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Gary Arthur Pearson
(Ph. D. Thesis)

March 4, 1965

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ABSTRACT

The stability of a current-carrying electron-proton plasma is studied by calculating the velocity distributions of the particles with the Lenard-Balescu form of the Fokker-Planck equations rather than with the familiar Landau form. It is known that the Landau equations yield velocity distributions that become unstable to longitudinal ion waves when the temperature ratio θ_e/θ_i is large and the electric field $\underline{E}_0$ that drives the current exceeds a critical magnitude E_{crit}. The Landau equations are then not adequate because they do not include the effect of the fluctuating electric fields associated with these ion waves upon the velocity distributions.

As E_0 is increased to E_{crit} and beyond, the Lenard-Balescu equations show that the fluctuations associated with certain ion waves increase rapidly, drive the electron velocity distribution towards isotropy in the ion frame, and thus prevent the plasma from becoming unstable to any ion wave. For E_0 greater than E_{crit}, these fluctuations are just sufficient to maintain the stability. By distorting the velocity distributions of the particles, these wave-particle interactions also alter the transport properties of the plasma in much the same way as an increased collision frequency would do.

The computer solutions obtained with $\theta_e/\theta_i = 70$ serve as examples. With $E_0 = E_{crit}$, the ion waves whose damping rate would vanish according to the Landau equations actually have a damping rate smaller by only a factor of 12 than with $E_0 = 0$. This damping rate decreases by another factor of 10 as E_0 is increased to $1.25\ E_{crit}$.

The energy of the corresponding fluctuations increases by these same factors as E_0 is increased from zero, but the total energy in fluctuations associated with all ion waves only doubles as E_0 increases from zero to 1.25 E_{crit}. The electrical conductivity is lower than the value from the Landau equations by 4.6% at $E_0 = 0$, by 8.1% at $E_0 = E_{crit}$, and by 12.1% at $E_0 = 1.25\ E_{crit}$; this correction becomes significant as E_0 is increased further.

The plasmas to which the results of this problem would directly apply are restricted by the conditions for validity of the Lenard-Balescu equations. When these conditions are not met, other physical processes may also be of importance. For example, even in a high-temperature low-density plasma, as E_0 is increased the fluctuations associated with ion waves will be affected by collisions and by mode coupling.

I. INTRODUCTION

The growth or damping of longitudinal ion waves in an unmagnetized current-carrying classical Coulomb plasma is considered. The discussion is restricted to a spatially uniforn low-density high-temperature plasma and often to an electron-proton plasma.

The velocity-distribution functions found by Spitzer and Härm[1] in calculating the linear electrical conductivity will support growing ion waves when the electric field exceeds a critical value. The Landau form of the Fokker-Planck equation, which Spitzer and Härm used, does not include the effect of waves upon the velocity distributions of the particles, and is therefore inadequate when the electric field exceeds this critical value. When the electron temperature is large compared to the ion temperature, this critical field is small enough that, if the Landau equation remained valid, its linearization would easily be justified.

In this case the velocity distributions of the particles are still determined primarily by the electric field and by the "ordinary" collisions contained in the Landau equation. Thus the quasi-linear theories, which are often used in the study of effects of wave-particle interactions, are not useful because they do not include the effects of "ordinary" collisions. Instead, the Lenard-Balescu kinetic equations are used because they include the effects of "ordinary" collisions and of the fluctuating electric fields associated with waves, although they are applicable only when the plasma can support only damped waves and so is stable.

In the Lenard-Balescu kinetic equations, the spectrum of electric-field fluctuations associated with waves arises as a balance of the continuous excitation and (Landau) damping of waves. This balance can occur only if all waves are damped. The continuous excitation can be described as spontaneous emission of longitudinal waves by the plasma particles, and the Landau damping is then interpreted as the net effect of induced emission and absorption by the particles.

These wave-particle interactions also affect the velocity distributions of the particles. In various examples that have been considered, the wave-particle interactions have qualitatively the same effect as "ordinary" collisions or particle-particle interactions. In fact it is often useful to consider the wave-particle interactions as the interactions of individual particles by the emission and absorption of waves.

In particular, the fluctuations associated with ion waves drive the velocity distributions towards isotropy, as do "ordinary" collisions, and this tends to stabilize the plasma. However, unlike "ordinary" collisions, the wave-particle interactions are altered as the plasma approaches instability. As the damping rate of certain waves decreases the fluctuations associated with them increase rapidly, so the effect of wave-particle interactions on the particle velocity distribution, which effect should include a stabilizing tendency, increases nonlinearly. The purpose of this thesis is to investigate this nonlinear stabilization.

The Lenard-Balescu kinetic equations are valid near and above the critical field only because of this nonlinear stabilization. These equations have other restrictions—such as the neglect of the effects that wave-wave interactions (mode coupling) and collisions have on waves—which limit their usefulness.

These concepts are presumably applicable to other problems involving micro-instabilities or velocity-space instabilities. In such cases, even though the plasma may be stable, the high level of fluctuations necessary to maintain this stability may be very important in altering certain transport rates such as diffusion across a magnetic field. Qualitatively the wave-particle interactions usually have the same effect as a higher collision rate.

In Secs. II through V, the concepts and equations used and the related work by others are reviewed. In Sec. VI we start with the Lenard-Balescu equations for an electron-proton plasma, discuss simplifying assumptions, and develop the equations for two model problems that are solved numerically. Section VII is a brief description of the numerical procedure used, and the results are discussed in Sec. VIII.

Sections II through V are a review and discussion of the physical basis for the Lenard-Balescu equation. The reader who is familiar with this material may prefer to ignore the appropriate sections, although Sec. III. E does contain one new and interesting result. Also, most readers are probably not familiar with the material in Sec. V. A, which may be read in conjunction with Sec. VI. B. The most useful general references for these review sections are the recent book by Montgomery and Tidman[2] and the review article on plasma waves by Bernstein, Trehan, and Weenink,[3] both of which contain extensive lists of appropriate references.

Appendix L contains a brief discussion of the notation that is used. All equations are valid in both electrostatic units and in Gaussian units. A plain letter like v denotes a scalar variable, a letter underlined like $\underset{\sim}{C}$ denotes a second-rank tensor quantity, and a letter underlined like $\underline{k}$ represents a vector quantity with magnitude k and direction $\underline{\hat{k}}$ so $\underline{k} = k\underline{\hat{k}}$.

II. SIMPLE COLLISIONAL MODELS OF A CURRENT-CARRYING PLASMA

A. The Spatially Uniform Classical Coulomb Plasma

We begin by making certain assumptions to obtain a set of equations that will approximately describe the behavior of a real plasma.

All quantum effects are ignored, so our plasma is "classical" and is treated as a collection of point particles with masses m_i, charges q_i, spatial positions $\underline{r}_i(t)$, and velocities $\underline{v}_i(t)$. This is a useful approximation only if the real plasma has high temperature and low density so that, for example, it remains nearly fully ionized.

We also treat the plasma as a "Coulomb" plasma in which the Maxwell equations are approximated as

$$\nabla \times \underline{E}(\underline{r}, t) = 0 \qquad \text{(II-1)}$$

$$\nabla \cdot \underline{E}(\underline{r}, t) = 4\pi \sum_i q_i \, \delta[\underline{r} - \underline{r}_i(t)] + 4\pi \rho_0(\underline{r}, t), \qquad \text{(II-2)}$$

and the equation of motion of a particle is approximated as

$$m_i \frac{d\underline{v}_i(t)}{dt} = q_i \underline{E}'(\underline{r}_i, t) . \qquad \text{(II-3)}$$

Equations (II-1) and (II-2) imply that

$$\underline{E}(\underline{r}, t) = \underline{E}_0(\underline{r}, t) + \sum_i q_i \frac{\underline{r} - \underline{r}_i(t)}{|\underline{r} - \underline{r}_i(t)|^3} \qquad \text{(II-4)}$$

and the prime in Eq. (II-3) reminds us that the particle being considered is to be left out of the sum. These approximations are not appropriate unless all speeds of interest are small compared to the speed of light, and the magnetic-field effects are negligible. With these approximations we can treat only longitudinal waves and so we cannot treat radiation.

The field $\underline{E}_0(\underline{r}, t)$ and the charge density $\rho_0(\underline{r}, t)$ are considered known and are related by

$$\nabla \times \underline{E}_0(\underline{r}, t) = 0 \qquad \text{(II-5)}$$

$$\nabla \cdot \underline{E}_0(\underline{r}, t) = 4\pi\rho_0(\underline{r}, t) . \quad \text{(II-6)}$$

We represent the real plasma by an ensemble in which the ensemble averages of all quantities except potentials are independent of the spatial position $\underline{r}$. We further suppose that each particle belongs to a species α with mass M_α and charge q_α, and we choose our ensemble so that all averages are unchanged by the interchange of any two particles of the same species. (Boltzmann statistics are appropriate because we treat a "classical" plasma.) Because $\underline{E}_0$ and ρ_0 are supposed known, we know immediately that their ensemble averages are

$$\left\langle \underline{E}_0(\underline{r}, t) \right\rangle = \underline{E}_0(t) \quad \text{(II-7)}$$

$$\left\langle \rho_0(\underline{r}, t) \right\rangle = 0 . \quad \text{(II-8)}$$

Because the particle flux of each species α

$$\underline{\gamma}_\alpha(t) = \left\langle \sum_{i \text{ in } \alpha} \underline{v}_i(t)\ \delta[\underline{r} - \underline{r}_i(t)] \right\rangle \quad \text{(II-9)}$$

must be independent of $\underline{r}$ and because particles are conserved, the number density of species α

$$n_\alpha = \left\langle \sum_{i \text{ in } \alpha} \delta[\underline{r} - \underline{r}_i(t)] \right\rangle \quad \text{(II-10)}$$

must be independent of both $\underline{r}$ and t. The distribution function $f_\alpha(\underline{v}, t)$, which is proportional to $\left\langle \sum_{i \text{ in } \alpha} \delta[\underline{r} - \underline{r}_i(t)]\, \delta[\underline{v} - \underline{v}_i(t)] \right\rangle$, must be independent of $\underline{r}$, so we normalize as

$$\int d^3v f_\alpha(\underline{v}, t) = 1 . \quad \text{(II-11)}$$

Because $\left\langle \underline{E}(\underline{r}, t) \right\rangle$ must be independent of $\underline{r}$ we must have charge neutrality

$$\sum_\alpha n_\alpha q_\alpha = 0 .$$

Similar statements can be made about other quantities. This choice of ensemble is appropriate only if the real plasma is "spatially uniform" on a distance scale large compared to distances that appear naturally in the problem, such as the collision mean free path. We thus use this ensemble to treat a limited region of a real plasma. The only influence of the region external to the region being considered is production of $\underline{E}_0(t)$. Notice that in this "spatially uniform" plasma

$$\left\langle \underline{E}(\underline{r}, t) \right\rangle = \underline{E}_0(t) . \qquad \text{(II-12)}$$

In Secs. III, IV. A, and IV. B we consider perturbations of our "spatially uniform" plasma that are not represented by an ensemble chosen as above.

A current-carrying plasma requires special consideration because the uniform current density

$$\underline{j}(t) = \sum_{\alpha} q_{\alpha} \gamma_{\alpha}(t) \qquad \text{(II-13)}$$

produces a magnetic field that cannot be neglected if one considers a sufficiently large cross-sectional area. Therefore, the cross sectional area of the volume of our real plasma must be small enough that this self-magnetic field can be neglected throughout. This requires (a) that the magnetic pressure be small compared to the kinetic pressure so the plasma remains uniform and (b) that the cyclotron frequency of the particles be low compared with other frequencies of interest, such as the collision frequency.

Throughout this work, the term "plasma" will usually imply a spatially uniform classical Coulomb plasma as discussed above. Although the real plasmas to which our results would apply directly are quite restricted, many of the qualitative features discussed are probably present in many real plasmas.

B. Collisional Models

In the collisional models only the electrons (mass m, charge -e) are treated in detail. The motion of the ions is neglected because of their relatively large masses. The electron-distribution function is affected only by the externally applied field $\underline{E}_0(t)$ and by collisions in the Boltzmann sense.[4,5,6] Because the Rutherford cross section does not yield finite results, one ignores the interaction of two particles when their impact parameter is larger than a certain "cutoff" distance b_{max}. This procedure is an attempt to account for the shielding produced by particles other than the two being considered. We show in Appendix A that the cross section for momentum transfer from an electron of speed v striking an ion of charge $Z_a e$ at rest is then

$$\sigma_m(v) = \frac{4\pi Z_a^2 e^4}{m^2 v^4} \ln\left(\frac{mv^2 b_{max}}{Ze^2}\right) . \qquad \text{(II-14)}$$

This and other results depend only weakly upon the cutoff b_{max} if the logarithm is large compared to unity. By comparison with Debye shielding, one usually chooses b_{max} of order $D_e \approx \sqrt{\theta_e/n_e e^2}$, where θ_e is a characteristic electron energy.[7,8] The uncertainty in $\sigma_m(v)$ is then of order unity compared with $\ln(n_e D_e^3)$ if we consider v to be of order $\sqrt{\theta_e/m}$.

The condition that $n_e D_e^3$ be very large can be considered the definition of a low-density high-temperature plasma. This condition ensures that, for impact parameters larger than D_e, there are many shielding particles between the particles being considered. However, even for impact parameters much smaller than D_e, the two particles interact simultaneously with many other particles. The reason this Boltzmann-like description yields useful results is that even for such impact parameters the deflections are very small and the simultaneous effects of many particles upon one being considered can be added linearly.[9]

The circumstance that most of the deflections are small suggests use of a Fokker-Planck equation. In fact, within the uncertainty mentioned above, identical results are found by means of the kinetic equation

$$\frac{\partial f_e(\underline{v}, t)}{\partial t} - \frac{e}{m} \underline{E}_0(t) \cdot \frac{\partial f_e(\underline{v}, t)}{\partial \underline{v}} = \left(\frac{\partial f_e}{\partial t}\right)_{coll} \qquad \text{(II-15)}$$

with either of the following as the right-hand side:[6]

(a) The Boltzmann collision term, made finite by the above cutoff procedure,

(b) The Fokker-Planck terms used by Spitzer and Härm,[1] by Rosenbluth, MacDonald, and Judd,[8] and others.

The Fokker-Planck terms have been derived by several methods.[10] The method developed by Landau in 1936 is expansion of the Boltzmann collision term in powers of the momentum transfer and retention of only the terms that diverge logarithmically, the so-called dominant terms.[11] The divergence is removed by the cutoff procedure discussed above and by also ignoring interactions with impact parameters less than e^2/θ_e, where the expansion is not valid. We obtain the same result by another method in Sec. V.

For simplicity, we call this kinetic equation the Landau equation. Because it is Markovian,[4] it can be used only to treat variations on a time scale long compared to the duration of a collision, which in this case is of the order ω_{pe}^{-1}, where the electron plasma frequency is

$$\omega_{pe} = \sqrt{\frac{4\pi n_e e^2}{m}} \;. \qquad \text{(II-16)}$$

1. Displaced-Maxwellian Model

In this model the electron velocity distribution has the form

$$f_e(\underline{v}, t) = \frac{1}{\pi^{3/2} a_e^{3}} \exp\left[-(\underline{v} - \underline{U})^2/a_e^{2}\right] \qquad \text{(II-17)}$$

where only $\underline{U}(t)$ depends upon time. Here the electron thermal speed a_e and the electron temperature in energy units θ_e are related by

$$a_e^{2} = 2\theta_e/m \;. \qquad \text{(II-18)}$$

We now define the electron Debye length

$$D_e = \sqrt{\theta_e/4\pi n_e e^2} \;, \qquad \text{(II-19)}$$

the minimum impact parameter

$$b_{min} = e^2/\theta_e \;, \qquad \text{(II-20)}$$

and the plasma parameter

$$\Lambda = D_e/b_{min} = 4\pi n_e D_e^{3} \qquad \text{(II-21)}$$

which is three times the number of electrons in a sphere of radius D_e (a Debye sphere). We also define a collision frequency

$$\nu_c = \frac{1}{3}\sqrt{\frac{2}{\pi}}\,\frac{\omega_{pe}}{\Lambda}\ln\Lambda , \qquad \text{(II-22)}$$

which carries with it the uncertainty of order $[\ln\Lambda]^{-1}$ as discussed before.

If we substitute Eq. (II-17) into the Landau equation, multiply by $\underline{v}$, and then integrate over $\underline{v}$, we find

$$\frac{d\underline{U}}{dt} + \frac{e}{m}\underline{E}_0(t) = -\nu_c \underline{U}(t)\left[\frac{1}{n_e}\sum_a{}' n_a Z_a^{\,2}\right]\Omega\left(\frac{U}{a_e}\right) \qquad \text{(II-23)}$$

where the prime signifies that the electrons are not included in the sum, and where

$$\Omega(x) = \frac{3}{x^3}\int_0^x u^2 e^{-u^2}\,du$$

$$\approx 1 \qquad \text{when } x \ll 1$$

$$\approx \frac{3\sqrt{\pi}}{4x^3} \qquad \text{when } x \gg 1 . \qquad \text{(II-24)}$$

In Appendix A we give an alternative derivation of Eq. (II-23) based simply upon the drag force exerted on the electrons by the ions through $\sigma_m(v)$.

If U(t) remains small compared to a_e so that $x \ll 1$, Eq. (II-23) becomes linear. If we assume $\underline{E}_0(t)$ and $\underline{j}(t) = -n_e e\underline{U}(t)$ are proportional to $e^{-i\omega t}$, we find the linear electrical conductivity on our displaced-Maxwellian model

$$\sigma_{DM}(\omega) = \frac{n_e e^2}{m}\,\frac{1}{\left[\frac{1}{n_e}\sum_a{}' n_a Z_a^{\,2}\right]\nu_c - i\omega} . \qquad \text{(II-25)}$$

When the frequency ω is small and the ions are singly charged, this reduces to

$$\sigma_{DM}(\omega) = \frac{n_e e^2}{m\nu_c} . \qquad \text{(II-26)}$$

high speeds. The electrical conductivity is not then well defined because this runaway current continues to increase in time unless it is limited by plasma boundaries.

4. Limitations of the Landau Equation

The cutoff procedure used to account crudely for the dielectric properties of the plasma does not recognize the possibility of long-range interactions taking place by means of propagating waves. In the case of an electron-proton plasma near thermal equilibrium, the results obtained from the Landau equation are actually correct within the uncertainties mentioned above, as we show in Sec. V. The reason is that there are no slightly damped waves, and therefore no corresponding large-amplitude fluctuations that interact strongly with an appreciable number of particles. In other words, very few particles are able to have long-range interactions by the emission and absorption of waves. One cannot expect the Landau equation to be so accurate in other cases.

III. LONGITUDINAL WAVES AND STABILITY

A. The Vlasov Equations

The Vlasov equations for a classical Coulomb plasma

$$\frac{\partial f_\alpha(\underline{r}, \underline{v}, t)}{\partial t} + \underline{v} \cdot \frac{\partial f_\alpha(\underline{r}, \underline{v}, t)}{\partial \underline{r}} + \frac{q_\alpha}{M_\alpha} \underline{E}(\underline{r}, t) \cdot \frac{\partial f_\alpha(\underline{r}, \underline{v}, t)}{\partial \underline{v}} = 0 \qquad \text{(III-1)}$$

$$\nabla \times \underline{E}(\underline{r}, t) = 0 \qquad \text{(III-2)}$$

$$\nabla \cdot \underline{E}(\underline{r}, t) = 4\pi \sum_\alpha n_\alpha q_\alpha \int f_\alpha(\underline{r}, \underline{v}, t) d^3v + 4\pi\rho_0(\underline{r}, t) \qquad \text{(III-3)}$$

are useful for treating certain problems in a low-density high-temperature plasma.[16] Here $\underline{E}(\underline{r}, t)$ also is an ensemble average. The ensemble is chosen so that each function is a smooth function of the arguments but not necessarily independent of $\underline{r}$. Here we also have

$$f_\alpha(\underline{r}, \underline{v}, t) = \frac{1}{n_\alpha} \left\langle \sum_{i \text{ in } \alpha} \delta[\underline{r} - \underline{r}_i(t)]\, \delta[\underline{v} - \underline{v}_i(t)] \right\rangle, \qquad \text{(III-4)}$$

where n_α is simply a normalization constant, which is often taken as unity.

The meaning and validity of all but Eq. (III-1) is clear. Equation (III-1) has the form of the Boltzmann equation except that the collision term is absent; it is sometimes called the collisionless Boltzmann equation. It is also called the correlationless kinetic equation because of a derivation we outline in Sec. IV. C.

Because $f_\alpha(\underline{r}, \underline{v}, t)$ and $\underline{E}(\underline{r}, t)$ are treated as smooth functions, the "ordinary" collisions or short-range interactions as discussed in Sec. II are not included so that particles can interact only through the self-consistent field $\underline{E}(\underline{r}, t)$. This model is appropriate only for treating variations on time scales that are short in comparison with collisional time scales. The Vlasov equations cannot be expected to treat correctly variations over distances of order or smaller than the distance between particles (actually in phase space) because the functions are treated as smooth.

Rostoker and Rosenbluth[17] have shown that the Vlasov equations become exact in the limit as q_α, M_α, and the particle density of each species approaches zero at the same rate, so the charge and mass densities of each species remains constant. Each species then is represented as a continuous fluid and all individual-particle aspects disappear. Since ν_c^{-1} and Λ become infinite, the collisional time scale is never reached and the distance between particles vanishes.

B. Plasma Response to a Perturbing Charge; The Dielectric Function

We consider a spatially uniform unperturbed plasma with particle densities n_α and with velocity distributions $f_\alpha(\underline{v})$ (normalized to unity) that do not vary on the time scales we consider, which are short compared to ν_c^{-1}. With E_0 smaller than E_{run}, the effects of $\underline{E}_0(t)$ can be ignored because they are important only on collisional time scales. In response to a perturbation $\rho_0(\underline{r}, t)$ applied after $t = 0$, the plasma will develop a small change $\delta f_\alpha(\underline{r}, \underline{v}, t)$ in the distribution functions, and the small field $\underline{E}(\underline{r}, t)$ will not be simply the field of $\rho_0(\underline{r}, t)$ itself.

The linearized Vlasov equations for this case are

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla\right) \delta f_\alpha(\underline{r}, \underline{v}, t) = -\frac{q_\alpha}{M_\alpha} \underline{E}(\underline{r}, t) \cdot \frac{\partial f_\alpha(\underline{v})}{\partial \underline{v}} \qquad \text{(III-5)}$$

$$\nabla \times \underline{E}(\underline{r}, t) = 0 \qquad \text{(III-6)}$$

$$\nabla \cdot \underline{E}(\underline{r}, t) = 4\pi \sum_\alpha n_\alpha q_\alpha \int d^3v \; \delta f_\alpha(\underline{r}, \underline{v}, t) + 4\pi\rho_0(\underline{r}, t) \; . \qquad \text{(III-7)}$$

Because $f_\alpha(\underline{v})$ does not depend upon $\underline{r}$ or t (on the time scales we consider), it is convenient to Fourier transform in space and Laplace transform in time so, for example,

$$\underline{E}(\underline{k}, \omega) = \int d^3r \, e^{-i\underline{k} \cdot \underline{r}} \int_0^\infty dt \, e^{i\omega t} \underline{E}(\underline{r}, t) \qquad \text{(III-8)}$$

and

$$\underline{E}(\underline{r}, t) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\underline{k} \cdot \underline{r}} \int_C \frac{d\omega}{2\pi} \, e^{-i\omega t} \, \underline{E}(\underline{k}, \omega). \qquad \text{(III-9)}$$

Here $\underline{k}$ is real and $\underline{E}(\underline{k}, \omega)$ is defined for all complex ω for which the defining integral converges. This requires that $\text{Im}\,\omega$ be larger than some number γ, and $\underline{E}(\underline{k}, \omega)$ is analytic for $\text{Im}\,\omega > \gamma$. The contour C runs from $\text{Re}\,\omega = -\infty$ to $\text{Re}\,\omega = \infty$ along a line parallel to the real ω axis and with $\text{Im}\,\omega > \gamma$.

Because of Eq. (III-6), we have $\underline{E}(\underline{k}, \omega) = E(\underline{k}, \omega)\hat{\underline{k}}$. With the initial condition $\delta f_a(\underline{r}, \underline{v}, t = 0) = 0$, the transformation of Eqs. (III-5) and (III-7) yields

$$i(\underline{k} \cdot \underline{v} - \omega)\,\delta f_a(\underline{k}, \underline{v}, \omega) = -\frac{q_a}{M_a} E(\underline{k}, \omega)\hat{\underline{k}} \cdot \frac{\partial f_a(\underline{v})}{\partial \underline{v}} \qquad \text{(III-10)}$$

and

$$ikE(\underline{k}, \omega) = 4\pi \sum_a n_a q_a \int d^3v\; \delta f_a(\underline{k}, \underline{v}, \omega) + 4\pi\rho_0(\underline{k}, \omega)\,. \qquad \text{(III-11)}$$

By solving Eq. (III-10) for δf_a and substituting it into Eq. (III-11), we find

$$\underline{E}(\underline{k}, \omega) = E(\underline{k}, \omega)\hat{\underline{k}} = \frac{-4\pi i \underline{k}\rho_0(\underline{k}, \omega)}{k^2\epsilon(\underline{k},\omega)} \qquad \text{(III-12)}$$

where

$$k^2\epsilon(\underline{k}, \omega) = k^2 - \sum_a \omega_{pa}{}^2 \int d^3v\, \frac{\hat{\underline{k}} \cdot [\partial f_a(\underline{v})/\partial \underline{v}]}{\hat{\underline{k}} \cdot \underline{v} - \omega/k}\,. \qquad \text{(III-13)}$$

Here we have defined the plasma frequency for each species as

$$\omega_{pa}{}^2 = \frac{4\pi n_a q_a{}^2}{m_a}\,. \qquad \text{(III-14)}$$

If the plasma were not present, Eq. (III-12) would be modified only by the Vlasov dielectric function $\epsilon(\underline{k}, \omega)$ being replaced by unity. The second term in Eq. (III-13) represents the polarization of the plasma. Notice that the substitution of Eq. (III-12) into Eq. (III-10) yields the expression for $\delta f_a(\underline{k}, \underline{v}, \omega)$.

The dielectric function depends only upon the unperturbed plasma. We can consider $\epsilon(\underline{k}, \omega)$ to be defined by Eq. (III-13) when $\text{Im}\,\omega > 0$, and we will be particularly interested in the case of $\text{Im}\,\omega$ approaching zero. Introducing the real variable V, we use the Plemelj formulas

$$\lim_{\epsilon\to 0} \frac{1}{x-(x'\pm i\epsilon)} = P\frac{1}{x-x'} \pm i\pi\,\delta(x-x') \qquad \text{(III-15)}$$

to find

$$\lim_{\epsilon'\to 0} k^2\epsilon(\underline{k}, kV+i\epsilon') = k^2 - R(\hat{\underline{k}}, V) - iI(\hat{\underline{k}}, V) \qquad \text{(III-16)}$$

where the real functions $R(\hat{\underline{k}}, V)$ and $I(\hat{\underline{k}}, V)$ are

$$R(\hat{\underline{k}}, V) = \sum_{a} \omega_{pa}^{\;2}\, P\int d^3v\, \frac{\hat{\underline{k}}\cdot[\partial f_a(\underline{v})/\partial(\underline{v})]}{\hat{\underline{k}}\cdot\underline{v} - V} \qquad \text{(III-17)}$$

and

$$I(\hat{\underline{k}}, V) = \sum_{a} \omega_{pa}^{\;2}\,\pi\int d^3v\, \delta(\hat{\underline{k}}\cdot\underline{v} - V)\hat{\underline{k}}\cdot\frac{\partial f_a(\underline{v})}{\partial(\underline{v})} \qquad \text{(III-18)}$$

and P denotes a principal-value integration. Because $\epsilon(\underline{k}, \omega)$ is analytic for $\mathrm{Im}\,\omega > 0$, $R(\hat{\underline{k}}, V)$ and $I(\hat{\underline{k}}, V)$ contain all the information that $\epsilon(\underline{k}, \omega)$ contains. They appear very frequently throughout our work. Notice that the symmetry relations

$$R(-\hat{\underline{k}}, -V) = R(\hat{\underline{k}}, V) \quad \text{and} \quad I(-\hat{\underline{k}}, -V) = -I(\hat{\underline{k}}, V) \qquad \text{(III-19)}$$

follow directly from the defining equations.

It is convenient to introduce the function

$$F(V; \hat{\underline{k}}) = \sum_{a} \omega_{pa}^{\;2}\int d^3v\, \delta(V - \hat{\underline{k}}\cdot\underline{v})f_a(\underline{v}) \qquad \text{(III-20)}$$

in terms of which

$$R(\hat{\underline{k}}, V) = P\int\frac{dV'}{V'-V}\,\frac{\partial F(V'; \hat{\underline{k}})}{\partial V'} \qquad \text{(III-21)}$$

and

$$I(\hat{\underline{k}}, V) = \pi\,\frac{\partial F(V; \hat{\underline{k}})}{\partial V} \quad . \qquad \text{(III-22)}$$

Notice that $F(V; \hat{\underline{k}})$ is a weighted sum of the projections of the velocity distributions onto the $\hat{\underline{k}}$ direction.

C. Longitudinal Plasma Waves

Landau first suggested that, for each $\underline{k}$, the inverse Laplace transform of expressions like Eq. (III-12) be evaluated as follows.[18] The functions of ω, including $\epsilon(\underline{k}, \omega)$, are defined by analytic continuation even for negative values of $\mathrm{Im}\,\omega$. The contour C in Eq. (III-9) is moved downward ($\mathrm{Im}\,\omega$ decreasing) and deformed so that each singularity that is encountered yields an explicit contribution. This leads directly to the interpretation of each solution of $\epsilon(\underline{k}, \omega) = 0$ as a mode of the plasma that depends only upon the unperturbed plasma.

These modes are not necessarily normal modes of the plasma in the sense of Van Kampen.[19] The normal modes are defined so that both $\underline{E}(\underline{r}, t)$ and $\delta f_{\alpha}(\underline{r}, \underline{v}, t)$ behave as $\exp i(\underline{k} \cdot \underline{r} - \omega t)$. The modes found by the Landau procedure correspond to normal modes when they are exponentially growing in time ($\mathrm{Im}\,\omega > 0$), but otherwise they do not. The reason is that δf_{α} also has a term with time behavior--$\exp(-i\underline{k} \cdot \underline{v}t)$. We will use the term "mode" to describe those found by the Landau procedure.

The highly damped modes (with $\mathrm{Im}\,\omega \lesssim - |\mathrm{Re}\,\omega|$) are very sensitive to the distribution functions $f_{\alpha}(\underline{v})$. This, plus the circumstance that they cannot be understood on any simply physical picture, limits their usefulness.

The weakly damped or slowly growing modes (with very small $|\mathrm{Im}\,\omega|$) can be discussed and understood in more detail. In Appendix B we write $\omega = kV + i\gamma$ where V and γ are real, and we obtain an expression for $k^2 \epsilon(\underline{k},\omega)$ by Taylor expanding in powers of γ/k. Separation of the real and imaginary parts of $k^2 \epsilon(\underline{k}, \omega) = 0$ then yields

$$k^2 - R(\hat{\underline{k}},V) + \mathcal{O}(\gamma/k) = 0 \qquad \text{(III-23)}$$

and

$$-I(\hat{\underline{k}}, V) - \frac{\gamma}{k} \frac{\partial}{\partial V} R(\hat{\underline{k}},V) + \mathcal{O}(\gamma^2/k^2) = 0, \qquad \text{(III-24)}$$

where $\mathcal{O}(x)$ represents terms that approach zero as fast or faster than x. When $|\gamma/k|$ is very small, $\mathrm{Re}\,\omega = kV$ is determined by the dispersion relation

$$k^2 = R(\hat{\underline{k}}, V) \qquad \text{(III-25)}$$

and $\mathrm{Im}\,\omega = \gamma$ is determined from

$$\gamma = - \frac{k\, I(\hat{\underline{k}}, V)}{\partial R(\hat{\underline{k}}, V)/\partial V} . \qquad \text{(III-26)}$$

Any mode for which these expressions are approximately correct we call a plasma wave.

1. Electron Plasma Waves

In Appendix B we obtain the asymptotic expansion for the contribution of species α to $R(\hat{\underline{k}}, V)$

$$R_\alpha(\hat{\underline{k}}, V) = \frac{\omega_{p\alpha}^2}{V^2}\left[1+2\left\langle v_{||}\right\rangle_\alpha V^{-1}+3\left\langle v_{||}^2\right\rangle_\alpha V^{-2}+\cdots\right] \quad \text{(III-27)}$$

where

$$\left\langle v_{||}^n\right\rangle_\alpha = \int d^3v\, f_\alpha(\underline{v})(\hat{\underline{k}}\cdot\underline{v})^n . \quad \text{(III-28)}$$

This series ordinarily does not converge and so is useful only when V is large. In terminating the series, one must use care to preserve Galilean invariance.

