rings. The answers depend on the particular fusion ring under consideration and are usually hard to answer (see the next section for a survey of some results). There is no known criterion for testing whether an abstract fusion ring admits a categorification. We bypass this problem by starting with a fusion ring that we know already admits at least one categorification, namely a quantum group category. The question then becomes, are there any other ribbon categories with the same fusion rules as the quantum group? Most people expect the answer to be no, at least modulo some "standard modification," and this is what we prove for the fusion rules of $SO(N)$.

Previously, a classification of tensor categories with the fusion rules of $SL(N, \mathbb{C})$ was determined by Kazhdan and Wenzl [KW93]. They find that tensor categories (rigid, semisimple $\mathbb{C}$-linear monoidal categories) with the fusion rules of $SL(N)$ are parametrized (up to monoidal equivalence) by a parameter $q \in \mathbb{C}$, which is either ± 1 or not a root of unity, and an N-th root of unity τ. When $\tau = 1$ the category corresponding to q is monoidally equivalent to $\mathbf{Rep}\ \ SL(N)_q$, the subcategory of $U_q(\mathfrak{sl}_n)$-representations generated by the N-dimensional defining (vector) representation X. It is well known this category is braided, and the braid operator $c_{X,X}$ has eigenvalues q and $-q^{-1}$. The other choices for τ result from twisting the associativity constraints of $\mathbf{Rep}\ SL(N)_q$ by a non-trivial 3-cocycle of $\mathbb{Z}_N$, and they are not braided in general. Kazhdan and Wenzl's strategy is to prove that the family of algebras $\mathrm{End}(X^{\otimes k})$ must be a certain quotient of a Hecke algebra (as is the case for the quantum group categories), and from this the entire category can be reconstructed.

The approach of Tuba and Wenzl to the classification of ribbon categories with the fusion rules of $O(N)$ or $Sp(N)$ is quite similar to that of Kazhdan and Wenzl: starting with a category $\mathcal{C}$ with the given fusion rule, analyze the family of algebras $\mathrm{End}(X^{\otimes k})$ and show they must be a certain quotient of the BMW algebra, another well known quotient of the braid group. An important part of the analysis is showing that the braid elements generate the whole endomorphism algebra. Tuba and Wenzl assume (as we do) that the category is braided to begin with. It is an interesting open problem to classify tensor categories of type BCD without the ribbon assumption, as was achieved by Kazhdan and Wenzl in type A.

Considering there is already a classification for ribbon categories with fusion rules coming from the full orthogonal groups one may wonder why there is not an immediate classification for $SO(N)$ type categories. It turns out this is the case for N odd. Indeed, the restriction rules from $O(2n{+}1)$ to $SO(2n{+}1)$, namely that every irrep stays irreducible, imply that the endomorphism algebras for the special orthogonal group are essentially the same as for the orthogonal group. Since the Tuba-Wenzl strategy is to study these endomorphism algebras, their classification method also applies for $SO(N)$ categories with N odd. We provide a formal proof of the classification in Sec. 4 as a direct consequence of Tuba and Wenzl's result for $O(2n+1)$ categories.

For $N = 2n$ it is a different story since even for quantum group categories, the braid group fails to generate the algebras $\mathrm{End}(X^{\otimes k})$ for $k \geq n$. In addition to the braid elements, one needs a minimal idempotent in $\mathrm{End}(X^{\otimes n})$ corresponding to the first $O(2n)$ irrep whose restriction to $SO(2n)$ is no longer irreducible. Hence there is no hope to recognize these as quotients of the braid group in the normal way. A somewhat similar situation is also confronted in recent work by Martirosyan and Wenzl on G_2 type categories ([MW17], [MW20]). Here the centralizer algebras are generated by braid elements, but the resulting representations of the braid group are not yet well understood like in the BMW and Hecke algebra cases. Their strategy, and ours, is to show more or less directly that the braid operators are uniquely determined by the fusion rules and one of the eigenvalues q of the braid on the 2nd tensor power. We specify a basis (the *path basis*) for modules of $\mathrm{End}(X^{\otimes k})$ and show that the matrix entries of the braid generators must be specific rational functions of q, and so are uniquely determined by q. In the language of Martirosyan and Wenzl we are proving the *path rigidity* for the braid representations attached to the fusion rules of $SO(N)$.

Showing that in the path basis the braid matrices are uniquely determined by q forms the main novel contribution of this thesis. Here we use an almost purely combinatorial approach based on the similarity of path bases with *Young's seminormal basis* for the irreps of the symmetric group [Gar03]. This goes back to Vershik and Okounkov's method of constructing Young's seminormal form using eigenvalues of *Jucys-Murphy* elements [VYO05].

Besides the actual result for $SO(N)$, we hope that the more flexible approach of path rigidity used in this thesis can be be modified to achieve classification results for other fusion rules coming from quantum groups, e.g. coming from exceptional Lie groups. Unfortunately, our results do not extend to these situations since we rely heavily on the fact that $X \otimes X$ decomposes into exactly three irreps–a condition not usually satisfied for the exceptional types. [5] We hope our classification results may be leveraged to work for fusion rules which include the spin representations, i.e. coming from the Lie algebra $\mathfrak{so}_N$. This problem is under investigation by Wenzl. Aside from other classification projects we hope our results could lead to a planar algebra presentation of $SO(N)$-type categories in a similar way that the BMW-algebras present the orthogonal and symplectic categories. This could be interpreted as a "fundamental theorem of quantum $SO(N)$ invariant theory" in comparison to classical FFTs of invariant theory [GW]. Such a presentation would be immediately useful for the important task of classifying autoequivalences of type D categories, as was recently done by Edie-Michell [EM20a] for Lie types A, B, C and G.

[5] We thank Wenzl for pointing out that certain simple objects in an E_6 category do have this property, but they are not self-dual.

1.1 Related work

Though this work is closest in spirit to the work of Wenzl and collaborators, there are many other nice classification results. In particular, much work has focused on the case of fusion categories, which are distinguished by having only finitely many isotypes. A famous result known as *Ocneanu rigidity* (see [ENO05]) says there are only finitely many equivalence classes of fusion categories with a fixed Grothendieck ring. However, a given fusion ring may not have any categorifications at all [Ost02] and simply deciding whether a given fusion ring admits any categorification can be very hard.

Ostrik ([Ost02], [Ost15]) and Larson [Lar14] classify categories with $2, 3$ and 4 simple objects respectively (with various assumptions), and pushing this classification to larger numbers of simple objects is an active area of research. In particular, *unitary modular* tensor categories (a special case of ribbon fusion categories) with ≤ 4 simple objects are classified by Rowell, Stong and Wang [RSW09].

One way to classify tensor categories is by the dimension of a generating object. This is motivated by the close connection between unitary fusion categories generated by an object X, finite index subfactors of the hyperfinite II_1 factor ([Jon83], [Müg03]) and subfactor planar algebras ([Jon99], [MPS10]). The categorical dimension of X is related to the index of the subfactor. Hence famous results on classification of subfactors by index may be interpreted as classification of singly-generated unitary fusion categories according to the dimension of a generating object. Recently Edie-Michell has obtained results for unitary fusion categories generated by objects of small dimension [EM19], [EM20b].

Another approach to classifying tensor categories, related to planar algebras, is via generating morphsims and relations. For instance, Liu [Liu16] has classified tensor categories that are generated by a morphism in $\mathrm{End}(X^{\otimes 2})$ satisfying a "Yang-Baxter relation" generalizing the braid relation. The BMW categories fall into this classification.

It is plausible that Liu's work can be used to classify BCD tensor categories without the braiding assumption. In a similar direction Morrison, Peters and Snyder [MPS16] achieve some classification results for categories generated by a *trivalent vertex*, which is a morphism $X \to X \otimes X$. In particular their results apply to G_2 categories, together with the assumption that the category is generated by the trivalent vertex. Martirosyan and Wenzl have shown that it suffices to assume that the image of the braid group generates $\mathrm{End}(X^{\otimes 3})$, and it is conjectured that this assumption is always true. Etingof and Ostrik [EO18] use their results to achieve a classification of $SO(3)$-type tensor catgories without any braiding assumption and so this result is stronger than the result we present for $SO(3)$-type categories.

In another direction, the portion of this thesis concerned with the study of braid matrices with respect to a path basis is closely related to a large body of work in combinatorial representation theory, in particular the study of *seminormal representations.* Much of this can be described as an outgrowth of Schur-Weyl duality to various classical and quantum algebraic structures. Some of the braid matrices we encounter (namely the actions on the "new stuff") are famous representations of the Hecke algebra first described by Hoefsmit [Hoe74] and used by Wenzl [Wen88] to construct subfactors. Nazarov [Naz90] used this approach to matrix representations for the Brauer algebra appearing in the classical setting of **Rep** $O(N)$. The work of Leduc and Ram [LR97] uses the Jucys-Murphy approach to write down braid matrices in path bases for quantum groups of type ABC. This is rather similar to what we do (since we are trying to show that the braid representations must agree with the ones coming from quantum groups), but Leduc and Ram do not address the subtleties with $SO(2n)$ and we do not provide explicit formulas for the braid matrices in all cases (only proving uniqueness). Martirosyan and Wenzl [MW20] also use a Jucys-Murphy approach, based on techniques developed by Wenzl for type E_N [Wen03] . Cyclotomic algebras provide a built-in approach to Jucys-Murphy theory, the most relevant example being the *cyclotomic affine BMW algebras* [Goo12]. The crucial facts about AB_2 representations established in Sec. 6.4 may be regarded as simple facts about the related 2-strand Ariki-Koike algebra [AK94]. However almost none of this literature directly addresses $SO(2n)$-type endomorphism algebras since in the $SO(2n)$ case the algebras are not generated by braid morphisms. Even in the classical case I do not believe it is completely understood what should be the analogue of the Brauer algebra for $SO(2n)$. In his original paper [Bra37], Brauer proposed a diagram algebra for $SO(2n)$, but the algebra is nonassociative. This algebra was further studied by Grood [Gro99].

1.2 Outline of thesis

We start with preliminary notation and background on ribbon categories in Sec. 2. In Sec. 2.2 we discuss conditions under which a self-dual object is symmetrically self dual. We think the results in this section are new to the literature, though they are a result of an argument due to Turaev. We give extra background on how to "twist"[6] $\mathbb{Z}_2$-graded ribbon categories with cohomological data as the construction plays an important role in the classification results (the modifications here account for the "standard normalization" in the statement of Thm. 1.0.1). We refer to this as the *cocycle construction* throughout.

In Sec. 3 we describe in detail the tensor product rules for $SO(N)$ (which has infinitely many simple objects) and associated fusion rings (which have only finitely many simple objects). This is standard material in the literature, but we provide a lot of detail so it is clear which tensor product rules we are considering. Readers who are not already acquainted or otherwise not interested in the Lie theoretic parametrization of irreps by highest weight can skip Sec. 3.1 and accept the fusion rules as a "black box". [7] Most of the thesis uses a combinatorial approach, namely a parametrization of simples by Young

[6]This is common terminology that we try to avoid in the thesis since we often give that name to the twist morhpism on an object in a ribbon category.

[7]Several computer algebra systems have built in algorithms for computing tensor product multiplicities, e.g. Sage and LiE.

diagram (or plus/minus variants in the N even case), as explained in Sec. 3.2. We call the simple isotypes $SO(N)$ *shapes*. The most important fusion rule will be that of tensoring with an object X corresponding to the *fundamental* or *defining* N-dimensional irrep of $SO(N)$ which is labeled by the 1-box Young diagram [1]. The "generic rule" (meaning when N is very large) says that whenever you tensor X with a Young diagram λ, you get a (multiplicity free) direct sum of simples corresponding to the Young diagrams obtained by adding or subtracting a box from λ. This generic rule has to be modified once the Young diagrams are large enough to have $N/2$ rows.

In the next parts of Sec. 3 we discuss the existence of categories with $SO(N)$ and $O(N)$ fusion rules, coming from quantum groups and Turaev-Wenzl skein theory, respectively. This discussion is critical for the classification of $SO(2n+1)$ type categories. For the $SO(2n)$ classification it is mostly unnecessary to know anything about the examples. However, there is a single point where the quantum group categories are used, namely in determining the dimensions of objects in an abstract $SO(2n)$ type category.

We proceed in Sec. 3 to apply the "twisting" of Sec. 2 to describe a normalization for the categories we consider. A nice argument by Morrison, Peters and Snyder [MPS11] shows that if X is a self-dual object whose tensor square splits into 3 distinct simples, then the framed link invariant associated to X must be the *Dubrovnik* or *Kauffman polynomial* [Kau90]. Furthermore, in a $\mathbb{Z}_2$ graded category one can switch between Dubrovnik and Kauffman types with a cocycle twist. We may thus "normalize" any category with $SO(2n)$ type fusion rules by requiring the link invariant to be Dubrovnik. With this choice, the eigenvalues for the braid element $c_{X,X}$ (an element of the 3-dimensional commutative algebra $\mathrm{End}(X^{\otimes 2})$) must be of the form $(q, -q^{-1}, r^{-1})$.

In Sec. 4 we precisely state the Tuba-Wenzl classification for $O(2n+1)$ type categories and show how this implies a classification of $SO(2n+1)$ type categories.

In Sec. 5 we lay the categorical framework that justifies reducing the study of a ribbon category to the study of the family of algebras $\mathrm{End}(X^{\otimes k})$. This family, together with morphisms between them coming from the tensor product, constitute the *diagonal* of the category. The idea that the category can be understood through a study of the centralizer algebras arguably goes back to Schur and Weyl. In particular, Weyl famously used *Schur duality* to construct the irreducible $SL(N)$ representations by identifying the algebras $\mathrm{End}(X^{\otimes k})$ as quotients of the group algebra of the symmetric group [Wey66]. We show that when a tensor category is $\mathbb{Z}_2$ graded and generated by a self-dual object X, the diagonal determines the category up to a cocycle modification. These results and methods are very similar to [TW05], Sec. 4. [8] Our results are slightly stronger because they do not use any assumption of braiding.

The next step is to study in detail the algebras $\mathrm{End}(X^{\otimes k})$ and the tensor product maps between them. In the proof of the main classification result, Thm. 8.0.1, we show how the diagonal is determined by the matrices for braid elements in a certain basis, the *path basis.* These correspond to minimal idempotents in the centralizer algebras, called *path idempotents.* These idempotents are indexed by paths through the *Bratteli diagram*, which encodes the rule for tensoring with X. In a ribbon category we are provided with special central elements in $\mathrm{End}(X^{\otimes k})$, called *full twists*, which are certain braids closely related to the twist on $X^{\otimes k}$ coming from the ribbon category structure. From these we define the *Jucys-Murphy* elements, which act diagonally in a path basis and satisfy simple relations with the braid generators. This is all detailed in Sec. 6 and consists of fairly well known material.

Sec. 7 is devoted to the goal of showing that the braid matrices are uniquely determined in the path basis by the braid eigenvalue q. This constitutes the most substantial novel contribution of this thesis. A subtask is to show the *restriction of parameters*, which states that the eigenvalues for the braid element $c_{X,X}$ must in fact be of the form $(q, -q^{-1}, q^{-(2n-1)})$, and that q must be a certain root of unity depending on the fusion

[8] The "tower of centralizer algebras" approach is broadly related to the *standard invariant* in subfactor theory.

rules (or not a root of unity if we are considering the generic fusion rules of $SO(2n)$ with infinitely many simples). The restriction of parameters is achieved in tandem with computing the eigenvalues of the JM elements in terms of q. The key is to focus on Young diagrams that are hook shaped. As with many arguments in this area, one proceeds by induction, propagating information through the Bratteli diagram.

Once the restriction of parameters and JM eigenvalue computation is complete, we continue in Sec. 7 to the uniqueness of braid representations. We advance by brute force. First we specify a path basis more completely. In general, the path basis is defined only up to rescalings, or in other words conjugation by a diagonal matrix, so we eliminate this ambiguity. We then show inductively that the entries of the braid elements can (in principle) be written as a rational expression in q. Except for a few cases, we do not write down the explicit formulas, and satisfy ourselves with inductive formulas.

In Sec. 8 we apply the results of the previous section to prove the classification result for $SO(2n)$ type categories. We conclude the thesis in Sec. 9 with a discussion of related unsolved problems and potential applications for our results.

2

Categorical preliminaries

2.1 Ribbon categories

This thesis is concerned with classifying $\mathbb{C}$-linear semisimple ribbon categories with fusion rules coming from $SO(N)$. In this section we review the terminology required to make sense of this problem. The definitions are standard and can be found in a textbook on tensor categories, e.g. [EGNO15] or [Kas95]. It is helpful to keep in mind the central examples **Rep** G (or more generally **Rep** H for a semisimple Hopf algebra H) as easy examples of tensor categories. **Rep** G will always denote either the finite-dimensional $\mathbb{C}$-representations of a finite group G or the finite-dimensional $\mathbb{C}$-representations of a classical Lie group G. However, from the quantum algebra perspective we are much more interested in similar categories that do not have a symmetric braiding, since symmetric braidings produce rather trivial link invariants. Quantum groups provide examples for these but are less familiar.

A category $\mathcal{C}$ consists of a set of objects, and for any two objects $X, Y \in \mathcal{C}$, a set of morphisms denoted $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ called the hom-set between X and Y. When $X = Y$ we use the special notation $\mathrm{End}_{\mathcal{C}}(X)$ and refer to this hom-set as the *endomorphism ring* of X.[1] We sometimes omit the subscript when $\mathcal{C}$ is established by context. Every

[1] Although at this level of generality $\mathrm{End}_{\mathcal{C}}(X)$ is merely a monoid, in a pre-additive category it is a ring.

object X has an identity morphism $1_X \in \mathrm{End}_{\mathcal{C}}(X)$. A category is $\mathbb{C}$-*linear* if all of its hom-sets are vector spaces over $\mathbb{C}$ and the composition of morphisms in the category is linear in each argument. In particular the rings $\mathrm{End}_{\mathcal{C}}(X)$ are $\mathbb{C}$-algebras. A *simple object* in a $\mathbb{C}$-linear category is one with no non-trivial subobjects. If $\mathcal{C}$ is an abelian category (so it contains finite direct sums, has a zero element, etc.) and every object in $\mathcal{C}$ is isomorphic to a finite direct sum of simple objects then $\mathcal{C}$ is a *semisimple* category. The set of simple objects modulo isomorphism are the *simple isotypes* of $\mathcal{C}$, denoted $\Gamma(\mathcal{C})$. Semisimplicity is an important assumption which tells us that objects are determined (up to isomorphism) by the multiplicities with which each simple isotype appears. In a $\mathbb{C}$-linear semisimple category, Schur's Lemma is an if and only if: an object X is simple if and only if $\mathrm{End}_{\mathcal{C}}(X) = \mathbb{C}$. Moreover, for any $X \in \mathcal{C}$ (not necessarily simple) the algebra $\mathrm{End}_{\mathcal{C}}(X)$ is a finite-dimensional semisimple $\mathbb{C}$-algebra. In other words a direct product of matrix rings with entries in $\mathbb{C}$. The simple two-sided ideals (i.e. full matrix blocks) of $\mathrm{End}_{\mathcal{C}}(X)$ correspond to the simple isotypes appearing in X.

The *Grothendieck semigroup* of a semisimple $\mathbb{C}$-linear category $\mathcal{C}$ is the free abelian semigroup generated by the set Γ of isomorphism types of simple objects. Hence elements of the Grothendieck semigroup are finite linear combinations

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma.$$

where $a_\gamma \in \mathbb{N}$ and $\gamma \in \Gamma$ is a simple isotype. We denote the Grothendieck semigroup by $\mathrm{Gr}^+(\mathcal{C})$. We reserve the simpler notation $\mathrm{Gr}(\mathcal{C})$ for the *Grothendieck group* of $\mathcal{C}$, which consists of all $\mathbb{Z}$-linear combinations of elements of Γ. The Grothendieck group is more frequently encountered than the semigroup but is slightly subtler as it should be considered a *based* object, meaning a free abelian group together with a fixed basis (in this case corresponding to the simple objects). There is a map from the simple objects of $\mathcal{C}$ to $\mathrm{Gr}^+(\mathcal{C})$, denoted $X \mapsto [X]$, which sends a simple object to its isotype. Since $\mathcal{C}$ is semisimple, this extends to a map defined on all objects of $\mathcal{C}$ which satisfies

$$[Y \oplus Y'] = [Y] + [Y']$$

for all $Y, Y' \in \mathcal{C}$. Note that $[X] = [Y]$ if and only if X is isomorphic to Y, so the elements of the Grothendieck semigroup parametrize the isomorphism types of all objects in $\mathcal{C}$. Any additive functor $F : \mathcal{C} \to \mathcal{D}$ between semisimple categories induces a morphism of Grothendieck semigroups, which can be described by a matrix with non-negative integer coefficients (describing how F of a simple object in $\mathcal{C}$ decomposes in $\mathcal{D}$). The functor F is determined up to natural isomorphism by this matrix.

Given two $\mathbb{C}$-linear semisimple categories $\mathcal{C}$ and $\mathcal{D}$ we can construct more. The direct product category $\mathcal{C} \times \mathcal{D}$ is defined in the usual way. This category is $\mathbb{C}$-linear but not semisimple (it is not closed under direct sums), but taking the direct sum completion we get a semisimple category whose Grothendieck semigroup is $\mathrm{Gr}(\mathcal{C}) \oplus \mathrm{Gr}(\mathcal{D})$. This direct sum completion (or matrix construction) is a general procedure to manually add in direct sums to $\mathcal{C}$ (see [TW05], [MPS10]). [2]

It turns out that it is more useful to have a way to combine $\mathcal{C}$ and $\mathcal{D}$ in a way that is reflected as the tensor product on the level of Grothendieck semigroups, to produce a category whose simple elements are indexed by the product $\Gamma(\mathcal{C}) \times \Gamma(\mathcal{D})$. Consider the category $\mathcal{C} \otimes \mathcal{D}$ with the same objects as $\mathcal{C} \times \mathcal{D}$ but hom-sets are given by $\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}} := \mathrm{Hom}_{\mathcal{C}}(X) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(Y)$. The *Deligne product* $\mathcal{C} \boxtimes \mathcal{D}$ is defined to be the direct sum completion of $\mathcal{C} \otimes \mathcal{D}$. As before, the Deligne product is semisimple if $\mathcal{C}$ and $\mathcal{D}$ are. There is a natural functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ which is $\mathbb{C}$-linear in both variables, and the image of an object (X, Y) is denoted $X \boxtimes Y$. The simple isotypes of $\mathcal{C} \boxtimes \mathcal{D}$ are all of the form

[2] To summarize the construction, one takes as objects all possible tuples of objects in $\mathcal{C}$, and a morphism between a n-tuple $(X_1, \ldots, X_n)$ and an m-tuple $(Y_1, \ldots Y_m)$ is a size $m \times n$ matrix whose (i, j)-th entry belongs to $\mathrm{Hom}_{\mathcal{C}}(X_j, Y_i)$. Composition of morphisms is defined via matrix multiplication (and the $\mathbb{C}$-linear structure of hom-sets).

$X \boxtimes Y$ where X and Y are simples in $\mathcal{C}$ and $\mathcal{D}$. Hence the Grothendieck semigroup of the product is $\mathrm{Gr}^+(\mathcal{C} \boxtimes \mathcal{D}) = \mathrm{Gr}^+(\mathcal{C}) \otimes_{\mathbb{N}} \mathrm{Gr}^+(\mathcal{D})$.

A *monoidal category* is a category $\mathcal{C}$ together with a (functorial) tensor product operation, denoted $X \otimes Y$, which is associative and unital up to a fixed natural isomorphism. The unit object is typically denoted $\mathbf{1}$ and satisfies $\mathbf{1} \otimes X \cong X \cong X \otimes \mathbf{1}$ for all $X \in \mathcal{C}$. To be precise, the category is equipped with fixed natural isomorphisms (the "unitors") which witness the above isomorphisms. Similarly, part of the data of $\mathcal{C}$ are *associators*, which are isomorphisms

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

that depend naturally on X, Y and Z, and satisfy certain coherence conditions (the "pentagon" and "triangle" axioms). For instance, the pentagon axiom says that the two ways of obtaining an isomorphism of $(((X \otimes) Y) \otimes Z) \otimes W)$ with $(X \otimes (Y \otimes (Z \otimes W)))$ using associators result in the same map. MacLane's coherence theorem [ML98] asserts that there is exactly one map between different parenthesizations of any number of objects which can be made using associators and unitors. As a result, in a monoidal category we can refer to morphisms between unparenthesized objects without ambiguity. This is critical for the graphical calculus used later. A monoidal category is *strict* if one has $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ on the nose and the associators are identity maps. Whether or not the category is strict we always assume w.l.o.g. that the unitor is strict, i.e. $X \otimes \mathbf{1} = \mathbf{1} \otimes X = X$ on the nose. The coherence theorem implies any monoidal category is (monoidally) equivalent to a strict one, called a *strictification* of $\mathcal{C}$. We will see this in more detail in Sec. 5.

A *monoidal functor* F between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a functor for which $F(X) \otimes F(Y)$ is naturally isomorphic to $F(X \otimes Y)$. More precisely, part of the data of F is a natural isomorphism which witnesses this, often denoted $F^2_{X,Y} : F(X) \otimes F(Y) \to$

$F(X \otimes Y)$. A *monoidal natural isomorphism* η between two monoidal functors F and G does not come with extra data (it is just a natural isomorphism) but it must satisfy

$$\eta_{X\otimes Y} \circ F^2_{X,Y} = G^2_{X,Y} \circ (\eta_X \otimes \eta_Y).$$

If $\mathcal{C}$ is a semisimple monoidal category then $\mathrm{Gr}^+(\mathcal{C})$ inherits a multiplication making it a semiring. The category $\mathcal{C}$ is said to *categorify* the semiring R if $\mathrm{Gr}^+(\mathcal{C}) \cong R$. The *Grothendieck ring* of $\mathcal{C}$ is $\mathrm{Gr}(\mathcal{C})$ with the inherited ring structure. In order not to forget the semiring we started with, $\mathrm{Gr}(\mathcal{C})$ should be considered a $\mathbb{Z}$*-based ring*, i.e. a ring whose underlying abelian group is free, with a specified basis satisfying a positivity condition (that the product of two basis elements is a positive $\mathbb{Z}$-combination in that basis). We sometimes refer to elements of this basis as *simple elements.* A morphism of $\mathbb{Z}$-based rings is required to send a simple element to a non-negative combination of simple elements. Any monoidal functor between semisimple monoidal categories induces a map of $\mathbb{Z}$-based rings. We say such a functor categorifies the corresponding ring map. As usual, the categorification of a given ring map is typically far from unique, if it exists.

If $\mathcal{C}$ and $\mathcal{D}$ are semisimple and monoidal then $\mathcal{C} \boxtimes \mathcal{D}$ is also semisimple monoidal and its Grothendieck ring is given by

$$\mathrm{Gr}(\mathcal{C} \boxtimes \mathcal{D}) \cong \mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathcal{D}) \tag{2.1}$$

and the simple objects of $\mathcal{C} \boxtimes \mathcal{D}$ are all of the form $X \boxtimes Y$ with X, Y simple. For instance, if $\mathcal{C} = \mathbf{Rep}\ G$ and $\mathcal{D} = \mathbf{Rep}\ H$ are the category of representations of finite groups G and H, then $\mathcal{C} \boxtimes \mathcal{D} \cong \mathbf{Rep}\ (G \times H)$.

A $\mathbb{C}$-linear monoidal category is *left rigid* if for every object X there exists an object X^* and *duality morphisms*

$$d_X : X^* \otimes X \to \mathbf{1}$$

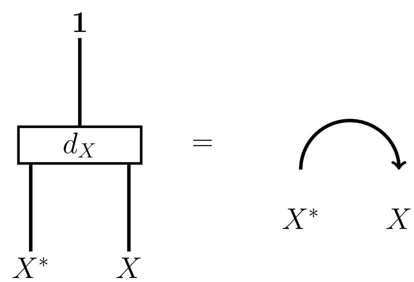
$$i_X : \mathbf{1} \to X \otimes X^*$$

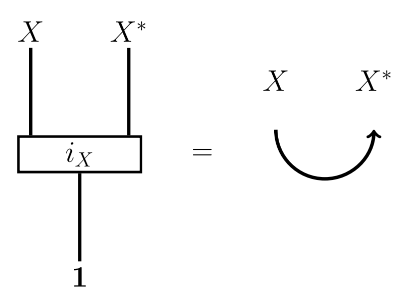
satisfying the S-bend identities:

$$(1_X \otimes d_X) \circ (i_X \otimes 1_X) = 1_X$$

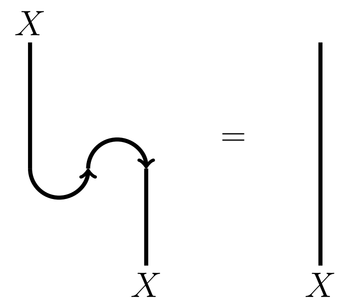
$$(d_X \otimes 1_{X^*}) \circ (1_{X^*} \otimes i_X) = 1_{X^*}.$$

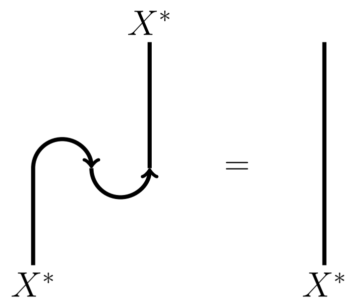
To explain why these are S-bend identities, we introduce the standard graphical notation for these duality morphisms:





We remark on several conventions. First, in our graphical calculus we read from bottom to the top, i.e. the labels on the bottom of the diagram describe the source of the morphism, and the labels on the top give the target of the morphism. By the coherence theorem we may erase the strand labeled by the trivial object **1** without any ambiguity. With these notations established, the S-bend identities read





X^* is called the *left dual* to X. There is a similar definition for right rigid, with the notation ${}^*X$ for the right dual. Such an object, together with the duality morphisms, is unique up to unique isomorphism. The category is *rigid* if both left and right duals exist. In a left rigid category we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*) \tag{2.2}$$

$$\mathrm{Hom}_{\mathcal{C}}(X, Y \otimes Z) \cong \mathrm{Hom}_{\mathcal{C}}(Y^* \otimes X, Z) \tag{2.3}$$

(and similar isomorphisms for right duals). In a (left) rigid monoidal category we may declare $(X \otimes Y)^* = Y^* \otimes X^*$ with the duality morphisms defined in the appropriate manner.

A *tensor category* is a $\mathbb{C}$-linear semisimple rigid monoidal category. A *Serre subcategory* ([EGNO15]) of a semisimple category is a full subcategory obtained as all possible direct sums made out of some subset of the simple isotypes. A *tensor subcategory* means a Serre subcategory closed under tensor product. Tensor subcategories are in bijection with subsets of simple isotypes closed under tensor product, and consequently can be enumerated by the fusion rules alone. We say a tensor category is *generated by the object* X if every simple object arises as a subobject of $X^{\otimes k}$, for some $k \geq 0$. A category with a simple generating object is called *singly-generated.*

We say $\mathcal{C}$ is G*-graded* (where G is a group) if $\mathcal{C}$ has a direct sum decomposition into full nonzero Serre subcategories $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ such that $\mathcal{C}_g \otimes \mathcal{C}_h \subset \mathcal{C}_{gh}$. The category is G-graded if and only if its Grothendieck ring is G-graded in an appropriate sense. Every category admits a *universal grading* by its *universal grading group* $U(\mathcal{C})$:

$$\mathcal{C} \cong \bigoplus_{g \in U(\mathcal{C})} \mathcal{C}_g.$$

For example, if H is a a finite or classical group then $U(\mathbf{Rep}\ (H))$ is isomorphic to $\widehat{Z(H)}$, the group of characters of the center of H. See Sec. 2.3 for the definition and more discussion of $U(\mathcal{C})$. We will only deal with the cases when $U(\mathcal{C}) = \mathbb{Z}_2$ or $U(\mathcal{C})$ is the trivial group.

A category is *pivotal* if there exists a natural isomorphism $\phi_X : X \to X^{**}$. This is equivalent to saying that for every object X, in addition to the right duality maps i_X and d_X, left duality maps $i'_X : \mathbf{1} \to X^* \otimes X$ and $d'_X : X \otimes X^* \to \mathbf{1}$ can be chosen such that the left and right duality functors ($^*-$ and $-^*$) coincide (see [TV17], Sec. 1.7).

In a pivotal category one can define the left and right categorical (or quantum) traces which are linear functionals $\mathrm{End}(X) \to \mathbb{C}$:

$$\mathrm{Tr}^l_X(f) = d'_X \circ (f \otimes \mathbf{1}) \circ i_X$$

$$\mathrm{Tr}^r_X(f) = d_X \circ (\mathbf{1} \otimes f) \circ i'_X.$$

A pivotal category is *spherical* if left and right traces coincide. In a spherical category one has a well defined notion of *categorical* (or *quantum*) *dimension* of an object X:

$$\dim_{\mathcal{C}}(X) = \mathrm{Tr}_X(\mathbf{1}) = d'_X \circ i_X = d_X \circ i'_X.$$

The dimension only depends on the isotype of X, so it induces a map on the Grothendieck ring $\dim_{\mathcal{C}} : \mathrm{Gr}(\mathcal{C}) \to \mathbb{C}$. We will often use the important fact that this map is a character of $\mathrm{Gr}(\mathcal{C})$, i.e.

$$\dim_{\mathcal{C}}(X \oplus Y) = \dim_{\mathcal{C}}(X) + \dim_{\mathcal{C}}(Y)$$
$$\dim_{\mathcal{C}}(X \otimes Y) = \dim_{\mathcal{C}}(X) \dim_{\mathcal{C}}(Y).$$

In a rigid semisimple tensor category duality induces an involution $-^* : \mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ that sends simple elements to simple elements. This map is determined by the tensor product rules, since for simple elements μ and λ, $\mu = \lambda^*$ if and only if $\mathbf{1}$ appears in $\mu \otimes \lambda$.

A $\mathbb{C}$-linear monoidal category is *braided* if there exists a natural isomorphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying the hexagon identity, namely that

$$c_{X \otimes Y, Z} = (\mathbf{1} \otimes c_{Y,Z}) \circ (c_{X,Z} \otimes \mathbf{1})$$

and similarly in the second coordinate (note these axioms require associators in the non-strict setting). The Grothendieck ring of a braided category is commutative.

