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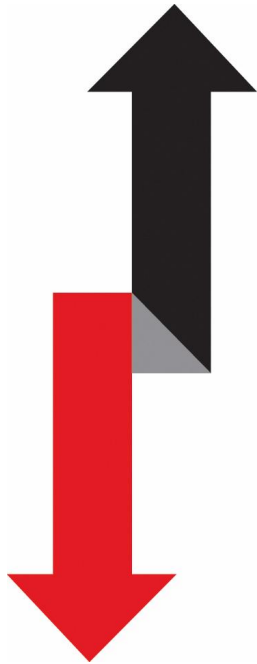
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# Equilibrium in Continuous-Time Financial Markets: Endogenously Dynamically Complete Markets

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**Abstract**

We prove existence of equilibrium in a continuous-time securities market in which the securities are potentially dynamically complete: the number of securities is at least one more than the number of independent sources of uncertainty. We prove that dynamic completeness of the candidate equilibrium price process follows from mild exogenous assumptions on the economic primitives of the model. Our result is universal, rather than generic: dynamic completeness of the candidate equilibrium price process and existence of equilibrium follow from the way information is revealed in a Brownian filtration, and of a mild exogenous nondegeneracy condition on the terminal security dividends. The nondegeneracy condition, which requires that finding one point at which a determinant of a Jacobian matrix of dividends is nonzero, is very easy to check. We find that the equilibrium prices, consumptions, and trading strategies are well-behaved functions of the stochastic process describing the evolution of information. We prove that equilibria of discrete approximations converge to equilibria of the continuous-time economy.

KEYWORDS: Dynamic completeness, convergence of discrete-time finance models, continuous-time finance, general equilibrium theory,

JEL Classification: D53, D52, G12

# 1  Introduction

In an Arrow-Debreu market, agents are allowed to shift consumption across states and time by trading a complete set of Arrow-Debreu contingent claims. Mas-Colell and Richard [73], Dana [32] and Bank and Riedel [11] prove existence of equilibrium in a continuous-time Arrow-Debreu market.

By contrast, in a securities market, agents are restricted to trading a pre-specified set of securities. The securities market is said to be dynamically complete if agents can, by rapidly trading the given set of securities, achieve all the consumption allocations they could achieve in an Arrow-Debreu market. Markets are said to be potentially dynamically complete if the number of securities is at least one more than the number of independent sources of uncertainty. A securities market which is potentially dynamically complete may or may not be dynamically complete.

Continuous-time finance models are securities market models. The vast majority of papers in continuous-time finance specify that the securities *prices* follow an exogenously specified stochastic process. But prices of securities are in fact determined day by day and minute by minute by the balancing of supply and demand. A complete economic model of continuous-time trading requires that the securities' *dividends* follow an exogenously specified stochastic process, and that the securities' *price* process be *derived* through *equilibrium* considerations. In particular, the securities' price process should be derived from the primitive economic data of the economy: the agents' information, utility functions, and endowments, and the securities' dividend process.

Existence of equilibrium in continuous-time finance models with a single agent has been established in a number of papers (Bick [13], He and Leland [47], Cox, Ingersoll and Ross [19], Duffie and Skiadis [43], Raimondo [86][1]).

Surprisingly, the existing literature does not contain a satisfactory theory of existence of equilibrium in continuous-time securities markets with more than one agent. With dynamic incompleteness, essentially nothing is known. While we hope that some of the techniques developed in this paper will help in addressing the dynamically incomplete case, we do not here present a result in that case.

This paper, and the existing literature, all deal with the case in which markets are potentially dynamically complete. Every previous paper *assumes*,[2] in one form or another, that the candidate equilibrium price process is dynamically complete, and shows that this

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[1]All of these papers except Raimondo [86] require one or more endogenous assumptions that are not expressed solely in terms of the primitives of the model.

[2]See Basak and Cuoco [12]; Cuoco [21]; Dana [33]; Detemple and Karatzas [34]; Duffie [37, 38, 39]; Duffie and Huang [40]; Duffie and Zame [44]; Karatzas Lehoczky and Shreve [56]; and Riedel [87]. The form of the assumption varies. Some assume directly that the dispersion matrix of the candidate equilibrium price process is almost surely nonsingular, which is well known to be equivalent to dynamic completeness. Others assume that the dispersion matrix of the candidate equilibrium price process is uniformly positive definite; this is stronger than dynamic completeness and it fails for an open set of primitives in our model. Still others assume that the candidate equilibrium price of consumption is uniformly bounded above and uniformly bounded away from zero, but this is inconsistent with unbounded consumption (found in the Black-Scholes and most other continuous-time finance models) and standard utility functions. Finally, some *construct* securities which suffice to complete the markets, rather than taking the securities' dividends as given by the model.

implies that the candidate equilibrium is in fact an equilibrium. But the candidate equilibrium price process is determined from the economic primitives of the model by a fixed point argument, which makes it impossible, except in knife-edge special cases, to determine from the primitives whether or not the assumption is satisfied. Thus, if we consider one of these models and choose specific utility functions, endowment and dividend processes, except in the rare cases where we can solve explicitly for the candidate equilibrium, we cannot apply any of the previous theorems to determine whether or not an equilibrium exists.

While dynamic completeness plays a role in proving existence of equilibrium, its main application in Finance is to derivative pricing. Given a dynamically complete securities price process, options and other derivatives can be uniquely priced by arbitrage arguments and can be replicated by trading the underlying securities. The options market has become very large and liquid, largely because securities firms can sell options and hedge their risk using replicating strategies. With dynamic incompleteness, arbitrage considerations do not determine a unique option price, and replication is not possible. The previous results provide no guarantee that equilibrium prices will support the standard theory of pricing and replicating options, which depends on dynamic completeness.

In this paper, we *prove* that the candidate equilibrium price process is *in fact* dynamically complete, and that the candidate equilibrium is in fact an equilibrium.[3] Dynamic completeness of the candidate equilibrium price process follows from mild exogenous assumptions on the economic primitives of the model. Our result implies that equilibrium generically exists, and is dynamically complete, but it is stronger. Generic existence results show that equilibrium exists apart from a small set of choices of the economic primitives, but it may be difficult to determine whether a particular choice of primitives lies in the generic set. Our result does support the standard theory of pricing and replicating options, which depends on dynamic completeness. In our result, dynamic completeness of the candidate equilibrium price process and existence of equilibrium follow from the way information is revealed in a Brownian filtration, and of a mild exogenous nondegeneracy condition on the terminal security dividends. Moreover, the condition we need to check—that a determinant of the securities dividends is nonsingular at one point—is of the same form but simpler, because it is evaluated at only one point, than the condition for dynamic completeness in the arbitrage-based theory, where prices are taken as given. There is no analogous condition on dividends sufficing to prove dynamic completeness in the previous equilibrium literature.

To motivate our nondegeneracy condition, suppose we are given a market with $K$ independent sources of uncertainty and $K + 1$ securities labelled $j = 0, \ldots, K$, so the market is potentially dynamically complete. Now suppose that two of the securities are perfect clones of each other, in that they pay exactly the same dividends at all nodes. Clearly, this is the same as a market with $K$ sources of uncertainty and $K$ securities, so it cannot possibly be dynamically complete. Similarly, if the dividend processes of the securities were linearly

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[3]More specifically, we show that any integrable consumption which is adapted to the Brownian filtration can be replicated by an admissible self-financing trading strategy. We allow the filtration in the original continuous-time economy to be larger than the Brownian filtration, and the filtration in the Loeb-measure economy we construct is *always* larger than the Brownian filtration. This does not pose a problem for existence of equilibrium because any consumption adapted to the larger filtration is a mean-preserving spread of a consumption adapted to the Brownian filtration, so agents' demands are always adapted to the Brownian filtration.

dependent, we could not possibly have dynamic completeness. Thus, we need to assume that the dividend processes are not linearly dependent. In our model, the dividend of security $j$ at the terminal date $T$ is given as a function $G_j(\beta(T, \omega))$ of the terminal realization $\beta(T, \omega)$ of the Brownian Motion in state $\omega$. In the important special case in which security 0 is a zero-coupon bond, our condition requires that $G_1, \ldots, G_K$ be $C^1$ functions on some open set $V \subset \mathbf{R}^K$, and that the Jacobian determinant of $(G_1, \ldots, G_K)$ be nonzero at one point $x \in V$. In particular, our assumption depends only on the securities dividends at the terminal date, and not on the other economic primitives; changing the utility function or endowment of an agent or the initial ownership of the securities has no effect on the existence of equilibrium. It is easy to check directly from the dividend process whether our nondegeneracy assumption holds; one simply needs to find one point where a determinant is nonzero. Thus, we give a simple condition on the dividends that guarantees that equilibrium exists, and that the equilibrium prices are dynamically complete, so options can be replicated and uniquely priced.

We obtain explicit formulas for the equilibrium price process and its dispersion matrix, each trader's equilibrium securities wealth, and the dispersion matrix for each traders' equilibrium trading strategy in terms of the equilibrium consumptions; each trader's equilibrium trading strategy can then be calculated using linear algebra. These formulas are expressed in terms of the equilibrium consumptions, which are not known *a priori*. However, since the equilibrium is Pareto Optimal, there is a vector of Negishi utility weights $\lambda$ such that, at each node, the equilibrium consumptions maximize the weighted sum of the utilities of the agents. Thus, the key features of the equilibrium can be calculated explicitly from knowledge of the primitives of the model (endowments and utility functions of the individuals, and the dividends of the securities) and the equilibrium utility weights. Moreover, even if the equilibrium utility weights are not known, the explicit nature of the formulas can potentially be used to establish general properties of equilibria, such as comparative statics results.

No previous result established convergence of equilibrium in the discrete setting to equilibrium in the continuous-time setting. We prove that all key elements of equilibria of discrete approximations converge to the corresponding elements of equilibria of the continuous-time model.[4] This is important for two reasons: First, many people regard the discrete models as the appropriate models for "real" economies. In particular, financial transaction data is very high-frequency, but nonetheless discrete. The theoretical literature has focused on continuous time because the formulas are much simpler in continuous-time than in the discrete case. Cox, Ross and Rubinstein [20] showed that one can compute the prices of options when the price of a stock is given by a geometric binomial random walk, and showed that the discrete option prices converge to those given by the Black-Scholes formula. Our convergence theorems tell us that the equilibrium pricing formulas from the continuous-time model apply asymptotically to equilibria of the discrete models as the discretization gets finer. This shows that the formulas are applicable to discrete models. It also shows that the formulas can be used in the econometric analysis of data which is high-frequency but

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[4]When the continuous-time model has multiple equilibria, it is possible that the equilibria of the discrete approximations will be near one continuous-time equilibrium for some approximations and near a different continuous-time equilibrium for other approximations, so the sequence of equilibria of the discrete approximations converges to the *set* of equilibria of the continuous-time economy, rather than to a single equilibrium of the continuous-time economy.

nonetheless discrete.[5] Second, it provides a means to compute equilibria of continuous-time economies. Since we show that the equilibria of the discretizations are close to equilibria of the continuous-time economy, algorithms to compute equilibria for discrete economies[6] provide a means to compute equilibria of the continuous-time economy.

Our starting point is a continuous-time model, on which we state our existence theorem, Theorem 2.1. A function is said to be real analytic if, at every point in its domain, there is a power series which converges to the function on an open set containing the point. We assume that the primitives of the economy are given by real analytic functions of time and the current value of the Brownian Motion for times $t \in (0, T)$, where $T$ is the terminal date.[7] Our nondegeneracy assumption is imposed on the dividends at the terminal date $T$.

We discretize the continuous-time model to construct a sequence of models; in each, we replace the Brownian Motion $\beta$ by a random walk $\hat{\beta}$. A naive discretization would approximate the $K$-dimensional Brownian Motion by a $K$-dimensional random walk in which each node has $2^K$ successors, and the random walk moves independently in each direction at each node. However, in a discrete random walk in which each node as $2^K$ successors, one needs at least $2^K$ securities to obtain dynamic completeness; for $K > 1$, $K + 1 < 2^K$ and the discrete model cannot possibly be dynamically complete. In order that the discrete approximations correspond closely to the continuous-time model, it is critical that the discrete model have the same number of securities as the continuous-time model, and that it be dynamically complete.[8] Thus, we constuct a $K$-dimensional random walk $\hat{\beta}$ in which each nonterminal node has exactly $K + 1$ successors. Endowments, dividends and utility functions are induced on the discrete economies from the specification of the continuous-time economy. These discrete economies are General Equilibrium Incomplete Markets (GEI) models; Magill and Quinzii [71] is an excellent reference on GEI models. Endowments and dividends are perturbed as necessary to ensure existence of a dynamically complete equilibrium, under the Duffie-Shafer [41, 42] or Magill-Shafer [72] Theorems. We then prove our convergence theorem, Theorem 3.2, that equilibria of the discrete approximations converge to equilibria of the continuous-time economy.

Since equilibrium prices are arbitrage-free, the equilibrium securities prices at the terminal date in the discrete model must be given by the exogenously specified dividends at

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[5]As noted above, if the continuous-time economy has multiple equilibria, equilibria of the discrete approximation economies converge to the set of continuous-time equilibria, rather than to a single equilibrium. Nonetheless, this tells us that the formulas given in Theorem 2.1 apply approximately to the equilibria of the discretizations. For example, Equation (6) gives a formula for the dispersion matrix of the securities prices in terms of the Brownian Motion, the future dividends, and the endogenously determined prices of consumption. The convergence results in Theorem 3.2 imply that Equation (6) holds asymptotically over any sequence of equilibria of the discrete approximations, using the prices of consumption in the given discrete approximations.

[6]See, for example, Brown, DeMarzo and Eaves [15] and Judd, Kubler and Schmedders [53, 54, 55].

[7]It is important to emphasize that the primitives of the model, and as we will find, the equilibrium, are analytic functions of time and the Brownian Motion, not analytic functions of time alone. Brownian Motion is almost surely nowhere differentiable, and almost surely of unbounded variation on every interval of time. Consequently, the equilibrium prices and trading strategies are nowhere differentiable, and of unbounded variation, as functions of time.

[8]If one were to extend these methods to the analysis of dynamically incomplete continuous-time models, it would be critical to ensure that the discrete approximation has the same degree of incompleteness as the continuous-time model, so the random walk would have to be chosen carefully in that context also.

the terminal date. It is well known that the equilibrium securities price process at time $t$ is the conditional expectation of future dividends, valued at the Arrow-Debreu equilibrium prices of consumption; this comes from the first order conditions for utility maximization (see, for example, Magill and Quinzii [71] in the discrete case). We show that because of the smooothness properties of the Gaussian distribution, such conditional expectations are given by real analytic functions of $(t, \hat{\beta}(t, \omega)) \in (0, T) \times \mathbf{R}^K$, so the equilibrium securities price process is a real analytic function of $(t, \hat{\beta}(t, \omega))$. Moreover, the dispersion matrix of the securities prices and its associated determinant is a real analytic function of $(t, \hat{\beta}(t, \omega))$; this is a property of the way information is revealed in a Brownian filtration, and does not depend on any specific assumptions on the functional form of the primitives.

We use nonstandard analysis to project the discrete model back into the continuous-time model, and find that the dispersion matrix of the continuous-time securities prices is the projection of the dispersion matrix in the discrete model, and hence is real analytic. A real analytic function cannot be zero on a set of positive measure unless it is identically zero. Since the determinant associated with the dispersion matrix is nonzero on a set of positive measure, it must be nonzero except on a set of measure zero. This implies that the dispersion matrix has rank $K$ except on a set of measure zero, but it is well known that this condition is equivalent to dynamic completeness; thus, we have shown that the candidate equilibrium prices in the continuous-time model are dynamically complete. We verify that the projected prices are equilibrium prices in the continuous-time model.

Our proof depends heavily on nonstandard analysis, and in particular the nonstandard theory of stochastic processes. Nonetheless, the statements of our main Theorems 2.1 and 3.2 can be understood without any knowledge of nonstandard analysis.

Nonstandard analysis provides powerful tools to move from discrete to continuous time, and from discrete distributions like the binomial to continuous distributions like the normal; in particular, it provides the ability to transfer computations back and forth between the discrete and continuous settings. Our sequence of discrete approximations extends to a hyperfinite approximation, one which is infinite but has all the formal properties of finite approximations. In particular, the hyperfinite approximation has a GEI equilibrium which is dynamically complete in the hyperfinite model. We then use nonstandard analysis to produce a candidate equilibrium in the continuous-time model, show that the equilibrium in the hyperfinite model is infinitely close to the candidate equilibrium in the continuous-time model, verify that the candidate prices are dynamically complete, and are in fact equilibrium prices.

Anderson [2] provided a construction for Brownian Motion and Brownian stochastic integration using Loeb measure–a measure in the usual standard sense produced by a nonstandard construction. Anderson's Brownian Motion is a hyperfinite random walk which can simultaneously be viewed as being a standard Brownian Motion in the usual sense of probability theory. While the standard stochastic integral is motivated by the idea of a Stieltjes integral, the actual standard definition of the stochastic integral is of necessity rather indirect because almost every path of Brownian Motion is of unbounded variation, and Stieltjes integrals are only defined with respect to paths of bounded variation. However, a hyperfinite random walk is of hyperfinite variation, and hence a Stieltjes integral with respect to it makes perfect sense. Anderson showed that the standard stochastic integral can be obtained readily from this hyperfinite Stieltjes integral.

This construction of Brownian Motion has been used to answer a number of questions in stochastic processes, and we are able to make do in this paper with slight extensions of it. However, we anticipate that extending this work to the dynamically incomplete case, or the case of dividends with jumps, will require using subsequent work in nonstandard stochastic analysis.[9]

We modify Anderson's construction of the hyperfinite random walk to a random walk with branching number equal to $K + 1$ and extend the results on stochastic integration to that random walk. We show that equilibrium consumptions are nonzero at all times and states. Consequently, we can use the first order conditions to characterize the equilibrium prices. Then, we use the Loeb measure construction to produce a candidate equilibrium of the original continuous-time model. The Central Limit Theorem then allows us to explicitly describe the candidate equilibrium prices as integrals with respect to a normal distribution; however, with more than one agent, the prices depend on the terminal distributions of wealth, which are not described in closed form. We show that the hyperfinite equilibrium is infinitely close to the candidate equilibrium, which implies that the equilibria of the discrete approximations converge to candidate equilibria of the continuous-time model. Finally, in a process analogous to that first used in Brown and Robinson [16], we show that the candidate equilibrium is an equilibrium of the continuous-time economy.[10]

Nonstandard analysis is a conservative extension of conventional analysis. In other words, the existence of nonstandard models follows from the conventional axioms of set theory and analysis, so the theorems presented here do not depend on any additional set theoretic or analytic axioms. In particular, the proofs presented here can be mechanically translated into standard proofs using only the usual axioms, but the resulting standard proofs would be exceedingly long and unintelligible. We believe it would be extremely difficult to provide tractable standard proofs of the convergence theorems.

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[9]The work of Keisler [57] on stochastic differential equations with respect to Brownian Motion; and the work by Albeverio and Herzberg [1], Hoover and Perkins [51], Lindstrøm [63, 64, 65, 66, 67, 68], and Ng [80] on stochastic integration with respect to more general martingales, including Lévy processes, is likely to be of particular relevance.

The nonstandard theory of stochastic integration has previously been applied to option pricing in Cutland, Kopp and Willinger [24, 25, 26, 27, 28, 29, 30]. Those papers, which are not in an equilibrium context, primarily concern convergence of discrete versions of options to continuous-time versions, and their methods can likely be used to establish convergence results for the equilibrium option pricing formulas developed in Anderson and Raimondo [6]. Cutland, Kopp and Willinger [27, 29] and Cutland [23] propose a convergence notion for stochastic processes, $D^2$-convergence, which is stronger than the topology of weak convergence. The statement of the convergence notion is nonstandard, and to the best of our knowledge no one has identified a standard topology on stochastic processes that corresponds to $D^2$-convergence. The equilibria of our discrete approximations $D^2$-converge to the set of equilibria of the continuous-time economy. Ng [78] provides a treatment of mathematical finance based on hyperfinite random walks, and in particular presents the Cox-Ross-Rubinstein [20] binomial model in a nonstandard context. Khan and Sun [59, 60] relate the Capital Asset Pricing Model and Arbitrage Pricing Theory in a single-period setting. Malliavin calculus provides an approach to continuous-time finance with asymmetric information; nonstandard treatments of Malliavin calculus are given by Cutland and Ng [31] and Osswald [83].

Keisler [58] presents a nonstandard stochastic process model of non-tatonnement price adjustment; agents are randomly chosen to come to the market, and their trades move the market price rapidly towards equilibrium.

[10]The argument is more complicated here because our continuous-time economy is more complicated than the economy in [16].

We think it is likely possible to use the analytic function techniques developed here to prove that, in the settings of the previous literature,[11] the candidate equilibrium prices are dynamically complete. This would allow one to remove the assumption of dynamic completeness of the candidate equilibrium price process from the papers cited, and would rely on the techniques used in those papers and our analytic function techniques, but would not require nonstandard analysis. We have chosen not to proceed in this direction, for the following three reasons: First, our ultimate goal is to extend these methods to obtain existence of equilibria in continuous-time financial markets which are dynamically *incomplete*. The approach in the previous literature begins with an Arrow-Debreu complete markets equilibrium. Since there is no known theorem asserting existence of equilibrium with more than one agent, an infinite-dimensional commodity space, and incomplete markets, there seems no hope of extending the approach in the previous literature to incomplete markets. By contrast, our approach begins with a GEI equilibrium in a discrete setting. Many steps in our argument work just as well for the dynamically incomplete case as for the dynamically complete case. While critical problems remain to be solved, our methods at least provide a way of attacking the dynamically incomplete case. Second, we feel the convergence theorem is at least as important as the existence theorem. As is often the case in nonstandard analysis, our convergence theorem is essentially an immediate corollary of the nonstandard existence proof; however, we doubt that a tractable standard proof of the convergence theorem can be given. Third, while our proof makes extensive use of nonstandard analysis and stochastic processes, it is completely independent of knowledge of functional analysis. Since most who work in continuous-time finance have a background in stochastic processes but not in functional analysis, it is desirable to have a proof that does not depend on functional analysis.

# 2    The Model

In this Section we define the continuous-time model.

Trade and consumption occur over a compact time interval $[0, T]$, endowed with a measure $\nu$ which agrees with Lebesgue measure on $[0, T)$ and such that $\nu(\{T\}) = 1$. Consumption and dividends on $[0, T)$ are flows; consumption at the terminal date $T$ is a lump. We choose this formulation to give a finite-horizon model in which the securities will have positive value at the terminal date $T$; if securities paid only a flow dividend on $[0, T]$, they would expire worthless at time $T$ and it would not be possible to formulate our nondegeneracy condition on their dividends. An alternative would be to take an infinite horizon model, but this would require addressing certain additional technical problems. We think of our finite time horizon $T$ as a truncation of an infinite-horizon model. The lump of consumption at time $T$ aggregates the flow consumption on the interval $(T, \infty)$ in the infinite-horizon model, conditional on the information available at time $T$. Our nondegeneracy assumption will be imposed on the lump dividend at the terminal date $T$.

