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Synge's Theorem, Systole, and Positive Intermediate Ricci Curvature

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Synge's Theorem, Systole, and Positive Intermediate Ricci Curvature

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Savanna Gail Gee

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To Colby and Brie.

Bella, too.

ABSTRACT OF THE DISSERTATION

Synge's Theorem, Systole, and Positive Intermediate Ricci Curvature

by

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Doctor of Philosophy, Graduate Program in Mathematics
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Dr. Frederick Wilhelm, Chairperson

In 1997, Wilhelm [15] proved the following generalization of Synge's Theorem: let (M, g_M) be a compact Riemannian n-manifold with $\mathrm{Ric}_k(M, g_M) \geq k$ and $\mathrm{sys}_1(M, g_M) > \pi\sqrt{\frac{k-1}{k}}$; if n is even and M is orientable, then M is simply connected; if n is odd, then M is orientable (Theorem 1.2). Furthermore, he proved that this lower bound on sys_1 is optimal when $k = n - 1$. In 2020, Mouillé [6] proved that $S^3 \times S^3$ admits a metric g_ℓ with $\mathrm{Ric}_2(S^3 \times S^3, g_\ell) > 0$ (Theorem 1.4).

In this dissertation, we first show that the metric g_ℓ (which is a Cheeger deformation) is canonical variation. This follows from a more general result we prove (Theorem 1.7), which is that if (M, g_ℓ) is a Cheeger deformation by (G, g_{bi}) that satisfies what we call the generalized Petersen-Wilhelm hypothesis (Definition 2.16), then for all $p \in M$, the orbit $G(p)$ is normal homogeneous and $g_\ell|_p$ is canonical variation with respect to the Riemannian submersion $\pi : (M, g_M) \xrightarrow{2.3} (M/G, \overline{g})$. Moreover, if $G(p)$ is totally geodesic for all $p \in M$, then g_ℓ is canonical variation (Theorem 1.8).

We then develop a technique for finding an optimal lower bound on Ric_k for any Riemannian manifold (M, g_M) with dimension $n \geq 4$. Specifically, for any $p \in M$, unit vector $x \in T_pM$, and $k \in \mathbb{N}$ such that $2 \leq k \leq n-2$, we prove that $\min \mathrm{Ric}_k(x; \bullet) = \mathrm{Ric}(x) - \max \mathrm{Ric}_{n-1-k}(x; \bullet)$ (Theorem 1.9).

From there, letting $\mathbb{Z}_2$ act on $S^3 \times S^3$ in two ways—as the antipodal map $a : (N_1, N_2) \mapsto (-N_1, -N_2)$ and as $f : (N_1, N_2) \mapsto (-N_1, N_2)$—we study for each value of $t \in (0,1)$ the manifold $\frac{S^3 \times S^3}{\mathbb{Z}_2}$ paired with the unique metric $\overline{g_t}$ that makes the quotient map $(S^3 \times S^3, g_t) \longrightarrow \left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_t}\right)$ a local isometry. In particular, we establish t-independent and t-dependent upper bounds on the product $\min \mathrm{Ric}_k \left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_t}\right) \cdot \left(\mathrm{sys}_1 \left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_t}\right)\right)^2$ when $k = 2, 3, 4$ (Theorems 1.5 and 1.6).

Finally, we notice that our upper bounds are smaller than Wilhelm's upper bounds. We conclude that when restricted to the family $\left\{\left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_t}\right) \mid 0 < t < 1\right\}$, we have established an improved upper bound on $\min \mathrm{Ric}_k \cdot (\mathrm{sys}_1)^2$ when $k = 2, 3, 4$.

Contents

Chapter 1:

Introduction

1.1 Motivation

In 1936, Synge proved the following theorem:

Theorem 1.1 (Theorem 6.3.6 in [10]). *Let* (M, g_M) *be a compact Riemannian* n*-manifold with* $\sec(M, g_M) > 0$*.*

1) *If* n *is even and* M *is orientable, then* M *is simply connected.*
2) *If* n *is odd, then* M *is orientable.*

In 1997 publication [15], Wilhelm proved that if a lower bound on the length of the shortest noncontractible closed curve in M—called the **first systole** of M and denoted $\mathrm{sys}_1(M, g_M)$—is imposed, the conclusions of Synge's Theorem hold when the assumption $\sec(M, g_M) > 0$ is replaced with the assumption $\mathrm{Ric}_k(M, g_M) \geq k$:

Theorem 1.2 (Main Theorem in [15]). *Let (M, g_M) be a compact Riemannian n-manifold with* $\mathrm{Ric}_k(M, g_M) \geq k$ *and* $\mathrm{sys}_1(M, g_M) > \pi\sqrt{\frac{k-1}{k}}$.

1) *If n is even and M is orientable, then M is simply connected.*

2) *If n is odd, then M is orientable.*

Remark: Theorem 1.2 is a generalization of Synge's Theorem. Indeed, when $k = 1$, Theorem 1.2 is Synge's Theorem.

Alternatively stated:

Theorem 1.3 (Alternative Version of Theorem 1.2). *Let (M, g_M) be a compact Riemannian n-manifold with* $\min \mathrm{Ric}_k(M, g_M) \cdot \Big(\mathrm{sys}_1(M, g_M)\Big)^2 > (k-1)\pi^2$.

1) *If n is even and M is orientable, then M is simply connected.*

2) *If n is odd, then M is orientable.*

Remark: The product $\min \mathrm{Ric}_k \cdot (\mathrm{sys}_1)^2$ is invariant under rescaling of the metric. That is, for all $\lambda \in \mathbb{R}$, $\min \mathrm{Ric}_k(M, \lambda^2 g_M) \cdot \Big(\mathrm{sys}_1(M, \lambda^2 g_M)\Big)^2 = \min \mathrm{Ric}_k(M, g_M) \cdot \Big(\mathrm{sys}_1(M, g_M)\Big)^2$.

Wilhelm proved that this bound on $\min \mathrm{Ric}_k \cdot (\mathrm{sys}_1)^2$ is optimal when $k = n-1$ (positive Ricci curvature). Example A in [15] proves optimality when n is even and $k = n-1$:

Example A *in* [15]: *Equip* S^m $(m \geq 2)$ *with its usual metric* g *and equip* $S^m \times S^m$ *with the product metric* $g+g$. *Let* $\mathbb{Z}_2$ *act as the antipodal map on both factors of* $S^m \times S^m$. *The quotient space* $\frac{S^m \times S^m}{\mathbb{Z}_2}$ *is even-dimensional, compact, orientable, and* not *simply connected* $(\pi_1 = \mathbb{Z}_2)$. *If we equip* $\frac{S^m \times S^m}{\mathbb{Z}_2}$ *with the unique metric* $\overline{g}$ *that makes the quotient map* $q : (S^m \times S^m, g+g) \longrightarrow \left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right)$ *a local isometry* (*see* Definition A.1 *and* Section 1.3.3 *in* [10]), *then*

$$\min \mathrm{Ric}\left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right) \cdot \left(\mathrm{sys}_1\left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right)\right)^2$$
$$= (m-1)\left(\sqrt{2}\pi\right)^2 = (2m-2)\pi^2 = \Big((2m-1)-1\Big)\pi^2 = (k-1)\pi^2.$$

This example does not work when $k = 2, 3, ..., 2m-2$. Indeed, when $k \leq m$,

$$\sec_{g+g}\Big(\left(v, \vec{0}\right), \left(\vec{0}, w\right)\Big) = 0 \quad \text{for all } v, w \in TS^m$$
$$\Longrightarrow \min \mathrm{Ric}_k(S^m \times S^m, g+g) = 0 \quad \text{when } k \leq m$$
$$\overset{A.2}{\Longrightarrow} \min \mathrm{Ric}_k\left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right) = 0$$
$$\Longrightarrow \min \mathrm{Ric}_k\left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right) \cdot \left(\mathrm{sys}_1\left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right)\right)^2 = 0 < (k-1)\pi^2$$

and when $m+1 \le k \le 2m-2$,

$$
\begin{aligned}
&\min \mathrm{Ric}_k(S^m \times S^m, g+g) = k-m \\
\overset{A.2}{\Longrightarrow}\ &\min \mathrm{Ric}_k\left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right) = k-m \\
\Longrightarrow\ &\min \mathrm{Ric}_k\left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right) \cdot \left(\mathrm{sys}_1\left(\frac{S^m \times S^m}{\mathbb{Z}_2}, \overline{g}\right)\right)^2 \\
&= (k-m)(\sqrt{2}\pi)^2 = 2(k-m)\pi^2 \le 2\left(k - \frac{k+2}{2}\right)\pi^2 = (k-2)\pi^2 < (k-1)\pi^2.
\end{aligned}
$$

A key result of Mouillé's dissertation [6] is a metric on $S^3 \times S^3$ that admits positive intermediate Ric_2 curvature:

Theorem 1.4 (Theorem A in [6])**.** *The manifold $S^3 \times S^3$ admits a metric g_ℓ such that $\mathrm{Ric}_2(S^3 \times S^3, g_\ell) > 0$. The metric g_ℓ is a Cheeger deformation of the usual product metric $g+g$ on $S^3 \times S^3$ with respect to the left diagonal action of S^3.*

In this dissertation, we study the Riemannian manifold $\frac{(S^3 \times S^3, g_\ell)}{\mathbb{Z}_2}$. Equipping $\frac{S^3 \times S^3}{\mathbb{Z}_2}$ with the unique metric $\overline{g_\ell}$ that makes the quotient map $q : (S^3 \times S^3, g_\ell) \longrightarrow \left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_\ell}\right)$ a local isometry, we explore how close we can get $\min \mathrm{Ric}_k\left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_\ell}\right) \cdot \left(\mathrm{sys}_1\left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_\ell}\right)\right)^2$ to $(k-1)\pi^2$ when $k = 2, 3, 4$.

RESEARCH QUESTION

How close can we get $\min \mathrm{Ric}_k\left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_\ell}\right) \cdot \left(\mathrm{sys}_1\left(\frac{S^3 \times S^3}{\mathbb{Z}_2}, \overline{g_\ell}\right)\right)^2$ to $(k-1)\pi^2$ when $k = 2, 3, 4$?

In Lemma 3.13, we prove that this Cheeger deformation $(S^3 \times S^3, g_\ell)$ is a canonical variation, which we denote by $(S^3 \times S^3, g_t)$. Our results are written in terms of g_t rather than g_ℓ.

1.2 Statement of Results

1.2.1 Main Theorems

Let

- $S^3 \times S^3$ be equipped with the canonical variation metric g_t.
- $a : S^3 \times S^3 \to S^3 \times S^3$ be defined by $(N_1, N_2) \mapsto (-N_1, -N_2)$.
- $f : S^3 \times S^3 \to S^3 \times S^3$ be defined by $(N_1, N_2) \mapsto (-N_1, N_2)$.
- $\mathbb{Z}_2^a \cong H_a = \{\mathrm{id}, a\} \subseteq \mathrm{Iso}\,\{S^3 \times S^3, g_t\}$.
- $\mathbb{Z}_2^f \cong H_f = \{\mathrm{id}, f\} \subseteq \mathrm{Iso}\,\{S^3 \times S^3, g_t\}$.
- $M = \frac{S^3 \times S^3}{\mathbb{Z}_2^a}$ be equipped with the unique metric g_t^a that makes the quotient map $\pi_t^a : (S^3 \times S^3, g_t) \longrightarrow (M, g_t^a)$ a local isometry.
- $N = \frac{S^3 \times S^3}{\mathbb{Z}_2^f}$ be equipped with the unique metric g_t^f that makes the quotient map $\pi_t^f : (S^3 \times S^3, g_t) \longrightarrow \left(N, g_t^f\right)$ a local isometry.

We proved the following two theorems (1.5 and 1.6):

Theorem 1.5. *For all* $t \in (0,1)$,

1a′) $\min \mathrm{Ric}_4 (M, g_t^a) \cdot \left(\mathrm{sys}_1 (M, g_t^a)\right)^2 \leq 2\pi^2$.

1b′) $\min \mathrm{Ric}_3 (M, g_t^a) \cdot \left(\mathrm{sys}_1 (M, g_t^a)\right)^2 \leq 2s^4\pi^2 \approx 0.4683\pi^2$.

1c′) $\min \mathrm{Ric}_2 (M, g_t^a) \cdot \left(\mathrm{sys}_1 (M, g_t^a)\right)^2 \leq s^4\pi^2 \approx 0.2341\pi^2$.

2a′) $\min \mathrm{Ric}_4 \left(N, g_t^f\right) \cdot \left(\mathrm{sys}_1 \left(N, g_t^f\right)\right)^2 \leq \left(\frac{-r^4+3r^2+4}{4}\right)\pi^2 \approx 1.3176\pi^2$.

2b′) $\min \mathrm{Ric}_3 \left(N, g_t^f\right) \cdot \left(\mathrm{sys}_1 \left(N, g_t^f\right)\right)^2 \leq \left(\frac{s^4+s^2}{2}\right)\pi^2 \approx 0.359\pi^2$.

2c′) $\min \mathrm{Ric}_2 \left(N, g_t^f\right) \cdot \left(\mathrm{sys}_1 \left(N, g_t^f\right)\right)^2 \leq \left(\frac{s^4+s^2}{4}\right)\pi^2 \approx 0.1795\pi^2$.

where $r \in (0,1)$ *satisfy* $r^6 + 6r^4 - 19r^2 + 8 = 0$ $(r \approx 0.7143)$ *and* $s \in (0,1)$ *satisfy* $s^6 - 2s^4 + 9s^2 - 4 = 0$ $(s \approx 0.6956)$.

Theorem 1.6 below displays less understandable but more precise bounds on the product $\min \mathrm{Ric}_k \cdot (\mathrm{sys}_1)^2$ for $k = 2, 3, 4$:

Theorem 1.6. *For all* $t \in (0,1)$,

1a) $\min \operatorname{Ric}_4 (M, g_t^a) \cdot \left(\operatorname{sys}_1 (M, g_t^a)\right)^2 \leq \min \left\{ \frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2} \right\} \cdot 2t^2\pi^2.$

1b) $\min \operatorname{Ric}_3 (M, g_t^a) \cdot \left(\operatorname{sys}_1 (M, g_t^a)\right)^2 \leq \min \left\{ t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2} \right\} \cdot 2t^2\pi^2.$

1c) $\min \operatorname{Ric}_2 (M, g_t^a) \cdot \left(\operatorname{sys}_1 (M, g_t^a)\right)^2 \leq \min \left\{ \frac{t^2}{2}, \frac{2t^4-4t^2+2}{(t^2+1)^2} \right\} \cdot 2t^2\pi^2.$

2a) $\min \operatorname{Ric}_4 \left(N, g_t^f\right) \cdot \left(\operatorname{sys}_1 \left(N, g_t^f\right)\right)^2 \leq \min \left\{ \frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2} \right\} \cdot \left(\frac{t^2+1}{2}\right) \pi^2.$

2b) $\min \operatorname{Ric}_3 \left(N, g_t^f\right) \cdot \left(\operatorname{sys}_1 \left(N, g_t^f\right)\right)^2 \leq \min \left\{ t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2} \right\} \cdot \left(\frac{t^2+1}{2}\right) \pi^2.$

2c) $\min \operatorname{Ric}_2 \left(N, g_t^f\right) \cdot \left(\operatorname{sys}_1 \left(N, g_t^f\right)\right)^2 \leq \min \left\{ \frac{t^2}{2}, \frac{2t^4-4t^2+2}{(t^2+1)^2} \right\} \cdot \left(\frac{t^2+1}{2}\right) \pi^2.$

RESEARCH CONCLUSION

We improved Wilhelm's bound on $\min \operatorname{Ric}_k \cdot (\operatorname{sys}_1)^2$ restricted to the families $\left\{(M, g_t^a) \;\middle|\; 0 < t < 1\right\}$ and $\left\{\left(N, g_t^f\right) \;\middle|\; 0 < t < 1\right\}$ of Riemannian manifolds.

1.2.2 When Cheeger Deformation and Canonical Variation Coincide

Under certain assumptions, Cheeger deformation and canonical variation coincide:

Theorem 1.7. *Let (M, g_M) be a Riemannian manifold. Let G be a compact Lie group that acts isometrically on M. Equip G with a bi-invariant metric g_{bi}, and let g_ℓ be the Cheeger deformed metric on M defined in* Definition 2.4*. Suppose the generalized Petersen-Wilhelm hypothesis is satisfied* (Definition 2.16)*. Then for each $p \in M$,*

1) *The intrinsic metric on $G(p)$ is normal homogeneous.*
2) *$g_\ell|_p$ is canonical variation with respect to $\pi : (M, g_M) \xrightarrow{2.3} (M/G, \overline{g})$ with rescaling factor $\frac{\ell^2}{\ell^2+\lambda_p^2}$ where λ_p is as in* Corollary 2.18.

When we add to Theorem 1.7 the assumption that the orbits of $G \curvearrowright M$ are totally geodesic, we get a more impressive result:

Theorem 1.8. *Assume the same setup as in* Theorem 1.7*. If the orbits of $G \curvearrowright M$ are totally geodesic, then λ_p in* Corollary 2.18 *is independent of p. That is, g_ℓ is canonical variation with respect to $\pi : (M, g_M) \xrightarrow{2.3} (M/G, \overline{g})$ with rescaling factor $\frac{\ell^2}{\ell^2+\lambda^2}$.*

1.2.3 An Optimal Lower Bound on Ric_k

In our effort to calculate $\min \mathrm{Ric}_k$, we discovered the following optimal inequality:

Theorem 1.9. *Let (M, g_M) be a Riemannian n-manifold with $n \geq 4$. Let $p \in M$. For all unit $x \in T_pM$, $k \in \mathbb{N}$ such that $2 \leq k \leq n-2$,*

$$\min \mathrm{Ric}_k(x; \bullet) = \mathrm{Ric}(x) - \max \mathrm{Ric}_{n-1-k}(x; \bullet)$$

where the minimum is taken over all orthonormal k-frames orthogonal to x and the maximum is taken over all orthonormal $n-1-k$ frames orthogonal to x.

Remark: It is notable that M need not be be compact nor complete for the conclusion of Theorem 1.9 to hold.

Proof. Let $p \in M$ and $x \in T_pM$ satisfy $|x|_{g_M} = 1$. Then $\mathrm{Ric}(x) = \sum_{i=1}^{n-1} \sec(x, e_i)$ where $\{e_i\}_{i=1}^{n-1}$ is an orthonormal basis for the orthogonal complement of x, which we denote $x^\perp$. For $1 \leq k \leq n-2$,

$$\sum_{i=1}^{n-1} \sec(x, e_i) = \sum_{j=1}^{k} \sec(x, v_j) + \sum_{\ell=1}^{n-1-k} \sec(x, u_\ell)$$

where $\{v_j\}_{j=1}^{k}$ is any collection of k vectors from $\{e_i\}_{i=1}^{n-1}$ and $\{u_\ell\}_{\ell=1}^{n-1-k}$ are the remaining vectors. Thus,

$$\begin{aligned}
\operatorname{Ric}(x) &= \sum_{j=1}^{k} \sec(x, v_j) + \sum_{\ell=1}^{n-1-k} \sec(x, u_\ell) \\
&= \operatorname{Ric}_k(x; v_1, v_2, ..., v_k) + \operatorname{Ric}_{n-1-k}(x; u_1, u_2, ..., u_{n-1-k}) \\
&\implies \operatorname{Ric}_k(x; v_1, v_2, ..., v_k) = \operatorname{Ric}(x) - \operatorname{Ric}_{n-1-k}(x; u_1, u_2, ..., u_{n-1-k}).
\end{aligned}$$

Fixing x, the equation above implies $\operatorname{Ric}_k(x; v_1, v_2, ..., v_k)$ decreases as $\operatorname{Ric}_{n-1-k}(x; u_1, u_2, ..., u_{n-1-k})$ increases. That is, for any particular x,

$$\operatorname{Ric}_k(x; v_1, v_2, ..., v_k) \geq \operatorname{Ric}(x) - \max \operatorname{Ric}_{n-1-k}(x; \bullet).$$

This bound is optimal for $k = 2, 3, ..., n-2$. Indeed, for all values of k, $\operatorname{Ric}_k(x; \bullet)$ is a continuous function on the compact space $\operatorname{Gr}_k\left(x^\perp\right)$ (see Definition A.3 and Theorem A.4), which implies $\max \operatorname{Ric}_{n-1-k}(x; \bullet)$ is attained by some $(n-1-k)$-frame $\{w_1, w_2, ..., w_{n-1-k}\} \subseteq x^\perp$ (see Theorem A.5). If we complete this to an orthonormal basis

$$\{x, w_1, w_2, ..., w_{n-1-k}, y_1, y_2, ..., y_k\}$$

for T_pM, then

$$\operatorname{Ric}_k(x; y_1, y_2, ..., y_k) = \operatorname{Ric}(x) - \operatorname{Ric}_{n-1-k}(x; w_1, w_2, ..., w_{n-1-k}).$$

It follows that

$$\min \operatorname{Ric}_k(x; \bullet) = \operatorname{Ric}(x) - \max \operatorname{Ric}_{n-1-k}(x; \bullet)$$

where the minimum is taken over all orthonormal k-frames orthogonal to x and the maximum is taken over all orthonormal $n-1-k$ frames orthogonal to x. ■

Chapter 2:

Background

2.1 Riemannian Submersions

Riemannian submersions play an important part in defining Cheeger deformation and canonical variation (see Section 2.2.1 [Step 3] and Definition 2.12).

Definition 2.1 (page 5 in [10]). *A map* $F : (M, g_M) \longrightarrow (B, g_B)$ *is a* **Riemannian submersion** *if and only if*

1) F *is a submersion*

2) *For each* $p \in M$, $dF_p|_{\ker(dF_p)^{\perp}}$ *is a linear isometry.*

Definition 2.2 (9.7 in [1]). *Let* $F : (M, g_M) \longrightarrow (B, g_B)$ *be a Riemannian submersion.*

1) $\underline{\boldsymbol{\mathcal{V}^F}} = \ker(dF)$ *is the vertical distribution of* F*.*

2) $\underline{\boldsymbol{\mathcal{H}^F}} = \ker(dF)^{\perp}$ *is the horizontal distribution of* F*.*

Remark: Vectors in $\mathcal{V}^F$ are tangent to the fibers of F, and vectors in $\mathcal{H}^F$ are perpendicular to the fibers of F.

Remark: When the submersion is clear, we sometimes denote $\mathcal{V}^F$ by $\mathcal{V}$ and $\mathcal{H}^F$ by $\mathcal{H}$.

Let G be a group acting on a Riemannian manifold (M, g_M) on the left. Recall that G **acts freely** on M if and only if $ap = p$ for some $p \in M$ implies $a = e$ (see page 162 of [4]), and G **acts isometrically** on M if and only if for all $a \in G$, $f_a : (M, g_M) \longrightarrow (M, g_M)$ defined by $p \mapsto ap$ is an isometry (see page 23 of [5]). Theorem 2.3 below is vital for defining Cheeger deformation (see Section 2.2.1 [Step 3]).

Theorem 2.3 (Theorem 5.6.21 in [10]). *Let (M, g_M) be a Riemannian manifold. If a compact Lie group G acts freely and isometrically on M, then the quotient manifold M/G can be given a Riemannian metric $\overline{g}$ so that the quotient map $F : (M, g_M) \longrightarrow (M/G, \overline{g})$ is a Riemannian submersion.*

2.2 Deformation of a Riemannian Manifold

2.2.1 Cheeger Deformation

Let (M, g_M) be a Riemannian manifold. Let G be a compact Lie group that acts isometrically on M. Equip G with a bi-invariant metric g_{bi} (see Theorem A.6). Let $\ell > 0$.

The following algorithm was developed in 1973 by Cheeger in [2]:

STEP 1: Equip $G \times M$ with the product metric $\ell^2 g_{\text{bi}} + g_M$.

STEP 2: Let $G \curvearrowright (G \times M)$ on the left by $a(b, m) = (ba^{-1}, am)$.

Remark: This action is free and isometric.

Remark: Every orbit of this action has a unique point of the form (e, p), so we can suppose vectors in $T(G \times M)$ are based at (e, p) for some $p \in M$.

STEP 3: Equip the quotient space $\frac{G\times M}{G}$ with the metric g_ℓ that makes the quotient map $q : (G \times M, \ell^2 g_{\text{bi}} + g_M) \longrightarrow \left(\frac{G\times M}{G}, g_\ell\right)$ a Riemannian submersion (see Theorem 2.3).

Remark: The quotient space $\frac{G\times M}{G}$ is diffeomorphic to M (see Theorem A.7).

Definition 2.4. $\Big\{(M, g_\ell) \mid \ell > 0\Big\}$ *is a family of* **Cheeger deformations** *of* (M, g_M).

