UC San Diego
UC San Diego Previously Published Works

Title
Regression of ranked responses when raw responses are censored

Permalink
https://escholarship.org/uc/item/0zr772g2

Authors
Donohue, Michael C
Gamst, Anthony C
Rissman, Robert A
et al.

Publication Date
2016-02-24

Peer reviewed

Regression of ranked responses when raw responses are censored

Michael C. Donohue[1], Anthony C. Gamst[2,3], Robert A. Rissman[3], and Ian Abramson[4]

[1]Alzheimer's Therapeutic Research Institute, University of Southern California

[2]Division of Biostatistics & Bioinformatics, University of California, San Diego

[3]Department of Neurosciences, University of California, San Diego

[4]Department of Mathematics, University of California, San Diego

February 25, 2016

Abstract

We discuss semiparametric regression when only the ranks of responses are observed. The model is $Y_i = F(\mathbf{x}_i' \boldsymbol{\beta}_0 + \varepsilon_i)$, where Y_i is the unobserved response, F is a monotone increasing function, $\mathbf{x}_i$ is a known $p-$vector of covariates, $\boldsymbol{\beta}_0$ is an unknown p-vector of interest, and ε_i is an error term independent of $\mathbf{x}_i$. We observe $\{(\mathbf{x}_i, R_n(Y_i)) : i = 1, \dots, n\}$, where R_n is the ordinal rank function. We explore a novel estimator under Gaussian assumptions. We discuss the literature, apply the method to an Alzheimer's disease biomarker, conduct simulation studies, and prove consistency and asymptotic normality.

Keywords: rank-based regression; censored observations; semiparametric; robust; asymptotics.

1 Introduction

Rank-based statistics are often attractive for their robustness properties. Occasionally, due to some practical measurement difficulties, we have no choice but to resort to ranks. For example, we might wish to analyze webpage or team ranks without access to an underlying continuous response variable. Suppose data arises from a monotone transformation of a linear model. How much information is lost when we observe ranks in place of raw data and how well can we estimate the linear parameters of the regression? One might think that the rank transformation, as depicted in Figure 1, causes a catastrophic loss of information about the target parameter. It is clear that the scale of the linear parameter and any intercept term are irrecoverable, however we can estimate the "direction" of the parameter (i.e. up to a scalar). Such an estimate can be useful for inference regarding the relative importance of effects, predicting ranks, and semiparametric estimation of response surfaces with *parallel* linear linear level sets. Response surfaces with linear (possibly non-parallel)

level sets exhibit "*asynergy*;" and synergy can be assessed by examination of the residuals of asynergistic fits [Donohue et al., 2007].

Semiparametric rank-based estimators introduced by Han [1987] and Sherman [1993] actually yield $\sqrt{n}$-consistent and asymptotically normal estimates for the direction of the parameter competitive with ordinary least squares methods using the raw observations. These rank based estimators all involve maximizing some form of rank correlation, and can be computationally complex in higher dimensions. We will explore another asymptotically normal estimate which is admittedly less robust than these estimators, but which is achieved by a simpler ordinary least squares computation. We prove consistency and asymptotic normality under strong Gaussian assumptions. Simulation studies demonstrate the methods sensitivity to the Gaussian assumptions.

We also apply the method and competitors to a dataset with an Alzheimer's disease blood plasma assay that is prone to batch effects [Donohue et al., 2014]. Bioassay florescence intensities are typically parametrically calibrated, plate by plate, using standards of know concentration (e.g. Davidian and Haaland [1990]). We explore the use of the rank transformation, also applied plate by plate, as an alternative nonparametric standardization under the assumption that there is negligible biological variation across plates. We then apply the rank-based regression methods under discussion.

1.1 A novel gaussian quantile rank-based regression

Consider the restricted rank-based regression problem under assumptions

(A1) $\mathbf{x}_i \sim \mathcal{N}_p(0, \Sigma)$

(A2) $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$

(A3) $Y_i = F(\mathbf{x}_i' \boldsymbol{\beta} + \varepsilon)$, F is monotone increasing

(A4) We observe $\mathbf{z}_i = (\mathbf{x}_i, R_n(Y_i))$ for $i = 1, \ldots, n$

However, we can assume without loss of generality that F is the identity map since F has no affect on the observed ranks and it is a nuisance parameter. Furthermore let G denote the distribution of $\mathbf{x}_i$ and H denote the distribution for Y_i (both of which are Gaussian). Let H_n be the empirical distribution function based on a sample $Y_1, \ldots, Y_n$. We seek a competing estimator to Spearmax (8), still based only on the observation $\{(\mathbf{x}_i, R_n(Y_i)) : i = 1 \ldots n\}$, that takes advantage of two facts:

$$n^{-1} R_n(Y_i) = H_n(Y_i) \tag{1}$$

and

$$\Phi^{-1}(H(Y_i)) = c(\mathbf{x}_i' \boldsymbol{\beta}_0 + \varepsilon_i) \tag{2}$$
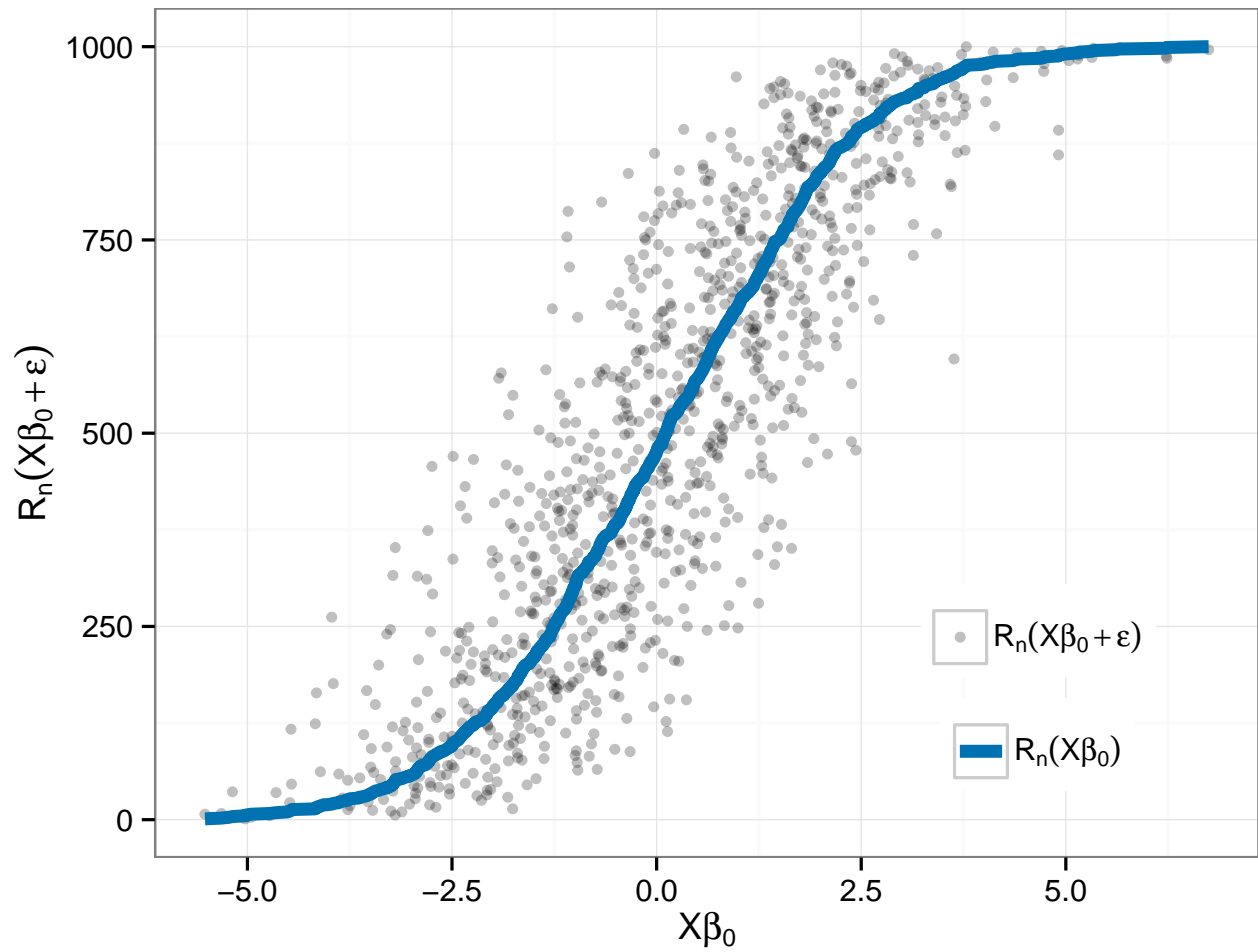
Figure 1: The rank transformation. Rank transformed responses (grey dots) retain information about the parameter of interest, $\boldsymbol{\beta}_0$.

for some constant, c, where Φ^{-1} is the standard normal quantile function (see Lemma 3.1 in Section 3). In light of these facts, we propose the Gaussian Quantile Regression:

$$\hat{\boldsymbol{\beta}}_n = \left(n^{-1}\sum_{i=1}^{n}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}n^{-1}\sum_{i=1}^{n}\mathbf{x}_i'\Phi^{-1}(H_n^*(Y_i)), \tag{3}$$

where $H_n^*(y) = (n+1)^{-1}R_n(y)$. In the appendix we show consistency of $\boldsymbol{\beta}_n$ and asymptotic normality of a version of $\hat{\boldsymbol{\beta}}_n$ where Φ^{-1} is replaced with a truncated quantile function, Φ_n^{-1}.

In the next section we will discuss related background literature. In Section 3 we discuss consistency and asymptotic normality. In Section 4 we describe simulations.

2 Background

2.1 R-Estimators

Note the distinction between our setting and that of classical "rank regression." *Rank estimators* (*R-estimators*) Jurečková and Sen [1996], are not useful in our setting since they require observations of the response. They utilize the ranks of the *residuals*, not the ranks of the responses. More specifically, *R*-estimators solve $\arg\min_{\boldsymbol{\beta}\in\mathcal{B}}\|\mathbf{L}_n(\boldsymbol{\beta})\|$, where

$$\mathbf{L}_n(\boldsymbol{\eta}) = \sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}}_n)a_n(R_n(Y_i - \mathbf{x}_i\boldsymbol{\eta})),$$

and a_n is an appropriate score function. Parzen et al. [1994] introduced a resampling method for inference regarding $\boldsymbol{\beta}_0$ using R-estimates.

2.2 Monotonic linear index models and "Spearmax"

Han [1987] introduced a "semiparametric monotonic linear index model" of the form

$$Y_i = D \circ F(\mathbf{x}_i'\boldsymbol{\beta}_0, \varepsilon_i) \tag{4}$$

where Y_i is an observed response, D is a monotone increasing function, F is strictly increasing in both arguments, $\mathbf{x}_i$ is a known p-vector, $\boldsymbol{\beta}_0$ is an unknown p-vector of interest, and ε_i is random error independent of $\mathbf{x}_i$. Han also introduced a maximum rank correlation estimator

$$\arg\max_{\boldsymbol{\beta}\in\mathcal{B}} \frac{1}{n(n-1)} \sum_{i\neq j} \{Y_i > Y_j\}\{\mathbf{x}_i'\boldsymbol{\beta} > \mathbf{x}_j'\boldsymbol{\beta}\} \tag{5}$$

where $\{\cdot\}$ represents the indicator function and $\mathcal{B}$ is an appropriate subset of $\mathbb{R}^p$, the unit ball say. The necessity of restricting to $\mathbb{R}^p$ becomes apparent in light of the fact that scalar multiples of a particular $\boldsymbol{\beta}$ yield the same rank correlation. The power of estimators of this type is that they exploit monotonicity without making assumptions about the particular form of D or F. Assumptions regarding the error distribution

are minimal as well. Sherman [1993] showed (5) is n-consistent and asymptotically normal. Cavanagh and Sherman [1998] proposed a class of consistent and asymptotically normal estimators of the form

$$\hat{\boldsymbol{\beta}}_n = \arg\max_{\boldsymbol{\beta}\in\mathcal{B}} \sum_{i=1}^{n} M(Y_i) R_n(\mathbf{x}_i'\boldsymbol{\beta}) \tag{6}$$

where R_n is the rank function

$$R_n(a_i) = \sum_{j=1}^{n} \{a_j \le a_i\}, \text{ for } (a_1, \ldots, a_n) \in \mathbb{R}^n \tag{7}$$

and M is either deterministic or R_n. When M is R_n, the quantity being maximized in (6) is a linear function of Spearman's rank correlation coefficient, and in this case we need only observe the ranks, not the raw responses. When $M = R_n$ we will refer to (6) as the *Spearmax* estimate of $\boldsymbol{\beta}_0$:

$$\hat{\boldsymbol{\beta}}_n = \arg\max_{\boldsymbol{\beta}\in\mathcal{B}} \sum_{i=1}^{n} R_n(Y_i) R_n(\mathbf{x}_i'\boldsymbol{\beta}) \tag{8}$$

The problem of estimating $\boldsymbol{\beta}_0$ becomes one of maximizing a step function over $\mathcal{B}$.

