UNIVERSITY OF CALIFORNIA
RIVERSIDE

Part I: Spectral Geometry of the Harmonic Gasket

Part II: Nonlinear Poisson Equation via a Newton-embedding Procedure

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

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June 2010

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Acknowledgments

I thank my advisor, Dr. Michel L. Lapidus for all of his encouragement, guidance, and support throughout my graduate career. I thank Dr. James P. Kelliher, Dr. Frederick Wilhelm, Dr. James Stafney, Dr. Scot Childress, and John Huerta for input in PDE and fractal geometry. I thank Dr. Vicente Alvarez for help with the majority of the figures in this dissertation. I thank Dr. Bruce Chalmers, Dr. Michael Anshelevich, and Dr. Marta Asaeda for sparking my interest in analysis. I thank Dr. Tim Ridenour for his explanation of power grid theory. I thank Ricky Han for his input on C^*-algebras. I thank Dr. Congming Li for initiating my research in PDE. I thank the Fractal Research Group (FRG) and Mathematical Physics and Dynamical Systems seminar (MPDS) at UC Riverside for providing support and a venue for sharing ideas and improving research. Finally, I thank all of my friends and family.

To my mother and father for all of their support.

ABSTRACT OF THE DISSERTATION

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Doctor of Philosophy, Graduate Program in Mathematics
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Dr. Michel L. Lapidus, Chairperson

Part I of this dissertation concerns the construction of a Dirac operator and spectral triple on the harmonic gasket in order to recover aspects of Jun Kigami's measurable Riemannian geometry. In particular, we recover Kigami's geodesic distance function on the gasket as an example of a fractal analog to Connes' theorem on a compact Riemannian manifold which states that from the spectral triple on a compact Riemannian manifold one can recover the geodesic distance on the manifold. Part I builds on results of Michel L. Lapidus in collaboration with Christina Ivan and Eric Christensen who have done work using Dirac operators and spectral triples to construct geometries of some fractal sets built on curves including the standard Sierpinski gasket. Part I is also related to the work of Jun Kigami who has constructed a prototype for a measurable Riemannian geometry using the harmonic gasket. Kigami has related measurable analogs of Riemannian energy, gradient, metric, volume, and geodesic distance in formulas which are analogous to their counterparts in Riemannian geometry. In addition to recovering Kigami's geometry, one of the spectral triple constructions we have used can be generalized to a certain class of sets built on countable unions of curves in $\mathbb{R}^n$.

Part II is an adaptation of an article I have written, NONLINEAR POISSON EQUATION VIA A NEWTON-EMBEDDING PROCEDURE, which has been accepted for publication in the journal *Complex Variables and Elliptic Equations*. Chapter 5 is a version of the article, augmented to include some background on second order elliptic equations and the Newton-embedding procedure.

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Part I

Spectral Geometry of the Harmonic Gasket

Chapter 1

Introduction

What is geometry? One might answer tautologically, stating that it is whatever it is defined to be. A metric space and a Riemannian manifold may represent extreme definitions, from the most general to the most restrictive, but certainly there is more to the question. Should a metric space really be called a geometry? Should there be meaningful paths in a space for it to be called a geometry? Should there be a shortest path between any two points in the space? Should it be unique?

For example, if you stood on a mountaintop and saw another peak in the distance and wanted to know how far it was to get there you may consider many geometries. Unless you are a bird, the geometry which views you as being in 3-dimensional Euclidean space is not very useful. If you were a kangaroo you might consider how best to hop your way to the other peak. The meaningful geometry for a hiker seems to come from considering how to walk from one peak to the other and then considering the shortest of all possible walking-paths. If there is a lake in the way, there may not be a unique shortest path.

To make things more complicated, suppose that two paths could have the same length yet one of them is much more physically exerting than the other. In this case, maybe the meaningful geometry is not the one you see with your eyes but something completely different—a distorted or even broken version of the original space. To find this geometry one must now consider human physiology in conjunction with the original topography as the necessary and sufficient information encoding the geometry. Naively, it seems reasonable that this information could be packaged into a suitable space of functions on the original topography—functions which coordinatize the original space into the desired space. This is one way of viewing Jun Kigami's measurable Riemannian geometry on the Sierpinski gasket.

The Sierpinski gasket is a fractal set which is not a smooth manifold nor even a topological manifold. It is shown below Figure 1.1 as it is usually viewed, in *Euclidean metric*. The Sierpinski gasket below has a natural metric structure induced by the Euclidean metric in $\mathbb{R}^2$, given by the existence of a shortest path (non unique) between any two points. These shortest paths are piecewise Euclidean segments and hence piecewise differentiable, but in general not differentiable.

Jun Kigami uses a theory of harmonic functions on the Sierpinski gasket to construct a new metric space that is example of the distortion mentioned above, but does not do any breaking. In fact, the new metric space is homeomorphic to the Sierpinski gasket. This new space shown below Figure 1.2, called the *harmonic gasket* or the *Sierpinski gasket in harmonic coordinates*, is actually given by a single harmonic coordinate chart for the Sierpinski gasket.

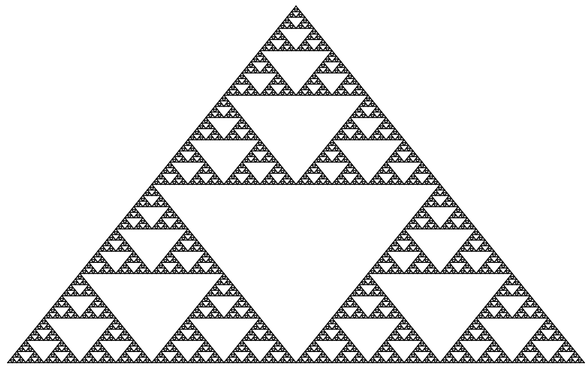

Figure 1.1: Sierpinski gasket

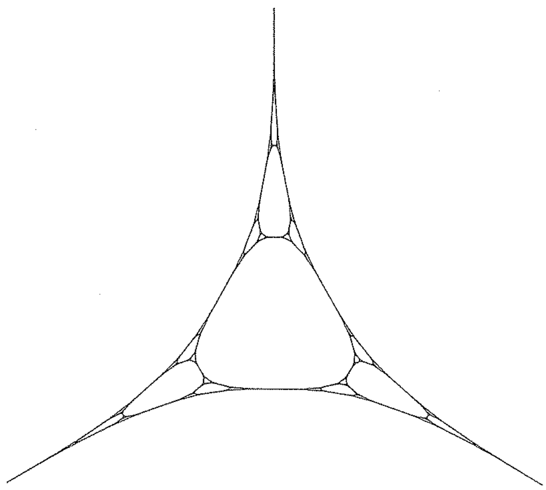

Figure 1.2: Harmonic gasket

The harmonic gasket has a C^1 shortest path between any two points. It is interesting to note that the harmonic coordinate chart smoothes out the Sierpinski gasket. Kigami, building on work by Kusuoka, has found several formulas in the setting of the harmonic gasket which are measurable analogous to their counterparts in Riemannian geometry. In particular, he has found formulas for energy and geodesic distance involving measurable analogs to Riemannian metric, Riemannian gradient, and Riemannian volume. For this reason, this geometry is appropriately called measurable Riemannian geometry.

Returning to the theme of defining a meaningful geometry, one might consider that at the outset there is no space waiting to be altered. In contrast, could one start with a space of functions and determine a geometry? If the function space is a commutative C^*-algebra, then one can tease from it a topological space. This is a consequence of Gelfand's theorem. If that topological space is metrizable then more information is needed to determine a metric. Knowledge of a certain Hilbert space of vector fields on the space and a particular differential operator is enough to determine a metric in many instances. This way of constructing a geometry is part of the broader theory of noncommutative geometry.

Alain Connes proved that for a compact spin Riemannian manifold, M, a triple of objects called a *spectral triple* encodes the geometry of M. The spectral triple consists of the C^*-algebra of complex-continuous functions on M, the Hilbert space of L^2-spinor fields, and a differential operator called the *Dirac operator*. The Dirac operator is constructed from the spin connection associated to M and can be thought of as the square root of the spin-Laplacian (mod scalar curvature).

Connes', through a very simple formula, uses the information from the spectral triple to recover the geodesic distance on M, and hence the geometry of M. In the absence of spin or even orientability, this result still holds, though the Dirac operator may not be uniquely defined. The reason for the name noncommutative geometry is that the arguments involved in this result do not rely on the commutivity of the C^*-algebra which opens the door to the possibility of defining geometries on noncommutative C^*-algebras.

The applications of noncommutative in Part I, however, stay within the context of the commutative C^*-algebras of complex-continuous functions on fractal sets.

The work I have done in Chapter 4 of Part I is to recover Kigami's measurable Riemannian geometry using spectral triples. My work builds on the work of my advisor, Michel L. Lapidus and his broader program to view fractals as generalized manifolds, and in particular, as suitable noncommutative spaces. In particular, I have built on his collaborative work with Christina Ivan and Eric Christensen which constructs geometries for several fractal sets built on curves using spectral triples [3]. In their work, they construct spectral triples for finite unions of curves in a compact Hausdorff space, parameterized graphs, infinite trees, and for the Sierpinski gasket in Euclidean metric. For the case of the Sierpinski gasket, they recover the geodesic distance, renormalized Hausdorff measure, and Hausdorff dimension from the spectral triple. The basis for the construction of these spectral triples is the spectral triple for a circle. Finite and countable direct sums of the circle triples are used to construct the desired spectral triples.

I have constructed several spectral triples for the harmonic gasket, all of which recover the geodesic distance on the harmonic gasket. As in [3], I use direct sums of circle triples to construct the desired spectral triples. One of the constructions I have used generalizes to a class of sets built on countable unions of curves in $\mathbb{R}^n$.

In the concluding remarks of Chapter 4, I discuss work in progress which includes an informal sketch of a vastly different construction of a Dirac operator and spectral triple from the ones built from direct sums. This *global* Dirac operator is defined directly from Kigami's measurable Riemannian metric and gradient, giving it a stronger resemblance to Connes' Dirac operator on a compact Riemannian manifold. The Hilbert space of the triple is constructed from Kigami's L^2-vector fields on the gasket, again giving a stronger fractal analog to Connes' theorem. In addition, the global construction may prove a better starting point for showing that the Dirac operator squares to the appropriate Laplacian in this setting, the Kusuoka Laplacian. I also discuss two open problems. These problems, which are inherently linked, are the computation of the spectral dimension and volume measure induced by the spectral triples for the harmonic gasket.

Remark 1. *The primary tool of noncommutative geometry used in Part I is the spectral triple and since the work involves only commutative algebras, I will refer to the geometry most often as spectral geometry.*

1.1 Structure of Part I

The body of Part I consists of three chapters. Chapter 2 is an exposition of measurable Riemannian geometry. Chapter 3 provides a description of methods in noncommutative geometry and spectral geometry, emphasizing their applications to fractal sets. Chapter 2 and Chapter 3 are in the service of giving context to the results in Chapter 4. One of the main results in Chapter 4 is the recovery of the measurable Riemannian geometry of the Sierpinski gasket using spectral triples.

Chapter 2

Measurable Riemannian Geometry of the Sierpinski Gasket

2.1 Overview

The Sierpinski gasket, an example of a post critically finite fractal, has as part of its definition, a metrizable compact topological structure. It is the particular example, for $N = 3$, of N-Sierpinski space which, via a collection of contraction mappings, can be embedded in $\mathbb{R}^{N-1}$. Sierpinski space is most commonly studied as the Sierpinski gasket in the plane. It is by no means smooth and is not even a continuous manifold (i.e. not locally Euclidean). So it is not possible to speak of Riemannian geometry on the gasket in the classical sense.

It is possible though, to abstract several notions from the Riemannian geometry: Riemannian energy, volume, metric, and gradient. Measurable analogs to these notions are constructed analytically from graph approximations of the gasket. Jun Kigami gives a prototype for a 'measurable' Riemannian geometry in [12]. Kigami utilizes earlier work by S. Kusuoka in [18] in which a quadratic 'energy' form $\mathcal{E}$, a 'volume' measure ν, a 'metric' realized by a non-negative symmetric matrix Z, and a 'gradient' $\widetilde{\nabla}$, operating on the domain of $\mathcal{E}$ (which consists of the continuous functions on the gasket that admit finite energy) are shown to satisfy,

$$\mathcal{E}(u) = \int_K (\widetilde{\nabla} u, Z\widetilde{\nabla} u) dv,$$

giving legitimacy to calling $(\nu, Z, \mathcal{E})$ a 'measurable' Riemannian structure on the Sierpinski gasket K [18]. Attaching a geodesic distance function to the above structure would further the analogy. As a way of achieving this, Kigami uses harmonic functions on K as a coordinate system for K. This results in a homeomorphism Φ, between K and its image under Φ, K_H, the harmonic gasket (also called the Sierpinski gasket in harmonic coordinates). With this definition, K_H is a subset of the plane $x + y + z = 0$ in $\mathbb{R}^3$. With respect to the $\mathbb{R}^2$-induced arclength metric on K_H, Kigami proves the existence of a C^1 shortest path γ, in this context called a *geodesic*, between any two points in K_H [12].

The length of this path yields a geodesic distance function on K_H which can be pulled back to K via Φ to yield for $x, y \in K$, with slight abuse of notation,

$$d_*(x, y) = \int_0^1 \left(\frac{d\gamma}{dt}, Z \frac{d\gamma}{dt} \right) dt$$

which is called the harmonic shortest path metric on K. This is the notion of measurable Riemannian geometry on K: *d_* is the geodesic distance associated to the measurable*

Riemannian structure $(\nu, Z, \widetilde{\nabla})$. One thing to pay attention to in the sections to follow is that the harmonic functions are constructed via graph approximations of K, so are independent of how K is initially embedded in $\mathbb{R}^2$. In this case, there is an intrinsic quality to the measurable Riemannian geometry. For M a smooth manifold, the following list compares Riemannian geometry on M with measurable Riemannian geometry on K.

Riemannian Geometry	Measurable Riemannian Geometry
metric g	metric Z
non-neg. sym. matrix A	non-neg. sym. matrix Z
varying smoothly on M	defined in the measurable sense on K
gradient ∇	gradient $\widetilde{\nabla}$
$(\nabla f, x)_g = D_x f$ all x	given by harmonic approximation
energy E	energy $\mathcal{E}$
$E[u,v] = \int_M (\nabla u, A\nabla v) d\nu_g$	$\mathcal{E}[u,v] = \int_K (\widetilde{\nabla} u, Z\widetilde{\nabla} v) dv$
u, v smooth on M	u, v continuous on K with finite energy
volume element $d\nu_g$	volume element $d\nu$
$d\nu_g = \sqrt{det(A)} dx$	ν is the Kusuoka measure on K
Laplacian Δ	Laplacian Δ_ν
$E[u,v] = -\int_M v \Delta u d v_g$	$\mathcal{E}[u,v] = -\int_K v \Delta_\nu u d\nu$
each smooth v with compact support	each $v \in dom\mathcal{E}$ vanishing on ∂K
geodesic distance d	geodesic distance d_*
$d(x,y) = \int_0^1 (\frac{d\gamma}{dt}, A\frac{d\gamma}{dt})^{\frac{1}{2}} dt$	$d_*(x,y) = \int_0^1 (\frac{d\gamma}{dt}, Z\frac{d\gamma}{dt})^{\frac{1}{2}} dt$
γ is a geodesic in M	γ is a geodesic in K

2.2 The Sierpinski Gasket

The most common and intuitive presentation of the Sierpinski gasket is as a solid equilateral triangle which has a smaller equilateral triangle removed from its center, and again an even smaller triangle removed from each of the three remaining triangles and so on as seen in Figure 2.2 below.

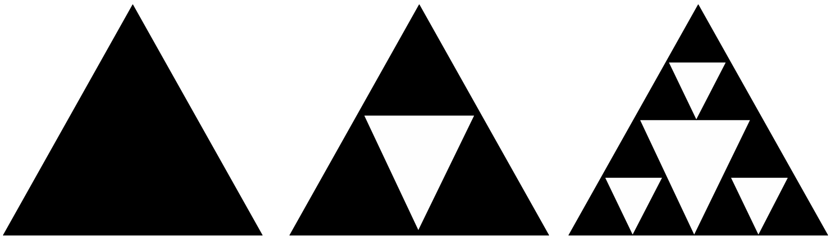

Figure 2.1: Construction of the Sierpinski gasket by the removal of triangles

This is done a countable number of times, and the result is called the Sierpinski gasket. The following Figure 2.2 is a high iteration approximation of the Sierpinski gasket.

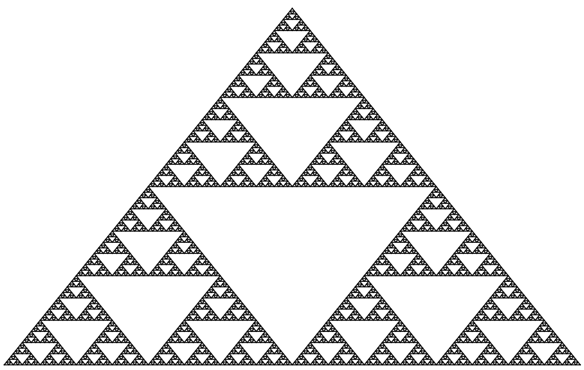

Figure 2.2: Sierpinski gasket, K

Considering the gasket in stages, or as approximations, is intuitive but also fundamental to defining additional structure on the gasket. *Graph approximations* will be the starting point for defining measure, operators, harmonic functions, etc. on the gasket.

The Sierpinski gasket is well described analytically as the unique fixed point of a certain contraction mapping on a metric space. The contraction mapping to be defined is composed of three contraction mappings of $\mathbb{R}^2$ that will allow for analysis, not just on graph approximations, but on arbitrarily small *portions* of the gasket, called cells.

Although continuity inherited from the Euclidean topology of the plane naturally connects with the analysis of the gasket, it is not critical to the definitions of measure, operators, harmonic functions, etc. (In fact, it turns out that harmonic functions, defined exclusively in terms of graphs, will necessarily be continuous functions in the Euclidean induced topology of the gasket.) To generate the desired structure on the gasket, Euclidean neighborhoods are replaced with graph neighborhoods. To begin, we define the following contractions on the plane:

$$F_i x = \frac{1}{2}(x - p_i) + p_i$$

$$i = 1, 2, 3 \quad ; \quad p_i \text{ is a vertex of a regular 3-simplex, } \mathcal{P}$$

Let Ξ be the set of nonempty compact subsets of $\mathbb{R}^2$. The Hausdorff metric,

$$d_H(C, D) = \max\left\{\sup_{x\in C}\inf_{y\in D} d(x,y), \sup_{y\in D}\inf_{x\in C} d(x,y)\right\}$$

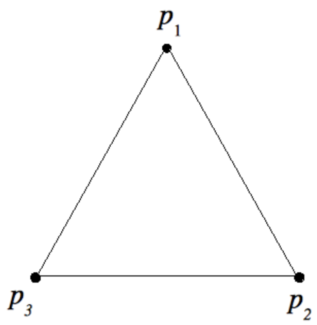

Figure 2.3: 3-simplex

for $C, D \in \Xi$ and Euclidean distance, d , makes (Ξ, d_H) a metric space. The assignment,

$$C \to \bigcup_{i=1}^{3} F_i(C)$$

from (Ξ, d_H) to itself, is a contraction mapping. The Sierpinski gasket, K, is then defined as the unique fixed point of the this mapping. In other words, it is the unique element $K \in \Xi$ such that

$$K = \bigcup_{i=1}^{3} F_i(K)$$

Remark 1. *N-Sierpinski space is defined analogously, using N contractions on R^{N-1}.*

Remark 2. *K is a subset of the closed equilateral triangle given by the simplex in Figure 2.2.*

Remark 3. *The fixed point of a contraction mapping F on a metric space can be found as the limit of the sequence $\{F^k(y)\}$ for any y in the metric space. It is therefore interesting to note that, starting with any nonempty compact subset of the plane, the iteration* *of our contraction mapping on that set will yield K as the limit. In other words, one could start with say, the iconical alien face of any size, pasted anywhere in the plane, and the iterations of the contraction map will successively transform the alien toward (in the Hausdorff metric) the gasket. For this reason, the fixed point of such a mapping is often called an attractor.*

For any integer $m \geq 1$, let w be the multi-index given by

$$w = (w_1, ..., w_m), \quad w_j \in \{1, 2, 3\}$$

and F_w be given by

$$F_w = F_{w_1} \circ \cdots \circ F_{w_m}.$$

Then K satisfies

$$K = \bigcup_{|w|=m} F_w(K).$$

This is called the decomposition of K into m-cells, with $F_w(K)$ being the m-cell given by w. $F_w(K)$ *is a subset of* K. A more intuitive description will be given, concluding the following discussion of graph approximations.

The graph approximations of K and their associated vertices are central to all further analysis of the gasket. The m^{th}*-level graph approximation* Γ_m, is given by,

$$\Gamma_m = \bigcup_{|w|=m} F_w(\mathcal{P}).$$

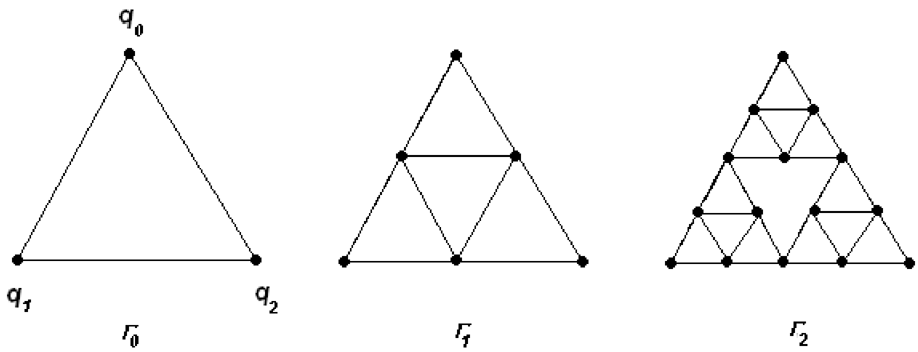

Figure 2.4: Graph approximations of the Sierpinski gasket

The graphs for $m = 1, 2, 3, 4$ are shown in Figure 2.2.

The graph approximations are precisely graphs, thus consisting only of edges and vertices. In contrast to the decomposition into m-cells, a graph approximation is *not regarded* as a subset of K. The vertices are points of K and the edges viewed as lines in the plane are also subsets of K, but the edges will not be used this way: The edges are used only for defining an equivalence relation on vertices. Two vertices are equivalent iff they are connected by an edge. Two vertices in the same equivalence class will be said to be *neighbors*. With the exception of the vertices, p_1, p_2, p_3, of the initial simplex, which each have two neighbors, all vertices will have four neighbors, making each graph approximation nearly a four-regular graph.

The transition from analysis on graphs to analysis on K is made easy by one special fact: The collection of vertices unioned over all graph approximations is a dense subset of K. This follows immediately from the definition of our contraction maps and the density of the dyadic numbers in the unit interval. The functions on K that we will consider will be continuous (in the Euclidean subspace topology) and therefore they will be completely determined by their values on the collection of vertices. It is common in analysis on a fractal to have the convienient addressing system for points, inherited from the contraction maps. In this way, points on the gasket will be given as *words* whose letters are in the set $1, 2, 3$. To formalize the above discussion, we have the following terminology [12]:

Definitions

- 1. $S \equiv${1,2,3}
- 2. $\Sigma \equiv S^{\mathbb{N}}$
- 3. $W_0 \equiv \{\emptyset\}$, $W_m \equiv S^m$ for $m > 0$, $W_* \equiv \bigcup_{m\geq 0} W_m$
- 4. $w \equiv w_1, ..., w_m \in W_*$
- 5. $|w| \equiv m$ for $w \in W_*$
- 6. $\Sigma_w \equiv$ sequences in Σ with initial segment $w_1, ..., w_m$
- 7. $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$
- 8. $V_0 \equiv \{p_1, p_2, p_3\} \equiv \partial K$, $V_m \equiv \bigcup_{w\in W_m} F_w(V_0)$, $V_* \equiv \bigcup_{m\geq 0} V_m$
- 9. $\partial_w \equiv$ the triangle with side lengths $\frac{1}{2^m}$ bordering the the m-cell K_w for $|w| = m > 0$ and $\partial_w = \partial_0 = \mathcal{P}$ for $|w| = 0$.
- 10. $K_w \equiv K$ for $|w| = 0$, $K_w \equiv F_w(K)$ for $|w| > 0$
- 11. $\Gamma_w \equiv \mathcal{P}$ for $|w| = 0$, $\Gamma_w \equiv F_w(\mathcal{P})$ for $|w| > 0$

- 12. For $x, y \in V_m$, $x \cong y$ iff x and y share an edge in Γ_m.

Note that Σ is the collection of sequences $\{a_i\}_{i=1}^{\infty}$ with $a_i \in \{1, 2, 3\}$. We refer to W_m as the *the set of words of length m* and V_m as *the set of level m vertices*. Note that V_* is the collection of all of the vertices from all of the graph approximations. As indicated above, V_0 is considered the boundary of K.

Recalling the map F_i, it is evident that it maps any point in the plane to the midpoint of segment joining that point to p_i. Then it is clear that F_1 maps Γ_0 to the subgraph Γ_1, the left-lower equalteral triangle of Γ_0. Similarly, F_{21} maps Γ_0 to the upper equilateral triangle of $F_1(\Gamma_0)$. In general, F_w, following the multi index $w = w_1, ..., w_m$ from right to left, maps Γ_0 to the according nested equilateral triangle. It should be noted that the vertices V_0 belong to Γ_0, the vertices V_m belong to the union of the subgraphs $F_w(\Gamma_0$ for $w = m$, and $V_m \subset V_{m+1}$.

As mentioned before, K is a subset of the solid equilateral bounded by Γ_0. In order to make K_w more intuitive, notice that K_w is indeed the solid equilateral triangle bounded by Γ_w intersected with the gasket K. In other words, it is that portion of the gasket that lives inside the smaller equilateral triangle given by w. The last item in the definition is the neighbor relation mentioned earlier.