The following two results describe the quotient map q and its differential $dq_{(e,p)}$:

Theorem 2.5. (1.2 in [11]) *The quotient map* $q : G \times M \longrightarrow M$ *can be identified with the action map from* $G \curvearrowright M$*. That is,* $q(a, p) = ap$ *for all* $a \in G$ *and* $p \in M$ (*see* Theorem A.7).

Theorem 2.6. (1.0.3 in [13]) *Let $p \in M$ and $v \in T_pM$. Let $\mathfrak{g}$ be the Lie algebra of G. Then for all $k \in \mathfrak{g}$,*

$$dq_{(e,p)}(k, v) = K_{M,p}(k) + v.$$

Next, we identify $\mathcal{V}^q_{(e,p)}$.

Definition 2.7 (1.0.1 in [13])**.** *Let $p \in M$ and $\mathfrak{g}$ be the Lie algebra of G. Define*

$$\underline{\boldsymbol{K_{M,p}}} : \mathfrak{g} \to TM \text{ by } k \mapsto \frac{d}{dt}\exp(tk)p\Big|_{t=0}.$$

Remark: $K_{M,p}$ is linear and takes $k \in \mathfrak{g}$ to the value at p of the Killing field generated by k.

Theorem 2.8 (3.0.2 in [14])**.** *Let $p \in M$ and $\mathfrak{g}$ be the Lie algebra of G. Then*

$$\mathcal{V}^q_{(e,p)} = \left\{ \Big(-k, K_{M,p}(k) \Big) \;\Big|\; k \in \mathfrak{g} \right\}.$$

To prove Lemma 4.1 later in this document, we rely on the following definitions:

Definition 2.9 (page 22 of [14])**.** *Let $p \in M$ and $v \in T_pM$. Let $\mathfrak{g}$ be the Lie algebra of G. Define $\underline{\boldsymbol{\widehat{v}_\ell}} \in \mathfrak{g} \times T_pM$ to be the vector satisfying*

1) *$\widehat{v}_\ell$ is horizontal with respect to $q : (G \times M, \ell^2 g_{\text{bi}} + g_M) \longrightarrow (M, g_\ell)$*

2) *$\widehat{v}_\ell$ projects to v under $d(\text{proj}_M)_{(e,p)} : \mathfrak{g} \times T_pM \longrightarrow T_pM$.*

To better understand $\widehat{v}_\ell$, write $\widehat{v}_1 = (\widehat{v}_\mathfrak{g}, \widehat{v}_M)$ where $\widehat{v}_\mathfrak{g}$ is the $\mathfrak{g}$-component of $\widehat{v}_1$ and $\widehat{v}_M$ is the T_pM-component of $\widehat{v}_1$. By Condition 2 in Definition 2.9, $\widehat{v}_M = v$. Denote $\widehat{v}_\mathfrak{g}$ by $\kappa_p(v)$. By Condition 1 in Definition 2.9 with $\ell = 1$, we have for all $k \in \mathfrak{g}$,

$$
\begin{aligned}
(g_{\mathrm{bi}} + g_M)\Big(\widehat{v}_1, \big(-k, K_{M,p}(k)\big)\Big) \overset{2.8}{=} 0 \iff & (g_{\mathrm{bi}} + g_M)\Big(\big(\kappa_p(v), v\big), \big(-k, K_{M,p}(k)\big)\Big) = 0 \\
\iff & g_{\mathrm{bi}}\Big(\kappa_p(v), k\Big) = g_M\Big(v, K_{M,p}(k)\Big).
\end{aligned}
$$

Definition 2.10 (page 22 of [14])**.** *Let* $\mathfrak{g}$ *be the Lie algebra of* G*. For all* $p \in M$*,* $\underline{\boldsymbol{\kappa_p}} : T_pM \longrightarrow \mathfrak{g}$ *is defined implicitly by the fact that for all* $k \in \mathfrak{g}$*,*

$$g_{\mathrm{bi}}\Big(\kappa_p(v), k\Big) = g_M\Big(v, K_{M,p}(k)\Big).$$

Remark: κ_p is linear (see Proposition 2.1 in [13]).

Then for all $\ell > 0$, $\widehat{v}_\ell = \left(\frac{\kappa_p(v)}{\ell^2}, v\right)$ since $g_{\mathrm{bi}}\Big(\kappa_p(v), k\Big) = g_M\Big(v, K_{M,p}(k)\Big) \iff$ $\ell^2 g_{\mathrm{bi}}\Big(\frac{\kappa_p(v)}{\ell^2}, k\Big) = g_M\Big(v, K_{M,p}(k)\Big)$.

Definition 2.11 (page 22 of [14])**.** *For* $p \in M$ $v \in T_pM$*, and* $\ell > 0$*,*

$$\underline{\boldsymbol{\widehat{v}_\ell}} = \left(\frac{\kappa_p(v)}{\ell^2}, v\right).$$

2.2.2 Canonical Variation

Definition 2.12 (Definition 9.67 in [1]). *Let* $F : (M, g_M) \longrightarrow (B, g_B)$ *be a Riemannian submersion. Let* $u, v \in \mathcal{V}^F$ *and* $x, y \in \mathcal{H}^F$. *Suppose* $t \in \mathbb{R}$ *satisfies* $0 < t < 1$. *The* **canonical variation** g_t *of the metric* g_M *is defined by setting*

1) $g_t(u, v) = t^2 g_M(u, v)$
2) $g_t(x, y) = g_M(x, y)$
3) $g_t(u, x) = 0.$

Canonical variation has a number of useful properties:

Theorem 2.13 (9.68 in [1]). *Let* $F : (M, g_M) \longrightarrow (B, g_B)$ *be a Riemannian submersion and* (M, g_t) *be the canonical variation of* (M, g_M) *with respect to* F. *Then for all* $t \in (0, 1)$,

1) $F : (M, g_t) \longrightarrow (B, g_B)$ *is a Riemannian submersion.*
2) $\mathcal{H}^F$ *is the same (independent of* t*).*
3) *If the fibers of* F *are totally geodesic with respect to* g_M*, then they are totally geodesic with respect to* g_t.

The following theorem describes how ∇^{g_t} compares to ∇^{g_M}:

Theorem 2.14 (Lemma 2.2 in [8]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion and (M, g_t) be the canonical variation of (M, g_M). Let $U, W \in \mathcal{V}$ and $X, Y \in \mathcal{H}$. Then for all $t \in (0, 1)$,*

1) $\mathcal{V}\nabla^{g_t}_X U = \mathcal{V}\nabla^{g_M}_X U$.

2) $\mathcal{H}\nabla^{g_t}_X U = t^2 \mathcal{H}\nabla^{g_M}_X U$.

3) $\nabla^{g_t}_X Y = \nabla^{g_M}_X Y$.

4) $\mathcal{V}\nabla^{g_t}_U W = \nabla^{g_M}_U W$.

2.2.3 Interpolating Between Cheeger and Canonical

For each $p \in M$, we can decompose the Lie algebra $\mathfrak{g}$ of G as $T_eG_p \oplus (T_eG_p)^{\perp_{g_{\mathrm{bi}}}} := \mathfrak{g}_p \oplus \mathfrak{m}_p$ (the notation here is adopted from [13]).

Theorem 2.15 (Proposition 2.1 in [13]). *For each $p \in M$, $K_{M,p}|_{\mathfrak{m}_p} : \mathfrak{m}_p \longrightarrow T_pG(p)$ is a linear isomorphism.*

Definition 2.16. *Let* (M, g_M) *be a Riemannian manifold. Let* G *be a compact Lie group that acts on* M *isometrically. Equip* G *with a bi-invariant metric* g_{bi} *and let* $\mathfrak{g}$ *be the Lie algebra of* G*. The* **generalized Petersen-Wilhelm hypothesis is satisfied** *if and only if for all* $p \in M$ *and for all* $k_1, k_2 \in \mathfrak{m}_p$*,* $g_{\text{bi}}(k_1, k_2) = 0 \implies g_M\Big(K_{M,p}(k_1), K_{M,p}(k_2)\Big) = 0$.

Remark: This is a generalization of assumption (1.5) in [11], which requires for all $p \in M$ and for all $k_1, k_2 \in \mathfrak{g}$ that $g_{\text{bi}}(k_1, k_2) = 0 \implies g_M\Big(K_{M,p}(k_1), K_{M,p}(k_2)\Big) = 0$ holds. Definition 2.16 is a generalization of (1.5) in the sense that every action $G \curvearrowright M$ satisfying (1.5) also satisfies Definition 2.16, but not the other way around.

Lemma 2.17. *Let* $(V, \langle,\rangle_V)$ *and* $(W, \langle,\rangle_W)$ *be inner product spaces of dimension* n*. Let* $L : (V, \langle,\rangle_V) \longrightarrow (W, \langle,\rangle_W)$ *be a linear isomorphism with matrix representation*

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

If for all $v_1, v_2 \in V$*, the implication* $\langle v_1, v_2\rangle_V = 0 \implies \big\langle L(v_1), L(v_2)\big\rangle_W = 0$ *holds, then* $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

Applying Lemma 2.17 to $K_{M,p}|_{\mathfrak{m}_p}$, we conclude that actions satisfying (1.5) in [11] must also satisfy for all $p \in M$ either $G_p = G$ $(K_{M,p} \equiv 0)$ or $G_p = \{e\}$ $\big(K_{M,p} \equiv K_{M,p}|_{\mathfrak{m}_p}\big)$. Indeed, if $\dim(\mathfrak{g}) = n$ and $\dim\Big(T_pG(p)\Big) = m < n$, then $K_{M,p}$ is an $m \times n$ matrix with $n - m$ zero columns $(K_{M,p}|_{\mathfrak{g}_p})$ and an $m \times m$ diagonal submatrix $(K_{M,p}|_{\mathfrak{m}_p})$:

$$\begin{bmatrix} 0 & \cdots & 0 & \lambda & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

Then $n \times 1$ vectors $(\underbrace{1, 0, \cdots, 0}_{n-m}, \underbrace{1, 2, 3, \cdots, m-1, m}_{m})$ and

$(\underbrace{-1-2-3-\cdots-(m-1)-m, 0, \cdots, 0}_{n-m}, \underbrace{1, 1, \cdots, 1}_{m})$ are perpendicular, but their

respective images $m \times 1$ images $(\lambda, 2\lambda, 3\lambda, \cdots, (m-1)\lambda, m\lambda)$ and $(\lambda, \lambda, \lambda, \cdots, \lambda, \lambda)$ under $K_{M,p}$ are not, so (1.5) is not satisfied.

Proof. (of Lemma 2.17) (by induction) Let $\mathcal{B}_V = \{v_1, v_2, ..., v_n\}$ be an orthonormal basis for V. By assumption, $L(v_i) \perp L(v_j)$ for all $i \neq j$, so $\{L(v_1), L(v_2), ..., L(v_n)\}$ is an orthogonal basis for W, and $\mathcal{B}_W = \{w_1, w_2, ..., w_n\}$ where $w_i = \frac{L(v_i)}{|L(v_i)|_W}$ is an orthonormal basis for W.

It follows that the matrix representation for L is

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_i = |L(v_i)|_W$. We proceed by induction.

<u>Base Case:</u> Take $v_1, v_2 \in \mathcal{B}_V$ and consider vectors $v_1 + v_2$ and $v_1 - v_2$. Since $\langle v_1 + v_2, v_1 - v_2 \rangle_V = 0$ implies $\langle L(v_1 + v_2), L(v_1 - v_2) \rangle_W = 0$, we get

$$\begin{aligned} & \langle L(v_1) + L(v_2), L(v_1) - L(v_2) \rangle_W = 0 \\ \implies & \langle \lambda_1 w_1 + \lambda_2 w_2, \lambda_1 w_1 - \lambda_2 w_2 \rangle_W = 0 \\ \implies & \langle \lambda_1 w_1, \lambda_1 w_1 \rangle_W - \langle \lambda_2 w_2, \lambda_2 w_2 \rangle_W = 0 \\ \implies & \lambda_1^2 - \lambda_2^2 = 0 \\ \implies & \lambda_2 = \pm \lambda_1 \implies \lambda_2 = \lambda_1 \text{ because we can replace } w_2 \text{ with } -w_2 \text{ if } \lambda_2 = -\lambda_1. \end{aligned}$$

<u>Induction Assumption (IA):</u> Assume $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda$.

Consider the orthonormal basis $\mathcal{B}_V = \{v_1, v_2, ..., v_n\}$ for V. Let $v \in \text{span}\{v_1, v_2, ..., v_{n-1}\}$ and consider vectors $\dfrac{v}{|v|_V} + v_n, \dfrac{v}{|v|_V} - v_n \in V$. These vectors are orthogonal, so their images $L\left(\frac{v}{|v|_V} + v_n\right)$ and $L\left(\frac{v}{|v|_V} - v_n\right)$ are also orthogonal by assumption.

Thus,

$$
\begin{aligned}
&\left\langle L\left(\frac{v}{|v|_V}+v_n\right), L\left(\frac{v}{|v|_V}-v_n\right)\right\rangle_W = 0 \\
\implies &\frac{1}{|v|_V^2}|L(v)|_W^2 - |L(v_n)|_W^2 = 0 \\
\overset{\text{(IA)}}{\implies} &\frac{1}{|v|_V^2}\cdot\lambda^2|v|_V^2 - |L(v_n)|_W^2 = 0 \\
\implies &\frac{1}{|v|_V^2}\cdot\lambda^2|v|_V^2 - |\lambda_n w_n|_W^2 = 0 \\
\implies &\lambda^2 - \lambda_n^2 = 0 \\
\implies &\lambda_n = \pm\lambda \implies \lambda_n = \lambda \text{ because we can replace } w_n \text{ with } -w_n \text{ if } \lambda_n = -\lambda.
\end{aligned}
$$

■

Corollary 2.18. *Assume the generalized Petersen-Wilhelm hypothesis is satisfied. Then for each $p \in M$, there exists a constant $\lambda_p \in \mathbb{R}^+$ such that for all $k \in \mathfrak{m}_p$,*

$$|K_{M,p}(k)|_{g_M} = \lambda_p |k|_{g_{\text{bi}}}.$$

Proof. Apply Lemma 2.17 with $(V, \langle,\rangle_V) = (\mathfrak{m}_p, g_{\text{bi}})$, $(W, \langle,\rangle_W) = (T_pG(p), g_M)$, and $L = K_{M,p}|_{\mathfrak{m}_p}$ (see Theorem 2.15). ■

We are now equipped to prove Theorems 1.7 and 1.8.

Proof. (of Theorem 1.7)

(1) Let $p \in M$. By Theorem A.8, $F_p : G/G_p \longrightarrow G(p)$ defined by $aG_p \mapsto ap$ is an equivariant diffeomorphism. By the definition of an equivariant map (see Definition A.9), there is a commutative diagram

$$\begin{array}{ccc} G/G_p & \xrightarrow{F_p} & G(p) \\ \theta_a \downarrow & & \downarrow \varphi_a \\ G/G_p & \xrightarrow[F_p]{} & G(p) \end{array}$$

where $a \in G$, θ_a is the map $bG_p \mapsto abG_p$ corresponding to the action $G \curvearrowright G/G_p$, and φ_a is the map $bp \mapsto abp$ corresponding to the action $G \curvearrowright G(p)$. Differentiating, we generate the following commutative diagram.

$$\begin{array}{ccc} (T_{[eG_p]}G/G_p, g_{\mathrm{nh},p}) & \xrightarrow{(dF_p)_{[eG_p]}} & (T_pG(p), g_M) \\ d(\theta_a)_{[eG_p]} \downarrow & & \downarrow d(\varphi_a)_p \\ (T_{[aG_p]}G/G_p, g_{\mathrm{nh},p}) & \xrightarrow[(dF_p)_{[aG_p]}]{} & (T_{ap}G(p), g_M) \end{array}$$

where $g_{\mathrm{nh},p}$ is the normal homogeneous metric on G/G_p induced by the submersion $(G, g_{\mathrm{bi}}) \longrightarrow G/G_p$. We can identify $(T_{[eG_p]}G/G_p, g_{\mathrm{nh},p}) \xrightarrow{(dF_p)_{[eG_p]}} (T_pG(p), g_M)$ with $(\mathfrak{m}_p, g_{\mathrm{bi}}) \xrightarrow{K_{M,p}|_{\mathfrak{m}_p}} (T_pG(p), g_M)$, so Theorem 2.15 and Corollary 2.18 together imply that—after a rescaling on g_{bi}—$(dF_p)_{[eG_p]}$ is a linear isometry. Since G/G_p and $G(p)$ are homogeneous Riemannian manifolds, $d(\theta_a)_{[eG_p]}$ and $d(\varphi_a)_p$ are linear isometries. By the commutative diagram above, $(dF_p)_{[aG_p]}$ is a composition of linear isometries and is hence itself a linear isometry for all $a \in G$. Therefore, F_p is an isometry.

(2) See (1.1) to (1.8) in [11].

Since $q : (G \times M, \ell^2 g_{\text{bi}} + g_M) \longrightarrow (M, g_\ell)$ is a Riemannian submersion, $dq_{(e,p)}|_{\mathcal{H}^q_{(e,p)}}$: $\Big(\mathcal{H}^q_{(e,p)}, \ell^2 g_{\text{bi}} + g_M\Big) \longrightarrow (T_pM, g_\ell)$ is an isometry. To understand $\mathcal{H}^q_{(e,p)}$, decompose $\mathfrak{g} \times T_pM \cong \mathfrak{g} \oplus T_pG(p) \oplus (T_pG(p))^{\perp_{g_M}} \cong \Big(\mathfrak{g} \times T_pG(p)\Big) \oplus \Big(\Big\{\vec{0}\Big\} \times (T_pG(p))^{\perp_{g_M}}\Big)$.

It follows from Theorem 2.8 that for all vectors $x \in (T_pG(p))^{\perp_{g_M}}$, $\Big(\vec{0}, x\Big) \in \mathcal{H}^q_{(e,p)}$. Indeed, for all $k \in \mathfrak{g}$, $(\ell^2 g_{\text{bi}} + g_M)\Big(\Big(\vec{0}, x\Big), \Big(-k, K_{M,p}(k)\Big)\Big) = -\ell^2 g_{\text{bi}}\Big(\vec{0}, k\Big) + g_M\Big(x, K_{M,p}(k)\Big) = 0$. That is, $\Big\{\vec{0}\Big\} \times (T_pG(p))^{\perp_{g_M}} \subseteq \mathcal{H}^q_{(e,p)}$.

To find $\mathcal{H}^q_{(e,p)} \cap \Big(\mathfrak{g} \oplus T_pG(p)\Big)$, let $k_1 \in \mathfrak{m}_p$ and $a, b \in \mathbb{R}$ and consider vector $\Big(ak_1, bK_{M,p}(k_1)\Big) \in \mathfrak{g} \oplus T_pG(p)$. For all $k \in \mathfrak{g}$,

$$
\begin{aligned}
& (\ell^2 g_{\text{bi}} + g_M)\Big(\Big(ak_1, bK_{M,p}(k_1)\Big), \Big(-k, K_{M,p}(k)\Big)\Big) = 0 \\
\iff & -\ell^2 g_{\text{bi}}\Big(ak_1, k\Big) + g_M\Big(bK_{M,p}(k_1), K_{M,p}(k)\Big) = 0.
\end{aligned}
$$

If $k \in \mathfrak{g}_p$, then $-\ell^2 g_{\text{bi}}\Big(ak_1, k\Big) + g_M\Big(bK_{M,p}(k_1), K_{M,p}(k)\Big) = 0$ for all $k_1 \in \mathfrak{m}_p$. If $k \in \mathfrak{m}_p$ and $k_1 \perp k$, then $-\ell^2 g_{\text{bi}}\Big(ak_1, k\Big) + g_M\Big(bK_{M,p}(k_1), K_{M,p}(k)\Big) = 0$ by Definition 2.16. If $k \in \mathfrak{m}_p$ and k_1 is proportional to k, then

$$
\begin{aligned}
& (\ell^2 g_{\text{bi}} + g_M)\Big(\Big(ak_1, bK_{M,p}(k_1)\Big), \Big(-k_1, K_{M,p}(k_1)\Big)\Big) = 0 \\
\iff & -\ell^2 g_{\text{bi}}\Big(ak_1, k_1\Big) + g_M\Big(bK_{M,p}(k_1), K_{M,p}(k_1)\Big) = 0 \\
\iff & -a\ell^2 |k_1|^2_{g_{\text{bi}}} + b|K_{M,p}(k_1)|^2_{g_M} = 0.
\end{aligned}
$$

Since $k_1 \in \mathfrak{m}_p \implies |K_{M,p}(k_1)|^2_{g_M} > 0$, the equation above does not hold when $a = 0$ (unless b is also zero, but then $\Big(ak_1, bK_{M,p}(k_1)\Big) = \vec{0}$, which is uninteresting). Thus,

$$\begin{aligned}
& -a\ell^2|k_1|^2_{g_{\mathrm{bi}}} + b|K_{M,p}(k_1)|^2_{g_M} = 0 \\
\iff & -\ell^2|k_1|^2_{g_{\mathrm{bi}}} + \frac{b}{a}|K_{M,p}(k_1)|^2_{g_M} = 0 \\
\iff & -\ell^2|k_1|^2_{g_{\mathrm{bi}}} + \lambda|K_{M,p}(k_1)|^2_{g_M} = 0 \text{ setting } \frac{b}{a} = \lambda \\
\iff & \lambda = \frac{\ell^2|k_1|^2_{g_{\mathrm{bi}}}}{|K_{M,p}(k_1)|^2_{g_M}}.
\end{aligned}$$

So with this value of λ, vector $\Big(k_1, \lambda K_{M,p}(k_1)\Big)$ is horizontal with respect to q. That is,

$$\left\{\left(k, \frac{\ell^2|k|^2_{g_{\mathrm{bi}}}}{|K_{M,p}(k)|^2_{g_M}} K_{M,p}(k)\right) \;\Big|\; k \in \mathfrak{m}_p\right\} \subseteq \mathcal{H}^q_{e,p}$$
$$\implies \left\{\Big(|K_{M,p}(k)|^2_{g_M} k, \ell^2|k|^2_{g_{\mathrm{bi}}} K_{M,p}(k)\Big) \;\mid\; k \in \mathfrak{m}_p\right\} \subseteq \mathcal{H}^q_{e,p}.$$

Let $X_1 = \left\{\Big(|K_{M,p}(k)|^2_{g_M} k, \ell^2|k|^2_{g_{\mathrm{bi}}} K_{M,p}(k)\Big) \;\mid\; k \in \mathfrak{m}_p\right\}$ and $X_2 = \left\{\vec{0}\right\} \times (T_pG(p))^{\perp_{g_M}}$. Then $X_1 \oplus X_2 \subseteq \mathcal{H}^q_{e,p}$ and

$$\begin{aligned}
\dim(X_1 \oplus X_2) &= \dim(X_1) + \dim(X_2) \\
&= \dim(\mathfrak{m}_p) + \dim\Big((T_pG(p))^{\perp_{g_M}}\Big) \\
&= \dim\Big(T_pG(p)\Big) + \dim\Big((T_pG(p))^{\perp_{g_M}}\Big) = \dim(M) = \dim(\mathcal{H}^q_{e,p}).
\end{aligned}$$

Therefore,

$$\mathcal{H}^q_{e,p} = \left\{\Big(|K_{M,p}(k)|^2_{g_M} k, \ell^2|k|^2_{g_{\mathrm{bi}}} K_{M,p}(k)\Big) \;\mid\; k \in \mathfrak{m}_p\right\} \oplus \left(\left\{\vec{0}\right\} \times (T_pG(p))^{\perp_{g_M}}\right).$$

So for all $k \in \mathfrak{m}_p$ and all $x \in \left(T_pG(p)\right)^{\perp_{g_M}}$,