2.3 M-Estimators

The Spearmax estimator is related to the classical *M-estimator* Jurečková and Sen [1996] for the linear regression model:

$$\arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \sum_{i=1}^{n} \rho(Y_i - \mathbf{x}_i'\boldsymbol{\beta})$$

for an appropriate absolutely continuous and differentiable $\rho : \mathbb{R} \to \mathbb{R}$. The Spearmax estimate can be written as a related minimization

$$\arg\max_{\boldsymbol{\beta}\in\mathcal{B}} \sum_{i=1}^{n} R_n(Y_i) R_n(\mathbf{x}_i'\boldsymbol{\beta}) = \arg\min_{\boldsymbol{\beta}\in\mathcal{B}} \|\mathbf{R}_n(\mathbf{Y})\|^2 + \|\mathbf{R}_n(\mathbf{X}\boldsymbol{\beta})\|^2 - 2\sum_{i=1}^{n} R_n(Y_i) R_n(\mathbf{x}_i'\boldsymbol{\beta})$$

$$= \arg\min_{\boldsymbol{\beta}\in\mathcal{B}} \|\mathbf{R}_n(\mathbf{Y}) - \mathbf{R}_n(\mathbf{X}\boldsymbol{\beta})\|^2$$

$$= \arg\min_{\boldsymbol{\beta}\in\mathcal{B}} \sum_{i=1}^{n} (R_n(Y_i) - R_n(\mathbf{x}_i'\boldsymbol{\beta}))^2$$

However, the rank functions involved make Spearmax distinct from classical M-estimation.

2.4 Current Status Data

Another interesting and related problem is *rank-based regression for current status data* Aragón and Quiróz [1995], Abrevaya [1999] with survival analysis applications. The setting is also linear regression with the response Y_i not observable. We observe (X_i, C_i, Δ_i), where Δ_i denotes the indicator on $\{Y_i \le C_i\}$. Aragón and Quiróz [1995] proposed and showed consistency of

$$\arg\max_{\boldsymbol{\beta}\in\mathcal{B}} \sum_{i=1}^{n} \Delta_i R_n(C_i - \mathbf{x}_i'\boldsymbol{\beta}) \tag{9}$$

and

$$\arg\max_{\boldsymbol{\beta}\in\mathcal{B}} \sum_{i=1}^{n} R_n(C_i - \mathbf{x}_i'\boldsymbol{\beta})\hat{F}_{\boldsymbol{\beta}}(C_i - \mathbf{x}_i'\boldsymbol{\beta})$$

where $\hat{F}_{\boldsymbol{\beta}}$ is a uniform strongly consistent estimate of the distribution of $(Y_i - \mathbf{x}_i'\boldsymbol{\beta})$. Abrevaya [1999] proved asymptotic normality of (9).

2.5 Concomitants

Statistics related to our Gaussian Quantile Regression (3) appear in the concomitant literature. David and Nagaraja [1998] provide a detailed review. We will summarize these results and provide a comparison to our proposed estimator.

Yang [1981a] discussed related statistics of a more general form. Let $(X_i, Y_i), i = 1, \ldots, n$ be a random sample from a bivariate distribution with cumulative distribution function $F(x, y)$. Let $Y_{i:n}$ denote the ith order statistic and $X_{[i:n]}$ denote the the so-called ith concomitant, that is the X variable associated with $Y_{i:n}$. The term *induced order statistics* has also been used in place of concomitant by Bhattacharya [1974]. Under mild conditions, Yang proved asymptotic normality of statistics of the form

$$S_n = n^{-1} \sum_{i=1}^{n} J(i/(n+1)) K(X_{[i:n]}, Y_{i:n}), \tag{10}$$

where J is a smooth bounded function (possibly depending on n) and K is some real valued function on $\mathbb{R}^2$. We can rewrite our Gaussian Quantile Regression (3) as

$$\hat{\boldsymbol{\beta}}_n = \left(n^{-1}\sum_{i=1}^{n}\mathbf{x}_i'\mathbf{x}_i\right)^{-1} n^{-1}\sum_{i=1}^{n}\mathbf{x}_{[i:n]}'\Phi^{-1}(i/(n+1)),$$

where we generalize the definition of ith concomitant here to be the $\mathbf{x}$ vector associated with $Y_{i:n}$. In the notation of Yang's statistics we have $J = \Phi^{-1}$ and $K(\mathbf{x}_{[i:n]}, Y_{i:n}) = \mathbf{x}_{[i:n]}$. Yang considers estimators of $\mathbf{E}(Y|X = x)$, $\mathbf{P}(Y \in A|X = x)$ and $\mathrm{var}(Y|X = x)$ which include observations of raw Y values to which we are not privy. Yang also only discusses the case $p = 1$. The method of proof used by Yang is based on earlier methods of Stigler [1969] and Hájek [1968], namely Hájek's projection lemma (see also Hettmansperger [1984] page 50). The idea is to show S_n (10) and its projection

$$\hat{S}_n = \sum_{i=1}^{n} \mathbf{E}(S_n|\mathbf{x}_i, Y_i) - (n-1)\mathbf{E}(S_n)$$

are asymptotically equivalent in terms of mean square. The asymptotic normality of the projection follows from the central limit theorem since the summands are i.i.d. We will deploy a similar strategy of approximating the dependent sum by an independent sum and invoking the central limit theorem. However, we will see that our approximation is of the form:

$$n^{-1}\sum_{i=1}^{n}\mathbf{x}_i\Phi^{-1}(H_n^*(Y_i)) \approx n^{-1}\sum_{i=1}^{n}\mathbf{x}_i\Phi^{-1}(H_n(Y_i)).$$

The result is also i.i.d. summands, but in our regression setting, this approximation allows the target regression parameter to emerge via the observation (2).

Yang [1981b] also proved asymptotic normality of

$$\sum_{i=1}^{n} J(t_{ni})X_{[i:n]}, \max_{1 \le i \le n} |t_{ni} - i/n| \to 0 \tag{11}$$

for some deterministic double indexed sequence $\{t_{ni}\}$. Statistics of this form have applications to tests for normality and independence. It is interesting to note that in an application of the rank-based regression, a statistic such as (11) would be useful since it offers a test for the Gaussian assumption without raw response observations.

2.6 L-Estimators and α-Regression Quantiles

Related to our proposed rank-based regression and the concomitant statistics such as (11) are the so called L-estimators (linear combinations of functions of order statistics) Jurečková and Sen [1996]. In the $p = 1$ location setting, these are statistics of the form

$$L_n = n^{-1}\sum_{i=1}^{n} X_{[n:i]}J_n(i/(n+1)).$$

L-estimators were first adapted to the linear regression problem with general p in 1973 by Bickel [1973]. Koenker and Bassett [1978] introduced α-regression quantiles $\hat{\boldsymbol{\beta}}(\alpha)$, $0 < \alpha < 1$ as the solution to

$$\hat{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta} \in \mathcal{B}}{\arg\min} \sum_{i=1}^{n} \rho_{\alpha}(Y_i - \mathbf{x}_i\boldsymbol{\beta}),$$

$$\rho_{\alpha}(x) = |x|((1 - \alpha)\{x < 0\} + \alpha\{x > 0\}),$$

where $\{A\}$ is the indicator on A. Koenker and Basset also posed the problem as a linear program and derived the asymptotic distribution.

3 Consistency and asymptotical normality

3.1 Consistency

In this section we will demonstrate $\Phi^{-1}(H(Y_i)) = c(\mathbf{x}_i'\boldsymbol{\beta}_0 + \varepsilon_i)$ and the consistency of $\hat{\boldsymbol{\beta}}_n$ (3). Table 1 collects the notation and assumptions. Again, we assume without loss of generality that D is the identity map.

The first lemma is an impetus for the estimator since the target parameter appears from a composition of a quantile function and a distribution function. The proof of the lemma relies on the assumption that all the random variables are Gaussian or a linear combination of Gaussian random variables.

Lemma 3.1.

$$\Phi^{-1}(H(Y_i)) = \sigma^*(\mathbf{x}_i'\boldsymbol{\beta}_0 + \varepsilon_i)$$

where $\sigma^* = (\sigma_{\boldsymbol{\beta}_0}^2 + \sigma^2)^{-1/2}$, $\sigma_{\boldsymbol{\beta}_0}^2 = \boldsymbol{\beta}_0'\Sigma\boldsymbol{\beta}_0$.

7

Proof.

$$H(Y_i) = \mathbf{P}[Y_j \leq Y_i] \qquad\qquad i \neq j$$

$$= \mathbf{P}\left[\frac{\mathbf{x}_j\boldsymbol{\beta}_0 + \varepsilon_j}{\sqrt{\sigma^2_{\boldsymbol{\beta}_0} + \sigma^2}} \leq \frac{\mathbf{x}_i\boldsymbol{\beta}_0 + \varepsilon_i}{\sqrt{\sigma^2_{\boldsymbol{\beta}_0} + \sigma^2}}\right]$$

$$= \Phi\left(\frac{\mathbf{x}_i\boldsymbol{\beta}_0 + \varepsilon_i}{\sqrt{\sigma^2_{\boldsymbol{\beta}_0} + \sigma^2}}\right).$$

Now observe

$$\Phi^{-1}(H(Y_i)) = \frac{\mathbf{x}_i\boldsymbol{\beta}_0 + \varepsilon_i}{\sqrt{\sigma^2_{\boldsymbol{\beta}_0} + \sigma^2}}.$$

$\square$

The following series of lemmas will be used to show L_1 convergence of the summands, ξ_{nij}, and the fact that $\mathrm{cov}(\boldsymbol{\xi}_{n1j}, \boldsymbol{\xi}_{n2j}) \to 0$ (proofs are in the appendix). These results and Chebychev's Inequality will be enough to yield consistency.

Lemma 3.2. $\xi_{nij} \overset{a.s.}{\to} \xi_{ij}$ *as* $n \to \infty$

(A1) $\mathbf{x}_i \sim \mathcal{N}_p(0, \Sigma)$, Σ is nonsingular

(A2) $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$

(A3) $Y_i = F(\mathbf{x}_i'\boldsymbol{\beta} + \varepsilon)$, F is monotone increasing

(A4) We observe $\mathbf{z}_i = (\mathbf{x}_i, R_n(Y_i))$ for $i = 1, \ldots, n$

$$\mathbf{x}_i \sim G \text{ for some distribution } G$$
$$Y_i \sim H \text{ for some distribution } H$$
$$\sigma_{\boldsymbol{\beta}_0}^2 = \boldsymbol{\beta}_0' \Sigma \boldsymbol{\beta}_0$$
$$\sigma^* = (\sigma_{\boldsymbol{\beta}_0}^2 + \sigma^2)^{-1/2}$$
$$R_n(y) = \sum_{j=1}^n \{Y_j \leq y\}$$
$$H_n(y) = n^{-1} R_n(y)$$
$$H_n^*(y) = (n+1)^{-1} R_n(y)$$
$$\boldsymbol{\xi}_i = \mathbf{x}_i \Phi^{-1}(H(Y_i))$$
$$\xi_{ij} = X_{ij} \Phi^{-1}(H(Y_i))$$
$$\boldsymbol{\xi}_{ni} = \mathbf{x}_i \Phi^{-1}(H_n^*(Y_i))$$
$$\xi_{nij} = X_{ij} \Phi^{-1}(H_n^*(Y_i))$$
$$\hat{\boldsymbol{\beta}}_n = (n^{-1} \sum_{i=1}^n \mathbf{x}_i' \mathbf{x}_i)^{-1} n^{-1} \sum_{i=1}^n \mathbf{x}_i' \Phi^{-1}(H_n^*(Y_i))$$

Table 1: Gaussian Quantile Regression assumptions and notation.

Lemma 3.3. $H_n^*(Y_i) \sim U_n$ where $U_n \sim \text{Unif}\{(n+1)^{-1}, \ldots, n(n+1)^{-1}\}$.

Lemma 3.4. $\mathbf{E}\Phi^{-1}(U_n)^4 \leq 6$.