The words described in the definition are useful also as an addressing scheme for points in the gasket. Just as words track subgraphs and subsets of K, they can easily be used to track points. Each vertex, $v \in V_* - V_0$, is given by a *finite* word $w \in W_*$ as follows: If $v \in \Gamma_m - \Gamma_{m-1}$ for $m > 1$, then there is a unique vertex $p_i \in V_0$ and vertex

$q \in \Gamma_{m-1}$, such that v is the midpoint of the segment joining p_i and q. In other words, $v = F_i(q)$. Then $i = w_1$.

Continuing in this manner, we determine $w_2, ...w_{m-1}$ uniquely, giving $v = F_{w_1,...w_{m-1}}(p)$ for some $p \in \Gamma_1$. At this point the uniqueness breaks down, as it is clear that $p = F_j(p_i) = F_k(p_l)$ for $p_i, p_l \in V_0$ with $j \neq k$ and thus $i \neq l$. Therefore v can be written correctly with the address $w_1, ..., w_{m-1}, j; i$ or with the address $w_1, ..., w_{m-1}, k; l$. The semicolon at the end of each address is used to emphasize the map indices from the index of the final argument. The final argument's index could be omitted as it is uniquely determined by the rightmost map index. In this case, a vertex $v \in \Gamma_m - \Gamma_{m-1}$ has an address w, with $|w| = m$.

Points in $K - V_*$ are given by *infinite* words in Σ that do not terminate in an infinite string of 1's, 2's, or 3's (any such repeating word is just an element of V_0). These points have a unique infinite sequence addressing them. Consider Σ to have the standard metric topology on sequences and K to have the Euclidean topology inherited from the plane. To summarize the above, there exists the following continuous surjection [12]:

$$\pi : \Sigma \to K$$

such that

$$\pi(w) = \bigcap_{m \geq 0} K_{w_1,...,w_m}$$

and

$$|\pi^{-1}(x)| = \begin{cases} 2, & x \in V_* - V_0 \\ 1, & otherwise \end{cases}$$

Remark 4. *π glues together the two different addresses for each $v \in V_* - V_0$.*

Remark 5. *π respects the contractions, F_i, on K. Formally, if*

$$\sigma_i : \Sigma \to \Sigma$$

is given by

$$\sigma_i(w_1 w_2 w_3 ...) = i w_1 w_2 w_3 ...,$$

then

$$\pi \circ \sigma_i = F_i \circ \pi.$$

2.3 Energy on K

Energy forms have roots in the field of partial differential equations. On one hand, they are often used as a platform for defining weak solutions to, in particular, elliptic boundary value problems. Taking the simple example of the Poisson equation,

$$-\Delta u = f(x)$$

on a domain Ω in $\mathbb{R}^n$ with Dirichlet boundary condition imposed on u, multiply both sides of the equation by a smooth function, v, with compact support. Integrating both sides, using Green's theorem on the left side yields,

$$E[u,v] \equiv \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx.$$

The left side of the equation, $E[u,v]$, is often called the *energy form* associated to the equation, or more precisely, in this case, associated to the Laplacian. Informally, the desired solution u is the function for which the above integral equation holds for *all* v. This integral equation has the property that a solution u only have first derivatives, and moreover, only weak first derivatives. Such a solution is a weak solution, which is a true generalization of a classical solution in that if the original equation has a classical $C^2(\Omega)$ solution, the weak solution coincides with the classical solution. For a formal discussion of energy and weak solutions, see [6].

Something interesting and curiously absent from many expositions in pure mathematics, is why $E[u,v]$ is called an *energy form*. To give a physical interpretation, first consider $E[u] \equiv E[u,u]$. Suppose the original Poisson equation is modeling an external force, f, applied to a taut membrane or drum where u is the displacement of the membrane (away from flat).

The Poisson equation is then a force equation and states that the deformation of the membrane is precisely given by the force applied. Now, to achieve the integral equation, we have spatially integrated the force equation. Naively speaking, the integral equation should be an energy equation. This is the case, and $E[u]$ is the potential energy due to the deformation of the membrane. The right side of the integral equation is the potential energy due to the external force. In other applications, it is useful to move the right side to the left side, in defining the energy form—in this case the energy becomes the total potential energy of the system.

For a domain Ω in $\mathbb{R}^n$ and $u, v \in C^1(\mathbb{R})$, the energy form and energy associated to the Laplacian is exactly as in the above example. Similarly, for a domain Ω in a Riemannian manifold (M,g) and $u, v \in C^1(\Omega)$,

$$E[u,v] = \int_\Omega (\nabla u, \nabla v)_g dv_g = \int_\Omega (\nabla u, A\nabla v) dv_g$$

where $(\cdot,\cdot)$ is Euclidean inner product and A is the matrix representation of g. As before, the energy of a function u is given by $E[u] \equiv E[u,u]$.

The energy form on K is constructed from *graph energies*, independent of a notion of Laplacian or differential operators of any kind. The *graph energy form on* Γ_m, $\mathcal{E}[u,v]$, is given by

$$\mathcal{E}[u,v] = \left(\frac{5}{3}\right)^m \sum_{p \cong q : p,q \in V_m} (u(p) - u(q))(v(p) - v(q))$$

with

$$dom\mathcal{E} = \{u \in C(K) \quad | \quad \lim_{m\to\infty} \mathcal{E}_m(u,u) < \infty\}.$$

The energy form, $\mathcal{E}[u,v]$, on K is then given by

$$\mathcal{E}[u,v] = \lim_{m\to\infty} \mathcal{E}_m(u,v)$$

with the *energy*, $\mathcal{E}(u)$, on K given by

$$\mathcal{E}(u) = \mathcal{E}(u,u).$$

Since $\mathcal{E}_m$ is a non-decreasing sequence, the limit exists and is finite by design, for all $u, v \in dom\mathcal{E}$. The form of the graph energies has several motivations. Kusuoka in [17], and Goldstein in [10], have independently constructed the Brownian motion on the Sierpinski gasket as a scaling limit of random walks. To view the energy as an analytic counterpart to Brownian motion on the gasket, see [13], [18], [14]. Other physical interpretations of the energy are seen electrical resistance networks as well as in systems of springs attached to point masses assigned to graph vertices. The former is described in Chapter 1, Section 5 of [30], by considering the energy,

$$E(u) = \sum_{p \cong q} c_{pq}(u(p) - u(q))^2$$

where c_{pq} are positive functions interpreted as conductances with their reciprocals, $r_{pq} = \frac{1}{c_{pq}}$, considered as resistances. The values of u are interpreted as voltages at the vertices. Here, a current of amperage $\frac{(u(p)-u(q))}{r_{pq}} = c_{pq}(u(p) - u(q))$ flow through the resistors, producing the energy of $c_{pq}(u(p) - u(q))^2$ from each resistor. The total energy is achieved by summing over the neighbor relation.

In this exposition, in particular, one can see a motivation for the $(\frac{5}{3})^m$ corresponding to the conductances c_{pq}, when p and q are neighbors in the level-m graph approximation. The key here is that resistors in series add their resistances, while resistors in parallel add their conductances, and imposing that the resistances are equal to 1, the energy is recovered, with the appropriate coefficients. For the similar motivation in spring systems, see Chapter 3, Section 3.1 of [27], where spring constants, replacing resistances, follow the same rules for springs in series and parallel, as do resistances.

In addition, in the following section, the $(\frac{5}{3})^m$ scaling in the graph energy will be motivated in terms of harmonic theory. The theory of harmonic functions, defined in terms of graph energies, will provide the platform for constructing the gasket in the 'harmonic metric', as well as for the quantities and formulas relating measurable Riemannian geometry to Riemannian geometry.

2.4 Harmonic Theory on the Gasket

The theory of harmonic functions on K is a generalization of classical harmonic theory in which there are the standard equivalences:

- u is harmonic
- u is an energy minimizer, for given boundary values
- u has the mean value property
- $\Delta u = 0$.

A suitable springboard for harmonic theory on K is that of energy minimization. It will be the case that a harmonic function defined in this way will enjoy a mean value property as well the Laplacian condition. Recall that the definition of the graph energy, $\mathcal{E}_m$, only involves a function's values at the vertices, V_m. The general idea is to consider the m^{th} graph energy of a function defined on V_m, and then extend that function to V_{m+1} without increasing the energy. To be precise, let

$$E_m(u) = \sum_{p \cong q : p,q \in V_m} (u(p) - u(q))^2.$$

Suppose u is defined on V_0 and we desire an extension $\hat{u}$ to V_1 which minimizes $E_1(u)$ over all extensions of u. A trivial minimization of the quadratic form yields a unique such $\hat{u}$ with

$$E_0(u) = \left(\frac{5}{3}\right) E_1(\hat{u}).$$

Similarly, extending u from V_m to V_{m+1} gives a unique minimizer $\hat{u}$ such that

$$E_m(u) = \left(\frac{5}{3}\right) E_{m+1}(\hat{u}),$$

and therefore extending u from V_0 to V_m gives a unique minimizer $\hat{u}$ such that

$$E_0(u) = \left(\frac{5}{3}\right)^m (\hat{u}).$$

We call $\hat{u}$ the *harmonic* extension of u. Given values of a function u on V_0, u can be uniquely extended harmonically to V_m for any m and therefore can be extended to V_*. $\hat{u}$, defined in this way, is (uniformly) continuous on V_* which is dense in K with respect to the Euclidean inherited topology. In this case, $\hat{u}$ extends uniquely to a function u on K, called a *harmonic function* on K.

Remark 6. *Note that the harmonic function u, is uniquely determined by its boundary value, $u|_{V_0}$.*

Remark 7. *Since $\mathcal{E}_m(u) = \left(\frac{5}{3}\right)^m E_m(u)$ and $\mathcal{E}(u) = \mathcal{E}_{m+1}(u) = \mathcal{E}_m(u) = \mathcal{E}_0(u)$, for u harmonic, u is indeed an energy minimizer.*

The values of $\hat{u}$ were omitted in the minimization problem above but are worth considering, at least in the extension from V_0 to V_1. Two perspectives on the minimization problem are of interest. First, Figure 2.4 below represents Γ_1 with the vertices labeled with their corresponding arbitrary function values.

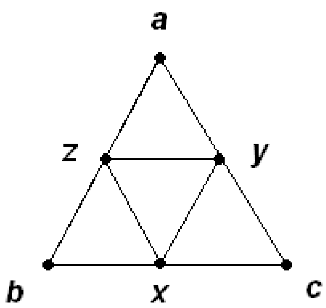

Figure 2.5: 1st graph approx. with function values

Let u be a function defined on V_0 with values a, b, c at the vertices p_1, 2,and p_3, respectively. The values at the vertices in $V_1 - V_0$ are labeled x, y, and z and are the unknown values of $\hat{u}$. Minimizing the quadratic form yields three linear equations:

$$x = \frac{1}{4}(b + c + y + z)$$

$$y = \frac{1}{4}(a + c + x + z)$$

$$z = \frac{1}{4}(a + b + x + y).$$

Thus $\hat{u}$, the harmonic extension, yields values at the vertices in $V_1 - V_0$ which are the average values of their four neighbors in Γ_1. Similarly, this holds from V_m to V_{m+1}.

Remark 8. *This average value property for harmonic functions on Γ_m is the analog of the standard mean value property of classical harmonic functions.*

Taking a second look at the minimization from V_0 V_1, with the particular values $1, 0$, and 0 at p_1, p_2, and p_3 respectively, yields the values $\frac{2}{5}, \frac{2}{5}$, and $\frac{1}{5}$ for $\hat{u}$. Since the minimizing equations are linear, $1, 0$, and 0 can be replaced with arbitrary a, b, and c to yield values

$$x = \frac{2}{5}a + \frac{2}{5}b + \frac{1}{5}c$$

$$y = \frac{2}{5}a + \frac{2}{5}c + \frac{1}{5}b$$

$$z = \frac{2}{5}b + \frac{2}{5}c + \frac{1}{5}a.$$

The same holds for harmonic functions on Γ_m. In other words, for an unknown function value at a vertex in $V_{m+1} - V_m$, it is only necessary to look at the graph cell, Γ_w ($|w| = m$), it belongs to in Γ_m, weighting the values at the vertices of Γ_w just as they were weighted above in Γ_0. This relationship is referred to as the '$\frac{2}{5} - \frac{1}{5}$' rule [30].

There are two quantities, the standard measure on K and the standard Laplacian on K, relevant to this line of discussion, but not integral to the goals of chapter 2, nor the chapters to follow. We will mention them briefly for completeness. First, the notion of a Laplacian on K is linked to the chosen measure on K and the energy form $\mathcal{E}[u, v]$, analogous to the earlier discussion of energy on a Riemannian manifold. Defining $\Delta_\mu(u)$ as a function, f, is exactly the Poisson equation. 'Integrating by parts' with measure μ, against a compactly supported function, v, yields a 'weak formulation' of the Poisson equation,

$$\int_K f v d\mu = \int_K v \Delta_\mu(u) d\mu = -\mathcal{E}(u,v),$$

and a suitable generalized definition (by the second equality) above of the Laplacian with respect to the measure μ when we impose that the above hold for all v (with zero boundary value) in the domain of the energy .

Remark 9. *The second equality above seems to skip a step—indeed, an integral (via integration by parts) with the integrand as the product of the 'gradients' of u and v would then make sense of the equality with the energy. Later we will see that due to Kusuoka, Kigami, and Teplyaev, there are notions of gradient on K fulfilling this condition.*

Note that each measure determines a different Laplacian. However, on page 41 in Chapter 2 of [30], one can find a proof of the fact that regardless of choice of measure, μ, a function u on K is harmonic (in the energy sense) iff $\Delta_\mu u = 0$, so long as $u \in dom\Delta_\mu$. In the proof of the converse, it is apparent that $\Delta_\mu(u) = 0$ implies

$$\sum_{p \cong q : p,q \in V_m} (u(p) - u(q)) = 0$$

for all m which leans toward a pointwise definition of the Laplacian which falls in line with classical harmonic theory. Define the m^{th}-*level graph laplacian*, Δ_m, by

$$\Delta_m = \sum_{p \cong q : p,q \in V_m} (u(p) - u(q)).$$

Δ_μ is then defined in the limit, with a μ-dependent renormalizing constant. Later on, the focus will be the Kusuoka measure but the most elementary measure is the standard (self-similar) measure.

The standard measure is the most common measure for self-similar fractals like the gasket. The standard measure is constructed by measuring the original simplex, or equilateral triangle, at 1. The self-similarity of the fractal K makes for a natural self-similar measure. The boundaries of the higher m-cells, ∂_w, which are smaller equilateral triangles, will all have measure $\frac{1}{2^m}$. Imposing additivity, the measure is extended to finite collections of ∂_w's. The finite collections of ∂_w's form an algebra of sets from which the Caratheodory extension theorem extends the measure uniquely to all of K. In the the case of the standard measure, the Laplacian Δ is then given by

$$\Delta u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x).$$

Remark 10. *With the construction of the Laplacian Δ_μ on K, in light of the result mentioned above, the analog of the equivalences of classical harmonic theory is complete.*

Let the space of harmonic functions be denoted by $\mathcal{H}$. Given the necessary condition of the '$\frac{2}{5} - \frac{1}{5}$' rule for a function $u \in \mathcal{H}$, it is clear that u is completely determined by its values on $V_0 = \{p_1, p_2, p_3\}$. In this case, $\mathcal{H}$ forms a 3-dimensional linear space which we can identify with $\mathbb{R}^3$ by associating $u \in \mathcal{H}$ to the triple $(u(p_1), u(p_2), u(p_3))$ in $\mathbb{R}^3$. Moreover, modding out $\mathcal{H}$ by the constant functions on K,

$$\mathcal{H}/\{\text{constant functions}\} \cong \mathbb{R}^3/\{span(1,1,1)\}.$$

Note that the right side is the 2-dimensional subspace of $\mathbb{R}^3$,

$$M_0 \equiv \{(x,,y,z) \mid x+y+z=0\}.$$

Remark 11. *1. $\mathcal{E}(\cdot,\cdot)$ gives $\mathcal{H}/\{$constant functions$\}$ an inner product structure.*

2. $(\cdot,\cdot)$, the Euclidean product on $\mathbb{R}^3$ (restricted to M_0), gives M_0 an inner product structure.

3. Though $\mathcal{H}/\{$constant functions$\}$ and M_0 are isomorphic as linear spaces, they are not, with the above products, identified as inner product spaces.

2.5 The Harmonic Gasket, K_H

In this section, we will define the Sierpinski gasket with harmonic metric, K_H. K_H is also referred to as the harmonic gasket. The two names refer to the same object but hint at distinct perspectives. The former implies that K is in some category of manifold, sans a metric. Seen this way, K is comparable to a sphere considered as a smooth manifold. In the absence of a metric, a sphere is no different than a larger sphere, or even an ellipsoid because there are diffeomorphisms between them.

The Sierpinski gasket is not a classical manifold but we can look at it as a space to be geometrized. Of course K carries notions of geometry by the default of how it is constructed in the Euclidean plane. But the point is that it can be much more general than that. The analysis carried out so far on K has been independent of 'how K sits' in the plane. Indeed, the analysis has been based on graphs and the neighbor relation so that the bending, stretching, and twisting of K away from how it sits in the flat plane, while preserving the neighbor relations of vertices, does not affect the analysis. So even though the standard visualization of K is in the plane, this perspective begs to see K as a more abstract object, *awaiting* a metric.

In this section K is assigned or geometrized by the *harmonic metric* to become the 'geometric' space K_H, a particular geometric realization of K. The latter perspective hints at K and K_H as distinct spaces equipped with their own geometries: K with the geometry implied by its specific manner of inclusion in the Euclidean plane, and K_H with the geometry implied by its configuration in the plane M_0 in $\mathbb{R}^3$. Both perspectives are explored in comparing measurable Riemannian geometry to the spectral geometries to come. In particular, they are useful in understanding the meaning of such a comparison.

K_H will be defined using the harmonic functions, $\mathcal{H}$. Recall that a harmonic function, h, is determined uniquely by its values on V_0. Identifying $\mathcal{H}$ with $\mathbb{R}^3$, take

$$h_1 = (1, 0, 0) \quad h_2 = (0, 1, 0) \quad h_3 = (0, 0, 1)$$

as a basis for $\mathcal{H}$. In terms of the evaluation of harmonic function, this is equivalent with $h_i(p_j) = \delta_{ij}$ for $j = 1, 2, 3$ and $p_j \in V_0$.

The final step in the construction of the harmonic gasket is to use h_1, h_2, and h_3 as a single 'coordinate chart' for K in the plane M_0. Kigami defines the following map,

$$\Phi : K \to M_0$$

$$\text{by} \qquad \Phi(x) = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right),$$

which is a homeomorphism onto its image [12]. Then $K \cong \Phi(K) \equiv K_H$ defines the *harmonic gasket* or *Sierpinski gasket in harmonic metric*. Though K_H is not a self-similar fractal, it is self-affine and can be given as fixed point of certain contraction mapping. Φ preserves compactness, so that K_H is a compact subset of M_0. To be precise, let P be the orthogonal projection from $\mathbb{R}^3$ to M_0. Let

$$q_i = \frac{P(e_i)}{\sqrt{2}} \text{ for } i = 1, 2, 3$$

where $\{e_i\}$ is the standard basis for R^3. The q_i's form a 3-simplex in M_0. For each i, choose f_i such that

$$\left\{ \frac{q_i}{|q_i|}, f_i \right\}$$

gives an orthonormal basis for M_0. Define the maps $J_i : M_0 \to M_0$ by

$$J_i(q_i) = \frac{3}{5} q_i \text{ and } J_i(f_i) = \frac{1}{5} f_i$$

Using the J_i's, define the following contractions $H_i : M_0 \to M_0$ by

$$H_i(x) = J_i(x - q_i) + q_i \text{ for } i = 1, 2, 3.$$

Remark 12. *The presence of two distinct scaling factors in the maps J_i, and therefore in the contractions H_i, breaks from the self-similar structure of K where only $\frac{1}{2}$ was present in the contractions.*

K_H is then given as the unique nonempty compact subset of M_0 such that

$$K_H = \bigcup_{i=1}^{3} H_i(K_H).$$

To see how the contractions H_i are related to, or perhaps *inherited* from, the contractions F_i used, for each $i = 1, 2, 3$, to define K, note that Φ commutes with the contractions in the sense that

$$\Phi \circ F_i = H_i \circ \Phi.$$

The graph approximations of K_H can be attained through Φ from the F_i's or directly from the H_i's as in the case of K. A picture of a graph approximation of K_H is given below in Figure 2.6.

2.6 Measurable Riemannian Geometry

The primary ingredients of Kigami's prototype for a *measurable Riemannian geometry* are the *measurable Riemannian structure* and geodesic distance. The measurable Riemannian structure is due to Kusuoka [18] and is a triple $(\nu, Z, \widetilde{\nabla})$, where ν is the Kusuoka measure, Z is a non-negative symmetric matrix, and $\widetilde{\nabla}$ is an operator

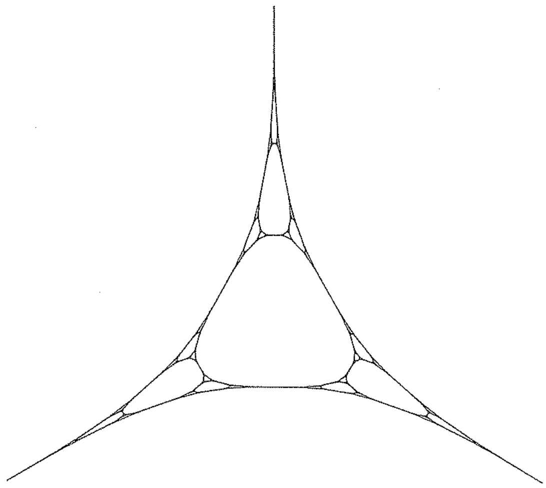

Figure 2.6: Harmonic gasket, K_H

analogous to the Riemannian gradient. More precisely, Kusuoka has shown that for any u and v in the domain of the energy on the Sierpinski gasket, K,

$$\mathcal{E}(u,v) = \int_K (\widetilde{\nabla} u, Z\widetilde{\nabla} v) d\nu,$$

where Z, $\widetilde{\nabla} u$, and $\widetilde{\nabla} v$, are ν-measurable functions defined ν-a.e. on K [12]. The equality above is analogous to its *smooth* counterpart in Riemannian geometry, and thus gives validity to the title 'measurable Riemannian structure' for $(\nu, Z, \widetilde{\nabla})$. Here, the Kusuoka measure ν is the analog to the Riemannian volume and Z is the analog to the Riemannian metric. In [12], Kigami furthers the likeness to Riemannian geometry by introducing a notion of smooth functions on K, as well as a theorem relating the Kusuoka gradient to the usual gradient on the Euclidean plane, and a notion of geodesic distance on K, which is realized by a C^1 path in the plane.

The Sierpinski gasket, in Euclidean or standard metric does not have C^1 paths between points, in general. In order to get C^1 paths, Kigami uses views the gasket in harmonic coordinates, as the harmonic gasket described earlier. The harmonic gasket, K_H, does have C^1 paths between any two points. Then via the homeomorphism, Ψ, a geodesic distance, realized by a C^1 path on K_H, is attached to K.

2.6.1 Volume (Kusuoka Measure)

The Kusuoka measure will be the measurable analog to Riemannian volume. The existence of the Kusuoka measure ν on K is due to Kusouka [18]. Specifically, for any $w \in W_*$, there exists a unique Borel regular probability measure measure ν on Σ such that

$$\nu(\Sigma_w) = \frac{1}{2}\left(\frac{5}{3}\right)^{|w|}(||J_w||_{HS})^2$$

for all $w \in W_*$, where $||X||_{HS}$ denotes the Hilbert-Schmidt norm on a linear operator from M_0 to itself (if (a_1, a_2) is an orthonormal basis of M_0, then $||X||_{HS} = \sqrt{|Xa_1|^2 + |Xa_2|^2}$). It also holds that ν is non-atomic. Following Kigami in [12], define $\pi_*\nu(A) = \nu(\pi^{-1}(A)$ for any Borel set $A \subset K$. Then $\pi_*\nu$ is a Borel probability measure on K and $\pi_*\nu(K_w)$ for any $w \in W_*$. Note $\nu(V_* = 0$, so that $(K, \pi_*\nu)$ is naturally identified with (Σ, ν) as a measure space (recall that $\pi^{-1}(x) = 1$ except when $x \in V_*$). For this reason, in a slight abuse of notation, we will use ν in place of $\pi_*\nu$ for the Kusuoka measure on K [12].

Another way of viewing the Kusuoka measure is as a specific *energy measure*. As defined in [32], an energy measure, ν_f, is defined for an open set $O \subset K$, by

$$\nu_f(O) = \lim_{n\to\infty}\left(\frac{5}{3}\right)^n \sum_{y\equiv x: x,y\in V_n\bigcap O} (f(y)-f(x))^2.$$

Thus $\mathcal{E}(f) = \nu_f(K)$, and the term energy measure is appropriate. Let μ be the standard self-similar measure on K. Theorem 4.1 in [32] cites the following results due to Kusuoka [17], [18]: For $\{h_1, h_2\}$, an $\mathcal{E}$-orthonormal basis of the two-dimensional space of harmonic functions mod (constants), and $\nu = \nu_{h_1} + \nu_{h_2}$ (independent of the choice of basis),

1. The measure ν_f is absolutely continuous with respect to ν for any $f \in Dom\mathcal{E}$.

2. The measures ν and μ are mutually singular.

2.6.2 Metric

The measurable analog to the Riemannian metric, or the *measurable Riemannian metric* Z, is also due to Kusuoka [18]. In Proposition 2.11 in [12], Kigami gives the definition of Z as the following: Let $w \in W_m$ and define $Z_m(w) = J_w^t(J_w)/||J_w||_{HS}^2$. Then $Z(w) = \lim_{m\to\infty} Z_m(w_1...w_m)$ exist ν-a.e. for $w \in \Sigma$, rank$Z(w) = 1$ and $Z(w)$ is the orthoganol projection onto its image for ν-a.e. $w \in \Sigma$.