[Calculation A]

$$\begin{aligned}
\left|\left(|K_{M,p}(k)|_{g_M}^2 k, \ell^2 |k|_{g_{\mathrm{bi}}}^2 K_{M,p}(k)\right)\right|_{\ell^2 g_{\mathrm{bi}}+g_M}^2 &= \left|dq_{(e,p)}\left(|K_{M,p}(k)|_{g_M}^2 k, \ell^2 |k|_{g_{\mathrm{bi}}}^2 K_{M,p}(k)\right)\right|_{g_\ell}^2 \\
&\overset{2.6}{=} \left|K_{M,p}\left(|K_{M,p}(k)|_{g_M}^2 k\right) + \ell^2 |k|_{g_{\mathrm{bi}}}^2 K_{M,p}(k)\right|_{g_\ell}^2 \\
&= \left|\left(\ell^2 |k|_{g_{\mathrm{bi}}}^2 + |K_{M,p}(k)|_{g_M}^2\right) K_{M,p}(k)\right|_{g_\ell}^2 \\
&= \left(\ell^2 |k|_{g_{\mathrm{bi}}}^2 + |K_{M,p}(k)|_{g_M}^2\right)^2 |K_{M,p}(k)|_{g_\ell}^2
\end{aligned}$$

$$\begin{aligned}
\implies |K_{M,p}(k)|_{g_\ell}^2 &= \frac{\left|\left(|K_{M,p}(k)|_{g_M}^2 k, \ell^2 |k|_{g_{\mathrm{bi}}}^2 K_{M,p}(k)\right)\right|_{\ell^2 g_{\mathrm{bi}}+g_M}^2}{\left(\ell^2 |k|_{g_{\mathrm{bi}}}^2 + |K_{M,p}(k)|_{g_M}^2\right)^2} \\
&= \frac{\ell^2 |K_{M,p}(k)|_{g_M}^4 |k|_{g_{\mathrm{bi}}}^2 + \ell^4 |k|_{g_{\mathrm{bi}}}^4 |K_{M,p}(k)|_{g_M}^2}{\left(\ell^2 |k|_{g_{\mathrm{bi}}}^2 + |K_{M,p}(k)|_{g_M}^2\right)^2} \\
&= \frac{\ell^2 |K_{M,p}(k)|_{g_M}^2 |k|_{g_{\mathrm{bi}}}^2 + \ell^4 |k|_{g_{\mathrm{bi}}}^4}{\left(\ell^2 |k|_{g_{\mathrm{bi}}}^2 + |K_{M,p}(k)|_{g_M}^2\right)^2} |K_{M,p}(k)|_{g_M}^2 \\
&= \frac{\ell^2 |k|_{g_{\mathrm{bi}}}^2 \left(|K_{M,p}(k)|_{g_M}^2 + \ell^2 |k|_{g_{\mathrm{bi}}}^2\right)}{\left(\ell^2 |k|_{g_{\mathrm{bi}}}^2 + |K_{M,p}(k)|_{g_M}^2\right)^2} |K_{M,p}(k)|_{g_M}^2 \\
&= \frac{\ell^2 |k|_{g_{\mathrm{bi}}}^2}{\ell^2 |k|_{g_{\mathrm{bi}}}^2 + |K_{M,p}(k)|_{g_M}^2} |K_{M,p}(k)|_{g_M}^2 \\
&\overset{2.18}{=} \frac{\ell^2 |k|_{g_{\mathrm{bi}}}^2}{\ell^2 |k|_{g_{\mathrm{bi}}}^2 + \lambda_p^2 |k|_{g_{\mathrm{bi}}}^2} |K_{M,p}(k)|_{g_M}^2 \\
&= \frac{\ell^2}{\ell^2 + \lambda_p^2} |K_{M,p}(k)|_{g_M}^2.
\end{aligned}$$

[Calculation B]

$$\left|\left(\vec{0},x\right)\right|^2_{\ell^2 g_{\mathrm{bi}}+g_M} = \left|dq_{(e,p)}\left(\vec{0},x\right)\right|^2_{g_\ell}$$
$$\overset{2.6}{=} \left|K_{M,p}\left(\vec{0}\right)+x\right|^2_{g_\ell}$$
$$= |x|^2_{g_\ell}$$
$$\implies |x|^2_{g_\ell} = \left|\left(\vec{0},x\right)\right|^2_{\ell^2 g_{\mathrm{bi}}+g_M} = |x|^2_{g_M}.$$

[Calculation C]

$$\begin{aligned}
&(\ell^2 g_{\mathrm{bi}}+g_M)\left(\left(|K_{M,p}(k)|^2_{g_M}k, \ell^2|k|^2_{g_{\mathrm{bi}}}K_{M,p}(k)\right), \left(\vec{0}, \frac{x}{\ell^2|k|^2_{g_{\mathrm{bi}}}}\right)\right)\\
&= g_\ell\left(dq_{(e,p)}\left(|K_{M,p}(k)|^2_{g_M}k, \ell^2|k|^2_{g_{\mathrm{bi}}}K_{M,p}(k)\right), dq_{(e,p)}\left(\vec{0}, \frac{x}{\ell^2|k|^2_{g_{\mathrm{bi}}}}\right)\right)\\
&\overset{2.6}{=} g_\ell\left(K_{M,p}\left(|K_{M,p}(k)|^2_{g_M}k\right)+\ell^2|k|^2_{g_{\mathrm{bi}}}K_{M,p}(k), K_{M,p}\left(\vec{0}\right)+\frac{x}{\ell^2|k|^2_{g_{\mathrm{bi}}}}\right)\\
&= g_\ell\left(\left(\ell^2|k|^2_{g_{\mathrm{bi}}}+|K_{M,p}(k)|^2_{g_M}\right)K_{M,p}(k), \frac{x}{\ell^2|k|^2_{g_{\mathrm{bi}}}}\right)\\
&= \frac{\ell^2|k|^2_{g_{\mathrm{bi}}}+|K_{M,p}(k)|^2_{g_M}}{\ell^2|k|^2_{g_{\mathrm{bi}}}} g_\ell(K_{M,p}(k),x)
\end{aligned}$$

which implies

$$\begin{aligned}
&g_\ell(K_{M,p}(k),x)\\
&= \frac{\ell^2|k|^2_{g_{\mathrm{bi}}}}{\ell^2|k|^2_{g_{\mathrm{bi}}}+|K_{M,p}(k)|^2_{g_M}}(\ell^2 g_{\mathrm{bi}}+g_M)\left(\left(|K_{M,p}(k)|^2_{g_M}k, \ell^2|k|^2_{g_{\mathrm{bi}}}K_{M,p}(k)\right), \left(\vec{0}, \frac{x}{\ell^2|k|^2_{g_{\mathrm{bi}}}}\right)\right)\\
&= \frac{\ell^2|k|^2_{g_{\mathrm{bi}}}}{\ell^2|k|^2_{g_{\mathrm{bi}}}+|K_{M,p}(k)|^2_{g_M}} g_M\left(K_{M,p}(k),x\right)\\
&= 0.
\end{aligned}$$

It follows that for all $p \in M$, there is a constant $\lambda_p \in \mathbb{R}^+$ (where λ_p is as in Corollary 2.18) such that

- $g_\ell(v,w) = \frac{\ell^2}{\ell^2+\lambda_p^2} g_M(v,w)$ for all $v,w \in \mathcal{V}_p^\pi$ [Calculation A]
- $g_\ell(x,y) = g_M(x,y)$ for all $x,y \in \mathcal{H}_p^\pi$ [Calculation B]
- $g_\ell(v,x) = g_M(v,x) = 0$ for all $v \in \mathcal{V}_p^\pi$ and $x \in \mathcal{H}_p^\pi$ [Calculation C].

That is, g_ℓ is canonical variation at each point $p \in M$ (see Definition 2.12). ■

Proof. (of Theorem 1.8) Let $p \in M$ and consider $(G(p), g_M|_{T_pG(p)})$. Cheeger deforming $(G(p), g_M|_{T_pG(p)})$ with respect to (G, g_{bi}) gives $\left(G(p), g_\ell|_{T_pG(p)}\right)$. Rescaling $g_M|_{T_pG(p)}$ by $\frac{\ell^2}{\ell^2+\lambda_p^2}$ (where λ_p is as in Corollary 2.18) gives $\frac{\ell^2}{\ell^2+\lambda_p^2} g_M|_{T_pG(p)}$. Both metrics $g_M|_{T_pG(p)}$ and $\frac{\ell^2}{\ell^2+\lambda_p^2} g_M|_{T_pG(p)}$ are homogeneous, and they agree at p since $g_\ell(v,w) = \frac{\ell^2}{\ell^2+\lambda_p^2} g_M(v,w)$ for all $v,w \in \mathcal{V}_p^\pi$ by Part 2 of 1.7. Therefore, they agree on the entirety of $G(p)$, i.e. $\lambda_{p'} = \lambda_p$ for all $p' \in G(p)$. Since $\lambda_p \overset{2.18}{=} \frac{|K_{M,p}(k)|_{g_M}}{|k|_{g_{\text{bi}}}}$, it follows that for any vertical vector field V tangent to M, $D_V |K_{M,p}(k)|_{g_M} = 0$.

Furthermore, if $x \in T_pM$ is π-horizontal and we extend x to a horizontal vector field X that is basic along an orbit of $G \curvearrowright M$ (i.e. a fiber of π), then for all $k \in \mathfrak{g}$,

$$
\begin{aligned}
& T_{K_{M,p}(k)} K_{M,p}(k) \overset{A.10}{=} 0 \\
\implies & \mathcal{H}\nabla^{g_M}_{K_{M,p}(k)} K_{M,p}(k) \overset{A.10}{=} 0 \\
\implies & 2g_M\left(\mathcal{H}\nabla^{g_M}_{K_{M,p}(k)} K_{M,p}(k), X\right) = 0 \\
\implies & D_X g_M\Big(K_{M,p}(k), K_{M,p}(k)\Big) = 0 \text{ by Kozsul's formula} \\
\implies & D_X |K_{M,p}(k)|^2_{g_M} = 0.
\end{aligned}
$$

So $D_V|K_{M,p}(k)|_{g_M} = 0$ for all vertical $V \in TM$ and $D_X|K_{M,p}(k)|^2_{g_M} = 0$ for all horizontal $X \in TM$ together imply that $|K_{M,p}(k)|_{g_M}$ is constant so that λ_p from Corollary 2.18 is independent of p. Hence, the rescaling factor $\frac{\ell^2}{\ell^2+\lambda_p^2}$ is independent of p, so g_ℓ is canonical variation with p-independent rescaling factor $\frac{\ell^2}{\ell^2+\lambda^2}$. ■

2.3 Curvature

2.3.1 Positive Intermediate Ricci Curvature

Definition 2.19. *Let (M, g_M) be a Riemannian n-manifold. Let $k \in \mathbb{N}$ satisfy $k \leq n-1$. Then M has positive kth-intermediate Ricci curvature, denoted* **$\mathrm{Ric}_k(M, g_M) > 0$**, *if and only for any $p \in M$, any unit vector $v \in T_pM$, and any orthonormal $(k+1)$-frame $\{v, e_1, e_2, ..., e_k\}$ in T_pM, $\sum_{i=1}^{k} \sec_{g_M}(v, e_i) > 0$.*

Remark: $\mathrm{Ric}_1(M, g_M) > 0 \iff \sec(M, g_M) > 0$.

Remark: $\mathrm{Ric}_{n-1}(M, g_M) > 0 \iff \mathrm{Ric}(M, g_M) > 0$.

Remark: $\mathrm{Ric}_k(M, g_M) > 0 \implies \mathrm{Ric}_{k+1}(M, g_M) > 0$.

Some examples of Riemannian manifolds with positive intermediate Ricci curvature: If (M, g_M) and (N, g_N) are non-negatively curved Riemannian manifolds, then $\mathrm{Ric}_k(M \times N, g_M + g_N) > 0$ for all $k \geq \max\{m, n\} + 1$. Specifically, for $m \geq 2$, $\mathrm{Ric}_k(S^m \times S^m, g+g) > 0$ for all $k \geq m+1$ so that $\mathrm{Ric}_4(S^3 \times S^3, g+g) > 0$. By Theorem 1.4, $\mathrm{Ric}_2(S^3 \times S^3, g_\ell) > 0$.

2.3.2 A-Tensor

The A-tensor is crucial to curvature calculations within a Riemannian submersion.

Definition 2.20 (page 460 in [9]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion. Let $E_1, E_2 \in TM$. Then $\underline{\boldsymbol{A_{E_1} E_2}} = \mathcal{V}\nabla_{\mathcal{H}E_1}(\mathcal{H}E_2) + \mathcal{H}\nabla_{\mathcal{H}E_1}(\mathcal{V}E_2)$.*

The A-tensor has several useful properties:

Theorem 2.21 ($2'$ in [9]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion. For any $E \in TM$, $A_E = A_{\mathcal{H}E}$.*

Theorem 2.22 (Lemma 2 in [9]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion. If $X, Y \in \mathcal{H}^F$, then $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$.*

Theorem 2.23 ($3'$ in [9]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion. If $X, Y \in \mathcal{H}^F$, then $A_X Y = -A_Y X$.*

Theorem 2.24 (9.21d in [1]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion. If $V \in \mathcal{V}^F$ and $X, Y \in \mathcal{H}^F$, then $g_M\Big(A_X Y, V\Big) = -g_M\Big(Y, A_X V\Big)$.*

The following lemma describes how the A-tensor changes under canonical variation:

Theorem 2.25 (Lemma 9.69a in [1]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion and g_t be the canonical variation of g_M with respect to F. Let $V \in \mathcal{V}^F$ and $X, Y \in \mathcal{H}^F$. Then*

1) $A^{g_t}_X Y = A^{g_M}_X Y$

2) $A^{g_t}_X V = t^2 A^{g_M}_X V.$

2.3.3 Formulas

We use several of O'Neill's curvature formulas stated in [9]. Some of these formulas experience a change in sign so they are consistent with the sign convention of the $(1, 3)$ curvature tensor as defined in Petersen's textbook [10]. Furthermore, the Riemannian submersion π that we study in this dissertation (see Section 3.2) has totally geodesic fibers (see Lemma 3.12), so we add this assumption to O'Neill's theorems whenever the T-tensor is involved ($T \equiv \vec{0}$; see Theorem A.10).

Vertizontal Curvature Equation

Theorem 2.26 (Corollary 1 Part 2 in [9]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion with totally geodesic fibers. Let $V \in \mathcal{V}^F$ and $X \in \mathcal{H}^F$. Then*

$$\sec_{g_M}(X, V) = \frac{|A^{g_M}_X V|^2_{g_M}}{|X|^2_{g_M} |V|^2_{g_M}}.$$

Horizontal Curvature Equation

Theorem 2.27 (Corollary 1 Part 3 in [9])**.** *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion. Let $X, Y \in TB$ have horizontal lifts $\widetilde{X}, \widetilde{Y} \in TM$. That is, $\widetilde{X}, \widetilde{Y} \in \mathcal{H}$ and $dF\left(\widetilde{X}\right) = X$, $dF\left(\widetilde{Y}\right) = Y$. Then*

$$\sec_{g_B}(X, Y) = \sec_{g_M}\left(\widetilde{X}, \widetilde{Y}\right) + \frac{3\left|A^{g_M}_{\widetilde{X}}\widetilde{Y}\right|^2_{g_M}}{\left|\widetilde{X}\right|^2_{g_M}\left|\widetilde{Y}\right|^2_{g_M} - g_M\left(\widetilde{X}, \widetilde{Y}\right)^2}.$$

Mixed Curvature Equations (3-1)

Theorem 2.28 (Theorem 1 Part {1} in [9])**.** *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion with totally geodesic fibers. Let $U, V, W \in \mathcal{V}^F$ and $X, Y, Z \in \mathcal{H}^F$. Then*

1) $R_{g_M}(U, V, W, X) = 0.$

2) $R_{g_M}(X, Y, Z, V)$

$$= -g_M\Big(\nabla^{g_M}_Z(A^{g_M}_X Y), V\Big) + g_M\Big(A^t_{\nabla^{g_M}_Z X} Y, V\Big) + g_M\Big(A^{g_M}_X(\nabla^{g_M}_Z Y), V\Big).$$

Mixed Curvature Equations (2-2)

Theorem 2.29 (Theorem 3 Parts $\{2\}$ and $\{2'\}$ in [9])**.** *Let $F:(M,g_M)\longrightarrow(B,g_B)$ be a Riemannian submersion with totally geodesic fibers. Let $V,W\in\mathcal{V}^F$ and $X,Y\in\mathcal{H}^F$. Then*

$$1)\ R_{g_M}(X,V,Y,W) = -g_M\Big(\nabla^{g_M}_V(A^{g_M}_X Y),W\Big) + g_M\left(A^{g_M}_{(\nabla^{g_M}_V X)}Y,W\right) + g_M\Big(A^{g_M}_X(\nabla^{g_M}_V Y),W\Big) - g_M\Big(A^{g_M}_X V, A^{g_M}_Y W\Big).$$

$$\begin{aligned}2)\ R_{g_M}(V,W,X,Y) = &-g_M\Big(\nabla^{g_M}_V(A^{g_M}_X Y),W\Big) + g_M\left(A^{g_M}_{(\nabla^{g_M}_V X)}Y,W\right)\\ &+ g_M\Big(A^{g_M}_X(\nabla^{g_M}_V Y),W\Big) + g_M\Big(\nabla^{g_M}_W(A^{g_M}_X Y),V\Big)\\ &- g_M\left(A^{g_M}_{(\nabla^{g_M}_W X)}Y,V\right) - g_M\Big(A^{g_M}_X(\nabla^{g_M}_W Y),V\Big)\\ &- g_M\Big(A^{g_M}_X V, A^{g_M}_Y W\Big) + g_M\Big(A^{g_M}_X W, A^{g_M}_Y V\Big).\end{aligned}$$

Theorem 2.29 (above) can be simplified if $X,Y\in\mathcal{H}^F$ are assumed to be *basic*. Recall that if $F:M\longrightarrow N$ is a smooth map between smooth manifolds, then $X\in TM$ and $Y\in TN$ are ***F*-related** if and only if for each $p\in M$, $dF_p(X_p)=Y_{F(p)}$ (see page 182 of [4]). Let $F:(M,g_M)\longrightarrow(B,g_B)$ be a Riemannian submersion. A vector field X on M is **basic** if and only if $X\in\mathcal{H}^F$ and is F-related to a vector field Y on B (see page 460 in [9]).

Theorem 2.30 (Theorem 2.29 with Basic Assumption)**.** *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion with totally geodesic fibers. Let $V, W \in \mathcal{V}^F$ and $X, Y \in \mathcal{H}^F$ be basic. Then*

1) $R_{g_M}(X, V, Y, W) = -g_M\Big(\nabla^{g_M}_V (A^{g_M}_X Y), W\Big) - g_M\Big(A^{g_M}_Y V, A^{g_M}_X W\Big).$

2)
$$R_{g_M}(V, W, X, Y) = -g_M\Big(\nabla^{g_M}_V (A^{g_M}_X Y), W\Big) - g_M\Big(A^{g_M}_Y V, A^{g_M}_X W\Big) + g_M\Big(\nabla^{g_M}_W (A^{g_M}_X Y), V\Big) + g_M\Big(A^{g_M}_Y W, A^{g_M}_X V\Big).$$

The proof of Theorem 2.30 uses the following fact:

Theorem 2.31 (Proposition 4.5.1 Part (1) in [10])**.** *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion. Let $V \in \mathcal{V}^F$ and $X \in \mathcal{H}^F$ be basic. Then $[V, X] \in \mathcal{V}^F$.*

Proof. (of Theorem 2.30) Since $X, Y \in \mathcal{H}^F$ are basic, $g_M\left(A^{g_M}_{(\nabla^{g_M}_V X)} Y, W\right)$ and $g_M\Big(A^{g_M}_X (\nabla^{g_M}_V Y), W\Big)$ can be simplified as follows.

$$\begin{aligned} g_M\left(A^{g_M}_{(\nabla^{g_M}_V X)} Y, W\right) &\overset{2.21}{=} g_M\left(A^{g_M}_{(\mathcal{H}\nabla^{g_M}_V X)} Y, W\right) \\ &\overset{2.23}{=} -g_M\left(A^{g_M}_Y \left(\mathcal{H}\nabla^{g_M}_V X\right), W\right) \\ &\overset{A.11}{=} -g_M\left(A^{g_M}_Y \left(\mathcal{H}\nabla^{g_M}_X V + \mathcal{H}[V, X]\right), W\right) \\ &\overset{2.31}{=} -g_M\left(A^{g_M}_Y \left(\mathcal{H}\nabla^{g_M}_X V\right), W\right) \\ &\overset{2.24}{=} g_M\left(\mathcal{H}\nabla^{g_M}_X V, A^{g_M}_Y W\right) \overset{2.20}{=} g_M\left(A^{g_M}_X V, A^{g_M}_Y W\right). \end{aligned}$$

$$
\begin{aligned}
g_M\Big(A^{g_M}_X(\nabla^{g_M}_V Y), W\Big) &= g_M\left(\mathcal{V}\left(A^{g_M}_X\left(\nabla^{g_M}_V Y\right)\right), W\right) \\
&\overset{2.20}{=} g_M\left(\mathcal{V}\left(A^{g_M}_X\left(\mathcal{H}\nabla^{g_M}_V Y\right)\right), W\right) \\
&= g_M\left(\mathcal{V}\left(A^{g_M}_X\left(\mathcal{H}\nabla^{g_M}_Y V\right)\right), W\right) \text{ by work above, which shows that} \\
&\qquad\qquad A^{g_M}_Y\Big(\mathcal{H}\nabla^{g_M}_V X\Big) = A^{g_M}_Y\Big(\mathcal{H}\nabla^{g_M}_X V\Big) \\
&= g_M\left(A^{g_M}_X\left(\mathcal{H}\nabla^{g_M}_Y V\right), W\right) \\
&\overset{2.24}{=} -g_M\left(\mathcal{H}\nabla^{g_M}_Y V, A^{g_M}_X W\right) \overset{2.20}{=} -g_M\left(A^{g_M}_Y V, A^{g_M}_X W\right).
\end{aligned}
$$

So
$$
\begin{aligned}
R_{g_M}(X,V,Y,W) &\overset{2.29}{=} -g_M\Big(\nabla^{g_M}_V(A^{g_M}_X Y), W\Big) + g_M\left(A^{g_M}_{\left(\nabla^{g_M}_V X\right)} Y, W\right) \\
&\quad + g_M\Big(A^{g_M}_X(\nabla^{g_M}_V Y), W\Big) - g_M\Big(A^{g_M}_X V, A^{g_M}_Y W\Big) \\
&= -g_M\Big(\nabla^{g_M}_V(A^{g_M}_X Y), W\Big) + g_M\left(A^{g_M}_X V, A^{g_M}_Y W\right) \\
&\quad - g_M\left(A^{g_M}_Y V, A^{g_M}_X W\right) - g_M\Big(A^{g_M}_X V, A^{g_M}_Y W\Big) \\
&= -g_M\Big(\nabla^{g_M}_V(A^{g_M}_X Y), W\Big) - g_M\left(A^{g_M}_Y V, A^{g_M}_X W\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{and } R_{g_M}(V,W,X,Y) &\overset{2.29}{=} -g_M\Big(\nabla^{g_M}_V\left(A^{g_M}_X Y\right), W\Big) + g_M\left(A^{g_M}_{\left(\nabla^{g_M}_V X\right)} Y, W\right) \\
&\quad + g_M\Big(A^{g_M}_X\left(\nabla^{g_M}_V Y\right), W\Big) + g_M\Big(\nabla^{g_M}_W\left(A^{g_M}_X Y\right), V\Big) \\
&\quad - g_M\left(A^{g_M}_{\left(\nabla^{g_M}_W X\right)} Y, V\right) - g_M\Big(A^{g_M}_X\left(\nabla^{g_M}_W Y\right), V\Big) \\
&\quad - g_M\Big(A^{g_M}_X V, A^{g_M}_Y W\Big) + g_M\Big(A^{g_M}_X W, A^{g_M}_Y V\Big) \\
&= -g_M\Big(\nabla^{g_M}_V\left(A^{g_M}_X Y\right), W\Big) + g_M\left(A^{g_M}_X V, A^{g_M}_Y W\right) \\
&\quad - g_M\left(A^{g_M}_Y V, A^{g_M}_X W\right) + g_M\Big(\nabla^{g_M}_W\left(A^{g_M}_X Y\right), V\Big) \\
&\quad - g_M\left(A^{g_M}_X W, A^{g_M}_Y V\right) + g_M\left(A^{g_M}_Y W, A^{g_M}_X V\right) \\
&\quad - g_M\Big(A^{g_M}_X V, A^{g_M}_Y W\Big) + g_M\Big(A^{g_M}_X W, A^{g_M}_Y V\Big) \\
&= -g_M\Big(\nabla^{g_M}_V\left(A^{g_M}_X Y\right), W\Big) - g_M\left(A^{g_M}_Y V, A^{g_M}_X W\right) \\
&\quad + g_M\Big(\nabla^{g_M}_W\left(A^{g_M}_X Y\right), V\Big) + g_M\left(A^{g_M}_Y W, A^{g_M}_X V\right). \qquad \blacksquare
\end{aligned}
$$

Chapter 3:

Deforming $(S^3 \times S^3, g + g)$

3.1 Cheeger Deformation of $(S^3 \times S^3, g + g)$

This section corresponds to Section 2.2.1. Let g be the usual metric on S^3 and $\ell > 0$.

STEP 1: Equip $S^3 \times (S^3 \times S^3)$ with the product metric $\ell^2 g + (g + g)$.