Lemma 3.5. $\mathbf{E}\xi_{nij}^2 \leq 3\sqrt{2}\sigma_j^2$, where $X_{ij} \sim \mathcal{N}(0, \sigma_j^2)$.

Lemma 3.6. $\mathbf{E}\xi_{nij} \to \mathbf{E}\xi_{ij}$ as $n \to \infty$

Lemma 3.7. $\text{cov}(\xi_{nsj}, \xi_{ntj}) \to 0$ as $n \to \infty$.

Theorem 3.8. *The estimate $\hat{\boldsymbol{\beta}}_n$ is consistent for a scalar factor of $\boldsymbol{\beta}_0$.*

Proof. Let $\bar{\xi}_{nj} = n^{-1} \sum_{i=1}^{n} \xi_{nij}$. By Chebychev's Inequality:

$$\mathbf{P}[|\bar{\xi}_{nj} - \mathbf{E}\xi_{nij}| \geq \delta] \leq \mathrm{var}(\bar{\xi}_{nj})/\delta^2$$

$$= (n\delta)^{-2}(n\,\mathrm{var}(\xi_{nij}) + n(n-1)\,\mathrm{cov}(\xi_{nsj}, \xi_{ntj}))$$

$$\leq \delta^{-2}(n^{-1}\mathbf{E}\xi_{nij}^2 + n^{-1}(n-1)\,\mathrm{cov}(\xi_{nsj}, \xi_{ntj}))$$

$$\leq \delta^{-2}(n^{-1}3\sqrt{2}\sigma_j^2 + n^{-1}(n-1)\,\mathrm{cov}(\xi_{nsj}, \xi_{ntj}))$$

$$\to 0$$

So for each $j \in \{1, \ldots, p\}$ by the above and Lemma 3.6 we have,

$$|\bar{\xi}_{nj} - \mathbf{E}\xi_{ij}| \leq |\bar{\xi}_{nj} - \mathbf{E}\xi_{nij}| + |\mathbf{E}\xi_{ij} - \mathbf{E}\xi_{nij}| \xrightarrow{P} 0,$$

and therefore

$$\bar{\boldsymbol{\xi}}_n \xrightarrow{P} \mathbf{E}\boldsymbol{\xi}_i = \mathbf{E}\left[\frac{\mathbf{x}_i(\mathbf{x}_i'\boldsymbol{\beta}_0 + \varepsilon_i)}{\sqrt{\sigma_{\boldsymbol{\beta}_0}^2 + \sigma^2}}\right] = \sigma_* \Sigma \boldsymbol{\beta}_0$$

We also have, as with the classical ordinary least squares setup,

$$\left(n^{-1} \sum_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \xrightarrow{P} \Sigma^{-1}.$$

Finally we have:

$$\hat{\boldsymbol{\beta}}_n = \left(n^{-1} \sum_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \bar{\boldsymbol{\xi}}_n \xrightarrow{P} \sigma_* \boldsymbol{\beta}_0.$$

$\square$

3.2 Asymptotic normality

In order to prove asymptotic normality of our estimator we are forced to truncate the Gaussian quantile function. For the extreme ranked data $(R_n(Y_i) \notin (1 - \alpha_n, \alpha_n))$, this results in $\mathbf{x}_i\Phi^{-1}(H_n(Y_i))$ being replaced with $\mathbf{x}_i\Phi^{-1}(1 - \alpha_n)$ or $\mathbf{x}_i\Phi^{-1}(\alpha_n)$ where

$$\alpha_n = \Phi\left(\sqrt{\frac{1}{2}\log n}\right) \to 1$$

More precisely, we define the truncated version of the Gaussian quantile function as

$$\Phi_n^{-1}(x) = \begin{cases} \Phi^{-1}(1 - \alpha_n), & \text{if } x \in (0, 1 - \alpha_n] \\ \Phi^{-1}(x), & \text{if } x \in (1 - \alpha_n, \alpha_n] \\ \Phi^{-1}(\alpha_n), & \text{if } x \in [\alpha_n, 1) \end{cases} \tag{12}$$

We will demonstrate that truncating at this particular α_n allows us to put a $cn^{1/4}$ bound on the first derivative of the quantile function. We prove asymptotic normality of the truncated version of the estimator

(3)

$$\tilde{\boldsymbol{\beta}}_n = \left(n^{-1}\sum_{i=1}^{n}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}n^{-1}\sum_{i=1}^{n}\mathbf{x}_i'\Phi_n^{-1}(H_n^*(Y_i))\tag{13}$$

by applying the Central Limit Theorem to an i.i.d. version of our estimator, then applying Slutsky's Theorem. The three series in discussion are

$$S_n^{(2)} = \sqrt{n}\left(n^{-1}\sum_{i=1}^{n}\mathbf{x}_i'\Phi^{-1}(H(Y_i)) - \sigma_*\Sigma\boldsymbol{\beta}_0\right) \Rightarrow \mathcal{N}_p(0, A),\tag{CLT}$$

$$S_n^{(1)} = \sqrt{n}\left(n^{-1}\sum_{i=1}^{n}\mathbf{x}_i'\Phi_n^{-1}(H(Y_i)) - \sigma_*\Sigma\boldsymbol{\beta}_0\right),\text{ and}$$

$$S_n = \sqrt{n}\left(n^{-1}\sum_{i=1}^{n}\mathbf{x}_i'\Phi_n^{-1}(H_n^*(Y_i)) - \sigma_*\Sigma\boldsymbol{\beta}_0\right).$$

In the first line above (CLT), A is the $p \times p$ dispersion matrix associated with

$$x_1'\Phi^{-1}(H(Y_1)) = \sigma_* x_1'(x_1'\boldsymbol{\beta}_0 + \varepsilon_1)$$

with entries

$$\begin{aligned}
a_{ij} &= \text{cov}(\sigma_* X_{1i}(x_1'\boldsymbol{\beta}_0 + \varepsilon_1), \sigma_* X_{1j}(\mathbf{x}_1'\boldsymbol{\beta}_0 + \varepsilon_1))\\
&= \sigma_*^2(\mathbf{E}(X_{1i}(\mathbf{x}_1\boldsymbol{\beta}_0 + \varepsilon_1)X_{1j}(\mathbf{x}_1'\boldsymbol{\beta}_0 + \varepsilon_1))\\
&\quad - \mathbf{E}(X_{1i}(\mathbf{x}_1'\boldsymbol{\beta}_0 + \varepsilon_1))\mathbf{E}(X_{1j}(\mathbf{x}_1'\boldsymbol{\beta}_0 + \varepsilon_1)))\\
&= \sigma_*^2(\mathbf{E}X_{1i}X_{1j}((\mathbf{x}_1'\boldsymbol{\beta}_0)^2 + \sigma^2) - (\mathbf{e}_i'\Sigma\boldsymbol{\beta}_0)(\mathbf{e}_j'\Sigma\boldsymbol{\beta}_0))\\
&= \frac{1}{\sigma_{\boldsymbol{\beta}_0}^2 + \sigma^2}(\mathbf{E}X_{1i}X_{1j}(\mathbf{x}_1'\boldsymbol{\beta}_0)^2 + \sigma^2\mathbf{e}_i'\Sigma\mathbf{e}_j - (\mathbf{e}_i'\Sigma\boldsymbol{\beta}_0)(\mathbf{e}_j'\Sigma\boldsymbol{\beta}_0))
\end{aligned}\tag{14}$$

Here $\mathbf{e}_i$ denotes the p-vector with 1 as its ith component and 0's elsewhere.

We show $\|S_n^{(2)} - S_n^{(1)}\| \xrightarrow{P} 0$, $\|S_n^{(1)} - S_n\| \xrightarrow{P} 0$, and apply Slutsky's Theorem to conclude S_n is asymptotically normal (proofs of lemmas are in the appendix).

Lemma 3.9. $\|S_n^{(2)} - S_n^{(1)}\| \xrightarrow{P} 0$

We will need two lemmas to prove $\|S_n^{(1)} - S_n\| \xrightarrow{P} 0$.

Lemma 3.10.

$$(\Phi_n^{-1}(H(Y_1)) - \Phi_n^{-1}(H_n(Y_1)))^2 \leq \sigma_* 2\pi\sqrt{n}(H(Y_1) - H_n^*(Y_1))^2$$

Lemma 3.11.

$$\mathbf{E}_1(H(Y_1) - H_n^*(Y_1))^2 \leq \frac{1}{n+1}$$

almost surely where $\mathbf{E}_1$ *is the conditional expectation* $\mathbf{E}_1(\cdot) = \mathbf{E}(\cdot|Y_1, \mathbf{x}_1)$.

Lemma 3.12.

$$\|S_n^{(1)} - S_n\| \xrightarrow{P} 0$$

Theorem 3.13.

$$S_n \Rightarrow \mathcal{N}_p(0, A)$$

Proof. From the preceding lemmas we have

$$\|S_n - S_n^{(2)}\| \leq \|S_n - S_n^{(1)}\| + \|S_n^{(1)} - S_n^{(2)}\| \xrightarrow{P} 0$$

Since $S_n^{(2)} \Rightarrow \mathcal{N}_p(0, A)$ by the Central Limit Theorem, Slutsky's Theorem gives us $S_n \Rightarrow \mathcal{N}_p(0, A)$. $\qquad\square$

Finally we have asymptotic normality of our estimate in the following sense (proofs are in the appendix).

Corollary.

$$\sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \sigma_* \Sigma_n^{-1} \Sigma \boldsymbol{\beta}_0) \Rightarrow \mathcal{N}_p(0, \Sigma^{-1} A \Sigma^{-1}),$$

where $\Sigma_n^{-1} = (n^{-1} \sum_{i=1}^n \mathbf{x}_i' \mathbf{x}_i)^{-1}$.

Corollary. *If the covariance matrix, Σ, is known then*

$$\sqrt{n}(\dot{\boldsymbol{\beta}}_n - \sigma_* \boldsymbol{\beta}_0) \Rightarrow \mathcal{N}_p(0, \Sigma^{-1} A \Sigma^{-1}),$$

where

$$\dot{\boldsymbol{\beta}}_n = n^{-1} \Sigma^{-1} \sum_{i=1}^n \mathbf{x}_i' \Phi_n^{-1}(H_n^*(Y_i))$$

4   Simulations

Our Truncated Gaussian Quantile Regression estimate (13) was pitted against the Spearmax estimate (8) and the usual ordinary least squares (OLS) estimate gotten from the full data with raw response. The Spearmax estimate was achieved by parameterizing $\boldsymbol{\beta} \in \mathcal{B}$, where $\mathcal{B}$ is the unit circle, by $\theta = \arctan(\beta_2/\beta_1)$. The solution to

$$\underset{\theta \in (0, 2\pi)}{\arg\max} \sum_{i=1}^n R_n(Y_i) R_n(\mathbf{x}_i(\cos\theta, \sin\theta)') \tag{15}$$

was achieved by bound constrained numerical optimization [Byrd et al., 1995]. The constrained region was centered on the true $\theta_0 = \arctan(2/1) = 26.6°$. Figure 2 demonstrates the function to be maximized for a particular sample of size twenty under the Gaussian conditions of the first simulation scenario (17).

We also consider an Empirical Quantile Regression estimate

$$\tilde{\boldsymbol{\beta}}_n = \left(n^{-1} \sum_{i=1}^n \mathbf{x}_i' \mathbf{x}_i \right)^{-1} n^{-1} \sum_{i=1}^n \mathbf{x}_i' \hat{\Phi}_{\mathbf{x}_i' \boldsymbol{\beta}_n}^{-1}(H_n^*(Y_i)) \tag{16}$$

where $\hat{\Phi}_{\mathbf{x}_i' \boldsymbol{\beta}_n}^{-1}$ is the inverse of the empirical distribution of $\mathbf{x}_i \tilde{\boldsymbol{\beta}}_n$.