We use a reference frame in which $\left\langle v_{||}\right\rangle_e$ vanishes, and we consider waves with phase speeds V very much larger than the speeds of nearly all ions and large enough that Eq. (III-27) can be used for the electrons. For the ions we use only the first term of Eq. (III-27), so the dispersion relation Eq. (III-25) becomes

$$(kV)^2 = \sum_\alpha \omega_{p\alpha}^2 + 3\left\langle v_{||}^2\right\rangle_e \omega_{pe}^2 V^{-2}+\cdots, \quad \text{(III-29)}$$

This expression is valid only when $V^2 >> \left\langle v_{||}^2\right\rangle_e$ so that the second term on the right side is only a small correction to the lowest order result

$$(kV)^2 \cong \sum_\alpha \omega_{p\alpha}^2 \cong \omega_{pe}^2 . \quad \text{(III-30)}$$

By using Eq. (III-30) in rewriting Eq. (III-29), we obtain the well-known result

$$(kV)^2 = \sum_\alpha \omega_{p\alpha}^2 + 3\left\langle v_{||}^2\right\rangle_e k^2 + \cdots \quad . \quad \text{(III-31)}$$

This result can be interpreted on the basis of moment equations and fluid concepts.[20,21] These electron plasma waves are primarily oscillations of the electrons, and the primary restoring force is the electric field produced by the perturbation in electron density. The ions also oscillate in response to the electric field, but the frequency is so high that the amplitudes of their oscillations are relatively small and they have little effect. Negative ions oscillate in phase with the electrons, and positive ions oscillate 180 deg out of phase with the electrons. The gradient in the electron pressure arising from one-dimensional adiabatic compression supplies a small additional restoring force that contributes the last term in Eq. (III-31). The justification for using one-dimensional adiabatic compression is provided by the kinetic-theory results given above.

We postpone our discussion of Eq. (III-26) for electron waves. Usually the approximation that $|\gamma/k|$ is small fails as V^2 decreases and becomes comparable to $\langle v_{||}^2 \rangle_e$, so Eq. (III-31) is ordinarily an adequate approximation of Eq. (III-25).

2. Ion Waves

We here restrict ourselves to a plasma with a single ion species of mass M and charge Ze, and we consider $f_e(\underline{v})$ to be Maxwellian with temperature θ_e in the reference frame in which $\langle v_{||} \rangle_i$ vanishes for all $\hat{\underline{k}}$. We find almost immediately from Eq. (III-21) that

$$R_e(\hat{\underline{k}}, V) = \frac{1}{2D_e^2} \, X\left(\frac{V}{a_e}\right) \qquad \text{(III-32)}$$

where D_e and a_e are defined as in Sec. II. B and

$$X(x) = -\frac{2}{\pi} P \int_{-\infty}^{\infty} du \, \frac{u}{u - x} \, e^{-u^2} . \qquad \text{(III-33)}$$

Here X(x) is an even function of x and is equivalent to the function Fried and Conte denote by Re Z′ (x, y= 0).[22] The series expansion

$$X(x) = -2 + 4x^2 - \frac{8}{3} x^4 + \cdots \qquad \text{(III-34)}$$

and the asymptotic expansion for large values of x

$$X(x) = \frac{1}{x^2} + \frac{3}{2x^4} + \cdots \qquad \text{(III-35)}$$

can be obtained from Eq. (III-33) or from Fried and Conte's book. Equation (III-35) is a special case of Eq. (III-27). The numerical results of Fried and Conte are shown in Fig. 1.

To consider ion waves, we suppose $|V|$ to be large compared to the speed of nearly all ions and very small compared to a_e. Then using the asymptotic expansion for the ion contribution and the series expansion for the electron contribution, we find

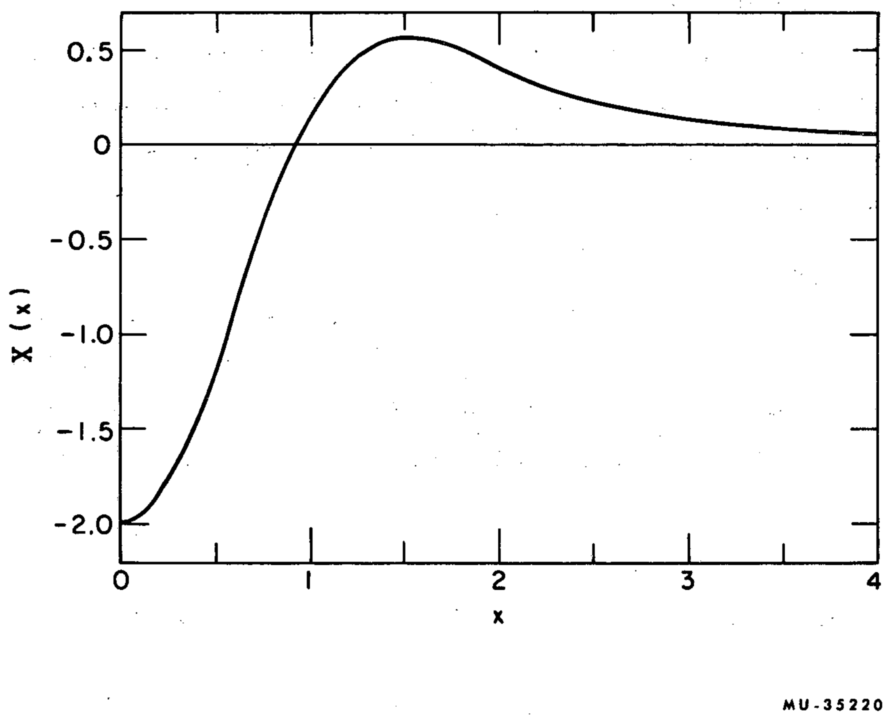


Fig. 1. The function X(x) defined by Eq. (III-33).

$$k^2 = \frac{\omega_{pi}^2}{V^2}\left[1 + 3\,\frac{\langle v_{||}^2\rangle_i}{V^2} + \cdots\right] - \frac{1}{D_e^2}\left[1 - 2\,\frac{V^2}{a_e^2} + \cdots\right]. \quad \text{(III-36)}$$

The lowest approximation is

$$k^2 = \frac{\omega_{pi}^2}{V^2} - \frac{1}{D_e^2}\,, \quad \text{(III-37)}$$

which can be rewritten as

$$V^2 = \frac{Z\,\theta_e}{M}\,\frac{1}{1 + k^2 D_e^2} \quad \text{(III-38)}$$

and is a well-known result.[20,21] The condition that V^2 be small compared with a_e^2 is very well satisfied. However, the condition that V^2 be large compared with $\langle v_{||}^2\rangle_i$ is not satisfied unless $\langle v_{||}^2\rangle_i$ is unusually small (or else Z is very large). For example, if the ion-velocity distribution is also Maxwellian, we must have $Z\theta_e$ very large compared to $2\theta_i$ and even then kD_e must not be too large.

Consideration of the damping or growth from Eq. (III-26) usually shows that the condition that $|\gamma/k|$ be small is not satisfied unless V^2 is somewhat larger than $\langle v_{||}^2\rangle_i$. Still, one should probably treat the ion contribution to Eq. (III-36) somewhat more carefully, because it tends to be cancelled by the electron contribution.

These results can also be discussed with moment equations and fluid concepts.[21] When the wavelength is short compared to D_e so that the frequency is near its maximum $(kV)^2 = \omega_{pi}^2$, the waves are oscillations of the ions that correspond almost exactly to the electron plasma waves; this is particularly apparent from a comparison of Eqs. (III-29) and (III-36). The electrons have little effect beyond providing a neutralizing uniform background. As the wavelength becomes longer and the frequency becomes smaller, the electrons tend to neutralize the charge-density perturbation of the ions and to provide an additional restoring force through the resulting electron pressure gradient. In fact for $k^2 D_e^2 << 1$ we have $V^2 = Z\theta_e/M$. This result follows immediately if one assumes that the electron and ion density perturbations are maintained nearly equal by the coupling electric field, the inertia is provided by the ions, and the restoring force is the electron-pressure gradient arising from isothermal compression. The justification of assuming isothermal compression is provided by our above results.

As k and ω become very small, the above description becomes incorrect because collisions become important. In fact, the ion waves will eventually become acoustic waves in a collision-dominated plasma. For this reason they are often called ion acoustic waves.

When several species of ions are present, there can be a variety of low-frequency ion waves. We will not discuss this case.

3. Stability and Landau Damping

The stability of the uniform Coulomb plasma according to the linearized Vlasov equations has been studied in some detail by Penrose,[23] who expresses his results in terms of $F(V;\hat{\underline{k}})$ defined in Eq. (III-20). He proves that when $F(V;\hat{\underline{k}})$ is a sufficiently smooth function of V, the plasma is unstable and $\epsilon(\underline{k}, \omega) = 0$ has roots with $\mathrm{Im}\,\omega > 0$ if, and only if, there is a minimum of $F(V;\hat{\underline{k}})$ as a function of V where $R(\hat{\underline{k}}, V) > 0$. The exception he cannot treat in detail occurs when $\epsilon(\underline{k}, \omega) = 0$ has roots with $\mathrm{Im}\,\omega = 0+$ but none with $\mathrm{Im}\,\omega > 0$, which we call the marginally stable case. Penrose's criterion allows us to determine stability or instability directly from $R(\hat{\underline{k}}, V)$ and $I(\hat{\underline{k}}, V)$, but it gives no direct information about the modes and waves involved.

One result that follows directly is that no plasma with isotropic velocity distributions can be unstable because in this case $F(V;\hat{\underline{k}})$ has no minimum.

To supplement these results, we consider the growth or damping of waves satisfying $k^2 = R(\hat{\underline{k}}, V)$ as given by

$$\gamma = -\frac{k\, I(\hat{\underline{k}}, V)}{\partial R(\hat{\underline{k}}, V)/\partial V} \qquad \text{(III-26)}$$

when $|\gamma/k|$ is sufficiently small. When the plasma is unstable, the Penrose criterion guarantees that the plasma can support a wave with $\gamma = 0$ and neighboring waves with γ very small and positive. When γ is negative, the wave is said to be Landau damped.

Dawson has provided a physical interpretation of wave growth or Landau damping as given by Eq. (III-26).[24] He shows that the extra energy per unit volume in the plasma due to a wave of amplitude E_1 is given by

$$W_w = \frac{E_1^2}{8\pi}\left[\frac{-V}{2k^2}\frac{\partial R(\hat{k}, V)}{\partial V}\right]. \qquad \text{(III-39)}$$

He also shows that the rate of energy change of the particles that have $\underline{k}\cdot\underline{v}$ about equal to V, and so move nearly in resonance with the waves, is

$$\frac{dW_{res}}{dt} = \frac{E_1^2}{8\pi}\left[\frac{-V}{k} I(\hat{k}, V)\right] \qquad \text{(III-40)}$$

per unit volume. If the wave amplitude varies as $e^{\gamma t}$, energy conservation yields

$$\frac{dW_w}{dt} = 2\gamma W_w = -\frac{dW_{res}}{dt} \qquad \text{(III-41)}$$

and substitution of Eqs. (III-39) and (III-40) yields Eq. (III-26) for γ. Notice that if we consider the same wave from a reference frame moving with a different velocity, the magnitude and the sign of V (and therefore of W_w and dW_{res}/dt) may be different but γ is unchanged.

We see that the growth or Landau damping of a wave arises from its interaction with resonant particles, which may gain or lose energy. This wave-particle interaction can be described as the absorption or induced emission of waves by particles. Depending upon the unperturbed plasma, the absorption may exceed the induced emission, giving Landau damping, or the induced emission may dominate, making the wave grow (in analogy with a LASER). Other descriptions based upon "phase mixing" or particle "bunching" are also used and are probably more appropriate in treating rapidly growing or highly damped modes.[25, 19]

In the remainder of this section we consider our results in various special cases.

D. The Two-Component Displaced-Maxwellian Plasma

We include ions of mass M, charge Ze, and with a Maxwellian velocity distribution with temperature θ_i. We define the ion Debye length and the ion thermal speed by

$$D_i^2 = \frac{\theta_i}{4\pi n_i Z^2 e^2} \qquad A^2 = \frac{2\theta_i}{M} \tag{III-42}$$

and we refer to the coordinate system moving with the mean velocity of the ions as the ion frame.

We make similar definitions and assumptions for the electrons, and we assume that the electron frame moves with velocity $\underline{U}$ relative to the ion frame. We denote the electron thermal speed by $a = a_e$.

Fried and Gould have reported and discussed numerical solutions of $\epsilon(\underline{k}, \omega) = 0$ for this case, including some of the highly damped modes.[26] However, we simply illustrate some of our previous discussion and obtain certain results that are useful in Sec. III. 3 and throughout the rest of the report by considering the plasma waves.

In the ion frame, we have from Eqs. (III-21), (III-22), and (III-32) that

$$R(\hat{\underline{k}}, V) = \frac{1}{2D_i^2} X\left(\frac{V}{A}\right) + \frac{1}{2D_e^2} X\left(\frac{V - U_{||}}{a}\right) \tag{III-43}$$

and

$$I(\hat{\underline{k}}, V) = -\frac{\sqrt{\pi}}{D_i^2} \frac{V}{A} \exp\left[\frac{-V^2}{A^2}\right] - \frac{\sqrt{\pi}}{D_e^2} \frac{V - U_{||}}{a} \exp\left[-\frac{(V - U_{||})^2}{a^2}\right]. \tag{III-44}$$

These are the basis of our entire discussion. Notice that the dependence on $\hat{\underline{k}}$ and $\underline{U}$ is only through

$$U_{||} = \hat{\underline{k}} \cdot \underline{U}. \tag{III-45}$$

1. With No Relative Drift

With $\underline{U} = 0$, all results are independent of $\hat{\underline{k}}$ if we work in the common ion and electron frame.

We need only consider the dispersion relation $k^2 = R(V)$ for nonnegative V. From the behavior of X(x) shown in Fig. 1, we see that R(V) is positive for V larger than a value of order a_e, which is the region that could contain electron plasma waves. In this region I(V) is small only when V^2 is much larger than a_e^2, so only under this condition are electron waves possible. The dispersion relation $k^2 = R(V)$ then yields the result we found before.

If $Z\theta_e > 3.5\,\theta_i$, then $R(V)$ is also positive for V^2 somewhat greater than A^2 but less than a value of order $Z\theta_e/M$. This is clearly the region that could contain ion waves. However, I(V) is small only when V^2 is large compared to A^2, so only under this condition are ion waves possible. Again, the dispersion relation is as we obtained before.

For both of these types of waves $\partial R(V)/\partial V$ and I(V) are both negative so, from Eq. (III-26), γ is negative for all waves. This, of course, agrees with the Penrose criterion since the plasma is isotropic. As V becomes much greater than a_e, I(V) decreases exponentially, so the Landau damping rate of the electron plasma wave decreases exponentially. However, when the damping time becomes comparable to the collision time, collisions will have an effect upon the actual damping rate.

2. With Equal Temperatures and Z = 1

In this case there are no ion waves when $\underline{U} = 0$. The effect of $Z \gg 1$ would be similar to the effect of $\theta_e \gg \theta_i$, (which we consider in Sec. III. D. 3) because both allow ion waves when $\underline{U} = 0$.

To apply the Penrose criterion we consider $\hat{\underline{k}}$ parallel to $\underline{U}$. From Eq. (III-44) we find that $F(V;\hat{\underline{k}})$ has a minimum if, and only if, U is large enough that

$$\frac{V'}{A} = \frac{U - V'}{a} > \frac{1}{\sqrt{2}} \qquad \text{(III-46)}$$

has a solution V', and the minimum is then at $V = V'$. At this minimum

$$R(\hat{\underline{k}},\ V') = \frac{1}{D_e^2}\, X\!\left(\frac{V'}{A}\right), \qquad \text{(III-47)}$$

and so $R(\hat{\underline{k}}, V')$ is positive when V'/A is larger than 0.925, the zero of $X(V'/A)$. Thus we find that the plasma is unstable if, and only if,

$$|\underline{U}| > U_{crit} = 0.925(a+A) = 0.925a\left(1 + \sqrt{\frac{m}{M}}\right). \qquad \text{(III-48)}$$

The first waves to begin to grow in this case cannot be classified as electron or ion waves because they have speeds of order A relative to the ion frame and of order a_e relative to the electron frame. Since $\partial R(\hat{\underline{k}}, V)/\partial V$ is positive in this region, the growing waves are those for which $R(\hat{\underline{k}}, V)$ is positive and $I(\hat{\underline{k}}, V)$ is negative. From Eq. (III-44) we see that the ion contribution to $I(\hat{\underline{k}}, V)$ is negative whereas the electron contribution is positive. Therefore we may say that the waves grow when the growth caused by the ions overcomes the damping by the electrons.

This is best classified as an example of a two-stream instability. Until $|\underline{U}|$ becomes comparable to U_{crit}, the electron plasma waves are not greatly affected in the electron frame.

Notice that if the temperatures were slightly unequal or if $Z \neq 1$, the quantitative analysis would be much more difficult.

3. Ion Waves in an Electron-Proton Plasma

Here we discuss the case $\theta_e \gg \theta_i$ in some detail because the results apply also in Sec. III. E. As $|\underline{U}|$ is increased from zero, certain ion waves will begin to grow when $U_{||}$ is still small compared to a_e, so the electron waves are hardly affected in the electron frame.

We work in the ion frame. As long as V^2 and U^2 are very small compared to a_e^2, the contribution of the electrons to $R(\hat{\underline{k}}, V)$ remains very nearly $-D_e^{-2}$, so the dispersion relation $k^2 = R(\hat{\underline{k}}, V)$ is nearly independent of $U_{||}$. In particular the region $V_{min} < V < V_{max}$, where $R(\hat{\underline{k}}, V)$ is positive, is very insensitive to U and is determined by $X(V/A) > -2\theta_i/\theta_e$. In the same approximation we have

$$\frac{D_i^2}{\sqrt{\pi}} I(\hat{\underline{k}}, V) = \frac{\theta_i}{\theta_e} \frac{U_{||} - V}{a} - \frac{V}{A} \exp(-V^2/A^2) . \qquad \text{(III-49)}$$

When $U_{||} = 0$, $I(\hat{\underline{k}}, V)$ is negative for positive V and all ion waves are damped. If $U_{||}$ exceeds a critical value U_{crit}, $I(\hat{\underline{k}}, V)$ will be positive somewhere within the region $V_{min} < V < V_{max}$ and the plasma will be unstable.

In an electron-proton plasma, two distinct cases occur in the application of the Penrose criterion. As $\underline{U}$ is increased with $\theta_e > 20\theta_i$, a minimum first appears in $F(V;\hat{k})$ for $\underline{k}$ parallel to $\underline{U}$ and at $V = V'$ with $V_{min} < V' < V_{max}$. In this case the plasma is unstable as soon as a minimum appears, so U_{crit} can be determined entirely from Eq. (III-49). Since $R(\hat{k}, V)$ is positive where the minimum appears, the first waves to grow have finite wavelength ($k \neq 0$).

With $\theta_e < 20\theta_i$, the minimum at $V = V'$ first appears with $V' > V_{max}$, and as $U_{||}$ increases further V' decreases. The plasma finally becomes unstable when, for $\hat{k}$ parallel to $\underline{U}$, V' decreases below V_{max}. In this case U_{crit} can be found from V_{max} and Eq. (III-49), and U_{crit} is quite sensitive to V_{max}. The first waves to begin growing have infinite wavelength ($k = 0$) because $R(\hat{k}, V_{max})$ vanishes.

Examples of the calculation of U_{crit} in both cases are given in Appendix C. The results are shown in Fig. 2, including the point at $\theta_e = \theta_i$ as determined in Sec. III. D. 2. The complete results for arbitrary θ_e/θ_i are given by Fried and Gould.[26] Notice that for $\theta_e > 20\theta_i$, U_{crit} is between three and four times the ion thermal speed A.

In both cases, when $U_{||}$ exceeds U_{crit}, the growing waves have V in the region where $F(V;\hat{k})$ slopes upward and $V_{min} < V < V_{max}$. These ion waves grow because the Landau damping by the ions is overcome by the effects of the electrons.

E. Waves in the Spitzer-Härm Problem

In the notation of the last section, the electron velocity distribution in the Spitzer-Härm problem is

$$f_e(\underline{v}) = \frac{1}{\pi^{3/2} a^3}\, e^{-V^2/a^2} + f_e^{(1)}(v)\cos\alpha, \qquad \text{(III-50)}$$

where α is the angle between $\underline{v}$ and $\underline{E}_0$, and $f_e^{(1)}(v)$ was calculated by Spitzer and Härm.[1] We here take the ions to be protons with a Maxwellian velocity distribution of temperature θ_i.

We show in Appendix G that in this case

$$I(\hat{k}, V) = I_0(V) + I_1(V)\cos\theta \qquad \text{(III-51)}$$

where θ is the angle between $\underline{k}$ and $\underline{E}_0$. Here $I_0(V)$ is given by Eq. (III-44) with $U_{||} = 0$ and

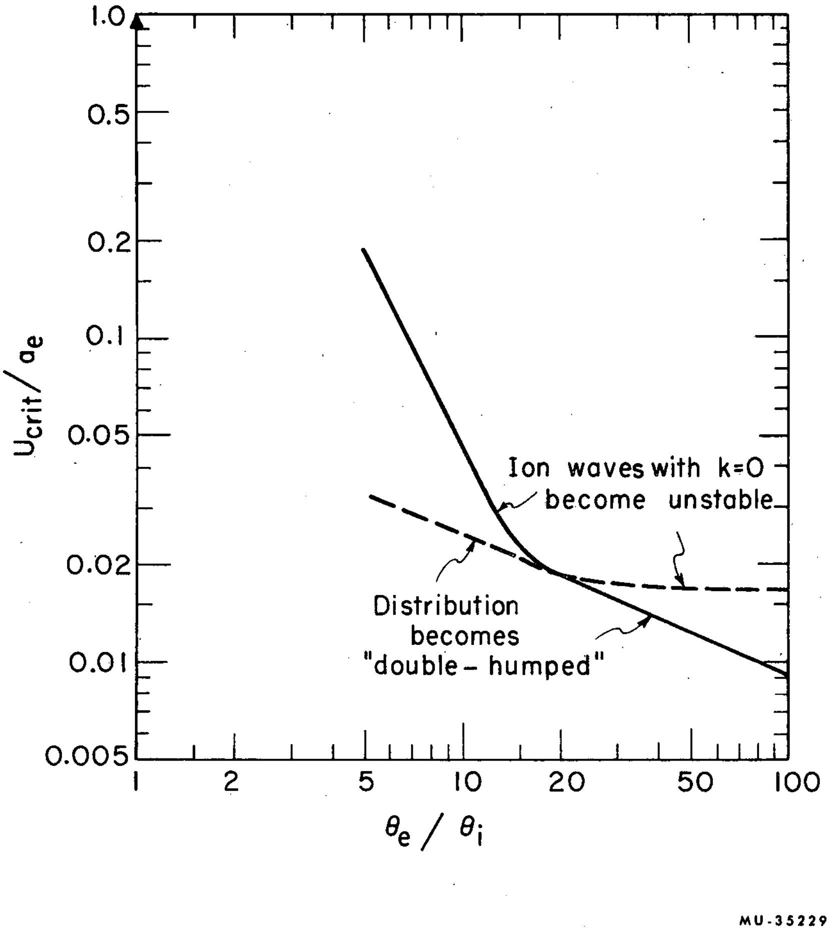


Fig. 2. The relative drift U_{crit} above which a displaced Maxwellian electron-proton plasma becomes unstable (solid line). The lower, nearly straight line shows the value of U at which $F(V;\hat{k})$ develops a minimum. The upper, curved line shows the value of U at which $I(\underset{\sim}{k}, V)$ first vanishes with $V = V_{max}$.

$$I_1(V) = 2\pi^2\omega_{pe}^{\ 2}\left[\int_V^\infty f_e^{(1)}(v)dv - Vf_e^{(1)}(V)\right] . \qquad \text{(III-52)}$$

As long as E_0 is small compared with E_{run}, the dispersion relation for electron plasma waves, Eq. (III-31), remains essentially unchanged in the reference frame where $\langle v_{||}\rangle_e$ vanishes for all $\hat{\underline{k}}$. However, because $f_e^{(1)}(v)$ decreases more slowly than the Maxwellian as v becomes large, there will always be a value V beyond which $|I_1(V)|$ exceeds $|I_0(V)|$ and certain electron plasma waves would grow. However, the linearization of the Landau kinetic equation is not justified where $f_e^{(1)}(v)$ is larger than the Maxwellian part, so this result cannot be taken seriously. Still, it indicates the difficulty caused by runaway electrons.

The presence of runaway electrons constitutes a two-stream situation in which certain high-speed electron plasma waves will grow. These waves are likely to have very small growth rates, and they can interact only with very fast electrons, so their effects are probably not important. This is fortunate because they would be very difficult to treat in detail.

In discussing the ion waves, we compare the results with those of the displaced-Maxwellian problem in an electron-proton plasma as discussed in Sec. III. D. In that case we can write

$$f_e(\underline{v}) = \frac{1}{\pi^{3/2}a^3}\, e^{-V^2/a^2} + f_e^{(2)}(v)\cos\alpha \qquad \text{(III-53)}$$

to a good approximation, where $f_e^{(2)}(v)$ is known from Taylor expansion of the displaced Maxwellian. Because the ion waves have such low speeds V, we have to a good approximation in both problems

$$k^2 = R_0(V), \qquad \text{(III-54)}$$

where $R_0(V)$ is given by Eq. (III-43) with $U_{||} = 0$, and

$$I(\hat{\underline{k}},\ V) = I_0(V) + I_1(0)\cos\alpha . \qquad \text{(III-55)}$$

To this approximation, all of the properties of the ion waves are the same in the two problems if the single number $I_1(0)$ is the same. This would require that

$$\int_0^\infty f_e^{(1)}(v)dv = \int_0^\infty f_e^{(2)}(v)dv. \qquad \text{(III-56)}$$

In both problems Eq. (A-7) of Appendix A must be satisfied, and by substituting into it Eqs. (III-50) and (III-53) we find that Eq. (III-56) actually is satisfied if the same $\underline{E}_0$ applies to both problems, and in fact

$$I_1(0) = -\frac{\sqrt{\pi} e E_0}{D_e^{\,2} \nu_c ma} \, . \qquad \text{(III-57)}$$

If we relate $\underline{E}_0$ to $\underline{U}$ by the displaced-Maxwellian conductivity, we recover Eq. (III-49).

We thus conclude that if the drift $\underline{U}$ in the displaced-Maxwellian problem is related to $\underline{E}_0$ by the displaced-Maxwellian linear conductivity, the ion waves have very nearly the same properties as in the Spitzer-Härm problem. This has not been pointed out before, although it is evident in the results of Bernstein and Kulsrud.[27]

The results found in Sec. III. D. 3, including Fig. 2, now apply directly to the Spitzer-Härm problem if we replace $\underline{U}/a$ by $-0.5064\underline{E}_0/E_{run}$. In particular we define

$$\frac{E_{crit}}{E_{run}} = \frac{1}{0.5064} \frac{U_{crit}}{a} \, . \qquad \text{(III-58)}$$

We see from Fig. 2 that when θ_e much exceeds θ_i, ion waves would grow according to the Spitzer-Härm model even when the linearization of the Landeau equation would still be justified because $E_0 << E_{run}$. Because the ion waves can interact with nearly all electrons, the wave-particle interactions would be important and the Landau equation would be inadequate.

The brief discussion of Ohmic heating in Appendix D indicates that θ_e is actually not likely to much exceed θ_i unless E_0 is $\geq 0.1E_{run}$, the ions are cooled by some process, or the electrons are heated by some process in addition to Ohmic heating.

IV. THE LENARD-BALESCU KINETIC EQUATION

We use the terms "collisons" and "ordinary collisions" to describe the effects included in the Landau equation as discussed in Sec. II. In order to study the case $E_{crit} < E_0 < E_{run}$, we use the Lenard-Balescu kinetic equation (the L-B equation), which includes the effects of collisions and of wave-particle interactions.[28, 29] In this chapter we discuss the L-B equation and its limitations.

A. Physical Processes of Possible Importance

In the spatially uniform classical Coulomb plasma we study, the particle motion and the velocity-distribution functions can be altered only by electric fields. These fields consist of the uniform $\underline{E}_0$ and of a fluctuating electric field with vanishing ensemble average. Part of this fluctuating field arises from the particle individuality and is responsible for direct particle-particle interactions or collisions. Another part involves the dielectric properties of the plasma and so is associated with the possibility of wave propagation; this part is responsible for the effects of wave-particle interactions.

The fluctuating electric fields associated with waves may be altered in various ways. Even if the actual plasma were a continuum as represented by the Vlasov equation, a wave could be altered by:

(a) Growth or Landau damping as described before. This we have chosen to interpret as induced emission or absorption by individual particles, although it appears in the Vlasov equation;

(b) Direct interactions with other waves through the nonlinearity of the plasma, which we call wave-wave interactions or mode coupling;

(c) Time variations in the dielectric properties of the plasma.

In addition, because the actual plasma consists of discrete particles, the fluctuating electric fields associated with waves are altered by:

(d) Changes in the propagation and particularly the damping of waves, which we call collisional effects upon the waves;

(e) Spontaneous emission of waves by individual particles. If spontaneous emission were not present in a stable plasma, the fluctuations associated with waves would decay to zero.

This classification scheme may not be complete, but it forms a useful basis for discussion.

1. Quasi-Linear Theories

These theories are usually derived from the Vlasov equations, so they include neither the effects of collisions upon the waves and the velocity-distribution functions nor the effects of spontaneous emission by the particles. These approximations are not valid in our problem, so we discuss and interpret only briefly the simplest of these theories and the results found with $\underline{E}_0 = 0$.[30, 31]

The slowly varying, spatially uniform part $f_a(\underline{v}, t)$ of each velocity distribution is separated, and the remainder is represented as modes that propagate, according to the linearized Vlasov equations, through this spatially uniform medium. The plasma is assumed to have no rapidly growing modes, and all highly damped modes are assumed to have small amplitudes and to be of no importance. The modes considered, then, are the waves, which have phase speeds $V(\underline{k}, t)$ and growth (or damping) rates $\gamma(\underline{k}, t)$ that vary slowly in time as the spatially uniform parts of the velocity distributions, and therefore the Vlasov dielectric funtion, vary slowly in time. These changes are assumed to be slow enough that the amplitude of a wave can be found by an adiabatic or WBK approximation. The energy $\mathcal{E}_{\underline{k}}$ in a wave is thus assumed to vary as

$$\frac{d\mathcal{E}_{\underline{k}}(t)}{dt} = 2\gamma(\underline{k}, t)\, \mathcal{E}_{\underline{k}}(t), \qquad \text{(IV-1)}$$

which can be integrated from some initial time if $\gamma(\underline{k}, t)$ is known. We see that mode coupling in the sense discussed before is not included, and that the effects of time variations in the dielectric properties of the plasma are treated in a simplified manner.

The effects of these waves upon the spatially uniform parts of the velocity distributions are treated with the nonlinear Vlasov equations and yield

$$\frac{\partial f_a(\underline{v}, t)}{\partial t} = \frac{\partial}{\partial \underline{v}} \cdot \left[\underset{\approx}{D}_a(\underline{v}, t) \cdot \frac{\partial f_a(\underline{v}, t)}{\partial \underline{v}} \right] \qquad \text{(IV-2)}$$

where the coefficients $\underset{\sim}{D}_a$ for diffusion in velocity space involve integrals (or sums) over $\underline{k}$. The integrands involve a known factor that multiplies $\mathcal{E}_{\underline{k}}(t)\,\delta[\underline{k}\cdot\underline{v} - kV(\underline{k},t)]$ where $\mathcal{E}_{\underline{k}}(t)$ and $V(\underline{k},t)$ must be determined as discussed above.

A quantum-mechanical derivation leads directly to the interpretation that waves are emitted and absorbed by individual particles, as we have suggested.[31] The δ function above indicates that only resonant particles interact strongly with a wave.

This quasi-linear theory is useful only when $\mathcal{E}_{\underline{k}}$ is large compared to the level of thermal fluctuations (so that spontaneous emission is not important), but small enough that $f_a(\underline{v},t)$ changes slowly and mode coupling can be ignored. This theory can be used only to treat variations on time scales short compared with the collision time ν_c^{-1}.

The theory has usually been applied to one-dimensional problems; in fact, there is some controversy concerning its applicability to three-dimensional problems. The result found in certain examples is that the system evolves towards a steady state in which $\mathcal{E}_{\underline{k}}(t)$ either vanishes or is constant in time.[30] Some waves grow or decay until they reach a constant amplitude and others decay entirely, but none continues to grow indefinitely. Thus in these one-dimensional examples that are initially unstable according to linearized theory, the nonlinear effect of wave-particle interactions eventually makes the plasma at least marginally stable, according to quasi-linear theory.

Recently these theories have been extended to include some of the effects of mode coupling.[32, 33]

2. The Equations of Field and Fried

Although they were not based upon the Vlasov equations, the equations developed by Field and Fried have the same structure, physical content, and limitations as those of the quasi-linear theory discussed above.[34] With a uniform field $\underline{E}_0$, the time derivative on the left of Eq. (IV-2) is replaced by

$$\frac{\partial}{\partial t} + \frac{q_a}{m_a}\,\underline{E}_0 \cdot \frac{\partial}{\partial \underline{v}}$$

as one might expect and, when $E_0 << \Lambda E_{run}$, no other modifications are necessary.