The uncertainty in the model is described by a standard $K$-dimensional Brownian Motion $\beta$ on a probability space $\Omega$; the components of $\beta$ are independent of each other, and the variance of $\beta_k(t, \cdot)$ equals $t$. We define $\mathcal{I}(t, \omega) = (t, \beta(t, \omega))$. The primitives of the economy—dividends, endowments and utility functions—will be described as functions of $\mathcal{I}(t, \omega)$.

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[11]See the references listed in footnote 2.

We are given a right-continuous filtration $\{\mathcal{F}_t : t \in [0,T]\}$ on $(\Omega, \mathcal{F}, \mu)$ such that $\mathcal{F}_0$ contains all null sets and $\beta$ is adapted to $\{\mathcal{F}_t\}$, i.e. for all $t$, $X(t, \cdot)$ is measurable with respect to $\mathcal{F}_t$.

Let $J = K$. There are $J + 1$ securities, indexed by $j = 0, \ldots, J$; each is in net supply either zero or one. Security $j$ pays dividends (measured in consumption units) at a flow rate $A_j(t, \omega) = g_j(\mathcal{I}(t, \omega))$ at times $t \in [0, T)$, and a lump dividend $A_j(T, \omega) = G_j(\mathcal{I}(T, \omega))$ at time $T$. We assume that $g : [0, T] \times \mathbf{R}^K \to \mathbf{R}_+^K$ is real analytic on $(0, T) \times \mathbf{R}^K$ and that $G_j$ is continuous almost everywhere on $\{T\} \times \mathbf{R}^K$. For example, $A_0$ could be a zero-coupon bond ($g_0(t, \omega) = 0$ for $t \in [0, T)$, $g_0(T, \omega)$ is constant), or $A_j(t, \omega) = e^{\sigma_j \cdot \beta(t, \omega)}$, where $\sigma_j$ is the $j^{th}$ row of a $J \times J$ matrix $\sigma$, for $j = 1, \ldots, J$.

There are $I$ agents $i = 1, \ldots, I$. Agent $i$ has a flow rate of endowment $e_i(t, \omega) = f_i(\mathcal{I}(t, \omega))$ at times $t \in [0, T)$, and a lump endowment $e_i(T, \omega) = F_i(\mathcal{I}(T, \omega))$, where $f_i$ is analytic on $(0, T) \times \mathbf{R}^K$ and $F_i$ and continuous almost everywhere on $\{T\} \times \mathbf{R}^K$. Let $e(t, \omega) = \sum_{i=1}^{I} e_i(t, \omega)$ denote the aggregate endowment.

The utility functions are von-Neumann Morgenstern utility functions, expectations of functions of the consumption and the process $\mathcal{I}$ which are analytic on $(0, T) \times \mathbf{R}^K$. More formally, given a measurable consumption function $c_i : [0, T] \times \Omega \to \mathbf{R}_{++}$, the utility function of the agent is

$$U_i(c) = E_\mu \left[ \int_0^T h_i(c_i(t, \cdot), \mathcal{I}(t, \cdot)) dt + H_i(c_i(T, \cdot), \mathcal{I}(T, \cdot)) \right]$$

where the functions $h_i : \mathbf{R}_{++} \times \left( [0, T) \times \mathbf{R}^K \right) \to \mathbf{R} \cup \{-\infty\}$ and $H_i : \mathbf{R}_{++} \times \left( \{T\} \times \mathbf{R}^K \right) \to \mathbf{R} \cup \{-\infty\}$ are analytic on $\mathbf{R}_{++} \times \left( (0, T) \times \mathbf{R}^K \right)$ and $C^2$ on $\mathbf{R}_{++} \times \left( \{T\} \times \mathbf{R}^K \right)$ respectively and satisfy

$$\begin{array}{llll}
\lim_{c \to 0+} \frac{\partial h_i}{\partial c} & = & \infty & \text{uniformly over } \left( [0, T] \times \mathbf{R}^K \right) \\
\lim_{c \to 0+} \frac{\partial H_i}{\partial c} & = & \infty & \text{uniformly over } \{T\} \times \mathbf{R}^K \\
\lim_{c \to \infty} \frac{\partial h_i}{\partial c} & = & 0 & \text{uniformly over } \left( [0, T] \times \mathbf{R}^K \right) \\
\lim_{c \to \infty} \frac{\partial H_i}{\partial c} & = & 0 & \text{uniformly over } \{T\} \times \mathbf{R}^K \\
\lim_{c \to 0+} h_i(c, (t, x)) & = & h_i(0, (t, x)) & \text{uniformly over } \left( [0, T] \times \mathbf{R}^K \right) \\
\lim_{c \to 0+} H_i(c, (T, x)) & = & H_i(0, (T, x)) & \text{uniformly over } \{T\} \times \mathbf{R}^K \\
\frac{\partial h_i}{\partial c} & > & 0 & \text{on } \mathbf{R}_{++} \times \left( [0, T] \times \mathbf{R}^K \right) \\
\frac{\partial H_i}{\partial c} & > & 0 & \text{on } \mathbf{R}_{++} \times \left( \{T\} \times \mathbf{R}^K \right) \\
\frac{\partial^2 h_i}{\partial c^2} & < & 0 & \text{on } \mathbf{R}_{++} \times \left( [0, T] \times \mathbf{R}^K \right) \\
\frac{\partial^2 H_i}{\partial c^2} & < & 0 & \text{on } \mathbf{R}_{++} \times \left( \{T\} \times \mathbf{R}^K \right)
\end{array}$$

Note that these conditions are satisfied by all state-independent CRRA utility functions. Note also that we allow quite general state-dependence of the utility function, as long as the state-dependence enters through the process $\mathcal{I}$. If the state-dependence were not measurable in the Brownian Motions, there would be no hope of obtaining effective dynamic completeness with securities whose dividends *are* measurable with respect to the Brownian filtration.

The assumption that the endowments and utility functions are analytic does not impose serious economic restrictions; for example, all conventional utility functions are in fact analytic. However, the dividend paid by an option is *not* analytic because of the kink that

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occurs when the stock price just equals the exercise price of the option on the exercise date. Moreover, shares of a limited liability corporation should not generally be thought of as analytic because they are, in effect, options to claim the corporation's stream of earnings, at an exercise price equal to the corporation's debt.[12] We require analyticity at the intermediate times $t \in (0, T)$, and not at $t = T$. Thus, our model allows us to include options and shares of limited liability corporations among our basic securities, provided that the exercise date is the terminal date $T$. In addition, as long as the basic securities pay dividends which are analytic functions of $\mathcal{I}$ over $(0, T)$, there is no problem in using the equilibrium prices derived from the basic securities to price options or other derivatives on those basic securities with exercise date $t \in (0, T)$.[13] The equilibrium price of any security, analytic or not, is equal to the expected value of its future dividends, evaluated at the equilibrium consumption prices. Since we prove that the basic securities are essentially dynamically complete, any option or other derivative on the basic securities can be replicated by an admissible self-financing trading strategy on the basic securities. Thus, the equilibrium price of any security is also given by the equilibrium value of the portfolio specified by the replicating strategy at that node. For more detail on equilibrium pricing of options, see Anderson and Raimondo [6, 7].

Note that we do *not* require $G_j$ to be analytic, or even differentiable. Standard option payoffs are not differentiable at the strike price, and other derivatives need not be continuous. However, because derivatives represent contracts that need to be understood and enforced, they tend to have relatively simple structures, such as piecewise linearity in the underlying securities; they may or may not be continuous at the boundaries between the regions of linearity. Since our security dividends are analytic functions of the Brownian Motion, derivatives should be piecewise analytic functions of the Brownian Motion. The assumption that the dividend at time $T$ is continuous almost everywhere allows for the possibility that security $A_j$ is a derivative, with exercise date $T$, on another security or securities, which may or may not be traded. Our formulation also allows $A_j$ to be a stock in a limited liability corporation, since shares in limited liability corporations are in effect options to buy the earnings flow of the firm at an exercise price equal to the firm's debt.

We assume the endowments and dividends satisfy the following growth conditions:

$$\exists_{r \in \mathbf{R}} \forall_{t \in [0,T], x \in \mathbf{R}^K} \begin{cases} |f_i(t,x)| & \leq & r + e^{r|x|} \\ |F_i(T,x)| & \leq & r + e^{r|x|} \\ |g_j(t,x)| & \leq & r + e^{r|x|} \\ |G_j(T,x)| & \leq & r + e^{r|x|} \\ \left| \frac{\partial f_i(t,x)}{\partial x} \right| & \leq & r + e^{r|x|} \\ \left| \frac{\partial g_j(t,x)}{\partial x} \right| & \leq & r + e^{r|x|} \end{cases} \tag{1}$$

These conditions are needed to show that if $\hat{\beta}_n$ is a sequence of random walks converging to Brownian Motion $\beta$, the distribution of $f_i(t, \hat{\beta}_n(t, \omega))$ converges to that of $f_i(t, \beta(t, \omega)$ and

---

[12]In particular, geometric Brownian Motion is problematic as a model of the price of shares in a limited liability corporation.

[13]We do not consider the payout of an option at times $t < T$ as a dividend. The payout of an option is a lump, which does not lie in the consumption set at time $t$. Rather, if the option is in the money at its expiration date $t < T$, one exercises it by selling securities to raise the money for the exercise price, takes delivery of the underlying security, then rebalances one's portfolio.

satisfies a uniform $L^2$ integrability condition. In fact, the distribution of $f_i(t, \hat{\beta}_n(t, \omega))$ will satisfy a uniform $L^p$ integrability condition for all $p \in [1, \infty)$.

We are grateful to the referees for pointing out Dana's [32] treatment of consumptions that are not uniformly bounded away from zero. We make the following joint assumption on the utility functions and the social consumption: there exists $r \in \mathbf{R}$ such that

$$
\begin{aligned}
\left. \frac{\partial h_i}{\partial c} \right|_{\left( \frac{\sum_{i=1}^I f_i(t,x) + \sum_{j=0}^J \eta_j g_j(t,x)}{I}, t, x \right)} &\leq r + e^{r|x|} \\
\left. \frac{\partial H_i}{\partial c} \right|_{\left( \frac{\sum_{i=1}^I F_i(t,x) + \sum_{j=0}^J \eta_j G_j(T,x)}{I}, T, x \right)} &\leq r + e^{r|x|}
\end{aligned}
\tag{2}
$$

Note that $\sum_{i=1}^I f_i(t, x) + \sum_{j=0}^J \eta_n g_j(t, x)$ is the social consumption at the node $(t, x)$ for $t < T$. Inequality (2) says if we give each of the $I$ agents an equal share of the social consumption, then every agent's marginal utility of consumption is bounded above by a linear exponential of the magnitude of the Brownian Motion. Note that it implies that the social endowment is strictly positive at almost all nodes. The assumption would follow without any additional assumptions on the utility functions if the social consumption were uniformly bounded away from zero. However, the assumption also allows for the social consumption to approach zero, as long as the marginal utility does not grow too quickly. For example, if each agent has state-independent CRRA utility and the social endowment is a geometric Brownian Motion, the condition is satisfied.

In continuous-time models, it is commonly assumed that the zero$^{th}$ security is a money-market account, in other words, it is instantaneously risk-free. Since we are determining securities prices endogenously, the assumption that a security is instantaneously risk-free is an endogenous assumption. For example, if we assume that $A_0$ is a zero-coupon bond (i.e. $G_0(t, \omega) = 0$ for $t < T$ and $G_0(T, \omega) = 1$, so the dividends of $A_0$ are risk-free), the *Arrow-Debreu equilibrium price* of $A_0$ will *not* be instantaneously risk-free except in degenerate situations. However, as long as the equilibrium securities prices are positive Itô Processes (and we shall show that they are), one is free to divide the securities price process and the consumption price process by the equilibrium price of the zero$^{th}$ security. Under this renormalization, relative prices are preserved and the price of the zero$^{th}$ security is identically one, which is obviously instantaneously risk-free. When one does this, the set of admissible self-financing trading strategies is left unchanged; see section 4.8 of Nielsen [81]. Consequently, the consumption processes that lie in the budget set remain invariant, and the renormalized prices are equilibrium prices. This motivates the form of our exogenous nondegeneracy condition on the terminal dividends of the securities. We assume that there is an open set $V \subset \mathbf{R}^K$ such that $G_0(T, x) > 0$ for all $x \in V$ and for $j = 1, \ldots, J$ and $i = 1, \ldots I$,

$$
G_j, F_i \in C^1(V) \text{ and } \exists_{x \in V} \text{ rank} \begin{pmatrix} \left. \frac{\partial(G_1/G_0)}{\partial \beta} \right|_{(T,x)} \\ \vdots \\ \left. \frac{\partial(G_J/G_0)}{\partial \beta} \right|_{(T,x)} \end{pmatrix} = K
\tag{3}
$$

Note that if $A_0$ is a bond, the rank condition is equivalent to assuming that the $K \times K$

matrix

$$\begin{pmatrix} \left.\dfrac{\partial G_1}{\partial \beta}\right|_{(T,x)} \\ \vdots \\ \left.\dfrac{\partial G_J}{\partial \beta}\right|_{(T,x)} \end{pmatrix}$$

is nonsingular. This simply says that there is some possible terminal value of the Brownian Motion so that the dividends of securities $A_1, \ldots, A_J$ are locally linearly independent.

Agent $i$ is initially endowed with deterministic security holdings $e_{iA} = (e_{iA_0}, \ldots, e_{iA_J}) \in \mathbf{R}^{J+1}$ satisfying

$$\sum_{i=1}^{I} e_{iA_j} = \eta_j$$

Note that the initial holdings are independent of the state $\omega$. We require the following condition for each $i = 1, \ldots, I$:

$$\forall_{(t,x) \in [0,T) \times \mathbf{R}^K} \ f_i(t,x) + e_{iA} g(t,x) \geq 0$$
$$\forall_{x \in \mathbf{R}^K} \ F_i(T,x) + e_{iA} G(T,x) \geq 0 \tag{4}$$

with strictly inequality on a set of positive measure (recall that the single time $T$ carries measure one). This condition says that, if an agent never traded the securities and simply consumed his/her endowment and dividends, consumption would be nonnegative at each node and strictly positive on a set of positive measure; it guarantees that each agent has strictly positive income at every candidate equilibrium price process.

In order to define the budget set of an agent, we need to have a way of calculating the capital gain the agent receives from a given trading strategy. In other words, we need to impose conditions on prices and strategies that ensure that the stochastic integral of a trading strategy with respect to a price process is defined. The essential requirements are that the trading strategy at time $t$ not depend on information which has not been revealed by time $t$, and the trading strategy times the variation in the price yields a finite integral. Specifically, a consumption price process is an Itô process $p_C(t, \omega)$. A securities price process is an Itô process $p_A = (p_{A_0}, \ldots, p_{A_J}) : \Omega \times [0, T] \to \mathbf{R}^{J+1}$ such that the associated cumulative gains process

$$\gamma_j(t, \omega) = p_{A_j}(t, \omega) + \int_0^t p_C(s, \omega) A_j(s, \omega) \, ds$$

is a martingale. Securities are priced *cum dividend* at time $T$. Given a securities price process $p_A$, an admissible trading strategy for agent $i$ is a process $z_i$ which is Itô integrable with respect to $\gamma$ (written $z_i \in \mathcal{L}^2(\gamma)$)[14] and such that $\int z_i \, d\gamma$ is a martingale.[15]

---

[14]Formally, $z_i \in \mathcal{L}^2(\gamma)$ if $z_i : [0, T) \times \Omega \to \mathbf{R}^{J+1}$; $z_i(t, \cdot)$ is $\mathcal{F}_t$-measurable for all $t \in [0, T)$; and $z_i$ is measurable on the product $[0, T) \times \Omega$. If the Itô Process $\gamma$ is given by $d\gamma = a \, dt + \sigma \, d\beta$, then since $\gamma$ is a martingale, $a$ must be zero almost surely. Itô integrability with respect to $\gamma$ requires two conditions: that $z_i(\cdot, \omega) \cdot a(\cdot, \omega) \in L^1([0, T])$ almost surely, which is trivially satisfied; and that $z_i(\cdot, \omega) \cdot \sigma(\cdot, \omega) \in L^2([0, T])$ almost surely. A stronger condition is that $z_i \in \mathcal{H}^2(\gamma)$; the definition of $\mathcal{H}^2(\gamma)$ is the same as that of $\mathcal{L}^2(\gamma)$, except that we strengthen the condition that $z_i(\cdot, \omega) \cdot \sigma(\cdot, \omega) \in L^2([0, T])$ almost surely to $z_i \cdot \sigma \in L^2$. We also define $\mathcal{L}^2 = \mathcal{L}^2(\beta)$ and $\mathcal{H}^2 = \mathcal{H}^2(\beta)$. For more details, see Nielsen [81].

[15]The requirement that $\int z_i \, d\gamma$ be a martingale is the standard "admissibility" condition ruling out arbitrage strategies such as the doubling strategy of Harrison and Kreps [46].

Given a securities price process $p_A$ and a consumption price process $p_C$, the budget set for agent $i$ is the set of all consumption plans $c_i$ such that there exists an admissible trading strategy so that $c_i$ and $t_i$ satisfy the budget constraint

$$p_A(t,\omega) \cdot z_i(t,\omega)$$

$$= p_A(0,\omega) \cdot e_{iA}(\omega) + \int_0^t z_i\, d\gamma + \int_0^t p_C(s,\omega)(e_i(s,\omega) - c_i(s,\omega))ds$$
$$\text{for almost all } \omega \text{ and all } t \in [0,T)$$

$$0 = p_A(0,\omega) \cdot e_{iA}(0,\omega) + \int_0^T z_i\, d\gamma + \int_0^T p_C(s,\omega)(e_i(s,\omega) - c_i(s,\omega))ds$$
$$+ p_C(T,\omega)(e_i(T,\omega) - c_i(T,\omega)))$$
$$\text{for almost all } \omega$$

Given a price process $p$, the demand of the agent is a consumption plan and an admissible trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

An equilibrium for the economy is a securities price process $p_A$, a consumption price process $p_C$, a profile of admissible trading strategies $z_1, \ldots, z_I$ and consumption plans $c_1, \ldots, c_I$ which lies in the demand set so that the securities and goods markets clear, i.e. for almost all $\omega$

$$\sum_{i=1}^I z_{iA_j}(t,\omega) = \eta_j \text{ for } j = 0, \ldots, J \text{ and almost all } (t,\omega)$$

$$\sum_{i=1}^I c_i(t,\omega) = \sum_{i=1}^I e_i(t,\omega) + \sum_{j=0}^J \eta_j A_j(t,\omega) \text{ for almost all } (t,\omega)$$

The following theorem is proved in Section 4.

**Theorem 2.1** *The continuous-time finance model just described has an equilibrium, which is Pareto optimal. Let $p_A$, $p_c$, $c_i$, and $z_i$ denote the equilibrium securities prices, consumption prices, consumptions, and trading strategies. The equilibrium prices are effectively dynamically complete: any integrable consumption process which is adapted to the* Brownian *filtration can be replicated by a unique admissible trading strategy. Let $\Sigma = \frac{\partial p_A}{\partial \beta}$ denote the dispersion matrix of the securities prices, $\Sigma_i = z_i\Sigma$ the dispersion matrix of agent $i$'s trading strategy, and $W_i = z_i \cdot p_A$ agent $i$'s securities wealth. Then[16]*

$$p_A(t,\beta) = E\left( p_C(T,\beta(T))A(T,\beta(T)) + \int_t^T p_C(s,\beta(s))A(s,\beta(s))\, ds \Big| \beta(t) = \beta \right) \quad (5)$$

$$\Sigma(t,\beta) = E\left( \frac{p_C(T,\beta(T))A(T,\beta(T))(\beta(T) - \beta)'}{T-t} \right.$$
$$\left. + \int_t^T \frac{p_C(s,\beta))A(s,\beta(s))(\beta(s) - \beta)'}{s-t} \Big| \beta(t) = \beta \right) \quad (6)$$

---

[16]In the following equations, the primes denote transpose.

*There exists a unique vector $\lambda \in \mathbf{R}_+^I$ of Negishi utility weights with $\sum_{i=1}^I \lambda_i = 1$ such that for all $(t, \beta)$, $c_1(t, \beta), \ldots, c_I(t, \beta)$ solve the problem*

$$\max \left\{ \sum_{i=1}^I \lambda_i h_i(c_i) : \sum_{i=1}^I c_i = \sum_{i=1}^I e_i(t, \beta) + \sum_{j=1}^J \eta_j A_j(t, \beta) \right\} \tag{7}$$

*for $t \in [0, T)$ and the analogous problem with $H_i$ substituted for $h_i$ for $t = T$;*

*Each $\Sigma_i$ and $W_i$ has a continuous version (still denoted $\Sigma_i$ and $W_i$)*

$$\Sigma_i(t, \beta) = E \left( \frac{p_C(T, \beta(T))\,(c_i(T, \beta(T)) - e_i(T, \beta(T)))\,(\beta(T) - \beta)'}{T - t} \right. \tag{8}$$

$$\left. + \int_t^T \frac{p_C(s, \beta(s))\,(c_i(s, \beta(s)) - e_i(s, \beta(s)))\,(\beta(s) - \beta)'}{s - t}\,ds \,\middle|\, (\beta(t) = \beta) \right)$$

$$W_i(t, \beta) = E\left( p_C(T, \beta(T))\,(c_i(T, \beta(T)) - e_i(T, \beta(T))) \right. \tag{9}$$

$$\left. + \int_{s=t}^T p_C(s, \beta(s))\,(c_i(s, \beta(s)) - e_i(s, \beta(s)))\,ds \,\middle|\, \beta(t) = \beta \right)$$

*There is a open set $B$ of full measure in $(0, T) \times \mathbf{R}^K$ and analytic functions $Z_1, \ldots, Z_I$ with domain $B$ such that $z_i(t, \omega) = Z_i(\mathcal{I}(t, \omega))$ whenever $\mathcal{I}(t, \omega) \in B$; $Z_i$ is uniquely determined on $B$ by the linear equations*

$$\Sigma_i = Z_i \Sigma, \quad W_i = Z_i \cdot p_A$$

*$p_A, p_C, c_1, \ldots, c_I, W_1, \ldots, W_I \in L^2$; $\Sigma, \Sigma_1, \ldots, \Sigma_I, \in \mathcal{H}^2$; $z_1, \ldots, z_I \in \mathcal{H}^2(\gamma)$; $p_A, p_C, W_1, \ldots, W_I$ are functions of $\mathcal{I}(t, \omega)$ which are continuous on $[0, T] \times \mathbf{R}^K$ and analytic on $(0, T) \times \mathbf{R}^K$; $\Sigma, \Sigma_1, \ldots, \Sigma_I$ are functions of $\mathcal{I}(t, \omega)$ which are continuous on $[0, T] \times V$ and analytic on $(0, T) \times \mathbf{R}^K$; $p_c$ and $c_i$ are given separately over $[0, T) \times \mathbf{R}^k$ and $\{T\} \times \mathbf{R}^K$ as functions of $\mathcal{I}(t, \omega)$; the functions over $[0, T) \times \mathbf{R}^K$ are analytic on $(0, T) \times \mathbf{R}^K$.*

**Remark 2.2** Equation (6) says that the $(j, k)$ entry of the dispersion matrix of the securities prices is the integral, over future times $s$ (including $s = T$), of the regression coefficient of dividends of the $j^{th}$ security at time $s$, valued at the equilibrium consumption prices, on the change in the $k^{th}$ component of the Brownian Motion up to time $s$. Equation (8) says that the $k^{th}$ element of $\Sigma_i$ is the integral, over future times $s$ (including $s = T$), of the regression coefficient of $i$'s consumption minus $i$'s endowment at time $s$, valued at the equilibrium consumption prices, on the change in the $k^{th}$ component of the Brownian Motion up to time $s$.