Covariates and errors were simulated under the Gaussian and stable distributions as described below. From each of the 10,000 simulated trials, the estimated angle of $\hat{\boldsymbol{\beta}}$, $\hat{\theta}$, was recorded using each of the methods. The sample standard deviation and bias from the 10,000 trials is graphically summarized for each scenario.
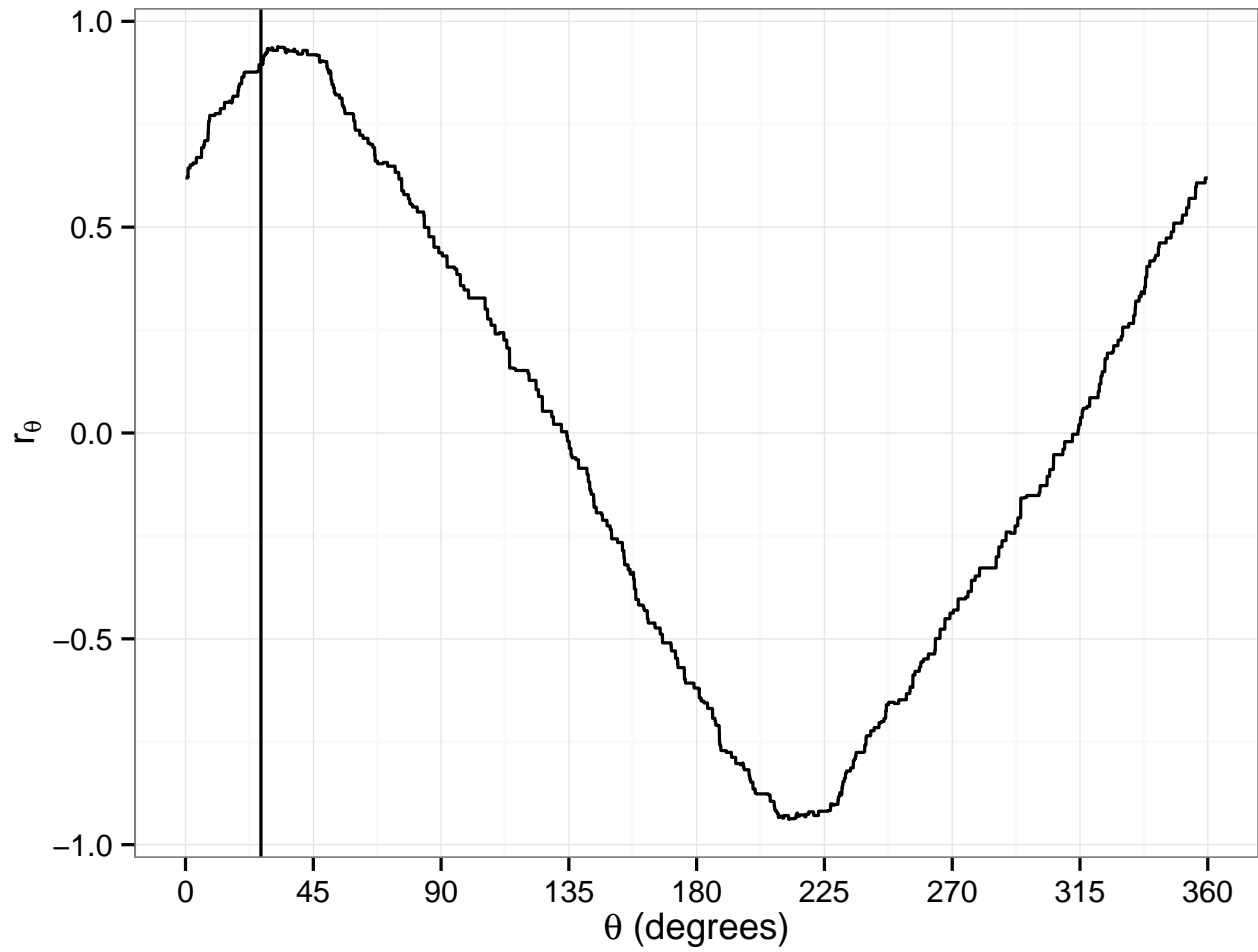
Figure 2: Spearman's correlation between Y and $\mathbf{x}'\boldsymbol{\beta}_\theta$, where $\boldsymbol{\beta}_\theta = (\cos(\theta), \sin(\theta))$ for a random sample of $n = 20$. The vertical line marks true $\theta_0 = \arctan(1/2) = 26.6°$.
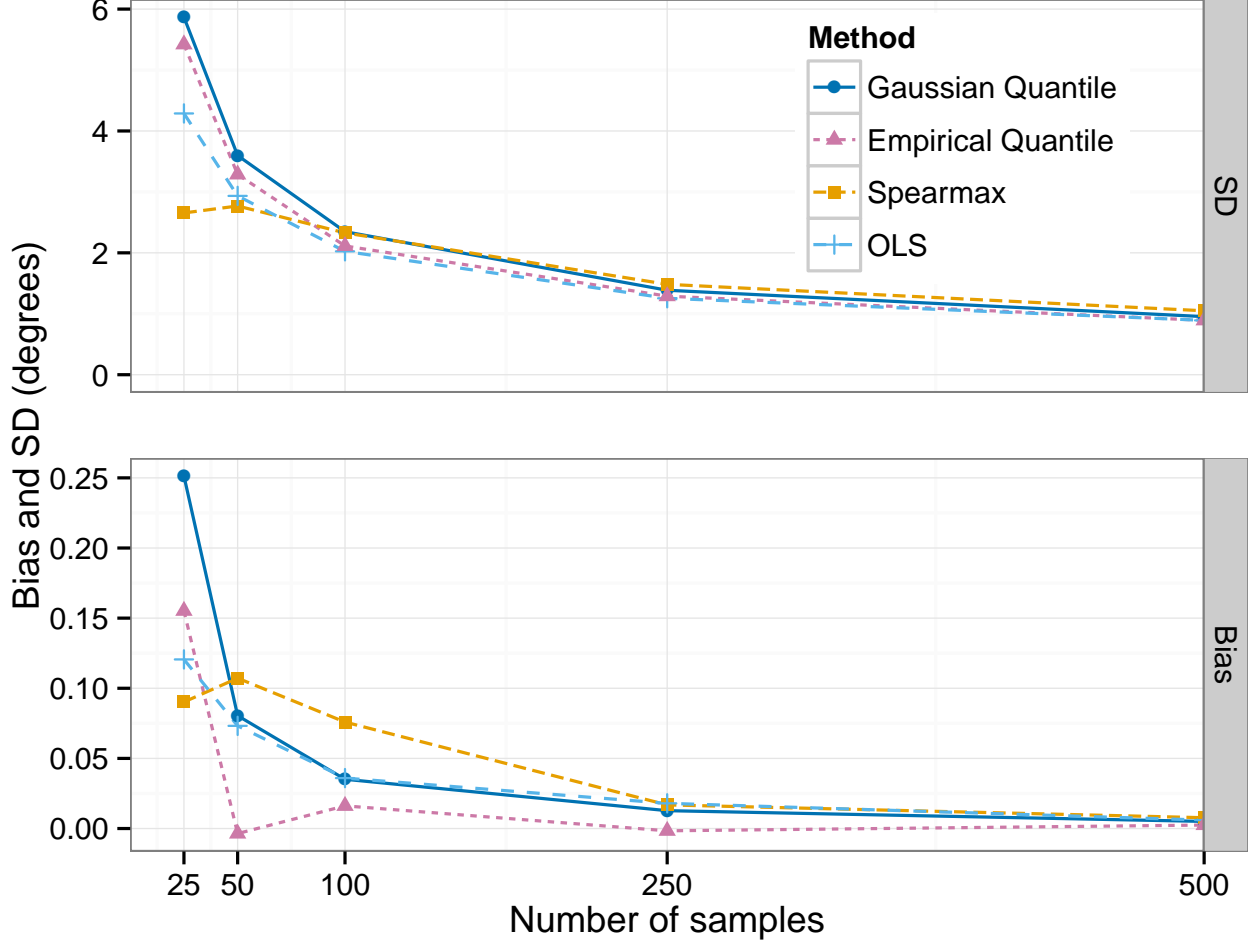
Figure 3: Gaussian simulations. Simulated standard deviation and bias of the point estimates from 10,000 simulations of varying sample size. Covariates and error are simulated under Gaussian distributions.

4.1 Gaussian simulations

The first run of simulations follow

$$\mathbf{x}_i \sim N \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right), \quad \boldsymbol{\beta}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \varepsilon_i \sim \mathcal{N}(0,1), i = 1, \ldots, n. \tag{17}$$

Simulated sample sizes ranged between $n = 25$ to $n = 3,000$. Results are summarized in Figure 3. Gaussian Quantile regression performs as well as OLS on the full data with moderate sample sizes. Spearmax performs best on small samples and the Empirical Quantile Regression performs best on moderate sample sizes.

4.2 Impact of skewness

The second scenario simulated the two covariates and errors, independently from stable distributions with fixed stability parameter $\alpha = 1$, sample size $n = 500$ and the skewness parameter β ranging in -1 to 1. Results of this scenario are summarized in Figure 4. Bias and SD of the Truncated Gaussian Quantile
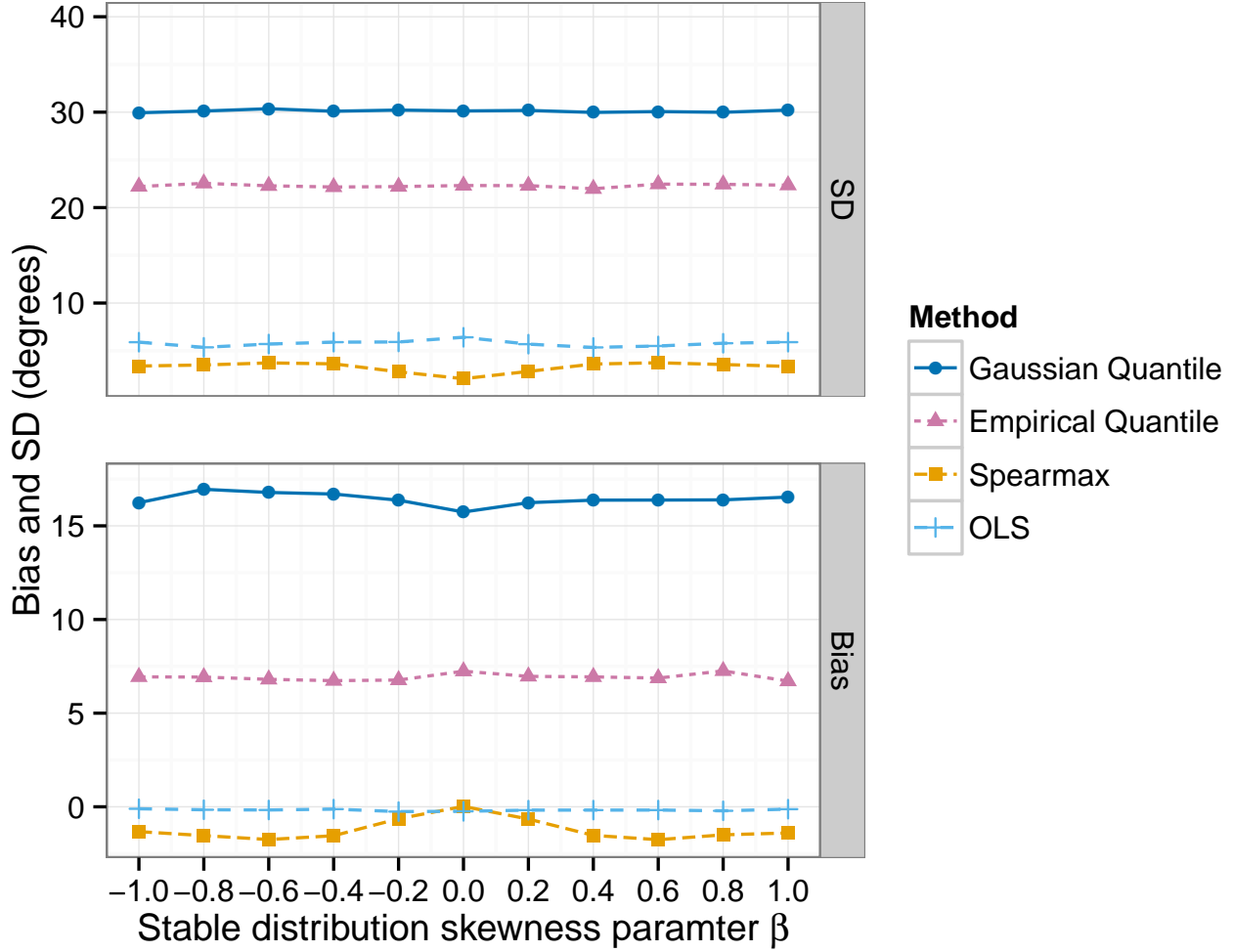
14

Figure 4: Impact of skewness. Simulated standard deviation and bias of the three point estimates from 10,000 simulations of size $n = 500$ drawing covariates and errors from the stable distribution with stability parameter $\alpha = 1$ and the given skewness parameter β.

estimator is greater with this heavy tailed distribution ($\alpha = 1$), but it is unaffected by skewness parameter. The Empirical Quantile distribution appears to mitigate some of the bias and SD induced by the skewness of the covariate distribution.