Field and Fried apply their equations to the problem of an electron-proton plasma with $E_0 >> E_{run}$ so the effects of collisions can be ignored. Initially the velocity distributions are Maxwellian with $\theta_i << \theta_e$, and the wave amplitudes correspond to the level of thermal fluctuations. Only the ion waves are included in their equations, and the behavior of the plasma in time is evaluated numerically. As the particles accelerate nearly freely in $\underline{E}_0$, the plasma becomes unstable almost immediately and some of the ion waves begin to grow. After a time of 10^3 to $10^4 \omega_{pi}^{-1}$ in their examples, the ion-wave amplitudes become so large that wave-particle interactions have important effects upon the velocity distribution of the electrons. The electrical current ceases to increase linearly in time and drops to a relatively low value. Some of the ion waves that originally grew become damped although other ion waves begin to grow. Field and Fried make certain simplifications that prevent them from accurately calculating the behavior at longer times. They believe that the assumptions of quasi-linear theory, which include the neglect of mode coupling, are justified throughout their problem.

In three-dimensional problems like this, the effects of wave-particle interactions are often similar to, and act in addition to, those of ordinary collisions. In this problem, these interactions yield a finite electrical conductivity in the absence of collisions; this possibility was first suggested by Buneman.[35] These interactions can greatly increase the diffusion of a plasma across a confining magnetic field.[36, 37] Several other examples are given in Sec. V.

3. The Lenard-Balescu Equation

The L-B equation differs in several respects from the equations of quasi-linear theories: (a) It includes the effects of collisions upon the velocity distributions but not upon the waves; (b) it includes spontaneous emission but does not include mode coupling; (c) the velocity-distribution functions are assumed to vary slowly compared with rates of emission and absorption of waves, so the amplitudes of fluctuations associated with waves are determined as a balance of spontaneous emission and absorption. The ensemble average of these amplitudes then varies only slowly as the particle-velocity distributions vary.

A basic limitation of the L-B equation is that it is meaningful only for a stable plasma, because in a marginally stable or unstable plasma the spontaneous emission cannot be balanced by absorption, for certain waves. Here stability is as determined from the linearized Vlasov equations.

The L-B equation is valid on a time scale of order or longer than the collision time ν_c^{-1}, which can be an advantage or a disadvantage in comparison with quasi-linear theories.

In Secs. IV. B and IV. C we outline two derivations of the L-B equation that will help to clarify its physical content and limitations as outlined above.

B. Derivation by Superposition of Test Particles

This method was suggested by Hubbard and Thompson.[38, 39, 40] It has been utilized and rigorously justified by Rostoker,[41, 42] but we do not give the full justification here. We consider $\underline{E}_0 = 0$ because, as we verify in Sec. IV. C, the effect of $\underline{E}_0$ can be inserted just as it was in the equations of Field and Fried.

The word description we give will usually apply to the fluctuations associated with waves. The L-B equation contains the effects of ordinary collisions as well, but the discussion of these is much like that in Sec. II.

1. The Test-Particle Problem

As a special case of the plasma response to a perturbing charge as given by Eq. (III-12), we consider $\rho_0(\underline{r}, t)$ to be a particle of charge q moving on the trajectory $\underline{r}'(t) = \underline{v}'t + \underline{r}_0$ with constant velocity $\underline{v}'$. Then

$$\underline{E}(\underline{k}, \omega) = \frac{4\pi q \underline{k}}{k^2 \epsilon(\underline{k}, \omega)} \; \frac{e^{-i\underline{k} \cdot \underline{r}_0}}{\omega - \underline{k} \cdot \underline{v}'} \qquad \text{(IV-3)}$$

where, because the uniform unperturbed plasma is assumed to be stable, the only pole with $\mathrm{Im}\,\omega \geq 0$ is the one at $\omega = \underline{k} \cdot \underline{v}'$. Evaluation of the inverse Laplace transform for very large t therefore yields

$$\underline{E}(\underline{k}, t \rightarrow \infty) = -i \frac{4\pi q \underline{k}}{k^2 \epsilon(\underline{k}, \underline{k} \cdot \underline{v}')} e^{-i\underline{k} \cdot \underline{r}'(t)} . \qquad \text{(IV-4)}$$

For sufficiently large t, the ensemble average or Vlasov field produced by the test particle is thus

$$\underline{E}(\underline{r} | q, \underline{r}'(t), \underline{v}') = \frac{q}{2\pi^2} \int d^2\hat{\underline{k}} \int_0^\infty k^3 dk \int_{-\infty}^{\infty} dV \frac{-i\hat{\underline{k}}\delta(V - \hat{\underline{k}} \cdot \underline{v}') e^{i\underline{k} \cdot [\underline{r} - \underline{r}'(t)]}}{k^2 - R(\hat{\underline{k}}, V) - iI(\hat{\underline{k}}, V)} . \qquad \text{(IV-5)}$$

This result depends upon $\underline{r}$, $\underline{r}'$, and t only through the combination $\underline{r} - \underline{r}'(t)$, and so represents a pattern that moves with the test particle and depends upon $\underline{v}'$. The test particle is often said to be "dressed" by the "shielding cloud" the plasma forms around it.

When $|\underline{r} - \underline{r}'|$ is small compared to D_e, only large k contribute significantly, so the plasma has little effect and the electric field is nearly that of the bare test particle. For $|\underline{r} - \underline{r}'|$ comparable to or larger than D_e, the shielding by the plasma is important. Where $|\underline{r} - \underline{r}'|$ is large compared to D_e, the only large contributions come from $\hat{\underline{k}}$ and V under conditions that $V = \hat{\underline{k}} \cdot v'$ is satisfied and that $R(\hat{\underline{k}}, V)$ is positive and $|I(\hat{\underline{k}}, V)|$ is small. Under this condition the test particle can move in resonance with a slightly damped plasma wave.

We thus interpret the long-range parts of Eq. (IV-5) as arising from plasma waves that are excited by, or spontaneously emitted by, the test particle. This emission is similar to Čerenkov radiation in that the particle moves in resonance with the wave being excited.

During the transient period when t is small, this spontaneous emission is not entirely balanced by absorption (Landau damping) by the plasma, but when Eq. (IV-5) is correct, this balance occurs. This suggests that Eq. (IV-5) is valid if t is large compared to the damping time of the least-damped wave with which the test particle moves in resonance. Of course this time must be short in comparison with ν_c^{-1} or else the Vlasov equation is not adequate.

Because the linearized Vlasov equation was used, the possibility of mode coupling has not been included. There appears to be no simple criterion for judging the validity of this approximation.

As the test particle emits waves, its energy must change. The average force exerted on the test particle by the plasma is given by $\underline{E}(\underline{r}'|q, \underline{r}', \underline{v}')$ with the field of the test particle removed. To evaluate this we may average $\underline{E}(\underline{r}'+\underline{\Delta}|q, \underline{r}', \underline{v}')$ and $\underline{E}(\underline{r}'-\underline{\Delta}|q, \underline{r}', \underline{v}')$. By using the symmetry of $R(\hat{\underline{k}}, V)$ to combine these and then setting $\underline{\Delta} = 0$, we find

$$\underline{E}_{drag}(q, \underline{v}') = \frac{q}{2\pi^2} \int d^2\hat{\underline{k}} \int_0^{k_m} k^3 dk \int_{-\infty}^{\infty} dV \, \frac{\delta(V - \hat{\underline{k}} \cdot \underline{v}') \, \hat{\underline{k}} I(\hat{\underline{k}}, V)}{[k^2 - R(\hat{\underline{k}}, V)]^2 + I^2(\hat{\underline{k}}, V)} \quad . \tag{IV-6}$$

The rate of energy loss $q\underline{v}' \cdot \underline{E}_{drag}$ is proportional to the square of the charge but independent of the mass of the test particle.

The integral over k diverges logarithmically at large k so we have provided a cutoff at k_m. This divergence arises from a failure in the linearization of the Vlasov equations at large k or small distances. It actually is connected with the large deflection suffered by particles that pass sufficiently close to the test particle, and so is similar to the divergence at small impact parameters in the Landau form of the Fokker-Planck equation.[39] This suggests that if $|q|$ is of order e we should choose k_m of order b_{min}^{-1}. As before, this will introduce an uncertainty of order unity compared with $\ln \Lambda$.

2. Autocorrelation Function of the Electric Field

The fluctuations in the microscopic electric field in our spatially uniform plasma can be described by the autocorrelation function[41]

$$\underset{\approx}{C}(\underline{R}, \tau; t) = \left\langle \underline{E}(r, t) \, \underline{E}(\underline{r} + \underline{R}, t + \tau) \right\rangle \; . \tag{IV-7}$$

From Eq. (II-4) for $\underline{E}(\underline{r}, t)$ we see that this will involve the correlation of two particles, which in general is very difficult to evaluate. However, Rostoker has provided a connection between the two-particle correlation function and the results of test particle problems that is quite general when Λ is sufficiently large.[42] In the picture he develops, two particles are correlated only because the first is part of the shielding cloud of the second, the second is part of the shielding cloud of the first, and both are parts of the shielding cloud of every other particle. Thus each plasma particle is considered a "dressed" test particle.

Rostoker also derives expressions for quantities like $\underset{\sim}{C}(\underline{R}, \tau, t)$ directly in terms of the test-particle results. This simplification is very important because the task of actually evaluating the two-particle correlation functions from the equations he gives turns out to be formidable. In our case his prescription is

$$\underset{\sim}{C}(R,\tau;t) = \sum_{\alpha} n_{\alpha} \int d^3\underline{r}' \int d^3\underline{v}' \underline{E}(\underline{r} \mid q_{\alpha}, \underline{r}', \underline{v}') \underline{E}(\underline{r}+\underline{R} \mid q_{\alpha}, \underline{r}'+\underline{v}'\tau, \underline{v}') f_{\alpha}(\underline{v}', t) \, . \tag{IV-8}$$

This result is interpreted as treating each particle as a "dressed" test particle that is otherwise uncorrelated with the other particles and moves with constant velocity $\underline{v}'$.[42]

We can evaluate Eq. (IV-8) directly by substitution of Eq. (IV-5). By evaluating the integral over $\underline{r}'$ and using the resulting δ function and the symmetry of $R(\hat{\underline{k}}, V)$ and $I(\hat{\underline{k}}, V)$, we find

$$\underset{\sim}{C}(\underline{R}, \tau;t) = \frac{1}{2\pi^3} \int d^2\hat{\underline{k}} \int_0^{\infty} k^4 dk \int_{-\infty}^{\infty} dV \frac{\hat{\underline{k}}\,\hat{\underline{k}}\,H(\hat{\underline{k}}, V)}{[k^2 - R(\hat{\underline{k}}, V)]^2 + I^2(\hat{\underline{k}}, V)} e^{i(\underline{k}\cdot\underline{R} - kV\tau)}, \tag{IV-9}$$

where

$$H(\hat{\underline{k}}, V) = \sum_{\alpha} \pi m_{\alpha} \omega_{p\alpha}{}^2 \int d^3v \, \delta(V - \hat{\underline{k}}\cdot\underline{v}) f_{\alpha}(\underline{v}, t) \, . \tag{IV-10}$$

The function $H(\hat{\underline{k}}, V)$ appears throughout our work, along with $R(\hat{\underline{k}}, V)$ and $I(\hat{\underline{k}}, V)$. As the plasma slowly changes in time, these three functions will change. We will not explicitly show the dependence of $\underset{\sim}{C}(R, \tau)$ upon t. These variations must be slow compared to the time required for a shielding cloud to form, which is the decay time of the least-damped wave of importance. The shielding cloud must also form in a time short compared with the time in which the velocity of a particle changes, since each particle is treated as a test particle.

The physical interpretation of the part of Eq. (IV-9) that corresponds to waves is clear. The spontaneous emission by all the particles is balanced by the (Landau) damping, which we interpret as induced emission and absorption by the particles. The function $H(\hat{\underline{k}}, V)$ sums the spontaneous emission by various particles; it is positive definite and gives the number of particles moving in resonance with a wave weighted by the squares of their charges but independent of their masses.

It is now even less clear how one can judge the validity of neglecting mode coupling.

3. The Fokker-Planck Equation

We may now evaluate the effects of these fluctuating electric fields and of $\underline{E}_{drag}(q, \underline{v}')$ upon the velocity distributions of the particles. Because we have been forced to assume that the particle velocities deviate only slightly from constancy during the time required for a shielding cloud to form, we seek a Fokker-Planck equation for each species. We write

$$\frac{\partial f_a(\underline{v}, t)}{\partial t} = \frac{\partial}{\partial \underline{v}} \cdot \underline{J}_a(\underline{v}, t), \tag{IV-11}$$

where the current of species a in velocity space is

$$\underline{J}_a(\underline{v}, t) = \underline{\mathfrak{F}}_a(\underline{v}, t) f_a(\underline{v}, t) - \frac{\partial}{\partial \underline{v}} \cdot [\underset{\sim}{\mathcal{D}}_a(\underline{v}, t) f_a(\underline{v}, t)] . \tag{IV-12}$$

The dynamic friction

$$\underline{\mathfrak{F}}_a(\underline{v}, t) = \frac{\langle \underline{\Delta v} \rangle}{\Delta t} \tag{IV-13}$$

and the velocity diffusivity

$$\underset{\sim}{\mathcal{D}}_a(\underline{v}, t) = \frac{\langle \underline{\Delta v}\, \underline{\Delta v} \rangle}{2 \Delta t} \tag{IV-14}$$

are defined in terms of the velocity change $\underline{\Delta v}$ in a small time Δt.

This formula for $\underline{J}_a(\underline{v}, t)$ represents the first two terms in an expansion in $\underline{\Delta v}$; we here drop the higher terms without providing the necessary justification. We must choose Δt short compared to the collisional time scales so that $\underline{\Delta v}$ is small. However, Δt must also be long compared to the correlation times for the electric-field fluctuations, which are comparable to the damping times of the waves, because the Fokker-Planck equation is derived with the assumption of Markovian behavior.[4]

We clearly have

$$\underline{\Delta v} = \frac{q_a}{m_a} \int_t^{t+\Delta t} \underline{E}\left[\underline{r}(t'), t'\right] dt' \qquad \text{(IV-15)}$$

where $\underline{r}(t')$ is the trajectory of the particle being considered.

In Appendix E, we derive directly from Eq. (IV-14) and (IV-15) that

$$\underset{\sim}{\mathcal{D}}_a(\underline{v}, t) = \frac{q_a^2}{2m_a^2} \int_{-\infty}^{\infty} d\tau \; \underset{\sim}{C}(\underline{v}\tau, \tau) \, . \qquad \text{(IV-16)}$$

In deriving this, we assume that during Δt the particle velocity is constant and we assume that Δt is long compared with the correlation time as discussed above. We also assume that the fluctuating field at $\underline{r}(t')$ is not significantly influenced by the presence of the particle under consideration.

In evaluating the dynamic friction in Appendix E, we evaluate the effect of $\underline{E}_{drag}$, which is present at $\underline{r}(t')$ only because the particle is there, by treating the particle velocity as constant during Δt. However, we must also take into account the first correction in the particle velocity during Δt, which of course is produced by the fluctuating electric field. The result is

$$\underline{\mathfrak{F}}_a(\underline{v}, t) = \frac{q_a}{m_a} \underline{E}_{drag}(q_a, \underline{v}) + \frac{\partial}{\partial \underline{v}} \cdot \underset{\sim}{\mathcal{D}}_a(\underline{v}, t) \, . \qquad \text{(IV-17)}$$

By substituting this into Eq. (IV-12) and using the fact that $\underset{\sim}{\mathcal{D}}_a(\underline{v}, t)$ is a symmetric tensor, we find

$$\underline{J}_a(\underline{v}, t) = \frac{q_a}{m_a} \underline{E}_{drag}(q_a, \underline{v}) f_a(\underline{v}, t) - \underset{\sim}{\mathcal{D}}_a(\underline{v}, t) \cdot \frac{\partial f_a(\underline{v}, t)}{\partial \underline{v}} \, . \qquad \text{(IV-18)}$$

The first term includes the effect of spontaneous emission by the particle being considered and does not involve $H(\hat{\underline{k}}, V)$. The second term represents the effects of fluctuating fields that are present everywhere in the plasma; it includes the absorption and induced emission by the particle under consideration, of fluctuations associated with waves. These fluctuations arise from spontaneous emission, and damping by all the particles, so $\mathcal{D}(\underline{v}, t)$ involves $H(\hat{\underline{k}}, V)$.

We may note that Eq. (IV-2) of quasi-linear theory has the same form as Eq. (IV-18) except that no term like the one involving $\underline{E}_{drag}$ appears. This is as we expect because spontaneous emission is not included in the quasi-linear theory. The fluctuating fields that yield $\mathcal{D}(\underline{v}, t)$ are determined in a different way in quasi-linear theory.

By substituting Eq. (IV-9) into Eq. (IV-16) and evaluating the integral over τ, we find

$$\mathcal{D}_a(\underline{v}, t) = \frac{q_a^{\,2}}{2\pi^2 m_a^{\,2}} \int d^2\hat{\underline{k}} \int_0^{k_m} k^3 dk \int_{-\infty}^{\infty} dV \frac{\hat{\underline{k}}\hat{\underline{k}}\, \delta(V - \hat{\underline{k}}\cdot\underline{v}) H(\hat{\underline{k}}, V)}{[k^2 - R(\hat{\underline{k}}, V)]^2 + I^2(\hat{\underline{k}}, V)} \tag{IV-19}$$

where again we have supplied a cutoff k_m to remove a logarithmic divergence at large k. This divergence appears for the same reason as before.[38] Substituting this result and Eq. (IV-6) for $\underline{E}_{drag}$ into Eq. (IV-18), we find

$$\underline{J}_a(\underline{v}, t) = \frac{q_a^{\,2}}{2\pi^2 m_a^{\,2}} \int d^2\hat{\underline{k}} \int_0^{k_m} k^3 dk \int_{-\infty} dV$$

$$\frac{\hat{\underline{k}}\delta(V - \hat{\underline{k}}\cdot \underline{v}) \left[m_a I(\hat{\underline{k}}, V) f_a(\underline{v}, t) - H(\hat{\underline{k}}, V)\hat{\underline{k}}\cdot \dfrac{\partial f_a(\underline{v}, t)}{\partial \underline{v}}\right]}{[k^2 - R(\hat{\underline{k}}, V)]^2 + I^2(\hat{\underline{k}}, V)} . \tag{IV-20}$$

This result combined with Eq. (IV-11) is the Lenard-Balescu kinetic equation for species a. As we verify in Sec. IV. C, we can include the effect of a sufficiently weak $\underline{E}_0(t)$ by using

$$\frac{\partial f_a(\underline{v}, t)}{\partial t} + \frac{q_a}{m_a} \underline{E}_0(t)\cdot \frac{\partial f_a(\underline{v}, t)}{\partial \underline{v}} = - \frac{\partial}{\partial \underline{v}} \cdot \underline{J}_a(\underline{v}, t) . \tag{IV-21}$$

This result follows directly if we suppose that $\underline{E}_0(t)$ has no effect upon the fluctuating fields as represented by $\underset{\approx}{C}(\underline{R}, \tau)$ and so simply adds to $\underline{E}_{drag}(q, \underline{v})$ in Eq. (IV-18).

4. Discussion

Although we have not provided the necessary justification for our steps, the procedure does yield the correct result and so can be used in interpreting the L-B equation.

The result Eq. (IV-20) was first derived rigorously by Balescu[29] and Lenard,[28] who worked independently and used quite different methods. It is usually written somewhat differently, but Eq. (IV-20) is a convenient form for our purposes.

Lenard showed that this result has the properties expected of a kinetic equation:[28]

(a) The distribution functions $f_a(\underline{v}, t)$ remain nonnegative for all t,

(b) The particle densities, the total momentum, and the total kinetic energy remain constant,

(c) As t becomes large, the velocity distributions become Maxwellian with equal temperatures and drift velocities.

The L-B equations are valid only in a spatially uniform classical Coulomb plasma that is stable according to the Penrose criterion and varies sufficiently slowly in time. The correlation time for electric-field fluctuations must be short compared to the collision time, so that the collisional effects on the waves are not important and so that the behavior is Markovian and can be represented by a Fokker-Planck equation.

The effects of mode coupling must also be unimportant, but we have no simple way of judging this. One sometimes uses the criterion that the energy in the fluctuating electric fields associated with waves must be small compared to the kinetic energy of the particles.[42] This is clearly necessary because, as Lenard showed, the total kinetic energy is rigorously conserved by the equations. However, whether this condition is sufficient and what is meant by "small" are open to question.

We also have some difficulty in testing carefully whether the effects of collisions on waves can be ignored. The difficulty is that these collisional effects on longitudinal plasma waves are a current topic of research and are not yet well understood.

C. Derivations from the BBGKY Hierarchy

Now that we have obtained the desired result, we will very briefly discuss another method of derivation of the L-B equation and of other equations. As is usually done with this method, we consider the special case of a classical Coulomb electron plasma in a uniform positive background.

1. The BBGKY Hierarchy

This set of equations is derived directly from the equations of motion or from the Liouville equation and so is essentially exact.[43] The only assumption made is that the system contains a very large number of particles and can be represented by an appropriate ensemble.

The first equation of the set (the f equation) gives the distribution function $f(\underline{r}, \underline{v}, t)$ in terms of the two-particle correlation function $g(\underline{r}_1, \underline{v}_1, \underline{r}_2, \underline{v}_2, t)$. The second equation (the g equation) gives the two-particle correlation function in terms of the three-particle correlation function $h(\underline{r}_1, \underline{v}_1, \underline{r}_2, \underline{v}_2, \underline{r}_3, \underline{v}_3, t)$, and it also involves the distribution function f. The third equation (the h equation) involves four-particle correlations, and so forth.

Such a hierarchy of equations is useful only when it can be terminated in some manner. We discuss examples of this.

2. The Vlasov Equations

Rostoker and Rosenbluth have shown that if e, m, and n_e^{-1} are considered to be of order ϵ, then the term involving g in the f equation is of order ϵ in comparison with the other terms.[17] Thus in the limit of small ϵ, in which the electrons are represented as a continuous charged fluid, the term involving g is unimportant and can be ignored. None of the other equations of the hierarchy is then needed, and the f equation becomes simply the Vlasov equations.

They show that their expansion is equivalent to using ω_{pe} as a characteristic frequency and D_e as a characteristic length, and ordering each term in $\epsilon \approx 1/\Lambda$ with $\Lambda \approx n_e D_e^3$. Here D_e is defined with θ_e as of order of the average electron energy. This means that in the expansion in e, m, and n_e^{-1}, θ_e is also considered of order ϵ and the electron thermal speed is considered of order ϵ^0.

3. Kinetic Equations

We here consider a method of ordering the terms in powers of $\epsilon \approx 1/\Lambda$; this method is best summarized by Frieman and Book.[44] The root-mean-square electron speed is used as a characteristic speed, and f, g, and h are assumed to be of the same orders as they are in thermal equilibrium. The plasma is assumed to be spatially uniform with $\underline{E}_0 = \langle \underline{E} \rangle = 0$, so according to the first equation of the hierarchy, $f(\underline{v}, t)$ changes only because of the term involving $g(\underline{r}_1 - \underline{r}_2, \underline{v}_1, \underline{v}_2, t)$, which therefore must represent the effects of fluctuating electric fields.

Various characteristic distances including D_e, $n_e^{-1/3}$, and $b_{min} = e^2/\theta_e$ and various characteristic times are considered. In all cases, the term involving h in the g equation is of higher order in ϵ than other terms and so is neglected.[44] This turns out to correspond to the neglect of mode coupling, collisional effects on waves, and three-body collisions in the Boltzmann sense, among other higher order effects. The first two equations of the hierarchy then form a complete set for determination of f and g. It is also shown that, when ϵ is sufficiently small, the equation for g can be solved asymptotically for large t with f considered constant in time. This is known as the Bogoliubov hypothesis and yields g as a functional of f.[44] When this result is substituted into the f equation, a kinetic equation results.

With the characteristic distance of order b_{min} or $n_e^{-1/3}$, and thus small compared with D_e, a certain set of terms involving f in the g equation are of higher order in ϵ and can be ignored. This corresponds to neglect of the dielectric properties of the plasma and yields the Boltzmann kinetic equation.

When the characteristic distance is of order $n_e^{-1/3}$ or D_e, and thus large compared to b_{min}, another set of terms in the g equation, involving g itself, can be ignored. This corresponds to ignoring large deflections of the particles and yields the Lenard-Balescu kinetic equation.

When the characteristic distance is of order $n_e^{-1/3}$, and thus both large compared with b_{min} and small compared with D_e, both sets of terms can be ignored. The result is the Landau form of the Fokker-Planck equations, which we thus find is an approximation to the L-B equations as well as to the Boltzmann equation. We make use of this in Sec. V.

In deriving the L-B equation by this method, one uses a characteristic distance D_e and finds that the lowest order terms have characteristic frequencies ν_c in the f equation and ω_{pe} in the g equation.[44] This is used as justification for solving the g equation for long times with the assumption that f varies slowly. If we do not require that $\underline{E}_0(t) = \langle \underline{E} \rangle$ vanish, the additional terms that appear have characteristic frequencies $(E_0/E_{run})\,\nu_c$ in both equations. Therefore, as long as E_0 is sufficiently less than ΛE_{run}, we can ignore the effect of $\underline{E}_0$ in the g equation. The f equation then becomes Eq. (IV-21). We must be sure, however, that $\underline{E}_0(t)$ does not introduce rapid changes in f which upset our solution of the g equation.

Dupree has presented a direct but tedious method of actually solving the g equation and obtaining the L-B equation.[45,46] The method is presented more clearly by Rutherford and Frieman.[47]

This derivation from the BBGKY hierarchy makes clear that the Lenard-Balescu kinetic equation correctly describes a stable spatially uniform classical Coulomb plasma if Λ is sufficiently large. However, in practical plasmas that are not near thermal equilibrium, it is difficult to determine how large Λ must be for mode coupling and collisional effects upon the waves to be unimportant.

V. APPLICATIONS OF THE LENARD-BALESCU EQUATION

The L-B equation is useful in two types of problems: (a) When the effects of wave-particle interactions are unimportant, the natural appearance of dielectric shielding eliminates the necessity for supplying a cutoff at large distances (small k), so the L-B equation can be used to test the validity of, and possibly improve the accuracy of, results obtained from the Landau equation. (b) The L-B equation can also be useful when the effects of wave-particle interactions are important and the Landau equation is inadequate.

A. Relative Importance of Wave-Particle Interactions; the Landau Equation

The integral

$$K(\hat{\underline{k}}, V) = \int_0^{k_m} \frac{k^3 dk}{[k^2 - R(\hat{\underline{k}}, V)]^2 + I^2(\hat{\underline{k}}, V)} \qquad \text{(V-1)}$$

appears in Eqs. (IV-6), (IV-19), and (IV-20) for $\underline{E}_{drag}(q, \underline{v}')$, $\underset{\sim}{\mathcal{D}}_a(\underline{v}, t)$, and $\underline{J}_a(\underline{v}, t)$, respectively.

This expression may be evaluated approximately, as follows. If we ignore $R(\hat{\underline{k}}, V)$ and $I(\hat{\underline{k}}, V)$ in comparison with k^2, the integral diverges logarithmically at small k, but since for a plasma near thermal equilibrium $|R(\hat{\underline{k}}, V)|$ and $|I(\hat{\underline{k}}, V)|$ are of order or less than D_e^{-2}, this approximation is not justified for k of order or smaller than D_e^{-1}. If we supply a second cutoff at $k \approx D_e^{-1}$ and we choose $k_m = [b_{min}]^{-1}$, we find

$$K(\hat{\underline{k}}, V) \approx \ln\left(\frac{D_e}{b_{min}}\right) = \ln \Lambda \ . \qquad \text{(V-2)}$$

When the approximate result is used in Eqs. (IV-20) and (IV-21), the result is one form of the Landau equation, as we should expect from the discussion in Sec. IV. C.

We see that when $R(\hat{\underline{k}}, V)$ is positive and $I(\hat{\underline{k}}, V)$ is small, corresponding to the possibility of a slightly damped wave, the integrand of Eq. (V-1) contains a resonance that could also contribute substantially to $K(\hat{\underline{k}}, V)$. Rather than evaluating the contribution of this resonance approximately, we may consider the exact evaluation of Eq. (V-1). Because we choose k_m^2 very large compared to $|R(\hat{k}, V)|$ and $|I(\hat{k}, V)|$, actually larger by a factor of order Λ^2, we find to a very good approximation

$$K(\hat{k}, V) = \left\{ \ln \left[k_m (R^2 + I^2)^{-1/4} \right] - \frac{1}{2} \right\} + \left[\frac{R}{2|I|} \left(\frac{\pi}{2} + \tan^{-1} \frac{R}{|I|} \right) + \frac{1}{2} \right]. \quad (V-3)$$

The second term here depends only upon $x = R(\hat{k}, V)/|I(\hat{k}, V)|$ and behaves as shown in Fig. 3; it is independent of the cutoff k_m. The first term is ordinarily about equal to $\ln \Lambda$ by the argument given above when we choose $k_m \approx [b_{min}]^{-1}$. We thus may approximate Eq. (V-3) by

$$K(\hat{k}, V) = \ln \Lambda + \begin{cases} \dfrac{\pi R(\hat{k}, V)}{2|I(\hat{k}, V)|} & \text{when} \quad R(\hat{k}, V) > 0 \\ 0 & \text{when} \quad R(\hat{k}, V) \leq 0 \end{cases} \quad (V-4)$$

where the error is ordinarily of order unity, which is the uncertainty introduced by the cutoff procedure itself. The error is larger only when $R(\hat{k}, V)$ and $I(\hat{k}, V)$ are both very small compared to D_e^{-2}. Since this relationship is not expected to occur over wide ranges of $\hat{k}$ and V where a significant number of particles can satisfy the resonance condition $V = \hat{k} \cdot v$, the overall error in the use of Eq. (V-4) will ordinarily be within the uncertainty introduced by the cutoff k_m.

The second term in Eq. (V-4) is exactly what one would obtain from an approximate evaluation of the integral over the resonance in Eq. (V-1), and so is interpreted as the contribution of wave-particle interactions. The first term is clearly to be interpreted as the contribution of "ordinary" collisions. We see that wave-particle interactions will be important if, and only if, we have $R(\hat{k}, V) \gg |I(\hat{k}, V)|$ over reasonably large and important regions of $\hat{k}$ and V. The Landau equation will be inadequate when a large fraction of the particles under consideration can move in phase with slightly damped longitudinal plasma waves.

One problem that was considered long before the appearance of the L-B equation, and yet illustrates the above argument, is the drag

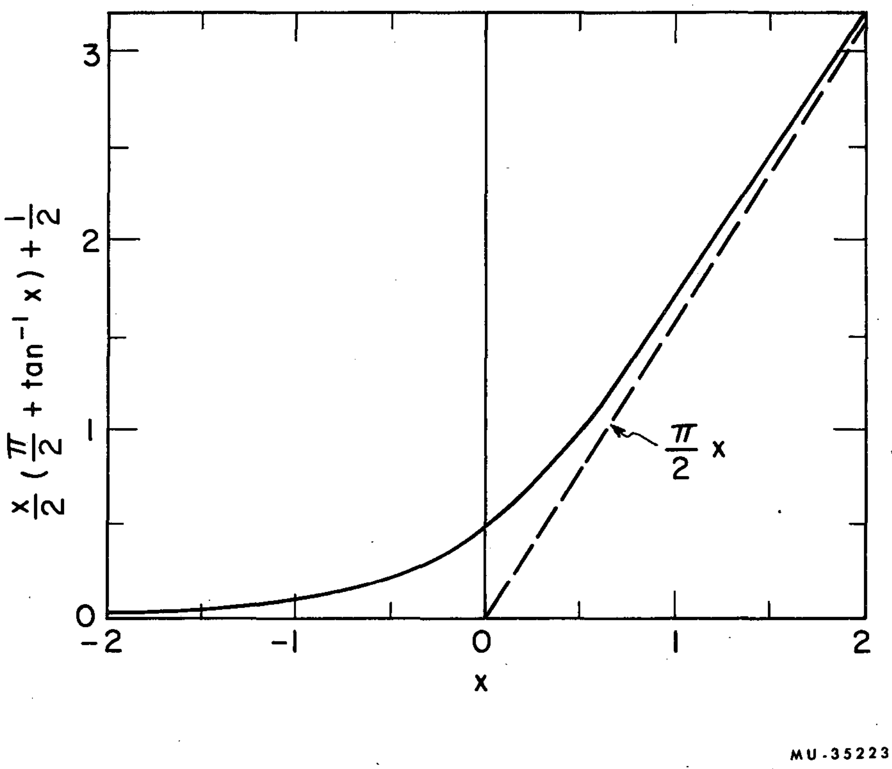


Fig. 3. Behavior of the second term in Eq. (V-3) as a function of $x = R(\hat{\underline{k}}, V)/|I(\hat{\underline{k}}, V)|$.

on a test particle. Using Eq. (V-4) for $\underline{E}_{drag}$ in Eq. (IV-6), we see that the collisional contribution depends upon $I(\hat{\underline{k}}, V)$, whereas the contribution of wave-particle interactions (in this case, spontaneous emission) depends upon $R(\hat{\underline{k}}, V)$ but is independent of $I(\hat{\underline{k}}, V)$. That is, the result involves the dispersion relation of the waves but not their damping. In an electron-proton plasma near thermal equilibrium, the contribution of wave-particle interactions is relatively small unless $|\underline{v}'|$ exceeds the electron thermal speed, because only then will $R(\hat{\underline{k}}, V)$ much exceed $|I(\hat{\underline{k}}, V)|$ for $\hat{\underline{k}}$ and V satisfying $V = \hat{\underline{k}} \cdot \underline{v}'$. When $|\underline{v}'|$ greatly exceeds the electron thermal speed, the contribution of wave-particle interactions to $\underline{E}_{drag}$ is comparable to the collisional contribution.[48]

B. Results When Wave-Particle Interactions Are Not Important

In an electron-proton plasma near thermal equilibrium, only a relatively few fast electrons can interact with slightly damped waves so the Landau equation should be adequate for most problems. This has been verified by solving the L-B equation in a few cases. The relaxation of an isotropic plasma toward thermal equilibrium was studied by Rosenberg and Wu.[49] The thermal conductivity was calculated by Sundaresan and Wu.[50] The relaxation of the velocity distribution in an isotropic electron plasma, not necessarily near equilibrium, to the equilibrium Maxwellian distribution was followed numerically by Dolinsky.[51] In these examples (Refs. 49-51), the deviations from the results found from the Landau equation were well within the uncertainty of order $[\ln \Lambda]^{-1}$, and sometimes fortuitously were within a few percent. Similar verification of the results from the Landau equation appears in the results of Kihara and his collaborators as discussed below.