In order to get the metric defined on K, let $Z_*(x) = Z(\pi^{-1}(x))$. Then Z_* is well defined, has rank 1 and is the orthoganol projection onto its image for ν-a.e. $x \in K$. Similar as with the Kusuoka measure, the $*$ is dropped and Z is used instead of Z_*. It also holds that Z is well defined on V_*, since for $x \in V_*$ and $\pi^{-1}(x) = \{w, \tau\}$, $Z(w) = Z(\tau)$ [12].

2.6.3 Gradient

There are a few characterizations of the gradient in the setting of the measurable Riemannian structure. The first we will mention is due to Kusuoka [18]. In Theorem 2.12 in [12], Kigami gives Kusuoka's result which is the existence of an assignment $\widetilde{\nabla} : Dom\mathcal{E} \to \{Y | Y : K \to M_0, Y \text{ is } \nu - measurable\}$ such that

$$\mathcal{E}(u,v) = \int_K (\widetilde{\nabla} u, Z\widetilde{\nabla} v) d\nu,$$

for any $u, v \in Dom\mathcal{E}$. To see the measure $(\widetilde{\nabla} u, Z\widetilde{\nabla} v)d\nu$ as *the* energy measure of u associated with the energy form $\mathcal{E}$, see [12], page 8.

Kigami's approach the gradient on K is to start with the usual gradient on open subsets of the plane M_0 which contain K_H. More precisely, fixing an orthonormal basis for M_0 and identifying M_0 with $\mathbb{R}^2$, the gradient on M_0 is given by $\nabla u =^t (\partial u/\partial x_1, \partial u/\partial x_2)$. In Proposition 4.6 in [12], Kigami has shown that if U is an open subset of M_0 which contains K_H, $v_1, v_2 \in C^1(U)$, and $v_1|_{K_H} = v_2|_{K_H}$, then $(\nabla v_1)|_{K_H} = (\nabla v_2)|_{K_H}$. In this sense the gradient of a *smooth function* on K_H is well defined by the restriction of the usual gradient to an open subset of M_0. Then using Φ, this theory can be pulled back to K. Precisely, in [12], Kigami defines

$$C^1(K) = \{u : u = (v|_{K_H}) \circ \Phi, \text{ where } v \text{ is } C^1 \text{ on an open subset of } M_0 \text{ containing } K_H\}$$

and for $u \in C^1(K)$,

$$\nabla u = (\nabla v|_{K_H}) \circ \Phi.$$

In Theorem 4.8 in [12], Kigami shows the following:

1. $C^1(K)$ is a dense subset of $dom\mathcal{E}$ under the norm $||u|| = \sqrt{\mathcal{E}(u,u)} + ||u||_\infty$;

2. $\widetilde{\nabla} u = Z\nabla u$ for any $u \in C^1(K)$;

3. $\mathcal{E}(u,v) = \int_K (\nabla u, Z\nabla v) d\nu \qquad$ for any $u, v \in C^1(K)$.

Thus Kigami shows that his gradient, ∇, 'essentially' coincides with the Kusuoka gradient, $\widetilde{\nabla}$—at least up to its role in the energy formula.

In [32], Teplyaev uses *harmonic tangents* to construct a gradient that essentially coincides with both the Kusuoka gradient and the Kigami gradient. Teplyaev defines the harmonic tangent of f at $x \in K$ as $T_x f = \lim_{n\to\infty} h_{n,x}$ where $h_{n,x}$ is a unique harmonic function which coincides with f on the vertices of a graph cell $\Gamma_{n,x}$ containing x, of size 2^{-n} (here the graph cells are indexed, in part, by n instead of a word w of length n, since the triangles are necessarily nesting around x if x is not in V_*, and basically doing the same, except from two 'directions' when x is a vertex.

It follows that if $T_x f$ exists, it is clearly harmonic, and thus as stated by Teplyaev, is a harmonic approximation to f at x. This is analogous to the usual derivative on $\mathbb{R}^n$ serving as a linear approximation. In fact, on the interval, harmonic approximation and linear approximation coincide! Indeed, the harmonic tangent corresponds to the usual tangent line to the graph of a differentiable function on the interval—the harmonic functions on the interval are exactly the affine function on the interval.

Note that $T_x f$, a harmonic function, can be viewed as a two dimensional vector living in M_0. Thinking about the directionality of the approximation, it can happen that there are two 'directional' harmonic tangents at $x \in V_*$ [32]. Teplyaev gives two theorems, 2.2 and 2.3, in [32] regarding the existence and regularity of $T_x f$. In particular, in 2.2, he has shown that if μ is the standard measure on K and f is in the domain of the standard Laplacian, then $T_x f$ exists for μ-a.e. $x \in K$ and that $T_x f$ is continuous in x for μ-a.e. $x \in K$. In 2.3, in particular, he has shown that if $\vec{h} = (h_1, ...h_2)$ where h_k are harmonic functions, $f = F(\vec{h}) : K \to \mathbb{R}$, and $F \in C^2(\mathbb{R}^2)$, then $T_x f$ exists for μ-a.e. $x \in K$, and that

$$T_x f(y) = f(x) + \nabla F(\vec{h}(x)) \bullet (\vec{h}(y) - \vec{h}(x)),$$

and hence that T_x is continuous in x.

Teplyaev discusses a class of fractals in harmonic coordinates in [31], of which the Sierpinski gasket in harmonic metric is an example. The following definitions and observations by Teplyaev in [31] will be interpreted as applying to the harmonic gasket, though they actually have a broader scope. In Definition 4.4 in [31], Teplyaev defines the space L^2_Z as the Hilbert space of M_0-valued functions on K with the norm given by

$$||u||^2_{L^2_Z} = \int_K (u, Zv) d\nu.$$

In [31] Teplyaev writes the Kigami energy formula as

$$\mathcal{E}(u, u) = ||\nabla u||^2_{L^2_Z},$$

for ∇u in the Kigami sense. This way of viewing the Kigami formula is a potential link between the effective resistance metric (see Section 2.7 below) and the metric derived from the spectral triples in Chapter 4 of this dissertation. The conjectured relationship will be stated in the concluding remarks of Chapter 4.

Teplyaev constructs another gradient [32]. In Remark 4.4 in [32] on page 141, Teplyaev defines the *essential gradient* on K as

$$\nabla_{ess} f(x) = P_x T_x f,$$

where P_x is an orthogonal projection. The motivation is that the metric $Z = Z_* \geq 0$ and $TrZ = 1$, so that ν-a.e. $P_x = Z_x$ is an orthogonal projection. Teplyaev makes note in Remark 4.4 [32] that the Kigami gradient is not equal to the essential gradient as the former is continuous and the latter everywhere discontinuous and that P_x can be interpreted as an orthogonal projection, in harmonic coordinates, onto the tangent line to K at x (see the reference to Teplyaev's Theorem 4.7 in the next subsection). Teplyaev does note, however, in Remark 4.4, that

$$\mathcal{E}(u,u) = \int_K ||\nabla_{ess} u||^2 d\nu,$$

showing that with respect to the energy form, ∇_{ess} coincides with both Kigami's and Kusuoka's gradients.

2.6.4 Geodesic Distance

The first important theorem regarding a geodesic, or segment, or shortest path between two points on K, in the context of K in harmonic coordinates, is due to Teplyaev. First, a boundary curve τ of the gasket in harmonic coordinates, is defined by Teplyaev as a parameterization of a connected component of $M_0 \backslash K_H$. In Theorem 4.7 in [32], Teplyaev states the following:

1. τ is concave and is a C^1 curve but *is not a* C^2 curve,

2. for any $x \in K$, such that $\Psi(x) \in \tau$, the projection P_x is, in harmonic coordinates, the orthogonal projection onto the tangent line to τ.

Let $h_*(p, q) = inf\{(\gamma)|\gamma$ is a rectifiable curve in K_H between p and $q\}$, where $l(\gamma)$ is the length of the curve γ. Kigami makes use of the above results to prove Theorem 5.1 in [12] which states that for any $p, q \in K_H$, there exists a C^1 curve $\gamma_* : [0, 1] \to K_H$ such that $\gamma_*(0) = p$, $\gamma_*(1) = q$, $Z(\Phi_{-1}(\gamma_*(t)))$ exists and $\frac{d\gamma_*}{dt} \in ImZ(\Phi_{-1}(\gamma_*(t)))$ for any $t \in [0, 1]$, and

$$h_*(\gamma_*(a), \gamma_*(b)) = \int_a^b \left(\frac{d\gamma_*}{dt}, Z(\Phi^{-1}(\gamma_*(t))) \frac{d\gamma_*}{dt} \right)^{\frac{1}{2}} dt$$

for any $a, b \in [0, 1]$ with $a < b$. Note that due to Kigami's result, the infimum in the definition of h_* can be replaced by the minimum. Kigami calls γ_* a *geodesic* between p and q. The proof of this theorem is lengthy with the majority of the work going into proving Kigami's Theorem 5.4 [12] which Kigami credits Teplyaev (Theorem 4.7 in [32]) for the result but gives his own proof.

Theorem 5.4 is a specific case of Theorem 5.1—precisely it gives the result of Theorem 5.1 on one arc of the boundary of the initial graph cell of K_H. From this he argues in Lemma 5.5 in [12] that since any arc of a boundary of a graph cell at any level together with the straight line segment connecting its endpoints is an affine transformation of one of the initial arcs (which forms a convex region with the straight line segment between its endpoints), that this region induced by the graph cell is also convex. He also states a well-know fact from convex geometry as Theorem 5.2 [12] which is that if C and D are compact subsets of $\mathbb{R}^2$ with $C \subset D$, with C convex and ∂D a rectifiable Jordan curve, then $l(\partial C) \leq l(\partial D)$.

To make sense of the existence of a geodesic of K_H, it is first easier to look at K. If p is a vertex of K and q an arbitrary point in K, then a shortest path to q from p is constructed by considering the lowest graph approximation which puts p and q in separate cells. Now by a connectedness argument (there are only two vertices connecting one cell to the other) and because a straight line exists from p (a side of the boundary of cell p is in) to the cell q is in, and a straight line is the shortest path between two points in $\mathbb{R}^2$, the *first leg* of the geodesic is the straight line segment from p to the cell q is in. We have thus arrived at a vertex of a cell at this graph approximation, and repeat this argument.

In the limit, since the vetices are dense in K, we arrive at q, and have constructed a geodesic, which is rectifiable since the sides of cells decrease at $\frac{1}{2^n}$ on K. If p and q are arbitrary points in K, they are connected by a vertex, and it is not hard to use the previous argument applied to a vertex to p and then from that vertex to q.

Kigami uses an analogous argument which replaces that fact that a straight line is the shortest distance in $\mathbb{R}^2$ with convexity. Indeed, for p a vertex and q arbitrary in K_H, consider the lowest graph approximation which puts the two points in different cells, and note that by convexity, the shortest path to the next cell is along the side of the boundary that p is in (of course the same connectedness argument is used here as well). This process is iterated as before yield a path a shortest path connecting p and q. For p and q arbitrary the same reasoning as with K holds. Since there is no simple contraction ratio for K_H, Kigami uses the fact that the highest eigenvalue of the maps J_i is $3/5$ to show that the path is rectifiable.

This is only a rough sketch of Kigami's proof as there is more machinery used, and several lengthy calculations. Kigami's Lemma 5.6 in [12] (which relies on Theorem 5.2 in [12]), is the technical foundation for the sketch I gave above. Kigami uses this distance function h_* to define the *harmonic shortest path metric* on K, $d_*(.,.)$, for $x, y \in K$, as

$$d_*(x,y) = h_*(\Phi(x), \Phi(y)).$$

In the next subsection is a summary of the measurable Riemannian formulas given by Kusuoka, Teplyaev, and Kigami, as well as a few other formulas involving the Laplacian.

2.6.5 Measurable Riemannian Formulas

The following are the three characterizations of the energy form on K in terms of the measurable Riemannian constructs, encountered in this Chapter:

$$\mathcal{E}(u,v) = \int_K (\widetilde{\nabla} u, Z \widetilde{\nabla} v) d\nu \qquad \text{(Kusuoka)}$$

$$\mathcal{E}(u,v) = \int_K (\nabla u, Z \nabla v) d\nu \qquad \text{(Kigami)}$$

$$\mathcal{E}(u,u) = \int_K ||\nabla_{ess} u||^2 d\nu \qquad \text{(Teplyaev)}$$

The harmonic shortest path (or geodesic) distance on K is given by

$$d_*(x,y) = h_*(\Phi(x), \Phi(y)) \qquad \text{(Kigami)},$$

where

$$h_*(\gamma_*(a), \gamma_*(b)) = \int_a^b \left(\frac{d\gamma_*}{dt}, Z(\Phi^{-1}(\gamma_*(t))) \frac{d\gamma_*}{dt} \right)^{\frac{1}{2}} dt. \qquad \text{(Kigami)}$$

Remark 13. *In the list of formulas presented in the beginning of Chapter 2, the mapping Φ was ignored in the representation of h_* as a slight abuse of notation for aesthetic purposes.*

Another important formula is characterization of the Laplacian, Δ_ν in terms of the energy form,

$$\int_K v \Delta_\nu(u) d\nu = -\mathcal{E}(u, v),$$

as the unique function $\Delta_\nu u$, such that this formula holds for all v (with boundary value zero) in the domain of the energy.

A more general formulation (not requiring v have zero boundary value) is given by Teplyaev in Theorem 4.10 in [32]. Also in this Theorem, Teplyaev shows that for $F \in C^2(\mathbb{R}^2)$, $f = F(h_1, h_2) : K \to \mathbb{R}$, and (h_1, h_2) a basis of the harmonic functions (mod constants), it holds that $f \in dom\Delta_\nu$ and

$$\Delta_\nu f = Tr(ZD^2 f),$$

where

$$D^2 f(x) = \left| \frac{\partial^2}{\partial h_i \partial h_j} F((h_1(x), h_2(x))) \right|_{i,j=1}^2.$$

2.7 Resistance Form on K

The energy form $\mathcal{E}$ is an example of a broader theory of resistance forms given by Kigami in [13]. In particular, such resistance forms admit an *effective resistance* between points of the underlying space. Applied to K, for any $p, q \in V_*$, the effective resistance [31] is given by

$$\sup\left\{\frac{(u(p)-u(q))^2}{\mathcal{E}(u,u)} : u \in Dom\mathcal{E}\right\}.$$

This quantity is finite for all p and q and the supremum denoted by $R(p,q)$, is called the *effective resistance between p and q*. R is a metric on V_* and any function in the domain of the energy is R-continuous [31]. If S is the R-completion of V_*, then any $u \in dom\mathcal{E}$ has a unique R-continuous extension to S [31].

A conjecture is that the metric $\sqrt{R}$ coincides with, or is at least equivalent with d_*, Kigami's geodesic distance on K. The motivation for this will be seen in the discussion of Dirac operators on K in Chapter 4 of this thesis.

Chapter 3

Spectral Triples, Noncommutative Geometry, and Fractals

3.1 Overview

Chapter 2 was dedicated to describing Jun Kigami's measurable Riemannian geometry. Chapter 3 surveys some of the tools of noncommutative geometry including the Dirac operator and its associated spectral triple. Chapter 3, like Chapter 2, presents necessary background for Chapter 4. As mentioned in the introductory chapter, the Dirac operator associated to a compact spin Riemannian manifold has been shown by Alain Connes [5] to encode much of geometric information of that manifold—in particular, the geodesic distance and the volume form.

In the context of the Riemannian manifold, it is known that the metric of the manifold can be recovered from the geodesic distance ([34], P.388) and it is in this sense that defining a Dirac operator on the manifold defines the geometry of the manifold. Indeed, any smooth manifold, M, can be assigned a Riemannian metric g. The *Meyers-Steenrod* theorem [23] is stated below:

Meyers-Steenrod Theorem. *If* (M, g) *and* (N, h) *are Riemannian manifolds and* $\phi : M \to N$ *is a distance preserving bijection, then* ϕ *is a Riemannian isometry.*

Suppose (M, g) and (M, h) admit the same geodesic distance function $d = d_g = d_h$. If we define $\phi = Id_M$, then ϕ is clearly a distance preserving bijection. The Meyers-Steenrod theorem says ϕ is an isometry, and therefore $h = g$ and it is clear that the geodesic distance function determines the metric and hence the geometry.

Thus, on one hand, if the manifold is spin compact and equipped with a Riemannian metric, the metric determines the Dirac operator (which in turn recovers the metric). On the other hand, any Dirac operator defined on a spin compact manifold will determine a Riemannian metric on the manifold. The observation that the Dirac operator defines the geometry is one of Connes' contributions to the field of geometry [34]. Indeed, it is a springboard for defining generalized manifolds and geometries in the context of spaces which admit a meaningful generalization of the Dirac operator, but not meaningful generalizations of smooth structure or metric or even paths in the space.

In the case of the Riemannian manifold, in the construction of the Dirac operator and spectral triple, one encounters a commutative C^*-algebra of functions on the manifold which act as 'coordinates' for the manifold. However, none of the arguments necessary to recover the geometry rely on the commutivity of the algebra. Thus, in considering the possible generalized geometries mentioned above, they can be partitioned into two camps—commutative and noncommutative. Commutative when their space admits a commutative C^*-algebra and noncommutative when their space admits a non-commutative C^*-algebra. The term *noncommutative geometry* refers to either case when the *tools of noncommutative geometry*—i.e. Dirac operators and spectral triples—are used to construct the geometry.

An application of the methods of noncommutative geometry, central to this thesis, is to fractals. In [20], Michel Lapidus began a program in noncommutative fractal geometry and in [3], Michel Lapidus, Christina Ivan, and Erik Christensen applied these noncommutative methods to some fractal sets built on curves—including trees, graphs, and the Sierpinski gasket. As will be seen in sections to follow, the work in [3] on the more complex sets is based largely on the Dirac operator and spectral triple on the circle.

It is important to note that the work in [3] on the Sierpinski gasket is with respect to the Sierpinski gasket in Euclidean metric as opposed to the treatment of the Sierpinski gasket in harmonic metric in Chapter 4 of this thesis. Of many results in [3], the application of noncommutative methods to the Sierpinski recovered the geodesic distance, volume measure, and metric *spectral* dimension.

This chapter, in conjunction with the previous chapter, provides the necessary framework for Chapter 4 which will be dedicated to exploring a *fractal* analog to Connes' theorem. Indeed, on one hand there is a *target* geometry that is a fractal analog of Riemannian geometry—Kigami's measurable Riemannian geometry. On the other hand there are the noncommutative methods applied to fractals discussed in this chapter which can be used to recover Kigami's geometry, namely through the Dirac operator.

The following exposition will be divided into five main parts: First, there will be a brief overview of C^*-algebras; Second, we discuss an abstraction of some of the elements of noncommutative geometry in the setting of Lipschitz seminorms and compact metric spaces. Third, we will define the unbounded Fredholm module and Spectral triple. Fourth, a summary of Connes' approach will be given. Finally, the methods and applications in [3] will be detailed.

3.2 C^*-algebras

A unital C^*-algebra is a Banach algebra, A, over $\mathbb{C}$, with multiplicative identity and an operation,

$$* : A \to A \quad \text{given by} \quad x \longmapsto x^*,$$

satisfying the following properties:

1. $x^{**} = x$
2. $(ax + by)^* = \hat{a}x^* + \hat{b}y^*$
3. $(xy)^* = y^*x^*$
4. $||x^*x|| = ||x^*||||x|| = ||x||^2$

The operation $(*)$ can be referred to as a conjugate-linear involution [5] satisfying (3) and (4). The easiest example to think about is the set of complex numbers itself. Indeed, $*$ given by complex conjugation $z \longmapsto z^* \equiv \hat{z}$ satisfies the conditions given above. Another important example is $C(X)$, the complex-continuous functions on a compact Hausdorff space X. The operation $*$ is referred to as the *adjoint* operation. A *representation* π of a C^*-algebra $\mathcal{A}$ on a Hilbert space H is a *linear* $*$*-homomorphism*,

$$\pi : \mathcal{A} \to B(H),$$

where $B(H)$ is the set of bounded linear operators from H to H. The *linear* condition on π is as usual; it must preserve linearity. To discuss what a $*$*-homomorphism* is, we note that for any Hilbert space, H, $B(H)$ has an adjoint operation, $*$. The following is a well known fact: If $(.,.)_H$ is the inner product on H, then for any $T \in B(H)$ and for all $x, y \in H$, there exists a unique $F \in B(H)$ such that

$$(Tx, y)_H = (x, Fy)_H.$$

We define $T^* = F$, and say T^* is the adjoint of T. For π to be a $*$-homomorphism, we require that π preserves the $*$ operation. Precisely, we require that if $x \longmapsto T$, then $x^* \longmapsto T^*$. It is also said that π is *faithful* if it is an injective map, and *unital* if $\pi(1_A) = Id_H$.

A theorem, due to Gelfand and Naimark, usually referred to as *Gelfand's Theorem* [5] (or the *Gelfand-Naimark Theorem* [34]) states that every commutative C^*-algebra is $*$-isomorphic (and homeomorphic) to $C(X)$ for some compact Hausdorff space X. It turns out that in proving this fact, one recovers an elegant description of X in terms of the algebra $\mathcal{A}$.

More precisely, X will be *determined* as the set of all *pure states* of $\mathcal{A}$ (also referred to as the *spectrum* or *set of characters* of $\mathcal{A}$), with the weak*-topology assigned. Thus, if one starts with $\mathcal{A} = C(X)$ for some compact Hausdorff space X, the Gelfand-Naimark theorem says that X is *recovered* as the space of pure states of $\mathcal{A}$. Note that if X is a compact metric space, then the weak*-topology on the set of pure states is metrizable.

The space of pure states will be denoted $\mathcal{P}(\mathcal{A})$. To give a brief sketch of the Gelfand-Naimark theorem, let $S(\mathcal{A})$ be the *states* of $\mathcal{A}$. By a state, T, we mean a *positive* complex valued linear functional on $\mathcal{A}$ with

$$||T|| = \sup_{||x||=1} ||Tx|| = 1 \quad \text{and} \quad T(1_{\mathcal{A}}) = 1_{\mathbb{C}}.$$

To define *positive*, we first define a *positive element* of $\mathcal{A}$: For $x \in \mathcal{A}$, x is positive if $spec(x) \subset \mathbb{R}_{+}$. The spectrum of x, $spec(x)$ is a generalization of the notion of eigenvalues of a matrix operator. For a matrix operator, its eigenvalues are defined in terms of where the determinant is zero. For a matrix, the determinant is zero *iff* the matrix is non-invertible. The definition of $spec(x)$ is generalized from non-invertibility. The set of eigenvalues of x, or $spec(x)$, is the collection $\{\lambda_i\}$ of elements of $\mathbb{C}$ such that $(\lambda_i Id_{\mathcal{A}} - x)$ is not invertible (i.e. does not have both left and right inverses).

A positive functional, T, on $\mathcal{A}$, is then defined as taking positive elements of $\mathcal{A}$ to the positive real line. Stated more precisely, if T is a positive functional then for any positive element $x \in \mathcal{A}$, $Tx = a + 0i$ with $a \in \mathbb{R}_{>0}$. For any $T \in S(\mathcal{A})$, T is a bounded linear functional with norm 1.

The next objective is to put a topology on $S(\mathcal{A})$. This can be done by using the weak*-topology of the functional dual of $\mathcal{A}$, denoted by $\mathcal{A}'$. Recall that there is a natural isometric isomorphism $\phi : \mathcal{A} \to \mathcal{A}''$ and thus $\phi[\mathcal{A}] \subset \mathcal{A}''$. The weak*-topology on $\mathcal{A}'$ is the topology containing the fewest open sets making every element of $\phi[\mathcal{A}]$ a continuous functional on $\mathcal{A}'$.

Since $S(\mathcal{A}) \subset \mathcal{A}'$, $S(\mathcal{A})$ inherits the weak*-topology from $\mathcal{A}'$. It can be shown that $\mathcal{A}'$ is closed in $\mathcal{A}'$ and convex and compact Hausdorff. By convexity, $S(\mathcal{A})$ contains extreme points. Let $P(\mathcal{A})$ denote the (nonempty) set of extreme points of $S(\mathcal{A})$. Consider the weak*-closure of $P(\mathcal{A})$, called the *pure state space of* $\mathcal{A}$ and denoted as $\mathcal{P}(\mathcal{A})$. The following are two facts regarding $\mathcal{P}(\mathcal{A})$:

- A linear functional, ρ, on a commutative C^*-algebra, is pure iff it is multiplicative, i.e. $\rho(xy) = \rho(x)\rho(y)$;
- $\mathcal{P}(\mathcal{A})$ is a closed subspace of $S(\mathcal{A})$.

Remark 1. *The first fact above, in particular, gives each pure state of $\mathcal{A}$ as a character of $\mathcal{A}$ and vice versa.*

Since a closed subspace of a compact Hausdorff space is also compact Hausdorff, it holds that $\mathcal{P}(\mathcal{A})$ is compact Hausdorff. The punchline is then to set $X = \mathcal{P}(\mathcal{A})$ and prove that there is a $*$-isomorphism between $\mathcal{A}$ and $C(X)$, which is linear, multiplicative, preserves the identity, and is a homeomorphism (actually an isometry when X is a compact metric space). This isomorphism (a specific case of the *Gelfand Representation*) can be defined from $\mathcal{A} \to C(X)$ as follows:

$$x \longmapsto \hat{x} \quad \text{where} \quad \hat{x}(\rho) = \rho(x).$$

The function $\hat{x} : \mathcal{P}(\mathcal{A}) \to \mathbb{C}$ is sometimes referred to as the *Gelfand Transformation* of x [34]. In particular, showing this map is onto can be done using the Stone-Weierstrass theorem and injectivity can be shown using the Krein-Millman theorem. So it is the case that the topology on X is recovered from $\mathcal{A}$ in the sense that the topology on X is the weak*-topology on X as a subset of $S(\mathcal{A}) \subset A'$. Another way to say this is that two unital commutative C^*-algebras are isomorphic iff their spaces of pure states (i.e. their spectra) are homeomorphic [5].