4.3 Impact of stability

The third scenario simulated the two covariates and errors, independently from stable distributions with fixed skewness parameter $\beta = 0$, sample size $n = 500$, and stability parameter α ranging in 0.2 to 2.0 (Gaussian). Results of this scenario are summarized in Figure 5. Here Spearmax shows an advantage over OLS, and again, the Empirical Quantile Regression seems to mitigate some of the bias and SD induced by the heavy tails of the distribution of $\mathbf{x}'\boldsymbol{\beta}_0$.
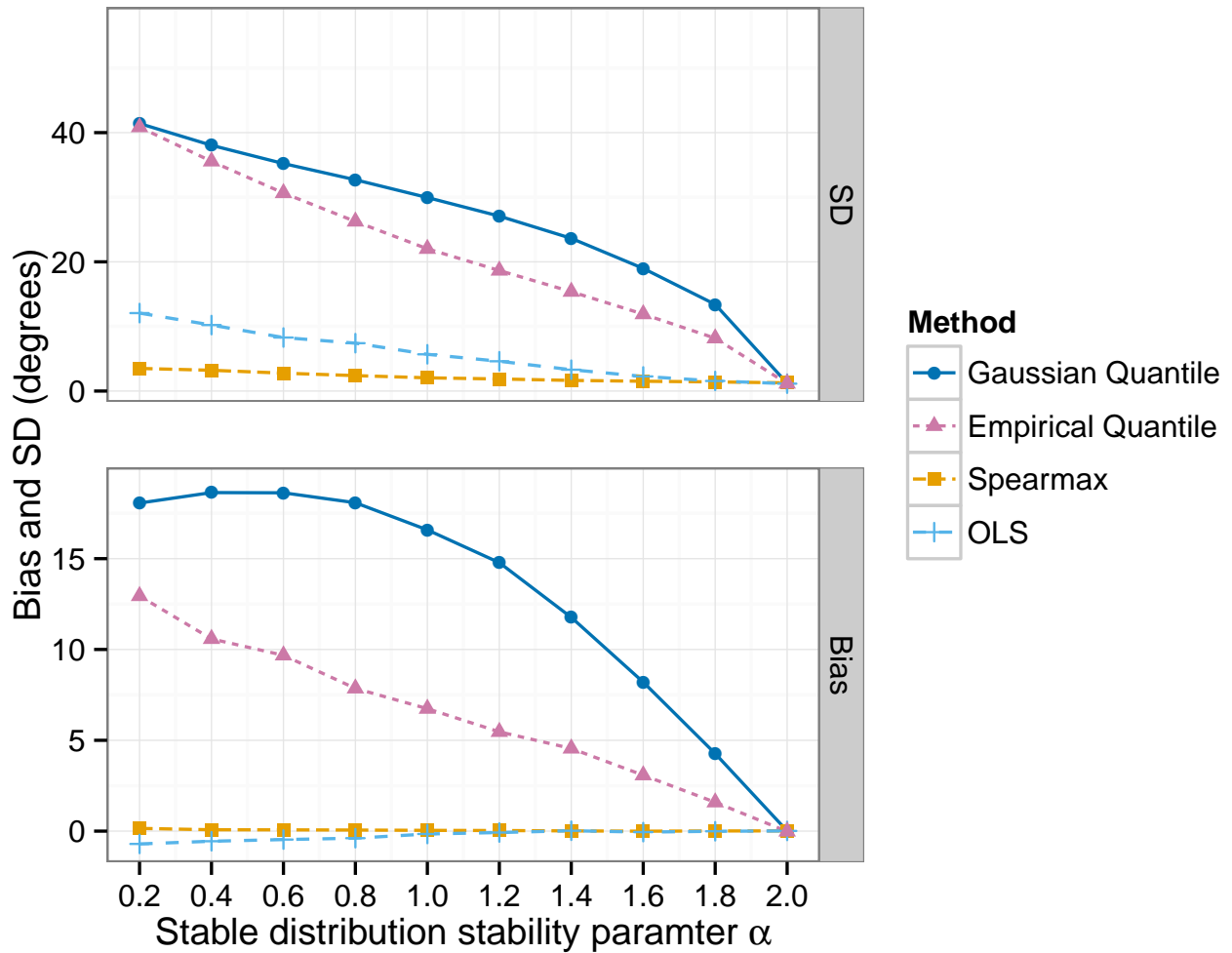
Figure 5: Impact of stability. Simulated standard deviation and bias of the three point estimates from 10,000 simulations of size $n = 500$ drawing covariates and errors from the stable distribution with skewness parameter $\beta = 0$ and the given stability parameter α.

coefficient	estimate	lower	upper	
Truncated Gaussian Quantile				
Age	0.1631	−0.114	0.4852	
ADAS	−0.9866	−1.964	−0.2361	*
Empirical Quantile				
Age	0.9969	0.981	1.0233	*
ADAS	0.0785	−0.121	0.2687	
Spearmax				
Age	0.7761	0.499	1.0278	*
ADAS	−0.6306	−1.499	−0.0702	*

Table 2: Estimates of the effect of age, Alzheimer's Disease Assessment Scale (ADAS), and their interaction on a blood plasma assay of amyloid-β_{1-42} using three rank-based regression methods. Bootstrap 95% confidence limits are estimated by 1,000 bootstrap resamples. Confidence intervals that exclude 0 are marked by "*".

5   Application to Alzheimer's blood plasma assay

We apply the three rank-based regression models to baseline blood plasma assays of amyloid-β_{1-42}. This data has been previously described and analyzed [Donohue et al., 2014]. For the present analysis, we focus on the association between response variable amyloid-β_{1-42} and predictors age at baseline and Alzheimer's Disease Assessment Scale (ADAS) in the cohort with Mild Cognitive Impairment (MCI). Assays were performed in duplicate for each participant. The means of duplicate florescence intensities were rank transformed plate by plate, under the assumption that the distribution of the assay on each plate should be similar. The original investigation, using random effects to model duplicate observations and plate effects, found that amyloid-β_{1-42} increased 0.16 pg/ml/year of age (SE=0.08, p=0.047), and in a separate model, found no significant association between ADAS and amyloid-β_{1-42}.

Table 2 shows the results of the alternative rank-based analyses. The Truncated Gaussian Quantile regression method found that increased amyloid-β_{1-42} was associated with better ADAS scores (ADAS increases with worsening); while the Empirical Quantile method found that increased amyloid-β_{1-42} was associated increased age. The age association is consistent with the original investigation. The Spearmax method found both coefficients to be significant at the 0.05 level. Notably, there was strong evidence that the distribution of $\mathbf{x}'\hat{\boldsymbol{\beta}}$ from the Truncated Gaussian Quantile regression was not Gaussian (Anderson-Darling $p < 0.001$). The R functions used for this analysis are provided in the Appendix.

6 Discussion

We present a novel regression estimator applicable to Han [1987] monotonic index model under Gaussian assumptions, when only the ranks of the responses and covariates are observed. The proposed estimator is shown in simulations to be competitive with the known alternative, Spearmax, when the covariates and errors are Gaussian. We prove the estimator is consistent and asymptotically normal. The appendix also demonstrate how to attain consistent bootstrap confidence regions in the bivariate case. An obvious drawback of our proposed rank-based estimate, compared to the Sherman [1993] class of estimates, is the strong Gaussian assumptions. The assumption, however, yields the key equality (2), which allows $\boldsymbol{\beta}_0$ to emerge asymptotically from the composition, $\Phi^{-1}(H_n^*(Y_i))$. This in turn allows for an estimate with computational advantages over Spearmax.

The rank-based regression estimator demonstrated advantages over the Spearmax estimate in terms of simulated standard error and bias in moderately sized ($n \geq 50$) fully Gaussian setting. Although Spearmax is more robust, when the dimension of the target parameter is large the computational benefits of the rank-based regression may make it an attractive alternative. Where the Spearmax estimator requires maximizing a step function over $p-$dimensions, the Gaussian quantile estimate requires only the ubiquitous numerical approximation to the Gaussian quantile function and the ordinary least squares machinery. In the Alzheimer's example, the Truncated Gaussian approach required about 50% of the system time required by Spearmax.

Simulations demonstrate that Gaussian Quantile Regression is sensitive to heavy tailed covariates and errors, but fairly robust to skewness. Empirical Quantile Regression offers some mitigation of effects of heavy tails, but Spearmax is recommended if computationally feasible and there is reason to believe the errors might be heavy tailed.

7 Appendix

7.1 Proof of Lemma 3.2

Lemma 3.2. $\xi_{nij} \overset{a.s.}{\to} \xi_{ij}$ as $n \to \infty$

Proof.

$$
\begin{aligned}
\sup_{Y_i} |H_n^*(Y_i) - H(Y_i)| &\leq \sup_y \left| \frac{n}{n+1} H_n(y) - H(y) \right| \\
&\leq \sup_y \left| \frac{n}{n+1} H_n(y) - H(y) \right| + \sup_y |H_n(y) - H(y)| \\
&\leq 1 - \frac{n}{n+1} + \sup_y |H_n(y) - H(y)| \qquad\qquad H_n \leq 1 \\
&\overset{a.s.}{\to} 0. \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Glivenko-Cantelli}
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
\xi_{nij} &= X_{ij}\Phi^{-1}(H_n^*(Y_i)) \\
&\overset{a.s.}{\to} X_{ij}\Phi^{-1}(H(Y_i)) \\
&= \frac{X_{ij}(\mathbf{x}_i'\boldsymbol{\beta}_0 + \varepsilon_i)}{\sqrt{\sigma_{\boldsymbol{\beta}_0}^2 + \sigma^2}} \\
&= \xi_{ij}.
\end{aligned}
$$

$\square$

7.2 Proof of Lemma 3.3

Lemma 3.3. $H_n^*(Y_i) \sim U_n$ where $U_n \sim \mathrm{Unif}\{(n+1)^{-1},\ldots,n(n+1)^{-1}\}$.

Proof. This is an expression of Renyi's Theorem [Resnick, 1999, p. 96] which states

$$
\mathbf{P}[R_n(Y_i) = k] = n^{-1} \text{ for } k \in \{1,\ldots,n\}
$$

$\square$

7.3 Proof of Lemma 3.4

Lemma 3.4. $\mathbf{E}\Phi^{-1}(U_n)^4 \le 6$.

Proof. For $n \ge 1$, note that $2n \ge n+1$ and therefore $n^{-1} \le 2(n+1)^{-1}$. So we have:

$$
\begin{aligned}
\mathbf{E}\Phi^{-1}(U_n)^4 &= \sum_{k=1}^{n} \Phi^{-1}(k(n+1)^{-1})^4 n^{-1} \\
&\le \sum_{k=1}^{n} \Phi^{-1}(k(n+1)^{-1})^4 2(n+1)^{-1} \\
&= \frac{2}{n+1}\left(\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} \Phi^{-1}(k(n+1)^{-1})^4 + \sum_{k=\lceil\frac{n}{2}\rceil}^{n} \Phi^{-1}(k(n+1)^{-1})^4\right) \\
&\le 2\int_1^{1/2} \Phi^{-1}(u)^4 du + 2\int_{1/2}^1 \Phi^{-1}(u)^4 du \\
&= 2\mathbf{E}\Phi^{-1}(U)^4 \\
&= 2\mathbf{E}Z^4 = 6
\end{aligned}
$$