Various methods have provided kinetic equations that converge without the insertion of cutoffs and so do not involve an uncertainty of order $[\ln \Lambda]^{-1}$.[44, 52-55] The results ordinarily involve combinations of the Boltzmann equation for close encounters and the L-B equation for distant encounters. For example, Frieman and Book[44] show that adding the Boltzmann and L-B equations and subtracting the Landau form of the Fokker-Planck equation yields a convergent result; the divergences of the Landau equation cancel the divergence of the Boltzmann equation at large impact parameters and the divergence of the L-B equation at large k, as was suggested by our discussion in Sec. IV. C.

The convergent kinetic equation derived by Kihara and his collaborators involves a proper matching of the Boltzmann and L-B equations in the region where both are valid.[54, 56, 57] This equation has been used to reduce the uncertainty of results obtained from the Landau equation. In the simpler problems that can be evaluated analytically, the results are just those found by using the Landau equation with $\ln \Lambda$ replaced by the logarithm of another quantity that is precisely determined.[58] This quantity depends upon the problem's being done, but it is of the same order of magnitude as Λ and the results can usually be given a physical interpretation in terms of results obtained from the Boltzmann equation by using an appropriately shielded Coulomb potential. For the more difficult problems that must be done numerically--such as calculations of the electrical conductivity, the thermal conductivity, and the viscosity--the results again are found to be well within the uncertainty of order $[\ln \Lambda]^{-1}$ in the results from the Landau equation.[59]

C. Results When Wave-Particle Interactions Are Important

A relatively few problems have been considered in which wave-particle interactions are important and the Landau equation is therefore inadequate. One such problem is the drag on a test particle, as mentioned before. Rand has recently considered a test particle with speed slow compared to the electron thermal speed.[60] He finds that the contribution of wave-particle interactions (the spontaneous emission of ion waves in this case) is important only if $\theta_e >> \theta_i$, and the speed of the test particle is fast compared to the thermal speed of the ion, as we would expect.

As recently shown by Akhiezer and Bolotin, the contribution of wave-particle interactions to the drag on a fast ion can also become large as the plasma approaches conditions where ion waves would become unstable.[61]

The various relaxation rates at which an electron-proton plasma approaches thermal equilibrium have been reconsidered by Ramazashvili, Rukhadze, and Silin by using the L-B equation.[62] With $\theta_e \gtrsim \theta_i$, they find that the modifications are small unless $\theta_e/\theta_i \gtrsim 100$. Even then the only rate that is significantly affected by the wave-particle interactions is the rate at which the electron velocity distribution becomes isotropic; this rate is increased by perhaps 30 per cent for $\theta_e/\theta_i = 100$ and by even more as this temperature ratio is increased further.

Gorbunov and Silin have calculated the electrical and thermal conductivities and the electron viscosity under similar conditions.[63] The wave-particle interactions tend to decrease these quantities below the results from the Landau equation by amounts that increase as θ_e/θ_i increases. For $\theta_e/\theta_i \approx 100$, the corrections amount to tens of percent, and for $\theta_e/\theta_i > 1000$, the effects of wave-particle interactions actually dominate.

Silin has also calculated the linear thermal conductivity by the electrons across a magnetic field.[64] With $\theta_e >> \theta_i$, the effect of the fluctuations associated with ion waves is to increase considerably this heat transport. Of course there are restrictions on the applicability of the Lenard-Balescu kinetic equations in a magnetized plasma.

D. General Conclusions

Beyond verifying and improving the results found from the Landau equation, the L-B equation has been used only for a few rather artificial problems. The form of Eq. (V-4) suggests that the effects of wave-particle interactions should add to the effects of "ordinary" collisions, and this is verified in the results obtained.

In an electron-proton plasma near thermal equilibrium the electron waves have little effect. However, if θ_e/θ_i is large, the fluctuations associated with ion waves can interact with nearly all electrons. The primary effect of these wave-particle interactions is to drive the electron velocity distribution toward isotropy; this by itself could account for the modifications of the transport coefficients found by Gorbunov and Silin.[63] Notice also that because an isotropic plasma is stable, these wave-particle interactions appear to have a stabilizing tendency, although this has not been directly investigated.

In all of the problems mentioned in Secs. IV. B and IV. C (except that of Dolinsky[51]), the dielectric function was evaluated from known zero-order or unperturbed distribution functions. Only Dolinsky followed the evolution of the distribution function and continually re-evaluated the dielectric function. He was able to do this because his problem was greatly simplified by the assumption of an isotropic plasma.

VI. APPLICATION TO A CURRENT-CARRYING PLASMA

We consider only an electron-proton plasma. The electrons are described by a velocity distribution $f(\underline{v}, t)$ and the ions by $F(\underline{v}, t)$. These are normalized as

$$\int f(\underline{v}, t)d^3v = \int F(\underline{v}, t)d^3v = 1 . \tag{VI-1}$$

The number densities of each species is n, and the masses are m and M. The plasma frequencies are

$$\omega_{pe}^2 = \frac{4\pi ne^2}{m} \quad \text{and} \quad \omega_{pi}^2 = \frac{4\pi ne^2}{M} . \tag{VI-2}$$

The Lenard-Balescu kinetic equations (L-B equations) can be written in terms of the three functions

$$R(\hat{\underline{k}}, V) = P\int \frac{d^3v}{\hat{\underline{k}} \cdot \underline{v} - V} \hat{\underline{k}} \cdot \frac{\partial}{\partial \underline{v}} [\omega_{pe}^2 f(\underline{v}, t) + \omega_{pi}^2 F(\underline{v}, t)] \tag{VI-3}$$

$$I(\hat{\underline{k}}, V) = \pi\int d^3v\, \delta(V - \hat{\underline{k}} \cdot \underline{v})\hat{\underline{k}} \cdot \frac{\partial}{\partial \underline{v}} [\omega_{pe}^2 f(\underline{v}, t) + \omega_{pi}^2 F(\underline{v}, t)] \tag{VI-4}$$

$$H(\hat{\underline{k}}, V) = \pi\int d^3v\, \delta(V - \hat{\underline{k}} \cdot \underline{v}) [m\omega_{pe}^2 f(\underline{v}, t) + M\omega_{pi}^2 F(\underline{v}, t)] . \tag{VI-5}$$

The first two include the dielectric properties of the plasma and therefore determine the properties of longitudinal plasma waves. In particular, if we consider a weakly damped wave traveling in the $\hat{\underline{k}}$ direction with real wave number k, the real part V of its phase velocity is given by $k^2 = R(\hat{\underline{k}}, V)$ and the damping rate is proportional to $I(\hat{\underline{k}}, V)$. (See Sec. III for further details.) The combination

$$K(\hat{\underline{k}}, V) = \int_0^{k_m} \frac{k^3 dk}{[k^2 - R(\hat{\underline{k}}, V)]^2 + I^2(\hat{\underline{k}}, V)} \tag{VI-6}$$

appears in the L-B equation. The cutoff k_m is necessary to remove a logarithmic divergence. Although the L-B equation includes the effects of "ordinary" collisions and wave-particle interactions, it is a Fokker-Planck equation and cannot treat large-angle scattering; this accounts for the divergence. The cutoff k_m is to be chosen of order e^2/θ_e where θ_e is a characteristic electron energy. (See Secs. IV and V for details.) The L-B equation is valid only for a plasma that is stable according to the linearized Vlasov equations; this insures that $I(\hat{\underline{k}}, V)$ never vanishes unless $R(\hat{\underline{k}}, V)$ is negative, so the denominator of Eq. (VI-6) never vanishes.

The function $H(\hat{\underline{k}}, V)$ is proportional to the number of particles moving in phase (with $V = \hat{\underline{k}} \cdot \underline{v}$) or in resonance with a wave. It determines the rate at which waves are spontaneously emitted by the particles. (See Sec. IV for further details.)

With the symmetry conditions $R(-\hat{\underline{k}}, -V) = R(\hat{\underline{k}}, V)$, $I(-\hat{\underline{k}}, -V) = -I(\hat{\underline{k}}, V)$, and $H(-\hat{\underline{k}}, -V) = H(\hat{\underline{k}}, V)$, the L-B equation for the electrons is

$$\frac{\partial f(\underline{v}, t)}{\partial t} = \frac{e}{m} \underline{E}_0(t) \cdot \frac{\partial f(\underline{v}, t)}{\partial \underline{v}} - \frac{\partial}{\partial \underline{v}} : \underline{J}_e(\underline{v}, t) \qquad \text{(VI-7)}$$

with

$$\underline{J}_e(\underline{v}, t) = \frac{e^2}{\pi^2 m} \int d^2\hat{\underline{k}} \int_0^\infty dV\, K(\hat{\underline{k}}, V)\hat{\underline{k}}\delta(V - \hat{\underline{k}} \cdot \underline{v})$$

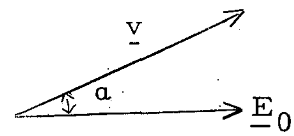
$$\times \left[I(\hat{\underline{k}}, V) f(\underline{v}, t) - \frac{H(\hat{\underline{k}}, V)}{m} \hat{\underline{k}} \cdot \frac{\partial f(\underline{v}, t)}{\partial \underline{v}} \right]. \qquad \text{(VI-8)}$$

The ion equation is obtained by replacing $f(\underline{v}, t)$, m, and e with $F(\underline{v}, t)$, M, and $-e$, respectively. These equations are derived for a spatially uniform classical Coulomb plasma, and ordinarily with $\underline{E}_0(t) = 0$, but if $\underline{E}_0(t)$ is not too large or rapidly varying it can be included. (See Sec. IV for details.)

We first discuss the general problem, but we soon see that it is impractical, even for numerical solution. We then discuss various model problems and select two for numerical study. The equations for these models are developed and simplified.

A. The General Problem

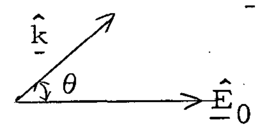
We consider only the case with $\underline{E}_0$ constant in time, so we naturally restrict our attention to solutions that are symmetric about the direction of $\underline{E}_0$. We describe $\underline{v}$ in spherical polar coordinates with $|\underline{v}| = v$ and $\underline{v} \cdot \underline{E}_0 = vE_0 \cos\alpha$ with $0 \leq \alpha \leq \pi$. Then $\underline{J}_e(\underline{v}, t)$ can have no component perpendicular to both $\underline{v}$ and $\underline{E}_0$ so we need to consider only the scalars $\hat{\underline{E}}_0 \cdot \underline{J}_e(\underline{v}, t)$ and $\underline{v} \cdot \underline{J}_e(\underline{v}, t)$. We show in Appendix F that Eq. (VI-7) then becomes



$$\frac{\partial f(v,\alpha,t)}{\partial t} = \frac{eE_0}{m}\left[\cos\alpha \frac{\partial f}{\partial v} - \frac{\sin\alpha}{v}\frac{\partial f}{\partial \alpha}\right] - \frac{1}{v^2}\frac{\partial}{\partial v}\left[v\underline{v}\cdot\underline{J}_e\right]$$

$$- \frac{1}{v\sin\alpha}\frac{\partial}{\partial\alpha}\left[\frac{\cos\alpha}{v}\underline{v}\cdot\underline{J}_e - \hat{\underline{E}}_0\cdot\underline{J}_e\right]. \qquad \text{(VI-9)}$$

We similarly describe $\hat{\underline{k}}$ in spherical polar coordinates with $\hat{\underline{k}} \cdot \hat{\underline{E}}_0 = \cos\theta$ and $0 \leq \theta \leq \pi$. Then $R(\hat{\underline{k}}, V) = R(\theta, V)$, $I(\hat{\underline{k}}, V) = I(\theta, V)$, $H(\hat{\underline{k}}, V) = H(\theta, V)$, and $K(\hat{\underline{k}}, V) = K(\theta, V)$. With these results



We can obtain explicit expressions for $\underline{v} \cdot \underline{J}_e$ and $\hat{\underline{E}}_0 \cdot \underline{J}_e$ in terms of scalar variables only.

We may use $\underline{v} \cdot \hat{\underline{k}}\, \delta(V - \hat{\underline{k}} \cdot \underline{v}) = V\delta(V - \hat{\underline{k}} \cdot \underline{v})$ and $\hat{\underline{E}}_0 \cdot \hat{\underline{k}} = \cos\theta$ in evaluating $\underline{v} \cdot \underline{J}_e$ and $\hat{\underline{E}}_0 \cdot \underline{J}_e$, respectively, from Eq. (VI-8). We also show in Appendix F that

$$\partial(V - \hat{\underline{k}}\cdot\underline{v})\hat{\underline{k}} \cdot \frac{\partial f}{\partial \underline{v}} = \delta(V - \hat{\underline{k}}\cdot\underline{v})\left[\frac{V}{v}\frac{\partial f}{\partial v} - \frac{1}{v}\left(\frac{V}{v}\cos\alpha - \cos\theta\right)\frac{\partial f}{\partial(\cos\alpha)}\right] \qquad \text{(VI-10)}$$

and that

$$\int d^2\hat{\underline{k}}\, \delta(V - \hat{\underline{k}}\cdot\underline{v}) = 2\int_0^{\pi} \sin\theta\, d\theta \int_0^{\pi} d\phi\, \delta(V - v\cos\alpha\cos\theta - v\sin\alpha\sin\theta\cos\phi). \qquad \text{(VI-11)}$$

By using these results in Eq. (VI-8), we find, for example, that

$$\underset{\sim}{v} \cdot \underset{\sim}{J}_e = \frac{2e^2}{\pi^2 m} \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} dV \, VK(\theta, V)$$

$$\times \left[\int_0^{\pi} d\phi \; \delta(V - v\cos\theta\cos\alpha - v\sin\alpha\sin\theta\cos\phi) \right]$$

$$\times \left\{ I(\theta, V) f(v, \alpha) - \frac{H(\theta, V)}{m} \left[\frac{V}{v} \frac{\partial f}{\partial v} - \frac{1}{v} \left(\frac{V}{v} \cos\alpha - \cos\theta \right) \frac{\partial f}{\partial (\cos\alpha)} \right] \right\}. \quad \text{(VI-12)}$$

To obtain $\hat{\underset{\sim}{E}}_0 \cdot \underset{\sim}{J}_e$, we need only to multiply this integrand by $(\cos\theta)/V$.

To complete the elimination of all vector quantities, we must consider $H(\theta, V)$, $I(\theta, V)$, and $R(\theta, V)$. We first separate the contributions of the electrons and the ions as, for example, $H(\theta, V) = H_e(\theta, V) + H_i(\theta, V)$. We show in Appendix F that

$$H_e(\theta, V) = 2\pi m \omega_{pe}^2 \int_0^{\infty} v^2 dv \int_0^{\pi} \sin\alpha \, d\alpha \, f(v, \alpha)$$

$$\times \left[\int_0^{\pi} d\phi \; \delta(V - v\cos\theta\cos\alpha - v\sin\theta\sin\alpha\cos\phi) \right] \quad \text{(VI-13)}$$

$$I_e(\theta, V) = 2\pi\omega_{pe}^2 \int_0^{\infty} v \, dv \int_0^{\pi} \sin\alpha \, d\alpha \left[\int_0^{\pi} d\phi \; \delta(V - v\cos\theta\cos\alpha \right.$$

$$\left. - v\sin\theta\sin\alpha\cos\phi) \right] \left[V \frac{\partial f}{\partial v} + \left(\cos\theta - \frac{V}{v}\cos\alpha\right) \frac{\partial f}{\partial(\cos\alpha)} \right] \quad \text{(VI-14)}$$

$$R_e(\theta, V) = 2\omega_{pe}^2 \int_0^{\infty} v \, dv \int_0^{\pi} \sin\alpha \, d\alpha \left\{ \pi \left(\frac{\partial f}{\partial v} - \frac{\cos\alpha}{v} \frac{\partial f}{\partial (\cos\alpha)} \right) \right.$$

$$+ \left[V \frac{\partial f}{\partial v} - \left(\frac{V}{v} \cos\alpha - \cos\theta \right) \frac{\partial f}{\partial (\cos\alpha)} \right]$$

$$\left. \times P \int_0^{\pi} \frac{d\phi}{v\cos\theta\cos\alpha + v\sin\theta\sin\alpha\cos\phi - V} \right\}. \quad \text{(VI-15)}$$

The integrals over ϕ are also evaluated in Appendix F. With $a \equiv v \cos\theta \cos\alpha - V$ and $b \equiv v \sin\alpha \sin\theta$, the results are

$$\int_0^{\pi} d\phi \; \delta(a + b\cos\phi) = \begin{cases} 0 & \text{if } b^2 < a^2 \\ (b^2 - a^2)^{-1/2} & \text{if } b^2 > a^2 \end{cases} \tag{VI-16}$$

and

$$P\int_0^{\pi} \frac{d\phi}{a + b\cos\phi} = \begin{cases} \frac{\pi}{a}\left(1 - \frac{b^2}{a^2}\right)^{-1/2} & \text{if } b^2 < a^2 \\ 0 & \text{if } b^2 > a^2 \end{cases} . \tag{VI-17}$$

One can easily verify that b^2 exceeds a^2 if, and only if, V is less than v and $\cos\alpha$ satisfies $r_- < \cos\alpha < r_+$. Here

$$r_{\pm} = \frac{V}{v}\cos\theta \pm \sin\theta \sqrt{1 - \frac{V^2}{v^2}} \tag{VI-18}$$

are the roots of $b^2 = a^2$ as a function of $\cos\alpha$ and, with $V < v$, are real with $|r_{\pm}| < 1$. For convenience in Eq. (V-12), this condition can be restated with α and θ interchanged.

The symmetry of the problem has enabled us to explicitly carry out all vector manipulations and to obtain equations that involve only scalar variables. In doing this we used all of the available δ functions, and no further general simplifications are possible. The expressions we have given apply to the electrons, but the corresponding ion equations can be obtained by simply replacing $f(\underline{v}, t)$, m, e, and the subscript e by $F(\underline{v}, t)$, M, $-e$, and the subscript i, respectively.

1. Reasons for Not Attacking the General Problem

The above equations are even more nonlinear and complicated than the corresponding Landau form of the Fokker-Planck equation. In the Landau equation, $K(\theta, V)$ is simply replaced by a constant, $\ln \Lambda$. (See Secs. II and V for details.) The problem could only be attacked as an initial-value problem and by numerical methods. One would specify the distribution functions at time t_0 and then calculate them at a short time interval later, using

$$f(v, \alpha, t_0 + \Delta t) = f(v, \alpha, t_0) + \Delta t \left[\frac{\partial f(v, \alpha, t)}{\partial t}\right]_{t=t_0} \qquad \text{(VI-19)}$$

and similarly for the ions; this would be repeated for as many time steps as desired.

The basic calculation would thus be the evaluation of $\partial f(v, \alpha, t)/\partial t$ and $\partial F(v, \alpha, t)/\partial t$ from known values of $f(v, \alpha)$ and $F(v, \alpha)$ by use of the equations we have given. Since $K(\theta, V)$ can be evaluated analytically from $R(\theta, V)$ and $I(\theta, V)$, as discussed in Sec. V. A, this basic calculation involves two major steps. The first is the calculation of $R(\theta, V)$, $H(\theta, V)$, and $I(\theta, V)$ from $f(v, \alpha)$ and $F(v, \alpha)$ by the evaluation of double integrals over v and α for each set of V and θ. The second is the evaluation of $\partial f(v, \alpha, t)/\partial t$ and $\partial F(v, \alpha, t)/\partial t$ by calculating double integrals over V and θ. We represent this schematically as

$$f(v, \alpha) \rightarrow I(\theta, V) \rightarrow \partial f(v, \alpha, t)/\partial t \qquad \text{(VI-20)}$$

where the arrows represent integrals over the variables to their left evaluated at each value of the variables to their right, and, of course, $f(v, \alpha)$ and $I(\theta, V)$ are only representative so each arrow actually involves several such integrals.

It is not feasible to carry out the process represented by (VI-20) even once, let alone for the hundreds or thousands of time steps that would presumably be needed. Even if this were possible, the results would depend upon the initial conditions, $\underline{E}_0$, and the time and so would be difficult to understand in any systematic manner. Our purpose is to demonstrate the nonlinear stabilizing effect of the wave-particle interactions associated with ion waves, and it might be difficult to separate this from other effects associated with the relaxation towards thermal equilibrium, Ohmic heating, or electron runaway.

We thus find it both necessary and desirable to consider only simplified model problems in order to demonstrate clearly the nonlinear stabilization.

2. Expansion in Legendre Polynomials

In preparation for discussing model problems, we may expand the distribution functions in Legendre polynomials of $\cos\alpha$ as

$$f(v, \alpha, t) = \sum_{\ell=0}^{\infty} f_\ell(v, t) P_\ell(\cos\alpha) \qquad \text{(VI-21)}$$

and the functions $H(\theta, V)$, $I(\theta, V)$, and $R(\theta, V)$ in Legendre polynomials of $\cos\theta$ as

$$H_e(\theta, V) = \sum_{p=0}^{\infty} H_{ep}(V) P_p(\cos\theta) . \qquad \text{(VI-22)}$$

To obtain equations for the $f_\ell(v, t)$, one may multiply Eq. (VI-9) by $(\ell + 1/2)P_\ell(\cos\alpha)\sin\alpha$ and then integrate over α; because of the orthogonality and normalization of the Legendre polynomials, this yields an expression for $\partial f_\ell(v, t)/\partial t$. This procedure is still completely general, but the set of equations obtained is not useful unless the expansions can be terminated after a finite, and preferably small, number of terms.

In Appendix G we prove the rather surprising result that the only term of Eq. (VI-21) that contributes to $H_{ep}(V)$, $I_{ep}(V)$, or $R_{ep}(V)$ is the one with $\ell = p$. We list below the results for $\ell = 0$ and $\ell = 1$ for future reference.

$$H_{e0}(V) = 2\pi^2 m\omega_{pe}^{\ 2} \int_V^{\infty} v f_0(v) dv \qquad \text{(VI-23a)}$$

$$I_{e0}(V) = -2\pi^2 \omega_{pe}^{\ 2} V f_0(V) \qquad \text{(VI-23b)}$$

$$R_{e0}(V) = 2\pi\omega_{pe}^{\ 2} \int_0^{\infty} v\, dv \frac{\partial f_0(v)}{\partial v} \left[2 + \frac{V}{v} \ln \frac{|V - v|}{V + v} \right] \qquad \text{(VI-23c)}$$

$$H_{e1}(V) = 2\pi^2 m\omega_{pe}^{\ 2} V \int_V^{\infty} f_1(v) dv \qquad \text{(VI-24a)}$$

$$I_{e1}(V) = 2\pi^2 \omega_{pe}^2 \left[\int_V^\infty f_1(v) dv - V f_1(V) \right] \qquad \text{(VI-24b)}$$

$$R_{e1}(V) = 2\pi\omega_{pe}^2 \int_0^\infty v\, dv \left\{ \frac{\partial f_1(v)}{\partial v} \left[2 \frac{V}{v} + \frac{V^2}{v^2} \ln \frac{|V - v|}{V + v} \right] \right.$$

$$\left. + \frac{f_1(v)}{v} \left[-2 \frac{V}{v} + \left(1 - \frac{V^2}{v^2} \right) \ln \frac{|V - v|}{V + v} \right] \right\} . \qquad \text{(VI-24c)}$$

Notice that H_{e0}, I_{e0}, H_{e1}, and I_{e1} are very simple to evaluate numerically since only two simple integrals are involved.

The expressions for $R_{e\ell}(V)$ do not involve principal value integrals, and although the integrands are singular at $v = V$, the singularity is very weak and so is usually integrable. This singularity could cause difficulty in numerical evaluation of these integrals, however. These expressions are also of value in deriving approximate analytic expressions. For example

$$R_{e0}(0) = -4\pi\omega_{pe}^2 \int_0^\infty f_0(v) dv \qquad \text{(VI-25)}$$

and by considering V larger than v and expanding the logarithm in powers of v/V, we find the asymptotic expansion

$$R_{e0}(V) = \frac{\omega_{pe}^2}{V^2} \left[1 + \frac{\langle v^2 \rangle}{V^2} + \frac{\langle v^4 \rangle}{V^4} + \cdots \right] \qquad \text{(VI-26)}$$

$$\langle v^n \rangle = 4\pi \int_0^\infty v^2 dv\, f_0(v)\, v^n \qquad \text{(VI-27)}$$

which is useful for large V. These results are in agreement with the expressions in Sec. III. C. Notice that Eq. (VI-26) gives a positive result although Eq. (VI-25) gives a negative result; thus there must be a region at small V where neither expression is useful. (For the Maxwellian case, see Sec. III. C.)

B. Various Models Considered

We must make many simplifying assumptions in order to obtain equations that can be solved numerically. Because the general equations are so highly nonlinear, we cannot check these assumptions by any direct expansion and ordering procedure. We are guided by the results discussed in the previous sections and by physical intuition, but certain assumptions are made out of sheer necessity.

1. Basic Simplifications

Here we discuss certain restrictions and approximations that are convenient and seem quite reasonable and so are used in all models we consider.

We suppose the velocity distributions are "basically" Maxwellian with the electron temperature θ_e much larger than the ion temperature θ_i. This statement is not precise, but if these "basic" distributions are assumed proportional to $\exp(-v^2/a^2)$ and $\exp(-v^2/A^2)$ for the electrons and the ions respectively, we have well-defined thermal speeds and temperatures related by

$$a = a_e = (2\theta_e/m)^{1/2} \qquad A = a_i = (2\theta_i/M)^{1/2}. \qquad \text{(VI-28)}$$

We may then define the electron Debye length

$$D_e = \sqrt{\theta_e/4\pi n e^2}\ , \qquad \text{(VI-29)}$$

the plasma parameter

$$\Lambda = 4\pi n D_e^{\ 3}, \qquad \text{(VI-30)}$$

a collision frequency

$$\nu_c = \frac{1}{3}\sqrt{\frac{2}{\pi}}\,\frac{\omega_{pe}}{\Lambda}\ln\Lambda, \qquad \text{(VI-31)}$$

and the runaway field

$$E_{run} = 0.5064\,\frac{m\nu_c a}{e} \qquad \text{(VI-32)}$$

as in Sec. II. B. These parameters prove useful in classifying the solutions obtained. As we discussed in Sec. III, with $\theta_e >> \theta_i$ the critical field E_{crit} at which ion waves would become unstable in the absence of the nonlinear stabilization we seek is small compared to E_{run}. Since we consider E_0 of order or smaller than E_{crit}, we can expect that if our present assumption is satisfied initially it will continue to be satisfied, although on a relatively long time scale the temperatures would change, as discussed in Appendix D.

We next use the approximation

$$K(\theta, V) = \ln \Lambda + \begin{cases} \dfrac{\pi R(\theta, V)}{2\,|I(\theta, V)|} & \text{if } R(\theta, V) > 0 \\ \\ 0 & \text{if } R(\theta, V) \leq 0 \end{cases} \tag{VI-33}$$

as discussed in Sec. V. A. The error is ordinarily within the uncertainty introduced by the cutoff k_m but can be larger if both $R(\theta, V)$ and $|I(\theta, V)|$ are extremely small. As we discussed in Sec. III. D, this tends to occur as E_0 approaches E_{crit} if $\theta_e < 20\theta_i$. The first term is interpreted as the contribution of "ordinary" collisions and the second as the effect of wave-particle interactions.

The function $R(\theta, V)$ is calculated with only the "basic" Maxwellian velocity distributions. This is justified because $R(\theta, V)$ is not sensitive to details of the distribution functions but instead depends upon quantities like certain moments of the distribution functions, as indicated by Eqs. (VI-25) and (VI-26). This removes the necessity for evaluating the rather complicated integrals that would yield $R(\theta, V)$. With this assumption, $R(V)$ is positive for $V > a_e$ and for $a_i < V < \sqrt{m/M}\, a_e$, which are the electron and ion wave regions, respectively, as discussed in Sec. III.

We ignore the effect of electron plasma waves by neglecting the second term of Eq. (VI-33) except in the ion wave region. This neglect corresponds to using

$$R(V) = \frac{1}{D_e^2}\left[\frac{\theta_e}{2\theta_i}\, X\left(\frac{V}{A}\right) - 1\right] \tag{VI-34}$$

in Eq. (VI-33). Here the function $X(x)$ is defined by Eq. (III-33) and discussed in Sec. III. C. As is illustrated by the examples in Sec. V, the electron waves ordinarily have little effect because they interact only with the fast electrons. As we discussed in Sec. III. E, the presence of runaway electrons implies unstable electron waves with high phase speeds, but with $E_0 << E_{run}$ these waves will affect only the relatively few very fast electrons. We would be unable to use the L-B equation for this problem because the plasma is actually unstable to these waves. The collisional effects upon these very slowly growing or weakly damped waves are probably important also.

Unfortunately, the assumptions we have discussed do not greatly simplify our problem. They are necessary, however, to make possible certain other simplifications.

2. Specific Models

Most of the assumptions we consider here cannot be justified in detail but are simply necessary to cut our problem to a manageable size. The resulting problems can be considered only as models.

We first assume that the ion velocity distribution is Maxwellian. This cuts the size of the problem considerably, and in certain models permits us to make use of the slowness of the ion waves compared to the electrons; no comparable assumption can be made in treating the ions. We may make rather convincing arguments that the "ordinary" collisions and the field E_0 do not make the ion distribution deviate greatly from being Maxwellian; the reason is that the collisional drag and the diffusion in velocity space caused by collisions with the electrons are very nearly independent of the ion velocity. The collisional drag is almost exactly balanced by the force of the field $\underline{E}_0$, and the diffusion tends merely to change θ_i. The ion-ion collisions, of course, tend to maintain the Maxwellian distribution. However, this argument is not enough to justify our assumption because in calculating the damping or growth of ion waves details of the "tail" of the ion distribution are important, and this is just the region most strongly affected by wave-particle interactions. With this assumption, we naturally work in the ion frame.

In treating the case $E_0 >> E_{run}$, Field and Fried made assumptions comparable to those we have considered, plus two additional ones.[34] The first is based upon the slowness of the ion-wave speeds and reduces the problem schematically to

$$f(v, \alpha) \rightarrow I(\theta) \rightarrow \frac{\partial f(v, \alpha)}{\partial t} \quad . \qquad \text{(VI-35)}$$

The second has little justification but was considered necessary; they factored $f(v, \alpha)$ as $f_{\perp}(v_{\perp}) f_{\parallel}(v_{\parallel})$ where $v_{\perp} = v \sin\alpha$, $v_{\parallel} = v \cos\alpha$, and $f_{\perp}(v_{\perp})$ is Maxwellian. The problem is then represented as

$$f_{\parallel}(v_{\parallel}) \rightarrow I(\theta) \rightarrow \frac{\partial f_{\parallel}(v_{\parallel})}{\partial t} \qquad \text{(VI-36)}$$

and was solved numerically in some detail.