A perspective to be gained in the above discussion of the Gelfand-Naimark theorem is the possibility of partitioning topologies (or geometries) roughly through the following correspondences:

1. *commutative topologies/geometries* $(X) \Longleftrightarrow$ commutative C^*-algebras $(C(X))$;

2. *noncommutative topologies/geometries* $\Longleftrightarrow$ noncommutative C^*-algebras.

Since $C(X)$ is commutative, we say X has a commutative topology or geometry. Thus we may consider noncommutative *rings of functions* on some 'noncommutative spaces'. Possibly the most mildly noncommutative example of a C^*-algebra is the space of $n \times n$ matrices over $\mathbb{C}$. Another example is $B(H)$ where H is a Hilbert space. A second, more general result due to Gelfand and Naimark is that *any* C^*-algebra can be faithfully represented in $B(H)$ for some Hilbert space H. The applications of noncommutative geometry in this thesis fall within the commutative case, and in particular are applications to fractal geometry. In this sense the realm of non-classical geometries is not restricted to the noncommutative side.

3.3 Compact Metrics, State Spaces, and Lipschitz Seminorms

Specifying a *natural* distance function on a set or space is central to noncommutative geometry. In the context of C^*-algebras, it was first suggested by Connes that from a suitable Lipschitz seminorm one obtains an ordinary metric on the state space of the C^*-algebra [25]. Let X be a compact metric space with metric ρ. Defined on real-valued or complex-valued functions on X, the Lipschitz seminorm, L_ρ, determined by ρ, is given by

$$L_\rho(f) = \sup\left\{\frac{|f(x) - f(y)|}{\rho(x,y)} : x \neq y\right\}. \tag{3.1}$$

The space of *ρ-Lipschitz functions on X* is comprised of those functions, f on X, with $L_\rho(f) < +\infty$. One can recover the metric ρ, in a simple way, from L_ρ, by the following formula [25]:

$$\rho(x,y) = \sup\{|f(x) - f(y)| : L_\rho(f) \leq 1\} \tag{3.2}$$

.

Indeed, it is obvious that $\rho \leq \sup\{|f(x) - f(y)| : L_\rho(f) \leq 1\}$. For fixed $y \in X$, the function $a(x) = \rho(x,y)$, by the triangle inequality, is such that $L_\rho(a) = 1$, and is therefore witness to $\rho \geq \sup\{|f(x) - f(y)| : L_\rho(f) \leq 1\}$. (The formula (2) above is an abstraction of what will be denoted Connes' *Formula 1* in Section 3.5 below.) Let $S(X)$ denote the space of probability measures on X. In [25], Rieffel notes that an extension of (2) defines $\hat{\rho}$ a metric on $S(X)$ by

$$\hat{\rho}(\mu,\nu) = \sup\{|\mu(f) - \nu(f)| : L_\rho(f) \leq 1\}, \tag{3.3}$$

and that the topology $\hat{\rho}$ defines on $S(X)$ is exactly the weak*-topology on $S(X)$ when $S(X)$ is viewed as the state space, $S(C(X))$, of $C(X)$. The formula (3) is referred to as the *Monge-Kantorovich* metric on probability measures or the *Hutchinson* metric in the theory of fractals [26]. Recall from the previous section, that via the Gelfand-Naimark theorem, the points of X can be identified with $\mathcal{P}(C(X))$, the pure state space of $C(X)$.

It is also noted in [25] that $\hat{\rho}$ and ρ coincide on $X = \mathcal{P}(C(X))$, and thus the formula 1 above can be interpreted as the fact that L_ρ is recovered from the restriction of $\hat{\rho}$ on $S(X)$ to $\mathcal{P}(C(X))$. In [25] and [26], Rieffel treats the extension of this idea to possibly noncommutative algebras, and characterizes when Lipschitz type seminorms can be recovered in such a manner. He also discusses the effective resistance metric associated to graphs, in the final section of [25]. Some of his ideas, as they might relate to the work on fractals in this thesis, will be discussed in the conclusion of the thesis.

3.4 Unbounded Fredholm Modules and Spectral Triples

In noncommutative geometry, a standard way to specify the *suitable* Lipschitz seminorm mentioned in the beginning of Section 3, is via a *Dirac* operator. Dirac operators have origin in Quantum mechanics, but will be described here in a branch of their evolution which parallels the evolution of noncommutative geometry. In particular, they will be defined in the context of *unbounded Fredholm modules* and *spectral triples*. Following [3], we will use the following definitions:

Definition 1. *let $\mathcal{A}$ be a unital C^*-algebra. An **unbounded Fredholm module** (H, D) over $\mathcal{A}$ consists of a Hilbert space H which carries a unital representation π of $\mathcal{A}$ and* *an unbounded, self-adjoint operator D on H such that*

i. the set $\{a \in \mathcal{A} : [D, \pi(a)]$is densely defined and extends to a bounded operator on $H\}$ is a dense subset of $\mathcal{A}$,

ii. the operator $(I + D^2)^{-1}$ is compact.

Definition 2. *Let $\mathcal{A}$ be a unital C^*-algebra and (H, D) an unbounded Fredholm module of $\mathcal{A}$. If the underlying representation π is faithful, then $(\mathcal{A}, H, D)$ is called a **spectral triple** [3]. In addition, D is called a* Dirac operator.

Remark 2. *We will refer to $(\mathcal{A}, H, D)$ as either a spectral triple or unbounded Fredholm module whether or no π is faithful.*

3.5 The Dirac operator and a Compact Spin Riemannian Manifold

In the setting of compact spin Riemannian manifolds, there is the notion of *the* Dirac operator. In other words as mentioned in the overview, such a manifold *determines* the Dirac operator. Proving this requires a wealth of technical machinery including Clifford algebras, spin geometry, and various connections on vector bundles. These topics, for the most part will not be treated here. There are many references on these topics, including [5] and [34]. In [5], Connes gives four formulas concerning the Dirac operator and a compact spin Riemannian manifold, (M, g). Two of them will be discussed in this section—*Formula 1* and *Formula 2.*

Let M be a compact spin Riemannian manifold and S its prescribed spinor module structure. If ∇^s is the spin connection on S, then the *Dirac operator* on S is the operator $\not{D}$ given by

$$\not{D} = -i(\hat{c} \circ \nabla^s),$$

where $\hat{c}$ is the Clifford action [34].

If M admits no spin structure, and even if M is not orientable, there is a more general notion of Dirac operator. Let $\mathcal{E}$ be a self-adjoint Clifford module over a compact Riemannian manifold M with Clifford connection ∇ and Clifford action $\hat{c}$. Following [34] the *generalized* Dirac operator associated to ∇ and $\hat{c}$, is given analogously by

$$D = -i(\hat{c} \circ \nabla).$$

Remark 3. *In the case of the generalized Dirac operator on a compact spin Riemannian manifold, Proposition 9.10 in [34] states that the Dirac operator is related to any generalized Dirac operator by a formula relating a Clifford connection to the spin connection. Suppose $\mathcal{E} = S \otimes F$ is a self-adjoint Clifford module over a compact spin Riemannian manifold and that $\nabla^{\mathcal{E}}$ is a Clifford connection on $\mathcal{E}$. Then there is a unique Hermitian connection $\nabla^{\mathcal{F}}$ on $\mathcal{F}$ such that*

$$\nabla^{\mathcal{E}} = \nabla^s \otimes 1_{\mathcal{F}} + 1_S \otimes \nabla^{\mathcal{F}} \quad [34].$$

In both cases, the Dirac operator is constructed using a connection, hence it is reasonable to think that it is a derivation (follows the Leibniz rule). Indeed, Proposition 9.11 on page 387 in [34] states for $a \in C^\infty(M)$ identified with its representation as a multiplication operator on the Clifford module $\mathcal{E}$, and D a generalized Dirac operator,

$$[D, a] = -ic(da),$$

where c is another way of writing the Clifford action and the relation between c and $\hat{c}$ is given by $c(\nu)s = \hat{c}(\nu \otimes s)$ for $s \in \mathcal{E}$ [34]. Here, da, is a section of the Clifford bundle over M. To emphasize the dependence on the Leibniz rule, we look at the proof of Proposition 9.11 in [34]:

$$i[D, a]s = \hat{c}(\nabla(as)) - a\hat{c}(\nabla s) = \hat{c}(\nabla(as) - a\nabla s) = \hat{c}(da \otimes s) = c(da)s.$$

The commutation relation, $[D, a] = -ic(da)$, is crucial to the recovery of the geodesic distance on M from the Dirac operator as well as giving the operator $[D, a]$ as multiplication by $c(da)$.

Going back to the case when M is spin compact, we have that (M, g) determines $\not{D}$ via its spin structure. To describe the rest of the spectral triple, the C^*-algebra $\mathcal{A}$ is chosen to be $C(M)$. The spinor module $S = \Gamma^\infty(S)$ is a pre-Hilbert space under the following positive definite Hermitian form,

$$\langle \Phi | \Psi \rangle = \int_M (\Phi | \Psi) |\nu_g|,$$

with $|\nu_g|$ as the Riemannian density on M ([34], p. 389). The completed Hilbert space, denoted by $L^2(M, S)$, will be called the *space of L^2-spinors* and is determined by the spinor bundle S over M [34]. The Hilbert space of the spectral triple for M will be chosen as this space of L^2 spinors. Finally, the representation π of $C(M)$ in $B(L^2(S, M))$ will be that which regards elements of $C(M)$ as bounded multiplication operators on $L^2(S, M)$ [34]. Then the triple,

$$(C(M), L^2(S, M), \not{D}),$$

is a spectral triple [5], [34]. In particular, see Lemma 1 on page 543 in [5].

Let d_g be the geodesic distance function on (M, g), a compact spin manifold. The first of Connes' formulas can now be stated ([5], p. 544):

Formula 1. *For any points $p, q \in M$,*

$$d_g(p, q) = \sup\{|a(p) - a(q)| : ||[\not{D}, a]|| \leq 1\}.$$

The norm $||.||$ in Formula 1 is the operator norm of $[\not{D}, a]$ as a multiplication operator on $L^2(S, M)$ and the supremum is taken over the elements $a \in C(M)$ (of course, identified as multiplication operators on $L^2(S, M)$).

An interesting point Connes makes is that the knowledge of the algebra of multiplication operators paired with the Hilbert space only yields information regarding the dimension of M. Also he notes that similarly, the knowledge of the Dirac operator and its pairing with the Hilbert space, yields information regarding the asymptotics of the eigenvalues, and hence the dimension of M. Connes then remarks that the spectral triple, however, is relevant to reconstructing the geometry on M ([5], p.543). This is seen immediately in Formula 1 since all components of the triple are at work in defining this distance function. In other words, the geometry of M is recovered from the interaction of the Dirac operator with the algebra of multiplication operators on the Hilbert space of spinors.

To see why Formula 1 holds, denote $\rho = d_g$ note that

$$||[\not{D}, a]|| = || - ic(da)|| = ||da|| = ||da||_\infty = Lip_\rho(a) \quad ([5],[34]) \ .$$

Then the proof of Formula 1 is exactly as with the analogous formula in Section 3.3. Also, as was mentioned in Section 3.3, Connes uses the right hand side of Formula 1 to define a metric,d, on the state space of the C^*-algebra as follows [5]:

$$d(\Phi, \Psi) = \sup\{|\Phi(a) - \Psi(a)| : ||[\not{D}, a]|| \leq 1\}.$$

An important point here is that, though the geodesic distance and the right hand side coincide for the Riemannian manifold, they are quite different as a springboard into the unknown. For the right hand side only involves the notion of functions on a space, as opposed to paths in a space. This is significant, for instance, in quantum mechanics, where there is no meaningful notion of a particle's path in a space, but there is a wave function assigned to that particle [5].

Remark 4. *Though the Dirac operator associated to the spin structure is used in Formula 1, it is important to note that the attribute of the Dirac operator essential to the validity of Formula 1, is the commutation relation from Theorem 9.11 in [34] mentioned above. Indeed, this relation holds for the generalized Dirac operators, and therefore Formula 1 holds in the absence of spin, and even in the absence of orientability [34].*

The second of Connes' formulas to be discussed is the recovery of the Riemannian volume form, from the data contained in the spectral triple defined above. It can be referred to as the *operator-theoretic* replacement for integration [5]. It requires a technical tool called the *Dixmier Trace*, Tr_w. We refer the reader to Chapter IV.2.β in [5] for background and a precise definition. Connes volume formula is given below:

Formula 2. *For any $f \in C(M)$,*

$$\int_M f d\nu = c(d) Tr_w(f|\not{D}|^{-d}),$$

where ν is the Riemannian volume measure, d is the dimension of M, and

$$c(d) = 2^{(d-(d/2))}\pi^{d/2}\Gamma\left(\frac{d}{2}+1\right).$$

As a convention, $\not{D}^{-1}$ is defined to be 0 on the finite dimensional subspace, $kerD$ [5].

In [5], Connes states that the right hand side of the integral formula in Formula 2 can be interpreted as the limit of the sequence,

$$\frac{1}{logN}\sum_{j=0}^{N}\lambda_j,$$

where the λ_j are the eigenvalues of $f|\not{D}|^{-d}$ (a compact operator). Connes also notes that equivalently the right hand side can be interpreted as the residue at the point $s = 1$, of

$$\zeta(s) = Tr(f|\not{D}|^{-ds}) \quad (Re(s) > 1).$$

Remark 5. *Formula 1 and Formula 2 are only part of theory described by Connes in [5]. In particular, the data contained in the spectral triple can describe cohomological aspects of M, curvature of certain vector bundles over M, and the Yang-Mills functional.*

We conclude this section with one more interesting formula which relates the Dirac operator to the Laplacian via the scalar curvature. The following is given as Theorem 9.16 in [34] and is due to Lichnerowicz:

. *On a compact spin Riemannian manifold, the Dirac operator* $\not{D}$*, the spinor Laplacian* Δ^s*, and the scalar curvature* s *are related by the following formula:*

$$\not{D}^2 = \Delta^s + \frac{1}{4}s.$$

Remark 6. *In the case of the generalized Dirac operator for a compact spin Riemannian manifold, Corollary 9.17 in [34] gives a similar relationship, though with an extra term coming from the curvature of the connection* $\nabla^{\mathcal{F}}$*, mentioned in Remark 3.*

3.6 Spectral Triples For Some Fractal Sets Built On Curves

The work to come in Chapter 4 of this thesis is part of a broader program, developed by my advisor Michel L. Lapidus, to view fractals as generalized manifolds, and in particular, as suitable noncommutative spaces. This program is described in [21], [20], and [3]. This section will summarize aspects of [3], *Dirac operators and spectral triples for some fractal sets built on curves*, as they relate to—and serve as a platform for—the work in Chapter 4.

The methods employed in [3] are those described in the previous section. In particular, the objective is to recover the *known* geometry of some fractal sets using a spectral triple. However, in contrast to the treatment of the general compact Riemannian manifold, in which the Dirac operator is defined using the machinery of Geometric algebra (i.e. Clifford modules, spin structures) and connections, the applications in [3] will use the spectral triple on a circle as the basis for spectral triples on more complex sets. As a result, the geometric algebra and connections are 'invisible' in the discourse to follow—indeed, these structures collapse to triviality on the circle (which is a one dimensional manifold).

In [3], an additional definition associated to a spectral triple is used to define the *metric* dimension of the spectral triple. This is a generalization of the dimension of a manifold—and indeed, in the case of the compact Riemannian manifold, recovers the dimension of the manifold [5]. As is noted in the previous section, this information is contained in the pairing of the Dirac operator and the Hilbert space, in the form of the asymptotics of the eigenvalues of the Dirac operator:

Definition 3. *Let D be the Dirac operator associated to the spectral triple in Definitions 1 and 2. If $Tr((I + D^2)^{-p/2}) < \infty$ for some positive real number p, then the spectral triple is called **p-summable** or just **finitely summable**. The number ∂_{ST}, given by*

$$\partial_{ST} = \inf\{p > 0 : tr(D^2 + I)^{\frac{-p}{2}} < \infty\},$$

*is called the **metric dimension** of the spectral triple ([3]).*

3.6.1 Circle Triple

Let C_r denote the circle with radius $r > 0$. In [3], the **natural spectral triple** for the circle $ST_n(C_r) = (AC_r, H_r, D_r)$ is defined as follows:

I. AC_r is the algebra of complex continuous $2\pi r$-periodic functions on $\mathbb{R}$;

II. $H_r = L^2([-\pi r, \pi r], (1/2\pi r)\mu)$;

III. $D_r = -i\frac{d}{dx}$;

IV. The representation π sends elements of AC_r to multiplication operators on H_r.

Note that H_r has a canonical orthonormal basis given by $exp\left(\frac{ikx}{r}\right)$. The operator D_r is actually defined as the closure of the restriction of the above operator to the linear span of the basis. Then D_r is self-adjoint and

$$[D_r, \pi_r(f)] = \pi_r(-if') \quad \text{or just} \quad -if'.$$

for any C^1 $2\pi r$-periodic function f on $\mathbb{R}$. Thus the natural spectral triple is a spectral triple, and the eigenvalues of the Dirac operator are given as $\lambda_i = k/r$ for $k \in \mathbb{Z}$.

To use the circle triple as basis for construction of spectral triples on more complex sets, it will be necessary to take countable sums of circle triples. To avoid the problem of having 0 as an eigenvalue with infinite multiplicity, the translated spectral triple is constructed [3]:

1. Let $D_r^t = D_r + \frac{1}{2r}I$.

2. $ST_t(C_r) = (AC_r, H_r, D_r^t)$ is called the **translated spectral triple** for the circle.

The set of eigenvalues becomes $\{(2k+1)/2r : k \in \mathbb{Z}\}$, but the domain of definition stays the same and most importantly, as to not change the effect of the spectral triple,

$$[D_r^t, \pi_r(f)] = [D_r, \pi_r(f)].$$

Let d_c be the geodesic distance function on the circle. Theorem 2.4 in [3] gives the following results:

- The metric induced by the spectral triple $ST_n(C_r)$ coincides with the geodesic distance on C_r, i.e.,

$$d_c(s,t) = \sup\{|f(t) - f(s)| : ||[D_r, \pi_r(f)]|| \leq 1\};$$

- The circle triple is p-summable for any real $s > 1$ but not summable for $s = 1$, thus the metric dimension of the spectral triple is 1, coinciding with the dimension of a circle.

3.6.2 Interval Triple

The interval is studied by means of the circle—by taking two copies of the interval and gluing the endpoints together. There is an injective homomorphism Φ from the continuous functions on an interval $[0, \alpha]$ to the continuous functions on $[-\alpha, \alpha]$ defined by

$$\Phi_\alpha(f)(t) = f(|t|).$$

The circle triple $(AC_{\alpha/\pi}, H_{\alpha/\pi}, D^t_{\alpha/\pi})$ is then used to describe the spectral triple for $C([0, \alpha])$. The fact that the following definition indeed defines a spectral triple follows immediately from the results on the circle:

For $\alpha > 0$, the **α-interval spectral triple** $ST_\alpha = (\mathcal{A}_\alpha, H_\alpha, D_\alpha)$ is given by the following:

i. $\mathcal{A}_\alpha = C([0, 1])$;

ii. $\mathcal{H}_\alpha = L^2([-\alpha, \alpha], m/2\alpha)$, where $m/2\alpha$ is the normalized Lebesgue measure;

iii. the representation $\pi_\alpha : \mathcal{A}_\alpha \to B(\mathcal{H}_\alpha)$ is defined for f in $\mathcal{A}_\alpha$ as the multiplication operator which multiplies by the function $\Psi_\alpha(f)$;

iv. an orthonormal basis $\{e_k : k \in \mathbb{Z}\}$ for $\mathcal{H}_\alpha$ is given by $e_k = exp(i\pi kx/\alpha)$ and D_α is the self-adjoint operator on $\mathcal{H}_\alpha$ which has all the vectors e_k as eigenvectors and such that $D_\alpha e_k = (\pi k/\alpha)e_k$ for each $k \in \mathbb{Z}$.

Let $d_\alpha(s,t) = |s-t|$ be the geodesic distance for the α-interval. Results for the α-interval spectral triple which follow immediately from the results for the circle are stated in Theorem 3.3 in [3]:

- The metric induced by the α-interval triple coincides with the geodesic distance for the α-interval, i.e.,

$$d_c(s,t) = \sup\{|f(t) - f(s)| : ||[D_\alpha, \pi_\alpha(f)]|| \leq 1\};$$

- The α-interval triple is p-summable for $s > 1$ but not summable for $s = 1$, thus it has metric dimension 1, coinciding with the dimension of the α-interval.

3.6.3 Curve Triple

Let T be a compact Hausdorff space and $r : [0,\alpha] \to T$ a continuous injective mapping. The image in T will be called the continuous curve and r the parameterization. The **r-curve triple**, ST_r, is given by the interval triple as follows:

Let r be as above and $(A_\alpha, H_\alpha, D_\alpha)$ be the α-interval spectral triple. Then $ST_r = (C(T), H_\alpha, D_\alpha)$ is an unbounded Fredholm module with representation $\pi_r : C(T) \to B(H_\alpha)$ defined via a homomorphism ϕ_r of $C(T)$ onto A_α given by

a. For all $f \in C(T)$, for all $t \in [0,\alpha]$: $\phi_r(f)(t) \equiv f(r(t))$;

b. For all $f \in C(T)$, $\pi_r(f) \equiv \pi_\alpha(\phi_r(f))$.

As is expected, the curve triple is summable for $s > 1$ but not $s = 1$ so the metric dimension is 1 (see Proposition 4.1 in [3]) One can recover a metric distance on the image of the curve in T, of course dependent of parameterization, and pending a metric already on T, and the desire or not to coincide with that metric (see Proposition 4.3 in [3]).

3.6.4 Sum of Curve Triples

The generalization of the result for a curve to a finite collection of curves in T is achieved by a direct sum of spectral triples, with one possible obstruction, and that is there may not exist a dense set of functions which have bounded commutators with the Dirac operator simultaneously. This cannot happen if the curves do not overlap except at finitely many points [3].

Suppose $\{r_i\}_{i=1}^h$ is a finite collection of curves in T such that for each $i \neq j$, the number of points in $r_i([0, \alpha_i]) \bigcap r_j([0, r_j([0, \alpha_j])$ is finite. The triple for the sum of curves is given by

$$\bigoplus_{i=1}^{h} ST_{r_i} = \left(C(T), \bigoplus_{i=1}^{h}, \bigoplus_{i=1}^{h} \right).$$

Proposition 5.1 in [3] states that $\bigoplus_{i=1}^h ST_{r_i}$ is an unbounded Fredholm module for $C(T)$. If in addition $\bigcup_{i=1}^h r_i([0, \alpha_i]) = T$, then the representation will be faithful.

3.6.5 Trees and Graphs

Definition 6.3 in [3], defines a spectral triple on a weighted graph with parameterization in a compact metric space T. If P is the collection of points of the graph then the direct sum of spectral triples for the edges is an unbounded Fredholm module over $C(T)$ and a spectral triple for $C(P)$.

Proposition 6.4 in [3] states that the distance function induced by the spectral triple/unbounded Fredholm module coincides with the geodesic distance function on the graph, inherited from the metric on T.

The authors of [3] point out that if the graph is not a tree, then it contains at least one cycle. A spectral triple can be added for each cycle without changing the the distance function induced by the sum of triples. This augmented triple induces an element in the K-homology of the graph—an element of the K-homology group will be able to measure the winding number of a nonzero continuous function around the circle—taking one such summand for each cycle allows one to keep track of the connectedness type of the graph [3], [5].

In [3], an analogous result holds for *finitely summable* or *p-summable* trees. Infinite trees in general can not be summed up as above, but the summability hypothesis allows for an analogous result as Proposition 6.4 in (See Theorem 7.9 in [3]).

3.6.6 Sierpinski Gasket

The construction of the spectral triple on the Sierpinski gasket comes directly from the circle triple. Indeed, the Sierpinski gasket is naturally decomposed into triangles. In fact, for each graph approximation, there are 3^n equilateral triangles, each of length $(\frac{1}{2})^n$. The spectral triple for each triangle can be obtained readily from the circle triple. In [3], Definition 8.1 gives the precise definition of each circle triple used, and the definition of the countable direct sum of these triples. Let the Sierpinski gasket be K. The idea is for each triangle in the gasket, $\Delta_{n,i}$ ($n \in \mathbb{N}$, $1 \le i \le 3^n$), to construct the unbounded Fredholm module for $C(K)$,

$$ST_{n,i}(K) = (C(K), H_{n,i}, D_{n,i}),$$

based on the isometry of the circle of radius 2^{-n} onto $\Delta_{n,i}$ for each i. Here $H_{n,i} = H_{2^{-n}\pi}$ for all i and $D_{n,i} = D^t_{2^{-n}\pi}$ for all i. As usual the continuous functions on this circle are given by $C([-2^{-n}\pi, 2^{-n}\pi])$ and the mapping from the interval onto the triangle induces a surjective homomorphism of $C(K)$ onto $C([-2^{-n}\pi, 2^{-n}\pi])$. This defines a representation $\pi_{n,i} : C(K) \to B(H_{n,i})$. Theorem 8.2 in [3] states that the countable sum of these triples,

$$\bigoplus_{n,i} ST_{n,i} = \left(C(K), \bigoplus_{n,i} H_{n,i}, \bigoplus_{n,i} D_{n,i} \right),$$

is a spectral triple for the Sierpinski gasket.

Dimension

Theorem 8.2 also states that ST_K is summable iff $s > log3/log2$ (see also Theorem 8.4 in [3]). Therefore its metric dimension is $log3/log2$ which coincides with the Minkowski and Hausdorff dimensions of the Sierpinski gasket.