# 3    Discrete Approximations of the Continuous-Time Model

In this section, we describe our process for discretizing a continuous-time model. We then state a theorem indicating that the equilibria of the discretized economy are close to those of the continuous-time economy. This result has two important consequences. First, it provides an effective computational method for computing equilibria of the continuous-time economy. All one has to do is compute, using standard algorithms, an equilibrium of a sufficiently fine

discretization. Second, actual securities markets are discrete in a number of important ways. Prices are restricted to lie in a grid, trades are carried out in integral numbers of shares, and trades take a certain amount of time to execute. Continuous-time models are useful because pricing formulas can be expressed more cleanly in continuous time than in discrete time. However, in order to know that the formulas obtained from continuous-time models are applicable to real markets, we need to know that the behavior of large discrete models is close to the behavior of large continuous-time models.

Here is the formal description of our discretization procedure:

Choose $n \in \mathbf{N}$. For $t \in [0, T]$, define $\hat{t} = \frac{\lfloor nt \rfloor}{n}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$; in particular, $\hat{T} = \frac{\lfloor nT \rfloor}{n}$. Define $\Delta T = \frac{1}{n}$ and $\mathcal{T} = \{0, \Delta T, 2\Delta T, \ldots, \hat{T}\}$. Define a measure $\hat{\nu}$ on $\mathcal{T}$ by $\hat{\nu}(\{t\}) = \Delta T$ if $t < \hat{T}$ and $\hat{\nu}(\{\hat{T}\}) = 1$.

**Lemma 3.1** *We can choose $K + 1$ vectors $v_0, \ldots, v_K \in \mathbf{R}^K$ such that*

$$v_j \cdot v_k = \begin{cases} K & if \ j = k \\ -1 & if \ j \neq k \end{cases}$$

$$\sum_{k=0}^{K} (v_k)_i \, (v_k)_j = \delta_{ij}(K+1)$$

$$\sum_{k=0}^{K} v_k = 0$$

**Proof:** See Appendix C. ∎

Now, we describe how we construct our probability space $\hat{\Omega}$, filtration $\{\mathcal{F}_t\}$, and random walk $\hat{\beta}$ with the property that every non-terminal node in $\hat{\beta}$ has exactly $K + 1$ successor nodes. Let $\hat{\Omega} = \{\omega : \mathcal{T} \setminus \{0\} \to \{0, 1, 2, \ldots, K\}\}$. The measure $\hat{\mu}$ on $\hat{\Omega}$ is given by $\hat{\mu}(A) = \frac{|A|}{|\hat{\Omega}|}$ for every $A \in \hat{\mathcal{F}}$, the algebra of all subsets of $\hat{\Omega}$; here, $|A|$ denotes the cardinality of $A$. For $t \in \mathcal{T}$, $\hat{\mathcal{F}}_t$ is the algebra of all subsets of $\hat{\Omega}$ that respect the equivalence relation $\omega \sim_t \omega' \Leftrightarrow \omega(s) = \omega'(s)$ for all $s \leq t$.

For $s \in \mathcal{T}$, define the random variable $v_s(\omega) = v_{\omega(s)}$, where $v_0, \ldots, v_k$ are the vectors chosen in Lemma 3.1. If $s \neq t$, the random variables $v_s$ and $v_t$ are independent. Moreover, for each $\ell \in \{1, \ldots, K\}$, the random variable $(v_s)_\ell$ has mean zero and standard deviation one. Define $\hat{\beta} : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}^K$ by

$$\hat{\beta}(t, \omega) = \sum_{0 < s \leq t, \, s \in \mathcal{T}} v_s(\omega)\sqrt{\Delta T}$$

$\hat{\beta}$ is a $K$-dimensional random walk. Note that $\hat{\beta}(t, \cdot)$ has variance-covariance matrix $tI$, where $I$ is the $K \times K$ identity matrix. Define

$$\hat{\mathcal{I}}(t, \omega) = (t, \hat{\beta}(t, \omega))$$

Given a consumption plan $\hat{c} : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}_+$, the agent's utility is

$$\hat{U}_i(\hat{c}) = E_{\hat{\mu}} \left( \left( \Delta T \sum_{s \in \mathcal{T}, s < \hat{T}} h_i(\hat{c}(t, \omega), \hat{\mathcal{I}}(t, \omega)) \right) + H_i(\hat{c}(\hat{T}, \omega), \hat{\beta}(\hat{T}, \omega)) \right)$$

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We use the Duffie-Shafer [41, 42] or Magill-Shafer Theorems [72] to perturb the endowments and security dividends to ensure the existence of a Pareto Optimal equilibrium. Let $\hat{e}_i(t,\omega) \geq f_i(\hat{\mathcal{I}}(t,\omega))$, $\left|\hat{e}_i(t,\omega) - f_i(\hat{\mathcal{I}}(t,\omega))\right| \leq (\Delta T)^2$ $(t < \hat{T})$ and $\hat{e}_i(\hat{T},\omega) \geq F_i(\hat{T},\omega)$, $\left|\hat{e}_i(\hat{T},\omega) - F_i(\hat{\mathcal{I}}(\hat{T},\omega))\right| \leq (\Delta T)^2$ denote the perturbed endowments. For all $\omega \in \hat{\Omega}$, let $\hat{A}(t,\omega)$ denote the perturbed dividends $\hat{A}(t,\omega) \geq g(\hat{\mathcal{I}}(t,\omega))$, $\left|\hat{A}(t,\omega) - g(\hat{\mathcal{I}}(t,\omega))\right| \leq (\Delta T)^2$ for all $t < \hat{T}$, and $\hat{A}(\hat{T},\omega) \geq G(\hat{\mathcal{I}}(\hat{T},\omega))$, $\left|\hat{A}(\hat{T},\omega) - G(\hat{\mathcal{I}}(\hat{T},\omega))\right| \leq (\Delta T)^2$. Notice that the social consumption $\hat{e}(t,\omega) + \sum_{j=0}^J \eta_j \hat{A}_j(t,\omega) > 0$.

Recall there are $J + 1$ securities indexed by $j = 0, \ldots, J$. A securities price process is a function $\hat{p}_A : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}^{J+1}$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t \in \hat{\mathcal{T}}}$. We will price securities *ex dividend* for $t < \hat{T}$; it will be convenient to price secuities *cum dividend* at $t = \hat{T}$.

A consumption price process is a function $\hat{p}_C : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}_+$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t \in \hat{\mathcal{T}}}$.

Given a securities price process $\hat{p}_A$ and a consumption price process $\hat{p}_C$, the associated total gains process $\hat{\gamma}$ for the securities is defined to be

$$\hat{\gamma}(t,\omega) = \begin{cases} \hat{p}_A(t,\omega) + (\Delta T) \sum_{s \in \mathcal{T}, s \leq t} \hat{p}_C(s,\omega)\hat{A}(s,\omega) & \text{if } t < \hat{T} \\ \hat{p}_A(t,\omega) + (\Delta T) \sum_{s \in \mathcal{T}, s < t} \hat{p}_C(s,\omega)\hat{A}(s,\omega) & \text{if } t = \hat{T} \end{cases}$$

Note that since securities are priced *ex dividend* at times $t < \hat{T}$, the dividend at time $t$ is included in the sum and not in the securities price $\hat{p}_A$; since securities are priced *cum dividend* at time $\hat{T}$, the dividend at time $\hat{T}$ is included in the securities price $\hat{p}_A$ and not in the sum.

A trading strategy[17] for agent $i$ is a function $\hat{z}_i : (\mathcal{T} \cup \{-\Delta T\}) \times \hat{\Omega} \to \mathbf{R}^{J+1}$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t \in \mathcal{T}}$ such that $\hat{z}_i(-\Delta T, \omega) = e_{iA}$ (this ensures that each agent arrives at time 0 holding the correct initial securities endowment; there is no trade in goods or securities at time $-\Delta T$) and $\hat{z}_i(\hat{T}, \omega) = \hat{z}_i(\hat{T} - \Delta T, \omega)$.

A consumption plan for agent $i$ is a function $\hat{c}_i : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}_+$. The budget set for agent $i$ is the set of all consumption plans $\hat{c}_i$ such that there exists a trading strategy $\hat{z}_i$ for which $\hat{c}_i$ satisfies the budget constraint

$$\left(\hat{c}_i(s,\omega) - \hat{e}_i(s,\omega) - \hat{z}_i(s - \Delta T, \omega) \cdot \hat{A}(s,\omega)\right) \cdot \hat{p}_C(s,\omega)$$
$$= (\hat{z}_i(s,\omega) - \hat{z}_i(s - \Delta T, \omega)) \cdot \hat{p}_A(s,\omega)$$

for all $s \in \mathcal{T}$ and all $\omega \in \hat{\Omega}$. The budget equation says that the value of the agent's consumption at every node equals the value of the agent's endowment at that node plus the value of the dividends generated by the agent's portfolio at that node plus the value of the securities sold by the agent at that node.[18] Note that since $\hat{z}$ is required to be adapted to $\{\hat{\mathcal{F}}_t\}_{t \in \hat{\mathcal{T}}}$, it follows that $\hat{c}_i$ is adapted to $\{\hat{\mathcal{F}}_t\}_{t \in \hat{\mathcal{T}}}$.

Given a security price $\hat{p}$ and a consumption price $\hat{p}_C$, the demand of the agent is a consumption plan and a trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

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[17]In the discrete model, stochastic integrals with zero drift are automatically martingales, so we do not need to require admissibility as a separate assumption.

[18]We take $\hat{z}_i(-\Delta T, \omega) = e_{iA}(0)$, so that agent enters period 0 holding the securities with which s/he is endowed. Since securities are priced *cum dividend* at $t = \hat{T}$, we require that $\hat{z}_i(\hat{T}, \omega) = \hat{z}_i(\hat{T} - \Delta T, \omega)$.

An equilibrium for the economy is a security price process $\hat{p}$, a consumption price process $\hat{p}_C$, trading strategies $\hat{z}_i$ and consumption plans $\hat{c}_i$ which lies in the demand sets of the agents so that the securities and goods markets clear, i.e. for all $t \in \mathcal{T}$ and all $\omega \in \hat{\Omega}$

$$\sum_{i=1}^{I} \hat{z}_i(t, \omega) = (\eta_0, \ldots, \eta_J)$$

$$\sum_{i=1}^{I} \hat{c}_i(t, \omega) = \sum_{i=1}^{I} \hat{e}_i(t, \omega) + \sum_{j=0}^{J} \eta_j \hat{A}_j(\omega, t)$$

Since short selling is not restricted, the first order conditions imply that the total gains process $\hat{\gamma}$ is a vector martingale (pages 230-231 of Magill and Quinzii [71]).

Now, we identify the equilibrium trading strategies and the discrete analogue $\hat{\sigma}$ of the dispersion matrix of the equilibrium securities price process and the dispersion matrix $\hat{\sigma}_i$ of agent $i$'s securities wealth. We have the equilibrium securities prices $\hat{p}_A(t, \omega)$ and at the $K+1$ successor nodes to $(t, \omega)$. Because the $K+1$ possible values of $\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)$ span $\mathbf{R}^K$, and $\hat{p}_A$ and $\hat{\beta}$ are both martingales, we will see that there is a unique $(K+1) \times K$ matrix process $\hat{\sigma}(t, \omega)$ adapted to $\hat{\mathcal{F}}_t$, satisfying

$$\hat{p}_A(t + \Delta T, \omega) + \hat{p}_C(t + \Delta T, \omega)A(t + \Delta T, \omega) - \hat{p}_A(t, \omega) = \hat{\sigma}(t, \omega)\left(\hat{\beta}(t + \Delta T), \omega) - \hat{\beta}(t, \omega)\right)$$

Because the equilibrium is dynamically complete, $\hat{\sigma}(t, \omega)$ has rank $K$ for all nodes $(t, \omega)$.

For each consumer $i = 1, \ldots, I$, define

$$\hat{D}_i(t, \omega)$$
$$= E\left(\left(\left(\hat{p}_C(\hat{T}, \cdot)\left(\hat{c}_i(\hat{T}, \cdot) - \hat{e}_i(\hat{T}, \cdot)\right)\right) + \Delta T \sum_{s=0}^{\hat{T} - \Delta T} \hat{p}_C(s, \cdot)\left(\hat{c}_i(s, \cdot) - \hat{e}_i(s, \cdot)\right)\right)\middle|(t, \omega)\right)$$

From the definition, $\hat{D}_i$ is a martingale. $\hat{D}_i$ is the conditional expectation of the difference in value, at the prices of consumption, between $i$'s consumption and $i$'s endowment. For each $i$, we will see that there is a unique $(K+1) \times K$ matrix process $\hat{\sigma}_i(t, \omega)$ adapted to $\hat{\mathcal{F}}_t$ such that

$$\hat{D}_i(t + \Delta T, \omega) - D_i(t, \omega) = \hat{\sigma}_i(t, \omega)\left(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)\right)$$

Since the securities wealth of agent $i$ must exactly equal the conditional expectation of the difference in value between $i$'s future consumption and $i$'s endowment, $\hat{z}_i(t, \omega)$ must be the unique $K \times 1$ row vector process adapted to $\hat{\mathcal{F}}_t$ which satisfies the budget constraint and the equation

$$\hat{\sigma}_i(t, \omega) = \hat{z}_i(t, \omega)\hat{\sigma}(t, \omega)$$

Thus, $\hat{z}_i$ is determined uniquely from $\hat{p}_A$ and $\hat{c}_i$ by linear algebra.[19]

In the following theorem, whose proof is given in Section 4, we show that equilibria of the discretized economies are close to equilibria of the continuous-time economy. If the continuous-time economy has a unique equilibrium, then $p_{nA}$, $p_{nC}$, $c_{ni}$ and $\lambda_n$ are given by

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[19]More formally, recall that $v_k$ is a $K \times 1$ column vector, so $v_k'$ is a $1 \times K$ row vector; $\hat{\gamma}$ is a $(K+1) \times 1$ column vector, and $\hat{D}_i$ is a scalar. Let $M$ be the $K \times K$ matrix, $\Gamma$ the $(K+1) \times K$ matrix, and $\Gamma_i (i = 1, \ldots, I)$

the unique equilibrium, and thus are independent of $n$. If the continuous-time economy has multiple equilibria, we could, for example, have the discrete-time equilibria close to one continuous-time equilibrium for $n$ odd and close to a different continuous-time equilibrium for $n$ even, so our continuous-time equilibrium will in general have to be chosen differently for different discretizations.

**Theorem 3.2** *Given a continuous-time economy satisfying the assumptions of Theorem 2.1, let $\hat{p}_{nA}$, $\hat{p}_{nC}$, $\hat{c}_{ni}$ and $\hat{z}_{ni}$ denote any equilibrium securities prices, consumption prices, consumptions and trading strategies of the discretized sequence of economies just described. Let $\hat{\lambda}_n$ denote the vector of Negishi utility weights maximized at that equilibrium. Then there are equilibria of the continuous-time economy with securities prices $p_{nA}$, consumption prices $p_{nC}$, consumptions $c_{ni}$, trading strategies $z_{ni}$ and utility weights $\lambda_n$ satisfying the conclusions of Theorem 2.1 such that*

$$\left|\hat{\lambda}_n - \lambda_n\right| \to 0 \tag{11}$$

$$\left\|\hat{p}_{nA} - p_{nA} \circ \hat{\mathcal{I}}\right\|_2 \to 0 \ \text{ and } \ \max_{t \in \mathcal{T}_n}\left|\hat{p}_{nA}(t, \cdot) - p_{nA}(\hat{\mathcal{I}}(t, \cdot))\right| \to 0 \ \text{ in probability} \tag{12}$$

$$\left\|\hat{p}_{nc} - p_{nc} \circ \hat{\mathcal{I}}\right\|_2 \to 0 \ \text{ and } \ \max_{t \in \mathcal{T}_n}\left|\hat{p}_{nc}(t, \cdot) - p_{nc}(\hat{\mathcal{I}}(t, \cdot))\right| \to 0 \ \text{ in probability} \tag{13}$$

the $1 \times K$ row vectors

$$M = \sqrt{\Delta T}\begin{pmatrix} v_0' \\ \vdots \\ v_{K-1}' \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} (\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega)) & \cdots & (\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega)) \\ \text{when } \omega(t + \Delta T) = 0 & \cdots & \text{when } \omega(t + \Delta T) = K - 1 \end{pmatrix}$$

$$\Gamma_i = \begin{pmatrix} \left(\hat{D}_i(t + \Delta T, \omega) - \hat{D}_i(t, \omega)\right) & \cdots & \left(\hat{D}_i(t + \Delta T, \omega) - \hat{D}_i(t, \omega)\right) \\ \text{when } \omega(t + \Delta T) = 0 & \cdots & \text{when } \omega(t + \Delta T) = K - 1 \end{pmatrix}$$

Thus, the $k^{th}$ row ($k = 0, \ldots, K - 1$) of $M$ is the value of $\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)$ when $\omega(t + \Delta T) = k$, the $k^{th}$ column ($k = 0, \ldots, K - 1$) of $\Gamma$ is the value of $\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega)$ when $\omega(t + \Delta T) = k$, and the $k^{th}$ entry ($k = 0, \ldots, K - 1$) of $\Gamma_i$ is the value of $\hat{D}_i(t + \Delta T, \omega) - \hat{D}_i(t, \omega)$ when $\omega(t + \Delta T) = k$; note that $\Gamma_i$ is neither a row nor a column of $\Gamma$. Since $v_0, \ldots, v_{K-1}$ are linearly independent, $M$ is nonsingular. Let

$$\hat{\sigma}(t, \omega) = \Gamma M^{-1}$$
$$\hat{\sigma}_i(t, \omega) = \Gamma_i M^{-1}$$

$\hat{\sigma}(t, \omega)$ is the unique $(K + 1) \times K$ matrix and $\hat{\sigma}_1(t, \omega), \ldots, \hat{\sigma}_I(t, \omega)$ the unique $1 \times K$ row vectors such that

$$\begin{aligned} \hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega) &= \hat{\sigma}(t, \omega)\left(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)\right) \\ \hat{D}_1(t + \Delta T, \omega) - \hat{D}_1(t, \omega) &= \hat{\sigma}_1(t, \omega)\left(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)\right) \\ &\vdots \\ \hat{D}_I(t + \Delta T, \omega) - \hat{D}_I(t, \omega) &= \hat{\sigma}_I(t, \omega)\left(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)\right) \end{aligned} \tag{10}$$

whenever $\omega(t + \Delta T) \in \{0, 1, \ldots, K - 1\}$. Since $\hat{\beta}$, $\hat{\gamma}$, and $\hat{D}_i, \ldots, \hat{D}_I$ are martingales, the sum (over the $K + 1$ successor nodes) of the change in $\hat{\beta}$, the change in $\hat{\gamma}$, and the change in $\hat{D}_i$ is zero, so the fact that Equation (10) holds for $\omega(t + \Delta T) \in \{0, 1, \ldots, K - 1\}$ implies it holds also for $\omega(t + \Delta T) = K$. Thus, the processes $\hat{\sigma}$ and $\hat{\sigma}_1, \ldots \hat{\sigma}_I$ are adapted.

$$\left\| \hat{\sigma}_n - \Sigma_n \circ \hat{\mathcal{I}} \right\|_2 \to 0 \ \text{and} \ \max_{t \in \mathcal{T}_n} \left| \hat{\sigma}_n(t,\cdot) - \Sigma_n(\hat{\mathcal{I}}(t,\cdot)) \right| \to 0 \ \text{in probability} \tag{14}$$

$$\left\| \hat{c}_{ni} - c_{ni} \circ \hat{\mathcal{I}} \right\|_2 \to 0 \ \text{and} \ \max_{t \in \mathcal{T}_n} \left| \hat{c}_{ni}(t,\cdot) - c_{ni}(\hat{\mathcal{I}}(t,\cdot)) \right| \to 0 \ \text{in probability} \tag{15}$$

$$\left| \hat{U}_{ni}\left(\hat{c}_{ni}\right) - U_{ni}\left(c_{ni}\right) \right| \to 0 \tag{16}$$

$$\left| \hat{z}_{ni} - z_{ni} \circ \hat{\mathcal{I}} \right| \to 0 \ \text{in probability} \tag{17}$$

$$\left\| \hat{\sigma}_{ni} - \Sigma_{ni} \circ \hat{\mathcal{I}} \right\|_2 \to 0 \ \text{and} \ \max_{t \in \mathcal{T}_n} \left| \hat{\sigma}_{ni}(t,\cdot) - \Sigma_{ni}(\hat{\mathcal{I}}(t,\cdot)) \right| \to 0 \ \text{in probability} \tag{18}$$

$$\left\| \hat{z}_{ni} \cdot \hat{p}_{An} - W_{ni} \circ \hat{\mathcal{I}} \right\|_2 \to 0 \ \text{and} \ \max_{t \in \mathcal{T}_n} \left| \hat{z}_{ni}(t,\cdot) \cdot \hat{p}_{An} - W_{ni}(\hat{\mathcal{I}}(t,\cdot)) \right| \to 0 \ \text{in probability} \tag{19}$$

# 4    Proofs of Theorems 2.1 and 3.2

Up to now, all of our definitions and results have been stated without any reference to nonstandard analysis. Our proof makes extensive use of nonstandard analysis, in particular Anderson's construction of Brownian Motion and the Itô Integral ([2]). It is beyond the scope of this paper to develop these methods; see Appendix B for material on nonstandard stochastic calculuus, and Anderson [4] and Hurd and Loeb [52] for references to nonstandard analysis.

We construct our probability space, filtration and Brownian Motion following Anderson's construction [2]. Choose $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. Using this hyperfinite $n$, define $\mathcal{T}$, $\hat{t}$, $\hat{T}$, $\hat{\nu}$, $\hat{\mu}$, $\hat{\Omega}$, $\hat{\beta}$, $\hat{\mathcal{F}}$, $\hat{\mathcal{F}}_t$, $\hat{U}_i$ and so on exactly as they were defined in Section 3; note that this involves perturbing endowments and dividends to ensure existence of a Pareto Optimal Equilibrium, and that the measures $\hat{\nu}$ and $\hat{\mu}$ are defined by internal cardinalities. By the Transfer Principle, the economy has an equilibrium, and the equilibrium is Pareto optimal.

Let $(\mathcal{T}, L(\hat{\nu}))$ denote the complete Loeb measure generated by $\hat{\nu}$ on $\mathcal{T}$; note that the Loeb measure is a standard countably additive measure, on the same underlying point set $\mathcal{T}$ as $\nu$. For $B \subset [0,T]$, let $\mathrm{st}^{-1}(B) = \{t \in \mathcal{T} : {}^\circ t \in B\}$. For any Lebesgue measurable set $B \subset [0,T]$, $\mathrm{st}^{-1}(B)$ is Loeb measurable and $L(\hat{\nu})\,(\mathrm{st}^{-1}(B)) = \nu(B)$ (Anderson [2]).

