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Three Essays in Operations and Marketing

by

Te Ke

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Abstract

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Doctor of Philosophy in Engineering – Industrial Engineering and Operations Research

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My thesis consists of three essays in the field of operations management and marketing.

In the first essay, I study the problem of consumer search for information on multiple products. When a consumer considers purchasing a product in a product category, the consumer can gather information sequentially on several products. At each moment the consumer can choose which product to gather more information on, and whether to stop gathering information and purchase one of the products, or to exit market with no purchase. Given costly information gathering, consumers end up not gathering complete information on all the products, and need to make decisions under imperfect information. I solve for the optimal search, switch, and purchase or exit behavior in such a setting, which is characterized by an optimal consideration set and a purchase threshold structure. It is shown that a product is only considered for search or purchase if it has a sufficiently high expected utility. Given multiple products in the consumer's consideration set, the consumer only stops searching for information and purchases a product if the difference between the expected utilities of the top two products is greater than some threshold. Comparative statics show that negative information correlation among products widens the purchase threshold, and so does an increase in the number of the choices. Under my rational consumer model, I show that choice overload can occur when consumers search or evaluate multiple alternatives before making a purchase decision. I also find that it is optimal for sellers of multiple products to facilitate information search for low-valuation consumers, while obfuscate information for those with high valuations.

In the second essay, I conduct an empirical study of peer effects of iPhone adoptions on social networks. I use a unique data set from a provincial capital city in China, in a span of over four years starting from iPhones first introduction to mainland China. I construct a social network using six month's call transactions between iPhone adopters and all other users on a carrier's network. Strength of social ties is measured by duration of calls. Based on the network structure, I test whether an individual's adoption decision is influenced by his friends' adoptions. A fixed-effect model shows that, on average, a friend's adoption increases

one's adoption probability in next month by 0.89%, and the marginal effect decreases in the size of his current neighboring adopters. To further control for potential time-varying correlated unobservables, I instrument adoptions of one's friends by their birthdays, based on the fact that consumers are more likely to adopt iPhones on birthdays. The IV estimation shows a slightly smaller peer effect at 0.75%. I also investigate how network structures modulate the magnitude of peer influence. My results show that peer effect is stronger when the influencer has more friends or has a stronger relationship with the influence.

In the third essay, I study the problem of coordination of operations and marketing decisions for new product introductions. In the industry with radical technology push or rapidly changing customer preference, it is firms' common wisdom to introduce high-end product first, and follow by low-end product line extensions. A key decision in this "down-market stretch" strategy is the introduction time. High inventory cost is pervasive in such industries, but its impact has long been ignored during the presale planning stage. This essay takes a first step towards filling this gap. I propose an integrated inventory (supply) and diffusion (demand) framework, and analyze how inventory cost influences the introduction timing of product line extensions, considering substitution effect among successive generations. I show that under low inventory cost or frequent replenishment ordering policy, the optimal introduction time indeed follows the well-known "Now " or "Never" rule. However, sequential introduction becomes optimal as the inventory holding gets more substantial or the product life cycle gets shorter. The optimal introduction timing can increase or decrease with the inventory cost depending on the marketplace setting, requiring a careful analysis.

To my parents,

Guomin Ke and Juying Hu.

To my wife and daughter,

Zhuqing Yang and Chloe Yunxi Ke.

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Chapter 1

Information Gathering on Multiple Alternatives

1.1 Introduction

When a consumer considers purchasing a product in a product category, she can gather information sequentially on several products.[1] Take the purchase of a car as an example. A consumer has some initial expected utilities for the cars in the market. She decides to start searching for information on one of the cars, and keeps gaining further information. Without having complete information on that car, she might decide to switch, and search for information on some other cars, and so on. At some point the consumer may decide to stop searching and purchase one of the cars, or stop searching and leave the market without making any purchase. This paper investigates what is a consumer's optimal search, switch and purchase or exit strategy. Two essential features of this problem are important to highlight: First, a consumer would never gain full information on any of the products given finite search, but will have to make a purchase or exit decision with imperfect information. Second, searching for information is costly to the consumer, so she will want to limit the extent of the search. These search costs could involve the physical cost of traveling to a store, the opportunity cost of time spent searching for information, or the psychological cost of processing information.

This general problem, in addition to applying to the case of a consumer searching for information to choose one product, applies to any setting where a decision-maker searches sequentially for information on multiple options. Search is costly and gradual, and any potential benefit is realized at the end of the search process. Individuals have to make this type of decision quite frequently: politicians seeking better public policies, managers choosing promising research and development projects, individuals looking for jobs, employers recruiting suitable job candidates, and firms considering alternative suppliers. In the consumer setting, the choice of almost any product or service can be seen through this per-

[1] Throughout the paper the consumer is referred to as "she".

spective, from the choice of a car, to that of a house, a coat, a restaurant for dinner, telephone service, etc. Proliferation of product information on the Internet and social media have made more visible and quantifiable the importance of modeling gradual search for information and purchase under imperfect information. While bearing in mind the generality of the problem, we take consumer search in a product category as the leading example in the presentation below.

Although the problem considered is central to choice in a market environment, it is quite under-researched when all its dimensions are included. For the simpler case where all information about an alternative could be learned in one search occasion, there is a large literature on optimal search and some of its market implications (e.g., McCall 1970; Diamond 1971; Rothschild 1974; Weitzman 1979). This literature, however, does not consider the possibility of gradual revelation of information throughout the search process. There is also some literature on gradual learning when a single alternative is considered (e.g., Roberts and Weitzman 1981, Moscarini and Smith 2001, Branco, Sun, and Villas-Boas 2012), and the choice there is between adopting the alternative or not. When faced with more than one alternative, as is the case considered in this paper, the problem becomes more complicated. This is because opting for one alternative in a choice set means giving up potential high payoffs from other alternatives about which the consumer has yet to learn more information. This paper can then be seen as combining these two literatures, with gradual search for information on multiple products.

Another related literature is the one on the multi-armed bandit problem (e.g., Gittins 1979; Whittle 1980; Bergemann and Välimäki 1996; Bolton and Harris 1999), where a decision-maker learns about different options by trying them one for each period, while earning some stochastic rewards along the way. This problem has an elegant result that the optimal policy is to choose the arm with the highest Gittins index, which for each arm only depends on what the decision-maker knows about that arm until then. However, the problem considered here is different from the bandit problem in one major aspect. In the case of gradual search for information considered here, a consumer optimally decides when to stop searching and make a purchase. Therefore, the decision horizon is endogenous, and optimally determined by the decision maker. In contrast, multi-arm-bandit problems generally presume an exogenously given decision horizon, which could be either finite or infinite. In fact, it has been shown that when a decision maker is allowed to choose the optimal stopping time, in general, the optimal policy does not include choosing the product with the highest Gittins index (Glazebrook 1979; Bergman 1981).[2]

[2] In the appendix we summarize the intuition on the role of the Gittins index in multi-armed bandits, and present a counter-example where the Gittins index policy is not the optimal policy in the gradual search for information case considered here. There is also a literature in computer science trying to find algorithms close to the optimal policy with multi-armed bandit problems with a limited budget (e.g., Guha and Munagala 2007, Hoffman, Shahriari, and Freitas 2013), which is close to gradual search for information if the shadow price of the budget constraint is interpreted as the search cost of the consumer. The setting considered here also enables us to solve the optimal search problem with information updates correlated across products, which has not been possible in the multi-armed bandit problem (Gittins, Glazebrook, and Weber 1989).

In this paper we present a framework where we compute the optimal policy for a consumer searching for information across multiple products. We consider a continuous setting where information about the product being searched changes according to a Brownian motion (interpreted as gathering information on different attributes). In this setting we completely characterize the optimal policy of a consumer in search for information in closed-form, by an *optimal consideration set* and *purchase threshold structure*. Given a set of products, a consumer will not consider them all for search or purchase under the optimal policy, because search is costly. We show that a consumer will optimally construct her consideration set by a simple rule: for a product to have a positive probability of being considered for purchase and to remain in the consideration set, its expected utility has to exceed a threshold. Unlike heuristics in previous studies (e.g., Hauser and Wernerfelt 1990; Feinberg and Huber 1996; Hauser 2014), the consideration set in our model is based on the optimal decision rule to a rational consumer model. Given a consumer's consideration set, we further show that, if the cost and informativeness of search are the same across products, a consumer always searches for information on the product with the highest expected utility. Given that there are multiple products in the consumer's consideration set, she should keep searching for information on the product until the difference in her expected utilities of the top two products is sufficiently large. This reflects the idea of a consumer continuing to search for information until one of the products clearly distinguishes itself as the best choice. This purchase threshold structure, formalizes one's intuition that consumers are looking at the *relative*, instead of absolute value of a product, compared with the alternative options.

We also consider the case with different costs and informativeness of search across products, and find that the purchase threshold structure is now different. We show that a consumer should first search on the product with the highest informativeness or lowest search cost, when both products have the same expected utility; and she should search on that product only, when her expected utility of the alternative product is sufficiently large. By searching for information on the product with the highest informativeness of attributes, the consumer learns more information per search cost incurred.

Based on the optimal search policy, we compute the purchase likelihood of a product given a consumer's initial expected utilities of all products, and the probability of no purchase at all. We find that a higher expected utility of one product may lead, under some conditions, to lower sales of all products combined. To understand this point, consider the case with two products. A product with a high expected utility is definitely bought if the alternative product has a low expected utility (such that the consumer does not search for information on any product). Suppose now that the expected utility of the alternative is increased. This encourages the consumer to continue searching on the two products. It is possible that positive information realizes after search, in which case the consumer can still buy at most one product; it is also possible that negative information realizes on both products,

Another related setting is considered in Callander (2011) where the search for the best alternative from a structured continuum of alternatives is done by trial and error, and where the mapping from choices to outcomes is represented as the realized path of a Brownian motion.

in which case no product will be bought at all. Therefore, as the expected utility of the alternative product gets higher, the total sales may decrease. Along the same logic, we find that *choice overload* can occur: more choices available may render a consumer to search more, which might lead to lower purchase likelihood. We also find that higher information availability or a lower consumer search cost leads to lower sales of products with high expected utilities. Therefore, a seller of multiple products should obfuscate information (e.g., increase search costs) on products with high expected utilities, or for high-valuation consumers. This finding parallels recent studies on obfuscation of price information from consumer search (e.g., Gabaix and Laibson 2006; Ellison and Ellison 2009), though under a rather different setting and rationale.

Information can be correlated across products: after a consumer obtains some information on one product, she may get some partial inferences on the alternatives without searching them. We consider the case of information correlation across products, and show that with positive correlation, the consumer requires a smaller difference in expected utilities of the products to choose one of the products, and a bigger difference for negative correlation. The rationale behind this result is that, if information is positively correlated across products, it is more difficult to get a big difference between expected utilities across products, and a small difference can make a consumer choose to purchase one of them. Consumers get higher expected utilities with negatively correlated products, due to a greater chance of one of the products leading to a higher expected payoff. We focus mostly on the two-product case, but also present results for the case with more than two products. We find that more choices of products will widen a consumer's purchase threshold.

The reminder of the paper is organized as follows. In the next section we present a basic model of the two-product case, where products have the same informativeness of attributes and search costs. Section 1.3 presents a consumer's optimal search policy in that case, and Section 1.4 presents results on the probabilities of purchase and no purchase. Section 1.5 considers the case of correlated information across products, and Section 1.6 presents what happens when the informativeness of attributes or search costs are different across products. Section 1.7 considers the case with more than two products. In Section 1.8 we present numerical simulations of a multi-product monopoly's pricing decisions given the consumer search behavior. In section 1.9, we consider discounting, the possibility of a finite mass of product attributes, and the possibility of decreasing informativeness of attributes for each product. Section 1.10 concludes. All proofs are presented in the Appendix.

1.2 Basic Model

Consumer Problem

A consumer gathers information sequentially on n products before making a purchase decision. Each product has many attributes that are uncertain to the consumer *a priori*.

Consider cars for example. Consumers obtain information, such as brand, model, safety design, fuel efficiency, warranty, and numerous other attributes, before deciding which car to buy. Specifically, we model a product as a collection of T attributes. A consumer's utility of product i, U_i is the sum of the utility derived from each attribute of the product.

$$U_i = v_i + \sum_{t=1}^{T} x_{it}, \tag{1.1}$$

where v_i is the consumer's initial expected utility, which is known before search, and x_{it} is the utility of attribute i, which is unknown before search. Without loss of generality, we assume that $\mathrm{E}[x_{it}] = 0$.[3] It is also assumed that x_{it} is independent identically distributed across attribute t for product i.

The independence assumption is based on the fact that only unexpected information changes one's belief, along the same line of Samuelson (1965)'s celebrated proof that properly anticipated stock prices fluctuate randomly. The identically distributed assumption implies that information revealed per search action stays constant over time, which facilitates the analysis and allows for the search problem to be stationary when $T \to \infty$. In the real world consumers may start with the most important attributes, and the longer a consumer spends searching for information, the less information per search she would expect to get. Simply put, a consumer may become more and more certain, as she gets more and more information. We abstract from this possibility in the main model. We discuss further this issue in Section 1.6, where we consider the case that different products can differ in information per search, and in Section 1.9, where we consider numerically the case in which the informativeness of each attribute decreases as more attributes are being checked. Allowing for constant informativeness of attributes permits us to focus on the situation where purchase decisions are done without full information. The search literature with one–step search (e.g., McCall 1970, Diamond 1971, Rothschild 1974) takes one extreme by assuming that a consumer learns everything by one search action, in which case, the information per search is a step function decreasing to zero after one search action. This model takes the other extreme by assuming that the information gained per step of search stays constant over time, and that after each step of search the variance of what is unknown remains unchanged. This model identifies the critical effect of making purchase decisions without full information, and can be seen as approximating situations where consumers have to make purchase decisions when there is substantial information about the products that is still unknown given the search costs, and the consumers make these decisions when checking product attributes that are of similar importance (potentially after the consumer having already checked the most crucial attributes).

Each time a consumer checks one attribute of product i, the consumer pays a search cost c_i, where we assume that the search costs for different products can be different, but are

[3]Suppose $\mathrm{E}[x_{it}] \neq 0$, then we can redefine $x'_{it} = x_{it} - \mathrm{E}[x_{it}]$ and $v'_i = v_i + \mathrm{E}[x_{it}]$. Then we can rewrite $U_i = v'_i + x'_{1i} + \cdots + x'_{it} + \cdots$, where now $\mathrm{E}[x'_{it}] = 0$.

the same across attributes for the same product. Search costs are sunk once paid. After checking t attributes on product i, a consumer's expected utility of the product u_i is:[4]

$$u_i(t) = \mathrm{E}_t\left[U_i\right] = v_i + \sum_{s=1}^{t} x_{is} + \mathrm{E}_t\left[\sum_{s=t+1}^{T} x_{is}\right] = v_i + \sum_{s=1}^{t} x_{is}. \tag{1.2}$$

Given initial expected utilities v_i, search costs c_i, and distribution of x_{it} for $i = 1, \cdots, n$ and $t = 1, \cdots, T$, a consumer's optimal search problem is to dynamically decide which product to search, and when to stop searching, and which product to buy or to buy none of them. In order to make the search problem tractable we consider the case where each attribute is increasingly subdivided into smaller attributes, and the search cost of each smaller attribute converges to zero at the rate that attributes are subdivided, such that in the limit we have a continuous-attribute analog of the discrete-attribute model, where the information in each attribute has infinitesimal importance and the number of attributes go to infinity (see also Bolton and Harris (1999) and Moscarini and Smith (2001) for a similar formulation). This enable us to get a sharp characterization of consumers' optimal search problem in closed form. As in our previous example, safety design, as a broad category, can consist of many minute attributes described in sellers' descriptions, or in thousands of online customer reviews, etc. Another way of thinking of an infinitesimal attribute of a product is as a quantum of valuable information that can be discovered by a consumer by an infinitesimal search. Specifically, under the continuous-attribute formulation, a consumer's utility and conditional expected utility of product i are respectively,

$$U_i = v_i + \int_{t=0}^{T} dB_i(t) = v_i + B_i(T) \tag{1.3}$$

$$u_i(t) = \mathrm{E}_t[U_i] = v_i + B_i(t). \tag{1.4}$$

where $B_i(t)$ is a Brownian motion with zero drift and volatility σ_i^2, where σ_i characterizes the *informativeness* of the consumer's search on product i.[5] The continuous fluctuation of a consumer's expected utility over search reflects the continuous flow of information amassed. The last assumption that we make is that the mass of attributes is infinite, $T \to \infty$, which allows the problem to be stationary. We consider the case with finite T, numerically, in Section 1.9.[6]

In this section, we develop a basic model of optimal search on multiple products, and develop generalizations in Sections 1.5-1.9. Let us consider a consumer, who has two products under consideration for purchase (i.e., $n = 2$), but is interested in buying at most one of

[4] The notation $\mathrm{E}_t[\cdot]$ is short for expectation conditioning on observed realized utilities $x_{i1}, \cdots, x_{it}$.

[5] Given that the x_{it} are independently distributed, by the law of large numbers we have that the change of expected utility follows a Brownian motion. For a detailed exposition of translating a discrete-attribute model to a continuous-attribute model see Branco, Sun, and Villas-Boas (2012).

[6] Alternatively, the solution that we present can be seen as the limit of the optimal solution for finite T when $T \to \infty$.

them. Before making a purchase decision, the consumer optimally chooses which product to search for information on over time. Let us name the two products as product 1 and 2. The two products are *homogeneous* in that they have the same informativeness of search σ, as well as the same unit search cost c. Heterogeneous products are discussed in Section 1.6.

We normalize the consumer's reservation utility without any purchase to be zero. At any point during the search process, the consumer has five choices: to search product 1; to search product 2; to purchase product 1 and leave the market; to purchase product 2 and leave the market; and to exit the market without making any purchase. She makes the decision based on her current expected utilities of the two products, u_1 and u_2.[7] It is assumed that the information updates for the two products are uncorrelated. Specifically, when a consumer searches information on product i, her expected utility of product i gets updated to $u_i + du_i$, with $du_i = dB_i(t)$; whereas her expected utility of the alternative remains unchanged. We relax the assumption to consider correlated information updates in Section 1.5. It is straightforward to show that u_1 and u_2 are sufficient statistics of the past observations, therefore, we can define $V(u_1, u_2)$ to be the consumer's maximum expected utility when she follows the optimal search policy in the future, given her current expected utilities u_1 and u_2. In the language of dynamic programming, u_1 and u_2 are state variables, and $V(u_1, u_2)$ is known as the value function. Given that there is an infinite mass of attributes to be checked, we have that $V(u_1, u_2)$ does not depend on t explicitly.

Note first that the maximum expected utility $V(u_1, u_2)$ is non-decreasing in either of the expected utilities u_1 or u_2, as expected. We state this result in the following lemma (the proof is provided in the appendix).

Lemma 1 *A consumer's maximum expected utility $V(u_1, u_2)$ is non-decreasing in her current expected utilities of the two products u_1 and u_2.*

We now consider the dynamic problem of consumer search.

Dynamics

Let us define the *search strategy* of a consumer as the mapping from her current expected utilities of the two products to her action. To determine a consumer's optimal search strategy we need to solve her maximum expected utility $V(u_1, u_2)$ for all u_1 and u_2. We characterize $V(u_1, u_2)$ by considering the following two cases below.

In one case, if a consumer's optimal decision is to leave the market immediately, with or without a purchase, her maximum expected utility can be obtained directly as

$$V(u_1, u_2) = \max\{0, u_1, u_2\}. \tag{1.5}$$

If her expected utilities of both products are negative, the consumer will exit without any purchase; otherwise she will purchase the product with higher expected utility.

[7] We drop the argument t of $u_i(t)$ below, when there is no confusion.

Consider now the other case, in which it is optimal for the consumer to continue searching for information. Given the continuation of search, a consumer determines which product to search on by expected utility maximization, and pays some search cost. Let us consider an infinitesimal search dt. A consumer's current maximum expected utility $V(u_1, u_2)$ should satisfy the following equation,

$$V(u_1, u_2) = -c\,dt + \max\left\{\mathrm{E}_{t_1}\left[V(u_1 + du_1, u_2)\right], \mathrm{E}_{t_2}\left[V(u_1, u_2 + du_2)\right]\right\}, \tag{1.6}$$

where t_i is the mass of attributes of product i that has been already searched. The first term on the right hand side is the search cost in time dt. The second term is the maximization between the expected utility from searching for information on product 1 and that from searching for information on product 2. Let us do a Taylor expansion of $\mathrm{E}_{t_1}\left[V(u_1 + du_1, u_2)\right]$ to get,

$$\begin{aligned} \mathrm{E}_{t_1}\left[V(u_1 + du_1, u_2)\right] &= \mathrm{E}_{t_1}\left[V(u_1, u_2) + V_{u_1} du_1 + \frac{1}{2} V_{u_1 u_1} du_1^2 + o\left(du_1^2\right)\right] \\ &= V(u_1, u_2) + \frac{\sigma^2}{2} V_{u_1 u_1} dt + o(dt), \end{aligned} \tag{1.7}$$

where V_{u_1} and $V_{u_1 u_1}$ are the first- and second-order partial derivatives with respect to u_1, respectively, and $o(dt)$ represents the terms that converge to zero faster than dt. In writing the second equality above, we have used the fact that $\mathrm{E}_{t_1}[du_1] = \mathrm{E}_{t_1}[dB_1(t_1)] = 0$, and $\mathrm{E}_{t_1}[du_1^2] = \mathrm{E}_{t_1}[dB_1(t_1)^2] = \sigma^2 dt$, which is due to the Ito's Lemma. Similarly, we can do a Taylor expansion of $\mathrm{E}_{t_2}\left[V(u_1, u_2 + du_2)\right]$, and substitute into equation (1.6) to obtain

$$V(u_1, u_2) = -cdt + \max\left\{V(u_1, u_2) + \frac{\sigma^2}{2} V_{u_1 u_1} dt, V(u_1, u_2) + \frac{\sigma^2}{2} V_{u_2 u_2} dt\right\} + o(dt), \tag{1.8}$$

By canceling out the same terms and dividing by dt on both sides of the equation, we obtain the following equality:

$$\max\left\{V_{u_1 u_1}, V_{u_2 u_2}\right\} = \frac{2c}{\sigma^2}. \tag{1.9}$$

This partial differential equation (1.9) completely characterizes a consumer's search behavior when she is willing to continue searching for information. The consumer optimally chooses to search product 1 if and only if

$$V_{u_1 u_1} = \frac{2c}{\sigma^2} \geq V_{u_2 u_2}, \tag{1.10}$$

and similarly for product 2. This optimality condition shows that a consumer optimally chooses which product to search on based on the curvature instead of the slope of her value function. This reflects the essence of information seeking: positive and negative information can occur with equal odds, and, therefore, one should focus on the second-order derivative.

Equation (1.9) determines $V(u_1, u_2)$ when it is optimal for a consumer to continue searching for information; equation (1.5) determines $V(u_1, u_2)$ when it is optimal for a consumer to stop searching. Now we need to determine a boundary that separates the two regimes. Within the boundary, it is optimal for a consumer to continue searching, with $V(u_1, u_2)$ determined by equation (1.9). Beyond the boundary, it is optimal for the consumer to stop searching for information and exit the market with or without a purchase, where $V(u_1, u_2)$ is given by equation (1.5).

Boundary Conditions

Intuitively, when a consumer's expected utility of product i is rather high, she will stop searching for information and purchase product i immediately. This is the upper boundary separating searching and purchasing. On the other hand, when a consumer's expected utilities of both products are rather low, she will stop searching for information, and exit the market without any purchase. This is the lower boundary condition separating searching and exiting. Bearing these ideas in mind, we can construct the boundary conditions.

Let us define $\overline{U}_i(u_j)$ as the *purchase boundary* for product i given the expected utility u_j for product j. Given u_j, when u_i is so high that it reaches $\overline{U}_i(u_j)$, the consumer will be indifferent between continuing searching for information and stopping to purchase product i. Correspondingly, we have the following *value matching* condition at the purchase boundary:

$$V(u_1, u_2)\,|_{u_i=\overline{U}_i(u_j)} = \overline{U}_i(u_j), \quad i \neq j = 1, 2. \tag{1.11}$$

The left-hand side is the utility a consumer expects if she continues searching for information; while the right-hand side is the expected utility a consumer can obtain right away by purchasing product i. The following lemma formalizes our intuition that as a consumer's expected utility of the alternative gets higher, the product under search must provide a correspondingly higher expected utility to incentivize the consumer to stop searching and purchase the product.

Lemma 2 *The purchase boundary of product i, $\overline{U}_i(u_j)$ is non-decreasing in a consumer's expected utility of its alternative, u_j.*

Equation (1.11) can be treated as the definition of the purchase boundary $\overline{U}_i(\cdot)$, but, *per se*, does not suffice to determine the locus of the boundary. The missing element is the *smooth-pasting* condition (e.g., Dixit 1993, p. 30). We make a technical assumption that $\overline{U}_i(\cdot)$ is continuous and piecewise differentiable. The smooth-pasting condition at the boundary of $u_i = \overline{U}_i(u_j)$ is then

$$V_{u_k}(u_1, u_2)\,|_{u_i=\overline{U}_i(u_j)} = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases} \qquad k = 1, 2;\; i \neq j = 1, 2. \tag{1.12}$$

The value matching condition can be thought of as a zero-order condition, and smooth-pasting would be seen as the first-order condition across the boundary. The appendix provides further intuition on the smooth-pasting conditions. Equations (1.11) and (1.12) together constitute the complete set of conditions to determine the upper boundary $\overline{U}_i(u_j)$.

Now let us turn our attention to the lower boundary conditions. Let us define $\underline{U}_i(u)$ as the *exit boundary* for product i. Given u_j, when u_i is so low that it touches $\underline{U}_i(u_j)$, the consumer will be indifferent between continuing searching and exiting the market with or without a purchase. Correspondingly we have the following value matching condition at the lower boundary of $u_i = \underline{U}_i(u_j)$:

$$V(u_1, u_2)\,|_{u_i=\underline{U}_i(u_j)} = \max\{0, u_j\}, \quad i \neq j = 1, 2. \tag{1.13}$$

Similarly we also need the following smooth-pasting conditions at the lower boundary:

$$V_{u_k}(u_1, u_2)\,|_{u_i=\underline{U}_i(u_j)} = 0, \quad k = 1, 2;\ i \neq j = 1, 2. \tag{1.14}$$

Equations (1.13) and (1.14) together constitute the complete set of conditions to determine the exit boundary $\underline{U}_i(u)$.

Since the two products have the same search costs and informativeness of search, they are symmetric in the search strategy space. Therefore, the purchase and exit boundaries should be the same for the two products, which are denoted as $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$ respectively in the discussion that follows.[8]

This completes the mathematical formulation of a consumer's optimal search problem. If a consumer's optimal decision is to stop searching and make a purchase decision, her maximum expected utility $V(u_1, u_2)$ is given by equation (1.5). If a consumer's optimal decision is to continue searching for information, her maximum expected utility $V(u_1, u_2)$ can be solved by combining equation (1.9) with boundary conditions (1.11)-(1.14). Correspondingly, the optimal search strategy can then be inferred from $V(u_1, u_2)$ by equations (1.5) and (1.10).

Technically, to solve equation (1.9) under boundary conditions (1.11)-(1.14) is not as straightforward as to solve a boundary value problem of a partial differential equation (PDE), due to the following two complexities: (1) Although equation (1.9) appears to be a common parabolic PDE, there is a maximization operator in the equation; (2) The purchase and exit boundaries are not given. A consumer needs to decide not only which product to search, which is characterized by the PDE, but also when to stop searching and make a purchase decision, which is characterized by the boundaries. We must solve the PDE and determine the boundaries simultaneously. This is a so-called *problem with ambiguous boundary conditions* (see, Peskir and Shiryaev 2006). We present an analytical solution to the problem in the next section.

1.3 Optimal Search for Information

In this section we solve the problem of optimal search on two products analytically, and characterize the comparative statics. Let us define $a \equiv \frac{\sigma^2}{4c}$, which serves as a natural

[8]When the two products have different search costs and informativeness of search, purchase and exit boundaries differ for different products. We analyze this case with heterogeneous products in Section 1.6.

scale for a consumer's expected utilities of the two products.[9] Let us also introduce the product logarithm function (also known as the Lambert W function): $W(z)$ defined as the upper branch of the inverse function of $z(W) = We^W$. The following theorem presents the solution, with proof in the appendix.

Theorem 1 *There exists a unique solution $V(u_1, u_2)$ along with boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$, which satisfies equations (1.5), (1.9) and (1.11)-(1.14). The value function is obtained as:*

$$V(u_1,u_2) = \begin{cases} \frac{1}{4a}\left[\overline{U}(u_2) - u_1\right]^2 + u_1 & \text{if } u_2 \le u_1 \le \overline{U}(u_2) \text{ and } u_1 \ge \underline{U}(u_2) \\ \frac{1}{4a}\left[\overline{U}(u_1) - u_2\right]^2 + u_2 & \text{if } u_1 \le u_2 \le \overline{U}(u_1) \text{ and } u_2 \ge \underline{U}(u_1) \\ u_1 & \text{if } u_1 > \overline{U}(u_2) \\ u_2 & \text{if } u_2 > \overline{U}(u_1) \\ 0 & \text{otherwise,} \end{cases} \tag{1.15}$$

and the purchase and exit boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$ are given as:

$$\overline{U}(u) = \begin{cases} u + \left[1 + W\left(e^{-(\frac{2u}{a}+1)}\right)\right] a & \text{if } u \ge -a \\ a & \text{otherwise.} \end{cases} \tag{1.16}$$

$$\underline{U}(u) = -a \quad (\text{relevant when } u \le -a)\,. \tag{1.17}$$

Note that the value function takes different forms in different regions. It actually belongs to the class of the so-called *viscosity solution*, a generalization of the classical concept of a solution to PDE, to allow for discontinuities and singularities (see Crandall, Ishii, and Lions 1992). The value function is quadratic in u_i and $\overline{U}(u_j)$ in each region for $i \neq j \in \{1, 2\}$. Note also that the value function, as well as the boundary conditions, is highly nonlinear, expressed in terms of product logarithm functions. Figure 1.1 presents the value function $V(u_1, u_2)$, as well as the payoff from search, which is defined by $V(u_1, u_2) - \max\{u_1, u_2, 0\}$, i.e., the difference between the maximum expected utility when search is allowed and that when search is not allowed.

We first note that the payoff from search is always non-negative. Although information is *ex ante* neutral, search indeed benefits consumers, because consumers have the option to learn the products first before committing to buy a potentially poor fit. Like a stock covered by its put option, search provides an upside possibility while protecting consumers from a downside risk. We also find that the payoff from search peaks at $u_1 = u_2 = 0$, which is the point where a consumer's three options – purchase 1, purchase 2 and exit without purchase – are most undistinguished. A consumer benefits most from search, when she is most uncertain about which option to take without search. It is not hard to show that,

$$\lim_{u\to\infty} V(u,u) - u = \frac{a}{4}. \tag{1.18}$$

[9] The term $\frac{\sigma^2}{4c}$ is the optimal purchase boundary in the single product case (Branco, Sun, and Villas-Boas 2012).

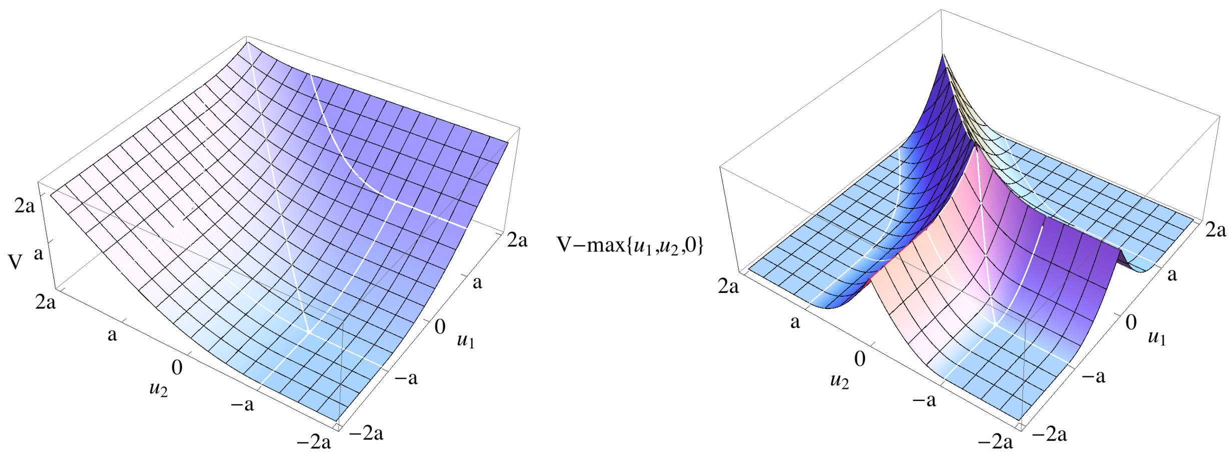

Figure 1.1: Maximum expected utility (left panel) and payoff from search (right panel), given a consumer's current expected utilities.

It implies that a consumer can always benefit from search no matter how high her current expected utilities are, as long as the two alternatives are not easily distinguished from each other.

Given a consumer's maximum expected utility $V(u_1, u_2)$, a consumer's optimal search strategy can be correspondingly determined, as presented in Figure 1.2. As delimited by solid lines, a consumer's expected utility space is segmented into five regions, corresponding to her optimal choice of five actions given her expected utilities of the two products.

As shown by Figure 1.2, roughly speaking, when u_1 is significantly greater than u_2, a consumer will purchase product 1 immediately and leave the market without any search; when u_1 is slightly greater than u_2, a consumer will search for more information on product 1 so as to distinguish between the two products; when u_1 and u_2 are both very low, a consumer will leave the market without any purchase. The following theorem completely characterizes a consumer's optimal search strategy rigorously.

Theorem 2 *Suppose that both products have the same cost and informativeness of search. Then, only products with expected utilities above $-a$ constitute a consumer's consideration set for search and purchase. Given two products in her consideration set, the consumer always searches for information on the one with higher expected utility. She stops searching and purchases the product if the difference in her expected utilities of the two products is above the purchase threshold of $\left[1 + W\left(e^{-(\frac{2u}{a}+1)}\right)\right] a$, where u is her expected utility of the alternative.*[10]

[10] If there is only one product in the consumer's consideration set, one can obtain from Branco et al. (2012) that the consumer stops searching for information and purchases the product when u hits a, and stops searching for information and exits the market when u hits $-a$.

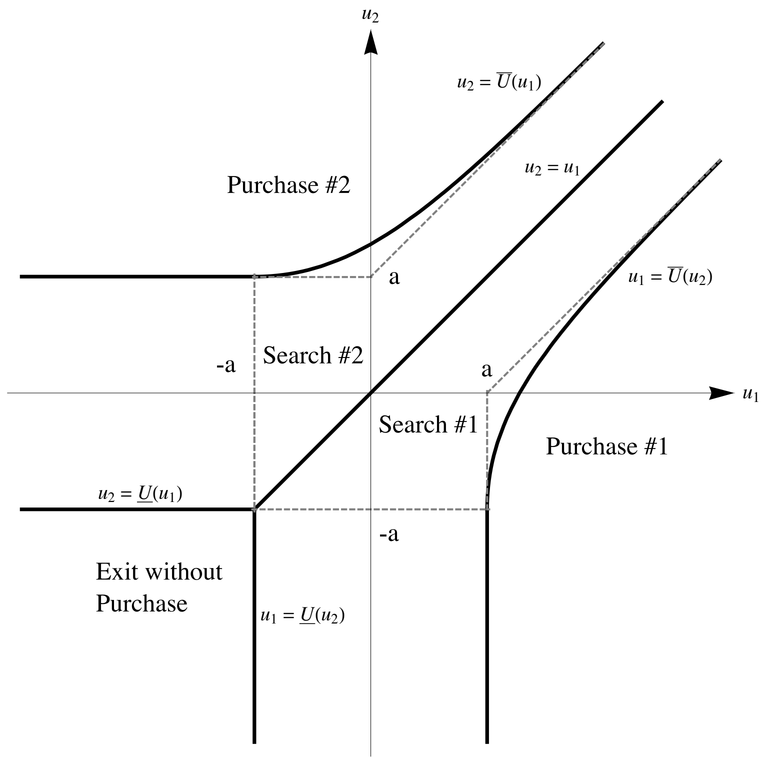

Figure 1.2: Optimal search strategy on two products.

Throughout the whole paper, when talking about "purchase threshold", we always mean the threshold imposed on the difference between the expected utilities of the two products. Note that a consumer's purchase threshold narrows as her expected utility of the alternative u increases, and converges to a relatively quickly. Therefore, a consumer with high expected utilities stops searching and purchases the product if her expected utility of the product exceeds that of the alternative by a. To summarize, we have the following corollary.

Corollary 1 *The purchase threshold on the expected utility difference between the two products decreases as the expected utility of the alternative product increases, and converges to* a.

Given a consumer's optimal search strategy, Figure 1.3 presents a simulation example of a consumer's dynamic search process. The consumer's initial expected utilities are $(.5a, .5a)$. She starts by searching on product 1, then switches to search on product 2 shortly afterwards, and then switches back and forth several times, before she finally decides to purchase product 2. The left panel in Figure 1.3 records the evolution of her expected utilities $u_1(t)$, $u_2(t)$, as well as her purchase boundaries $\overline{U}\left(u_2(t)\right)$ and $\overline{U}\left(u_1(t)\right)$ over time. It shows that when the consumer searches on one product, her expected utility of this product follows a Brownian motion, and her expected utility of the alternative stays constant. The right panel shows the trajectory of her expected utilities in the utility space.

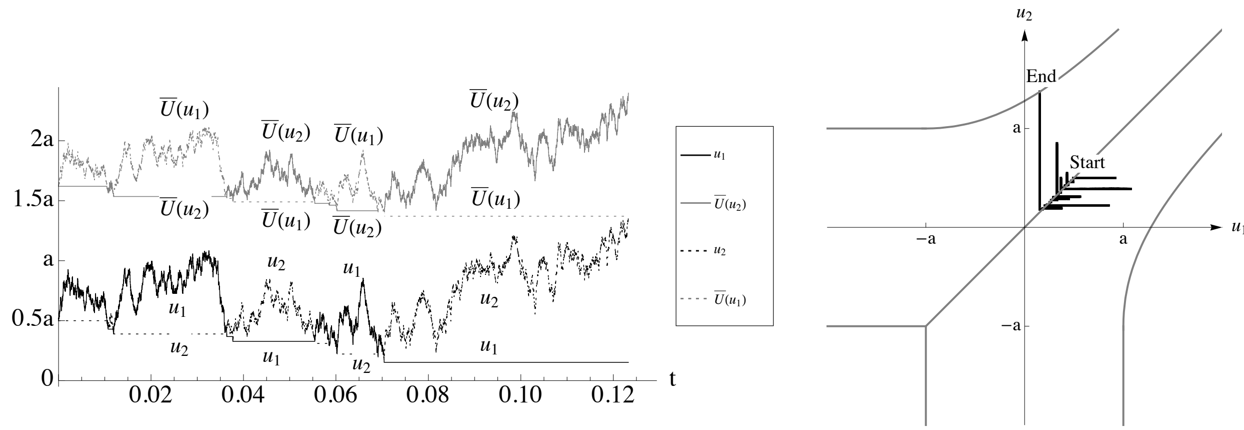

Figure 1.3: An example of a consumer's optimal search process.