STEP 2: Let $S^3 \curvearrowright \Big(S^3 \times (S^3 \times S^3)\Big)$ on the left by $x\Big(y, (p, m)\Big) = \Big(yx^{-1}, (xp, xm)\Big)$.

STEP 3: Equip the quotient space $\frac{S^3 \times (S^3 \times S^3)}{S^3} \overset{A.7}{\cong} S^3 \times S^3$ with the metric g_ℓ that makes the quotient map $q : \Big(S^3 \times (S^3 \times S^3), \ell^2 g + (g + g)\Big) \to \Big(\frac{S^3 \times (S^3 \times S^3)}{S^3}, g_\ell\Big) \cong (S^3 \times S^3, g_\ell)$ a Riemannian submersion (see Theorem 2.3).

Definition 3.1 (see Definition 2.4)**.** *Then* $\Big\{ (S^3 \times S^3, g_\ell) \ \big| \ \ell > 0 \Big\}$ *is a family of Cheeger deformations of* $(S^3 \times S^3, g + g)$.

Definition 3.2 (see Definition 2.7)**.** *Let* $\mathfrak{s} = \mathrm{Im}(\mathbb{H})$ *be the Lie algebra of* S^3 *and* $(N_1, N_2) \in S^3 \times S^3$.

$$K_{S^3\times S^3,(N_1,N_2)} : \mathfrak{s} \to T_{(N_1,N_2)}(S^3 \times S^3)$$
$$\alpha \mapsto \frac{d}{dt}\exp(t\alpha)(N_1, N_2)\Big|_{t=0} = (\alpha N_1, \alpha N_2).$$

Lemma 3.3 (see Theorem 2.8)**.** *Let* $(N_1, N_2) \in S^3 \times S^3$. *Then*

$$\mathcal{V}^q_{(1,(N_1,N_2))} = \left\{ \Big(-\alpha, (\alpha N_1, \alpha N_2)\Big) \;\Big|\; \alpha \in \mathfrak{s} = \mathrm{Im}(\mathbb{H}) \right\}.$$

Lemma 3.4 (see Definition 2.10)**.** *Let* $(N_1, N_2) \in S^3 \times S^3$ *and* $\alpha, \beta \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$. *Then* $\kappa_{(N_1,N_2)}\Big((\alpha N_1, \beta N_2)\Big) = \alpha + \beta$.

Proof. For $(N_1, N_2) \in S^3 \times S^3$ and $\alpha, \beta \in \mathfrak{s}$,

$$\kappa_{(N_1,N_2)}\Big((\alpha N_1, \beta N_2)\Big) = \kappa_{(N_1,N_2)}\Big(\Big(\alpha N_1, \vec{0}\Big)\Big) + \kappa_{(N_1,N_2)}\Big(\Big(\vec{0}, \beta N_2\Big)\Big).$$

Let $\gamma \in \mathfrak{s} = \operatorname{Im}(\mathbb{H})$. Then

$$\begin{aligned}
& g\Big(\kappa_{(N_1,N_2)}\Big(\Big(\alpha N_1, \vec{0}\Big)\Big), \gamma\Big) \overset{2.10}{=} (g+g)\Big(\Big(\alpha N_1, \vec{0}\Big), (\gamma N_1, \gamma N_2)\Big) \\
\iff & g\Big(\kappa_{(N_1,N_2)}\Big(\Big(\alpha N_1, \vec{0}\Big)\Big), \gamma\Big) = g(\alpha N_1, \gamma N_1) \\
\iff & g\Big(\kappa_{(N_1,N_2)}\Big(\Big(\alpha N_1, \vec{0}\Big)\Big), \gamma\Big) = \begin{cases} |\alpha|^2 & \text{if } \gamma = \alpha \\ 0 & \text{if } \gamma \perp \alpha \end{cases} \\
\iff & \kappa_{(N_1,N_2)}\Big(\Big(\alpha N_1, \vec{0}\Big)\Big) = \alpha.
\end{aligned}$$

Similarly, $\kappa_{(N_1,N_2)}\Big(\Big(\vec{0}, \beta N_2\Big)\Big) = \beta$. ■

Definition 3.5 (see Definition 2.11)**.** *Let* $(N_1, N_2) \in S^3 \times S^3$ *and* $\alpha, \beta \in \mathfrak{s} = \operatorname{Im}(\mathbb{H})$. *Then for all* $\ell > 0$, $\underline{\widehat{\boldsymbol{v}}_{\boldsymbol{\ell}}} \overset{3.4}{=} \left(\dfrac{\alpha+\beta}{\ell^2}, (\alpha N_1, \beta N_2)\right)$.

3.2 Canonical Variation of $(S^3 \times S^3, g + g)$

This section corresponds to Section 2.2.2. Consider the left diagonal action of S^3 on $S^3 \times S^3$ given by $p(N_1, N_2) = (pN_1, pN_2)$. Since

- S^3 is a closed Lie group
- $f_p : (S^3 \times S^3, g+g) \longrightarrow (S^3 \times S^3, g+g)$

  given by $(N_1, N_2) \mapsto (pN_1, pN_2) = L_p\Big((N_1, N_2)\Big)$ is an isometry for all $p \in S^3$, and
- the diagonal S^3 action on $S^3 \times S^3$ is free,

there is a Riemannian metric $\overline{g}$ on the quotient manifold $\frac{S^3 \times S^3}{\Delta S^3} \cong S^3$ that makes the quotient map $\pi : (S^3 \times S^3, g+g) \longrightarrow \left(\frac{S^3 \times S^3}{\Delta S^3} \cong S^3, \overline{g}\right)$ a Riemannian submersion (see Theorem 2.3).

Lemma 3.6. *Let $\mathfrak{s} = \mathrm{Im}(\mathbb{H})$ denote the Lie algebra of S^3. Then for all $(N_1, N_2) \in S^3 \times S^3$, $\mathcal{V}^{\pi}_{(N_1,N_2)} = \Big\{(\alpha N_1, \alpha N_2) \mid \alpha \in \mathfrak{s}\Big\}$.*

Proof. Let $(N_1, N_2) \in S^3 \times S^3$.

$$\begin{aligned} \mathcal{V}^{\pi}_{(N_1,N_2)} &= \ker\left(d\pi_{(N_1,N_2)}\right) \\ &= \left\{v \in T_{(N_1,N_2)}(S^3 \times S^3) \;\middle|\; d\pi_{(N_1,N_2)}(v) = \vec{0}\right\} \supseteq T_{(N_1,N_2)}\Big(S^3(N_1, N_2)\Big). \end{aligned}$$

Since $d\pi_{(N_1,N_2)} : T_{(N_1,N_2)}(S^3 \times S^3) \longrightarrow T_{\pi\left((N_1,N_2)\right)}S^3$ is linear, the Rank-Nullity Theorem implies $\dim\left(\mathcal{V}^{\pi}_{(N_1,N_2)}\right) = 6 - 3 = 3$. So $T_{(N_1,N_2)}\Big(S^3(N_1, N_2)\Big)$ is a 3-dimensional subspace of a 3-dimensional space $\mathcal{V}^{\pi}_{(N_1,N_2)}$, which means $T_{(N_1,N_2)}\Big(S^3(N_1, N_2)\Big) = \mathcal{V}^{\pi}_{(N_1,N_2)}$. ∎

Lemma 3.7. *Let $\mathfrak{s} = \mathrm{Im}(\mathbb{H})$ denote the Lie algebra of S^3. Then for all $(N_1, N_2) \in S^3 \times S^3$, $\mathcal{H}^{\pi}_{(N_1,N_2)} = \Big\{(\alpha N_1, -\alpha N_2) \mid \alpha \in \mathfrak{s}\Big\}$.*

Proof. Theorem 2.13 Part 2 tells us that $\mathcal{H}$ is independent of t, so we will calculate $\mathcal{H}^{\pi}_{(N_1,N_2)}$ with respect to $g+g$.

Let $W = \Big\{(\alpha N_1, -\alpha N_2) \mid \alpha \in \mathfrak{s}\Big\}$. Recall that $\mathcal{V}^{\pi}_{(N_1,N_2)} \overset{3.6}{=} \Big\{(\beta N_1, \beta N_2) \mid \beta \in \mathfrak{s}\Big\}$. Since $(g+g)\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2)\Big) = 0$ for all $\alpha, \beta \in \mathfrak{s}$, $W \subseteq \mathcal{H}^{\pi}_{(N_1,N_2)}$.

Then $\dim(W) = \dim(\mathfrak{s}) = 3 = \dim\left(\mathcal{H}^{\pi}_{(N_1,N_2)}\right) \implies W = \mathcal{H}^{\pi}_{(N_1,N_2)}$. ■

Definition 3.8 (see Definition 2.12)**.** *The canonical variation g_t of $g+g$ on $S^3 \times S^3$ with respect to the Riemannian submersion π is defined for all $(N_1, N_2) \in S^3 \times S^3$ and $\alpha, \beta \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$ by*

1) $g_t\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big) = t^2 (g+g)\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big)$
2) $g_t\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big) = (g+g)\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big)$
3) $g_t\Big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2)\Big) = 0.$

Lemmas 3.9 and 3.10 below are necessary for curvature computations:

Lemma 3.9. *Equip S^3 with its usual metric g. Let $\alpha, \beta \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$. Consider the vector fields on S^3 defined by $V_\alpha : N \mapsto \alpha N$ and $V_\beta : N \mapsto \beta N$. Then*

$$\nabla^g_{V_\beta} V_\alpha = \nabla^g_{\beta N} \alpha N = (\alpha \times \beta) N.$$

Proof.

$$
\begin{aligned}
\nabla^g_{V_\beta} V_\alpha &= \left(\nabla^{(\mathbb{R}^4, g_{\text{std}})}_{V_\beta} V_\alpha\right)^{TS^3} \\
&= \left(\nabla^{(\mathbb{R}^4, g_{\text{std}})}_{\beta N} \alpha N\right)^{TS^3} \\
&= \left(\alpha \nabla^{(\mathbb{R}^4, g_{\text{std}})}_{\beta N} N\right)^{TS^3} \\
&= \Big(\alpha(\beta N)\Big)^{TS^3} \\
&\overset{A.12}{=} \Big((\alpha\beta)N\Big)^{TS^3} \\
&= \Big(\big(\text{Re}(\alpha\beta) + \text{Im}(\alpha\beta)\big)N\Big)^{TS^3} \\
&= \big(\underbrace{\text{Re}(\alpha\beta)N}_{\perp S^3}\big)^{TS^3} + \big(\underbrace{\text{Im}(\alpha\beta)N}_{\text{tan. to } S^3}\big)^{TS^3} = \vec{0} + \text{Im}(\alpha\beta)N \overset{A.13}{=} (\alpha \times \beta)N.
\end{aligned}
$$

■

Corollary 3.10. *Equip* $S^3 \times S^3$ *with its usual product metric* $g+g$. *Let* $\alpha, \beta, \gamma, \delta \in \mathfrak{s} = \text{Im}(\mathbb{H})$. *Consider the vector fields on* $S^3 \times S^3$ *defined by* $V_{\alpha\beta} : (N_1, N_2) \mapsto (\alpha N_1, \beta N_2)$ *and* $V_{\gamma\delta} : (N_1, N_2) \mapsto (\gamma N_1, \delta N_2)$. *Then*

1) $\nabla^{g+g}_{V_{\gamma\delta}} V_{\alpha\beta} = \nabla^{g+g}_{(\gamma N_1, \delta N_2)}(\alpha N_1, \beta N_2) = \Big((\alpha \times \gamma)N_1, (\beta \times \delta)N_2\Big)$

2) $[V_{\gamma\delta} V_{\alpha\beta}] = \Big[(\gamma N_1, \delta N_2), (\alpha N_1, \beta N_2)\Big] = \Big(2(\alpha \times \gamma)N_1, 2(\beta \times \delta)N_2\Big)$.

Proof.

$$
\begin{aligned}
\nabla^{g+g}_{V_{\gamma\delta}} V_{\alpha\beta} &= \nabla^{g+g}_{(\gamma N_1, \delta N_2)}(\alpha N_1, \beta N_2) \\
&= \Big(\nabla^g_{\gamma N_1} \alpha N_1, \nabla^g_{\delta N_1} \beta N_2\Big) \overset{3.9}{=} \Big((\alpha \times \gamma)N_1, (\beta \times \delta)N_2\Big).
\end{aligned}
$$

$$
\begin{aligned}
[V_{\gamma\delta}V_{\alpha\beta}] &= \Big[(\gamma N_1, \delta N_2), (\alpha N_1, \beta N_2)\Big] \\
&\overset{A.11}{=} \nabla^{g+g}_{(\gamma N_1,\delta N_2)}(\alpha N_1, \beta N_2) - \nabla^{g+g}_{(\alpha N_1,\beta N_2)}(\gamma N_1, \delta N_2) \\
&= \Big((\alpha \times \gamma)N_1, (\beta \times \delta)N_2\Big) - \Big((\gamma \times \alpha)N_1, (\delta \times \beta)N_2\Big) \text{ by calculation above} \\
&= \Big(2(\alpha \times \gamma)N_1, 2(\beta \times \delta)N_2\Big).
\end{aligned}
$$

■

Lemma 3.11. *The quotient S^3 at the base of the Riemannian submersion $\pi : (S^3 \times S^3, g + g) \longrightarrow \left(\frac{S^3 \times S^3}{\Delta S^3} \cong S^3, \overline{g}\right)$ has constant curvature 2.*

Proof. Let $v_1, v_2 \in T_pS^3$. Then v_1, v_2 have horizontal lifts $\widetilde{v_1} = (\alpha N_1, -\alpha N_2)$ and $\widetilde{v_2} = (\beta N_1, -\beta N_2)$ in $T_{(N_1,N_2)}(S^3 \times S^3)$ and

$$
\begin{aligned}
&\sec_{\overline{g}}(v_1, v_2) \\
&= \sec_{\overline{g}}\bigg(d\pi_{(N_1,N_2)}\Big((\alpha N_1, -\alpha N_2)\Big), d\pi_{(N_1,N_2)}\Big((\beta N_1, -\beta N_2)\Big)\bigg) \\
&\overset{2.27}{=} \sec_{g+g}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big) \\
&\quad + \frac{3\left|A^{g+g}_{(\alpha N_1,-\alpha N_2)}(\beta N_1, -\beta N_2)\right|^2_{g+g}}{|(\alpha N_1, -\alpha N_2)|^2_{g+g} \cdot |(\beta N_1, -\beta N_2)|^2_{g+g} - (g+g)\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big)^2} \\
&= \frac{1}{4}\mathrm{curv}_{g+g}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big) + \frac{3}{4}\left|A^{g+g}_{(\alpha N_1,-\alpha N_2)}(\beta N_1, -\beta N_2)\right|^2_{g+g} \\
&= \frac{1}{4}(2) + \frac{3}{4}\left|\mathcal{V}\nabla_{(-\alpha N_2,-\alpha N_2)}(-\beta N_2, -\beta N_2)\right|^2_{g+g} \\
&\overset{3.10,\ 3.6}{=} \frac{1}{2} + \frac{3}{4}\left|\Big((\beta \times \alpha)N_1, (\beta \times \alpha)N_2\Big)\right|^2_{g+g} = \frac{1}{2} + \frac{3}{4}(2) = 2.
\end{aligned}
$$

■

Lemma 3.12. *The fibers of π are totally geodesic with respect to g_t for all $t \in (0,1)$.*

Proof. Since the fibers of π are submanifolds of $S^3 \times S^3$ (see page 459 of [9]), we can apply Theorem A.10. Let E, F be arbitraty vector fields on $S^3 \times S^3$. Then for some $\gamma, \alpha, \beta \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$ and $(N_1, N_2) \in S^3 \times S^3$, $\mathcal{V}E \overset{3.6}{=} (\gamma N_1, \gamma N_2)$, and

$$\begin{aligned} F = \mathcal{V}F + \mathcal{H}F &\overset{\substack{3.6\\3.7}}{=} (\alpha N_1, \alpha N_2) + (\beta N_1, -\beta N_2) \\ &= (\alpha N_1, \alpha N_2) + \Big((\beta^{\alpha} + \beta^{\perp\alpha})N_1, -(\beta^{\alpha} + \beta^{\perp\alpha})N_2\Big) \\ &= (\alpha N_1, \alpha N_2) + (\beta^{\alpha} N_1, -\beta^{\alpha} N_2) + (\beta^{\perp\alpha} N_1, -\beta^{\perp\alpha} N_2). \end{aligned}$$

So $T_E^{g+g} F \overset{A.10}{=} \mathcal{H}\nabla^{g+g}_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla^{g+g}_{\mathcal{V}E}(\mathcal{H}F)$

$$\begin{aligned} &= \mathcal{H}\nabla^{g+g}_{(\gamma N_1, \gamma N_2)}(\alpha N_1, \alpha N_2) + \mathcal{V}\nabla^{g+g}_{(\gamma N_1, \gamma N_2)}\Big((\beta^{\alpha} N_1, -\beta^{\alpha} N_2) + (\beta^{\perp\alpha} N_1, -\beta^{\perp\alpha} N_2)\Big) \\ &= \mathcal{H}\nabla^{g+g}_{(\gamma N_1, \gamma N_2)}(\alpha N_1, \alpha N_2) + \mathcal{V}\nabla^{g+g}_{(\gamma N_1, \gamma N_2)}(\beta^{\alpha} N_1, -\beta^{\alpha} N_2) \\ &\quad + \mathcal{V}\nabla^{g+g}_{(\gamma N_1, \gamma N_2)}(\beta^{\perp\alpha} N_1, -\beta^{\perp\alpha} N_2) \\ &\overset{3.10}{=} \mathcal{H}\Big((\alpha \times \gamma)N_1, (\alpha \times \gamma)N_2\Big) + \mathcal{V}\Big((\beta^{\alpha} \times \gamma)N_1, -(\beta^{\alpha} \times \gamma)N_2\Big) \\ &\quad + \mathcal{V}\Big((\beta^{\perp\alpha} \times \gamma)N_1, -(\beta^{\perp\alpha} \times \gamma)N_2\Big) \\ &\overset{\substack{3.6\\3.7}}{=} \vec{0}. \end{aligned}$$

Then by Theorem 2.13 Part 3, we get that $T = \vec{0}$ with respect to g_t for all $t \in (0,1)$. ■

3.3 Cheeger to Canonical $(S^3 \times S^3, g+g)$

This section corresponds to Section 2.2.3.

Lemma 3.13 (see Theorem 1.8)**.** *The Cheeger deformation of* $(S^3 \times S^3, g+g)$ *defined in* Section 3.1 *is canonical variation with rescaling parameter* $t^2 = \frac{\ell^2}{\ell^2+2}$. *That is, for all* $(N_1, N_2) \in S^3 \times S^3$ *and* $\alpha, \beta \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$,

1) $g_\ell\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big) = \frac{\ell^2}{\ell^2+2}(g+g)\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big)$

2) $g_\ell\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big) = (g+g)\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big)$

3) $g_\ell\Big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2)\Big) = 0.$

Proof. This follows from Theorem 1.8. The Cheeger deformation of $(S^3 \times S^3, g+g)$ defined in Section 3.1 satisfies the generalized Petersen-Wilhelm hypothesis (Definition 2.16) since for all $\alpha, \beta \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$ such that $\alpha \perp_g \beta$,

$$(g+g)\Big(K_{S^3\times S^3,(N_1,N_2)}(\alpha), K_{S^3\times S^3,(N_1,N_2)}(\beta)\Big) \overset{3.2}{=} (g+g)\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big)$$
$$= g(\alpha N_1, \beta N_1) + g(\alpha N_2, \beta N_2) = 0.$$

The fibers of π are totally geodesic by 3.12.

Furthermore, for all $\gamma \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$ satisfying $|\gamma|_g = 1$,

$$\begin{aligned}|K_{S^3\times S^3,(N_1,N_2)}(\gamma)|^2_{g+g} &\overset{3.2}{=} |(\gamma N_1, \gamma N_2)|^2_{g+g} \\ &= g(\gamma N_1, \gamma N_1) + g(\gamma N_2, \gamma N_2) = |\gamma N_1|^2_g + |\gamma N_2|^2_g = 2. \qquad \blacksquare\end{aligned}$$

Chapter 4:

Lemmas and Formulas for Curvature Calculations

4.1 Zero Curvature

Lemma 4.1. *Let* $\mathfrak{s} = \mathrm{Im}(\mathbb{H})$ *denote the Lie algebra of* S^3*. Let* $(N_1, N_2) \in S^3 \times S^3$ *and* $\mathcal{P}$ *be a plane in* $T_{(N_1,N_2)}(S^3 \times S^3)$*. Then for all* $t \in (0,1)$*,* $\sec_{g_t}(\mathcal{P}) = 0 \iff \mathcal{P} = \mathrm{span}\Big\{(\alpha N_1, \alpha N_2), (\alpha N_1, -\alpha N_2)\Big\}$ *for some* $\alpha \in \mathfrak{s}$*.*

The proof of Lemma 4.1 requires the following result:

Theorem 4.2 (Lemma 3.6 in [6])**.** *Let* (M, g_M) *and* (N, g_N) *be positively curved Riemannian manifolds. A plane* $\mathcal{P}$ *tangent to* $M \times N$ *has curvature zero with respect to the product metric* $g_M + g_N$ *if and only if it can be written as* $\mathcal{P} = \mathrm{span}\Big\{\Big(v, \vec{0}\Big), \Big(\vec{0}, w\Big)\Big\}$ *for some* $v \in TM$ *and* $w \in TN$*.*

Proof. (of Lemma 4.1) $\boxed{\implies}$ Recall Lemma 3.13, which states that $(S^3 \times S^3, g_t)$ is a Cheeger deformation $(S^3 \times S^3, g_\ell)$ with $\ell^2 = \frac{2t^2}{1-t^2}$. With this in mind, assume plane $\mathcal{P}$ in $T_{(N_1,N_2)}(S^3 \times S^3)$ satisfies $\text{curv}_{g_\ell}(\mathcal{P}) = 0$. Let $\widetilde{\mathcal{P}}_\ell$ be the horizontal lift of $\mathcal{P}$ with respect to $q : \left(S^3 \times (S^3 \times S^3), \ell^2 g + g + g\right) \longrightarrow (S^3 \times S^3, g_\ell)$. Then $dq\left(\widetilde{\mathcal{P}}_\ell\right) = \mathcal{P}$. Decompose $\widetilde{\mathcal{P}}_\ell = \text{proj}_{S^3}\left(\widetilde{\mathcal{P}}_\ell\right) + \text{proj}_{S^3 \times S^3}\left(\widetilde{\mathcal{P}}_\ell\right) := \mathcal{P}_G + \mathcal{P}_M$. Then $\widehat{(\mathcal{P}_M)_\ell} \overset{2.11}{=} \widetilde{\mathcal{P}}_\ell$ and if $\mathcal{P}_M = \text{span}\{v_1, v_2\}$, then $\mathcal{P}_G = \text{span}\left\{\kappa_{(N_1,N_2)}(v_1), \kappa_{(N_1,N_2)}(v_2)\right\}$.

$$
\begin{aligned}
\text{curv}_{g_\ell}(\mathcal{P}) = 0 &\overset{2.27}{\implies} \text{curv}_{\ell^2 g+g+g}\left(\widetilde{P}_\ell\right) = 0 \\
&\implies \text{curv}_{\ell^2 g}\left(\mathcal{P}_G\right) + \text{curv}_{g+g}\left(\mathcal{P}_M\right) = 0 \\
&\implies \text{curv}_{\ell^2 g}\left(\kappa_{(N_1,N_2)}(v_1), \kappa_{(N_1,N_2)}(v_2)\right) = 0 \text{ and } \text{curv}_{g+g}\left(v_1, v_2\right) = 0 \\
&\qquad \text{since } \text{curv}(S^3, g) \geq 0 \text{ and } \text{curv}(S^3 \times S^3, g+g) \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
&\text{curv}_{g+g}\left(v_1, v_2\right) = 0 \\
\overset{4.2}{\implies} \; &\text{span}\{v_1, v_2\} = \text{span}\left\{\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \beta N_2\right)\right\} \text{ for some } \alpha, \beta \in \mathfrak{s} = \text{Im}(\mathbb{H}).
\end{aligned}
$$