Let $(\Omega, \mathcal{F}, L(\hat{\mu}))$ be the (complete) Loeb measure generated by $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$ (Loeb [69]). Although $(\Omega, \mathcal{F}, L(\hat{\mu}))$ is generated by a nonstandard construction, Loeb showed that it is a probability space in the usual standard sense. Let

$$\mathcal{F}_t = \{B \in \mathcal{F} : L(\hat{\mu})(B \Delta C) = 0 \text{ for some } C \text{ which respects the}$$
$$\text{equivalence relation } \omega \sim \omega' \Leftrightarrow \omega \sim_s \omega' \text{ for all } s \simeq t\}$$

Let $\beta : [0,T] \times \Omega \to \mathbf{R}^K$ be defined by $\beta(t,\omega) = {}^\circ\left(\hat{\beta}(\hat{t},\omega)\right)$. $\beta$ is a $K$-dimensional Brownian Motion in the usual standard sense, and $\beta(t,\cdot) = E(\beta(T,\cdot)|\mathcal{F}_t)$.[20] Let $\mathcal{I} : [0,T] \times \Omega \to \mathbf{R}^{K+1}$ be defined by

$$\mathcal{I}(t,\omega) = (t, \beta(t,\omega))$$

---

[20]Anderson [2] proved that $\beta$ is a standard Brownian Motion provided that $\hat{\beta}$ moves up or down $\frac{1}{\sqrt{n}}$ independently in each coordinate at every node; this would require each node in the tree for $\hat{\beta}$ to have $2^K$ successor nodes, precluding dynamic completeness in the hyperfinite model for $K > 1$. Neither Anderson nor Keisler [57] quite covers the random walk $\hat{\beta}$ considered here because the coordinates of $\hat{\beta}$ are uncorrelated but not independent. However, the coordinates of $\beta$ *are* independent. To see this, fix $x \in \mathbf{R}^K$. Then

Since the equilibrium is Pareto Optimal, the marginal utility of consumption is infinite at zero, the aggregate consumption is strictly positive at each node, and all agents have strictly positive income by Equation (4), the equilibrium consumptions of all agents are strictly positive at each node; they may be infinitesimal. Let $\Delta$ be the open $I-1$-dimensional simplex in $\mathbf{R}_{++}^I$. Pareto optimality implies that there exists $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_I) \in {}^*\Delta$ such that at each node $(t, \omega)$, the consumptions maximize

$$\sum_{i=1}^{I} \hat{\lambda}_i {}^* h_i(\hat{c}_i(t, \omega), \hat{\mathcal{I}}(t, \omega)) \text{ and } \sum_{i=1}^{I} \hat{\lambda}_i {}^* H_i(\hat{c}_i(\hat{T}, \omega), \hat{\mathcal{I}}(\hat{T}, \omega))$$

Since each $\hat{c}_i(t, \omega) > 0$, each $\hat{\lambda}_i > 0$, so there is a positive constant $\hat{\iota}(t, \omega)$ such that

$$\hat{\lambda}_1 {}^* \frac{\partial h_1}{\partial c_1}\Big|_{(\hat{c}_1(t,\omega), \hat{\mathcal{I}}(t,\omega))} = \cdots = \hat{\lambda}_I {}^* \frac{\partial h_I}{\partial c_I}\Big|_{(\hat{c}_I(t,\omega), \hat{\mathcal{I}}(t,\omega))} = \hat{\iota}(t, \omega) \qquad \text{for } t < \hat{T}$$
$$\hat{\lambda}_1 {}^* \frac{\partial H_1}{\partial c_1}\Big|_{(\hat{c}_1(\hat{T},\omega), \hat{\mathcal{I}}(\hat{T},\omega))} = \cdots = \hat{\lambda}_I {}^* \frac{\partial H_I}{\partial c_I}\Big|_{(\hat{c}_I(\hat{T},\omega), \hat{\mathcal{I}}(\hat{T},\omega))} = \hat{\iota}(\hat{T}, \omega)$$

Let $\hat{c}(t, \omega) = \sum_{i=1}^{I} \hat{c}_i(t, \omega)$.

We claim that $\hat{\iota}(\cdot, \cdot) \in SL^p$ for each $p \in [1, \infty)$, and that $\hat{\iota}(t, \omega) \not\simeq 0$ for Loeb-almost all $(t, \omega)$. For every node $(t, \omega)$ there is an agent $i(t, \omega)$ such that $\hat{c}_{i(t,\omega)}(t, \omega) \geq \frac{\hat{c}(t,\omega)}{I}$. The following calculation assumes $t < \hat{T}$; if $t = \hat{T}$, substitute $H_i$ for $h_i$. Since $\hat{\lambda}_i \leq 1$ for each $i$:

$$
\begin{aligned}
\hat{\iota}(t, \omega) &= \hat{\lambda}_{i(t,\omega)} {}^* \frac{\partial h_{i(t,\omega)}}{\partial c_{i(t,\omega)}}\Big|_{\left(\hat{c}_{i(t,\omega)}(t,\omega), \hat{\mathcal{I}}(t,\omega)\right)} \\
&\leq {}^* \frac{\partial h_{i(t,\omega)}}{\partial c_{i(t,\omega)}}\Big|_{\left(\frac{\hat{c}(t,\omega)}{I}, \hat{\mathcal{I}}(t,\omega)\right)} \\
&\leq {}^* \frac{\partial h_{i(t,\omega)}}{\partial c_{i(t,\omega)}}\Big|_{\left(\frac{\sum_{i=1}^{I} {}^* f_i(\hat{\mathcal{I}}(t,\omega)) + \sum_{j=0}^{J} \eta_j {}^* g_j(\hat{\mathcal{I}}(t,\omega))}{I}, \hat{\mathcal{I}}(t,\omega)\right)} \\
&\leq r + e^{r|\hat{\beta}(t,\omega)|}
\end{aligned}
$$

by Inequality (2). Therefore $(\hat{\iota}(t, \omega))^p \leq \bar{r} + e^{\bar{r}|\hat{\beta}(t,\omega)|}$ for some $\bar{r} \in \mathbf{R}$, so $(\hat{\iota})^p \in SL^1$ by Proposition 3.1 of Raimondo [86], so $\hat{\iota} \in SL^p$. Since $\sum_{i=1}^{I} \hat{\lambda}_i = 1$, we can assume without loss of generality that $\lambda_1 \not\simeq 0$. Then

$$\hat{\iota}(t, \omega) = \hat{\lambda}_1 {}^* \frac{\partial h_i}{\partial c_i}\Big|_{(\hat{c}_i(t,\omega), \hat{\mathcal{I}}(t,\omega))}$$

---

$\{x \cdot v_s : s \in \mathcal{T}\}$ is a family of IID random variables with standard distribution, mean zero and finite variance $\sigma_x$ so Anderson's Theorem 21 implies that $x \cdot \beta(t, \cdot)$ is Normal mean zero variance $t\sigma_x$. Since $x \cdot \beta(t, \cdot)$ is Normal for all $x \in \mathbf{R}^K$, it is well known that $\beta(t, \cdot)$ is system Normal (see, for example, Bryc [17], Theorem 2.2.4). Since $\beta(t, \cdot)$ is system Normal with variance-covariance matrix $tI$, where $I$ is the $K \times K$ identity matrix, the components of $\beta$ are independent. Each component of the random walk $\hat{\beta}$ is a hypermartingale (martingale with respect to the hyperfinite filtration), so satisfies the $S$-continuity property by Keisler's continuity theorem for hypermartingales, and it follows from Anderson's proof that $\beta$ is almost surely continuous. Anderson's proof that $\beta$ has independent increments goes through without change in the present setting. Anderson used a slightly smaller filtration than the one considered here, while Keisler used the filtration considered here.

$$\geq \quad \hat{\lambda}_1 * \left.\frac{\partial h_i}{\partial c_i}\right|_{(\hat{c}(t,\omega),\hat{\mathcal{I}}(t,\omega))}$$

$$\geq \quad \hat{\lambda}_1 * \left.\frac{\partial h_i}{\partial c_i}\right|_{\left(\sum_{i=1}^{I} {}^* f_i(\hat{\mathcal{I}}(t,\omega)) + \sum_{j=0}^{J} \eta_j {}^* g_j(\mathcal{I}(t,\omega) + O((\Delta T)^2)\right)}$$

$$\not\simeq 0$$

whenever the social consumption is finite, i.e. for Loeb-almost every $(t,\omega)$. If we set $\hat{p}_C(t,\omega) = \hat{\iota}(t,\omega)$ for $t \in \mathcal{T}$, $\hat{p}_C$ gives the Arrow-Debreu prices of consumption.

We claim that $\hat{\lambda}_i \not\simeq 0$ for each $i = 1,\ldots,I$. If not, we may assume without loss of generality that $\hat{\lambda}_I \simeq 0$. For every node for which $\hat{\iota}(t,\omega) \not\simeq 0$, we have

$$* \left.\frac{\partial h_I}{\partial c_I}\right|_{(\hat{c}_I(t,\omega),\hat{I}(t,\omega))} = \frac{\hat{\iota}(t,\omega)}{\hat{\lambda}_I} \simeq \infty \text{ or } * \left.\frac{\partial H_I}{\partial c_I}\right|_{(\hat{c}_I(\hat{T},\omega),\hat{I}(\hat{T},\omega))} = \frac{\hat{\iota}(\hat{T},\omega)}{\hat{\lambda}_I} \simeq \infty$$

as appropriate, so $\hat{c}_I(t,\omega) \simeq 0$ which shows that $\hat{c}_I(t,\omega) \simeq 0$ Loeb-almost everywhere. Since $\hat{c}_I(t,\omega) \leq \hat{c}(t,\omega) \leq r + e^{r|\hat{\beta}(t,\omega)|} + O((\Delta T)^2)$, then as above, $\hat{c}_I \in SL^p$ for all $p \in [1,\infty)$ by Proposition 3.1 of Raimondo [86], and hence $\hat{p}_C \times \hat{c}_I \in SL^p$ for all $p \in [1,\infty)$ by the Cauchy-Schwarz Inequality and Anderson [2];

$$^\circ E\left(\hat{p}_C(\hat{T},\omega)\hat{c}_I(\hat{T},\omega) + \Delta T \sum_{t=0}^{\hat{T}-\Delta T} \hat{p}_c(t,\omega)\hat{c}_I(t,\omega)\right)$$

$$= E\left({}^\circ\hat{p}_C(\hat{T},\omega)^\circ\hat{c}_I(\hat{T},\omega) + \int_0^T {}^\circ\hat{p}_c(t,\omega)^\circ\hat{c}_I(t,\omega)\,dt\right)$$

$$= 0$$

However, since the hyperfinite market is dynamically complete, agent $I$ can afford any bundle satisfying the Arrow-Debreu budget constraint. In the following calculation, Equation (20) follows because $\hat{e}_I(t,\omega) \leq r + e^{r|\hat{\beta}(t,\omega)|} + O\left((\Delta T)^2\right)$ and $\hat{A}_j(t,\omega) \leq r + e^{r|\hat{\beta}(t,\omega)|} + O\left((\Delta T)^2\right)$, so $\hat{e}_I, \hat{A} \in SL^p$ for all $p \in [1,\infty)$, so $\hat{p}_c\hat{e}_I, \hat{p}_c\hat{A} \in SL^p$ for all $p \in [1,\infty)$. Agent I's income is

$$\hat{e}_{IA} \cdot \hat{p}_A(0,0) + E\left(\hat{p}_C(\hat{T},\cdot)\hat{e}_I(\hat{T},\cdot) + \Delta T \sum_{t=0}^{\hat{T}-\Delta T} \hat{p}_C(t,\cdot)\hat{e}_I(t,\cdot)\right)$$

$$= E\left(\hat{p}_C(\hat{T},\cdot)\left(\hat{e}_{IA}A(\hat{T},\cdot) + \hat{e}_I(\hat{T},\cdot)\right) + \Delta T \sum_{t=0}^{\hat{T}-\Delta T} \hat{p}_C(t,\cdot)\left(\hat{e}_{IA}A(t,\cdot) + \hat{e}_I(t,\cdot)\right)\right)$$

$$\simeq E\left({}^\circ\hat{p}_C(\hat{T},\cdot)\left(e_{IA}A(\hat{T},\cdot) + e_I(\hat{T},\cdot)\right) + \int_0^T {}^\circ p_C(t,\cdot)\left(e_{IA}A(t,\cdot) + e_I(t,\cdot)\right)\right) \quad (20)$$

$$\neq 0$$

since the expectation is of a function which is nonnegative and strictly positive on a set of positive measure by Inequality (4). Thus, $\hat{c}_I$ lies strictly inside the Arrow-Debreu budget set; since preferences are strictly monotonic, it cannot be the demand of agent $I$, a contradiction which shows that $\hat{\lambda}_i \not\simeq 0$ for each $i = 1,\ldots,I$.

We now show that the equilibrium consumptions and consumption prices are given by standard analytic functions. Consider the standard analytic function

$$\rho : (0,\infty)^I \times \mathbf{R}_{++} \times \Delta \times \mathbf{R}_{++} \times \left((0,T) \times \mathbf{R}^K\right) \to \mathbf{R}^{I+1}$$

defined by

$$\rho\left((c_1,\ldots,c_I),\iota,\lambda,c,(t,\beta)\right)$$
$$= \left(\lambda_1 \left.\frac{\partial h_1}{\partial c_1}\right|_{(c_1,t,\beta)} - \iota, \ldots, \lambda_I \left.\frac{\partial h_I}{\partial c_I}\right|_{(c_I,t,\beta)} - \iota, c_1 + \cdots + c_I - c\right)$$

$\rho$ encodes the first order conditions for the equilibrium consumptions. The determinant of the Jacobian matrix of partial derivatives of $\rho$ with respect to $((c_1,\ldots,c_I),\iota)$ is given by

$$\det\left(\frac{\partial\rho}{\partial((c_1,\ldots,c_I,),\iota)}\right)$$

$$= \det \begin{pmatrix} \lambda_1 \frac{\partial^2 h_1}{\partial c_1^2} & 0 & 0 & \cdots & 0 & -1 \\ 0 & \lambda_2 \frac{\partial^2 h_2}{\partial c_2^2} & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_I \frac{\partial^2 h_I}{\partial c_I^2} & -1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

$$= \det \begin{pmatrix} \lambda_1 \frac{\partial^2 h_1}{\partial c_1^2} & 0 & 0 & \cdots & 0 & -1 \\ 0 & \lambda_2 \frac{\partial^2 h_2}{\partial c_2^2} & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_I \frac{\partial^2 h_I}{\partial c_I^2} & -1 \\ 0 & 0 & 0 & \cdots & 0 & \sum_{i=1}^I \frac{1}{\lambda_i \frac{\partial^2 h_i}{\partial c_i^2}} \end{pmatrix}$$

$$= \lambda_1 \cdots \lambda_I \frac{\partial^2 h_1}{\partial c_1^2} \cdots \frac{\partial^2 h_I}{\partial c_I^2} \sum_{i=1}^I \frac{1}{\lambda_i \frac{\partial^2 h_i}{\partial c_i^2}}$$

$$\neq 0$$

since $\lambda \in \Delta$ and $\frac{\partial^2 h_i}{\partial c_i^2} < 0$. By the Analytic Implicit Function Theorem (Theorem A.2 in Appendix A) and the Transfer Principle, there exist standard analytic functions $\hat{\pi}$, $\hat{\psi}_i :$ $\Delta \times \mathbf{R}_{++} \times ((0,T) \times \mathbf{R}^K) \to \mathbf{R}$ for $i = 1, \ldots, I$ such that

$$\hat{\iota}(t,\omega) = {}^*\hat{\pi}(\hat{\lambda}, \hat{c}(t,\omega), \hat{\mathcal{I}}(t,\omega)) \text{ for } t < \hat{T}$$
$$\hat{c}_i(t,\omega) = {}^*\hat{\psi}_i(\hat{\lambda}, \hat{c}(t,\omega), \hat{\mathcal{I}}(t,\omega)) \text{ for } t < \hat{T}$$

Using a similar calculation at $t = \hat{T}$ and the ordinary Implicit Function Theorem, by Equation (3), there are standard $C^1$ functions $\hat{\Pi}$, $\hat{\Psi}_i : \Delta \times \mathbf{R}_{++} \times (\{T\} \times \mathbf{R}^K) \to \mathbf{R}$ such that

$$\hat{\iota}(\hat{T},\omega) = {}^*\hat{\Pi}(\hat{\lambda}, \hat{c}(\hat{T},\omega), \hat{\mathcal{I}}(\hat{T},\omega))$$
$$\hat{c}_i(\hat{T},\omega) = {}^*\hat{\Psi}_i(\hat{\lambda}, \hat{c}(\hat{T},\omega), \hat{\mathcal{I}}(\hat{T},\omega))$$

Let $\lambda \in \Delta$, $\pi : \mathbf{R}_{++} \times \mathbf{R}^K \times [0, T) \to \mathbf{R}$, $\Pi : \mathbf{R}_{++} \times \mathbf{R}^K \to \mathbf{R}$, $\psi_i : \mathbf{R}_{++} \times \mathbf{R}^K \times [0, T) \to \mathbf{R}$ and $\Psi_i : \mathbf{R}_{++} \times \mathbf{R}^K \to \mathbf{R}$ be defined by

$$
\begin{aligned}
\lambda &= {}^\circ\hat{\lambda} \\
\pi(c, \mathcal{I}) &= \hat{\pi}(\lambda, c, \mathcal{I}) \\
\Pi(c, \beta) &= \hat{\Pi}(\lambda, c, \beta) \\
\psi_i(c, \mathcal{I}) &= \hat{\psi}_i(\lambda, c, \mathcal{I}) \\
\Psi_i(c, \beta) &= \hat{\Psi}_i(\lambda, c, \beta)
\end{aligned}
$$

Because $\hat{\pi}$ and $\hat{\psi}$ are standard analytic functions, $\pi$ and $\Psi$ are standard analytic functions; because $\hat{\Pi}$ and $\hat{\Psi}$ are standard $C^1$ functions, $\Pi$ and $\Psi$ are standard $C^1$ functions. Let $c(t, \omega) = {}^\circ\hat{c}(t, \omega)$, $c_i(t, \omega) = {}^\circ\hat{c}_i(t, \omega)$. Since $c(t, \omega)$ and $\psi_i(c, \beta, t)$ are analytic functions for $t \in (0, T)$, each $c_i(t, \omega)$ is an analytic function of $\mathcal{I}(t, \omega)$. Each $c_i(T, \omega)$ is a $C^1$ function of $c(T, \omega)$; however, since $c(T, \omega)$ is a continuous function of $\mathcal{I}(T, \omega)$ almost everywhere, $c_i(T, \omega)$ is a continuous function of $\mathcal{I}(T, \omega)$ almost everywhere.

Let

$$
p_C(t, \omega) = {}^\circ\hat{p}_C(t, \omega)
$$

$$
\begin{aligned}
p_C(t, \omega) &= {}^\circ\hat{p}_C(t, \omega) \\
&= {}^\circ\left( {}^*\hat{\pi}\left(\hat{\lambda}, \hat{c}(t, \omega), \hat{\mathcal{I}}(t, \omega)\right)\right) \\
&= \hat{\pi}\left({}^\circ\hat{\lambda}, {}^\circ\hat{c}(t, \omega), {}^\circ\hat{\mathcal{I}}(t, \omega)\right) \\
&= \hat{\pi}\left(\lambda, c(t, \omega), \mathcal{I}(t, \omega)\right) \\
&= \pi\left(c(t, \omega), \mathcal{I}(t, \omega)\right)
\end{aligned}
$$

is an analytic function of $\mathcal{I}(t, \omega)$ on $(0, T) \times \mathbf{R}^K$, since $\pi$ is analytic and $c(t, \omega)$ is an analytic function of $\mathcal{I}(t, \omega)$. Since $c(T, \omega) = \Pi(c(T, \omega), \beta(T, \omega))$ is a $C^1$ function of $\beta(T, \omega) \in V$, $p_C(T, \cdot)$ is a $C^1$ function of $\beta(T, \omega) \in V$.

In the following calculation, Equation (21) holds because the first order conditions for demand imply the price of a security is the expected value of future dividends times the weighted marginal utility of consumption, which equals the expected value of the dividend times the price of consumption. Equation (22) follows because $\hat{p}_A(\hat{T}, \cdot) = \hat{p}_C(\hat{T}, \omega)A(\hat{T}, \omega)$, $\hat{p}_C \in SL^p$ for all $p \in [1, \infty)$, and $A_j(t, \omega) \le r + e^{r|\hat{\beta}(t, \omega)|} + O\left((\Delta T)^2\right)$, so as above $\hat{p}_A(\hat{T}, \cdot) \in SL^p$ for all $p \in [1, \infty)$, by Proposition 3.1 of Raimondo [86], so $\hat{p}_C(\cdot, \cdot)\hat{A}(\cdot, \cdot) \in SL^p$ for all $p \in [1, \infty)$ (Anderson [2]); and the internal integrands are $S$-continuous functions of $\hat{\mathcal{I}}(t, \omega)$ whenever ${}^\circ\hat{\mathcal{I}}(t, \omega)$ is at a point at which $c$ and $G$ are continuous. Equation (23) follows because $\hat{\mathcal{I}}(\cdot, \omega)$ is almost surely $S$-continuous, and for any such $\omega$ and any $s \in [{}^\circ t, T]$, the conditional distribution of $\mathcal{I}(s, \omega)$ is the same given $({}^\circ t, \omega)$ as it is given $(t, \omega)$.

$$
\begin{aligned}
&{}^\circ\hat{p}_A(t, \omega) \\
&= {}^\circ E\left(\left.\left(\left(\hat{p}_C(\hat{T}, \cdot)\hat{A}(\hat{T}, \cdot) + \Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} \hat{p}_C(s, \cdot)\hat{A}(s, \cdot)\right)\right) \right| (t, \omega)\right)
\end{aligned}
$$

$$= \; {}^\circ E\left(\left.{}^*\hat{\Pi}(\hat{\lambda}, \hat{c}(\hat{T}, \cdot), \hat{\mathcal{I}}(\hat{T}, \cdot))\hat{A}(\hat{T}, \cdot)\right| (t, \omega)\right) \tag{21}$$

$$+ {}^\circ E\left(\left.\Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} \left({}^*\hat{\pi}(\hat{\lambda}, \hat{c}(s, \cdot), \hat{\mathcal{I}}(s, \cdot))\hat{A}(s, \cdot)\right)\right| (t, \omega)\right)$$

$$= \; {}^\circ E\left(\left.{}^*\hat{\Pi}(\hat{\lambda}, \hat{c}(\hat{T}, \cdot), \hat{\mathcal{I}}(\hat{T}, \cdot)){}^*G(\hat{\mathcal{I}}(\hat{T}, \cdot)) + O\left((\Delta T)^2\right)\right| (t, \omega)\right)$$

$$+ {}^\circ E\left(\left.\Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} {}^*\hat{\pi}(\hat{\lambda}, \hat{c}(s, \cdot), \hat{\mathcal{I}}(s, \cdot))\left({}^*g\left(\hat{\mathcal{I}}(s, \cdot)\right) + O\left((\Delta T)^2\right)\right)\right| (t, \omega)\right)$$

$$= \; E\left(\left.\Pi(c(T, \cdot), \mathcal{I}(T, \cdot))G(\mathcal{I}(T, \cdot))\right| (t, \omega)\right) + E\left(\left.\int_{{}^\circ t}^T \pi(c(s, \cdot), \mathcal{I}(s, \cdot))g\left(\mathcal{I}(s, \cdot)\right)\, ds\right| (t, \omega)\right) \tag{22}$$

$$= \; E\left(\left.\Pi(c(T, \cdot), \mathcal{I}(T, \cdot))G\left(\mathcal{I}(T, \cdot)\right) + \int_{{}^\circ t}^T \pi(c(s, \cdot), \mathcal{I}(s, \cdot))\left(g\left(\mathcal{I}(s, \cdot)\right)\, ds\right)\right| ({}^\circ t, \omega)\right) \tag{23}$$

Let

$$p_A(t, \omega) = E\left(\left.\Pi(c(T, \cdot), \mathcal{I}(T, \cdot))G\left(\mathcal{I}(T, \cdot)\right) + \int_{{}^\circ t}^T \pi(c(s, \cdot), \mathcal{I}(s, \cdot))\left(g\left(\mathcal{I}(s, \cdot)\right)\, ds\right)\right| (t, \omega)\right)$$

We have shown that ${}^\circ\hat{p}_A(t, \omega)$ is an $S$-continuous function of $(t, \hat{\beta}(t, \omega)) \in \text{st}^{-1}\left([0, T) \times \mathbf{R}^K\right)$, and that ${}^\circ\hat{p}_A(t, \omega) = p_A(t, \omega)$ whenever the path $\hat{\beta}(\cdot, \omega)$ is $S$-continuous, i.e. Loeb-almost everywhere. For almost all $\omega$, $\hat{p}_A(\cdot, \omega)$ is an $S$-continuous function of $t \in \mathcal{T}$. Moreover, $p_A$ is an analytic function of $(t, \beta(t, \omega)) \in (0, T) \times \mathbf{R}^K$ by Theorem A.4 and Proposition 2.2.3 of Krantz and Parks [62].