The comparative statics are summarized in Proposition 1. We defer the proofs to Section 1.5, where we prove the proposition under a more general model setting.

Proposition 1 *Given a consumer's expected utility of the alternative product as u, her purchase threshold of the product increases in a, i.e., increases in the informativeness of search* σ*, and decreases in the search costs c. Given a consumer's expected utilities of the two products as* u_1 *and* u_2*, her maximum expected utility* $V(u_1, u_2)$ *increases in a, i.e., increases in the informativeness of search* σ*, and decreases in the search costs c. As a goes to infinity,* $V(u_1, u_2)$ *goes to infinity; as a goes to zero,* $V(u_1, u_2)$ *converges to* $\max\{u_1, u_2, 0\}$.

As search costs decrease, or informativeness of search increases, the purchase threshold gets higher, and consequently a consumer searches more, and correspondingly gets more benefit from information. Finally, note that the solution presented, and correspondingly our basic model, is extremely parsimonious in parameterization, with essentially only one parameter, a, given the complexity of the problem.

1.4 Purchase Likelihood

Given a consumer's optimal search strategy, we can infer her purchase likelihood of each product, starting from any initial state (u_1, u_2). Let us define the purchase likelihood of product i as $P_i(u_1, u_2)$. Then, according to symmetry, the purchase likelihood of product 2 starting from (u_1, u_2) would be $P_2(u_2, u_1) = P_1(u_1, u_2)$. The function $P_1(u_1, u_2)$ can be calculated by invoking the Optional Stopping Theorem (see, Williams 1991, page 100) and solving an ordinary differential equation (see details in the appendix).

$$P_1(u_1,u_2)=\begin{cases}0 & \text{if } u_1\le -a \text{ or } u_2\ge \overline{U}(u_1)\\ 1-\frac{\overline{U}(u_2)-u_1}{2a} & \text{if } u_2\le u_1<\overline{U}(u_2) \text{ and } u_1>-a\\ \frac{\overline{U}(u_1)-u_2}{\overline{U}(u_1)-u_1}-\frac{\overline{U}(u_1)-u_2}{2a} & \text{if } -a<u_1<u_2<\overline{U}(u_1)\\ 1 & \text{if } u_1\ge \overline{U}(u_2).\end{cases} \tag{1.19}$$

The left panel in Figure 1.4 presents an illustration of $P_1(u_1,u_2)$. From the figure, we can see the intuitive result (proof is straightforward, thus omitted) that a consumer is more likely to buy one product if her expected utility of the product is higher, or her expected utility of the alternative is lower.

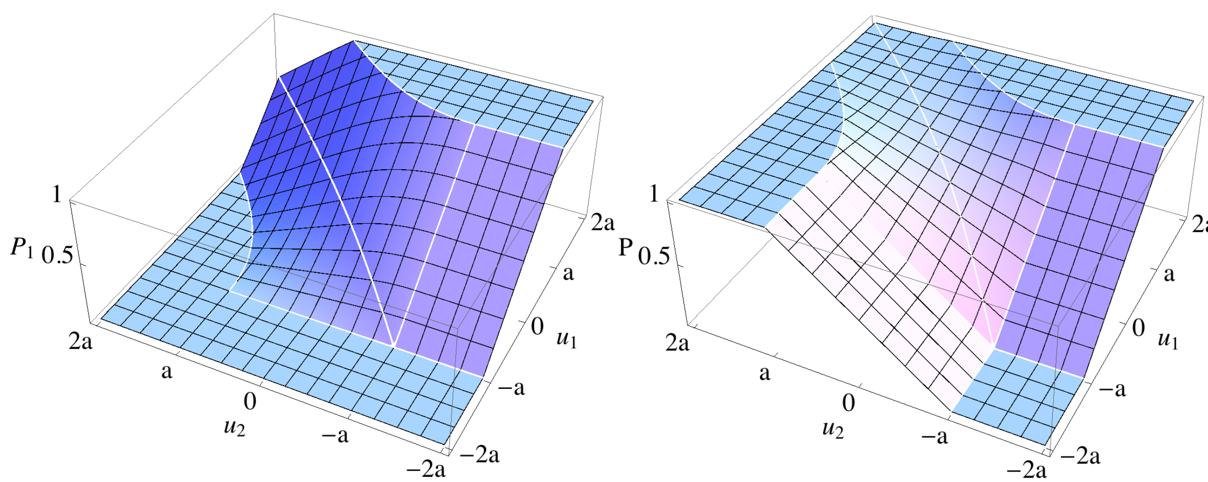

Figure 1.4: Purchase likelihood of product 1 (left), and of at least one product (right).

Let us define the purchase likelihood of at least one product to be $P(u_1,u_2)\equiv P_1(u_1,u_2)+P_2(u_1,u_2)$. The right panel in Figure 1.4 presents an illustration of $P(u_1,u_2)$. It is interesting to note that $P(u_1,u_2)$ does not always increase with u_1 or u_2. This means that a higher expected utility of one product may lead to a lower purchase likelihood of the two products combined. This will never happen in a classical setup without considering consumer search behavior. To understand the intuition, let us consider a special case. Given a consumer's expected utilities of the two products as u_1 and u_2, if u_2 is high enough such that the difference between u_2 and u_1 is greater than the purchase threshold, the consumer will purchase product 2 immediately. In this case, the purchase likelihood is one. Now suppose that for some reason (for example, promotions), the seller increases the consumer's expected utility of product 1. As a result, the difference between u_2 and u_1 is now below the purchase threshold. In this case, the consumer will optimally search for more information before making a purchase decision. After search, it is possible that the consumer likes the products more, in which case, she will buy at most one of them; it is also possible that the consumer gets some negative information on both products, and decides to buy nothing. In general, the purchase likelihood will be lower than one after the increase of u_1. At an aggregated level, a higher expected utility of one product might decrease the total sales. By the same argument, we can show that more available alternatives may decrease the purchase likelihood.[11] In this way, we provide a rational explanation to consumer *choice overload*,[12] under the circumstance that the consumer engages in gradual evaluations or information search before making a choice. More options to choose from may lead a consumer to exert a greater effort level to distinguish the best from the rest, and result in lower probability of choosing anything. It is also noteworthy that more alternatives will never decrease a consumer's *ex ante* welfare in this case, because a consumer can simply ignore the added alternatives; however, it is possible that more alternatives decrease a consumer's *ex post* welfare.

It is also interesting to study the comparative statics of the consumers' purchase likelihood. We describe the results in Figure 1.5, which characterizes how search costs and informativeness of search influence a consumer's purchase likelihoods (see proofs in the appendix). The left panel plots the sign of $\partial P_1(u_1, u_2)/\partial a$ as a function of u_1 and u_2, and the right panel plots the sign of $\partial P(u_1, u_2)/\partial a$. Grayness indicates the sign: if the sign is positive, it is dark gray; if the sign is zero, it is light gray; if the sign is negative, it is white. Thus, the purchase likelihood increases with informativeness and decreases with search costs in the dark gray area; it decreases with informativeness and increases with search costs in the white area; it stays constant in the light gray area. The dashed lines in both plots replicate the boundaries of optimal search strategy shown in Figure 1.2.

Figure 1.5 can lead to the following observations.[13] First, when a consumer's expected utilities of the two products are positive, her purchase likelihood of the product with high (low) expected utility decreases (increases) when informativeness of search increases or search costs decrease. Otherwise, her purchase likelihood of the product with positive (negative) expected unity decreases (increases) when informativeness of search increases or search costs decrease. Therefore, it is not always a wise decision for the seller to facilitate consumer search by increasing informativeness of search or decreasing search costs. In particular, higher informativeness of search or lower search costs may lead to lower purchase likelihood of the high-valuation products.

Second, when a consumer's expected utilities of at least one of the products is relatively high, her purchase likelihood of the two products combined decreases when the informativeness of search increases or search costs decrease. To summarize, to increase information

[11]Introduction of a new product can be equivalently viewed as increasing its expected utility from negative infinity to some positive level.

[12]For lab and field experiments on choice overload, see, e.g., a meta-analytic review by Scheibehenne, Greifeneder, and Todd (2010). See also Kuksov and Villas-Boas (2010) for an alternative explanation of choice overload.

[13]The "protrusion" in the right panel of Figure 1.5 can be understood by considering the case with only one product. It can be shown that given a consumer's expected utility of u, her purchase likelihood $P(u) = \frac{1}{2}\left(1 + \frac{u}{a}\right)$ if $-a < u < a$, $P(u) = 0$ if $u \leq -a$ and $P(u) = 1$ if $u \geq a$. One can easily verify that $\frac{\partial P(u)}{\partial a}$ is discontinuous at $u = -a$. Now in the case of two products, one can similarly show that $\frac{\partial P_i(u_i,u_j)}{\partial a}$ is discontinuous at $u_i = -a$ $(i = 1, 2)$, and thus $\frac{\partial P(u_i,u_j)}{\partial a}$ is discontinuous at $u_i = -a$ $(i = 1, 2)$.

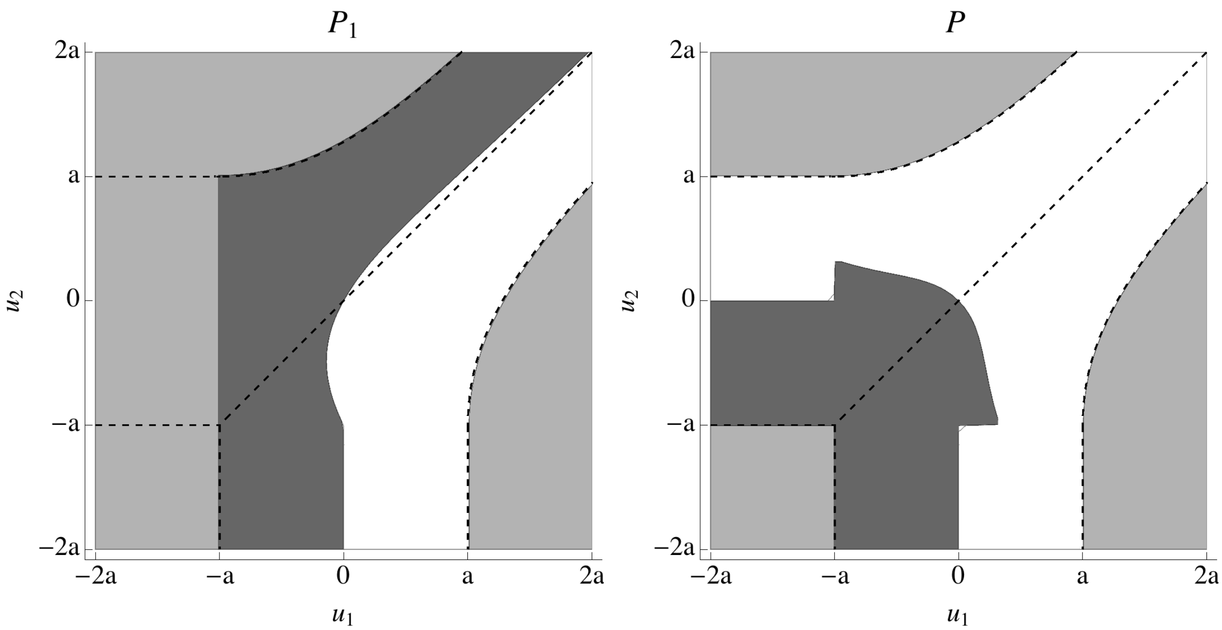

Figure 1.5: Comparative statics of purchase likelihoods $P_1(u_1, u_2)$ and $P(u_1, u_2)$.

availability and to facilitate consumer searching behavior will deteriorate sales for consumers who have a high-valuation of at lead one product, while enhance sales for consumers who have a low-valuation for both products. Therefore, a seller should carefully manage the information accessability of its products, even though information is ex ante neutral. If both products are from the same seller, who cares about the total sales, then he should obfuscate product information from consumer search if currently consumers already have a relatively high valuation of either the two products.

1.5 Products with Correlated Information

Two houses in the same neighborhood share similar characteristics in transportation accessibility, quality of schools, crime statistics, climate, etc. Two car models under the same brand share similar information in engine technology, driving performance, safety design, warranty, etc. In general, two products under purchase consideration may share common attributes. When a consumer searches for information on one product, she will get some partial information on the other at the same time. Sometimes, however, positive information from one product speaks negatively of the other. For example, when searching for information on electric vehicles, consumers may get reviews of disadvantages of traditional gasoline vehicles. That is, information can be correlated either positively or negatively between the two products under consideration.

Possible information correlation among products has so far not been considered in our basic model in Section 1.2. In this section, we extend our basic model to study the problem of optimal search on two products with correlated information. In particular, instead of assuming uncorrelated utility updates, we consider the following utility updating dynamics in a consumer's search process. When a consumer searches for information on product 1, she gets a utility update for product 1 as $du_1 = dB_1(t_1)$; meanwhile she also gets some partial information on product 2, with utility update as $du_2 = \rho\, du_1$. Similarly, when a consumer spends dt in searching for information on product 2, she gets utility update du_2 for product 2, and $du_1 = \rho\, du_2$ for product 1. The constant ρ characterizes the information correlation between the two products. Intuitively, searching one product should not consistently reveal more information about others, hence it is stipulated that $|\rho| < 1$. When $\rho = 0$, we go back to our basic model without inter-product information correlation. As above, we can construct the Bellman equation as well as the boundary conditions for the problem of optimal search on two informationally correlated products.

By taking dt ahead, we have the following iterative relationship:

$$V(u_1, u_2) = -cdt + \max\left\{\mathrm{E}_{t_1}\left[V(u_1 + du_1, u_2 + \rho du_1)\right], \mathrm{E}_{t_2}\left[V(u_1 + \rho du_2, u_2 + du_2)\right]\right\}. \quad (1.20)$$

Similarly we can reduce the equation above as the following partial differential equation:

$$\max\left\{V_{u_1u_1} + \rho^2 V_{u_2u_2}, V_{u_2u_2} + \rho^2 V_{u_1u_1}\right\} + 2\rho V_{u_1u_2} = \frac{2c}{\sigma^2}. \quad (1.21)$$

Despite the slightly increased complexity, one can still obtain that a consumer optimally chooses to search product 1, if and only if

$$V_{u_1u_1} \geq V_{u_2u_2}, \quad (1.22)$$

and *vice versa* for product 2, as long as $|\rho| < 1$.

As for boundary conditions, it turns out that equations (1.11)-(1.14) still apply here exactly. It may appear straightforward at first glance, but the smooth-pasting condition for the general case here with $\rho \neq 0$ is not a trivial result. One should note that we now have a constrained multi-dimensional Brownian motion: a consumer's expected utility can only move along the direction with a slope equal to either ρ or $\frac{1}{\rho}$. We provide the derivation of the smooth-pasting conditions in the appendix. The following theorem presents the solution for the value function.

Theorem 3 *There exists a unique solution $V(u_1, u_2)$ along with boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$, which satisfies equation (1.5) and (1.21) under boundary conditions (1.11)-(1.14). The value function is*

$$V(u_1, u_2) = \begin{cases} \frac{1}{4a}\left[\widehat{U}(u_1, u_2) - u_1\right]^2 + u_1 & \textit{if } u_2 \leq u_1 \leq \overline{U}(u_2) \textit{ and } u_1 \geq \underline{U}(u_2) \\ \frac{1}{4a}\left[\widehat{U}(u_2, u_1) - u_2\right]^2 + u_2 & \textit{if } u_1 \leq u_2 \leq \overline{U}(u_1) \textit{ and } u_2 \geq \underline{U}(u_1) \\ u_1 & \textit{if } u_1 > \overline{U}(u_2) \\ u_2 & \textit{if } u_2 > \overline{U}(u_1) \\ 0 & \textit{otherwise,} \end{cases} \quad (1.23)$$

where $\widehat{U}(u_i, u_j)$*, with support on* $\left\{(u_i, u_j)|u_j \leq u_i \leq \overline{U}(u_j) \text{ and } u_i \geq -a\right\}$*, is defined as*

$$\widehat{U}(u_i, u_j) \equiv \begin{cases} \frac{u_j-\rho u_i}{1-\rho} + (1-\rho)\left[1 + W\left(\frac{1+\rho}{1-\rho}e^{-\frac{2(u_j-\rho u_i)}{(1-\rho)^2(1+\rho)a}-\frac{1-2\rho-\rho^2}{1-\rho^2}}\right)\right]a & u_j \geq \rho u_i - (1-\rho)a \\ a & \text{otherwise.} \end{cases} \tag{1.24}$$

The purchase and exit boundaries $\overline{U}(\cdot)$ *and* $\underline{U}(\cdot)$ *are given as*

$$\overline{U}(u) = \begin{cases} u + (1-\rho^2)\left[W\left(e^{-\frac{2u}{(1-\rho^2)a}-\frac{1-4\rho+\rho^2}{1-\rho^2}}\right)+1\right]a & \text{if } u \geq -(1-2\rho)a \\ a & \text{otherwise.} \end{cases} \tag{1.25}$$

$$\underline{U}(u) = -a \quad (\text{relevant when } u \leq -a)\,. \tag{1.26}$$

The value function above is similar to its counterpart in the uncorrelated case in Theorem 1, except that $V(u_1, u_2)$ is no longer quadratic in the purchase boundary $\overline{U}(u_i)$, rather it is quadratic in $\widehat{U}(u_i, u_j)$. In fact, $\widehat{U}(u_i, u_j)$ is also related to the concept of purchase boundary. Given a consumer's current expected utilities of the two products $u_1 \geq u_2$, Theorem 3 states that she will search for information on product 1. During the search process, she gets new information on product 1 as well as some partial new information on product 2. If she has accumulated enough positive information on product 1, she will purchase product 1 at some point. The term $\widehat{U}(u_1, u_2)$ is her expected utility of product 1 at the boundary when she is indifferent between continuing searching for information on product 1 and purchasing product 1, given that she starts from (u_1, u_2). The model is still quite parsimonious, parameterized by a and ρ only. Figure 1.6 presents an illustration of the value function $V(u_1, u_2)$.

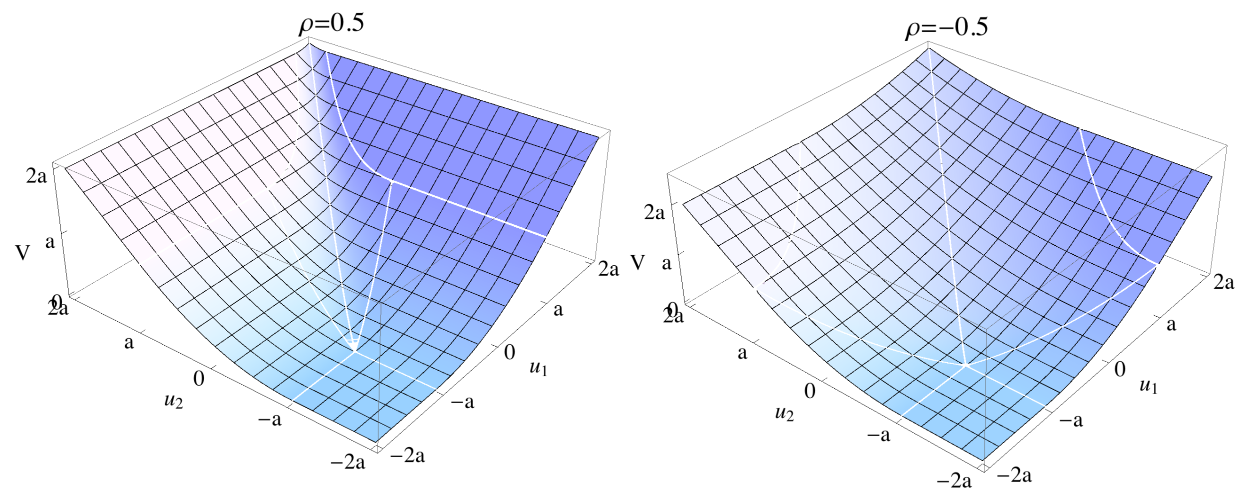

Figure 1.6: Maximum expected utility of two products with correlated information.

With information spillovers between products, a consumer's optimal search strategy is similar to the case without information correlation. Inter-product information correlation impacts both a consumer's consideration set and the purchase threshold. The following theorem characterizes a consumer's optimal search strategy. The corresponding corollary describes the optimal search strategy when the expected utilities of the two products are relatively high.

Theorem 4 *With information correlated between two products, a consumer considers a product for search and purchase if and only if her expected utility of the product is above* $-a + \max\{\rho(u+a), 0\}$*, where* u *is her expected utility of the alternative product, and* ρ *is the information correlation coefficient. Given two products in her consideration set, the consumer always searches for information on the product with higher expected utility. She stops searching for information and purchases the product if the difference in her expected utilities of the two products is above the purchase threshold of* $(1-\rho^2)W\left(e^{-\frac{2u}{(1-\rho^2)a}-\frac{1-4\rho+\rho^2}{1-\rho^2}}\right)a + (1-\rho)^2 a$.

Corollary 2 *The purchase threshold on the expected utility difference between the two products decreases as the expected utility of the alternative product increases, and converges to* $(1-\rho)^2 a$.

Figure 1.7 illustrates a consumer's optimal search strategy given her current expected utilities of the two products, under both positive and negative information correlation.

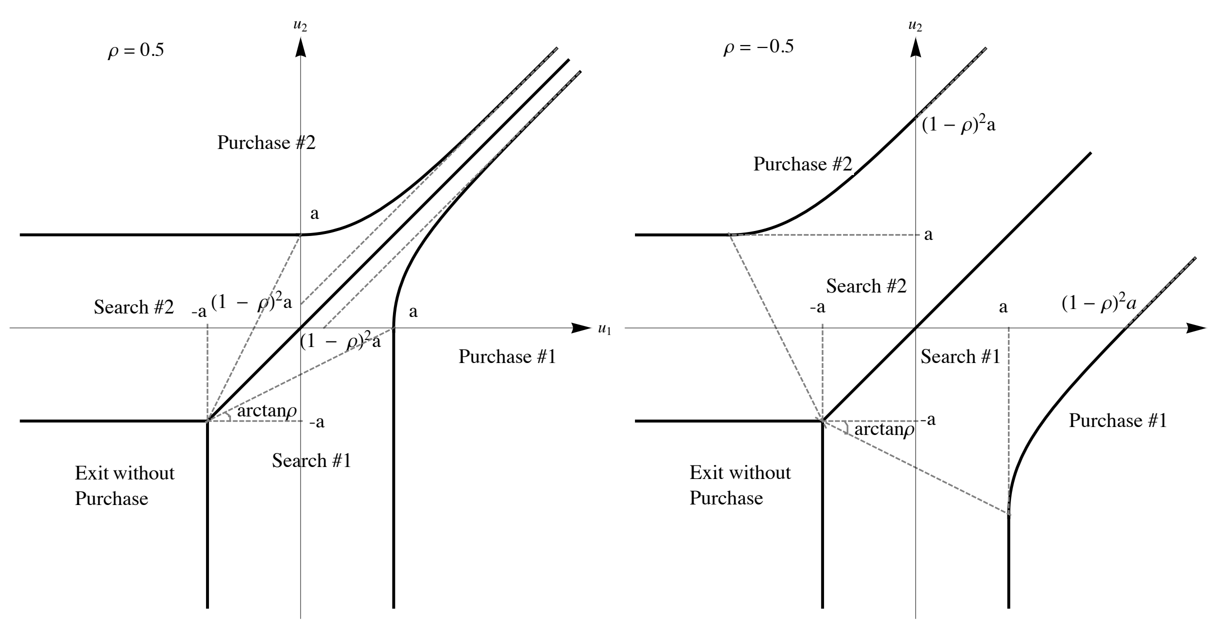

Figure 1.7: Optimal search strategy on two products with correlated information.

The comparative statics are summarized in Proposition 2.

Proposition 2 *Given a consumer's expected utility of the alternative product u, her purchase threshold of the product increases in a and decreases in the information correlation ρ. Given a consumer's expected utilities of the two products u_1 and u_2, her maximum expected utility $V(u_1, u_2)$ increases in a and decreases in the information correlation ρ.*

As information correlation gets higher, a consumer will impose a narrower purchase threshold on the difference between her expected utilities of the two products. Therefore, two products with positive information correlation compete with each other more fiercely: a small informational advantage can render a consumer to choose one product over the other. Interestingly, a consumer expects higher expected utility when searching for information over two products with negative information correlation. In fact, negative information correlation benefits consumers by playing a role of insurance. During the search process, as a consumer is downgrading one product, she favors the other product more at the same time. This increases a consumer's likelihood of purchase, and thus her expected utility.

For a firm selling two products, it would then be better to sell products with negative correlation in attribute fit than positive correlation, as products with a negative correlation lead to a greater probability of one of the products being bought by any given consumer. Furthermore, in terms of obfuscation strategies, obfuscation would be even more beneficial in the case of positive correlation if the expected valuations are high, as bad information on one product also means a negative shock on the other product. On the other hand, the firm would tend to reduce obfuscation and facilitate search in the case of negatively correlated products, as in that case bad news about one product means good news about the other product.

1.6 Heterogeneous Products

Another natural extension to our basic model is to consider heterogeneous products, where searching cost c_i and informativeness coefficient σ_i are different across products. We restrict our discussion on two products with uncorrelated information only.

The problem formulation is similar to the homogeneous case. Given c_i and σ_i for product i $(i = 1, 2)$, equation (1.9) now would be

$$\max\left\{-2c_1 + \sigma_1^2 V_{u_1u_1}, -2c_2 + \sigma_2^2 V_{u_2u_2}\right\} = 0. \tag{1.27}$$

A consumer optimally chooses to search product 1, if and only if

$$V_{u_1u_1} = \frac{2c_1}{\sigma_1^2} \text{ and } V_{u_2u_2} \leq \frac{2c_2}{\sigma_2^2}, \tag{1.28}$$

and *vice versa* for product 2. The boundary conditions (1.11)-(1.14) apply directly here by recognizing that the purchase boundary $\overline{U}_i(u)$ and exit boundary $\underline{U}_i(u)$ are specific for each product i.

By defining $a_i \equiv \frac{\sigma_i^2}{4c_i}$ $(i = 1, 2)$, the optimal search problem is, in fact, completely characterized by only two parameters: a_1 and a_2. From a mathematical perspective, the optimal search problem with heterogeneous products is a nontrivial extension of that in the homogeneous case. The new complexity comes from the difficulty of pinning down the boundary between searching product 1 and searching product 2. Nevertheless, the problem can still be solved analytically. The purchase boundary $\overline{U}_i(u)$ now cannot be written down explicitly, and instead is given implicitly. Theorem 9, provided in the appendix, presents the solution of $V(u_1, u_2)$. The consumer's optimal search strategy is described by the following theorem.

Theorem 5 *Let a consumer's expected utility of product i be u_i. Product i will be considered for search and purchase if and only if $u_i \geq -a_i$. Suppose that two products are in a consumer's consideration set with $a_1 > a_2$. If $u_2 \geq -\frac{\sqrt{a_1 a_2}}{2} \ln\left(\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\right)$, the consumer will keep searching for information on product 1 only, until either she purchases product 1 when u_1 exceeds u_2 by a_1, or she purchases product 2 when u_2 exceeds u_1 by a_1. Otherwise, a consumer will keep searching for information on product i if her expected utility of product i plus the purchase threshold of product i exceeds that of the alternative product, i.e., $u_i + \overline{U}_i(u_j) \geq u_j + \overline{U}_j(u_i)$, until either she switches to search for information on the alternative when $u_i + \overline{U}_i(u_j) < u_j + \overline{U}_j(u_i)$, or she purchases product i when her expected utility of product i exceeds that of the alternative by some threshold.*

Figure 1.8 presents a consumer's optimal search strategy for $a_1 = 2$ and $a_2 = 1$. We find that the optimal consideration set still applies for the case with heterogeneous products. When a consumer's expected utility of product i is lower than $-a_i$, she will never consider this product. However, the purchase threshold structure is new and different. Consumers who have high expected utilities for both products only search for information on one product, the one with highest a_i. Denote $i^* = \arg\max_i a_i$. During the search process, the consumer imposes a constant purchase threshold of a_{i^*} on the expected utility difference of the two products. When her expected utility of product i^* exceeds that of the alternative by a_{i^*}, she purchases product i^* right away; otherwise, when her expected utility of product i^* is below that of the alternative by a_{i^*}, she purchases the alternative product right away. Therefore, the alternative product only serves as a reservation option, and the consumer will never search for information on it. With sufficiently high expected utilities of the two products, a consumer will not exit the market without a purchase, so her primary objective is to decide which product is a better choice. To achieve this goal, it is optimal for her to search on the product with the highest information per search cost, which is exactly the one with the highest a_i.

Note that in this case, the purchase threshold is greater for the product that has the highest informativeness of search than for the other product. Therefore, it is easier to get immediate purchase when the product with the lowest informativeness of search has a high expected valuation and the alternative product has a sufficiently low expected valuation, than when the product with the highest informativeness has a high expected valuation and the alternative product has a sufficiently low expected valuation. That is, in order to get immediate purchase it is easier to reduce the expected utility of the product with the highest informativeness of search, than to reduce the expected utility of the product with lowest informativeness of search. This would then lead to a benefit for the firm to try to sell the product with the lowest informativeness of search, if both expected valuations are relatively high. On the other hand, if the expected valuations are relatively low, the product with the highest informativeness of search has a greater advantage, because the exit threshold is lower.

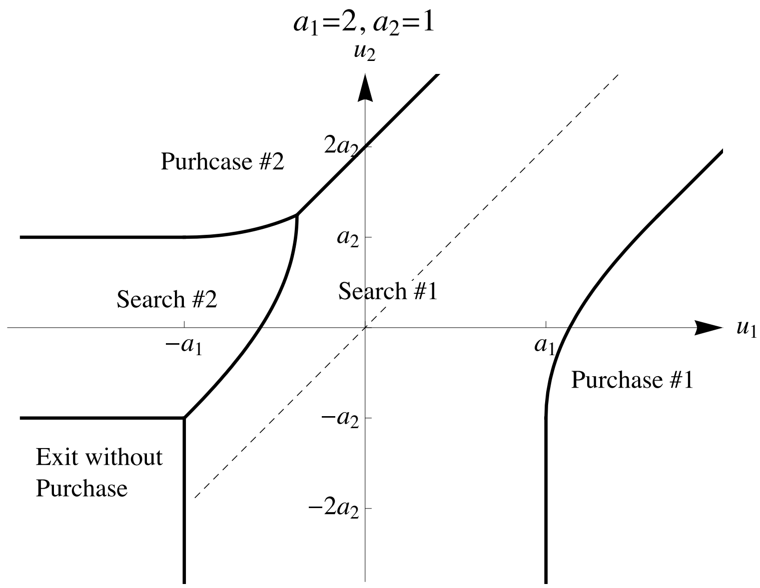

Figure 1.8: Optimal search strategy on two heterogeneous products.

The result that consumers with sufficiently high expected utilities only search for information on one of the products (the one that delivers more information per search cost) should be interpreted with caution. As preluded, this result depends on the assumption of identical distribution of utilities of attributes, which in our continuous-time model, is equivalent to the assumption that the informativeness stays constant during the search process. In a setting where informativeness decreases as a consumer accumulates more information, the result above will no longer hold. Intuitively, a consumer would search first on the product that provides more information per search cost initially, but then after some time the informativeness of that product decreases, and then the consumer will optimally switch to search for information on the alternative product, which now provides higher information per search cost.

The following proposition also comes from Theorem 9 in the Appendix. It states that a consumer prefers to search for information on the product with lower search costs or higher informativeness of search, given her expected utilities of the two products being equal. Low search costs and high informativeness of search can prioritize a product with low expected utility for being searched.

Proposition 3 *Given her expected utilities of the two products being equal, it is optimal for a consumer to search product* i*, if* $a_i > a_j$*, i.e., if the search cost for product* i *is smaller or the informativeness of search for product* i *is greater than for the other product.*

1.7 More than Two Products

In this section, we extend our basic model of optimal search on two products to the case of more than two products. We first solve the problem of optimal search on three products analytically, and find that in general, the consideration set and purchase threshold structures extend robustly to the case with three products. This case allows us also to obtain new insights regarding the purchase threshold.

Let us consider the optimal search problem with three products that have the same informativeness of search and search costs, and without information correlation. At any time, a consumer optimally chooses which product to search on, based on her current expected utilities of the three products as (u_1, u_2, u_3). A consumer's maximum expected utility is defined as $V(u_1, u_2, u_3)$. If a consumer's optimal decision is to stop searching and make a purchase decision right away, we have

$$V(u_1, u_2, u_3) = \max\{u_1, u_2, u_3, 0\}. \tag{1.29}$$

If the consumer chooses to continue searching for information, we have that for $i, j = 1, 2, 3$,

$$\begin{array}{rl} & \max\left\{V_{u_1u_1}, V_{u_2u_2}, V_{u_3u_3}\right\} = \frac{1}{2a} \\ \textit{Value Matching at Upper Boundary:} & V\left(u_1, u_2, u_3\right)|_{u_i=\overline{U}(u_{i+1},u_{i+2})} = \overline{U}(u_{i+1}, u_{i+2}) \\ \textit{Smooth-Pasting at Upper Boundary:} & V_{u_j}\left(u_1, u_2, u_3\right)|_{u_i=\overline{U}(u_{i+1},u_{i+2})} = \delta_{ij} \\ \textit{Value Matching at Lower Boundary:} & V\left(u_1, u_2, u_3\right)|_{u_i=\underline{U}(u_{i+1},u_{i+2})} = \max\{0, u_{i+1}, u_{i+2}\} \\ \textit{Smooth-Pasting at Lower Boundary:} & V_{u_j}\left(u_1, u_2, u_3\right)|_{u_i=\underline{U}(u_{i+1},u_{i+2})} = 0 \end{array} \tag{1.30}$$

where we have used the cyclic indexing rule, with $u_i \equiv u_{i \bmod 3}$ for $i > 3$, and where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. The function $\overline{U}(u_i, u_j)$ is the purchase boundary. Given u_i and u_j, when u_k hits $\overline{U}(u_i, u_j)$, the consumer will purchase product k right away. The function $\underline{U}(u_i, u_j)$ is the exit boundary, defined accordingly. The following results present the solution to the optimal search problem with three products.

Theorem 6 *There exists a unique solution* $V(u_1, u_2, u_3)$*, which satisfies equations (1.29) and (1.30). For* $i = 1, 2, 3$*,*

$$V(u_1, u_2, u_3) = \begin{cases} \frac{1}{4a}\left[\overline{U}(u_{i+1}, u_{i+2}) - u_i\right]^2 + u_i & \textit{if } -a, u_{i+1}, u_{i+2} \leq u_i \leq \overline{U}(u_{i+1}, u_{i+2}) \\ u_i & \textit{if } u_i > \overline{U}(u_{i+1}, u_{i+2}) \\ 0 & \textit{otherwise.} \end{cases} \tag{1.31}$$

The purchase and exit boundaries, $\overline{U}(\cdot)$ *and* $\underline{U}(\cdot)$ *for product* k*, with* $u_k > \max\{u_i, u_j\}$*, are given as*

$$\begin{aligned}
\overline{U}(u_i,u_j) &= \begin{cases} u_{i\vee j} + \left[1 + W\left(e^{-2-\frac{u_i+u_j+u_{i\vee j}}{a}} \frac{1+4W\left(\frac{1}{2}e^{-\frac{7}{4}-\frac{9u_{i\wedge j}}{4a}}\right)}{6\times 2^{1/3}W\left(\frac{1}{2}e^{-\frac{7}{4}-\frac{9u_{i\wedge j}}{4a}}\right)^{4/3}}\right)\right]a & \text{if } u_i, u_j \geq -a \\ \overline{U}(u_i) & \text{if } u_i \geq -a > u_j \\ \overline{U}(u_j) & \text{if } u_j \geq -a > u_i \\ a & \text{otherwise.} \end{cases} \\
\underline{U}(u_i,u_j) &= -a \quad (u_i, u_j \leq -a),
\end{aligned}$$

where $u_{i\vee j} \equiv \max\{u_i, u_j\}$ *and* $u_{i\wedge j} \equiv \min\{u_i, u_j\}$*.*

The solution structure for the three-product case looks similar to the one for the two-product case. The maximum expected utility $V(u_1, u_2, u_3)$ is still quadratic in the purchase boundary. In fact, this can be shown to be true for any number of products. However, the purchase boundary $\overline{U}(u_i, u_j)$ now becomes more complicated. We provide intuition on $\overline{U}(u_i, u_j)$ below. A consumer's optimal search strategy is characterized by the following theorem, also illustrated in Figure 1.9.

Theorem 7 *Only products with expected utility above* $-a$ *constitute a consumer's consideration set for search and purchase. Given three products in her consideration set, the consumer always searches for information on the one with the highest expected utility. She stops searching and makes a purchase if the difference in her expected utilities of the top two is above some purchase threshold, which depends on the consumer's expected utilities of the alternatives.*

The following corollary presents the monotonicity and asymptotics of the purchase threshold with respect to the expected utilities of the alternative products.

Corollary 3 *Suppose* $u_k > u_{i\vee j}$*. The purchase threshold of product* k *with respect to the other two alternatives,* $\overline{U}(u_i, u_j) - u_{i\vee j}$ *decreases with* $u_{i\vee j}$ *and increases with* $u_{i\wedge j}$*, and satisfies that,*

$$\overline{U}(u_i,u_j) - u_{i\vee j} \to \left[1 + W\left(\frac{1}{3}e^{\frac{1}{3}-\frac{2\left(u_{i\vee j}-u_{i\wedge j}\right)}{a}}\right)\right]a, \text{ as } u_i, u_j \to +\infty. \tag{1.32}$$

Recall that, in the two-product case, a consumer imposes a purchase threshold on the difference between her expected utilities of the two products, and the purchase threshold gets narrower as her expected utility of the alternative product gets higher, and converges to a. Now with three products, we show that a consumer imposes a purchase threshold on the difference between her expected utilities of the *top* two products, and the purchase threshold still gets narrower as her expected utility of the *second* alternative product gets higher, but gets *wider* as her expected utility of the *third* alternative product gets higher. As the

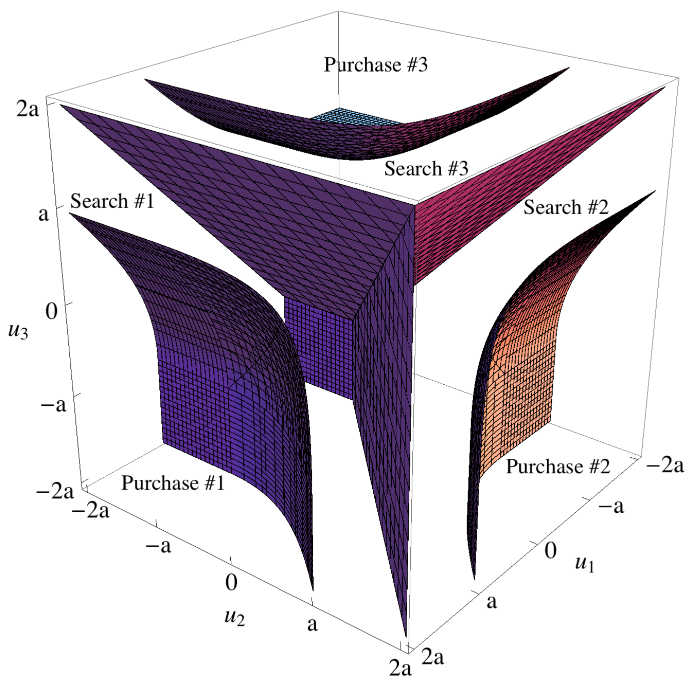

Figure 1.9: Optimal search strategy with three products.

expected utility of the third alternative gets higher, a consumer has a higher "reservation", therefore she needs to see a bigger difference between the top two to convince her to buy one product over the other. Moreover, the asymptotics show that as the expected utility of the second alternative goes to infinity, the purchase threshold converges to $\left[1 + W\left(\frac{1}{3}e^{\frac{1}{3}-\frac{2\Delta u}{a}}\right)\right]a$, which is greater than a. Consequently, more alternatives widen a consumer's purchase threshold, as more alternatives provoke more search efforts, and a consumer needs to see a bigger difference between the top two to convince her to buy one product over the other.