The third line in the above follows because $\Phi^{-1}(1/2)^4 = 0$. This allows the Riemann approximation to be left-handed where $\Phi^{-1}(\cdot)^4$ is decreasing ($k < n/2$), and right-handed where $\Phi^{-1}(\cdot)^4$ is increasing ($k > n/2$). Figure 6 (with $n = 20$) demonstrates that the approximation can always be kept below the curve $\Phi^{-1}(\cdot)^4$. $\square$
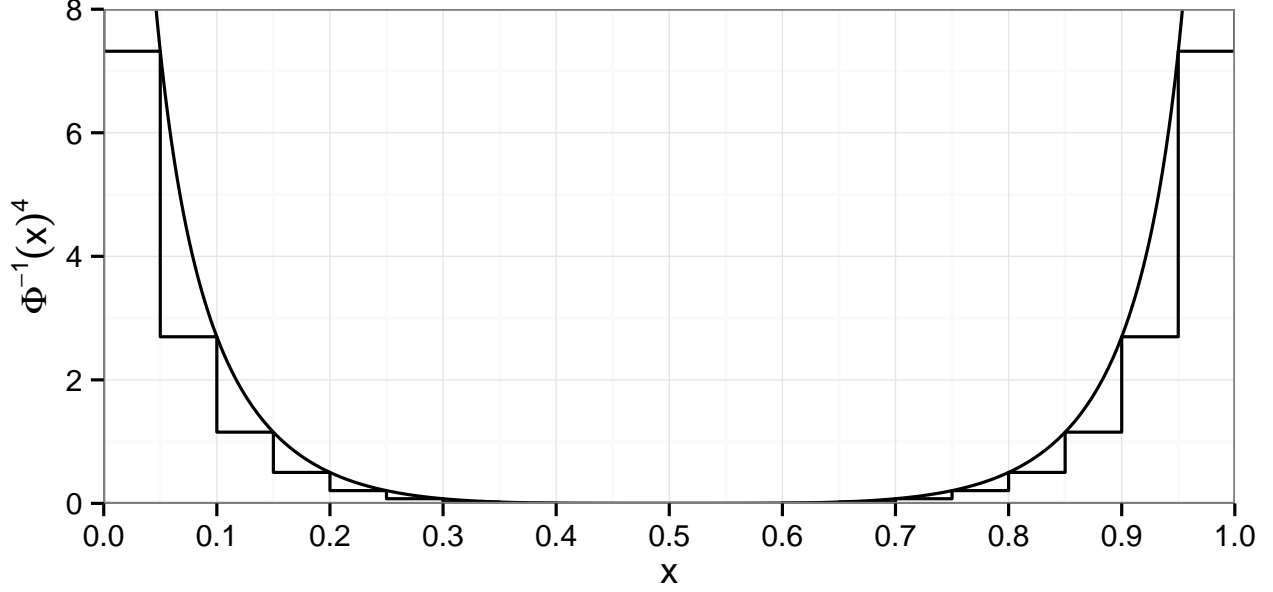
19

Figure 6: Depiction of the Riemann approximation of $\Phi^{-1}(x)^4$ used in Lemma 3.4

7.4 Proof of Lemma 3.5

Lemma 3.5. $\mathbf{E}\xi_{nij}^2 \leq 3\sqrt{2}\sigma_j^2$, where $X_{ij} \sim \mathcal{N}(0, \sigma_j^2)$.

Proof.

$$
\begin{aligned}
\mathbf{E}\xi_{nij}^2 &= \mathbf{E}[X_{ij}\Phi^{-1}(H_n^*(Y_i))]^2 \\
&\leq \sqrt{\mathbf{E}X_{ij}^4 \mathbf{E}\Phi^{-1}(H_n^*(Y_i))^4} && \text{Schwarz Inequality} \\
&= \sqrt{\mathbf{E}X_{ij}^4 \mathbf{E}\Phi^{-1}(U_n)^4} && \text{Lemma 3.3} \\
&= \sqrt{3\sigma_j^4 \cdot 6} = 3\sqrt{2}\sigma_j^2 && \text{Lemma 3.4}
\end{aligned}
$$

$\square$

7.5 Proof of Lemma 3.6

Lemma 3.6. $\mathbf{E}\xi_{nij} \to \mathbf{E}\xi_{ij}$ *as* $n \to \infty$

Proof. First, $\{\xi_{nij}\}$ is uniformly integrable via the Crystal Ball Condition [Resnick, 1999, p. 184] since $\sup_n \mathbf{E}|\xi_{nij}| \leq 3\sqrt{2}\sigma_j^2$ by Lemma 3.5. Uniform integrability and $\xi_{nij} \overset{a.s.}{\to} \xi_{ij}$ gives us $\mathbf{E}\xi nij \to \mathbf{E}\xi_{ij}$ [Resnick, 1999, p. 191] $\square$

7.6 Proof of Lemma 3.7

Lemma 3.7. $\text{cov}(\xi_{nsj}, \xi_{ntj}) \to 0$ *as* $n \to \infty$.

Proof. Now $\{\xi_{nsj}\xi_{ntj}\}$ is also u.i. since

$$\mathbf{E}|\xi_{nsj}\xi_{ntj}| \leq \sqrt{\mathbf{E}\xi_{nsj}^2 \mathbf{E}\xi_{ntj}^2} \leq 3\sqrt{2}\sigma_j^2 < \infty$$

We also have $\xi_{nsj}\xi_{ntj} \overset{a.s.}{\to} \xi_{sj}\xi_{tj}$ and so $\mathbf{E}[\xi_{nsj}\xi_{ntj}] \to \mathbf{E}[\xi_{sj}\xi_{tj}]$.

$$\mathrm{cov}(\xi_{nsj}, \xi_{ntj}) = \mathbf{E}[\xi_{nsj}\xi_{ntj}] - \mathbf{E}\xi_{nsj}\mathbf{E}\xi_{ntj}$$
$$\to \mathbf{E}[\xi_{sj}\xi_{tj}] - \mathbf{E}\xi_{sj}\mathbf{E}\xi_{tj} = 0,$$

where the last equality follows by independence. $\square$

7.7   Proof of Lemma 3.9

Lemma 3.9. $\|S_n^{(2)} - S_n^{(1)}\| \overset{P}{\to} 0$

Proof. For any $\delta > 0$ we have by the Markov Inequality

$$\mathbf{P}[\|S_n^{(2)} - S_n^{(1)}\| > \delta] \leq \delta^{-2}\mathbf{E}\|S_n^{(2)} - S_n^{(1)}\|^2$$
$$= \delta^{-2}n^{-1}\mathbf{E}\left\|\sum_{i=1}^n \mathbf{x}_i'(\Phi^{-1}(H(Y_i)) - \Phi_n^{-1}(H(Y_i)))\right\|$$
$$\leq \delta^{-2}n^{-1}\mathbf{E}\sum_{i=1}^n \left\|\mathbf{x}_i'(\Phi^{-1}(H(Y_i)) - \Phi_n^{-1}(H(Y_i)))\right\|$$
$$= \delta^{-2}\mathbf{E}\left\|\mathbf{x}_i'(\Phi^{-1}(H(Y_i)) - \Phi_n^{-1}(H(Y_i)))\right\|$$
$$\leq \delta^{-2}\mathbf{E}\left[\|\mathbf{x}_i\|^2 \Phi^{-1}(H(Y_i))^2\{H(Y_i) \in (0, 1-\alpha) \cup (\alpha_n, 1)\}\right]$$
$$= \delta^{-2}\sigma_*^2\mathbf{E}\left[\|\mathbf{x}_i(\mathbf{x}_i'\boldsymbol{\beta}_0 + \varepsilon_i)\|^2\{H(Y_i) \in (0, 1-\alpha) \cup (\alpha_n, 1)\}\right]$$
$$\to 0. \tag{DCT}$$

The last line follows from the Dominated Convergence Theorem since

$$\|\mathbf{x}_i(\mathbf{x}_i'\boldsymbol{\beta}_0 + \varepsilon_i)\|^2\{H(Y_1) \in (0, 1-\alpha_n) \cup (\alpha_n, 1)\} \overset{a.s.}{\to} 0$$

and

$$\|\mathbf{x}_i(\mathbf{x}_i'\boldsymbol{\beta}_0 + \varepsilon_i)\|^2\{H(Y_1) \in (0, 1-\alpha_n) \cup (\alpha_n, 1)\} \leq \|\mathbf{x}_i(\mathbf{x}_i'\boldsymbol{\beta}_0 + \varepsilon_i)\|^2 \in L_1$$

by the Gaussian assumptions on $\mathbf{x}_i$. $\square$

7.8   Proof of Lemma 3.10

Lemma 3.10.

$$(\Phi_n^{-1}(H(Y_1)) - \Phi_n^{-1}(H_n(Y_1)))^2 \leq \sigma_* 2\pi\sqrt{n}(H(Y_1) - H_n^*(Y_1))^2$$

Proof. Note that the slope of Φ^{-1} at x is the reciprocal of the slope of Φ at $\Phi^{-1}(x)$, which can be evaluated in terms of the density as $\phi(\Phi^{-1}(x))$. That is, the first derivative of Φ^{-1} for $x \in (0,1)$ is

$$
\begin{aligned}
\frac{d}{dx}\Phi^{-1}(x) &= \frac{1}{\phi(\Phi^{-1}(x))} \\
&= \sqrt{2\pi}\exp(\Phi^{-1}(x)^2/2).
\end{aligned}
$$

Recall Φ^{-1} is symmetric about $1/2$ and Φ_n^{-1} preserves that symmetry. Also, the slope of Φ_n^{-1} is maximized at the truncation points, i.e.

$$
\begin{aligned}
\frac{d}{dx}\Phi_n^{-1}(x) &< \frac{1}{\phi(\Phi^{-1}(\alpha_n))} \\
&= \sqrt{2\pi}\exp\left(\frac{1}{2}\Phi^{-1}\left(\Phi\left(\sqrt{\frac{1}{2}\log n}\right)\right)^2\right) \\
&= \sqrt{2\pi}n^{1/4}.
\end{aligned}
$$

By a first order Taylor series approximation we have

$$
\begin{aligned}
(\Phi_n^{-1}(H(Y_1)) - \Phi_n^{-1}(H_n^*(Y_1)))^2 &\leq (\sqrt{2\pi}n^{1/4}(H(Y_1) - H_n^*(Y_1)))^2 \\
&= 2\pi\sqrt{n}(H(Y_1) - H_n^*(Y_1))^2
\end{aligned}
$$

as desired. $\qquad\square$

7.9   Proof of Lemma 3.11

Lemma 3.11.

$$
\mathbf{E}_1(H(Y_1) - H_n^*(Y_1))^2 \leq \frac{1}{n+1}
$$

almost surely where $\mathbf{E}_1$ is the conditional expectation $\mathbf{E}_1(\cdot) = \mathbf{E}(\cdot|Y_1, \mathbf{x}_1)$.

Proof. We have

$$\mathbf{E}_1(H(Y_1) - H_n^*(Y_1))^2$$

$$= H(Y_1)^2 - 2H(Y_1)\mathbf{E}_1 H_n^*(Y_1) + \mathbf{E}_1 H_n^*(Y_1)^2$$

$$= H(Y_1)^2 - 2H(Y_1)\left(\frac{1}{n+1} + \frac{n-1}{n+1}H(Y_1)\right)$$

$$+ \mathbf{E}_1\left(\frac{1}{n+1} + \frac{1}{n+1}\sum_{i=2}^n \{Y_i \leq Y_1\}\right)^2$$

$$= \frac{3-n}{n+1}H(Y_1)^2 - \frac{2}{n+1}H(Y_1) + \frac{1}{(n+1)^2}$$

$$+ \mathbf{E}_1\left(\frac{2}{(n+1)^2}\sum_{i=2}^n \{Y_i \leq Y_1\} + \frac{1}{(n+1)^2}\left(\sum_{i=2}^n \{Y_i \leq Y_1\}\right)^2\right)$$

$$= \frac{3-n}{(n+1)^2}H(Y_1)^2 - \frac{4}{(n+1)^2}H(Y_1) + \frac{1}{(n+1)^2}$$

$$+ \frac{1}{(n+1)^2}\left((n-1)H(Y_1) + (n-1)(n-2)H(Y_1)^2\right)$$

$$= \frac{5-n}{(n+1)^2}H(Y_1)^2 - \frac{n-5}{(n+1)^2}H(Y_1) + \frac{1}{(n+1)^2}$$

$$\leq \frac{1}{n+1}$$

almost surely. The last inequality follows from $H(Y_1) \in [0,1]$ almost surely. □

7.10 Proof of Lemma 3.12

Lemma 3.12.

$$\|S_n^{(1)} - S_n\| \xrightarrow{P} 0$$

Proof.

$$\mathbf{P}[\|S^{(1)} - S_n\| > \delta] \leq \delta^{-2}\mathbf{E}\|S_n^{(1)} - S_n\|^2$$

$$= \delta^- 2n^{-1}\mathbf{E}\left\|\sum_{i=1}^n \mathbf{x}_i(\Phi_n^{-1}(H(Y_i)) - \Phi_n^{-1}(H_n^*(Y_i)))\right\|^2$$

$$\leq \delta^- 2n^{-1}\mathbf{E}\sum_{i=1}^n \left\|\mathbf{x}_i(\Phi_n^{-1}(H(Y_i)) - \Phi_n^{-1}(H_n^*(Y_i)))\right\|^2$$

$$= \delta^- 2\mathbf{E}\left\|\mathbf{x}_i(\Phi^{-1}(H(Y_i)) - \Phi_n^{-1}(H(Y_i)))\right\|^2$$

$$\leq \delta^- 2\sigma_*^2 2\pi\sqrt{n}\mathbf{E}\left[\|\mathbf{x}_i\|^2(H(Y_i) - H_n^*(Y_i))^2\right]$$

$$= \delta^- 2\sigma_*^2 2\pi\sqrt{n}\mathbf{E}\left[\|\mathbf{x}_i\|^2\mathbf{E}_1(H(Y_i) - H_n^*(Y_i))^2\right]$$

$$\leq \frac{\sigma_*^2 2\pi\sqrt{n}}{\delta^2(n+1)}\mathbf{E}\|\mathbf{x}_i\|^2$$

$$\to 0.$$

□

23

7.11 Proof of Lemma

Corollary.

$$\sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \sigma_* \Sigma_n^{-1} \Sigma \boldsymbol{\beta}_0) \Rightarrow \mathcal{N}_p(0, \Sigma^{-1} A \Sigma^{-1}),$$

where $\Sigma_n^{-1} = (n^{-1} \sum_{i=1}^n \mathbf{x}_i' \mathbf{x}_i)^{-1}$.