In the following calculation of $\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega)$, note that we do not calculate how the future dividend *values* are affected by the information revealed by $\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)$. Instead, we calculate how the *probability* of each possible future dividend is affected by the information revealed by $\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega) = \sqrt{\Delta T}v_{\omega(t+\Delta T)}$. The calculation uses the fact that the distribution of the random walk is multinomial. In order to know the value of $\hat{\beta}(s, \omega) - \hat{\beta}(t, \omega)$, it is enough to count how many of the $\frac{s-t}{\Delta T}$ steps of the random walk in this time interval lie in each of the possible directions $v_0, \ldots, v_K$; it does not depend on the order in which the steps occur. In Equation (24), we write $\hat{p}_C\hat{A}\left(s, \hat{\beta}(t, \omega) + \sum_{i=0}^K k_iv_i\sqrt{\Delta T}\right)$ as an abbreviation for

$$\hat{p}_C\left(s, \hat{\beta}(t, \omega) + \sum_{i=0}^K k_iv_i\sqrt{\Delta T}\right)\hat{A}\left(s, \hat{\beta}(t, \omega) + \sum_{i=0}^K k_iv_i\sqrt{\Delta T}\right)$$

Equation (25) holds because for each $s > t, s \in \mathcal{T}$, including $s = \hat{T}$, and for $k_0, \ldots, k_K \in {}^*(\mathbf{N} \cup \{0\})$ satisfying $k_0 + \cdots + k_K = (s-t)/\Delta T$, letting $v_i'$ denote the row vector transpose of the column vector $v_i$,

$$\frac{\sqrt{\Delta T}\left(\sum_{i=0}^K k_iv_i'\sqrt{\Delta T}\right)v_{\omega(t+\Delta T)}}{s-t} = \frac{\Delta T}{s-t}\sum_{i=0}^K k_iv_i \cdot v_{\omega(t+\Delta T)}$$

$$= \frac{\Delta T}{s-t}\left(Kk_{\omega(t+\Delta T)} + (-1)\sum_{i\neq\omega(t+\Delta T)} k_i\right)$$

$$= \frac{\Delta T}{s-t}\left(Kk_{\omega(t+\Delta T)} - \left(\frac{s-t}{\Delta T} - k_{\omega(t+\Delta T)}\right)\right)$$

$$= \frac{\Delta T}{s-t}\left((K+1)k_{\omega(t+\Delta T)} - \frac{s-t}{\Delta T}\right)$$

$$= \frac{(K+1)k_{\omega(t+\Delta T)}}{(s-t)/\Delta T} - 1$$

Equation (27) follows because $\hat{A}(t,\omega) \le r + e^{r|\hat{\beta}(t,\omega)|} + O\left((\Delta T)^2\right)$, $|\hat{A}\hat{\beta}| \le |\hat{\beta}|\left(r + e^{r|\hat{\beta}|}\right) \le \left(r|\hat{\beta}| + e^{(r+1)|\hat{\beta}|}\right) \le 2e^{(r+1)|\hat{\beta}|}$, so as above $\hat{A}, \hat{A}\hat{\beta} \in SL^p$ for all $p \in [1,\infty)$ by Proposition 3.1 of Raimondo [86], and we previously showed $\hat{p}_C \in SL^p$ for all $p \in [0,\infty)$; it follows that $\hat{p}_A(\hat{T},\cdot) = \hat{p}_C(\hat{T},\cdot)\hat{A}(\hat{T},\cdot) \in SL^p$ and $\hat{p}_C\hat{A}\hat{\beta} \in SL^p$ for all $p \in [1,\infty)$ (Anderson [2]); and the internal integrands are $S$-continuous functions of $\hat{\mathcal{I}}(t,\omega)$ whenever $^\circ\hat{\mathcal{I}}(t,\omega)$ is at a point at which $c$ and $G$ are continuous. Equation (28) follows because $\hat{\mathcal{I}}(\cdot,\omega)$ is almost surely $S$-continuous, and for any such $\omega$ and any $s \in [^\circ t, T]$, the conditional distribution of $\mathcal{I}(s,\omega)$ is the same given $(^\circ t, \omega)$ as it is given $(t,\omega)$.

$\hat{\gamma}(t+\Delta T,\omega) - \hat{\gamma}(t,\omega)$

$$= \hat{p}_A(t+\Delta T,\omega) + (\Delta T)\sum_{s\in\mathcal{T},s\le t+\Delta T}\hat{p}_C(s,\omega)\hat{A}(s,\omega) - \left(\hat{p}_A(t,\omega) + (\Delta T)\sum_{s\in\mathcal{T},s\le t}\hat{p}_C(s,\omega)\hat{A}(s,\omega)\right)$$

$$= \hat{p}_A(t+\Delta T,\omega) + \hat{p}_c(t+\Delta T,\omega)\hat{A}(t+\Delta T,\omega) - \hat{p}_A(t,\omega)$$

$$= E\left(\left(\hat{p}_A(\hat{T},\cdot)\right)\Big|(t+\Delta T,\omega)\right) - E\left(\left(\hat{p}_A(\hat{T},\cdot)\right)\Big|(t,\omega)\right)$$

$$+ \Delta T\sum_{s=t+\Delta T}^{\hat{T}-\Delta T}\left(E\left(\left(\hat{p}_C(s,\cdot)\hat{A}(s,\cdot)\right)\Big|(t+\Delta T,\omega)\right) - E\left(\left(\hat{p}_C(s,\cdot)\hat{A}(s,\cdot)\right)\Big|(t,\omega)\right)\right)$$

$$= \frac{1}{(K+1)^{(\hat{T}-t)/\Delta T}}\sum_{k_0+\ldots+k_K=(\hat{T}-t)/\Delta T}\hat{p}_A\left(\hat{T},\hat{\beta}(t,\omega)+\sum_{i=0}^{K}k_iv_i\sqrt{\Delta T}\right)\cdot\left(\frac{(K+1)(((\hat{T}-t)/\Delta T)-1)!}{k_0!k_1!(k_{\omega(t+\Delta T)}-1)!\ldots k_K!} - \frac{((\hat{T}-t)/\Delta T)!}{k_0!k_1!\ldots k_K!}\right)$$

$$+ \Delta T\sum_{s=t+\Delta T}^{\hat{T}-\Delta T}\frac{1}{(K+1)^{(s-t)/\Delta T}}\sum_{k_0+\ldots+k_K=(s-t)/\Delta T}\hat{p}_C\hat{A}\left(s,\hat{\beta}(t,\omega)+\sum_{i=0}^{K}k_iv_i\sqrt{\Delta T}\right)\cdot$$

$$\left(\frac{(K+1)(((s-t)/\Delta T)-1)!}{k_0!k_1!(k_{\omega(t+\Delta T)}-1)!\ldots k_K!} - \frac{((s-t)/\Delta T)!}{k_0!k_1!\ldots k_K!}\right) \tag{24}$$

$$= \frac{1}{(K+1)^{(\hat{T}-t)/\Delta T}}\sum_{k_0+\ldots+k_K=(\hat{T}-t)/\Delta T}\hat{p}_A\left(\hat{T},\hat{\beta}(t,\omega)+\sum_{i=0}^{K}k_iv_i\sqrt{\Delta T}\right)\left(\frac{((\hat{T}-t)/\Delta T)!}{k_0!k_1!\ldots k_K!}\right)\left(\frac{(K+1)k_{\omega(t+\Delta T)}}{((\hat{T}-t)/\Delta T)} - 1\right)$$

$$+ \Delta T\sum_{s=t+\Delta T}^{\hat{T}-\Delta T}\frac{1}{(K+1)^{(s-t)/\Delta T}}\sum_{k_0+\ldots+k_K=(s-t)/\Delta T}\hat{p}_C\hat{A}\left(s,\hat{\beta}(t,\omega)+\sum_{i=0}^{K}k_iv_i\sqrt{\Delta T}\right)\cdot$$

$$\left(\frac{((s-t)/\Delta T)!}{k_0!k_1!\ldots k_K!}\right)\left(\frac{(K+1)k_{\omega(t+\Delta T)}}{((s-t)/\Delta T)} - 1\right)$$

$$= \frac{1}{(K+1)^{(\hat{T}-t)/\Delta T}}\sum_{k_0+\ldots+k_K=(\hat{T}-t)/\Delta T}\hat{p}_A\left(\hat{T},\hat{\beta}(t,\omega)+\sum_{i=0}^{K}k_iv_i\sqrt{\Delta T}\right)\left(\frac{((\hat{T}-t)/\Delta T)!}{k_0!k_1!\ldots k_K!}\right)\cdot$$

$$\left(\frac{\sqrt{\Delta T}\left(\sum_{i=0}^{K}k_iv_i\sqrt{\Delta T}\right)\cdot v_{\omega(t+\Delta T)}}{\hat{T}-t}\right)$$

$$+ \Delta T\sum_{s=t+\Delta T}^{\hat{T}-\Delta T}\frac{1}{(K+1)^{(s-t)/\Delta T}}\sum_{k_0+\ldots+k_K=(s-t)/\Delta T}\left(\hat{p}_C\hat{A}\left(s,\hat{\beta}(t,\omega)+\sum_{i=0}^{K}k_iv_i\sqrt{\Delta T}\right)\cdot\left(\frac{((s-t)/\Delta T)!}{k_0!k_1!\ldots k_K!}\right)\right)\cdot$$

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$$\left(\frac{\sqrt{\Delta T}\left(\sum_{i=0}^{K}k_i v_i\sqrt{\Delta T}\right)\cdot v_{\omega(t+\Delta T)}}{s-t}\right)\right) \tag{25}$$

$$= \quad E\left(\left.\frac{\hat{p}_A(\hat{T},\hat{\beta}(\hat{T},\cdot))(\hat{\beta}(\hat{T},\cdot)-\hat{\beta}(t,\cdot))'}{\hat{T}-t}+\frac{\Delta T\sum_{s=t+\Delta T}^{\hat{T}-\Delta T}\hat{p}_C\hat{A}(s,\hat{\beta}(s,\cdot))(\hat{\beta}(s,\cdot)-\hat{\beta}(t,\cdot))'}{s-t}\right|(t,\omega)\right)v_{\omega(t+\Delta T)}\sqrt{\Delta T} \tag{26}$$

$$= \quad E\left(\left.\frac{p_A(T,\beta(T,\cdot))(\beta(T,\cdot)-\beta(t,\cdot))'}{T-t}+\int_{\circ t}^{T}\frac{p_C(s,\beta(s,\cdot))A(s,\beta(s,\cdot))(\beta(s,\cdot)-\beta(t,\cdot))'}{s-\circ t}ds\right|(t,\omega)\right)\left(\hat{\beta}(t+\Delta T,\omega)-\hat{\beta}(t,\omega)\right) \tag{27}$$

$$+o\left(\sqrt{\Delta T}\right)$$

$$\hat{\sigma}(t,\omega)$$

$$\simeq \quad E\left(\left.\frac{p_A(T,\beta(T,\cdot))(\beta(T,\cdot)-\beta(t,\cdot))'}{T-t}+\int_{\circ t}^{T}\frac{p_C(s,\beta(s,\cdot))A(s,\beta(s,\cdot))(\beta(s,\cdot)-\beta(t,\cdot))'}{s-\circ t}ds\right|(t,\omega)\right)$$

$$= \quad E\left(\left.\frac{p_C(T,\beta(T,\cdot))A(T,\beta(T,\cdot))(\beta(T,\cdot)-\beta(t,\cdot))'}{T-t}+\int_{\circ t}^{T}\frac{p_C(s,\beta(s,\cdot))A(s,\beta(s,\cdot))(\beta(s,\cdot)-\beta(t,\cdot))'}{s-\circ t}ds\right|(\circ t,\omega)\right) \tag{28}$$

This shows that $\hat{\sigma}$ is an $S$-continuous function of $(t,\hat{\beta})\in\mathrm{st}^{-1}\left([0,T)\times\mathbf{R}^K\right)$. Note that the integrand is the product of the $(K+1)\times 1$ column vector $A$ and the $1\times K$ row vector $\beta'$, so it is a $(K+1)\times K$ matrix. Letting $\Sigma:[0,T)\times\mathbf{R}^K\to\mathbf{R}^{(K+1)\times K}$ be given by

$$\Sigma(t,\beta) \quad = \quad E\left(\frac{p_C(T,\beta(T))A(T,\beta(T))(\beta(T)-\beta(t))'}{T-t}\right.$$

$$\left.+\int_t^T\frac{p_C(s,\beta(s))A(s,\beta(s))(\beta(s)-\beta(t))'}{s-t}ds\right|\beta(t)=\beta\right)$$

we have

$$\circ\hat{\sigma}(t,\omega)=\Sigma\left(\mathcal{I}(t,\omega)\right)$$

for every $(t,\omega)$ such that $\hat{\mathcal{I}}(t,\omega)$ is finite. Since $\hat{\sigma}$ is $SL^2$ and adapted to $\left\{\hat{\mathcal{F}}_t\right\}$, and is an $SL^2$ lifting of $\Sigma\circ\mathcal{I}$, $\sigma\circ\mathcal{I}$ is $L^2$ and adapted to $\{\mathcal{F}_t\}$ (Anderson [2]). Therefore, by Theorem B.1, we have

$$p_A(t,\omega) \quad = \quad \circ\hat{p}_A(\hat{t},\omega)$$

$$= \quad \circ\hat{p}_A(0,\omega)+\circ\sum_{s=0}^{\hat{t}-\Delta T}\left(\hat{\sigma}(s,\omega)\left(\hat{\beta}(s+\Delta T,\omega)-\beta(s,\omega)\right)-(\Delta T)\,\hat{p}_C(s,\omega)\hat{A}(s,\omega)\right)$$

$$= \quad p_A(0,\omega)+\int_0^t\Sigma\left(\mathcal{I}(s,\omega)\right)d\beta(s,\omega)-\int_0^t p_C(s,\omega)A(s,\omega)\,ds$$

Since $p_A$ is an analytic (hence $C^2$) function of $(t,\beta)$, by Itô's Lemma and the uniqueness of Itô Coefficients,

$$\Sigma(t,\beta)=\frac{\partial p_A(t,\beta)}{\partial\beta}$$

is a partial derivative of an analytic function, so $\Sigma$ is analytic on $(0,T)\times\mathbf{R}^K$.

Extend $\Sigma$ to $\{T\}\times V$ by

$$\Sigma(T,\beta)=\frac{\partial p_A(T,\beta)}{\partial\beta}$$

for $\beta\in V$; note that here, we are using $\beta\in\mathbf{R}^K$ to denote a particular value that the Brownian Motion might take. We claim that $\Sigma$ is a continuous function of $(t,\beta)\in[0,T]\times V$.

Notice that the second term in the definition of $\Sigma$ tends to zero as $t \to T$, uniformly over $\beta$ ranging over compact subsets of $\mathbf{R}^K$, so we can restrict attention to the first term. Suppose $\beta_0 \in V$. Fix $\varepsilon > 0$. Since $A(T, \cdot)$ is $C^1$ on $V$ and $p_C(T, \cdot)$ is $C^1$ on $V$, $p_A(T, \cdot) = A(T, \cdot)p_C(T, \cdot)$ is $C^1$ on $\{T\} \times V$. Since $V$ is open, we may find $\delta > 0$ such that $B(\beta_0, 2\delta) \subset V$ and

$$\beta, y \in B(\beta_0, 2\delta) \Rightarrow \left\| \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,y)} - \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta)} \right\| < \frac{\varepsilon}{3}$$

For any $\beta \in B(\beta_0, \delta)$ and $y \in B(\beta, \delta)$, we have $y \in B(\beta_0, 2\delta)$. Using the Mean Value Theorem one component at a time, we find that

$$\left| p_A(T, y) - p_A(T, \beta) - \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta)} (y - \beta) \right| < \frac{\varepsilon |y - \beta|}{3}$$

so

$$\left\| \left( p_A(T, y) - p_A(T, \beta) - \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta)} (y - \beta) \right) (y - \beta)' \right\| < \frac{\varepsilon |y - \beta|^2}{3}$$

Since $p_C \le r + e^{r|\beta|}$ and $|A_j(T, y)| \le r + e^{r|y|}$, and $\left\| \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta)} \right\|$ is uniformly bounded over $\beta \in B(\beta_0, \delta)$, there is a constant $\bar{r} \in \mathbf{R}$ such that for every $\beta \in B(\beta_0, \delta)$, for all $y \in \mathbf{R}^K$,

$$\left\| \left( p_A(T, y) - p_A(T, \beta) - \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta)} (y - \beta) \right) (y - \beta)' \right\| \le \bar{r} + e^{\bar{r}|y|}$$

Find $t_0 < T$ such that for all $t \in [t_0, T)$,

$$\frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K \setminus B(\beta, \delta)} \frac{\bar{r} + e^{\bar{r}|y|}}{T - t} e^{-|y-\beta|^2/2(T-t)} \, dy < \frac{\varepsilon}{3}$$

If $t \in [t_0, T)$ and $\beta \in B(\beta_0, \delta)$,

$$\left\| \frac{E\left( p_A(T, \beta(T, \cdot))(\beta(T, \cdot) - \beta(t, \cdot))' \mid \beta(t, \omega) = \beta \right)}{T - t} - \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta_0)} \right\|$$

$$= \left\| \frac{1}{(2\pi)^{K/2}} \int_{\mathbf{R}^K} \frac{p_A\left(T, \beta + \sqrt{T-t}\,x\right)\sqrt{T-t}\,x'}{T - t} e^{-|x|^2/2} \, dx - \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta_0)} \right\|$$

$$= \left\| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{p_A(T, y)(y - \beta)'}{T - t} e^{-|y-\beta|^2/(2(T-t))} \, dy - \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta_0)} \right\|$$

$$\le \left\| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{\left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta)}(y - \beta)(y - \beta)'}{T - t} e^{-|y-\beta|^2/(2(T-t))} \, dy - \left.\frac{\partial p_A}{\partial \beta}\right|_{(T,\beta_0)} \right\|$$

$$+ \left\| \frac{1}{2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{p_A(t, \beta)(y - \beta)'}{T - t} e^{-|y-\beta|^2/(2(T-t))} \, dy \right\|$$

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$$+ \left\| \frac{1}{(2\pi(T-t))^{K/2}} \int_{B(\beta,\delta)} \frac{\varepsilon(y-\beta)(y-\beta)'}{3(T-t)} e^{-|y-\beta|^2/(2(T-t))} \, dy \right\|$$

$$+ \left| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K \backslash B(\beta,\delta)} \frac{\bar{r} + e^{\bar{r}|y|}}{T-t} e^{-|y-\beta|^2/(2(T-t))} \, dy \right|$$

$$= \left\| \frac{\partial p_A}{\partial \beta} \bigg|_{(T,\beta)} - \frac{\partial p_A}{\partial \beta} \bigg|_{(T,\beta_0)} \right\| + 0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

which shows that $\Sigma$ is continuous on $[0, T] \times V$.

Now, we show that the pricing process is dynamically complete. Let $\Sigma_j$ $(j = 0, \ldots, J)$ denote the $j^{th}$ row of $\Sigma$, and let $\bar{\Sigma}(t, \beta)$ be the $K \times K$ matrix whose $j^{th}$ row $(j = 1, \ldots, J)$ is

$$\bar{\Sigma}_j(t, \beta) = \frac{p_{A_0}(t, \beta)\Sigma_j(t, \beta) - p_{A_j}(t, \beta)\Sigma_0(t, \beta)}{(p_{A_0}(t, \beta))^2}$$

$$= \frac{\partial \frac{p_{A_j}}{p_{A_0}}}{\partial \beta}$$

Notice that

$$\text{rank } \bar{\Sigma}(t, \beta) \leq \text{rank } \Sigma(t, \beta) \tag{29}$$

Let

$$B = \{\mathcal{I} \in (0, T) \times \mathbf{R}^K : \det \bar{\Sigma}(\mathcal{I}) = 0\}$$

Suppose that $B$ has positive Lebesgue measure. Then $\det \bar{\Sigma}(\mathcal{I}) = 0$, for every $\mathcal{I} \in B$. The determinant is a polynomial function of the entries of the matrix, hence is an analytic function of $\mathcal{I} \in \mathbf{R}^K \times (0, T)$, so $\det \bar{\Sigma}(\mathcal{I})$ must be identically zero on $(0, T)$;[21] since it is continuous on $[0, T] \times V$, it is identically zero on $\{T\} \times V$. Using the nondegeneracy assumption (Equation (3)), choose $\omega_0$ such that $\hat{\beta}(\cdot, \omega_0)$ is finite, $^\circ\hat{\beta}(\hat{T}, \omega_0) \in V$, and

$$\det \begin{pmatrix} \frac{\partial (G_1/G_0)}{\partial \beta} \big|_{\mathcal{I}(T,\omega_0)} \\ \vdots \\ \frac{\partial (G_J/G_0)}{\partial \beta} \big|_{\mathcal{I}(T,\omega_0)} \end{pmatrix} \neq 0$$

Since the securities prices are equilibrium prices, they must be arbitrage-free, so we have for $j = 1, \ldots, J$ and any $\omega$ such that $^\circ\hat{\beta}(\hat{T}, \omega) \in V$,

$$\frac{p_{A_j}(T, \omega)}{p_{A_0}(T, \omega)} \simeq \frac{\hat{p}_{A_j}(\hat{T}, \omega)}{\hat{p}_{A_0}(\hat{T}, \omega)}$$

$$= \frac{\hat{A}_j(\hat{T}, \omega)}{\hat{A}_0(\hat{T}, \omega)}$$

---

[21]See Theorem A.3 in Appendix A.

$$= \frac{{}^*G_j\left(\hat{T}, \hat{\beta}(\hat{T}, \omega)\right) + O((\Delta T)^2)}{{}^*G_0\left(\hat{T}, \hat{\beta}(\hat{T}, \omega)\right) + O((\Delta T)^2)}$$

$$\simeq \frac{{}^*G_j\left(\hat{T}, \hat{\beta}(\hat{T}, \omega)\right)}{{}^*G_0\left(\hat{T}, \hat{\beta}(\hat{T}, \omega)\right)}$$

$$\simeq \frac{G_j\left(T, \beta(T, \omega)\right)}{G_0\left(T, \beta(T, \omega)\right)}$$

so

$$\det \bar{\Sigma}(T, \beta(T, \omega_0)) = \det \begin{pmatrix} \left.\frac{\partial\left(p_{A_1}/p_{A_0}\right)}{\partial \beta}\right|_{\mathcal{I}(T, \omega_0)} \\ \vdots \\ \left.\frac{\partial\left(p_{A_J}/p_{A_0}\right)}{\partial \beta}\right|_{\mathcal{I}(T, \omega_0)} \end{pmatrix} = \det \begin{pmatrix} \left.\frac{\partial(G_1/G_0)}{\partial \beta}\right|_{\mathcal{I}(T, \omega_0)} \\ \vdots \\ \left.\frac{\partial(G_J/G_0)}{\partial \beta}\right|_{\mathcal{I}(T, \omega_0)} \end{pmatrix} \neq 0$$

a contradiction which proves that $B$ is a set of measure zero.