The problem of optimal search for information on four or more products can be stated and obtained in a similar way, with increased computational complexity. Yet it is interesting to revisit Bergman (1981)'s findings for the case of an infinite number of products with equal initial expected utilities. For this case Bergman (1981) shows that the optimal search strategy is to search information on the product with the highest Gittins index.

Consider a consumer's expected utilities of infinite number of products being equal as u_0 initially. If she has an outside option with value K, the maximum expected utility of search for information when only one product is available can be obtained as follows

$$V(u_0; K) = \frac{1}{4a}(a + K - u_0)^2 + u_0. \tag{1.33}$$

The Gittins index for a product can then be obtained as the value of the outside option that equates the maximum expected utility of choosing one arm (i.e., searching information on one product) with the value of the outside option, $V(u_0; K) = K$ (Whittle 1980). Solving for K, we obtain the Gittins index $K = u_0 + a$.

The consumer's optimal search strategy is then to continue searching information on one product, until her expected utility of the product either decreases below u_0, or increases above $u_0 + 2a$. In the former case, the consumer picks another product to search information on. In the latter case, she purchases the product and leaves the market. That is, given an infinite number of products with equal initial expected utilities, a consumer imposes a constant purchase threshold of $2a$ on the difference of her expected utilities between the product under search and the remaining unsearched products. In contrast, a high-valuation consumer imposes a purchase threshold of a for two products, and a purchase threshold of $\frac{4}{3}a$ for three products (if the two other products have the same expected utility). The purchase threshold widens as a consumer takes more products under consideration, as she has more options to acquire a higher payoff, but that purchase threshold on the difference of her expected utilities is bounded from above by $2a$.

1.8 Firm's Pricing Decision

In this section, we present some numerical simulations on a multi-product monopoly's pricing decisions given that consumers search for product information before making a purchase decision. Consider a seller of two products, based on our basic model. We assume that consumers observe the seller's prices before engaging in any search. Consumers are homogeneous in their initial valuations of the two products, as q_1 and q_2. Consumers' initial expected utility of product i is thus, $v_i = q_i - p_i$. Because all consumers' preferences are aligned, the two products can be considered as *ex ante* vertically differentiated.[14] It is interesting to notice that we are able to study the vertical differentiation problem under *ex ante* homogeneous consumers, as consumers will become heterogenous in their valuations after search.

Without loss of generality, we assume the marginal costs of both products to be zero.[15] The seller chooses prices so as to maximize the expected total profit

$$\max_{p_1,p_2} p_1 P_1(q_1 - p_1, q_2 - p_2) + p_2 P_2(q_1 - p_1, q_2 - p_2). \tag{1.34}$$

where $P_i(u_1, u_2)$ has been defined in Section 1.4, as the purchase likelihood of product i given a consumer's current expected utilities of the two products as u_1 and u_2. Let us denote the optimal prices as p_1^* and p_2^*. Without solving the profit optimization problem, we can show the following lemma, with proof in the appendix.

Lemma 3 *If $q_1 > q_2 \geq -a$, we have $q_1 - p_1^* \geq q_2 - p_2^*$.*

[14] For consumer search on horizontally differentiated products, see, e.g., Wernerfelt (1994).

[15] In the case with marginal cost for product i, $g_i > 0$, we can redefine $q_i' = v_i - g_i$ and $p_i' = p_i - g_i$, and then we get back to the profit optimization problem with zero marginal costs.

Under the optimal pricing policy, consumers expect higher expected utility from the product with higher valuation. From Theorem 2, we know that a consumer always searches on the product with higher expected utility, therefore given the optimal pricing policy, homogeneous consumers alway search on the product with higher valuation. Figure 1.10 presents a numerical simulation of a consumer's optimal search strategy in her valuation space, under the seller' optimal pricing policy.[16]

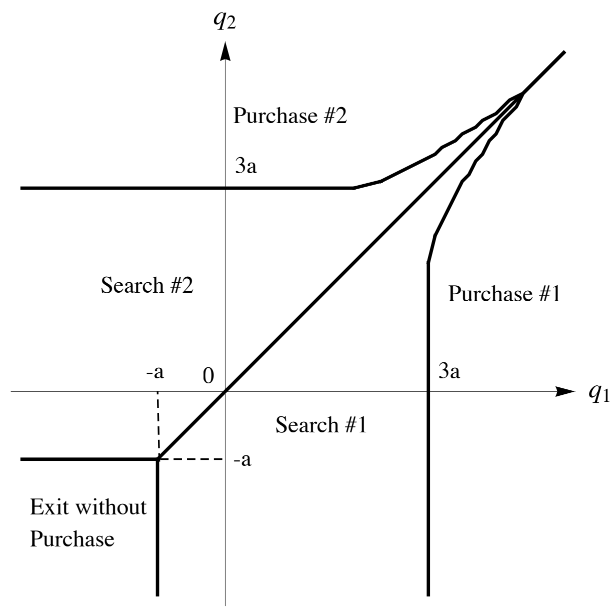

Figure 1.10: Homogeneous consumers' optimal search strategy on two products, given a monopolistic seller's optimal pricing policy.

Compared with Figure 1.2, a clear feature is that consumers with high valuations will be incentivized to purchase directly without any search. Consistent with our previous observations in Section 1.4, high-valuation consumers' search behavior will harm the seller's profit, thus are deterred from search by the firm offering a sufficiently low price such that those consumers choose to purchase immediately without search. Figure 1.11 shows the seller's optimal price for product 1 and the maximum profit.[17] The optimal price for product 2 can be obtained by symmetry, $p_2^*(q_1, q_2) = p_1^*(q_2, q_1)$. We can see that when a consumer's valuations of the two products are relatively high and close to each other, the seller deters her search behavior and incentivizes her to purchase immediately by imposing a price difference between the two products.

[16]We cannot solve the optimization problem (1.34) analytically. This problem involves a constrained non-convex global optimization problem that makes it hard to obtain analytical solutions. We explain our approach in the Appendix.

[17]When a product is neither searched nor purchased, its price is not uniquely determined. In this case, we stipulate the price to be its infimum. See the Appendix for more details.

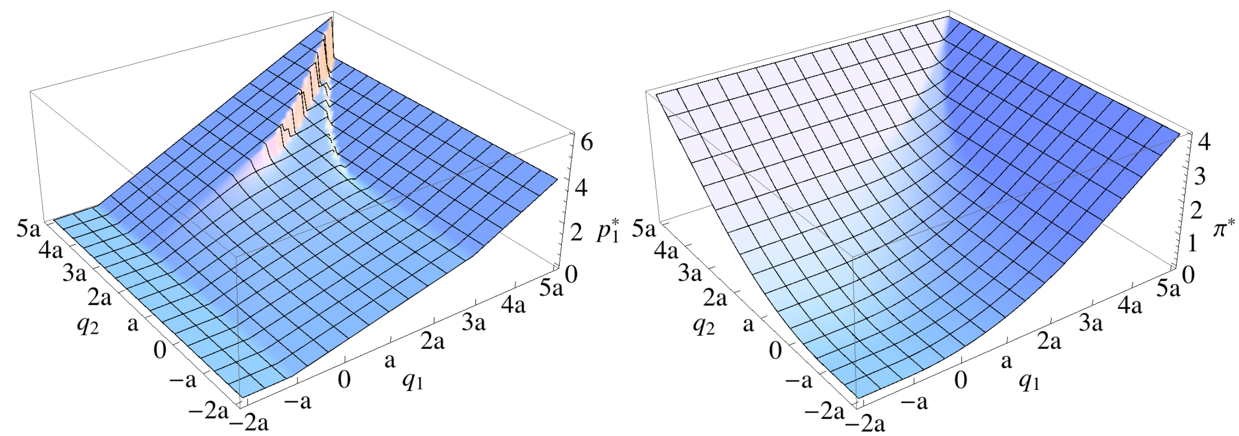

Figure 1.11: A monopolistic seller's optimal price for product 1, p_1^* and maximum profit, π^*.

We can also check how the optimal prices and maximum profits vary with a. We find that $\partial\pi^*(q_1, q_2)/\partial a$ is similar to $\partial P(u_1, u_2)/\partial a$ shown in Figure 1.5. The seller's profit increases with search costs while it decreases with informativeness of search if and only if q_1 and q_2 are relatively high. Therefore, in the case that a seller's objective is to maximize profit instead of sales, we obtain again our previous managerial implications that a seller should deter search for high-valuation consumers, while facilitate search for the low-valuation consumers.

1.9 Discounting, Finite Mass of Attributes, and Decreasing Informativeness

Discounting

In this section, we consider three more extensions to the basic model: discounting, finite mass of attributes, and decreasing informativeness of attributes. We have so far implicitly assumed that a consumer searches fairly fast and there is no time discounting in the search process. In some cases a consumer can search for information for longer time horizons, and it may be interesting in those cases to consider discounting the consumer's future search efforts as well as the payoff from purchase. To incorporate discounting, we can reformulate equation (1.6) as,

$$V(u_1, u_2) = -c\,dt + e^{-rdt} \max\left\{\mathrm{E}_{t_1}\left[V(u_1 + du_1, u_2)\right], \mathrm{E}_{t_2}\left[V(u_1, u_2 + du_2)\right]\right\}, \tag{1.35}$$

where r is the time discounting factor. Using the same technique as above, we can rewrite the above equation as the following partial differential equation,

$$\max\left\{V_{u_1u_1}, V_{u_2u_2}\right\} = \frac{2c}{\sigma^2} + \frac{2r}{\sigma^2}V. \tag{1.36}$$

which looks almost the same as equation (1.9), except that now we have an extra term $\frac{2r}{\sigma^2}V$ on the right hand side of the equation. The boundary conditions are still exactly given by equations (1.11)-(1.14), which, together with equation (1.36) and (1.5), constitute the mathematical problem of optimal search with time discounting. Theorem 10 in the appendix completely characterizes the optimal solution, where the value function $V(u_1, u_2)$ can be explicitly expressed as a function of the purchase boundary $\overline{U}(u)$, which is no longer in quadratic form as in the basic model. However, $\overline{U}(u)$ cannot be expressed explicitly. It is determined by an ordinary differential equation with a boundary condition. The following theorem characterizes a consumer's optimal search strategy (the proof is straightforward given Theorem 10, thus omitted).

Theorem 8 *Only products with expected utilities above $\sqrt{\frac{c^2}{r^2}+\frac{\sigma^2}{2r}}-\frac{c}{r}-\frac{\sigma}{\sqrt{2r}}\ln\left[\sqrt{\frac{r\sigma^2}{2c^2}}+\sqrt{\frac{r\sigma^2}{2c^2}+1}\right]$ constitute a consumer's consideration set for search and purchase. Given two products in her consideration set, the consumer always searches for information on the one with higher expected utility. She stops searching and purchases the product if the difference in her expected utilities of the two products is above some purchase threshold, which depends on her current expected utility of the alternative.*

From the theorem above, we find that the way for a consumer to optimally constitute her consideration set is almost the same as in the basic model, except that the consumer now has a higher bar for selection. In fact, we can show that $\sqrt{\frac{c^2}{r^2}+\frac{\sigma^2}{2r}}-\frac{c}{r}-\frac{\sigma}{\sqrt{2r}}\ln\left[\sqrt{\frac{r\sigma^2}{2c^2}}+\sqrt{\frac{r\sigma^2}{2c^2}+1}\right]$ increases with r. The more impatient a consumer is, the higher a bar she would impose on the expected utilities when selecting products into her consideration set. The purchase threshold structure is almost the same (consumers still search on the product with higher expected utility), but the asymptotics are different, as shown by the following corollary (with proof in the appendix).

Corollary 4 *With time discounting $r > 0$, the purchase threshold on the expected utility difference between the two products decreases as the expected utility of the alternative product increases, and converges to zero.*

As before, the purchase threshold decreases with the expected utility of the alternative, but now converges to zero, instead of a positive constant as in the basic model. This is easy to understand from equation (1.36): with time discounting, a consumer essentially bears two kinds of costs: an explicit search cost modeled by c, and an implicit cost due to delays of the purchase rV. Therefore, impatient high-valuation consumers will search less before making a purchase. Figure 1.12 illustrates a consumer's optimal search strategy with time discounting, which seems to suggest that discounting does not affect too much the optimal search strategy.

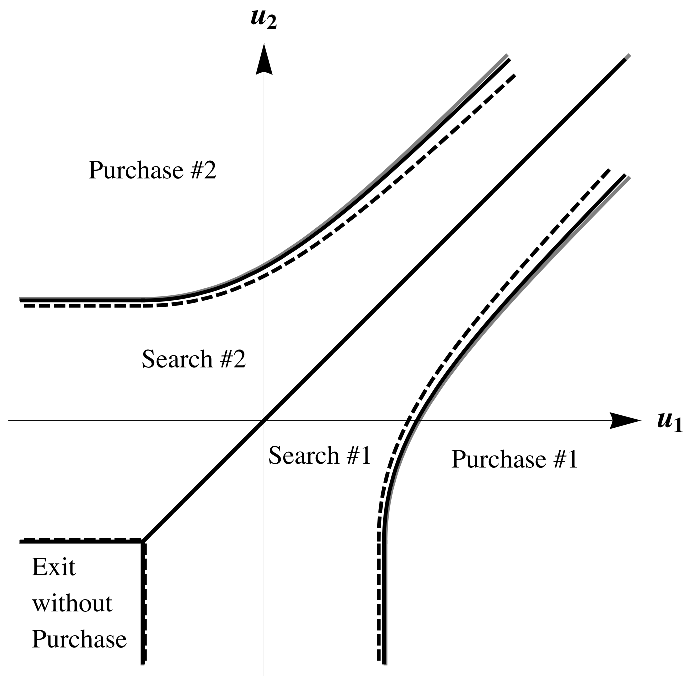

Figure 1.12: Optimal search strategy on two products with time discounting. The black and dashed lines represent the case with $r = .1$ and $r = .5$ respectively. The original case of $r = 0$ is presented by the gray lines.

Finite Mass of Attributes

Consider now the possibility of a finite mass of attributes, i.e., T is finite. Given finite T, the optimal search problem becomes intractable analytically, but we can use numerical simulations to consider the consumers' optimal search behavior. With finite T, as a consumer searches attributes of the different products the consumer becomes less demanding on the difference of expected utilities to make a choice. At the beginning of the search process it is also interesting to consider how the optimal search process for finite mass of T compares with the case of infinite T. Figure 1.13 presents a comparison of the optimal strategies between the analytical solution with infinite T and a numerical solution with finite T, for $T = 10, c = 1$, and $\sigma = 10$. We can see that even for a relatively big a (large σ, small c) and relatively small T, our analytical solution with infinite T seems to approximate the numerical solution with finite T relatively well.

Decreasing Informativeness

A natural framework to incorporate decreasing informativeness is to model a consumer search process as sequential costly acquisitions of independent noisy signals of the unknown true product utility. As above, a consumer's utility of product i is denoted as U_i, unknown to

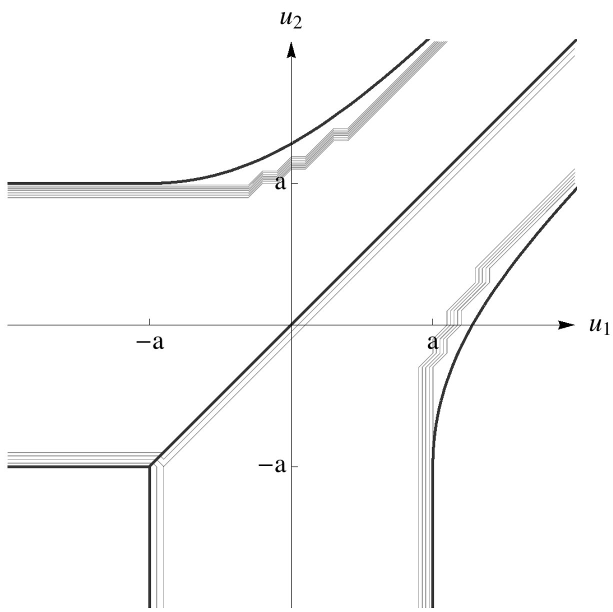

Figure 1.13: Optimal search strategy on two products with finite mass of attributes. The grey lines represents the numerical solution with finite T, and the black lines represents the analytical solution with infinite T.

the consumer. At any time t, a consumer's current belief of U_i follows $N(u_i, \sigma_i^2)$. Now, σ_i^2 is no longer a constant. In fact, if a consumer spends an infinitesimal time dt to search for information on product i, she pays a search cost $c_i dt$, and gets a noisy signal $\widetilde{U}_i | U_i \sim N(U_i, \kappa_i^2/dt)$, where κ_i^2 is a measure of the noisiness of the signal. Upon receiving the signal, the consumer updates her belief of product i's utility, by Bayes' rule, as $N\left(\frac{1/\sigma_i^2}{1/\sigma_i^2+dt/\kappa_i^2} u_i + \frac{dt/\kappa_i^2}{1/\sigma_i^2+dt/\kappa_i^2} \widetilde{U}_i, \frac{1}{1/\sigma_i^2+dt/\kappa_i^2}\right)$. To simplify the notation, let us define $s_i \equiv 1/\sigma_i^2$ and $k_i \equiv 1/\kappa_i^2$. Let us consider a model of two products with zero information correlation. A consumer's maximum expected utility is denoted as $V(u_1, u_2, s_1, s_2)$, which now depends not only on her current expected utility of each product, but also the variance, or the uncertainty of her current belief. Similarly, a consumer's optimal search problem can be formulated by the following iterative relationship:

$$
\begin{aligned}
V(u_1, u_2, s_1, s_2) &= \max\Big\{0, u_1, u_2, \\
&\quad -c_1 dt + \mathrm{E}_t\left[V\left(\frac{s_1}{s_1 + k_1 dt} u_1 + \frac{k_1 dt}{s_1 + k_1 dt} \widetilde{U}_1, u_2, s_1 + k_1 dt, s_2\right)\right], \\
&\quad -c_2 dt + \mathrm{E}_t\left[V\left(u_1, \frac{s_2}{s_2 + k_2 dt} u_2 + \frac{k_2 dt}{s_2 + k_2 dt} \widetilde{U}_2, s_1, s_2 + k_2 dt\right)\right]\Big\} \\
&= \max\Big\{0, u_1, u_2, V(u_1, u_2, s_1, s_2) + \left[k_1 V_{s_1} + \frac{k_1}{2 s_1^2} V_{u_1 u_1} - c_1\right] dt,
\end{aligned}
$$

$$V(u_1, u_2, s_1, s_2) + \left[k_2 V_{s_2} + \frac{k_2}{2s_2^2} V_{u_2 u_2} - c_2\right] dt \Big\} \tag{1.37}$$

The conditional expectation E_t in the first equality above is over $\widetilde{U}_i \sim N\left(u_i, \sigma_i^2 + \kappa_i^2/dt\right)$ from the consumer's perspective. The second equality above is due to Taylor expansions, and $o(dt)$ terms have been omitted in the limit. As before, we can formulate the above problem as an ambiguous-boundary PDE problem. However, now we have two more arguments s_1 and s_2 besides u_1 and u_2, which makes the problem difficult to solve analytically. The problem can still be solved numerically.

Figure 1.14 shows a consumer's optimal search strategy at some time point with $c_1/k_1 = c_2/k_2 = 1$, and the consumer's current variances of the two products' utilities, σ_1^2 and σ_2^2, are not equal, given by $s_1 = 0.5$, $s_2 = 1$. We can see that in general, Figure 1.14 is similar to Figure 1.8 in terms of the structure of the boundaries. We still have the optimal consideration set and the purchase threshold structures. However, because the parametric frameworks are different, we cannot compare the locus of the boundaries in the two figures directly. As expected with decreasing informativeness, we can also get that when a product is searched, the purchase threshold for that product falls, and that the boundary separating "Search #1" and "Search #2" moves in the direction of being more likely for the other product to be searched next.

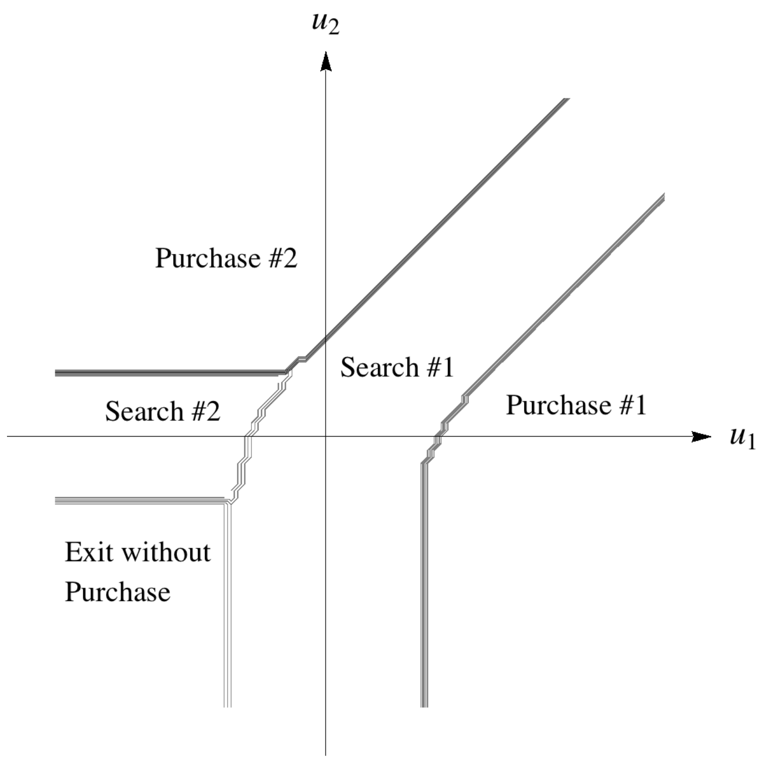

Figure 1.14: Optimal search strategy on two products with decreasing informativeness.

1.10 Conclusion

Gradual search for information is important for understanding numerous economic activities with imperfect competition and market frictions. We consider this possibility, presenting a parsimonious model on continuous search for information on a choice set of multiple options. Although the paper has taken consumer search in a product market as the leading example, the model can be applied generally to other cases of gradual search for information on multiple alternatives.

The paper solves for the optimal search, switch, and purchase or exit behavior in such a setting, which is characterized by an optimal consideration set and purchase thresholds. A consumer always searches for information on the product with the highest expected utility if the informativeness of search per search cost is the same across products, and only stops to make a purchase if her expected utility of a product is sufficiently greater than those of the alternatives. Positive correlation across products narrows the purchase threshold, while negative correlation widens it. More product alternatives also widen the purchase threshold. With heterogeneous products, if the informativeness of search is constant through time, the consumer only searches on the product with the highest informativeness of search or lowest search costs if her expected utility of the alternative is sufficiently high, and she will always first search for information on that product, when both products have the same expected utility. The model also presents several implications that are empirically testable.

Understanding consumers' search behavior for information also helps to explain some seemingly puzzling results: more alternatives might lead to a lower purchase likelihood, when consumers engage in search for information. Also, information availability decreases sales of products for high-valuation consumers, while it increases sales for low-valuation consumers. Therefore, sellers of multiple products may want to facilitate information search for low-valuation consumers, while obfuscate information for high-valuation consumers.

The set-up considered may motivate further studies on the economics of search for information. One interesting possibility to consider is to allow consumers to search on multiple products at the same time, known as parallel search (Vishwanath 1988). It would also be interesting to investigate what happens in terms of vertical differentiation under oligopolistic competition when there is a correlation of information across products.

Chapter 2

Peer Effects on Social Networks

2.1 Introduction

Peer effect occurs when the action of one agent directly affects others' choices, usually those that are socially close to him, as opposed to via the intermediation of market. Peer effects are ubiquitous in life. Teenagers look to their peers when deciding what songs to listen and what movies to watch, consumers consult their friends before shopping for cars or houses, and even firms and organizations try to learn from others before deciding whether or not to adopt new technologies.

These peer effects are of primary importance to corporate managers as well as policy-makers since they allow a stimulus to one individual to be multiplied through the network. Management has long been aware of the importance of peer effect in launching a successful new product. Firms frequently give out free samples to selected customers, and consciously design effective marketing campaigns to leverage peer effects on social medias (Aral and Walker 2011). Policy interventions, such as school desegregation and busing, have used social interactions as the major goal to alleviate stratification by income, education, race, and to improve social equality (Moffitt et al. 2001). Quantifying the magnitude of peer effect therefore is critical to constructing sound network interventions in both the public and private sectors.

In this paper, I study the peer effect in adoption of a new consumer technology—iPhones—using individual-level iPhone adoption data from a provincial capital city Xining in northwestern China. My sample spans a period when the mobile phone carrier China Unicom has the exclusive right to sell iPhones in mainland China, hence my data includes almost all iPhone users adopted during the time period[1]. I construct a social network using half a year's call transactions between iPhone adopters and all other users on a carrier's network. Based on the network structure, I test whether or not an individual's iPhone adoption decision was affected by his/her friends' decisions. I quantify the peer effect using both fixed effect and instrumental variable approaches, and investigate how network structures

[1]Except a very few people who bought iPhones from overseas and brought it back to China to use.

modulate the magnitude of peer influence.

On a fundamental level, iPhone adoption could be subject to peer effect due to informational, behavioral, social, or network externality reasons. Friends' recommendations and user reviews give a consumer much information about how good iPhone is and whether it fits their need—this is the informational channel, so-called word-of-mouth. It is also possible that a consumer observe other people's purchase and usage decisions and infer iPhone is a good product—this is the behavior channel, so-called behavior learning. Alternatively, iPhone could also be viewed as a fashionable product. Using an iPhone speaks something about one's personality and taste; and observing other people using it changes directly the utility one can get from using the same product, either positively or negatively—this is the social channel, e.g., snob effect. Lastly, iPhone is a communication device with many add-on applications. Having one more person in the network makes it more attractive for others to join since they will have more people to communicate with[2]. In this paper, I do not attempt to distinguish these channels. Rather, I focus on quantifying the peer effect, which could be an aggregate of all the channels.

Identification of peer effect has long been a challenge to economists. Peer effect implies that the behavior of connected agents on a network tends to be correlated. However, correlation in the behavior *per se* does not necessarily imply that any agent's action has a causal effect on that of others. Other factors besides peer effect could also give rise to such behavioral correlation. From a policy and strategy point of view, only causal peer effects are of primary interest because it impacts the outcome of individual-level policy interventions. Moffitt et al. (2001) summarized that the primary factors confounding the identification of peer effects are: simultaneity, endogenous group formation and correlated unobservables. Simultaneity problem arises if one person's action influences the others, and *vice versa* (Manski 1993). Fortunately, my setting does not suffer from this simultaneity problem, because of the natural sequence of individual adoptions across-time in the panel data. Endogenous group formation problem arises when the outcome variable also affects the likelihood of two agents being connected, which in my case, means two strangers starting calling each other because both of them use iPhone. Arguably, it is true that this might happen theoretically; yet, I believe it would be too subtle an effect to intervene with the identification, especially since using a smart phone or not is irrelevant to the quality and cost of phone call service in the market where my data is collected.

Therefore, the only serious potential confounding factor remained in my setting is correlated unobservables. Adoption decisions of one's peers can be endogenous for his adoption decision, because people who know each other tend to face similar unobserved environment to adopt the technology. To address this issue, I come up with two approaches. To control for time-invariant correlated unobservables, I apply an individual fixed-effect model. To further control for time-varying correlated unobservables, I come up with an instrumental variable

[2]Strictly speaking, the network externality effect operates via add-on applications but not phone calls. This is because, during the time when we collected our data, using a smart phone or not is irrelevant to the quality and cost of phone call services. Hence, we do not expect using iPhone to change one's preference of phone calls.

approach. I use an individual's friends' birthdays as an instrumental variable for his friends' adoption decisions, and see how these birthday-induced adoptions by his friends affects his own adoption decision.

Both my fixed effect (FE) and instrumental variable (IV) models show results similar in magnitudes. A friend's adoption decision indeed has a positive impact on one's own. According to the FE model, having one more friend adopting iPhone increases an individual's probability of adoption by 0.89% in the next month. In comparison, the IV regression gets a slightly smaller estimate of 0.75%, after clearing away effects of potential time-varying correlated unobservables. I also show that this peer effect decreases in the number of current adopters. In other words, as more friends have already adopted, the marginal impact of an additional friend becomes smaller. Using my estimates, a firm looking to promote iPhone sales in a setting similar to mine would be able to compute the average external peer effect of a new user[3]. And these numbers would be of great interests for managers when designing optimal promotion schemes.

I also investigate how heterogeneity in network structure impacts the magnitude of peer influence. It is one of the frontier questions to study the role that individual and relationship attributes play in social influence processes (e.g. Aral and Walker 2014; Banerjee et al. 2013). My results show both a "popularity" and an "intimacy" effect. The more popular an individual is, as measured by the number of his first-degree contacts, the greater his peer effect would be on each of his peers. The peer effect is also stronger between "closer" friends. My results show that the more time a pair of friends spent on talking to each other during the six months, the greater the peer effect is between them.

The paper unfolds itself as the following. Section 2.2 summarizes relevant literature and our connections to previous studies. Section 2.3 gives the background and basic patterns of our data sample. Major empirical results are provided in section 2.4. Section 2.5 explores the impact of network heterogeneity on peer effect and various robustness checks. And section 2.6 concludes.

2.2 Literature Review

Individuals make decisions in almost every social aspect under the influence of friends, neighbors, or professional peers: from education (Sacerdote 2001; Epple and Romano 2011), criminal activities (Glaeser, Sacerdote, and Scheinkman 1996; Bayer, Hjalmarsson, and Pozen 2009), welfare program participation (Bertrand, Luttmer, and Mullainathan 2000; Duflo and Saez 2003), to physicians' prescriptions (Manchanda, Xie, and Youn 2008; Nair, Manchanda, and Bhatia 2010; Iyengar, Bulte, and Valente 2011), etc. In product market especially, there is widely recorded phenomenon of peer influence on purchasing behaviors: from computers

[3]For example, at the initial stage of iPhone diffusion when an average individual have about one user friend, the direct impact of a new iPhone user on his peers would be about 1.01%, compared to a much smaller effect of 0.65% at a later stage when an average individual have about 100 adopted friends.

(Goolsbee and Klenow 2002), online groceries (Bell and Song 2007), TV service (Nam, Manchanda, and Chintagunta 2010), insurance plans (Cai, De Janvry, and Sadoulet 2013), to solar panels (Bollinger and Gillingham 2012), etc.

It is worth mentioning that there is a subtle difference between peer effect and network effect (for the latter, see a survey by Birke 2009). Network effect relies on network externality, which captures the phenomena that having more people in a certain group makes the utility of later joiners even higher. Typical examples include adoption of industry standards (e.g. David 1985; Augereau, Greenstein, and Rysman 2006), choice of business platforms (Brown and Morgan, 2009; Hendel, Nevo, and Ortalo-Magné, 2009; Cantillon and Yin, 2008), and membership of social media websites such as Facebook and LinkedIn. Peer effect, on the other hand, encompasses a much broader meaning. In addition to being triggered by network externality, peer effect could also be due to informational, behaviorial or social reasons: consumers could learn about a product from others' comments and choices, or simply find it fashionable to go for what is "hot". Regardless of the underlying mechanism, peer effect manifests itself as a causal influence of one's action upon his peers.

The peer influence that we study in this paper can be seen as a special case of general social interaction effects (Manski 1993; Moffitt et al. 2001). Social interaction effects usually include both contextual effects—the direct influence of others' characteristics on one's choice—and peer effects—the influence by others' actions[4]. Many attempts have been made to demonstrate and quantify the peer effects. The early literature on aggregate diffusion has been trying to quantify "peer effects" by treating the entire population of past adopters as the reference group (Bass 1969, Mahajan, Muller, and Bass 1990). With access to more micro-level data, recent studies have taken on a more subtle view of reference groups, emphasizing the role of social structures in channeling peer effects based on geographic locations (e.g. Bollinger and Gillingham 2012), ethnic or culture proximity (e.g. Bandiera and Rasul 2006), friend or family relationships (e.g. Conley and Udry 2010), or some combination of these factors.

However, a closer look at these heterogenous peer effects poses an identification challenge aforementioned. Some of the studies have tried controlling for detailed individual-level information to alleviate the correlated unobservable problem[5]. Unsatisfied with these ap-

[4]A literature somewhat relates to ours are those that use identification strategies to study the influence of an individual's social activities or characteristics on his *own* behavior. For example, Shriver, Nair, and Hofstetter (2013) studies whether online users' (surfing-related) content-generation activity affects their social ties and vice versa, by exploiting changes to wind speeds at various surfing locations. Our question here is different from and conceptually harder than theirs, because peer effect captures the spread of the same behavior among individuals and it is usually harder to find exogenous shocks only to some people but not to their friends for the same behavior.

[5]Several case-specific identification strategies have also been used to study peer effect. Bollinger and Gillingham (2012) identified the peer effects in adoption of solar photovoltaic panels, by leveraging the time delay of installations after the initial request. However, their analysis is based on zip code level data without network structures; also the validity of their empirical strategies hinges critically on the assumption that there is no covariate that influences two adoption decisions that are separated by the installation delay or longer. Nam, Manchanda, and Chintagunta (2010) studies the adoption of a video-on-demand service, where

proaches, more recent studies have been utilizing randomized field experiments to get clean identifications of peer effects (e.g., Sacerdote 2001; Duflo and Saez 2003; Cai, De Janvry, and Sadoulet 2013, among others). But oftentimes the population under study is limited due to feasibility constraints of experiments, and the social network structure is either unavailable, or measured by subjective surveys. In contrast, our sample includes all people in a metropolis, and we construct social network from objective measurements, using duration of phone calls among consumers to approximate social tie strength. Such a communication network would be able to preserve more detailed and nuanced information than location networks. And we also use an instrumental variable approach to get a clean identification of peer effect.

The research that is closest to ours is Tucker (2008). In her paper, she identified the network externality in adoption of a video-messaging technology, by utilizing a stand-alone use of the technology (watching local TV programs) as an instrument. Methodologically, her approach is close to ours. Nonetheless, her study is on a very specific setting: adoptions of a technology occur in a corporation instead of the marketplace, and individuals do not incur any pecuniary cost to adopt the technology[6].

To summarize, our paper studies peer effect of a mainstream consumer product (iPhone) on a social network, which is constructed from objective phone calls among consumers. In contrast to the large literature on prediction of product diffusion using network structures (e.g., Hill, Provost, Volinsky, et al. 2006; Katona, Zubcsek, and Sarvary 2011, among others), the main goal of this paper is not to make predictions of individual adoptions. However, based on a consistent estimate of peer effects, our findings could indeed help predict future sales of similar products, and would be of great interest to business practitioners, who are designing marketing strategies in regions that are similar to ours.

2.3 Background and Data Description

Data Background

IPhone was first introduced in China on October 30, 2009. For a rather long time, iPhone was offered to subscribers of China Unicom exclusively, until January 17, 2014, when China Mobile started to offer iPhone on its network. There are three players in the mobile phone telecommunication market in China: China Unicom, China Mobile, and China Telecom; and all of which are state-owned public companies. Currently, China Mobile owns roughly 70% market share of mobile telecommunications in China, whereas China Unicom about 20% and China Telecom the rest 10%.

random fluctuation in the signal quality adds exogenous shocks to the content of message communicated from friends to friends, but not the initial adoption decisions.

[6]As a technical point, Tucker (2008) used a pooled probit regression model outlined by Allison (1982), which is only valid if errors are not correlated over time. For a setting of individual technology adoption, we feel that this might be too strong an assumption. Hence, in this paper we opt for a panel fixed effect model.

Our data set includes monthly call transactions and iPhone adoption records of all iPhone adopters in China Unicom network who adopted before the end of October 2013 in capital city Xining of Qinghai province in northwestern China. The data on call transaction is available from May 2013 to October 2013. Whereas the dataset of iPhone adoption is complete: running from the first adoption in November 2009 to October 2013.

We also get complimentary data containing individual information of the adopters, such as cell phone monthly usage charge, service plan subscription, and most importantly, individual's birthday, which we will use as an instrumental variable for adoption time. Data on individual information is available for a subgroup (72.3% of the entire sample population) of adopters that adopted between May 2012 and October 2013.

Adoption Pattern

There are in total 82,471 adoption instances from November 2009 to October 2013. Some adopters later stopped using iPhone (by either dropping out of the carrier's network or by replacing it with a phone of other brands), and there are 47,727 (57.9%) active iPhone users by the end of October 2013.[7] The monthly adoption and usage trend is shown in Figure 2.1. We can see that the adoption rate grows exponentially during the sample period.

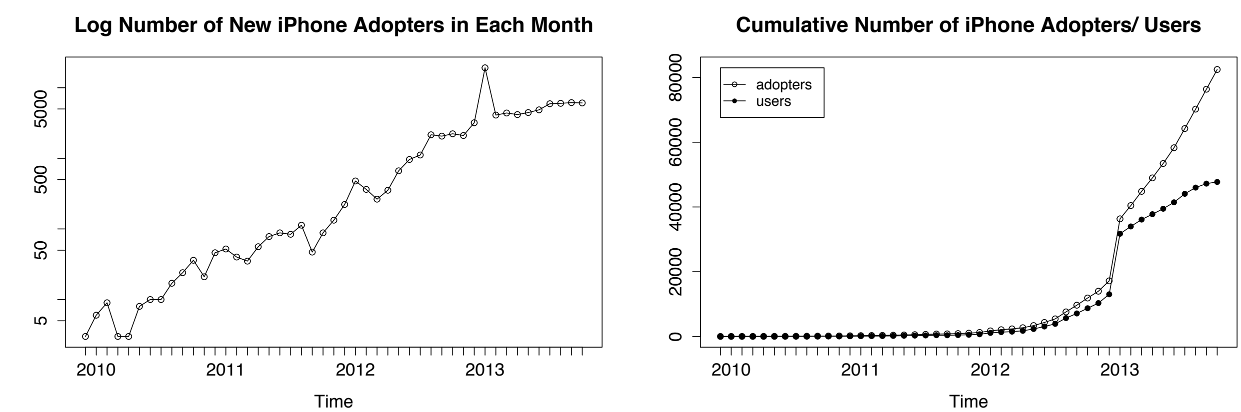

Figure 2.1: Adoption and usage trend of iPhone after introduction in Nov-2009.