Proof. Since $\Sigma_n^{-1} \xrightarrow{P} \Sigma$, we have

$$\sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \sigma_* \Sigma_n^{-1} \Sigma \boldsymbol{\beta}_0) = \Sigma_n^{-1} S_n \Rightarrow \mathcal{N}_p(0, \Sigma^{-1} A \Sigma^{-1})$$

$\square$

7.12 Proof of Lemma

Corollary. *If the covariance matrix, Σ, is known then*

$$\sqrt{n}(\dot{\boldsymbol{\beta}}_n - \sigma_* \boldsymbol{\beta}_0) \Rightarrow \mathcal{N}_p(0, \Sigma^{-1} A \Sigma^{-1}),$$

where

$$\dot{\boldsymbol{\beta}}_n = n^{-1} \Sigma^{-1} \sum_{i=1}^n \mathbf{x}_i' \Phi_n^{-1}(H_n^*(Y_i))$$

Proof. Since $\Sigma_n^{-1} \xrightarrow{P} \Sigma$, we have

$$\sqrt{n}(\dot{\boldsymbol{\beta}}_n - \sigma_* \boldsymbol{\beta}_0) = \Sigma_n^{-1} S_n \Rightarrow \mathcal{N}_p(0, \Sigma^{-1} A \Sigma^{-1})$$

$\square$

7.13 Confidence intervals

Based on the asymptotic normality of our estimator, we can produce confidence regions by estimating its bias and dispersion matrix. Alternatively we can use the weak dependence property that accompanies the established $\sqrt{n}$-consistency to construct jackknife confidence intervals. This amounts to a bootstrap for triangular arrays which are identically distributed across rows with a weak dependence condition. The theory is similar in spirit to the stationary time series bootstrap, except the dependence here is not serial. Recall the setup is $Y = X\boldsymbol{\beta}_0 + \varepsilon$ along with our Gaussian assumptions and that we observe

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} \quad \text{and } \mathbf{R}_n(\mathbf{Y}) = \begin{pmatrix} R_n(Y_1) \\ R_n(Y_2) \\ \vdots \\ R_n(Y_3) \end{pmatrix}.$$

Our asymptotically normal estimator is

$$\tilde{\boldsymbol{\beta}}_n = \left(n^{-1} \sum_{i=1}^n \mathbf{x}_i' \mathbf{x}_i\right)^{-1} n^{-1} \sum_{i=1}^n \mathbf{x}_i' \Phi_n^{-1}(H_n^*(Y_i)).$$

and we have shown $\sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \sigma_* \Sigma_n^{-1} \Sigma \boldsymbol{\beta}_0) = \Sigma_n^{-1} S_n \Rightarrow \mathcal{N}_p(0, \Sigma^{-1} A \Sigma^{-1})$. Suppose $p = 2$ and let

$$\theta_{0n} = \arctan\left(\frac{\mathbf{e}_2 \Sigma_n^{-1} \Sigma \boldsymbol{\beta}_0}{\mathbf{e}_1 \Sigma_n^{-1} \Sigma \boldsymbol{\beta}_0}\right) \xrightarrow{P} \arctan(\boldsymbol{\beta}_{02}/\boldsymbol{\beta}_{01})$$

and

$$\tilde{\theta}_n = \arctan(\tilde{\boldsymbol{\beta}}_{n2}/\tilde{\boldsymbol{\beta}}_{n1})$$

By the bivariate delta method Lehmann [1999] we have

$$\sqrt{n}(\tilde{\theta}_n - \theta_{0n}) \Rightarrow \mathcal{N} \tag{18}$$

with zero mean provide $\boldsymbol{\beta}_{n1}$ is bounded away from zero.

In the previous section we explored the covariance structure (14) of our estimate, namely the limiting dispersion matrix A. Since $\sqrt{n}(\tilde{\theta}_n - \theta_{0n})$ is admittedly not a pivot, we are forced to estimate its variance if confidence intervals using the normal approximation are desired. Adapting the standard methodology from the stationary bootstrap literature Kunsch [1989], Lahiri [1991], Politis and Romano [1992], as well as the order statistics literature Sen [1998], we describe an appropriate functional jackknife procedure.

Observe the n leave-one-out jackknife subsamples of the form

$$\tilde{\boldsymbol{\beta}}_{-i}^* = (n-1)^{-1} \left(\sum_{\{j:j \neq i\}} \mathbf{x}_j' x_j\right) \sum_{\{j:j \neq i\}} \mathbf{x}_j' \Phi_{(n-1)}^{-1}(H_{-i}^*(Y_j)),$$

$$\tilde{\theta}_{-i}^* = \arctan(\tilde{\boldsymbol{\beta}}_{-i2}^*/\tilde{\boldsymbol{\beta}}_{-i1}^*),$$

where $H_{-i}^*(Y_j) = n^{-1} \sum_{\{k:k \neq i\}} \{Y_k \leq Y_j\}$. Now our estimate of the variance is the sample variance of the subsample estimates

$$\mathrm{var}(\tilde{\theta}_n) \approx \tilde{\sigma}_n^2 = n^{-1} \sum_{i=1}^n \tilde{\theta}_{-i}^{*2} - \left(n^{-1} \sum_{i=1}^n \tilde{\theta}_{-i}^*\right)^2. \tag{19}$$

If the bias is negligible, we have an approximate $(1 - \alpha)100\%$ confidence interval:

$$\left[\tilde{\theta}_n - z_\alpha \tilde{\sigma}_n, \tilde{\theta}_n + z_\alpha \tilde{\sigma}_n\right],$$

where $z_\alpha = \Phi^{-1}(1 - \alpha/2)$. We can also include a bias correction by estimating the mean of our estimate:

$$\mathbf{E}(\tilde{\theta}_n) \approx \tilde{\mu}_n = n^{-1} \sum_{i=1}^n \tilde{\theta}_{-i}^*.$$

The bias corrected approximate confidence interval then takes the form

$$\left[\tilde{\theta}_n - \tilde{\mu}_n - z_\alpha \tilde{\sigma}_n, \tilde{\theta}_n - \tilde{\mu}_n + z_\alpha \tilde{\sigma}_n\right].$$

To justify the consistency of these estimates consider, by Chebychev's Inequality and (18)

$$\mathbf{P}[|\tilde{\mu}_n - \mathbf{E}(\tilde{\mu}_n)| > \delta] \leq \delta^{-2} \mathrm{var}(\tilde{\mu}_n) \leq \delta^{-2} \mathrm{var}(\tilde{\theta}_{-1}^*) = \delta^{-2} \mathrm{var}(\tilde{\theta}_{n-1}) \to 0.$$

We also have

$$|\mathbf{E}(\tilde{\theta}_n) - \mathbf{E}(\tilde{\mu}_n)| = |\mathbf{E}(\tilde{\theta}_n) - \mathbf{E}(\tilde{\theta}_1^*)| = |\mathbf{E}(\tilde{\theta}_n) - \mathbf{E}(\tilde{\theta}_{n-1})| \to 0$$

so that

$$|\mathbf{E}(\tilde{\theta}_n) - \tilde{\mu}_n| \leq |\mathbf{E}(\tilde{\theta}_n) - \mathbf{E}(\tilde{\mu}_n)| + |\mathbf{E}(\tilde{\mu}_n) - \tilde{\mu}_n| \xrightarrow{P} 0.$$

Therefore, $\tilde{\mu}_n$ is a consistent estimate for $\mathbf{E}(\tilde{\theta}_n)$.

By another application of the delta method we have $\sqrt{n}(\tilde{\theta}_n^2 - \theta_{0n}^2) \Rightarrow \mathcal{N}$ with mean zero. We then have

$$\mathbf{P}\left[\left|n^{-1}\sum_{i=1}^n \tilde{\theta}_{-i}^{*2} - \mathbf{E}(\tilde{\theta}_1^{*2})\right| > \delta\right] \leq \delta^{-2} \operatorname{var}\left(n^{-1}\sum_{i=1}^n \tilde{\theta}_{-i}^{*2}\right)$$

$$\leq \delta^{-2} \operatorname{var}(\tilde{\theta}_{-1}^{*2})$$

$$\leq \delta^{-2} \operatorname{var}(\tilde{\theta}_{n-1}^2)$$

$$\to 0.$$

Additionally, we have

$$|\mathbf{E}(\tilde{\theta}_n^2) - \mathbf{E}(\tilde{\theta}_{-1}^{*2})| = |\mathbf{E}(\tilde{\theta}_n^2) - \mathbf{E}(\tilde{\theta}_{n-1}^2)| \to 0,$$

so that

$$\left|\mathbf{E}(\tilde{\theta}_n^2) - n^{-1}\sum i = 1^n \tilde{\theta}_{-i}^{*2}\right| \leq \left|\mathbf{E}(\tilde{\theta}_n^2) - \mathbf{E}(\tilde{\theta}_{-1}^{*2})\right| + \left|\mathbf{E}(\tilde{\theta}_{-1}^{*2}) - n^{-1}\sum i = 1^n \tilde{\theta}_{-i}^{*2}\right| \to 0.$$

Finally we have consistency of our variance estimate, $|\tilde{\sigma}_n^2 - \operatorname{var}(\tilde{\theta}_n)| \xrightarrow{P} 0$.

Recall that when the dispersion, Σ, is known, Corollary gives us asymptotic normality of the form

$$\sqrt{n}(\dot{\boldsymbol{\beta}}_n - \sigma_*^{-1}\boldsymbol{\beta}_0) \Rightarrow \mathcal{N}_p(0, \Sigma^{-1}A\Sigma^{-1}),$$

where

$$\dot{\boldsymbol{\beta}}_n = n^{-1}\sum_{i=1}^n \mathbf{x}_i' \Phi^{-1}(H_n^*(Y_i)).$$

In this case, with $p = 2$ we compute statistics of the form

$$\dot{\theta}_n = \arctan(\dot{\boldsymbol{\beta}}_{n2}/\boldsymbol{\beta}_{n1})$$

such that,

$$\sqrt{n}(\dot{\theta}_n - \theta_0) \Rightarrow \mathcal{N}, \theta_0 = \arctan(\boldsymbol{\beta}_{02}/\boldsymbol{\beta}01).$$

We can derive consistent estimators of the mean and variance similar to the unknown dispersion case based on the jackknife observations

$$\boldsymbol{\beta}_{-i}^* = (n-1)^{-1}\Sigma^{-1}\sum_{\{j:j\neq i\}} \mathbf{x}_j' \Phi_{(n-1)}^{-1}(H_n^*(Y_j)),$$

$$\dot{\theta}_{-i}^* = \arctan(\dot{\boldsymbol{\beta}}_{-i2}/\dot{\boldsymbol{\beta}}-i1).$$

Additionally, with known Σ we can estimate the distribution of $\dot{\theta}_n$ directly by subsampling and using the quantiles of this approximation to form confidence intervals, rather than the normal quantile approximations. Specifically we have

$$\operatorname{Dist}_{\dot{\theta}_n}(x) \approx \operatorname{Dist}_{\dot{\theta}_n}^*(x) = n^{-1}\sum_{i=1}^n \{\dot{\theta}_{-i}^* \leq x\}.$$

Let $\dot{\theta}^*_{(1)} \leq \dot{\theta}^*_{(2)} \leq \cdots \leq \dot{\theta}^*_{(n)}$, denote the order statistics. then,

$$Q^*(\alpha/2) \approx \dot{\theta}^*_{(k_1)}, Q^*(1-\alpha/2) \approx \dot{\theta}^*_{(k_2)},$$

where $k_1 = \lfloor B\alpha/2 \rfloor, k_2 = \lfloor B(1-\alpha/2) \rfloor + 1$. We get a percentile jackknife confidence interval for $\boldsymbol{\beta}_0$ of the form

$$[\dot{\theta}^*_{(k_1)}, \dot{\theta}^*_{(k_2)}].$$

The confidence interval can be shown to be consistent by an argument similar to that for the mean and variance estimators.