Volume

Let μ is the Hausdorff probability measure on K and $D_K = \bigoplus_{n,i} D_{n,i}$. To recover the volume form or integration on K, the Dixmier trace is used and Theorem 8.7 in [3] states that if τ is the functional on $C(K)$ given by

$$\tau(f) = Tr_\omega \left(\pi_K(f) |D_K|^{-\frac{log3}{log2}} \right),$$

then for any $f \in C(K)$,

$$\tau(f) = \left[\frac{4}{log3} \zeta \left(\frac{log3}{log2} \right) \right] \int_K f(x) d\mu(x).$$

Distance

Let d_K be the geodesic distance on K (i.e. the Euclidean induced distance on K). Theorem 8.13 in [3] states that

$$d_K(p,q) = \sup\{|f(p) - f(q)| : ||[D_K, \pi_K(f)]|| \leq 1\},$$

and hence the metric induced by the spectral triple coincides with the geodesic distance metric.

Chapter 4

Spectral Triples and Measurable Riemannian Geometry

4.1 Overview and Notation

The objective of this chapter is to extend the methods of Chapter 3 to K_H, the harmonic gasket. In particular, we want to recover Kigami's geodesic distance on K_H. There will be two distinct constructions of a spectral triple on K_H, both of which will recover Kigami's geodesic distance. One construction will be analogous to that on the Sierpinski gasket in Chapter 3—that is a construction which is essentially a countable sum of circle triples. The other will be countable sum of curve triples. The curve triple construction will generalize readily to a certain class of sets built on curves in $\mathbb{R}^2$. Chapter 4 will conclude with a discussion of work in progress and future directions.

In order to construct a Dirac operator and Spectral triple on the harmonic gasket as a countable sum of triples, it is necessary to recall some notation as well as add some more notation. For q_1, q_2, and q_3, the boundary points of K_H, and $x \in K_H$, our integrated function system is given by

$$H_i(x) = T_i(x - q_i) + q_i \text{ for } i = 1, 2, 3.$$

which yields the relation,

$$K_H = \bigcup_{i=1}^{3} H_i(K_H).$$

Equivalently, we have K_H given by the harmonic coordinate chart for the Sierpinski gasket, K,

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right),$$

where $K_H = \Phi[K]$. Recall that Γ_0 is the initial graph approximation to K. For any word $w \in (1, 2, 3)^{\mathbb{N}}$, $\Gamma_w = F_w[\Gamma_0]$ is the w^{th}-graph cell of K. Let the initial *graph* approximation to K_H be $\Phi[\Gamma_0]$ and the w^{th}-*graph* cell of K_H be given by $\Phi[\Gamma_w]$, or equivalently by $H_w[\Phi[\Gamma_0]]$.

The term *graph* is used less restrictively in this setting as the curves which are the edges carry more information than abstract graph edges. Nevertheless, we will refer to these objects as graph cells of K_H and define the following notation: $\mathcal{T}_0 = \Phi[\Gamma_0]$, $\mathcal{T}_w = H_w[\Phi[\Gamma_0]]$. For $|w| = n$, the union of the 3^n graph cells of K_H form the n^{th}-*level graph approximation to* K_H. Analogously to K, K_H is then given by the closure of the union of the graph cells over $n \in \mathbb{N}$ and $1 \leq i \leq 3^n$. This decomposition of K_H will be mirrored in the construction of the spectral triple on K_H from the countable sum of spectral triples on the graph cells of K_H.

There is an aesthetic to this type of construction as mentioned in [3] with respect to the analogous construction of a spectral triple on K in that it *keeps track of all of the holes* in K. Since K_H is homeomorphic to K, the same will be true for the spectral triple on K_H. This being said, it will be convenient to decompose K_H a little further due to the lack of self-similarity of K_H. For Γ_0, identifying a left side L, a right side R, and a bottom side B, yields the natural notation $\Gamma_{0,s}$ for $s \in \{L, R, B\}$. Similarly, $\Gamma_{w,s}$ will be the s^{th} side of Γ_w.

Note that $F_w[\Gamma_{0,s}] = \Gamma_{w,s}$ for any side s. This observation allows one to label the three constituent curves connecting the three vertices of $\mathcal{T}_w$ by defining $\mathcal{T}_{w,s} = \Phi[\Gamma_{w,s}]$ (or equivalently by $\mathcal{T}_{w,s} = H_w[\Phi[\Gamma_{0,s}]]$). Tracking the vertices is therefore useful. Let $U_0 = \{q_1, q_2, q_3\}$, $U_w = H_w(U_0)$, and $U_n = \bigcup_{|w|=n} U_w$. Then U_w is exactly the set of vertices of $\mathcal{T}_w$, so that the curve $T_{w,s}$ can be associated to two of the three vertices of $\mathcal{T}_w$, say $H_w(q_i)$ and $H_w(q_j)$, where $i \neq j$.

Finally, since several of Jun Kigami's results in [12] will be used during this chapter, it is useful to include some of his notation relevant to those results. Let J be the convex hull of U_0 (i.e. the regular triangle with vertices U_0) and $J_w = H_w(J_w)$ and $K_{H,w} = H_w(K_H)$, called a w-cell of K_H (note that in contrast with a graph cell, $K_{H,w}$ is an affine transformation of the entire harmonic gasket). Let $p, q \in U_w$ with $p \neq q$ and define $\widehat{pq}$ by $\widehat{pq} = \Phi(\overline{p_* q_*})$, where $p_* = \Phi^{-1}(p)$ and $q^* = \Phi^{-1}(q)$ and $\overline{xy}$ is the (Euclidean) line segment between x and y. Note that $\mathcal{T}_{w,s} = \widehat{pq}$ for some $p, q \in U_w$.

4.2 Unbounded Fredholm Module on the Graph Cell $\mathcal{T}_w$

The spectral triple on $\mathcal{T}_w$ will be constructed from spectral triples on its three constituent *edge* curves. Namely, $\mathcal{T}_w = \bigcup_{s\in\{L,R,B\}} \mathcal{T}_{w,s}$ and we consider the curve $\mathcal{T}_{w,s}$ parameterized by its arclength, $\alpha_{w,s}$, in M_0 (i.e. the copy of $\mathbb{R}^2$ where K_H lives). The collection of edges is countable, so when it is convenient, we will refer to $T_{w,s}$ as T_{E_i} for $i \in \mathbb{N}$. Since $\mathcal{T}_{w,s} = \widehat{pq}$ for $p, q \in U_w$, we have that $\alpha_{w,s}$ is finite by Lemma 5.5 in [12].

Remark 1. *Lemma 5.5 follows immediately from Theorem 5.4 in [12], which shows that $\widehat{q_1q_2}$ (where $q_1, q_2 \in U_0$), is such that $\widehat{q_1q_2} \bigcup \overline{q_1q_2}$ is a closed curve which bounds a convex region. Since the region associated to arbitrary vertices p and q is an affine transformation of the former region, the result follows. Theorem 5.4 is proven in [12] but it is noted by Kigami that the results were previously announced by Teplyaev in [32].*

Following the construction in Proposition 4.1 in [3], let $r_{w,s}$ be a paremateriza-tion of $\mathcal{T}_{w,s}$, by its arclength, $\alpha_{w,s}$. The (w, s) subscripts will be suppressed temporarily for simplicity. In this case, $r : [0, \alpha] \to K_H$ is a continuous (in fact, C^1; again by Theorem 5.4 in [12]) and injective mapping. Note that K_H is a compact metric space. Let $(\mathcal{A}_\alpha, \mathcal{H}_\alpha, D_\alpha)$ be the α-interval spectral triple in [3]. Recall that for $\alpha > 0$, the **α-interval spectral triple** as defined in Definition 3.1 in [3] is given by:

1. $\mathcal{A}_\alpha = C([0, 1])$;
2. $\mathcal{H}_\alpha = L^2([-\alpha, \alpha], m/2\alpha)$, where $m/2\alpha$ is the normalized Lebesgue measure;
3. the representation $\pi_\alpha : \mathcal{A}_\alpha \to B(\mathcal{H}_\alpha)$ is defined for f in $\mathcal{A}_\alpha$ as the multiplication operator which multiplies by the function $\Psi_\alpha(f)$;

4. an orthonormal basis $\{e_k : k \in \mathbb{Z}\}$ for $\mathcal{H}_\alpha$ is given by $e_k = exp(i\pi kx/\alpha)$ and D_α is the self-adjoint operator on $\mathcal{H}_\alpha$ which has all the vectors e_k as eigenvectors and such that $D_\alpha e_k = (\pi k/\alpha)e_k$ for each $k \in \mathbb{Z}$.

Remark 2. $\Psi_\alpha : C([0,\alpha]) \to C([-\alpha,\alpha])$ *is an injective homomorphism given by* $\Psi_\alpha(f)(t) = f(|t|)$, *for each* $f \in C([0,\alpha])$ *and each* $t \in [-\alpha,\alpha]$. *The Dirac operator,* D_α, *is the closure of the restriction of the operator* $\frac{1}{i}\frac{d}{dx}$ *to the linear span of* $\{e_k : k \in \mathbb{Z}\}$. *Also note in (4) above that the set of eigenvalues of the* D_α, *is* $\{\pi k/\alpha : k \in \mathbb{Z}\}$.

Consider the triple defined by $ST_r = (C(K_H), \mathcal{H}_\alpha, D_\alpha)$ with representation $\pi_r : C(K_H) \to B(\mathcal{H}_\alpha)$ defined via a homomorphism ϕ_r of $C(K_H)$ onto $\mathcal{A}_\alpha$ as follows:

1. for all $f \in C(K_H)$, for all $t \in [0,\alpha] : \phi_r(f)(t) \equiv f(r(t))$;

2. for all $f \in C(K_H)$, $\pi_r(f) \equiv \pi_\alpha(\phi_r(f))$.

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By Proposition 4.1 in [3], ST_r is a spectral triple, with a slight abuse–the representation is not faithful since two continuous functions can agree on $\mathcal{T}$ but not on all of K_H. Technically, it is an unbounded Fredholm module. This could be addressed by using $C(r[0,\alpha])$ instead of $C(K_H)$ but is not necessary. Ultimately, the images of these curves will be K_H and the resulting representation will be faithful.

By Proposition 4.3 in [3], the metric, d_r, induced by the spectral triple on $\mathcal{T}$, is given for $x \neq y \in \mathcal{T}$, by

$$d_r(x,y) = |r^{-1}(x) - r^{-1}(y)|.$$

Suppose p and q are the vertices in U_w such that $\mathcal{T}=\widehat{pq}$. Let D_{pq} be the compact convex region bounded by $\widehat{pq} \bigcup \overline{pq}$. Let $\widetilde{pq}$ be any rectifiable Jordan curve in K_H, connecting p and q and take D'_{pq} to be the compact region bounded by $\widetilde{pq} \bigcup \overline{pq}$. Since Φ is a homeomorphism and thus preserves the holes, and hence the interior and exterior of K—we have that $(D_{pq} - \widehat{pq}) \bigcap K_H$ is empty.

It therefore holds that $D_{pq} \subset D'_{pq}$ and by Theorem 5.2 in [12], with respect to $\mathbb{R}^2$, the arclength of $\widehat{pq} \bigcup \overline{pq}$ is less than or equal to the arclength of $\widetilde{pq} \bigcup \overline{pq}$. Subtracting off the segment, $\overline{pq}$, the two boundaries have in common, yields the result that the arclength of $\widehat{pq}$ is less than or equal to the arclength of $\widetilde{pq}$.

The same argument holds for arbitrary p and q on $\mathcal{T}$. In short, Kigami's harmonic shortest path or *geodesic* between any two points on $\mathcal{T}$ *is a path on* $\mathcal{T}$. By design, r is a parameterization of $\mathcal{T}$ by its arclength α. Therefore, for any points $x, y \in \mathcal{T}$

$$|r^{-1}(x) - r^{-1}(y)| = l_{\mathcal{T}}(x, y) = d_{geo}(x, y),$$

where $l_{\mathcal{T}}$ is the $\mathbb{R}^2$-induced arclength between points on $\mathcal{T}$ and d_{geo} is Kigami's geodesic distance metric. It is worth noting that reproducing the arclength metric of $\mathcal{T}$ in $\mathbb{R}^2$ using a spectral triple readily gives the first equality but it is only due to the special convexity properties of the cells of K_H that the second equality holds.

Regarding the eigenvalues of D_α, we note that 0 is an eigenvalue of D_α. We will eventually have a countable sum of Dirac operators, all of which have 0 for an eigenvalue, leading to 0 having infinite multiplicity as an eigenvalue of the Dirac operator on K_H. Following [3], this can be avoided by *translating* the Dirac operator to $D^t_\alpha = D_\alpha + (1/2\alpha)I$.

Since the commutator eliminates the $(1/2\alpha)I$, we have $[D^t_\alpha, \pi_\alpha(f)] = [D_\alpha, \pi_\alpha]$ and thus $||[D^t_\alpha, \pi_\alpha(f)]|| = ||[D_\alpha, \pi_\alpha]||$, so that the only effect on the triple the translate has is that the set of eigenvalues of D^t_α is given by $\{(2k+1)\pi/2\alpha : k \in \mathbb{Z}\}$. In this case 0 is not an eigenvalue.

Recalling that the definition of *metric* or *spectral* dimension, ∂_{ST}, in [3] is

$$\partial_{ST} = \inf\{p > 0 : tr(D^2 + I)^{\frac{-p}{2}} < \infty\},$$

it follows from Proposition 4.1 in [3] the spectral dimension of $\mathcal{T}_{w,s}$ is 1. Of course, this is clear since the eigenvalues of D^t_α have multiplicity one so that the trace of their reciprocals forms a harmonic series.

Remark 3. *Earlier, to emphasize the relationship of the length α of a curve, we kept the α subscript and suppressed the (w, s) subscripts. Now we will do the opposite, since (w, s) determines $\alpha_{w,s}$, so long as it is understood that we are always parameterizing by arclength.*

Summarizing the results for the edge curves, $\mathcal{T}_{w,s}$, we have the following:

Proposition 1. *The triple $ST_{\mathcal{T}w,s} = (C(K_H), \mathcal{H}_{w,s}, D^t_{w,s})$ associated to $\mathcal{T}_{w,s}$ is an unbounded Fredholm module with the following properties:*

1. *The spectrum of the Dirac operator, $D^t_{w,s}$ is given by $\{(2k+1)\pi/2\alpha_{w,s} : k \in \mathbb{Z}\}$.*
2. *The metric induced by $ST_{\mathcal{T}w,s}$ on $\mathcal{T}_{w,s}$ coincides with the geodesic distance on K_H.*

3. The spectral dimension of $\mathcal{T}_{w,s}$ is 1.

For the first of two constructions of a triple on the graph cell $\mathcal{T}_w$, we will use the three edge triples, whose associated edges form $\mathcal{T}_w$. To construct the unbounded Fredholm module on $\mathcal{T}_w$, we define the triple

$$ST_{\mathcal{T}w} = \left(C(K_H), \bigoplus_{s\in\{L,R,B\}} \mathcal{H}_{w,s}, \bigoplus_{s\in\{L,R,B\}} D^t_{w,s} \right),$$

with representation $\pi_w = \bigoplus_{s\in\{L,R,B\}} \pi_{w,s}$. By Proposition 5.1 in [3], $ST_{\mathcal{T}w}$ is an unbounded Fredholm module, but it also follows from Proposition 1 above. Indeed, the real-valued linear functions on $\mathbb{R}^2$, restricted to K_H separate points of K_H and therefore are dense in $C(K_H)$. By the Stone-Wierstrass theorem, these functions are dense in $C(K_H)$.

Let $f(x, y) = ax + by$ for $a, b \in \mathbb{R}$ and $x, y \in R^2$. Let $\mathcal{T}_w$ be parameterized by its arclength by $r : [0, \Sigma_{s\in\{L,R,B\}}\alpha_{w,s}] \to \mathcal{T}_w$ and be given by $r(\tau) = (x(\tau), y(\tau))$. Then r is C^1, except at possibly the three points where the curves are glued together. Thus $||r'(\tau)||$ is defined and $||r'(\tau)|| = 1$ except at possibly those three points. Let $D_w = \bigoplus_{s\in\{L,R,B\}} D^t_{w,s}$ and $||.||_\infty$ be the essential supremum of functions on $[0, \Sigma_{s\in\{L,R,B\}}\alpha_{w,s}]$.

Using the fact that the operator norm of a multiplication operator is given by its essential supremum as a function, we compute

$$||[D_w, \pi_w(f)]|| = ||D_w(f)|| = ||D_w(f)||_\infty = \left|\left|\frac{1}{i}\frac{df}{d\tau}\right|\right|_\infty = ||a(x'(\tau)) + b(y'(\tau))||_\infty$$

$$\leq |a| + |b|,$$

where the last inequality follows from $|x'(\tau)| \leq 1$ and $|y'(\tau)| \leq 1$. Therefore the linear real-valued functions are dense in $C(K_H)$ and have bounded commutators with D_w. To see that the operator $(D_w^2 + I)^{-1}$ is compact we note that the eigenvalues of D_w are the disjoint union of the eigenvalues of $D_{w,s}^t$. Applying the spectral theorem, the eigenvalues of $(D_w^2 + I)^{-1}$ are given by

$$\left[\left(\frac{(2k+1)\pi)}{2\alpha_{w,s}} \right)^2 + 1 \right]^{-1},$$

for $k \in \mathbb{Z}$ and $s \in \{L, R, B\}$. Arranged in nonincreasing order, these eigenvalues go to zero(the multiplicities are at most 3). In this case the image of the unit ball of our Hilbert space under $(D_w^2 + I)^{-1}$ is totally bounded and therefore pre-compact. Hence, $(D_w^2 + I)^{-1}$ is compact and $ST_{\mathcal{T}w}$ is the unbounded Fredholm module associated to the graph cell $\mathcal{T}_w$.

We now consider the metric distance on $\mathcal{T}_w$ induced by the spectral triple, given by

$$d_w(p, q) = \sup\{|f(p) - f(q)| : ||[D_w, \pi_w(f)]|| \leq 1\}.$$

By previous calculations, this can be rewritten as

$$d_w(p, q) = \sup\{|f(p) - f(q)| : ||[D_w(f)||_\infty \leq 1\}.$$

Let $l_{\mathcal{T}w}(p, q)$ for $p, q \in \mathcal{T}_w$ be the $\mathbb{R}^2$ induced arclength distance on $\mathcal{T}_w$, or in other words the geodesic distance on $\mathcal{T}_w$. Let $\mathrm{Lip}_l(f)$ be the Lipschitz norm on the restriction of functions in $C(K_H)$ to $\mathcal{T}_w$ with respect to the metric $l_{\mathcal{T}w}(p, q)$, i.e.,

$$\mathrm{Lip}_l(f) = \sup\left\{\frac{|f(p)-f(q)|}{l_{\mathcal{T}w}(p,q)} : p,q \in \mathcal{T}_w\right\}.$$

Note that $||D_w(f)||_\infty = \mathrm{Lip}_l(f)$. The distance induced by the spectral triple is then given by

$$d_w(p,q) = \sup\{|f(p)-f(q)| : \mathrm{Lip}_l(f) \leq 1\},$$

and it is immediately clear that $d_w \leq l_{\mathcal{T}w}$. Conversely, fix any $p \in \mathcal{T}_w$ and let $f(x) = l_{\mathcal{T}w}(p,x)$. Then $\mathrm{Lip}_l(f) \leq 1$ by the triangle inequality for $l_{\mathcal{T}w}$ (also note that letting $p = x$, it is clear that $\mathrm{Lip}_l(f) = 1$). Therefore f is witness to $d_w \geq l_{\mathcal{T}w}$ and it is shown that $d_w = l_{\mathcal{T}w}$.

To compute the spectral dimension of $\mathcal{T}_w$, we note that it is the same situation as with $\mathcal{T}_{w,s}$, except that the eigenvalues may have multiplicities up to 3. In this case the harmonic sum becomes at most 3 harmonic sums, and therefore the spectral dimension of $\mathcal{T}_w$ is 1. Letting $\mathcal{H}_w = \bigoplus_{s\in\{L,R,B\}} \mathcal{H}_{w,s}$, we have the following result:

Proposition 2. *The triple $ST_{\mathcal{T}w} = (C(K_H), \mathcal{H}_w, D_w)$ associated to $\mathcal{T}_w$ is an unbounded Fredholm module with the following properties:*

1. *The spectrum of the Dirac operator, D_w, is given by*

$$\left\{\left[\left(\frac{(2k+1)\pi}{2\alpha_{w,s}}\right)\right] : k \in \mathbb{Z}, s \in \{L,R,B\}\right\}.$$

2. *The metric d_w induced by $ST_{\mathcal{T}w}$ on $\mathcal{T}_w$ coincides with the $\mathbb{R}^2$ induced arclength metric $l_{\mathcal{T}w}$ on $\mathcal{T}_w$;*

3. *The spectral dimension of $\mathcal{T}_w$ is 1.*

Remark 4. *Proposition 2 does not state that the d_w coincides with Kigami's geodesic metric on K_H, because in general it will not. Points on different sides of $\mathcal{T}_w$ will be connected by a geodesic that does not lie completely on $\mathcal{T}_w$ and thus $d_w \geq d_{geo}$. However, when p_1 and p_2 belong to the same edge of a graph cell, it is clear from Proposition 1 and Proposition 2, that $d_w(p_1, p_2) = d_{geo}(p_1, p_2)$.*

4.3 Alternate Construction of the $\mathcal{T}_w$ Triple

Another way to construct a spectral triple on $\mathcal{T}_w$ is to carry the spectral triple on a circle directly to $\mathcal{T}_w$ as is done in [3] for an arbitrary graph cell of the Sierpinski gasket. Let r be the radius of a circle. Since it is the complex continuous functions on the circle that are of interest, we make the natural identification with the complex continuous $2\pi r$-periodic functions on the real line. Let the $\mathbb{R}^2$ induced arclength of $\mathcal{T}_w$ be α_w.

Considering a circle of radius α_w, the appropriate algebra of functions are the complex continuous $2\pi\alpha_w$-periodic functions on the real line. Let $r_w : [-\pi\alpha_w, \pi\alpha_w] \to \mathcal{T}_w$ be an arclength parameterization of $\mathcal{T}_w$, counterclockwise, with $r_w(0)$ equal to the vertex joining the bottom and right sides of $\mathcal{T}_w$. The mapping r_w induces a surjective homomorphism Ψ_w of $C(K_H)$ onto $C([-\pi\alpha_w, \pi\alpha_w])$ by

$$\Psi_w(f)(\tau) = f(r_w(\tau)),$$

for $f \in C(K_H)$ and $\tau \in [-\pi\alpha_w, \pi\alpha_w]$ by Definition 8.1 in [3]. Let

$$\mathfrak{H}_w = L^2([-\pi\alpha_w, \pi\alpha_w], (1/2\pi\alpha_w)m),$$

where m is the Lebesgue measure on $[-\pi\alpha_w, \pi\alpha_w]$ and let $\Pi_w : C(K_H) \to B(\mathfrak{H}_w)$ be the representation of f in $C(K_H)$ as the multiplication operator which multiplies by $\Psi_w(f)$. We will again use the translated Dirac operator and define $\mathfrak{D}_w = D^t_{\alpha w}$.

The triple $\mathfrak{ST}_{\mathcal{T}w} = (C(K_H), \mathfrak{H}_w, \mathfrak{D}_w)$ is an unbounded Fredholm module with representation Π_w. In addition it has properties analogous $ST_{\mathcal{T}w}$ that are mentioned in the following proposition. The results in the proposition follow from the results regarding the spectral triple on a circle in Section 2 of [3]. In fact, the interval spectral triple used above is built in [3] from the circle triple, by gluing two copies of the interval together at the endpoints.