In the data, we observe dives and surges of new adoptions in some months. This is mainly due to three reasons: consumers' strategic waiting before launching of new models, occasional limited supply capacity in certain months, and seasonalities, such as the Spring

[7]The four-year cumulative attrition rate of over 40% seems relatively high, which we think is an idiosyncratic feature of China's market. As iPhone's exclusive carrier in the sample period, China Unicom owns only 20% share of the telecommunication market, compared with China Mobile's 70% market share. We expect a significant amount of people stops using iPhone because they switches from China Unicom to Chine Mobile.

Festival around the beginning of each year. We include monthly fixed effect in all our regressions to control for these time trends.

Calling Pattern

Our call transaction data set includes all people who have adopted iPhone from November 2009 to October 2013 and remained active in May to November 2013, as well as all of their friends, who have either made or answered a call with an iPhone adopter between May 2013 and October 2013. IPhone users must be China Unicom subscribers, while their friends are not necessarily China Unicom subscribers. Out of 82,471 iPhone adopters, 74,967 (90.9%) remained active in the carrier's network in the period from May to October 2013, regardless of what types of phones they used at the time.[8] There are in total, 4,030,156 friends of all active iPhone adopters. Call transactions are aggregated by month. Each transaction consists of the following information: phone number identifier of the caller, phone number identifier of the receiver, and their monthly call duration. Between two users on the network, if they did not make a single call during the sample period, their (null) transaction is not included in the dataset.

There are 10,762,428 call transaction records in total between May 2013 and October 2013. We use these call transactions to construct social network for iPhone adopters and their friends. Therefore our social network embeds 90.9% of the entire sample population who have ever adopted an iPhone between Novermber 2009 and October 2013 in Xining. The total number of people who made calls is 82,420, and the total number of people who received calls is 4,328,013 during this period. Combining both callers and receivers (iPhone adopters and their friends), there are 4,105,123 individuals on the phone-call network. The huge difference between numbers of callers and receivers comes from calls from outside the carrier's network. We do not have information on incoming calls from outside network, and can only observe outgoing calls to outside network. The average monthly call duration for each pair of contacts, who at least made one call in that month, is roughly 11 minutes.

The following Table 2.1 provides the summary statistics for our sample. As we can see, the phone call network proves to be quite stable over the sample period.

2.4 The Empirical Model

Our empirical model follows the linear-in-mean model of social interactions (Manski 1993), which we interpret as a reduced form of the behavioral process generating adoption decision across the population network.

Strictly speaking, there are two variations of the linear-in-mean model, one of which has the absolute number of adopters as the explanatory variable whereas the other has the

[8] By "being active", we mean a consumer made or received at least one call in the period. In China, when a consumer switches mobile phone carrier, he has to change his phone number. Our record of a consumer discontinued when he left China Unicom.

Table 2.1: Summary Statistics for the Data Sample

	# Observations	Mean	Std Dev	p25	p50	p75
Panel A: Adoption (November 2009 to October 2013)						
adoption	82,471					
individual birthday	54,171					
Panel B: Call Transaction (May 2013 to October 2013)						
duration-05	2,911,112	11.49	52.65	1.00	2.57	7.57
duration-06	2,943,157	11.29	51.29	1.00	2.60	7.60
duration-07	3,201,634	10.98	49.22	0.98	2.55	7.45
duration-08	3,290,485	10.97	50.03	1.00	2.57	7.50
duration-09	3,232,633	11.21	51.47	1.00	2.55	7.47
duration-10	3,238,310	11.25	52.50	0.98	2.52	7.40
total duration	10,762,428	19.57	130.05	.0.97	2.65	8.97

Note: The panel consists of all iPhone adopters in China Unicome network in a provincial capital city Xining in northwestern China by the end of October 2013. Monthly (total) call duration includes aggregated call transactions of all pairs with non-zero call duration in that month.

fraction of adopters out of all friends. Both specifications are theoretically justifiable, and researchers usually make their own choices as which one to use. In this paper, we use absolute number of adopters as the explanatory variable for both the main models and the extension on network heterogeneity. Similar results using the fraction of adopters as explanatory variable are discussed in the robustness check section 2.

The Network Structure

We index consumers (network nodes) by i. We first construct a social network by aggregating all call transactions from May 2013 to October 2013. If there is a call from Alice to Bob in the six months, we establish a directed link from Alice to Bob. In this way we get a directed social network, which has 443 weakly connected maximal components in the network, among which the largest one consists of $4,102,936$ individuals (99.95% of the whole population).

Given the network, we define inward neighbors for consumer i as all other consumers that have a directed link to i; and similarly we define the outward neighbors for i as all others that have a directed link from i. Then, we construct the panel dataset with the following variables

$$(i, t, ADOPT_{it}, INSTALLBASE\_IN_{it}, INSTALLBASE\_OUT_{it})$$

where $ADOPT_{it}$ is the adoption indicator of individual i in month t: $ADOPT_{it} = 1$ if i first adopted iPhone in month t, and $ADOPT_{it} = 0$ otherwise.[9] $INSTALLBASE\_IN_{it}$

[9]Aforementioned, we consider adoption instead of usage decisions, therefore, with $ADOPT_{it} = 1$, all records of individual i after t are dropped from our panel.

is the cumulative number of i's inward neighbors who have adopted iPhone by month t[10]; and similarly $INSTALLBASE\_OUT_{it}$ is the cumulative number of i's outward neighbors who have adopted iPhone by month t. By combining our adoption dataset and the social network constructed from transaction data set, we get a panel dataset of 3,100,442 individual-year observations. The summary statistics for the panel is provided in appendix A.23.

In this section, we use absolute number of (inward/outward) neighbouring iPhone users as a measure of peer effects on the network, by assuming homogeneous peer effect. In section 2, we discuss various extension of the basic model and try incorporating more network structure into the measures.

Fixed Effect Model

In this paper, we define a focal consumer's *peers* as those who are directly linked to him on the phone call network. In other words, we are estimating a *local* network effect. Arguably, a consumer's adoption decision could also be affected by macro-level network features, such as iPhone diffusion on the overall network. We would not be able to test such macro-level features in this paper since we only have data on one city (and hence, one network). But we do control for these variables by including time trends in the regression.

The two-way fixed effect (FE) regression of peer effect on adoption is

$$ADOPT_{it} = \beta_1 INSTALLBASE_{it-1} + \alpha_i + \gamma_t + \varepsilon_{it}, \tag{2.1}$$

where ε_{it} is assumed i.i.d. across individuals i and time t, and α_i, γ_t are individual and time fixed effect respectively[11]. The variables $INSTALLBASE\_IN_{it}$ and $INSTALLBASE\_OUT_{it}$ are highly correlated, with a correlation coefficient at 0.998, hence we include only one of them in the regression equation as variable $INSTALLBASE_{it-1}$ to avoid collinearity.

In this paper, we choose a linear probability model over a logit for its simplicity in incorporating instrumental variable and a flexible fixed effect structure with a panel structure[12]. Another reason that makes it specifically difficult to implement a logit model in our setting is quasi-separation issue of our data, which leads to a non-convergence and potential bias of

[10] The variable $INSTALLBASE\_IN_{it}$ does not account for the people who stopped using iPhone after initial adoption. The idea behind the model is that anyone who have used iPhone before could share with a new comer his experience and personal opinion about the product, thus influencing the new comer's decision.

[11] Slightly different from most other studies on the topic, our *INSTALLBASE* does not include previous actions of one's own. This does not derive from an assumption from the fact that we only include an individual in the sample up till the point he adopted. Hence, our fixed effect model does not suffer the inconsistency problem pointed out by Narayanan and Nair (2013), which is essentially an inconsistency problem of a dynamic panel FE model.

[12] As pointed out by Narayanan and Nair (2013), having a flexible fixed effect structure is vital to getting a consistent estimate when individual characteristics correlate with install bases, while choosing a linear probability model over a non-linear one does not compromise the uncovering of the true value even when underlying process is a non-linear one.

logit maximum likelihood estimator (Albert and Anderson, 1984)[13]. Hence, in this paper, we use a linear probability model to estimate the peer effect on iPhone adoption averaged across the population.

By including only past values of *INSTALLBASE* in the regression, our model specification features a one-direction influence (usually termed as *passive* social interaction in the literature, see Hartmann et al., 2008) instead of a feedback loop. Theoretically, a setting would best fit a one-direction framework if the action of one focal agent affects his neighbours but not the other way around, or, more commonly, if the agent is concerned only about the *realized* actions of his neighbors and is myopic enough to not foresee how his current decision will impact his future self by influencing others around him. In this paper, we implicitly make the latter assumption that an agent is concerned only about the past actions of his neighbors and is myopic. Hence, we quantify peer effect via equation (2.1).

Table 2.2 summarizes the estimation results. As we can see that having one more friend adopting iPhone, on average, increases an individual's probability of adoption by 0.89% in the next month. This holds true for either inward- or outward-phone-call definition of friend. A quadratic model shows that this peer effect decreases in the number of current adopters. In other words, as more friends have already adopted, the marginal impact of an additional friend becomes smaller.

Figure 2.2 plots the coefficients of peer effect by month from January 2010 to October 2013. Due to very few adoptions in the earlier stage and small variation in the explanatory variable, the estimates of peer effect coefficient appear insignificant with very wide confidence intervals before 2012. However, iPhones underwent an acceleration of diffusion towards the end of 2012. And a positive peer effect started to manifest itself. From then on, the estimate of peer effect remained significant and stable throughout the sample period. Figure , plotting the monthly fixed effects, shows a macro-trend for increased likelihood of adoption over our sample period.

Instrumental Variable Model

As discussed earlier, the only potential confounding factor remained in our setting is correlated unobservables. Our fixed effect model in the previous section controls for time-invariant unobservables, but the concern for time-variant correlated unobservables remains. In other words, if there exist omitted variables that both correlate with the network structure and are time-varying in nature, our identification with a FE model could be compromised. As an example, let us consider the following situation with only two types of people in the population, fashion-followers, who are more likely to purchase a product when the overall sales are high (compare to peer effect that depends on adopters in the local friends network), and

[13]Heinze and Schemper (2002) proposed using a penalized maximum likelihood estimation originally developed by Firth (1993) to solve the separation problem. However, such a method would pose much complexity to a panel discrete choice model, which is already subject to the incidental parameters problem due to adding fixed effect to a non-linear logistic or probit model. Hence, in this paper, we try to shield from these technical difficulties by using a linear probability model.

Table 2.2: Estimation of Peer Effect with FE Model

VARIABLES	(1) $ADOPT_{it}$	(2) $ADOPT_{it}$	(3) $ADOPT_{it}$	(4) $ADOPT_{it}$
INSTALLBASE_IN	0.00890***		0.0114***	
	(0.000636)		(0.000353)	
INSTALLBASE_OUT		0.00886***		0.0114***
		(0.000629)		(0.000354)
$INSTALLBASE\_IN^2$			-4.30e-05***	
			(6.79e-06)	
$INSTALLBASE\_OUT^2$				-4.29e-05***
				(6.80e-06)
Constant	-0.00232***	-0.00232***	-0.00229***	-0.00229***
	(8.54e-05)	(8.54e-05)	(8.57e-05)	(8.57e-05)
Individual FE	Y	Y	Y	Y
Monthly FE	Y	Y	Y	Y
Observations	3,100,442	3,100,442	3,100,442	3,100,442
R-squared	0.275	0.275	0.277	0.277
Number of i	74,967	74,967	74,967	74,967

Note: * denotes significance at 10% level, ** at 5% level, and *** at 1% level. All estimations above use robust standard error to control for heteroscedasticity in linear probability models.

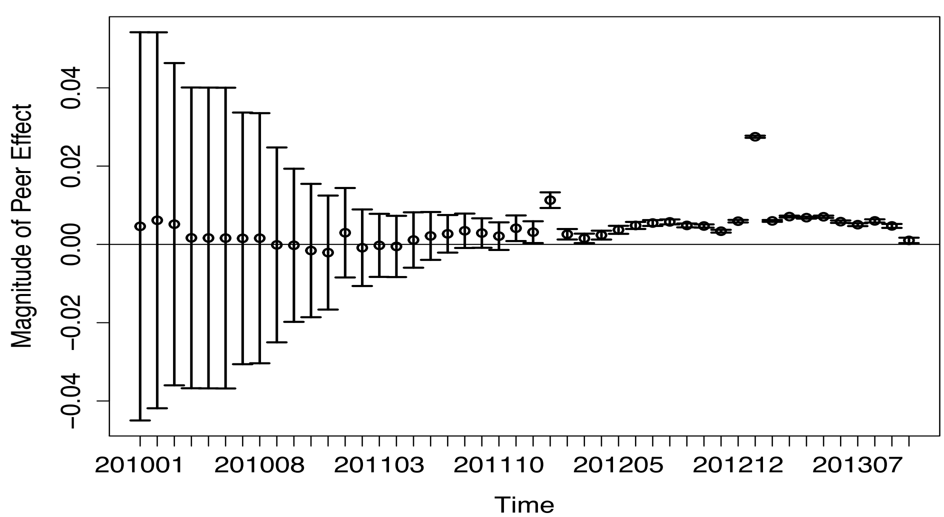

Figure 2.2: Peer Effect on IPhone Adoption by Month

non-followers. Fashion-followers are naturally more likely to adopt in the later stage of product diffusion, hence cannot be controlled by individual FE. Moreover, if fashion-followers are also more likely to become friends, then it cannot be controlled by monthly FE either and will induce a spurious peer effects among friends. In this section, we use an instrumental variable (IV) approach to further control for such spurious behavioral correlations.

We use an individual's birthday as an IV for his adoption decision, and test how this affects his neighbors' subsequent adoptions. The basic idea is that it is more likely for people to adopt iPhones on their birthdays, because either they are more likely to reward themselves with a long fancied product, such as an iPhone, or they are more likely to receive one as a gift on their birthdays. So we expect to see a higher probability of iPhone adoption around an individual's birthday. Being totally random, birthdays would satisfy the exclusion restriction automatically[14]. All we need to check is the inclusion requirement to make it a legitimate instrumental variable, which we test with weak instrument tests below.

For individual i, we define $BDAY_{it}$ as a dummy variable which equals to one if i's birthday is in month t and zero otherwise. We assume that individuals' monthly adoption decisions depend on their birthday dummies in the following sense:

$$ADOPT_{it} = \delta_{0i} + \delta_{1i} BDAY_{it} + \eta_{it}, \tag{2.2}$$

where we allow for heterogenous birthday impact on individuals' adoption decisions. We denote individual i's adoption time as $\tau(i)$. Individual i's installed base can be constructed by aggregating adoptions among his neighbours of $\mathcal{N}(i)$ up till time $t-1$:

$$\begin{aligned} INSTALLBASE_{it-1} &= \sum_{j \in \mathcal{N}(i)} \sum_{s=1}^{\min\{\tau(j),t-1\}} ADOPT_{js} \\ &= \sum_{j \in \mathcal{N}(i)} \delta_{0j} \min\{\tau(j), t-1\} + \sum_{j \in \mathcal{N}(i)} \delta_{1j} \sum_{s=1}^{\min\{\tau(j),t-1\}} BDAY_{js} \\ &+ \sum_{j \in \mathcal{N}(i)} \sum_{s=1}^{\min\{\tau(j),t-1\}} \eta_{js}. \end{aligned}$$

From the equation, we know for each j, $\sum_{s=1}^{\min\{\tau(j),t-1\}} BDAY_{js}$ can be used to instrument $INSTALLBASE_{it-1}$. Therefore, $INSTALLBASE_{it-1}$ is over-identified, and the most effective IV can be obtained by GMM estimation. Here, we take a first step by assuming a homogeneous impact of birthdays on adoptions, i.e., $\delta_{1j} \equiv \delta_1, \delta_{0j} \equiv \delta_0$. In this case, $INSTALLBASE_{it-1}$ is exactly identified by the following IV:

$$IV\_BDAY_{it-1} \stackrel{\text{def}}{=} \sum_{j \in \mathcal{N}(i)} \sum_{s=1}^{\min\{\tau(j),t-1\}} BDAY_{js}. \tag{2.3}$$

[14]To further justify the usage of birthday as an IV, China Unicom did not have advertisements or promotions based on customers' birthdays in the sample period.

By further defining

$$A_{it-1} \stackrel{\text{def}}{=} \sum_{j \in \mathcal{N}(i)} \min\{\tau(j), t-1\}, \tag{2.4}$$

the first stage regression becomes the following:

$$INSTALLBASE_{it-1} = \delta_1 IV\_BDAY_{it-1} + \delta_0 A_{it-1} + \omega_i + \pi_t + \epsilon_{it}. \tag{2.5}$$

Here, ω_i and π_t are the individual and time fixed effects. By estimating the above regression equation, we have $\widehat{INSTALLBASE}_{it-1}$ as the independent variable of interest in the second stage regression:

$$ADOPT_{it} = \beta_1 \widehat{INSTALLBASE}_{it-1} + \alpha_i + \gamma_t + \varepsilon_{it} \tag{2.6}$$

Here, again, ε_{it} is assumed i.i.d. across i and t, and α_i, γ_t are individual and time fixed effect respectively.

Table 2.3 gives the estimation results for the first stage IV regression. We get an estimate of δ_1 at about 0.038 with a significant p-value as 0.000. Similar results hold for both inward- and outward- definition of linked friends. This shows that friends' adoption decisions are indeed affected by their birthdays[15]. In the birthday month, an average individual would be 3.8% more likely to adopt the iPhone.

Table 2.4 shows the IV regression of peer effect on iPhone adoptions. Here we get estimate of peer effects similar in magnitude but slightly smaller than that of the basic fixed effect model. According to the IV estimates, having one additional friend adopting iPhone increases an individual's probability of adoption by about 0.75% in the next month (compared to the 0.89% estimate by FE model). This smaller effect could be due to the fact that our IV estimates eliminate the effect by some of the time-varying correlated unobservables. Similar to the previous section, our IV estimates also find this marginal impact of newly adopted friends decreasing when the size of already adopted user base gets bigger. Both two-stage least square and GMM robust estimators yield very similar estimates.

2.5 Robustness Check

Network Structure and Heterogenous Peer Effect

Heterogeneous social interactions have important implications for policy design and for firms' allocation of marketing efforts. Peer effect, as one of many influences that channeled through social interactions, could be very sensitive to the social and structural conditions under

[15] Arguably, there might be some heterogeneity in the effectiveness of the IV. For example, some people, like teenagers, are more likely to get an iPhone as a gift on their birthdays than others. Since we have strong reasons to believe in the monotonicity of such effect, i.e. no person will be *less* likely to adopt iPhone on his birthday, this would not compromise our identification. It might, however, add some subtleties to the interpretation of the results. As in the usual case of heterogeneous treatment effect, our IV regression estimates the average causal effect for those that are affected by the instrument.

Table 2.3: First Stage IV Regression

VARIABLES	(1) *INSTALLBASE_IN*	(2) *INSTALLBASE_OUT*	(3) *INSTALLBASE_IN*	(4) *INSTALLBASE_OUT*
IV_BDAY_IN	0.0227***		0.0227**	
	(0.000513)		(0.00904)	
IV_BDAY_OUT		0.0229***		0.0229**
		(0.000514)		(0.00904)
A	0.00511***	0.00512***	0.00511***	0.00512***
	(3.33e-05)	(3.34e-05)	(0.000454)	(0.000455)
Constant	0.0226***	0.0226***	0.0226***	0.0226***
	(0.00465)	(0.00466)	(0.00334)	(0.00335)
Individual FE	Y	Y	Y	Y
Monthly FE	Y	Y	Y	Y
Error	OLS	OLS	Robust	Robust
Observations	3,100,442	3,100,442	3,100,442	3,100,442
R-squared	0.412	0.412	0.412	0.412
Number of i	74,967	74,967	74,967	74,967

Note: * denotes significance at 10% level, ** at 5% level, and *** at 1% level. The above results are estimated using standard OLS error.

Table 2.4: Estimation of Peer Effect with Birthday IV

VARIABLES	(1) $ADOPT_{it}$	(2) $ADOPT_{it}$	(3) $ADOPT_{it}$	(4) $ADOPT_{it}$	(5) $ADOPT_{it}$	(6) $ADOPT_{it}$	(7) $ADOPT_{it}$
INSTALLBASE_IN	0.00753***		0.00753***	0.00711***		0.0110***	
	(6.00e-05)		(6.00e-05)	(0.00181)		(7.57e-05)	
INSTALLBASE_OUT		0.00751***			0.00710***		0.0110***
		(5.99e-05)			(0.00181)		(7.58e-05)
$INSTALLBASE\_IN^2$						-6.62e-05***	
						(9.63e-07)	
$INSTALLBASE\_OUT^2$							-6.63e-05***
							(9.62e-07)
BDAY			0.000135				
			(0.000320)				
Individual FE	Y	Y	Y	Y	Y	Y	Y
Monthly FE	Y	Y	Y	Y	Y	Y	Y
Error				HAC/Clu(i,t)	HAC/Clu(i,t)		
Estimator	IV-2SLS	IV-2SLS	IV-2SLS	IV-GMM	IV-GMM	IV-2SLS	IV-2SLS
Observations	3,100,442	3,100,442	3,100,442	3,100,442	3,100,442	3,100,442	3,100,442
R-squared	0.275	0.275	0.275	0.266	0.266	0.276	0.276
Number of i	74,967	74,967	74,967	74,967	74,967	74,967	74,967
Weak IV Test							
CD Wald F-stat	1.20e+06	1.20e+06	1.20e+06	1.20e+06	1.20e+06	4.00e+05	4.00e+05
KP Wald F-stat				37.80	37.63		

Note: * denotes significance at 10% level, ** at 5% level, and *** at 1% level. Here, HAC stands for heteroskedasticity-autocorrelation (HAC) robust, and Clu(i,t) stands for 2-way clustered standard error (Cameron et al. 2006, Thompson 2009) that are robust to arbitrary heteroskedasticity and intra-group correlation with respect to both time and individual dimensions. For weak instrument tests, CD Wald F-stat stands for Cragg-Donald Wald F-stat, and KP Wald F-stat for Kleibergen-Paap Wald F-stat (Stock and Yogo, 2010).

which the interaction happens. Studies, especially in sociology, have long recognized that early adopters with different "status" or level of "popularity" would have varied influence over later new comers, and hence different impact on the speed and pattern of diffusion (e.g. Nair, Manchanda, and Bhatia 2010; Banerjee et al. 2013). In this section, we investigate whether these more "important" agents would have greater impacts on the adoption decisions of their neighbours.

We use standard network indices, including inward and outward degrees and tie strength to measure importance of a friend to the focal agent (Banerjee et al. 2013; Tucker 2008). Degree index approximates the popularity of an individual; while tie strength, measured by total duration of phone calls, evaluates the extent of the friendship between a pair of contacts[16]. And for each individual, we aggregate his neighbors' adoptions weighted by these network indices.

Figure 2.3 gives a histogram of the logarithmic degree of iPhone adopters on our network. It is worth noting that our data includes all iPhone users and their friends, but not their friends' friends. In other words, it enables us to calculate degree of iPhone adopters precisely but not that of the non-users, the latter of which, luckily, is not among what we need for the empirical purpose of this paper[17]. Figure 2.4 shows the histogram of logarithmic total call duration over the six months among all pairs of contacts in the network. Again, figure 2.4 leaves out calls among non-users, which we do not need.

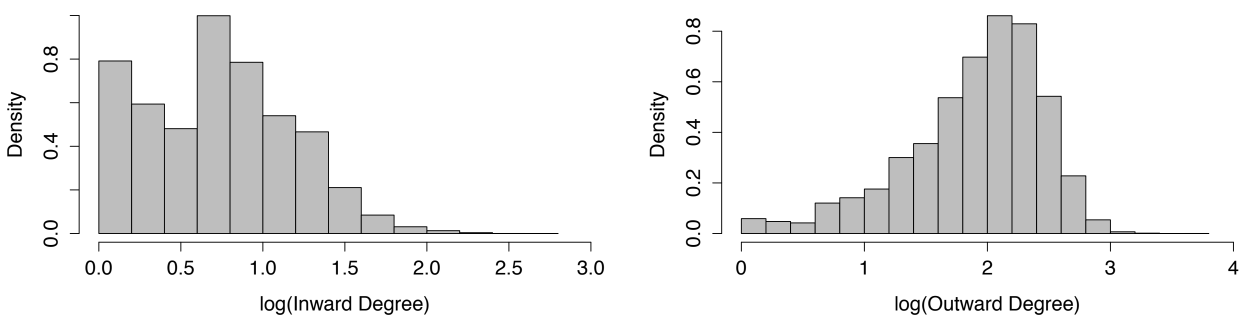

Figure 2.3: Histogram of degrees of iPhone adopters on the social network.

[16]Ideally, we would like to explore more network-based indices, such as betweenness, closeness, and centrality, among others (Banerjee et al. 2013; Tucker 2008). However, our data set includes only iPhone users and their friends, but not their friends' friends. In other words, our network is not complete in a way that would make the other network indices precisely calculable. Hence, we can only include degree and tie strength in the current paper.

[17]This is because while weighting neighbourhood adoption dummies with degree values, the non-adopters have *ADOPTION* equals to zero. Hence, leaving out the degree of non-adopters does not impact our empirical analysis.

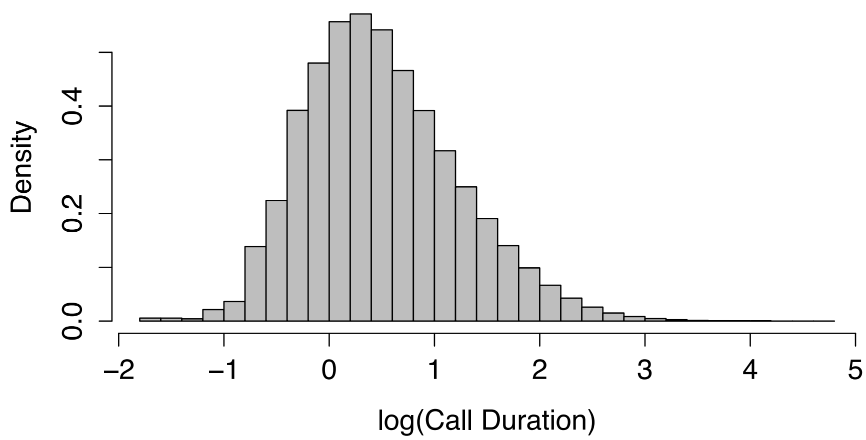

Figure 2.4: Histogram of six months' total call duration for pairs of contacts.

For individual i, let m_j^i be the network index for his neighbor $j \in \mathcal{N}(i)$. m_j^i could be either degree of j or tie strength between i and j. Hence the fixed effect regression with network indices would be

$$ADOPT_{it} = \beta_0 + \beta_1^m \sum_{j \in N(i)} m_j^i \sum_{s=1}^{\min\{\tau(j),t-1\}} ADOPT_{js} + \beta_2 X_i + \alpha_i + \gamma_t + \varepsilon_{it}. \tag{2.7}$$

Table 2.5 gives the estimation results for regression (2.7). The first four columns investigate the peer effect of degree-weighted neighbourhood adoptions. As is generally recognized in the network literature, a person with more friends (i.e. higher degree) is often considered to be more important, either because of his perceived "social status" and "popularity" or the fact that he might have better information inflows due to more contacts. In either way, a friend with higher degree is often expected to be more of an "opinion leader" and to have a bigger influence over the people around him. Our empirical results support this hypothesis. On average, increasing the inward-degree of an adopted friend by one enhances an individual's subsequent probability of adoption by about 0.01%.

Column (5) to (8) investigate the peer effect of tie-strength-weighted neighbourhood adoptions. Theories have not yet agreed on the relative magnitudes of peer effect from a good friend compared to that of a casual acquaintance. Some believe that individuals are more readily to be influenced by their close friends, while others argue that information from a "weak tie" contact might prove to be more useful (Granovetter, 1973). In our setting of iPhone diffusion, individual adoption decisions are more affected by strong-tie contacts (friends that they communicated more with). As we can see, increasing the call duration between an individual and his adopter friend by 100 minutes a month would lead to an increase in his probability of adoption by about 1.3% as estimated by an IV model.

Table 2.5: Heterogeneity of Peer Effect with Network Indices

VARIABLES	(1) $ADOPT_{it}$	(2) $ADOPT_{it}$	(3) $ADOPT_{it}$	(4) $ADOPT_{it}$	(5) $ADOPT_{it}$	(6) $ADOPT_{it}$	(7) $ADOPT_{it}$	(8) $ADOPT_{it}$
DEG_IN	0.0001***		9.86e-05***					
	(5.75e-06)		(1.62e-06)					
logDEG_IN		0.0024***		0.0024***				
		(0.000128)		(2.69e-05)				
TIE_IN					4.53e-05***		9.79e-05***	
					(4.65e-06)		(1.57e-06)	
logTIE_IN						0.0060***		0.0067***
						(0.000156)		(6.29e-05)
Constant	-0.0024***	-0.0023***			-0.0024***	-0.0023***		
	(8.61e-05)	(8.57e-05)			(8.62e-05)	(8.57e-05)		
Individual FE	Y	Y	Y	Y	Y	Y	Y	Y
Monthly FE	Y	Y	Y	Y	Y	Y	Y	Y
Estimator	FE	FE	IV-2SLS	IV-2SLS	FE	FE	IV-2SLS	IV-2SLS
Observations	3,100,442	3,097,225	3,100,442	3,097,225	3,100,442	3,100,442	3,100,442	3,100,442
R-squared	0.271	0.274	0.271	0.274	0.270	0.277	0.267	0.277
Number of i	74,967	74,886	74,967	74,886	74,967	74,967	74,967	74,967

Note: $*$ denotes significance at 10% level, $**$ at 5% level, and $***$ at 1% level. Here, *DEG* and *logDEG* are, respectively, neighbourhood adoption dummies weighted by friends' degrees or logarithmic degrees. *TIE* and *logTIE* are similar adoption dummies weighted by tie strengths. Tie strength between any pair of contacts on the phone-call network is measured by the total duration of their phone calls over the seven month period. The FE estimation above uses robust standard error.

Absolute Number of Adopters vs. Fraction of Adopters

In previous sections, we use absolute number of adopters as the explanatory variable. As mentioned earlier, there are two variations of the linear-in-mean model, and in this section, we present results as a robustness check using the ratio of adopters as the explanatory variable.

Variable *FRACTION_IN* is defined as total number of adopters among one's friends *INSTALLBASE_IN* divided by his total number of friends *DEGREE_IN*. The subfix *IN* indicates that these variables are defined over the inward-phone-call network, and similar definition holds for variables with subfix *OUT*. Using the adopter ratio *FRACTION* instead of adopter number *INSTALLBASE* as the explanatory variable, column (1) and (2) in table 2.6 give the results for the FE model as in equation (2.1), while column (3) shows that for IV model as in equation (2.6). As we can see, the IV model gives an estimate of peer effect at about 52.8% (for inward friends), meaning that having an extra one tenth of one's friends adopt iPhone would increase his probability of adoption by about 5.3%.

Overall, however, the results using *FRACTION* are much less robust than that using *INSTALLBASE*. This is mainly due to the small fraction of adopters compare to the large number of contacts on the network. This is especially true for outward friends, since our population of phone call receivers includes all land-lines and users of other mobile phone carriers that China Unicom users ever called. Those users could never adopt iPhone (unless

Table 2.6: Peer Effect with Fraction of Adopters

VARIABLES	(1) $ADOPT_{it}$	(2) $ADOPT_{it}$	(3) $ADOPT_{it}$	(4) $ADOPT_{it}$	(5) $ADOPT_{it}$
FRACTION_IN	0.0734***	0.0820***		0.528***	0.250***
	(0.000793)	(0.00155)		(0.102)	(0.0212)
FRACTION_IN			0.121***		
			(0.00454)		
Constant	-0.00240***	1.053***	-0.00243***		
	(0.000540)	(0.00364)	(0.000486)		
Individual FE	Y	Y	Y	Y	Y
Monthly FE	Y	Y	Y	Y	Y
Estimator	FE	FE	FE	IV-2SLS	IV-2SLS
Sample Start	2010	2012	2010	2010	2012
Observations	2,501,265	920,066	3,099,985	2,501,265	919,709
R-squared	0.270	0.279	0.268	0.172	59,931
Number of i	60,933	60,288	74,957	60,933	0.269

Note: $*$ denotes significance at 10% level, $**$ at 5% level, and $***$ at 1% level. Variable *FRACTION* is the ratio of adopters among one's friends, defined as the total number of adopted friends (*INSTALLBASE*) divided by the number of friends (*DEGREE*). The FE estimation above uses robust standard error.

change carrier to China Unicom first); and their existence in the network greatly decreases the identifying variation in the explanatory variable *FRACTION*, a point that is shown clearly in the table of panel data summary statistics A.1.

2.6 Conclusion

A Peer effect occurs when the action of one agent directly affects its peers choices outside the market channel. Understanding peer influence is critical to estimating product demand and diffusion, creating effective viral marketing, and designing "network interventions" to promote positive social changes.

In this paper we study whether the adoption of a consumer technology, in our case an iPhone, is affected causally by his network neighbours' decisions and network characteristics of the other adopters. The empirical setting is to measure the peer effect of iPhone adoption in a provincial capital city in China, during a four-year period starting from the introduction of the first iPhones to Mainland China. We use a unique panel dataset of phone call records by person and by time, that allows us to construct each iPhone adopter's social network, by using half a years call transactions between iPhone adopters and all other users on a carriers network. We measure strength of a social network pairwise tie by the duration of calls. Based on the network structure, We quantify the peer effect of iPhone adoptions, and investigate how the network structure modulates the magnitude of peer influence.

The main specification to identify peer effects is to see how the probability of an individual adopting an iPhone is affected by the measures of networks we create. Of course network size and strength is not randomly assigned. Identification of peer effects, therefore, is a challenge. Peer effect implies that the behavior of connected agents on a network tends to be correlated. However, other factors besides peer effect could also give rise to such behavioral correlation. For example, adoption decisions of ones neighbors can be endogenous for his adoption decision, because people who know each other tend to face similar unobserved environment to adopt the technology.

The identification in our paper has two approaches. To control for time-invariant correlated unobservables, we apply a fixed-effect model, and shows that a friend's adoption increases one's adoption probability in next month by 0.89%. To further control for potentially time-varying unobservables, we instrument adoptions of one's friends by their birthdays, based on the fact that consumers are more likely to adopt iPhones on birthdays. The IV estimation shows a slightly smaller peer effect at 0.75%, after clearing away impacts of potential correlated unobservables. Both models show that the marginal effect of peer influence decreases in the number of current peer adopters. In other words, as more friends have already adopted, the marginal impact of an additional friend becomes smaller.

We also investigate how heterogeneity in network structure impacts the magnitude of peer influence. Our results show both a "popularity" and an "intimacy" effect. It is shown that the "popularity" of an individual, as measured by the number of his first-degree contacts, will affect how much influence he can exert on his fellow peers. The higher an individual's degree is, the greater his peer effect would be on his neighbours. The peer effect is also stronger between "closer" friends. The more time a pair of friends spent on talking to each other during the six months, the greater the peer effect is between them.

Our results could provide useful insights for managers. We studied the diffusion of a mainstream consumer product, the iPhone. The empirical setting is based in a provincial capital city in China, Xining. The city, with a population of 2.3 million and GDP per capita at $6999 in 2013, is a good representation of a typical city in China as well as that of a mid-level developing country. Business practitioners launching promotions for similar products might find our results useful in designing optimal marketing strategies in regions comparable to mine.

Chapter 3

Inventory Management for New Products

3.1 Introduction

Many firms introduce new products that are variants of their existing products in a given category to target different customer segments and satisfy customers' different desires Krishnan and Ulrich, 2001; Ramdas, 2003. Because the launch of a new product with successive (and differentiated) generations always commands a large commitment of resources in production and marketing, the introduction strategy requires careful planning Dobson and Kalish, 1988. A key element in the introduction strategy is the introduction time. Depending upon the product category, firms choose to time the introductions of product line extensions differently. We describe three examples below:

- In the publishing industry, hardcover books are introduced to the market first, while paperback generations are released about one year later Mcdowell, 1989; Shapiro and Varian, 1999.
- In the fashion industry, fashion houses such as Armani first introduce new top-of-the-line designs at very high price points and only several months later do they introduce their lower-priced lines Pesendorfer, 1995.
- In the automobile industry, "Volvo of North America released its 6-cylinder 760 model in Oct 1983 and the 4-cylinder 740 model 17 months later even though both cars share the same chassis and the 4-cylinder engine was available earlier." Moorthy and Png, 1992.

In all of these examples, firms chose different times for launching line extensions even though no technology constraints prevented them from making simultaneous releases. On one hand, as the successive generations are substitutes, delaying the introduction of one generation leads to less cannibalization of the existing generation. On the other hand, a large body of empirical marketing research Bass, 1969 suggests that demand diffusion begins slowly, speeds up and slows down after maturity, so if the firm waits too long, sales may have slowed considerably as the product has already diffused through the market Druehl, Schmidt, and Souza, 2009, especially considering rapidly changing customer preference. According to Wilson and Norton, 1989, "the timing of the introduction of the line extension affects the subsequent sales pattern for both products and the total profit to be made within the planning period", so the decision of when to introduce a new variant of an existing product is a critical tactical decision. In this paper, we use the terms "new variant", "new generation" and "line extension" interchangeably.

Several papers in the marketing literature have addressed the strategy for timing the release of two successive (and somewhat differentiated) generations of the same product when both generations could be offered, however inventory cost has been the missing factor, which can actually help align some discrepancies between the conclusions derived from diffusion theories and industry practice Wilson and Norton, 1989. Despite of relative ignorance in the main stream of literature, inventory cost plays a crucial role in industrial practice. Firms tend to manufacture or order products in large batches to achieve efficiency and minimize cost. In the publishing industry example, new books are often produced in large quantities, partly due to economies of scale in printing. In industries with relatively short product life cycles, such as apparel and consumer electronics, where rapidly-changing consumer preferences and frequent innovations have reduced product life cycles from years to months, a capacity-constrained business that offers many product variants will produce each variant only once in the planning horizon to avoid large setup costs associated with changeovers Kurawarwala and Matsuo, 1996; Bitran, Haas, and Matsuo, 1986. Besides, in the global economy, many firms have outsourced their supply chains to Asia with big orders. As a result, inventory cost will be non-negligible in those industry practices. Firms thus have to weigh the instantaneous profit from the new product line extensions against the inventory holding cost resulting from a slowed demand rate of the older generation Bayus and Putsis Jr, 1999. Incorporating the inventory aspect into an integrated model has been considered as an intractable problem to date, given that most models only accounting for diffusion and substitution are already very difficult to analyze Wilson and Norton, 1989. To the best of our knowledge, this paper takes a first step toward filling this gap.

We propose an integrated model that considers the S-curve market penetration of new products, substitution between generations, as well as inventory cost in order to decide the launch-time of a new generation to maximize total profits. Our paper belongs to the research stream that tries to coordinate the decisions of operations management and marketing science Eliashberg and Steinberg, 1987; Ho, Savin, and Terwiesch, 2002; Malhotra and Sharma, 2002; Hausman, Montgomery, and Roth, 2002; Chopra, Lovejoy, and Yano, 2004; Jerath, 2007. Our contributions to the marketing and operations management research are three-fold. First, we bring an operations management perspective into the introduction timing decision through a focus on inventory holding cost that arises from a simple ordering policy. By assuming only one replenishment occurs during the entire planning horizon, we incorporate inventory cost into the revenue optimization and characterize the optimal introduction strategy. Second, we revisit the optimal introduction policies proposed in marketing literature. Our model not only covers their guidelines, but also renders the implications of how inventory holding cost impacts the introduction timing decision. Third, by developing an integrated model accounting for both demand and supply sides, we suggest that the decisions of marketing and operations management should be coordinated not only at the operational level, such as match between demand and supply Ho and Tang, 2004, but at the tactical level as well, for example the introduction timing decision.