If $f(v, \alpha, t)$ is represented by a small number of terms of a Legendre polynomial expansion, the problem is represented by

$$f_{\ell}(v) \rightarrow I_{\ell}(V) \qquad I(\theta, V) \rightarrow \frac{\partial f_{\ell}(v)}{\partial t} \qquad \text{(VI-37)}$$

where now each arrow represents several integrals. Because the θ dependence of $I(\theta, V)$ is known, the integral over θ can be evaluated analytically if the number of terms in the expansion is small enough, and we then have

$$f_{\ell}(v) \rightarrow I_{\ell}(V) \rightarrow \frac{\partial f_{\ell}(v)}{\partial t} \quad . \qquad \text{(VI-38)}$$

An extremely crude model that we consider numerically involves only $\ell = 0$ and $\ell = 1$ and will be called Model A. We assume $f_0(v)$ is Maxwellian with known temperature θ_e, so this model strongly resembles that in the Spitzer-Härm problem, except that we use the L-B equation and we do not linearize in E_0 and $f_1(v, t)$. Schematically, we have

$$f_1(v) \rightarrow I_1(V) \rightarrow \frac{\partial f_1(v)}{\partial t} \quad , \qquad \text{(VI-39)}$$

so the complexity of our Model A is roughly the same as that of the problems solved by Dolinsky[51] and by Field and Fried.[34]

The second model we consider numerically is based upon

$$f(v, \alpha, t) = f_0(v) + f_1(v) \cos\alpha + \delta f(v, \alpha, t) \qquad \text{(VI-40)}$$

where $f_0(v)$ and $f_1(v)$ are known from the linearized version of Model A, and $\delta f(v, \alpha, t)$ is expected to be small. Without further assumptions we still have the fairly general problem

$$\delta f(v, \alpha) \rightarrow \delta I(\theta, V) \qquad I(\theta, V) \rightarrow \frac{\partial \, \delta f(v, \alpha, t)}{\partial t} . \qquad \text{(VI-41)}$$

Because $\delta f(v, \alpha)$ is considered small, we neglect $\delta H(\theta, V)$ and $\delta I(\theta, V)$ wherever possible, which means everywhere except in the denominator of Eq. (VI-33) for $K(\theta, V)$, where $\delta I(\theta, V)$ is important because the other contributions to $I(\theta, V)$ tend to cancel. This approximation amounts to a somewhat unsystematic linearization in δf. In the contribution of ion waves, we make approximations based upon the slowness of the ion waves; then $\delta I(\theta, V)$ is evaluated at $V=0$, for example, so we have

$$\delta f(v, \alpha) \rightarrow \delta I(\theta) \qquad I(\theta, V) \rightarrow \frac{\partial \, \delta f(v, \alpha)}{\partial t} . \qquad \text{(VI-42)}$$

We call this problem our Model B. Formally, it is much more complicated than our Model A and the problems done by Dolinsky and by Field and Fried. Numerical solution is possible only because detailed knowledge of $\delta f(v, \alpha)$ and $\delta I(\theta)$ are not needed; this is a consequence of the simplifications based upon the slowness of the ion waves in comparison with the electrons.

C. Reduction to Equations for the Model Problems

We can develop the equations for Models A and B together by using

$$f(v, \alpha, t) = f_0(v) + f_1(v, t) \cos\alpha + \delta f(v, \alpha, t) . \qquad \text{(VI-43)}$$

In Model A, δf vanishes; whereas in Model B, $f_1(v)$ is known and is independent of time. In both models $f_0(v)$ and the ion velocity distribution are Maxwellian.

1. Introduction of Dimensionless Variables

The remaining development is quite formal and we can introduce the dimensionless variables

$$x = \frac{v}{a}, \qquad u = \frac{V}{a}, \qquad y = \cos\alpha, \qquad \tau = \nu_c t \qquad \text{(VI-44)}$$

and the parameters

$$\delta = \sqrt{\frac{m}{M}} << 1, \quad \gamma = \frac{\theta_e}{\theta_i} >> 1, \quad \epsilon = \frac{A}{a} = \frac{\delta}{\gamma^{1/2}} << 1, \quad E = E_0/E_{run} . \qquad \text{(VI-45)}$$

For the electron velocity distribution we define

$$a^3 f(v, \alpha, t) = h(x, y) = \pi^{-3/2} e^{-x^2} + y f(x) + g(x, y) \qquad \text{(VI-46)}$$

with $\quad f(x) = a^3 f_1(v, t) \qquad$ and $g(x, y) = a^3 \delta f(v, \alpha, t)$.

We introduce the dimensionless function

$$R(u) = \frac{\pi D_e^{\,2}}{2 \ln \Lambda} \begin{cases} R(V) & \text{if} \quad R(V) > 0 \\ 0 & \text{if} \quad R(V) \leqslant 0 \end{cases} \qquad \text{(VI-47)}$$

where $R(V)$ is, according to our assumptions, given by Eq. (VI-34). We therefore have explicitly

$$R(u) = \frac{\pi}{2 \ln \Lambda} \left[\frac{\gamma}{2} X \left(\frac{u}{\epsilon} \right) - 1 \right] \qquad \text{(VI-48)}$$

when $u_{min} < u < u_{max}$, and $R(u) = 0$ otherwise. Here u_{min} and u_{max} are the values of u for which the quantity in brackets vanishes.

Similarly we define

$$H(\theta, u) = \frac{D_e^{\,2}}{ma} H(\theta, V) \qquad \text{(VI-49)}$$

where, by our assumptions, the contribution of δf is to be ignored. From Eqs. (VI-23a) and (VI-24a), we find explicitly that

$$H(u, \theta) = H_0(u) + H_1(u) \cos\theta \qquad \text{(VI-50)}$$

with

$$H_0(u) = \frac{\sqrt{\pi}}{2} \left[e^{-u^2} + \frac{1}{\epsilon} e^{-u^2/\epsilon^2} \right] \qquad \text{(VI-51)}$$

$$H_1(u) = \pi^2 u \int_u^\infty f(x) dx \ . \qquad \text{(VI-52)}$$

Because $I(\theta, V)$ is negative in the ion-wave region, we define

$$I(\theta, u) = -D_e^{\,2} I(\theta, V), \qquad \text{(VI-53)}$$

which is positive for $u_{min} < u < u_{max}$ (when the plasma is stable to ion waves). By our assumptions, the contribution of δf is not to be included except when $|I(\theta, V)|$ appears in a denominator and then is to be evaluated with u = 0, so we find

$$I(\theta, u) = I_0(u) + I_1(u) \cos\theta + I_g(\theta) \tag{VI-54}$$

where from Eqs. (VI-23b) and (VI-24b)

$$I_0(u) = \sqrt{\pi}\, u \left[e^{-u^2} + \frac{\gamma}{\epsilon} e^{-u^2/\epsilon^2} \right] \tag{VI-55}$$

$$I_1(u) = \pi^2 \left[uf(u) - \int_u^\infty f(x)dx \right], \tag{VI-56}$$

and, as is shown in Appendix H,

$$I_g(\theta) = -\pi \cos\theta \int_0^{\pi/2} d\phi \left[\rho(\sin\theta \cos\phi) + \rho(-\sin\theta \cos\phi) \right] \tag{VI-57}$$

with

$$\rho(y) = \frac{\partial}{\partial y} \int_0^\infty g(x, y)dx .$$

The ions appear in our equations only through the terms involving ϵ and γ in Eqs. (VI-48), (VI-51), and (VI-55).

In terms of these dimensionless variables and functions, Eq. (VI-9) can be written as

$$\frac{\partial h(x, y)}{\partial \tau} = 0.5064E \left[y \frac{\partial h}{\partial x} + \frac{1 - y^2}{x} \frac{\partial h}{\partial y} \right] - \frac{1}{x^2} \frac{\partial}{\partial x} \left[x \left(\frac{a}{\nu_c} \underset{-}{v} \cdot \underset{-}{J}_e \right) \right]$$

$$+ \frac{1}{x} \frac{\partial}{\partial y} \left[\frac{y}{x} \left(\frac{a}{\nu_c} \underset{-}{v} \cdot \underset{-}{J}_e \right) - \left(\frac{a^2}{\nu_c} \underset{-}{\hat{E}}_0 \cdot \underset{-}{J}_e \right) \right] . \tag{VI-58}$$

Similarly, by using Eqs. (VI-16) and (VI-33), we find that Eq. (VI-12) yields

$$\left(\frac{a}{\nu_c} \underset{-}{v} \cdot J_e \right) = \frac{-3}{\pi^{3/2}} \int_0^x du\, u \int_{r_-}^{r_+} d(\cos\theta) \frac{1 + R(u)/I(\theta, u)}{x \sqrt{(r_+ - \cos\theta)(\cos\theta - r_-)}}$$

$$\times \left\{ I(\theta, u)h(x, y) + H(\theta, u) \frac{u}{x} \frac{\partial h}{\partial x} + H(\theta, u) \frac{1}{x} \left(\cos\theta - \frac{uy}{x} \right) \frac{\partial h}{\partial y} \right\}, \tag{VI-59}$$

where

$$r_{\pm} = \frac{uy}{x} \pm \sqrt{1-y^2}\,\sqrt{1-u^2/x^2} \quad .$$

To find $\left[(a^2/\nu_c)\,\underline{\hat{E}}_0 \cdot \underline{J}_e\right]$ we must simply multiply the integrand by $(\cos\theta)/u$.

We have already made considerable use of the assumptions discussed in Sec. VI. B. Further use of these assumptions will apply specifically for either Model A or Model B.

2. Contributions of "Ordinary" Collisions

Our purpose in this section is to write the terms in

$$-\frac{1}{x^2}\frac{\partial}{\partial x}\left[x\left(\frac{a}{\nu_c}\underline{v}\cdot\underline{J}_e\right)\right] + \frac{1}{x^2}\frac{\partial}{\partial y}\left[y\left(\frac{a}{\nu_c}\underline{v}\cdot\underline{J}_e\right)\right] - \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{a^2}{\nu_c}\underline{\hat{E}}_0\cdot\underline{J}_e\right) \qquad \text{(VI-60)}$$

that do not involve R(u) in the form

$$A(x,y)\frac{\partial^2 h}{\partial x^2} + B(x,y)\frac{\partial^2 h}{\partial x\partial y} + C(x,y)\frac{\partial^2 h}{\partial y^2} + D(x,y)\frac{\partial h}{\partial x}$$

$$+ E(x,y)\frac{\partial h}{\partial y} + F(x,y)h(x,y) \quad . \qquad \text{(VI-61)}$$

We define the integrals

$$\mathcal{J}_n\left(y, \frac{u}{x}\right) = \frac{1}{\pi}\int_{r_-}^{r_+} d(\cos\theta)\,\frac{\cos^n\theta}{\sqrt{(r_+ - \cos\theta)(\cos\theta - r_-)}} \quad , \qquad \text{(VI-62)}$$

which can easily be evaluated with the substitution of variables used in Appendix I. The results are

$$\mathcal{J}_0 = 1, \quad \mathcal{J}_2 = \frac{u^2y^2}{x^2} + \frac{1}{2}\left(1-y^2\right)\left(1-\frac{u^2}{x^2}\right)$$

$$\mathcal{J}_1 = \frac{uy}{x}, \quad \mathcal{J}_3 = \frac{u^3y^3}{x^3} + \frac{3uy}{2x}\left(1-y^2\right)\left(1-\frac{u^2}{x^2}\right) \quad . \qquad \text{(VI-63)}$$

In terms of these, the collisional contributions are

$$\left(\frac{a}{\nu_c}\underline{v}\cdot\underline{J}_e\right)_{coll} = \frac{-3}{\sqrt{\pi}\,x}\int_0^x du\,u\left\{h(x,y)\left[I_0\mathcal{J}_0+I_1\mathcal{J}_1\right]+\frac{u}{x}\frac{\partial h}{\partial x}\left[H_0\mathcal{J}_0+H_1\mathcal{J}_1\right]\right.$$

$$\left.+\frac{1}{x}\frac{\partial h}{\partial y}\left[H_0\mathcal{J}_1+H_1\mathcal{J}_2\right]-\frac{uy}{x^2}\frac{\partial h}{\partial y}\left[H_0\mathcal{J}_0+H_1\mathcal{J}_1\right]\right\}, \qquad \text{(VI-64)}$$

and

$$\left(\frac{a^2}{\nu_c}\hat{\underline{E}}_0\cdot\underline{J}_e\right)_{coll} = \frac{-3}{\sqrt{\pi}\,x}\int_0^x du\left\{h(x,y)\left[I_0\mathcal{J}_1+I_1\mathcal{J}_2\right]+\frac{u}{x}\frac{\partial h}{\partial x}\left[H_0\mathcal{J}_1+H_1\mathcal{J}_2\right]\right.$$

$$\left.+\frac{1}{x}\frac{\partial h}{\partial y}\left[H_0\mathcal{J}_2+H_1\mathcal{J}_3\right]-\frac{uy}{x^2}\frac{\partial h}{\partial y}\left[H_0\mathcal{J}_1+H_1\mathcal{J}_2\right]\right\} \qquad \text{(VI-65)}$$

where we have used Eqs. (VI-50) and (VI-54), ignoring $I_g(\theta)$ by assumption as discussed before.

The procedure is now to substitute these expressions into Eq. (VI-60), carry out the differentiations, and collect terms, which is quite straightforward but tedious. We simply give the results, which are expressed in terms of the five rather simple functions:

$$\Phi(x) = \int_0^x e^{-u^2}du = \frac{\sqrt{\pi}}{2}\,\text{erf}\,x \qquad \text{(VI-66a)}$$

$$Q(x) = 2\int_0^x u^2e^{-u^2}\,du = \Phi(x) - xe^{-x^2} \qquad \text{(VI-66b)}$$

$$G_0(x) = \int_x^\infty f(u)du \qquad \text{(VI-67a)}$$

$$G_3(x) = \int_0^x u^3f(u)du \qquad \text{(VI-67b)}$$

$$G_5(x) = \int_0^x u^5f(u)du\ . \qquad \text{(VI-67c)}$$

In the form of Eq. (VI-61), the results are

$$A(x, y) = A_0(x) + A_1(x)y \tag{VI-68}$$

$$A_0(x) = \frac{3}{4x^3}\left[Q(x) + \epsilon^2 Q\left(\frac{x}{\epsilon}\right)\right]$$

$$A_1(x) = \frac{3}{5}\pi^{3/2}\left[\frac{1}{x^4}G_5(x) + xG_0(x)\right],$$

$$B(x, y) = B_1(x)(1 - y^2) \tag{VI-69}$$

$$B_1(x) = \pi^{3/2}\left[\frac{1}{x^3}G_3(x) - \frac{3}{5x^5}G_5(x) + \frac{2}{5}G_0(x)\right],$$

$$C(x, y) = C_0(x)(1 - y^2) + C_1(x)(y - y^3) \tag{VI-70}$$

$$C_0(x) = \frac{3}{4x^3}\left[\Phi(x) + \Phi\left(\frac{x}{\epsilon}\right)\right] - \frac{1}{2x^2}A_0(x)$$

$$C_1(x) = \frac{1}{2x}B_1(x),$$

$$D(x, y) = D_0(x) + D_1(x)y \tag{VI-71}$$

$$D_0(x) = \frac{3}{2x}\left[e^{-x^2} + \frac{1}{\epsilon}e^{-x^2/\epsilon^2}\right] + \frac{3}{2x^2}\left[Q(x) + \gamma\epsilon^2 Q\left(\frac{x}{\epsilon}\right)\right] - \frac{1}{x}A_0(x)$$

$$D_1(x) = B_1(x),$$

$$E(x, y) = E_0(x)y + E_1(x) \tag{VI-72}$$

$$E_0(x) = -2C_0(x)$$

$$E_1(x) = -2C_1(x),$$

$$F(x, y) = F_0(x) + F_1(x)y \tag{VI-73}$$

$$F_0(x) = 3e^{-x^2} + 3\frac{\gamma}{\epsilon}e^{-x^2/\epsilon^2}$$

$$F_1(x) = 3\pi^{3/2}f(x).$$

The y dependence of these quantities is simple. The functions with subscript 1 depend only upon f(x), whereas both the ions and the Maxwellian part of the electron-velocity distribution contribute to the functions with subscript 0.

Two problems that have been considered by means of the Landau equation serve as checks on our above results. With $E = f(x) = g(x, y) = 0$ and the wave contributions neglected, Eq. (VI-58) reduces to

$$\frac{\partial h(x)}{\partial \tau} = \pi^{-3/2} e^{-x^2} \left[(4x^2 - 2)A_0(x) - 2xD_0(x) + F_0(x)\right]$$

$$\equiv \Gamma_0(x) . \qquad \text{(VI-74)}$$

By direct substitution from above we find

$$\Gamma_0(x) = 3(\gamma - 1) \left[\frac{1}{\epsilon} e^{-x^2/\epsilon^2} - \frac{\epsilon^2}{x} Q\left(\frac{x}{\epsilon}\right)\right] \pi^{-3/2} e^{-x^2} , \qquad \text{(VI-75)}$$

so when $\gamma = 1$ $(\theta_e = \theta_i)$ we find $\partial h/\partial\tau = 0$ as we expect. Otherwise the rate at which the kinetic energy of the electrons, per electron, is changing reduces to

$$\frac{\partial}{\partial t} \int d^3v f_e(\underline{v}) \left(\frac{1}{2} mv^2\right) = \theta_e \frac{\partial}{\partial t} 4\pi \int_0^{\infty} x^4 h(x) dx = \theta_e \nu_c 4\pi \int_0^{\infty} x^4 \Gamma_0(x) dx .$$

The integral over $\Gamma_0(x)$ can be evaluated exactly and yields

$$\frac{\partial}{\partial t} \int d^3v f_e(\underline{v}) \left(\frac{1}{2} mv^2\right) = - \frac{3m}{M} \nu_c (\theta_e - \theta_i)(1 + \epsilon^2)^{-3/2} \qquad \text{(VI-76)}$$

which is exactly the result given by Spitzer.[5] With the approximation $\epsilon^2 << 1$, this is the result we used in Appendix D.

In the second special case we assume $\theta_e = \theta_i$ so $\Gamma_0(x) = 0$ and we linearize in f(x) and E, neglecting g(x, y) entirely. This is exactly the problem solved by Spitzer and Härm in evaluating the linear electrical conductivity.[1] We find from Eq. (VI-58)

$$\frac{\partial f(x, \tau)}{\partial \tau} = - 0.1819\, E\, x\, e^{-x^2} + \Gamma_1(x) \qquad \text{(VI-77)}$$

(VI-77 cont.)

$$\Gamma_1(x) = A_0(x)\,\frac{\partial^2 f}{\partial x^2} + A_1(x)(4x^2 - 2)\,\pi^{-3/2}e^{-x^2}$$

$$+ D_0(x)\,\frac{\partial f}{\partial x} + D_1(x)\,(-2x)\pi^{-3/2}e^{-x^2}$$

$$+ \left[E_0(x) + F_0(x)\right] f(x) + F_1(x)\pi^{-3/2}e^{-x^2} .$$

Substitution of $A_1(x)$, $D_1(x)$, and $F_1(x)$ into Eq. (VI-77) yields

$$\Gamma_1(x) = A_0(x)\,\frac{\partial^2 f}{\partial x^2} + D_0(x)\,\frac{\partial f}{\partial x} + \left[E_0(x) + F_0(x) + 3e^{-x^2}\right] f(x)$$

$$+ \left[\frac{12}{5x^2}\,G_5(x) - \frac{2}{x^2}\,G_3(x) + \left(\frac{12}{5}\,x^3 - 2x\right) G_0(x)\right] e^{-x^2} . \qquad \text{(VI-78)}$$

If we consider M to be infinite so $\epsilon = 0$ [and $\Gamma_0(x)$ vanishes for any finite γ], Eq. (VI-77) reduces exactly to the equation solved by Spitzer and Härm. In this approximation the ion contributions vanish except in $E_0(x)$.

Only the functions $B_1(x)$, $C_0(x)$, $C_1(x)$, and $E_1(x)$ are not checked by these special cases. We have some confidence in these results because $C_0(x)$ is simply related to $E_0(x)$, whereas $B_1(x)$, $C_1(x)$, and $E_1(x)$ are simply related to each other.

3. Model A

In this model f(x) depends upon τ and we ignore g(x, y) entirely. To obtain an equation for $f(x, \tau)$, we multiply Eq. (VI-58) by 3y/2 and integrate over y from -1 to 1 to find

$$\frac{\partial f(x,\tau)}{\partial \tau} = -0.1819E\, x\, e^{-x^2} - \frac{1}{x^2}\,\frac{\partial}{\partial x}\left[x\int_{-1}^{1}\frac{3}{2}\,y\left(\frac{a}{\nu_c}\,\underline{v}\cdot\underline{J}_e\right) dy\right]$$

$$+ \frac{1}{x}\int_{-1}^{1}\frac{3}{2}\,y\,dy\,\frac{\partial}{\partial y}\left\{\frac{y}{x}\left(\frac{a}{\nu_c}\,\underline{v}\cdot\underline{J}_e\right) - \left(\frac{a^2}{\nu_c}\,\underline{\hat{E}}_0\cdot\underline{J}_e\right)\right\} .$$

(VI-79)

By referring to the previous section, we find that the "ordinary" collisions [the terms not involving R(u)] contribute simply $\Gamma_1(x)$. Thus if the effects of wave-particle interactions were ignored, our Model A would be simply the Spitzer-Härm problem and would be subject to the same limitation of small E. We notice that the above procedure does not include the energy exchange between electrons and ions as represented by $\Gamma_0(x)$, but this would only be important on time scales of order $(M/m)\,\nu_c^{-1}$, which are long compared with the time scales we consider.

To determine the wave contribution we represent the expression in braces by $\alpha(y)$ and use

$$\int_{-1}^{1} y\,dy\,\frac{\partial\alpha}{\partial y} = \alpha(1) + \alpha(-1) - \int_{-1}^{1} \alpha(y)dy.$$

But from Eq. (VI-58) and the corresponding equation for $[(a^2/\nu_c)\hat{\underline{E}}_0 \cdot \underline{J}_e]$, we see that $\alpha(y)$ can be written as an integral where the integrand contains a factor $(uy/x - \cos\theta)$, which implies that $\alpha(1) = \alpha(-1) = 0$. To see this, we note that as y approaches ± 1, $r_\pm$ both approach uy/x, so the only value of $\cos\theta$ that contributes is $\cos\theta = r_\pm = uy/x$, and the factor $(uy/x - \cos\theta)$ then vanishes. The same conclusion follows from the δ function in Eq. (VI-12) as $\sin\alpha$ approaches zero. We therefore have

$$\frac{\partial f(x,\tau)}{\partial\tau} = -0.1819E\,x\,e^{-x^2} + \Gamma_1(x) - \frac{1}{x^2}\frac{\partial}{\partial x}[xJ_1(x)] - \frac{1}{x^2}J_1(x) + \frac{1}{x}J_2(x) \tag{VI-80}$$

where

$$J_1(x) = \frac{3}{2}\int_{-1}^{1} y\left(\frac{a}{\nu_c}\,\underline{v}\cdot\underline{J}_e\right)_{\text{wave}} dy$$

$$J_2(x) = \frac{3}{2}\int_{-1}^{1}\left(\frac{a^2}{\nu_c}\,\hat{\underline{E}}_0\cdot\underline{J}_e\right)_{\text{wave}} dy. \tag{VI-81}$$

In Appendix I we show that

$$J_1(x) = \frac{-9}{2\sqrt{\pi}\,x}\int_0^x du\,u\,\frac{R(u)}{I_0(u)}\left\{\frac{-2u^2}{\pi^{3/2}x}\,e^{-x^2}\left[H_0V_1+H_1V_2\right]+\frac{uf}{x^2}\left[H_0V_2+H_1V_3\right]\right.$$

$$+\frac{2}{3}f(x)I_0+\frac{u}{x}\left(\frac{\partial f}{\partial x}-\frac{f}{x}\right)H_0\left[\frac{u^2}{x^2}V_2+\frac{1}{2}\left(1-\frac{u^2}{x^2}\right)\left(V_0-V_2\right)\right]$$

$$\left.+\frac{u}{x}\left(\frac{\partial f}{\partial x}-\frac{f}{x}\right)H_1\left[\frac{u^2}{x^2}V_3+\frac{1}{2}\left(1-\frac{u^2}{x^2}\right)\left(V_1-V_3\right)\right]\right\} \qquad \text{(VI-82)}$$

$$J_2(x) = \frac{-9}{2\sqrt{\pi}\,x}\int_0^x du\,\frac{R(u)}{I_0(u)}\left\{\frac{-2u}{\pi^{3/2}}\,e^{-x^2}\left[H_0V_1+H_1V_2\right]+\frac{f}{x}\left[H_0V_2+H_1V_3\right]\right.$$

$$\left.+\frac{2u}{3x}f(x)I_0+\frac{u^2}{x^2}\left(\frac{\partial f}{\partial x}-\frac{f}{x}\right)\left[H_0V_2+H_1V_3\right]\right\}, \qquad \text{(VI-83)}$$

where

$$V_n(u) = \int_{-1}^{1} d(\cos\theta)\,\frac{\cos^n\theta}{1+(\cos\theta)/b(u)} \qquad \text{(VI-84)}$$

with

$$b(u) = \frac{I_0(u)}{I_1(u)} \;. \qquad \text{(VI-85)}$$

If we substitute

$$\frac{\cos\theta}{1+(\cos\theta)/b(u)} = b(u)\left[1-\frac{1}{1+(\cos\theta)/b(u)}\right]$$

into Eq. (VI-84), we find the identities

$$V_1(u) = b(u)[2 - V_0(u)]$$

$$V_2(u) = -b(u)V_1(u)$$

$$V_3(u) = b(u)\left[\frac{2}{3} - V_2(u)\right] \;. \qquad \text{(VI-86)}$$

When $|b(u)|$ is large, we may expand $[1 + (\cos\theta)/b(u)]^{-1}$ in powers of $[(\cos\theta)/b(u)]$ to find

$$V_3(u) = -\frac{2}{5b(u)} - \frac{2}{7b^3(u)} - \frac{2}{9b^5(u)} \cdots \qquad \text{(VI-87)}$$

and $V_2(u)$, $V_1(u)$, and $V_0(u)$ from the above identities. For arbitrary $|b(u)|$ (larger than unity) we find directly

$$V_0(u) = b(u) \ln \frac{b(u) + 1}{b(u) - 1} \qquad \text{(VI-88)}$$

and $V_1(u)$, $V_2(u)$, and $V_3(u)$ from the identities.

With a stable plasma, $|b(u)|$ is, and must be, larger than unity for $u_{min} < u < u_{max}$, so that $I_0(u) + I_1(u) \cos\theta$ does not vanish for any θ. As E is increased toward the critical value, $|b(u)|$ will approach unity for certain u, and $V_0(u)$ will become large. This is expected to give the nonlinear stabilization we seek, but in this model the effect becomes stronger only logarithmically.

Again we must face the straightforward but tedious step of substituting these expressions into Eq. (VI-80) and collecting terms. We express the result as

$$\frac{\partial f(x,\tau)}{\partial \tau} = A'(x) \frac{\partial^2 f}{\partial x^2} + B'(x) \frac{\partial f}{\partial x} + C'(x)f + [D'(x) - 0.1819\, Ex]\, e^{-x^2} \qquad \text{(VI-89)}$$

where

$$A'(x) = A_0(x) + A'_{wave}(x) \qquad \text{(VI-90)}$$

$$B'(x) = D_0(x) + B'_{wave}(x) \qquad \text{(VI-91)}$$

$$C'(x) = E_0(x) + F_0(x) + 3e^{-x^2} + C'_{wave}(x) \qquad \text{(VI-92)}$$

$$D'(x) = \frac{12}{5x^2} G_5(x) - \frac{2}{x^2} G_3(x) + \left[\frac{12}{5} x^3 - 2x\right] G_0(x) + D'_{wave}(x). \qquad \text{(VI-93)}$$

For the wave contributions we give only the results, which may be expressed in terms of

$$\chi_0(u) = \frac{9}{4\sqrt{\pi}} R(u) \left[H_0(u)V_0(u) + H_1(u)V_1(u)\right]$$

$$\chi_1(u) = \frac{9}{\pi^2} R(u) \left[H_0(u)V_1(u) + H_1(u)V_2(u)\right]$$

$$\chi_2(u) = \frac{9}{4\sqrt{\pi}} R(u) \left[H_0(u)V_2(u) + H_1(u)V_3(u)\right], \qquad \text{(VI-94a)}$$

$$S(x) = \int_0^x du\, u^2 \left[\chi_0(u) - \chi_2(u)\right]$$

$$T(x) = \int_0^x du\, u^4 \left[3\chi_2(u) - \chi_0(u)\right]$$

$$U(x) = \int_0^x du\, \chi_2(u)$$

$$W(x) = \int_0^x du\, (2u^3 + u)\, \chi_1(u). \qquad \text{(VI-94b)}$$

The results are

$$A'_{wave}(x) = \frac{S(x)}{x^3} + \frac{T(x)}{x^5} \qquad \text{(VI-95)}$$

$$B'_{wave}(x) = \frac{2\chi_2(x)}{x} + \frac{3}{\sqrt{\pi}\, x^2}\int_0^x du\, u\, R(u) - \frac{S(x)}{x^4} - \frac{3T(x)}{x^6}$$

(VI-96)

$$C'_{wave}(x) = \frac{3R(x)}{\sqrt{\pi}\, x} + \frac{3T(x)}{x^7} + \frac{S(x)}{x^5} - \frac{2U(x)}{x^3} \qquad \text{(VI-97)}$$

$$D'_{wave}(x) = -\chi_1(x) + \frac{W(x)}{x^2}. \qquad \text{(VI-98)}$$

We must also supply appropriate initial and boundary conditions on $f(x, \tau)$. We are particularly interested in the asymptotic solution as τ becomes large; this solution represents a steady state on the time scale of interest. This steady state is very convenient because for specified mass ratio δ^2 and specified $\ln \Lambda$ the solution will depend only upon $\gamma = \theta_e/\theta_i$ and $E = E_0/E_{run}$ in our dimensionless units; this is a large simplification over the general problem. For this purpose the initial condition is not crucial; we could use $f(x, 0) = 0$ for example, although some other choice may provide more rapid convergence to $f(x, \tau \to \infty)$. We use the boundary condition $f(0, \tau) = 0$ so that the electron velocity distribution is continuous, as we expect from the diffusion nature of our equations. For numerical purposes we also must supply a boundary condition at some large value of x. We simply set $f(x_{max}, \tau) = 0$ and choose x_{max} large compared with the values of x that contribute significantly. Our model is not correct for very large x anyway.

4. Model B

In this model f(x) is considered to be known from solving Model A with a very small value of E. In this case f(x) is proportional to E so we make a convenient change in notation. We write

$$h(x, y, \tau) = \pi^{-3/2} e^{-x^2} + Ef(x)y + g(x, y, \tau) \qquad \text{(VI-99)}$$

so that f(x) itself does not depend upon E. We continue to define $I_1(u)$, $H_1(u)$, $G_0(x)$, $G_3(x)$, $G_5(x)$, $A_1(x)$, $B_1(x)$, $C_1(x)$, $D_1(x)$, $E_1(x)$, $F_1(x)$, and $\Gamma_1(x)$, which depend linearly upon f(x) and are now considered known, by the expressions given previously; these then are also independent of E. With our simplifying assumptions concerning the "ordinary" collisions, Eq. (VI-58) becomes

$$\frac{\partial g(x, y)}{\partial \tau} = 0.5064E \left[y \frac{\partial g}{\partial x} + \frac{1 - y^2}{x} \frac{\partial g}{\partial y} \right] + \left[A_0(x) + EyA_1(x) \right] \frac{\partial^2 g}{\partial x^2}$$

$$+ (1 - y^2) EB_1(x) \frac{\partial^2 g}{\partial x \partial y} + (1 - y^2) \left[C_0(x) + EyC_1(x) \right] \frac{\partial^2 g}{\partial y^2}$$

$$+ \left[D_0(x) + EyD_1(x)\right] \frac{\partial g}{\partial x} + \left[yE_0(x) + EE_1(x)\right] \frac{\partial g}{\partial y}$$

$$+ \left[F_0(x) + EyF_1(x)\right] g(x, y) + \Gamma_0(x) + Ey\Gamma_1'(x)$$

$$+ E^2\Gamma_2(x) + E^2y^2\Gamma_3(x) + \left[\frac{\partial g}{\partial \tau}\right]_{waves} . \qquad \text{(VI-100)}$$

Here everything but g(x, y) and $[\partial g/\partial \tau]_{waves}$ is presumed known. We have defined

$$\Gamma_1'(x) = \Gamma_1(x) - 0.1819xe^{-x^2} \qquad \text{(VI-101)}$$

$$\Gamma_2(x) = 0.5064f(x)/x + B_1(x)\partial f/\partial x + E_1(x)f(x) \qquad \text{(VI-102)}$$

$$\Gamma_3(x) = 0.5064[\partial f/\partial x - f/x] + A_1(x)\partial^2 f/\partial x^2 + F_1(x)f(x). \qquad \text{(VI-103)}$$

The full expression for $[\partial g/\partial \tau]_{waves}$ includes all of the terms in Eq. (VI-58) that involve R(u). To simplify this we use the circumstance that most of the electrons have $x >> u_{max}$. By assumption, we neglect all terms that involve positive powers of u/x. Then, according to Eq. (VI-59), the contribution of the wave part of $\underline{v} \cdot \underline{J}_e$ is neglected, and in $\hat{\underline{E}}_0 \cdot \underline{J}_e$ the expression in braces reduces to

$$\left\{I(\theta, u)h(x, y) + \frac{\cos\theta}{x} H(\theta, u) \frac{\partial h(x, y)}{\partial y}\right\} .$$

Therefore, we have

$$\left[\frac{\partial g}{\partial \tau}\right]_{waves} = \frac{3}{\pi^{3/2}x^2} \frac{\partial}{\partial y}\int_{u_{min}}^{u_{max}} R(u)du \int_{r_-}^{r_+} \frac{\cos\theta \, d(\cos\theta)}{\sqrt{(r_+ - \cos\theta)(\cos\theta - r_-)}}$$

$$\times \left\{h(x, y) + \frac{\cos\theta}{x} \frac{H(\theta, u)}{I(\theta, u)} \frac{\partial h}{\partial y}\right\} \qquad \text{(VI-104)}$$

where now $r_\pm = \pm(1-y^2)^{1/2}$. The term not involving $H(\theta, u)$, which corresponds to the direct effect of spontaneous emission upon the particle being considered, vanishes because the integrand is odd in $\cos\theta$. We therefore find

$$\left[\frac{\partial g}{\partial \tau}\right]_{\text{waves}} = \frac{1}{x^3} \frac{\partial}{\partial y}\left[S(y) \frac{\partial h(x,y)}{\partial y}\right] \qquad \text{(VI-105)}$$

where, if we define

$$W_n(u,y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} d(\cos\theta) \frac{\cos^n\theta}{\sqrt{1-y^2-\cos^2\theta}} \frac{1}{I(\theta,u)}, \qquad \text{(VI-106)}$$

we have

$$S(y) = \frac{3}{\pi^{3/2}} \int_{u_{min}}^{u_{max}} du\, R(u)\left[H_0(u)W_2(u,y) + EH_1(u)W_3(u,y)\right]. \qquad \text{(VI-107)}$$

Notice that since $H(\theta, U)$ is always positive and $R(u)$ and $I(\theta, u)$ are positive for $u_{min} < u < u_{max}$, $S(y)$ will be positive everywhere except at $y = \pm 1$, where it is proportional to $1 - y^2$ and so vanishes. The evaluation of $W_n(u, y)$ from Eq. (VI-106) is discussed in Appendix J.