A braided category is *ribbon* if it is right rigid and there is a natural isomorphism

$$\theta_X : X \to X$$

satisfying the equation

$$c_{Y,X}c_{X,Y}(\theta_X \otimes \theta_Y) = \theta_{X\otimes Y} \tag{2.4}$$

as well as the compatibility conditions

$$\theta_X^* = \theta_{X^*}$$

and

$$(\theta_X \otimes \mathbf{1}) \circ i_X = (\mathbf{1} \otimes \theta_{X^*}) \circ i_X.$$

A braided category is ribbon if and only if it is also spherical and it is sometimes useful to consider the braiding and spherical structures rather than the twist. We say that two ribbon categories are *ribbon equivalent* if there is a monoidal equivalence between them which is compatible with the braiding and spherical structures.

Any morphism in a ribbon category can be represented by the graphical calculus (see for example [Kas95], [TV17]), which consists of planar diagrams composed of coupons and oriented ribbons[3] whose ends either attach to a coupon or the bottom or top of the diagram. The ribbons are labeled with objects of $\mathcal{C}$, the coupons are labeled by morphisms of the category, and the orientation determines whether cups and caps are left or right duality maps. Diagrams are always read bottom to top. They are invariant under any isotopies of the ribbon/coupon configuration which keep the boundary of the diagram fixed. We will usually deal with symmetrically self dual objects (see the discussion in the next section), which will allow us to employ unoriented diagrams. Any ribbon can be "flattened" by taking the *blackboard framing* [4], which allow us to represent ribbons by (unframed) tangle diagrams invariant only up to regular isotopy (i.e. only invariant under Reidemeister II and III). We use this convention throughout since ribbons are harder to

[3]A ribbon is a slight thickening of a curve to a 2-dimensional band. Thinking of our diagrams in 3-space this corresponds to a choice of normal vector at every point on the curve, which is called a *framing*.

[4]The blackboard framing corresponds to a normal vector lying in the plane of the diagram.

draw, but be warned – these diagrams are not invariant under Reidemeister I.

In a left rigid ribbon category we can define morphisms

$$i'_X = (\mathbf{1} \otimes \theta_X) \circ c_{X,X^*} \circ i_X : \mathbf{1} \to X^* \otimes X \tag{2.5}$$

$$d'_X = d_X \circ c_{X,X^*} \circ (\theta_X \otimes \mathbf{1}) : X \otimes X^* \to \mathbf{1}. \tag{2.6}$$

This turns X^* into a right dual for X as well as a left dual, so the category is rigid. By direct computation one can check that the left and right duality functors coincide, so the left duality maps above define a pivotal structure. In fact it is spherical, so we may write $\mathrm{Tr}_X(f)$ for the left (or right) trace of $f \in \mathrm{End}_{\mathcal{C}}(X)$. Given any $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ we have

$$\mathrm{Tr}(fg) = \mathrm{Tr}(gf).$$

In particular, if p_λ is a minimal idempotent in a $\mathrm{End}_{\mathcal{C}}(X)$ with image isomorphic to V_λ then

$$\mathrm{Tr}(p_\lambda) = \dim_{\mathcal{C}}(V_\lambda).$$

This follows since semisimplicity ensures the existence of $a \in \mathrm{Hom}(V_\lambda, X)$ and $b \in \mathrm{Hom}(X, V_\lambda)$ such that $ba = \mathbf{1}$ and $ab = p_\lambda$.

If $\mathcal{C}$ is a semisimple spherical category then $\dim_{\mathcal{C}}(X) \neq 0$ whenever X is a simple object of $\mathcal{C}$. Consequently for any object X the quantum trace $\mathrm{Tr}_X(-)$ is non-degenerate.

The *normalized trace* of $f \in \mathrm{End}_{\mathcal{C}}(X)$ is defined by

$$\mathrm{tr}_X(f) = \frac{1}{\dim_{\mathcal{C}} X} \mathrm{Tr}_X(f).$$

Using the normalized trace the inclusion $\mathrm{End}_{\mathcal{C}}(X) \xrightarrow{-\otimes 1} \mathrm{End}(X \otimes Y)$ is trace preserving. The non-degeneracy of the trace allows us to define the *conditional expectation* (or *partial* *trace*) $\epsilon_Y : \mathrm{End}(X \otimes Y) \to \mathrm{End}(X)$, characterized by

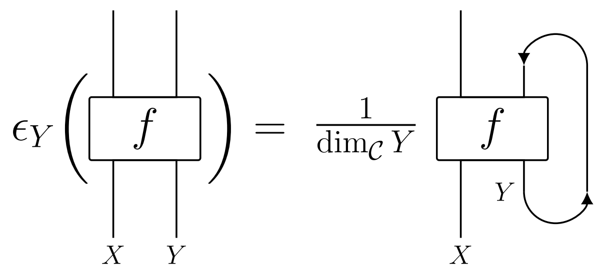
$$\mathrm{tr}_X(\epsilon_Y(f)g) = \mathrm{tr}_{X\otimes Y}(f(g \otimes 1)) \tag{2.7}$$

for all $f \in \mathrm{End}(X \otimes Y)$, $g \in \mathrm{End}(X)$. We can write an explicit formula for $\epsilon_Y(f)$ using left duality maps as follows.

Lemma 2.1.1.

$$\epsilon_Y(f) = \frac{1}{\dim_{\mathcal{C}} Y}(1 \otimes d'_Y)(f \otimes 1)(1 \otimes i_Y) \tag{2.8}$$

Hence we can express ϵ_k in the graphical calculus by



If X is simple then for $f \in \mathrm{End}(X \otimes Y)$ we have

$$\epsilon_Y(f) = \mathrm{tr}_{X\otimes Y}(f)1_X \tag{2.9}$$

by taking traces. This implies that the conditional expectations satisfy what is known as the *Markov property*:

Lemma 2.1.2. *Suppose X is simple and Y and Z are arbitrary objects. Let $f \in End(Z \otimes X)$ and $m \in End(X \otimes Y)$. Then*

$$\epsilon_Y((f \otimes 1)(1 \otimes m)) = tr_{X\otimes Y}(m)f. \tag{2.10}$$

Both the trace and normalized trace are multiplicative over tensor product, e.g.

$$\mathrm{tr}_{X\otimes Y}(f\otimes g) = \mathrm{tr}_X(f)\mathrm{tr}_Y(g). \tag{2.11}$$

2.2 The universal grading group and spherical structures

Here we settle some questions regarding whether a self-dual generating object X is *symmetrically self dual* or not. For such an object the left and right duality morphisms agree, and consequently one may consider any wires labeled by that object in the graphical calculus to be unoriented. Our main results follow from an argument of Turaev [Tur16] but seem to be new. We thank Sachin Valera for questioning a previous mistake by the author, and for bringing to attention Wang's conjecture, mentioned below.

We recall some definitions (see [EGNO15] Secs. 3.6 and 4.14). Let $\mathcal{C}$ be a semisimple tensor category. The *universal grading group* of $\mathcal{C}$ is the group $U(\mathcal{C})$ generated by symbols $\deg(X)$ for every simple object X, subject to relations $\deg(X)\deg(Y) = \deg(Z)$ if Z appears in $X \otimes Y$. For convenience in the formulas below we sometimes write $x = \deg(X), y = \deg(Y)$, etc.The universal grading group depends only on the fusion rules of $\mathcal{C}$. The map $X \to \deg(X) \in U(\mathcal{C})$ defined on simple objects yields a grading of $\mathcal{C}$:

$$\mathcal{C} = \bigoplus_{g\in U(\mathcal{C})} \mathcal{C}_g$$

and every other grading comes from a quotient of $U(\mathcal{C})$.[5] Furthermore, the character group $\mathrm{Hom}(U(\mathcal{C}), \mathbb{C}^\times)$ is naturally isomorphic to $\mathrm{Aut}_\otimes(1_\mathcal{C})$, the group of monoidal automorphisms of the identity functor. If $\mathcal{C}$ is pivotal then the set of pivotal structures forms a torsor for the group $\mathrm{Aut}_\otimes(1_\mathcal{C})$. In terms of characters, if $\chi : U(\mathcal{C}) \to \mathbb{C}^\times$ is a character

[5]Recall that we require all gradings to be faithful, meaning every graded piece is nontrivial.

and j is a pivotal structure on $\mathcal{C}$, then we can define a new pivotal structure $\chi \cdot j$ by

$$(\chi \cdot j)_X = \chi(x) j_X. \tag{2.12}$$

If the original structure j is spherical, then $\chi \cdot j$ is spherical iff $\chi^2 = 1$. Switching the spherical structure changes quantum dimensions by

$$\dim_{\mathcal{C}}^{\chi} X = \chi(x) \dim_{\mathcal{C}} X.$$

Note that modifying the spherical structure can only change q-dims by factors of ± 1.

Now suppose X is self-dual and let $i_X : \mathbf{1} \to X \otimes X$ and $d_X : X \otimes X \to \mathbf{1}$ be right duality morphisms with left duality morphisms i'_X and d'_X defined using the ribbon structure as in Eqs. 2.5. Since $\dim \mathrm{Hom}(\mathbf{1}, X^{\otimes 2}) = 1$, we have $i'_X = \alpha i_X$ and $d'_X = \tilde{\alpha} d_X$ for some scalars $\alpha, \tilde{\alpha} \in \mathbb{C}^{\times}$. The S-bend relations imply $\tilde{\alpha} = \alpha^{-1}$ and sphericality implies $\alpha \in \{\pm 1\}$. When $\alpha = 1$ the self-dual object X is called *symmetrically self-dual.* The scalar α is also known as the *2nd Frobenius Schur indicator* of the object X, denoted $\nu_2(X)$.[6]

Being symmetrically self-dual is equivalent to admitting left duality morphisms i_X, d_X and right duality morphsims i'_X, d'_X such that $i_X = i'_X$ and $d_X = d'_X$. In the graphical calculus this means that the orientation of any strands labeled X are irrelevant (they do not affect the morphism); we may consider them unoriented.

In a ribbon category the FS indicator is given by

$$\nu_2(X) = \theta_X r^{-1} \in \{\pm 1\}. \tag{2.13}$$

where θ_X is the twist on X and r^{-1} is the eigenvalue of the braid $c_{X,X} : X \otimes X \to X \otimes X$

[6]Classically, an irreducible self-dual representation of G has FS indicator +1 if the representation preserves a non-degenerate symmetric form, and FS indicator -1 if it admits a non-degenerate skew-symmetric form.

on the object $\mathbf{1} \subset X \otimes X$. The spherical structure on $\mathcal{C}$ is called *unimodal* if every self-dual object is symmetrically self dual.

Lemma 2.2.1. *If X is symmetrically self dual then $\mathcal{C}$ is unimodal.*

Proof. Turaev presented an argument for particular categories in Thm. XII.7.2 of [Tur16], and the argument applies to this more general setting. It goes like this. If Y is a simple object in $\mathcal{C}$ then let f be an integer so that $Y \subset X^{\otimes f}$. Let p_Y be a minimal idempotent in $\mathrm{End}(X^{\otimes f})$ of type Y, such that

$$p_Y^* = p_Y.$$

Here's why such an idempotent exists: the duality functor $-^*$ acts by an involutive anti-automorphism on the semisimple algebra $\mathrm{End}(X^{\otimes f})$ and it preserves the ideal corresponding to the isotype of Y. This ideal is isomorphic to a full matrix algebra. Hence by the Skolem-Noether theorem, $-^*$ is given by transposition of matrices for some choice of basis. Now any symmetric rank-1 idempotent will do.

Let $a : Y \to \mathrm{End}^{X^{\otimes f}}$ and $b : \mathrm{End}^{X^{\otimes f}}$ be morphisms such that $a \circ b = p_Y$ and $b \circ a = 1_Y$. Then the morphisms

$$i_Y := (b \otimes b) \circ i_{X^{\otimes f}},\ d_Y := d_{X^{\otimes f}} \circ (a \otimes a)$$

define left duality maps for Y. Here the duality maps on $X^{\otimes f}$ are defined using the duality maps for X. The S-bend relations follow from $p_Y^* = p_Y$.

Now the FS indicator for Y can be computed using Eq. (2.13) and naturality of twist and braiding:

$$\begin{aligned}\nu_2(Y)i_Y = (1\otimes\theta_Y)\circ c_{Y,Y}\circ i_Y &= (1\otimes\theta_Y)\circ c_{Y,Y}\circ(b\otimes b)\circ i_{X^{\otimes f}}\\ &= (b\otimes b)(1\otimes\theta_{X^{\otimes f}})\circ c_{X^{\otimes f},X^{\otimes f}}\circ i_{X^{\otimes f}}\\ &= (b\otimes b)i'_{X^{\otimes f}}\\ &= (b\otimes b)i_{X^{\otimes f}}.\end{aligned}$$

In the last equality we used that $i_{X^{\otimes f}} = i'_{X^{\otimes f}}$, which is true since they are defined in terms of i_X and i'_X, and we assume $i_X = i'_X$. This shows $\nu_2(Y) = 1$ and Y is symmetrically self-dual. □

Since $\mathcal{C}$ is generated by X and X is self-dual, the universal grading group $U(\mathcal{C})$ is either trivial or equal to $\mathbb{Z}_2$. The following lemma is easy to prove.

Lemma 2.2.2. *$U(\mathcal{C})$ is trivial if and only if the trivial object appears in some odd tensor power of X.*

Example 2.2.3. Suppose $\mathcal{C}$ is a tensor category with the fusion rules of $O(N), Sp(N)$ or $SO(2n)$. Then $\mathcal{C}$ is generated by an object X corresponding to the defining representation. It is self-dual. For these groups, since the trivial object never appears in an odd tensor power of X, we conclude $U(\mathcal{C}) \cong \mathbb{Z}_2$ with the grading on simple objects $\deg X_\lambda = 0$ if X_λ appears in an even tensor power of X and $\deg X_\lambda = 1$ if X_λ appears in an odd tensor power of X. If $\mathcal{C}$ has the fusion rules of $SO(2n+1)$ then $U(\mathcal{C})$ is the trivial group, since the trivial object appears in $X^{\otimes 2n+1}$.

Let $j_Y : Y \to Y^{**}$ denote the spherical structure on $\mathcal{C}$. Recall that given a character χ of the finite group $U(\mathcal{C})$ one can modify j to get a new spherical structure on the (tensor category) $\mathcal{C}$. It is defined by

$$j'_Y = \chi(y)j_Y$$

where $y \in U(\mathcal{C})$ denotes the degree of the simple object Y in the universal grading. Every possible spherical structure on $\mathcal{C}$ arises in this way. Hence if $U(\mathcal{C})$ is trivial then there is a unique spherical structure and if $U(\mathcal{C}) = \mathbb{Z}_2$ there are two.

Lemma 2.2.4. *This modification changes the FS indicator of a simple object by*

$$\nu_2'(Y) = \chi(y)\nu_2(Y).$$

Proof. This is because changing the spherical structure changes the twist by a factor of $\chi(y)$ but leaves the braid unchanged. Then apply Eq. (2.13). □

Theorem 2.2.5. *If $U(\mathcal{C}) = \mathbb{Z}_2$ then X is symmetrically self-dual for one of the spherical structures and anti-symmetrically self-dual for the other. If $U(\mathcal{C})$ is trivial then X is symmetrically self-dual.*

Proof. First, if $U(\mathcal{C}) = \mathbb{Z}_2$, then the category is $\mathbb{Z}_2$-graded and the object X has odd degree. Therefore changing the spherical structure with the nontrivial character of $\mathbb{Z}_2$ affects $\nu_2(X)$ by a minus sign (by the second lemma).

If $U(\mathcal{C})$ is trivial, then consider the ribbon category $\mathcal{C} \boxtimes \textbf{Rep}\ \mathbb{Z}_2$. The self dual object $X \boxtimes -1$ generates $\mathcal{C} \boxtimes \textbf{Rep}\ \mathbb{Z}_2$ (this is implied by the first lemma) and $U(\mathcal{C} \boxtimes \mathbb{Z}_2)$ is $\mathbb{Z}_2$. Hence there are two distinct spherical structures j and j' on $\mathcal{C} \boxtimes \textbf{Rep}\ \mathbb{Z}_2$. For one of them $X \boxtimes -1$ is symmetrically self-dual. By Turaev's Lemma $\mathcal{C}$ is unimodal, so $X \boxtimes 1$ is also symmetrically self-dual. However, if $X \boxtimes 1$ is symmetrically self-dual for one of the spherical structures j, j' then it is for both, since $X \boxtimes 1$ lives in the even graded part of the category and the spherical structures j and j' agree there. Hence X was symmetrically self-dual to begin with. □

Hence in the $\mathbb{Z}_2$ graded cases we may always choose the unique spherical structure for which X is symmetrically self-dual. In the $SO(2n+1)$ case, there is only one spherical structure compatible with the underlying tensor category, and we have deduced that X is symmetrically self-dual.

Finally, we present some corollaries of this theorem which may be of independent interest. The next result generalizes Lemma 2.2 of [MPS16].

Corollary 2.2.6. *If X is a self-dual object in a ribbon category (not necessarily generated by X) such that $\mathbf{1}$ appears in an odd tensor power of X, then X is symetrically self-dual.*

Proof. Indeed just apply the theorem (and the lemma about $\mathbf{1}$ appearing in an odd tensor power) to the subcategory generated by X. □

Wang conjectured that in a ribbon category the adjoint subcategory (the subcategory tensor generated by all objects of the form $Y \otimes Y^*$) is unimodal ([Wan10], Conjecture 4.26). We can prove a special case of this.

Corollary 2.2.7. *Suppose $\mathcal{C}$ is a ribbon category generated by a self-dual object X. Then the adjoint subcategory is unimodal.*

Proof. There are two cases. If $U(\mathcal{C})$ is trivial then then the adjoint subcategory is equal to the whole category, which is unimodal by the theorem. If $U(\mathcal{C}) = \mathbb{Z}_2$ then the adjoint subcategory is exactly the 0 graded piece of $\mathcal{C}$ (it is generated by X^2). Now we can do the same argument as in the proof of the theorem. Namely, for one of the spherical structures X is symmetrically self dual. Hence for this spherical structure the adjoint subcategory is unimodal. However, changing the spherical structure doesn't affect the 0-graded piece, so the adjoint subcategory is unimodal regardless of the choice of spherical structure. □

2.3 The cocycle construction

Given a group G a *3-cocycle* is a function $\omega : G^3 \to \mathbb{C}^\times$ satisfying

$$\omega(a, b, c)\omega(a, bc, d)\omega(b, c, d) = \omega(ab, c, d)\omega(a, b, cd) \tag{2.14}$$

for all $a, b, c, d \in G$. The group of 3-cocycles is denoted $Z^3(G)$. A 3-coboundary is a function of the form $\delta(\mu) : G^3 \to \mathbb{C}^\times$ where $\mu : G^2 \to \mathbb{C}^\times$ is an arbitrary function and

$$\delta(\mu)(a, b, c) = \mu(a, b)\mu(a, bc)^{-1}\mu(ab, c)\mu(b, c)^{-1}. \tag{2.15}$$

The set of coboundaries is a subgroup $B^3(G)$ of $Z^3(G)$ and $H^3(G) = Z^3(G)/B^3(G)$ is the third cohomology group of G (we omit mention of the coefficient ring $\mathbb{C}^\times$ since it is fixed throughout). Every cohomology class contains a *normalized* 3-cocycle, which additionally satisfies

$$\omega(e, a, b) = \omega(a, e, b) = \omega(a, b, e) = 1 \text{ for all } a, b \in G. \tag{2.16}$$

In the case $G = \mathbb{Z}_N$ we have $H^3(\mathbb{Z}_N) \cong \{\tau \in \mathbb{C} : \tau^N = 1\}$. Explicitly, given an Nth root of unity $\tau \in \mathbb{C}$ the formula

$$\omega^\tau(a, b, c) = \begin{cases} 1 & \text{if } a + b < N \\ \tau^c & \text{if } a + b \geq N \end{cases} \tag{2.17}$$

yields an isomorphism $\{\tau \in \mathbb{C} : \tau^N = 1\} \cong H^3(G)$.

Given a tensor category whose simple objects are all invertible and form a group G, one can obtain new tensor categories with the same objects and fusion rules by twisting the associativity constraints with a 3-cocycle of G. The same construction works for tensor categories, which are graded by a group G. In the following we use the notation $x = \deg(X), y = \deg(Y)$, etc.

Definition 2.3.1. (*Cocycle construction.*) Given a 3-cocycle $\omega : U(\mathcal{C})^3 \to \mathbb{C}^\times$ we can define $\mathcal{C}$ twisted by ω, denoted $\mathcal{C}(\omega)$, as the tensor category with the same objects and morphisms as $\mathcal{C}$, but with a new associativity morphism defined on simple objects X, Y, Z by

$$\alpha'_{X,Y,Z} = \omega(x, y, z)\alpha_{X,Y,Z}$$

and new unit constraints

$$l'_X = \omega(1,1,g)^{-1} l_X \quad \text{and} \quad r'_X = \omega(g,1,1) r_X$$

where α, l, r denote the original data of $\mathcal{C}$.

Two 3-cocyles ω and ω' are cohomologous if and only if the identity functor $\mathcal{C}(\omega) \to \mathcal{C}(\omega')$ can be equipped with a monoidal structure. Hence the cocycle construction only depends (up to monoidal equivalence) on the cohomology class of ω. In particular, we always assume ω is a normalized cocycle, so $\mathcal{C}$ and $\mathcal{C}(\omega)$ have the same unit constraints.

If $\mathcal{C}$ is rigid then $\mathcal{C}(\omega)$ is also rigid but the duality morphisms must be altered to satisfy the S-bend identities. For any simple object X we fix the following left duality morphisms in $\mathcal{C}(\omega)$:

$$\begin{aligned} i(\omega)_X &= \omega(x^{-1}, x, x^{-1}) i_X \\ d(\omega)_X &= d_X. \end{aligned}$$

Similarly we define right duality morphisms by

$$\begin{aligned} i'(\omega)_X &= i'_X \\ d'(\omega)_X &= \omega(x, x^{-1}, x) d'_X. \end{aligned}$$

If $\mathcal{C}$ is pivotal, we may assume the duality morphisms are chosen so that the left and right duality functors coincide. A computation shows that with the definitions above, the left and right duality functors in $\mathcal{C}(\omega)$ also coincide, so $\mathcal{C}(\omega)$ is pivotal (in fact, the pivotal structure $j : \mathrm{id} \to -^{**}$ in $\mathcal{C}$ is also a pivotal structure on $\mathcal{C}(\omega)$). Furthermore, any normalized 3-cocycle ω satisfies $\omega(x, x^{-1}, x) = \omega(x^{-1}, x, x^{-1})^{-1}$ which implies that the left trace of $\mathcal{C}(\omega)$ coincides with the left trace of $\mathcal{C}$, and similarly for the right trace.

Therefore if $\mathcal{C}$ is spherical then so is $\mathcal{C}(\omega)$ (using the pivotal structure provided by the duality morphisms above) and we have

$$\dim_{\mathcal{C}(\omega)} X = \dim_{\mathcal{C}} X \tag{2.18}$$

for all objects X in $\mathcal{C}$.

Next suppose $\mathcal{C}$ is braided. In this case $U(\mathcal{C})$ is abelian, and we can construct braided twists of $\mathcal{C}$ using *abelian cocycles*, defined below (this is explained in the setting of pointed categories by Sec. 3 of [JS93]). Note that $\mathcal{C}(\omega)$ is not necessarily braided via the morphisms $c_{X,Y}$ coming from $\mathcal{C}$. However, if we are provided with a function $a : U(\mathcal{C})^2 \to \mathbb{C}^\times$ which satisfies both

$$a(x, y+z) = \omega(y,z,x)^{-1}\omega(y,x,z)\omega(x,y,z)^{-1}a(x,y)a(x,z) \tag{2.19}$$

$$a(x+y, z) = \omega(z,x,y)\omega(x,z,y)^{-1}\omega(x,y,z)a(x,z)a(y,z) \tag{2.20}$$

then we can define a braiding on $\mathcal{C}(\omega)$ by

$$c(\omega, a)_{X,Y} = a(x,y)c_{X,Y}.$$

Indeed, the conditions above are equivalent to the hexagon axioms for $c(\omega, a)$. A pair (ω, a) where ω is a normalized 3-cocycle of an abelian group G and a satisfies the above conditions is called an *abelian 3-cocycle.* The abelian 3-cocycles form a group $Z_3(G)$. As we've seen, given an abelian 3-cocyle (ω, a) we can modify the associativity and braiding morphisms in $\mathcal{C}$ using the formulas above to form a new braided tensor category which we denote $\mathcal{C}(\omega, a)$.

Let $k : G^2 \to \mathbb{C}^\times$ be a function satisfying

$$k(x, 0) = 1 = k(0, y).$$

Then δk is defined to be the abelian cocycle (ω, a) defined by

$$\omega(x, y, z) = k(y, z)k(x + y, z)^{-1}k(x, y + z)k(x, y)^{-1}$$
$$a(x, y) = k(x, y)k(y, x)^{-1}.$$

The set of abelian cocyles of the form δk form a subgroup $B^3_{\mathrm{ab}}(G) \leq Z^3_{\mathrm{ab}}(G)$ and the quotient is the 3rd abelian cohomology group of G, $H^3_{\mathrm{ab}}(G)$. Similarly to before, two abelian cocycles (ω, γ) and (ω', γ') are cohomologous if and only if the identity functor $\mathcal{C}(\omega, a) \to \mathcal{C}(\omega', a')$ can be equipped with monoidal structure, and is braided.

When $\mathcal{C}$ is braided and spherical, i.e. ribbon, the pivotal structure can be transferred as before to $\mathcal{C}(\omega, a)$ to produce another ribbon category. The q-dims of objects in $\mathcal{C}$ are unchanged. However, the twist of $\mathcal{C}(\omega, a)$ differs from that of $\mathcal{C}$ on a simple object X by

$$\theta(\omega, a)_X = a(x, x)\theta_X.$$

Another modification is to change the spherical structure given a character $\chi : U(\mathcal{C}) \to \mathbb{C}^\times$ such that $\chi^2 = 1$, as discussed earlier. The resulting category is denoted $\mathcal{C}(\omega, a)^\chi$ and the passage from $\mathcal{C}$ to $\mathcal{C}(\omega, a)^\chi$ affects the q-dims and twists of a simple object X by

$$\dim X \mapsto \chi(x) \dim X$$
$$\theta_X \mapsto \chi(x)a(x, x)\theta_X.$$

The notation $\mathcal{C}(\omega, a)^\chi$ is justified as the modifications of the spherical structure by χ and braiding/associativity by ω are independent, i.e. $(\mathcal{C}(\omega, a))^\chi = (\mathcal{C}^\chi)(\omega, a)$.

We summarize the discussion in a proposition.

Proposition 2.3.2. *Suppose $\mathcal{C}$ is a semisimple ribbon category with universal grading group $U(\mathcal{C})$. Then for any abelian 3-cocycle (ω, a) and character $\chi : U(\mathcal{C}) \to \mathbb{C}^\times$ such* *that* $\chi^2 = 1$ *we can modify the associativity, braiding, and twist of* $\mathcal{C}$ *on simple objects* X, Y, Z *by*

$$\alpha_{X,Y,Z} \mapsto \omega(x,y,z)\alpha_{X,Y,Z}$$
$$c_{X,Y} \mapsto a(x,y)c_{X,Y}$$
$$\theta_X \mapsto \chi(x)a(x,x)\theta_X$$

to produce a new ribbon category $\mathcal{C}(\omega, a)^\chi$ *with the same fusion rules. The q-dims of simple objects* X *change by*

$$\dim_{\mathcal{C}(\omega,a)^\chi} X = \chi(x) \dim_{\mathcal{C}} X.$$

The identity functor $\mathcal{C}(\omega, a)^\chi \to \mathcal{C}(\omega', a')^{\chi'}$ *can be equipped with monoidal structure if and only if* ω *and* ω' *are cohomologous (as 3-cocyles). The identity functor can be equipped with braided monoidal structure if and only if* (ω, a) *and* (ω', a') *are cohomologous (as abelian 3-cocycles) and preserves the spherical structure if and only if* $\chi = \chi'$*.*

2.3.1 Abelian 3-cocyles of $\mathbb{Z}_2$

All the categories we will consider have $U(\mathcal{C}) = \{e\}$ or $U(\mathcal{C}) = \mathbb{Z}_2$ so here we describe the abelian 3-cocyles of $\mathbb{Z}_2$.

As a representative set of normalized 3-cocyles for $H^3(\mathbb{Z}_2) \cong \mathbb{Z}_2$ we choose $\{1, \omega\}$ where $\omega : \mathbb{Z}_3 \to \mathbb{C}^\times$ is given by

$$\omega(a,b,c) = \begin{cases} 1 & \text{if } (a,b,c) \neq (1,1,1) \\ -1 & \text{if } (a,b,c) = (1,1,1) \end{cases}$$

One computes $H^3_{\mathrm{ab}}(\mathbb{Z}_2) \cong \mathbb{Z}_4$. For $z \in \mathbb{C}$ let $f_z : \mathbb{Z}_2^2 \to \mathbb{C}^\times$ be the function

$$f_z(a,b) = \begin{cases} 1 & \text{if } (a,b) \neq (1,1) \\ z & \text{if } (a,b) = (1,1) \end{cases}$$

Then $\{(1, f_1), (1, f_{-1}), (\omega, f_i), (\omega, f_{-i})\}$ are a representative set of abelian 3-cocycles of $\mathbb{Z}_2$.

Remark 2.3.3. Including the spherical structure and mirror modifications, we end up with 16 total ways to modify a $\mathbb{Z}_2$-graded ribbon category to get a new one. It would be nice to connect this phenomenon to the *16-fold way* for minimal modular extensions of super-modular categories [BGH$^+$17].

3

Tensor product rules for $SO(N)$

The fusion ring of **Rep** $\mathrm{SO}(N)$ is a $\mathbb{Z}$-based ring indexed by the set of non-isomorphic simple objects. The multiplication in this ring comes from the tensor product of representations. The finite dimensional (always assumed polynomial) irreducible representations of classical groups are parametrized by their highest weight.

3.1 Lie theory

The special orthogonal group $G = SO(N)$ consists of the linear transformations $\mathbb{C}^N \to \mathbb{C}^N$ which preserve a non-degenerate symmetric bilinear form, with determinant 1. Over $\mathbb{C}$ the isomorphism type of the group doesn't depend on the form, so for concreteness we choose

$$v \cdot J_N w \tag{3.1}$$

where $\cdot$ is the usual dot product on $\mathbb{C}^N$ and

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{3.2}$$

$$J_{2n+1} = \begin{pmatrix} 0 & & I_n \\ & 1 & \\ I_n & & 0 \end{pmatrix} \tag{3.3}$$

where I_n is the $n \times n$ identity matrix. This choice of form is convenient since the diagonal matrices in G now form a maximal torus, denoted T. The upper triangular matrices of G form a subgroup B called a Borel subgroup.

Let $\mathfrak{g} = \mathfrak{so}_N$ denote the Lie algebra of $SO(N)$ and $\mathfrak{h}$ the Lie algebra of T. Both $\mathfrak{g}$ and $\mathfrak{h}$ act faithfully on $\mathbb{C}^N$. Since T consists of diagonal matrices, so does $\mathfrak{h}$, and more precisely

$$\mathfrak{h} = \begin{cases} \{\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)\} & \text{for } G = SO(2n) \\ \{\mathrm{diag}(a_1, \ldots, a_n, 0, -a_1, \ldots, -a_n)\} & \text{for } G = SO(2n+1). \end{cases} \tag{3.4}$$

On any finite-dimensional representation V of $SO(N)$ the abelian algebra $\mathfrak{h}$ acts by diagonalizable matrices and the simultaneous eigenvector decomposition gives V as a direct sum of *weight spaces*:

$$V = \bigoplus_{\mu} V_\mu.$$

Here $\mu \in \mathfrak{h}^*$ is a linear functional and

$$V_\mu = \{v \in V \mid Av = \mu(A)v \text{ for all } A \in \mathfrak{h}\}.$$

The dimension of V_μ is the *multiplicity* of the weight μ in V, denoted $\mathrm{mult}_V(\mu)$. The *character* of V is the formal sum

$$\chi_V = \sum_{\mu} \text{mult}_V(\mu) e^{\mu}$$

where e^{μ} denotes a standard basis element of the group algebra $\mathbb{Z}[\mathfrak{h}^*]$. This means we can multiply formal sums using $e^{\mu}e^{\lambda} = e^{\mu+\lambda}$. Any two finite-dimensional representations are equivalent if and only if they have the same character. We also have

$$\chi_{V \oplus W} = \chi_V + \chi_W$$
$$\chi_{V \otimes W} = \chi_V \chi_W.$$

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ denote the usual basis of $\mathfrak{h}^*$ coming from Eq. (3.4), i.e.

$$\varepsilon_i(\text{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)) = a_i$$

in the $SO(2n)$ case. Let $\mathfrak{h}^*_{\mathbb{R}}$ denote the $\mathbb{R}$-span of these vectors and let P be the *weight lattice* of G, here defined as the $\mathbb{Z}$-span of the ε_i. The weights occurring in any finite-dimensional $SO(N)$-representation belong to P, hence in this basis we can describe any weight by an n-tuple of integers, for instance $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$.

A weight vector which is also an eigenvector for the Borel subgroup B is a *highest weight vector* of V. The theorem of highest weight states that f.d. irreps of $SO(N)$ always contain a unique highest weight vector (up to scalar multiplication) and isotypes of irreps are in 1-1 correspondence with highest weights belonging to the set $\Gamma(G)$ where

$$\Gamma(SO(2n)) = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \cdots \geq |\lambda_n| \geq 0\} \tag{3.5}$$

$$\Gamma(SO(2n+1)) = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}. \tag{3.6}$$

For $\lambda \in \Gamma(G)$ let $V(\lambda)$ denote the irreducible module with highest weight λ. We now see that the fusion ring of **Rep** $SO(N)$ is equipped with a distinguished basis labeled by elements of $\Gamma(SO(N))$. Since every representation of **Rep** $SO(N)$ is self-dual, the involution in Gr(**Rep** $SO(N)$) is trivial. To describe the fusion rule we return to the Lie theoretic background.

The group G acts by conjugation on $\mathfrak{g}$; this is the *adjoint representation.* The non-zero weights of the adjoint representation are the *roots* of G and their $\mathbb{Z}$-span in $\mathfrak{h}^*_{\mathbb{R}}$ is the *root lattice*, denoted Q. We specify a choice of *simple roots*, given below for $SO(2n)$ and $SO(2n+1)$ in Table 3.1, which form a $\mathbb{Z}$-basis for the root lattice.

Table 3.1: Simple roots for $SO(N)$, even and odd

$SO(2n)$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1} = (\dots, 0, 1, -1, 0, \dots)$ for $1 \leq i \leq n-1$ $\alpha_n = \epsilon_{n-1} + \epsilon_n = (0, \dots, 1, 1)$.
$SO(2n+1)$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1} = (\dots, 0, 1, -1, 0, \dots)$ for $1 \leq i \leq n-1$ $\alpha_n = \varepsilon_n = (0, \dots, 0, 1)$.