The rest of the paper is organized as follows. We review relevant literature in Section 2. In section 3, we characterize the optimal introduction strategy under the one-replenishment ordering policy, and compare our findings with previous guidelines. We proceed to enrich the model in Section 4, by considering extensions, such as finite planning horizon and multiple-replenishment ordering policy. Finally in section 5, we conclude the paper with a summary of key insights and suggestions for future research. All proofs and mathematical details are relegated to the Appendix.

3.2 Literature Review

In this section, we first review the literature that center on the research of introduction timing of product line extensions, and then review some related work that lies at the interface between marketing and operations management.

There have been many studies about product line management Quelch and Kenny, 1994; Dobson and Kalish, 1988; Krishnan and Ulrich, 2001, but not enough attention has been given to considering time dynamics in this process Ramdas, 2003. We broadly classify the existing literature on introduction timing into two categories: (1) continuous-time models in the diffusion of innovation context, and (2) two-period models for comparing simultaneous and sequential strategies.

Research in the continuous-time category often relates to the seminal Bass diffusion model Bass, 1969, which initiates the stream of examining demand diffusion for a single new product. Many studies have extended the Bass model into multi-product diffusion literature Peterson and Mahajan, 1978; Bayus, Kim, and Shocker, 2000. A subset of this group of work concentrates on modeling the diffusion paths of successive product generations, where most entry timing research arises. Norton and Bass (1987) proposed a model of adoption and substitution for successive generations. They assume independent demand dynamics for different generations, and a uniform adoption rate for all customers who have (not) entered the market, and who have (not) adopted the old generation product. In another seminal work, Wilson and Norton (1989) address demand dynamics over the product life cycle in the same context, and the optimal time to introduce the second generation is shown to be "Now or Never" (i.e., it's optimal to introduce the new generation either immediately or never). However, this result is not consistent with the industry practices that were cited above. Following the same line, but based on a little bit more complicated demand substitution assumptions, Mahajan and Muller (1996) reconsidered the optimal introduction timing problem for successive generations of products. By incorporating the discounting effect on seller's revenue, they show in contrast, that the optimal policy is "Now or Maturity", where the new generation product is introduced immediately or when the present generation product has reached sufficient sales. However, the complexity of their demand model forbids them to give a clear definition of when is "maturity", and they focus on technological innovations for successive generations. As technology improvement is a key ingredient of this branch of research, many researchers have addressed dynamic technology improvement in these kinds of problems. Krankel *et al.* (2006) incorporate technology improvement into the multi-generation diffusion demand context and provide a state-dependent threshold policy governing introduction timing decisions. Krishnan and Ramachandran (2008) study the trade-offs in timing product launches when the core technology available is improving rapidly. Druehl *et al.* (2009) analyze the impact of product development cost, the rate of margin decline and the cannibalization across generations on a firm's time-pacing decision. However, the progression of product technology is not the demand driver in our model setting, in fact we focus on the case of releasing two successive (and somewhat differentiated) generations of the same product in the absence of development constraints.

Research of the two-period model category is mainly to address the comparison of sequential and simultaneous introduction strategies. Moorthy and Png (1992) analyze the introduction strategy of a high-end product and its low-end variant. Their results suggest that if the firm can commit in advance to the subsequent prices and product designs, the introduction of low-end product should be delayed to alleviate cannibalization. In contrast, Bhattacharya *et al.* (2003) show that the strategy of introducing a low-end product before its high-end variant might be optimal if technological improvement is taken into account. None of the papers we have reviewed consider the impact of inventory on the introduction timing decisions.

Another relevant stream of literature studies the interface between marketing and operations management. In the literature of operations management, the classic approach often ignores the nonstationarity in demand inherent in the new product diffusion Shen and Su, 2007. On the other hand, marketing researchers typically focus on developing accurate characterizations of the demand process, and they seldom take supply side factors into consideration. Only recently have we seen some attempts to bridge the two areas. For example, Kurawarwala and Matsto (1998) present a model of procurement in which the demand process follows a Bass-type diffusion. Their model corresponds to an extension of a conventional newsvendor model and provides an example of how procurement policy can be influenced by new product diffusion dynamics. Ho *et al.* (2002) provide a joint analysis of demand and sales dynamics in a constrained new product diffusion context. Their analysis generalizes the Bass model to include backordering and customer losses, and determines the diffusion dynamics when the firm actively makes supply-related decisions to influence the diffusion process. Savin and Terwiesch (2005) present a model describing the demand dynamics of two new products competing for a limited target market, in which the demand trajectories of the two products are driven by a market saturation effect and an imitation effect reflecting the product experience of previous adopters. Schmidt and Druehl (2005) explore the influence of progressive improvements in product attributes and continual cost reduction on the new product diffusion process. Hopp and Xu (2005) analyze the cost and revenue trade-off of choosing optimal product line length and pricing decisions.

As inventory cost has been largely ignored in the introduction timing research, we are more interested in finding out how the inventory cost influences introduction timing of product line extensions. We start building an integrated model that considers joint decision on production line introduction and inventory management below.

3.3 Integrated Framework of Inventory and Introduction

Demand Model

Bass model Bass, 1969 formulates the aggregated adoption rate of a new product, which has received support from many empirical studies Mahajan, Muller, and Wind, 2000. Let $F(t)$ be the proportion of customers in the target market who have adopted the new product. Bass argues that the hazard rate $h(t) \equiv \frac{f(t)}{1-F(t)}$ ie. adoption rate for people who haven't adopted yet satisfies:

$$h(t) = p + qF(t) \tag{3.1}$$

where p is the innovation parameter describing the self-driven adoption, and q is the imitation parameter describing the word-of-mouth effect. While Bass model deals with diffusion dynamics for single product, Wilson and Norton (1989) extended Bass' seminal work to model demand dynamics for a set of product line extensions. They assumed that (1) adoption of different product generations contribute to a single information flow; (2) sales of different generations are proportional to the information flow; and (3) potential customers make purchase decisions as soon as they become informed. Under these assumptions, Bass model is used for characterizing the information flow

$$\frac{dF(t;T)}{dt} = \Big[1 - F(t;T)\Big]\Big[p + qS(t;T)\Big] \tag{3.2}$$

where T denotes the release time of second generation. $F(t;T)$ and $S(t;T)$ are respectively the proportion of population who have been aware of and who have purchased the product, regardless of the generation. $S(t;T)$ can be further expressed as the sum of $S_i(t;T)$ $(i = 1, 2)$ representing the proportion who have purchased ith generation of products respectively.

Different with Wilson and Norton's original setting, we're interested in the case where it is the customer differentiation rather than technological improvement that drives the firm to provide different generations of the product. The first generation usually targets at high-valuation customers while the second generation is usually designed to further reap revenue from low-valuation customers. By assuming two market segments, we propose a new interpretation of Wilson/Norton model below. In fact, we assume that out of the entire population, m_1 fraction are high-valuation customers and the remaining $(1 - m_1)$ fraction are low-valuation customers. Before the release of second generation, it is assumed that all customers with high valuation will purchase the first high-end generation after they become informed. Only those high-valuation customers who have purchased the product will spread the product information via the word-of-mouth recommendations. Before the release of low-end products, all low-valuation customers will leave the market immediately without purchasing the high-end products. After release of the second low-end generation, only α_1 portion out of m_1 customers with high valuation retain for the first high-end generation; α_2 portion out of m_1 customers with high valuation switch to the second low-end generation; and the rest of α_3 portion of m_1 customers leave the market. $\alpha_3 = 0$ represents the perfect high-valuation customer retention, in which case all high-valuation customers either stick to the first generation, or migrate to the second generation after release of the second generation. Otherwise when $\alpha_3 > 0$, customer churn occurs, perhaps due to the backslash effect from the second low-end generation to the high-end products.

Coefficient α_1 characterizes the *stickiness* of high-valuation customers to the first generation of high-end products; α_2 characterizes the *compatibility* of the second generation of low-end products to the first generation of high-end products; while α_3 quantify the *customer attrition* due to the introduction of the second low-end generation. We have $\alpha_1+\alpha_2+\alpha_3 = 1$. Additionally all customers with low valuation are assumed to purchase the second low-end generation after they become informed. Consistent with Wilson and Norton's parameter setting, we have normalized the total market size for both segments to be unity. Thus the market size is m_1 in the sole presence of the first high-end generation. With the coexistence of both product generations, the market size for first high-end generation shrinks to $m_2 = \alpha_1 m_1$ due to cannibalization from the low-end products; and the market size for second low-end generation is $m_3 = \alpha_2 m_1 + (1-m_1)$, contributed from both high- and low-valuation customers. To summarize, we have provided a new perspective of the classical Wilson and Norton's product line extension model. All parameters on market size in the original Wilson and Norton's setting can be one-to-one mapped into our parameter setting characterizing customer segmentation, stickiness, compatibility and attrition.

With the definitions of market size for different product generations before and after the release, we have the cumulative sales dynamics

$$S_1(t;T) = \begin{cases} m_1 F(t;T) & (t < T) \\ m_1 F(T;T) + m_2[F(t;T) - F(T;T)] & (t \geq T) \end{cases} \tag{3.3}$$

$$S_2(t;T) = \begin{cases} 0 & (t < T) \\ m_3[F(t;T) - F(T;T)] & (t \geq T) \end{cases} \tag{3.4}$$

Substituting sales (3.3) and (3.4) into diffusion dynamics (3.2) and noticing $S(t;T) = S_1(t;T) + S_2(t;T)$, we can solve $F(t;T)$ as

$$F(t;T) = \begin{cases} \frac{1-e^{-(p+m_1 q)t}}{1+\frac{m_1 q}{p}e^{-(p+m_1 q)t}} & (t \leq T) \\ \frac{1-Ce^{-(p'+q')(t-T)}}{1+C\frac{q'}{p'}e^{-(p'+q')(t-T)}} & (t > T) \end{cases} \tag{3.5}$$

where $p' = p + q(m_1 - m_2 - m_3)F(T;T)$, $q' = (m_2 + m_3)q$ and $C = \frac{1-F(T;T)}{1+\frac{q'}{p'}F(T;T)}$. Correspondingly the cumulative sales for both generations $S_1(t;T)$ and $S_2(t;T)$ can be obtained by substituting $F(t;T)$ back into (3.3) and (3.4).

Now or Never

Seller's revenue consist of sales of the first generation to high-valuation customers, and the sales of the second generation to both high- and low-valuation customers:

$$\begin{aligned} \pi_0(T) &= r_1 m_1 F(T;T) + r_1 m_2[1 - F(T;T)] + r_2 m_3[1 - F(T;T)] \\ &= (r_1 m_1 - r_1 m_2 - r_2 m_3)F(T;T) + (r_1 m_2 + r_2 m_3) \end{aligned} \tag{3.6}$$

where r_1 and r_2 are unit profit for the first and second product generations respectively. As we generally consider books or products with short life cycles (i.e. apparel, toys, consumer electronics, personal computers), r_i $(i = 1, 2)$ are treated as fixed during product life cycle Kurawarwala and Matsuo, 1996; Bitran, Haas, and Matsuo, 1986; Wilson and Norton, 1989; Mahajan and Muller, 1996. Without consideration of inventory cost, we tradeoff the sales revenue of the first and second generations by choosing an optimal introduction time.

Proposition 4 [Now or Never] *To maximize the total revenue, the optimal introduction policy is "now-or-never": we introduce the second generation right now when* $r_1 m_1 - r_1 m_2 - r_2 m_3 < 0$*, or equivalently when* $\left[(\frac{r_1}{r_2}+1)\alpha_3 + \frac{r_1}{r_2}\alpha_2 + \alpha_1\right]m_1 < 1$*; while we never introduce the second generation when* $r_1 m_1 - r_1 m_2 - r_2 m_3 \geq 0$*, or equivalently when* $\left[(\frac{r_1}{r_2}+1)\alpha_3 + \frac{r_1}{r_2}\alpha_2 + \alpha_1\right]m_1 \geq 1$.

Consistent with Wilson and Norton, 1989, the "now-or-never" rule is quite intuitive: the second generation should be introduced as early as needed if the market potential for the first generation is relatively small; or the unit profit from the first generation is relatively low; or the stickiness of high-valuation customers to the first generation is relatively high; or the compatibility of second generation to the first generation product is relatively high (noticing $\alpha_3 = 1 - \alpha_1 - \alpha_2$). Moreover since r_2 as the unit profit for low-end product is usually smaller than r_1 as the unit profit for high-end product, $\frac{r_1}{r_2}$ should be larger than unity. This means when it comes to determine the optimal introduction time in the absence of inventory cost, customer attrition is the most important factor, followed by the product compatibility. Customer stickiness should be of least consideration.

Inventory Cost

Inventory cost is actually an important missing factor in the above formulation, as motivated by various industrial practice in the introduction section. As a consequence, the "Now-or-Never" policy may turn out to be suboptimal or even bad after incorporating inventory cost in the total revenue. Intuitively speaking, to delay the introduction of the second generation will increase the inventory cost for the first generation of products, because we keep the inventory not only for longer time but also at a larger quantity. Especially when the sales rate for the first generation slows down in the late stage, the effective inventory cost for the remaining products can be very expensive. Therefore, it seems a better idea to produce less first generation product, and to introduce the second generation earlier, so that we can save the inventory cost for holding the first generation products. However, to expedite the introduction of the second generation will increase the inventory cost for the second generation of products, because we not only keep a larger quantity of inventory, but also face a less-developed market, a corresponding slower product diffusion speed and thus a longer sales season. In a nutshell, inventory cost is an important but rather complicated factor when considering the product line extension introduction. It's hard to predict how the increase or decrease of inventory holding cost accelerates or decelerates the introduction. In fact, we feel the need to systematically study the integration of timing the product line extension introduction and the inventory management. As a first step toward the integrated framework, we introduce a simple inventory model in this section, and proceed to enrich it in the following sections. The results from our simple model verifies our intuitions that the inventory cost indeed plays a complicated and vital role.

Under the diffusion demand model described above, we consider a one-replenishment ordering policy. At the beginning of the entire sale season ($t = 0$), we make an order (or complete the production) of certain amount of the first-generation products, which satisfy all future demand for the first-generation products; and then right before the introduction ($t = T$), we make an order (or complete the production) of certain amount of the second-generation products, which satisfy all future demand for the second-generation products. Despite of its simple nature, this one-replenishment policy is also reasonable in practice, especially when considering large setup cost, relatively short product life cycles, international outsourcing, rapidly-changing consumer preferences and frequent innovations in the industry of publishing, fashion and high-tech electronics, etc. We choose to build our model under the infinite planning horizon, for two reasons: (1) Demand diffusion given by Bass model decreases exponentially over time, so any relatively long planning horizons are actually equivalent to the infinite planning horizon. (2) It keeps the formulation in a relatively simple form, thus computationally manageable. We'll analyze finite planning horizons in the next section. Under the one-replenishment ordering policy, seller's inventory cost can be expressed as the sum of the inventory holding costs for each generation

$$\pi_I(T) = -h\int_0^\infty [S_1(\infty;T) - S_1(t;T)]dt - h\int_T^\infty [S_2(\infty;T) - S_2(t;T)]dt \tag{3.7}$$

$$= -hm_1\int_0^T [F(T;T) - F(t;T)]dt - hm_2\int_0^T [1 - F(T;T)]dt$$
$$-h(m_2+m_3)\int_T^\infty [1 - F(t;T)]dt \tag{3.8}$$

where we have assumed that the annual unit holding cost h is the same for both generations.

Seller's total profit $\pi(T)$ as a function of release time T is the sum of the sales revenue $\pi_0(T)$ and the inventory cost $\pi_I(T)$:

$$\begin{aligned}
\pi(T) &= -\frac{\left(1+m_1\frac{q}{p}\right)\left(r_1m_1-r_1m_2-r_2m_3-h(m_1-m_2)T\right)e^{-(p+m_1q)T}}{1+m_1\frac{q}{p}e^{-(p+m_1q)T}} \\
&+ \frac{h}{q}\ln\left(1+m_1\frac{q}{p}e^{-(p+m_1q)T}\right)-\frac{h}{q}\ln\left(1+(m_2+m_3)\frac{q}{p}e^{-(p+m_1q)T}\right) \\
&+ m_1r_1-\frac{h}{q}\ln\left(1+m_1\frac{q}{p}\right) \qquad (3.9)
\end{aligned}$$

All terms with h as a multiplier in the above formulation come from the inventory holding cost. We can see that the inventory holding cost is a highly nonlinear function of the introduction time T, originating from the complicated nature of the Bass diffusion process. For a given set of marketplace parameters of market sizes m_i $(i = 1, 2, 3)$, diffusion rates p and q, unit profits r_i $(i = 1, 2)$ and unit holding cost h, we can in principle solve the optimal introduction time T^* by maximizing the total profit subjected to $T \geq 0$. However we find it in fact impossible to get a full analytical characterization of the optimal solution in the eight-dimension parameter space. Instead we come up with some strong sufficient conditions elaborated in next subsection, which in fact can cover a wide array of marketplace parameter settings we're interested in.

Optimal Introduction Strategy

In this subsection, we introduce a series of propositions to characterize the optimal introduction strategy (with all proofs in the appendix).

Proposition 5 *It's never optimal to never introduce the second generation.*

To understand why theoretically this becomes the case after incorporating the inventory holding cost, we consider the total profit with initially very late introduction time T, in which case the sales of second generation become negligible. If we further delay the introduction, the additional revenue savings from the second generation are very slim, because they come from the change of the sales of second generation. However, the inventory cost for the first generation increases proportionally with the delay of the introduction time, which dominates the total profit change. Thus it's always better to at lease introduce the second generation, because it saves our inventory holding cost for the first generation product. However, one need to notice that Proposition 5 hinges on the infinite planning horizon assumption. In the late stage when diffusion almost saturates, it makes tiny gains to introduce a second generation, and in practice we'll never introduce the second generation because of other managerial considerations. In the following subsection, when we present results on the optimal introduction strategy, we reasonably use a relatively long planning horizon as a cutoff, so that all optimal introduction time beyond this cutoff is degenerated as "to never introduce". Next, we discuss when sequential and simultaneous introduction become optimal. For sake of simplicity, let's first define

$$A \equiv \frac{1}{p}\frac{m_3 p + (m_1 - m_2)(m_2 + m_3)q}{p + (m_2 + m_3)q} > 0, \tag{3.10}$$

which is a function of diffusion parameters and market potentials.

Proposition 6 **[Optimality of Sequential Introduction]** *If $r_1 m_1 - r_1 m_2 - r_2 m_3 > -hA$, it's optimal to introduce the second generation sequentially. The optimal introduction time T^* can be obtained by solving the following equation*

$$\left(1 + \frac{q}{p}m_1\right)\left(1 + \frac{q}{p}(m_2 + m_3)e^{-T^*(p+qm_1)}\right)\left[r_1 m_1 - r_1 m_2 - r_2 m_3 - (m_1 - m_2)hT^*\right] + \frac{h}{p}\left(1 + \frac{q}{p}m_1 e^{-T^*(p+qm_1)}\right)\left(m_3 + \frac{q}{p}(m_1 - m_2)(m_2 + m_3)e^{-T^*(p+qm_1)}\right) = 0. \tag{3.11}$$

The proposition reveals the optimality conditions for sequential introduction policy in two aspects: sale revenue and inventory cost. From the perspective of sales revenue, it's optimal to introduce the second generation sequentially, if the market potential or unit cost of the first generation is relatively large. In this case, sequential introduction reduces the cannibalization between successive generations. The implications from inventory cost is fully explored by the following corollary.

Corollary 5 *If it's optimal to introduce two generations sequentially with the inventory holding cost h, it's optimal to introduce them sequentially in the case of any holding cost $h' > h$, with other things the same.*

By this corollary, we highlight the influence of inventory holding cost on the sequential introduction strategy. High unit inventory cost will drive the seller to introduce product line extension sequentially. This looks counter-intuitive at first glance, because delay of the introduction seems to increase the inventory cost. But the fact is just the other way around. Different from traditional research on inventory management, where demand can be time-varying but must be fixed ahead. In our model setting, demand is not purely exogenous, but can be influenced by the extension introduction decisions. Sequential introduction enables the sales of the second generation to start at a relatively high rate, thus a lower holding cost. Under sequential introduction policy, higher unit inventory cost results in higher savings from total inventory holding cost, which can justify the potential loss in sales revenue.

A remaining problem for proposition 6 is the possible occurrence of multiple optima. As numerically searched and studied below, there indeed exist cases when multiple local maxima and minima coexist. It's particularly difficult to find a full characterization for these cases in the full parameter space. This difficulty happens, since we put very few constraints on our parameter setting, so that our parameters vary in very broad ranges that can characterize rather different markets and industries in a unified framework. However, when the total market size is stagnant or diminishing, or the market size for the second generation doesn't overweigh the primal market size of the first generation too much, we are able to introduce a sufficient condition to guarantee the uniqueness of the solution as the global maximum.

Corollary 6 *When $r_1m_1 - r_1m_2 - r_2m_3 > -hA$ and $m_1 \geq m_2 + \frac{m_3^2}{3m_2+4m_3}$, it's optimal to introduce the second generation sequentially, and the optimal introduction time T^* can be uniquely determined by solving equation (3.11).*

Now let's turn to the simultaneous introduction policy by introducing the following proposition. For sake of convenience, we introduce

$$0 < B \equiv \min\left\{\frac{pqm_3}{p^2 + pq(2m_2 + 3m_3) + q^2m_1(m_2 + m_3)}, \frac{m_3}{3m_2 + 4m_3}\right\} < \frac{1}{4}, \tag{3.12}$$

which is also a function of diffusion parameters and market potentials.

Proposition 7 [Optimality of Simultaneous Introduction] *When $m_1 \leq m_2 + B \cdot m_3$, it's optimal to introduce both generations simultaneously if $r_1m_1 - r_1m_2 - r_2m_3 \leq -\frac{m_2+m_3}{m_1}hA$; otherwise when $m_1 \geq m_2 + B \cdot m_3$, it's optimal to introduce both generations simultaneously if $r_1m_1 - r_1m_2 - r_2m_3 \leq -hA$.*

The intuition behind the proposition is as below. First, we always require the condition $r_1m_1 - r_1m_2 - r_2m_3 \leq -hA$, which is just the complementary condition of the inequality in proposition 6. As a necessary condition for the optimality of simultaneous introduction policy, condition $r_1m_1 - r_1m_2 - r_2m_3 \leq -hA$, generally restricts the unit sales profit from the second generation not to be too small, and the unit holding cost not to be too high. Otherwise, the sequential introduction dominates, as elaborated in proposition 6. However when $m_1 \leq m_2 + B \cdot m_3$, condition $r_1m_1 - r_1m_2 - r_2m_3 \leq -hA$ is not enough to guarantee the optimality of the simultaneous introduction. We end up with a stronger condition by multiplying $\frac{m_2+m_3}{m_1}$ on the right hand side of the inequality.

Proposition 6 together with proposition 7 almost characterize all scenarios except for the case when $m_1 \leq m_2 + B \cdot m_3$ plus $-\frac{m_2+m_3}{m_1} \cdot hA < r_1m_1 - r_1m_2 - r_2m_3 \leq -hA$. This is the case when the total profit may look like a "tilted S" curve: first decreases then increases and finally decreases as the introduction time T goes from 0 to $+\infty$. So the optimal introduction time can be either now or some time later. In a different setting without incorporating the inventory cost, Mahajan and Muller considered the introduction for successive generations driven by technological innovations, instead of customer differentiation. They found the total discounted profit as a function of introduction time is just a "tilted S", and further concluded the "now-or-maturity" rule Mahajan and Muller, 1996. In our numerical examples with various marketplace parameter settings illustrated below, we found the optimal introduction policy for all cases with "tilted S"-curved total profit turn out to be simultaneous introduction.

Numerical Studies

We conduct numerical studies under some typical parameter settings in this subsection. Firstly, we analyze how inventory cost influences the optimal introduction strategy jointly with other marketplace factors. Then we revisit "Now-or-Never" Wilson and Norton, 1989 as well as "Now-or-Maturity" rules Mahajan and Muller, 1996 in these settings. And lastly we highlight the impact of the inventory holding cost on the optimal introduction strategy.

Figure 3.1 summarizes the optimal introduction strategies under various marketplace settings. We normalize the unit sale profit of the first generation r_1 to be unity. Then for each plot in the array, x-axis represents the unit sale profit of the second product generation r_2, and y-axis represents the annual unit inventory holding cost h. As motivated before, $r_2 \leq r_1$, so x-coordinate ranges from 0 to 1. We consider up-to-25% annual inventory holding cost, so y-coordinate ranges from 0 to 0.25. Please notice that different from the traditional definition, the unit annual inventory holding cost here is relative to the unit profit of first-generation products. For each plot, we divide the whole plane into a 17×17 lattice, and find the optimal introduction time T^* by maximizing the total profit function for each lattice point numerically. The optimal introduction time T^* at each lattice point is then represented by the darkness of the point: as T^* goes from 0 to 5 *year*, the color gradually change from dark to light. All points with $T^* \geq 5$ are in light color. We adopt the innovation coefficient $p = 0.15\ year^{-1}$ and imitation coefficient $q = 2\ year^{-1}$ for all plots. A natural diffusion timescale $\tau = \frac{\ln(q/p)}{q+p} \simeq 1.2\ year$ is the time when single-product sale peaks. When $T \geq 5\ year \simeq 4\tau$, the product diffusion is almost saturated, and it makes little impact on the total profit to introduce a new generation. As motivated previously, we can effectively adopt a relatively long effective planning horizon $T_p = 5\ year$. Therefore in the plots, all light-colored points actually corresponds to the cases that we "never" introduce the second generation. As a comment to clarify, the timescale (in unit *year*) of the dynamics is entirely determined by diffusion parameters p and q, which can be inferred from sale data.

Array of plots in Figure 3.1 are organized with respect to the high-valuation customer characteristics (α's) and the market size for the first-generation products (m_1). Plots in the same row have the same customer characteristics, including *stickiness* α_1, *compatibility* α_2, and correspondingly *customer attrition* α_3. Plots in the first row represent the case when after release of the low-end second generation, most high-valuation customers stick to the high-end first-generation products with $\alpha_1 = 75\%$; only a few migrate to the low-end second-generation products with $\alpha_2 = 25\%$; and little customer attrition $\alpha_3 = 0$. This can be a good representation of the high-tech consumer electronics industry. Plots in the second row represent the case when most high-valuation customers stick to the high-end first-generation products with $\alpha_1 = 75\%$; little migrate to the low-end second-generation products with $\alpha_2 = 0$; and a small portion of customers leave the market after the release of the low-end products $\alpha_3 = 25\%$. This can be a good representation of the fashion or luxury goods industry. Finally plots in the third row represent the case when only a few high-valuation customers stick to the high-end first-generation products with $\alpha_1 = 25\%$; most migrate to the low-end second-generation products with $\alpha_2 = 75\%$; and little customer

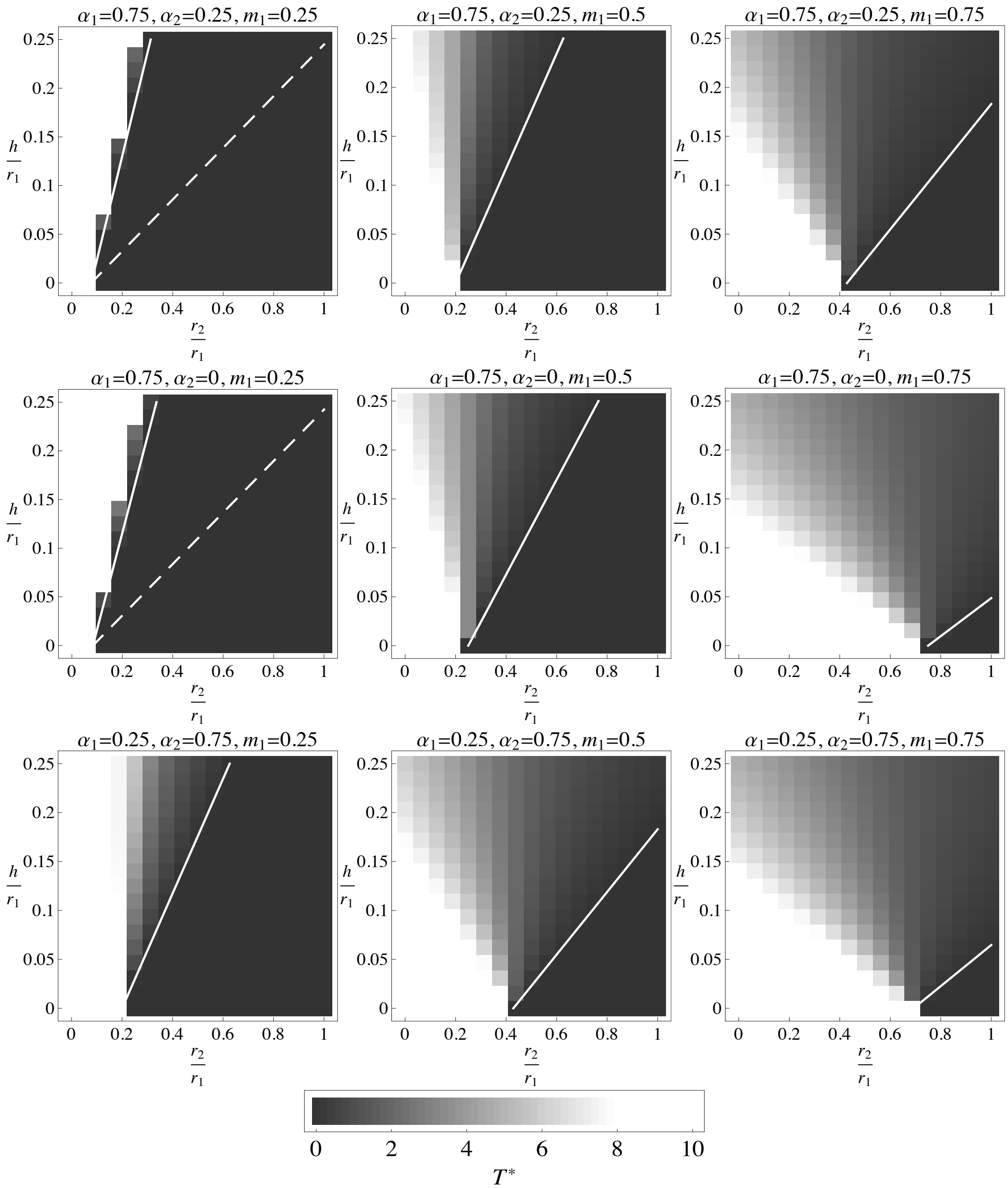

Figure 3.1: Plots of Optimal Introduction Strategy under Various Marketplace Settings.

attrition $\alpha_3 = 0$. This can be a good representation of the publishing industry. Plots in the same column have the same market potential of the first-generation m_1. From left to right, $m_1 = 0.25,\ 0.5,\ 0.75$, respectively represent the case when the first generation is a prelude to the second-generation products, an equally important market to the second generation, and the primal target market out of both generations.

For each plot corresponding to a specific marketplace setting, we apply the theoretical results in the last subsection to analyze the optimal introduction strategy, as marked by two white lines. The solid line corresponds to proposition 6, implying that it's optimal to introduce generations sequentially for all points above it; the dashed line corresponds to proposition 7, implying that simultaneous introduction is optimal for all points below it. The region between the two lines is undetermined by the analytical propositions. Out of nine plots, there are seven plots where the solid line and dashed line are exactly overlapped, in which cases the optimal introduction strategy can be entirely determined by the analytical propositions. There are also two plots where the dashed line is lower than the solid line, in which case the undetermined region is non-empty. However, our numerical result implies that simultaneous introduction policy is optimal within this region for both plots. In conclusion, it's optimal to introduce both generations simultaneously for all points below the solid white lines; it's optimal to never introduce the second generation in the light-colored region; and it's optimal to introduce second generation sequentially in the dark-colored region above the white solid line. The optimal introduction time is suggested by the degree of darkness, as standardized in the palette below the plot array.

It's not hard to notice several interesting trends in Figure 3.1. Firstly, for every industry, when the market size of the first generation increases, regions for simultaneous introduction strategy shrink, and sequential introduction strategy becomes more and more dominant. This is because the cannibalization from the second low-end generation becomes more and more costly. Secondly, for both high-tech and fashion industries, when the market size for the first generation is small ($m_1 = 0.25$), the optimal introduction policy is indeed "now-or-never"; however for publishing industry, "now-or-never" is never a good guideline, consistent with the common wisdom. Finally, roughly speaking, high unit inventory holding cost drives the seller to sequentially introduce the successive generations of products; while high profitability of the second generation justify the simultaneous introduction. The guideline is formalized in Table 3.1.

Table 3.1: Optimal Introduction Strategies with respect to Inventory Holding Cost and Profitability of Low-End Products.

		Profitability of Successive Low-End Products		
		Low	Medium	High
Inventory Holding Cost	High	Never	Sequential	Simultaneous
	Low	Never	Simultaneous	Simultaneous

Now we turn to reevaluate "now-or-maturity" policy in our model setting. Proposed by Mahajan and Muller, 1996, this influential rule states that "The optimal decision rule for a firm introducing a new generation of a technological durable product is either to introduce the product as soon as possible or delay its introduction to the maturity stage in the life-cycle of the first generation." This proposition originates from their numerical finding that the optimal introduction time usually makes a quantum leap to the ninth period, when "now" is no longer optimal. However, back to our setting, Figure 3.1 has revealed that the optimal introduction time can vary continuously from "now" to "never". Moreover, they have given a qualitative criterion on optimal introduction strategy: "If either the market potential of the second generation is large, or the profits gained from the second generation are large, the firm introduces the second generation now." This is consistent with our result. Actually in Figure 3.1, when m_1 is small and r_2 is high, simultaneous introduction is indeed optimal.

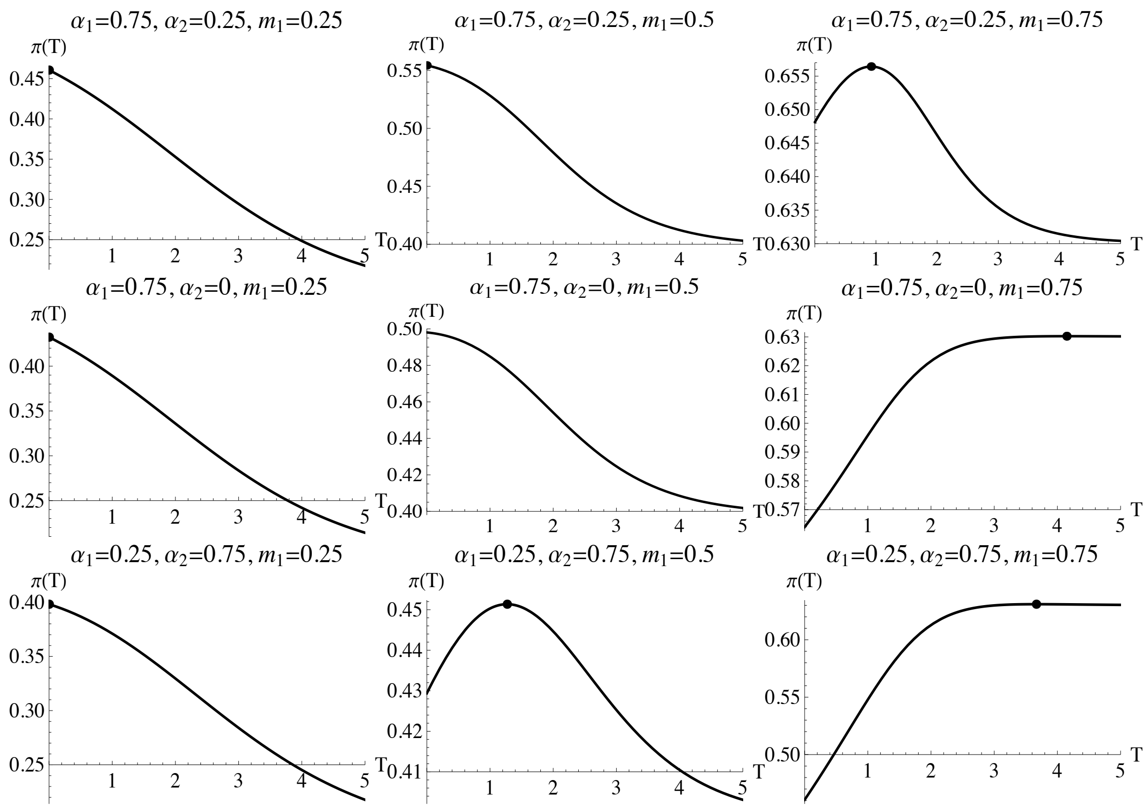

Figure 3.2: Impact of Introduction Timing of Product Line Extension on Total Profit. Points on the peak mark the optimal introduction time and corresponding the maximal total profit. Annual inventory holding cost $h = 10\%$; the ratio of unit sale profit from the second generation and that of the first generation $\frac{r_2}{r_1} = 0.5$.

To get a direct understanding of the applicability of "now-or-maturity" rule in our arena, we take a set of specific parameter settings as examples to show the impact of introduction time on the total profit below. In fact, maturity of the first generation market can be measured by the peak sale timing $T_m = \frac{\ln(m_1 q)-\ln(p)}{p+m_1 q}$, which stays relatively constant (ranges within 1.4 and 1.8), as m_1 ranges from 0.25 to 0.75, given $p = 0.15$ and $q = 2$. So "now-or-maturity" can be translated as to introduce the second generation now or around time $T_m \simeq 1.5$ *year* (or with some constant factor), given the diffusion rates. As shown in Figure 3.2, our model generates all possible optimal introduction timings ranging from 0 to T_p, with significant deviations from "now-or-maturity". The discrepancy also casts questions to the "now-or-maturity" rule: (1) When is the maturity exactly? (2) What's the applicable range for the rule? Our model tries to provide a recipe to explicitly answer these questions.

Finally, we discuss how the optimal introduction time depends on inventory holding cost. In Figure 3.1, by fixing x-axis and looking at how the darkness changes along y-axis, we observe that the optimal introduction timing can vary with inventory cost in different ways. We start formalizing the ideas in Figure 3.3. As usual, we present a plot array organized by different market potentials and high-valuation customer characteristics. For each plot, omitting the dashed lines at this moment, three solid lines in different colors correspond to three different unit sale profits of the second generation r_2. From this figure, we can summarize three general rules regarding the relationship of the optimal introduction time and the unit inventory holding cost. (1) The optimal introduction time relies on the unit inventory holding cost in a nonlinear complex way. Depending on different marketplace settings, particularly unit sale profits, the optimal introduction time can increase or decrease in the inventory holding cost. (2) When the unit inventory holding cost is relatively high, the optimal introduction timing tends to converge to a constant value, which is irrelevant with the inventory holding cost. (3) There exists a certain combination of marketplace setting, in which case the optimal introduction time doesn't depend on the inventory cost.