According to Eq. (VI-105), the primary effect of the ion waves upon the electron velocity distribution is to produce a diffusion in the angular direction in velocity space. By considering an "H-theorem" in Appendix K, we show that in our approximation the ion waves always tend to make the electron velocity distribution isotropic (even if the ion velocity distribution is not Maxwellian or even isotropic). This certainly agrees with the conclusion reached by Ramazashvili, Rukhadze, and Silin.[62] The effect of the ion waves vanishes only when $\partial h(x,y)/\partial y$ vanishes everywhere, so the electron velocity distribution is isotropic.

With an isotropic ion velocity distribution as we have assumed, this is a stabilizing effect. When $I(\theta, u)$ becomes small in the ion-wave region, $S(y)$ becomes large, at least for a certain range of y. This should provide the nonlinear stabilization we seek.

In the present notation, the problem of Field and Fried[34] with all of their assumptions except the factorization in $v_\perp$ and $v_\parallel$ reduces to

$$\frac{\partial h(x,y,\tau)}{\partial \tau} = 0.5064E\left[y\frac{\partial h}{\partial x} + \frac{1-y^2}{x}\frac{\partial h}{\partial y}\right] + \frac{1}{x^3}\frac{\partial}{\partial y}\left[\mathcal{S}(y)\frac{\partial h}{\partial y}\right], \qquad \text{(VI-108)}$$

and $I(\theta, u)$ is approximated by the result of Eq. (VI-57) with $g(x, y)$ replaced by $h(x, y)$. Of course $\mathcal{L}(y)$ is determined in the spirit of quasilinear theory, but this still requires knowledge of $R(u)$ and $I(\theta)$ only. It is quite clear from the forms of Eq. (VI-57) and Eq. (VI-108) that the factorization in $v_\perp$ and $v_\parallel$ is very unnatural. It also seems that this factorization is not necessary.

To complete the definition of our problem we must consider the initial condition and boundary conditions. It would be very convenient to remove the effect of slow energy changes due to Ohmic heating and collisional transfer to the ions so that we could solve for an asymptotic solution in time, as in Model A. We may do this by considering $g(x, y)$ to be expanded in Legendre polynomials of y and then requiring that the coefficient of $P_0(y)$ vanish. The isotropic part of the electron velocity distribution is then simply the Maxwellian. This simplification will also be useful when we consider the boundary condition at $x = 0$. Notice that the isotropic part of $g(x, y)$ would not directly affect the electrical current, the heat flow, or the ion waves [according to Eq. (VI-57)].

The initial condition is of no real importance as we calculate $g(x, y, \tau \to \infty)$, but it should be chosen so that the plasma is stable.

The boundary condition at x_{max} will be $g(x_{max}, y, \tau) = 0$ just as in Model A. The boundary condition at $x = 0$ is somewhat more difficult. We expect from the diffusion nature of our equations and from physical intuition that the electron velocity distribution will be continuous, finite, and reasonably smooth. We suppose that a Taylor-series expansion

$$h(x, y) = \sum_{n=0}^{\infty} A_n(y)x^n \qquad \text{(VI-109)}$$

is valid for small x. If the velocity distribution is to be continuous, we must have $A_0(y) = h(0, y)$ be a constant. But then as x approaches zero, Eq. (VI-100) takes the form

$$\frac{\partial g}{\partial \tau} = \frac{1}{x^2} \frac{\partial}{\partial y} \left[S(y) \frac{\partial A_1}{\partial y}\right] + \mathcal{O}\left(\frac{1}{x}\right) \qquad \text{(VI-110)}$$

so we must have $S(y)\partial A_1/\partial y$ equal to a constant. Since $S(y)$ behaves as $1 - y^2$ as y approaches ± 1, this condition can only be satisfied, with $A_1(y)$ finite for all y, if $A_1(y)$ is constant. Thus

$$h(x, y) = A_0 + A_1 x + A_2(y)x^2 + \cdots \qquad \text{(VI-111)}$$

where A_0 and A_1 are constant. This implies

$$g(x, y) = C + A_1 x - f(x)y + \mathcal{O}(x^2) \qquad \text{(VI-112)}$$

where C is another constant. We have no way of determining C and A_1, but fortunately, we have decided to remove the isotropic part of $g(x, y)$, so we have

$$g(x, y) = -f(x)y + \mathcal{O}(x^2) \ . \qquad \text{(VI-113)}$$

Actually, we realize that Eq. (VI-105) is only an approximation and is not valid for small x. This approximation is useful only if the region of small x is relatively unimportant, but in this case the boundary condition at $x = 0$ should not be important.

We also must consider the boundary condition at $y = \pm 1$. To be specific we consider y near +1, or α near 0. We consider the region of velocity space with $0 \leq \alpha < \alpha_0$. The surface area of this region is proportional to α_0, when α_0 is small, while its volume is proportional to α_0^2. The rate at which electrons enter this volume per unit v is given by the area $2\pi \sin\alpha_0$ times the normal flux

$$-\hat{\underline{\alpha}}_0 \cdot [\underline{J}_e(\underline{v}, t) - \frac{e}{m}\underline{E}f(\underline{v}, t)] = -\hat{\underline{\alpha}}_0 \cdot \underline{J}_e(\underline{v}, t) - \frac{e}{m} E \sin\alpha_0 \, f(\underline{v}, t)$$

and must vanish as fast at the volume as α_0 approaches zero if $\underline{J}_e(v, t)$, $f(\underline{v}, t)$ and $\partial f(\underline{v}, t)/\partial t$ are to be finite. This requires that the above expression vanish as fast as α_0 or $\sin\alpha_0$, which implies that $\hat{\underline{\alpha}} \cdot \underline{J}_e(\underline{v}, t)$ vanishes as fast as $\sin\alpha$. By writing out $\hat{\underline{\alpha}} \cdot \underline{J}_e(\underline{v}, t)$ from Sec. VI. A, one finds that this implies $\partial f(\underline{v}, t)/\partial(\cos\alpha)$ either vanishes or approaches a finite constant as α approaches zero. Thus we find that $\partial g(x, y)/\partial y$ is finite at $y = \pm 1$. This would be satisfied automatically if $g(x, y)$ were represented by any finite number of terms in an expansion in Legendre polynomials of y.

With these boundary and initial conditions, the definition of Model B is complete.

VII. NUMERICAL PROCEDURE

Because we have no need for high numerical accuracy, all integrals are evaluated by the simple trapezoidal method. Also, since we seek the steady-state solution that is approached asymptotically as t (or τ) becomes large, we are not particularly concerned with following the time development in detail.

Special procedures are useful in preventing the computational instabilities that tend to arise in solving diffusion-like equations such as ours.

A. Model A

The functions of x and of u are determined at the discrete values $x_j = u_j$. Since both the slow ion waves and the fast electrons must be well represented, we ordinarily use 271 values of x_j with $0 \leq x_j \leq 0.02$ for $1 \leq j \leq 140$; $0.02 < x_j < 1$ for $141 \leq j \leq 211$; and $1 \leq x_j \leq 7$ for $212 \leq j \leq 271$.

The numerical solution of

$$\frac{\partial f}{\partial \tau} = A \frac{\partial^2 f}{\partial x^2} \qquad \text{(VII-1)}$$

by the explicit method suggested in Sec. VI. A develops computational instabilities unless the time step $\Delta \tau$ is chosen somewhat smaller than $(\Delta x)^2/A$.[65] This suggests that the solution of Eq. (VI-89) by explicit methods may develop computational instabilities unless $\Delta\tau$ were sufficiently small. Because Δx and A vary widely in our problem, it is not clear how small $\Delta\tau$ would have to be, but $\Delta\tau$ would probably be so small that an impractically large number of time steps would be needed to reach the steady-state solution.

Equation (VII-1) can also be solved by an implicit procedure. Denoting $f(x_j, \tau = \tau_0)$ by f_j and $f(x_j, \tau = \tau_0 + \Delta\tau)$ by f_j', we write

$$\frac{f'_j - f_j}{\Delta\tau} = \rho \frac{A}{(\Delta x)^2}\left[f_{j+1} - 2f_j + f_{j-1}\right] + (1-\rho)\frac{A}{(\Delta x)^2}\left[f'_{j+1} - 2f'_j + f'_{j-1}\right], \tag{VII-2}$$

where $\Delta x = x_j - x_{j-1}$ is chosen independent of j, and ρ is chosen in the range $0 \leqslant \rho \leqslant 1$. Here $\partial f/\partial\tau$ is not simply evaluated at the "old" time τ_0 at which f is known, but instead is a weighted average of the values at the "old" time τ_0 and the "new" time $\tau_0 + \Delta\tau$ at which f is unknown. The Eqs. (VII-2) form a set of linear, algebraic, simultaneous equations for the f'_j that can be solved as follows. One writes

$$f'_{j-1} = e_{j-1} f'_j + d_{j-1} \tag{VII-3}$$

where, of course, e_j and d_j are unknown. By eliminating f'_{j-1} between Eq. (VII-2) and Eq. (VII-3), solving for f'_j, and comparing the result to Eq. (VII-3) with j replaced by j + 1, one finds recursion relations that give e_j and d_j in terms of e_{j-1}, d_{j-1}, and the known quantities in Eq. (VII-2). With $j_{min} \leqslant j \leqslant j_{max}$, $e_{j_{min}}$ and $d_{j_{min}}$ are determined by the boundary condition at j_{min}, so these recursion relations yield e_j and d_j for j from $j_{min} + 1$ through $j_{max} - 1$. By using the boundary condition at j_{max} along with Eq. (VII-3), one finds f'_j for j from j_{max} down through j_{min}.

This implicit procedure is computationally stable for any $\Delta\tau$ if $\rho \leqslant 0.5$.[65] The "new" time is weighted more heavily than the "old" time; in fact with $\rho = 0$, the "old" time appears only on the left in Eq. (VII-2). We use the same implicit procedure to solve our Eq. (VI-89), which contains additional terms that can be treated in the same way. A minor difference is that we have chosen an uneven spacing for x_j, which makes the difference equations appear more complicated. A more important difference is that the coefficients $A'(x)$, $B'(x)$, $C'(x)$, and $D'(x)$ depend upon $f(x, \tau)$ and so are not known at the "new" time. Therefore one must choose trial values of these coefficients at the "new" time, calculate f'_j, and use f'_j to compute improved values for these coefficients. This iteration procedure can be repeated as many times as desired before one continues to the next time step. We determine the trial values of the coefficients by linear extrapolation from earlier times.

Another difficulty is that, although our equations are valid only for a stable plasma, the computer may suddenly find $|I_1(u)| \geq I_0(u)$ for $u_{min} < u < u_{max}$, which corresponds to an unstable plasma. In this case we attempt to force the plasma toward stability by simply prescribing that the fluctuations associated with the unstable waves have very large amplitudes. Thus we replace I_1/I_0 by $1 - 2\exp(-C)$ if $I_1/I_0 \geq 1$, and by $-1 + 2\exp(-C)$ if $I_1/I_0 \leq -1$. [Actually, the term $2\exp(-C)$ is neglected except in the logarithm in Eq. (VI-86), which becomes $V_0(u) = C$.] Ordinarily we choose C equal to 40. The number of times this prescription is used on each time step is monitored, and, of course, our solution is incorrect unless as the steady state is approached closely the plasma is stable, so this prescription is not needed.

This numerical procedure works quite well for E_0 smaller that E_{crit}. Computational instabilities do not appear for $\Delta\tau$ as large as 8, although in this case the solution oscillates about the final steady state and the oscillations decay rather slowly. (We are not surprised that the physical time development is not followed when the time step is $8\,\nu_c^{-1}$.) Ordinarily we use $\Delta\tau = 2$ because this yields convergence to the steady-state solution in the smallest number of time steps. The convergence and stability are equally good for $\rho = 0$, 0.1, 0.2, 0.3, and 0.4, but with $\rho = 0.5$ the convergence at small x is relatively slow; ordinarily we use $\rho = 0.4$. We use two iterations on each time step (that is, the f_j' are calculated three times), but this is not necessarily the optimum. We ordinarily use the initial condition $f(x, 0) = 0$, and the convergence is complete (within the eight-figure computer accuracy) within less than 40 time steps. For most purposes 10 time steps would be sufficient.

When E_0 approaches or exceeds E_{crit}, we modify our procedure. From a previous run with a value of E_0 we denote by E_A, we have a solution $f_A(x)$. From this we find that even if the problem were linear with $f(x) = E_0 f_A(x)/E_A$, the plasma would remain stable until E exceeds E_A', with $E_A' > E_{crit}$. We then run with $E_0 = E_B$, choosing $E_A < E_B < E_A'$ and using as our initial condition $f(x) = E_B f_A(x)/E_A$. Even with these precautions to ensure that a stable steady-state solution exists, we often must choose $\Delta\tau$ rather small to avoid computational instabilities. This procedure soon becomes impractical because of the small amounts that E_0 is increased and because of the small time steps. On runs for which a steady-state solution is found, the prescription for handling unstable waves is ordinarily needed only on the first few time steps.

On an IBM-7044 computer, each time step requires about 6 seconds, and most runs last between 3 and 8 minutes. We refer to the computer program for Model A as Program 1.

B. Model B

This two-dimensional problem is feasible numerically only because of the various simplifying assumptions, especially the ones based upon the slowness of the ion waves. Detailed knowledge of g(x, y) is not needed; for example, the properties of the ion waves are determined entirely by $\int_0^\infty g(x, y)\,dx$, as shown in Appendix H. We calculate functions of x at only the 28 values x_i with $0.1 \leq x_i \leq 3.1$ for $1 \leq i \leq 15$, and $3.4 \leq x_i \leq 7$ for $16 \leq i \leq 28$. The range $u_{min} \leq u \leq u_{max}$ of ion-wave speeds is divided into 50 equal intervals with endpoints u_ℓ.

The purpose of Program 2 is to use the results from Program 1 obtained with very small E (usually 10^{-4}) to calculate the "known" coefficients in Eq. (VI-100) and other functions needed in Model B. The results are given at the values of x_i and u_ℓ indicated above. Each run of Program 2 requires about 40 seconds on an IBM-7044 computer.

1. Procedure in Program 3

This is the main program for Model B. The range $-1 \leqslant y \leqslant 1$ is divided symmetrically about $y = 0$ into as many as 70 intervals with endpoints y_j. For convenience in evaluating Eq. (VI-57) and Eq. (VI-106), the functions of θ are determined at the values of θ within $0 \leqslant \theta \leqslant \pi$ that satisfy $\sin\theta = y_j$ for some value of j.

Since substitution of Eq. (VI-105) into Eq. (VI-100) yields a diffusion-like equation, we again attempt to avoid computational difficulties by using a known implicit method.[66] This is similar to the method in Model A with $\rho = 0$ and without corrective iterations on each time step. On the odd-numbered time steps, we used the previously described implicit procedure to determine the x dependence of g(x, y) at each y_j; the terms in Eq. (VI-100) involving $\partial g/\partial x$ and $\partial^2 g/\partial x^2$ are evaluated at the "new" time and all other terms are evaluated at the "old" time. On the even-numbered time steps, the y dependence of g(x, y) for each x_i is found similarly; only the terms involving $\partial g/\partial y$ and $\partial^2 g/\partial y^2$ are evaluated at the "new" time. The coefficients needed at the "new" time are simply evaluated at the "old" time and no corrective iterations are performed. The amplitudes of fluctuations associated with unstable ion waves are again specified artificially, here by setting $I(\theta, u)$ equal to Ξ_{min} whenever the computer finds $I(\theta, u) \leqslant 0$ within $u_{min} < u < u_{max}$; ordinarily we choose $\Xi_{min} = 0.001$. Because of the discussion in Sec. VI. C, we remove the isotropic part of g(x, y) after each time step so that

$$\int_{-1}^{1} dy\, g(x, y) = 0. \qquad \text{(VII-4)}$$

Our procedure is based upon an implicit method that is known to give computationally stable solutions of

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \qquad \text{(VII-5)}$$

for arbitrarily large $\Delta\tau$ if Δx and Δy are constant and equal.[66] However, our scheme does not work well with Eq. (VI-100). Ordinarily we must still use $\Delta\tau \leq 0.01$, which would seem to be impractically small. Fortunately, each time step requires only 0.5 second on an IBM-7094 computer, and the solution converges to the steady state quite well within a physical time ν_c^{-1} and extremely well within $5\nu_c^{-1}$. The latter is not expected physically and is apparently the result of our numerical procedure.

The method of choosing the initial value of $g(x,y)$ and the value of E is the same as in Model A for the two cases $E_0 < E_{crit}$ and $E_0 \gtrsim E_{crit}$. Again, as E_0 becomes large, $\Delta\tau$ must be reduced to ensure convergence of the solution, but $\Delta\tau$ smaller than 0.001 is impractical. The longest runs involve 1000 time steps and require 8.5 minutes on an IBM-7094 computer.

2. Boundary Conditions and the Choice of y_j

The boundary conditions at $x = 0$ and $x = 7$, which are $g(0, y) = g(7, y) = 0$, cause no difficulty. The boundary condition at $x = 0$ can be changed to $\partial g(x, y)/\partial x = 0$ with an almost undetectable change in the results. This is desirable because of the simplifying assumptions that are not valid at small x and because of the rather widely spaced values of x_j at small x.

The boundary condition at $y = \pm 1$ was shown at the end of Sec. VI to be that $\partial g/\partial y$ be finite, which implies $\partial g/\partial \alpha = 0$. If we were using α as a variable, we would equate the values of g at $\alpha = 0$ and at the point closest to 0 but positive (and similarly at $\alpha = \pi$). This would be adequate if the spacing of the points in α were sufficiently small. We see that this is equivalent to using the boundary condition $\partial g/\partial y = 0$ at $y = \pm 1$ with an appropriate choice of y_j. This we do because the actual condition, that $\partial g/\partial y$ be finite at $y = \pm 1$, is too indefinite for use.

To test this procedure, the choice of y_j has been varied. With the range $-1 \leq y \leq 1$ divided into either 40 or 70 equal intervals, all results are nearly the same except in the immediate neighborhoods of $y = \pm 1$. The important waves have θ near π and are affected only by $g(x, y)$ with y near zero; as we expected, these are very insensitive to the choice of y_j. The quantities that are most sensitive to the choice of y_j are the small corrections to the electrical current and the heat flow, both of which are sensitive to $g(x,y)$ near $y = \pm 1$. Any large error resulting from the boundary condition at $y = \pm 1$ or from the choice of y_j should have appeared with this change of Δy by nearly a factor of two. We also defined y_j by dividing $0 \leq \alpha \leq \pi$ into 70 equal intervals; this division gives very closely spaced values near $y = \pm 1$. However, the numerical error in evaluating $\partial/\partial y$ and $\partial^2/\partial y^2$ by difference equations is apparently excessive in this case; still, the ion waves are hardly affected by the choice of y_j.

Ordinarily we have chosen y_j by dividing the range $-1 \leq y \leq 1$ into 40 equal intervals rather than 70 simply because the computation is faster and the volume of output is smaller. Notice that with the equal intervals of $\sin\theta$, the values of θ are most closely spaced near $\theta = \pi$ (the region of most interest to us) and near $\theta = 0$.

3. Calculation of the Fluctuation Spectrum

From Eq. (IV-9) we find

$$\left\langle \frac{E^2}{8\pi} \right\rangle = \int d^2\hat{\underline{k}} \int_0^\infty dV\, W(\hat{\underline{k}}, V) \qquad \text{(VII-6)}$$

with
$$W(\hat{\underline{k}}, V) = \frac{1}{8\pi^4} \int_0^\infty dk \frac{k^4 H(\hat{\underline{k}}, V)}{[k^2 - R(\hat{\underline{k}}, V)]^2 + I^2(\hat{\underline{k}}, V)} . \qquad \text{(VII-7)}$$

Notice that $W(\hat{\underline{k}}, V)$ is defined somewhat differently than Rostoker's "energy per mode."[41] The integral over k diverges at large k because of the self energy of each particle, which could be subtracted out. When $R(\hat{\underline{k}}, V)$ greatly exceeds $|I(\hat{\underline{k}}, V)|$, the resonance yields

$$W_{res}(\underline{\hat{k}}, V) = \frac{[R(\underline{\hat{k}}, V)]^{3/2}}{16\pi^3} \frac{H(\underline{\hat{k}}, V)}{|I(\underline{\hat{k}}, V)|} , \qquad \text{(VII-8)}$$

which represents the distribution of electric-field energy of the fluctuations associated with waves. It is now convenient to define, by means of the dimensionless quantities of Sec. VI. C,

$$W(\theta, u) = \frac{\Lambda a_e}{n_e \theta_e} W_{res}(\underline{\hat{k}}, V) = \frac{1}{2\pi^2} \left[\frac{2 \ln \Lambda}{\pi} R(u)\right]^{3/2} \frac{H(\theta, u)}{I(\theta, u)} . \qquad \text{(VII-9)}$$

We next define

$$\frac{\Lambda}{n_e \theta_e} \left\langle \frac{E^2}{8\pi} \right\rangle_{\text{ion waves}} = 2\pi \int_{-1}^{1} d(\cos\theta) \int_{u_{min}}^{u_{max}} du\, W(\theta, u), \qquad \text{(VII-10)}$$

even though the condition $R(\underline{\hat{k}}, V) \gg |I(\underline{\hat{k}}, V)|$ is not satisfied everywhere within $-1 \leqslant \cos\theta \leqslant 1$ and $u_{min} \leqslant u \leqslant u_{max}$. We do have $R(u)$ and $I(\theta, u)$ positive throughout these ranges so $W(\theta, u)$ is a positive quantity, and when $I(\theta, u)$ is not sufficiently small, $W(\theta, u)$ will hopefully not be large. There is no way to define the energy in the ion waves without some such arbitrary choice.

The purpose of Program 4 is to evaluate Eq. (VII-9) and Eq. (VII-10) for Model B with results from Programs 2 and 3. A typical run requires 1 minute on an IBM-7044 computer.

VIII. NUMERICAL RESULTS

After the results are presented, they will be used in evaluating the validity of the models considered. The plasma parameter Λ appears in our dimensionless equations only through $\ln \Lambda$ in the definition of R(u) by Eq. (VI-47), so we choose $\ln \Lambda = 10$ throughout the numerical work. We also choose the mass ratio δ^2 as appropriate for an electron-proton plasma.

A. Results Obtained with Model A

This model correctly gives the linear solution valid for $E_0 \ll E_{crit}$. The two cases -- $E_0 \ll E_{crit}$ and $E_0 \gtrsim E_{crit}$ -- are considered separately.

1. With $E_0 \ll E_{crit}$

Here Model A reduces to the Spitzer-Härm[1] problem except that in Model A the effects of the fluctuations associated with ion waves are included. Although the cutoff procedure and the convenient separation of "collisions" and "wave-particle interactions" introduces an uncertainty of order $1/\ln \Lambda = 10\%$ in the results, there is no serious objection to the model in this case.

The electrical conductivity σ is defined by

$$j = -2\pi(n_e e a_e)\int_0^\infty x^3 dx \int_{-1}^{1} y\, dy\, h(x, y) = \sigma E_0, \qquad \text{(VIII-1)}$$

and with our definition of E_{run}, the Spitzer-Härm result is

$$\sigma_{SH} = \frac{n_e e a_e}{E_{run}} . \qquad \text{(VIII-2)}$$

Figure 4 shows the effect of the ion waves upon σ. With $\theta_e = \theta_i$ the ion waves have no effect with our approximations, and we find $\sigma/\sigma_{SH} = 0.99939$ when an appropriate point spacing is used. As θ_e/θ_i is increased, the ion waves become important and reduce σ, as expected from the discussion in Sec. VI. C. 4. The values cannot be taken

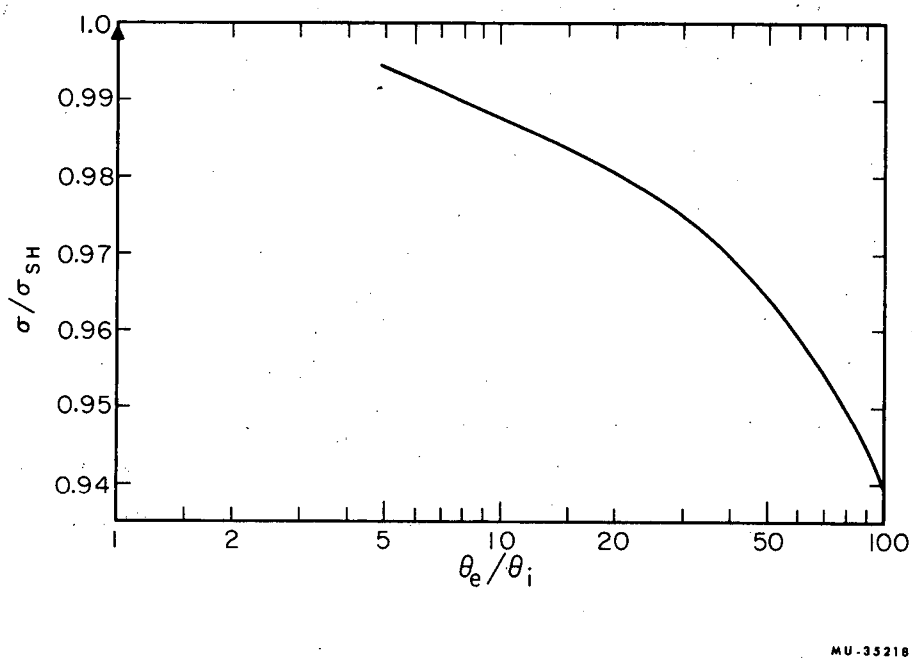


Fig. 4. Effect of ion waves on the linear electrical conductivity σ in an electron-proton plasma with $\ln \Lambda = 10$. Here σ_{SH} is the Spitzer-Härm[1] value. At $\theta_e = \theta_i$, σ/σ_{SH} is 0.99939.

literally because of the inherent 10% uncertainty, but the trend is significant. Actually, the decreases in σ by 1.2% at $\theta_e/\theta_i = 10$ and by 6.1% at $\theta_e/\theta_i = 100$ agree well with the values 1.1% and 5.3% found by Gorbunov and Silin.[63]

The thermoelectric coefficient β is defined by

$$Q = 2\pi(n_e a_e \theta_e) \int_0^\infty x^5 \, dx \int_{-1}^{1} y \, dy \, h(x, y) = - \beta E_0, \qquad \text{(VIII-3)}$$

and with our definition of E_{run}, the result obtained by Shkarofsky, Bernstein, and Robinson is[12]

$$\beta_{SBR} = \frac{0.5064}{0.3951} \frac{5}{2} \frac{n_e a_e \theta_e}{E_{run}} . \qquad \text{(VIII-4)}$$

Figure 5, which is strikingly similar to Fig. 4, shows the effect of ion waves upon the linear β. With $\theta_e = \theta_i$, we find $\beta/\beta_{SBR} = 0.99771$. As Gorbunov and Silin define the heat flow relative to the electron frame rather than to the ion frame, to compare our results with theirs we must use the connection formula

$$\frac{\beta}{\beta_{SBR}} = \frac{\beta_{GS}}{\beta_{SBR}} + \frac{0.3951}{0.5064} \frac{\sigma}{\sigma_{SH}} . \qquad \text{(VIII-5)}$$

Then the decreases in β by 1.65% at $\theta_e/\theta_i = 10$ and by 6.4% at $\theta_e/\theta_i = 100$ agree fairly well with the values 2.35% and 7.8% found by use of the results of Gorbunov and Silin in Eq. (VIII-5).

Because this model is linear in E_0 when E_0 is small compared to E_{crit}, we may calculate E_{crit} by defining E_0/E_{crit} as the largest value of $I_1(u)/I_0(u)$ within $u_{min} < u < u_{max}$. The results shown in Fig. 6 agree very well with those of Fig. 2 when the relationship $u_{crit}/a_e = 0.5064 \, E_{crit}/E_{run}$ is used, as was suggested in Sec. III. E. For θ_e/θ_i less than about 20, the agreement is within about 2%, and even at larger values of θ_e/θ_i, for which the ion waves increase E_{crit}

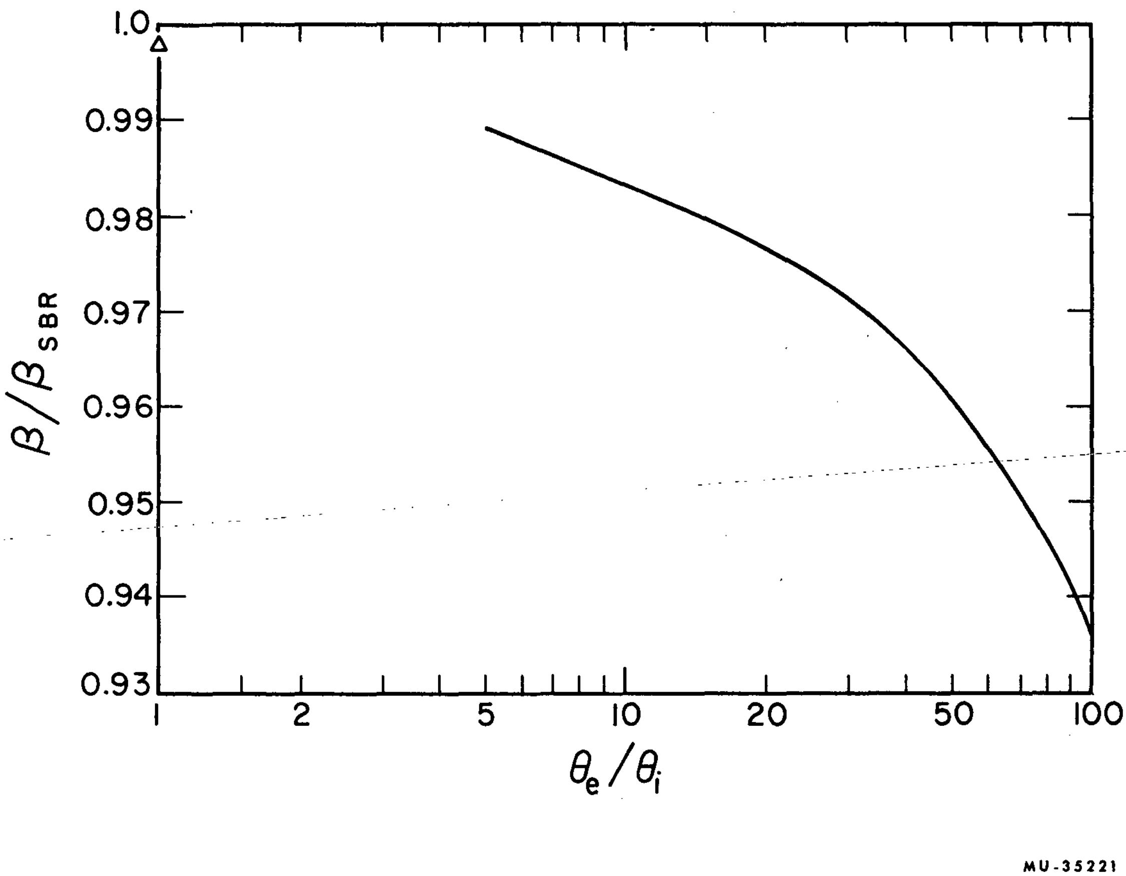


Fig. 5. Effect of ion waves on the linear thermoelectric coefficient β. Here β_{SBR} is the value given by Shkarofsky, Bernstein, and Robinson.[12] At $\theta_e = \theta_i$, β/β_{SBR} is 0.99771.