Then $\operatorname{curv}_{\ell^2 g}\left(\kappa_{(N_1,N_2)}(v_1), \kappa_{(N_1,N_2)}(v_2)\right) = 0$

$$\overset{}{\Longrightarrow} \operatorname{curv}_{\ell^2 g}\left(\kappa_{(N_1,N_2)}\left(\left(\alpha N_1, \vec{0}\right)\right), \kappa_{(N_1,N_2)}\left(\left(\vec{0}, \beta N_2\right)\right)\right) = 0$$

$$\overset{3.4}{\Longrightarrow} \operatorname{curv}_{\ell^2 g}(\alpha, \beta) = 0$$

$$\Longrightarrow \beta \text{ is proportional to } \alpha$$

$$\Longrightarrow \mathcal{P}_M = \operatorname{span}\left\{\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \alpha N_2\right)\right\} \text{ for some } \alpha \in \mathfrak{s} = \operatorname{Im}(\mathbb{H})$$

$$\Longrightarrow \mathcal{P}_M = \operatorname{span}\{(\alpha N_1, \alpha N_2), (\alpha N_1, -\alpha N_2)\} \text{ for some } \alpha \in \mathfrak{s} = \operatorname{Im}(\mathbb{H}).$$

It follows that $\mathcal{P} = \operatorname{span}\{(\alpha N_1, \alpha N_2), (\alpha N_1, -\alpha N_2)\}$ for some $\alpha \in \mathfrak{s} = \operatorname{Im}(\mathbb{H})$ since $\mathcal{P} = dq\left(\widetilde{\mathcal{P}}_\ell\right) = dq\left(\widehat{(\mathcal{P}_M)_\ell}\right) = dq\left(\widehat{(\alpha N_1, \alpha N_2)_\ell}\right) + dq\left(\widehat{(\alpha N_1, -\alpha N_2)_\ell}\right)$ where

$$dq\left(\widehat{(\alpha N_1, \alpha N_2)_\ell}\right) \overset{3.5}{=} dq\left(\frac{2\alpha}{\ell^2}, (\alpha N_1, \alpha N_2)\right)$$
$$\overset{2.6}{=} K_{S^3 \times S^3, (N_1,N_2)}\left(\frac{2\alpha}{\ell^2}\right) + (\alpha N_1, \alpha N_2)$$
$$\overset{3.2}{=} \frac{2}{\ell^2}(\alpha N_1, \alpha N_2) + (\alpha N_1, \alpha N_2) = \frac{\ell^2 + 2}{\ell^2}(\alpha N_1, \alpha N_2)$$

and $dq\left(\widehat{(\alpha N_1, -\alpha N_2)_\ell}\right) \overset{3.5}{=} dq\left(\vec{0}, (\alpha N_1, -\alpha N_2)\right)$

$$\overset{2.6}{=} K_{S^3 \times S^3, (N_1,N_2)}\left(\vec{0}\right) + (\alpha N_1, -\alpha N_2) \overset{3.2}{=} (\alpha N_1, -\alpha N_2).$$

$\boxed{\Longleftarrow}$ By Theorem 2.26,

$$\sec_{g_t}\Big((\alpha N_1, \alpha N_2), (\alpha N_1, -\alpha N_2)\Big) = \frac{\Big|A^{g_t}_{(\alpha N_1, -\alpha N_2)}(\alpha N_1, \alpha N_2)\Big|^2_{g_t}}{|(\alpha N_1, -\alpha N_2)|^2_{g_t}|(\alpha N_1, \alpha N_2)|^2_{g_t}} \text{ where}$$

$$\begin{aligned}
A^{g_t}_{(\alpha N_1, -\alpha N_2)}(\alpha N_1, \alpha N_2) &\overset{2.13}{=} t^2 A^{g+g}_{(\alpha N_1, -\alpha N_2)}(\alpha N_1, \alpha N_2) \\
&\overset{2.20}{=} \mathcal{H}\nabla^{g+g}_{(\alpha N_1, -\alpha N_2)}(\alpha N_1, \alpha N_2) \\
&\overset{3.10}{=} \mathcal{H}\Big((\alpha \times \alpha)N_1, -(\alpha \times \alpha)N_2\Big) = \vec{0} \\
&\implies \Big|A^{g_t}_{(\alpha N_1, -\alpha N_2)}(\alpha N_1, \alpha N_2)\Big|^2_{g_t} = 0 \\
&\implies \sec_{g_t}\Big((\alpha N_1, \alpha N_2), (\alpha N_1, -\alpha N_2)\Big) = 0.
\end{aligned}$$

■

4.2 Vertical Curvature

Lemma 4.3. *Let g be the usual metric on S^3 and $(N_1, N_2) \in S^3 \times S^3$. Let $\mathfrak{s} = \mathrm{Im}(\mathbb{H})$ denote the Lie algebra of S^3. Let $\alpha, \beta \in \mathfrak{s}$ satisfy $\alpha \perp_g \beta$. Then for all $t \in (0, 1)$,*

$$\sec_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big) = \frac{1}{2t^2}.$$

Proof. Since the fibers of π are totally geodesic for all values of t (see Lemma 3.12), the intrinsic curvature computed in the fibers of π and the extrinsic curvature computed in the ambient manifold $S^3 \times S^3$ are equal. For this reason, all curvature calculations in this proof are made in $S^3 \times S^3$.

Suppose $|\alpha|_g = |\beta|_g = 1$. Then

$$
\begin{aligned}
&\sec_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big) \\
&= \frac{1}{t^2}\sec_{g+g}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big) \\
&= \frac{1}{t^2}\left(\frac{\operatorname{curv}_{g+g}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big)}{|(\alpha N_1, \alpha N_2)|^2_{g+g}\cdot|(\beta N_1, \beta N_2)|^2_{g+g} - g\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big)^2}\right) \\
&= \frac{1}{t^2}\left(\frac{\operatorname{curv}_{g+g}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big)}{2\cdot 2 - 0}\right) \\
&= \frac{1}{4t^2}\Big(\operatorname{curv}_g(\alpha N_1, \beta N_1) + \operatorname{curv}_g(\alpha N_2, \beta N_2)\Big) = \frac{1}{4t^2}(2) = \frac{1}{2t^2}.
\end{aligned}
$$

■

4.3 Horizontal Curvature

Lemma 4.4. *Let g be the usual metric on S^3 and $(N_1, N_2) \in S^3 \times S^3$. Let $\mathfrak{s} = \mathrm{Im}(\mathbb{H})$ denote the Lie algebra of S^3. Let $\alpha, \beta \in \mathfrak{s}$ satisfy $\alpha \perp_g \beta$. Then for all $t \in (0,1)$,*

$$\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big) = 2 - \frac{3}{2}t^2.$$

Proof. Suppose $|\alpha|_g = |\beta|_g = 1$. Then

$$\begin{aligned}
&\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big) \\
&\overset{2.27}{=} \sec_{\overline{g}}\bigg(d\pi_{(N_1,N_2)}\Big((\alpha N_1, -\alpha N_2)\Big), d\pi_{(N_1,N_2)}\Big((\beta N_1, -\beta N_2)\Big)\bigg) \\
&\quad - \frac{3|A^{g_t}_{(\alpha N_1, -\alpha N_2)}(\beta N_1, -\beta N_2)|^2_{g_t}}{|(\alpha N_1, -\alpha N_2)|^2_{g_t} \cdot |(\beta N_1, -\beta N_2)|^2_{g_t} - g_t\Big((\alpha N_1, -\alpha N_2)\ ,\ (\beta N_1, -\beta N_2)\Big)^2}
\end{aligned}$$

where $\sec_{\overline{g}}\bigg(d\pi_{(N_1,N_2)}\Big((\alpha N_1, -\alpha N_2)\Big), d\pi_{(N_1,N_2)}\Big((\beta N_1, -\beta N_2)\Big)\bigg) \overset{3.11}{=} 2$

and $$\frac{3\left|A^{g_t}_{(\alpha N_1, -\alpha N_2)}(\beta N_1, -\beta N_2)\right|^2_{g_t}}{|(\alpha N_1, -\alpha N_2)|^2_{g_t} \cdot |(\beta N_1, -\beta N_2)|^2_{g_t} - g_t\Big((\alpha N_1, -\alpha N_2)\ ,\ (\beta N_1, -\beta N_2)\Big)^2}$$

$$\begin{aligned}
&= \frac{3}{4}\left|A^{g_t}_{(\alpha N_1, -\alpha N_2)}(\beta N_1, -\beta N_2)\right|^2_{g_t} \\
&\overset{2.25}{=} \frac{3}{4}\left|A^{g+g}_{(\alpha N_1, -\alpha N_2)}(\beta N_1, -\beta N_2)\right|^2_{g_t} \\
&= \frac{3}{4}\left|A^{g+g}_{(\alpha N_1, -\alpha N_2)}(\beta N_1, -\beta N_2)\right|^2_{g_t} \\
&= \frac{3}{4}\left|\Big((\beta \times \alpha)N_1, (\beta \times \alpha)N_2\Big)\right|^2_{g_t} \\
&= \frac{3}{4}t^2\left|\Big((\beta \times \alpha)N_1, (\beta \times \alpha)N_2\Big)\right|^2_{g+g} = \frac{3}{4}t^2(2) = \frac{3}{2}t^2.
\end{aligned}$$

■

4.4 Vertizontal Curvature

Lemma 4.5. *Let g be the usual metric on S^3 and $(N_1, N_2) \in S^3 \times S^3$. Let $\mathfrak{s} = \mathrm{Im}(\mathbb{H})$ denote the Lie algebra of S^3. Let $\alpha, \beta \in \mathfrak{s}$ satisfy $\alpha \perp_g \beta$. Then for all $t \in (0,1)$,*

$$\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2)\Big) = \frac{t^2}{2}.$$

Proof. Suppose $|\alpha|_g = |\beta|_g = 1$. Then

$$\begin{aligned}
\sec_{g_t}\Big((\alpha N_1, -\alpha N_2)\ ,\ (\beta N_1, \beta N_2)\Big) &\overset{2.26}{=} \frac{\left|A^{g_t}_{(\alpha N_1, -\alpha N_2)}(\beta N_1, \beta N_2)\right|^2_{g_t}}{|(\alpha, -\alpha)|^2_{g_t} \cdot |(\beta, \beta)|^2_{g_t}} \\
&\overset{2.25}{=} \frac{\left|t^2 A^{g+g}_{(\alpha N_1, -\alpha N_2)}(\beta N_1, \beta N_2)\right|^2_{g_t}}{|(\alpha, -\alpha)|^2_{g_t} \cdot |(\beta, \beta)|^2_{g_t}} \\
&\overset{2.20}{=} \frac{t^4 \left|\mathcal{H}\nabla^{g+g}_{(\alpha N_1, -\alpha N_2)}(\beta N_1, \beta N_2)\right|^2_{g_t}}{|(\alpha, -\alpha)|^2_{g_t} \cdot |(\beta, \beta)|^2_{g_t}} \\
&\overset{3.10}{\underset{3.7}{=}} \frac{t^4 \left|\Big((\beta \times \alpha)N_1, -(\beta \times \alpha)N_2\Big)\right|^2_{g_t}}{|(\alpha, -\alpha)|^2_{g_t} \cdot |(\beta, \beta)|^2_{g_t}} \\
&\overset{2.12}{=} \frac{t^4 \left|\Big((\beta \times \alpha)N_1, -(\beta \times \alpha)N_2\Big)\right|^2_{g+g}}{|(\alpha, -\alpha)|^2_{g+g} \cdot t^2 |(\beta, \beta)|^2_{g+g}} \\
&= \frac{t^4(2)}{2 \cdot t^2(2)} = \frac{t^2}{2}.
\end{aligned}$$

■

4.5 The 3-1 Rules

Lemma 4.6. *Let $U, V, W \in \mathcal{V}^\pi$ and $X \in \mathcal{H}^\pi$. Then for all $t \in (0,1)$,*

$$R_{g_t}(U, V, W, X) = 0.$$

Proof. This follows from Part 1 of Theorem 2.28 and Theorem 3.12. ∎

Lemma 4.7. *Let $V \in \mathcal{V}^\pi$ and $X, Y, Z \in \mathcal{H}^\pi$. Then for all $t \in (0,1)$,*

$$R_{g_t}(X, Y, Z, V) = 0.$$

Proof.

$$R_{g_t}(X,Y,Z,V) \overset{\substack{2.28\\3.12}}{=} -\underbrace{g_t(\nabla^t_Z(A^t_X Y), V)}_{\text{Part (1)}} + \underbrace{g_t(A^t_{\nabla^t_Z X} Y, V)}_{\text{Part (2)}} + \underbrace{g_t(A^t_X(\nabla^t_Z Y), V)}_{\text{Part (3)}}.$$

Calculation of Part (1)

$$\begin{aligned}
g_t\Big(\nabla^{g_t}_Z(A^{g_t}_X Y), V\Big) &\overset{2.22}{=} g_t\left(\nabla^{g_t}_Z\left(\frac{1}{2}[X,Y]^{\mathcal{V}}\right), V\right)\\
&= \frac{1}{2}\cdot 2g_t\left(\nabla^{g_t}_Z\left(\frac{1}{2}[X,Y]^{\mathcal{V}}\right), V\right)\\
&= \frac{1}{2}\cdot 2t^2(g+g)\left(\nabla^{g+g}_Z\left(\frac{1}{2}[X,Y]^{\mathcal{V}}\right), V\right) \text{ by Koszul's formula}\\
&\overset{2.22}{=} t^2(g+g)\left(\nabla^{g+g}_Z(A^{g+g}_X Y), V\right).
\end{aligned}$$

Calculation of Part (2)

$$
\begin{aligned}
g_t\left(A^{g_t}_{\nabla^{g_t}_Z X}Y, V\right) &\overset{2.21}{=} g_t\left(A^{g_t}_{\mathcal{H}\nabla^{g_t}_Z X}Y, V\right) \\
&\overset{2.22}{=} g_t\left(\frac{1}{2}\mathcal{V}\left[\left(\nabla^{g_t}_Z X\right)^{\mathcal{H}}, Y\right], V\right) \\
&\overset{2.14}{=} g_t\left(\frac{1}{2}\mathcal{V}\left[\left(\nabla^{g+g}_Z X\right)^{\mathcal{H}}, Y\right], V\right) \\
&\overset{2.12}{=} t^2(g+g)\left(\frac{1}{2}\mathcal{V}\left[\left(\nabla^{g+g}_Z X\right)^{\mathcal{H}}, Y\right], V\right) \\
&\overset{2.22}{=} t^2(g+g)\left(A^{g+g}_{\mathcal{H}(\nabla^{g+g}_Z X)}Y, V\right) \overset{2.21}{=} t^2(g+g)\left(A^{g+g}_{\nabla^{g+g}_Z X}Y, V\right).
\end{aligned}
$$

Calculation of Part (3)

$$
\begin{aligned}
g_t\left(A^{g_t}_X\left(\nabla^{g_t}_Z Y\right), V\right) &= g_t\Big(\mathcal{V}\Big(A^{g_t}_X(\nabla^{g_t}_Z Y)\Big), V\Big) \\
&\overset{2.20}{=} g_t\Big(\mathcal{V}\Big(\nabla^{g_t}_X(\mathcal{H}\nabla^{g_t}_Z Y)\Big), V\Big) \\
&\overset{2.14}{=} g_t\Big(\mathcal{V}\Big(\nabla^{g_t}_X(\mathcal{H}\nabla^{g+g}_Z Y)\Big), V\Big) \\
&\overset{2.14}{=} g_t\Big(\mathcal{V}\Big(\nabla^{g+g}_X(\mathcal{H}\nabla^{g+g}_Z Y)\Big), V\Big) \\
&\overset{2.12}{=} t^2(g+g)\Big(\mathcal{V}\Big(\nabla^{g+g}_X(\mathcal{H}\nabla^{g+g}_Z Y)\Big), V\Big) \\
&\overset{2.20}{=} t^2(g+g)\Big(\mathcal{V}\Big(A^{g+g}_X(\nabla^{g+g}_Z Y)\Big), V\Big) = t^2(g+g)\Big(A^{g+g}_X(\nabla^{g+g}_Z Y), V\Big).
\end{aligned}
$$

Let $\alpha, \beta, \gamma, \nu \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$. Define vector fields on $S^3 \times S^3$ by $X : (N_1, N_2) \mapsto (\alpha N_1, -\alpha N_2)$, $Y : (N_1, N_2) \mapsto (\beta N_1, -\beta N_2)$, $Z : (N_1, N_2) \mapsto (\gamma N_1, -\gamma N_2)$, and $V : (N_1, N_2) \mapsto (\nu N_1, \nu N_2)$.

$$
\begin{aligned}
&R_{g_t}(X, Y, Z, V) \\
&= -t^2(g+g)\left(\nabla_Z^{g+g}(A_X^{g+g}Y), V\right) + t^2(g+g)\left(A^{g+g}_{\nabla_Z^{g+g}X}Y, V\right) \\
&\quad + t^2(g+g)\left(A_X^{g+g}(\nabla_Z^{g+g}Y), V\right) \\
&\overset{2.28}{=} R_{g+g}(X, Y, Z, V) \\
&\overset{A.14}{=} \frac{1}{4}(g+g)([X, V], [Y, Z]) - \frac{1}{4}(g+g)([X, Z], [Y, V]) \\
&\overset{3.10}{=} \frac{1}{4}(g+g)\Big((2(\nu \times \alpha)N_1, -2(\nu \times \alpha)N_2), (2(\gamma \times \beta)N_1, 2(\gamma \times \beta)N_2)\Big) \\
&\qquad\qquad - \frac{1}{4}(g+g)\Big((2(\gamma \times \alpha)N_1, 2(\gamma \times \alpha)N_2), (2(\nu \times \beta)N_1, -2(\nu \times \beta)N_2)\Big) \\
&\overset{3.8}{=} 0.
\end{aligned}
$$

■

4.6 2V-2H Curvatures

Lemma 4.8. *Let $\alpha, \beta, \gamma, \nu \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$ be unit with respect to the usual metric g on S^3. Let $(N_1, N_2) \in S^3 \times S^3$ and define vectors in $T_{(N_1,N_2)}(S^3 \times S^3)$ by $x = (\alpha N_1, -\alpha N_2)$, $y = (\beta N_1, -\beta N_2)$, $v = (\nu N_1, -\nu N_2)$, and $w = (\omega N_1, \omega N_2)$. Let $\cdot$ be the usual dot product on $\mathbb{R}^3$. Then for all $t \in (0, 1)$,*

1) $R_{g_t}(x, v, y, w) = -2t^2(\alpha \times \beta) \cdot (\nu \times \omega) - 2t^4(\nu \times \beta) \cdot (\omega \times \alpha)$.

2) $R_{g_t}(v, w, x, y) = -4t^2(\alpha \times \beta) \cdot (\nu \times \omega) - 2t^4(\nu \times \beta) \cdot (\omega \times \alpha) + 2t^4(\omega \times \beta) \cdot (\nu \times \alpha)$.

Proof. If $V, W \in \mathcal{V}^\pi$ and $X, Y \in \mathcal{H}^\pi$ are basic, then

$R_{g_t}(X, V, Y, W) \overset{2.30}{=} -g_t\Big(\nabla^{g_t}_V(A^{g_t}_X Y), W\Big) - g_t\Big(A^{g_t}_Y V, A^{g_t}_X W\Big)$ where

$$\begin{aligned} g_t(\nabla^{g_t}_V(A^{g_t}_X Y), W) &= g_t\Big(\mathcal{V}(\nabla^{g_t}_V(A^{g_t}_X Y)), W\Big) \\ &\overset{2.25}{=} g_t\Big(\mathcal{V}(\nabla^{g_t}_V(A^{g+g}_X Y)), W\Big) \\ &\overset{2.20}{=} g_t\Big(\mathcal{V}(\nabla^{g_t}_V(\nabla^{g+g}_X Y)^{\mathcal{V}}), W\Big) \\ &\overset{2.14}{=} g_t\Big(\mathcal{V}(\nabla^{g+g}_V(\nabla^{g+g}_X Y)^{\mathcal{V}}), W\Big) \\ &\overset{2.20}{=} g_t\Big(\mathcal{V}(\nabla^{g+g}_V(A^{g+g}_X Y)), W\Big) \\ &\overset{2.12}{=} t^2(g+g)\Big(\mathcal{V}(\nabla^{g+g}_V(A^{g+g}_X Y)), W\Big) = t^2(g+g)\Big(\nabla^{g+g}_V(A^{g+g}_X Y), W\Big) \end{aligned}$$

and $g_t\Big(A^{g_t}_Y V, A^{g_t}_X W\Big) \overset{2.25}{=} g_t\Big(t^2 A^{g+g}_Y V, t^2 A^{g+g}_X W\Big)$

$$\overset{2.20}{=} t^4 g_t\Big(\mathcal{H}\nabla^{g+g}_Y V, \mathcal{H}\nabla^{g+g}_X W\Big) \overset{2.12}{=} t^4(g+g)\Big(\mathcal{H}\nabla^{g+g}_Y V, \mathcal{H}\nabla^{g+g}_X W\Big).$$

Extend vectors v, w to π-vertical vector fields on $S^3 \times S^3$ defined by $V : (N_1, N_2) \mapsto (\nu N_1, \nu N_2)$ and $W : (N_1, N_2) \mapsto (\omega N_1, \omega N_2)$ and extend vectors x, y to basic π-horizontal vector fields $X : (pN_1, pN_2) \mapsto (p\alpha N_1, -p\alpha N_2)$ and $Y : (pN_1, pN_2) \mapsto (p\beta N_1, -p\beta N_2)$. These vector fields are basic since $\Big\{(p\alpha N_1, -p\alpha N_2) \mid \ \in S^3\Big\}$ and $\Big\{(p\beta N_1, -p\beta N_2) \mid \ \in S^3\Big\}$ are orbits under the left diagonal action of S^3 on $S^3 \times S^3$, so $d\pi$ maps each of these orbits to a single vector, making X and Y each π-related to constant vector fields.

By Theorem A.15, $\nabla_V^{g+g}(A_X^{g+g}Y)$ only depends on $A_X^{g+g}Y$ along a curve $c_V(t)$ in $S^3 \times S^3$ that satisfies $c_V(0) = (N_1, N_2)$ and $c_V'(0) = (\nu N_1, \nu N_2)$. For $s \in [0, 2\pi)$, define $c_V(s) = \Big(\big(\cos s + (\sin s)\nu\big)N_1, \big(\cos s + (\sin s)\nu\big)N_2\Big)$.