Every root which can be expressed as a non-negative sum of simple roots is a *positive root*. The set of positive roots is Φ^+, and $\Phi = \Phi^+ \cup -\Phi^+$, meaning every root can be written (uniquely) as a positive sum or negative sum of simple roots.

We equip $\mathfrak{h}^*_{\mathbb{R}}$ with the inner product $\langle , \rangle$ for which the ϵ_i are orthonormal. Note that $\langle \alpha, \alpha \rangle = 2$ for short roots. For every root α there is orthogonal reflection $s_\alpha : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ about the hyperplane perpendicular to α. The *Weyl group* W is the group generated by these reflections. W acts faithfully on the set of roots so in particular it is a finite group. It also preserves the weight lattice. For $G = SO(2n+1)$ the Weyl group is $S_n \ltimes \mathbb{Z}_2^n$, where S_n is the symmetric group, acting on the weight lattice by permutations, and $\mathbb{Z}_2^n$ acts by sign changes on the coordinates. For $G = SO(2n)$ we have $W \cong S_n \ltimes \mathbb{Z}_2^{n-1}$, where the $\mathbb{Z}_2^{n-1}$ part acts by sign changes affecting an even number of entries (this $\mathbb{Z}_2^{n-1}$ can be seen as the subgroup of $\mathbb{Z}_2^n$ consisting of strings with an even number of 1's). We denote the action of W on the weight lattice by $w(\lambda)$. There is another important action of W, namely by shifted reflections:

$$w \cdot \lambda := w(\lambda + \rho) - \rho$$

where ρ is half the sum of the positive roots. The *dominant Weyl chamber* is a cone emanating from the origin, bounded by hyperplanes orthogonal to the simple roots. The sets $\Gamma(G)$ parametrizing irreps correspond exactly to weights which lie in the closure of the Weyl chamber. These are identical to the weights which lie in the interior of the *shifted Weyl chamber*, which is the translation of the Weyl chamber by $-\rho$ and is an analogous fundamental domain for the shifted Weyl group action.

We can now give Steinberg's formula for the tensor product coefficients. Let $N_{\lambda\mu}^{\nu}$ denote the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$, and $\mathrm{mult}_\lambda(\mu)$ the multiplicity of the weight μ in $V(\lambda)$. Then we have

Theorem 3.1.1. *(Steinberg, [Ste61])*

$$N_{\lambda,\mu}^{\nu} = \sum_{w \in W} (-1)^w mult_\lambda(w \cdot \nu - \mu). \tag{3.7}$$

This formula has a pleasing geometric interpretation for how to decompose $V(\lambda) \otimes V(\mu)$: take the weight diagram for λ and translate it to be centered at μ. Whenever a portion of the diagram extends outside the shifted Weyl chamber, one can "fold" it back towards the dominant Weyl chamber using a shifted reflection. Once all the weights have been folded into the dominant Weyl chamber the total amount at each weight ν (counted with signs according to how many reflections were performed) gives the multiplicity of $V(\nu)$ in $V(\mu) \otimes V(\lambda)$. The weights which land on the boundary planes of the Weyl chamber contribute 0 to the decomposition.

We compute several explicit examples we need in the $SO(2n)$ case. In this case

$$\rho = (n-1, n-2, \dots, 1, 0)$$

and the shifted Weyl chamber is given by the inequalities

$$\lambda_i - \lambda_{i+1} \geq -1, \qquad \text{for } 1 \leq i \leq n-1 \tag{3.8}$$

$$\lambda_{n-1} + \lambda_n \geq -1. \tag{3.9}$$

Let s_i denote the simple reflection about the hyperplane orthogonal to the simple root α_i. Under the shifted action s_i transforms a weight λ by affecting the i and $i+1$ coordinates of λ:

$$s_i \cdot (\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_{i+1} - 1, \lambda_i + 1, \dots, \lambda_n), \qquad \text{for } 1 \leq i \leq n-1$$

$$s_n \cdot (\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, -\lambda_n - 1, -\lambda_n - 1)$$

Example 3.1.2. Let $X = \mathbb{C}^{2n}$ denote the defining representation of $SO(2n)$. The standard basis provide a basis of weight vectors for X, and the weights which appear in X are $\{\pm\epsilon_i : 1 \leq i \leq n\}$, each with multiplicity one. Since X has a unique highest weight, namely $\epsilon_1 = (1, 0, \dots, 0)$, it is irreducible and $X \cong V(\epsilon_1)$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a highest weight for $SO(2n)$. Consider Steinberg's formula for $\lambda \otimes X$, using the weight diagram for X. If $\mu = \pm\epsilon_i$ is a weight which appears in X, then $\lambda + \mu$ must lie in the shifted Weyl chamber. If $\lambda + \mu$ lies on a bounding hyperplane then it contributes 0 to $\lambda \otimes X$. If it lies in the interior of the shifted Weyl chamber then it is a highest weight, obtained by adding or subtracting 1 from one of the coordinates of λ. Therefore

$$X \otimes V(\lambda) \cong \bigoplus_{\nu \leftrightarrow \lambda} V(\nu)$$

where the sum is over all highest weights ν which are of the form $\lambda \pm \epsilon_i$ for some i.

Example 3.1.3. Consider $\bigwedge^2(X)$, the second exterior power of $X = \mathbb{C}^{2n}$. Assume $n \geq 3$; then $\bigwedge^2(X)$ is known to be an irreducible $SO(2n)$ representation. The usual basis of $\bigwedge^2(X)$ arising from the standard basis of X gives a basis of weight vectors. The weights which appear are of the form $\pm\epsilon_i \pm \epsilon_j$ for $i \neq j$, appearing with multiplicity 1, and 0, with multiplicity n. It has highest weight $\epsilon_1 + \epsilon_2 = (1, 1, 0, \dots, 0)$. Hence $\bigwedge^2(X) \cong V(\epsilon_1 + \epsilon_2)$.

Proposition 3.1.4. *Let λ be a highest weight for $SO(2n)$ with $n \geq 3$. Let d denote the number of distinct integers in the list $(\lambda_1, \dots, \lambda_{n-1}, \lambda_n, -\lambda_n)$. Then $N^{\lambda}_{\lambda, \epsilon_1+\epsilon_2} = d - 1$ and*

$$V(\epsilon_1 + \epsilon_2) \otimes V(\lambda) \cong \bigoplus_{\nu} V(\nu) \oplus (d-1)V(\lambda)$$

where the sum is over all highest weights ν of the form $\lambda \pm \epsilon_i \pm \epsilon_j$ with $i \neq j$.

Proof. Let μ be a non-zero weight of $V(\epsilon_1 + \epsilon_2)$. Then $\lambda + \mu$ is either a highest weight, or lies on a bounding hyperplane of the shifted Weyl chamber, or lies outside the shifted Weyl chamber. We claim that whenever $\lambda + \mu$ lies outside, it can be folded back with a single reflection, and the resulting weight is λ. Indeed, if $\lambda + \mu$ lies outside then it must violate one of the inequalities (3.8) or (3.9). In the first case this means

$$(\lambda + \mu)_i - (\lambda + \mu)_{i+1} < -1.$$

Then we find that

$$0 \leq \lambda_i - \lambda_{i+1} < \mu_{i+1} - \mu_i - 1 \leq 1,$$

where the second inequality is true since μ_i and μ_{i+1} belong to $\{0, \pm 1\}$. Hence $\lambda_i = \lambda_{i+1}$ and $\mu = -\epsilon_i + \epsilon_{i+1}$. Therefore $\lambda + \mu = (\lambda_1, \dots, \lambda_i - 1, \lambda_{i+1} + 1, \dots \lambda_n)$ and $s_i \cdot (\lambda + \mu) = \lambda$, as claimed. Furthermore, the number of μ for which $\lambda + \mu$ breaks one of the inequalities (3.8) is equal to $|\{i : \lambda_i = \lambda_{i+1}\}|$. In a similar way one finds that $\lambda + \mu$ breaks the inequality

(3.9) if and only if $\lambda_n = -\lambda_{n-1}$ and $\mu = -\epsilon_{n-1} - \epsilon_n$, in which case $s_n \cdot (\lambda + \mu) = \lambda$.

Hence, by Steinberg's formula, every highest weight ν of the form $\lambda + \mu$ appears with multiplicity 1 in $V(\epsilon_1 + \epsilon_2) \otimes \lambda$, since none of these weights are cancelled out by folding. The only question is the multiplicity of λ. Let r denote the number of weights μ of X such that $\lambda + \mu$ lies outside the shifted Weyl chamber. Since the weight 0 appears with multiplicity n in X, the previous analysis shows $N^{\lambda}_{\lambda,\epsilon_1+\epsilon_2}$ is equal to $n - r$. Using the conditions on λ and μ derived in the previous paragraph, one finds that $n - r = d - 1$. □

Remark 3.1.5. Note that $N^{\lambda}_{\lambda,\epsilon_1+\epsilon_2} \geq 1$ for all non-zero λ.

We will also need a well known general fact concerning tensor product rules:

Proposition 3.1.6. *([GW], Prop. 5.5.19) The tensor product $V(\mu) \otimes V(\lambda)$ decomposes as*

$$V(\mu) \otimes V(\lambda) \cong V(\mu + \lambda) \oplus \bigoplus_{\nu} N^{\nu}_{\lambda,\mu} V(\nu)$$

where the sum is taken over those highest weights ν for which $\lambda + \mu - \nu$ is a positive root.

In other words, we have $N^{\lambda+\mu}_{\lambda,\mu} = 1$ and $N^{\nu}_{\lambda,\mu} = 0$ unless $\nu = \lambda + \mu - \alpha$ for some positive root α.

3.2 Representations and Young diagrams

For most of the thesis we will take a combinatorial approach and parametrize irreps using Young diagrams. Already we've seen that irreps of $SO(N)$ are parametrized by highest weights, which correspond to a sequence of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_n|$. Such a sequence is a *partition* if $\lambda_n \geq 0$. We identify a partition with its *Young diagram*, which is a collection of boxes arranged in rows, the ith row containing λ_i boxes. We use exponential notation to denote Young diagrams, so for instance $[1^k]$ denotes the one column Young diagram with k boxes while $[k]$ denotes the one row diagram of size k.

The Young diagram associated to any sequence λ is obtained by replacing λ_n with $|\lambda_n|$. The size of λ, denoted $|\lambda|$, is the number of boxes in λ. Given a Young diagram λ let λ' denote the *transpose diagram* whose ith column has λ_i many boxes. Note that λ'_i is equal to the number of boxes in the ith column of λ. Let $\mathcal{Y}$ denote the set of all Young diagrams and $\mathcal{Y}_k$ diagrams of size k. Then the irreducible representations of $O(N)$ and $Sp(N)$ are parametrized by the following sets:

(a) $\Gamma(O(N)) = \{\lambda \in \mathcal{Y} : \lambda'_1 + \lambda'_2 \leq N\}$

(b) $\Gamma(Sp(N)) = \{\lambda \in \mathcal{Y} : \lambda_1 \leq N\}$

The corresponding sets for the special orthogonal groups depend on whether N is even or odd. We define

(c) $\Gamma(SO(2n+1)) = \{\lambda \in \mathcal{Y} : \lambda'_1 \leq n\}$

(d) $\Gamma(SO(2n)) = \{\lambda \in \mathcal{Y} : \lambda'_1 < n\} \cup \{\lambda^{\pm} : \lambda \in \mathcal{Y},\ \lambda'_1 = n\}$

Here the definition of $\Gamma(SO(2n))$ means that for every Young diagram λ with exactly n rows there are two distinct elements labelled λ^+ and λ^-, corresponding to λ_n positive or negative. This is the only case in which the irreps are not exactly parametrized by Young diagrams. We refer to elements of $\Gamma(G)$ as *G-shapes*.

Define an involution $r : \Gamma(O(N)) \to \Gamma(O(N))$ where $r(\lambda)$ is the Young diagram which is obtained from λ by replacing the first column of λ with $N - \lambda'_1$ many boxes. [1] The condition $\lambda \in \Gamma(O(N))$ ensures that $r(\lambda)$ is a Young diagram belonging to $\Gamma(O(N))$. When $r(\lambda) \neq \lambda$ we say they are *associate* diagrams. This involution coincides with the map $\lambda \mapsto \lambda \otimes \det$ where det is the determinant representation of $O(N)$ (corresponding to

[1] This corresponds to the automorphism of the Dynkin diagram for type D which swaps the two endpoints connected to the triple point.

the Young diagram $[1^N]$). Then the restriction rule can be stated as

$$\lambda_{|SO(N)} \cong \begin{cases} \lambda \text{ or } r(\lambda) & \text{if } r(\lambda) \neq \lambda \\ \lambda^+ \oplus \lambda^- & \text{if } r(\lambda) = \lambda \end{cases}$$

where in the first case we choose between λ or $r(\lambda)$, taking the unique diagram with $< n$ boxes in its first column. Note the condition $r(\lambda) = \lambda$ occurs exactly when $N = 2n$ and λ has n boxes in its first column.

In the case of N odd, the restriction map is onto, ie

$$\mathrm{Gr}(O(2n+1))_{|SO(2n+1)} = \mathrm{Gr}(SO(2n+1)) \tag{3.10}$$

and in fact

$$\Gamma(O(2n+1)) \cong \Gamma(SO(2n+1)) \times \{\pm 1\}. \tag{3.11}$$

via the bijection

$$\lambda \mapsto \begin{cases} (\lambda, (-1)^{|\lambda|}) & \text{if } \lambda_1' \leq n \\ (r(\lambda), (-1)^{|\lambda|}) & \text{if } \lambda_1' > n. \end{cases} \tag{3.12}$$

This map is a bijection thanks to the decomposition $O(2n+1) \cong SO(2n+1) \times \{\pm I\}$ and the fact that $-I$ acts on the isotype λ by $(-1)^{|\lambda|}$. The map extends to (or is categorified by) an equivalence of symmetric monoidal categories:

$$\mathbf{Rep}\ (O(2n+1)) \cong \mathbf{Rep}\ (SO(2n+1)) \boxtimes \mathbf{Rep}\ \mathbb{Z}_2.$$

Under this correspondence, $\mathbf{Rep}\ SO(2n+1) \cong \mathbf{Rep}\ SO(2n+1) \boxtimes 1$ appears as the tensor subcategory of $\mathbf{Rep}\ O(2n+1)$ spanned by Young diagrams with an even number of boxes. This identification sends an $SO(2n+1)$-shape λ to itself if $|\lambda|$ is even, and to $r(\lambda)$ if $|\lambda|$

is odd. On the level of Grothendieck rings we have (by Eq. (2.1))

$$\mathrm{Gr}(O(2n+1)) \cong \mathrm{Gr}(SO(2n+1)) \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathbb{Z}_2). \tag{3.13}$$

In the even case, the restriction map is not onto but the image can be described as follows. We have an involution $\sigma : \Gamma(SO(2n)) \to \Gamma(SO(2n))$ which fixes λ for every λ with $\lambda'_1 < n$ and exchanges λ^+ with λ^-. This involution coincides with the action on characters given by conjugation with a certain element of $O(2n)$ (which gives an outer automorphism of $SO(2n)$). This shows that σ is an involutive automorphism of $\mathrm{Gr}(SO(N))$. Let $\mathrm{Gr}(SO(2n))^\sigma$ denote the fixed ring of σ. Then

$$\mathrm{Gr}(O(2n))_{|SO(2n)} = \mathrm{Gr}(SO(2n))^\sigma. \tag{3.14}$$

This fixed ring consists of $\mathbb{Z}$-linear combinations of simple elements such that the coefficients for any pair λ^+ and λ^- are equal.

Lemma 3.2.1. *The fixed ring $Gr(SO(2n))^\sigma$ is generated algebraically by the elements*

$$[1], [1^2], \ldots, [1^{n-1}], [1^n]^+ + [1^n]^-.$$

Proof. See [KT], Prop. 1.2.6. In fact they show that the fixed ring is a free polynomial ring with the above elements as generators. □

Lemma 3.2.2. *The ring $Gr(SO(2n))$ is generated algebraically by the elements*

$$[1], [1^2], \ldots, [1^{n-1}], [1^n]^+, [1^n]^-. \tag{3.15}$$

Proof. Let S denote the (unital) subring generated by these elements. We can prove that every simple element belongs to S using double induction with respect to (1) the size of the corresponding diagram and (2) the partial order $\ll$ on highest weights defined by

$\mu \ll \lambda$ if $\lambda - \mu$ is a positive root.[2] Note that here $\lambda - \mu$ does not denote subtraction in the Grothendieck ring, but subtraction as elements of the weight lattice. In the following we use the notation $\lambda \oplus \mu$ to denote the operation of plus in the Grothendieck ring. By the previous lemma, the only simple elements that can't already be written as a polynomial in the elements in (3.15) are of the form λ^+ or λ^-. Any $SO(2n)$-shape of the form λ^+ has an associated shape μ obtained by removing the first column from λ^+. Then by Prop. 3.1.6,

$$[1^n]^+ \otimes \mu = \lambda^+ \oplus \bigoplus_{\nu \ll \lambda^+} N^{\nu}_{[1^n]^+,\mu} \nu.$$

By induction, all the simple elements ν which appear in the sum belong to S. Similarly, since μ has fewer boxes than λ, μ also belongs to S by induction. Hence λ^+ also belongs to S. Finally, $\lambda^+ \oplus \lambda^-$ already belongs to S by the previous lemma, so $\lambda^- \in S$ as well, which completes the proof. □

For any classical group G there is a distinguished irrep which is the defining representation (or fundamental module) X whose underlying vector space is $\mathbb{C}^N$. It corresponds to $\lambda = [1]$, the one-box Young diagram. Multiplying a simple module X_λ by X is easily described:

$$X \otimes X_\lambda \cong \sum_{\mu \leftrightarrow \lambda} X_\mu \tag{3.16}$$

where the notation $\mu \leftrightarrow \lambda$ means the sum is taken over every $\mu \in \Gamma(G)$ obtained from λ by adding or removing a box. In the case of $SO(N = 2n)$ the fusion rule is slightly more delicate. For λ with $\lambda'_1 < n$ we have

$$X \otimes X_\lambda \cong \sum_{\mu \leftrightarrow \lambda} \begin{cases} X_\mu & \text{if } \mu'_1 < n \\ X_{\mu^+} \oplus X_{\lambda^-} & \text{if } \mu'_1 = n \end{cases} \tag{3.17}$$

[2]Note that for each λ, the set of highest weights μ with $\mu \ll \lambda$ is finite so the induction is valid.

while for λ with $\lambda'_1 = n$ we have

$$X \otimes X_{\lambda^\pm} \cong \sum_{\mu \leftrightarrow \lambda} \begin{cases} X_\mu & \text{if } \mu'_1 < n \\ X_{\mu^\pm} & \text{if } \mu'_1 = n \end{cases} \tag{3.18}$$

where again the sum is always taken over all those μ obtained from λ by adding or removing a box. For $SO(2n)$ we abuse language a bit and say λ is obtained from μ by adding (removing) a box whenever λ is a component of $X \otimes [\mu]$ with one more (fewer) boxes than μ, even when λ and/or μ have n rows and are labeled with plus/minuses.

Remark 3.2.3. Proposition 3.1.4 gives the $SO(2n)$ rule for tensoring with $[1^2]$, when $n \geq 3$. A particularly useful example is

$$[1^2] \otimes [K] = [K, 1^2] + [K+1, 1] + [K-1, 1] + [K]. \tag{3.19}$$

This equation is valid for $n \geq 4$; for $n = 3$ the term $[K, 1^2]$ should be replaced by $[K, 1^2]^+ + [K, 1^2]^-$. Aside from the multiplicity of λ in $[1^2] \otimes \lambda$, the rule is simple, and we restate it in terms of $SO(2n)$ shapes:

Lemma 3.2.4. *Suppose λ, ν are distinct $SO(2n)$ shapes, with $n \geq 3$. Then ν appears in $[1^2] \otimes \lambda$ if and only if ν can be obtained by adding or removing a box from two distinct rows of λ. If it appears then it has multiplicity 1.*

The rule for tensoring by [2] can be obtained via

$$[2] \otimes \lambda = X^{\otimes 2} \otimes \lambda - [1^2] \otimes \lambda - \lambda, \tag{3.20}$$

due to the decomposition $X^{\otimes 2} = [2] + [1^2] + \mathbf{1}$.

Lemma 3.2.5. *Suppose λ, ν are distinct $SO(2n)$ shapes with $n \geq 3$. Then ν appears in*

$[2] \otimes \lambda$ if and only if ν can be obtained by adding or removing a box from two distinct columns of λ. If it appears then it has multiplicity 1.

Proof. If ν is obtained from λ by adding or removing boxes in the same row but different columns, then ν appears only once in $X^{\otimes 2} \otimes \lambda$ (due to the unique shape in $X \otimes \lambda$ which differs from both λ and ν by one box). On the other hand ν does not appear in $[1^2] \otimes \lambda$, so it must appear in $[2] \otimes \lambda$, by Eq. 3.20.

If ν is obtained by adding or removing boxes in different rows and different columns then the multiplicity of ν in $X^{\otimes 2} \otimes \lambda$ is 2, owing to the two shapes in $X \otimes \lambda$ which are exactly one box different from both ν and λ. Since ν appears in $[1^2] \otimes \lambda$ with multiplicity 1 by Lemma 3.2.4, the same is true for $[2] \otimes \lambda$ by Eq. 3.20.

Finally, if ν is any other shape in $X^{\otimes 2} \otimes \lambda$ then it must be obtained from λ by adding or removing boxes from the same column. In this case ν appears only once in $X^{\otimes 2} \otimes \lambda$, and it also appears in $[1^2] \otimes \lambda$. Hence it does not appear in $[2] \otimes \lambda$. □

3.3 Fusion rings of $SO(N)$-type

There also exist ribbon categories with only finitely many simple objects (i.e. fusion categories)[3] whose fusion ring is a quotient of that of SO(N). The simple objects are parametrized by integer points in a *Weyl alcove*, which is a linear simplex formed by truncating the Weyl chamber by a hyperplane orthogonal to a certain positive root. Combinatorially we can characterize these simple objects by a condition on the length of the first row or first two rows of a Young diagram. If the condition is just that $\lambda_1 \leq K$ then the fusion ring is said to be of $SO(N) - Sp(K)$ type. If the condition is $\lambda_1 + \lambda_2 \leq K$ then the fusion ring is of $SO(N) - O(K)$ type. The terminology comes from the fact that the condition on the *rows* is the same condition put on the *columns* of Young diagrams

[3]In general a semisimple category is *fusion* if it has finitely many simple isotypes. A fusion ribbon category is also called a *premodular* category.

parametrizing irreps in $O(K)$ (resp. $Sp(K)$), see Sec. 3.2, (a) and (b). [4] For certain values of N and K there are also fusion rings of type $SO(N) - SO(K)$ arising from de-equivariantizations of $SO(N)-O(K)$ type categories, see e.g. [BB01]. We hope to address these in future work. The categories considered now arise in several contexts, including quantum groups at roots of unity ([AP95], [Saw06]), affine Lie algebras (e.g. [Fei02]) and Turaev-Wenzl style skein theory coming from the BMW algebras ([Wen90], [TW97]). Here we just describe the relevant fusion rings. We review the basic language of affine Weyl groups and alcoves, for instance as in [Hum90]. See [BK01], [Sch18] for a discussion specialized to our context.

3.3.1 $SO(N) - O(K)$ type fusion rings

First let us describe the $SO(N)-O(K)$ rings. These have simple elements indexed by

$$\Gamma(SO(N) - O(K)) = \{\lambda \in \Gamma(SO(N)) : \lambda_1 + \lambda_2 \leq K\}. \tag{3.21}$$

In terms of highest weights, these correspond to weights in the (<u>closure</u> of the) dominant Weyl chamber which satisfy the extra condition

$$\langle \lambda, \theta \rangle \leq K$$

where $\theta = (1, 1, 0, \dots 0)$ is the highest root of $SO(N)$. It is often more convenient to think of these as the weights inside the <u>interior</u> of the *shifted Weyl alcove*, which is the subset of $\mathfrak{h}^*_{\mathbb{R}}$ bounded by the shifted Weyl chamber and the additional hyperplane

$$\{\lambda : \langle \lambda, \theta \rangle = K + 1\}.$$

[4] The notation is inspired by a general symmetry called *level-rank duality*, which arises from the transpose operation on Young diagrams.

If we translate the shifted Weyl alcove to emanate from the origin (instead of $-\rho$), we get the *Weyl alcove*, which is the truncation of the dominant Weyl chamber by the additional hyperplane

$$\{\lambda : \langle \lambda, \theta \rangle = \varkappa\}.$$

where $\varkappa = K + \langle \rho, \theta \rangle + 1$.

We define the *affine Weyl group* $\widehat{W}$ to be the group generated by the reflections about the hyperplanes bounding the Weyl alcove. This includes the reflections orthogonal to simple roots, so $\widehat{W}$ contains the Weyl group W as a subgroup. In fact we have $\widehat{W} \cong W \ltimes L$ where L is the subgroup of translation in $\widehat{W}$. We may identify L with a discrete subgroup of $\mathfrak{h}^*_{\mathbb{R}}$: if t_v denotes translation by v then $L = \{v \in \mathfrak{h}^*_{\mathbb{R}} : t_v \in \widehat{W}\}$. Then one can show that $L = \varkappa Q^\vee = \{\varkappa x : x \in Q^\vee\}$ where

$$Q^\vee := \{\lambda \in \mathbb{Z}^n : \sum_i \lambda_i \text{ is even}\}. \tag{3.22}$$

This important W-invariant sublattice of P is called the *coroot lattice.* Just like the Weyl group, the affine Weyl group also acts on $\mathfrak{h}^*_{\mathbb{R}}$ by its shifted (or dot action). Under both the normal action and shifted action the weight lattice P is preserved by $\widehat{W}$. Now the Weyl alcove (resp. shifted Weyl alcove) are fundamental domains for the action (resp. shifted action) of $\widehat{W}$. The set we're interested in are the weights contained in the <u>interior</u> of the shifted Weyl alcove. [5]

Now we present the fusion ring of type $SO(N)-O(K)$ as a quotient of $\mathrm{Gr}(SO(N))$. It is well known that $\mathrm{Gr}(SO(N))$ can be identified with $\mathbb{Z}[P]^W$, the invariants (under the Weyl group) of the group algebra of P, by sending a simple element to its character (see e.g. [KT]). Since $L = \varkappa Q^\vee$ is a W-invariant sublattice of P, the quotient P/L is a W-

[5] This no longer coincides with the weights in the <u>closure</u> of the Weyl alcove because the extra bounding hyperplane for the Weyl alcove is located further away from the origin than the corresponding hyperplane for the shifted Weyl alcove.

module and we can define the invariant ring $(\mathbb{Z}[P/L])^W$. We have a ring homomorphism

$$\phi_K : (\mathbb{Z}[P])^W \to (\mathbb{Z}[P/L])^W$$

obtained by extending the quotient map $P \to P/L$. The following theorem is presumably well known but hard to pin down in the literature (see e.g. [Wen11], Prop. 1.1 for the argument).

Theorem 3.3.1. *Let $\overline{\chi_\lambda}$ denote the image of χ_λ under ϕ_K.*

1. *(Modification rule.) For any $w \in \widehat{W}$ and $\lambda \in \Gamma$ such that $w.\lambda \in \Gamma$, we have*

$$\overline{\chi_{w.\lambda}} = (-1)^w \overline{\chi_\lambda}.$$

2. *The image of ϕ_K has a $\mathbb{Z}$-basis given by $\{\overline{\chi_\lambda} : \lambda \in \Gamma(SO(N) - O(K))\}$.*

3. *ϕ_K is surjective.*

It is critical to note that $w.\lambda$ refers to the shifted action of the affine Weyl group. The formula implies that the character of any irrep whose highest weight belongs to one of the walls of the shifted Weyl alcove gets sent to 0 by ϕ_K. Let $\mathcal{I}_K$ denote the kernel of ϕ_K, considered as a map on $\mathrm{Gr}(SO(N))$.

Definition 3.3.2. The *fusion ring of type $SO(N) - O(K)$*, denoted $\mathrm{Gr}(SO(N) - O(K))$, is the quotient of $\mathrm{Gr}(SO(N))$ by $\mathcal{I}_K$, with simple elements corresponding to the images of λ belonging to $\Gamma(SO(N) - O(K))$.

By the theorem we can explicitly take the fusion ring to be $\mathbb{Z}[P/L]^W$ with simple elements $\overline{\chi_\lambda}$, which we usually refer to just as λ. It is well known that there exist ribbon categories whose Grothendieck rings are isomorphic to $\mathrm{Gr}(SO(N) - O(K))$. From this we deduce the non-obvious fact that the structure coefficients of $\mathrm{Gr}(SO(N)-O(K))$ are non-negative, so it is indeed a $\mathbb{Z}$-based ring in the precise sense of Sec. 2.1.

The fusion rule of this ring has a simple description in terms of the affine Weyl group and the $SO(N)$ tensor product multiplicities $N^{\nu}_{\lambda,\mu}$, thanks to the previous theorem:

Theorem 3.3.3. *(Kac-Walton formula.) Let λ, μ, ν be simple elements of the $SO(N)-O(K)$ type fusion ring and let $\mathcal{N}^{\nu}_{\lambda,\mu}$ denote the multiplicity of ν in $\lambda \otimes \mu$. Then*

$$\mathcal{N}^{\nu}_{\lambda,\mu} = \sum_{w\in\widehat{W}} (-1)^w N^{w.\nu}_{\lambda,\mu}.$$

Indeed, both the Kac-Walton formula and Part (1) of Thm. 3.3.1 are equivalent to a geometrical algorithm for computing $\lambda \otimes \mu$ in $\mathrm{Gr}(SO(N)-O(K))$: first decompose $V(\lambda)\otimes V(\mu)$ in **Rep** $SO(N)$, and then use the shifted action of the affine Weyl group to "fold" the constituents back into the set $\Gamma(SO(N)-O(K))$, where they contribute with a sign according to the number of reflections used in folding. Note the similarity of this algorithm with Steinberg's Rule (Thm. 3.1.1). In fact Steinberg's Rule can be combined with the Kac-Walton formula to obtain

Theorem 3.3.4. *(Quantum Racah Rule.)*

$$\mathcal{N}^{\nu}_{\lambda,\mu} = \sum_{w\in\widehat{W}} (-1)^w mult_{\lambda}(w.\nu - \mu),$$

Here $\mathrm{mult}_{\lambda}(\mu)$ denotes the classical multiplicity of μ in the $SO(N)$-irrep $V(\lambda)$. Often the Kac-Walton formula is used to define the fusion ring directly, as an alternate to the definition provided here. We use this definition just to emphasize the fusion ring as a quotient of the classical Grothendieck ring.

Example 3.3.5. We compute $[1^2]\otimes[K]$ in the $SO(2n)-O(K)$ fusion rules for $n \geq 3$

and $K \geq 3$. By Eq. 3.19, $[1^2] \otimes [K]$ decomposes in $\mathrm{Gr}(SO(2n))$ as

$$[1^2] \otimes [K] = [K, 1^2] + [K+1, 1] + [K-1, 1] + [K] \tag{3.23}$$

where $[K, 1^2]$ should be replaced with $[K, 1^2]^+ + [K, 1^2]^-$ when $n = 3$. We have to see what happens to each simple element under the quotient map $\phi_K : \mathrm{Gr}(SO(2n)) \to \mathrm{Gr}(SO(2n) - O(K))$. The shifted Weyl alcove has the bounding inequality

$$\lambda_1 + \lambda_2 \leq K + 1$$

and the corresponding simple reflection s_0 of the affine Weyl group acts by

$$s_0 \cdot (\lambda_1, \lambda_2, \ldots, \lambda_n) = (K + 1 - \lambda_2, K + 1 - \lambda_1, \lambda_3, \ldots, \lambda_n).$$

Hence $[K, 1^2]$ (resp. $[K, 1^2]^+$ and $[K, 1^2]^-$ for $n = 3$) lies on the boundary of the shifted Weyl alcove, so $\phi_K([K, 1^2]) = 0$. Since $s_0 \cdot [K+1, 1] = [K]$ we have $\phi_K([K+1, 1]) = -\phi_K([K])$ by the modification rule of Thm. 3.3.1. Therefore in $\mathrm{Gr}(SO(2n) - O(K))$ we have

$$[1^2] \otimes K = [K-1, 1]. \tag{3.24}$$

In fact, it can be seen that $[K]$ is an *invertible* simple element, which means $\lambda \otimes [K]$ is simple whenever λ is simple.

Since the classical Grothendieck ring is generated (algebraically) by the shapes corresponding to columns (see Lemma 3.2.2), the same is true in the quotient:

Lemma 3.3.6. *$Gr(SO(N) - O(K))$ is generated by the simple elements corresponding to the 1-column $SO(2n)$ shapes:* $[1], [1^2], \ldots, [1^{n-1}], [1^n]^+, [1^n]^-$.

In the $SO(2n)$ case the involution σ on $\mathrm{Gr}(SO(2n))$ (which interchanges plus/minus labeled simple elements and fixes the rest) descends to the quotient. Hence by Lemma 3.2.1 we have

Lemma 3.3.7. *The fixed ring $Gr(SO(2n) - O(K))^\sigma$ is generated by the elements*

$$[1], [1^2], \ldots, [1^{n-1}], [1^n]^+ + [1^n]^-.$$

3.3.2 $SO(2n+1) - Sp(2k)$ type fusion rings

In the special case $N = 2n+1$ we can form another family of fusion rings, arising from having roots of different lengths type B. These have simple elements indexed by

$$\Gamma(SO(2n+1) - Sp(2k)) := \{\lambda \in \Gamma(SO(2n+1)) : \lambda_1 \leq k\}.$$

The discussion concerning the fusion rules is nearly identical to the $SO(N) - O(K)$ case. The difference is modifying the Weyl alcove and affine Weyl group. The new shifted Weyl alcove is the truncation of the shifted dominant Weyl chamber by the hyperplane $\{\lambda : \langle \lambda, \beta \rangle = k + 1/2\}$ where $\beta = (1, 0, \ldots, 0)$ is the *highest short root* of the $SO(2n+1)$-type root system. The affine Weyl group is the semidirect product $W \ltimes L$ where now $L = \varkappa P$ with $\varkappa = 2k + 2\langle \beta, \rho \rangle = 2k + 2n - 1$. As before, $\mathrm{Gr}(SO(N) - Sp(K))$ may be defined as the image of $\mathrm{Gr}(SO(N))$ under the map

$$\mathrm{Gr}(SO(N)) \cong (\mathbb{Z}[P])^W \to (\mathbb{Z}[P/L])^W.$$

The simple elements are the characters which correspond to shapes in $\Gamma(SO(2n+1) - Sp(2k))$. The Kac-Walton formula and quantum Racah rule still work as stated.