If we let $B_t = \{\beta \in \mathbf{R}^K : (t, \beta) \in B\}$ denote the $t$-section of $B$, then by Fubini's Theorem, $\lambda(\{t : B_t \text{ has positive Lebesgue measure}\}) = 0$. Since the distribution of $\beta(t, \cdot)$ is absolutely continuous with respect to Lebesgue measure,

$$\begin{aligned}
&(\lambda \times L(\hat{\mu}))\left(\{(t, \omega) : (\mathcal{I}(t, \omega)) \in B\}\right) \\
&= \int_{[0,T]} L(\hat{\mu})\left(\{\omega : (\mathcal{I}(t, \omega)) \in B\}\right) d\lambda \\
&= \int_{[0,T]} L(\hat{\mu})\left(\{\omega : \beta(t, \omega) \in B_t\}\right) d\lambda \\
&= 0
\end{aligned}$$

Therefore, the normalized securities price process $p_A/p_{A_0}$ is essentially dynamically complete ([81], Theorem 5.6). Since $p_A$ is a positive Itô Process, the renormalization does not change the set of admissible self-financing trading strategies ([81], Section 4.8). Consequently, the securities price process $p_A$ is essentially dynamically complete; and the budget sets under $p_A$ are the same as the budget sets under $p_A/p_{A_0}$.

We now derive the formulas for the hyperfinite equilibrium trading strategies, and show that they are sufficiently regular to extract a candidate trading strategy in continuous time. Let

$$\begin{aligned}
&\hat{W}_i(t, \omega) \\
&= E\left(\hat{p}_C(\hat{T}, \hat{\beta}(\hat{T}))(\hat{c}_i(\hat{T}, \hat{\beta}(\hat{T})) - \hat{e}_i(\hat{T}, \hat{\beta}(\hat{T}))) \right. \\
&\left. + \Delta T \sum_{s=t}^{\hat{T}-\Delta T} \hat{p}_C(s, \hat{\beta}(s))(\hat{c}_i(s, \hat{\beta}(s)) - \hat{e}_i(s, \hat{\beta}(s))) \middle| (t, \omega)\right)
\end{aligned}$$

Since $\hat{z}_i$ finances $\hat{c}_i$, we must have

$$\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) = \hat{W}_i(t, \omega)$$

By the same arguments we used to derive the fomulas for $\hat{p}_A$ and $\hat{\sigma}$, we have for every $\omega$ such that $\hat{\beta}(\cdot, \omega)$ is $S$-continuous, for all $t \in \mathcal{T}$,

$$
\begin{aligned}
\hat{W}_i&(t, \omega) \\
&\simeq \; E\left(p_C(T, \beta(T))\left(c_i(T, \beta(T)) - e_i(T, \beta(T))\right)\right. \\
&\qquad \left. + \int_{\circ t}^{T} p_C(s, \beta(s))\left(c_i(s, \beta(s)) - e_i(s, \beta(s))\right) ds \,\middle|\, (^{\circ}t, \omega)\right)
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\hat{D}_i&(t + \Delta T, \omega) - \hat{D}_i(t, \omega) \\
&= \; E\left(\frac{p_C(T, \beta(T))\left(c_i(T, \beta(T)) - e_i(T, \beta(T))\right)\left(\beta(T) - \beta(t)\right)}{T - t}\right. \\
&\qquad \left. + \int_{\circ t}^{T} \frac{p_C(s, \beta(s))\left(c_i(s, \beta(s)) - e_i(s, \beta(s))\right)\left(\beta(s) - \beta(t)\right)}{s - {}^{\circ}t} ds \,\middle|\, (^{\circ}t, \omega)\right) \\
&\quad \times \left(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)\right) + o\left(\sqrt{\Delta T}\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\sigma}_i&(t, \omega) \\
&\simeq \; E\left(\frac{p_C(T, \beta(T))\left(c_i(T, \beta(T)) - e_i(T, \beta(T))\right)\left(\beta(T) - \beta(t)\right)}{T - {}^{\circ}t}\right. \\
&\qquad \left. + \int_{\circ t}^{T} \frac{p_C(s, \beta(s))\left(c_i(s, \beta(s)) - e_i(s, \beta(s))\right)\left(\beta(s) - \beta(t)\right)}{s - {}^{\circ}t} ds \,\middle|\, (^{\circ}t, \omega)\right)
\end{aligned}
\tag{31}
$$

Define standard functions $\Sigma_1, \ldots, \Sigma_I : [0, T) \times \mathbf{R}^K \to \mathbf{R}^K$, and $W_1, \ldots, W_I : [0, T] \times \mathbf{R}^K \to \mathbf{R}^K$ by

$$
\begin{aligned}
\Sigma_i&(t, \beta) \\
&= \; E\left(\frac{p_C(T, \beta(T))\left(c_i(T, \beta(T)) - e_i(T, \beta(T))\right)\left(\beta(T) - \beta(t)\right)}{T - t}\right. \\
&\qquad \left. + \int_{t}^{T} \frac{p_C(s, \beta(s))\left(c_i(s, \beta(s)) - e_i(s, \beta(s))\right)\left(\beta(s) - \beta(t)\right)}{s - t} ds \,\middle|\, \beta(t) = \beta\right)
\end{aligned}
$$

$$
\begin{aligned}
W_i&(t, \beta) \\
&= \; E\left(p_C(T, \beta(T))\left(c_i(T, \beta(T)) - e_i(T, \beta(T))\right)\right. \\
&\qquad \left. + \int_{t}^{T} p_C(s, \beta(s))\left(c_i(s, \beta(s)) - e_i(s, \beta(s))\right) ds \,\middle|\, \beta(t) = \beta\right)
\end{aligned}
$$

$\Sigma_1, \ldots, \Sigma_I$ are continuous on $[0, T)$ and analytic on $(0, T)$, $W_1, \ldots, W_I$ are continuous on $[0, T]$ and analytic on $(0, T)$. Equations (30) and (31) show that

$$
\begin{aligned}
\hat{\sigma}_i(t, \omega) &\simeq \Sigma_i(\mathcal{I}(^{\circ}t, \omega)) \\
\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) &\simeq W_i\left(\mathcal{I}(^{\circ}t, \omega)\right)
\end{aligned}
$$

for every $\omega$ such that $\hat{\beta}(\cdot, \omega)$ is $S$-continuous. Summing up, we have

$$
\begin{aligned}
\hat{p}_A(t, \omega) &\simeq p_A(^\circ t, \omega) \\
\hat{p}_c(t, \omega) &\simeq p_c(^\circ t, \omega) \\
\hat{\sigma}(t, \omega) &\simeq \Sigma\left(\mathcal{I}(^\circ t, \omega)\right) \\
\hat{c}_i(t, \omega) &\simeq c_i(^\circ t, \omega) \\
\hat{z}_i(t, \omega) \cdot \hat{\sigma}(t, \omega) = \hat{\sigma}_i(t, \omega) &\simeq \Sigma_i\left(\mathcal{I}(^\circ t, \omega)\right) \\
\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) &\simeq W_i\left(\mathcal{I}(^\circ t, \omega)\right)
\end{aligned}
\tag{32}
$$

for all $t \in \mathcal{T}$, for every $\omega$ such that $\hat{\beta}(\cdot, \omega)$ is $S$-continuous. Thus, for every $\omega$ such that $\hat{\beta}(\cdot, \omega)$ is $S$-continuous, we have for *every $t \in \mathcal{T}$*[22]

$$
\begin{aligned}
\hat{p}_A(t, \omega) &\simeq p_A(^\circ t, \omega) \simeq {}^*p_A(\hat{\mathcal{I}}(t, \omega)) \\
\hat{p}_c(t, \omega) &\simeq p_c(^\circ t, \omega) \simeq {}^*p_c(\hat{\mathcal{I}}(t, \omega)) \\
\hat{\sigma}(t, \omega) &\simeq \Sigma\left(\mathcal{I}(^\circ t, \omega)\right) \simeq {}^*\Sigma\left(\hat{\mathcal{I}}(t, \omega)\right) \\
\hat{c}_i(t, \omega) &\simeq c_i\left(\mathcal{I}(^\circ t, \omega)\right) \simeq {}^*c_i\left(\hat{\mathcal{I}}(t, \omega)\right) \\
\hat{z}_i(t, \omega) \cdot \hat{\sigma}(t, \omega) = \hat{\sigma}_i(t, \omega) &\simeq \Sigma_i\left(\mathcal{I}(^\circ t, \omega)\right) \simeq {}^*\Sigma_i\left(\hat{\mathcal{I}}(t, \omega)\right) \\
\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) &\simeq W_i\left(\mathcal{I}(^\circ t, \omega)\right) \simeq {}^*W_i\left(\hat{\mathcal{I}}(t, \omega)\right)
\end{aligned}
\tag{33}
$$

Given $\omega$ such that $\hat{\beta}(\cdot, \omega)$ is $S$-continuous,

$$
\max_{t \in \mathcal{T}} \left| \hat{p}_A(t, \omega) - {}^*p_A(\hat{\mathcal{I}}(t, \omega)) \right| \simeq 0
$$

since $\mathcal{T}$ is hyperfinite, and every hyperfinite set contains its maximum by the Transfer Principle. But for $L(\hat{\mu})$-almost all $\omega$, $\hat{\beta}(t, \omega)$ is $S$-continuous (see footnote 20), so for $L(\hat{\mu})$-almost all $\omega$,

$$
\max_{t \in \mathcal{T}} \left| \hat{p}_A(t, \omega) - {}^*p_A(\hat{\mathcal{I}}(t, \omega)) \right| \simeq 0
$$

Fix $\varepsilon \in \mathbf{R}_{++}$.

$$
\left\{ \omega : \max_{t \in \mathcal{T}} \left| \hat{p}_A(t, \omega) - {}^*p_A(\hat{\mathcal{I}}(t, \omega)) \right| < \varepsilon \right\}
$$

is an internal set which contains a set of Loeb measure 1, so

$$
\hat{\mu}\left(\left\{ \omega : \max_{t \in \mathcal{T}} \left| \hat{p}_A(t, \omega) - {}^*p_A(\hat{\mathcal{I}}(t, \omega)) \right| < \varepsilon \right\}\right) > 1 - \varepsilon
\tag{34}
$$

We have already noted that $\hat{p}_C, \hat{p}_A \in SL^p$ for all $p \in [1, \infty)$. Equation(26) and the growth conditions (Equation (1)) imply that, regardless of whether $\hat{\mathcal{I}}(t, \omega)$ is finite, $|\hat{\sigma}(t, \omega)| \le r + e^{r|\hat{\beta}(t, \omega)|}$ and $|\hat{\sigma}_i(t, \omega)| \le r + e^{r|\hat{\beta}(t, \omega)|}$, which implies as above by Proposition 3.1 of Raimondo [86] that $\hat{\sigma}, \hat{\sigma}_1, \ldots, \hat{\sigma}_I \in SL^p$ for all $p \in [1, \infty)$, in particular for $p = 2$. $\hat{z}_i \cdot \hat{p}_A$ is bounded below by minus the value (at $\hat{p}_c$) of $i$'s future endowment, and above by the value of total

---

[22]In Equation (33), when we write ${}^*p_A(\hat{\mathcal{I}}(t, \omega))$, we mean $p_A$ is a standard function of $\mathcal{I}(t, \omega) \in [0, T] \times \mathbf{R}^K$; take the nonstandard extension of this standard function defined on $[0, T] \times \mathbf{R}^K$, and evaluate it at $\hat{\mathcal{I}}(t, \omega)$. The analogous definition is used for ${}^*p_c, {}^*\Sigma, {}^*\Sigma_1, \ldots, {}^*\Sigma_I, {}^*W_1, \ldots, {}^*W_I$.

market future consumption, hence there exists $\bar{\bar{r}} \in \mathbf{R}$ such that $\hat{z}_i \cdot \hat{p}_A(t, \omega) \leq \bar{\bar{r}} + e^{\bar{\bar{r}}\left(|\hat{\beta}(t,\omega)|+T\right)}$, so $\hat{z}_i \cdot \hat{p}_A$ is $SL^p$ for all $p \in [1, \infty)$. $\hat{c}_i(t, \omega) \leq r + e^{r|\hat{\beta}(t,\omega)|}$, so $\hat{c}_i$ is $SL^p$ for all $p \in [1, \infty)$.

$$
\begin{aligned}
\hat{p}_{Aj}(t, \omega) \leq r + e^{r(|\hat{\beta}(t,\omega)|+T)} &\Rightarrow p_{Aj}(\mathcal{I}(t, \omega)) \leq r + e^{r(|\beta(t,\omega)|+T)} \\
&\Rightarrow {}^*p_{Aj}(\hat{\mathcal{I}}(t, \omega)) \leq r + e^{r(|\hat{\beta}(t,\omega)|+T)} \\
&\Rightarrow {}^*p_{Aj} \circ \hat{\mathcal{I}} \in SL^p \ (1 \leq p < \infty)
\end{aligned}
$$

Similarly, ${}^*p_c \circ \hat{\mathcal{I}}, {}^*\Sigma \circ \hat{\mathcal{I}}, {}^*c_i \circ \hat{\mathcal{I}}, {}^*\Sigma_i \circ \hat{\mathcal{I}}, {}^*W_i \circ \hat{\mathcal{I}} \in SL^p$ for all $p \in [1, \infty)$. Since $\hat{p}_A$ and ${}^*p_A \circ \mathcal{I}$ are $SL^p$ for all $p \in [1, \infty)$ and ${}^\circ\left(\hat{p}_A(\cdot, \cdot) - {}^*p_A(\hat{\mathcal{I}}(\cdot, \cdot))\right)$ is zero on a set of full Loeb measure,

$$
\left\|\hat{p}_A - {}^*p_A \circ \hat{\mathcal{I}}\right\|_2 \simeq \left\|{}^\circ\left(\hat{p}_A - {}^*p_A(\circ\hat{\mathcal{I}})\right)\right\|_2 = 0 \tag{35}
$$

Now, we consider the form of the equilibrium trading strategies $\hat{z}_i$. $\hat{\beta}(t, \omega)$ is finite and $\det \bar{\Sigma}(t, \beta(t, \omega)) \neq 0$ at (Loeb) almost every node; fix such a node $(t_0, \omega_0)$. Then $\hat{\beta}(t_0 + \Delta T, \omega_0)$ is also finite. From the growth condition on dividends, it follows that $\hat{A}(t_0 + \Delta T, \omega_0)$ is finite. Let $\omega_1, \ldots, \omega_K$ be elements of $\Omega$ such that $\omega_k(s) = \omega_0(s)$ for $s \leq t_0$ and $\omega_k(t_0 + \Delta T) = k$; we can assume without loss of generality that $\omega_0(t_0 + \Delta T) = 0$ (recall that $\omega(t_0 + \Delta T) = k$ means that $\hat{\beta}(t_0 + \Delta T, \omega) - \hat{\beta}(t_0, \omega) = v_k \sqrt{\Delta T}$). Since $\hat{z}_i$ finances $\hat{c}_i$, we have for $k = 0, \ldots, K$

$$
\begin{aligned}
&\hat{\sigma}_i(t_0, \omega_0)\left(\hat{\beta}(t_0 + \Delta T, \omega_k) - \hat{\beta}(t_0, \omega_0)\right) \\
&= \hat{D}_i(t_0 + \Delta T, \omega_k) - \hat{D}_i(t_0, \omega_0) \\
&= \hat{z}_i(t_0, \omega_0) \cdot (\hat{\gamma}(t_0 + \Delta T, \omega_k) - \hat{\gamma}(t_0, \omega_0)) \\
&= \hat{z}_i(t_0, \omega_0) \cdot \left(\hat{\sigma}(t_0, \omega_0)\left(\hat{\beta}(t_0 + \Delta T, \omega_k) - \hat{\beta}(t_0, \omega_0)\right)\right) \\
&= (\hat{z}_i(t_0, \omega_0)(\hat{\sigma}(t_0, \omega_0)))\left(\hat{\beta}(t_0 + \Delta T, \omega_k) - \hat{\beta}(t_0, \omega_0)\right)
\end{aligned}
$$

Since $\{v_0, \ldots, v_K\}$ spans $\mathbf{R}^K$, we have

$$
\hat{z}_i(t_0, \omega_0)\hat{\sigma}(t_0, \omega_0) = \hat{\sigma}_i(t_0, \omega_0)
$$

In addition, we have

$$
\hat{z}_i(t_0, \omega_0) \cdot \hat{p}_A(t_0, \omega_0) = \hat{W}_i(t_0, \omega_0)
$$

Let

$$
\tilde{\sigma}(t_0, \omega_0) = (\hat{\sigma}(t_0, \omega_0)|\hat{p}_A(t_0, \omega_0)), \ \tilde{\sigma}_i(t_0, \omega_0) = (\hat{\sigma}_i(t_0, \omega_0)|\hat{W}_i(t_0, \omega_0))
$$

denote the matrix whose first $K$ columns are the columns of $\hat{\sigma}(t_0, \omega_0)$ and whose $(K+1)^{st}$ column is $\hat{p}_A(t_0, \omega_0)$; and the row vector whose first $K$ entries are the entries of $\hat{\sigma}_i(t_0, \omega_0)$ and whose $(K+1)^{st}$ entry is $\hat{W}_i(t_0, \omega_0)$. Then

$$
\hat{z}_i(t_0, \omega_0)\tilde{\sigma}(t_0, \omega_0) = \tilde{\sigma}_i(t_0, \omega_0)
$$

$\tilde{\sigma}(t_0, \omega_0)$ is $(K+1) \times (K+1)$.

We claim that ${}^\circ\tilde{\sigma}(t_0, \omega_0)$ is nonsingular. ${}^\circ\hat{\sigma}(t_0, \omega_0) = \Sigma\left(\mathcal{I}({}^\circ t, \omega)\right)$ is $(K+1) \times K$ and has rank $K$ by Equation (29). If $p_A({}^\circ t_0, \omega_0) = {}^\circ\hat{p}_A(t_0, \omega_0)$ lies in the span of the columns

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of $\Sigma(\mathcal{I}(^\circ t_0, \omega_0))$, there is a vector $x \in \mathbf{R}^K$ such that $\Sigma\left(\mathcal{I}\left(^\circ t_0, \omega_0\right)\right) x = p_A(^\circ t_0, \omega_0)$. Since $\Sigma\left(\mathcal{I}\left(^\circ t_0, \omega_0\right)\right) = \frac{\partial p_A}{\partial \beta}$, we have for $\alpha \in \mathbf{R}$,

$$
\begin{aligned}
p_A\left(^\circ t_0, \beta(^\circ t_0, \omega_0) + \alpha x\right) &= p_A(\mathcal{I}(^\circ t_0, \omega_0)) + \alpha \Sigma\left(\mathcal{I}\left(^\circ t_0, \omega_0\right)\right) x + o(\alpha) \\
&= p_A(\mathcal{I}(^\circ t_0, \omega_0)) + \alpha p_A(\mathcal{I}(^\circ t_0, \omega_0)) + o(\alpha) \\
&= (1 + \alpha) p_A(\mathcal{I}(^\circ t_0, \omega_0)) + o(\alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\alpha} \left( \frac{p_{A_j}(^\circ t_0, \beta(^\circ t_0, \omega_0) + \alpha x)}{p_{A_0}(^\circ t_0, \beta(^\circ t_0, \omega_0) + \alpha x)} - \frac{p_{A_j}(\mathcal{I}(^\circ t_0, \omega_0))}{p_{A_0}(\mathcal{I}(^\circ t_0, \omega_0))} \right) \\
&= \frac{1}{\alpha} \left( \frac{(1 + \alpha) p_{A_j}(\mathcal{I}(^\circ t_0, \omega_0)) + o(\alpha)}{(1 + \alpha) p_{A_0}(\mathcal{I}(^\circ t_0, \omega_0)) + o(\alpha)} - \frac{p_{A_j}(\mathcal{I}(^\circ t_0, \omega_0))}{p_{A_0}(\mathcal{I}(^\circ t_0, \omega_0))} \right) \\
&= \frac{1}{\alpha} \left( \frac{p_{A_j}(\mathcal{I}(^\circ t_0, \omega_0))}{p_{A_0}(\mathcal{I}(^\circ t_0, \omega_0))} + o(\alpha) - \frac{p_{A_j}(\mathcal{I}(^\circ t_0, \omega_0))}{p_{A_0}(\mathcal{I}(^\circ t_0, \omega_0))} \right) \\
&= o(1) \text{ as } \alpha \to 0
\end{aligned}
$$

so the directional derivative of $\frac{p_{A_j}}{p_{A_0}}$ in the direction $x$ is zero. Since $\bar{\Sigma}(\mathcal{I}(^\circ t_0, \omega_0))$ is the Jacobian of $\frac{(p_{A_1}, \dots, p_{A_J})}{p_{A_0}}$

$$
\bar{\Sigma}\left(\mathcal{I}\left(^\circ t_0, \omega_0\right)\right) x = 0
$$

contradicting the fact that $\det \bar{\Sigma}\left(\mathcal{I}\left(^\circ t_0, \omega_0\right)\right) \neq 0$. Accordingly, $^\circ \tilde{\sigma}(t_0, \omega_0)$ has rank $K + 1$, and thus is invertible. Since the formula for the inverse of a matrix is a polynomial function of the matrix components, the coefficients of the inverse matrix are given by a standard analytic function of $\mathcal{I}\left(^\circ t_0, \omega_0\right)$. $\tilde{\sigma}(t_0, \omega_0)$ is also invertible and

$$
^\circ \left( \tilde{\sigma}(t_0, \omega_0)^{-1} \right) = \left( ^\circ \tilde{\sigma}(t_0, \omega_0) \right)^{-1}
$$

Then

$$
\hat{z}_i(t_0, \omega_0) = \tilde{\sigma}_i(t_0, \omega_0) \left( \tilde{\sigma}(t_0, \omega_0) \right)^{-1}
$$

so there are standard analytic functions[23] $Z_1, \dots, Z_I : \left( \left( (0, T) \times \mathbf{R}^K \right) \setminus B \right) \to \mathbf{R}^{K+1}$ such that

$$
\hat{z}_i(t_0, \omega) \simeq Z_i\left(\mathcal{I}(^\circ t_0, \omega)\right) \simeq {}^* Z_i\left( \hat{\mathcal{I}}(t_0, \omega) \right)
$$

whenever

$$
\mathcal{I}(^\circ t_0, \omega_0) \in \left( (0, T) \times \mathbf{R}^K \right) \setminus B
$$

Define $z_i(t, \omega) = {}^\circ \hat{z}_i(\hat{t}, \omega)$; $\hat{z}_i \hat{\sigma} = \hat{\sigma}_i$ is an $SL^2$ lifting of $z_i \sigma$, which belongs to $\mathcal{H}^2$.