Now we will try to find out whether our analytical framework can confirm these observations. First we notice the following proposition:

Proposition 8 *If* $r_1 m_1 - r_1 m_2 - r_2 m_3 = 0$*, the optimal introduction time doesn't depend on the unit inventory holding cost.*

This is the case when the markets for two generations are perfectly balanced, so that the inventory exerts the same impact on each market, and doesn't affect the introduction strategy. In general, to further characterize the relationship between unit inventory holding cost and the optimal introduction time turns out to be rather complicated, without a closed-form expression. However, by looking at a specific case when the total market potential stays the same before and after the introduction of the second generation, we are able to grasp the main idea behind. The following proposition formalizes the rules discovered above (with proof in appendix):

Proposition 9 *In the case of* $m_2 + m_3 = m_1$*, the optimal introduction time* T^* *is a(n) increasing (decreasing) function of unit inventory holding cost* h *iff* $r_1 m_1 - r_1 m_2 - r_2 m_3 < 0$

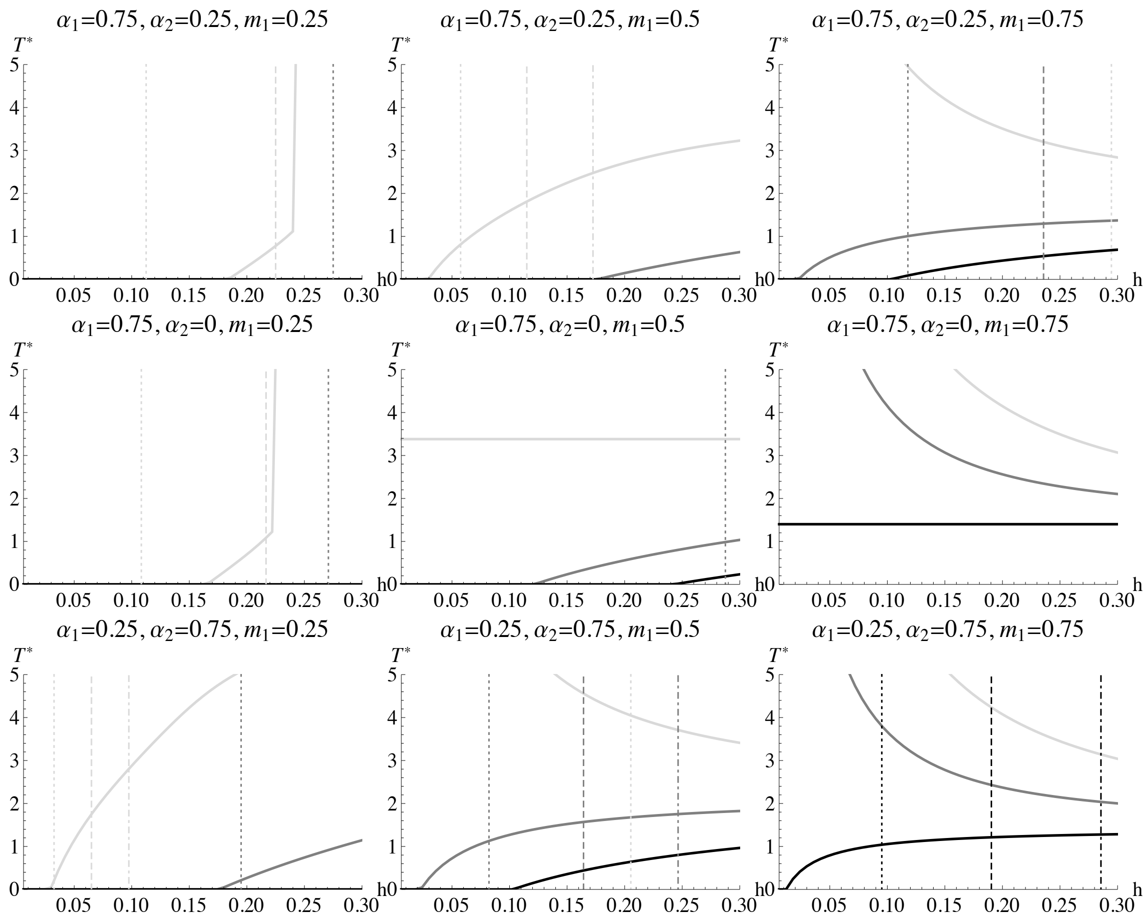

Figure 3.3: Optimal Introduction Time As A Function of Inventory Holding Cost. Different colors represent different unit sale profits of the second-generation products r_2. Light gray, gray and black solid lines correspond to $\frac{r_2}{r_1} = 0.25,\ 0.5,\ 0.75$ respectively. Vertical lines mark the convergence scale of T^* in h. Dotted line marks $h = h^0 = |r_1 \frac{m_1 - m_2}{m_3} - r_2|(p + qm_1)$; dashed line marks $h = 2h^0$ and dotted-dashed line marks $h = 3h^0$.

(> 0). *Moreover, when* $h \gg h^0 = |r_1 \frac{m_1 - m_2}{m_3} - r_2|(p + qm_1)$, *the optimal introduction time* T^* *doesn't depend on* h.

Generally without $m_2 + m_3 = m_1$, the optimal introduction time can be a non-monotonic function of unit inventory holding cost, yet Proposition 9 can still apply most of the time. As shown in Figure 3.3, none of the plots satisfy $m_2 + m_3 = m_1$, but actually proposition 9 apply to all cases there: all decreasing solid lines satisfy $r_1 m_1 - r_1 m_2 - r_2 m_3 > 0$, and all increasing solid lines satisfy $r_1 m_1 - r_1 m_2 - r_2 m_3 < 0$. Also h^0 roughly serves as a scale to measure the convergence of $T^*(h)$.

3.4 Extensions

So far we have fully explored the optimal introduction strategies under the one-replenishment ordering policy in an infinite planning horizon. We proceed to enrich the basic model in this section. Particularly, we extend the model in the following two aspects to make it more flexible. Suggested by previous research Wilson and Norton, 1989; Mahajan and Muller, 1996, we first consider a finite planning horizon. As shown below, this extension actually doesn't impair our analysis in the last section. Our main findings regarding the optimal introduction strategy are rather robust. Especially under short planning horizons, "now-or-never" rule is never optimal. We then generalize the simple ordering policy to include multiple replenishment. We find that when the replenishment gets frequent, the "now-or-never" rule dominates again.

Finite Planning Horizon

Under a finite planning horizon T_p, the objective function of seller's total profit $\pi_p(T)$ can be reformulated as,

$$\begin{aligned}
\pi_p(T) &= r_1 m_1 F(T;T) + r_1 m_2 [F(T_p;T) - F(T;T)] + r_2 m_3 [F(T_p;T) - F(T;T)] \\
&- hm_1 \int_0^T [F(T;T) - F(t;T)]dt - hm_2 \int_0^T [F(T_p;T) - F(T;T)]dt \\
&- h(m_2 + m_3) \int_T^{T_p} [F(T_p;T) - F(t;T)]dt \qquad (3.13) \\
&= [r_1 m_1 - r_1 m_2 - r_2 m_3 + h(m_2 - m_1)T]\, F(T;T) \\
&+ [r_1 m_2 + r_2 m_3 + hm_3 T - h(m_2 + m_3)T_p]\, F(T_p;T) \\
&+ h \left[m_1 T - \frac{1}{q} \ln(1 + \frac{q}{p} m_1) + \frac{1}{q} \ln(1 + \frac{q}{p} m_1 e^{-(p+qm_1)T}) \right] \\
&+ h(m_2 + m_3) \left[(T_p - T) - \frac{1}{q'} \ln(1 + C\frac{q'}{p'}) + \frac{1}{q'} \ln(1 + C\frac{q'}{p'} e^{-(p'+q')(T_p - T)}) \right] \qquad (3.14)
\end{aligned}$$

where $F(t;T)$, C, p' and q' have been defined in (3.5). $\pi_p(T)$ goes to $\pi(T)$ in (3.9) as the planning horizon T_p goes to infinity. In principle we can come up with analytical characterizations of the optimal introduction timing by maximizing the total profit $\pi_p(T)$, as T varies between 0 and T_p. However, the expression of the objective function turns out to be rather complicated to analyze, even to express. Thus we use numerical studies instead, to inspect the impact of the finite planning horizon below.

We consider the case when the annual inventory holding cost $h = 10\%$ (of the unit profit of the first-generation product r_1), and the unit profit of the second-generation product $r_2 = 0.5r_1$. Along the line with the previous sections, we consider nine different marketplace parameter settings, differentiated by α_i $(i = 1, 2, 3)$ and m_1. To illustrate the impact of planning horizon T_p on the optimal timing T^*, we consider four different lengths of planning horizon with $T_p = \infty$, 5 *year*, 2.5 *year*, 1 *year*. $T_p = \infty$ corresponds to the case of infinite planning horizon, as fully analyzed in the previous section. We denote the optimal introduction timing under the infinite horizon as T_0^*. To get an understanding from the perspective of market penetration rate, let's look at the market with $m_1 = 0.5$. There is a natural diffusion timescale $T_m \simeq 1.65$ *year* under our parameter setting. In fact $T_p =$ 5 *year* $\simeq 3T_m$ means that we choose the planning horizon as the time when the adoption fraction of the first-generation products under its sole presence is around 98%. Similarly $T_p = 2.5$ *year* $\simeq 1.5T_m$ corresponds to a fraction of around 69%, and $T_p = 1$ *year* $\simeq 0.6T_m$ corresponds to a fraction of around 22%, which is a rather short planning horizon.

Table 3.2: Optimal Introduction Timing As A Function of Planning Horizon.

α_1	α_2	m_1	$T_p = \infty$	$T_p = 5$ *year*		$T_p = 2.5$ *year*		$T_p = 1$ *year*	
			T_0^*	T^* ($\Delta T^*\%$)	$\Delta\pi^*\%$	T^* ($\Delta T^*\%$)	$\Delta\pi^*\%$	T^* ($\Delta T^*\%$)	$\Delta\pi^*\%$
0.75	0.25	0.25	0	0 (0%)	0%	0 (0%)	0%	0 (0%)	0%
		0.5	0	0 (0%)	0%	0 (0%)	0%	0 (0%)	0%
		0.75	0.922	0.911 (1.2%)	0%	0 (>100%)	-2.8%	0 (>100%)	-20.9%
0.75	0	0.25	0	0% (0%)	0%	0 (0%)	0%	0 (0%)	0%
		0.5	0	0 (0%)	0%	0 (0%)	0%	0 (0%)	0%
		0.75	4.15	3.99 (4.0%)	0%	2.5 (65.9%)	-0.4%	1 (>100%)	1.6%
0.25	0.75	0.25	0	0 (0%)	0%	0 (0%)	0%	0 (0%)	0%
		0.5	1.27	1.24 (2.2%)	0%	0.116 (9.94%)	-10.3%	0 (>100%)	-43.3%
		0.75	3.66	3.57 (2.6%)	0%	2.5 (46.6%)	-1.2%	1 (>100%)	11.5%

As shown in Table 3.2, let's first compare the optimal introduction timing T^* under different planning horizons. We define $\Delta T^*\% \equiv \frac{T_0^* - T^*}{T^*}$ as the relative difference between the optimal introduction timing under infinite planning horizon and the one under planing horizon of length T_p. When $T_p = 5$ *year*, $\Delta T^*\%$ is mostly zero with the largest relative difference as 4%. Moreover, if we take a look at the total profit, we'll find that the approximation of infinite planning horizon by $T_p = 5$ *year* is even better. We define $\Delta\pi^*\% \equiv \frac{\pi_G(T_0^*) - \pi_G(T^*)}{\pi_G(T^*)}$ as the relative difference from the maximum total profit, when under planning horizon of length T_p, we adopt the optimal introduction timing T_0^*, which is the optimal solution obtained under the infinite horizon. We find that $\Delta\pi^*\%$ is all zero for $T_p = 5$ *year*. This implies that in the case with relatively long planning horizon $T_p = 5$ *year*, we can capture 100% profit by using the optimal introduction timing obtained under infinite horizon.

Then let's turn to the case with $T_p = 2.5$ *year*. Firstly, we notice that $\Delta T^*\%$ can be large. However, a scrutiny at T^* reveals that for both cases with $\Delta T^*\% = 65.9\%$ and 46.6%, the optimal introduction timing $T^* = 2.5$ *year*. This indicates "never" as the optimal introduction strategy, which is consistent with the relatively late introduction obtained under the infinite horizon, with $T_0^* = 4.15$ *year* and 3.66 *year*. Therefore the only discrepancy of the optimal introduction strategies between infinite horizon and $T_p = 2.5$ *year* is the case with $\alpha_1 = 0.75$, $\alpha_2 = 0.25$ and $m_1 = 0.75$. It's optimal to introduce the product line extension "now" under 2.5 *year* planning horizon; while the sequential introduction is optimal for infinite horizon. Nevertheless, by further taking a look at the difference in profit, we find all $\Delta\pi^*\%$ is within 10%. Therefore $T_p = 2.5$ *year* can still be well approximated by infinite horizon, in that by using the optimal introduction strategy obtained under infinite planning horizon, we are still able to capture 90% profit under a 2.5 *year* planning horizon.

By applying similar analysis to the case of $T_p = 1$ *year*, we observe that there exist significant differences between the infinite horizon and $T_p = 1$ *year*, in terms of the optimal introduction timing as well as the profit. We generate the plot array of optimal introduction strategy under planning horizon $T_p = 1$ *year* in Figure 3.4. Please first notice that "never" refers to $T^* = T_p = 1$ *year* in the current scenario. We find that (1) Under a short planning horizon, "Now-or-Never" is never an optimal introduction strategy under all marketplace settings. (2) Low inventory holding cost together with high profitability of the product line extension justify the optimality of simultaneous introduction. (3) It becomes optimal to sequentially introduce the product line extension, when inventory holding cost is high and the profitability of the second-generation products is medium. Therefore the idea formalized in Table 3.1 is still applicable in the short planning horizon.

Many other factors come into play when determining the planning horizon. For many slow-paced industries, it's always reasonable to guarantee a moderate market penetration fraction. Therefore we expect normally the planning horizon $T_p \gtrsim 2.5$ *year* in our setting, so that the market penetration fraction is over 70%. This means our analysis under infinite horizon is directly applicable to the practical finite-horizon situations: to apply the optimal introduction strategies obtained under the infinite horizon guarantees around 90% of the total profit. Moreover, with no surprise, we find that as the planning horizon gets shorter, we tends to expedite the introduction of the product line extension.

Multiple Replenishments

To include multiple replenishments in our inventory model, we consider a simple scheduled ordering policy: fixed interval ordering Graves, 1996; Cachon, 1999. In reality it is often impossible to replenish inventory continuously, and thus the fixed interval ordering policy is motivated and widely used in practice. Delivery of orders is assumed to be instantaneous. We assume an exogenous ordering interval O_i for ith generation products. Then our most

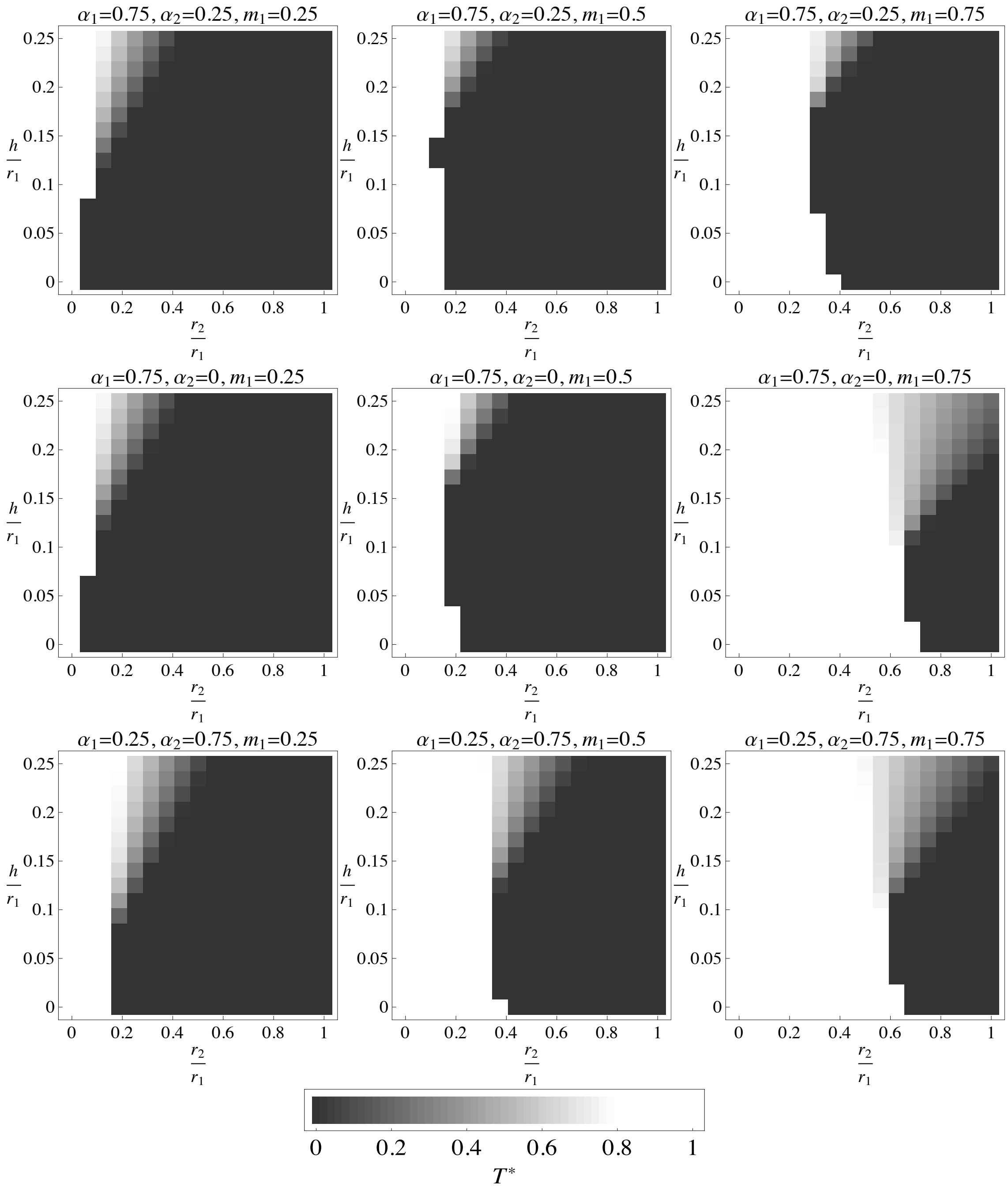

Figure 3.4: Plots of Optimal Introduction Strategy under Short Finite Planning Horizon $T_p = 1$ *year*.

generalized form of the total profit objective function $\pi_G(T)$ can be expressed as:

$$\begin{aligned}
\pi_G(T) &= r_1m_1F(T;T) + r_1m_2[F(T_p;T) - F(T;T)] + r_2m_3[F(T_p;T) - F(T;T)] \\
&- hm_1 \sum_{j=1}^{\lfloor T/O_1 \rfloor} \int_{(j-1)O_1}^{jO_1} [F(jO_1;T) - F(t;T)]dt - hm_1 \int_{\lfloor T/O_1 \rfloor \cdot O_1}^{T} [F(T;T) - F(t;T)]dt \\
&- hm_2 \int_0^T [F(T_p;T) - F(T;T)]dt \\
&- hm_2 \sum_{j=1}^{\lfloor (T_p-T)/O_1 \rfloor} \int_{T+(j-1)O_1}^{T+jO_1} [F(T+jO_1;T) - F(t;T)]dt \\
&- hm_2 \int_{\lfloor (T_p-T)/O_1 \rfloor \cdot O_1 + T}^{T_p} [F(T_p;T) - F(t;T)]dt \\
&- hm_3 \sum_{j=1}^{\lfloor (T_p-T)/O_2 \rfloor} \int_{T+(j-1)O_2}^{T+jO_2} [F(T+jO_2;T) - F(t;T)]dt \\
&- hm_3 \int_{\lfloor (T_p-T)/O_2 \rfloor \cdot O_2 + T}^{T_p} [F(T_p;T) - F(t;T)]dt \qquad (3.15) \\
&= [r_1m_1 - r_1m_2 - r_2m_3 + h(m_2 - m_1)T]\, F(T;T) \\
&+ [r_1m_2 + r_2m_3 + hm_3T - h(m_2+m_3)T_p]\, F(T_p;T) \\
&+ hm_1T - hm_1 \sum_{j=1}^{\lfloor T/O_1 \rfloor} \left[O_1F(jO_1;T) + \frac{1}{m_1q} \ln\Big(\frac{p + m_1qe^{-(p+m_1q)(j-1)O_1}}{p + m_1qe^{-(p+m_1q)jO_1}} \Big) \right] \\
&+ hm_1 \left[\lfloor T/O_1 \rfloor O_1F(T;T) - \frac{1}{m_1q} \ln\Big(\frac{p + m_1qe^{-(p+m_1q)\lfloor T/O_1 \rfloor O_1}}{p + m_1qe^{-(p+m_1q)T}} \Big) \right] \\
&+ hm_2(T_p - T) - hm_2 \sum_{j=1}^{\lfloor (T_p-T)/O_1 \rfloor} \left[O_1F(T+jO_1;T) + \frac{1}{q'} \ln\Big(\frac{p' + Cq'e^{-(p'+Cq')(j-1)O_1}}{p' + Cq'e^{-(p'+Cq')jO_1}} \Big) \right] \\
&+ hm_2 \left[\lfloor (T_p-T)/O_1 \rfloor O_1F(T_p;T) - \frac{1}{q'} \ln\Big(\frac{p' + Cq'e^{-(p'+Cq')\lfloor (T_p-T)/O_1 \rfloor O_1}}{p' + Cq'e^{-(p'+Cq')(T_p-T)}} \Big) \right] \\
&+ hm_3(T_p - T) - hm_3 \sum_{j=1}^{\lfloor (T_p-T)/O_2 \rfloor} \left[O_2F(T+jO_2;T) + \frac{1}{q'} \ln\Big(\frac{p' + Cq'e^{-(p'+Cq')(j-1)O_2}}{p' + Cq'e^{-(p'+Cq')jO_2}} \Big) \right] \\
&+ hm_3 \left[\lfloor (T_p-T)/O_2 \rfloor O_2F(T_p;T) - \frac{1}{q'} \ln\Big(\frac{p' + Cq'e^{-(p'+Cq')\lfloor (T_p-T)/O_2 \rfloor O_2}}{p' + Cq'e^{-(p'+Cq')(T_p-T)}} \Big) \right] \qquad (3.16)
\end{aligned}$$

where again $F(t;T)$, C, p' and q' have been defined in (3.5). Despite of its complicated form, the idea behind the total profit function is similar to the simple case discussed before. The key extension is that we need to count for the inventory holding cost by replenishment cycles. The first replenishment cycle starts from 0 for the first-generation products, while it starts from T for the second-generation. Using numerical searching, we find the optimal introduction timing under multiple-replenishment policy with replenishment intervals $O_1 = 0.5\ year$, $O_2 = 1\ year$, and planning horizon $T_p = 3\ year$. Figure 3.5 summarizes the results under various marketplace settings. We find that (1) "Now-or-Never" rule applies again. (2) Simultaneous introduction becomes more dominant, especially for the case when the market size for the first-generation is relatively small.

To have a close look at the impact of multiple replenishments, we take a specific example with $\alpha_1 = 0.25$, $\alpha_2 = 0.75$, $m_1 = 0.5$, $r_2 = 0.5r_1$, $h = 0.1r_1$ and as usual $p = 0.15$, $q = 2$. As shown in Figure 3.6, as O_2 the replenishment interval for the second-generation products gets shorter, we save inventory holding cost for the post-introduction period $[T, T_p]$. This saving gets more substantial as the post-introduction period gets longer, or equivalently the introduction time T gets earlier. On the other hand, as O_1 the replenishment interval for the first-generation products gets shorter, we save inventory holding cost for both the pre- and post-introduction periods, because we sell first-generation products in both periods. However, since we target first-generation products at high-valuation customers, we expect the sale during the pre-introduction period is the majority. Thus the saving from the first-generation products usually gets more substantial as the pre-introduction period gets longer, or equivalently the introduction time T gets larger. Moreover, we know from the inventory theory, the savings from introducing multiple replenishment is quadratically proportional to the length of the period. As a result, the saving for cases with rather early or rather late introduction is more significant than that with median introduction timing. Consequentially, "now" and "never" become more preferable as the replenishment gets more frequent, and "now-or-never" rule dominates the optimal introduction strategy again.

3.5 Summary and Future Research

Among earlier studies of introduction timing for product line extensions, researchers address this research question primarily in the marketing discipline Wilson and Norton, 1989; Moorthy and Png, 1992, neglecting important factors from operations management, such as inventory. This assumption is valid if the firm can continuously replenish inventory. However, inventory holding is often unavoidable in many industry practices, and this has led to a clear call in academia to develop more comprehensive models addressing the timing decisions from both operations management and marketing science perspectives, with the hope to design methodologies to improve a firm's profit or enhance the decision maker's performance. The purpose of this paper is to take a first step towards understanding the implications of timing introductions of product lines by coordinating decisions of marketing and operations management.

We study the problem that a firm plans to introduce a low-end product line extension given a high-end product has been introduced, with a primary focus on the decision of when to introduce the low-end generation. We propose an integrated model that considers important factors from areas of both operation management and marketing. On the demand side, we provide a new perspective of the classical Wilson and Norton's product line ex-

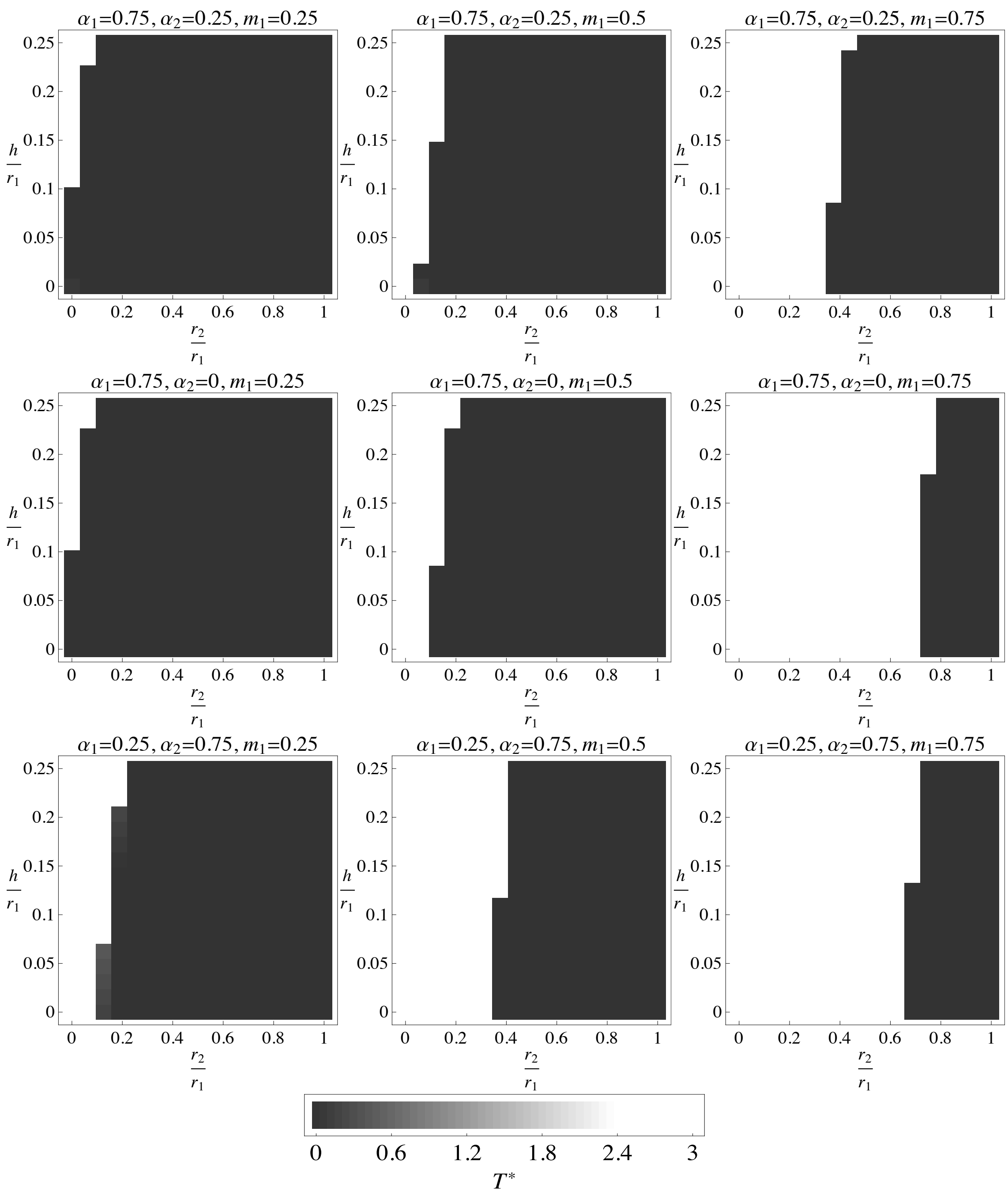

Figure 3.5: Plots of Optimal Introduction Strategy under Multiple-Replenishment Ordering Policy with Replenishment Intervals $O_1 = 0.5$ *year*, $O_2 = 1$ *year*, and Planning Horizon $T_p = 3$ *year*.

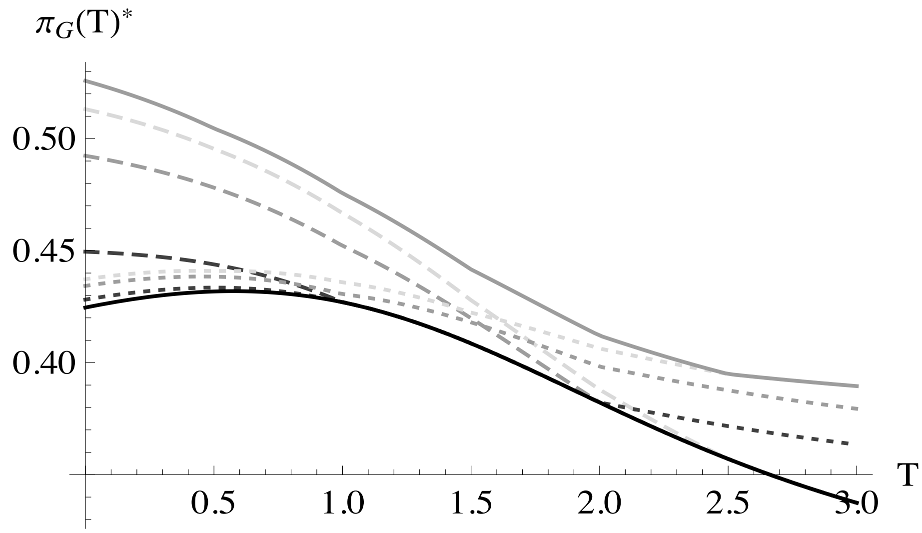

Figure 3.6: Impact of Replenishment Interval on Optimal Introduction Strategy. Each line shows the total profit $\pi_G(T)$ as a function of the introduction timing T. The solid black line is the baseline case with only one replenishment for each product generation, i.e. $O_1 = O_2 = T_p = 3$ *year*. When we decrease O_1 gradually to $2,\ 1,\ 0.5$ *year*, the total profit are shown respectively as dark gray, gray and light grey dotted lines. When we decrease O_2 gradually to $2,\ 1,\ 0.5$ *year*, the total profit are shown respectively as dark gray, gray and light grey dashed lines. The solid grey line shows the case with $O_1 = O_2 = 0.5$ *year*.

tension model Wilson and Norton, 1989. In our setting, the market is segmented into two groups of customers by their different valuations of the product. All low-valuation customers can only afford the low-end products; high-valuation customers can choose to stick to the original product generation, or to migrate to the low-end product line extension; or leave the market immediately. On the supply side, we focus on the impact of inventory holding cost under the one-replenishment ordering policy. Under our integrated framework, we determine the optimal introduction strategy and present analytical characterizations for the one-replenishment ordering schedule. We also conduct numerical studies by different industries and market potentials. Our results show that sequential introduction can be optimal under substantial inventory cost, medium profitability of the product line extension, or relatively short planning horizon. The optimal introduction timing can increase or decrease with the inventory cost depending on the marketplace setting. Moreover, we extend our model to allow for multiple replenishments, and find the "now-or-never" rule applies under frequent replenishments.

Our work shows that an interdisciplinary decision-making approach of both operations management and marketing science will help firms achieve an improved profit. These two aspects of a firm should be synchronized not only at the operational level, but at the tactical level as well, so managers should understand both sides and then make a recipe that is right for their company's particular situation. We also suggest that operational variables, such as order policy and inventory cost, can impact the introduction timing decision, thus might be considered in future empirical efforts.

Our analysis opens up several opportunities for future research. In our model, the demand is partially endogenous but deterministic. We expect demand uncertainty to be an interesting factor to explore, as it will combine the inventory management and the introduction timing decision more tightly. Firms tend to determine the introduction timing for the second generation based on the inventory level of the first-generation product. Thus a careful systematic analysis on the inventory decision beforehand becomes even more vital. Moreover, we haven't considered strategic customer behaviors in our model. We assumed all low-valuation customers are myopic and impatient in waiting. In reality, customers can be forward-looking. Their estimation of the introduction time can have an impact on the firm's timing decision Prasad, Bronnenberg, and Mahajan, 2004. With her own expectation of the introduction time, a rational customer compares the net present values between the two products to make the purchase. Finally, to explore other marketing mix decisions is also of interest, especially to include the pricing strategies for the successive generations.

Appendix A

Appendix

A.1 Multi-armed Bandits, Gittins Index, and Search for Information:

In the standard multi-armed bandit, a decision-maker chooses which arm to pull in each period, where the reward obtained in the pulled arm is stochastic with unknown distribution for each arm (with independent distributions across arms). Gittins (1979) show that the optimal policy in this problem is to pull the arm that has the highest dynamic allocation index, as defined by Gittins, also known as the Gittins index, which is obtained for each arm and only depends on the information on the distribution of rewards that the decision-maker has at that time for that arm. One interpretation of the Gittins index (provided by Whittle 1980) is that it is the value at which the decision-maker would be indifferent between getting that value for sure, or playing the arm with the option of taking that retirement value at any time. That is, if we define $V(I_i, K_i)$ as the value of playing arm i, with information I_i about arm i, with the possibility of retiring and getting K_i, then the Gittins index can be seen as the K_i such that $K_i = V(I_i, K_i)$. Bergman (1981) showed that, in general, the optimal policy in the problem of gradual search for information does not involve playing the arm with the highest index (i.e., highest Gittins index).

In order to see this we present a counter-example (adapted from Bergman for rational expectations in search for information). Suppose that there are two possible products that a consumer can purchase, A and B. Prior to any search the expected value of product A is 10, and the expected value of product B is 4. The first time a consumer searches on any of the products she does not learn anything about the value of any product. The second time the consumer searches for information on product A the consumer learns that the value of product A is either 20 or zero with equal probability. The second time that the consumer searches for information on product B the consumer learns that the value of product B is either 18 or -10 with equal probability. The search cost of each time the consumer searches for information is 1. The Gittins index for product A is obtained by making $K_A = -2 + \frac{20}{2} + \frac{K_A}{2}$, which yields $K_A = 16$. The Gittins index for product B is obtained by making $K_B = -2 + \frac{18}{2} + \frac{K_B}{2}$, which yields $K_B = 14$. That is, the Gittins index policy would suggest to check information on product A first, which yields an expected payoff of $-2 + \frac{20}{2} + \frac{1}{2}(-2 + \frac{18}{2}) = 11.5$. However, by checking information on product B first, the consumer is able to get the higher expected payoff of $-2 + \frac{18}{2} + \frac{10}{2} = 12$. By checking product B first the consumer keeps product A in reserve which she can choose to buy without checking further.

As discussed in Section 1.7, Bergman (1981) shows that the Gittins index policy is optimal when there are an infinite number of products that are *ex ante* equal in distribution. In this case, the Gittins index is a direct extension to the "reservation price" in classical sequential search (McCall 1970), to allow for multiple search actions on one alternative.

A.2 Proof of Lemma 1:

According to the symmetry between u_1 and u_2, it suffices to show for $\forall u_2$, $V(u_1'', u_2) \geq V(u_1', u_2)$ for $\forall u_1'' > u_1'$. In fact, let $x(u_1, u_2)$ be the consumer's optimal action given her current expected utilities of the two products as u_1 and u_2. Any other strategy, including $x'(u_1, u_2) \equiv x(u_1 + u_1' - u_1'', u_2)$ must be suboptimal to x. By the definition of x', we know that, to follow strategy x' starting from (u_1'', u_2) will always generate the same action sequence as to follow strategy x starting from (u_1', u_2). Any search process will end up with purchasing product 1, purchasing product 2, or exiting market without any purchase. Because the same action sequence is followed for both random searching processes starting from (u_1'', u_2) and (u_1', u_2), they will end up with the same choice of actions. In any case, the consumer will be no worse off by following x' in the search process starting from (u_1'', u_2), because $u_1'' > u_1'$. As a result, $V(u_1', u_2)$, as the expected utility by following x for the search process starting from (u_1', u_2), will be no larger than the expected utility by following x' for the search process starting from (u_1'', u_2), which in turn is no larger than $V(u_1'', u_2)$ according to the suboptimality of x'. ■

A.3 Proof of Lemma 2:

To simplify notation, we drop the subscript i in $\overline{U}_i(u)$. For $\forall u'' > u'$, we know $V(\overline{U}(u'), u'') \geq V(\overline{U}(u'), u') = \overline{U}(u')$, according to the monotonicity of $V(u_1, u_2)$ by Lemma 1. So given a consumer's expected utility of product 2 as u'', when her expected utility of product 1 reaches $\overline{U}(u')$, she has the maximum expected utility of continuing searching for information as $V(\overline{U}(u'), u'')$, which is greater than the expected utility of purchasing product 1 right away as $\overline{U}(u')$. Her optimal decision is then to continue searching, until she hits a higher expected utility of product 1 as $\overline{U}(u'')$. Therefore, we have $\overline{U}(u'') \geq \overline{U}(u')$. ■

A.4 Derivation of the Smooth-Pasting Condition in Equation (1.12):

We prove the smooth-pasting condition at the purchase boundary of product 1. The proof for product 2 can be constructed similarly according to symmetry. Let us consider an extra search in dt on product 1 at the boundary $\left(\overline{U}(u_2), u_2\right)$. The corresponding utility update du_1 can be positive or negative, with equal odds. If $du_1 \geq 0$, the consumer will purchase product 1 immediately, and leave the market; otherwise if $du_1 < 0$, the consumer's expected utility of product 1 decreases, and she will stay in the market searching for more information. Therefore, her value function upon the extra search on product 1 would be:

$$\begin{aligned} V_1\left(\overline{U}(u_2), u_2\right) &\equiv -cdt + \frac{1}{2}\left(\overline{U}(u_2) + \mathrm{E}[du_1|du_1 \geq 0]\right) + \frac{1}{2}\mathrm{E}\left[V(\overline{U}(u_2) + du_1, u_2|du_1 < 0)\right] \\ &= V\left(\overline{U}(u_2), u_2\right) + \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}\left[1 - V_{u_1}\left(\overline{U}(u_2), u_2\right)\right] + o(\sqrt{dt}), \end{aligned} \tag{A.1}$$

where we have used the fact that $E[du_1|du_1 \geq 0] = -E[du_1|du_1 < 0] = \sigma\sqrt{\frac{dt}{2\pi}}$.