3.4 Fusion rings associated to orthogonal groups

While our main interest is in fusion rules associated to the special orthogonal groups, it is useful to briefly discuss fusion rings coming from the full orthogonal groups. These are the rings which are considered in the classification by Tuba and Wenzl. The close relationship between $SO(2n+1)$ and $O(2n+1)$ (namely, that every $O(2n+1)$ irrep stays irreducible upon restriction to $SO(2n+1)$) means that the Tuba-Wenzl classification for $O(2n+1)$ categories immediately implies a classification for $SO(2n+1)$ categories; this is explained in Sec. 4. For the classification of $SO(2n)$ type categories we do not need to know anything about orthogonal type categories so here we restrict our attention to the $N = 2n+1$ case. This is convenient since every $SO(2n+1)$ shape is a bonafide Young diagram, so we don't have to worry about the pesky plus/minus shapes as in $SO(2n)$.

Fusion rings associated with the full orthogonal group $O(2n+1)$ have elements in bijection with the following sets of Young diagrams:

$$\Gamma(O(2n+1) - O(K)) := \{\lambda : \lambda'_1 + \lambda'_2 \leq 2n+1, \lambda_1 + \lambda_2 \leq K\} \cup \{[K, 1^{2n-1}]\} \quad (3.25)$$

$$\Gamma(O(2n+1) - Sp(2k)) := \{\lambda : \lambda'_1 + \lambda'_2 \leq 2n+1, \lambda_1 \leq k\}. \quad (3.26)$$

In the first case, the hook shape $[K, 1^{2n-1}]$ is the only Young diagram appearing whose transpose is not a valid $O(K)$ shape. However, it is a valid $O(2n+1)$ shape since its first two columns add up to $2n+1$. In the second case, the transpose of every Young diagram which appears is a valid $Sp(2k)$ shape.

As in the generic case, there is a bijection

$$\Gamma(O(2n+1) - G) \cong \Gamma(SO(2n+1) - G) \times \{\pm 1\}.$$

Indeed, it is given by the same formula Eq. 3.12. Under this map, an $O(2n+1) - G$ shape

λ is sent to either $(\lambda, (-1)^{|\lambda|})$ or $(r(\lambda), (-1)^{|\lambda|})$, depending on whether λ or its associate diagram $r(\lambda)$ has fewer than n rows. Recall $r(\lambda)$ just differs from λ in the first column, and that the sum of their first columns is $2n+1$. In particular, the "extra" diagram in Eq. (3.25) is associate to the row diagram $[K]$. We will use the following facts, which serve to define the fusion ring of $O(2n+1)-G$ type:

Fact 1. There exists a $\mathbb{Z}$-based ring $\mathrm{Gr}(O(2n+1)-G)$ (with $G = O(K)$ or $Sp(2k)$) with a set of simple elements parametrized by $\Gamma(O(2n+1)-G)$.

Fact 2. These are the fusion rings considered in the Tuba-Wenzl classification [TW05].

Fact 3. The bijection between simple elements extends to a $\mathbb{Z}$-based ring isomorphism

$$\mathrm{Gr}(O(2n+1)-G) \cong \mathrm{Gr}(SO(2n+1)-G) \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathbb{Z}_2). \tag{3.27}$$

The first fact follows from the stronger fact that there are ribbon categories whose simple objects are parametrized by the set $\Gamma(O(2n+1)-G)$ (more detail on this in Sec. 3.6.2).

Definition 3.4.1. An $O(2n+1)-G$ *type category* is a ribbon category whose Grothendieck ring is identified with $\mathrm{Gr}(O(2n+1)-G)$.

These are also known as examples of *BCD categories* [BB01, TW05]. They are perhaps most easily described as coming from quotients of the BMW algebras specialized at certain roots of unity [Wen90], [TW00]; this is the Turaev-Wenzl skein theory referred to above. We sketch a proof of Facts (2) and (3). By [TW05], Prop. 8.6, the fusion rules for an $O(2n+1)-G$ category are completely determined by the labeling of simple isotypes and the rule for tensoring with an object X corresponding to the Young diagram [1]. The rule is simple and given by

$$[1] \otimes \lambda \cong \bigoplus_{\lambda \leftrightarrow \mu} \mu$$

where the sum is taken over all Young diagrams μ obtained by adding or removing a box from λ. Hence (2) and (3) may be proved at the same time by showing that the corresponding object in the ring $\mathrm{Gr}(SO(2n+1)-G)\otimes_{\mathbb{Z}}\mathrm{Gr}(\mathbb{Z}_2)$ satisfies this rule, which reduces to a routine check using the known fusion rules of $\mathrm{Gr}(SO(2n+1)-G)$.

Remark 3.4.2. It is interesting to note that $\mathrm{Gr}(SO(2n+1)-G)$ embeds in $\mathrm{Gr}(O(2n+1)-G)$ as the subring spanned by diagrams with an even number of boxes. For instance the $SO(2n+1)-G$ shape $[1]$ corresponds to the $O(2n+1)-G$ shape $[1^{2n}]$. In categorical language, this amounts to saying that the adjoint subcategory of an $O(2n+1)-G$ category is an $SO(2n+1)-G$ category. Furthermore, any $O(2n+1)-G$ category is $\mathbb{Z}_2$ graded according to whether a simple object has odd or even many boxes in its diagram.

3.5 Normalization of BCD-type categories

As we've seen in 2.2, a ribbon category $\mathcal{C}$ with the fusion rules of $SO(2n), Sp(2n)$ and $O(2n+1)$ are $\mathbb{Z}_2$ graded, with the grading on a simple object λ appearing in $X^{\otimes k}$ given by $(-1)^k$. By the results of Section 2.2, there is a unique spherical structure so that X is a symmetrically self-dual object. This means the left and right duality morphisms coincide, and we may represent wires labeled X by unoriented strands.

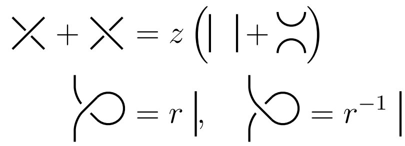
In graphical notation we represent $i_X = i'_X$ by $\smile$ and $d_X = d'_X$ by $\frown$. Let $e = i_X \circ d_X = \genfrac{}{}{0pt}{}{\smile}{\frown}$. Likewise the braid morphism $c_{X,X}$ is denoted $c = \times$. Thus $c^{-1} = \times$.

For the categories considered above, $X^{\otimes 2}$ splits into 3 simples labeled $\mathbf{1}, [2]$ and $[1^2]$. The following proposition is well known and proves that the framed link invariant coming from any symmetrically self-dual simple object whose tensor square splits into 3 simples is either the Kauffman or Dubrovnik polynomial [Kau90, TW05, MPS11].

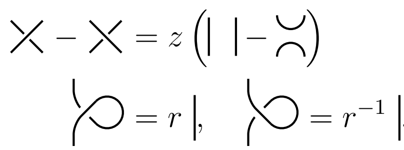
Proposition 3.5.1. *([MPS11], Thm. 3.1) Suppose X is a symmetrically self-dual simple object in a ribbon category $\mathcal{C}$ whose tensor square decomposes into three distinct simples.*

Then there are complex numbers z, r such that one of the following sets of relations hold:

- *The Kauffman relations:*



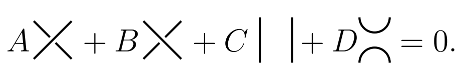
- *The Dubrovnik relations:*



Proof. First, note that r is the scalar associated to the twist $\theta = \theta_X$:

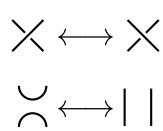
$$r = \theta.$$

Now since $\dim \mathrm{End}(X^{\otimes 2}) = 3$ there must be a non-zero linear relation of the form



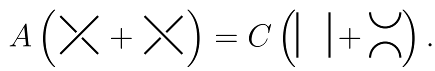
(3.28)

Consider the operator on $\mathrm{End}(X^{\otimes 2})$ which sends $f \mapsto (1 \otimes d_X)(1 \otimes f \otimes 1)(i_X \otimes 1)$. Using the braiding and rigidity axioms one checks that this operator performs the swaps



We call it the *1-click* operator since its effect on these diagrams is to rotate them by $\pi/2$

clockwise. The 1-click has order two on the space of relations of the form (3.28), and so it admits an eigenrelation with eigenvector 1 or -1. In the eigenvalue $+1$ case we have a relation of the form



Note that A must be nonzero, as otherwise  is both non-invertible and proportional to the identity. Hence we can divide by A and take $z = C/A$ (so we are in the Kauffman case). The case of a -1 eigenvalue leads in the same way to the Dubrovnik relation. □

Corollary 3.5.2. *Suppose X is a symmetrically self-dual object in a ribbon category $\mathcal{C}$ whose tensor square decomposes into three distinct simples. If $c_{X,X}$ satisfies the Dubrovnik relation, there exist $q, r \in \mathbb{C}^\times$ such that*  *has eigenvalues $\{q, -q^{-1}, r^{-1}\}$ where $r = \theta_X$ and $z = q - q^{-1}$. In the Kauffman relation case there exist $q, r \in \mathbb{C}^\times$ such that the eigenvalues of*  *are $\{iq, -iq^{-1}, ir^{-1}\}$ where $r = i\theta_X$ and $z = iq - iq^{-1}$.*

Ribbon categories with the fusion rules of one of these groups are classified by the eigenvalues of the braid element $c_{X,X}$, which we denote (q, q', r^{-1}) where q is the eigenvalue on the simple object $[2] \subset X^{\otimes 2}$, q' is the eigenvalue on $[1^2]$ and r^{-1} is the eigenvalue on the trivial irrep $\mathbf{1} \subset X_\lambda$. Twisting $\mathcal{C}$ by the cocycle $(1, f_{-1})$ changes the eigenvalues of $c_{X,X}$ by

$$(q, q', r^{-1}) \mapsto (-q, -q', -r^{-1}).$$

The resulting category is monoidally equivalent to $\mathcal{C}$. On the other hand, twisting $\mathcal{C}$ by an abelian cocycle (ω, a) modifies the braiding by $c_{X,X}(\omega, f_{\pm i}) = \pm i c_{X,X}$. If $c_{X,X}$ satisfies the Kauffman relation then $c_{X,X}(\omega, f_{\pm i})$ satisfies the Dubrovnik relation and vice versa.

Summarizing, for a ribbon tensor category which is $\mathbb{Z}_2$ graded and generated by a self-dual simple object X with $X^{\otimes 2}$ we may modify the spherical structure and/or twist by an abelian 3-cocycle to guarantee the braid element $c_{X,X}$ is symmetrically self-dual, satisfies the Dubrovnik relation and has eigenvalues $(q, -q^{-1}, r^{-1})$ for some $r, q \in \mathbb{C}^\times$.

A similar construction (but unrelated to the cocycle construction) is the mirror. Given a braided tensor category $\mathcal{C}$, the mirror $\overline{\mathcal{C}}$ is the braided tensor category obtained by defining new braid morphisms $\overline{c}_{X,Y} = c_{Y,X}^{-1}$. When $\mathcal{C}$ is ribbon, the mirror is also, with twist given by $\overline{\theta}_X = \theta_X^{-1}$. Switching from $\mathcal{C}$ to its mirror affects the eigenvalues of $c_{X,X}$ by

$$(q, q', r^{-1}) \mapsto (q^{-1}, (q')^{-1}, r)$$

without changing the monoidal equivalence class of the category.

3.6 Examples of $SO(N)$ and $O(N)$ type categories

Here we summarize the various facts regarding existence of $SO(N)$ type categories.

3.6.1 $SO(N)$ categories from quantum groups

We briefly describe $SO(N)$ categories coming from Drinfel'd-Jimbo quantum groups [Dri86, Jim86, Jan96, Kas95, CP95]. These are q-deformations $U_q\mathfrak{so}_N$ of the universal enveloping algebra of the Lie algebra $\mathfrak{so}(N)$. In the generic case, when q is not a root of unity, one can extract out of the category of finite dimensional representations a semisimple ribbon category **Rep** $U_q\mathfrak{so}_N$ whose Grothendieck ring is isomorphic to **Rep** $\mathfrak{so}_N$. Since this category was built from the Lie algebra $\mathfrak{so}_N$ rather than $SO(N)$ itself, it includes simple objects corresponding to spin representations (i.e. highest weights with half-integer coordinates), which we don't consider in the thesis.

Definition 3.6.1. For q not a root of unity, **Rep** $SO(N)_q$ is the subcategory of **Rep** $U_q\mathfrak{so}(N)$ spanned by simple objects with integer highest weights.

This category has the same fusion rules as $SO(N)$ but the eigenvalues of the braid element $c_{X,X}$ are given by $(q, -q^{-1}, q^{2n})$. There are well known Lie-theoretic formulas for the dimensions of simples and twists which depend only on q (we give the dimension formula below.) We have intermediate results in Sec. 7 which show that any ribbon category with $SO(N)$ fusion rules must have the same dimensions and twists. This will be an important step towards classification (but does not immediately imply a classification since in general the fusion rules, dimensions and twists are not a full set of invariants for a ribbon category).

When q is a root of unity the category of finite-dimensional representations is no longer semisimple. However if one instead looks at the category of *tilting modules* [AP95, Saw06] and takes the quotient by negligible morphisms, one obtains a semisimple ribbon category $(\textbf{Rep } U_q\mathfrak{so}(N))^{ss}$. The simple objects are parametrized by the highest weights of $\mathfrak{so}_N$ which are contained in a Weyl alcove, exactly as in Sec. 3.3. The only difference from the $SO(N)$-type fusion rules is the category of tilting modules includes spin representations (half integer weights) in addition to the $SO(N)$ shapes. Nevertheless the fusion rules are given by the Kac-Walton formula (now generalized to include all highest weights of $\mathfrak{so}_N$). Using this it is easily checked that the simples corresponding to integer highest weights are closed under tensor product, hence span a full tensor subcategory.

Definition 3.6.2. When q is a root of unity, $\textbf{Rep } SO(N)_q$ denotes the ribbon subcategory of $(\textbf{Rep } U_q\mathfrak{so}(N))^{ss}$ spanned by simples with integer highest weights.

The Grothendieck ring of $\textbf{Rep } SO(N)_q$ is isomorphic to $\mathrm{Gr}(SO(N)-G)$ for some G depending on q (this is the same dependence as described in Table 3.2 below), and every ring $\mathrm{Gr}(SO(N)-G)$ occurs for some value of q. The eigenvalues of the braid operator are $(q, -q^{-1}, q^{-(N-1)})$.

The promised and well-known dimension formula (valid for any $q \neq \pm 1$) is a q-version of Weyl's dimension formula:

$$\dim_{\textbf{Rep } SO(N)_q}(\lambda) = \prod_{\alpha \in \Phi^+} \frac{[\langle \alpha, \lambda + \rho \rangle]_q}{[\langle \alpha, \rho \rangle]_q}. \tag{3.29}$$

Here $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ and the product is over all positive roots.

Remark 3.6.3. One can use the formula to check

$$\dim_{\mathbf{Rep}\ SO(2n)_q}(\lambda^+) = \dim_{\mathbf{Rep}\ SO(2n)_q}(\lambda^-) \tag{3.30}$$

when λ^+ and λ^- differ just by a sign in their last entry.

Remark 3.6.4. The Lie superalgebra $osp(1|2n)$ has semisimple representation theory and its tensor product rules are the same as for $SO(2n+1)$ [RS82]. For generic q there exists a q-deformation $U_q\ osp(1|2n)$ and the associated representation category $\mathbf{Rep}\ U_q\ osp(1|2n)$ has the same fusion rules as $SO(2n+1)$ [Zha92b]. By ([Zha92a], Sec. C.1) the fundamental object X has braid eigenvalues [6]

$$(-q, q^{-1}, q^{-2n}).$$

Presumably there are associated fusion categories when q is not a root of unity which have fusion rules of $SO(2n+1) - G$ type. If this is the case then they are classified by our $SO(2n+1)$ theorem. Zhang [Zha92a] and Blumen [Blu05] have studied finite-dimensional versions of $U_q\ osp(1|2n)$ when q is a root of unity, from which one should be able to extract a semisimple category which likely has fusion rules of $SO(2n+1) - G$ type.

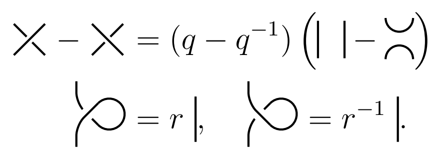
3.6.2 $O(N)$ categories from BMW algebras

The simplest construction to get ribbon categories associated to the full orthogonal group $O(N)$ is to take a quotient of a tangle category by a Dubrovnik (or Kauffman) skein relation. Turaev and Wenzl introduced these categories by this method in [TW97]. They are further studied by Beliakova and Blanchet [BB01]. The connection between BMW

[6] Actually Zhang's q is our $-q^{-1}$. See also Blumen's work, e.g. ([Blu06], Cor. 7.1), whose q agrees with ours.

algebras and quantum groups is explained by Wenzl in [Wen90], who first observed that certain endomorphism algebras for quantum group representations are quotients of the BMW algebras. In the classical setting this is the role played by the Brauer algebra and indeed the BMW algebras form a deformation of the Brauer algebras.

Roughly speaking, one considers the tangle category whose objects are natural numbers and the morphism space $\mathrm{Hom}(n, m)$ is the $\mathbb{C}$-space spanned by (n, m)-ribbons connecting n lower points to m upper points. Given non-zero complex numbers q, r we have the Dubrovnik skein relation



The $\mathbb{C}$-space of $(0, 0)$-tangles (i.e. linear combinations of framed links) modulo these relations is 1-dimensional, and yields an invariant of framed links called the *Kauffman bracket* [Kau90]. Using the tangles $\cup$ and $\cap$ one can close any (n, n)-tangle to a $(0, 0)$-tangle and compute its invariant using the Kauffman bracket. This yields so-called *Markov traces* $\mathrm{Hom}_{\mathcal{T}}(n, n) \to \mathbb{C}$. Upon quotienting by the tensor ideal of negligible morphisms and idempotent completing we obtain a ribbon category $\mathcal{V}(q, r)$.[7] The Markov traces carry over to Markov traces on the endomorphism algebras of $\mathcal{V}(q, r)$. These endomorphism algebras are well known quotients of *BMW algebras* which are the algebras one gets by quotienting the braid group algebra by the Dubrovnik relation (so named for Birman-Murakami-Wenzl). The construction of the categories by Turaev and Wenzl uses detailed results on the structure of the BMW algebras and certain semisimple quotients by Wenzl [Wen11]. We need detailed information only in the odd case $O(2n + 1)$.

1. The category is generated by a single self dual object X (the “1-strand” object).

[7]With respect to a trace tr, a *negligible morphism* f is one which satisfies $\mathrm{tr}(f \circ g) = \mathrm{tr}(h \circ f) = 0$ for all g, h which compose with f.

Its tensor square splits into 3 simple objects and the braid element $c_{X,X}$ has the eigenvalues $(q, -q^{-1}, r^{-1})$.

2. For q not a root of unity and $r = \pm q^{2n}$ for some $n \geq 1$ the category $\mathcal{V}(q, r)$ is semisimple with the fusion rules of $O(2n+1)$. The braid morphism $c_{X,X}$ has eigenvalues $(q, -q^{-1}, r^{-1})$. Explicit formulas for the q-dims were derived by Wenzl [Wen90] and are listed in this context by [BB01], Eqs. (6) and (7). We need information concerning the simple object corresponding to the determinant representation, labeled by the Young diagram $[1^{2n+1}]$. When $r = q^{2n}$ the q-dims and twists are given by

$$\dim_{\mathcal{V}(q,q^{2n})}[1^{2n+1}] = 1$$
$$\theta_{[1^{2n+1}]} = 1$$

whereas when $r = -q^{2n}$ they are given by

$$\dim_{\mathcal{V}(q,-q^{2n})}[1^{2n+1}] = 1$$
$$\theta_{[1^{2n+1}]} = -1.$$

3. When q^2 is a primitive l-th root of unity and $r = \pm q^{2n}$ with $l \geq 2n+2$, the category $\mathcal{V}(q, r)$ is semisimple with the fusion rules of $O(2n+1) - G$, where G depends on q according to the table:

Table 3.2: Relation between fusion rule and order of q^2 for $O(2n+1)$ categories.

	fusion rule
l even	$O(2n+1) - O(l-2n+1)$
l odd, $q^l = -1$	$O(2n+1) - O(l-2n+1)$
l odd, $q^l = 1$	$O(2n+1) - Sp(l-2n-1)$

The braid morphism $c_{X,X}$ again has eigenvalues $(q, -q^{-1}, r^{-1})$. The q-dims and twists are given by the same formula, in particular for the simple object corresponding to the Young diagram $[1^{2n+1}]$.

4. For $q = \pm 1$ the Kauffman relation does not suffice to reduce the $(0,0)$-tangles to a 1-dimensional space. Upon adding the relation $\bigcirc = d$ for some $d \in \mathbb{Z} \setminus \{0\}$ one can define a trace as before and quotienting by negligibles yields a semisimple symmetric category. For $d = 2n+1$ or $d = -2n+1$ for some $n \geq 1$ this category has the fusion rules of $O(2n+1)$. If $d = 2n$ then the category has $O(2n)$ fusion rules while $d = -2n$ yields $Sp(2n)$ fusion rules.

Remark 3.6.5. The full orthogonal categories corresponding to $r = q^{2n}$ can be obtained from the quantum group categories **Rep** $SO(N)_q$ by taking the Deligne product with **Rep** $\mathbb{Z}_2$ (c.f. the proof of Thm. 4.0.3). The categories with $r = -q^{2n}$ are **Rep** $SO(2n+1)_{-q} \boxtimes \mathbb{Z}_2$, twisted by the cocycle $(1, -1)$. Hence it is natural to view these as coming from the Lie superalgebra $osp(1|2n)$ (see Remark 3.6.4 and 4.0.7).

4

Classification of $SO(2n+1)$ type categories

The classification result of Tuba and Wenzl implies a classification of ribbon categories with the fusion rules of $SO(N)$ for $N = 2n+1$ odd and at least 3, as explained now.

For a ribbon category $\mathcal{C}$ of type $SO(2n+1)$ or $SO(2n+1) - G$ and $\lambda \in \Gamma(\mathcal{C})$, X_λ refers to a simple object of $\mathcal{C}$ corresponding to the partition λ. Let $X = X_{[1]}$ be an object corresponding to the fundamental irrep. The basic fusion rule (Eq. 3.16) implies

$$X \otimes X \cong \mathbf{1} \oplus X_{[1^2]} \oplus X_{[2]}.$$

Here we require $2n+1 \geq 3$ since for $2n+1 = 1$ the Young diagram $[1^2]$ is not a valid shape. To ensure the diagrams are all valid in the $SO(2n+1) - G$ fusion case we also assume the rank of G is at least 3. In other words G must be $O(2k), O(2k+1)$ or $Sp(2k)$ with $k \geq 3$.

Remark 4.0.1. The quantity k is often called the *level* of the fusion ring. The level 1 and level 2 cases contain several interesting families of categories that we are omitting in this discussion (and the pending result).

Let us precisely state the special case of Tuba and Wenzl's result on $O(2n+1)$ and $O(2n+1) - G$ type categories. By the Sec. 3.5 the braid eigenvalues of $c_{X,X}$ are either of the form $(q, -q^{-1}, r^{-1})$ (the Dubrovnik case) or (q, q^{-1}, r^{-1}) (the Kauffman case). In either case q denotes the eigenvalue of $c_{X,X}$ on the object [2]. All the $O(2n+1)$ type categories are $\mathbb{Z}_2$ graded so in the next theorem we assume all categories are equipped with the unique spherical structure so that the generating object X is symmetrically self dual.

Theorem 4.0.2. *([TW05], Thm. 9.4) Ribbon categories with the fusion rules of $O(2n+1)$ or $O(2n+1) - G$ for some G with rank ≥ 3 are determined, up to equivalence, by the eigenvalues of the braid operator $c_{X,X}$, which must be of the form $(q, -q^{-1}, \varepsilon q^{-2n})$ (the Dubrovnik case) or $(q, q^{-1}, \varepsilon q^{-2n})$ (the Kauffman case), where $\varepsilon \in \{\pm 1\}$ and the order of q as a root of unity (possibly ∞) is determined by the fusion rules as in Table 3.2. More precisely, two such categories $\mathcal{C}$ and $\mathcal{C}'$ with the same fusion rules and braid eigenvalues q and q' are monoidally eqivalent if and only if they are both Dubrovnik (or both Kauffman), $q' \in \{q^{\pm 1}\}$ and $\varepsilon' = \varepsilon$, or $q' \in \{-q^{\pm 1}\}$ and $\varepsilon' = -\varepsilon$. They are ribbon equivalent if and only if $q = q'$ and $\varepsilon = \varepsilon'$.*

The "if" part of the theorem may be restated as: every $O(2n+1)$ or $O(2n+1) - G$ type category is a cocycle/mirror modification of one of the ribbon categories $\mathcal{V}(q, q^{2n})$ or $\mathcal{V}(q, -q^{2n})$ of Sec. 3.6.2 with q belonging to a fundamental domain for the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $q \mapsto \pm q^{\pm 1}$. The idea of $SO(2n+1)$ classification is very simple. If you start with an $SO(2n+1)$ type category $\mathcal{C}$ then $\mathcal{C} \boxtimes \textbf{Rep}\ \mathbb{Z}_2$ has the fusion rules of $O(2n+1)$, so we can apply the Tuba-Wenzl classification to this category.

Theorem 4.0.3. *(a) Ribbon categories of type $SO(2n+1)$ with non-symmetric braidings are determined (up to monoidal equivalence) by the eigenvalues of $c_{X,X}$, which are of the form $(q, -q^{-1}, q^{-2n})$ for q not a root of unity. More precisely, two such categories with eigenvalues $(q, -q^{-1}, q^{-2n})$ and $(q', -q'^{-1}, q'^{-2n})$ are monoidally equivalent if and only if $q' \in \{q^{\pm 1}\}$. They are ribbon equivalent if $q = q'$.*

(b) Ribbon categories of type $SO(2n+1) - G$ with $n \geq 1$ and rank$(G) \geq 3$ must have non-symmetric braiding and are determined (up to monoidal equivalence) by the eigenvalues of $c_{X,X}$ which are of the form $(q, -q^{-1}, q^{-2n})$ where q is a root of unity whose order is determined by G (just as in Table 3.2). More precisely, two such categories with eigenvalues $(q, -q^{-1}, q^{-2n})$ and $(q', -q'^{-1}, q'^{-2n})$ are monoidally equivalent if and only if $q' \in \{q^{\pm 1}\}$. They are ribbon equivalent if $q = q'$.

Remark 4.0.4. Recall that $SO(2n+1)$ categories refer to ribbon categories with a *fixed* identification of their Grothendieck ring with $\text{Gr}(SO(2n+1))$ or $\text{Gr}(SO(2n+1) - G)$. When we say "two categories are monoidally/ribbon equivalent" in the theorems above and proof below we mean equivalent via an equivalence which is the identity map on the level of Grothendieck rings.

Proof. First we consider the generic case, with infinitely many simple isotypes. Given a ribbon category $\mathcal{C}$ with the tensor product rules of $SO(2n+1)$, form the Deligne product $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ where $\mathbf{Rep}\ \mathbb{Z}_2$ is considered as a ribbon category with the standard symmetric braiding and trivial twists. $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ becomes a ribbon category with straightforward braiding and twist, e.g.

$$c_{X \boxtimes \epsilon, Y \boxtimes \epsilon'} = c_{X,Y} \otimes 1$$

$$\theta_{X \boxtimes \epsilon} = \theta_X \otimes 1.$$

Then $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ has the tensor product rules of $O(2n+1)$ by Eq. (3.13). If X is the object corresponding to the fundamental representation of $SO(2n+1)$ then $X \boxtimes -1$ is the object of $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ corresponding to the fundamental irrep of $O(2n+1)$. Note that by Thm. 2.2.5 the object X in $\mathcal{C}$ is symmetrically self-dual, so the same is true for $X \boxtimes -1$ in $\mathbb{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ without having to change the spherical structure. To apply the Tuba-Wenzl classification we examine the eigenvalues of $c_{X \boxtimes -1, X \boxtimes -1}$ which are equal to the eigenvalues of $c_{X,X}$.

Let q denote the eigenvalue of X on $[2] \subset X^{\otimes 2}$. First we show that the all the eigenvalues of $c_{X,X}$ are given by $(q, -q^{-1}, q^{-2n})$, i.e. the categories must satisfy the Dubrovnik (and not the Kauffman) relation and the third parameter r is equal to q^{2n}. If $q \notin \{\pm 1, \pm i\}$ then the Tuba-Wenzl classification states that q is not a root of unity and $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ is ribbon equivalent to a cocycle or mirror modification of $\mathcal{V}(q', q'^{2n})$ or $\mathcal{V}(q', -q'^{2n})$ for some $q' \in \{q, q^{-1}\}$ (see Secs. 3.6 and 3.5). In $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ the q-dim and twist of the object $1 \boxtimes -1$ corresponding to the determinant representation (with Young diagram $[1^{2n+1}]$) are both 1. The same is true for $\mathcal{V}(q', q'^{2n})$. Furthermore, we see from the cocycle modification rules (Prop. 2.3.2) that modifying $\mathcal{V}(q', q'^{2n})$ by any non-trivial cocycle and character alters (at least one of) the q-dim or twist of $1 \boxtimes -1$. Hence if $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ is ribbon equivalent to a twist of $\mathcal{V}(q', q'^{2n})$, then the twist is trivial, $q' \in \{q^{\pm 1}\}$, and the eigenvalues of $c_{X,X}$ are of the form $(q, -q^{-1}, q^{-2n})$. On the other hand consider what happens if $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ is ribbon equivalent to a twist of $\mathcal{V}(q', -q'^{2n})$. There is a unique cocycle modification which makes both the q-dim and twist of $[1^{2n+1}]$ equal to 1, namely the abelian cocycle $(1, f_{-1})$. Hence if $\mathcal{C} \boxtimes \mathbf{Vect}_{\mathbb{Z}_2}$ is equivalent to $\mathcal{V}(q', -q'^{2n})$ then $q' \in \{-q^{\pm 1}\}$ and $c_{X,X}$ again has eigenvalues $(q, -q^{-1}, q^{-2n})$.

Now suppose two categories $\mathcal{C}, \mathcal{C}'$ have the same braid eigenvalues $(q, -q^{-1}, q^{-2n})$. Then the above discussion shows $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ is ribbon equivalent to $\mathcal{C}' \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ via an equivalence which induces the identity on Grothendieck rings. In particular this restricts to a ribbon equivalence between the subcategories $\mathcal{C} \boxtimes 1$ and $\mathcal{C}' \boxtimes 1$ so $\mathcal{C}$ and $\mathcal{C}'$ are ribbon equivalent. Now if $q' = q^{-1}$ then the mirror of $\mathcal{C}$ has the same eigenvalues as $\mathcal{C}'$ so $\mathcal{C}'$ is ribbon equivalent to $\overline{\mathcal{C}}$, which is monoidally equivalent to $\mathcal{C}$.

Conversely, if $\mathcal{C}$ and $\mathcal{C}'$ are monoidally equivalent then $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ and $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ are monoidally equivalent and from the "only if" part of the Tuba-Wenzl thereom we conclude that $q' \in \{\pm q^{\pm 1}\}$. It suffices to show that if $q' = -q$ then actually $\mathcal{C}$ and $\mathcal{C}'$ are not monoidally equivalent. The previous discussion shows $\mathcal{C} \boxtimes \mathbf{Rep}\ \mathbb{Z}_2$ is ribbon equivalent to (possibly the mirror of) either $\mathcal{V}(q, q^{2n})$ or $\mathcal{V}(-q, -q^{2n})^{(1,f_{-1})}$. Similarly if $q' = -q$ then $\mathcal{C}' \boxtimes$ **Rep** $\mathbb{Z}_2$ is ribbon equivalent to one of $\mathcal{V}(-q, q^{2n})$ or $\mathcal{V}(q, -q^{2n})^{(1,f_{-1})}$ (or a mirror of these). In any case $\mathcal{C}' \boxtimes$ **Rep** $\mathbb{Z}_2$ is not monoidally equivalent to $\mathcal{C} \boxtimes$ **Rep** $\mathbb{Z}_2$ by the Tuba-Wenzl theorem. Hence $\mathcal{C}$ and $\mathcal{C}'$ are themselves not monoidally equivalent.

The case of a fusion ring (with finitely many isotypes) is virtually identical thanks to the facts stated in Sec. 3.4. In particular, by Fact (3), $\mathcal{C} \boxtimes$ **Rep** $\mathbb{Z}_2$ has $O(2n+1)$ type fusion rules and by Fact (2) the Tuba-Wenzl classification applies to $\mathcal{C}\boxtimes$**Rep** $\mathbb{Z}_2$. The same considerations concerning the twist and dimension of the object $X \boxtimes -1$ (corresponding to the Young diagram $[1^{2n+1}]$) are identical and lead to the same conclusion. □

The theorem may be rephrased in terms of quantum groups (c.f. [EO18], Thms A.1 and A.3 for the $SO(3)$ case without the ribbon assumption).

Corollary 4.0.5. *Every $SO(2n+1)$ or $SO(2n+1) - G$ type ribbon category with braid eigenvalue q is ribbon equivalent to* **Rep** $SO(2n+1)_q$.

Proof. The quantum group categories achieve every possible value for q (see Sec. 3.6). □

Remark 4.0.6. We see that on one hand there are "fewer" $SO(2n+1)$ categories than $O(2n+1)$ categories. In particular, every $SO(2n+1)$ category must be Dubrovnik, all self-dual objects are symmetrically self-dual, and we must have $r = q^{2n}$ and not $r = -q^{2n}$. However the latter family did not really disappear since the $SO(2n+1)$ categories corresponding to $\pm q$ are not equivalent (at least by an equivalence which is the identity on the Grothendieck ring). They come from the two families of $O(2n+1)$ categories with $\varepsilon = \pm 1$. In the $q = 1$ limit they produce distinct symmetric tensor categories, namely **Rep** $SO(2n+1)$ and **Rep** $osp(1|2n)$.

We also obtain the following result (see Remark 3.6.4):

Corollary 4.0.7. *For q not a root of unity, we have*

$$\textbf{Rep } U_q\ osp(1|2n) \cong \textbf{Rep } SO(2n+1)_{-q}$$

as ribbon categories.

In particular, the two categories give identical link invariants, reproducing a result of Clark [Cla15].