We only need to understand $A_X^{g+g}Y$ along $c_V(s)$, so we only need $X|_{c_V(s)}$ and $Y|_{c_V(s)}$. For notational simplicity, let $\cos s + (\sin s)\nu = p_\nu(s)$. Then $X|_{c_V(s)} : (p_\nu(s)N_1, p_\nu(s)N_2) \mapsto (p_\nu(s)\alpha N_1, -p_\nu(s)\alpha N_2)$ and $Y|_{c_V(s)} : (p_\nu(s)N_1, p_\nu(s)N_2) \mapsto (p_\nu(s)\beta N_1, -p_\nu(s)\beta N_2)$.

$$
\begin{aligned}
&A^{g+g}_{X|_{c_V(s)}} Y|_{c_V(s)} \\
&\overset{2.20}{=} \mathcal{V}\nabla^{g+g}_{\left(p_\nu(s)\alpha N_1, -p_\nu(s)\alpha N_2\right)} \left(p_\nu(s)\beta N_1, -p_\nu(s)\beta N_2\right) \\
&= \mathcal{V}\left(\left(\nabla^{\mathbb{R}^4\times\mathbb{R}^4}_{\left(p_\nu(s)\alpha N_1, -p_\nu(s)\alpha N_2\right)} \left(p_\nu(s)\beta N_1, -p_\nu(s)\beta N_2\right)\right)^{T(S^3\times S^3)}\right) \\
&= \mathcal{V}\nabla^{\mathbb{R}^4\times\mathbb{R}^4}_{\left(p_\nu(s)\alpha N_1, -p_\nu(s)\alpha N_2\right)} \left(p_\nu(s)\beta N_1, -p_\nu(s)\beta N_2\right) \\
&\quad \text{since } \mathcal{V} \subseteq T(S^3 \times S^3) \implies \text{proj}_{\mathcal{V}} \circ \text{proj}_{T(S^3\times S^3)} = \text{proj}_{\mathcal{V}} \\
&\overset{A.16}{=} \mathcal{V}\left(p_\nu(s)\nabla^{\mathbb{R}^4\times\mathbb{R}^4}_{\left(\alpha N_1, -\alpha N_2\right)} \left(\beta N_1, -\beta N_2\right)\right) \\
&= p_\nu(s)\left(\nabla^{\mathbb{R}^4\times\mathbb{R}^4}_{\left(\alpha N_1, -\alpha N_2\right)} \left(\beta N_1, -\beta N_2\right)\right)^{\mathcal{V}} \text{ since left multiplication by } p_\nu(s) \text{ preserves } \mathcal{V} \\
&= p_\nu(s)\left(\nabla^{g+g}_{\left(\alpha N_1, -\alpha N_2\right)} \left(\beta N_1, -\beta N_2\right)\right)^{\mathcal{V}} \text{ since } \mathcal{V} \subseteq T(S^3 \times S^3) \\
&\overset{3.10}{\underset{3.6}{=}} p_\nu(s)\Big((\beta \times \alpha)N_1, (\beta \times \alpha)N_2\Big) \\
&= \left(\left(p_\nu(s)(\beta \times \alpha)\overline{p_\nu(s)}\right) p_\nu(s)N_1, \left(p_\nu(s)(\beta \times \alpha)\overline{p_\nu(s)}\right) p_\nu(s)N_2\right)
\end{aligned}
$$

where

$$p_\nu(s)(\beta \times \alpha)\overline{p_\nu(s)}$$

$$= \Big(\cos s + (\sin s)\nu\Big)(\beta \times \alpha)\Big(\cos s - (\sin s)\nu\Big)$$

$$= (\cos s)^2(\beta \times \alpha) + (\cos s)(\sin s)\Big(\nu(\beta \times \alpha) - (\beta \times \alpha)\nu\Big) - (\sin s)^2\nu(\beta \times \alpha)\nu$$

$$= (\cos s)^2(\beta \times \alpha) + (\cos s)(\sin s)\Big(\nu(\beta \times \alpha) - (\overline{\beta \times \alpha})(\overline{\nu})\Big) - (\sin s)^2\nu(\beta \times \alpha)\nu$$

$$= (\cos s)^2(\beta \times \alpha) + (\cos s)(\sin s)\Big(\nu(\beta \times \alpha) - \overline{\nu(\beta \times \alpha)}\Big) - (\sin s)^2\nu(\beta \times \alpha)\nu$$

$$= (\cos s)^2(\beta \times \alpha)$$

$$+ (\cos s)(\sin s)\Big(\mathrm{Re}\big(\nu(\beta \times \alpha)\big) + \mathrm{Im}\big(\nu(\beta \times \alpha)\big) - \Big(\mathrm{Re}\big(\nu(\beta \times \alpha)\big) - \mathrm{Im}\big(\nu(\beta \times \alpha)\big)\Big)\Big)$$

$$- (\sin s)^2\nu(\beta \times \alpha)\nu$$

$$= (\cos s)^2(\beta \times \alpha) + 2(\cos s)(\sin s)\mathrm{Im}\big(\nu(\beta \times \alpha)\big) - (\sin s)^2\nu(\beta \times \alpha)\nu$$

$$\overset{A.13}{=} (\cos s)^2(\beta \times \alpha) + \sin(2s)\big(\nu \times (\beta \times \alpha)\big) - (\sin s)^2\nu(\beta \times \alpha)\nu.$$

To simplify notation, let $(\nu N_1, \nu N_2) = \nu N$ and

$$\left(\Big(p_\nu(s)(\beta \times \alpha)\overline{p_\nu(s)}\Big)\, p_\nu(s)N_1, \Big(p_\nu(s)(\beta \times \alpha)\overline{p_\nu(s)}\Big)\, p_\nu(s)N_2\right)$$
$$= \Big(p_\nu(s)(\beta \times \alpha)\overline{p_\nu(s)}\Big)\, p_\nu(s)N.$$

Then $\quad \nabla^{g+g}_V A^{g+g}_X Y$

$$= \nabla^{g+g}_{\nu N}\Big(p_\nu(s)(\beta \times \alpha)\overline{p_\nu(s)}\Big)\, p_\nu(s)N$$

$$= \nabla^{g+g}_{\nu N}\Big((\cos s)^2(\beta \times \alpha) + \sin(2t)\big(\nu \times (\beta \times \alpha)\big) - (\sin s)^2\nu(\beta \times \alpha)\nu\Big)\, p_\nu(s)N$$

$$\overset{A.11}{=} \nabla^{g+g}_{\nu N}(\cos s)^2(\beta \times \alpha)p_\nu(s)N + \nabla^{g+g}_{\nu N}\sin(2t)\big(\nu \times (\beta \times \alpha)\big)p_\nu(s)N$$

$$- \nabla^{g+g}_{\nu N}(\sin s)^2\nu(\beta \times \alpha)\nu p_\nu(s)N$$

where $\nabla^{g+g}_{\nu N}(\cos s)^2(\beta \times \alpha)p_\nu(s)N$

$$\overset{A.11}{=} \left(\left(\frac{d}{ds}(\cos s)^2 \right) (\beta \times \alpha)p_\nu(s)N + (\cos s)^2 \nabla^{g+g}_{\nu N}(\beta \times \alpha)p_\nu(s)N \right) \Bigg|_{s=0}$$

$$= ((\beta \times \alpha) \times \nu)N$$

$$\nabla^{g+g}_{\nu N} \sin(2s)(\nu \times (\beta \times \alpha))p_\nu(s)N$$

$$\overset{A.11}{=} \left(\left(\frac{d}{ds} \sin(2s) \right) (\nu \times (\beta \times \alpha))p_\nu(s)N + \sin(2s) \nabla^{g+g}_{\nu N}(\nu \times (\beta \times \alpha))p_\nu(s)N \right) \Bigg|_{s=0}$$

$$= 2(\nu \times (\beta \times \alpha))N$$

and $\nabla^{g+g}_{\nu N}(\sin s)^2 \nu(\beta \times \alpha)\nu p_\nu(s)N$

$$\overset{A.11}{=} \left(\left(\frac{d}{ds}(\sin s)^2 \right) \nu(\beta \times \alpha)\nu p_\nu(s)N + (\sin s)^2 \nabla^{g+g}_{\nu N} \nu(\beta \times \alpha)\nu p_\nu(s)N \right) \Bigg|_{s=0}$$

$$= \vec{0}.$$

Therefore, $\nabla^{g+g}_V A^{g+g}_X Y = ((\beta \times \alpha) \times \nu)N + 2(\nu \times (\beta \times \alpha))N$

$$= (\nu \times (\beta \times \alpha))N = \big((\nu \times (\beta \times \alpha))N_1, (\nu \times (\beta \times \alpha))N_2\big).$$

Thus, $t^2(g+g)\Big(\nabla_V^{g+g}(A_X^{g+g}Y), W\Big)$

$$= t^2 g\left((\nu\times(\beta\times\alpha))N_1, \omega N_1\right) + t^2 g\left((\nu\times(\beta\times\alpha))N_2, \omega N_2\right)$$

$$\overset{A.17}{=} t^2 g_{\mathbb{R}^4}\left((\nu\times(\beta\times\alpha))N_1, \omega N_1\right) + t^2 g_{\mathbb{R}^4}\left((\nu\times(\beta\times\alpha))N_1, \omega N_1\right)$$

$$\overset{A.18}{=} 2t^2(\nu\times(\beta\times\alpha))\cdot\omega$$

$$\overset{A.19}{=} -2t^2(\beta\times\alpha)\cdot(\nu\times\omega)$$

$$= 2t^2(\alpha\times\beta)\cdot(\nu\times\omega).$$

Also, $t^4(g+g)\Big(A_Y^{g+g}V, A_X^{g+g}W\Big)$

$$\overset{2.20}{=} t^4(g+g)\Big(\mathcal{H}\nabla_Y^{g+g}V, \mathcal{H}\nabla_X^{g+g}W\Big)$$

$$\overset{3.10}{\underset{3.7}{=}} t^4 g\Big((\nu\times\beta)N_1, (\omega\times\alpha)N_1\Big) + t^4 g\Big((\nu\times\beta)N_2, (\omega\times\alpha)N_2\Big)$$

$$\overset{A.17}{=} t^4 g_{\mathbb{R}^4}\Big((\nu\times\beta)N_1, (\omega\times\alpha)N_1\Big) + t^4 g_{\mathbb{R}^4}\Big((\nu\times\beta)N_2, (\omega\times\alpha)N_2\Big)$$

$$\overset{A.18}{=} 2t^4(\nu\times\beta)\cdot(\omega\times\alpha).$$

So $R_{g_t}(x,v,y,w) = -2t^2(\alpha\times\beta)\cdot(\nu\times\omega) - 2t^4(\nu\times\beta)\cdot(\omega\times\alpha)$.

Similarly, $R_{g_t}(v,w,x,y)$

$$
\begin{aligned}
&\overset{2.30}{=} -g_t\Big(\nabla^{g_t}_V(A^{g_t}_X Y), W\Big) - g_t\Big(A^{g_t}_Y V, A^{g_t}_X W\Big)\\
&\quad + g_M\Big(\nabla^{g_t}_W(A^{g_t}_X Y), V\Big) + g_t\Big(A^{g_t}_Y W, A^{g_t}_X V\Big)\\
&= -t^2(g+g)\Big(\nabla^{g+g}_V(A^{g+g}_X Y), W\Big) - t^4(g+g)\Big(A^{g+g}_Y V, A^{g+g}_X W\Big)\\
&\quad + t^2(g+g)\Big(\nabla^{g+g}_W(A^{g+g}_X Y), V\Big) + t^4(g+g)\Big(A^{g+g}_Y W, A^{g+g}_X V\Big)\\
&= -2t^2(\alpha\times\beta)\cdot(\nu\times\omega) - 2t^4(\nu\times\beta)\cdot(\omega\times\alpha)\\
&\quad + 2t^2(\alpha\times\beta)\cdot(\omega\times\nu) + 2t^4(\omega\times\beta)\cdot(\nu\times\alpha)\\
&= -4t^2(\alpha\times\beta)\cdot(\nu\times\omega) - 2t^4(\nu\times\beta)\cdot(\omega\times\alpha) + 2t^4(\omega\times\beta)\cdot(\nu\times\alpha). \qquad \blacksquare
\end{aligned}
$$

4.7 The Three Quaternion Rule

Lemma 4.9. *Let $\alpha, \beta, \gamma \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$ be perpendicular with respect to the usual metric g on S^3. Consider the basis*

$$\left\{ \begin{array}{ccc} (\alpha N_1, \alpha N_2) & (\beta N_1, \beta N_2) & (\gamma N_1, \gamma N_2) \\ (\alpha N_1, -\alpha N_2) & (\beta N_1, -\beta N_2) & (\gamma N_1, -\gamma N_2) \end{array} \right\}$$

for $T_{(N_1,N_2)}(S^3 \times S^3)$. Then for all $t \in (0,1)$, the $(0,4)$ curvature tensor R_{g_t} evaluated on combinations of these basis vectors such that all three of α, β, and γ are included is equal to zero.

Proof. Case 1 $(4V\ 0H)$

Case 1a: $R_{g_t}\Big((\alpha,\alpha),(\alpha,\alpha),(\beta,\beta),(\gamma,\gamma)\Big) \overset{A.20}{=} 0.$

Case 1b: $R_{g_\ell}\Big((\alpha N_1, \alpha N_2),(\beta N_1, \beta N_2),(\gamma N_1, \gamma N_2),(\alpha N_1, \alpha N_2)\Big) \overset{A.21}{=} 0.$

Case 2 $(3V\ 1H)$

These curvatures are zero by Lemma 4.6.

Case 3 $(2V\ 2H)$

According to Lemma 4.8, mixed curvatures of this type depend on dot products in $\mathbb{R}^3$ of the form $(q_1 \times q_2) \cdot (q_3 \times q_4)$ where $q_i \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$. Since there are four positions for quaternions in $(q_1 \times q_2) \cdot (q_3 \times q_4)$, it must be the case that exactly one of α, β, or γ is repeated. Suppose (WLOG) the repeated vector is α and $\gamma = \alpha \times \beta$.

Case 3a: $(\alpha \times \alpha) \cdot (\beta \times \gamma) = 0$ since $\alpha \times \alpha = 0$.

Case 3b: $(\alpha \times \beta) \cdot (\alpha \times \gamma) = \gamma \cdot (\alpha \times \gamma) = 0$ since $\gamma \perp_g (\alpha \times \gamma)$.

Case 4 $(1V\ 3H)$

These curvatures are zero by Lemma 4.7.

Case 5 $(0V\ 4H)$

Case 5a: $R_{g_\ell}\Big((\alpha N_1, -\alpha N_2), (\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2), (\gamma N_1, -\gamma N_2)\Big) \overset{A.20}{=} 0.$

Case 5b: $R_{g_\ell}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2), (\gamma N_1, -\gamma N_2), (\alpha N_1, -\alpha N_2)\Big) \overset{A.21}{=} 0.$ ■

4.8 Curvature of Product Planes

Lemma 4.10. *Let $(N_1, N_2) \in S^3 \times S^3$ and g be the usual metric on S^3. Let $\alpha, \beta \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$ satisfy $\alpha \perp_g \beta$. Then for all $t \in (0,1)$,*

1) $\sec_{g_t}\Big(\big(\alpha N_1, \vec{0}\big), \big(\vec{0}, \beta N_2\big) \Big) = \dfrac{2t^4 - 4t^2 + 2}{(t^2+1)^2}.$

2) $\sec_{g_t}\Big(\big(\alpha N_1, \vec{0}\big), \big(\beta N_1, \vec{0}\big) \Big) = \dfrac{2}{t^2+1}.$

Proof. Suppose $|\alpha|_g = |\beta|_g = 1$.

$\boxed{\sec_{g_t}\Big(\big(\alpha N_2, \vec{0}\big), \big(\vec{0}, \beta N_2\big)\Big)}$ To use the curvature formulas we derived in the previous sections, we must write each of these vectors as a linear combination of a π-vertical and a π-horizontal vector (see Lemma 3.6 and Lemma 3.7).

$\big(\alpha N_1, \vec{0}\big) = \frac{1}{2}(\alpha N_1, \alpha N_2) + \frac{1}{2}(\alpha N_1, -\alpha N_2)$ and $\big(\vec{0}, \beta N_2\big) = \frac{1}{2}(\beta N_1, \beta N_2) - \frac{1}{2}(\beta N_1, -\beta N_2)$.

So

$$
\begin{aligned}
&\operatorname{curv}_{g_t}\Big((\alpha N_1, \vec{0}), (\vec{0}, \beta N_2)\Big) \\
= &\operatorname{curv}_{g_t}\left(\frac{1}{2}(\alpha N_1, \alpha N_2) + \frac{1}{2}(\alpha N_1, -\alpha N_2), \frac{1}{2}(\beta N_1, \beta N_2) - \frac{1}{2}(\beta N_1, -\beta N_2)\right).
\end{aligned}
$$

There are $2^4 = 16$ terms in this calculation, but half of them are zero by Lemma 4.6, Lemma 4.7, Lemma 4.1, and Theorem A.22, giving

$$
\begin{aligned}
&= \frac{1}{16} R_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, \alpha N_2)\Big) \\
&\quad - \frac{1}{16} R_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, -\alpha N_2)\Big) \\
&\quad - \frac{1}{16} R_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, -\alpha N_2)\Big) \\
&\quad + \frac{1}{16} R_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, \alpha N_2)\Big) \\
&\quad + \frac{1}{16} R_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, -\alpha N_2)\Big) \\
&\quad - \frac{1}{16} R_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, \alpha N_2)\Big) \\
&\quad - \frac{1}{16} R_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, \alpha N_2)\Big) \\
&\quad + \frac{1}{16} R_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, -\alpha N_2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{16} \mathrm{curv}_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big) + \frac{1}{16} \mathrm{curv}_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2)\Big) \\
&\quad + \frac{1}{16} \mathrm{curv}_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2)\Big) + \frac{1}{16} \mathrm{curv}_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2)\Big) \\
&\quad - \frac{1}{8} R_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, \alpha N_2)\Big) \\
&\quad - \frac{1}{8} R_{g_t}\Big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, -\alpha N_2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{16}|(\alpha N_1,\alpha N_2)|^2_{g_t}|(\beta N_1,\beta N_2)|^2_{g_t}\sec_{g_t}\Big((\alpha N_1,\alpha N_2),(\beta N_1,\beta N_2)\Big)\\
&\quad+\frac{1}{16}|(\alpha N_1,-\alpha N_2)|^2_{g_t}|(\beta N_1,-\beta N_2)|^2_{g_t}\sec_{g_t}\Big((\alpha N_1,-\alpha N_2),(\beta N_1,-\beta N_2)\Big)\\
&\quad+\frac{1}{16}|(\alpha N_1,\alpha N_2)|^2_{g_t}|(\beta N_1,-\beta N_2)|^2_{g_t}\sec_{g_t}\Big((\alpha N_1,\alpha N_2),(\beta N_1,-\beta N_2)\Big)\\
&\quad+\frac{1}{16}|(\alpha N_1,-\alpha N_2)|^2_{g_t}|(\beta N_1,\beta N_2)|^2_{g_t}\sec_{g_t}\Big((\alpha N_1,-\alpha N_2),(\beta N_1,\beta N_2)\Big)\\
&\quad-\frac{1}{8}R_{g_t}\Big((\alpha N_1,-\alpha N_2),(\beta N_1,\beta N_2),(\beta N_1,-\beta N_2),(\alpha N_1,\alpha N_2)\Big)\\
&\quad-\frac{1}{8}R_{g_t}\Big((\alpha N_1,\alpha N_2),(\beta N_1,\beta N_2),(\beta N_1,-\beta N_2),(\alpha N_1,-\alpha N_2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{16}(2t^2)(2t^2)\underbrace{\left(\frac{1}{2t^2}\right)}_{4.3}+\frac{1}{16}(2)(2)\underbrace{\left(2-\frac{3}{2}t^2\right)}_{4.4}+\frac{1}{16}(2t^2)(2)\underbrace{\left(\frac{t^2}{2}\right)}_{4.5}\\
&\quad+\frac{1}{16}(2)(2t^2)\underbrace{\left(\frac{t^2}{2}\right)}_{4.5}-\frac{1}{8}\underbrace{(2t^2)}_{4.8}-\frac{1}{8}\underbrace{\left(4t^2-2t^4\right)}_{4.8}
\end{aligned}
$$

$$
= \frac{1}{2}t^4 - t^2 + \frac{1}{2}.
$$

$$
\begin{aligned}
\Big|\Big(\alpha N_1,\vec{0}\Big)\Big|^2_{g_t} &= \left|\frac{1}{2}(\alpha N_1,\alpha N_2)+\frac{1}{2}(\alpha N_1,-\alpha N_2)\right|^2_{g_t}\\
&= \frac{1}{4}\left|(\alpha N_1,\alpha N_2)\right|^2_{g_t}+\frac{1}{4}\left|(\alpha N_1,-\alpha N_2)\right|^2_{g_t}\\
&= \frac{1}{4}t^2\left|(\alpha N_1,\alpha N_2)\right|^2_{g+g}+\frac{1}{4}\left|(\alpha N_1,-\alpha N_2)\right|^2_{g+g}\\
&= \frac{1}{4}t^2(\sqrt{2})^2+\frac{1}{4}(\sqrt{2})^2\\
&= \frac{t^2+1}{2} = \Big|\Big(\vec{0},\beta N_2\Big)\Big|^2_{g_t}.
\end{aligned}
$$

So $\sec_{g_t}\Big(\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \beta N_2\right)\Big) = \dfrac{\operatorname{curv}_{g_t}\Big(\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \beta N_2\right)\Big)}{\left|\left(\alpha N_1, \vec{0}\right)\right|_{g_t}^2 \left|\left(\vec{0}, \beta N_2\right)\right|_{g_t}^2}$

$$= \left(\frac{1}{2}t^4 - t^2 + \frac{1}{2}\right)\left(\frac{2}{t^2+1}\right)\left(\frac{2}{t^2+1}\right) = \frac{2t^4 - 4t^2 + 2}{(t^2+1)^2}.$$

$\boxed{\sec_{g_t}\Big(\left(\alpha N_1, \vec{0}\right), \left(\beta N_2, \vec{0}\right)\Big)}$

We calculated $\operatorname{curv}_{g_t}\left(\frac{1}{2}(\alpha N_1, \alpha N_2) + \frac{1}{2}(\alpha N_1, -\alpha N_2), \frac{1}{2}(\beta N_1, \beta N_2) - \frac{1}{2}(\beta N_1, -\beta N_2)\right).$

We need $\operatorname{curv}_{g_t}\left(\frac{1}{2}(\alpha N_1, \alpha N_2) + \frac{1}{2}(\alpha N_1, -\alpha N_2), \frac{1}{2}(\beta N_1, \beta N_2) + \frac{1}{2}(\beta N_1, -\beta N_2)\right).$

To make this second calculation, we can use the work we did for the first as follows:

$$\begin{aligned}
&\tfrac{1}{16}R_{g_t}\big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, \alpha N_2)\big)\\
&\boxed{-}\tfrac{1}{16}R_{g_t}\big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, -\alpha N_2)\big)\\
&\boxed{-}\tfrac{1}{16}R_{g_t}\big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, -\alpha N_2)\big)\\
&+\tfrac{1}{16}R_{g_t}\big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, \alpha N_2)\big)\\
&+\tfrac{1}{16}R_{g_t}\big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, -\alpha N_2)\big)\\
&\boxed{-}\tfrac{1}{16}R_{g_t}\big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, \alpha N_2)\big)\\
&\boxed{-}\tfrac{1}{16}R_{g_t}\big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, \alpha N_2)\big)\\
&+\tfrac{1}{16}R_{g_t}\big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, -\alpha N_2)\big)
\end{aligned}$$

$$\longrightarrow$$

$$\begin{aligned}
&\tfrac{1}{16}R_{g_t}\big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, \alpha N_2)\big)\\
&\boxed{+}\tfrac{1}{16}R_{g_t}\big((\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, -\alpha N_2)\big)\\
&\boxed{+}\tfrac{1}{16}R_{g_t}\big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, -\alpha N_2)\big)\\
&+\tfrac{1}{16}R_{g_t}\big((\alpha N_1, \alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, \alpha N_2)\big)\\
&+\tfrac{1}{16}R_{g_t}\big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, -\alpha N_2)\big)\\
&\boxed{+}\tfrac{1}{16}R_{g_t}\big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, \alpha N_2)\big)\\
&\boxed{+}\tfrac{1}{16}R_{g_t}\big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, \beta N_2), (\alpha N_1, \alpha N_2)\big)\\
&+\tfrac{1}{16}R_{g_t}\big((\alpha N_1, -\alpha N_2), (\beta N_1, -\beta N_2), (\beta N_1, -\beta N_2), (\alpha N_1, -\alpha N_2)\big)
\end{aligned}$$

So

$$\begin{aligned}
&\operatorname{curv}_{g_t}\Big(\left(\alpha N_1, \vec{0}\right), \left(\beta N_1, \vec{0}\right)\Big)\\
&= \frac{1}{16}(2t^2)(2t^2)\left(\frac{1}{2t^2}\right) + \frac{1}{16}(2)(2)\left(2 - \frac{3}{2}t^2\right) + \frac{1}{16}(2t^2)(2)\left(\frac{t^2}{2}\right)\\
&\quad + \frac{1}{16}(2)(2t^2)\left(\frac{t^2}{2}\right) + \frac{1}{8}(2t^2) + \frac{1}{8}\left(4t^2 - 2t^4\right)\\
&= \frac{2}{t^2+1}.
\end{aligned}$$

■

Chapter 5:

Proofs of Main Theorems

5.1 Minimal Displacement Calculations

Definition 5.1 (page 247 of [10]). *Let (M, g_M) be a Riemannian manifold and $f : (M, g_M) \to (M, g_M)$ be an isometry. The* **displacement of f** *with respect to g_M, denoted* $\text{Displ}_{g_M}(f) : M \to \mathbb{R}$*, is defined by* $p \mapsto \text{dist}_{g_M}\Big(p, f(p)\Big)$.

Definition 5.2. *Let (M, g) be a compact Riemannian manifold and $f : (M, g_M) \to (M, g_M)$ be an isometry. The* **minimal displacement of f** *with respect to g_M, denoted* $\text{minDispl}_{g_M}(f)$*, is defined by* $\text{minDispl}_{g_M}(f) = \min\limits_{p \in M} \Big\{\text{Displ}_{g_M}(f)\big|_p\Big\}$.

Lemma 5.3. *Let g be the usual metric on S^3 and g_t be the metric on $S^3 \times S^3$ from Definition 3.8. Define $a : S^3 \times S^3 \to S^3 \times S^3$ by $(N_1, N_2) \mapsto (-N_1, -N_2)$. Then*

1) $\text{minDispl}_{g+g}(a) = \sqrt{2}\pi$.

2) $\text{minDispl}_{g_t}(a) = \sqrt{2}\pi t$ *for all* $t \in (0, 1)$.