On the other hand, let us consider a consumer who spends dt in searching for information on product 2 at the boundary $\left(\overline{U}(u_2), u_2\right)$. If $du_2 = dB_2(t_2) \geq 0$, according to Lemma 2, the consumer's purchase threshold for product 1 increases, so she will stay in the market continuing the search for information; otherwise, if $du_2 < 0$, the consumer will purchase product 1 immediately. Therefore, her value function upon the extra search on product 2 would be:

$$\begin{aligned} V_2\left(\overline{U}(u_2), u_2\right) &\equiv -cdt + \frac{1}{2}\mathrm{E}\left[V(\overline{U}(u_2), u_2 + du_2|du_2 < 0)\right] + \frac{1}{2}\overline{U}(u_2) \\ &= V\left(\overline{U}(u_2), u_2\right) - \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}V_{u_2}\left(\overline{U}(u_2), u_2\right) + o(\sqrt{dt}). \end{aligned} \tag{A.2}$$

A consumer chooses which product to search for information on based on expected utility maximization. Therefore, her value function upon the extra search should satisfy:

$$V\left(\overline{U}(u_2), u_2\right) = \max\{V_1\left(\overline{U}(u_2), u_2\right), V_2\left(\overline{U}(u_2), u_2\right)\}. \tag{A.3}$$

By substituting the expression of $V_1\left(\overline{U}(u_2), u_2\right)$ and $V_2\left(\overline{U}(u_2), u_2\right)$ into the above equation, we have

$$\max\{1 - V_{u_1}\left(\overline{U}(u_2), u_2\right), V_{u_2}\left(\overline{U}(u_2), u_2\right)\} = 0. \tag{A.4}$$

Meanwhile, by taking derivative of both sides of equation (1.11) with respect to u_2, we have

$$\overline{U}'(u_2)\left[1 - V_{u_1}\left(\overline{U}(u_2), u_2\right)\right] = V_{u_2}\left(\overline{U}(u_2), u_2\right). \tag{A.5}$$

Combining the above two equations, we obtain $V_{u_1}\left(\overline{U}(u_2), u_2\right) = 1$ and $V_{u_2}\left(\overline{U}(u_2), u_2\right) = 0$.

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A.5 Proof of Theorem 1:

The solution is not easy to obtain, but it is fairly straightforward to verify that it satisfies equation (1.9) along with all boundary conditions (1.11)-(1.14). Actually, as a viscosity solution, $V(u_1, u_2)$ takes different forms in different regions in our solution. We also need a set of conditions at the boundaries separating different regions. Say $V(u_1, u_2)$ takes the form of $V^1(u_1, u_2)$ in region 1 and $V^2(u_1, u_2)$ in region 2. At each "internal" boundary $\mathcal{C}$ separating region 1 and 2, we need to impose

(1) Value Matching Condition: $V^1(u_1, u_2)|_{\{u_1,u_2\}\in\mathcal{C}} = V^2(u_1, u_2)|_{\{u_1,u_2\}\in\mathcal{C}}$;

(2) Smooth-Pasting Condition: $V^1_{u_i}(u_1, u_2)|_{\{u_1,u_2\}\in\mathcal{C}} = V^2_{u_i}(u_1, u_2)|_{\{u_1,u_2\}\in\mathcal{C}}$ $(i = 1, 2)$.

One can verify that $V(u_1, u_2)$ in equation (A.64) satisfies the two conditions above at all internal boundaries: $\mathcal{C}_1 \equiv \{(u_1, u_2)|u_1 = u_2 \geq -a\}$, $\mathcal{C}_2 \equiv \{(u_1, u_2)|u_1 = -a, -a \leq u_2 \leq a\}$ and $\mathcal{C}_3 \equiv \{(u_1, u_2)|u_2 = -a, -a \leq u_1 \leq a\}$.

The uniqueness of the solution is guaranteed by the generic uniqueness of viscosity solution to Hamilton-Jacobi-Bellman equation (1.9) (Bardi and Capuzzo-Dolcetta 2008, page 6). ■

A.6 Derivation of Purchase Likelihood in Equation (1.19):

If $u_1 \geq \overline{U}(u_2)$, the consumer will purchase product 1 right away, therefore $P_1(u_1, u_2) = 1$. If $u_1 \leq -a$ or $u_2 \geq \overline{U}(u_1)$, the consumer will never purchase product 1, therefore $P_1(u_1, u_2) = 0$. Otherwise if $-a < u_1 < \overline{U}(u_2)$ and $u_2 < \overline{U}(u_1)$, there are two cases, depending on the value of u_2.

In the first case with $u_2 \leq -a$, the consumer will search on product 1 only. Given her current expected utility u_1, she will either hit $-a$ first or hit a first. According to the Optional Stopping Theorem, we have $u_1 = P_1(u_1, u_2)a + [1 - P_1(u_1, u_2)](-a)$, i.e.,

$$P_1(u_1, u_2) = \frac{1}{2} + \frac{u_1}{2a}, \quad -a < u_1 \leq a, u_2 \leq -a. \tag{A.6}$$

In the second case, $u_2 > -a$. When $u_1 \geq u_2$, the consumer searches on product 1, and either hits $\overline{U}(u_2)$ first or hits u_2 first. Let us define the probability of hitting $\overline{U}(u_2)$ first as $q_1(u_1, u_2)$. Then by invoking the Optional Stopping Theorem, we similarly get

$$q_1(u_1, u_2) = \frac{u_1 - u_2}{\overline{U}(u_2) - u_2}. \tag{A.7}$$

According to symmetry, the probability of hitting $\overline{U}(u_1)$ first, staring from (u_1, u_2) with $u_1 < u_2$ would be $q_1(u_2, u_1)$. Let us further define $P_0(u)$ as the probability of exiting the market without any purchase, given current expected utilities as (u, u). Let us consider an infinitesimal search on product 1 at (u, u), with utility update as du. By conditioning on du, we have the following equality

$$\begin{aligned} P_0(u) &= \frac{1}{2}\Pr[\text{exit}|du \geq 0] + \frac{1}{2}\Pr[\text{exit}|du < 0] && \text{(A.8)} \\ &= \frac{1}{2}\left[1 - q_1(u + du, u)\right] P_0(u) + \frac{1}{2}\left[1 - q_1(u, u - du)\right] P_0(u - du) && \text{(A.9)} \\ &= P_0(u) - \frac{du}{2}\left[P_0'(u) - \left(\frac{\partial q_1(u,u)}{\partial u_2} - \frac{\partial q_1(u,u)}{\partial u_1}\right) P_0(u)\right], && \text{(A.10)} \end{aligned}$$

where the last equality is obtained by doing a Taylor expansion o $q_1(u+du, u)$, $q_1(u, u-du)$, and $P_0(u - du)$. Then we have,

$$\frac{P_0'(u)}{P_0(u)} = \frac{\partial q_1(u,u)}{\partial u_2} - \frac{\partial q_1(u,u)}{\partial u_1} = -\frac{2}{a\left[1 + W\left(e^{-(\frac{2u}{a}+1)}\right)\right]}. \tag{A.11}$$

Combining the differential equation above with the initial condition $P_0(-a) = 1$, we can solve $P_0(u)$ as

$$P_0(u) = W\left(e^{-(\frac{2u}{a}+1)}\right). \tag{A.12}$$

Starting from (u_1, u_2) with $u_1 \geq u_2$, the consumer searches for information on product 1. With probability $q_1(u_1, u_2)$, she hits the boundary $\overline{U}(u_2)$ first, and purchases product 1 right away. With probability $1 - q_1(u_1, u_2)$, she hits u_2 first. And then starting from (u_2, u_2), she eventually purchases product 1 with probability $\frac{1}{2}[1 - P_0(u_2)]$. Therefore, we have,

$$P_1(u_1, u_2) = q_1(u_1, u_2) + [1 - q_1(u_1, u_2)]\,\frac{1}{2}[1 - P_0(u_2)], \quad -a < u_2 < u_1 < \overline{U}(u_2). \tag{A.13}$$

Similarly starting from (u_1, u_2) with $u_1 < u_2$, the consumer searches on product 2. With probability $1 - q_1(u_2, u_1)$, she hits u_1 first. And then starting from (u_1, u_1), she eventually purchases product 1 with probability $\frac{1}{2}[1 - P_0(u_1)]$.

$$P_1(u_1, u_2) = [1 - q_1(u_2, u_1)]\,\frac{1}{2}[1 - P_0(u_1)], \quad -a < u_1 < u_2 < \overline{U}(u_1). \tag{A.14}$$

By combining all the scenarios above, we have equation (1.19). ■

A.7 Comparative Statics of Purchase Likelihoods in Figure 1.5:

We prove the comparative statics of $P_1(u_1, u_2)$ first and then those of $P(u_1, u_2)$. We only focus on the region where $-a < u_1 < \overline{U}(u_2)$ and $-a < u_2 < \overline{U}(u_1)$. For other regions, the proof is straightforward, thus omitted.

To prove the comparative statics of $P_1(u_1, u_2)$, we consider two cases. In the first case with $u_2 > u_1$, we have

$$P_1(u_1, u_2) = \frac{\overline{U}(u_1) - u_2}{\overline{U}(u_1) - u_1} - \frac{\overline{U}(u_1) - u_2}{2a}. \tag{A.15}$$

Given u_1 and u_2,

$$\frac{\partial P_1(u_1, u_2)}{\partial a} = -\frac{u_1 + u_2}{2a^2} + \frac{2u_1(u_2 - u_1)\left(\overline{U}(u_1) - u_1 - a\right)}{\left(\overline{U}(u_1) - u_1\right)^3 a} + \frac{u_2}{\left(\overline{U}(u_1) - u_1\right) a}. \tag{A.16}$$

If $u_1 \geq 0$, it is easy to verify that

$$\frac{\partial P_1(u_1, u_2)}{\partial a} > 0 \Leftrightarrow u_2 > \frac{u_1\left(\overline{U}(u_1) - u_1\right)^3 + 4au_1^2\left(\overline{U}(u_1) - u_1 - a\right)}{2a\left(\overline{U}(u_1) - u_1\right)^2 - \left(\overline{U}(u_1) - u_1\right)^3 + 4au_1\left(\overline{U}(u_1) - u_1 - a\right)}. \tag{A.17}$$

Otherwise if $u_1 < 0$, one can show that $\frac{\partial P_1(u_1,u_2)}{\partial a} > 0$. In fact, $\frac{\partial P_1(u_1,u_2)}{\partial a}$ is a linear function of u_2. It suffices to verify that

$$\left.\frac{\partial P_1(u_1, u_2)}{\partial a}\right|_{u_2=0} = -\frac{u_1}{2a^2} - \frac{2u_1^2\left(\overline{U}(u_1) - u_1 - a\right)}{\left(\overline{U}(u_1) - u_1\right)^3 a} > 0, \tag{A.18}$$

$$\left.\frac{\partial P_1(u_1, u_2)}{\partial a}\right|_{u_2=\overline{U}(u_1)} = -\frac{u_1 + \overline{U}(u_1)}{2a^2} + \frac{2u_1\left(\overline{U}(u_1) - u_1 - a\right)}{\left(\overline{U}(u_1) - u_1\right)^2 a} + \frac{\overline{U}(u_1)}{\left(\overline{U}(u_1) - u_1\right) a} > 0 \tag{A.19}$$

In the second case with $u_2 \leq u_1$, we have

$$P_1(u_1, u_2) = 1 - \frac{\overline{U}(u_2) - u_1}{2a}. \tag{A.20}$$

Given u_1 and u_2,

$$\frac{\partial P_1(u_1, u_2)}{\partial a} = -\frac{u_1 + u_2}{2a^2} + \frac{u_2}{\left(\overline{U}(u_2) - u_2\right) a} > 0 \Leftrightarrow u_1 < u_2 + \frac{2au_2}{\overline{U}(u_2) - u_2}. \tag{A.21}$$

Now let us turn to the comparative statics of $P(u_1, u_2)$. Because of symmetry, we only need to consider the case with $u_1 \geq u_2$. We have

$$P(u_1, u_2) = 1 - \frac{\overline{U}(u_2) - u_1}{a} + \frac{\overline{U}(u_2) - u_1}{\overline{U}(u_2) - u_2}. \tag{A.22}$$

Given u_1 and u_2,

$$\frac{\partial P(u_1, u_2)}{\partial a} \propto 2au_2(u_1 - u_2) - (u_1 + u_2)a^2 - (u_1 + u_2)\left(\overline{U}(u_2) - u_2 - a\right)\left(\overline{U}(u_2) - u_2 + a\right) \tag{A.23}$$

If $u_1 \geq 0$, it is easy to verify that

$$\frac{\partial P(u_1,u_2)}{\partial a} > 0 \Leftrightarrow u_1 < -\frac{u_2 a(2u_2+a) + u_2\left(\overline{U}(u_2)-u_2-a\right)\left(\overline{U}(u_2)-u_2+a\right)}{a^2+\left(\overline{U}(u_2)-u_2-a\right)\left(\overline{U}(u_2)-u_2+a\right)-2au_2} \tag{A.24}$$

Otherwise, if $u_1 < 0$, one can show that $\frac{\partial P(u_1,u_2)}{\partial a} > 0$. In fact, $\frac{\partial P(u_1,u_2)}{\partial a}$ is a linear function of u_1. It suffices to verify that for $-a \leq u_2 < 0$ we have

$$\left.\frac{\partial P(u_1,u_2)}{\partial a}\right|_{u_1=0} \propto -3au_2^2 - u_2\left(\overline{U}(u_2)-u_2-a\right)\left(\overline{U}(u_2)-u_2+a\right) > 0, \tag{A.25}$$

$$\left.\frac{\partial P(u_1,u_2)}{\partial a}\right|_{u_1=-a} \propto a^3 - 3a^2u_2 - 2au_2^2 - (u_2-a)\left(\overline{U}(u_2)-u_2-a\right)\left(\overline{U}(u_2)-u_2+a\right) > 0. \tag{A.26}$$

■

A.8 Smooth-Pasting Conditions for Correlated Products:

We derive the smooth pasting condition (1.12) for product 1, with the two products informationally correlated. We focus on the case with $0 < \rho < 1$ below (the case of $\rho < 0$ can be obtained similarly). Similarly to the proof of the case with $\rho = 0$, let us consider an extra infinitesimal search at the boundary $\left(\overline{U}(u_2), u_2\right)$. By searching for information on product 1 for extra time dt, the consumer earns an extra utility update du for product 1 and $\rho\, du$ for product 2. The utility update du can be either greater or less than zero with the same probability $\frac{1}{2}$. Let us first consider the scenario with $du \geq 0$. Now the consumer has a higher expected utility of product 1 with $du_1 = du$, which may drive the consumer to purchase product 1 and leave the market right away. However, at the same time, the consumer's expected utility of product 2 also increases by $du_2 = \rho\, du$, which rises the $\overline{U}(u_2)$ by $\rho\overline{U}'(u_2)du$. As a result, it is also possible for her to continue staying in the market. The choice between immediate purchase and continuation of search depends on the comparison between the utility update $du_1 = du$ and the update $\rho\overline{U}'(u_2)du$.

Consequently, if $\overline{U}'(u_2) < \frac{1}{\rho}$, the consumer will purchase the product 1 and leave the market with utility $\overline{U}(u_2) + \mathrm{E}[du|du \geq 0]$; otherwise, if $\overline{U}'(u_2) \geq \frac{1}{\rho}$, the consumer continues searching for information with expected utility $\mathrm{E}\left[V(\overline{U}(u_2)+du, u_2+\rho du|du \geq 0)\right]$.

Similarly, we have the following assertions for the case with $du < 0$. If $\overline{U}'(u_2) \leq \frac{1}{\rho}$, the consumer will continue searching with expected utility $\mathrm{E}\left[V(\overline{U}(u_2)+du, u_2+\rho\, du|du < 0)\right]$; otherwise, if $\overline{U}'(u_2) > \frac{1}{\rho}$, the consumer purchases the product 1 and leaves the market with expected utility $\overline{U}(u_2) + \mathrm{E}[du|du < 0]$.

To summarize, if $\overline{U}'(u_2) < \frac{1}{\rho}$, the consumer's expected utility of searching extra dt on product 1 is

$$\begin{aligned}
& V_1\left(\overline{U}(u_2), u_2\right) \\
\equiv\ & -cdt + \frac{1}{2}\left(\overline{U}(u_2) + \mathrm{E}[du|du \geq 0]\right) + \frac{1}{2}\mathrm{E}\left[V(\overline{U}(u_2) + du, u_2 + \rho du|du < 0)\right] \\
=\ & V\left(\overline{U}(u_2), u_2\right) + \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}\left[1 - V_{u_1}\left(\overline{U}(u_2), u_2\right) - \rho V_{u_2}\left(\overline{U}(u_2), u_2\right)\right] + o(\sqrt{dt}). \quad (A.27)
\end{aligned}$$

If $\overline{U}'(u_2) > \frac{1}{\rho}$, the consumer's expected utility of searching extra dt on product 1 is

$$\begin{aligned}
& V_1\left(\overline{U}(u_2), u_2\right) \\
\equiv\ & -cdt + \frac{1}{2}\left(\overline{U}(u_2) + \mathrm{E}[du|du < 0]\right) + \frac{1}{2}\mathrm{E}\left[V(\overline{U}(u_2) + du, u_2 + \rho\, du|du \geq 0)\right] \\
=\ & V\left(\overline{U}(u_2), u_2\right) - \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}\left[1 - V_{u_1}\left(\overline{U}(u_2), u_2\right) - \rho V_{u_2}\left(\overline{U}(u_2), u_2\right)\right] + o(\sqrt{dt}). \quad (A.28)
\end{aligned}$$

Finally, if $\overline{U}'(u_2) = \frac{1}{\rho}$, the consumer's expected utility of searching extra dt on product 1 is

$$\begin{aligned}
& V_1\left(\overline{U}(u_2), u_2\right) \\
\equiv\ & -cdt + \frac{1}{2}\mathrm{E}\left[V(\overline{U}(u_2) + du, u_2 + \rho du|du < 0)\right] + \frac{1}{2}\mathrm{E}\left[V(\overline{U}(u_2) + du, u_2 + \rho du|du \geq 0)\right] \\
=\ & V\left(\overline{U}(u_2), u_2\right) + o(\sqrt{dt}), \quad (A.29)
\end{aligned}$$

which is the same as equation (1.9).

On the other hand, when the consumer searches product 2 for extra dt at the boundary $\left(\overline{U}(u_2), u_2\right)$, we apply the same analysis above, and conclude that the consumer's expected utility of searching extra dt on product 2 is

$$V_2\left(\overline{U}(u_2), u_2\right) \equiv$$
$$\begin{cases}
V\left(\overline{U}(u_2), u_2\right) + \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}\left[\rho - \rho V_{u_1}\left(\overline{U}(u_2), u_2\right) - V_{u_2}\left(\overline{U}(u_2), u_2\right)\right] + o(\sqrt{dt}) & \text{if } \overline{U}'(u_2) < \rho \\
V\left(\overline{U}(u_2), u_2\right) - \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}\left[\rho - \rho V_{u_1}\left(\overline{U}(u_2), u_2\right) - V_{u_2}\left(\overline{U}(u_2), u_2\right)\right] + o(\sqrt{dt}) & \text{if } \overline{U}'(u_2) > \rho \\
V\left(\overline{U}(u_2), u_2\right) + o(\sqrt{dt}) & \text{otherwise.}
\end{cases}$$

So far, we have the expected utility of searching extra dt on product 1 and 2. The consumer will choose to search for information on the product with greater expected utility, so her expected utility at the boundary $\left(\overline{U}(u_2), u_2\right)$ is $\max\left\{V_1\left(\overline{U}(u_2), u_2\right), V_2\left(\overline{U}(u_2), u_2\right)\right\}$. At the same time, the consumer's expected utility at $\left(\overline{U}(u_2), u_2\right)$ is given exactly by $V\left(\overline{U}(u_2), u_2\right)$. To make the above two expressions identical, the $\sqrt{dt}$-order term must vanish. We obtain the following set of equations.

$$\begin{cases} \max\{1-V_{u_1}-\rho V_{u_2}, \rho-\rho V_{u_1}-V_{u_2}\}=0 & \text{if } \rho > \overline{U}'(u_2) \geq 0 \\ \max\{1-V_{u_1}-\rho V_{u_2}, 0\}=0 & \text{if } \overline{U}'(u_2)=\rho \\ \max\{1-V_{u_1}-\rho V_{u_2}, -\rho+\rho V_{u_1}+V_{u_2}\}=0 & \text{if } \frac{1}{\rho} > \overline{U}'(u_2) > \rho \\ \max\{0, -\rho+\rho V_{u_1}+V_{u_2}\}=0 & \text{if } \overline{U}'(u_2)=\frac{1}{\rho} \\ \max\{-1+V_{u_1}+\rho V_{u_2}, -\rho+\rho V_{u_1}+V_{u_2}\}=0 & \text{if } \overline{U}'(u_2)>\frac{1}{\rho} \end{cases} \tag{A.30}$$

In order to simplify notation, we have dropped $\left(\overline{U}(u_2), u_2\right)$ in writing the value function V. Let us first have a look at the case with $\rho > \overline{U}'(u_2) \geq 0$. The first equation in (A.30) implies either $\rho-\rho V_{u_1}-V_{u_2}=0 \geq 1-V_{u_1}-\rho V_{u_2}$, or $1-V_{u_1}-\rho V_{u_2}=0 \geq \rho-\rho V_{u_1}-V_{u_2}$. In the latter case with $\rho-\rho V_{u_1}-V_{u_2}=0$, we have $V_{u_1}=1-\frac{1}{\rho}V_{u_2}$. Then $0 \geq 1-V_{u_1}-\rho V_{u_2}=\left(\frac{1}{\rho}-\rho\right)V_{u_2} \geq 0$. So we must have $1-V_{u_1}-\rho V_{u_2}=0$ in either case. Therefore, the first equation in (A.30) is equivalent to $1-V_{u_1}-\rho V_{u_2}=0$. With a similar argument, we can show that the above set of equations can be equivalently rewritten as

$$\begin{cases} V_{u_1}+\rho V_{u_2}=1 & \text{if } \rho > \overline{U}'(u_2) \geq 0 \\ V_{u_1}+\rho V_{u_2} \geq 1 & \text{if } \overline{U}'(u_2)=\rho \\ V_{u_1}=1 \text{ and } V_{u_2}=0 & \text{if } \frac{1}{\rho} > \overline{U}'(u_2) > \rho \\ \rho V_{u_1}+V_{u_2} \leq \rho & \text{if } \overline{U}'(u_2)=\frac{1}{\rho} \\ \rho V_{u_1}+V_{u_2}=\rho & \text{if } \overline{U}'(u_2)>\frac{1}{\rho} \end{cases} \tag{A.31}$$

Now, by taking derivative of both sides of equation (1.11) with respect to u_2, we have $(1-V_{u_1})\overline{U}'(u_2)=V_{u_2}$. If $\rho > \overline{U}'(u_2) \geq 0$ and $V_{u_1} \neq 1$, we have $\overline{U}'(u_2)=\frac{V_{u_2}}{1-V_{u_1}}=\frac{1}{\rho}>\rho$ by equation (A.31), which is a contradiction. Therefore, if $\rho > \overline{U}'(u_2) \geq 0$ there must be $V_{u_1}=1$ and $V_{u_2}=0$. Similarly, we can show that if $\overline{U}'(u_2)>\frac{1}{\rho}$, or $\overline{U}'(u_2)=\rho$ or $\overline{U}'(u_2)=\frac{1}{\rho}$, there must be $V_{u_1}=1$ and $V_{u_2}=0$ too. In summary, we have obtained equation (1.12) for the general case with $0<\rho<1$. ■

A.9 Proof of Proposition 2:

We prove the comparative statics for the purchase threshold first. From equation (1.25), we know that we only need to show that when $u \geq -(1-2\rho)a$, $\overline{U}(u)$ increases in a and decreases in ρ.

In fact, when $u \geq -(1-2\rho)a$, we have $0 < W\left(e^{-\frac{2u}{(1-\rho^2)a}-\frac{1-4\rho+\rho^2}{1-\rho^2}}\right) \leq 1$. Let us define the notation $w \equiv 1 + W\left(e^{-\frac{2u}{(1-\rho^2)a} - \frac{1-4\rho+\rho^2}{1-\rho^2}}\right)$. Then, we have $1 < w \leq 2$, and $\ln(w-1) \leq 0$.

$$
\begin{aligned}
\frac{\partial \overline{U}(u)}{\partial a} &= (1-\rho)^2 + (1-\rho^2) W\left(e^{-\frac{2u}{(1-\rho^2)a} - \frac{1-4\rho+\rho^2}{1-\rho^2}}\right) + \frac{2u}{a}\left[1 - \frac{1}{1 + W\left(e^{-\frac{2u}{(1-\rho^2)a} - \frac{1-4\rho+\rho^2}{1-\rho^2}}\right)}\right] \\
&= (\rho^2 - 2\rho)\frac{2-w}{w} + 1 - (1-\rho^2)\frac{w-1}{w}\ln(w-1) \\
&\geq -\frac{2-w}{w} + 1 - (1-\rho^2)\frac{w-1}{w}\ln(w-1) \\
&= 2\frac{w-1}{w} - (1-\rho^2)\frac{w-1}{w}\ln(w-1) > 0. \qquad \text{(A.32)}
\end{aligned}
$$

Similarly, one can show that

$$
\frac{\partial \overline{U}(u)}{\partial \rho} = -2(1-\rho)a\frac{2-w}{w} + 2\rho a\frac{w-1}{w}\ln(w-1) \leq 0. \qquad \text{(A.33)}
$$

Now, we start proving the comparative statics for the maximum expected utility $V(u_1, u_2)$. The function $V(u_1, u_2)$ is continuous and symmetric with respect to $u_1 = u_2$. It suffices to show that when $\rho u_1 - (1-\rho)a \leq u_2 \leq u_1 \leq \overline{U}(u_2)$, $V(u_1, u_2)$ increases in a and decreases in ρ.

Let us define the notation $\widetilde{w} \equiv 1 + W\left(\frac{1+\rho}{1-\rho}e^{-\frac{2(u_2-\rho u_1)}{(1-\rho)^2(1+\rho)a} - \frac{1-2\rho-\rho^2}{1-\rho^2}}\right)$. When $\rho u_1 - (1-\rho)a \leq u_2 \leq u_1 \leq \overline{U}(u_2)$, we have $u_1 \leq \widehat{U}(u_1, u_2)$ and $1 < \widetilde{w} \leq \frac{2}{1-\rho}$.

$$
\begin{aligned}
\frac{\partial V(u_1, u_2)}{\partial a} &= \frac{\widehat{U}(u_1, u_2) - u_1}{4a(1-\rho)} \cdot \left[(u_1 - u_2) + (1-\rho)^2 a\widetilde{w} + \frac{4(u_2 - \rho u_1)}{1+\rho}\left(1 - \frac{1}{\widetilde{w}}\right)\right] \\
&\propto (u_1 - u_2) + (1-\rho)^2 a\widetilde{w} + \frac{4(u_2 - \rho u_1)}{1+\rho}\left(1 - \frac{1}{\widetilde{w}}\right) \\
&= (u_1 - u_2) + 2(1-\rho)^2 a + \frac{4\rho(1-\rho)a}{1+\rho} - 2(1-\rho)^2 a(1 - \frac{1}{\widetilde{w}})\ln\left[\frac{1-\rho}{1+\rho}(\widetilde{w} - 1)\right] \\
&- \left[(1-\rho)^2 a\widetilde{w} + \frac{4\rho(1-\rho)a}{1+\rho}\frac{1}{\widetilde{w}}\right]. \qquad \text{(A.34)}
\end{aligned}
$$

In the last equality, each term in the first line is non-negative, while the term in the second line is negative. Let us define an auxiliary function $h(\widetilde{w}) \equiv (1-\rho)^2 a\widetilde{w} + \frac{4\rho(1-\rho)a}{1+\rho}\frac{1}{\widetilde{w}}$. It is easy to show that $h(\widetilde{w})$ is uni-modal with a minimum at $\widetilde{w}^* = 2\sqrt{\frac{\rho}{1-\rho^2}} < \frac{2}{1-\rho}$. Thus, for $\widetilde{w} \in \left[1, \frac{2}{1-\rho}\right]$, $h(\widetilde{w})$ must reach a maximum at either $\widetilde{w} = 1$ or $\widetilde{w} = \frac{2}{1-\rho}$. At the same time, $h\left(\frac{2}{1-\rho}\right) - h(1) = (1-\rho)^2 a \geq 0$. Then for $\widetilde{w} \in (1, \frac{2}{1-\rho}]$, $h(\widetilde{w})$ must reach a maximum at

$\widetilde{w} = \frac{2}{1-\rho}$. Consequently, $h(\widetilde{w}) \leq h\left(\frac{2}{1-\rho}\right)$, for $\forall \widetilde{w} \in \left(1, \frac{2}{1-\rho}\right]$. Therefore, we have

$$\begin{aligned}
\frac{\partial V(u_1,u_2)}{\partial a} &\geq (u_1-u_2) + 2(1-\rho)^2 a + \frac{4\rho(1-\rho)a}{1+\rho} - 2(1-\rho)^2 a(1-\frac{1}{\widetilde{w}})\ln\left[\frac{1-\rho}{1+\rho}(\widetilde{w}-1)\right] \\
&- h\left(\frac{2}{1-\rho}\right) \\
&= (u_1-u_2) - 2(1-\rho)^2 a(1-\frac{1}{\widetilde{w}})\ln\left[\frac{1-\rho}{1+\rho}(\widetilde{w}-1)\right] \geq 0. \qquad (A.35)
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\frac{\partial V(u_1,u_2)}{\partial \rho} &= -\frac{\widehat{U}(u_1,u_2)-u_1}{2(1+\rho)(1-\rho)^3 a\widetilde{w}}\Big\{ \left[(u_1-u_2)+2(1-\rho)^2 a(2-\widetilde{w})\right]\left(\frac{2}{1-\rho}-\widetilde{w}\right) \\
&- 2\rho(1-\rho)a(\widetilde{w}-1)\ln\left[\frac{1-p}{1+p}(\widetilde{w}-1)\right]\Big\} \\
&\leq -\frac{\widehat{U}(u_1,u_2)-u_1}{2(1+\rho)(1-\rho)^3 a\widetilde{w}}\Big\{ \left[(u_1-u_2)+2(1-\rho)^2 a(2-\widetilde{w})\right]\left(\frac{2}{1-\rho}-\widetilde{w}\right) \\
&- 2\rho(1-\rho)a(\widetilde{w}-1)\left[\frac{1-p}{1+p}(\widetilde{w}-1)-1\right]\Big\} \\
&= -\frac{\widehat{U}(u_1,u_2)-u_1}{2(1+\rho)(1-\rho)^3 a\widetilde{w}}\left[(u_1-u_2)+\frac{(1-\rho)^3}{1+\rho}\left(\frac{2}{1-\rho}-\widetilde{w}\right)\right]\left(\frac{2}{1-\rho}-\widetilde{w}\right) \\
&\leq 0. \qquad (A.36)
\end{aligned}$$

■

A.10 Optimal Search with Heterogeneous Products:

Theorem 9 *There exists a unique solution $V(u_1,u_2)$ along with boundaries $\overline{U}_i(\cdot)$ and $\underline{U}_i(\cdot)$ $(i=1,2)$, which satisfies equations (1.5), (1.28), (1.11)–(1.14). The value function is*

$$V(u_1,u_2) = \begin{cases} \frac{1}{4a_1}\left[\overline{U}_1(u_2)-u_1\right]^2 + u_1 & \textit{if } u_2+\overline{U}_2(u_1)-\overline{U}_1(u_2) \leq u_1 \leq \overline{U}(u_2) \textit{ and } u_1 \geq \underline{U}(u_2) \\ \frac{1}{4a_2}\left[\overline{U}_2(u_1)-u_2\right]^2 + u_2 & \textit{if } u_1+\overline{U}_1(u_2)-\overline{U}_2(u_1) \leq u_2 \leq \overline{U}(u_1) \textit{ and } u_2 \geq \underline{U}(u_1) \\ \max\{0,u_1,u_2\} & \textit{otherwise.} \end{cases} \qquad (A.37)$$

Without loss of generality, assume $a_1 > a_2$. The purchase boundary $\overline{U}_1(\cdot)$ is given as

$$\overline{U}_1(u) = \begin{cases} u + a_1 & \textit{if } u > u^* \\ u + \frac{a_1 - a_2 Z_1(u)}{1-Z_1(u)} & \textit{if } -a_2 < u \leq u^* \\ a_1 & \textit{otherwise,} \end{cases} \qquad (A.38)$$

where $u^* \equiv -\frac{\sqrt{a_1 a_2}}{2}\ln\left(\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\right) > 0$. *The purchase boundary* $\overline{U}_2(\cdot)$ *is supported in* $(-\infty, u^* - a_1]$ *and is given as*

$$\overline{U}_2(u) = \begin{cases} u + \frac{a_1 Z_2(u) - a_2}{Z_2(u)-1} & \textit{if } -a_1 < u \le u^* - a_1 \\ a_2 & \textit{if } u \le -a_1. \end{cases} \tag{A.39}$$

The functions $Z_1(u) < 1$ *and* $Z_2(u) > 1$ *are defined implicitly by the following two equations, respectively:*

$$\frac{\sqrt{\frac{a_2}{a_1}} - \sqrt{Z_1(u)}}{1 - Z_1(u)} + \frac{1}{2}\ln\left(\frac{1-\sqrt{Z_1(u)}}{1+\sqrt{Z_1(u)}}\right) = \frac{u}{\sqrt{a_1 a_2}} + \sqrt{\frac{a_2}{a_1}} + \frac{1}{2}\ln\left(\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\right) \tag{A.40}$$

$$\frac{\sqrt{Z_2(u)} - \sqrt{\frac{a_1}{a_2}}}{Z_2(u) - 1} + \frac{1}{2}\ln\left(\frac{\sqrt{Z_2(u)}-1}{\sqrt{Z_2(u)}+1}\right) = \frac{u}{\sqrt{a_1 a_2}} + \sqrt{\frac{a_1}{a_2}} + \frac{1}{2}\ln\left(\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\right). \tag{A.41}$$

The exit boundaries $\underline{U}_i(\cdot)$ $(i = 1, 2)$ *are given as*

$$\underline{U}_1(u) = \begin{cases} -a_1 & \textit{if } u \le -a_2 \\ u - a_1 & \textit{if } u \ge u^* \end{cases} \tag{A.42}$$

$$\underline{U}_2(u) = -a_2 \quad (\textit{relevant when } u \le -a_1) \tag{A.43}$$

Proof. It is straightforward to verify that the solution satisfies equations (1.5), (1.28), (1.11)–(1.14). The more difficult part comes from the verification of the value matching and smooth-pasting conditions at internal boundaries.[1] There are four internal boundaries: $\mathcal{C}_1 \equiv \{(u_1, u_2) | \overline{U}_1(u_2) + u_1 = u_2 + \overline{U}_2(u_1)$ and $-a_2 \le u_2 \le u^*\}$, $\mathcal{C}_2 \equiv \{(u_1, u_2) | u_1 = -a_1, -a_2 \le u_2 \le a_2\}$, $\mathcal{C}_3 \equiv \{(u_1, u_2) | u_2 = -a_2, -a_1 \le u_1 \le a_1\}$ and $\mathcal{C}_4 \equiv \{(u_1, u_2) | u_2 = u^*, u^* - a_1 \le u_1 \le u^* + a_1\}$. Verifications of the boundary conditions at $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are straightforward, thus omitted here. We focus on the value matching and smooth-pasting conditions at boundary $\mathcal{C}_1$ below, which is the boundary separating "search product 1" from "search product 2".

Given $-a_2 \le u_2 \le u^*$ implied from $\mathcal{C}_1$, the purchase boundaries can be written as

$$\overline{U}_1(u) = u + \frac{a_1 - a_2 Z_1(u)}{1 - Z_1(u)} \tag{A.44}$$

$$\overline{U}_2(u) = u + \frac{a_1 Z_2(u) - a_2}{Z_2(u) - 1} \tag{A.45}$$

where $Z_i(u)$ $(i = 1, 2)$ are given in equations (A.40) and (A.41). It is straightforward to show that the left-hand side of the two equations (A.40) and (A.41) as a function of $Z_1(u)$ and $Z_1(u)$, respectively, is monotonic. Therefore, $Z_1(u)$ and $Z_1(u)$ are well defined. One can

[1]See the proof of Theorem 1 for explanation of internal boundaries.

verify that $\overline{U}_1(u)$ and $\overline{U}_2(u)$ satisfy the following two ordinary differential equations (ODEs) subject to the boundary conditions:[2]

$$\overline{U}_1'(u) = \frac{\sqrt{a_1a_2\left(a_1+u-\overline{U}_1(u)\right)\left(a_2+u-\overline{U}_1(u)\right)}-a_1a_2}{a_2\left(u-\overline{U}_1(u)\right)}, \quad \overline{U}_1(-a_2)=a_1 \quad \text{(A.46)}$$

$$\overline{U}_2'(u) = \frac{\sqrt{a_1a_2\left(a_1+u-\overline{U}_2(u)\right)\left(a_2+u-\overline{U}_2(u)\right)}-a_1a_2}{a_1\left(u-\overline{U}_2(u)\right)}, \quad \overline{U}_2(-a_1)=a_2 \quad \text{(A.47)}$$

Given $(u_1,u_2)\in\mathcal{C}_1$, our objective is to verify that u_1 and u_2 satisfy the following value matching and smooth-pasting conditions:

$$\frac{1}{4a_1}\left(\overline{U}_1(u_2)-u_1\right)^2+u_1 = \frac{1}{4a_2}\left(\overline{U}_2(u_1)-u_2\right)^2+u_2 \quad \text{(A.48)}$$

$$-\frac{1}{2a_1}\left(\overline{U}_1(u_2)-u_1\right)+1 = \frac{1}{2a_2}\left(\overline{U}_2(u_1)-u_2\right)\overline{U}_2'(u_1) \quad \text{(A.49)}$$

$$-\frac{1}{2a_2}\left(\overline{U}_2(u_1)-u_2\right)+1 = \frac{1}{2a_1}\left(\overline{U}_1(u_2)-u_1\right)\overline{U}_1'(u_2) \quad \text{(A.50)}$$

By substituting the expressions of $\overline{U}_1'(u)$ and $\overline{U}_2'(u)$ in equations (A.46) and (A.47) into the three equations above, one can show that they are not independent—only two of the three equations are independent. By substituting the expressions of $\overline{U}_1(u)$ and $\overline{U}_2(u)$ in equations (A.44) and (A.45), we can rewrite the three equations equivalently as follows:

$$\sqrt{Z_1(u_2)} = \frac{\sqrt{a_1a_2}-\sqrt{(u_1-u_2)(a_1-a_2+u_1-u_2)}}{a_2-u_1+u_2} \quad \text{(A.51)}$$

$$\sqrt{Z_2(u_1)} = \frac{\sqrt{a_1a_2}+\sqrt{(u_1-u_2)(a_1-a_2+u_1-u_2)}}{a_1+u_1-u_2} \quad \text{(A.52)}$$

To reiterate, our equivalent objective now is to verify that given $(u_1,u_2)\in\mathcal{C}_1$, u_1 and u_2 satisfy the two equations above. In fact, because $(u_1,u_2)\in\mathcal{C}_1$, we know that $\overline{U}_1(u_2)+u_1 = u_2+\overline{U}_2(u_1)$, which implies

$$Z_1(u_2)Z_2(u_1)=1, \quad \text{(A.53)}$$

which implies $\ln\left(\frac{1-\sqrt{Z_1(u_2)}}{1+\sqrt{Z_1(u_2)}}\right) = \ln\left(\frac{\sqrt{Z_2(u_1)}-1}{\sqrt{Z_2(u_1)}+1}\right)$. Based on this fact, we take $u=u_2$ in equation (A.40) and $u=u_1$ in equation (A.41), and subtract these two equations to get:

$$\frac{\sqrt{\frac{a_2}{a_1}}-\sqrt{Z_1(u_2)}}{1-Z_1(u_2)} - \frac{\sqrt{Z_2(u_1)}-\sqrt{\frac{a_1}{a_2}}}{Z_2(u_1)-1} = \frac{u_2-u_1}{\sqrt{a_1a_2}}+\sqrt{\frac{a_2}{a_1}}-\sqrt{\frac{a_1}{a_2}} \quad \text{(A.54)}$$

By combining and solving equations (A.53) and (A.54), we actually prove that $Z_1(u_2)$ and $Z_2(u_1)$ satisfy equations (A.51) and (A.52). ∎

[2] In fact, we obtain $\overline{U}_1(u)$ and $\overline{U}_2(u)$ in equations (A.44) and (A.45) by solving these two ODEs.