5

Monoidal algebras and their diagonals

The classification of $SO(2n)$-categories will not directly rely on the results on Tuba and Wenzl but it follows their strategy. The strategy of [KW93] and [TW05] is to describe the family of algebras $\mathrm{End}(X^{\otimes n})$ where X is a simple object corresponding to the fundamental (or vector) representation of the underlying Lie group. For this to work we need some understanding of the relationship between a category generated by an object X and the family of algebras $\mathrm{End}(X^{\otimes n})$.

Our approach here is very similar to [TW05], Section 4. The main difference in exposition is an emphasis on the cocycle construction. Our results are slightly stronger since they do not require any sort of braiding assumption.

Definition 5.0.1. A *monoidal algebra* is a strict $\mathbb{C}$-linear monoidal category $\mathcal{A}$ with objects $\mathbf{1}, X, X^2, X^3, \ldots$ in bijection with $\mathbb{N}$ with tensor product rules $X^k \otimes X^l = X^{k+l}$.

Sometimes we denote a monoidal algebra as a pair $(\mathcal{A}, X)$ to emphasize that the choice of the object X is part of the definition (i.e. monoidal algebras are "based"). Accordingly, a morphism between monoidal algebras $(\mathcal{A}, X)$ and $(\mathcal{B}, Y)$ is a monoidal functor $F : \mathcal{A} \to \mathcal{B}$ satisfying $F(X) = Y$. It turns out that any monoidal functor between monoidal algebras is (monoidally) naturally isomorphic to a strict monoidal functor. Furthermore, any monoidal equivalence between monoidal algebras is naturally isomorphic to a strict monoidal isomorphism (these facts are proved in Lemma 5.0.7 below). Hence to classify monoidal algebras up to monoidal equivalence it suffices to use the following simpler notion of isomorphism:

Definition 5.0.2. An *isomorphism* of monoidal algebras is a morphism of monoidal algebras $F : \mathcal{A} \to \mathcal{B}$ which is strict monoidal:

$$F(f \otimes g) = F(f) \otimes F(g),$$

and bijective on Hom-spaces.

Definition 5.0.3. Given a strict tensor category $\mathcal{C}$ and object X of $\mathcal{C}$ the *monoidal algebra generated by* X, denoted $\langle X \rangle$, is the full (tensor) subcategory with objects $\mathbf{1}, X, X^{\otimes 2}, \dots$.

Remark 5.0.4. When $\mathcal{C}$ is not strict, one can form a strictification $\mathcal{C}^{\text{str}}$ and define the monoidal algebra generated by X in $\mathcal{C}^{\text{str}}$ (the equivalence class of this monoidal algebra does not depend on the choice of strictification). More directly, we can form $\langle X \rangle$ as follows. When $\mathcal{C}$ is not a strict category we use the notation X^n for the object $(\dots((X \otimes X) \otimes X) \cdots \otimes X)$. Then $\langle X \rangle$ is defined (as an abelian category) to be the full subcategory of $\mathcal{C}$ with objects $\mathbf{1}, X, X^2, \dots$. To define the tensor structure let $f \in \mathrm{Hom}_{\mathcal{C}}(X^k, X^p)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X^l, X^q)$. Then $f \otimes_{\langle X \rangle} g \in \mathrm{Hom}_{\mathcal{C}}(X^{k+l}, X^{p+q})$ is given by

$$f \otimes_{\langle X \rangle} g = \alpha_{p,q}^{-1}(f \otimes_{\mathcal{C}} g)\alpha_{k,l} \tag{5.1}$$

where $\alpha_{i,j} : X^{i+j} \to X^i \otimes_{\mathcal{C}} X^j$ is the appropriate associator (in $\mathcal{C}$). This defines a strict monoidal structure on $\langle X \rangle$ and the resulting monoidal algebra is equivalent to the one obtained by first strictifying $\mathcal{C}$.

With strictification in mind we occasionally abuse terminology, using the notation $X^{\otimes k}$ to refer to X^k.

Definition 5.0.5. An object X of a semisimple tensor category $\mathcal{C}$ *generates* $\mathcal{C}$ if every simple object of $\mathcal{C}$ appears in some tensor power of X.

As one might expect, if X generates $\mathcal{C}$ then the category $\mathcal{C}$ can be recovered from the monoidal algebra $\langle X \rangle$. To recover $\mathcal{C}$ from $\langle X \rangle$ we add subobjects using idempotent completion, and direct sums using the additive completion (also called the matrix construction). The result, denoted $\overline{\langle X \rangle}$, is always a strict category, and is rigid, spherical, braided, ribbon etc. whenever $\mathcal{C}$ is. The construction is functorial, so equivalent monoidal algebras give rise to equivalent completions. This is well known and a proof can be found in [TW05], Sec. 3.

Theorem 5.0.6. *Suppose $\mathcal{C}$ is a semisimple tensor category generated by X. Then $\mathcal{C}$ is monoidally equivalent to $\overline{\langle X \rangle}$. If $\mathcal{C}$ is braided (ribbon) then the equivalence is braided (ribbon).*

This theorem indicates that to identify (up to monoidal equivalence) a category $\mathcal{C}$ generated by an object X, it suffices to identify the (monoidal equivalence class of) the monoidal algebra $\langle X \rangle$. In fact it is enough to consider a seemingly stronger notion of equivalence for monoidal algebras, which is a consequence of various well known strictification results:

Lemma 5.0.7. *Suppose $\mathcal{C}, \mathcal{D}$ are semisimple tensor categories generated by objects X, Y respectively. Then there exists a monoidal equivalence $(F, \gamma) : \mathcal{C} \to \mathcal{D}$ for which $F(X)$ is isomorphic to Y if and only if there is a strict monoidal isomorphism $\langle X \rangle \to \langle Y \rangle$ taking X to Y.*

Proof. On one hand, the functoriality of the construction $\langle X \rangle \to \overline{\langle X \rangle}$ shows that a strict monoidal isomorphism $\langle X \rangle \to \langle Y \rangle$ gives rise to a monoidal equivalence from $\overline{\langle X \rangle} \simeq \mathcal{C}$ to $\overline{\langle Y \rangle} \simeq \mathcal{D}$ (which sends X to Y).

For the other direction, we must show that a monoidal equivalence (F, γ) gives rise to a strict monoidal isomorphism between the monoidal algebras. Fix an isomorphism $u : F(X) \to Y$. First we construct a strict tensor functor $F' : \langle X \rangle \to \langle Y \rangle$ from F in the following manner. It is clear what F' does on objects and on a morphsim $f : X^i \to X^j$ define $F'(f)$ to be the unique morphism making the following diagram commute:

$$\begin{array}{ccc} F(X^i) & \xrightarrow{F(f)} & F(X^j) \\ \downarrow \Gamma & & \downarrow \Gamma \\ F(X)^i & & F(X)^j \\ \downarrow u^{\otimes i} & & \downarrow u^{\otimes j} \\ Y^i & \xrightarrow{F'(f)} & Y^j \end{array}$$

Here the maps Γ stand for any morphism with the given source and target which are obtained through applications of the monoidal structure axioms for (F, γ). Coherence for monoidal functors ensures there is a unique such map (so it doesn't matter how you reduce $F(X^i)$ to $F(X)^i$ using γ and associators, see [JS93], Cor. 1.8 and [ML98], Sec. XI.2). This defines a ($\mathbb{C}$-linear) functor. Using coherence for monoidal functors it can be checked that F' is strict monoidal.

Hence quasi-inverse monoidal equivalences $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ give rise to strict monoidal equivalences F' and G' between $\langle X \rangle$ and $\langle Y \rangle$. Let $\eta : G'F' \to \mathrm{id}_{\langle X \rangle}$ and $\mu : \mathrm{id}_{\langle Y \rangle} \to F'G'$ be monoidal natural isomorphisms. We can modify G' to produce a new functor G'' which is identical to G' on objects and defined on a morphism $g : Y^i \to Y^j$ by the commutative diagram

$$\begin{array}{ccc} G'F'(X^i) = X^i & \xrightarrow{G'(g)} & X^j = G'F'(X^j) \\ \eta_{X^i} \downarrow & & \eta_{X^j} \downarrow \\ X^i & \xrightarrow{G''(g)} & X^j \end{array}$$

Since η is a monoidal natural isomorphsim, G'' is a strict monoidal functor. By construction it is a left inverse to F, ie $G''F' = 1_{\mathcal{C}}$. Similarly, one can use μ to produce a right inverse to F' from G'. Since F' has a left and right inverse, the inverses are equal and F' is an isomorphism. $\square$

5.1 Reconstruction for $\mathbb{Z}_2$-graded monoidal algebras

We now examine to what extent a $\mathbb{Z}_2$-graded monoidal algebra is determined by its diagonal subalgebra.

Definition 5.1.1. A monoidal algebra $\mathcal{A} = \langle X \rangle$ is $\mathbb{Z}_2$-*graded* if $\mathcal{C} = \overline{\mathcal{A}}$ is rigid, $\mathbb{Z}_2$-graded, and X is self-dual in $\mathcal{C}$.

Definition 5.1.2. The *diagonal* of a monoidal algebra $\mathcal{A}$, denoted $\Delta\mathcal{A}$, is the monoidal algebra with the same objects as $\mathcal{A}$ and

$$\mathrm{Hom}_{\Delta\mathcal{A}}(X^{\otimes k}, X^{\otimes l}) = \begin{cases} \mathrm{End}_{\mathcal{A}}(X^{\otimes k}) & \text{if } k = l \\ 0 & \text{if } k \neq l. \end{cases}$$

Definition 5.1.3. The *adjoint subalgebra* of a $\mathbb{Z}_2$-graded monoidal alegbra $\mathcal{A}$, denoted $\mathrm{Ad}\mathcal{A}$, is the full subcategory of $\mathcal{A}$ with objects $\mathbf{1}, X^2, X^4, \ldots$.

The diagonal algebra is the main invariant we will use to classify ribbon categories with the tensor product rules of $SO(2n)$. Its data consists of a family (or "tower") of semisimple algebras $\mathrm{End}(X^0), \mathrm{End}(X^1), \mathrm{End}(X^2), \ldots$ together with a bilinear tensor product operation

$$\mathrm{End}(X^i) \times \mathrm{End}(X^j) \to \mathrm{End}(X^{i+j})$$

taking (f, g) to $f \otimes g$. It is required that the tensor product is associative, unital, and satisfies the interchange axiom:

$$(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g').$$

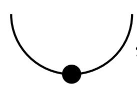
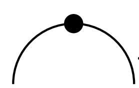
One might expect that the diagonal contains all the information of a $\mathbb{Z}_2$-graded monoidal algebra, since the only missing information is the off-diagonal Hom spaces

$\mathrm{Hom}_{\mathcal{A}}(X^i, X^j)$, which are only non-zero when i and j have the same parity. But if $i + j$ is even, then in $\mathcal{C}$ the left duality provides a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X^i, X^j) \cong \mathrm{End}_{\mathcal{C}}(X^{(i+j)/2}).$$

This provides a way to recover the off-diagonal Hom spaces from the endomorphism algebras. We follow this strategy but find that up to isomorphism there are precisely two monoidal algebras with the same diagonal and they differ by a 3-cocycle twist. To prove this, we first describe the additional piece of data needed to reconstruct a monoidal algebra from its diagonal.

Suppose $\mathcal{A} = \langle X \rangle$ is a $\mathbb{Z}_2$-graded monoidal algebra and let $\mathcal{C} = \overline{\mathcal{A}}$. By assumption X is self-dual so in $\mathcal{C}$ the trivial object $\mathbf{1}$ appears (with mulitplicity 1) in $X^{\otimes 2}$. Let $\iota : \mathbf{1} \to X^2$ and $\pi : X^2 \to \mathbf{1}$ be chosen such that $\pi \circ \iota = 1$ and $\Pi_{\mathcal{A}} = \pi \circ \iota$ is an idempotent with image $\mathbf{1}$. In the graphical calculus for $\mathcal{A}$ we introduce the following symbols for ι and π:

$\iota =$  , $\pi =$  .

We use the dots to distinguish these cups and caps from those which are normalized to satisfy the S-bend relations. Note that π and ι may be rescaled but $\Pi_{\mathcal{A}}$ is independent of this choice.

If $\Psi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of $\mathbb{Z}_2$-graded monoidal algebras then $\Psi(\Pi_{\mathcal{A}}) = \Pi_{\mathcal{A}'}$, since both $\Psi(\Pi_{\mathcal{A}})$ and $\Pi_{\mathcal{A}'}$ project onto the trivial object which appears with multiplicity 1 in the second tensor power.

Definition 5.1.4. Suppose $\mathcal{A} = \langle X \rangle$ and $\mathcal{A}' = \langle Y \rangle$ are $\mathbb{Z}_2$-graded monoidal algebras. We say $\mathcal{A}'$ is an *extension of the diagonal* $\Delta\mathcal{A}$ if there is an isomorphism (of monoidal algebras) $\psi : \Delta\mathcal{A} \to \Delta\mathcal{A}'$ such that $\psi(\Pi_{\mathcal{A}}) = \Pi_{\mathcal{A}'}$. We say $\mathcal{A}$ and $\mathcal{A}'$ are *diagonally isomorphic* if the isomorphism ψ can be extended to an isomorphism of monoidal algebras $\mathcal{A} \to \mathcal{A}'$.

Given a $\mathbb{Z}_2$-graded monoidal algebra $\mathcal{A}$ we use the cocycle construction of Sec. 2.3 to twist the associator by a normalized 3-cocycle $\omega \in H^3(\mathbb{Z}_2, \mathcal{C}^\times)$ and obtain a new tensor category $\mathcal{A}(\omega)$ whose objects coincide with those of $\mathcal{A}$. This is not a monoidal algebra (it is not strict) but the object X still generates a monoidal algebra, following the construction of Remark 5.0.4.

Definition 5.1.5. The *twist of* $\mathcal{A}$ *by* $\omega \in H^3(\mathbb{Z}_2, \mathbb{C}^\times)$ is the monoidal algebra $\mathcal{A}^\omega$ generated by X in $\mathcal{A}(\omega)$.

Proposition 5.1.6. *Suppose* $\mathcal{A} = \langle X \rangle$ *is a* $\mathbb{Z}_2$*-graded monoidal algebra. Then for each* 3*-cocycle* $\omega \in H^3(\mathbb{Z}_2, \mathbb{C}^\times)$ *the monoidal algebra* $\mathcal{A}^\omega$ *extends the diagonal of* $\mathcal{A}$*.*

Proof. Using the construction of Remark 5.0.4, the objects and morphism spaces of $\mathcal{A}$ and $\mathcal{A}^\tau$ are the same (the difference between the monoidal algebras lies in the tensor products of morphisms). Thus we can define the identity functor from $\mathcal{A}$ to $\mathcal{A}^\tau$, and we claim its restriction to $\Delta\mathcal{A}$ is a diagonal isomorphism. We check this is a strict monoidal functor, i.e. $f \otimes_{\mathcal{A}} g = f \otimes_{\mathcal{A}^\tau} g$ whenever f and g are endomorphisms. Indeed, the definition of tensor product in $\mathcal{A}^\tau$ dictates that for $f \in \mathrm{End}(X^{\otimes k})$ and $g \in \mathrm{End}(X^{\otimes l})$, we have

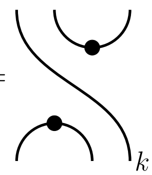
$$f \otimes_{\mathcal{A}^\tau} g = (\alpha'_{k,l})^{-1}(f \otimes_{\mathcal{A}} g)\alpha'_{k,l}.$$

Now the associator $\alpha'_{k,l}$ is a composition of smaller associators $\alpha'_{r,s,t}$ in $\mathcal{A}^\tau$ which are all scalar multiples of the identity of $X^{\otimes k+l}$. Therefore $\alpha'_{k,l}$ commutes with $f \otimes_{\mathcal{A}} g$ and so $f \otimes_{\mathcal{A}^\tau} g = f \otimes_{\mathcal{A}} g$ as needed.

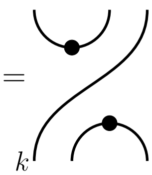
As $\mathcal{A}$ and $\mathcal{A}^\tau$ have the same Hom-spaces, it is clear that $\Pi_{\mathcal{A}} = \Pi_{\mathcal{A}'}$ (in fact, we can pick the same ι's and π's). □

Now we consider when an isomorphism $\psi : \Delta\mathcal{A} \to \Delta\mathcal{A}'$ can be extended to an isomorphism between $\mathcal{A}$ and $\mathcal{A}'$. Suppose $\mathcal{A}$ is generated by X and $\mathcal{A}'$ by Y and fix morphisms ι_X, π_X and ι_Y, π_Y as above. For each $k = 1, 2, \ldots$ define an element $\tau_{X,k} \in$

End(X^{k+2}) by $\tau_X = (1_{X^k} \otimes \iota_X) \circ (\pi_X \otimes 1_{X^k})$:

$\tau_{X,k} =$ 

Similarly we define $\upsilon_{X,k}$:

$\upsilon_{X,k} =$ 

Note that $\tau_{X,k}$ and $\upsilon_{X,k}$ do not depend on the choice of ι_X and π_X, which implies that if ψ extends to an isomorphism $\mathcal{A} \to \mathcal{A}'$ then $\psi(\tau_{X,k}) = \tau_{Y,k}$ and similarly for $\upsilon_{X,k}$. We will soon show that ψ extends if and only if $\psi(\tau_{X,1}) = \tau_{Y,1}$.

Lemma 5.1.7. *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are $\mathbb{Z}_2$-graded monoidal algebras and suppose $\psi : \Delta\mathcal{A} \to \Delta\mathcal{A}'$ is an isomorphism of diagonals. Then*

$$\psi(\tau_{X,1}) = \pm\tau_{Y,1}.$$

Proof. Since $\psi(\Pi_{\mathcal{A}}) = \Pi_{\mathcal{A}'}$, we see that $\psi(\tau_{X,1})$ belongs to the space $(1_Y \otimes \Pi_{\mathcal{A}'})\,\mathrm{End}(Y^3)(\Pi_{\mathcal{A}'} \otimes 1_Y)$. This space is 1-dimensional since $1_Y \otimes \Pi_{\mathcal{A}}$ and $\Pi_{\mathcal{A}} \otimes 1_Y$ are minimal idempotents in End(Y^3). Thus $\psi(\tau_{X,1}) = \lambda\tau_{Y,1}$ for some scalar λ. Now the equality $\tau_{X,2} = (1_X \otimes \tau_{X,1}) \circ (\tau_{X,1} \otimes 1_X)$ implies $\psi(\tau_{X,2}) = \lambda^2\tau_{Y,2}$. If we precompose this equation with $1_{Y^2} \otimes \Pi_{\mathcal{A}'}$ we obtain $\psi(\Pi_{\mathcal{A}} \otimes \Pi_{\mathcal{A}}) = \lambda^2\Pi_{\mathcal{A}'} \otimes \Pi_{\mathcal{A}'}$. Hence $\lambda^2 = 1$. □

Recall that twisting a monoidal algebra by a 3-cocycle produces a diagonal extension with the same objects. For convenience in the next lemma we use the symbol X_ω to refer to the generating object of $\mathcal{A}^\omega$.

Lemma 5.1.8. *Suppose $\mathcal{A}$ is a $\mathbb{Z}_2$-graded monoidal algebra and $\mathcal{A}^\omega$ the monoidal alegbra obtained by twisting with a nontrivial cocycle ω. Let $\psi : \Delta\mathcal{A} \to \Delta\mathcal{A}^\omega$ denote the identity functor between diagonals. Then*

$$\psi(\tau_{X,1}) = -\tau_{X_\omega,1}.$$

Proof. Since $\mathcal{A}$ and $\mathcal{A}^\omega$ differ only in how morphsims are tensored, they have the same compositional rules so we may choose $\iota_{X_\omega} = \iota_X$ and $\pi_{X_\omega} = \pi_X$. Let's examine the factors in the definition $\tau_{X_\omega,1} = (1_{X_\omega} \otimes_{\mathcal{A}^\omega} \iota_{X_\omega}) \circ (\pi_{X_\omega} \otimes_{\mathcal{A}^\omega} 1_{X_\omega})$. From the definition of tensor product in $\mathcal{A}^\omega$ (see Remark 5.0.4) we have

$$\pi_{X_\omega} \otimes_{\mathcal{A}^\omega} 1_{X_\omega} = \pi_X \otimes_{\mathcal{A}} 1_X$$

and

$$1_{X_\omega} \otimes_{\mathcal{A}^\omega} \iota_{X_\omega} = \omega(1,1,1)^{-1} 1_X \otimes_{\mathcal{A}} \iota_X = -(1_X \otimes_{\mathcal{A}} \iota_X).$$

Hence $\tau_{X_\omega,1} = -\tau_{X,1}$ as desired. □

Proposition 5.1.9. *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are $\mathbb{Z}_2$-graded monoidal algebras and suppose $\psi : \Delta\mathcal{A} \to \Delta\mathcal{A}'$ is an isomorphism of diagonals. Then the following are equivalent:*

1. *ψ extends to an isomorphism $\Psi : \mathcal{A} \to \mathcal{A}'$ of monoidal algebras.*
2. *$\psi(\tau_{X,1}) = \tau_{Y,1}$.*
3. *$\psi(\upsilon_{X,1}) = \upsilon_{Y,1}$.*
4. *$\psi(\tau_{X,k}) = \tau_{Y,k}$ and $\psi(\upsilon_{X,k}) = \upsilon_{Y,k}$ for all $k = 1, 2, 3, \ldots$*

Proof. That 1 implies 2 and 3 is straightforward. To show 2 implies 3, note that $\upsilon_{X,1}$ can be described in terms of $\tau_{X,1}$ as the unique element of the 1-dimensional space $(\Pi_{\mathcal{A}} \otimes 1_X) \operatorname{End}(X^3)(1_X \otimes \Pi_{\mathcal{A}})$ which satisfies $\tau_{X,1} \circ \upsilon_{X,1} = 1_X \otimes \Pi_{\mathcal{A}}$. Hence it suffices to check that $\tau_{Y,1} \circ \psi(\upsilon_{X,1}) = 1_Y \otimes \Pi_{\mathcal{A}'}$, which follows from the assumption and the fact that (by definition) $\psi(\Pi_{\mathcal{A}}) = \Pi_{\mathcal{A}'}$. Proving 3 implies 2 is identical.

To show that 2 and 3 imply 4, we use induction and the formula $\tau_{X,k} = (1_{X^{k-1}} \otimes \tau_{X,1}) \circ (\tau_{X,k-1} \otimes 1_X)$ (the formula is easily verified graphically).

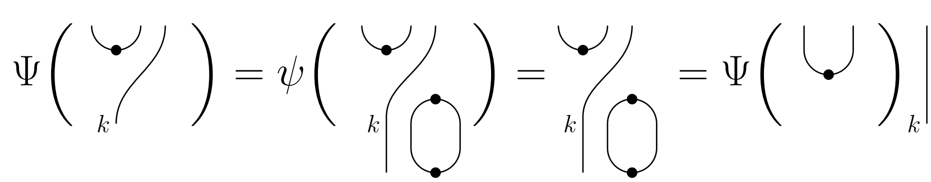
Now we suppose $\psi(\tau_{X,k}) = \tau_{Y,k}$ and $\psi(\upsilon_{X,k}) = \upsilon_{Y,k}$ for all k and prove 1. To that end we must define Ψ on a morphism $f \in \mathrm{Hom}(X^i, X^j)$ for $i \neq j$. Since i and j have the same parity we have $i = j \pm 2r$ for some $r \in \mathbb{N}$. Let $f^\Delta \in \mathrm{End}(X^{\max(i,j)})$ denote the morphism obtained by "padding" f on the right by ι_X's or π_X's to produce an endomorphism, ie

$$f^\Delta := \begin{cases} f \otimes \iota_X^{\otimes r} & i > j \\ f \otimes \pi_X^{\otimes r} & i < j. \end{cases}$$

Then we define Ψ by applying ψ to f^Δ and "unpadding":

$$\Psi(f) := \begin{cases} (1_{Y^j} \otimes \pi_Y^{\otimes r}) \circ \psi(f^\Delta) & i > j \\ \psi(f^\Delta) \circ (1_{Y^i} \otimes \iota_Y^{\otimes r}) & i < j \end{cases}$$

The fact that this defines a functor can be checked case-by-case and uses the assumption $\psi(\Pi_{\mathcal{A}}) = \Pi_{\mathcal{A}'}$. It is more critical to check that $\Psi(f \otimes g) = \Psi(f) \otimes \Psi(g)$. Since Ψ is known to be a functor it suffices to show that $\Psi(f \otimes 1_{X^k}) = \Psi(f) \otimes 1_{Y^k}$ and $\Psi(1_{X^k} \otimes g) = 1_{Y^k} \otimes \Psi(g)$. The second equation follows quickly from the definition of Ψ. For the first equation we start by checking it for $f = \iota_X$ and $f = \pi_X$. For $f = i_X$ we have the following graphical proof:



The first equality is the definition of Ψ, the second uses the assumption $\psi(\upsilon_{X,k}) = \upsilon_{X,k}$ and the third equality follows from the observation that $\Psi(i_X) = i_Y$. The proof for $f = \pi_X$ is similar and uses $\tau_{X,k}$ instead. Now suppose $f : X^i \to X^j$ and $i < j$ so $j = i+2r$. Then we prove $\Psi(f \otimes 1_{X^k}) = \Psi(f) \otimes 1_{Y^k}$ by induction on r, the base case $r = 0$ is clear since ψ is an isomorphism on the diagonal. For the inductive step, let $f' = f \otimes \pi_X$. Then $f = f' \circ (1_{X^i} \otimes \iota_X)$. Now using the calculation above for ι_X and the inductive hypothesis for f' we deduce the result is true for f. The case for $i > j$ is handled in the same manner. This completes the proof that Ψ is an isomorphism of monoidal algebras. $\square$

We can now prove the main theorem of this section, which states that a $\mathbb{Z}_2$-graded monoidal algebra is determined by its diagonal up to a cocycle twist. Let ω denote a nontrivial normalized 3-cocycle of $\mathbb{Z}_2$.

Theorem 5.1.10. *Suppose $\mathcal{A} = \langle X \rangle$ is a $\mathbb{Z}_2$-graded monoidal algebra. Then up to isomorphsim there are exactly two monoidal algebras with diagonal isomorphic to $\Delta\mathcal{A}$, namely $\mathcal{A}$ and $\mathcal{A}^\omega$.*

Proof. By Prop. 5.1.6, $\Delta\mathcal{A}^\omega$ is isomorphic to $\Delta\mathcal{A}$. On the other hand, if $\mathcal{B} = \langle Y \rangle$ is another $\mathbb{Z}_2$-graded monoidal algebra and $\psi : \Delta\mathcal{B} \to \Delta\mathcal{A}$ is an isomorphism then $\psi(\tau_{X,1}) = \pm\tau_{Y,1}$ by Lemma 5.1.7. If the sign is +1 then by the previous proposition ψ extends to an isomorphism so $\mathcal{B} \cong \mathcal{A}$. If the sign is -1 then we can compose ψ with a diagonal isomorphism $\phi : \Delta\mathcal{A}^\omega \to \Delta\mathcal{A}$ to obtain $\psi \circ \phi : \Delta\mathcal{A}^\omega \to \Delta\mathcal{A}$ which satisfies $(\psi \circ \phi)(\tau_{X_\omega}) = \tau_{Y,1}$ by Lemma 5.1.8. So in this case $\psi \circ \phi$ extends to an isomorphism, proving $\mathcal{B} \cong \mathcal{A}^\omega$. $\square$

6

Jucys-Murphy theory for ribbon categories

Consider a semisimple ribbon category $\mathcal{C}$ generated by a simple object X. We've seen to what degree $\mathcal{C}$ can be reconstructed from the diagonal subcategory $\Delta\mathcal{C}$, at least when $\mathcal{C}$ is $\mathbb{Z}_2$-graded. The diagonal is described by the family of algebras $\mathrm{End}(X^{\otimes k})$, together with tensor product maps between them:

$$\mathrm{End}(X^{\otimes k}) \times \mathrm{End}(X^{\otimes l}) \to \mathrm{End}(X^{\otimes k+l})$$

sending (f, g) to $f \otimes g$. Ultimately the goal is to show that the diagonal is uniquely determined by the fusion rules and the braid parameter q. First we see how much information the fusion rules alone give us. The algebras $\mathrm{End}(X^{\otimes k})$ are direct sums of matrix algebras since $\mathcal{C}$ is a semisimple category. The full matrix blocks correspond to the simple isotypes appearing in $X^{\otimes k}$. Hence the structure of each $\mathrm{End}(X^{\otimes k})$, as an individual algebra, is determined by the fusion rules.

At the next level of scrutiny the fusion rules give us information about the inclusions $\mathrm{End}(X^{\otimes k-1}) \to \mathrm{End}(X^{\otimes k})$ given by $f \mapsto f \otimes 1$ and $f \mapsto 1 \otimes f$. If p is a minimal idempotent of type λ then $p \otimes 1$ (and $1 \otimes p$) has image isomorphic to $\lambda \otimes X$. Thus when $p \otimes 1$ is written as a sum of minimal idempotents in $\mathrm{End}(X^{\otimes k})$, the number of type μ is equal to the multiplicity of μ in $\lambda \otimes X$.

The rule for tensoring with X is encoded in a Bratteli diagram. It represents combinatorial scaffolding that the diagonal subcategory must rest on. In the cases we study, the multiplicities for tensoring with X are always 1, so the Bratteli diagram is *multiplicity free.* In this case we may define a complete set of minimal idempotents for $\mathrm{End}(X^{\otimes k})$, adapted to the inclusions $f \mapsto f \otimes 1$ and labeled by paths of length k in the Bratteli diagram. These minimal idempotents give a natural basis for simple modules of $\mathrm{End}(X^{\otimes k})$ labeled by paths. Eventually we will show that the representation of the braid elements in this basis is uniquely determined by q. Then, this will be enough to identify the diagonal subcategory.

6.1 Path idempotents for centralizer algebras

Let Γ denote a set of labels for the distinct simple objects of $\mathcal{C}$. With some abuse of notation we also identify Γ with a representative set of simple objects, so expressions such as $X \otimes \lambda$ denotes an object in $\mathcal{C}$. We assume $\mathbf{1} \in \Gamma$ corresponds to the trivial object.

Definition 6.1.1. Given an object $X \in \mathcal{C}$, the *fusion graph*) for X is a graph $\mathcal{B}$ with vertices

$$\mathcal{B} = \bigsqcup_{k \geq 0} \mathcal{B}(k)$$

where

$$\mathcal{B}(k) = \{\lambda \mid \lambda \text{ appears in } X^{\otimes k}\}.$$

If $x \in \mathcal{B}(k)$ then we say x is in *level* k, denoted $l(x) = k$. The only edges are between elements in adjacent levels. The number of edges connecting λ in level k and μ in level $k-1$ is the equal to the multiplicity of μ in $X \otimes \lambda$.

Example 6.1.2. Suppose $\mathcal{C}$ has the fusion rules of $SO(2n)$ and X corresponds to the defining representation. According to the fusion rules described for $SO(2n)$, in Sec. 3 the kth level of the fusion graph consists of all those $SO(2n)$ shapes whose diagrams have

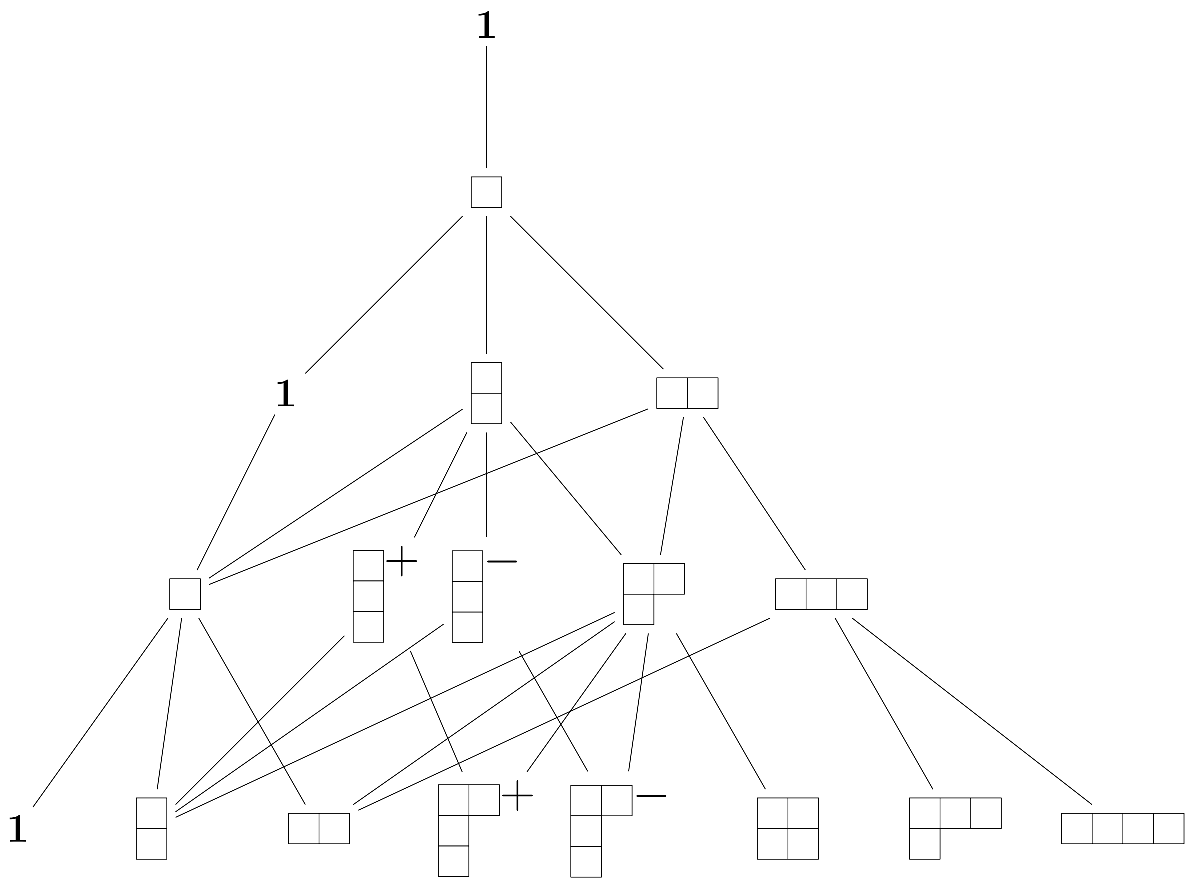


Figure 6.1: The first 4 levels of the Bratteli diagram for $SO(6)$.

at most k boxes, and the parity of the number of boxes agrees with k. An edge occurs whenever a shape is obtained from another by adding or removing a box. See Figure 6.1 for the case of $SO(6)$.