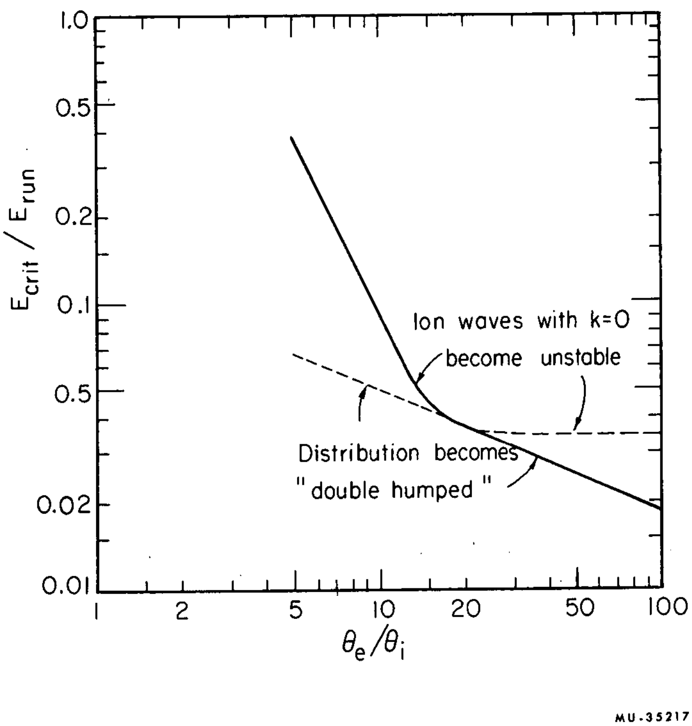


Fig. 6. The critical field above which (according to the linearized kinetic equation including the effect of ion waves) an electron-proton plasma would be unstable.

slightly, the values of E_{crit} from Fig. 6 do not exceed those from Fig. 2 by more than 5%.

The results of this linear problem are also used as input data for Model B.

2. With $E_0 \gtrsim E_{crit}$

In this case Model A is not satisfactory. With $\theta_e/\theta_i = 50$, only a very weak nonlinear stabilization was found. To maintain stability with $E_0 = 0.02519 = 1.005\, E_{crit}$, for example, some of the ion waves have damping rates smaller than with $E_0 = 0$ by a factor 10^{-3}; the corresponding fluctuations have energies 1000 times as large as with $E_0 = 0$. No stable solutions were found with E_0 exceeding E_{crit} by more than 1%.

As verified with Model B (Sec. VIII. C), Model A is inadequate primarily because the angular dependences of the electron velocity distribution and of $I_0(u) + I_1(u)\cos\theta$ are too inflexible. We noted in Sec. VI. C. 3 that as $|I_1(u)|$ approaches $I_0(u)$, the effect of the fluctuations associated with ion waves grows only logarithmically.

B. Survey of Results from Model B

This model demonstrates the nonlinear stabilization very well. The solution

$$h(x, y) = \frac{1}{\pi^{3/2}} e^{-x^2} + \left(\frac{E_0}{E_{run}}\right) f(x)y + g(x, y) \qquad \text{(VIII-6)}$$

implies

$$I(u, \theta) = I_0(u) + \left(\frac{E_0}{E_{run}}\right) I_1(u)\cos\theta + I_g(\theta). \qquad \text{(VIII-7)}$$

We will refer to these with $g(x, y)$ and $I_g(\theta)$ neglected as the linear solution, which is the solution of Model A with $E_0 << E_{crit}$. The corrections $g(x, y)$ and $I_g(\theta)$ that are found in the full nonlinear solutions of Model B primarily result from the nonlinear effect of fluctuations associated with ion waves.

Figure 7 shows the damping rate $-\gamma$ of the ion wave that would first become unstable as E_0 increases, according to the linear solution. Here $-\gamma_0$ designates this damping rate when E_0 vanishes. From the linear solution, γ/γ_0 would decrease linearly along the dashed line until, at $E_0 = E_{crit}$, γ would change sign and the plasma would become unstable. The solid curves show the nonlinear solution; as γ approaches zero the nonlinear effect becomes large and prevents γ from approaching zero too closely. This is a very clear demonstration of the nonlinear stabilization. The value of E_0 was not increased beyond the values shown in Fig. 7 simply because of the cost of computer time. Extrapolation suggests that the plasma would remain stable at much larger values of E_0.

This stabilization is most effective with large θ_e/θ_i. The reason is indicated in Fig. 8, which shows the energy in fluctuations associated with ion waves. Although in Fig. 7 γ/γ_0 is larger with large θ_e/θ_i, Fig. 8 shows that the amount by which the energy in the fluctuations increases is much greater when θ_e/θ_i is large. This is partly because $|\gamma_0|$ decreases and the fluctuation energy with $E_0 = 0$ increases as θ_e/θ_i increases. In fact $\gamma_0 = -0.928\times10^{-4}\,\omega_{pe}$ at $\theta_e/\theta_i = 40$, $\gamma_0 = -0.559\times10^{-4}\,\omega_{pe}$ at $\theta_e/\theta_i = 70$ and $\gamma_0 = -0.349\times10^{-4}\,\omega_{pe}$ at $\theta_e/\theta_i = 100$. Notice that with $\ln\Lambda = 10$ we find that $\nu_c = 2.4\times10^{-4}\,\omega_{pe}$ is of the same order; this difficulty is discussed in Sec. VIII. D.

The energy in fluctuations associated with ion waves as shown in Fig. 8 was calculated from the somewhat arbitrary definition given in Sec. VII. B. However, discussion in Sec. VIII. C shows that this amounts to choosing the position of the baseline for the curves in Fig. 8; the increments are unaffected by this definition.

Because the rate of spontaneous emission remains nearly unchanged, the decrease in $|\gamma|$ in Fig. 7 by a factor of order 10^{-2} as E_0 increases from zero to its largest value indicates that the energy in fluctuations associated with the particular ion wave considered increases by a factor of 100. However, according to Fig. 8, the energy in the fluctuations associated with all ion waves increases by a factor

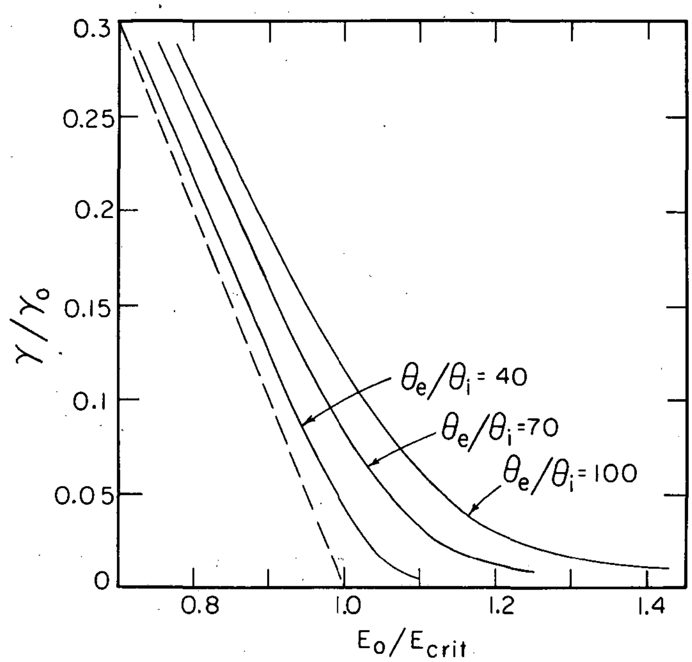


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Fig. 7. The damping rate $-\gamma$ of the ion wave that would first become unstable according to a linearized kinetic equation.

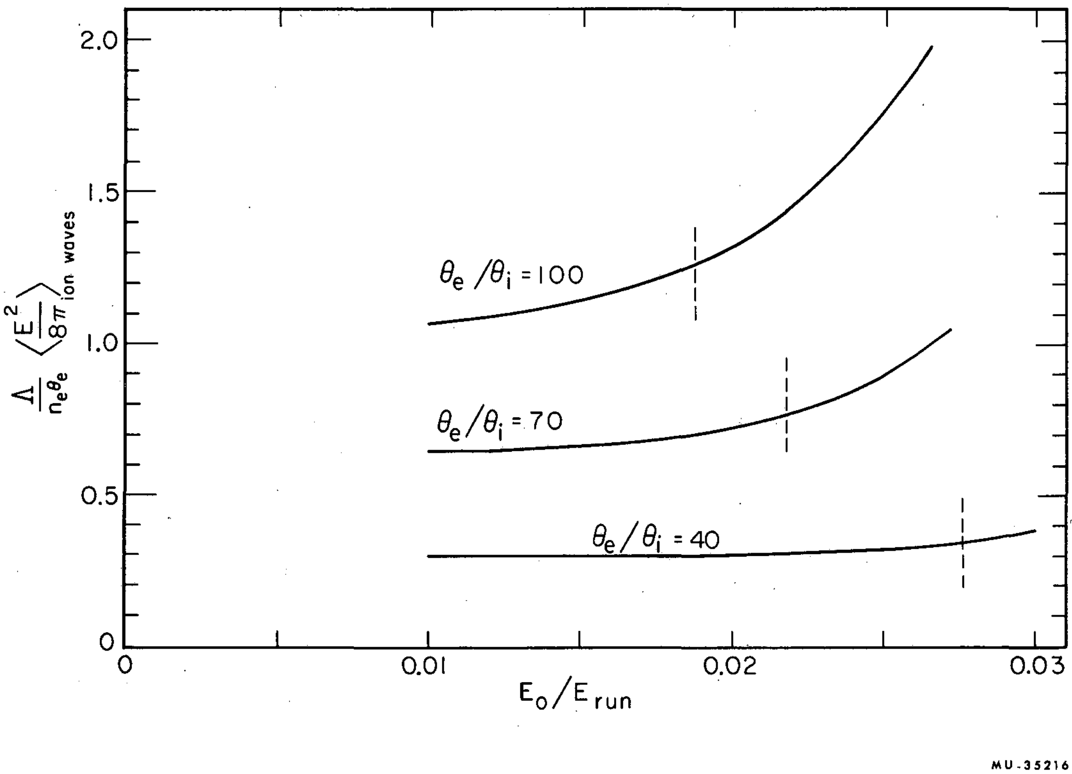

Fig. 8. The energy in fluctuations associated with ion waves. The vertical dashed lines denote E_{crit} in the various cases.

of less than two, and remains smaller than the kinetic energy of the particles by a factor of about Λ^{-1}.

Similar calculations were attempted with $\theta_e/\theta_i = 10$, but no nonlinear stabilization was found. In this case the plasma approaches instability by $I(u, \theta)$ approaching zero where $R(u)$ vanishes (see Sec. III. D), so according to Eq. (VII-9) the increase in the energy of the fluctuations is very small. Actually, nonlinear stabilization is still expected in this case, but one probably should use Eq. (V-3) rather than the approximation (V-4), which is poor when both $|R(u)|$ and $|I(u, \theta)|$ are very small. Even then, the nonlinear stabilization would probably be weak and thus difficult to study numerically.

Figures 9 and 10 show the nonlinear electrical conductivity σ and the thermoelectric coefficient β as found from Eqs. (VIII-1) and (VIII-3). By comparing Figs. 9 and 10 with Fig. 8, we see that the effect of the fluctuations associated with ion waves is roughly proportional to their energy. With E_0 larger than E_{crit}, the effect upon σ and β is significant, and as E_0 increases further, the corrections may become important. With the displaced-Maxwellian model of Sec. II, the first nonlinear effect on σ is an increase when E_0 approaches E_{run}; this effect is quite different from the results shown in Fig. 9.

Figures 8, 9, and 10 contain no results for $E_0/E_{run} < 0.01$ because of a systematic error that becomes larger as E_0 decreases. The origin of this error is not known. The error is not seriously large until E_0 becomes very small. It is nonphysical because, for example, σ tends to decrease as E_0 decreases, which does not agree with Model A or with our expectations.

C. Samples of Detailed Results from Model B

To present further details, we consider only $\theta_e/\theta_i = 70$, so $E_{crit}/E_{run} = 0.0217$. Often only E_0/E_{run} equal to 0.01, 0.0228, or 0.02715 are considered, since these represent the regions of E_0 below, near, and above E_{crit}.

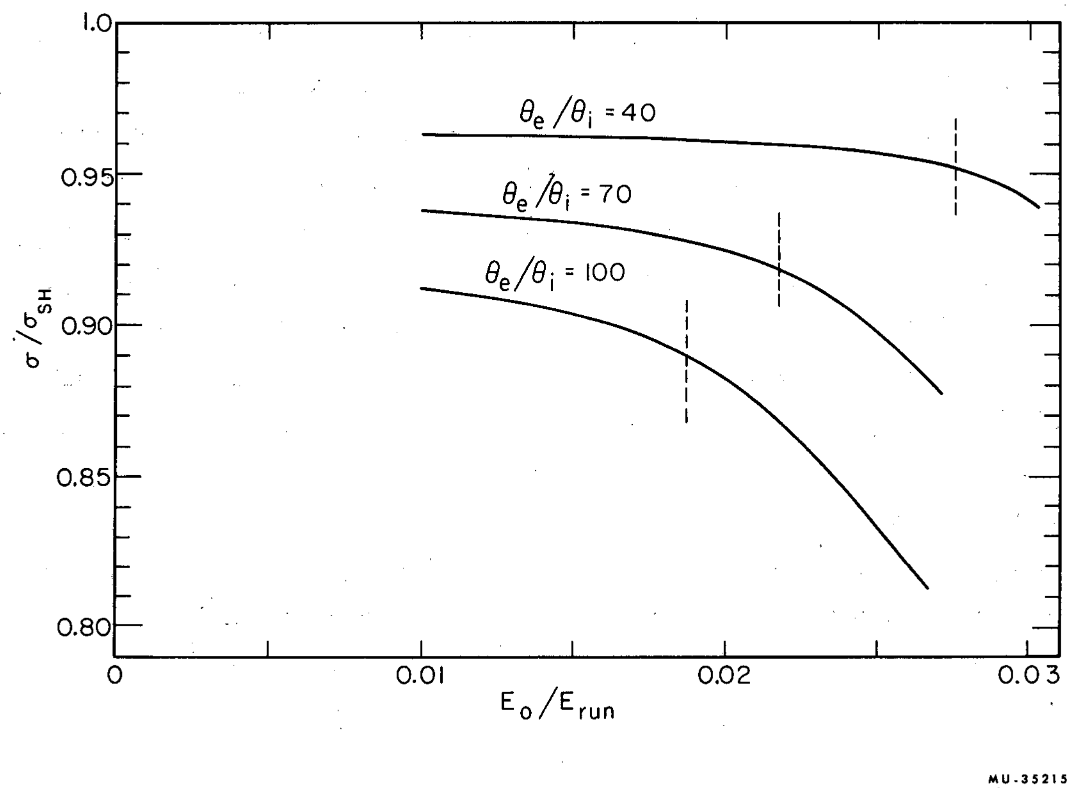

Fig. 9. The nonlinear electrical conductivity σ. The vertical dashed lines denote E_{crit} in the various cases.

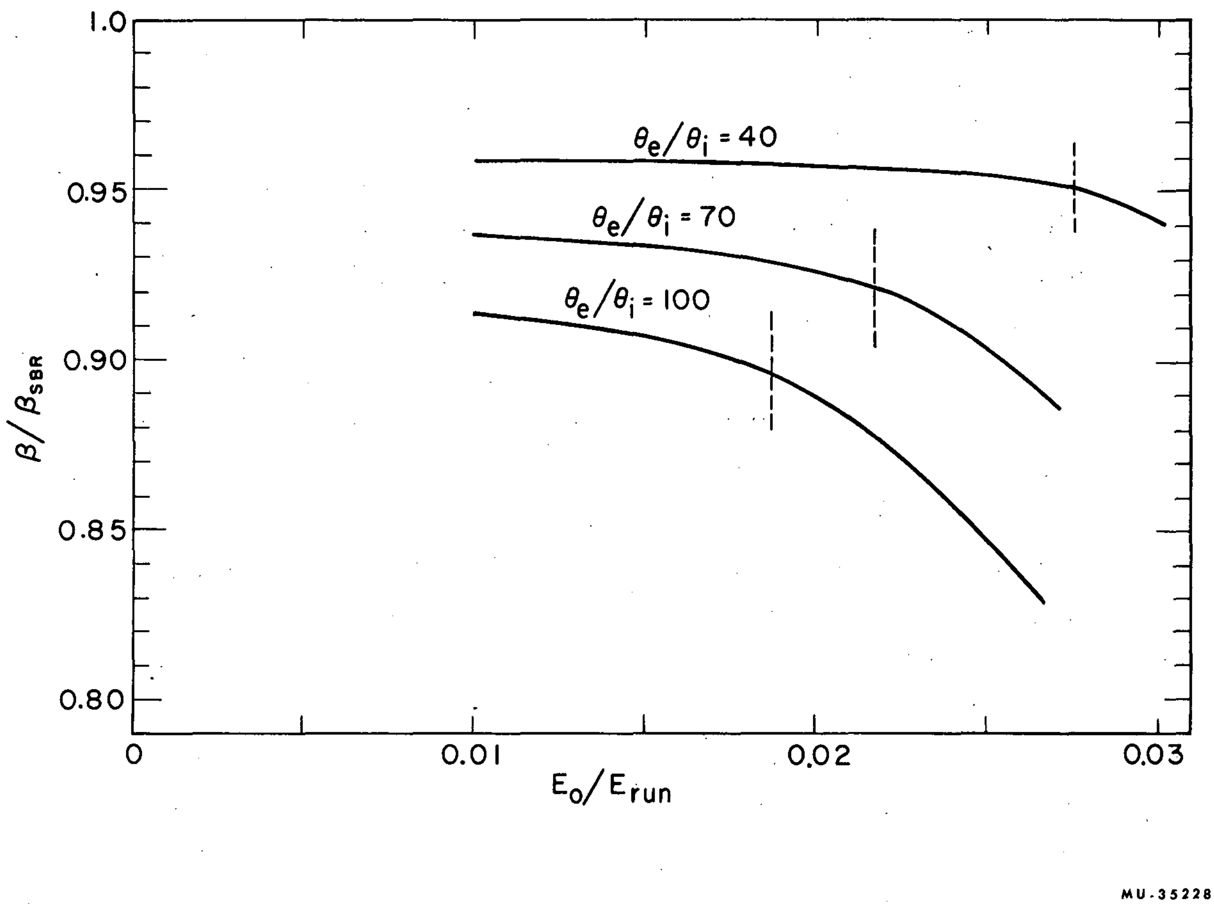

Fig. 10. The nonlinear thermoelectric coefficient β. The vertical dashed lines denote E_{crit} in the various cases.

1. Damping Rate of Ion Waves

We consider waves with phase speed equal to that of the wave that would first become unstable according to the linear solution, but we consider all directions of propagation. Figure 11 shows the damping rate $-\gamma$ of such waves for all directions θ and various values of E_0. With $E_0 = 0$ the plasma is isotropic, and $\gamma = \gamma_0$ by definition. According to the linear solution (shown by the dashed lines), as E_0 is increased the anisotropic correction is proportional to $\cos\theta$ and to E_0. When E_0 exceeds E_{crit}, the waves with θ near π (so $\hat{\underline{k}}$ is near the direction the electrons are accelerated by $\underline{E}_0$) would grow and the plasma would be unstable.

The nonlinear solution, as shown by the solid lines of Fig. 11, behaves similarly until E_0 becomes comparable to or exceeds E_{crit}. Then the nonlinear correction adjusts itself to be just sufficient to maintain the plasma stability, as shown by the curve with $E_0/E_{run} = 0.02715$ in Fig. 11. With the simplifying assumptions of Model B, the nonlinear correction is an odd function of $\theta - \pi/2$ as indicated. The intercept of such curves at $\theta = \pi$ as E_0 is varied provides the data for Fig. 7.

Notice in Fig. 11 that with $E_0/E_{run} = 0.02715$ the nonlinear correction deviates considerably from being proportional to $\cos\theta$, the form to which it was restricted in Model A. In fact $|\gamma|$ remains small over a considerable region of θ near π. This is significant because solid-angle considerations weight the importance of the various regions of θ by a factor $\sin\theta$. Thus, in Model A, even when the intercept at $\theta = \pi$ is very near $\gamma = 0$, the shape of the curve is so restricted that the effect of the fluctuations associated with ion waves does not become really large. This is apparently the major reason for the failure of Model A.

If one considers a phase speed different from that of Fig. 11, the primary effect is to lower all curves in Fig. 11 so that all waves are more highly damped. This follows from Eq. (III-26) for γ and from Eq. (VIII-7) for $I(u, \theta)$ since it turns out that $I_1(u)$ is very nearly

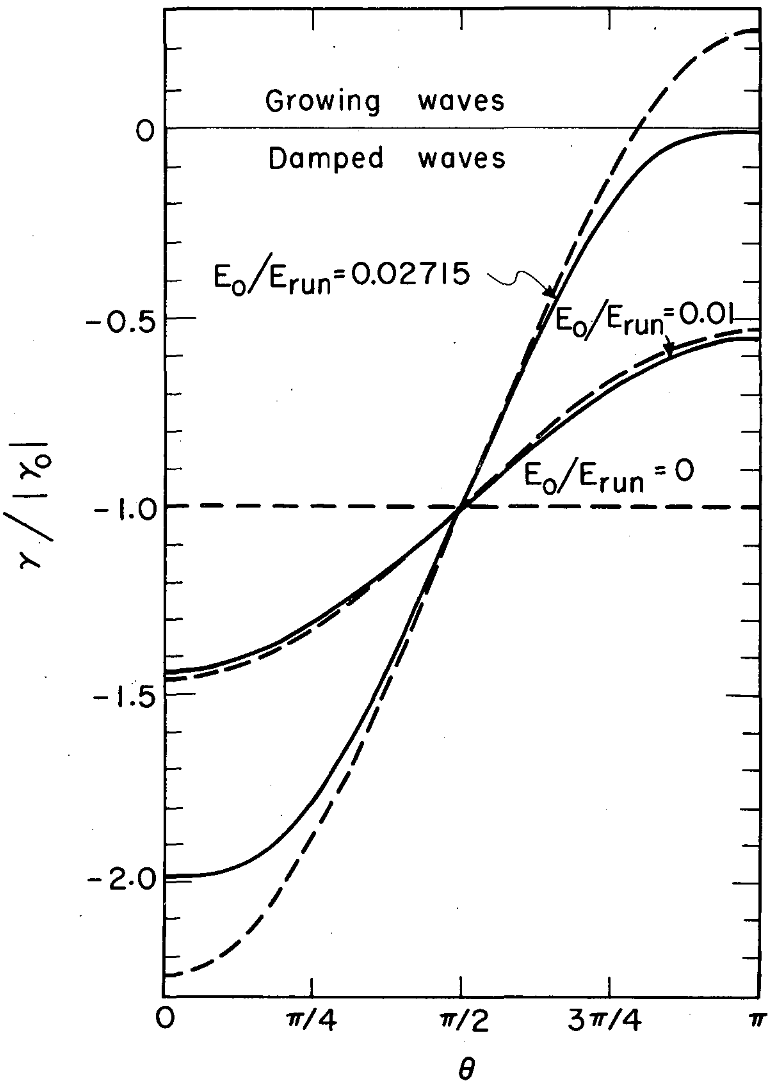

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Fig. 11. The damping rate $-\gamma$ of ion waves propagating in any direction θ with phase speed $u = V/a_e = 0.01031$. Here $\theta_e/\theta_i = 70$ so $E_{crit}/E_{run} = 0.0217$. The dashed curves show the results from a linearized kinetic equation, and the solid lines show the nonlinear results.

independent of u and $\partial R(u)/\partial u$ is a slowly varying function of u. The amount the curves are lowered is determined by the variation of $I_0(u)$.

We conclude that the nonlinear effect of wave-particle interactions does not greatly affect the damping rate $-\gamma$ of ion waves that would be stable even from the linear solution, but it changes the damping rate of the waves that would be unstable just enough to stabilize the plasma.

2. Energy in Fluctuations Associated with Ion Waves

As mentioned before, the energy of fluctuations that correspond to very weakly damped ion waves is large, because in the Lenard-Balescu kinetic equation the fluctuations arise from a balance of spontaneous emission and Landau damping.

The energy density $\langle E^2/8\pi \rangle$ of fluctuations associated with ion waves per unit $u = V/a_e$ and per unit solid angle of $\hat{\underline{k}}$ is defined by Eq. (VII-9) and denoted by $W(\theta, u)$, in units of $n_e\theta_e/\Lambda$. The data for Fig. 8 are obtained by integrating $2\pi \sin\theta \; W(\theta, u)$ over $0 \leqslant \theta \leqslant \pi$ and $u_{min} \leqslant u \leqslant u_{max}$. The factor $2\pi \sin\theta$ is the solid-angle factor mentioned in Sec. VIII. C. 1.

Figures 12, 13, and 14 show examples of the behavior of $W(\theta, u)$. Notice that the results are given at values of θ that are not quite evenly spaced.

With E_0 much smaller than E_{crit}, $W(\theta, u)$ is nearly independent of θ as was suggested by Fig. 11 and is shown in Fig. 12. In Fig. 12 the peak at small phase speed actually does not correspond to weakly damped ion waves, so the definition of $W(\theta, u)$ is somewhat arbitrary in this case. This peak could have been omitted in the integral that yields the data for Fig. 8, but since it does not change as E_0 is varied, its contribution to the quantity shown in Fig. 8 is simply an additive constant for each value of θ_e/θ_i.

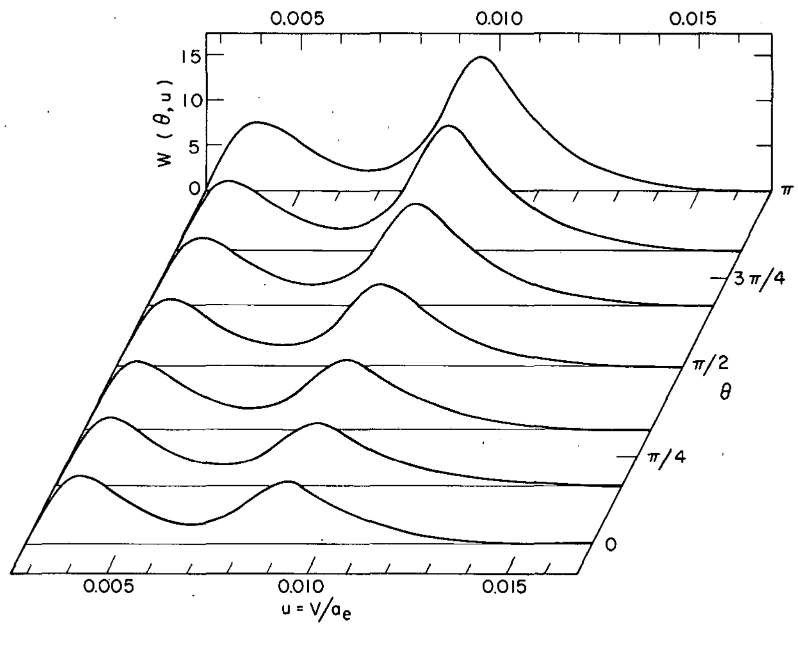

Fig. 12. Distribution of energy in fluctuations associated with ion waves over direction of propagation θ and phase speed u with $E_0/E_{run} = 0.01$. Here $\theta_e/\theta_i = 70$, so $E_{crit}/E_{run} = 0.0217$, $u_{min} = 0.002617$, and $u_{max} = 0.01684$.

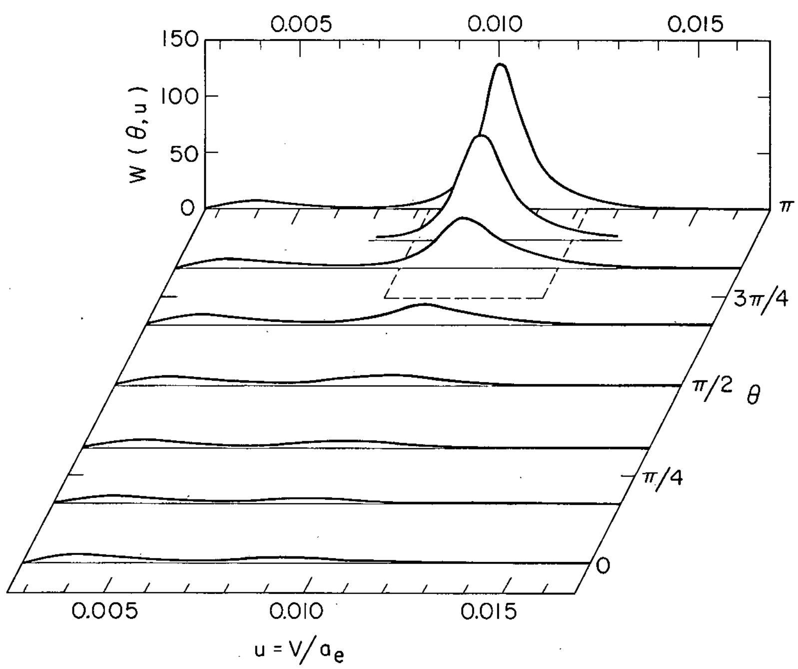

Fig. 13. Distribution of energy in fluctuations associated with ion waves with $E_0/E_{run} = 0.0228$ and $\theta_e/\theta_i = 70$. The dashed line encloses the region shown in Fig. 14.

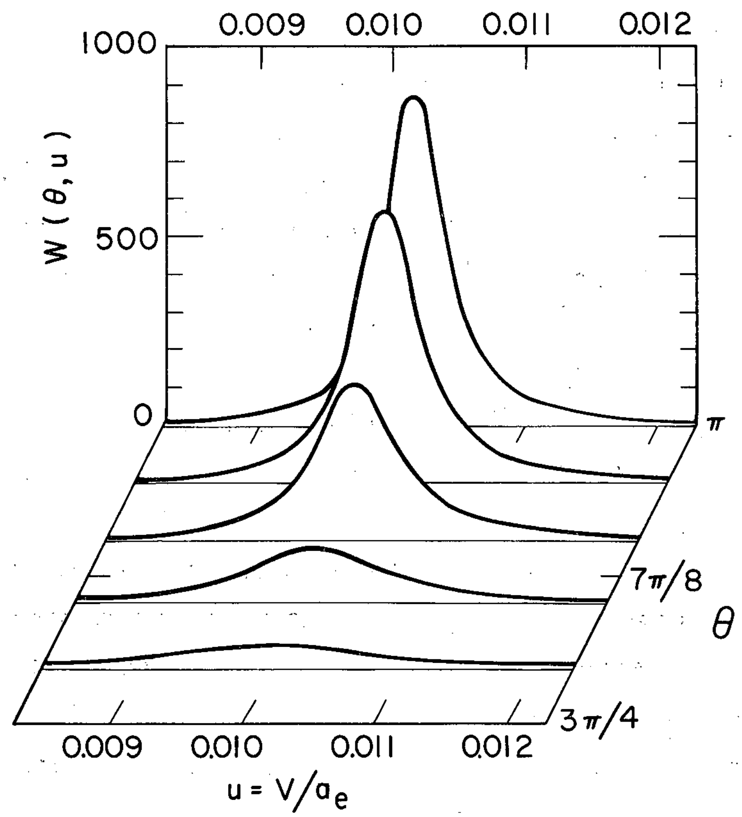

Fig. 14. Distribution of energy in fluctuations associated with ion waves with $E_0/E_{run} = 0.02715$ and $\theta_e/\theta_i = 70$. The region shown is outlined by a dashed line in Fig. 13.

With E_0 slightly larger than E_{crit}, the ion-wave peak becomes much larger for θ near π. This is shown in Fig. 13, where the vertical scale has been changed by a factor of 8 so the peak at small phase speed does not appear so large, although it actually is unchanged. The ion-wave peak is somewhat smaller for θ near 0 than in Fig. 12, but it is much higher and narrower for θ near π.

As E_0 is increased even further, the ion-wave peak becomes even higher and narrower for θ near π. With another change in the vertical scale, Fig. 14 shows this for the region enclosed by the dashed line in Fig. 13. In this case the peak is nearly 100 times higher than when E_0 vanishes, in agreement with the damping rates of Figs. 7 and 11.

Because of the solid-angle factor $2\pi \sin\theta$, the large ion-wave peak in Figs. 13 and 14 does not contribute as much to the quantity in Fig. 8 as one might expect otherwise.

The large peak in the fluctuations associated with ion waves could have important effects that we have not mentioned. The scattering of light by the plasma would be modified; in principle this could be used to measure the fluctuation spectrum quite directly. All transport coefficients would be modified, including spatial diffusion if the plasma were slightly nonuniform.

3. Modification of the Electron Velocity Distribution

As discussed in Sec. VI. C. 4, the fluctuation spectrum shown in Figs. 12, 13, and 14 produces a diffusion of the electron velocity distribution in the angular direction. In Eq. (VI-105) the diffusion coefficient is $S(y)/x^3$ where $y = \cos\alpha$ and $x = v/a_e$. In Fig. 15 we see that as E_0 increases, the ion-wave peak of Figs. 13 and 14 becomes large and the even function $S(y)$ increases, mainly in the region of y near zero. This is as expected because the resonant particles for an ion wave with θ near π are in the region of velocity space with α near $\pi/2$ or y near zero. This diffusion always tends to make the electron velocity distribution more nearly isotropic, and from Fig. 15, the effect should be strongest for y near zero.