We now show that $(p_A, p_C, (c_1, \dots, c_I), (Z_1, \dots, Z_I))$ is an equilibrium of the Loeb continuous-time economy with utility weights $\lambda$ and induces an equilibrium with utility weights $\lambda$ for the original continuous-time economy. For Loeb-almost all $(t, \omega)$,

$$
\sum_{i=1}^I c_i(t, \omega) = \sum_{i=1}^I {}^\circ \hat{c}_i(\hat{t}, \omega)
$$

---

[23] Recall that $B = \{\mathcal{I} \in (0, T) \times \mathbf{R}^K : \det \bar{\Sigma}(\mathcal{I}) = 0\}$ is a set of measure zero.

$$= {}^\circ\left(\sum_{i=1}^{I}\hat{c}_i(\hat{t},\omega)\right)$$

$$= {}^\circ\left(\sum_{i=1}^{I}\hat{e}_i(\hat{t},\omega)+\sum_{j=0}^{J}\eta_j\hat{A}_j(\hat{t},\omega)\right)$$

$$= \sum_{i=1}^{I}{}^\circ\hat{e}_i(\hat{t},\omega)+\sum_{j=0}^{J}\eta_j\,{}^\circ\hat{A}_j(\hat{t},\omega)$$

$$= \sum_{i=1}^{I}e_i(t,\omega)+\sum_{j=0}^{J}\eta_j A_j(t,\omega)$$

$$\sum_{i=1}^{I}z_i(t,\omega) = \sum_{i=1}^{I}{}^\circ\hat{z}_i(\hat{t},\omega)$$

$$= {}^\circ\left(\sum_{i=1}^{I}\hat{z}_i(\hat{t},\omega)\right)$$

$$= {}^\circ\left(\eta_0,\ldots,\eta_J\right)$$

Thus, the consumptions $c_1,\ldots,c_I$ clear the goods market and the trading strategies $z_1,\ldots,z_I$ clear the securities market.

By Theorem B.3, for all $t\in[0,T]$

$$p_A(t,\omega) = {}^\circ\hat{p}_A(\hat{t},\omega)$$

$$= {}^\circ\left(\hat{\gamma}(\hat{t},\omega)-\Delta T\sum_{s\le\hat{t},s<\hat{T}}\hat{p}_C(s,\omega)\hat{A}(s,\omega)\right)$$

$$= {}^\circ\left(\hat{\gamma}(0,\omega)+\int_0^{\hat{t}}\hat{\sigma}\,d\hat{\beta}-\Delta T\sum_{s\le\hat{t},s<\hat{T}}\hat{p}_C(s,\omega)\hat{A}(s,\omega)\right)$$

$$= \gamma(0,\omega)+\int_0^{t}\sigma\,d\beta-\int_0^{t}p_C(s,\omega)A(s,\omega)\,ds$$

Given any hyperfinite trading strategy $\hat{z}_i$ and $t\in\mathcal{T}$, we have

$$\int_0^{t}\hat{z}_i d\hat{\gamma}$$

$$= \sum_{s\in\mathcal{T},s<t}\hat{z}_i(s,\omega)\cdot(\hat{\gamma}(s+\Delta T,\omega)-\hat{\gamma}(s,\omega))$$

$$= \sum_{s\in\mathcal{T},s<t}(\hat{z}_i(s,\omega)\hat{\sigma}(s,\omega))\cdot\left(\hat{\beta}(s+\Delta T,\omega)-\hat{\beta}(s,\omega)\right)$$

$$= \int_0^{t}(\hat{z}_i\hat{\sigma})\,d\hat{\beta}$$

Since $\hat{c}_1,\ldots,\hat{c}_I,z_1,\ldots,z_I$ are adapted to the hyperfinite filtration, $\hat{c}_1,\ldots,\hat{c}_I,z_1,\ldots,z_I$ are adapted to the Loeb filtration; since $\hat{c}_1,\ldots,c_I\in SL^p$ for all $p\in[1,\infty)$, $c_1,\ldots,c_I\in L^p$ for all $p\in[1,\infty)$ (Anderson [2]). For all $t\in[0,T)$,

$$p_A(t,\omega)\cdot z_i(t,\omega)-p_A(0,\omega)\cdot e_{iA}(\omega)$$

$$-\int_0^t z_i\, d\gamma - \int_0^t p_C(s,\omega)\left(e_i(s,\omega) - c_i(s,\omega)\right) ds$$
$$= \quad p_A(t,\omega)\cdot z_i(t,\omega) - p_A(0,\omega)\cdot e_{iA}(\omega)$$
$$-\int_0^t z_i\sigma\, d\beta - \int_0^t p_C(s,\omega)\left(e_i(s,\omega) - c_i(s,\omega)\right) ds$$
$$\simeq \quad \hat{p}_A(\hat{t},\omega)\cdot \hat{z}_i(t,\omega) - \hat{p}_A(0,\omega)\cdot \hat{e}_{iA}(\omega)$$
$$-\int_0^{\hat{t}} \hat{z}_i\hat{\sigma}\, d\hat{\beta} - \int_0^{\hat{t}} \hat{p}_C(s,\omega)\left(\hat{e}_i(s,\omega) - \hat{c}_i(s,\omega)\right) ds$$
$$= \quad \hat{p}_A(\hat{t},\omega)\cdot \hat{z}_i(\hat{t},\omega) - \hat{p}_A(0,\omega)\cdot \hat{e}_{iA}(\omega)$$
$$-\int_0^{\hat{t}} \hat{z}_i\, d\hat{\gamma} - \int_0^{\hat{t}} \hat{p}_C(s,\omega)\left(\hat{e}_i(s,\omega) - \hat{c}_i(s,\omega)\right) ds$$
$$= \quad 0$$

since the trading strategy $\hat{z}_i$ finances the consumption $\hat{c}_i$. Thus, the trading strategy $z_i$ finances the consumption $c_i$.

For all nodes $(t,\omega)$ in the hyperfinite economy,
$$\hat{\lambda}_i * h_i'(c_i(t,\omega), \mathcal{I}(t,\omega)) = \hat{p}_C(t,\omega)$$
for each $i$. Recalling that $\hat{\lambda}_i \not\simeq 0$, so $\lambda_i \neq 0$, and $°\hat{p}_C \in (0,\infty)$ Loeb almost-everywhere,

$$\lambda_i \left.\frac{\partial h_i}{\partial c_i}\right|_{(c_i(t,\omega),\mathcal{I}(t,\omega))} = \quad °\left(\hat{\lambda}_i\right) ° \left(*\left.\frac{\partial h_i}{\partial c_i}\right|_{(\hat{c}_i(t,\omega),\hat{\mathcal{I}}(t,\omega))}\right)$$
$$= \quad °\left(\hat{\lambda}_i * \left.\frac{\partial h_i}{\partial c_i}\right|_{(\hat{c}_i(t,\omega),\hat{\mathcal{I}}(t,\omega))}\right)$$
$$= \quad °\hat{p}_C(\hat{\mathcal{I}}(t,\omega))$$
$$= \quad p_C(\mathcal{I}(t,\omega))$$

holds for almost all nodes in the Loeb continuous-time economy. This shows that for almost all nodes, the consumptions $c_1(t,\omega), \ldots, c_I(t,\omega)$ satisfy the necessary conditions for the Negishi weight problem (maximize $\sum_{i=1}^I \lambda_i h_i\left(c_i(t,\omega), \mathcal{I}(t,\omega)\right)$ subject to $\sum_{i=1}^I c_i(t,\omega) = c(t,\omega)$), as well as the necessary conditions for demand.

Since preferences are strictly concave, the necessary conditions are in fact sufficient. Here, we present the details for the demand; the argument for the Negishi weight problem is similar. Suppose that $c_i'$ lies in the budget set. Then

$$E\left(p_C(T)c_i'(T) + \int_0^T p_C(t)c_i'(t)\, dt\right) \leq e_{iA}\cdot p_A(0) + E\left(\int_0^T p_C(t)e_i(t)\, dt\right)$$
$$= E\left(p_C(T)c_i(T) + \int_0^T p_C(t)c_i(t)\, dt\right)$$

Since $h_i$ and $H_i$ are concave

$$h_i(c_i'(t,\beta),(t,\beta)) \leq h_i(c_i(t,\beta),(t,\beta)) + \left.\frac{\partial h_i}{\partial c_i}\right|_{(c_i(t,\beta),(t,\beta))}(c_i'(t,\beta) - c_i(t,\beta))$$

$$H_i(c_i'(T,\beta),(T,\beta)) \leq H_i(c_i(T,\beta),(T,\beta)) + \left.\frac{\partial H_i}{\partial c_i}\right|_{(c_i(T,\beta),(T,\beta))}(c_i'(T,\beta) - c_i(T,\beta))$$

so

$$
\begin{aligned}
U(c_i') &- U(c_i) \\
&= E\left( H_i(c_i'(T)) - H_i(c_i(T)) + \int_0^t h_i(c_i'(t)) - h_i(c_i(t))\, dt \right) \\
&\leq E\left( \left.\frac{\partial H_i}{\partial c_i}\right|_{(c_i(T),T,(\beta))} (c_i'(T) - c_i(T)) + \int_0^t \left.\frac{\partial h_i}{\partial c_i}\right|_{(c_i(t),(t,\beta))} (c_i'(t) - c_i(t))\, dt \right) \\
&= \frac{1}{\lambda_i} E\left( p_C(T)(c_i'(T) - c_i(T)) + \int_0^t p_C(t)(c_i'(t) - c_i(t))\, dt \right) \\
&= \frac{1}{\lambda_i}\left( E\left( p_C(T)c_i'(T) + \int_0^T p_C(t)c_i'(t)\, dt \right) - E\left( p_C(T)c_i(T) + \int_0^T p_C(t)c_i(t)\, dt \right) \right) \\
&\leq 0
\end{aligned}
$$

so $c_i$ lies in agent $i$'s demand set. This shows that we have an equilibrium for the Loeb continuous-time economy, and that $\lambda$ is a vector of Negishi utility weights for the Loeb continuous-time economy.

Now, consider the original continuous-time economy specified in the statement of Theorem 2.1. To distinguish the original economy from the Loeb economy, we will denote the Browian Motion in the original economy by $\bar{\beta}$ and let $\bar{\mathcal{I}}(t,\omega) = (t, \bar{\beta}(t,\omega))$. The equilibrium prices, consumptions and (except for a set of measure zero) trading strategies of the Loeb economy are given by functions of $\mathcal{I}(t,\omega)$:

$$
\begin{aligned}
p_A(t,\omega) &= P_A(\mathcal{I}(t,\omega)) \\
p_C(t,\omega) &= P_C(\mathcal{I}(t,\omega)) \\
c_i(t,\omega) &= C_i(\mathcal{I}(t,\omega)) \\
z_i(t,\omega) &= Z_i(\mathcal{I}(t,\omega))
\end{aligned}
$$

Using these functions, define a candidate equilibrium for the original continuous-time economy:

$$
\begin{aligned}
\bar{p}_A(t,\omega) &= P_A(\bar{\mathcal{I}}(t,\omega)) \\
\bar{p}_C(t,\omega) &= P_C(\bar{\mathcal{I}}(t,\omega)) \\
\bar{c}_i(t,\omega) &= C_i(\bar{\mathcal{I}}(t,\omega)) \\
\bar{z}_i(t,\omega) &= Z_i(\bar{\mathcal{I}}(t,\omega))
\end{aligned}
$$

Fix $\bar{\omega}_0 \in \bar{\Omega}$ and suppose that $\omega_0 \in \Omega$ satisfies $\beta(t,\omega_0) = \bar{\beta}(t,\bar{\omega}_0)$.

$$
\begin{aligned}
\bar{z}_i(t,\bar{\omega}_0)\bar{p}_A(t,\bar{\omega}_0) &= Z_i(t,\bar{\beta}(t,\bar{\omega}_0))P_A(t,\bar{\beta}(t,\bar{\omega}_0)) \\
&= Z_i(t,\beta(t,\omega_0))P_A(t,\beta(t,\omega_0)) \\
&= W_i(t,\beta(t,\omega_0)) \\
&= W_i(t,\bar{\beta}(t,\bar{\omega}_0))
\end{aligned}
$$

where $W_i$ is an analytic function of $(t,\beta) \in (0,T) \times \mathbf{R}^K$. By Itô's Lemma, $W_i(t,\bar{\beta}(t))$ must satisfy the same stochastic differential equation with respect to $\bar{\beta}$ as $W_i(t,\beta(t))$ satisfies with

respect to $\beta$, and the coefficients of the stochastic differential equation are functions of $(t, \bar\beta)$ and $(t, \beta)$ respectively:

$$
\begin{aligned}
d(W_i(t, \beta(t))) &= a(t, \beta(t))\, dt + b(t, \beta(t))\, d\beta \\
d(W_i(t, \bar\beta(t))) &= a(t, \bar\beta(t))\, dt + b(t, \bar\beta(t))\, d\bar\beta
\end{aligned}
$$

Since $z_i$ finances $c_i$, we have

$$
\begin{aligned}
a(t, \beta) &= (C_i(t, \beta) - f_i(t, \beta)) \\
b(t, \beta) &= \frac{\partial D_i(t, \beta)}{\partial \beta}
\end{aligned}
$$

for all $(t, \beta) \in (0, T) \times \mathbf{R}^K$ so

$$
\begin{aligned}
a(t, \bar\beta) &= (C_i(t, \bar\beta) - f_i(t, \bar\beta)) \\
b(t, \bar\beta) &= \frac{\partial D_i(t, \bar\beta)}{\partial \bar\beta}
\end{aligned}
$$

so $\bar z_i$ finances $\bar c_i$. Since $z_i$ is admissible, $\bar z_i$ is admissible and

$$
E\left(\bar p_C(T)\bar c_i(T) + \int_0^T \bar p_C(t)\bar c_i(t)\, dt\right) = \bar e_{iA} \cdot \bar p_A(t) + E\left(\int_0^T \bar p_C(t)\bar e_i(t)\, dt\right)
$$

$$
\begin{aligned}
\sum_{i=1}^I \bar c_i(t, \beta) &= \sum_{i=1}^I c_i(t, \beta) \\
&= \sum_{i=1}^I e_i(t, \beta) + \sum_{j=0}^J \eta_j A_j(t, \beta) \\
&= \sum_{i=1}^I \bar e_i(t, \beta) + \sum_{j=0}^J \eta_j \bar A_j(t, \beta) \\
\sum_{i=1}^I \bar z_i(t, \beta) &= \sum_{i=1}^I z_i(t, \beta) \\
&= \sum_{j=0}^J \eta_j
\end{aligned}
$$

Thus, the markets for consumption and securities clear at almost all nodes in the original continuous-time economy.

For almost all nodes in the original continuous time economy,

$$
\begin{aligned}
\lambda_i \left.\frac{\partial h_i}{\partial c_i}\right|_{(\bar c_i(t, \bar\beta), (t, \bar\beta))} &= \lambda_i \left.\frac{\partial h_i}{\partial c_i}\right|_{(c_i(t, \beta), (t, \beta))} \\
&= p_C(\mathcal{I}(t, \omega)) \\
&= \bar p_C(\bar{\mathcal{I}}(t, \omega))
\end{aligned}
$$

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This shows that for almost all nodes, the consumptions $c_1(t, \omega), \ldots, c_I(t, \omega)$ satisfy the necessary conditions for the Negishi weight problem maximize $\sum_{i=1}^{I} \lambda_i h_i \left( c_i(t, \omega), \mathcal{I}(t, \omega) \right)$ subject to $\sum_{i=1}^{I} c_i(t, \omega) = c(t, \omega)$, as well as the necessary conditions to be the demand.

Since preferences are strictly concave, the necessary conditions are in fact sufficient; the argument is exactly the same as the argument above for the Loeb continuous-time economy. This shows that we have an equilibrium for the original continuous-time economy, and that $\lambda$ is a vector of Negishi utility weights for the original continuous-time economy.

This completes the proof of Theorem 2.1.

Let $\mathcal{E}$ denote the set of all equilibria of the original continuous-time economy. Recall that we have been working in a discretization indexed by a particular $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, but suppressing the index $n$. Putting the index back into the notation, we have shown that for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$,

$$(\lambda_n, p_{nA}, p_{nC}, (c_{n1}, \ldots, c_{nI}), (Z_{n1}, \ldots, Z_{nI})) \in \mathcal{E}$$

so[24]

$$(\lambda_n, {}^*p_{nA}, {}^*p_{nC}, ({}^*c_{n1}, \ldots, {}^*c_{nI}), ({}^*Z_{n1}, \ldots, {}^*Z_{nI})) \in {}^*\mathcal{E}$$

Fix $\varepsilon \in \mathbf{R}_{++}$. Equation (34) tells us that

$$\exists_{m \in {}^*\mathbf{N}} \; \forall_{n \geq m} \; \exists_{(\lambda_n, p_{nA}, p_{nC}, (Z_{n1}, \ldots, Z_{nI}), (c_{n1}, \ldots, c_{nI})) \in {}^*\mathcal{E}}$$
$$\hat{\mu} \left( \left\{ \omega : \max_{t \in \mathcal{T}_n} \left| \hat{p}_{nA}(t, \omega) - {}^*p_{nA}(\hat{\mathcal{I}}(t, \omega)) \right| < \varepsilon \right\} \right) > 1 - \varepsilon$$

(any $m \in {}^*\mathbf{N} \setminus \mathbf{N}$ will do). Since the construction of the hyperfinite economy and the price process $\hat{p}_{nA}$ is just the transfer of the construction for finite $n$, the Transfer Principle tells us that

$$\exists_{m \in \mathbf{N}} \; \forall_{n \geq m} \; \exists_{(\lambda_n, p_{nA}, p_{nC}, (Z_{n1}, \ldots, Z_{nI}), (c_{n1}, \ldots, c_{nI})) \in \mathcal{E}}$$
$$\hat{\mu} \left( \left\{ \omega : \max_{t \in \mathcal{T}_n} \left| \hat{p}_{nA}(t, \omega) - p_{nA}(\hat{\mathcal{I}}(t, \omega)) \right| < \varepsilon \right\} \right) > 1 - \varepsilon$$

which is the convergence in probability part of Equation (12); Equation (17) and the convergence in probability parts of Equations (13, 14, 15, 18, 19) in the statement of Theorem 3.2 follow in exactly the same way. Equation (35) tells us that

$$\exists_{m \in {}^*\mathbf{N}} \; \forall_{n \geq m} \; \exists_{(\lambda_n, p_{nA}, p_{nC}, (Z_{n1}, \ldots, Z_{nI}), (c_{n1}, \ldots, c_{nI})) \in {}^*\mathcal{E}} \; \left\| \hat{p}_{nA} - {}^*p_{nA} \circ \hat{\mathcal{I}} \right\|_2 < \varepsilon$$

As before, the Transfer Principle tells us that

$$\exists_{m \in \mathbf{N}} \; \forall_{n \geq m} \; \exists_{(\lambda_n, p_{nA}, p_{nC}, (Z_{n1}, \ldots, Z_{nI}), (c_{n1}, \ldots, c_{nI})) \in \mathcal{E}} \; \left\| \hat{p}_{nA} - p_{nA} \circ \hat{\mathcal{I}} \right\|_2 < \varepsilon$$

This is the convergence in norm part of Equations (12); Equations (11, 16) and the convergence in norm parts of Equations (13, 14, 15, 18, 19) in the statement of Theorem 3.2 follow in exactly the same way. This completes the proof of Theorem 3.2.∎

---

[24]Since $\lambda_n \in \mathbf{R}$, ${}^*\lambda_n = \lambda_n$.

# A    Real Analytic Functions of Several Variables

In this Appendix, we summarize the results on real analytic functions of several variables used in our proof.

**Definition A.1** Let $U \subset \mathbf{R}^n$ be open. A function $F : U \to \mathbf{R}^m$ is *real analytic* if, for every $x_0 \in U$, there is a power series $G_{x_0}(x)$ centered at $x_0$ with a positive radius of convergence $\delta_{x_0}$ such that $F(x) = G_{x_0}(x)$ whenever $x \in U$ and $|x - x_0| < \delta_{x_0}$.

**Theorem A.2 (The Analytic Implicit Function Theorem)** *Suppose* $X \subset \mathbf{R}^n, Y \subset \mathbf{R}^m$ *are open,* $F : X \times Y \to \mathbf{R}^n$ *is real analytic,* $x_0 \in X$, $y_0 \in Y$, $F(x_0, y_0) = 0$, *and*

$$\det \frac{\partial F}{\partial x} \neq 0$$

*at* $(x_0, y_0)$. *Then there are neighborhoods* $U$ *of* $x_0$ *and* $V$ *of* $y_0$ *and a real analytic function* $\phi : V \to U$ *such that for all* $y \in V$,

$$F(\phi(y), y) = 0$$

*Moreover, if* $x \in U$, $y \in V$, *and* $F(x, y) = 0$, *then* $x = \phi(y)$.

**Proof:** The conventional statement of the Implicit Function Theorem establishes all the claims except the claim that the implicit function $\phi$ is real analytic. For this, see Theorem 2.3.5 of Krantz and Parks [62].■

**Theorem A.3** *Let* $U \subset \mathbf{R}^n$ *be open and convex,*[25] $F : U \to \mathbf{R}$ *real analytic. If* $\{x \in U : F(x) = 0\}$ *has positive Lebesgue measure, then* $F$ *is identically zero on* $U$.