Proof. $\boxed{\text{minDispl}_{g+g}(a)}$ Let $\gamma : [0,1] \to S^3 \times S^3$ be an arbitrary, constant speed curve in $S^3 \times S^3$ connecting point (N_1, N_2) to its antipode $(-N_1, -N_2)$. Then $\gamma(t)$ is given by $t \mapsto (\gamma_1(t), \gamma_2(t))$ where $\gamma_1(t), \gamma_2(t)$ are constant speed curves in S^3 satisfying $\gamma_1(0) = N_1, \gamma_1(1) = -N_1, \gamma_2(0) = N_2, \gamma_2(1) = -N_2$.

$$
\begin{aligned}
E_{g+g}(\gamma) &= \frac{1}{2}\int_0^1 \left|(\gamma_1'(t), \gamma_2'(t))\right|_{g+g}^2 dt \\
&= \frac{1}{2}\int_0^1 \left|\gamma_1'(t)\right|_g^2 dt + \frac{1}{2}\int_0^1 \left|\gamma_2'(t)\right|_g^2 dt \\
&= E_g(\gamma_1) + E_g(\gamma_2) \overset{A.23}{=} \frac{1}{2}\Big(L_g(\gamma_1)\Big)^2 + \frac{1}{2}\Big(L_g(\gamma_2)\Big)^2 \geq \frac{1}{2}\pi^2 + \frac{1}{2}\pi^2 = \pi^2 \\
&\implies L_{g+g}(\gamma) \overset{A.23}{=} \sqrt{2E_{g+g}(\gamma)} \geq \sqrt{2}\pi.
\end{aligned}
$$

That is, for any smooth, constant speed curve $\gamma(t)$ in $S^3 \times S^3$ connecting points (N_1, N_2) and $(-N_1, -N_2)$, $L_{g+g}(\gamma) \geq \sqrt{2}\pi$. This implies $\text{dist}_{g+g}\Big((N_1, N_2), a(N_1, N_2)\Big) \geq \sqrt{2}\pi$, which implies $\text{minDispl}_{g+g}(a) \geq \sqrt{2}\pi$. If we choose $\gamma_1(t)$ and $\gamma_2(t)$ to be geodesics, then $L_{g+g}(\gamma) = \sqrt{2}\pi$. Therefore, $\text{minDispl}_{g+g}(a) = \sqrt{2}\pi$.

$\boxed{\text{minDispl}_{g_t}(a)}$ Let $\alpha \in \mathfrak{s} = \text{Im}(\mathbb{H})$ satisfy $|\alpha|_g = 1$. Define $\gamma : [0, \pi] \to S^3 \times S^3$ by $s \mapsto (\cos s)(N_1, N_2) + (\sin s)(\alpha N_1, \alpha N_2)$. Then $\gamma(s)$ is a curve in $S^3 \times S^3$ connecting (N_1, N_2) to $(-N_1, -N_2)$, and its length with respect to g_t is

$$
\begin{aligned}
L_{g_t}(\gamma) &= \int_0^\pi |\gamma'(s)|_{g_t} ds \\
&= \int_0^\pi \left| -(\sin s)(N_1, N_2) + (\cos s)(\alpha N_1, \alpha N_2)\right|_{g_t} ds \\
&= \int_0^\pi \left|(\alpha, \alpha)\gamma(s)\right|_{g_t} ds = \int_0^\pi t\left|(\alpha, \alpha)\gamma(s)\right|_{g+g} ds = \int_0^\pi t\sqrt{2}\, ds = \sqrt{2}\pi t.
\end{aligned}
$$

Curve $\gamma(s)$ is minimal with respect to $g+g$ since $L_{g+g}(\gamma) = \sqrt{2}\pi$. If we suppose $c(s)$ is another constant speed curve in $S^3 \times S^3$ parametrized on $[0, \pi]$, then $|c'(s)|_{g+g} \geq \sqrt{2} = |\gamma'(s)|_{g+g}$. Furthermore,

$$
\begin{aligned}
|c'(s)|^2_{g_t} &= \left|c'(s)^{\mathcal{V}} + c'(s)^{\mathcal{H}}\right|^2_{g_t} \\
&= \left|c'(s)^{\mathcal{V}}\right|^2_{g_t} + \left|c'(s)^{\mathcal{H}}\right|^2_{g_t} \\
&\overset{2.12}{=} t^2\left|c'(s)^{\mathcal{V}}\right|^2_{g+g} + \left|c'(s)^{\mathcal{H}}\right|^2_{g+g} \\
&= \underbrace{t^2\left|c'(s)^{\mathcal{V}}\right|^2_{g+g} + t^2\left|c'(s)^{\mathcal{H}}\right|^2_{g+g}}_{t^2|c'(s)|^2_{g+g}} - t^2\left|c'(s)^{\mathcal{H}}\right|^2_{g+g} + \left|c'(s)^{\mathcal{H}}\right|^2_{g+g} \\
&= t^2|c'(s)|^2_{g+g} + (1-t^2)\left|c'(s)^{\mathcal{H}}\right|^2_{g+g} \geq t^2(2) + (1-t^2)\left|c'(s)^{\mathcal{H}}\right|^2_{g+g} \geq t^2(2) = |\gamma'(s)|^2_{g_t} \\
&\implies |c'(s)|^2_{g_t} \geq |\gamma'(s)|^2_{g_t} \implies L_{g_t}(c) \geq L_{g_t}(\gamma).
\end{aligned}
$$

Thus, $\gamma(s)$ is minimal with respect to g_t for all $t \in (0,1)$. Therefore, $\text{minDispl}_{g_t}(a) = L_{g_t}(\gamma) = \sqrt{2}\pi t$. ■

Lemma 5.4. *Let g be the usual metric on S^3 and g_t be the metric on $S^3 \times S^3$ from Definition 3.8. Define $f : S^3 \times S^3 \to S^3 \times S^3$ by $(N_1, N_2) \mapsto (-N_1, N_2)$. Then*

1) $\text{minDispl}_{g+g}(f) = \pi$.

2) $\text{minDispl}_{g_t}(f) = \pi\sqrt{\frac{t^2+1}{2}}$ *for all* $t \in (0,1)$.

Proof. $\boxed{\text{minDispl}_{g+g}(f)}$ The proof that $\text{minDispl}_{g+g}(a) = \sqrt{2}\pi$ in Lemma 5.3 can easily be adapted to show $\text{minDispl}_{g+g}(f) = \pi$.

$\boxed{\text{minDispl}_{g_t}(f)}$ Let $\alpha \in \mathfrak{s} = \text{Im}(\mathbb{H})$ satisfy $|\alpha|_g = 1$. Define $\gamma : [0,\pi] \to S^3 \times S^3$ by $s \mapsto \Big((\cos s)N_1 + (\sin s)\alpha N_1, N_2\Big)$. Then $\gamma(s)$ is a curve in $S^3 \times S^3$ connecting (N_1, N_2) to $(-N_1, N_2)$, and its length with respect to $g+g$ is

$$\begin{aligned} L_{g+g}(\gamma) &= \int_0^\pi |\gamma'(s)|_{g+g} ds \\ &= \int_0^\pi \big|\big(-(\sin s)N_1 + (\cos s)\alpha N_1, 0\big)\big|_{g+g} ds \\ &= \int_0^\pi \big|(\alpha\gamma_1(s), 0)\big|_{g+g} ds \quad \text{where } \gamma_1(s) = (\cos s)N_1 + (\sin s)\alpha N_1 \\ &= \int_0^\pi \big|\alpha\gamma_1(s)\big|_g ds = \pi. \end{aligned}$$

Curve $\gamma(s)$ is minimal with respect to $g+g$ since $L_{g+g}(\gamma) = \pi$. If we suppose $c(s)$ is another constant speed curve in $S^3 \times S^3$ parametrized on $[0,\pi]$, then $L_{g+g}(c) \geq L_{g+g}(\gamma)$, which implies by Theorem A.23 that $E_{g+g}(c) \geq E_{g+g}(\gamma)$. More specifically,

$$\begin{aligned} 2E_{g+g}(c) &= \int_0^\pi |c'(s)|^2_{g+g} ds \\ &= \int_0^\pi |c'(s)^{\mathcal{V}}|^2_{g+g} ds + \int_0^\pi |c'(s)^{\mathcal{H}}|^2_{g+g} ds \\ &\overset{A.23}{\geq} 2E_{g+g}(\gamma) = \int_0^\pi |\gamma'(s)|^2_{g+g} ds = \int_0^\pi |\gamma'(s)^{\mathcal{V}}|^2_{g+g} ds + \int_0^\pi |\gamma'(s)^{\mathcal{H}}|^2_{g+g} ds \\ &\implies \int_0^\pi |c'(s)^{\mathcal{V}}|^2_{g+g} ds + \int_0^\pi |c'(s)^{\mathcal{H}}|^2_{g+g} ds \geq \int_0^\pi |\gamma'(s)^{\mathcal{V}}|^2_{g+g} ds + \int_0^\pi |\gamma'(s)^{\mathcal{H}}|^2_{g+g} ds. \end{aligned}$$

Consider the Riemannian submersion $\pi : S^3 \times S^3 \to \frac{S^3 \times S^3}{\Delta S^3}$ defined in Section 3.2. The curve $\pi(\gamma)$ in the quotient space is minimal since it is a geodesic of length $\pi\sqrt{2}$ in a sphere with constant curvature 2 (see Lemma 3.11). Indeed,

$$\begin{aligned}
\nabla^{\overline{g}}_{(\pi\circ\gamma)'}(\pi\circ\gamma)' &\overset{A.24}{=} \nabla^{\overline{g}}_{d\pi(\gamma')} d\pi(\gamma') \\
&\overset{2.1}{=} \nabla^{\overline{g}}_{d\pi(\mathcal{H}\gamma')} d\pi(\mathcal{H}\gamma') \\
&\overset{A.16}{=} d\pi\left(\nabla^{g+g}_{\mathcal{H}\gamma'}\mathcal{H}\gamma'\right) \\
&= d\pi\left(\nabla^{g+g}_{\frac{1}{2}(\alpha,-\alpha)\gamma_1(s)}\frac{1}{2}(\alpha,-\alpha)\gamma_1(s)\right) \\
&\overset{3.10}{=} d\pi\left(\frac{1}{2}\alpha\gamma_1(s)\times\frac{1}{2}\alpha\gamma_1(s), \frac{1}{2}\alpha\gamma_1(s)\times\frac{1}{2}\alpha\gamma_1(s)\right) = \vec{0} \\
&\implies \pi\circ\gamma \text{ is a geodesic}
\end{aligned}$$

$$\begin{aligned}
2E_{\overline{g}}(\pi\circ\gamma) &= \int_0^\pi |(\pi\circ\gamma)'(s)|^2_{\overline{g}}\, ds \\
&\overset{A.24}{=} \int_0^\pi \left|d\pi(\gamma'(s))\right|^2_{\overline{g}}\, ds \\
&= \int_0^\pi \left|d\pi(\gamma'(s)^{\mathcal{V}}) + d\pi(\gamma'(s)^{\mathcal{H}})\right|^2_{\overline{g}}\, ds \\
&\overset{2.2}{=} \int_0^\pi \left|d\pi(\gamma'(s)^{\mathcal{H}})\right|^2_{\overline{g}}\, ds \\
&\overset{2.1}{=} \int_0^\pi |\gamma'(s)^{\mathcal{H}}|^2_{g+g}\, ds \\
&= \int_0^\pi \left|\frac{1}{2}(\alpha,-\alpha)\gamma_1(s)\right|^2_{g_t} ds \;\text{ since } \gamma'(s) = \frac{1}{2}(\alpha,\alpha)\gamma_1(s) + \frac{1}{2}(\alpha,-\alpha)\gamma_1(s) \\
&= \int_0^\pi \frac{1}{4}(2)ds = \frac{\pi}{2} \\
&\overset{A.23}{\implies} L_{\overline{g}}(\pi\circ\gamma)^2 = \frac{\pi^2}{2} \implies L_{\overline{g}}(\pi\circ\gamma) = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

The fact that $\pi(\gamma)$ is minimal implies

$$
\begin{aligned}
& E_{\overline{g}}(\pi \circ c) \geq E_{\overline{g}}(\pi \circ \gamma) \\
\Longrightarrow\ & \int_0^\pi |(\pi \circ c)'(s)|_{\overline{g}}^2 \, ds \geq \int_0^\pi |(\pi \circ \gamma)'(s)|_{\overline{g}}^2 \, ds \\
\overset{A.24}{\Longrightarrow}\ & \int_0^\pi |d\pi(c'(s))|_{\overline{g}}^2 \, ds \geq \int_0^\pi |d\pi(\gamma'(s))|_{\overline{g}}^2 \, ds \\
\Longrightarrow\ & \int_0^\pi \left| d\pi\left(c'(s)^{\mathcal{V}}\right) + d\pi\left(c'(s)^{\mathcal{H}}\right) \right|_{\overline{g}}^2 ds \geq \int_0^\pi \left| d\pi\left(\gamma'(s)^{\mathcal{V}}\right) + d\pi\left(\gamma'(s)^{\mathcal{H}}\right) \right|_{\overline{g}}^2 ds \\
\overset{\substack{2.2 \\ 2.1}}{\Longrightarrow}\ & \int_0^\pi \left| c'(s)^{\mathcal{H}} \right|_{\overline{g}}^2 ds \geq \int_0^\pi \left| \gamma'(s)^{\mathcal{H}} \right|_{\overline{g}}^2 ds \\
\overset{2.12}{\Longrightarrow}\ & \int_0^\pi \left| c'(s)^{\mathcal{H}} \right|_{g+g}^2 ds \geq \int_0^\pi \left| \gamma'(s)^{\mathcal{H}} \right|_{g+g}^2 ds.
\end{aligned}
$$

Thus, for some constant $a \geq 0$, $\int_0^\pi \left| c'(s)^{\mathcal{H}} \right|_{g+g}^2 ds \overset{(*)}{=} \int_0^\pi \left| \gamma'(s)^{\mathcal{H}} \right|_{g+g}^2 ds + a.$ Then

$$
\begin{aligned}
& \int_0^\pi \left| c'(s)^{\mathcal{V}} \right|_{g+g}^2 ds + \int_0^\pi \left| c'(s)^{\mathcal{H}} \right|_{g+g}^2 ds \geq \int_0^\pi |\gamma'(s)^{\mathcal{V}}|_{g+g}^2 ds + \int_0^\pi |\gamma'(s)^{\mathcal{H}}|_{g+g}^2 ds \\
\overset{(*)}{\Longrightarrow}\ & \int_0^\pi \left| c'(s)^{\mathcal{V}} \right|_{g+g}^2 ds + \int_0^\pi \left| \gamma'(s)^{\mathcal{H}} \right|_{g+g}^2 ds + a \geq \int_0^\pi \left| \gamma'(s)^{\mathcal{V}} \right|_{g+g}^2 ds + \int_0^\pi \left| \gamma'(s)^{\mathcal{H}} \right|_{g+g}^2 ds \\
\Longrightarrow\ & \int_0^\pi \left| c'(s)^{\mathcal{V}} \right|_{g+g}^2 ds + a \geq \int_0^\pi \left| \gamma'(s)^{\mathcal{V}} \right|_{g+g}^2 ds \\
\Longrightarrow\ & t^2 \left(\int_0^\pi \left| c'(s)^{\mathcal{V}} \right|_{g+g}^2 ds + a \right) \geq t^2 \int_0^\pi \left| \gamma'(s)^{\mathcal{V}} \right|_{g+g}^2 ds \\
\Longrightarrow\ & \int_0^\pi t^2 \left| c'(s)^{\mathcal{V}} \right|_{g+g}^2 ds + t^2 a \overset{(\bullet)}{\geq} \int_0^\pi t^2 \left| \gamma'(s)^{\mathcal{V}} \right|_{g+g}^2 ds.
\end{aligned}
$$

Thus, for all $t \in (0,1)$,

$$
\begin{aligned}
2E_{g_t}(c) &= \int_0^\pi |c'(s)|^2_{g_t} ds \overset{2.2}{=} \int_0^\pi |c'(s)^{\mathcal{V}}|^2_{g_t} ds + \int_0^\pi |c'(s)^{\mathcal{H}}|^2_{g_t} ds \\
&\overset{2.12}{=} \int_0^\pi t^2 |c'(s)^{\mathcal{V}}|^2_{g+g} ds + \int_0^\pi |c'(s)^{\mathcal{H}}|^2_{g+g} ds \\
&\overset{(\bullet)}{\geq} \int_0^\pi t^2 |\gamma'(s)^{\mathcal{V}}|^2_{g+g} ds - t^2 a + \int_0^\pi |c'(s)^{\mathcal{H}}|^2_{g+g} ds \\
&\overset{(*)}{=} \int_0^\pi t^2 |\gamma'(s)^{\mathcal{V}}|^2_{g+g} ds - t^2 a + \int_0^\pi |\gamma'(s)^{\mathcal{H}}|^2_{g+g} ds + a \\
&= \int_0^\pi |\gamma'(s)^{\mathcal{V}}|^2_{g_t} ds + \int_0^\pi |\gamma'(s)^{\mathcal{H}}|^2_{g_t} ds + (1-t^2)a \\
&= \int_0^\pi |\gamma'(s)|^2_{g_t} ds + (1-t^2)a = 2E_{g_t}(\gamma) + (1-t^2)a \geq 2E_{g_t}(\gamma) \\
&\overset{A.23}{\Longrightarrow} L_{g_t}(c) \geq L_{g_t}(\gamma).
\end{aligned}
$$

So

$$
\begin{aligned}
\left(\operatorname{minDispl}_{g_t}(\gamma)\right)^2 &= \left(L_{g_t}(\gamma)\right)^2 \overset{A.23}{=} 2\pi E_{g_t}(\gamma) = \pi \int_0^\pi |\gamma'(s)|^2_{g_t} ds \\
&= \pi \int_0^\pi \left|(\alpha\gamma_1(s), 0)\right|^2_{g_t} ds \quad \text{where } \gamma_1(s) = (\cos s)N_1 + (\sin s)\alpha N_1 \\
&= \pi \int_0^\pi \left(\left|\frac{1}{2}(\alpha,\alpha)\gamma_1(s)\right|^2_{g_t} + \left|\frac{1}{2}(\alpha,-\alpha)\gamma_1(s)\right|^2_{g_t} \right) ds \\
&= \frac{\pi}{4} \int_0^\pi \left(\left|(\alpha,\alpha)\gamma_1(s)\right|^2_{g_t} + \left|(\alpha,-\alpha)\gamma_1(s)\right|^2_{g_t} \right) ds \\
&= \frac{\pi}{4} \int_0^\pi \left(t^2\left|(\alpha,\alpha)\gamma_1(s)\right|^2_{g+g} + \left|(\alpha,-\alpha)\gamma_1(s)\right|^2_{g+g} \right) ds \\
&= \frac{\pi}{4} \int_0^\pi \left(t^2(2) + 2 \right) ds = \pi^2 \left(\frac{t^2+1}{2} \right) \\
&\Longrightarrow L_{g_t}(\gamma) = \pi \sqrt{\frac{t^2+1}{2}}.
\end{aligned}
$$

■

5.2 Proofs of Theorem 1.5 and Theorem 1.6

Proof. (of 1.6) Let g be the usual metric on S^3 and $\alpha, \beta, \gamma \in \mathfrak{s} = \mathrm{Im}(\mathbb{H})$ be perpendicular with respect to g. Let $(N_1, N_2) \in S^3 \times S^3$.

— **Proof of 1a and 2a** —

$$\begin{aligned}
&\mathrm{Ric}_4\Big((\alpha N_1, -\alpha N_2); (\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\gamma N_1, \gamma N_2), (\beta N_1, -\beta N_2)\Big)\\
&= \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\alpha N_1, \alpha N_2)\Big)}_{4.1} + \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2)\Big)}_{4.5}\\
&\quad + \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\gamma N_1, \gamma N_2)\Big)}_{4.5} + \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_2, -\beta N_2)\Big)}_{4.4}\\
&= 0 + \frac{t^2}{2} + \frac{t^2}{2} + \left(2 - \frac{3t^2}{2}\right) = \frac{4 - t^2}{2}.
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Ric}_4\left(\left(\alpha N_1, \vec{0}\right); \left(\vec{0}, \alpha N_2\right), \left(\vec{0}, \beta N_2\right), \left(\vec{0}, \gamma N_2\right), \left(\beta N_1, \vec{0}\right)\right)\\
&= \underbrace{\sec_{g_t}\left(\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \alpha N_2\right)\right)}_{4.1} + \underbrace{\sec_{g_t}\left(\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \beta N_2\right)\right)}_{4.10}\\
&\quad + \underbrace{\sec_{g_t}\left(\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \gamma N_2\right)\right)}_{4.10} + \underbrace{\sec_{g_t}\left(\left(\alpha N_1, \vec{0}\right), \left(\beta N_1, \vec{0}\right)\right)}_{4.10}\\
&= 0 + \frac{2t^4 - 4t^2 + 2}{(t^2+1)^2} + \frac{2t^4 - 4t^2 + 2}{(t^2+1)^2} + \frac{2}{t^2+1} = \frac{4t^4 - 6t^2 + 6}{(t^2+1)^2}.
\end{aligned}$$

Therefore, $\min \mathrm{Ric}_4(S^3 \times S^3, g_t) \leq \min \left\{ \frac{4-t^2}{2}, \frac{4t^4 - 6t^2 + 6}{(t^2+1)^2} \right\}$.

— Calculation 1a —

$$\min \mathrm{Ric}_4(S^3 \times S^3, g_t) \cdot \left(\mathrm{minDispl}_{g_t}(a)\right)^2 \leq \min \left\{ \frac{4-t^2}{2}, \frac{4t^4 - 6t^2 + 6}{(t^2+1)^2} \right\} \cdot \underbrace{2t^2\pi^2}_{5.3}.$$

(∗∗∗) The subgroup $H_a = \{\mathrm{id}, a\} \subseteq \mathrm{Iso}(S^3 \times S^3, g_t)$ acts properly discontinuously on $S^3 \times S^3$. Thus, the quotient map $\pi_a : S^3 \times S^3 \longrightarrow \frac{S^3 \times S^3}{H_a}$ is a covering map (see Theorem A.25). Furthermore, $\frac{S^3 \times S^3}{H_a}$ can be equipped with a smooth structure such that π_a is a smooth covering map (see Theorem A.26). Finally, for each $t \in (0,1)$, there is unique metric g_t^a on $\frac{S^3 \times S^3}{H_a}$ such that $\pi_a : (S^3 \times S^3, g_t) \longrightarrow \left(\frac{S^3 \times S^3}{H_a}, g_t^a\right)$ is a Riemannian covering map (see Section 1.3.3 in [10] and Definition A.27). Then $\pi_a : (S^3 \times S^3, g_t) \longrightarrow \left(\frac{S^3 \times S^3}{H_a}, g_t^a\right)$ is a local isometry for all $t \in (0,1)$, so curvature is preserved (see Theorem A.2). Thus, $\min \mathrm{Ric}_4\left(\frac{S^3 \times S^3}{H_a}, g_t^a\right) = \min \mathrm{Ric}_4(S^3 \times S^3, g_t)$ for all $t \in (0,1)$. Consider the curve $\gamma : [0, \pi] \longrightarrow S^3 \times S^3$ connecting (N_1, N_2) to $(-N_1, -N_2)$ defined by $s \mapsto (\cos s)(N_1, N_2) + (\sin s)(\alpha N_1, \alpha N_2)$. This γ is a segment in $S^3 \times S^3$ with length $L_{g_t}(\gamma) = \mathrm{minDispl}_{g_t}(a) = \sqrt{2}\pi t$ (see proof of Lemma 5.3 for details). In $\frac{S^3 \times S^3}{H_a}$, the projection $\pi_a(\gamma)$ is a loop since $\pi_a(\gamma(0)) = \pi_a(N_1, N_2) = \pi_a(a(N_1, N_2)) = \pi_a(-N_1, -N_2) = \pi_a(\gamma(\pi))$. Furthermore, $\pi_a(\gamma)$ is noncontractible (see Theorem A.28). Thus, (∗) $\mathrm{sys}_1\left(\frac{S^3 \times S^3}{H_a}, g_t^a\right) \leq L_{g_t^a}(\pi_a \circ \gamma) \overset{A.2}{=} L_{g_t}(\gamma) = \mathrm{minDispl}_{g_t}(a)$.