The Bratteli diagram for the $SO(2n) - O(K)$ fusion rules (with $n \geq 3$ and $K \geq 3$) is the subggraph obtained by excluding all shapes with more than K boxes in their first two rows.

Using the semi-simplicity of $\mathcal{C}$ we see $\mathrm{End}(X^{\otimes k})$ is a direct sum of simple matrix algebras labeled by the irreps of $\mathcal{C}$ appearing in $X^{\otimes k}$. Such a simple component labeled by λ we denote $\mathrm{End}(X^{\otimes k})_\lambda$. A minimal idempotent $p \in \mathrm{End}(X^{\otimes k})$ must be contained in a unique simple component, which is the *type* of p. The isotype of p is λ if and only if

Im $p \cong \lambda$ in $\mathcal{C}$. The projection onto the λ-isotypic component of $\mathrm{End}(X^{\otimes k})$ is a central idempotent denoted z_λ (or $z_{\lambda,k}$ to emphasize it is an endomorphism of $X^{\otimes k}$). A minimal idempotent has type λ if and only if $z_\lambda p = p$.

Recall that if $A = \bigoplus_\alpha A_\alpha$ and $B = \bigoplus_\beta B_\beta$ are direct sums of matrix algebras and $A \subset B$ is a unital inclusion then we can define the *Bratteli diagram* of the inclusion. It is a bipartite graph with one layer indexed by the simple isotypes of A and the other indexed by the simple isotypes of B. To describe the edges in the graph, start with a minimal idempotent of type α in A. Then use the inclusion into B to decompose it into a sum of minimal idempotents in B. This decomposition is not unique, but the number of idempotents of type β which appear doesn't depend on the decomposition. This multiplicity is equal to the number of edges between nodes α and β in the Bratteli diagram. We can extend this notion to the Bratteli diagram for an arbitrary sequence of unital inclusions $A_1 \subset A_2 \subset \dots$

The fusion graph coincides with the Bratteli diagram of the inclusions

$$\cdots \to \mathrm{End}_{\mathcal{C}}(X^{\otimes n}) \xrightarrow{-\otimes 1} \mathrm{End}_{\mathcal{C}}(X^{\otimes n+1}) \to \dots \tag{6.1}$$

Indeed, if $p_\lambda \in \mathrm{End}(X^{\otimes k-1})$ is a minimal idempotent of type λ then $p_\lambda \otimes 1 \in \mathrm{End}(X^{\otimes k})$ is a minimal idempotent and Im $(p_\lambda \otimes 1) \cong X \otimes X_\lambda$ in $\mathcal{C}$. If we decompose $p_\lambda \otimes 1$ into an orthogonal sum of minimal idempotents of $\mathrm{End}(X^{\otimes k})$, there will be $m_{X\otimes\lambda,\mu}$ of type μ in the sum. Hence we also sometimes refer to the fusion graph as the Bratteli diagram.

Note that the Bratteli diagram for $SO(2n)$ or $SO(2n) - O(K)$ is *multiplicity free*. In other words, this means for any simple object X_λ in $\mathcal{C}$, $X \otimes X_\lambda$ decomposes as a sum of simple objects, each with multiplicity 1. In the general discussion below we also assume X is a simple object, so level 1 of $\mathcal{B}$ has a single vertex (as does level 0, which has the single vertex $\mathbf{1}$).

We now introduce the language of paths which is well developed in [GdLHJ89] and [LR97]. A *path* of length k in $\mathcal{B}$ is a sequence

$$S = S(0) \to S(1) \to S(2) \to \cdots \to S(k)$$

such that $S(i+1)$ is connected by an edge to $S(i)$ for each $i = 1, 2, \ldots, n-1$. In other words, $S(i+1)$ labels an irrep appearing in $X \otimes S(i)$. All paths we consider increase the level at every step. The *shape* of S is the final term in the path, i.e. $\mathrm{sh}(S) = S(k)$. Given a path S, the symbol S' (resp. S'') denotes the smaller path obtained by removing the last (resp. last 2) elements of S. For collections of paths we use the following notation:

$\mathcal{P}_k$ is the set of paths of length k starting at $\mathbf{1} \in \mathcal{B}(0)$.

$\mathcal{P}_k^\lambda$ is the set of paths of length k starting at $\mathbf{1}$ and ending at $\lambda \in \mathcal{B}(k)$.

$\mathcal{P}_\mu^\lambda$ is the set of paths starting at $\mu \in \mathcal{B}(i)$ and ending at $\lambda \in \mathcal{B}(k)$.

If S and T are paths of length k and l respectively for which $S(k) = T(0) = \lambda$ then $S \to T$ is the concatenated path of length $k + l$ given by

$$S(0) \to \cdots \to S(k-1) \to \lambda \to T(1) \to \cdots \to T(l).$$

In particular, if μ is a shape in the Bratteli diagram connected to $S(k)$ then $S \to \mu$ is the path of length $k+1$ obtained by concatenating μ to S. We inductively define the *path idempotents*, a complete set of minimal idempotents for $\mathrm{End}(X^{\otimes k})$ which are labeled by paths starting at $\mathbf{1}$ and are compatible with the inclusions $\mathrm{End}(X^{k-1}) \mapsto \mathrm{End}(X^k)$ via $f \mapsto f \otimes 1$.[1] There is a unique path of length 0, namely $(\mathbf{1})$ and we set

$$p_{(\mathbf{1})} = 1 \in \mathrm{End}(X^0) = \mathrm{End}(\mathbf{1}).$$

[1]You could just as easily define a different set of idempotents compatible with the other inclusion $f \mapsto 1 \otimes f$. However, figuring out how to express one set of idempotents in terms of the other is as hard as determining the category.

Now suppose $p_T \in \mathrm{End}(X^{\otimes k-1})$ is defined for every path T of length $k-1$. Then for a path S of length k and shape λ, the path idempotent p_S is defined by

$$p_S = z_\lambda(p_{S'} \otimes 1)$$

where z_λ is the central idempotent of $\mathrm{End}(X^{\otimes k})$ of type λ. By the multiplicity-freeness of the Bratteli diagram, p_S is a minimal idempotent. By induction, we can write p_S solely in terms of central idempotents by

$$p_S = \prod_{i=1}^{k} z_{S(i)} \otimes 1_{X^{\otimes k-i}} \tag{6.2}$$

For a path of length k we have

$$p_S \otimes 1 = \sum_{T:\ T'=S} p_T,$$

the sum over all paths of length $k+1$ which start with the path S. The path idempotents allow us to specify (up to scalars) bases for the simple modules V_λ of $\mathrm{End}(X^{\otimes k})$, which are indexed by $\lambda \in \mathcal{B}(k)$. A path idempotent p_T of shape λ acts as a rank-1 projection on V_λ. Choose a non-zero vector $v_T \in p_T V_\lambda$. Then the collection $\{v_T : T \in \mathcal{P}_k^\lambda\}$ forms a basis of V_λ, uniquely determined up to scalar multiples. Any such basis is called a *path basis* for V_λ. Note that V_λ is an $\mathrm{End}(X^{\otimes i})$-module for any $i \leq k$ by restriction along the inclusion $\mathrm{End}(X^{\otimes i}) \otimes 1 \subset \mathrm{End}(X^{\otimes k})$. For a path $T = T(0) \to T(1) \to \cdots \to T(k)$, the path basis vector v_T is determined (up to scalars) by the property

$$\mathrm{End}(X^{\otimes i})v_T \cong V_{T(i)} \text{ for all } 1 \leq i \leq k$$

as $\mathrm{End}(X^{\otimes i})$-modules.

Path bases are particularly well suited to studying "local" endomorphisms of $X^{\otimes k}$, meaning those morphisms which are the identity away from some small number of consecutive factors in $X^{\otimes k}$. The precise result is

Proposition 6.1.3. *Suppose $f \in End(X^{\otimes k})$ centralizes $End(X^{\otimes i}) \otimes 1$ for some $i < k$. Let V_λ be a simple $End(X^{\otimes k})$-module and let v_T be a path basis vector corresponding to $T = T(0) \to \cdots \to T(k) \in \mathcal{P}_k^\lambda$. Let $\underline{T}$ and $\overline{T}$ denote the truncated paths $T(0) \to \cdots \to T(i)$ and $T(i) \to \cdots \to T(k)$ respectively. Then*

$$fv_T = \sum_{S \in \mathcal{P}_{T(i)}^\lambda} \alpha_{S,\overline{T}} v_{\underline{T} \to S}.$$

where the scalars $\alpha_{S,\overline{T}}$ depend only on S (which is a path from $T(i)$ to λ) and the path $\overline{T}$.

Proof. See [GdLHJ89] or [LR97]. □

For example we may apply the proposition to morphisms of the form $1^i \otimes f$ which commute with $\mathrm{End}(X^{\otimes i}) \otimes 1$. In this case it says that when $1^i \otimes f$ is applied to a path basis vector the result is a sum of path vectors which are identical to the original path in the first i levels. Another similar use is applying a morphism of the form $1^i \otimes a \otimes 1^j$ with $a \in \mathrm{End}(X^2)$. In this case the proposition says that an application of such a morphism can only affect the $i+1$-th level of a path.

Remark 6.1.4. More precisely we can use the proposition to define subsidiary path modules which are modules for the centralizer of $\mathrm{End}(X^{\otimes i})$ in $\mathrm{End}(X^{\otimes k})$, which we denote

$$Z(i,k) := \{a \in \mathrm{End}(X^{\otimes k}) : a(b \otimes 1) = (b \otimes 1)a, \ \forall\, b \in \mathrm{End}(X^{\otimes i})\}.$$

Given $\mu \in \mathcal{B}(i)$ and $\lambda \in \mathcal{B}(k)$ (with $i \leq k$) let W_μ^λ denote the $\mathbb{C}$-space with basis $\{w_R : R \in \mathcal{P}_\mu^\lambda\}$. Fix any path T_0 of length i ending at μ. According to the proposition, if

$f \in Z(i,k)$ then for any $R \in \mathcal{P}_\mu^\lambda$ we have

$$f v_{T_0 \to R} = \sum_{S \in \mathcal{P}_\mu^\lambda} \alpha_{S,R} v_{T_0 \to S}$$

where the coefficients $\alpha_{S,R}$ do not depend on T_0. Thus they can be used to define the action of $Z(i,k)$ on W_μ^λ:

$$f w_R = \sum_{S \in \mathcal{P}_\mu^\lambda} \alpha_{S,R} w_S.$$

It is clear that knowing the action of f on W_μ^λ in the $\{w_S\}$ basis is the same as knowing the action of f on V_λ in the $\{v_S\}$ basis. Proposition 6.1.3 also implies a decomposition of V_λ as an $\mathrm{End}(X^{\otimes i}) \otimes Z(i,k)$-module:

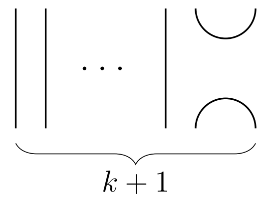
$$V_\lambda = \sum_{\mu \in \mathcal{B}(i)} V_\mu \otimes W_\mu^\lambda. \tag{6.3}$$

The path bases provide a natural basis in which to compute the representations of the braid group appearing in $\mathrm{End}(X^{\otimes k})$. We will use the Jucys-Murphy approach to explicitly compute matrices for braid elements in a path basis when $\mathcal{C}$ has the fusion rules of $SO(2n)$. Matrix representations of $\mathrm{End}(X^{\otimes k})$ using different path bases of V_λ differ only by conjugation by a diagonal matrix. In particular, the path idempotents are always rank-1 diagonal matrices in any path basis.

6.2 Matrix representations for Jones projections

We now assume X is a symmetrically self-dual simple object. In this case the Bratteli diagram has a special structure, obtained by repeated reflections. It accounts for shapes in $\mathcal{B}$ with fewer than k boxes. This phenomenon is an expression of the *Jones basic construction*. In this setting we have unoriented cups/caps and may define

$e = \cup \circ \cap = \asymp \in \mathrm{End}(X^{\otimes 2})$ and e_k to be the element $1_{X^{\otimes k-1}} \otimes e \in \mathrm{End}(X^{\otimes k+1})$:

$$e_k =$$


The self-duality implies $X^{\otimes k} \cong X^{\otimes k-2} \otimes X^2 \supset X^{\otimes k-2} \otimes \mathbf{1} \cong X^{\otimes k-2}$. Hence every simple appearing in $X^{\otimes k-2}$ also appears in $X^{\otimes k}$. Following the notation of [TW05] we write

$$X^{\otimes k} \cong X_{(k-2)} \oplus X_k$$

where $X_{(k-2)}$ consists of simples which appear in $X^{\otimes k-2}$ and X_k only consists of simples not appearing in $X^{\otimes k-2}$. Clearly $X_{(k-2)} \neq 0$, but it is possible $X_k = 0$. Colloquially we say that $X_{(k-2)}$ is the "old stuff" of X and X_k is the "new stuff". [2] Since the simples appearing in $X_{(k-2)}$ and X_k are mutually distinct, we have a canonical decomposition

$$\mathrm{End}(X^{\otimes k}) \cong \mathrm{End}(X_{(k-2)}) \oplus \mathrm{End}(X_k).$$

Lemma 6.2.1. *The ideal generated by $e_i \in End(X^{\otimes k})$ $(i \leq k-1)$ is equal to $End(X_{(k-2)})$.*

Proof. This follows from the semisimplicity of $\mathrm{End}(X^{\otimes k})$. Note that since the e_i are all conjugate, the ideal is generated by any one of them. □

Note that, in particular, every e_i acts by 0 in any simple $\mathrm{End}(X^{\otimes k})$-module labeled by a new isotype. Setting $x = \dim_{\mathcal{C}} X$, we see $E_k := (1/x)e_k$ is an idempotent, called *the k-th Jones projection.* Let us denote by ϵ_k the partial trace $\epsilon_k : \mathrm{End}(X^{\otimes k}) \to \mathrm{End}(X^{\otimes k-1})$. This is an $\mathrm{End}(X^{\otimes k-1})-\mathrm{End}(X^{\otimes k-1})$-bimodule map, and by Eq. 2.8 the Jones projections

[2]This terminology can occasionally be misleading since X_k may contain simples which appear in a lower tensor power even though they are not in $X^{\otimes k-2}$. In our setting this occurs only with $SO(2n+1)$ and not with $SO(2n)$.

are related to the partial traces via

$$E_k a E_k = \epsilon_{k+1}(a) E_k \tag{6.4}$$

for all $a \in \mathrm{End}(X^{\otimes k+1})$.

Let $\langle \mathrm{End}(X^{\otimes k}) \otimes 1, E_k \rangle$ denote the subalgebra of $\mathrm{End}(X^{\otimes k+1})$ generated by $\mathrm{End}(X^{\otimes k}) \otimes 1$ and E_k. By Eq. 6.4 the sequence of inclusions

$$\mathrm{End}(X^{\otimes k-1}) \otimes 1 \subset \mathrm{End}(X^{\otimes k}) \otimes 1 \subset \langle \mathrm{End}(X^{\otimes k}) \otimes 1, E_k \rangle$$

which contain an instance of the *Jones basic construction* (see [Wen88], Sec. 1 and [RW92] for the basics). Critical for the basic construction is the fact that the inclusions respect the non-degenerate (normalized) traces $\mathrm{tr} : \mathrm{End}(X^{\otimes k}) \to \mathbb{C}$, coming from the duality in the category. [3] It can be shown that the old stuff $\mathrm{End}(X_{(k-1)})$ is contained in the latter algebra. We will need the following facts:

1. The unital algebra homomorphism $\mathrm{End}(X^{\otimes k}) \otimes 1 \to \mathrm{End}(X^{\otimes k+1}) \to \mathrm{End}(X_{(k-1)})$ is injective.

2. The Bratteli diagram for the resulting inclusion $\mathrm{End}(X^{\otimes k}) \hookrightarrow \mathrm{End}(X_{(k-1)})$ is the reflection (about a horizontal axis) of the diagram for $\mathrm{End}(X^{\otimes k-1}) \hookrightarrow \mathrm{End}(X^{\otimes k})$.

Fact (1) can be proved by showing the image of any minimal idempotent is non-zero and minimal. In Fact (2) we are identifying the isotypes which are old in $\mathcal{B}(k+1)$ with the full collection of isotypes in $\mathcal{B}(k-1)$. Hence the portion of the Bratteli diagram concerned with edges between $\mathcal{B}(k)$ and the old istoypes in $\mathcal{B}(k+1)$ (i.e. whose shapes have fewer than $k+1$ boxes), is just a reflected copy of the full Bratteli diagram between levels $k-1$

[3] Recall tr is non-degenerate by the semisimplicity and rigidity of $\mathcal{C}$, which is equivalent to the non-vanishing of the q-dims of all simple objects.

and k. The reader is invited to examine Fig. 6.1 and confirm this, e.g. for levels $2, 3, 4$.[4]

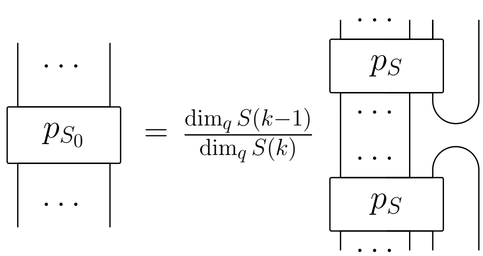
Now we explain how to compute matrix representations for E_k in the path bases in terms of the q-dims of simple objects. As before, this material is well known (see e.g. [LR97], [RW92]) but we the statements as a convenience to the reader. To compute the action of E_k on a path basis it suffices to consider the action of E_k on the path modules W_μ^λ where $\mu \in \mathcal{B}(k-1)$ and $\lambda \in \mathcal{B}(k+1)$. In fact, if μ and λ are not the same shape then E_k acts by 0 on W_μ^λ. Indeed, any idempotent of the form $p_\mu \otimes E_k$ has image isomorphic to $\mu \otimes \mathbf{1} \cong \mu$ in the category, where p_μ is a minimal idempotent of type μ. Hence $p_\mu \otimes E_k$ is a minimal idempotent of type μ and hence acts by 0 on any $\mathrm{End}(X^{\otimes k+1})$ module of type λ.

We will use the following lemmas to compute the action of E_k on a path module W_λ^λ. The proofs can be found in [RW92].[5]

Lemma 6.2.2. *Let S be a path of length k starting at $\mathbf{1}$. Then*

$$\epsilon_k(p_S) = \frac{\dim_{\mathcal{C}} S(k)}{x \dim_{\mathcal{C}} S(k-1)} p_{S'}.$$

Lemma 6.2.3. *Let S be a path of length k starting at $\mathbf{1}$ and $S \to S(k-1)$ the path of length $k+1$ obtained moving to the shape $S(k-1)$ after traversing S. Then*



Now we examine the action of E_{k+1} on a path module W_λ^λ (with $\lambda \in \mathcal{B}(k-1), \mathcal{B}(k+1)$). A path basis for W_λ^λ is indexed by such paths.

[4] By Fact (2), the entire Bratteli diagram could be reconstructed just from the subdiagram of new stuff. Borrowing terminology from subfactor theory, this subdiagram is the *principal graph* of X.

[5] In subfactor language the following facts are statements about the *Ocneanu-Sunder path model*.

Proposition 6.2.4. *Let λ be the shape of a simple object appearing in both $X^{\otimes k-1}$ and $X^{\otimes k+1}$. Then E_k acts as a rank-1 projection on W_λ^λ and in any path basis has diagonal entries $\{(E_k)_{SS} : \ S \in \mathcal{P}_\lambda^\lambda\}$, where for a path S,*

$$(E_k)_{SS} = \frac{\dim_{\mathcal{C}} S(k)}{x \dim_{\mathcal{C}} \lambda}. \tag{6.5}$$

The off-diagonal entries of E_k depend on a choice of scaling of the path basis, but since E_k is rank-1 with all non-zero diagonal entries we see that there is always a scaling for the path basis for which the rows of E_k are identical, i.e. the matrix entries are constant down the columns:

$$E_{S,T} = E_{T,T}. \tag{6.6}$$

We will address this rescaling in detail when computing matrices for braid elements.

6.3 The full twist and Jucys-Murphy elements

To compute the braid representations in the path bases we follow the Jucys-Murphy approach. This approach has been used in various settings to compute seminormal representations of various towers of algebras. For instance, it is applied by Okounkov and Vershik to the symmetric groups [VYO05], by Ram to other Weyl groups [Ram97], by Nazarov to the Brauer algebras [Naz90] and by Leduc and Ram to quantum groups of type ABC [LR97]. The Jucys-Murphy approach is also used by Wenzl [Wen03] to study the centralizer algebras coming from E_n-type quantum groups, and by Martirosyan and Wenzl [MW17] to study those of a G_2-type quantum group. We define the Jucys-Murphy elements, which form a large commutative subalgebra of $\mathrm{End}(X^{\otimes k})$ and act diagonally on path bases. The eigenvalues of these elements are independent of the choice of scaling for path bases and are determined by the eigenvalues of $c_{X,X}$ and the fusion rules. Once the eigenvalues are known, we can compute the matrix entries of the braid generators in the path basis.

Definition 6.3.1. Let $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$ denote standard generators of the k-strand braid group. The *full twist* is a central element Δ_k^2 defined by

$$\Delta_k^2 = (\sigma_1\sigma_2 \ldots \sigma_{k-1})^k.$$

In terms of braids, this element corresponds to a full counter-clockwise twist of the k-strands. By convention, $\Delta_0^2 = \Delta_1^2 = 1$.

If $\mathcal{C}$ is a ribbon category then the assignment $\sigma_i \mapsto c_i$ where

$$c_i = 1^{\otimes i-1} \otimes c_{X,X} \otimes 1^{k-i-1}$$

defines a homomorphism from the braid group to $\mathrm{End}(X^{\otimes k})$. The image of the full twist in $\mathrm{End}(X^{\otimes k})$ is also denoted Δ_k^2. The following fact is a critical application of the graphical calculus:

Lemma 6.3.2. *Suppose X is an object in a ribbon category $\mathcal{C}$. Then Δ_k^2 is central in $End(X^{\otimes k})$.*

Proof. For any f the diagrams representing $f \circ \Delta_k^2$ and $\Delta_k^2 \circ f$ can be isotoped into each other by sliding f through the twist. □

For $i \leq k$ we consider Δ_i^2 as an element of $\mathrm{End}(X^{\otimes k})$ via the usual embedding $\mathrm{End}(X^{\otimes i}) \to \mathrm{End}(X^{\otimes k})$. Since Δ_k^2 is central in $\mathrm{End}(X^{\otimes k})$ it acts as a diagonal matrix in the path basis for a simple $\mathrm{End}(X^{\otimes k})$-module V_λ. Its eigenvalues are clearly independent of the choice of path basis. The full twist is related to the twist coming from the ribbon structure by the well known equation

$$\Delta_k^2 = \theta_X^{-k} \theta_{X^{\otimes k}}. \tag{6.7}$$

Definition 6.3.3. Suppose $\mathcal{C}$ is a non-symmetric ribbon category generated by X. The kth *Jucys-Murphy element* is

$$J_k = \Delta_k^2 \Delta_{k-1}^{-2}.$$

For $i \leq k$ we can consider J_i an element of $\mathrm{End}(X^{\otimes k})$ via the usual embedding $\mathrm{End}(X^{\otimes i}) \to \mathrm{End}(X^{\otimes k})$ (i.e. we identify D_i with $D_i \otimes 1$). As braids the JM elements look as in Figure 6.2.

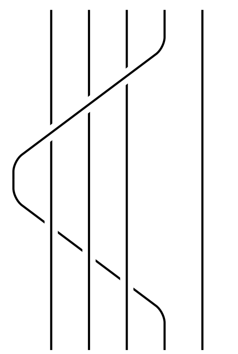


Figure 6.2: The Jucys-Murphy element $J_4 \in \mathrm{End}(X^{\otimes 6})$.

Lemma 6.3.4. *The Jucys-Murphy elements J_k centralize End($X^{\otimes k-1}$) and satisfy the relations*

$$J_k = c_{k-1} J_{k-1} c_{k-1}. \tag{6.8}$$

Furthermore, they are diagonal in the path basis of any simple End($X^{\otimes k}$)-module.

Proof. Since Δ_k^2 and Δ_{k-1}^2 centralize $\mathrm{End}(X^{\otimes k-1})$, so too does J_k. The equation above is easily verified using the graphical representation of J_k (Figure 6.2). Finally, since Δ_k^2 and Δ_{k-1}^2 are both diagonal in a path basis, so is J_k. □

Our aim is to determine the matrix entries of a braid element c_k when expressed in the path basis. Since $c_k \in Z(k-2, k)$ it suffices (by Remark 6.1.4) to consider the action of c_k on the path modules W_μ^λ spanned by paths from $\mu \in \mathcal{B}(k-2)$ to $\lambda \in \mathcal{B}(k)$. In fact, J_k and J_{k-1} also belong to $Z(k-2, k)$. When the Bratteli diagram is that of a classical

Lie group, these spaces are often 1 or 2 dimensional and it is not hard to determine the action of the algebra generated by c_k, J_k and J_{k-1} on this space, which is what we do next. By Lemma 6.3.4, the algebra generated by c_k, J_k and J_{k-1} is a quotient of the group algebra of the affine braid group AB_2 which we describe below.

6.4 Low dimensional representations of AB_2

Let AB_2 (the 2-strand affine braid group) denote the group generated by elements A, D_1, D_2 satisfying the relations

$$D_1 D_2 = D_2 D_1 \tag{6.9}$$

$$A D_1 A = D_2. \tag{6.10}$$

Note that these relations imply $D_1 D_2$ is central. As noted above we have

Proposition 6.4.1. *Given $\mu \in \mathcal{B}(k-2)$ and $\lambda \in \mathcal{B}(k)$, the assignments $A \mapsto c_{k-1}, D_1 \mapsto J_{k-1}, D_2 \mapsto J_k$ defines an action of AB_2 on the path module W_μ^λ. Under this assignment $D_1 D_2 \mapsto \Delta_k^2 \Delta_{k-2}^{-2}$ which acts a scalar on W_μ^λ.*

Proof. This follows from Lemma 6.3.4 and the definition of the JM elements. □

The following lemma allows us to transfer information about the fusion rules to information about the action of A on the path module W_μ^λ:

Lemma 6.4.2. *Suppose $\mathcal{C}$ is a ribbon category generated by a simple object X, with multiplicity-free Bratteli diagram B. Suppose $Y \in \mathcal{B}(2)$ and $c_{X,X}$ has corresponding eigenvalue q_Y. If $\mu \in \mathcal{B}(k-2)$ and $\lambda \in \mathcal{B}(k)$ are such that λ appears in $\mu \otimes Y$ then q_Y is an eigenvalue of $A = c_{k-1}$ on the path module W_μ^λ.*

Proof. Let p_Y be a minimal idempotent in $\mathrm{End}(X^{\otimes 2})$ of type Y. Let S be an arbitrary path of length $k-2$ ending at μ. As $Z(k-2,k)$ modules we have $W_\mu^\lambda \cong \mathrm{End}(X^{\otimes k}) z_\lambda (p_S \otimes 1)$

where z_λ is the central idempotent in $\mathrm{End}(X^{\otimes k})$ of type λ. Then by assumption $z_\lambda p_S \otimes p_Y$ is a non-zero element of $\mathrm{End}(X^{\otimes k}) z_\lambda (p_S \otimes 1)$, and it is an eigenvector for c_{k-1} with eigenvalue q_Y. □

We study low-dimensional representations of AB_2 for which A is diagonalizable. First, note that the 1-dimensional representations are classified by pairs of non-zero scalars (λ, d) which determine the action of AB_2 via the mapping

$$\begin{aligned} A &\mapsto \lambda \\ D_1 &\mapsto d \\ D_2 &\mapsto \lambda^2 d. \end{aligned} \tag{6.11}$$

For convenience we denote these representations $\phi_\lambda^d : AB_2 \to \mathbb{C}$. Next we discuss 2-dimensional representations. As for any 2-dimensional representation of a finite-dimensional $\mathbb{C}$-algebra, there are 3 possibilities: it either splits as the direct sum of two 1-dimensional irreps, it is irreducible, or it is indecomposable but not irreducible (so there is exactly one shared eigenvector for the algebra operators). Note the latter case is the only non-semisimple possibility. [6]

Proposition 6.4.3. *Let V be a 2-dimensional complex representation of AB_2 for which A is diagonalizable. Then either V splits into the direct sum of two 1-dimensional reps labelled (λ_1, d_1) and (λ_2, d_2) for some $\lambda_1, \lambda_2, d_1, d_2 \in \mathbb{C}$, or there exists a basis for V and*

[6] We couldn't find a way to rule this case out just by using the axioms of a ribbon category. In the next section it takes work to establish that a 2-dimensional path module W_μ^λ must be irreducible, so in particular we never get the non-semisimple representation. We conjecture that this is always the case, i.e. that in a (semisimple) ribbon category all path modules for the braid group are semisimple.

non-zero scalars a, b and $\lambda_1 \neq \lambda_2$ such that

$$A \mapsto \begin{pmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{pmatrix}$$

$$D_1 \mapsto \begin{pmatrix} a & -b(\lambda_1 + \lambda_2) \\ 0 & b \end{pmatrix}$$

$$D_2 \mapsto \begin{pmatrix} -b\lambda_1\lambda_2 & -b\lambda_1\lambda_2(\lambda_1 + \lambda_2) \\ 0 & -a\lambda_1\lambda_2 \end{pmatrix}$$

Proof. Suppose V does not split as a sum of two 1-dimensional irreps. Note this implies $\lambda_1 \neq \lambda_2$ since otherwise A is a scalar times 1_V (by the diagonalizability of A) and the representation splits. There must be a simultaneous eigenvector v of D_1 and D_2 which is not an eigenvector of A. Take a, β to be the scalars given by $D_1 v = av$ and $D_2 v = \beta v$. Since v and Av are linearly independent they are a basis of V and upon writing A, D_1, D_2 in this basis and setting $b = -(\lambda_1\lambda_2)^{-1}\beta$ we arrive at the matrices above. □

By looking at the entries of D_1 and D_2 we deduce:

Corollary 6.4.4. *Suppose V is a 2-dimensional indecomposable representation for which A is diagonalizable, parametrized by $(\lambda_1, \lambda_2, a, b)$. Suppose $\lambda_1 + \lambda_2 \neq 0$. Then D_1 and D_2 are diagonalizable if and only if $b \neq a$.*

Suppose $\lambda_1 + \lambda_2 \neq 0$ (as will always be the case for the representations appearing in a non-symmetric $SO(N)$ category) and V is a 2-dimensional indecomposable representation for which D_1, D_2 are diagonalizable. Pick a basis so that A, D_1, D_2 have the matrices above. Then the quantity

$$\alpha := \frac{b-a}{-b(\lambda_1+\lambda_2)}$$

is non-zero and the vectors $\{(\alpha^{-1},0)^T,(1,\alpha)^T\}$ form an eigenbasis for D_1, D_2. In this basis we obtain the matrix representation

$$D_1 \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$D_2 \mapsto \begin{pmatrix} -b\lambda_1\lambda_2 & 0 \\ 0 & -a\lambda_1\lambda_2 \end{pmatrix}$$

$$A \mapsto \begin{pmatrix} -\alpha^{-1} & -(\lambda_1+\alpha^{-1})(\lambda_2+\alpha^{-1}) \\ 1 & \alpha^{-1}+\lambda_1+\lambda_2 \end{pmatrix} \tag{6.12}$$

If the upper right entry of A is non-zero then the representation is irreducible and if the entry is 0 then the representation is indecomposable but admits a shared eigenvector. Note that the upper right entry of A vanishes exactly when $\alpha = -\lambda_1^{-1}$ or $\alpha = -\lambda_2^{-1}$, which by the definition of α is equivalent to $b = -(\frac{\lambda_1}{\lambda_2})^{\pm 1}a$. In summary we have

Proposition 6.4.5. *Suppose V is a 2-dimensional AB_2 representation such that A is diagonalizable and D_1, D_2 act by diagonal matrices. Suppose A has eigenvalues λ_1, λ_2 with $\lambda_1 \neq \pm\lambda_2$ and $D_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then either:*

1. *V splits as the sum of two 1-dimensional representations and*

$$D_2 = \begin{pmatrix} \lambda_1^2 a & 0 \\ 0 & \lambda_2^2 b \end{pmatrix} \quad or \quad \begin{pmatrix} \lambda_2^2 a & 0 \\ 0 & \lambda_1^2 b \end{pmatrix}.$$

2. V is indecomposable, $a \neq b$, and

$$D_2 = \begin{pmatrix} -b\lambda_1\lambda_2 & 0 \\ 0 & -a\lambda_1\lambda_2 \end{pmatrix}.$$

If in addition $b \neq -(\frac{\lambda_1}{\lambda_2})^{\pm 1}a$ then V is an irreducible representation.

Remark 6.4.6. If V is indecomposable then D_1D_2 acts by the scalar $-ab\lambda_1\lambda_2$ on V (according to the proposition). In the case V splits then D_1D_2 acts by scalar only if $\lambda_1^2a^2 = \lambda_2^2b^2$ in the first case, or $\lambda_2^2a^2 = \lambda_1^2b^2$ in the second case.

Remark 6.4.7. The results in this section are essentially a special case of Ariki's result [Ari94] on the conditions for semisimplicity of Ariki-Koike algebras [AK94]. As usual these algebras are generically semsisimple and Ariki gives rational functions in the parameters whose zero sets are exactly the parameters where semisimplicity fails.

7

Computation of braid representations and restriction of parameters

The goal of this section is to explicitly compute the matrix representations of the braid elements in the path modules for an $SO(2n)$-type category. Along the way we will prove additional restrictions on the eigenvalues $q, -q^{-1}, r^{-1}$ of the braid element $c_{X,X}$.

Let $\mathcal{C}$ be a ribbon category of $SO(2n)$ or $SO(2n) - O(K)$ type with $n \geq 3$ and $K \geq 3$, and fix a simple object X corresponding to the fundamental representation. These fusion rules are $\mathbb{Z}_2$-graded, so modifying $\mathcal{C}$ by a cocycle construction we may assume the braid element $c_1 = c_{X,X}$ has eigenvalues $(q, -q^{-1}, r^{-1})$ on the simple objects $[2], [1^2], \mathbf{1}$ respectively. Similarly we may assume X is symmetrically self-dual, meaning we can pick the left and right duality maps i_X, d_X, i'_X, d'_X such that $i_X = i'_X$ and $d_X = d'_X$. Under these circumstances the twist on X coming from the ribbon structure is $\theta_X = r$. Let $e = 1/\dim(X) i_X \circ d_X$ denote projection onto $\mathbf{1} \subset X^{\otimes 2}$. With these parameters the braid element must satisfy the Dubrovnik relation. We infer from this that $\mathcal{C}$ is symmetric if and only if $q = \pm 1$. In this thesis we will only address $SO(2n)$ type categories with non-symmetric braidings, so we assume henceforth $q \neq \pm 1$.