Proposition 3. *The triple $\mathfrak{ST}_{\mathcal{T}w} = (C(K_H), \mathfrak{H}_w, \mathfrak{D}_w)$ associated to $\mathcal{T}_w$ is an unbounded Fredholm module with the following properties:*

1. *The spectrum of the Dirac operator, $\mathfrak{D}_w$, is given by*

$$\left\{ \left[\left(\frac{(2k+1)\pi}{2\alpha_w} \right) \right] : k \in \mathbb{Z} \right\}.$$

2. *The metric d_w induced by $\mathfrak{ST}_{\mathcal{T}w}$ on $\mathcal{T}_w$ coincides with the $\mathbb{R}^2$ induced arclength metric $l_{\mathcal{T}w}$ on $\mathcal{T}_w$.*

3. *The spectral dimension of $\mathcal{T}_w$ is 1.*

Remark 5. *Please note that these results mimic the results in Proposition 2, except for the eigenvalues of the Dirac operator in each case. D_w has eigenvalues that are proportional to the reciprocals of the side lengths, $\alpha_{w,s}$, of $\mathcal{T}_w$, while $\mathfrak{D}_w$ has eigenvalues that are proportional to the reciprocal of α_w, the length of $\mathcal{T}_w$. It is not known if the two sets of eigenvalues coincide—i.e. the ratios of the side lengths to each other is not known. The lack of self-similarity of K_H makes this question hard to answer.*

Remark 6. *Note that the remark following Proposition 2 holds here, and in particular, by Proposition 1 and Proposition 3, the spectral distance is greater than or equal to Kigami's geodesic distance but they coincide when restricted to edges of graph cells.*

4.4 Spectral Triple on K_H via the Countable Sum of the $ST_{\mathcal{T}w}$ Triples

In this section, we construct the countable sum of the $ST_{\mathcal{T}w}$ triples. More precisely, for each $n \in \mathbb{N}$ there are 3^n words such that $|w| = n$, each word w corresponding to a graph cell triple $ST_{\mathcal{T}w}$. For the countable sum of triples we will use the following notation:

1. $\mathcal{H}_{K_H} = \bigoplus_{|w|=n}^{n\in\mathbb{N}} \mathcal{H}_w$

2. $\pi_{K_H} = \bigoplus_{|w|=n}^{n\in\mathbb{N}} \pi_w$

3. $D_{K_H} = \bigoplus_{|w|=n}^{n\in\mathbb{N}} D_w$

The countable sum of graph cell triples is defined as $ST_{K_H} = (C(K_H), \mathcal{H}_{K_H}, D_{K_H})$. To see that ST_{K_H} is a spectral triple we first note that a function in the image of π_{K_H} is densely defined on K_H, so that the representation is faithful. It is clear, as before, that the real-valued linear functions on $\mathbb{R}^2$ restricted to K_H, are dense in $C(K_H)$. Also, from Proposition 2, it is clear that for any real-valued linear function, $f(x, y) = ax + by$, restricted to the graph cell $\mathcal{T}_w$, has a bounded commutator with D_w. In particular the bound is

$$||[D_w, \pi_w(f)]|| \leq |a| + |b|.$$

Note that the bound $|a|+|b|$ is *independent* of w. Therefore f has a bounded commutator with all of the Dirac operators, D_w, *simultaneously*. This shows that $||[D_{K_H}, \pi_{K_H}(f)]|| \leq |a|+|b|$ and hence the real-valued linear functions on $\mathbb{R}^2$, restricted to K_H, form a dense subset of $C(K_H)$ which have bounded commutators with D_{K_H}. To see that the operator $(D_{K_H}^2 + I)^{-1}$ is compact, we look at the collection of eigenvalues of D_{K_H} which are the disjoint union of eigenvalues of all of the D_w's, which in turn are the disjoint union of eigenvalues of all of the $D_{w,s}$'s whose eigenvalues are

$$\{(2k+1)\pi/2\alpha_{w,s} : k \in \mathbb{Z}\},$$

for w with $|w| = n$, $n \in \mathbb{N}$, $s \in \{L, R, B\}$, and $k \in \mathbb{Z}$. Thus the eigenvalues of $(D_{K_H}^2 + I)^{-1}$ are readily arranged in a non-increasing sequence which goes to zero and therefore $(D_{K_H}^2 + I)^{-1}$ is compact. In addition, one verifies that the operator is symmetric on its eigenvectors, so that it is self-adjoint.

Let d_{geo} be Kigami's geodesic distance metric on K_H and Lip_g be the Lipschitz norm on $C(K_H)$ with respect to d_{geo}. The spectral triple induced distance, d_{spec} is given by

$$d_{spec}(p, q) = \sup\{|f(p) - f(q)| : ||[D_{K_H}, \pi_{K_H}(f)]|| \leq 1\}.$$

As before, we have that the multiplication operator norm of the commutator $||[D_{K_H}, \pi_{K_H}(f)]|| = ||Df||$ coincides with $||D_{K_H} f||_{\infty, K_H}$ so that d_{spec} is equivalently given by

$$d_{spec}(p, q) = \sup\{|f(p) - f(q)| : ||D_{K_H} f||_{\infty, K_H} \leq 1\}.$$

The following lemma will be useful in comparing d_{geo} with d_{spec}:

Lemma 1. *For any function f in the domain of D_{K_H},*

$$||D_{K_H}||_{\infty,K_H} = Lip_g(f).$$

Proof. Let E represent the side of a cell given by w, s. Then

$$||D_{K_H} f||_{\infty,K_H} = \sup_E \{||D_E f||_{\infty,E}\} = \sup_E \left\{ \left|\left| \frac{1}{i} \frac{df}{dx} \right|\right|_{\infty,E} \right\}$$

$$= \sup_E \left\{ \sup_{p,q \in E} \left\{ \frac{|f(p) - f(q)|}{d_{geo}(p,q)} \right\} \right\} \leq \mathrm{Lip}_g(f).$$

The last inequality is clear since $\mathrm{Lip}_g(f)$ is the supremum over all possible non-diagonal pairs of points on the gasket which includes the non-diagonal pairs of points which are restricted to belonging to the same side of a given graph cell.

To get the inequality in the other direction, note that for $p \neq q$, and for $p \in V_*$, the set of vertices of K_H, we have a sequence of points (possibly infinite), (p_n), such that $p_0 = p$ and either $p_N = q$ for some $N \in \mathbb{N}$, or $\lim(p_n) = q$ with the property that the geodesic from p to q is a concatenation of the geodesic from p_0 to p_1 with the geodesic from p_1 to p_2 and so on. Moreover, with the exception of possibly q, the p_n's can be taken to be in V_*, the set of vertices of K_H. Let E_i denote the edge connecting p_i to p_{i+1}. Then

$$|f(p) - f(q)| = \left| \sum_{i \in \mathbb{N}} f(p_i) - f(p_{i+1}) \right| \leq \sum_{i \in \mathbb{N}} |f(p_i) - f(p_{i+1})|$$

$$\leq \sum_{i \in \mathbb{N}} (d_{geo}(p_i, p_{i+1}) ||D_{E_i} f||_{\infty, E_i}) \leq (||D_{K_H} f||_{\infty, K_H}) \sum_{i \in \mathbb{N}} d_{geo}(p_i, p_{i+1})$$

$$= ||D_{K_H} f||_{\infty, K_H} d_{geo}(p, q).$$

Therefore,

$$\frac{|f(p) - f(q)|}{d_{geo}(p, q)} \leq ||D_{K_H} f||_{\infty, K_H}.$$

For the case when neither p nor q are assumed to be in V_*, the geodesic between p and q contains as an intermediate point, a vertex $r \in V_*$. Moreover, we have a sequence $(r_i) \in V_*$ lying on the geodesic between p and q such that $r_0 = r$ and $\lim(r_i) = q$. Then by the result for a vertex to an arbitrary point we have

$$\frac{|f(p) - f(r_i)|}{d_{geo}(p, r_i)} \leq ||D_{K_H} f||_{\infty, K_H} \text{ for all } i \in \mathbb{N}.$$

By the continuity of the functions $f(x)$ and $a(x) = d_{geo}(p, x)$, we have

$$\frac{|f(p) - f(q)|}{d_{geo}(p, q)} \leq ||D_{K_H} f||_{\infty, K_H}.$$

and since p and q were arbitrary distinct points in K_H,

$$\mathrm{Lip}_g(f) \leq ||D_{K_H} f||_{\infty, K_H}.$$

■

Let $p, q \in K_H$. To compare $d_{geo}(p, q)$ with $d_{spec}(p, q)$, note that for any f such that $||D_{K_H} f||_{\infty, K_H} \leq 1$, we have by the lemma above, that $\mathrm{Lip}_g(f) = ||D_{K_H} f||_{\infty, K_H}$ and therefore,

$$\frac{|f(p) - f(q)|}{d_{geo}(p, q)} \leq 1.$$

In this case $|f(p) - f(q)| \leq d_{geo}(p, q)$, and it holds that $d_{spec}(p, q) \leq d_{geo}(p, q)$. To get the inequality in the other direction, define the the function $h(x) = d_{geo}(p, x)$. Then $\mathrm{Lip}_g(h) = 1$ and

$$|h(p) - h(q)| = |0 - d_{geo}(p, q)| = d_{geo}(p, q).$$

Therefore h is witness to $d_{geo}(p, q) \leq d_{spec}(p, q)$, and we have shown that $d_{spec}(p, q) = d_{geo}(p, q)$.

The following theorem summarizes the results for the spectral triple $ST_{K_H} = (C(K_H), \mathcal{H}_{K_H}, D_{K_H})$. Let $\mathfrak{d}_{K_H}$ denote the spectral dimension of K_H with respect to ST_{K_H}.

Theorem 1. *The triple $ST_{K_H} = (C(K_H), \mathcal{H}_{K_H}, D_{K_H})$ associated to K_H is a spectral triple with the following properties:*

1. *The spectrum of the Dirac operator, D_{K_H}, is given by*

$$\bigcup_{|w|=n}^{n \in \mathbb{N}} \left\{ \left[\left(\frac{(2k+1)\pi}{2\alpha_{w,s}} \right) \right] : k \in \mathbb{Z}, s \in \{L, R, B\} \right\}.$$

2. *The metric distance d_{spec} induced by ST_{K_H} coincides with Kigami's geodesic distance, d_{geo}.*

3. *The spectral dimension $\mathfrak{d}_{K_H}$ is the infimum of all $p > 0$ such that*

$$\sum_{w,s,k} \left[\left(\frac{(2k+1)\pi}{2\alpha_{w,s}} \right) \right]^{-p} < \infty.$$

4.5 Spectral Triple on K_H via the Countable Sum of the $\mathfrak{ST}_{\mathcal{T}w}$ Triples

We now consider a similar construction of a spectral triple on K_H, this time using the countable sum of triples $\mathfrak{ST}_{\mathcal{T}w} = (C(K_H), \mathfrak{H}_w, \mathfrak{D}_w)$. Recall that this triple comes directly from the spectral triple on the circle, as opposed to $ST_{\mathcal{T}w}$ which is built from its three edge triples. Also recall that Proposition 3 shows that the triples induce the same 'geometry' of the graph cell, though the eigenvalues of $\mathfrak{D}_w$ are related to the eigenvalues of D_w, but are not the same. In a sense, this is the natural construction of with respect to its *holes and connectedness*.

To be precise, this construction yields a spectral triple for each closed path, or *cycle*, in the space. Following the line of reasoning on page 23 of [3], each of these spectral triples associated to a cycle induces an element in the K-homology of each graph approximation of K_H. Each of these members of the K-homology group measures the winding number of a nonzero continuous function around the cycle to which it is associated, keeping track of the connectedness type of the graph approximation.

To formally construct the countable sum of $\mathfrak{ST}_{\mathcal{T}w}$ triples, we will use the following notation:

1. $\mathfrak{H}_{K_H} = \bigoplus_{|w|=n}^{n\in\mathbb{N}} \mathfrak{H}_w$
2. $\Pi_{K_H} = \bigoplus_{|w|=n}^{n\in\mathbb{N}} \Pi_w$
3. $\mathfrak{D}_{K_H} = \bigoplus_{|w|=n}^{n\in\mathbb{N}} \mathfrak{D}_w$.

The countable sum of the $\mathfrak{ST}_{\mathcal{T}w}$ triples is defined as $\mathfrak{ST}_{K_H} = (C(K_H), \mathfrak{H}_{K_H}, \mathfrak{D}_{K_H})$.

Showing that $\mathfrak{ST}_{K_H}$ is a spectral triple follows the argument used to show that ST_{K_H} is a spectral triple. Indeed, a function in the image of Π_{K_H} is densely defined on K_H, so that we have a faithful representation.

Again, the real-valued linear functions on $\mathbb{R}^2$ restricted to K_H, are dense in $C(K_H)$ and for any real-valued linear function, $f(x, y) = ax+by$, restricted to the graph cell $\mathcal{T}_w$, has a bounded commutator with $\mathfrak{D}_w$ with bound $|a| + |b|$, independent of w. Thus $||[\mathfrak{D}_{K_H}, \pi_{K_H}(f)]|| \le |a| + |b|$ and hence the real-valued linear functions on $\mathbb{R}^2$, restricted to K_H, form a dense subset of $C(K_H)$ which have bounded commutators with $\mathfrak{D}_{K_H}$.

To see that the operator $(\mathfrak{D}_{K_H}^2 + I)^{-1}$ is compact, we look at the eigenvalues of $\mathfrak{D}_{K_H}$ which are given by the disjoint union of eigenvalues of all of the $\mathfrak{D}_w$'s,

$$\{(2k + 1)\pi/2\alpha_w\},$$

for w with $|w| = n$, $n \in \mathbb{N}$, and $k \in \mathbb{Z}$. As mentioned before, the α_w's are the lengths of the boundaries of the w-cells. In this case, the eigenvalues of $(\mathfrak{D}_{K_H}^2 + I)^{-1}$ go to zero and therefore it is compact. In addition, one verifies that $\mathfrak{D}_{K_H}$ is symmetric on its eigenvectors, so that it is self adjoint.

To compare the spectral distance function induced by $\mathfrak{ST}_{K_H}$ with d_{geo}, in light of previous arguments, we have the following analog to Lemma 1, which characterizes $||\mathfrak{D}_{K_H}||_{\infty,K_H}$ in terms of d_{geo}.

Lemma 2. *For any function f in the domain of $\mathfrak{D}_{K_H}$,*

$$||\mathfrak{D}_{K_H}||_{\infty,K_H} = Lip_g(f).$$

Proof. To prove the first inequality, note that

$$||\mathfrak{D}_{K_H} f||_{\infty,K_H} = \sup_w \{||\mathfrak{D}_w f||_{\infty,\mathcal{T}w}\} = \sup_w \left\{ \left|\left| \frac{1}{i}\frac{df}{dx} \right|\right|_{\infty,\mathcal{T}w} \right\}$$

$$= \sup_w \left\{ \sup_{p,q\in\mathcal{T}w} \left\{ \frac{|f(p)-f(q)|}{d_w(p,q)} \right\} \right\} \leq \sup_w \left\{ \sup_{p,q\in\mathcal{T}w} \left\{ \frac{|f(p)-f(q)|}{d_{geo}(p,q)} \right\} \right\}$$

$$\leq \mathrm{Lip}_g(f),$$

since $d_w \geq d_{geo}$. The last inequality holds since $\mathrm{Lip}_g(f)$ is the supremum over all possible non-diagonal pairs of points on the gasket which includes the non-diagonal pairs of points restricted to belonging to the same graph cell. To achieve the reverse inequality, we note that the critical inequality used to get this direction in Lemma 1 was

$$|f(p_i) - f(p_{i+1})| \leq d_{geo}(p_i, p_{i+1})||D_{E_i} f||_{\infty,E_i}$$

where the p_i's represent the decomposition of the geodesic constructed in Lemma 1, and E_i is the edge connecting p_i to p_{i+1}. Recalling the second remark following Proposition 3, we have that the spectral distance induced on $\mathcal{T}_w$ by $\mathfrak{S}\mathfrak{T}_w$ coincides with d_{geo} when restricted to edges. This is of course a sufficient condition to replace E_i with w in the above inequality. Indeed, for p_i and p_{i+1} belonging to the same edge,

$$\frac{|f(p_i) - f(p_{i+1})|}{d_{geo}(p_i, p_{i+1})} = \frac{|f(p_i) - f(p_{i+1})|}{d_w(p_i, p_{i+1})} \leq ||\mathfrak{D}_w f||_{\infty,\mathcal{T}w}.$$

Therefore, as promised,

$$|f(p_i) - f(p_{i+1})| \leq d_{geo}(p_i, p_{i+1})||\mathfrak{D}_w f||_{\infty, \mathcal{T}w}.$$

Now it follows from the argument in Lemma 1 that for an arbitrary point q and a vertex p,

$$\frac{|f(p) - f(q)|}{d_{geo}(p,q)} \leq ||\mathfrak{D}_{K_H} f||_{\infty, K_H}.$$

The extension to the case when p and q are both allowed to be arbitrary points in K_H is exactly as in Lemma 1 so it follows that

$$\mathrm{Lip}_g(f) \leq ||\mathfrak{D}_{K_H} f||_{\infty, K_H}.$$

■

Let h_{spec} be the distance function induced by $\mathfrak{ST}_{K_H}$ and let $\mathfrak{d}_{K_H}$ be the spectral dimension of K_H with respect to $\mathfrak{ST}_{K_H}$. Just as Lemma 1 gives $d_{spec} = d_{geo}$, Lemma 2 gives $h_{spec} = d_{geo}$. The following theorem, an analog to Theorem 1, summarizes the results for the spectral triple $\mathfrak{ST}_{K_H} = (C(K_H), \mathfrak{H}_{K_H}, \mathfrak{D}_{K_H})$.

Theorem 2. *The triple $\mathfrak{ST}_{K_H} = (C(K_H), \mathfrak{H}_{K_H}, \mathfrak{D}_{K_H})$ associated to K_H is a spectral triple with the following properties:*

1. *The spectrum of the Dirac operator, $\mathfrak{D}_{K_H}$, is given by*

$$\bigcup_{|w|=n}^{n\in\mathbb{N}} \left\{ \left[\left(\frac{(2k+1)\pi}{2\alpha_w} \right) \right] : k \in \mathbb{Z} \right\}.$$

2. *The metric distance h_{spec} induced by $\mathfrak{ST}_{K_H}$ coincides with Kigami's geodesic distance, d_{geo}.*

3. *The spectral dimension* $\mathfrak{b}_{K_H}$ *is the infimum of all* $p > 0$ *such that*

$$\sum_{w,k} \left[\left(\frac{(2k+1)\pi}{2\alpha_w} \right) \right]^{-p} < \infty.$$

The corollary to follow compares the geometries of the Sierpinski gasket induced by ST_{K_H} and $\mathfrak{S}\mathfrak{T}_{K_H}$:

Corollary 1. *For* d_{spec}, h_{spec}, $\mathfrak{d}_{K_H}$, *and* $\mathfrak{b}_{K_H}$, *as defined above, the following equalities hold:*

1. $d_{spec} = h_{spec}$.

2. $\mathfrak{d}_{K_H} = \mathfrak{b}_{K_H}$.

Proof. The first fact follows immediately from Theorem 1 and Theorem 2. The second fact is true since

$$\alpha_w = \sum_{s \in \{L,R,B\}} \alpha_{w,s}$$

.

■

4.6 The Direct Sum of ST_{K_H} and $\mathfrak{S}\mathfrak{T}_{K_H}$

In this section we note the the two spectral triples on K_H, ST_{K_H} and $\mathfrak{S}\mathfrak{T}_{K_H}$, constructed above can be summed together giving a spectral triple that also recovers Kigami's distance on K_H. This construction has the refinement of the curve triple construction and also keeps track of the holes in K_H.

Theorem 3. *Let* $ST_{\oplus} = ST_{K_H} \oplus \mathfrak{S}\mathfrak{T}_{K_H}$. *Then* $ST_{\oplus}$ *is a spectral triple for* K_H *and the distance,* $d_{\oplus}$, *induced by* $ST_{\oplus}$ *on* K_H *coincides with Kigami's distance.*

Proof. It is clear from the proofs of Theorems 1 and 2 that the for any real-valued linear function, $f(x,y) = ax + by$ on K_H,

$$||[D_{K_H}, \pi_{K_H}(f)]|| \leq |a| + |b| \text{ and } ||[\mathfrak{D}_{K_H}, \pi_{K_H}(f)]|| \leq |a| + |b|.$$

Since $D_\oplus = D_{K_H} \oplus \mathfrak{D}_{K_H}$, we have

$$||D_\oplus, \pi_\oplus(f)]|| \leq |a| + |b|.$$

Thus the real-valued linear functions on K_H have bounded commutators with $D_\oplus$ and hence the dense subalgebra condition is satisfied.

The operator, $(D_\oplus^2 + I)^{-1}$ is compact, as the eigenvalues of $D_\oplus$ are the disjoint union of the eigenvalues of D_{K_H} and $\mathfrak{D}_{K_H}$. In short, the union is countable and can be arranged in a nonincreasing order in which eigenvalues go to zero and the same argument as in Theorem 1 holds. The self-adjointedness of $D_\oplus$ is also clearly inherited from its summands.

To prove the claim of recovery of Kigami's distance we need to verify that

$$||D_\oplus f||_{\infty, K_H} = \mathrm{Lip}_g(f).$$

Indeed, by Lemma 1 and Lemma 2,

$$||D_\oplus f||_{\infty, K_H} = \max\{||D_{K_H}||_{\infty, K_H}, ||\mathfrak{D}_{K_H}||_{\infty, K_H}\} = \mathrm{Lip}_g(f).$$

It follows immediately that $d_\oplus = d_{geo}$.

■

4.7 Sets Built on Curves in $\mathbb{R}^n$

Let $\mathcal{R} = \bigcup_{j \in \mathbb{N}} R_j$ where R_j is a curve in $\mathbb{R}^n$. Let $\overline{\mathcal{R}}$ denote the closure of $\mathcal{R}$ in $\mathbb{R}^n$. Suppose the following conditions on $\mathcal{R}$ hold:

R1. $\mathcal{R}$ is a pre-compact space in $\mathbb{R}^n$;

R2. R_j is a rectifiable C^1 curve in $\mathbb{R}^n$;

R3. The arclengths of the R_j's go to zero at a geometric rate;

R4. For any two points in $\overline{\mathcal{R}}$, there exists a rectifiable piecewise-C^1 path, γ, connecting the points, whose arclength, $L(\gamma)$, is a minimum of the arclengths of all paths connecting them. Moreover, γ can be given as a (possibly infinite) concatenation of the R_j's.

For $p, q \in \overline{\mathcal{R}}$ and γ as in (R4), we will define the geodesic distance, d_{geo}, by $d_{geo}(p, q) = L(\gamma)$.

Theorem 4. *Suppose $\mathcal{R} = \bigcup_{j \in \mathbb{N}} R_j$ satisfies conditions R1-R4. Then the countable sum of R_j-curve triples is a spectral triple for $\overline{\mathcal{R}}$ and if $d_{\mathcal{R}}$ is the distance function induced by the spectral triple then $d_{\mathcal{R}} = d_{geo}$.*

Proof. The construction of the spectral triple from the R_j-curve triples is exactly as in Theorem 2. Let each of the R_j's be parameterized by its arclength and let $ST_{\mathcal{R}}$ be given by

$$ST_{\mathcal{R}} = \left(C(\overline{\mathcal{R}}), \bigoplus_{j \in \mathbb{N}} H_j, \bigoplus_{\jmath \in \mathbb{N}} D_j \right),$$

where H_j and D_j represent the Hilbert space and Dirac operator corresponding to the R_j-curve triple. We will have the representation π as the sum of the representations for each R_j. The argument for a dense subset of $C(\overline{\mathcal{R}})$ having bounded commutators with the Dirac operator $D = \bigoplus_{\mathrm{J} \in \mathbb{N}} D_j$, is again given by the real functionals on $\mathbb{R}^n$. $C(\overline{\mathcal{R}})$. Let $f(x_1, ..., x_n) = a_1 x_1 + ... + a_n x_n$ an arbitrary real functional. Let R_j be parameterized by arclength in the variable τ. Then

$$||[D_j, \pi_j(f)]|| = ||D_j(f)|| = ||D_j(f)||_\infty = \left|\left|\frac{1}{i}\frac{df}{d\tau}\right|\right|_\infty = ||a_1(x_1'(\tau)) + ... + a_n(x_n'(\tau))||_\infty$$

$$\leq |a_1| + ... + |a_n|.$$

Since this bound is not dependent on j, we have

$$||[D, \pi(f)]|| = \sup_j \{||[D_j, \pi_j(f)]||\} \leq |a_1| + ... + |a_n|.$$

Therefore the real linear functionals on $\mathbb{R}^n$ suffice as our dense subset of $C(\overline{\mathcal{R}})$ which have bounded commutators with D.

The eigenvalues of D_j are given in Proposition 1—in particular by $L(R_j)$. Since the eigenvalues of D are are the disjoint union of the eigenvalues for the D_j's, and $L(R_j) \to 0$, we have that $(D^2 + I)^{-1}$ is a compact operator. The self-adjointedness of D follows from D_j being self-adjoint for each j.

Let $\mathrm{Lip}_g(f)$ denote the Lipschitz norm with respect to d_{geo}. To show that the spectral distance, $d_{\mathcal{R}} = d_{geo}$, it suffices to show $||Df||_{\infty,\overline{\mathcal{R}}} = \mathrm{Lip}_g(f)$. (The remainder of this proof is exactly as in the proof of Lemma 1). Note that

$$||Df||_{\infty,\overline{\mathcal{R}}} = \sup_j \left\{||D_j f||_{\infty,R_j}\right\} = \sup_j \left\{\left|\left|\frac{1}{i}\frac{df}{dx}\right|\right|_{\infty,R_j}\right\}$$

$$= \sup_j \left\{ \sup_{p,q \in R_j} \left\{ \frac{|f(p) - f(q)|}{d_{geo}(p,q)} \right\} \right\} \leq \mathrm{Lip}_g(f).$$

The last inequality is clear, just as in Lemma 1, since Lip_g is the supremum over all $p \neq q$, not just those $p \neq q$ restricted to being in the same R_j.

The inequality in the other direction will come from $R4$—precisely from the fact that the *geodesic* γ between p and q is a concatenation of the R_j's. Let $\{p_j\}$ the sequence of endpoints tracking the R_j curves along γ and note,

$$|f(p) - f(q)| = |\sum_{j \in \mathbb{N}} f(p_j) - f(p_{j+1})| \leq \sum_{i \in \mathbb{N}} |f(p_j) - f(p_{j+1})|$$

$$\leq \sum_{j \in \mathbb{N}} \left(d_{geo}(p_j, p_{j+1}) ||D_j f||_{\infty, R_j} \right) \leq \left(||Df||_{\infty, \overline{\mathcal{R}}} \right) \sum_{j \in \mathbb{N}} d_{geo}(p_j, p_{j+1})$$

$$= ||Df||_{\infty, \overline{\mathcal{R}}} d_{geo}(p, q).$$

Therefore $\mathrm{Lip}_g(f) \leq ||Df||_{\infty, \overline{\mathcal{R}}}$.

The spectral distance, $d_{\mathcal{R}}$, is thus given by

$$d_{\mathcal{R}} = \sup\{|f(p) - f(q)| : \mathrm{Lip}_g(f) \leq 1\}.$$

It follows as before that $d_{\mathcal{R}} \leq d_{geo}$ and for fixed p, the function $f(x) = d_{geo}(p, x)$ is witness to $d_{\mathcal{R}} \geq d_{geo}$.

■

4.8 Work in Progress and Future Directions

4.8.1 Uniqueness of Dirac Operators

One of the main observations of the results of this chapter is that there are multiple Dirac operators which induce the same geometry of the harmonic gasket. In particular, they induce Kigami's measurable Riemannian geometry. This gives rise to questions of *the* Dirac operator versus *a* Dirac operator, as well as to what suitable equivalences of Dirac operator might be. Part of my current work is to shed light on such questions.