We can further improve on the percentile jackknife confidence intervals by considering a studentized bootstrap-t confidence interval. Consider

$$S_\theta = \frac{\dot{\theta}_n - \theta_0}{\sqrt{\mathrm{var}(\dot{\theta}_n)}}.$$

We can estimate $\mathrm{Dist}_{S_\theta}(x) = \mathbf{P}(S_\theta \leq x)$ by a nested jackknife

$$\mathrm{Dist}^*_{S_\theta}(x) = n^{-1} \sum i = 1^n \{\dot{\theta}^*_i \leq x\sqrt{\hat{\mathrm{var}}(\dot{\theta}^*_i)} + \dot{\theta}_n\},$$

where $\hat{\mathrm{var}}(\dot{\theta}^*_i)$ is estimated in the same manner as (19). The estimate $\hat{\mathrm{var}}(\dot{\theta}^*_i)$ is gotten from the sample variance of $(n-1)$ jackknife subsamples of size $(n-2)$.

7.14   R Functions

```r
# General Truncated Gaussian Quantile function
tgq <- function(x, cut)
{
  alpha <- pnorm(cut)
  ifelse(x < 1-alpha, -cut,
  ifelse(x > alpha, cut, qnorm(x)))
}


# H (rank) transformation
fH <- function(x)
{
  rank(x, na.last='keep')/(length(x[!is.na(x)])+1)
}


# Truncated Gaussian Quantile utility function for regression
ftgq <- function(x){
```

```
  n <- length(x[!is.na(x)])
  tgq(rank(x, na.last='keep')/(n+1), cut=sqrt(log(n)/2))
}


Length <- function(x) sqrt(x%*%x)[1]
Norm <- function(x) x/Length(x)


# Truncated Gaussian Quantile Regression
tgqr <- function (formula, data, subset, FUN = ftgq, ...)
{
  cl <- match.call()
  mf <- match.call(expand.dots = FALSE)
  m <- match(c("formula", "data", "subset", "weights", "na.action",
      "offset"), names(mf), 0L)
  mf <- mf[c(1L, m)]
  mf$drop.unused.levels <- TRUE
  mf[[1L]] <- quote(stats::model.frame)
  mf <- eval(mf, parent.frame())
  mf[,1] <- FUN(model.response(mf, "numeric"))
  b <- lm(formula, data=mf, ...)$coef
  Norm(b)
}


# Empirical Quantile Regression
eqr <- function (formula, data, subset, tol=1e-5,
  maxiter=100, truncate = FALSE, ...)
{
  cl <- match.call()
  mf <- match.call(expand.dots = FALSE)
  m <- match(c("formula", "data", "subset", "weights", "na.action",
      "offset"), names(mf), 0L)
  mf <- mf[c(1L, m)]
  mf$drop.unused.levels <- TRUE
  mf[[1L]] <- quote(stats::model.frame)
  mf <- eval(mf, parent.frame())
  mm <- model.matrix(formula, mf)
```

```r
  R <- rank(model.response(mf, "numeric"), na.last='keep')
  #initial value
  n <- nrow(mm)
  mf[,1] <- R/(n+1)
  beta_eq <- Norm(lm(formula, data=mf, ...)$coef)
  p <- length(beta_eq)
  beta_eq0 <- Norm(rep(1, p))
  if(truncate){
    alpha <- pnorm(sqrt(log(n)/2))
  }
  i <- 1
  while(i < maxiter){
    Finv0 <- approxfun(ecdf(mm%*%beta_eq)(mm%*%beta_eq),
      mm%*%beta_eq, rule = 2)
    if(truncate){
      tFinv <- function(x){
        ifelse(x < 1-alpha, Finv0(1-alpha),
        ifelse(x > alpha, Finv0(alpha), Finv0(x)))
      }
      Finv <- tFinv
    }else{
      Finv <- Finv0
    }
    mf[,1] <- Finv(R/(n+1))
    beta_eq <- Norm(lm(formula, data=mf, ...)$coef)
    if(Length(beta_eq - beta_eq0)<tol){
      break()
    }else{
      beta_eq0 <- beta_eq
      i <- i+1
    }
  }
  beta_eq
}


# Spearmax
```

```
spearmax <- function (formula, data, subset, tol=1e-5,
  maxiter=100, truncate = FALSE, ...)
{
  cl <- match.call()
  mf <- match.call(expand.dots = FALSE)
  m <- match(c("formula", "data", "subset", "weights", "na.action",
      "offset"), names(mf), 0L)
  mf <- mf[c(1L, m)]
  mf$drop.unused.levels <- TRUE
  mf[[1L]] <- quote(stats::model.frame)
  mf <- eval(mf, parent.frame())
  mm <- model.matrix(formula, mf)
  R <- rank(model.response(mf, "numeric"), na.last='keep')
  p <- ncol(mm)
  ncorfun <- function(b){
    -cor(R, rank(mm%*%b, na.last='keep'))
  }
  res <- constrOptim(Norm(rep(1,p)), ncorfun, method = "Nelder-Mead",
    ui = -Norm(rep(1,p)), ci = -Norm(rep(1,p))*2)
  b <- Norm(res$par)
  names(b) <- colnames(mm)
  b
}
```

References

Jason Abrevaya. Rank regression for current-status data: asymptotic normality. *Statistics & Probability Letters*, 43(3):275–287, 1999. ISSN 0167-7152. URL http://econpapers.repec.org/article/eeestapro/v_3a43_3ay_3a1999_3ai_3a3_3ap_3a275-287.htm.

Jorge Aragón and Adolfo J. Quiróz. Rank regression for current status data. *Statistics & Probability Letters*, 24(3):251–256, August 1995. ISSN 0167-7152. doi: 10.1016/0167-7152(94)00180-G. URL http://www.sciencedirect.com/science/article/pii/016771529400180G.

P. K. Bhattacharya. Convergence of Sample Paths of Normalized Sums of Induced Order Statistics. *The Annals of Statistics*, 2(5):1034–1039, September 1974. ISSN 0090-5364. URL http://www.jstor.org/stable/2958068.

P. J. Bickel. On Some Analogues to Linear Combinations of Order Statistics in the Linear Model. *The Annals of Statistics*, 1(4):597–616, July 1973. ISSN 0090-5364. URL http://www.jstor.org/stable/2958306.

R. Byrd, P. Lu, J. Nocedal, and C. Zhu. A Limited Memory Algorithm for Bound Constrained Optimization. *SIAM Journal on Scientific Computing*, 16(5):1190–1208, September 1995. ISSN 1064-8275. doi: 10.1137/0916069. URL http://epubs.siam.org/doi/abs/10.1137/0916069.

Christopher Cavanagh and Robert P. Sherman. Rank estimators for monotonic index models. *Journal of Econometrics*, 84(2):351–381, June 1998. ISSN 0304-4076. doi: 10.1016/S0304-4076(97)00090-0. URL http://www.sciencedirect.com/science/article/pii/S0304407697000900.

H. A. David and H. N. Nagaraja. 18 Concomitants of order statistics. In N. Balakrishnan and C. R. Rao, editor, *Handbook of Statistics*, volume 16 of *Order Statistics: Theory & Methods*, pages 487–513. Elsevier, 1998. URL http://www.sciencedirect.com/science/article/pii/S0169716198160200.

Marie Davidian and Perry D. Haaland. Regression and calibration with nonconstant error variance. *Chemometrics and Intelligent Laboratory Systems*, 9(3):231–248, December 1990. ISSN 0169-7439. doi: 10.1016/0169-7439(90)80074-G. URL http://www.sciencedirect.com/science/article/pii/016974399080074G.

M. C. Donohue, S. H. Moghadam, A. D. Roe, C. K. Sun, S. D. Edland, R. G. Thomas, R. C. Petersen, M. Sano, D. Galasko, P. S. Aisen, and R. A. Rissman. Longitudinal plasma amyloid beta in Alzheimer's disease clinical trials. *Alzheimer's & dementia : the journal of the Alzheimer's Association*, (Journal Article), 2014. ISSN 1552-5279; 1552-5260. doi: S1552-5260(14)02769-1[pii].

Michael Donohue, Ian Abramson, and Anthony Gamst. Asynergistic Regression Based on Maximized Rank Correlation. *Communications in Statistics - Simulation and Computation*, 37(1):92–105, December 2007. ISSN 0361-0918. doi: 10.1080/03610910701723716. URL http://dx.doi.org/10.1080/03610910701723716.

Jaroslav Hájek. Asymptotic Normality of Simple Linear Rank Statistics Under Alternatives. *The Annals of Mathematical Statistics*, 39(2):325–346, April 1968. ISSN 0003-4851. URL http://www.jstor.org/stable/2239025.

Aaron K. Han. Non-parametric analysis of a generalized regression model: The maximum rank correlation estimator. *Journal of Econometrics*, 35(23):303–316, July 1987. ISSN 0304-4076. doi: 10.1016/0304-4076(87)90030-3. URL http://www.sciencedirect.com/science/article/pii/0304407687900303.

Thomas P Hettmansperger. *Statistical inference based on ranks*. Wiley, New York, 1984. ISBN 047188474X 9780471884743.

Jana Jurečková and Pranab Kumar Sen. *Robust Statistical Procedures: Asymptotics and Interrelations*. John Wiley & Sons, April 1996. ISBN 9780471822219.

Roger Koenker and Gilbert Bassett, Jr. Regression Quantiles. *Econometrica*, 46(1):33–50, January 1978. ISSN 0012-9682. doi: 10.2307/1913643. URL `http://www.jstor.org/stable/1913643`.

Hans R. Kunsch. The Jackknife and the Bootstrap for General Stationary Observations. *The Annals of Statistics*, 17(3):1217–1241, September 1989. ISSN 0090-5364. URL `http://www.jstor.org/stable/2241719`.

Soumendra Nath Lahiri. Second order optimality of stationary bootstrap. *Statistics & Probability Letters*, 11(4):335–341, April 1991. ISSN 0167-7152. doi: 10.1016/0167-7152(91)90045-S. URL `http://www.sciencedirect.com/science/article/pii/016771529190045S`.

E. L. Lehmann. *Elements of Large-Sample Theory*. Springer Science & Business Media, 1999. ISBN 9780387985954.

M. I. Parzen, L. J. Wei, and Z. Ying. A resampling method based on pivotal estimating functions. *Biometrika*, 81(2):341–350, June 1994. ISSN 0006-3444, 1464-3510. doi: 10.1093/biomet/81.2.341. URL `http://biomet.oxfordjournals.org/content/81/2/341`.

Dimitris N. Politis and Joseph P. Romano. A General Resampling Scheme for Triangular Arrays of -Mixing Random Variables with Application to the Problem of Spectral Density Estimation. *The Annals of Statistics*, 20(4):1985–2007, December 1992. ISSN 0090-5364. URL `http://www.jstor.org/stable/2242377`.

Sidney I. Resnick. *A Probability Path*. Springer Science & Business Media, November 1999. ISBN 9780817684099.

Pranab Kumar Sen. 12 Order statistics: Asymptotics in applications. In N. Balakrishnan and C. R. Rao, editor, *Handbook of Statistics*, volume 16 of *Order Statistics: Theory & Methods*, pages 335–373. Elsevier, 1998. URL `http://www.sciencedirect.com/science/article/pii/S0169716198160145`.

Robert P. Sherman. The Limiting Distribution of the Maximum Rank Correlation Estimator. *Econometrica*, 61(1):123, January 1993. ISSN 00129682. doi: 10.2307/2951780. URL `http://www.jstor.org/discover/2951780?sid=21105499836393&uid=3739256&uid=4&uid=2`.

Stephen Mack Stigler. Linear Functions of Order Statistics. *The Annals of Mathematical Statistics*, 40(3):770–788, June 1969. ISSN 0003-4851. URL `http://www.jstor.org/stable/2239628`.

Shie-Shien Yang. Linear Functions of Concomitants of Order Statistics with Application to Nonparametric Estimation of a Regression Function. *Journal of the American Statistical Association*, 76(375):658–662, September 1981a. ISSN 0162-1459. doi: 10.1080/01621459.1981.10477700. URL `http://www.tandfonline.com/doi/abs/10.1080/01621459.1981.10477700`.

Shie-Shien Yang. Linear combination of concomitants of order statistics with application to testing and estimation. *Annals of the Institute of Statistical Mathematics*, 33(1):463–470, December 1981b. ISSN 0020-3157, 1572-9052. doi: 10.1007/BF02480956. URL `http://link.springer.com/article/10.1007/BF02480956`.