Our method of computing braid representations hinges on knowing the eigenvalues of the full twist Δ_k^2 (or equivalently the Jucys-Murphy operators) on the path modules. If V^λ denotes the path module of $\mathrm{End}(X^{\otimes k})$ equipped with a path basis $\{v_T\}$, then the action of c_{k-1} is determined by its action on the collection of path modules W^λ_μ. For $\mu \neq \lambda$ there are exactly 1 or 2 paths from μ to λ, so W^λ_μ is 1 or 2 dimensional. In case W^λ_μ is 1-dimensional, the action of c_{k-1} is determined by the fusion rules and the braid eigenvalues $(q, -q^{-1}, r^{-1})$.

On the other hand, if W^λ_μ is 2-dimensional then we can show with more effort that eigenvalues of c_{k-1} are still determined by the fusion rules and braid eigenvalues. By Prop. 6.4.5, there are 3 possibilities for W^λ_μ as an AB_2-module: it either splits into the direct sum of two irreps, it is irreducible, or it is indecomposable but not irreducible. If we know which we have then the action of c_k in the path basis is determined by the eigenvalues of c_k and of the Jucys-Murphy element J_k by Prop. 6.4.1. Our aim is to show that the eigenvalues of the JM elements can be expressed in terms of q and r, and to determine whether each W^λ_μ splits into 2 one-dimensional irreps, is irreducible, or is not semisimple. Whether or not this happens depends on whether q and r satisfy certain additional relations, e.g., r is an integer power of q. Ultimately we will prove that for $\mu \neq \lambda$, W^λ_μ is an irreducible AB_2-module, and the eigenvalues of the JM elements are given by a certain equation in q, that $r = q^{2n-1}$, and that q is a certain root of unity in the fusion case.

7.1 Jucys-Murphy eigenvalues and restriction of parameters

Recall that the Jucys-Murphy elements are defined by $J_k = \Delta^2_{k-1}\Delta^{-2}_k$ where Δ^2_k is the full twist in $\mathrm{End}(X^{\otimes k})$. Since the full twist is central, it acts by a scalar on any path module V^λ for $\lambda \in \mathcal{B}(k)$. We denote this scalar $\alpha_{k,\lambda}$:

$$\Delta^2_k|_{V^\lambda} = \alpha_{k,\lambda} 1_{V^\lambda} \tag{7.1}$$

Clearly the eigenvalues of the Jucys-Murphy elements are determined by these scalars: if S is a path with $S(k-1) = \mu, S(k) = \lambda$ then

$$J_k v_S = \alpha_{k,\lambda} \alpha_{k-1,\mu}^{-1} v_S.$$

We will derive a combinatorial formula for these scalars below. For $\lambda \in \mathcal{B}(k)$, $|\lambda|$ denotes the number of boxes in the Young diagram corresponding to λ. Given a box b in the (i, j) position of a Young diagram, the *content* of b is $\mathrm{cn}(b) = j - i$ (the positions of a Young diagram are listed like matrices, increasing from left to right and top to bottom). To get started we need some auxiliary statements.

Lemma 7.1.1. *Let $\lambda = [2, 1]$ and consider the path module V^λ. Then the braid elements c_1, c_2 do not commute on V^λ. Therefore $q \neq -q^{-1}$.*

Proof. This is proved in [TW05], Lemma 6.1.

□

Recall that W_μ^λ is the path module for AB_2 spanned by paths of length 2 between $SO(2n)$ shapes $\mu \in \mathcal{B}(k-2)$ and $\lambda \in \mathcal{B}(k)$ which are 2 levels apart in the Bratteli diagram. Unless $\mu = \lambda$ this space is 1 or 2-dimensional.

Lemma 7.1.2. *If W_μ^λ is 2-dimensional and $\lambda \neq \mu$ then c_{k-1} has eigenvalues q and $-q^{-1}$, and none of the operators c_{k-1}, J_k, J_{k-1} act as a scalar times the identity on W_μ^λ.*

Proof. Note that by the fusion rules, if $\lambda \neq \mu$ then λ appears in both $\mu \otimes [2]$ and $\mu \otimes [1^2]$. Hence by Lemma 6.4.2, c_{k-1} acts with eigenvalues q and $-q^{-1}$. We have $q \neq -q^{-1}$ by the previous lemma, so c_{k-1} cannot act by a scalar times the identity. Now if either J_{k-1} or J_k act by a scalar, then the other does too, since $J_{k-1} J_k$ always acts a scalar on W_μ^λ. But from the equation $J_k = c_{k-1} J_k c_{k-1}$ this implies c_{k-1}^2 acts by a scalar. This can only happen if the eigenvalues of c_{k-1} are negative of each other, which can't happen since we assume $q \neq q^{-1}$. □

It turns out that to compute the eigenvalues of the full twist in terms of r and q it suffices to determine the twist on "new stuff", i.e. shapes in $\mathcal{B}(k)$ with k many boxes (and not fewer).

Lemma 7.1.3. *Suppose $\lambda \in \mathcal{B}(k-2)$ with Δ_{k-2}^2 acting by $\alpha_{k-2,\lambda}$. Then the eigenvalue of the full twist on λ as an object of $\mathcal{B}(k)$ is given by*

$$\alpha_{k,\lambda} = r^{-2}\alpha_{k,\lambda-2}.$$

Proof. Let p_λ be a minimal idempotent of shape λ in $\mathrm{End}(X^{\otimes k-2})$. Let $E = (1/x)e_{k-1} = (1/x)1^{k-2} \otimes (i_X \circ d_X)$ denote the $k-1$-th Jones projection (see Sec. 6.2). Then $p_\lambda \otimes E$ is a minimal idempotent in $\mathrm{End}(X^{\otimes k})$ of type λ. We can compute $\Delta_k^2(p_\lambda \otimes E)$ as follows:[1]

$$\begin{aligned}
\Delta_k^2(p_\lambda \otimes E) &= r^{-k}\Theta_{X^{\otimes k}}(p_\lambda \otimes E) \\
&\qquad \text{(by Eq. 6.7)} \\
&= (1/x)r^{-k}c_{X^{\otimes k-2},X^{\otimes 2}}c_{X^{\otimes 2},X^{\otimes k-2}}\theta_{X^{\otimes k-2}}\theta_{X^{\otimes 2}}(p_\lambda \otimes (i_X \circ d_X)) \\
&\qquad \text{(by the ribbon property Eq. 2.4)} \\
&= (1/x)r^{-k}c_{X^{\otimes k-2},X^{\otimes 2}}c_{X^{\otimes 2},X^{\otimes k-2}}(\theta_{X^{\otimes k-2}} \otimes 1)(p_\lambda \otimes (i_X \circ d_X)) \\
&\qquad \text{(by naturality of } \theta \text{ and } \theta_{\mathbf{1}} = 1) \\
&= r^{-k}(\theta_{X^{\otimes k-2}} \otimes 1)(p_\lambda \otimes E) \\
&\qquad \text{(by naturality of the braiding)} \\
&= r^{-2}(\Delta_{k-2}^2 \otimes 1)(p_\lambda \otimes E) \\
&\qquad \text{(by Eq. 6.7)} \\
&= r^{-2}\alpha_{k-2,\lambda}(p_\lambda \otimes E).
\end{aligned}$$

Hence the result. ∎

[1] These steps are also easily verified in the graphical calculus.

Now we examine path modules corresponding to hook shapes. Recall that a Young diagram is a *hook* if it is of the form $[l, 1^m]$ for some $l, m \geq 1$. A hook is *nontrivial* if it is not a row or column. As usual, an $SO(2n)$-shape is also called a hook if its underlying Young diagram is a hook. In the following formula the notation $b \in \lambda$ means b is a box in the Young diagram of λ. In particular we get the same value for $SO(2n)$-shapes which differ only in the sign of their nth entry.

Proposition 7.1.4. *Suppose $\lambda \in \mathcal{B}(k)$ is a hook. Then*

1. $\alpha_{k,\lambda} = r^{|\lambda|-k} q^{2\sum_{b\in\lambda} cn(b)}$.

2. *q^2 is not an l-th root of unity for $l = 1, \dots k-1$.*

Proof. By Lemma 7.1.3, it suffices to prove the formula (1) when $|\lambda| = k$, which we do by induction. For $k = 1$ we have $\Delta_1^2 = 1$ and $\lambda = [1]$ has only one box with content 0 so (1) follows and (2) is vacuous. For $k = 2$, we note that by our labeling convention c_1 has the eigenvalue q on the shape [2], so $\Delta_2^2 = c_1^2$ acts by q^2 on $V^{[2]}$ which agrees with the formula to prove. Similarly for the shape $[1^2]$, so formula (1) is true. We cannot have $q^2 = 1$ as the Dubrovnik relation would imply $\mathcal{C}$ is symmetric.

Next we consider $k = 3$ and $\lambda = [2, 1]$. Consider the path module $W_{[1]}^{[2,1]}$, on which c_2 has eigenvalues $q, -q^{-1}$. By the previous paragraph the Jucys-Murphy element $J_2 = \Delta_2^2$ has eigenvalues (q^2, q^{-2}) on the paths going through [2] and $[1^2]$, respectively. Hence by Lemma 7.1.2, $q^2 \neq q^{-2}$, i.e. $(q^2)^2 \neq 1$, which proves (2) in this case. Now consider $W_{[1]}^{[2,1]}$ as an AB_2-module. Since we have established $q \neq \pm q^{-1}$, we may apply Prop. 6.4.5 and deduce that either $W_{[1]}^{[2,1]}$ is indecomposable or it splits as a sum of 2 1-dimensional representations. In the latter case we see that both c_1 and c_2 act diagonally in the path basis, which contradicts Lemma 7.1.1. Hence $W_{[1]}^{[2,1]}$ is indecomposable and by Prop. 6.4.5, the eigenvalues of J_3 are (q^{-2}, q^2) and $\Delta_3^2 = J_3 J_2$ acts as the identity on $V^{[2,1]}$, establishing (1).

We prove the result for the remaining hooks by induction (note that removing boxes from a hook always results in a hook). Suppose $k \geq 4$ or $k = 3$ and $\lambda = [3]$ or $\lambda = [1^3]$. In this situation it is always possible to remove 2 boxes from λ in the same row or column. Let ν be obtained in such a way from λ. Then the path module W_ν^λ is 1-dimensional, and the fusion rules imply that c_{k-1} acts by q or $-q^{-1}$, depending on whether λ is obtained by adding two boxes in the same row or column, respectively. Let b_1 denote the first box added to ν en route to λ and b_2 the second. By induction, J_{k-1} acts on W_ν^λ by $q^{2\mathrm{cn}(b_1)}$. Since W_ν^λ is 1-dimensional, J_k must act by $q^{2(\mathrm{cn}(b_1)\pm 1)}$, the sign depending on whether b_1, b_2 are added in the same row/column. But in the first case $\mathrm{cn}(b_2) = \mathrm{cn}(b_1) + 1$ and in the second $\mathrm{cn}(b_2) = \mathrm{cn}(b_1) - 1$, so in either case we have that J_k acts by the scalar $q^{2\mathrm{cn}(b_2)}$. So once again using the inductive hypothesis, we have that $\Delta_k^2|_{W_\mu^\lambda} = J_k J_{k-1} \Delta_{k-2}^2|_{W_\mu^\lambda}$ acts by $q^{2(\mathrm{cn}(b_1)+\mathrm{cn}(b_2))} q^{2\sum_{b\in\mu}\mathrm{cn}(b)} = q^{2\sum_{b\in\lambda}\mathrm{cn}(b)}$, which establishes (1).

To prove (2), let λ be a nontrivial hook of size $k \geq 4$. We can write $\lambda = [l, 1^{k-l}]$. Let $\mu = [l-1, 1^{k-l-1}]$ denote the shape obtained by removing boxes from both the first row and first column of λ. Then the fusion rules imply that c_{k-1} has eigenvalues $q, -q^{-1}$ on W_μ^λ and by induction, J_{k-1} has the eigenvalues $q^{2(l-1)}, q^{-2(k-l)}$ on W_μ^λ. By Lemma 7.1.2 these cannot be equal so $q^{2(k-1)} \neq 1$. □

Corollary 7.1.5. *If $\mathcal{C}$ is a ribbon category with the fusion rules of $SO(2n)$ (not a fusion category) then q^2 is not a root of 1.*

Proof. Indeed, there are isotypes corresponding to hooks of arbitrary size so by part (2) of the previous proposition q^2 is not a root of unity. □

To determine a restriction on the parameter r, we consider another family of 2-dimensional path modules.

Lemma 7.1.6. *Suppose $\mathcal{C}$ is an $SO(2n)$ or $SO(2n) - O(K)$ category, with n and $K \geq 3$, and non-symmetric braiding. Then $r \neq \pm q^{\pm 1}$.*

Proof. Since $\dim_{\mathcal{C}}(X) = 1 + \frac{r-r^{-1}}{q-q^{-1}}$ must be non-zero we find $r \neq -q$ and $r \neq q^{-1}$. It can be shown that $r \neq q$ and $r \neq -q^{-1}$ by examining the (level 3) path module $W_{[1]}^{[1]}$ (see [TW05], Lemma 6.4). □

Proposition 7.1.7. *Suppose $\mathcal{C}$ is an $SO(2n)$-category or $SO(2n) - O(K)$ category with n and $K \geq 3$. Then $r = q^{2n-1}$.*

Proof. Consider the $SO(2n)$-shape $\lambda = [1^n]^+$. By the fusion rules of $SO(2n)$ we have $X \otimes [1^n]^+ \cong [1^{n-1}] \oplus [2, 1^{n-1}]^+$, so the path module W_λ^λ (acted on by $c_{n+1}, J_{n+1}, J_{n+2}$) is 2-dimensional. According to the fusion rules, $[1^n]^+ \subset [1^n]^+ \otimes [1^2]$, so c_{n+1} acts on W_λ^λ with eigenvalues r^{-1} and $-q^{-1}$. By Lemma 7.1.6 we know $r \neq \pm(-q^{-1})$ so Prop. 6.4.5 applies and we conclude W_λ^λ is either indecomposable or splits as two 1-dimensional irreps and as usual we aim to rule out the latter possibility. Using the calculation of JM eigenvalues on hook shapes (Prop. 7.1.4), we compute the eigenvalues of J_{n+1} on the basis $v_{(\lambda,[1^{n-1}],\lambda)}, v_{(\lambda,[2,1^{n-1}]^+,\lambda)}$ to be $(r^{-2}q^{2(n-1)}, q^2)$ and the eigenvalues of J_{n+2} to be $(q^{-2(n-1)}, r^{-2}q^{-2})$. If the representation splits then looking at the eigenvalue corresponding to the path $(\lambda, [2, 1^{n-1}]^+, \lambda)$ yields $r^{-2}q^{-2} = q^2r^{-2}$ or $r^{-2}q^{-2} = q^2q^{-2} = 1$, both of which are absurd thanks to Prop. 7.1.4 part (2) and Lemma 7.1.6. Therefore W_λ^λ is indecomposable and by Prop. 6.4.5 the eigenvalue of J_{n+2} on the $(\lambda, [1^n - 1], \lambda)$-path is equal to $r^{-1}q^{-1}q^2$. Comparing this with our previously computed value of $q^{-2(n-1)}$ from Prop. 7.1.4 we derive $r = q^{2n-1}$. □

In the fusion category case a similar analysis applied to row shapes gives another restriction on r and allows us to determine the order of q as a root of unity (the answer depends on the fusion rules). This is the main result on restriction of parameters for fusion categories.

Proposition 7.1.8. *Suppose $\mathcal{C}$ is an $SO(2n) - O(K)$ category with $n \geq 3$ and $K \geq 3$.*

1. *If K is even then $r = -q^{-(K-1)}$ and q^2 is a primitive root of order $2n + K - 2$.*

2. *If K is odd then either $r = q^{-(K-1)}$ and q is a primitive root of order $l' = 2n+K-2$, or $r = -q^{-(K-1)}$ and q is a primitive root of order $2(2n+K-2)$.*

Remark 7.1.9. In any case q^2 is always a primitive root of order $2n + K - 2$.

Proof. Consider the shapes $\mu = [K] \in \mathcal{B}(K)$ and $\lambda = [K-1, 1] \in \mathcal{B}(K+2)$. There is a single path from μ to λ in the Bratteli diagram (first remove the last box from μ and then add one in the second row). By the fusion rule given by Eq. 3.24 for $SO(2n) - O(K)$ categories, λ appears in $\mu \otimes [1^2]$. Therefore c_{k+1} acts by $-q^{-1}$ on this 1-dimensional space. On the other hand, by Prop. 7.1.4, J_{k-1} and J_k have eigenvalues $r^{-2}q^{2(K-1)}$ and q^{-2} respectively. Then the AB_2 relation implies $r^{-2}q^{2(K-1)} = q^{-2}(-q^{-1})^2$ so $r = \pm q^{-(K-1)}$.

We combine this with $r = q^{2n-1}$ to find

$$q^{2n+K-2} = \pm 1.$$

If K is even and this is equal to $+1$ then q^2 would be a $n + K/2 - 1$-th root of unity which contradicts the previous restriction on q, Prop. 7.1.4 part (2). Hence $r = -q^{-(K-1)}$ in the K even case and the equation above shows that the order of q^2 divides $2n + K - 2$; however the order of q^2 must be greater than half that by Prop. 7.1.4 so the order of q^2 must equal $2n + K - 2$.

If K is odd and $r = q^{-(K-1)}$ then the equation above shows that the order of q must divide $2n + K - 2$ and as before this implies its order must in fact equal $2n + K - 2$. In the case $r = -q^{-(K-1)}$ the order of q^2 must similarly be $2n + K - 2$, and then the equation above implies q has order $2(2n + K - 2)$. □

Now the restriction on q is enough for us to compute all of the eigenvalues of the full twist. The following expression was derived for the quantum group categories by Wenzl ([Wen93], Sec. 3.2), where the full-twist eigenvalue is called the *framing anomaly.* Note in particular that the formula only depends on the shape of λ as a Young diagram.

Hence if λ^+ and λ^- are $SO(2n)$ shapes just differing by the sign in their nth entry, we have $\alpha_{k,\lambda^+} = \alpha_{k,\lambda^-}$.

Theorem 7.1.10. *Suppose $\mathcal{C}$ is of $SO(2n)-type$ and is not a fusion category. Suppose $\lambda \in \mathcal{B}(k)$ is an $SO(2n)$-shape of size $|\lambda| \leq k$. Then Δ_k^2 acts on V^λ by*

$$\alpha_{k,\lambda} = r^{|\lambda|-k} q^{2\sum_{b\in\lambda} cn(b)}.$$

Proof. Again by Lemma 7.1.3 it suffices to consider λ with $|\lambda| = k$. If it is possible to remove two boxes from the same row or same column of λ then the exact same argument as used in the previous proposition for hook shapes works to compute $\alpha_{k,\lambda}$. The only time this procedure cannot be used to compute $\alpha_{k,\lambda}$ is when λ is a staircase partition, i.e. $k = \binom{d+1}{2}$ for some $d \geq 2$ and $\lambda = [d, d-1, \ldots, 1]$ (possibly with a plus/minus if $d = n$). In this case λ has exactly d outside boxes $b_1, b_2, \ldots, b_d$ with contents $d-1, d-3, \ldots, 1-d$ respectively. Let ν_i denote the shape obtained by removing b_i from λ and $\mu_{i,j}$ the shape of size $k-2$ obtained by removing both i and j boxes. By induction, the matrix J_{k-1} acts on the 2-dimensional path-module $W^\lambda_{\mu_{i,j}}$ with basis indexed by paths $(\mu_{i,j} \to \nu_j \to \lambda)$ and $(\mu_{i,j} \to \nu_i \to \lambda)$ by the matrix

$$J_{k-1} \mapsto \begin{pmatrix} q^{2\mathrm{cn}(b_j)} & 0 \\ 0 & q^{2\mathrm{cn}(b_i)} \end{pmatrix}$$

Let x_i denote the eigenvalue of J_k on a path ending with $\nu_i \mapsto \lambda$. Then J_k has the matrix

$$J_k \mapsto \begin{pmatrix} x_i & 0 \\ 0 & x_j \end{pmatrix}.$$

Since $\Delta_k^2 = J_{k-1}J_k$ must act as a scalar on $W^\lambda_{\mu_{i,j}}$ we have

$$q^{2\mathrm{cn}(b_j)}x_i = x_j q^{2\mathrm{cn}(b_i)}$$

for every $1 \leq i < j \leq d$. These homogeneous equations are satisfied iff $(x_1, \ldots, x_d) = \gamma(q^{2\mathrm{cn}(b_1)}, \ldots, q^{2\mathrm{cn}(b_d)})$ for some $\gamma \in \mathbb{C}$ and it suffices to prove $\gamma = 1$. To that end, let $i = \lfloor d/2 \rfloor$ and $j = i+1$. Then $\mathrm{cn}(b_i) = \mathrm{cn}(b_j) + 2 \in \{1, 2\}$. Consider $W^\lambda_{\mu_{i,j}}$. If $W_{\mu_{i,j}}$ is indecomposable then by Prop. 6.4.5 $\gamma = 1$. The other possibility is $W_{\mu_{i,j}}$ splits as a sum of 1-dimensional irreps. This leaves two possibilities: either $\gamma q^{-2} = q^4$ and $\gamma q^2 = 1$ or $\gamma q^{-2} = 1$ and $\gamma q^2 = 1$. Neither case is possible since q^2 is either not a root of unity, or it is a root with order $2n + K - 2 > 3$ (by the assumption $n, K \geq 3$). □

Since $J_k = \Delta_k^2 \Delta_{k-1}^{-2}$, we obtain the following formula for the diagonal entries of the Jucys-Murphy elements, valid in any path basis:

Corollary 7.1.11.

$$(J_k)_{S,S} = \begin{cases} q^{2cn(b)} & \textit{if } S(k-1) \to S(k) \textit{ involves adding a box } b \\ r^{-2}q^{-2cn(b)} & \textit{if } S(k-1) \to S(k) \textit{ involves removing the box } b. \end{cases} \tag{7.2}$$

We can now identify the path modules W^λ_μ for which $\mu \neq \lambda$.

Proposition 7.1.12. *Suppose λ and μ are distinct $SO(2n)$-shapes with $\lambda \in \mathcal{B}(k), \mu \in \mathcal{B}(k-2)$ and $\lambda \subset \mu \otimes X^{\otimes 2}$. Then W^λ_μ is irreducible as an AB_2-module.*

Proof. The path module W^λ_μ is either 1 or 2 dimensional, and there is nothing to prove in the first case. Suppose it is 2 dimensional. By Lemma 7.1.2, we know c_{k-1} acts by the scalars $q, -q^{-1}$ and these eigenvalues are not equal. If W^λ_μ is indecomposable then it's automatically irreducible by Prop. 6.4.5. Hence it suffices to rule out the possbility that W^λ_μ splits into two 1-dimensional irreps.

First suppose μ is obtained from λ by adding boxes b_1, b_2. We may assume these boxes are not in the same row or column, as otherwise W_μ^λ is 1-dimensional. Then by Theorem 7.1.10 the JM element J_{k-1} has eigenvalues $(q^{2\text{cn}(b_1)}, q^{2\text{cn}(b_2)})$ on the path basis of W_μ^λ. Also J_k has the same eigenvalues $(q^{2\text{cn}(b_2)}, q^{2\text{cn}(b_1)})$ while b_{k-1} has eigenvalues $q, -q^{-1}$. If the representation splits then we have a relation $q^{2\text{cn}(b_2)} = q^{\pm 2} q^{2\text{cn}(b_1)}$, equivalently $q^{2(\text{cn}(b_1)-\text{cn}(b_2)\pm 1)} = 1$. Since b_1 and b_2 are not in the same row or column, we have that $\text{cn}(b_1) - \text{cn}(b_2) \neq \pm 1$. Hence the relation above implies q^2 is a root of unity with order at most $|\text{cn}(b_1) - \text{cn}(b_2) \pm 1|$. This is in contradiction with the fact that q^2 has order $2n + K - 2$ (by Prop. 7.1.8) since $|\text{cn}(b_1) - \text{cn}(b_2)| + 1$ is at most the maximum hook length of λ which is bounded by $n + K - 1$.

The case in which μ is obtained by removing 2 boxes is quite similar and we omit it. We will discuss the case that μ is obtained from λ by adding a box (say b_1) and removing a different box (say b_2). Using Prop. 7.1.4 we compute the eigenvalues of J_{k-1} to be $(q^{2\text{cn}(b_1)}, r^{-2}q^{-2\text{cn}(b_2)})$ and the eigenvalues of J_k to be $(r^{-2}q^{-2\text{cn}(b_2)}, q^{2\text{cn}(b_1)})$. Suppose for contradiction that W_μ^λ splits into 1-dimensional irreps. Then we would obtain a relation $r^{-2}q^{-2\text{cn}(b_2)} = q^{2(\text{cn}(b_1)\pm 1)}$. Using $r = q^{2n-1}$ this implies

$$q^{2(\text{cn}(b_1)+\text{cn}(b_2)+2n-1\pm 1)} = 1. \tag{7.3}$$

However since any $SO(2n)$-shape has at most n rows and K columns the content of any box is at least $-(n-1)$ and at most $K-1$. Furthermore, since b_1 and b_2 must not belong to the same row or same column, their contents differ by at least two, which implies

$$\text{cn}(b_1) + \text{cn}(b_2) \geq -(n-1) - (n-3) = -2n + 4$$

and similarly

$$\text{cn}(b_1) + \text{cn}(b_2) \leq 2K - 4.$$

Therefore

$$3 \pm 1 \leq \mathrm{cn}(b_1) + \mathrm{cn}(b_2) + 2n - 1 \pm 1 \leq 2K + 2n - 5 \pm 1.$$

This shows that Eq. (7.3) contradicts that q^2 is a primitive root of order $2n + K - 2$. □

7.2 Uniqueness of dimensions

In the next section we will identify W_λ^λ as AB_2-modules. The matrix representations will be expressed in terms of the (categorical) dimensions in $\mathcal{C}$, so next we show that the dimensions are determined by the parameter q and the fusion rules. Important for the proof are the following quantities:

$$s_k := \mathrm{Tr}_q(J_{k+1}(p_S \otimes 1))$$

where p_S is a path idempotent of length k ending at $[1^k]$ and $k = 1, \ldots, n-1$. In addition we define

$$s_0 = \dim_{\mathcal{C}} X.$$

In the language of ribbon categories, these numbers are the coefficients of the S-matrix in the column corresponding to the object X, for instance s_k is often denoted $s_{[1^k],X}$. We have the following relation:

Lemma 7.2.1. *Suppose $\mathcal{C}$ is an $SO(2n)$ or $SO(2n) - O(K)$ type category and c_1 has eigenvalues $(q, -q^{-1}, q^{2n-1})$. Then the coefficients s_k satisfy the following recursion:*

$$s_k = \dim_{\mathcal{C}}[1^k] \left(\frac{s_{k-1}}{\dim_{\mathcal{C}}[1^{k-1}]} + (q - q^{-1}) q^{2n-1} (J_k)_{S,S} - (q - q^{-1}) q^{-(2n-1)} (J_k)^{-1}_{S,S} \right).$$

Proof. We use the BMW skein relation in conjunction with the AB_2-relation. Let S be the unique path of length k ending at $[1^k]$. Note that S' ends at $[1^{k-1}]$. We have

$$\begin{aligned}
s_k &= \mathrm{Tr}_q(J_{k+1}(p_S \otimes 1)) \\
&= \mathrm{Tr}_q(c_k J_k c_k (p_S \otimes 1)) \\
&= \mathrm{Tr}_q(c_k J_k (c_k^{-1} + (q - q^{-1})(1 - e_k))(p_S \otimes 1)) \\
&= \mathrm{Tr}_q(c_k J_k c_k^{-1}(p_S \otimes 1)) + (q - q^{-1})\mathrm{Tr}_q(c_k J_k (p_S \otimes 1)) - (q - q^{-1})\mathrm{Tr}_q(c_k J_k e_k (p_S \otimes 1)).
\end{aligned}$$

Now examine each term separately. Using the diagrammatic calculus we check that

$$\begin{aligned}
\mathrm{Tr}_q(c_k J_k c_k^{-1}(p_S \otimes 1)) &= \frac{\dim_{\mathcal{C}}[1^k]}{\dim_{\mathcal{C}}[1^k - 1]} s_{k-1} \\
\mathrm{Tr}_q(c_k J_k (p_S \otimes 1)) &= q^{2n-1}(J_k)_{S,S} \dim_{\mathcal{C}}[1^k] \\
\mathrm{Tr}_q(c_k J_k e_k (p_S \otimes 1)) &= q^{-(2n-1)}(J_k)_{S,S} \dim_{\mathcal{C}}[1^k].
\end{aligned}$$

This gives us the desired equation. □

Recall that for an integer n, $[n]_q$ denotes the *quantum number* given by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)}.$$

Clearly $[n]_q$ is a rational expression in q.

Proposition 7.2.2. *Suppose $\mathcal{C}$ is an $SO(2n)$ or $SO(2n) - O(K)$ type ribbon category and c_1 has eigenvalues $(q, -q^{-1}, q^{2n-1})$. Then the q-dims of simple objects are uniquely determined and can be expressed as a rational function of q.*

Proof. First we show that we can compute $\dim_{\mathcal{C}}[1^k]$ and $\dim_{\mathcal{C}}[2, 1^{k-2}]$ for $k = 1, \ldots, n-1$ by induction. For $k = 1$ we can compute $x := \dim_{\mathcal{C}}[1] = \dim_{\mathcal{C}} X$ by attaching caps to the Dubrovnik relation to obtain $\dim[1] = 1 + [2n-1]_q$. Now suppose we have a formula for

$\dim_{\mathcal{C}}[1^i]$ and $\dim_{\mathcal{C}}[2, 1^{i-2}]$ for $i = 1, \ldots, k-1$. By the $SO(2n)$ fusion rules we have

$$\dim_{\mathcal{C}}[1] \dim_{\mathcal{C}}[1^{k-1}] - \dim_{\mathcal{C}}[1^{k-2}] = \dim_{\mathcal{C}}[2, 1^{k-2}] + \dim_{\mathcal{C}}[1^k]. \tag{7.4}$$

We can use the Jucys-Murphy elements to obtain another linear equation for the unknowns $\dim_{\mathcal{C}}[2, 1^{k-2}]$ and $\dim_{\mathcal{C}}[1^k]$. Let $p_S \in \mathrm{End}(X^{\otimes k-1})$ denote a path idempotent of type $[1^{k-1}]$. We will compute $s_{k-1} = \mathrm{Tr}_q(J_k(p_S \otimes 1))$ in two ways. Define the paths

$$\begin{aligned} Q &= S \to [1^{k-2}] \\ R &= S \to [1^k] \\ T &= S \to [2, 1^{k-2}] \end{aligned}$$

Then $p_S \otimes 1 = p_Q + p_R + p_T$. Hence

$$\begin{aligned} s_{k-1} &= \mathrm{Tr}_q(J_k(p_Q + p_R + p_T)) \\ &= \mathrm{Tr}_q((J_k)_{QQ} p_Q + (J_k)_{RR} p_R + (J_k)_{TT} p_T) \\ &= (J_k)_{QQ} \dim_{\mathcal{C}}[1^{k-2}] + (J_k)_{RR} \dim_{\mathcal{C}}[1^k] + (J_k)_{TT} \dim_{\mathcal{C}}[2, 1^{k-2}]. \end{aligned} \tag{7.5}$$

On the other hand repeatedly applying the previous lemma shows that s_{k-1} can be expressed solely in terms of $\dim_{\mathcal{C}}[1], \dim_{\mathcal{C}}[1^2], \ldots \dim_{\mathcal{C}}[1^{k-1}]$, which are themselves known to be rational functions in q by induction. Hence Eqs. (7.4) and (7.5) give linear equations in the unknowns $\dim_{\mathcal{C}}[1^k]$ and $\dim_{\mathcal{C}}[2, 1^{k-2}]$. Furthermore, $(J_k)_{RR} = q^{-2(k-1)}$ and $(J_k)_{TT} = q^2$ so $(J_k)_{RR} \neq (J_k)_{TT}$ since q^2 is not a root of unity in the non-fusion case, and in the fusion case by Prop. 7.1.8 the order of q^2 is $2n + K - 2$ which is greater than k. Hence the system of equations admits a unique solution for $\dim_{\mathcal{C}}[1^k]$ and $\dim_{\mathcal{C}}[2, 1^{k-2}]$, and both are expressible as rational functions of q.

Next we consider $\dim_{\mathcal{C}}[1^n]^{\pm}$. Now the fusion rules give an equation

$$\dim_{\mathcal{C}}[1]\dim_{\mathcal{C}}[1^{n-1}] = \dim_{\mathcal{C}}[1^{n-2}] + \dim_{\mathcal{C}}[2,1^{n-2}] + \dim_{\mathcal{C}}[1^n]^+ + \dim_{\mathcal{C}}[1^n]^-. \tag{7.6}$$

We again get another linear equation by computing s_{n-1} in the two different ways. On one hand the previous lemma again shows s_{n-1} can be written as a rational expression of q. Now $p_S \otimes 1$ decomposes as a sum of 4 minimal idempotents corresponding to paths

$$\begin{aligned} Q &= S \to [1^{n-2}] \\ R^+ &= R \to [1^n]^+ \\ R^- &= R \to [1^n]^- \\ T &= S \to [2, 1^{k-2}]. \end{aligned}$$

Using that $(J_k)_{R^+,R^+} = (J_k)_{R^-,R^-}$ we arrive at an equation

$$s_{n-1} = (J_k)_{QQ}\dim_{\mathcal{C}}[1^{n-2}] + (J_k)_{R^+,R^+}(\dim_{\mathcal{C}}[1^n]^+ + \dim_{\mathcal{C}}[1^n]^-) + (J_k)_{TT}\dim_{\mathcal{C}}[2,1^{n-2}]. \tag{7.7}$$

As before, the equations (7.6) and (7.7) uniquely determine $\dim_{\mathcal{C}}[2,1^{n-2}]$ and $\dim_{\mathcal{C}}[1^n]^+ + \dim_{\mathcal{C}}[1^n]^-$.