4.8.2 Self-Affinity, Spectral Dimension, and Volume Measure

Another observation is that self-affinity of the harmonic gasket has posed a challenge in computing its spectral dimension explicitly. This is in contrast to the case of the Sierpinski gasket which is a self-similar fractal whose spectral dimension is computed in [20] using the fact that every triangle in the Sierpinski gasket is a copy of the original simplex, scaled by $1/2^n$.

The spectral dimension of the harmonic gasket is an open question, but it is conjectured by Michel L. Lapidus to be 1 [19]. There is *evidence* of this fact, including Vicente Alvarez's numerical analysis of the spectrum of the Kusuoka Laplacian in [2]. A closely related question to the spectral dimension is the volume measure induced by the spectral triple. For the Sierpinski gasket, the volume measure induced by the spectral triple was computed as the renormalized Hausdorff measure in [20]. I am currently working to better understand the asymptotics of eigenvalues of the Dirac operators for the harmonic gasket in order to compute the spectral dimension as well as the volume measure induced by the spectral triples. This direction of research is in part in the service of building connections with work done by Jun Kigami and Michel L. Lapidus in [15] and [16].

4.8.3 Global Dirac Operator

Computing the spectral dimension and volume measure induced by the spectral triples constructed is still only a portion of the story arc I wish to complete. I would like to have a Dirac operator on the harmonic gasket which recovers Kigami's geometry and whose square is the Kusuoka Laplacian. In this case the spectral dimension associated to the Dirac operator on the harmonic gasket is 1 if and only if the spectral dimension associated to the Kusuoka Laplacian is 1. Assuming that any suitable notion of scalar curvature of the harmonic gasket yields zero scalar curvature, then this construction of a Dirac operator would be a stronger analog to Connes' theorem on a Riemannian manifold. Using the countable sums, it is not obvious how to connect the Dirac operators with the Kusuoka Laplacian.

One possible way around the countable sum constructions is to construct a *global* Dirac operator and spectral triple using elements of Kigami's measurable Riemannian geometry—in particular a Dirac operator which is constructed from a derivation on Kigami's L^2_Z Hilbert space of vector fields on the harmonic gasket. Morally, this derivation, D, restricted to Kigami's C^1 functions on the gasket would be given by $Df = -iZ\widetilde{\nabla}f$, where $\widetilde{\nabla}$ is the harmonic tangent operator and Z is the measurable Riemannian metric. Then, informally, $||[D,f]|| = ||Df||$ as a multiplication operator on L^2_Z and thus $||[D,f]|| = ||Df||_\infty$.

Using Kigami's theorems with the Kigami gradient ∇ it follows that $Z\widetilde{\nabla}f =$

∇f, the restriction of the usual gradient of a function on the plane to the harmonic gasket. Thus $||[D, f]||$ coincides with $||\nabla f||_\infty$, which in turn coincides with the Lipschitz (with respect to Kigami's geodesic distance) norm of f. It would remain to prove that this Dirac operator squares to the Kusuoka Laplacian. I am currently working to formalize these arguments, which includes a notion of a *measurable* vector bundle and co-bundle and a *measurable connection* on the bundle. It should be noted that aside from squaring to the Kusuoka Laplacian, this Dirac operator would be constructed directly from the measurable Riemannian metric and therefore be the basis of an even stronger measurable Riemannian analog to Connes' theorem on a Riemannian manifold.

4.8.4 Effective Resistance Metric

If the argument in the previous section can be formalized, the spectral geometric approach may prove a link between the effective resistance metric on the Sierpinski gasket and the measurable Riemannian geometry of the Sierpinski gasket. The effective resistance metric is described with oscillating springs between vertices of the gasket in an identical manner as which it is described with an electrical network on the gasket. Since the harmonic gasket is the Sierpinski gasket in a harmonic coordinate chart, it reasonable to think that the effective resistance metric on the Sierpinski gasket must be related to Kigami's geodesic distance. The following conjectured relationship is inspired by conversations with Michel L. Lapidus in [19] and comments by Marc Rieffel in [25].

Connes' Formula 1 and Kigami's effective resistance metric formula in conjunction with the proposed global Dirac operator mentioned above may be a way of formalizing this relationship. Indeed, the square root of the effective resistance metric is

$$\sqrt{R(p,q)} = \sup\left\{ \frac{|u(p)-u(q)|}{\sqrt{\mathcal{E}(u,u)}} : u \in Dom\mathcal{E} \right\}.$$

Using Kigami's theorem, this becomes

$$\sqrt{R(p,q)} = \sup\left\{ \frac{|u(p)-u(q)|}{||\nabla u||_{L^2_Z}} : u \in Dom\mathcal{E} \right\}.$$

Connes's Formula 1 with the global Dirac operator becomes

$$d(p,q) = \sup\{|u(p)-u(q)| : ||\nabla u||_\infty \leq 1\}.$$

The formulas are similar. I am working to formulate a precise relationship between d and $\sqrt{R}$.

4.8.5 Countable Unions of Curves

Theorem 4 in this chapter is a generalization of the methods used for the harmonic gasket. Though it applies to a large class of sets including the Sierpinski gasket and the harmonic gasket, my goal is to replace condition R4 with a more geometric condition. The concatenation condition which applies to both gaskets is the result of connectedness and a certain type of convexity for the harmonic gasket and connectedness and straight line segments for the Sierpinski gasket. The goal is to abstract something from these conditions in order to eliminate the concatenation condition.

4.8.6 Quantum Graphs and Ecology

The spectral triple approach to these sets built on curves shares many similarities with the theory of quantum graphs. Quantum graphs essentially share two parts in common with spectral triples—the Hilbert spaces and the differential operator. The difference is that in quantum graphs, choices of boundary conditions replace the algebra of continuous functions on the space. The quantum graph is constructed to directly admit a theory of differential equations on the space. I am working to apply such quantum graphs to model population dynamics in ecological systems in dendritic stream and river systems.

Part II

Nonlinear Poisson Equation via a Newton-embedding Procedure

Chapter 5

The Nonlinear Poisson Equation via a Newton-embedding Procedure

5.1 Introduction

In Chapter 5 we solve the nonlinear Poisson boundary value using a Newton-embedding procedure. The focal point of the project is, of course, the collection of hypotheses imposed on the nonlinear forcing term. As will be discussed later in this chapter, the Newton-embedding procedure yields a linear elliptic partial differential boundary value problem at each iteration in the procedure. Convergence in the procedure relies on, in particular, some uniform control over the norm of the solution at each iteration and most of the hypotheses on the nonlinear forcing term are influenced by the necessity of this norm control.

5.1.1 Structure of Part II

The two sections to follow are, respectively, brief primers on second order elliptic equations and the Newton-embedding procedure. In Section 3 of this chapter, we state the problem formally and list the results. The remainder of Chapter 5 is primarily dedicated to proving those results. The chapter concludes with a discussion of a class of nonlinear functions allowable in the procedure and possible future projects based on certain observations.

5.2 Second Order Elliptic Equations and the Newton-embedding Procedure

5.2.1 Second Order Elliptic Equations

Let Ω be a domain in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. Consider the following boundary value problem,

$$\begin{cases} Lu = f & \text{in } \Omega \\ u|_\Gamma = 0 & \text{on } \Gamma = \partial\Omega, \end{cases}$$

where L is a second order differential operator given by

$$Lu = -\Sigma_{i,j}^n a_{ij}(x) u_{x_i x_j} + \Sigma_{i=1}^n b^i(x) u_{x_i} + c(x) u$$

Assume for now that $a^{ij}, b^i, c \in L^\infty(\Omega)$. We say that L is *elliptic* if there exists a constant $\alpha > 0$ such that

$$\Sigma_{i,j=1}^n a^{i,j}(x) \beta_i \beta_j \geq \alpha |\beta|^2$$

for Lebesgue-a.e. $x \in \Omega$ and all $\beta \in \mathbb{R}^n$ [6]. In other words, the matrix given by $(a^{ij}(x))$ is positive definite with smallest eigenvalue greater than or equal to α [6]. Second order elliptic equations often model physical phenomena, and this is true of the simplest elliptic problem given by $-\Delta u = f$, $u = 0$, called the Poisson problem.

One of many applications is when the Poisson equation is modeling an external force, f, applied to a taut membrane or drum where u is the displacement of the membrane (away from flat). The Poisson equation is then a force equation and states that the deformation of the membrane is precisely given by the force applied. In the exposition to follow, we will address the nonlinear form of this problem where the forcing function f is replaced with $f(u) = f \circ u$. We will do this using using an approximation method which in essence linearizes the problem, or more specifically, allows for a solution via iteratively solving linear second order elliptic problems. This linear problem has a fairly simple form as well and is given by

$$\begin{cases} -\Delta u + q(x)u &= g(x) \quad \text{in } \Omega \\ u|_\Gamma &= 0 \quad \text{on } \Gamma, \end{cases}$$

It is necessary to explain what is meant by a *solution* in this setting. Suppose $f \in L^2(\Omega)$ and we multiply both sides of $Lu = f$ by a smooth function v with compact support in Ω. Integrating by parts reduces this second order equation to first order, with boundary terms vanishing because of the compact support of v, to find

$$\int_\Omega \Sigma_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \Sigma_{i=1}^n b^i u_{x_i} v + cuvdx = \int_\Omega fvdx$$

It is clear by density that u need only be in the Sobolev space $H_0^1(\Omega)$ for this process to make sense. The energy form B associated to the operator L is given by

$$B[u,v] = \int_\Omega \Sigma_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \Sigma_{i=1}^n b^i u_{x_i} v + cuvdx$$

which is defined for $u, v \in H_0^1$. We say that u is a *weak solution* to $Lu = f$ if $B[u,v] = (f, v)$ for all $v \in H_0^1(\Omega)$, where $(/,/)$ is the inner product on $L^2(\Omega)$ [6]. In this work, we will be concerned with elliptic operators where $c > 0$, $b^i \equiv 0$, and $a^{ij} \equiv \delta_{ij}$. In this case $B[u,v]$ is an inner product on $H_0^1(\Omega)$ and the Riesz representation theorem provides existence and uniqueness. This frames the notion of a solution in the linear elliptic equations. I have avoided a *weak solution* formulation for the nonlinear equation. *What I will call a solution to the nonlinear equation is an* $H_0^1 \bigcap H^2$ *which solves the nonlinear problem almost everywhere.*

Another important feature of elliptic equations, especially important in the Newton-embedding procedure, is regularity. In other words, what can the structure of an elliptic operator say about the solution to the associated boundary value problem? L. Evans in [6] gives an elegant motivation for regularity. The problem $-\Delta u = f$ in Ω with zero on the boundary condition, has a unique solution in $H_0^1(\Omega)$, as discussed above. But as the following computation illustrate, there is reason to believe the solution is at least *two weak derivatives smoother* than f. To see this, assume that u is smooth and vanishes on the boundary of Ω. In [6], L. Evans computes

$$\int_\Omega f^2 dx = \int_\Omega (\Delta u)^2 dx = \Sigma_{i,j=1}^n \int_\Omega u_{x_i x_i} u_{x_j x_j} dx = -\Sigma_{i,j=1}^n \int_\Omega u_{x_i x_i x_j} u_{x_j} dx$$

$$= \Sigma_{i,j=1}^n \int_\Omega u_{x_i x_j} u_{x_i x_j} dx = \int_\Omega |D^2 u|^2$$

In this case the second derivatives of u are controlled by the L^2-norm of f. Substituting an H_0^1 assumption for smooth sets up a technically more difficult argument, but the intuition is correct. Theorem 1 in 6.3.1 and Theorem 4 in 6.3.2 in [6] show that, under suitable assumptions, the H_0^1 solution of an elliptic problem, is indeed in H^2, with its H^2-norm controlled by the L^2-norm of f. This norm control is what we refer to as a *regularity estimate.*

For the purpose of the Newton-embedding procedure, such regularity estimates are crucial as will be seen in Section 7 addressing the convergence results in the procedure. Note also, that given the regularity lifting to H^2, the weak solution will also solve the linear differential equation almost everywhere. This will extend, via the Newton-embedding procedure, to an $H_0^1 \bigcap H^2$ function which solves the nonlinear Poisson problem almost everywhere.

5.2.2 Newton-embedding Procedure

The Newton-embedding procedure is a variant of the classical Newton's method used to approximate a zero of a function, taught in a first year calculus course. One makes an initial guess, evaluates the derivative at the point, and then intersects the tangent line at that point with the x-axis. The derivative is then evaluated at the intersection point with the new tangent line intersecting the x-axis and so on. This forms a sequence of points on the x-axis whose limit is hopefully a zero of the function.

One problem with the classical Newton's method is that the initial guess must be *close* enough to a zero for the iterations to converge. One way of getting around this obstacle is to embed the equation to be solved, say, $f(x) = 0$ into a one-parameter family of equations, $f_t(x) = 0$ for $t \in [0, 1]$ where a solution is known at $t = 0$ and the desired solution is the one corresponding to $t = 1$. An example would be $f(x) = x^2 - 2^x = 0$ embedded into the family $f_t(x) = x^2 - t2^x = 0$. Here the solution $x = 0$ is known for $t = 0$. The idea is then to *push the solution along* with a *finite number* of increasing times, performing Newton's method at each time. With convergence in Newton's method at $t_j \in [0, 1]$, the limit point is then used as the initial guess for Newton's method at time $t_{j+1} \in [0, 1]$.

This is exactly the method of the Newton-embedding procedure used in the work to follow, with the exception that we are not performing calculus on the real line, but on a Hilbert space of functions. Because of this, each iteration in Newton's method at time t will be given by a differential equation and the limit point will be given by a function as will be seen in Section 6. Also, the classical derivative is replaced with the Frechet derivative, which is explained and computed in Section 6.

5.3 Statement of the Problem and Theorems

The goal of this chapter is to find suitable hypotheses on a function $f \in C^2(\mathbb{R})$ related to attaining a solution to the semilinear boundary value problem given by

$$(*) \begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega \\ u &= \phi \quad \text{on } \Gamma = \partial\Omega, \end{cases}$$

using the Newton-embedding procedure that is applied in [11]. Here, Ω is assumed to be a bounded domain in $\mathbb{R}^n$ $(n > 2)$ with smooth boundary Γ and $f(u)$ is defined as $f \circ u$. In order to distinguish between f as a real-valued function on $\mathbb{R}$ and as a map from a space of real-valued functions on Ω to another space of real-valued functions on Ω via composition, we will define the *derived* function as $\widetilde{f}$, an operator between spaces of real-valued functions on Ω. More precisely, if f is a real-valued function on $\mathbb{R}$ and A and B are spaces of real-valued functions on Ω, then

$$\widetilde{f} : A \to B \quad \text{is defined by} \quad \widetilde{f}(u) = f \circ u.$$

for $u \in A$. Note that the definition of $\widetilde{f}$ is consistent with the notation used in $(*)$, since $\widetilde{f}(u) = f \circ u = f(u)$. In addition, $H^k(\Omega)$ is defined as the L^2 functions on Ω having (weak) derivatives up to order k which are L^2 functions on Ω. This is the Hilbert space notation substituted for the Sobolev space notation $W^{k,2}(\Omega)$. The space of real-valued functions on Ω which are Hölder continuous with exponent α will be denoted $C^\alpha(\bar{\Omega})$. The author of [11] constructs an H^2 solution when Ω is a domain in $\mathbb{R}^3$ and Γ is smooth, provided the following assumptions on f hold:

(i) $\widetilde{f}$ is a continuous map from $H^2(\Omega)$ to $L^2(\Omega)$;

(ii) $\widetilde{f'}$ and $\widetilde{f''}$ are continuous maps from $H^1(\Omega)$ to $C^\alpha(\bar{\Omega})$, $\alpha \in (0, \frac{1}{2}]$;

(iii) $\widetilde{f}$, $\widetilde{f'}$, and $\widetilde{f''}$ are bounded maps;

(iv) $f' < 0$ on $\mathbb{R}$.

An additional condition in [11] is the choice of a *uniform width* of time intervals in the procedure that ensures convergence, which exists as a consequence of the above assumptions. However, we prove the following theorems in Sections 2 and 3 of this article:

Theorem 1. *If $f : \mathbb{R} \to \mathbb{R}$ and $\tilde{f} : H^1(\Omega) \to C^0(\bar{\Omega})$ and Ω is a domain in $\mathbb{R}^n$ $(n > 2)$, then f is a constant function.*

Theorem 2. *Let $h : \mathbb{R} \to \mathbb{R}$, $1 \le p \le \infty$, and Ω be a domain in $\mathbb{R}^n$. If $\widetilde{h}$ is a bounded map from $H^2(\Omega) \bigcap H^1_0(\Omega)$ to $L^p(\Omega)$, then h is a bounded map from $\mathbb{R}$ to $\mathbb{R}$.*

By Theorem 1, the assumption in (ii) that $\widetilde{f'}$ maps H^1 to C^α forces f' to be a constant function. Theorem shows that the bound on $\widetilde{f}$ in assumption (iii) forces f to be a bounded function on $\mathbb{R}$. Thus f is shown to be linear and bounded on $\mathbb{R}$, and is therefore a constant function, reducing the scope of the procedure in [11] to the family of problems given by $-\Delta u = const$.

In Section 2, Theorem 1 is proven. In Section 3, Theorem 2 is proven using a sequence of smooth 'bump' functions in H^2. As a consequence of Theorem 2, the bounds imposed in (iii) on $\widetilde{f'}$ and $\widetilde{f''}$ imply that f' and f'' are also bounded functions on $\mathbb{R}$.

In the remaining sections of this article, we apply the Newton-embedding procedure to the zero boundary value case of the problem $(*)$, given the following hypotheses on $f \in C^2(\mathbb{R})$:

(I) $\widetilde{f}$ is a continuous map from $H^2(\Omega)$ to $L^2(\Omega)$;

(II) $\widetilde{f'}$ and $\widetilde{f''}$ are continuous maps from $H^1(\Omega)$ to $L^n(\Omega)$;

(III) there exists a constant $M > 0$ such that

$$|f| \leq M, \quad |f'| \leq M, \quad \text{and} \quad |f''| \leq M;$$

(IV) $f' < 0$.

In Section 7, given $(I) - (IV)$, the main result of this article is proven:

Theorem 4. *With Ω a bounded domain in $\mathbb{R}^3$ with smooth boundary and assumptions (I)-(IV), the semilinear boundary value problem,*

$$(*')\begin{cases} -\Delta u &= f(u) \qquad in\ \Omega \\ u &= 0 \qquad on\ \Gamma = \partial\Omega, \end{cases}$$

has a unique solution in $H^2(\Omega)\bigcap H_0^1(\Omega)$*, and hence a continuous solution, which can be approximated by the Newton-embedding procedure.*

In our application of the Newton-embedding procedure used to construct the solution to $(*')$, each k-th step of the procedure yields a linear boundary value problem of the form

$$(**)\begin{cases} -\Delta u + q(x)u &= g(x) \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \Gamma. \end{cases}$$

The functions g and q will be different at each step, but ultimately derive from f. Due to the assumptions $(I)-(IV)$ on f, it is the case that g is in $L^2(\Omega)$, q is in $L^n(\Omega)$, and $q > 0$—and this is sufficient to make $(**)$ well-posed:

Theorem 3. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ $(n > 2)$ *with smooth boundary* Γ*. Then for* $g \in L^2(\Omega)$*,* $q \in L^n(\Omega)$*, and* $q > 0$*, the linear boundary value problem*

$$(**)\begin{cases} -\Delta u + q(x)u &= g(x) \qquad in\ \Omega \\ u &= 0 \qquad on\ \Gamma, \end{cases}$$

has a unique solution $u \in H^2(\Omega)\bigcap H_0^1$ *with*

$$||u||_{H^2(\Omega)} \leq C(||g||_{L^2(\Omega)}),$$

where C *depends only on* Ω*,* n*, and* q*.*

Remark 1. *Theorem 3 is a standard result, however, for completeness and lack of an explicit reference it will be proved in Sections 5 and 6.*

5.4 The Mesa Function

Let Ω be a domain in $\mathbb{R}^n$ with $n > 2$ and let $c \in \Omega$. We start by constructing a function that is radially symmetric about c, which we will use in the proof of Theorem 1, and call a *mesa function*. The notation '$\subset\subset$', used throughout Sections 2 and 3, denotes *compactly contained*. Let $T > 0$ such that $B(c, T) \subset\subset \Omega$, where $B(c, T)$ denotes the open ball of radius T about c. Also let $a, b \in \mathbb{R}$ with $a < b$, and $\alpha \in (0, \frac{n-2}{2})$. In order to define the function, it is necessary to decompose the interval $[0, T]$ as follows:

If we let $r_1^+ = \dfrac{T}{2}$, then there is an s_1^+ such that

$$\frac{1}{(s_1^+)^\alpha} - \frac{1}{(r_1^+)^\alpha} = b - a.$$

In particular, $0 < s_1^+ < r_1^+$. Setting $s_1^- = \dfrac{s_1^+}{2}$ allows for an r_1^- such that

$$\frac{1}{(r_1^-)^\alpha} - \frac{1}{(s_1^-)^\alpha} = b - a.$$

In particular, $0 < r_1^- < s_1^-$. Continuing in this manner, set $r_{m+1}^+ = \dfrac{r_m^-}{2}$.

Note that $r_{m+1}^+ > 0$ for all m and r_{m+1}^+ goes to zero with $\dfrac{1}{2^m}$.

Using the above notation, let $U : \Omega \to \mathrm{R}$ be the radially symmetric piecewise function defined inductively by

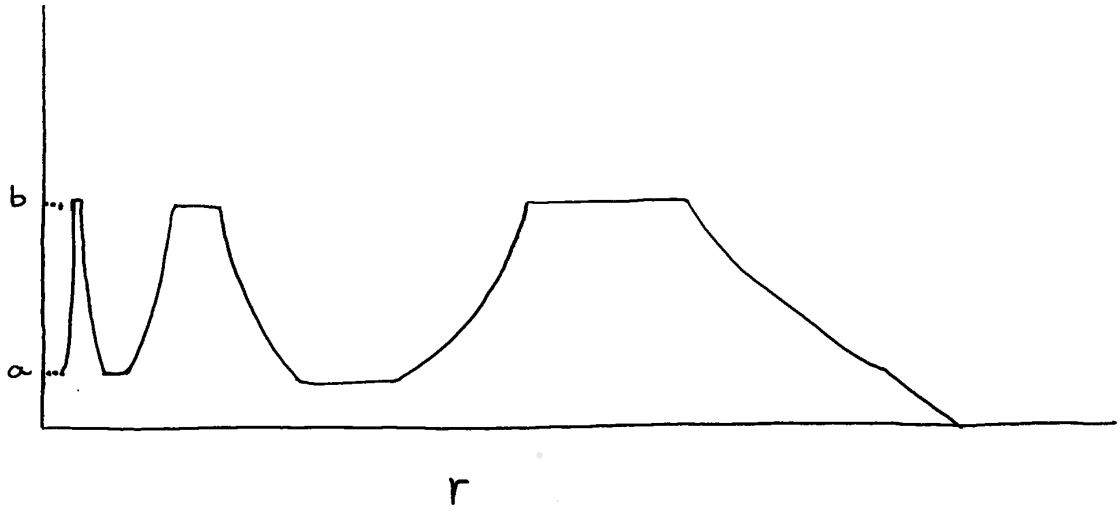

Figure 5.1: Artist's depiction of a mesa function with $r = |x - c|$

$$U(x) = \begin{cases} 0 & , \quad |x-c| \geq T \\ (\frac{-2a}{T})|x-c| + 2a & , \quad r_1^+ \leq |x-c| \leq T \\ \frac{1}{|x-c|^\alpha} - \frac{1}{(r_m^+)^\alpha} + a & , \quad s_m^+ \leq |x-c| \leq r_m^+ \\ b & , \quad s_m^- \leq |x-c| \leq s_m^+ \\ b - (\frac{1}{|x-c|^\alpha} - \frac{1}{(s_m^-)^\alpha}) & , \quad r_m^- \leq |x-c| \leq s_m^- \\ a & , \quad r_{m+1}^+ \leq |x-c| \leq r_m^-. \end{cases}$$

We will call $U(x)$ a *mesa function with exponent* α. Figure Figure 5.1 is a sketch of a mesa function whose partition points have been altered to show more 'mesas'.

U is bounded and has compact support, so is trivially in $L^2(\Omega)$. It remains to show that it has (weak) first derivatives in $L^2(\Omega)$. The proposed first derivatives are given by

$$U_{x_i}(x) = \begin{cases} 0 & , \quad |x-c| \geq T \\ \frac{-2a}{T} & , \quad r_1^+ \leq |x-c| \leq T \\ \frac{-\alpha x_i}{|x-c|^{\alpha+2}} & , \quad s_m^+ \leq |x-c| \leq s_m^+ \\ 0 & , \quad s_m^- < |x-c| \leq s_m^+ \\ \frac{\alpha x_i}{|x-c|^{\alpha+2}} & , \quad r_m^- < |x-c| \leq s_m^- \\ 0 & , \quad r_{m+1}^+ < |x-c| \leq r_m^-. \end{cases}$$

Away from zero, on each annulus of the decomposed Ω, the expressions in U_{x_i} are classical derivatives of their corresponding expressions in $U(x)$. Let $\phi \in C_0^\infty(\Omega)$ and fix N. Integrating $U\phi_{x_i}$ by parts over the annuli given by $[r_1^+, T]$, $[s_m^+, r_m^+]$, $[s_m^-, s_m^+]$, $[r_m^-, s_m^-]$, and $[r_{m+1}^+, r_m^-]$ for $m = 1, ..., N$ and recalling that $U \equiv 0$ for $|x-c| \geq T$, gives

$$\int_{\Omega - B(c, r_{N+1}^+)} U\phi_{x_i} dx = -\int_{\Omega - B(c, r_{N+1}^+)} U_{x_i}\phi dx + \int_{\partial B(c, r_{N+1}^+)} U\phi\rho^i dS,$$

where $\rho = (\rho^1, ..., \rho^n)$ is the inward pointing normal on $\partial B(c, r_{N+1}^+)$.