Therefore, for all $t \in (0,1)$,

$$\begin{aligned}
&\min \operatorname{Ric}_4\left(\frac{S^3\times S^3}{H_a}, g_t^a\right)\cdot\left(\operatorname{sys}_1\left(\frac{S^3\times S^3}{H_a}, g_t^a\right)\right)^2\\
&\le \min \operatorname{Ric}_4(S^3\times S^3, g_t)\cdot\Big(\operatorname{minDispl}_{g_t}(a)\Big)^2 \le \min\left\{\frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2}\right\}\cdot 2t^2\pi^2.
\end{aligned}$$

— **Calculation 2a** —

$$\min \operatorname{Ric}_4(S^3\times S^3, g_t)\cdot\Big(\operatorname{minDispl}_{g_t}(f)\Big)^2 \le \min\left\{\frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2}\right\}\cdot \underbrace{\left(\frac{t^2+1}{2}\right)\pi^2}_{5.4}.$$

Adapt $(***)$. Use $\gamma : [0,\pi] \to S^3 \times S^3$ defined by $s \mapsto \Big((\cos s)N_1 + (\sin s)\alpha N_1, N_2\Big)$ from the proof of Lemma 5.4. Conclude that for all $t \in (0,1)$,

$$\begin{aligned}
&\min \operatorname{Ric}_4\left(\frac{S^3\times S^3}{H_f}, g_t^a\right)\cdot\left(\operatorname{sys}_1\left(\frac{S^3\times S^3}{H_f}, g_t^f\right)\right)^2\\
&\overset{(\star\star)}{\le} \min \operatorname{Ric}_4(S^3\times S^3, g_t)\cdot\Big(\operatorname{minDispl}_{g_t}(f)\Big)^2\\
&\le \min\left\{\frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2}\right\}\cdot\left(\frac{t^2+1}{2}\right)\pi^2.
\end{aligned}$$

— **Proof of 1b and 2b** —

$$\begin{aligned}
&\mathrm{Ric}_3\Big((\alpha N_1, -\alpha N_2); (\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\gamma N_1, \gamma N_2)\Big) \\
&= \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\alpha N_1, \alpha N_2)\Big)}_{4.1} + \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2)\Big)}_{4.5} \\
&\quad + \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\gamma N_1, \gamma N_2)\Big)}_{4.5} \\
&= 0 + \frac{t^2}{2} + \frac{t^2}{2} = t^2.
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Ric}_3\Big(\Big(\alpha N_1, \vec{0}\Big); \Big(\vec{0}, \alpha N_2\Big), \Big(\vec{0}, \beta N_2\Big), \Big(\vec{0}, \gamma N_2\Big)\Big) \\
&= \underbrace{\sec_{g_t}\Big(\Big(\alpha N_1, \vec{0}\Big), \Big(\vec{0}, \alpha N_2\Big)\Big)}_{4.1} + \underbrace{\sec_{g_t}\Big(\Big(\alpha N_1, \vec{0}\Big), \Big(\vec{0}, \beta N_2\Big)\Big)}_{4.10} \\
&\quad + \underbrace{\sec_{g_t}\Big(\Big(\alpha N_1, \vec{0}\Big), \Big(\vec{0}, \gamma N_2\Big)\Big)}_{4.10} \\
&= 0 + \frac{2t^4 - 4t^2 + 2}{(t^2+1)^2} + \frac{2t^4 - 4t^2 + 2}{(t^2+1)^2} = \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2}.
\end{aligned}$$

Therefore, $\min \mathrm{Ric}_3(S^3 \times S^3, g_t) \leq \min \left\{ t^2, \dfrac{4t^4 - 8t^2 + 4}{(t^2+1)^2} \right\}$.

— **Calculation 1b** —

$$\min \mathrm{Ric}_3(S^3 \times S^3, g_t) \cdot \Big(\mathrm{minDispl}_{g_t}(a)\Big)^2 \leq \min \left\{ t^2, \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2} \right\} \cdot \underbrace{2t^2\pi^2}_{5.3}.$$

Then for all $t \in (0,1)$,

$$\min \mathrm{Ric}_3\left(\frac{S^3\times S^3}{H_a}, g_t^a\right)\cdot\left(\mathrm{sys}_1\left(\frac{S^3\times S^3}{H_a}, g_t^a\right)\right)^2$$
$$\overset{(*)}{\leq} \min \mathrm{Ric}_3(S^3\times S^3, g_t)\cdot\Big(\mathrm{minDispl}_{g_t}(a)\Big)^2 \leq \min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot 2t^2\pi^2.$$

— **Calculation 2b** —

$$\min \mathrm{Ric}_3(S^3\times S^3, g_t)\cdot\Big(\mathrm{minDispl}_{g_t}(f)\Big)^2 \leq \min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot \underbrace{\left(\frac{t^2+1}{2}\right)\pi^2}_{5.4}.$$

Then for all $t \in (0,1)$,

$$\min \mathrm{Ric}_3\left(\frac{S^3\times S^3}{H_f}, g_t^f\right)\cdot\left(\mathrm{sys}_1\left(\frac{S^3\times S^3}{H_f}, g_t^f\right)\right)^2$$
$$\overset{(\star\star)}{\leq} \min \mathrm{Ric}_3(S^3\times S^3, g_t)\cdot\Big(\mathrm{minDispl}_{g_t}(f)\Big)^2 \leq \min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot \left(\frac{t^2+1}{2}\right)\pi^2.$$

— **Proof of 1c and 2c** —

$$\mathrm{Ric}_2\Big((\alpha N_1, -\alpha N_2); (\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2)\Big)$$
$$= \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\alpha N_1, \alpha N_2)\Big)}_{4.1} + \underbrace{\sec_{g_t}\Big((\alpha N_1, -\alpha N_2), (\beta N_1, \beta N_2)\Big)}_{4.5}$$
$$= 0 + \frac{t^2}{2} = \frac{t^2}{2} = \frac{1}{2}\mathrm{Ric}_3\Big((\alpha N_1, -\alpha N_2); (\alpha N_1, \alpha N_2), (\beta N_1, \beta N_2), (\gamma N_1, \gamma N_2)\Big).$$

$$
\begin{aligned}
&\operatorname{Ric}_2\Big(\left(\alpha N_1, \vec{0}\right); \left(\vec{0}, \alpha N_2\right), \left(\vec{0}, \beta N_2\right) \Big) \\
&= \underbrace{\sec_{g_t}\Big(\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \alpha N_2\right) \Big)}_{4.1} + \underbrace{\sec_{g_t}\Big(\left(\alpha N_1, \vec{0}\right), \left(\vec{0}, \beta N_2\right) \Big)}_{4.10} \\
&= 0 + \frac{2t^4 - 4t^2 + 2}{(t^2+1)^2} = \frac{2t^4 - 4t^2 + 2}{(t^2+1)^2} \\
&= \frac{1}{2}\operatorname{Ric}_3\Big(\left(\alpha N_1, \vec{0}\right); \left(\vec{0}, \alpha N_2\right), \left(\vec{0}, \beta N_2\right), \left(\vec{0}, \gamma N_2\right) \Big).
\end{aligned}
$$

— Calculation 1c —

$$
\begin{aligned}
\min \operatorname{Ric}_2(S^3 \times S^3, g_t) \cdot \left(\operatorname{minDispl}_{g_t}(a)\right)^2 &= \frac{1}{2} \min \operatorname{Ric}_3(S^3 \times S^3, g_t) \cdot \left(\operatorname{minDispl}_{g_t}(a)\right)^2 \\
&\leq \frac{1}{2} \min \left\{ t^2, \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2} \right\} \cdot \underbrace{2t^2\pi^2}_{5.3} \\
&= \min \left\{ t^2, \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2} \right\} \cdot t^2\pi^2.
\end{aligned}
$$

Then for all $t \in (0,1)$,

$$
\begin{aligned}
&\min \operatorname{Ric}_2\left(\frac{S^3 \times S^3}{H_a}, g_t^a \right) \cdot \left(\operatorname{sys}_1\left(\frac{S^3 \times S^3}{H_a}, g_t^a \right) \right)^2 \\
&\overset{(***)}{\leq} \min \operatorname{Ric}_2(S^3 \times S^3, g_t) \cdot \left(\operatorname{minDispl}_{g_t}(a)\right)^2 \leq \min \left\{ t^2, \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2} \right\} \cdot t^2\pi^2.
\end{aligned}
$$

— **Calculation 2c** —

$$\begin{aligned}\min \mathrm{Ric}_2(S^3\times S^3, g_t)\cdot\left(\mathrm{minDispl}_{g_t}(f)\right)^2 &= \frac{1}{2}\min \mathrm{Ric}_3(S^3\times S^3, g_t)\cdot\left(\mathrm{minDispl}_{g_t}(f)\right)^2\\ &\le \frac{1}{2}\min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot \underbrace{\left(\frac{t^2+1}{2}\right)\pi^2}_{5.4}\\ &= \min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot\left(\frac{t^2+1}{4}\right)\pi^2.\end{aligned}$$

Then for all $t\in(0,1)$,

$$\begin{aligned}&\min \mathrm{Ric}_2\left(\frac{S^3\times S^3}{H_f}, g_t^f\right)\cdot\left(\mathrm{sys}_1\left(\frac{S^3\times S^3}{H_f}, g_t^f\right)\right)^2\\ &\overset{(\star\star)}{\le} \min \mathrm{Ric}_2(S^3\times S^3, g_t)\cdot\left(\mathrm{minDispl}_{g_t}(f)\right)^2\\ &\le \min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot\left(\frac{t^2+1}{4}\right)\pi^2.\end{aligned}$$

■

Proof. (of 1.5)

— **Proof of 1a′ and 2a′** —

By 1a and 1b in Theorem 1.6,

$$\min \mathrm{Ric}_4\left(M, g_t^a\right)\cdot\left(\mathrm{sys}_1\left(M, g_t^a\right)\right)^2 \le \min\left\{\frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2}\right\}\cdot 2t^2\pi^2.$$

$$\min \mathrm{Ric}_4\left(N, g_t^f\right)\cdot\left(\mathrm{sys}_1\left(N, g_t^f\right)\right)^2 \le \min\left\{\frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2}\right\}\cdot\left(\frac{t^2+1}{2}\right)\pi^2.$$

$$\frac{4-t^2}{2} = \frac{4t^4-6t^2+6}{(t^2+1)^2} \implies t^6+6t^4-19t^2+8=0.$$

By viewing https://www.desmos.com/calculator/tj1cpczd9g, we see that the degree-six polynomial $t^6 + 6t^4 - 19t^2 + 8$ has a real root in $(0, 1)$ approximately equal to 0.7143. Let r be this solution to the equation $t^6 + 6t^4 - 19t^2 + 8 = 0$.

Then, by viewing https://www.desmos.com/calculator/b0go9j32b2, we see

$$\begin{aligned}
\min\left\{\frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2}\right\} \cdot 2t^2 &= \begin{cases} \frac{4-t^2}{2} \cdot 2t^2 & \text{when } 0 < t \leq r \\ \frac{4t^4-6t^2+6}{(t^2+1)^2} \cdot 2t^2 & \text{when } r < t < 1 \end{cases} \\
&= \begin{cases} 4t^2 - t^4 & \text{when } 0 < t \leq r \\ \frac{8t^6-12t^4+12t^2}{(t^2+1)^2} & \text{when } r < t < 1 \end{cases} \\
&\leq 2
\end{aligned}$$

and by viewing https://www.desmos.com/calculator/acbt0sxvlt, we see

$$\begin{aligned}
\min\left\{\frac{4-t^2}{2}, \frac{4t^4-6t^2+6}{(t^2+1)^2}\right\} \cdot \frac{t^2+1}{2} &= \begin{cases} \frac{4-t^2}{2} \cdot \frac{t^2+1}{2} & \text{when } 0 < t \leq r \\ \frac{4t^4-6t^2+6}{(t^2+1)^2} \cdot \frac{t^2+1}{2} & \text{when } r < t < 1 \end{cases} \\
&= \begin{cases} \frac{-t^4+3t^2+4}{4} & \text{when } 0 < t \leq r \\ \frac{2t^4-3t^2+3}{t^2+1} & \text{when } r < t < 1 \end{cases} \\
&\leq \frac{-r^4+3r^2+4}{4} \approx 1.3176.
\end{aligned}$$

— **Proof of 1b′ and 2b′** —

By 2a and 2b in Theorem 1.6,

$$\min \mathrm{Ric}_3\left(M, g_t^a\right) \cdot \left(\mathrm{sys}_1\left(M, g_t^a\right)\right)^2 \leq \min\left\{t^2, \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2}\right\} \cdot 2t^2\pi^2.$$

$$\min \mathrm{Ric}_3\left(N, g_t^f\right) \cdot \left(\mathrm{sys}_1\left(N, g_t^f\right)\right)^2 \leq \min\left\{t^2, \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2}\right\} \cdot \left(\frac{t^2+1}{2}\right)\pi^2.$$

$$t^2 = \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2} \implies t^6 - 2t^4 + 9t^2 - 4 = 0.$$

By viewing https://www.desmos.com/calculator/vnotpxutz7, we see that the degree-six polynomial $t^6 - 2t^4 + 9t^2 - 4$ has a real root in $(0, 1)$ approximately equal to 0.6956. Let s be this solution to the equation $t^6 - 2t^4 + 9t^2 - 4 = 0$.

Then, by viewing https://www.desmos.com/calculator/dr02mmdk5k, we see

$$\begin{aligned}
\min\left\{t^2, \frac{4t^4 - 8t^2 + 4}{(t^2+1)^2}\right\} \cdot 2t^2 &= \begin{cases} t^2 \cdot 2t^2 & \text{when } 0 < t \leq s \\ \frac{4t^4-8t^2+4}{(t^2+1)^2} \cdot 2t^2 & \text{when } s < t < 1 \end{cases} \\
&= \begin{cases} 2t^4 & \text{when } 0 < t \leq s \\ \frac{8t^6-16t^4+8t^2}{(t^2+1)^2} & \text{when } s < t < 1 \end{cases} \\
&\leq 2s^4 \approx 0.4683
\end{aligned}$$

and by viewing https://www.desmos.com/calculator/ms5lsqdyjf, we see

$$\min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot\frac{t^2+1}{2} = \begin{cases} t^2\cdot\frac{t^2+1}{2} & \text{when } 0<t\le s \\ \frac{4t^4-8t^2+4}{(t^2+1)^2}\cdot\frac{t^2+1}{2} & \text{when } s<t<1 \end{cases}$$

$$= \begin{cases} \frac{t^4+t^2}{2} & \text{when } 0<t\le s \\ \frac{2t^4-4t^2+2}{t^2+1} & \text{when } s<t<1 \end{cases}$$

$$\le \frac{s^4+s^2}{2} \approx 0.359.$$

— **Proof of 1c′ and 2c′** —

$$\min \mathrm{Ric}_2\left(M, g_t^a\right)\cdot\left(\mathrm{sys}_1\left(M, g_t^a\right)\right)^2 \overset{1.6}{\le} \min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot t^2\pi^2$$

$$= \frac{1}{2}\min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot 2t^2\pi^2$$

$$\le \frac{1}{2}(2s^4\pi^2) \text{ for all } t\in(0,1) \text{ by 2a}'$$

$$= s^4\pi^2 \approx 0.2341\pi^2$$

$$\min \mathrm{Ric}_2\left(M, g_t^a\right)\cdot\left(\mathrm{sys}_1\left(M, g_t^a\right)\right)^2 \overset{1.6}{\le} \min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot\left(\frac{t^2+1}{4}\right)\pi^2$$

$$= \frac{1}{2}\min\left\{t^2, \frac{4t^4-8t^2+4}{(t^2+1)^2}\right\}\cdot\left(\frac{t^2+1}{2}\right)\pi^2$$

$$\le \frac{1}{2}\left(\frac{s^4+s^2}{2}\right)\pi^2 \text{ for all } t\in(0,1) \text{ by 2b}'$$

$$= \left(\frac{s^4+s^2}{4}\right)\pi^2 \approx 0.1795\pi^2. \quad \blacksquare$$

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Appendix A:

Other Definitions and Theorems Referenced

Definition A.1 (page 12 and page 196 in [10]). *A map* $f : (M, g_M) \longrightarrow (N, g_N)$ *is a* **local isometry** *if and only if for each* $p \in M$*, there is a neighborhood* $U \subseteq M$ *of* p *such that* $f|_p : U \longrightarrow f(U)$ *is an isometry. Alternatively,* f *is a local isometry if and only if for all* $p \in M$*, the differential* $df_p : T_pM \longrightarrow T_{f(p)}N$ *is a linear isometry.*

Theorem A.2 (Proposition 5.6.1 in [10] and Proposition 7.6 in [5]). *Let* $f : (M, g_M) \longrightarrow (N, g_N)$ *be a local isometry. Then*

1) *F maps geodesics to geodesics.*

2) *If f is a bijection, then f is distance preserving.*

3) *The Riemannian curvature tensor is invariant under f.*

Definition A.3 (Example 1.36 in [4]). *Let* V *be an* n*-dimensional real vector space. For any integer* $0 \leq k \leq n$*, the* **Grassmanian** *is the set* $G_k(V)$ *of all* k*-dimensional linear subspaces of* V*. It is a* $k(n-k)$*-dimensional smooth manifold.*

Theorem A.4 (Problem 21-13 in [4]). *Let V be an n-dimensional real vector space. The Grassmannian $G_k(V)$ is compact for each integer $0 \leq k \leq n$.*

Theorem A.5 (Theorem 27.4 in [7]). *Let X, Y be topological spaces. Let $f : X \longrightarrow Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exists points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

Theorem A.6 (Exercise 1.6.24 in [10]). *Every compact Lie group admits a bi-invariant metric, i.e. both left and right translations are isometries.*

Theorem A.7 (Theorem 4.31 in [4]). *Suppose N, N_1, N_2 are smooth manifolds, and $f_1 : N \longrightarrow N_1$ and $f_2 : N \longrightarrow N_2$ are surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $f : N_1 \longrightarrow N_2$ such that $f \circ f_1 = f_2$.*

Remark: Our use of Theorem A.7 to show $\frac{G \times M}{G} \cong M$ sets $N = G \times M$, $N_1 = \frac{G \times M}{G}$, $N_2 = M$, f_1 equal to the quotient map $G \times M \longrightarrow \frac{G \times M}{G}$, and f_2 equal to the action map of $G \curvearrowright M$.

Theorem A.8 (Theorem 2.18 in [4]). *Let G be a Lie group, let M be a homogeneous space (with respect to G), and let p be any point of M. The isotropy group G_p is a closed subgroup of G, and the map $F : G/G_p \longrightarrow M$ defined by $F(aG_p) = a \cdot p$ is an equivariant diffeomorphism.*

Definition A.9 (page 164 in [4]). *Suppose G is a Lie group and M and N are smooth manifolds endowed with smooth left or right G-actions. Let θ be the action of G on M and φ be the action of G on N. A map $F : M \longrightarrow N$ is equivariant with respect to the given G-actions if and only if the following diagram commutes for each $a \in G$:*

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ {\scriptstyle \theta_a}\downarrow & & \downarrow{\scriptstyle \varphi_a} \\ M & \xrightarrow[F]{} & N \end{array}$$

Theorem A.10 (page 460 in [9] and Exercise 5.9.20 in [10]). *Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion. Let $E, F \in TM$. Then $T_E F := \mathcal{H}_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F)$, and if N is a submanifold of M, then N is totally geodesic $\iff$ $T \equiv 0$.*

Theorem A.11 (Theorem 2.2.2 in [10]). *The assignment $X \mapsto \nabla X$ on (M, g_M) is uniquely defined by the following properties.*

1) $\nabla_{\alpha v+\beta w} Y = \alpha \nabla_v Y + \beta \nabla_w Y$ *and* $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$.

2) *For functions* $f : M \to \mathbb{R}$, $\nabla_X (fY) = (D_X f)Y + f \nabla_X Y$.

3) $\nabla_X Y - \nabla_Y X = [X, Y]$.

4) $D_Z g_M (X, Y) = g_M (\nabla_Z X, Y) + g_M (X, \nabla_Z Y)$.

Theorem A.12 (Exercise 7-22 in [4]). *Quaternionic multiplication is associative.*

Theorem A.13 (Exercise 7-22 in [4]). *Let $a_i, b_i, c_i, d_i \in \mathbb{R}$. Quaternionic multiplication is defined by*

$$\begin{aligned}(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) = \quad & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ & + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\ & + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j \\ & + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k.\end{aligned}$$

Theorem A.14 (Corollary 3.19 in [3]). *Let (M, g_{bi}) be a Riemannian manifold paired with a bi-invariant metric. Let $X, Y, Z, W \in TM$. Then*

$$R_{g_{\text{bi}}}(X, Y, Z, W) = \tfrac{1}{4} g_{\text{bi}}\Big([X, W], [Y, Z]\Big) - \tfrac{1}{4} g_{\text{bi}}\Big([X, Z], [Y, W]\Big).$$

Theorem A.15 (Lemma 2.2.4 in [10]). *Let M be a manifold and ∇ an affine connection on M. If X is a vector field on M and $c : I \longrightarrow M$ a smooth curve with $\dot{c}(0) = v \in T_pM$, then $\nabla_v X$ depends only on the values of X along c, i.e., if $X \circ c = Y \circ c$, then $\nabla_{\dot{c}} X = \nabla_{\dot{c}} Y$.*

Theorem A.16 (Exercise 2.5.12 in [10]). *Let (M, g_M) be a Riemannian manifold and $f : (M, g_M) \longrightarrow (M, g_M)$ be an isometry. Then $df(\nabla_X Y) = \nabla_{df(X)} df(Y)$.*

Definition A.17 (Example 1.1.3 in [10]). *The* **standard Riemannian metric on S^n** *is defined for all $p \in S^n$ by $g_{S^n}((p, v), (p, w)) = g_{\mathbb{R}^{n+1}}((p, v), (p, w)) \overset{A.18}{=} v \cdot w$.*

Definition A.18 (Example 1.1.1 in [10]). *The* **standard Riemannian metric on $\mathbb{R}^n$** *is defined for all $p \in \mathbb{R}^n$ by $g_{\mathbb{R}^n}((p, v), (p, w)) = v \cdot w$.*

Theorem A.19. *Let $a, b, c \in \mathbb{R}^3$.*

$$\begin{aligned} a \cdot (b \times c) &= -a \cdot (c \times b) \\ &= -b \cdot (a \times c) \\ &= -c \cdot (b \times a). \end{aligned}$$

Theorem A.20 (Proposition 3.1.1 in [10]). *The $(0, 4)$ Riemannian curvature tensor $R(X, Y, Z, W)$ is skew-symmetric in the first two and last two entries. That is,*

$$R(X, Y, Z, W) = -R(Y, X, Z, W).$$

$$R(X, Y, Z, W) = -R(X, Y, W, Z).$$

Theorem A.21 (page 84 in [10]). *Let (M, g_M) be a Riemannian manifold with constant sectional curvature k. Let $p \in M$ and $v_1, v_2, v_3, v_4 \in T_pM$. Then*

$$R(v_1, v_2, v_3, v_4) = kg_M(v_2, v_3)g_M(v_1, v_4) - kg_M(v_1, v_3)g_M(v_2, v_4).$$

Theorem A.22 (page 522 in [12]). *Suppose (M, g) is a Riemannian manifold with $\mathrm{curv}_g(M) \geq 0$. If $x, y \in TM$ such that $\mathrm{curv}_g(x, y) = 0$, then*

$$R(y, x)x = R(x, y)x = \vec{0}.$$

Theorem A.23 (Proposition 5.4.1 in [10]). *Let* (M, g_M) *be a Riemannian manifold and* $c(t) : [a,b] \to M$ *be a constant speed curve. Then* $c(t)$ *is length minimizing if and only if it is energy minimizing. Furthermore,* $L(c) = \sqrt{2(b-a)E(c)}$.

Theorem A.24 (Proposition 3.24 in [4]). *Let* $F : M \longrightarrow N$ *be a smooth map between smooth manifolds, and let* $\gamma : I \longrightarrow M$ *be a smooth curve. For any* $t_0 \in I$*, the velocity at* $t = t_0$ *of the composite curve* $F \circ \gamma : I \longrightarrow N$ *is given by*

$$(F \circ \gamma)'(t_0) = dF\Big(\gamma'(t_0)\Big).$$

Theorem A.25 (Theorem 81.5 in [7]). *Let* X *be path connected and locally path connected. Let* G *be a group of homeomorphisms of* X*. The quotient map* $F : X \longrightarrow X/G$ *is a covering map if and only if the action of* G *is properly discontinuous. In this case, the covering map is regular and* G *is its group of covering transformations.*

Theorem A.26 (Proposition 4.40 in [4]). *Suppose* M *is a connected smooth* n*-manifold, and* $F : E \longrightarrow M$ *is a topological covering map. Then* E *is a topological* n*-manifold and has a unique smooth structure such that* π *is a smooth covering map.*

Definition A.27 (page 12 in [10]). *Let* (M, g_M) *and* (N, g_N) *be Riemannian manifolds. A map* $F : M \longrightarrow N$ *is a* **Riemannian covering map** *if and only if*

1) F *is a smooth covering map*

2) F *is a local isometry.*

Theorem A.28 (Lemma 54.2 in [7]). *Let E, B be topological spaces and $F : E \longrightarrow B$ be a covering map. Let $F(e_0) = b_0$. Let the map $F : I \times I \longrightarrow B$ be continuous with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map $\widetilde{F} : I \times I \longrightarrow E$ such that $\widetilde{F}(0,0) = e_0$. If F is a path homotopy, then $\widetilde{F}$ is a path homotopy.*