**Proof:** Lojasiewicz's Structure Theorem for Varieties (Theorem 6.3.3 of Krantz and Parks [62]) states that if $U$ is an open set in $\mathbf{R}^n$ and $F : U \to \mathbf{R}$ is analytic, then for every $x_0 \in U$, there exists a neighborhood $V_{x_0}$ of $x_0$ such that either $F(x) = 0$ for all $x \in V_{x_0}$ or $\{x \in V_{x_0} : F(x) = 0\}$ is a finite union of real algebraic varieties of dimension $< n$. If $F(x) = 0$ for all $x \in V_{x_0}$ and $y \in U$, there is a ray that passes through $V_{x_0}$ and through $y$; the restriction of $F$ to the ray is an analytic function of a single variable, and it vanishes on an interval (the intersection of the ray with the set $V_0$); since it is well known that an analytic function of one variable that vanishes on an interval is identically zero, we must have $F(y) = 0$ for all $y \in U$ and we are done. On the other hand, if $\{x \in V : F(x) = 0\}$ is a finite union of algebraic varieties of dimension $< n$, $\{x \in V_{x_0} : F(x) = 0\}$ has Lebesgue measure zero. There is a countable collection $\{x_n : n \in \mathbf{N}\}$ such that $\cup_{n \in \mathbf{N}} V_{x_n} \supset U$, so $\{x \in U : F(x) = 0\}$ has Lebesgue measure zero.■

Finally, we show that the conditional expectation of a function of $G(\beta(T))$, conditional on $(t, \beta(t))$ is, under mild hypotheses on $G$, an analytic function of $(t, \beta(t))$. This fact is widely known among probabilists, but we were unable to find a specific reference that shows that the analyticity is *joint* in $(t, \beta(t))$, and we need the joint analyticity. Therefore, we provide a proof.

---

[25]It is sufficient to assume that $U$ is connected and open. We prove the easier case when $U$ is convex, since that is the case we use.

**Theorem A.4** *Suppose $F$ is measurable on $\mathbf{R}^K$ and there exists $r \in \mathbf{R}$ such that*

$$|F(x)| \leq r + e^{r|x|}$$

*Let $\beta$ be a standard $K$-dimensional Brownian Motion, and let*

$$G(t, \beta) = E\left(F(\beta(T))\,|\,\beta(t) = \beta\right)$$

*Then $G(t, \beta)$ is an analytic function of $(t, \beta) \in (0, T) \times \mathbf{R}^K$.*

**Proof:** Fix $t < T$. Then

$$
\begin{aligned}
G(t, \beta) &= \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} F(\beta + x) e^{-|x|^2/2(T-t)}\, dx \\
&= \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} F(y) e^{-|y-\beta|^2/2(T-t)}\, dy \\
&= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} e^{\beta \cdot y/(T-t)} F(y) e^{-|y|^2/2(T-t)}\, dy \\
&= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \sum_{k=0}^{\infty} \frac{\left(\frac{\beta \cdot y}{T-t}\right)^k}{k!} F(y) e^{-|y|^2/2(T-t)}\, dy \\
&= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k k!} \sum_{k_1+\ldots+k_K=k} \\
&\quad \begin{pmatrix} k \\ k_1 \ \cdots \ k_K \end{pmatrix} (\beta_1 y_1)^{k_1} \cdots (\beta_K y_K)^{k_K} F(y) e^{-|y|^2/2(T-t)}\, dy \\
&= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k} \sum_{k_1+\ldots+k_K=k} \\
&\quad \frac{(\beta_1 y_1)^{k_1} \cdots (\beta_K y_K)^{k_K}}{k_1! \cdots k_K!} F(y) e^{-|y|^2/2(T-t)}\, dy \qquad (36)
\end{aligned}
$$

We view the two sums as an integral over

$$\{(k, k_1, \ldots, k_K) : k, k_1, \ldots, k_K \in \mathbf{N} \cup \{0\}, k = k_1 + \cdots + k_K\}$$

where each element has mass 1, so the expression becomes the integral of a product measure space. We want to use Fubini's Theorem to interchange the order of the integral and the sums. In order to do this, we need to show that the integrand is $L^1$ on the product. In the following calculation, Equation (37) follows because the integral of $|y_1^{k_1} \cdots y_K^{k_K}|$ over any orthant equals the integral of $y_1^{k_1} \cdots y_k^{k_K}$ over the positive orthant, and there are $2^K$ orthants. Equation (39) follows because the component normal random variables $y_1, \ldots, y_K$ are independent, and from the formula for the $k_i^{th}$ moment of the absolute value of a normal random variable. In Equation (40), the integrand is greater than or equal to the integrand in Equation (38) on the set $\{y \in \mathbf{R}_+^K : |y| \leq 4r(T-t)\}$. In Equation (41), the integrand is greater than or equal to the integrand in Equation (38) on the set $\{y \in \mathbf{R}_+^K : |y| > 4r(T-t)\}$; integrating over all of $\mathbf{R}_+^K$ makes the integral still larger. In Equation (42), the second term

comes from the fact that Equation (40) is the integral of a constant over $\{y \in \mathbf{R}_+^K : |y| \leq 4r(T-t)\}$ times the density of a probability distribution over $\mathbf{R}^K$; the third term comes from the formula for the $k_i^{th}$ moment of the absolute value of a normal random variable.

$$\left| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} F(y) e^{-|y|^2/2(T-t)} \, dy \right|$$

$$\leq \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{(r + e^{r|y|})|y_1^{k_1} \cdots y_K^{k_K}|}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} \, dy$$

$$= \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}_+^K} \frac{(r + e^{r|y|}) y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} \, dy \tag{37}$$

$$= \frac{2^K r}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}_+^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} \, dy$$

$$+ \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}_+^K} \frac{y_1^{k_1} \cdots y_K^{k_K} e^{r|y|}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} \, dy \tag{38}$$

$$\leq r \frac{\sqrt{k_1!}(T-t)^{k_1/2} \cdots \sqrt{k_K!}(T-t)^{k_K/2}}{k_1! \cdots k_K!} \tag{39}$$

$$+ \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\{y \in \mathbf{R}_+^K : |y| \leq 4r(T-t)\}} \frac{(4(T-t))^k e^{4r(T-t)}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} \, dy \tag{40}$$

$$+ \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}_+^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} e^{-|y|^2/4(T-t)} \, dy \tag{41}$$

$$\leq \frac{r(T-t)^{k/2}}{\sqrt{k_1! \cdots k_K!}} + \frac{(4(T-t))^k e^{4r(T-t)}}{k_1! \cdots k_K!} + \frac{r(2(T-t))^{k/2}}{\sqrt{k_1! \cdots k_K!}} \tag{42}$$

Consider the power series

$$\sum_{k=0}^{\infty} \frac{1}{(T-t)^k} \sum_{k_1 + \cdots + k_K = k} \beta_1^{k_1} \cdots \beta_K^{k_K}$$

$$\frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} F(y) e^{-|y|^2/2(T-t)} \, dy \tag{43}$$

By Equation (42), the absolute value of the coefficient of $\beta_1^{k_1} \cdots \beta_K^{y_K}$ is bounded by

$$\frac{1}{(T-t)^k} \left( \frac{r(T-t)^{k/2}}{\sqrt{k_1! \cdots k_K!}} + \frac{(4(T-t))^k e^{4r(T-t)}}{k_1! \cdots k_K!} + \frac{r(2(T-t))^{k/2}}{\sqrt{k_1! \cdots k_K!}} \right)$$

$$\leq \frac{3r(T-t)^{-k/2}}{\sqrt{k_1! \cdots k_K!}} + \frac{4^k e^{4r(T-t)}}{k_1! \cdots k_K!}$$

$$\leq \frac{3r(T-t)^{-k/2} + 4^k e^{4r(T-t)}}{\sqrt{k_1! \cdots k_K!}}$$

$$\leq \frac{\left(3r + e^{4r(T-t)}\right) \sqrt{\max\{\frac{1}{T-t}, 16\}}^k}{\sqrt{k_1! \cdots k_K!}}$$

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which shows that the power series in Equation (43) converges absolutely within a positive radius of convergence, by Proposition 2.2.10 of Krantz and Parks [62]. Thus, we can apply Fubini's Theorem to obtain

$$e^{-|\beta|^2/2(T-t)} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k} \sum_{k_1+\cdots+k_K=k} \beta_1^{k_1} \cdots \beta_K^{k_K}$$

$$\frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} F(y) e^{-|y|^2/2(T-t)} \, dy$$

$$= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k} \sum_{k_1+\cdots+k_K=k}$$

$$\frac{(\beta_1 y_1)^{k_1} \cdots (\beta_K y_K)^{k_K}}{k_1! \cdots k_K!} F(y) e^{-|y|^2/2(T-t)} \, dy$$

$$= G(t,\beta)$$

by Equation(36). Therefore, for fixed $t < T$, $G(t,\beta)$ is a product of two analytic functions, hence is an analytic function of $\beta \in \mathbf{R}^K$. For any $s \in (0,t)$, we have

$$G(s,\beta) = \frac{1}{(2\pi(t-s))^{K/2}} \int_{\mathbf{R}^K} G(t,\beta+x) e^{-|x|^2/2(t-s)} \, dx$$

The right side is an integral with respect to $x$ of an analytic function of $(s,\beta,x)$, and hence $G$ is analytic on $(0,t) \times \mathbf{R}^K$ by Proposition 2.2.3 of Krantz and Parks [62]. Since this is true for every $t < T$, $G$ is analytic on $(0,T) \times \mathbf{R}^K$.∎

# B  Nonstandard Stochastic Integration

In order to show that the equilibrium of the hyperfinite economy generates an equilibrium of the standard continuous-time economy, one needs to show that capital gains are the same in the two settings. Capital gains are given by Stieltjes Integrals with respect to securities prices in the hyperfinite setting and by Itô Integrals with respect to securities prices in the continuous-time setting.

Anderson [2] showed that the Itô Integral with respect to Brownian Motion is the standard part of a Stieltjes Integral with respect to a hyperfinite random walk. Anderson's theorem covers hyperfinite random walks which move independently in each component by an amount $\pm 1/\sqrt{n}$. In that random walk, each node in the tree has $2^K$ successor nodes. As discussed above, in order to obtain dynamic completeness in the hyperfinite model, we need to use a random walk in which each node has $K+1$ successor nodes. Thus, Anderson's theorem does not cover the case considered here.

Lindstrøm [63, 64, 65, 66] showed that the stochastic integral with respect to a square integrable martingale is the standard part of a Stieltjes Integral with respect to a hyperfinite $SL^2$ martingale. Lindstrom's theorem is limited to one-dimensional martingales. Because the components of a vector Brownian Motion are uncorrelated, a process is Itô Integrable with respect to a vector Brownian Motion if and only if it is integrable with respect to each component. However, the components of a vector martingale can be correlated and

consequently, a process can be integrable with respect to a vector martingale even if it is not integrable with respect to the individual components. This fact has economic significance. If two components of the vector martingale are instantaneously nearly perfectly correlated at some point, then the equilibrium trading strategy may well require taking a nonstandard infinite long position in one security and a nonstandard infinite short position in the other. In both the hyperfinite and continuous-time model, the capital gain is well-defined and finite when computed with respect to the vector martingale. However, the hyperfinite capital gain may be a positive nonstandard infinite number in one component and a negative nonstandard infinite number in the other components; they add up to a well-defined finite integral when both components are considered. The continuous-time capital gain may be undefined with respect to the two components when considered separately, but well-defined and finite when the integral is computed with respect to the vector martingale.

Thus, we need to extend either Anderson's theorem or Lindstrøm's theorem. The more general approach would be to extend Lindstrøm's theorem to vector martingales; such an extension is probably needed to tackle the dynamically incomplete markets case. However, it is considerably easier, and sufficient for our purposes in this paper, to extend Anderson's theorem to the particular kind of random walk considered here. This is the approach we follow.

For definitions of standard terms in stochastic integration (such as $\mathcal{H}^2$ and $\mathcal{L}^2$), see Nielsen [81]. For definitions of nonstandard terms such as $SL^2$ lifting, see Anderson [2].

**Theorem B.1** *Let $\hat{\beta}$ be the hyperfinite random walk defined above, and $\beta = {}^\circ\hat{\beta}$ the standard Brownian Motion it generates. Suppose $Z \in \mathcal{H}^2$. Then there is an $SL^2$ lifting $\hat{Z}$ of $Z$. Given any $SL^2$ lifting $\hat{Z}$ of $Z$, for every $t \in \mathcal{T}$ we have*

$$ {}^\circ \int_0^t \hat{Z} d\hat{\beta} = \int_0^{{}^\circ t} Z d\beta $$

**Proof:** Lemma 31 in Anderson [2] proves the existence of an $SL^2$ lifting; the hyperfinite probability space is slightly different from the one considered here, but the proof goes through without change.

Theorem 33 in Anderson [2] shows that, with respect to the hyperfinite random walk and Brownian Motion considered there, the Itô Integral is the standard part of the hyperfinite Stieltjes Integral. The proof of Theorem 33 depends on the specific form of the hyperfinite random walk only in establishing the Itô Isometry, so we show that the Itô Isometry holds for the random walk $\hat{\beta}$. $\hat{Z}$ may be a $1 \times 1$ scalar process, $1 \times K$ vector process, or a $(J+1) \times K$ matrix process. The proofs in the vector and matrix cases are virtually identical apart from notation, while the proof in the scalar case is easier, so we assume that $\hat{Z}$ is a $1 \times K$ vector process with $k^{th}$ component $\hat{Z}_k$.

$$ \left\| \int_0^t \hat{Z} d\hat{\beta} \right\|_2 $$
$$ = \left\| \int_0^t \hat{Z}_k d\hat{\beta}_k \right\|_2 $$
$$ = \left\| \sum_{k=1}^K \sum_{s \in \mathcal{T}, s < t} \hat{Z}_k(s, \cdot) \left( v_{s+\Delta T} \right)_k \sqrt{\Delta T} \right\|_2 $$

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$$= \sum_{k=1}^{K} \sum_{s \in \mathcal{T}, s<t} \left\| \hat{Z}_k(s, \cdot) \, (v_{s+\Delta T})_k \, \sqrt{\Delta T} \right\|_2$$

(because the terms $\hat{Z}_k(s, \omega) v_{s+\Delta T}$ are uncorrelated across $s, k$)

$$= \sum_{k=1}^{K} \sum_{s \in \mathcal{T}, s<t} \Delta T \left\| \hat{Z}_k(s, \cdot) \right\|_2$$

$$= \sum_{k=1}^{K} \left\| \hat{Z}_k |_{\{s \in \mathcal{T} : s<t\}} \right\|_2$$

$$= \left\| \hat{Z} |_{\{s \in \mathcal{T} : s<t\}} \right\|_2$$

which establishes the Itô Isometry.

Since we have already shown that $\beta$ is a standard Brownian Motion, the rest of the proof goes through unchanged.∎

**Definition B.2** Suppose $Z \in \mathcal{L}^2$. An $\mathcal{SL}^2$ lifting of $Z$ is an internal nonanticipating process $\hat{Z}$ such that $\hat{Z}(\cdot, \omega) \in SL^2(\mathcal{T} \setminus \{\hat{T}\})$ for almost all $\omega$ and such that $^\circ\hat{Z}(t, \omega) = Z(^\circ t, \omega)$ for almost all $(t, \omega) \in \left( \mathcal{T} \setminus \{\hat{T}\} \right) \times \Omega$.

**Theorem B.3** *Let $\hat{\beta}$ be the hyperfinite random walk defined above, and $\beta = {}^\circ\hat{\beta}$ the standard Brownian Motion it generates. Suppose $Z \in \mathcal{L}^2$. Then there exists an $\mathcal{SL}^2$ lifting of $Z$. If $\hat{Z}$ is any $\mathcal{SL}^2$ lifting of $Z$, then for every $t \in \mathcal{T}$ we have*

$$^\circ \int_0^t \hat{Z} d\hat{\beta} = \int_0^{^\circ t} Z d\beta$$

**Proof:** Let $f(\omega) = \|Z(\cdot, \omega)\|_2$ and find $\bar{f}$ internal such that $^\circ\bar{f}(\omega) = f(\omega)$ $L(\hat{\mu})$-almost surely. Let $\bar{Z}$ be an internal nonanticipating process such that $^\circ\bar{Z}(t, \omega) = Z(^\circ t, \omega)$ $L(\hat{\nu}) \times L(\hat{\mu})$-almost everywhere, so for $L(\hat{\mu})$-almost all $\omega$, $^\circ\bar{Z}(\cdot, \omega) = Z(^\circ t, \omega)$ for $\nu$-almost all $t \in \mathcal{T} \setminus \{\hat{T}\}$. For $m \in {}^*\mathbf{N}$, and let

$$(\bar{Z}_m)_{ij}(t, \omega) = \max \left\{ -m, \min \left\{ m, \bar{Z}_{ij}(t, \omega) \right\} \right\}$$
$$\bar{f}_m(\omega) = \|\bar{Z}_m(\cdot, \omega)\|_2$$

For all $\omega$, we have $\bar{f}_{m+1}(\omega) \geq \bar{f}_m(\omega)$. For $L(\hat{\mu})$ almost all $\omega$, we have $\lim_{m \in \mathbf{N}, n \to \infty} {}^\circ\bar{f}_m(\omega) = f(\omega) < \infty$. Therefore, there exists $m_0 \in \mathbf{N}$ such that for all $m \in \mathbf{N}$, $m \geq m_0$

$$\hat{\mu} \left( \left\{ \omega : \left| \bar{f}_m(\omega) - \bar{f}(\omega) \right| < \frac{1}{m} \right\} \right) > 1 - \frac{1}{m}$$

so we may find $m \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that

$$\hat{\mu} \left( \left\{ \omega : \left| \bar{f}_m(\omega) - \bar{f}(\omega) \right| < \frac{1}{m} \right\} \right) > 1 - \frac{1}{m}$$

so $^\circ\bar{f}_m(\omega) = \bar{f}(\omega)$ $L(\hat{\mu})$-almost surely; for any such $\omega$, $\bar{Z}_m(\cdot, \omega) \in SL^2$ (Anderson [2], Theorem 11), so if we define $\hat{Z} = \bar{Z}_m$, $\hat{Z}$ is an $\mathcal{SL}^2$ lifting of $Z$.

Now, suppose $\hat{Z}$ is any $\mathcal{SL}^2$ lifting of $Z$. For $m \in \mathbf{N}$, define the internal stopping time

$$\hat{\tau}_m(\omega) = \max\left\{t \in \mathcal{T} : \int_0^t \|\hat{Z}(s,\omega)\|_2^2 \, d\hat{\nu} \le m\right\}$$

and define

$$\hat{Z}_m(t,\omega) = \begin{cases} \hat{Z}(t,\omega) & \text{if} \quad t \le \tau_m(\omega) \\ 0 & \text{if} \quad t > \tau_m(\omega) \end{cases}$$

Let $Z_m(t,\omega) = {}^\circ\hat{Z}_m(\hat{t},\omega)$. For $L(\hat{\mu})$-almost all $\omega$, there exists $m(\omega)$ such that $\tau_{m(\omega)} = \hat{T}$, in which case $\hat{Z}_m(\cdot,\omega) = \hat{Z}(\cdot,\omega)$ and therefore

$$\int_0^t \hat{Z}_m \, d\hat{\beta} = \int_0^t \hat{Z}(\cdot,\omega) \, d\hat{\beta}$$

By the definition of the standard stochastic integral (see, for example, page 96 of Steele [88]) and Theorem B.1,

$$\begin{aligned}
\int_0^{\circ t} Z \, d\beta &= \lim_{m\to\infty} \int_0^{\circ t} Z_m \, d\beta \\
&= \lim_{m\to\infty} {}^\circ \int_0^t \hat{Z}_m \, d\hat{\beta} \\
&= {}^\circ \int_0^t \hat{Z} \, d\hat{\beta}
\end{aligned}$$

∎

# C  Proof of Lemma 3.1

**Proof:** [of Lemma 3.1] Suppose first $K = 1$; let $v_0 = (1)$, $v_1 = (-1)$. Then $v_0 \cdot v_0 = v_1 \cdot v_1 = 1$ and $v_0 \cdot v_1 = -1$; $(v_0)_1(v_0)_1 + (v_1)_1(v_1)_1 = 1 + 1 = 2$; and $(v_0)_1 + (v_1)_1 = 1 - 1 = 0$.

Now suppose we have chosen $v_0, \ldots, v_K \in \mathbf{R}^K$ such that $v_j \cdot v_k = -1$ if $j \ne k$ and $v_k \cdot v_k = K$; $\sum_{k=0}^K (v_k)_i (v_k)_j = \delta_{ij}(K+1)$; and $\sum_{k=0}^K v_k = 0$. Let

$$\tilde{v}_k = \begin{cases} \left(\sqrt{\frac{K+2}{K+1}}(v_k)_1, \ldots, \sqrt{\frac{K+2}{K+1}}(v_k)_K, -\frac{1}{\sqrt{K+1}}\right) & \text{for } k = 0, \ldots, K \\ \left(0, \ldots, 0, \sqrt{K+1}\right) & \text{for } k = K+1 \end{cases}$$

For $j, k \in \{0, \ldots, K\}$,

$$\begin{aligned}
\tilde{v}_j \cdot \tilde{v}_k &= \frac{K+2}{K+1} v_j \cdot v_k + \frac{1}{K+1} \\
&= \begin{cases} \frac{K+2}{K+1}(-1) + \frac{1}{K+1} = -1 & \text{if} \quad j \ne k \\ \frac{K+2}{K+1}K + \frac{1}{K+1} = \frac{K^2+2K+1}{K+1} = K+1 & \text{if} \quad j = k \end{cases} \\
\tilde{v}_j \cdot \tilde{v}_{K+1} &= -1 \\
\tilde{v}_{K+1} \cdot \tilde{v}_{K+1} &= K+1
\end{aligned}$$

For $i, j \in \{1, \ldots, K\}$,

$$
\begin{aligned}
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j &= \sum_{k=0}^{K} (\tilde{v}_k)_i (\tilde{v}_k)_j + 0 \\
&= \frac{K+2}{K+1} \sum_{k=2}^{K} (v_k)_i (v_k)_j \\
&= \frac{K+2}{K+1} (K+1) \delta_{ij} \\
&= \delta_{ij} (K+2)
\end{aligned}
$$

For $i \in \{1, \ldots, K\}$ and $j = K+1$,

$$
\begin{aligned}
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j &= \sum_{k=0}^{K} \sqrt{\frac{K+2}{K+1}} (v_k)_i \left( -\frac{1}{\sqrt{K+1}} \right) + 0 \cdot \sqrt{K+1} \\
&= -\frac{\sqrt{K+2}}{K+1} \sum_{k=0}^{K} (v_k)_i + 0 \\
&= 0 \\
&= \delta_{ij} (K+2)
\end{aligned}
$$

For $i = K+1$ and $j \in \{1, \ldots, K\}$, note that the formula is invariant under switching $i$ and $j$. Finally, for $i = j = K+1$

$$
\begin{aligned}
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j &= (K+1) \frac{1}{K+1} + (K+1) \\
&= 1 + (K+1) \\
&= K+2 \\
&= \delta_{ij} (K+2)
\end{aligned}
$$

For $j = 1, \ldots, K$,

$$
\sum_{k=0}^{K+1} (\tilde{v}_k)_j = \sqrt{\frac{K+2}{K+1}} \sum_{k=0}^{K} (v_k)_j = \sqrt{\frac{K+2}{K+1}} \cdot 0 = 0
$$

Moreover,

$$
\sum_{k=0}^{K+1} (\tilde{v}_k)_{K+1} = -\frac{K+1}{\sqrt{K+1}} + \sqrt{K+1} = 0
$$

This shows that $\sum_{k=0}^{K+1} \tilde{v}_k = 0$.

Thus, by induction, we can choose vectors $v_0, \ldots, v_K \in \mathbf{R}^K$ with the specified properties.∎

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