Let $u(x) = \dfrac{1}{|x-c|^\alpha}$. Note that $|U_{x_i}| \leq |u_{x_i}|$, so that $|DU| \leq |Du|$. Following the line of argument [1, p.246] given by L. Evans, since $\alpha < n-1$, $|Du| = \dfrac{\alpha}{|x-c|^{\alpha+1}} \in L^1(\Omega)$ and therefore $|DU| \in L^1(\Omega)$. Letting $N \to \infty$ (and thus $r_{N+1}^+ \to 0$),

$$\left| \int_{\partial B(c, r_{N+1}^+)} U\phi\rho^i dS \right| \leq \| U\phi \|_\infty \int_{\partial B(c, r_{N+1}^+)} \rho^i dS \ \leq \ M(r_{N+1}^+)^{n-1} \to 0.$$

This implies

$$\int_\Omega U\phi_{x_i}dx = -\int_\Omega U_{x_i}\phi dx,$$

and therefore U_{x_i} is a (weak) derivative of U. Moreover, since $\alpha < \dfrac{n-2}{2}$, following the argument in [1, p.246], $\mid Du \mid \in L^2(\Omega)$ and thus $|DU| \in L^2(\Omega)$ and $U(x) \in H^1(\Omega)$. The following lemma summarizes the above discussion:

Lemma 1. *If Ω is a domain in $\mathbb{R}^n$ with $n > 2$, and $U(x)$ is a mesa function with exponent $\alpha \in \left(0, \dfrac{n-2}{2}\right)$, then $U(x) \in H^1(\Omega)$.*

Using Lemma 1, we give the proof of Theorem 1:

Proof of Theorem 1. Suppose on the contrary, that f is not constant and assumes distinct values at a and b. Without loss of generality, assume that $a < b$. Let $c \in \Omega$ and T be such that $B(c,T) \subset\subset \Omega$. Since $n > 2$, there exists α such that $0 < \alpha < \frac{n-2}{2}$. Let $U(x)$ be the mesa function centered at c, with exponent α, support in $B(c,T)$, and prescribed maximum and minimum, b and a, respectively. By Lemma 1, $U(x)$ is in $H^1(\Omega)$. Using the notation in the previous section for the domain of $U(x)$, it holds that for any $\delta > 0$ there exists an N such that $[s_N^-, s_N^+] \subset B(c,\delta)$ and $[r_{N+1}^+, r_N^-] \subset B(c,\delta)$. Note that $f \circ U \equiv f(b)$ on $[s_N^-, s_N^+]$ and $f \circ U \equiv f(a)$ on $[r_{N+1}^+, r_N^-]$. Since the measure of the above intervals is strictly positive, $f \circ U$ has no continuous representative. In other words, the oscillations of $f \circ U$ do not diminish in any neighborhood of c. This contradicts the hypothesis that $\widetilde{f}$ maps U to a continuous function. ■

5.5 Uniform bounds

In this section we prove Theorem 2, showing that bounding $\widetilde{h}$ results in bounding h:

Proof of Theorem 2. Let $p < \infty$. Suppose on the contrary, that h is not bounded red and that the bound on $\widetilde{h}$ is M. Then there exists a sequence, $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}$ such that $|h(x_k)| > k$. Let $y_0 \in \Omega$ and r be such that $B = B(y_0, r) \subset\subset \Omega$. Set $B_{\frac{1}{2}} = B(y_0, \frac{r}{2})$. Choose a smooth function, γ, such that $\gamma \equiv 1$ on $B_{\frac{1}{2}}$, $\gamma \equiv 0$ on $\Omega - B$, and $0 \leq \gamma \leq 1$. Define the smooth function u_k on Ω by $u_k = x_k\gamma$. Then $u_k \in H^2(\Omega) \bigcap H_0^1(\Omega)$ for all k and

$$||\widetilde{h}(u_k)||_{L^p(\Omega)} = ||h(u_k)||_{L^p(\Omega)} \geq ||h(u_k)||_{L^p(B_{\frac{1}{2}})} = ||h(x_k)||_{L^p(B_{\frac{1}{2}})} > k|B_{\frac{1}{2}}|^{\frac{1}{p}}.$$

Choosing k_0 large enough such that $k_0|B_{\frac{1}{2}}|^{\frac{1}{p}} > M$ gives a contradiction. If $p = \infty$, a similar computation holds, choosing $k_0 > M$. ■

Remark 2. *Since C^α is embedded in L^∞, Theorem 2 with $p = \infty$ suffices to show that a uniform bound on $||h(u)||_{C^\alpha}$ implies h is bounded. Therefore the assumptions made in [11], imply that f, f', and f'' are bounded functions. Moreover, under the same assumptions, as shown in the previous section, f is linear. In this case f is a constant, reducing the scope of the procedure to problems given by $-\Delta u = const$.*

5.6 Newton-embedding Procedure

The Newton-embedding procedure we wish to apply to

$$(*')\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega, \end{cases}$$

has two parts. It is well described in [11], but recalled here for clarity. The procedure first embeds the problem in a one-parameter family of problems,

$$-\Delta u = tf(u) \qquad \text{in } \Omega$$

with $u = 0$ on Γ and parameter $t \in [0,1]$. We set

$$F_t(u) = \Delta u + tf(u).$$

Solving $(*')$ is then a matter of solving $F_1(u) = 0$.

Let $u(x,t)$ be the solution to $F_t(u) = 0$. Starting with $t_0 = 0$, the problem is solved with solution $u(x,0)$ in Ω. Observe that with boundary value zero imposed, $u(x,0)$ is uniquely determined as $u(x,0) \equiv 0$. To solve $F_{t_1}(u) = 0$, $u(x,0)$ is taken as an initial approximation and the standard Newton's method is applied. With convergence, the solution $u(x,t_1)$ to $F_{t_1}(u) = 0$ is achieved. The function $u(x,t_1)$ is then used as an initial approximation for $F_{t_2}(u) = 0$ and so on for increasing times t_j. Thus the solutions are pushed along with increasing times using Newton's method with the goal of reaching $t = 1$ in finitely many time shifts.

Let $u_0(x,t_j) = u(x,t_{j-1})$, the initial approximation for $F_{t_j}(u) = 0$, and $u_m(x,t_j)$ be the m^{th} iteration of Newton's method at time t_j. In the following discussion, the argument of the u_m's will be suppressed. We will also temporarily use the symbol D for the Frechet derivative in contrast to its usual use as the gradient. The Frechet derivative is a generalization of the derivative of an operator from $\mathbb{R}^k$ to $\mathbb{R}^l$ that is used to derive an operator between Banach spaces. In this setting we have an operator F which will map from $L^2(\Omega)$ to $L^2(\Omega)$. The Frechet derivative is given, for $u, w \in L^2$ and $h \in \mathbb{R}$, by,

$$DF(u)[w] = \lim_{h \to 0} \frac{F(u + hw) - F(u)}{h}.$$

Note that

$$DF_{t_j}(u_m)[w] = \Delta w + t_j D\widetilde{f}(u_m)[w] \quad \text{and} \quad D\widetilde{f}(u_m)[w] = \widetilde{f}'(u_m)w = f'(u_m)w$$

for $w \in H^2(\Omega)$ and that the $(m+1)^{th}$ iterate in the Newton approximation is given by

$$DF_{t_j}(u_m)[u_{m+1} - u_m] = -F_{t_j}(u_m).$$

To see this we can compute the Frechet derivative as follows (when there is no confusion, we will drop the 'tilde' notation on f):

$$DF(u)[w] = \lim_{h \to 0} \frac{\Delta(u + hw) + tf(u + hw) - \Delta u - tf(u)}{h}$$

$$= \lim_{h \to 0} \frac{h\Delta w + t(f(u + hw) - f(u))}{h} = \Delta w + t \lim_{h \to 0} \frac{f(u + hw) - f(u)}{h}$$

$$= \Delta w + tDf(u)[w].$$

To compute $Df(u)[w]$, let $x \in \Omega$ and recall that $f : \mathbb{R} \to \mathbb{R}$:

$$(Df(u)[w])(x) = \lim_{h \to 0} \left(\frac{f(u(x) + hw(x)) - f(u(x))}{h} \right) \left(\frac{w(x)}{w(x)} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(u(x) + hw(x)) - f(u(x))}{hw(x)} \right) w(x) = f'(u(x))w(x)$$

where f' is used to denote the derivative of f as a function from $\mathbb{R}$ to $\mathbb{R}$. Thus $Df(u) = f'(u)$ acts on $L^2(\Omega)$ through multiplication. Applied to u_m and t_j we have,

$$DF_{t_j}(u_m)[w] = \Delta w + t_j f'(u_m)w.$$

As a consequence, the $(m+1)^{th}$ iterate in the Newton approximation at time t_j is given by

$$DF_{t_j}(u_m)[u_{m+1} - u_m] = -F_{t_j}(u_m).$$

Computing each side of the equation above, we have

$$\Delta u_{m+1} - \Delta u_m + t_j f'(u_m)(u_{m+1} - u_m) = -\Delta u_m - t_j f(u_m).$$

In this case, the $(m+1)^{th}$ iteration at time t_j yields the following linear problem:

$$(**) \begin{cases} -\Delta u_{m+1} + (-t_j f'(u_m))(u_{m+1}) &= t_j(f(u_m) - f'(u_m)u_m) \quad \text{in } \Omega \\ u_{m+1} &= 0 \quad \text{on } \Gamma. \end{cases}$$

This is the problem

$$(**) \begin{cases} -\Delta u + q(x)u &= g(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma, \end{cases}$$

stated in the introduction with

$$q = -t_j f^{'}(u_m), \quad g = t_j[f(u_m) + f^{'}(u_m)u_m], \quad \text{and} \quad u = u_{m+1}.$$

Initially, a weak solution in H_0^1 is desired, so it makes sense that u be in H_0^1 and that f and f' should be defined on H_0^1. However, as will be shown in Section 6, an

H_0^1 solution to $(**)$ is also in $H^2 \bigcap H_0^1$. In light of this, f and f' need only be defined on $H^2 \bigcap H_0^1$. Note that if f maps H^2 to L^2 and f' maps H^2 to L^n, then g is in L^2 for all dimensions $n > 2$, via the Sobolev embedding theorem. Indeed, since u is in H^1, u is again in $L^{\frac{2n}{n-2}}$ and the Hölder inequality gives

$$\int_\Omega [f'(u)u]^2 \leq C||f'(u)||^2_{L^n}||u||^2_{L^{\frac{2n}{n-2}}}.$$

To fulfill the positivity condition on q in $(**)$, we impose that $f' < 0$. Now, at each time $t_j > 0$ and for all m, the m^{th} step in the iteration at time t_j is a model for $(**)$.

For the remainder of the article, we assume Ω is a bounded domain in $\mathbb{R}^n$ $(n > 2)$ with smooth boundary Γ and make the following assumptions (I)-(IV) on the nonlinear function $f \in C^2(\mathbb{R})$:

(I) $\widetilde{f}$ is a continuous map from $H^2(\Omega)$ to $L^2(\Omega)$;

(II) $\widetilde{f'}$ and $\widetilde{f''}$ are continuous maps from $H^1(\Omega)$ to $L^n(\Omega)$;

(III) there exists a constant $M > 0$ such that

$$|f| \leq M, \quad |f'| \leq M, \quad \text{and} \quad |f''| \leq M;$$

(IV) $f' < 0$.

Remark 3. *There is a redundancy and lack of 'sharpness' in assumptions (I) and (II), given (III). Indeed, if the functions f, f', and f'' are bounded, they naturally map to bounded functions on Ω, and hence to $L^\infty(\Omega)$ which is contained in $L^p(\Omega)$ for all $p \geq 1$ since Ω is bounded. The reason for stating L^2 explicitly is that it is a familiar assumption for framing weak solutions to linear elliptic problems. The bounds on the functions are* *not necessary to existence and uniqueness in* $(**)$*, nor to the regularity lifting of the* H_0^1 *solution to* H^2 *. Moreover, the* L^2 *hypothesis on* f *and the* L^n *hypothesis on* f' *are sufficient for existence and uniqueness and the regularity lifting. For a more general treatment of elliptic equations with measurable coefficients, see [33].*

5.7 Existence and Uniqueness

For this section, we assume (I), (II), and (IV). To prove existence and uniqueness for $(**)$ in $H_0^1(\Omega)$ (H^1 functions with zero on the boundary), the Riesz Representation theorem is sufficient. We seek a unique solution in $H_0^1(\Omega)$. The associated energy form for $(**)$ is

$$B(u,v) = \int_\Omega DuDv + quv.$$

It is well defined on $H_0^1(\Omega)$. Indeed, since $n > 2$ and $u, v \in H_0^1(\Omega)$, then $u, v \in L^{\frac{2n}{n-2}}(\Omega)$ by the Sobolev embedding theorem. Also since Ω is bounded, if $q \in L^n(\Omega)$, then $q \in L^{\frac{n}{2}}(\Omega)$. Note that

$$\frac{2}{n} + \frac{n-2}{2n} + \frac{n-2}{2n} = 1.$$

Therefore by Hölder's inequality, quv is integrable over Ω with

$$\int_\Omega |quv| \leq ||q||_{L^{\frac{n}{2}}} ||u||_{L^{\frac{2n}{n-2}}} ||v||_{L^{\frac{2n}{n-2}}}.$$

This inequality combined with the Sobolev inequality

$$|||u||_{L^{\frac{2n}{n-2}}} \leq C||u||_{H_0^1}$$

gives

$$|B(u,v)| \leq C||u||_{H_0^1}||v||_{H_0^1}$$

where $C > 0$ is dependent on Ω, n, and $||q||_{L^{\frac{n}{2}}}$ but not on u and v. By the Poincaré inequality and the positivity of q, we have

$$||u||^2_{H_0^1} \leq C\int_\Omega |Du|^2 \leq C\int_\Omega |Du|^2 + qu^2 = CB(u,u) \tag{5.1}$$

where $C > 0$ is dependent on n and Ω but not on u. Since $f \in L^2(\Omega)$, it is a bounded linear functional on $H_0^1(\Omega)$ [6]. Since $B(u,v)$ is an inner product on H_0^1, the Riesz Representation theorem provides a unique $u^* \in H_0^1(\Omega)$ such that

$$B(u^*, v) = \int_\Omega fv \qquad \text{for all } v \in H_0^1.$$

In other words, u^* is the unique weak solution to $(**)$ in H_0^1.

5.8 Regularity

With the same hypotheses as in the previous section, we wish to lift the regularity of the unique solution to $(**)$ from H_0^1 to H^2, with the estimate controlled by the L^2 norm of $g(x)$. Theorem 6.3.4 (Boundary H^2-regularity) in [6] gives the desired regularity lifting of a solution to $(**)$ when $q \in L^\infty$. However, the L^∞ condition is only used in factoring out $||q||_{L^\infty}$ from the following integral to find, for $u, v \in H^1$ and $\epsilon > 0$ in Cauchy's inequality,

$$\int |quv| \leq ||q||_{L^\infty}\int |uv| \leq C\left(\frac{1}{2\epsilon}||u||^2_{L^2} + \frac{\epsilon}{2}||v||^2_{L^2}\right).$$

The L^n hypothesis on q provides,

$$\int |quv| \leq \frac{1}{2\epsilon}||cu||_{L^2}^2 + \frac{\epsilon}{2}||v||_{L^2}^2 \leq \frac{1}{2\epsilon}\left(||q||_{L^n}^2 ||u||_{L^{\frac{2n}{n-2}}}^2\right) + \frac{\epsilon}{2}||v||_{L^2}^2$$

$$\leq C\left(\frac{\epsilon}{2}||u||_{H^1}^2 + \frac{\epsilon}{2}||v||_{L^2}^2\right) \leq C\left(\frac{\epsilon}{2}||Du||_{L^2}^2 + \frac{\epsilon}{2}||v||_{L^2}^2\right)$$

by Hölder's inequality, the Sobolev embedding theorem and Poincaré's inequality. By the above estimates, we have also

$$\int (cu)^2 \leq M||Du||_{L^2}^2.$$

Following the line of reasoning in [6], the result for $q \in L^n$ is a sufficient replacement for the estimate for L^∞ to get the regularity estimate,

$$||u||_{H^2(\Omega)} \leq C\left(||g||_{L^2(\Omega)} + ||u||_{H^1(\Omega)}\right)$$

where C depends only on Ω and n and q. Now, recalling the second energy estimate (1) in Section 5,

$$||u||_{H^1(\Omega)}^2(\Omega) \leq CB(u,u) = C\int_\Omega gu \leq C\left(\frac{1}{2}||g||_{L^2(\Omega)}^2 + \frac{1}{2}||u||_{L^2(\Omega)}^2\right),$$

since u is a weak solution to $(**)$. The last inequality is given by Cauchy's inequality with $\epsilon = 1$. Also since u is a unique solution, the L^2 norm of u is controlled by the L^2 norm of g by Theorem 6.2.6 in [6]. Therefore,

$$||u||_{H^2(\Omega)} \leq C||g||_{L^2(\Omega)},$$

where C depends only on Ω, n, and more *significantly*, q. The above estimate, in conjunction with the existence and uniqueness result in Section 5, completes the proof of Theorem 3.

5.9 Convergence

In this section we complete the proof of Theorem 4. In the previous two sections, it was shown that $(**)$ is uniquely solvable in H^1 and the solution is *a priori* in H^2 with estimate controlled by the forcing term g. Recalling that $(**)$ represents an arbitrary iteration of Newton's method at time t_j, the linear equation solved by the difference, $u_{m+1} - u_m$ for $m > 1$, is given by

$$-\Delta(u_{m+1} - u_m) + (-t_j f'(u_m))(u_{m+1} - u_m)$$

$$= t_j(f(u_m) - f(u_{m-1}) - f'(u_{m-1})(u_m - u_{m-1})) \qquad \text{in } \Omega$$

$$u_{m+1} - u_m = 0 \qquad \text{on } \Gamma.$$

This is $(**)$ with

$$u = u_{m+1} - u_m, \quad q = -t_j f'(u_m),$$

$$g = t_j(f(u_m) - f(u_{m-1}) - f'(u_{m-1})(u_m - u_{m-1})),$$

and a zero boundary condition. Indeed, using the same argument as at the end of Section 5, it is clear that $g \in L^2$. For $m = 0$, by the definition of u_0 at time t_j, the problem satisfied by $u_1 - u_0$ is

$$-\Delta(u_1 - u_0) + (-t_j f'(u_0))(u_1 - u_0)$$

$$= (t_j - t_{j-1})f(u_0) \qquad \text{in } \Omega$$

$$u_1 - u_0 = 0 \qquad \text{on } \Gamma,$$

and is again a model for (**). To facilitate the convergence estimates to follow, it will be helpful to use Taylor's theorem to simplify g. Similar to the application of a mean value theorem used in [11], for $m > 1$, g can be written as

$$g = t_j(u_m - u_{m-1})^2 \int_{(0,1)} f''(\tau u_m + (1-\tau)u_{m-1})(1-\tau)d\tau.$$

Theorem 3 and the boundedness of f'' give the estimate,

$$||u_{m+1} - u_m||_{H^2} \leq C||t_j(u_m - u_{m-1})^2 \int_{(0,1)} f''(\tau u_m + (1-\tau)u_{m-1})(1-\tau)d\tau||_{L^2}$$

$$\leq \frac{Ct_jM}{2}||(u_m - u_{m-1})^2||_{L^2}$$

$$\leq \frac{Ct_jM}{2}||(u_m - u_{m-1})||^2_{L^4}.$$

Before progressing with the estimate, it is important to discuss the dependence on dimension. For dimensions $n = 3$ and $n = 4$, the L^4 norm is controlled by the H^1 norm, by the Sobolev embedding theorem, which in turn is controlled by the H^2 norm. For dimensions $n = 5, 6, 7,$and 8, the L^4 norm is controlled by the H^2 norm, via the more general Sobolev inequality [1,p.270]. The subsequent calculations do not depend on which dimension $n \in (3, 4, 5, 6, 7, 8)$ is assumed. However, only in dimension $n = 3$ does the general Sobolev theorem assure that our H^2 solution is indeed continuous. For $n = 5, 6, 7,$ and 8, the H^2 solution is respectively, L^{10}, L^6, $L^{\frac{14}{3}}$, and L^4. To continue with the convergence estimate, for $n \in (3, 4, 5, 6, 7, 8)$, we have

$$\frac{Ct_jM}{2}||(u_m - u_{m-1})||_{L^4}^2 \leq \frac{Ct_jMC_s}{2}||(u_m - u_{m-1})||_{H^2}^2$$

where C_s is the constant from the Sobolev theorem and only depends on Ω and n. Since in Theorem 3, C depends on $||f'(u_m(x,t_j))||_{L^n}$ and hence m and t_j, we invoke the boundedness of f'. Therefore $||f'(u_m(x,t_j))||_{L^n}$ is bounded by some constant $C > 0$, uniformly over m and t_j. Let $K = \frac{CMC_s}{2}$. Inductively,

$$||u_{m+1} - u_m||_{H^2} \leq (t_jK||u_1 - u_0||_{H^2})^{2^m - 1}||u_1 - u_0||_{H^2}$$

and therefore for $s \in \mathbb{N}$,

$$||u_{m+s} - u_m||_{H^2} \leq [a^{2^{m+s-1}-1} + ... + a^{2^m - 1}]||u_1 - u_0||_{H^2}$$

where $a = t_jK||u_1 - u_0||_{H^2}$. If t_j is chosen such that $a < 1$, then the *positive* expression in brackets above is bounded from above by the tail end of a convergent geometric series, and therefore goes to zero as $m \to \infty$. We have now shown that u_m is a Cauchy sequence in the Banach space $H^2(\Omega)$, and therefore converges to some $u^* \in H^2(\Omega)$. As stated in [11], due to the continuity of $\widetilde{f}$ and the boundedness of f', it is clear that u^* satisfies

$$(*')\begin{cases} -\Delta u = t_j f(u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

almost everywhere and that the uniqueness of the solution u^* follows from the uniqueness of the solution $u_m(x,t_j)$ to $(**)$ for each m and t_j.

We now show that, given hypotheses $(I)-(IV)$, we can choose t_j sufficient for convergence as above, for each j, in a manner such that for some $N \in \mathbb{N}$, $t_N = 1$. In other words, the solution to $(*')$ can be constructed after a finite number of applications of Newton's method. To make this precise we look at the problem satisfied by $u_1 - u_0$ at time t_j and apply Theorem 3 and the boundedness of f and f' to estimate,

$$||u_1 - u_0||_{H^2} \leq C||(t_j - t_{j-1})f(u_0)||_{L^2}$$

$$\leq C(t_j - t_{j-1})||f(u_0)||_{L^2} \leq MC(t_j - t_{j-1}).$$

If $A = MC$, then A depends on the bounds on f and f', the volume of Ω, and n, but not on t_j. In the following inequality,

$$Kt_j||u_1 - u_0||_{H^2} \leq KAt_j(t_j - t_{j-1}) < 1,$$

the condition for convergence at time t_j was that the leftmost expression be < 1. Since $t_j \leq 1$ for all j, it suffices to impose, for each $j \geq 1$, that

$$t_j - t_{j-1} < \frac{1}{KA}.$$

As KA only depends on Ω, $p = 2$, n, and M, (and in particular, not j), KA gives a uniform bound on the time intervals, and therefore $t = 1$ is attainable after finitely many applications of Newton's method. When Ω is a domain in $\mathbb{R}^3$, the H^2 solution is then continuous by the general Sobolev embedding theorem, and thus the proof of Theorem 4 is completed.

5.10 Conclusion

The goal for improving this procedure is to weaken the assumptions on f and f'. In particular, we would like to eliminate the boundedness or equivalently the uniform boundedness of $f(u)$ and $f'(u)$. To do this requires a function f such that $f(u_m(x,t))$ does not grow too fast in L^2 norm as t increases and such that $f'(u_m(x,t))$ does not grow too fast in L^n norm as m and t increase. If the boundedness of f is dropped from the assumptions, a linear function would be allowed, but assumption (IV) would force it to be decreasing. Since the spectrum of $-\Delta$ is positive, $(*')$ is then solved uniquely with $u \equiv 0$ (which is achieved vacuously in the procedure). An example of a function satisfying (I)-(IV) is

$$f(x) = cot^{-1}(x)$$

whose derivatives are

$$f'(x) = \frac{-1}{1+x^2} \quad \text{and} \quad f''(x) = \frac{2x}{(1+x^2)^2}.$$

Similarly, if $\epsilon > 0$, $A > 0$, and $h, k \in \mathbb{R}$, then

$$Acot^{-1}\left(\frac{x-h}{\epsilon}\right) + k$$

represents a family of functions, each of which satisfy (I)-(IV). A subset of this family, given by

$$f_\epsilon(x) = \frac{1}{\pi}cot^{-1}(\frac{x}{\epsilon}) - 1,$$

is of interest since

$$f_\epsilon(x) \to -H \quad \text{as} \quad \epsilon \to 0$$

$$f'_\epsilon(x) = \frac{-\epsilon}{\epsilon^2 + x^2} \to -\delta \quad \text{as} \quad \epsilon \to 0,$$

where H is the Heaviside function and δ is the Dirac delta function and the arrows imply at least pointwise convergence and possibly a more refined limit. It is natural to ask whether the Newton-embedding procedure can be carried out in a distributional setting with $f = -H$ and whether f_ϵ produces a meaningful approximation to the Heaviside function for small ϵ. More generally, if $\mathcal{P}$ is the class of functions which satisfy (I)-(IV), it is of interest as to which functions exist in a suitable closure of $\mathcal{P}$. In this case, 'suitable closure' can be taken to mean one whose functions allow for the application of the Newton-embedding procedure in possibly a distributional or more general setting, and produce a solution which can be approximated by applying the procedure to a function in $\mathcal{P}$.

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