Recall the Dynkin automorphism σ of the fusion ring which interchanges + and − labelled shapes. By Lemmas 3.3.7 and 3.2.1, the fixed subring $\mathrm{Gr}(\mathcal{C})^\sigma$ is generated algebraically by $[1], [1^2], \ldots, [1^{n-1}], [1^n]^+ + [1^n]^-$. Hence the character $\dim_{\mathcal{C}} : \mathrm{Gr}(\mathcal{C})^\sigma \to \mathbb{C}$ is fully determined by its value on these elements, which we've seen to depend only on q. In particular $[1^n]^+ \otimes [1^n]^-$ belongs to the fixed subring so $\dim_{\mathcal{C}}([1^n]^+ \otimes [1^n]^-)$ is uniquely determined by q. Thus we know that the sum and the product of $\dim_{\mathcal{C}}[1^n]^+$ and $\dim_{\mathcal{C}}[1^n]^-$ are given as certain rational expressions in q, say f_1 and f_2 respectively. In general there are two solutions to the system of equations

$$\dim_{\mathcal{C}}[1^n]^+ + \dim_{\mathcal{C}}[1^n]^- = f_1$$
$$\dim_{\mathcal{C}}[1^n]^- \dim_{\mathcal{C}}[1^n]^- = f_2.$$

However in the quantum group case the q-dims are equal (see Remark 3.6.3) and satisfy the same equations. Hence the system admits a unique solution,

$$\dim_{\mathcal{C}}[1^n]^+ = \dim_{\mathcal{C}}[1^n]^- = \frac{1}{2}(\dim_{\mathcal{C}}[1^n]^+ + \dim_{\mathcal{C}}[1^n]^-).$$

We have proved that $\dim_{\mathcal{C}}[1], \dim_{\mathcal{C}}[1^2], \ldots, \dim_{\mathcal{C}}[1^n]^\pm$ can all be written as a rational expression in q, depending only on the fusion rules of $\mathcal{C}$. Since these objects algebraically generate the fusion ring of $\mathcal{C}$, the remaining q-dims are also determined uniquely. □

Corollary 7.2.3. *Suppose $\mathcal{C}$ is an $SO(2n)$ category and λ^+, λ^- are shapes with n rows. Then*

$$\dim_{\mathcal{C}} \lambda^+ = \dim_{\mathcal{C}} \lambda^-.$$

Proof. The q-dims are independent of the category $\mathcal{C}$, so it suffices to note this is true when $\mathcal{C}$ is a quantum group category. □

Remark 7.2.4. The proof above marks the only time we use additional information about the existing quantum group categories. Everything else has just been a consequence of the fusion rules and ribbon axioms.

7.3 Uniqueness of braid representations

So far we have computed the Jucys-Murphy elements in any path basis as well as the diagonal entries of the braid matrices. However the off-diagonal elements of the braid representations depend on a particular choice of path basis. We now describe such a choice and then prove that in this basis the braid matrices are uniquely determined.

Since the arguments here use heavily the combinatorics of the Bratteli diagram, we review the situation. The $SO(2n)$-shapes are all Young diagrams with fewer than n rows, and two shapes λ^+ and λ^- for every Young diagram with exactly n rows. The $SO(2n) - O(K)$ shapes are those $SO(2n)$ shapes which additionally satisfy $\lambda_1 + \lambda_2 \leq K$. The kth level $\mathcal{B}(k)$ of the Bratteli diagram consists of all $SO(2n)$-shapes (or $SO(2n) - O(K)$ shapes) for which the size of the Young diagram is congruent to k mod 2. There is an edge connecting any two shapes which can be obtained by adding or removing a box, with the additional rules that adding a box to a + (resp. $-$) labeled diagram results in a + (resp. $-$) labeled diagram. We write $\mu \prec \lambda$ to mean λ is one level after μ and they are connected.

Let $<$ be any total ordering of the set of $SO(2n)$-shapes which refines size, meaning $\mu < \lambda$ if $|\mu| < |\lambda|$. We use this and the last-letter convention to define a total ordering $<_{LL}$ on paths of equal length, defined by $S <_{LL} T$ if $S(i) < T(i)$ as $SO(2n)$-shapes, where i is the last-letter of disagreement for S and T, meaning $S(j) = T(j)$ for all $j > i$.

Proposition 7.3.1. *Let $\mathcal{C}$ be an $SO(2n)$-type ribbon category generated by X, with $n \geq 3$. Then for each λ appearing in X^k there exists a path basis for V^λ such that for all $j = 1, \ldots, k-1$*

(a) If $|\lambda| = k$ and $S \neq T$ are paths ending at λ which only differ in the j-th position such that $S < T$ then $(c_j)_{S,T} = 1$.

(b) If $|\lambda| < k$ and S, T are paths ending at λ which only differ in the jth position and $S(j+1) = S(j-1) = T(j+1) = T(j-1)$ then

$$(e_j)_{S,T} = \frac{\dim_{\mathcal{C}} T(j-1)}{\dim_{\mathcal{C}} S(j)}.$$

Up to an overall rescaling this basis is unique.

Proof. The proof is by induction on k. First suppose $|\lambda| = k$. That there exists a path basis for V^λ which satisfies (a) follows from the existence of seminormal representations for irreps of the Hecke algebra (see, e.g. [Wen88] or [KT08], Ch. 5). We give an argument here for the reader's convenience. Using the restriction rules we have

$$V^\lambda \cong \bigoplus_{\mu \prec \lambda} V^\mu$$

as $\mathrm{End}(X^{k-1})$-modules, with the sum going over all μ obtained by removing a box from λ. By induction we may fix bases for each V^μ satisfying (a) for $c_1, \ldots, c_{k-2}$, and the basis for V^μ is unique up to an overall rescaling on V^μ. We will show how to pick these scalings so that c_{k-1} satisfies (a). Let T_0 denote the least path (w.r.t $<$) ending at λ and let $\mu_0 = T_0(k-1)$. Let $v_{T_0} \in V^{\mu_0}$ be chosen arbitrarily (this choice determines every other path basis vector in V^{μ_0}). For each $\mu \prec \lambda$ distinct from μ_0 there is a unique shape $\nu \in \mathcal{B}(k-2)$ such that $\nu \prec \mu_0$ and $\nu \prec \mu$. Let P be a path of length $k-2$ ending at ν and consider the paths Q, R ending at λ which agree with P up to level $k-2$ and satisfy $Q(k-1) = \mu_0$, $R(k-1) = \mu$. Now define

$$v_R := p_R c_{k-1} v_Q.$$

This fixes the overall scaling on V^μ, and the scaling doesn't depend on the path P. Hence this procedure uniquely defines a path basis in V^λ. It remains to check that (a) holds for all paths S, T which differ only at the $(k-1)$-th position and satisfy $S < T$. If $S(k-1) = \mu_0$ then $(c_k)_{S,T} = 1$ by the definition of v_T. It is not possible for $T(k-1) = \mu_0$ because this would contradict $S < T$. On the other hand, if both $S(k-1) = \mu_1$ and $T(k-1) = \mu_2$ are distinct from μ_0 with $\mu_0 < \mu_1 < \mu_2$ then we can find unique shapes

$\tau \in \mathcal{B}(k-3)$ and $\nu_0 < \nu_1 < \nu_2 \in \mathcal{B}(k-2)$ such that

$$\tau \prec \nu_1,\ \nu_2,\ \nu_3$$

$$\nu_0 \prec \mu_0,\ \mu_1,\ \nu_1 \prec \mu_0,\ \mu_2,\ \nu_2 \prec \mu_1,\ \mu_2.$$

Also fix an arbitrary path Q of length $k-3$ ending at τ. For simplicity we refer to paths $Q \to \nu_i \to \mu_j \to \lambda$ by (i,j). This gives us 6 paths ordered by

$$(0,0) < (1,0) < (0,1) < (2,1) < (1,2) < (2,2).$$

To prove (a) for S, T it suffices to show $(c_k)_{(2,1),(2,2)} = 1$. We consider the braid relation $c_{k-1}c_{k-2}c_{k-1} = c_{k-2}c_{k-1}c_{k-2}$ evaluated at the matrix entry $((0,0),(2,2))$. Recall that $(c_j)_{S,T}$ is only non-zero when $S = T$ or S, T differ only at position j. Using this we have

$$(c_k c_{k-1} c_k)_{(0,0),(2,2)} = (c_k)_{(0,0),(0,1)}(c_{k-1})_{(0,1),(2,1)}(c_k)_{(2,1),(2,2)}$$
$$(c_{k-1} c_k c_{k-1})_{(0,0),(2,2)} = (c_{k-1})_{(0,0),(1,0)}(c_k)_{(1,0),(1,2)}(c_{k-1})_{(1,2),(2,2)}.$$

On the other hand

$$(c_k)_{(0,0),(0,1)} = 1 \qquad \text{(by the definition of } v_{(0,1)})$$
$$(c_{k-1})_{(0,1),2,1)} = (c_{k-1})_{(0,0),(1,0)} = (c_{k-1})_{(1,2),(2,2)} = 1 \qquad \text{(by the inductive hypothesis.)}$$

Hence the braid relation gives $(c_k)_{(2,1),(2,2)} = 1$ as well.

Now suppose $|\lambda| < k$. Again we restrict to $\mathrm{End}(X^{k-1})$ to get the decomposition

$$V^\lambda \cong \bigoplus_{\mu \prec \lambda} V^\mu$$

where now the sum is over all μ obtained by adding or removing a box from λ. We may assume by induction that each V^μ is equipped with a basis (unique up to a global rescaling of V^μ) satisfying (a) and (b) for $c_1, \ldots, c_{k-2}$ and $e_1, \ldots, e_{k-2}$. We've seen that $E_{k-1} = (1/x)e_{k-1}$ is a rank-1 idempotent on the path module W_λ^λ and has diagonal entries

$$(E_{k-1})_{S,S} = \frac{\dim_{\mathcal{C}} S(k-1)}{x \dim_{\mathcal{C}} \lambda}$$

for each length 2 path $S = \lambda \to S(k-1) \to \lambda$. Hence up to a global factor there is a unique choice of path basis for W_λ^λ for which $(E_{k-1})_{S_1,S_2} = \frac{\dim_{\mathcal{C}} T(k-1)}{x \dim_{\mathcal{C}} \lambda}$ (we are choosing a normalization so that the rank 1-idempotent E has identical entries in every column). In turn this determines the scaling of V^λ, as follows: let T_0 be the least path with $T_0(k-2) = T(k) = \lambda$ and arbitrarily choose the path basis vector v_{T_0}. The above comments concerning W_λ^λ imply that for all length 2 paths S from λ to λ there is a unique choice for the vector $v_{T_0'' \to S}$ such that e_{k-1} has the desired matrix representation on

$$\mathrm{span}\{\mathrm{v}_{\mathrm{T}_0'' \to \mathrm{S}} \mid \mathrm{S} = \lambda \to \mathrm{S}(\mathrm{k}-1) \to \lambda\} \cong \mathrm{W}_\lambda^\lambda.$$

Since $S(k-1)$ ranges over all the μ in the decomposition for V^λ, we have fixed a scaling for at least one path basis vector in each V^μ, which determines the scaling for every path basis vector by induction. By construction, (b) holds for this choice of scaling. □

We can restate this proposition in terms of matrix units for $\mathrm{End}(X^k)$:

Theorem 7.3.2. *Suppose $\mathcal{C}$ is an $SO(2n)$ or $SO(2n) - O(K)$ type ribbon category generated by X, with $n \geq 3$ and $K \geq 3$. Then there exists a unique choice of matrix units*

$$\{F_{R,S} \mid R, S \text{ paths of length } k \text{ ending at the same shape}\}$$

for End(X^k) corresponding to path bases (i.e. satisfying $F_{R,R} = p_R$) and satisfying (a) *and (b) above. Furthermore, if R, S are paths of length $k-1$ ending at μ then*

$$F_{R,S} \otimes 1 = \sum_{\mu \to \lambda} F_{R\to\lambda, S\to\lambda}. \tag{7.8}$$

Proof. If we fix path basis vectors v_S for V^λ then we can define matrix units via the conditions

$$F_{R,S} \in p_R \operatorname{End}(X^k) p_S$$
$$F_{R,S} v_S = v_R.$$

These matrix units do not change if we modify the overall rescaling of the path basis, which proves uniqueness. It only remains to check Eq. (7.8). For this, suppose $\mu \prec \lambda$. If R and S are of length $k-1$ ending at μ then in $\operatorname{End}(X^{\otimes k})$ we have the elements $F_{R\to\lambda,S\to\lambda}$. Letting R and S vary, this gives a full set of matrix units acting on the simple $\operatorname{End}(X^{k-1})$-invariant subspace V^μ of V^λ. They satisfy (a) and (b), so by uniqueness (applied to the matrix units in $\operatorname{End}(X^{\otimes k-1})$), they must agree with the action of $F_{R,S}$ on V^μ.

Hence on the path module V^λ, $F_{R,S} \otimes 1$ and $F_{R\to\lambda,S\to\lambda}$ act the same, from which Eq. (7.8) follows.

□

Remark 7.3.3. We can explicitly write down matrices for the braid generators in the "new stuff", i.e. on path modules V^λ with $|\lambda| = k$. Since the e_i vanish on these modules they afford representations of the Hecke algebra. In fact they are the well known semi-normal representations. Explicitly, if S is a path ending at λ such that λ is obtained from $S(k-2)$ by adding two boxes in the same row or same column then $c_{k-1} = q$ or $-q^{-1}$, respectively. If S, T are two paths differing only in the $(k-1)$-th position and $S < T$ (i.e.

$S(k-1) < T(k-1)$ in the ordering of $SO(2n)$-shapes) then we can define

$$d = \mathrm{cn}(b_2) - \mathrm{cn}(b_1)$$

where b_1 is the box added to $S(k-2)$ to get $S(k-1)$ and b_2 the box added to $S(k-1)$ to get λ. According to the normalization chosen in the above proposition, c_{k-1} acts on $\mathrm{span}\{v_S, v_T\}$ by

$$c_{k-1} \mapsto \begin{pmatrix} \frac{q^d}{[d]_q} & 1 - \frac{1}{[d]_q^2} \\ 1 & \frac{q^{-d}}{[-d]_q} \end{pmatrix}. \tag{7.9}$$

If we put $q = 1$ in these formulae we reduce to Young's seminormal representations of the symmetric group. Note that if λ is a Young diagram with n rows then V^{λ^+} and V^{λ^-} afford the same representations for the braid matrices.

The main result of the section is to show that all the matrix entries for the braid elements are determined by q (and otherwise independent of the category $\mathcal{C}$).

Theorem 7.3.4. *In the basis described by the previous proposition, all the matrix entries $(c_{k-1})_{R,S}$ of the braid elements are uniquely determined by q (and in fact can be written as certain rational functions of q with coefficients in $\mathbb{Q}$).*

Proof. We will inductively construct rational functions $f_{R,S} \in \mathbb{Q}(v)$ for every pair of paths R, S of length k, such that $(c_k)_{R,S} = f_{R,S}(q)$. For $k = 2$ the matrix of c_1 in the (ordered) path basis must be $\mathrm{diag}(q^{2n-1}, -q^{-1}, q)$. Now suppose the matrix entries of c_{k-1} can all be written as rational functions of q. Consider an entry $(c_k)_{R,S}$ with R, S paths of length $k+1$. We may assume R and S are equal, or differ only in the k-th position, since otherwise $(c_k)_{R,S} = 0$. Let $\lambda = R(k+1) = S(k+1)$ and $\mu = S(k-1) = R(k-1)$.

Case 1. $|\lambda| = k+1$.

In this case $(c_{k-1})_{R,S}$ is given by Eq. (7.9) and hence can be written as a rational function of q, for instance $(c_{k-1})_{R,R} = f_{R,R}(q)$ where $f_{R,R}(v) = \frac{v^d}{[d]_v}$.

Case 2. $|\lambda| < k+1$ and $\mu = \lambda$.

The following argument has previously been used by Leduc-Ram [LR97] (in the generic q case). Use the BMW relation

$$c_k - c_k^{-1} = (q - q^{-1})(1 - e_k).$$

and examine the action of c_k on

$$\operatorname{span}\{v_T : T \text{ differs from } S \text{ only in position } k\} \cong W_\lambda^\lambda.$$

If this space is just 1-dimensional then c_k must act by the scalar $r{=}q^{-(2n-1)}$ so its (single) matrix entry is certainly determined by q. Hence we may assume the space is at least 2 dimensional. On this subspace $J_k J_{k+1}$ acts by the scalar $q^{-2(2n-1)}$. Also $J_{k+1} = c_k J_k c_k$, so $c_k^{-1} = q^{2(2n-1)} J_k c_k J_k$ and the BMW relation reads

$$c_k - q^{2(2n-1)} J_k c_k J_k = (q - q^{-1})(1 - e_k).$$

Taking the (R, S) matrix entry of this equation we obtain

$$(c_k)_{R,S}(1 - q^{2(2n-1)}(J_k)_{R,R}(J_k)_{S,S}) = (q - q^{-1})(\delta_{R,S} - (e_k)_{R,S}).$$

By the choice of normalization $(e_k)_{R,S} = \frac{\dim_{\mathcal{C}} S(k)}{\dim_{\mathcal{C}} \lambda}$ (which is a rational function of q) and $(J_k)_{R,R}$ and $(J_k)_{S,S}$ are known integral powers of q^2. If $R \neq S$ then the right hand side above is non-zero, so we have

$$(c_k)_{R,S} = -(q - q^{-1}) \frac{(e_k)_{R,S}}{1 - q^{2(2n-1)}(J_k)_{R,R}(J_k)_{S,S}}. \tag{7.10}$$

If $R = S$ then we use the following fact then we have the following equation:

$$(c_k)_{R,R}(1 - q^{2(2n-1)}(J_k)^2_{R,R}) = -(q - q^{-1})(1 - (e_k)_{R,R}). \tag{7.11}$$

If $1 - q^{2(2n-1)}(J_k)^2_{R,R}$ is non-zero then we may again write

$$(c_k)_{R,R} = (q - q^{-1}) \frac{1 - (e_k)_{R,R}}{1 - q^{2(2n-1)}(J_k)^2_{R,R}}. \tag{7.12}$$

which shows $(c_k)_{R,R}$ is uniquely determined by q. When q^2 is not a root of unity, it is easy to check that $q^{2(2n-1)}(J_k)^2_{R,R} \neq 1$ (one just checks that the exponent of q^2 is non-zero) so this is a valid formula for $(c_k)_{R,R}$.

However, if q is a root of unity it may happen that both sides of Eq. 7.11 are 0 so we need a different formula. Since $(e_k)_{R,R} = \frac{\dim_{\mathcal{C}} R(k)}{\dim_{\mathcal{C}} \lambda}$, this happens exactly when $\dim_{\mathcal{C}} R(k) = \dim_{\mathcal{C}} \lambda$. Nevertheless, there must always exist some other isotype μ_0 appearing in $X \otimes \lambda$ with

$$\dim_{\mathcal{C}} \mu_0 \neq \dim_{\mathcal{C}} \lambda. \tag{7.13}$$

Indeed, if this wasn't the case, then every isotype appearing in $\lambda \otimes X$ would have dimension equal to $\dim_\lambda$, which would imply $\dim_{\mathcal{C}}(X \otimes \lambda) = e \dim_{\mathcal{C}} \lambda$, where $e \geq 2$ is an integer equal to the number of successors of λ, i.e. the dimension of W^λ_λ. But this implies $\dim_{\mathcal{C}}(X) = e$, a contradiction since $\dim_{\mathcal{C}}(X) = 1 + [2n-1]_q$ and q^2 is a root of unity of order $2n + K - 2 > 2n - 1$.

Therefore, let R_0 denote the path $\lambda \to \mu_0 \to \lambda$. since $\mu = R_0(k)$ satisfies Eq. 7.13, the diagonal entry $(c_k)_{R_0,R_0}$ is known from Eq. 7.12, as well as all the off-diagonal entries in the R_0-labeled column of the matrix from Eq. 7.10. Now we look at the (R, R_0) entry of the matrix equation $J_{k+1} = c_k J_k c_k$:

$$0 = \sum_{T \in \mathcal{P}_\lambda^\lambda} (c_k)_{R,T} (J_k)_{T,T} (c_k)_{T,R_0}.$$

In this sum, all of the terms except $(c_k)_{R,R}$ are known, and its coefficient is the non-zero quantity $(J_k)_{R,R}(c_k)_{R,R_0})$. Hence we can write

$$(c_k)_{R,R} = \frac{\sum_{T \in \mathcal{P}_\lambda^\lambda} (c_k)_{R,T} (J_k)_{T,T} (c_k)_{T,R_0}}{(J_k)_{R,R}(c_k)_{R,R_0}}. \tag{7.14}$$

The equations (7.10), (7.12) and (7.14) tell us how to define the rational functions $f_{R,S}$ and $f_{R,R}$.

Case 3. $|\lambda| < k+1$ and $\mu \neq \lambda$.

Suppose first there is only one path R of length 2 between μ and λ. Then λ is contained in $\mu \otimes x$ for a unique shape $x \in \mathcal{B}(2)$, and

$$(c_k)_{R,R} = \begin{cases} q & \text{if } x = [2] \\ -q^{-1} & \text{if } x = [1^2] \\ q^{-2(2n-1)} & \text{if } x = \mathbf{1}. \end{cases} \tag{7.15}$$

Hence the expression for $(c_k)_{R,R}$ depends only on the fusion rules. Now we may assume there are exactly two paths of length 2 between μ and λ, say R and S with intervening shapes ν_1, ν_2 respectively. Then W_μ^λ is an irreducible AB_2-module with D_1 having eigenvalues (q^{2e_1}, q^{2e_2}) on the paths R, S respectively (for some integers e_1, e_2 depending only on R, S, computed in Theorem 7.1.10). By the results on irreducible AB_2-modules, we have

$$(c_k)_{R,R} = \frac{q^e}{[e]_q} \tag{7.16}$$

$$(c_k)_{S,S} = \frac{q^{-e}}{[-e]_q} \tag{7.17}$$

$$(c_k)_{R,S}(c_k)_{S,R} = 1 - \frac{1}{[e]_q^2} \tag{7.18}$$

This provides rational expressions for the diagonal entries. The third equation indicates that it suffices to find an expression for one of the off-diagonal entries, which we do now.

Upon examining the Bratteli diagram we see that at least one of ν_1, ν_2 must have no greater than $k-2$ boxes, so in particular $\nu_1 \in \mathcal{B}(k-2)$. We consider the path module $W_{\nu_1}^{\lambda}$ spanned by paths of length 3 from ν_1 to λ. We consider the following paths (which don't generally span $W_{\nu_1}^{\lambda}$):

$$R = \nu_1 \to \mu \to \nu_1 \to \lambda$$

$$S = \nu_1 \to \mu \to \nu_2 \to \lambda$$

$$Q = \nu_1 \to \lambda \to \nu_1 \to \lambda$$

$$T = \nu_1 \to \lambda \to \nu_2 \to \lambda.$$

Then we must show that the coefficient $(c_k)_{R,S}$ can be written as a rational function of q (depending only on R, S). We use the relation

$$c_k c_{k-1} e_k = e_{k-1} e_k \tag{7.19}$$

evaluated at the matrix entry (S, Q). Since $S(k-2) \neq S(k)$, we have $p_S e_{k-1} = 0$ so

$$(e_{k-1} e_k)_{SQ} = 0. \tag{7.20}$$

On the other hand,

$$(c_k c_{k-1} e_k)_{SQ} = (c_k)_{SS}(c_{k-1}e_k)_{SQ} + (c_k)_{SR}(c_{k-1}e_k)_{RQ}. \tag{7.21}$$

Note that T is the unique path which only differs from Q in the kth position (or not at all) and only possibly differs from S in the $(k-1)$-st position. Hence

$$(c_{k-1}e_k)_{SQ} = (c_{k-1})_{ST}(e_k)_{TQ}. \tag{7.22}$$

In a similar way

$$(c_{k-1}e_k)_{RQ} = (c_{k-1})_{RQ}(e_k)_{QQ}. \tag{7.23}$$

Putting together the last five equations we have

$$(c_k)_{SR} = -\frac{(c_k)_{SS}(c_{k-1})_{ST}(e_k)_{TQ}}{(c_{k-1})_{RQ}(e_k)_{QQ}.}$$

Note that $(e_k)_{TQ} = (e_k)_{QQ}$ by our choice of matrix units, and we have already found expressions for all of the other terms appearing on the right hand side, via induction and Eq. (7.17). More precisely we may define

$$(f_k)_{SR} = -\frac{(f_k)_{SS}(f_{k-1})_{S'T'}}{(f_{k-1})_{R'Q'}}.$$

The other off-diagonal entry is given by the expression

$$(f_k)_{RS} = (1 - 1/[e]_v^2)/(f_k)_{SR}.$$

This completes the proof. □

8

Classification of $SO(2n)$ type categories

Finally we can use the uniqueness of the braid representations to prove the classification theorem for non-symmetric $SO(2n)$-type ribbon categories. Recall that any $SO(2n)$ or $SO(2n) - O(K)$ type category may be normalized with a cocycle construction so that the fundamental object X is symmetrically self-dual and the braids satisfy the Dubrovnik relation. In this situation the braid operator $c_{X,X}$ has eigenvalues $(q, -q^{-1}, q^{-(2n-1)})$ for some $q \in \mathbb{C}^\times$ by Prop. 7.1.7.

Theorem 8.0.1.

1. *Suppose $\mathcal{C}$ is a non-symmetric ribbon category with the fusion rules of $SO(2n)$, with $n \geq 3$, normalized in the usual way to satisfy the Dubrovnik relation, with braid eigenvalues $(q, -q^{-1}, q^{-(2n-1)})$. Then q is not a root of unity and q determines the category. More precisely, if two such categories have braid eigenvalues q_1, q_2 then the categories are monoidally equivalent if and only if $q_1 = \pm q_2^{\pm 1}$. They are ribbon equivalent if and only if $q_1 = q_2$.*

2. *Suppose $\mathcal{C}$ is a non-symmetric ribbon category of $SO(2n)-O(K)$ type with $n \geq 3$ and $K \geq 3$, normalized to be Dubrovnik and with braid eigenvalues $(q, -q^{-1}, q^{-(2n-1)})$. Then q^2 is a primitive $2n+K-2$-th root of unity, and q determines the category. More precisely, if two such categories have braid eigenvalues q_1, q_2 then the categories are monoidally equivalent if and only if $q_1 = \pm q_2^{\pm 1}$. They are ribbon equivalent if and only if $q_1 = q_2$.*

Remark 8.0.2. As for the $SO(2n+1)$ result, here "equivalence" refers to an equivalence of categories which is the identity on the level of Grothendieck rings.

Proof. We have already proved the statements regarding the restrictions on q and the other braid eigenvalues in Cor. 7.1.5, Prop. 7.1.7 and Prop. 7.1.8.

The only if statements are identical to the argument of Tuba and Wenzl ([TW05], Thm. 9.3). The argument for the direction works for both the fusion and non-fusion case. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are categories with fundamental objects X_1 and X_2 and braid eigenvalues q_1, q_2. Suppose $q_1 = \pm q_2^{\pm 1}$. By replacing the braiding on $\mathcal{C}_1$ by its negative and/or its mirror we may assume $q_1 = q_2$. By Thm. 7.3.2 we can define matrix units $(F_1)_{R,S}$ and $(F_2)_{R,S}$ in $\mathrm{End}(X_1^k)$ and $\mathrm{End}(X_2^k)$ in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. This gives us algebra isomorphisms

$$\psi_k : \mathrm{End}(X_1^k) \to \mathrm{End}(X_2^k)$$
$$\psi_k((F_1)_{RS}) := (F_2)_{RS}$$

and by Thm. 7.3.2 these ψ_k satisfy

$$\psi_k(a \otimes 1) = \psi_{k-1}(a) \otimes 1 \tag{8.1}$$

for all $a \in \mathrm{End}(X_1^{k-1})$. Furthermore, by the uniqueness of braid matrices theorem (Thm. 7.3.4) these maps satisfy

$$\psi_k(c_{k-1}^1) = c_{k-1}^2$$

where c_{k-1}^i is a simple crossing in $\mathcal{C}_i$. Hence ψ_k preserves all braids. In turn this implies ψ_k defines an isomorphism of diagonals. Indeed, it suffices to check that for all $f \in \mathrm{End}(X^k)$ and $g \in \mathrm{End}(X^l)$ we have

$$\psi_{k+l}(f \otimes g) = \psi_k(f) \otimes \psi_l(g).$$

If we let $c^i_{k,l}$ denote the braiding $X_i^k \otimes X_i^l \to X_i^l \otimes X_i^k$ then

$$\begin{aligned}\psi_{k+l}(f \otimes g) &= \psi_{k+l}((c^1_{k,l})^{-1}(g \otimes 1^k)c^1_{k,l}(f \otimes 1^l)) \\ &= (c^2_{k,l})^{-1}(\psi_l(g) \otimes 1^k)c^2_{k,l}(\psi_k(f) \otimes 1) \\ &= \psi_k(f) \otimes \psi_l(g).\end{aligned}$$

Hence ψ is an isomorphism of diagonals, and by Thm. 5.1.10, $\mathcal{C}_1$ and $\mathcal{C}_2$ differ at most by a cocycle twist. However, we assumed both are Dubrovnik categories so they do not differ by a twist, and hence are monoidally equivalent. □

We can rephrase the classification result in terms of quantum groups.

Corollary 8.0.3. *Let $\mathcal{C}$ be an $SO(2n)$ or $SO(2n) - O(K)$ type category (not necessarily Dubrovnik). Then $\mathcal{C}$ is ribbon equivalent to a cocycle twist of* **Rep** *$SO(2n)_q$ for some q.*

Proof. Indeed, the quantum group categories exhaust all possibilities for q (see Sec. 3.6.1).

□

9

Conclusion

In this thesis we classified ribbon categories with fusion rules coming from $SO(N)$. The main ideas were along the same lines as Tuba and Wenzl's classification program. In conclusion we review some outstanding problems.

9.1 Other classification problems

Our results do not apply to categories with a symmetric braiding. One can define analogues of the JM-elements and full-twist, but now the ribbon axioms do not guarantee *a priori* the centrality of the full-twist. This is a stumbling block to directly applying our techniques to symmetric categories. In the case of orthogonal and symplectic categories, Tuba and Wenzl find there is essentially a unique symmetric category with generic fusion rules, and no symmetric tensor category with a finite fusion ring. We expect the same to hold here.

There are many interesting categories excluded by our assumptions $n \geq 3$ and $K \geq 3$. These families are fundamentally different from the familiar ones studied here in that $X^{\otimes 2}$ generally does not split into 3 simples. For instance, categories connected to $SO(4)$ (i.e. $n = 2$) are generated by X whose tensor square splits into 4 simples. However, $SO(4)$ tensor product rules arise as the adjoint subring for $SL(2) \times SL(2)$ type fusion rules (this comes from the fact that the Dynkin diagram D_2 is $A_1 \times A_1$). Using

Kazhdan and Wenzl's result on $SL(N)$ type categories we can classify the $SL(2) \times SL(2)$ type fusion rules. From there one could classify $SO(4)$ categories by relating the adjoint subcategory to the full category. Due to the close relationship of the diagonal and adjoint subcategories, we expect similar reconstruction theorems to work for the adjoint. Possibly the general extension theory of Etingof, Nikshych and Ostrik [ENO10] would provide such a result for fusion categories. We expect results concerning reconstruction from the adjoint subcategory may be useful in Wenzl's program to classify $\mathfrak{so}_N$ rules (i.e. what we did in this thesis but including spin representations).

The categories with $K = 2$ provide a wealth of interesting examples. This level seems to present an exceptional situation. For instance, the classification of tensor subcategories of quantum group categories at level 2 has many differences from higher levels [Saw06]. If including spin representations, these are called *metaplectic categories* and have been the target of active research, e.g. [ACRW16], [HNW14]. I don't know if the $K = 2$ categories with integer weights (as in this thesis) have been addressed but they are a good target for classification. Finally, we expect that categories with $n = 1$ or $K = 1$ should be relatively easy to classify. In particular, since $SO(2)$ is abelian, all of its simple objects are invertible and form a group. Now one could apply known classification results for so-called *pointed categories* [EGNO15].

There is yet another family of fusion rings associated to $SO(N)$. These arise from *deequivariantizing* $SO(N) - O(K)$ fusion rules [BB01], and it would make sense to call them $SO(N) - SO(K)$ fusion rules. When K is odd we expect to leverage the $SO(N) - O(K)$ classification here to achieve such a result for $SO(N) - SO(K)$ categories, just as we did for the N odd case. The K even case is much trickier and seems to require some more insight. In particular, one encounters path modules W_μ^λ for $\mu \neq \lambda$ which can be 4-dimensional, a situation never encountered in this thesis. We achieved some partial results in the study of these categories, including resolving a question of [BB01] regarding the parametrization of simple objects, which agrees with other unpublished work by Rowell and Deaton [Row19].

A more lofty goal is to carry out the classification program for fusion rules of exceptional type. Wenzl [Wen03] has done computations of braid matrices for E_N type quantum groups which are relevant for the Jucys-Murphy approach to classification.

9.2 Applications of $SO(N)$ classification

A useful application of the $SO(N)$ classification results would be to describe the endomorphism algebras $\mathrm{End}(X^{\otimes k})$ via generators and relations. This would enable an elementary construction of the $SO(N)$ categories without the use of quantum groups in the same vein as the Turaev-Wenzl construction for the orthogonal type categories. This already exists for $SO(2n+1)$ since the endomorphism algebras agree with those of $O(2n+1)$. For $SO(2n)$, it was already known to Brauer in the classical case [Bra37] that the centralizer algebras are generated by the braid elements and an additional path idempotent in the nth tensor power of X. This is the first fundamental theorem of invariant theory for $SO(2n)$. With our results on braid matrices, one can prove that the same is true for any $SO(2n)$ type ribbon category. The next step is to write down enough relations to give a presentation for $\mathrm{End}(X^{\otimes k})$ using this new element.

This is closely related to the task of writing down the associated *planar algebra* [Jon99] by generators and relations. Actually, fewer relations are needed to describe the planar algebra since the axioms of a planar algebra imply more consequences for the algebras $\mathrm{End}(X^{\otimes k})$ than a usual algebra presentation. The results in this thesis imply that every $SO(2n)$ category is a *deequivariantization* of a certain $O(2n)$ category [BB01]. Since the $O(2n)$ category has a planar algebra presentation (namely the BMW relations), it should be possible to describe the planar algebra for $SO(2n)$ by adding an extra "2n-box" generator. An entirely analogous procedure was carried out by Morrison, Peters and Snyder for a different class of categories [MPS10]. We hope to figure this out in future work.

Such a presentation would be very useful. For instance, Edie-Michell recently used planar algebra presentations to compute braided auto-equivalences of ribbon categories of types A, B, C and G [EM20a]. This gives critical information about the category in the form of the *Brauer-Picard group*, a fundamental but hard to compute invariant, which is the central ingredient for the extension theory of [ENO10].

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