A.11 Proof of Corollary 3:

The monotonicity of $\overline{U}(u_i,u_j)-u_{i\vee j}$ with respect to $u_{i\vee j}$ and $u_{i\wedge j}$ is straightforward to show by taking derivatives, thus omitted here. Suppose $u_1>u_2\to+\infty$, then

$$\begin{aligned}
\overline{U}(u_1,u_2) &= \lim_{u_1>u_2\to+\infty} u_1+\left[1+W\left(e^{-2-\frac{2u_1+u_2}{a}}\frac{1+4W\left(\frac{1}{2}e^{-\frac{7}{4}-\frac{9u_2}{4a}}\right)}{6\times 2^{1/3}W\left(\frac{1}{2}e^{-\frac{7}{4}-\frac{9u_2}{4a}}\right)^{4/3}}\right)\right]a \\
&= u_1+\left[1+W\left(e^{-2-\frac{2\Delta u}{a}}\lim_{u_2\to+\infty}\frac{e^{-\frac{3u_2}{a}}\left[1+4W\left(\frac{1}{2}e^{-\frac{7}{4}-\frac{9u_2}{4a}}\right)\right]}{6\times 2^{1/3}W\left(\frac{1}{2}e^{-\frac{7}{4}-\frac{9u_2}{4a}}\right)^{4/3}}\right)\right]a \\
&= u_1+\left[1+W\left(e^{-2-\frac{2\Delta u}{a}}\lim_{x\equiv e^{-\frac{u_2}{a}}\to 0}\frac{x^3\left[1+4W\left(\frac{1}{2}e^{-\frac{7}{4}}x^{\frac{9}{4}}\right)\right]}{6\times 2^{1/3}W\left(\frac{1}{2}e^{-\frac{7}{4}}x^{-\frac{9}{4}}\right)^{4/3}}\right)\right]a \\
&= u_1+\left[1+W\left(e^{-2-\frac{2\Delta u}{a}}\lim_{x\to 0}\frac{x^3+o(x^3)}{3e^{-\frac{7}{3}}x^3+o(x^3)}\right)\right]a \\
&= u_1+\left[1+W\left(\frac{1}{3}e^{\frac{1}{3}-\frac{2(u_1-u_2)}{a}}\right)\right]a. \qquad \text{(A.55)}
\end{aligned}$$

■

A.12 Proof of Lemma 3:

Proof. We prove by contradiction. Suppose $q_1>q_2\geq -a$, but $q_1-p_1^*<q_2-p_2^*$. From the expression of $P_i(u_1,u_2)$, we can easily get $P_1(q_1-p_1^*,q_2-p_2^*)<P_2(q_1-p_1^*,q_2-p_2^*)$. Let us define,

$$p_1' \equiv q_1-q_2+p_2^* \qquad \text{(A.56)}$$

$$p_2' \equiv \max\{q_2-q_1+p_1^*,0\}. \qquad \text{(A.57)}$$

By definition $p_1',p_2'\geq 0$. Let's first consider the case $q_2-q_1+p_1^*>0$. Then, we have $P_1(q_1-p_1',q_2-p_2')=P_1(q_2-p_2^*,q_1-p_1^*)=P_2(q_1-p_1^*,q_2-p_2^*)$, where the second equality is due to the symmetry of $P_i(u_1,u_2)$. Similarly, we have $P_2(q_1-p_1',q_2-p_2')=P_1(q_1-p_1^*,q_2-p_2^*)$. Let's denote the profit under the pricing policy $p_i=p_i^*$ as π^*, and that under the pricing policy $p_i=p_i'$ as π'. We have,

$$\begin{aligned}
\pi'-\pi^* &= [p_1'P_1(q_1-p_1',q_2-p_2')+p_2'P_2(q_1-p_1',q_2-p_2')] \\
&- [p_1^*P_1(q_1-p_1^*,q_2-p_2^*)+p_2^*P_2(q_1-p_1^*,q_2-p_2^*)] \qquad \text{(A.58)} \\
&= [(q_1-q_2+p_2^*)P_2(q_1-p_1^*,q_2-p_2^*)+(q_2-q_1+p_1^*)P_1(q_1-p_1^*,q_2-p_2^*)] \\
&- [p_1^*P_1(q_1-p_1^*,q_2-p_2^*)+p_2^*P_2(q_1-p_1^*,q_2-p_2^*)] \qquad \text{(A.59)} \\
&= (q_1-q_2)\left[P_2(q_1-p_1^*,q_2-p_2^*)-P_1(q_1-p_1^*,q_2-p_2^*)\right]>0. \qquad \text{(A.60)}
\end{aligned}$$

In the second case with $q_2 - q_1 + p_1^* \geq 0$ and $p_2' = 0$, it is easy to show that the first equality (A.59) above will instead take $\geq$, because $P_1(u_1, u_2)$ decreases with u_2. Therefore we still have $\pi' > \pi^*$. This contradicts the optimality of p_i^*. ■

A.13 Numerical Profit Optimization in Equation (1.34):

If $q_i \leq -a$, $u_i = q_i - p_i \leq -a$, product i will never be considered. In this case, for optimal pricing of a single product, the profit optimization problem is straightforward and is given by Branco, Sun, and Villas-Boas (2012). By symmetry the only case we need to consider is that $q_1 > q_2 \geq -a$. In this case, Lemma 3 implies that $u_1 = q_1 - p_1^* > q_2 - p_2^* = u_2$. There are two cases. In the first case when q_1 is much greater than q_2, and corresponding $u_1 \geq \overline{U}(u_2)$, the consumer will purchase product 1 immediately without any search. In this case, the seller's objective is to maximize p_1. We know that

$$p_1 = q_1 - u_1 \leq q_1 - \overline{U}(u_2) = q_1 - \overline{U}(q_2 - p_2) \leq q_1 - a. \tag{A.61}$$

The equal sign in the above equality holds when $p_2 \geq q_2 + a$. Therefore, the optimal price $p_1^* = q_1 - a$ and $p_2^* \in \{p_2 : p_2 \geq q_2 + a\}$. In the second case, q_1 is greater than u_2 but not by a lot, and correspondingly $\overline{U}(u_2) \geq u_1 > u_2$. By equation (1.19), we have

$$P_1(u_1, u_2) = 1 - \frac{\overline{U}(u_2) - u_1}{2a}, \tag{A.62}$$

$$P_2(u_1, u_2) = \frac{\overline{U}(u_2) - u_1}{\overline{U}(u_2) - u_2} - \frac{\overline{U}(u_2) - u_1}{2a}. \tag{A.63}$$

By substituting these purchase likelihood functions into the optimization problem (1.34), we can numerically obtain the optimal prices by solving the first-order necessary conditions.

A.14 Comparative Statics of A Monopoly's Optimal Price and Profit:

Comparative statics of the optimal prices and maximum profit with respect to a are shown in Figure A.1. The left panel plots the sign of $\partial p_1^*(q_1, q_2)/\partial a$ as a function of q_1 and q_2, and the right panel plots the sign of $\partial \pi^*(q_1, q_2)/\partial a$. Grayness indicates the sign: if it is positive, it is dark gray; if it is negative, it is white. The dashed lines in both plots replicate the boundaries of the optimal search strategy shown in Figure 1.10.

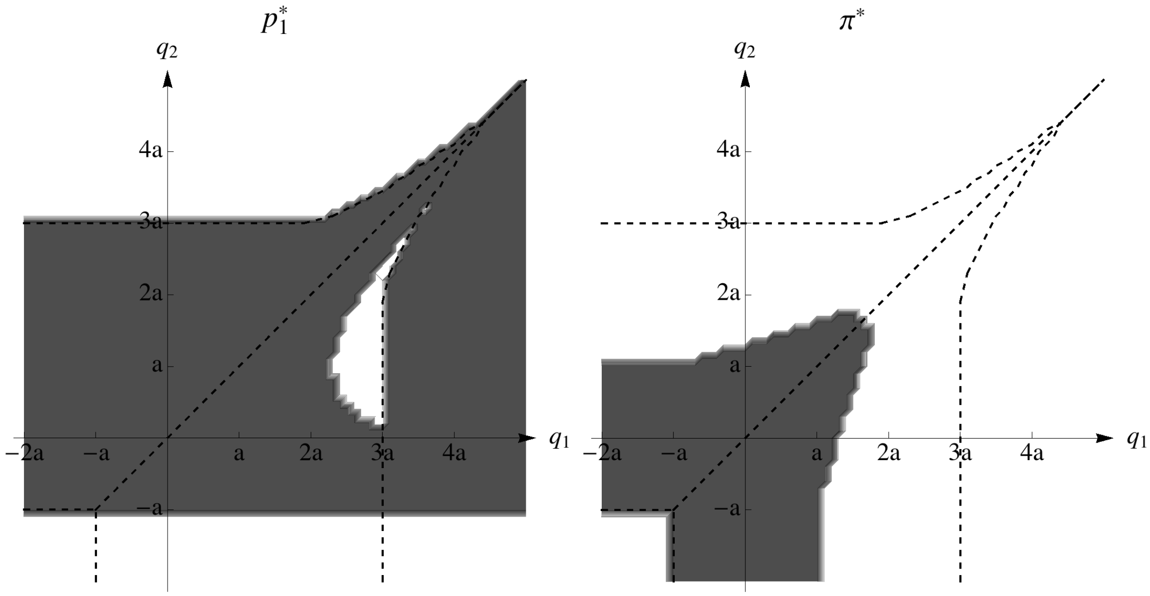

Figure A.1: Comparative statics of the optimal price for product 1, p_1^* and maximum profit, π^*.

A.15 Optimal Search with Time Discounting:

Theorem 10 *There exists a unique solution $V(u_1, u_2)$ along with boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$, which satisfies equations (1.5), (1.36) and (1.11)-(1.14). The value function is obtained as:*

$$V(u_1,u_2) = \begin{cases} \left(\overline{U}(u_2)+\frac{c}{r}\right)\cosh\left[\frac{\sqrt{2r}}{\sigma}(\overline{U}(u_2)-u_1)\right]-\frac{\sigma}{\sqrt{2r}}\sinh\left[\frac{\sqrt{2r}}{\sigma}(\overline{U}(u_2)-u_1)\right]-\frac{c}{r} & u_2\le u_1\le \overline{U}(u_2), u_1\ge \underline{U}(u_2) \\ \left(\overline{U}(u_1)+\frac{c}{r}\right)\cosh\left[\frac{\sqrt{2r}}{\sigma}(\overline{U}(u_1)-u_2)\right]-\frac{\sigma}{\sqrt{2r}}\sinh\left[\frac{\sqrt{2r}}{\sigma}(\overline{U}(u_1)-u_2)\right]-\frac{c}{r} & u_1\le u_2\le \overline{U}(u_1), u_2\ge \underline{U}(u_1) \\ u_1 & u_1>\overline{U}(u_2) \\ u_2 & u_2>\overline{U}(u_1) \\ 0 & \textit{otherwise,} \end{cases} \tag{A.64}$$

and the purchase and exit boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$ are given as:

$$\overline{U}(u) = \begin{cases} X(u) & \textit{if } u\ge \underline{U} \\ \overline{U} & \textit{otherwise.} \end{cases} \tag{A.65}$$

$$\underline{U}(u) = \underline{U} \quad (\textit{relevant when } u\le \underline{U}), \tag{A.66}$$

where $\overline{U}$ and $\underline{U}$ are the purchase and exit boundaries respectively for the optimal search problem with only one product.

$$\overline{U} = \sqrt{\frac{c^2}{r^2}+\frac{\sigma^2}{2r}}-\frac{c}{r} \tag{A.67}$$

$$\underline{U} = \overline{U}-\frac{\sigma}{\sqrt{2r}}\ln\left[\sqrt{\frac{r\sigma^2}{2c^2}}+\sqrt{\frac{r\sigma^2}{2c^2}+1}\right]. \tag{A.68}$$

$X(u)$ is given by the following ordinary differential equation with a boundary condition,

$$\sqrt{\frac{r}{2}}\sigma \coth\left[\frac{\sqrt{2r}}{\sigma}\left(X(u)-u\right)\right] = \left(1+X'(u)\right)\left(c+rX(u)\right) \tag{A.69}$$

$$X(\underline{U}) = \overline{U}. \tag{A.70}$$

The solution is by construction, and the proof is omitted here.

A.16 Proof of Corollary 4:

Proof. We first show that,

$$\lim_{u\to+\infty} \overline{U}(u) - u = 0. \tag{A.71}$$

In fact, we only need to show that $\lim_{u\to+\infty} X(u) - u = 0$, where $X(u)$ is defined in Theorem 10. By definition, we know $X(u) \geq u$. By Lemma 2, we know $X'(u) \geq 0$. Therefore, as $u \to +\infty$, $\left(1+X'(u)\right)\left(c+rX(u)\right) \to +\infty$, which implies $\coth\left[\frac{\sqrt{2r}}{\sigma}\left(X(u)-u\right)\right] \to 0$ by equation (A.69). This implies that $X(u) - u \to 0$.

Next, to show $\overline{U}(u) - u$ decreases with u, we need to prove that,

$$\overline{U}'(u) \leq 1. \tag{A.72}$$

We only need to show that $X'(u) \leq 1$ for $u \geq \underline{U}$, where $u \geq \underline{U}$ is defined in Theorem 10. In fact, by contradiction, suppose there exists $u_0 \geq \underline{U}$ such as $X'(u_0) > 1$. By equation equation (A.69), we have

$$X'(u) = \sqrt{\frac{r}{2}}\sigma \frac{\coth\left[\frac{\sqrt{2r}}{\sigma}\left(X(u)-u\right)\right]}{c+rX(u)} - 1 \tag{A.73}$$

Taking derivatives on both sides of the equation above, we have,

$$\begin{aligned} X''(u) &= -\frac{r(X'(u)-1)}{\left(c+rX(u)\right)\sinh^2\left[\frac{\sqrt{2r}}{\sigma}\left(X(u)-u\right)\right]} \\ &\quad - \frac{r^{3/2}\sigma}{\sqrt{2}(c+rX(u))^2}\coth\left[\frac{\sqrt{2r}}{\sigma}\left(X(u)-u\right)\right]X'(u) \end{aligned} \tag{A.74}$$

As $X'(u_0) > 1$, we have $X''(u_0) < 0$ by the equation above. This implies that for any small positive number ε, $X'(u_0-\varepsilon) \simeq X'(u_0) - X''(u_0)\varepsilon > X'(u_0) > 1$, which in turn implies that $X''(u_0-\varepsilon) < 0$ by using the expression of $X''(u)$ above. By mathematical induction, we can show that for all $u_0 \geq u \geq \underline{U}$, we should have $X'(u) > 1$ and $X''(u) < 0$. However, we know that $X'(\underline{U}) = 0$, which is a contradiction. ∎

A.17 Proof of Proposition 5

To show it's never optimal to never introduce the second generation, it's enough to show $T = +\infty$ cannot be a maximizing point. In fact,

$$\begin{aligned}\pi'(T) &= \left(1 + m_1\frac{q}{p}\right)\left(\frac{(m_2+m_3)h}{e^{(p+m_1q)T} + (m_2+m_3)\frac{q}{p}} - \frac{m_2h}{e^{(p+m_1q)T} + m_1\frac{q}{p}}\right. \\ &+ \left.\frac{e^{(p+qm_1)T}(p+m_1q)[(m_1r_1 - m_2r_1 - m_3r_2) - hT(m_1-m_2)]}{(e^{(p+m_1q)T} + m_1\frac{q}{p})^2}\right) \qquad \text{(A.75)}\end{aligned}$$

As T goes to infinity, $\pi'(T)$ approximates as $-\left(1 + m_1\frac{q}{p}\right)(m_1 - m_2)hTe^{-(p+m_1q)T}$, which goes to 0^-. Then there must exist a large number $T_M > 0$ such that $\pi'(T) < 0$ for all $T > T_M$. So $\pi(T)$ decreases at $[T_M, +\infty]$, $\pi(+\infty) < \pi(T_M)$. □

A.18 Proof of Proposition 6

Notice equation (3.11) is equivalent to the F.O.N.C. $\pi'(T^*) = 0$. What we are trying to prove is the global maximum can and must be reached at a point satisfying F.O.N.C. In fact, $r_1m_1 - r_1m_2 - r_2m_3 > -hA$ is equivalent to $\pi'(0) > 0$, so $T = 0$ cannot be the optimum. Since we've already shown there exists $T_M > 0$ such that $\pi'(T_M) < 0$, there must exist $T^* \in (0, T_M)$ such that $\pi'(T^*) = 0$ and $\pi'(T^*) \leq 0$, according to mean value theorem. □

A.19 Proof of Corollary 6

Let's first introduce a useful lemma, which consider the S.O.N.C. to guarantee all points satisfying F.O.N.C. to be local maxima.

Lemma 4 *When* $m_1 \geq m_2 + \frac{m_3^2}{3m_2+4m_3}$, $\pi''(T^*) < 0$ *for any* T^* *satisfying* $\pi'(T^*) = 0$.

Proof. Given $\pi'(T) = 0$,

$$\begin{aligned}\pi''(T) &= e^{-T(p+qm_1)}(p+qm_1)\Big\{-\frac{hp(p+qm_1)(m_1-m_2)}{(p+e^{-T(p+qm_1)}qm_1)^2} + \frac{e^{-T(p+qm_1)}hqm_1(-p-qm_1)m_2}{(p+e^{-T(p+qm_1)}qm_1)^2} \\ &\quad -\frac{e^{-T(p+qm_1)}hq(-p-qm_1)(m_2+m_3)^2}{(p+e^{-T(p+qm_1)}q(m_2+m_3))^2} \\ &\quad +\frac{2e^{-T(p+qm_1)}pqm_1(p+qm_1)(p+qm_1)(-hT(m_1-m_2)+m_1r_1-m_2r_1-m_3r_2)}{(p+e^{-T(p+qm_1)}qm_1)^3}\Big\} \\ &= e^{-T(p+qm_1)}(p+qm_1)\Big\{-\frac{hp(p+qm_1)(m_1-m_2)}{(p+e^{-T(p+qm_1)}qm_1)^2} + \frac{e^{-T(p+qm_1)}hqm_1(-p-qm_1)m_2}{(p+e^{-T(p+qm_1)}qm_1)^2}\end{aligned}$$

$$
\begin{aligned}
& -\frac{e^{-T(p+qm_1)}hq(-p-qm_1)(m_2+m_3)^2}{(p+e^{-T(p+qm_1)}q(m_2+m_3))^2} \\
& +\frac{2e^{-T(p+qm_1)}qm_1(p+qm_1)}{(p+e^{-T(p+qm_1)}qm_1)}\Big(\frac{hm_2}{p+e^{-T(p+qm_1)}qm_1}-\frac{h(m_2+m_3)}{p+e^{-T(p+qm_1)}q(m_2+m_3)}\Big)\Big\} \\
= \;& e^{-T(p+qm_1)}(p+qm_1)^2h\Big\{-\frac{pm_1}{(p+xqm_1)^2}+\frac{m_2}{(p+xqm_1)} \\
& +\frac{xq(m_2+m_3)^2}{(p+xq(m_2+m_3))^2}-\frac{xqm_1}{(p+xqm_1)}\frac{2(m_2+m_3)}{p+xq(m_2+m_3)}\Big\} \\
= \;& -\frac{(p+qm_1)^2hx}{(p+xqm_1)^2(p+xq(m_2+m_3))^2}\cdot g(x)
\end{aligned}
$$

where $x \equiv e^{-(p+m_1q)T} \in (0,1]$ for $T \in [0,+\infty)$ and

$$
\begin{aligned}
g(x) \;\equiv\; & q^3m_1(m_2+m_3)^2(m_1-m_2)x^3+pq^2(2m_1+m_2+m_3)(m_2+m_3)(m_1-m_2)x^2 \\
& +p^2q\Big(m_1(3m_2+4m_3)-(m_2+m_3)(3m_2+m_3)\Big)x+p^3(m_1-m_2) \qquad \text{(A.76)}
\end{aligned}
$$

$\pi''(T^*) < 0$ is equivalent to $g(x^*) > 0$, which holds for any x^* (corresponding any T^*) as long as $m_1 \geq \frac{(m_2+m_3)(3m_2+m_3)}{(3m_2+4m_3)} = m_2 + \frac{m_3^2}{3m_2+4m_3}$. ■

The existence of points satisfying F.O.N.C. has been shown in the proof of proposition 6. Under the lemma, the second condition in the corollary guarantees the the point satisfying F.O.N.C. to be unique and thus the global maximum. Otherwise, if multiple local optima exist, there must exist at least one local minimum, resulting in a contradiction. □

A.20 Proof of Proposition 7

We first prove under the conditions $m_1 \geq m_2 + \frac{pqm_3^2}{p^2+pq(2m_2+3m_3)+q^2m_1(m_2+m_3)}$ and $r_1m_1 - r_1m_2 - r_2m_3 > -hA$, $T = 0$ is the global maximum of the total profit function $\pi(T)$. In fact,

$$
\begin{aligned}
\pi'(T) \;=\; & (p+m_1q)e^{-(p+m_1q)T}\Big\{\frac{(m_2+m_3)h}{p+(m_2+m_3)qe^{-(p+m_1q)T}}-\frac{m_2h}{p+m_1qe^{-(p+m_1q)T}} \\
+ \;& \frac{p(p+m_1q)\big[(m_1r_1-m_2r_1-m_3r_2)-hT(m_1-m_2)\big]}{(p+m_1qe^{-(p+m_1q)T})^2}\Big\} \\
\leq \;& (p+m_1q)e^{-(p+m_1q)T}\Big\{\frac{(m_2+m_3)h}{p+(m_2+m_3)qe^{-(p+m_1q)T}}-\frac{m_2h}{p+m_1qe^{-(p+m_1q)T}} \\
+ \;& \frac{p(p+m_1q)}{(p+m_1qe^{-(p+m_1q)T})^2}\Big[-\frac{h}{p}\frac{m_3p+(m_1-m_2)(m_2+m_3)q}{p+(m_2+m_3)q}-h(m_1-m_2)\frac{1-e^{-(p+m_1q)T}}{p+m_1q}\Big]\Big\} \\
= \;& (p+m_1q)x\Big\{\frac{(m_2+m_3)h}{p+(m_2+m_3)qx}-\frac{m_2h}{p+m_1qx} \\
- \;& \frac{h}{(p+m_1qx)^2}\Big[\frac{(p+m_1q)\big(m_3p+(m_1-m_2)(m_2+m_3)q\big)}{\big(p+(m_2+m_3)q\big)}+(m_1-m_2)p(1-x)\Big]\Big\}
\end{aligned}
$$

where in the last equation we define $x \equiv e^{-(p+m_1q)T} \in (0,1]$ as $T \in [0,+\infty)$. In order to prove $\pi'(T) \leq 0$ for all $T \in [0,+\infty)$, it's sufficient (but unnecessary) to show that

$$\begin{aligned} &\frac{(m_2+m_3)h}{p+(m_2+m_3)qx} - \frac{m_2h}{p+m_1qx} \\ - &\frac{h}{(p+m_1qx)^2}\Big[\frac{(p+m_1q)(m_3p+(m_1-m_2)(m_2+m_3)q)}{(p+(m_2+m_3)q)} + (m_1-m_2)p(1-x)\Big] \leq 0 \end{aligned}$$

for all $x \in (0,1]$, or equivalently

$$\begin{aligned} f(x) \equiv \quad & q(p+qm_1)(m_1-m_2)(m_2+m_3)\big(p+q(m_2+m_3)\big)x^2 \\ & -\Big[q^3m_1^2(m_2+m_3)^2 + p\Big(m_2\big(p^2+qm_2(p-qm_2)\big) + 2qm_2(p-qm_2)m_3 + q(p-qm_2)m_3^2\Big) \\ & -m_1\Big(p^3 + q\big(q^2m_2^3 + m_2(p-qm_3)^2 + pm_3(2p-qm_3) + qm_2^2(-p+2qm_3)\big)\Big)\Big]x \\ & -p\Big((m_1-m_2)\big(p^2+q(2p+qm_1)m_2\big) + q(3p+qm_1)(m_1-m_2)m_3 - pqm_3^2\Big) \leq 0 \qquad \text{(A.77)} \end{aligned}$$

for all $x \in (0,1]$. In fact $f(1) = 0$, $f(0) = -p\Big((m_1-m_2)(p^2+q(2p+qm_1)m_2) + q(3p+qm_1)(m_1-m_2)m_3 - pqm_3^2\Big) \leq 0$ by assumption. Because the coefficient for the second-order term $q(p+qm_1)(m_1-m_2)(m_2+m_3)(p+q(m_2+m_3))$ is positive, the parabola $f(x)$ is convex. Thus $f(x) \leq 0$ for all $x \in (0,1]$. We have proved that $\pi'(T) \leq 0$ for all $T \in [0,+\infty)$, so $\pi(T)$ is decreasing over $[0,+\infty)$ with the maximum at $T^* = 0$.

Then we're to prove under the condition $m_1 \geq m_2 + \frac{m_3^2}{3m_2+4m_3}$ and $r_1m_1 - r_1m_2 - r_2m_3 > -hA$, $\pi'(T) \leq 0$ for all $T \geq 0$ so that $T = 0$ is also guaranteed to be the global maximum of the total profit function $\pi(T)$. In fact, condition $r_1m_1 - r_1m_2 - r_2m_3 > -hA$ is equivalent to $\pi'(0) \leq 0$. When $\pi'(0) < 0$, we have $\pi'(T) \leq 0$ for all $T \geq 0$. Otherwise, suppose there exists $T_1 > 0$ such that $\pi'(T_1) > 0$. Then there must exist $T_\xi \in (0, T_1)$ such that $\pi'(T_\xi) = 0$ and $\pi''(T_\xi) > 0$, according to the mean value theorem. However, under condition $m_1 \geq m_2 + \frac{m_3^2}{3m_2+4m_3}$, lemma 4 dictates $\pi''(T) < 0$ for any T satisfying $\pi'(T) = 0$, which ends up with a contradiction. When $\pi'(0) = 0$, we have $\pi'(\varepsilon) < 0$ for arbitrarily small $\varepsilon > 0$, as a result from condition $m_1 \geq m_2 + \frac{m_3^2}{3m_2+4m_3}$, we can repeat the above logic to show a contradiction.

Thus we have shown that when $m_1 \geq m_2 + B \cdot m_3$ and $r_1m_1 - r_1m_2 - r_2m_3 > -hA$, $T = 0$ is the global maximizer of function $\pi(T)$ over $[0,+\infty)$. Finally we show that under the conditions $m_1 \leq m_2 + m_3$ and $r_1m_1 - r_1m_2 - r_2m_3 > -\frac{m_2+m_3}{m_1}hA$, $\pi(0) \geq \pi(T)$ for all $T \geq 0$ so that $T = 0$ maximizes the total profit function $\pi(T)$. In fact,

$$\begin{aligned} \pi(T) - \pi(0) \quad = \quad & -\frac{h}{q}\ln\left[\frac{(p+qm_1)(p+qe^{-T(p+qm_1)}(m_2+m_3))}{(p+e^{-T(p+qm_1)}qm_1)(p+q(m_2+m_3))}\right] \\ & +(m_1r_1 - m_2r_1 - m_3r_2)\frac{p(1-e^{-T(p+qm_1)})}{p+e^{-T(p+qm_1)}qm_1} \end{aligned}$$

$$
\begin{aligned}
& +\frac{h(m_1-m_2)(p+qm_1)Te^{-T(p+qm_1)}}{p+e^{-T(p+qm_1)}qm_1} && \text{(A.78)}\\
\leq\ & -\frac{h}{q}\ln\left[\frac{(p+qm_1)(p+qe^{-T(p+qm_1)}(m_2+m_3))}{(p+e^{-T(p+qm_1)}qm_1)(p+q(m_2+m_3))}\right] \\
& -\frac{m_2+m_3}{m_1}\cdot\frac{h}{p}\cdot\frac{m_3p+(m_1-m_2)(m_2+m_3)q}{p+(m_2+m_3)q}\frac{p(1-e^{-T(p+qm_1)})}{p+e^{-T(p+qm_1)}qm_1} \\
& +\frac{h(m_1-m_2)(1-e^{-T(p+qm_1)})}{p+e^{-T(p+qm_1)}qm_1} && \text{(A.79)}
\end{aligned}
$$

where in the last expression, we used the conditions $r_1m_1 - r_1m_2 - r_2m_3 > -\frac{m_2+m_3}{m_1}hA$ and the generic inequality $x \leq e^x - 1$. By further applying $1 \leq \frac{m_2+m_3}{m_1}$, we have

$$
\begin{aligned}
\pi(T)-\pi(0)\ \leq\ & -\frac{h}{q}\ln\left[\frac{(p+qm_1)(p+qe^{-T(p+qm_1)}(m_2+m_3))}{(p+e^{-T(p+qm_1)}qm_1)(p+q(m_2+m_3))}\right] \\
& -\frac{m_2+m_3}{m_1}\cdot\frac{h}{p}\cdot\frac{m_3p+(m_1-m_2)(m_2+m_3)q}{p+(m_2+m_3)q}\frac{p(1-e^{-T(p+qm_1)})}{p+e^{-T(p+qm_1)}qm_1} \\
& +\frac{m_2+m_3}{m_1}\cdot\frac{h(m_1-m_2)(1-e^{-T(p+qm_1)})}{p+e^{-T(p+qm_1)}qm_1} \\
=\ & -\frac{h}{q}\Big(\ln(1-y)-\frac{m_2+m_3}{m_1}\cdot y\Big) && \text{(A.80)}
\end{aligned}
$$

where y takes the form below, by defining $x \equiv e^{-T(p+qm_1)} \in (0,1]$ as usual,

$$
\begin{aligned}
y\ \equiv\ & \frac{pq(m_2+m_3-m_1)(1-x)}{(p+xqm_1)(p+q(m_2+m_3))} && \text{(A.81)}\\
=\ & \frac{(m_2+m_3-m_1)(1-x)}{\frac{p}{q}+\frac{q}{p}m_1(m_2+m_3)x+m_1x+(m_2+m_3)} \\
\leq\ & \frac{(m_2+m_3-m_1)(1-x)}{2\sqrt{m_1(m_2+m_3)x}+m_1x+(m_2+m_3)} \\
=\ & \frac{(m_2+m_3-m_1)(1-x)}{(\sqrt{m_1x}+\sqrt{m_2+m_3})^2} \\
\leq\ & 1-\frac{m_1}{m_2+m_3} && \text{(A.82)}
\end{aligned}
$$

From inequalities (A.80) and (A.82), we know that in order to show $\pi(T)-\pi(0) \leq 0$ for any $T > 0$, it suffices to show $J(y) \equiv -\ln(1-y) - \frac{m_2+m_3}{m_1}\cdot y \leq 0$ for all $0 \leq y \leq 1-\frac{m_1}{m_2+m_3}$. In fact, $J'(y) = \frac{1}{1-y} - \frac{m_2+m_3}{m_1} \leq 0$ is equivalent to $y \leq 1-\frac{m_1}{m_2+m_3}$. Thus $y^* = 1-\frac{m_1}{m_2+m_3}$ is the only minimum, $J(y)$ is decreasing over $[0, y^*]$, and $J(y) \leq J(0) = 0$ for any $y \in [0, 1-\frac{m_1}{m_2+m_3}]$. Therefore, we have shown $\pi(T)-\pi(0) \leq 0$ for any $T > 0$, so that $T = 0$ is the global maximizer of the total profit.

In conclusion, we have proved that when $m_1 \geq m_2 + m_3$ and $r_1 m_1 - r_1 m_2 - r_2 m_3 > -hA$, or when $m_1 \leq m_2 + m_3$ and $r_1 m_1 - r_1 m_2 - r_2 m_3 > -\frac{m_2+m_3}{m_1} hA$, the simultaneous introduction policy is optimal in that $T = 0$ maximizes the total profit. □

A.21 Proof of Proposition 8

As shown in equation (A.75), when $r_1 m_1 - r_1 m_2 - r_2 m_3 = 0$, h becomes a common factor in the expression of $\pi'(T)$, which does not determine the optimal introduction time. So the optimal introduction time should not depend on h. □

A.22 Proof of Proposition 9

In the case of $m_2 + m_3 = m_1$,

$$\begin{aligned} \pi'(T) &= \frac{e^{-T(p+qm_1)}(p+qm_1)}{(p+e^{-T(p+qm_1)}qm_1)^2} \Big\{ (r_1 m_1 - r_1 m_2 - r_2 m_3) p(p+qm_1) \\ &\quad + h[m_3 p + e^{-T(p+qm_1)} q(m_1 - m_2)(m_2 + m_3) - p(m_1 - m_2)(p+qm_1)T] \Big\} \end{aligned}$$

By defining $\varphi(T) \equiv e^{-T(p+qm_1)} q(m_1 - m_2)(m_2 + m_3) - p(m_1 - m_2)(p + qm_1)T$, the F.O.N.C. of local optimality $\pi'(T) = 0$ can be translated as

$$\varphi(T) = -\frac{(r_1 m_1 - r_1 m_2 - r_2 m_3) p(p+qm_1)}{h} - m_3 p. \tag{A.83}$$

Meanwhile $\varphi(T)$ is a strictly decreasing function, and as T varies from $-\infty$ to $+\infty$, $\varphi(T)$ ranges from $+\infty$ to $-\infty$. So there must exist a unique solution to equation (A.83), named after T^0. When $T^0 \geq 0$, optimal introduction time $T^* = T^0$, which maximizes the total profit function $\pi(T)$; otherwise when $T^0 < 0$, optimal introduction time $T^* = 0$. Therefore the optimal introduction time T^* can be treated as a increasing function on T^0. Equation (A.83) regulates the dependent relationship of T^0 on h. In fact, according to the rule of derivative of implicit functions,

$$\frac{dT^0}{dh} = -\frac{(m_1 - m_2)[p + e^{-T(p+qm_1)} q(m_2 + m_3)] h^2}{(r_1 m_1 - r_1 m_2 - r_2 m_3) p} \tag{A.84}$$

When $r_1 m_1 - r_1 m_2 - r_2 m_3 < 0$, $\frac{dT^0}{dh} > 0$, T^0 is strictly increasing in h, as a result, the optimal introduction time T^* is also increasing in h. Similarly we get when $r_1 m_1 - r_1 m_2 - r_2 m_3 > 0$, the optimal introduction time T^* is decreasing in h. Lastly, if $h \gg |r_1 \frac{m_1 - m_2}{m_3} - r_2|(p + qm_1)$, then $|\frac{(r_1 m_1 - r_1 m_2 - r_2 m_3) p(p+qm_1)}{h}| \ll m_3 p$, the equation (A.83) approximates $\varphi(T) = -m_3 p$, which has nothing to do with h. Consequently, T^0 and thus T^* doesn't depend on h. □

A.23 Summary Statistics for the Panel

The following table A.1 provides summary statistics for the panel used in section 2.4, combining both phone-call network and adoption data set.

Table A.1: Summary Statistics for the Panel

	Observation	Mean	Std Dev	p25	p50	p75
i (individual)	74,967					
t (month)	2009/11 – 2013/10					
Panel A: Basic Panel						
ADOPT	3,100,442	0.02	0.15	0	0	0
INSTALLBASE_IN	3,100,442	0.35	1.80	0	0	0
INSTALLBASE_OUT	3,100,442	0.35	1.81	0	0	0
$INSTALLBASE\_IN^2$	3,100,442	3.37	168.39	0	0	0
$INSTALLBASE\_OUT^2$	3,100,442	3.39	168.84	0	0	0
Panel B: IV Panel for Section 2						
IV_BDAY_IN	3,100,442	7.91	16.24	0	3	9
IV_BDAY_OUT	3,100,442	7.91	16.25	0	3	9
$IV\_BDAY\_IN^2$	3,100,442	326.17	3,932.44	0	9	81
$IV\_BDAY\_OUT^2$	3,100,442	326.68	3,939.51	0	9	81
Panel C: Adoptor Fraction Panel for Section 2						
FRACTION_IN	2,501,265	0.06	0.18	0	0	0
FRACTION_OUT	3,099,985	0.00	0.02	0	0	0
Panel D: Network Heterogeneity Panel for Section 2						
USER_DEG_IN	3,100,442	10.63	96.20	0	0	0
USER_DEG_OUT	3,100,442	114.95	656.98	0	0	0
IV_BDAY_DEG_IN	3,100,442	223.06	971.13	0	32	159
IV_BDAY_DEG_OUT	3,100,442	2,360.78	5,876.02	0	573	2,357
USER_TIE_IN	3,100,442	13.33	162.40	0	0	0
USER_TIE_OUT	3,100,442	14.72	168.86	0	0	0
IV_BDAY_TIE_IN	3,100,442	358.29	1,323.26	0	17.34	195.61
IV_BDAY_TIE_OUT	3,100,442	358.29	1,323.14	0	17.34	195.72
USER_logDEG_IN	3,097,225	1.03	6.03	0	0	0
USER_logDEG_OUT	3,098,960	1.97	10.22	0	0	0
IV_BDAY_logDEG_IN	3,097,225	22.01	55.82	0	6.09	22.32
IV_BDAY_logDEG_OUT	3,098,960	42.73	91.30	0	14.45	47.72
USER_logTIE_IN	3,100,442	0.55	2.86	0	0	0
USER_logTIE_IN	3,100,442	0.56	2.90	0	0	0
IV_BDAY_logTIE_IN	3,100,442	13.02	26.56	0	3.37	15.30
IV_BDAY_logTIE_OUT	3,100,442	13.02	26.55	0	3.37	15.31

Note: The panel consists of (almost) all iPhone adoptions in the city of Xining from Nov-2009 to Oct-2013. The social network is constructed from call transactions between May-2013 and Nov-2013.

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