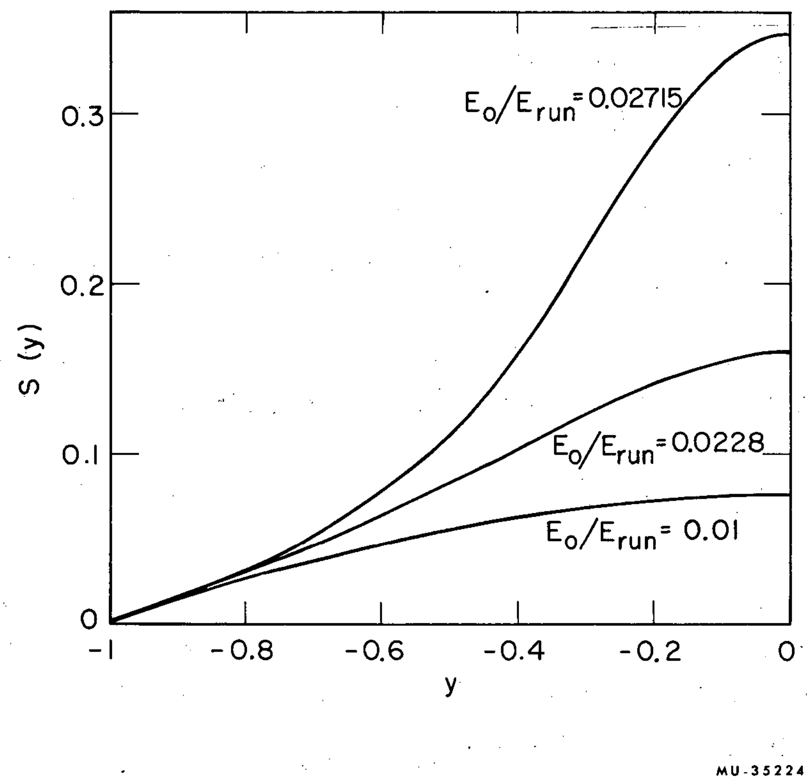

Fig. 15. The function S(y) that determines the effect of fluctuations associated with ion waves upon the electron-velocity distribution. Here $\theta_e/\theta_i = 70$.

Figure 16 shows the anisotropic part of h(x, y) from Eq. (VIII-6) with $E_0/E_{run} = 0.02715$, which corresponds to Fig. 14 and the upper curves of Figs. 11 and 15. For each constant value of x the solid curve shows the linear solution $(E_0/E_{run})\, f(x)\, y$ and the dashed curve gives the nonlinear solution which includes g(x, y). The nonlinear correction g(x, y) is small, although with small x it very nearly cancels the anisotropy of the linear solution, as expected from the discussion in Sec. VI. C. 4.

Close inspection of Fig. 16 reveals that the correction that g(x, y) makes on the slope of the curves is largest for y near zero, as expected. This change most strongly affects the Landau damping of ion waves with θ near 0 or near π, the latter being the region of θ where the stabilization is necessary.

At larger values of x than those shown in Fig. 16, g(x,y) is unimportant in determining the Landau damping of the ion waves and the various transport coefficients, and the model actually fails because the anisotropic parts of h(x, y) become comparable with the isotropic Maxwellian part. However, the behavior of g(x, y) is still interesting, as is illustrated by Fig. 17. Clearly g(x, y) does not remain an odd function of y, but in Fig. 17(a) actually resembles the parabolic Legendre polynomial $P_2(y)$. Presumably the ion waves have little effect at this large value of x, so g(x, y) simply represents the small next term in a Legendre polynomial expansion. At even larger x, a definite peak forms at $y = -1$, as shown in Fig. 17(b). This represents a distortion of the velocity distribution that eventually blends into the region of runaway electrons. The effect of our incorrect boundary conditions at $y = \pm 1$ is apparent in Fig. 17. The important region of y near zero is unaffected by this.

D. Validity of the Models

The assumptions and approximations fall into two groups: those that are necessary for the Lenard-Balescu kinetic equations to be applicable and the ones that are convenient in finding approximate solutions

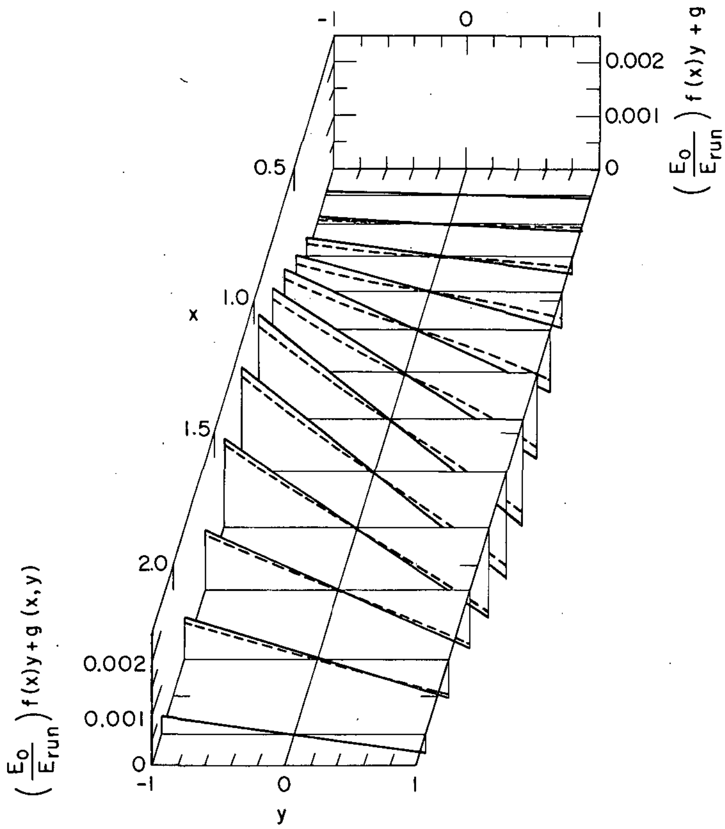

Fig. 16. Anisotropic part of the electron-velocity distribution with E_0/E_{run} = 0.02715 and θ_e/θ_i = 70. Here $x = V/a_e$ and $y = \cos\alpha$. The solid lines show the result from a linearized kinetic equation, and the dashed lines give the nonlinear result.

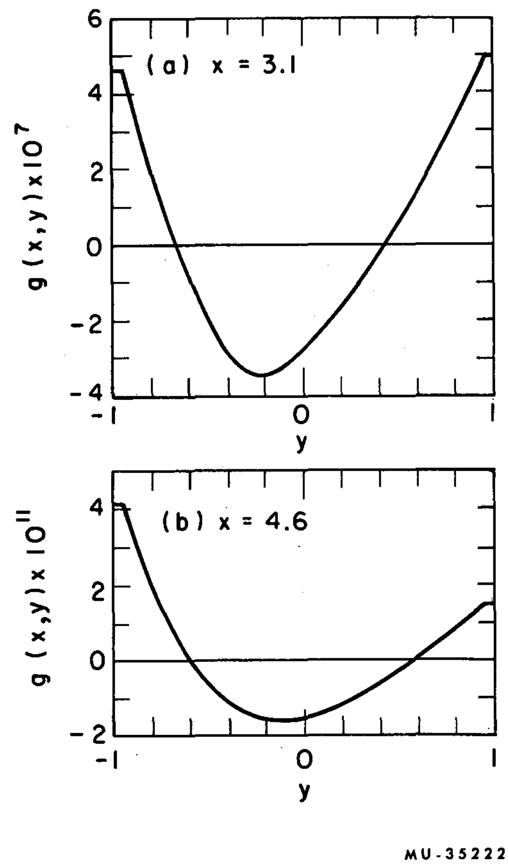

Fig. 17. Nonlinear correction to the electron velocity distribution at high electron speed. Here $E_0/E_{run} = 0.02715$ and $\theta_e/\theta_i = 70$.

of the Lenard-Balescu kinetic equations. These are discussed separately and evaluated in terms of the results obtained.

1. Applicability of the Lenard-Balescu Kinetic Equation

The restriction to a spatially uniform classical Coulomb plasma defines the problem being studied. If these simplifications are not justified for the real plasma being studied, the problem must be modified. For example, the effect of transverse waves may be important.[67] The problem might also be quite different in a magnetized plasma.[37]

The conditions that the plasma must be stable according to the linearized Vlasov equation and must vary "slowly" are certainly satisfied in the numerical solutions obtained. These conditions might be violated during the transient as the plasma approaches the quasi-stationary state calculated, but this transient presumably lasts no longer than a few collision times.

The derivation in Sec. IV. C indicated that if Λ were sufficiently large, the neglect of mode coupling and collisional effects on the waves would be justified. Here we present further verification of this in our particular problem. Notice again that in the dimensionless variables of Sec. VI. C, Λ appears in our equations only through a weighting of the relative importance of wave-particle interactions as inversely proportional to $\ln \Lambda$. Therefore the solutions obtained would also depend very weakly upon Λ, in the variables used. In fact the numerical solutions were obtained only for $\ln \Lambda = 10$ because no qualitative changes are expected when Λ is varied.

As Λ becomes larger, $|\nu_c/\gamma|$ in our solutions will vary roughly as Λ^{-1}, so for sufficiently large Λ the collisional effects on waves can certainly be ignored. As we noted in Sec. VIII. B, with $\ln \Lambda = 10$ so $\Lambda \approx 2\times 10^4$, $|\nu_c/\gamma|$ is as large as 10^2 or 10^3 in some of our solutions, so presumably collisional effects would not be negligible. But with $\ln \Lambda = 20$, for example, the solutions would differ little from those we obtained, and the neglect of collisional effects on the waves would certainly be justified.

According to Fig. 8, the ratio of the energy in waves to the kinetic energy of the particles is of order Λ^{-1}. The simple criterion that mode coupling can be ignored when this ratio is "sufficiently small" indicates that when Λ is "sufficiently large" the neglect of mode coupling is justified. We have no good way of estimating how large Λ must actually be. However, we notice that from Fig. 8 the energy of the waves is not much greater than when E_0 vanishes.

Even if Λ is not large enough to justify the neglect of collisional effects on the waves, the qualitative features of the Lenard-Balescu kinetic equations and of our solutions should remain valid. The dispersion relation and the expression for the damping rate $-\gamma$ would be different, but the fluctuation spectrum should still arise as a balance of spontaneous emission and damping. The plasma would probably continue to stabilize itself, but the stability would not be determined with the linearized Vlasov equations. The effect of collisions upon ion waves is a current topic of research; recently it has been shown that a slow collision rate can either increase or decrease γ, under different conditions.[68]

The situation when mode coupling cannot be ignored is not so clear. It again seems very unlikely that the fluctuations associated with any particular ion wave can continue to grow in time, except slowly as the plasma slowly changes.

2. The Models Solved Numerically

Certain approximations were made to further restrict the generality of the problem and to simplify the interpretation of the results. These include

(a) Considering an electron-proton plasma;

(b) Assuming that the velocity distributions are almost Maxwellian;

(c) Approximately separating collisions and wave-particle interactions;

(d) Calculating $R(\theta, V)$ using only the Maxwellian parts of the velocity distributions;

(e) Ignoring the effects of electron waves;

(f) Restricting the isotropic part of the electron velocity distribution to be Maxwellian with constant temperature θ_e.

These were adequately discussed in Sec. VI. Our numerical results show that these approximations served their purpose well. We expect that these simplifications are quite reasonable, except possibly for (c) when θ_e/θ_i is less than about 20.

Unfortunately, the equations remain very complicated even with the symmetry of our problem and with the above approximations. Therefore only certain model problems were studied numerically. Model A was useful only when E_0 is small compared to E_{crit}, and it requires no further justification in that case. To obtain the more useful Model B, three further approximations were made.

(g) The terms in the Lenard-Balescu equations arising from wave-particle interactions were simplified on the basis that the phase speed of the ion waves is small compared to the electron speeds. This approximation fails for the small fraction of electrons with small speeds, but, as is verified in our numerical solution, the region of $V \ll a_e$ makes no significant contribution to any quantity of interest. Thus, this approximation seems to be well justified.

(h) Because no similar approximation can be made in evaluating the ion velocity distribution, this distribution was chosen to be Maxwellian with constant temperature θ_i. This has not been fully justified because the wave-particle interactions might introduce significant distortions. However, we notice that even if the ion velocity distribution is distorted, the fluctuations associated with ion waves will tend to make the electron velocity distribution isotropic. Unless the ion velocity distribution is so distorted that ion waves would be unstable due to it alone, this will be a stabilizing effect and it will become large nonlinearly as instability is approached. Thus a Maxwellian velocity distribution for the ions can be considered a typical example.

(i) The terms that remain too complicated for numerical evaluation are those corresponding to "collisions" as represented by the Landau form of the Fokker-Planck equation. With anisotropic velocity distributions, these terms have been evaluated only in very special cases, and it is certainly not our purpose here to pursue this problem. We have thus been forced to make simplifications that amount to evaluating the Fokker-Planck coefficients by using the "known" parts of the velocity distribution and neglecting the contributions of the "unknown" and small $g(x, y)$. This resembles the approximation made in test-particle problems and Brownian motion problems, but in this case $g(x, y)$ does not represent the distribution of a different species of particle. No attempt has been made to evaluate the effect of this assumption on the results found.

We conclude that our model problems can be considered fairly realistic examples, except that assumption (i) has not been justified.

IX. CONCLUSIONS

According to the Lenard-Balescu kinetic equations, the primary effect of fluctuations associated with longitudinal ion waves upon the electron velocity distribution is to tend to make it isotropic, which is a stabilizing tendency unless the velocity distribution of the ions is very distorted. As the plasma is forced toward conditions where it would become unstable to certain ion waves, this nonlinear stabilization becomes stronger and prevents instability.

In our example of a current-carrying plasma with electron temperature high compared to ion temperature, we have demonstrated with a model problem that the plasma remains stable to ion waves for electric fields considerably above the critical field of the Spitzer-Härm problem. The fluctuations necessary to provide this stabilization have the same qualitative effects as "ordinary" collisions and substantially reduce the electrical conductivity and the thermoelectric coefficient.

Kinetic equations such as the Lenard-Balescu equations are useful in such problems because they include the effects of "ordinary" collisions and of wave-particle interactions and because the time scale involved is the relatively long collisional time scale. However, the Lenard-Balescu equations can be used only when the plasma parameter Λ, which is proportional to the number of particles in a Debye sphere, is large enough that the effects of mode coupling and collisions upon the waves can be ignored. In our examples, this requires a very hot low-density plasma.

ACKNOWLEDGMENTS

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APPENDICES

A. The Momentum-Transfer Cross Section

The differential cross section for an electron of speed v being deflected through an angle θ by stationary heavy ions of charge Ze is

$$\sigma_\theta = \frac{Z^2 e^4}{m^2 v^4 (1 - \cos\theta)^2} . \tag{A-1}$$

On this deflection, the component of momentum parallel to the initial electron velocity that is transferred to the ion is $m\underline{v}(1 - \cos\theta)$, so one ordinarily defines the momentum-transfer cross section by

$$\sigma_m(v) = 2\pi \int_0^\pi \sigma_\theta \sin\theta \ (1 - \cos\theta)\, d\theta, \tag{A-2}$$

but when σ_θ is given by Eq. (A-1), this integral diverges. Inserting a cutoff impact parameter b_{max} as discussed in Sec. II. B is equivalent to replacing the lower limit by $\theta_{min} = 2Ze^2/(mv^2 b_{max})$. We then obtain

$$\sigma_m(v) = \frac{2\pi Z^2 e^4}{m^2 v^4} \ln \frac{2}{1 - \cos\theta_{min}} . \tag{A-3}$$

When θ_{min} is very small, this reduces to Eq. (II-14).

The drag force exerted on an ion by an electron velocity distribution $f_e(\underline{v})$ (normalized to unity) is now given by

$$\underline{F}_i = \int d^3v \ n_e \ v\sigma_m(v) \ f_e(\underline{v}) \, m\underline{v} . \tag{A-4}$$

One usually ignores the logarithmic dependence of σ_m upon v in such integrals and evaluates the logarithmic term at some characteristic electron speed. The total drag force on the electrons per unit volume is then

$$\underline{F} = -\omega_{pe}{}^2 e^2 \ln \Lambda \sum_{\alpha}{}' n_\alpha Z_\alpha{}^2 \int d^3v \frac{\underline{v}}{v^3} f_e(\underline{v}) . \qquad \text{(A-5)}$$

The mean velocity of the electrons $\underline{U}$ then satisfies the equation of motion

$$\frac{d\underline{U}}{dt} + \frac{e}{m} \underline{E}_0(t) = \frac{1}{n_e m} \underline{F} . \qquad \text{(A-6)}$$

In the special case of steady state in an electron-proton plasma, this becomes

$$\underline{E}_0 = -\omega_{pe}{}^2 e \ln \Lambda \int d^3v \frac{1}{v^3} \underline{v} f_e(\underline{v}) . \qquad \text{(A-7)}$$

We evaluate the integral in Eqs. (A-5) and (A-7) for the special case where $f_e(\underline{v})$ is a displaced-Maxwellian distribution. To do this we first notice that the integral has the same form as the integral that gives the electric field at $\underline{r} = 0$ due to a charge distribution $\rho(\underline{r})$. In this case the "charge density" $f_e(\underline{v})$ is symmetric about $\underline{v} = \underline{U}$ so we know that the "electric field" at $\underline{v} = 0$ is the same as if all of the "charge" within $|\underline{v} - \underline{U}| < |\underline{U}|$ were at $\underline{v} = \underline{U}$ and the remaining "charge" were absent. Thus we have

$$\int d^3v \frac{\underline{v}}{v^3} f_e(\underline{v}) = \left[4\pi \int_0^{U/a_e} y^2 dy \frac{1}{\pi^{3/2}} e^{-y^2} \right] \frac{\underline{U}}{U^3} = \frac{4}{3\sqrt{\pi}} \frac{\underline{U}}{a_e{}^3} \Omega(U/a_e), \qquad \text{(A-8)}$$

where $\Omega(x)$ is given in Eq. (II-24). When we combine Eqs. (A-5), (A-6), and (A-8) and use Eq. (II-22), we find Eq. (II-23).

B. Expansions of the Dielectric Function

For any particular $\hat{\underline{k}}$ and positive k, we write

$$k^2 \epsilon(\underline{k}, \omega) = k^2 - G(V + i\Gamma) \qquad \text{(B-1)}$$

where we have defined $\omega/k = V + i\Gamma$, and V and Γ are real. When Γ is positive, $G(V + i\Gamma)$ is defined by Eq. (III-13), but otherwise it is defined by analytic continuation. When Γ is sufficiently small

$$G(V + i\Gamma) = G(V) + \Gamma \left[\frac{\partial G}{\partial \Gamma}\right]_{\Gamma=0} + \frac{\Gamma^2}{2}\left[\frac{\partial^2 G}{\partial \Gamma^2}\right]_{\Gamma=0} + \cdots . \qquad \text{(B-2)}$$

Because G is analytic, it satisfies the Cauchy-Riemann conditions and we have

$$\left[\frac{\partial^n G}{\partial \Gamma^n}\right]_{\Gamma=0} = (i)^n \frac{\partial^n G(V)}{\partial V^n} , \qquad \text{(B-3)}$$

where in our case

$$G(V) = R(\hat{\underline{k}}, V) + iI(\hat{\underline{k}}, V) . \qquad \text{(B-4)}$$

Therefore Eq. (B-1) can be written for sufficiently small $\Gamma = \gamma/k$ as

$$k^2 \epsilon(\underline{k}, kV + i\gamma) = \left[k^2 - R(\hat{\underline{k}}, V) + \Gamma \frac{\partial I(\hat{\underline{k}}, V)}{\partial V} + \mathcal{O}(\Gamma^2)\right]$$

$$+ i\left[-I(\hat{\underline{k}}, V) - \Gamma \frac{\partial R(\hat{\underline{k}}, V)}{\partial V} + \mathcal{O}(\Gamma^2)\right] \qquad \text{(B-5)}$$

which is the basis for discussing plasma waves.

We next consider the contribution of species α to $R(\hat{\underline{k}}, V)$, which from Eqs. (III-20) and (III-21) is

$$R_\alpha(\hat{\underline{k}}, V) = P \int \frac{dV'}{V'-V} \frac{\partial F_\alpha(V'; \hat{\underline{k}})}{\partial V'} \qquad \text{(B-6)}$$

with

$$F_\alpha(V'; \hat{\underline{k}}) = \omega_{p\alpha}{}^2 \int d^3v \;\; \delta(V' - \hat{\underline{k}}\cdot\underline{v}) f_\alpha(\underline{v}). \qquad \text{(B-7)}$$

We now assume that $F_\alpha(\hat{\underline{k}}, V')$ vanishes for $|V'| > |V|$ for the V being considered. Then Eq. (B-6) can be integrated by parts to yield

$$R_\alpha(\hat{\underline{k}}, V) = - \int \frac{dV'}{(V' - V)^2} \; F_\alpha(V'; \hat{\underline{k}}) \; . \qquad \text{(B-8)}$$

But

$$\frac{1}{(V - V')^2} = \frac{1}{V^2} \left[1 + 2\frac{V'}{V} + 3\frac{V'^2}{V^2} + 4\frac{V'^3}{V^3} + \cdots \right] . \qquad \text{(B-9)}$$

By substituting Eqs. (B-7) and (B-9) into Eq. (B-8) and evaluating the integral over V' by using the δ function, we find Eqs. (III-28). Even when our above assumption is not satisfied, these results represent an asymptotic expansion of $R_\alpha(\hat{\underline{k}}, V)$ that does not converge but is useful when V is sufficiently large. We will not prove this statement.

C. Calculation of U_{crit}

In the following, V_{min} and V_{max} are the roots of $X(V/A) = 2\theta_i/\theta_e$ and can be determined from the values of $X(x)$ given by Fried and Conte. We consider $\hat{\underline{k}}$ parallel to $\underline{U}$ so

$$\frac{D_i^2}{\sqrt{\pi}} \, I(\hat{\underline{k}}, V) = \frac{\theta_i}{\theta_e} \; \frac{U - V}{a} \; - \; \frac{V}{A} \; e^{-V^2/A^2} \; . \qquad \text{(C-1)}$$

In the curves we discuss, the electron contribution and the negative of the ion contribution to Eq. (C-1) are plotted separately. Where the curves intersect, $I(\hat{\underline{k}}, V)$ vanishes.

With $\theta_e = 10\theta_i$, we find $V_{min} = 1.03A$ and $V_{max} = 2.63A$ so the curves appear as in Fig. 18(a), where we use a greatly expanded vertical scale. Notice that the ion curve passes nearly vertically upward through the origin. With $U = 4.5A$ as shown, the curves intersect at three places; the minimum of $F(V; \hat{\underline{k}})$ is the intersection near $V = 3A$ and thus occurs outside the range $V_{min} < V < V_{max}$ where R is positive, so the plasma is stable. As U is increased, the curve of the electron contribution slides to the right. Finally when Eq. (C-1) vanishes for $V = V_{max}$, we have reached $U = U_{crit}$. In our example we find $U_{crit} = 6.13A$. Notice that this result is quite sensitive to the exact value of V_{max}.

With $\theta_e = 40\theta_i$, we find $V_{min} = 0.95A$ and $V_{max} = 4.65A$ so the curves appear as in Fig. 18(b), where we have used an even more expanded vertical scale. With $U = 2.5A$ as shown, $F(V; \hat{\underline{k}})$ has no minimum. The minimum will form when U is increased until the curves become tangent, and since $R(\hat{\underline{k}}, V)$ is positive there, the corresponding U will be U_{crit}. The point V_0 where the curves will be tangent is where they have the same slope and so is given by

$$e^{V_0^2/A^2} = \frac{a\theta_e}{A\theta_i}\left(2\frac{V_0^2}{A^2} - 1\right), \qquad \text{(C-2)}$$

which can be solved by iteration. In our example we find $V_0 = 3.53A$. When Eq. (C-1) vanishes for $V = V_0$, we have reached $U = U_{crit}$. In our example, we have $U_{crit} = 3.68A$.

The curves of Fig. 2 were constructed by the above procedure. The function $F(V; \hat{\underline{k}})$ develops a minimum when $I(\hat{\underline{k}}, V_0) = 0$, and the waves with $\dot{k} = 0$ begin to grow when $I(\underline{k}, V_{max}) = 0$.

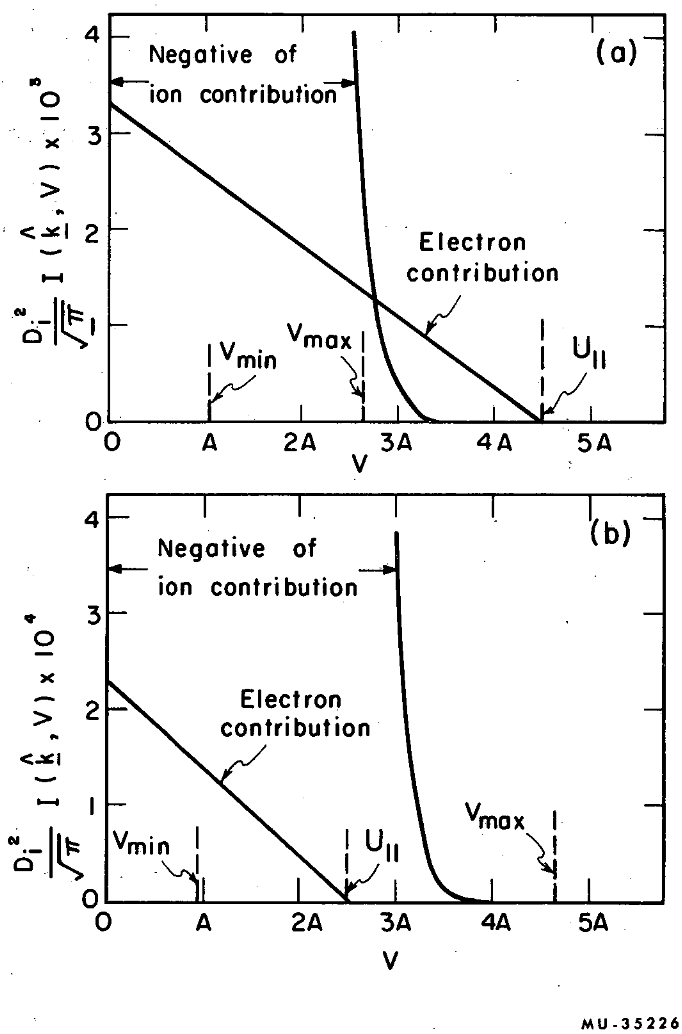


Fig. 18. Illustrations for the calculation of U_{crit} in an electron-proton plasma. The temperature ratio θ_e/θ_i is 10 in (a) and 40 in (b).

D. Ohmic Heating

Unless E_0 is smaller than E_{run}, the electrical power supplied will primarily go into accelerating the runaway electrons. With $E_0 << E_{run}$, the power is effective in heating the plasma, although as θ_e increases, E_{run} decreases; so, in practice, the temperatures attainable are limited to a few hundred electron volts.

We consider an electron-proton plasma and assume the velocity distributions remain nearly Maxwellian. The power transfer from electrons to ions by collisions is then $3\nu_c(\theta_e - \theta_i)m/M$ per electron so we find when $E_0 << E_{run}$

$$\frac{d}{dt}\left(\frac{3}{2}\theta_e\right) = \frac{\sigma_{S-H}}{n_e} E_0^{\,2} - \frac{3m}{M}\nu_c(\theta_e - \theta_i) \qquad \text{(D-1)}$$

and

$$\frac{d}{dt}\left(\frac{3}{2}\theta_i\right) = \frac{3m}{M}\nu_c(\theta_e - \theta_i). \qquad \text{(D-2)}$$

We consider only $\theta_e > \theta_i$ so the ions are always heated. Whether the electrons cool or heat depends upon the sign of

$$\frac{1}{\nu_c\theta_e}\frac{d\theta_e}{dt} = \frac{4}{3}(0.5064)\left(\frac{E_0}{E_{run}}\right)^2 - \frac{2m}{M}\left(1 - \frac{\theta_i}{\theta_e}\right). \qquad \text{(D-3)}$$

Whether θ_e/θ_i increases or decreases depends upon the sign of

$$\frac{\theta_i}{\nu_c\theta_e}\frac{d}{dt}\left(\frac{\theta_e}{\theta_i}\right) = \frac{4}{3}(0.5064)\left(\frac{E_0}{E_{run}}\right)^2 - \frac{2m}{M}\left(\frac{\theta_e}{\theta_i} - \frac{\theta_i}{\theta_e}\right) \qquad \text{(D-4)}$$

The various cases for an electron-proton plasma are shown in Fig. 19.

We see that although any point on this diagram could be reached by a suitable choice of the time dependence of $E_0(t)$, large temperature ratios are not likely to be produced or maintained by Ohmic heating alone unless $E_0 > 0.1E_{run}$. With E_0 constant in time, a plasma would

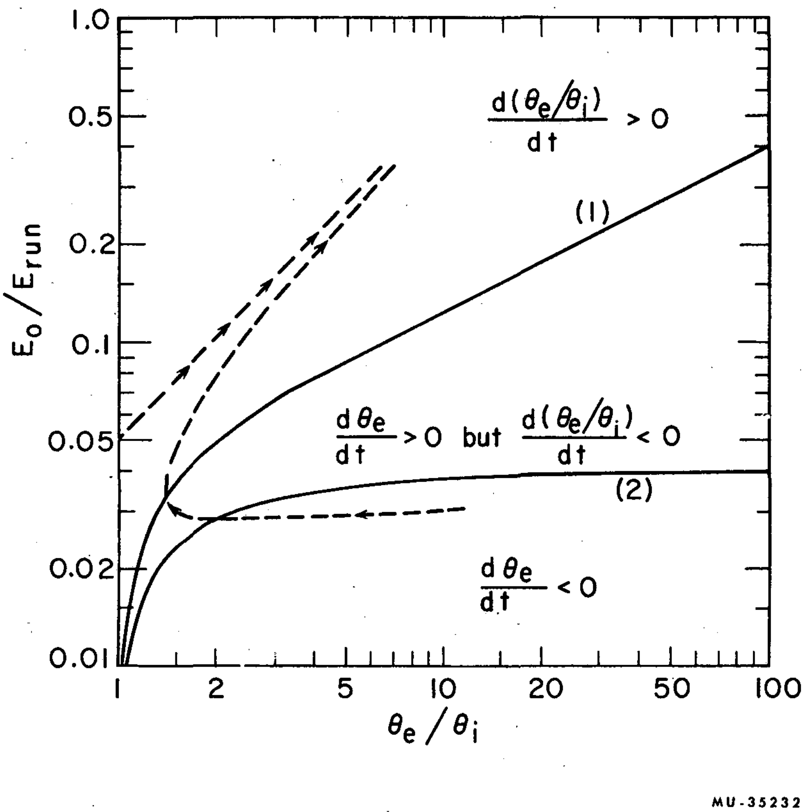


Fig. 19. Diagram describing the time variation of the temperatures θ_e and θ_i in an electron-proton plasma if only Ohmic heating and collisional transfer from electrons to ions are important. The dashed lines suggest the time evolution if E_0 is constant in time.

evolve along a trajectory like the ones sketched in Fig. 19.

E. Derivation of Fokker-Planck Coefficients

Using Eqs. (IV-14) and (IV-15), we have

$$\mathcal{D}_a(\underline{v}, t) = \frac{q_a^2}{2m_a^2 \Delta t} \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \left\langle \underline{E}[\underline{r}(t'), t'] \, \underline{E}[\underline{r}(t''), t''] \right\rangle . \tag{E-1}$$

We introduce $\tau = t'' - t'$ and we use the approximation $\underline{r}(t'') = \underline{r}(t') + \underline{v}\tau$ so

$$\mathcal{D}_a(\underline{v}, t) = \frac{q_a^2}{2m_a^2 \Delta t} \int_t^{t+\Delta t} dt' \int_{t-t'}^{t-t'+\Delta t} d\tau \left\langle \underline{E}[\underline{r}(t'), t'] \underline{E}[\underline{r}(t') + \underline{v}\tau, t' + \tau] \right\rangle . \tag{E-2}$$

We assume the quantity in brackets is $\underset{\sim}{C}(\underline{v}\tau, \tau)$, although the ensemble here cannot be the same as in our derivation of $\underset{\sim}{C}(\underline{R}, \tau)$ because here we know that a particle exists on the trajectory $\underline{r}(t')$. Because Δt is chosen long compared with the correlation time, we make little error by extending the limits on τ to $-\infty$ and ∞. Also, since Δt is chosen short compared with the time scale over which $\underset{\sim}{C}(\underline{v}\tau, \tau)$ varies, the integral over t' yields Δt. Thus we find

$$\mathcal{D}_a(\underline{v}, t) = \frac{q_a^2}{2m_a^2} \int_{\infty}^{\infty} d\tau \, \underset{\sim}{C}(\underline{v}\tau, \tau). \tag{E-3}$$

From Eqs. (IV-13) and (IV-15), we find

$$\mathcal{F}_a(\underline{v}, t) = \frac{q_a}{m_a \Delta t} \int_0^{\Delta t} \left\langle \underline{E}[\underline{r}(t+\tau), t+\tau] \right\rangle d\tau . \tag{E-4}$$

We now use the approximation

$$\underline{E}\left[\underline{r}(t+\tau),\ t+\tau\right] = \underline{E}\left[\underline{r}(t)+\underline{v}\tau,\ t+\tau\right] + \left[\underline{r}(t+\tau) - \underline{r}(t) - \underline{v}\tau\right]\cdot\left[\frac{\partial}{\partial \underline{x}}\underline{E}(\underline{x}, t+\tau)\right]_{\underline{x}=\underline{r}(t)+\underline{v}\tau}, \quad \text{(E-5)}$$

and by integrating the particle equation of motion twice, we find

$$\underline{r}(t+\tau) - \underline{r}(t) - \underline{v}\tau = \frac{q_\alpha}{m_\alpha}\int_0^\tau d\tau'' \int_0^{\tau''} d\tau'\ \underline{E}[\underline{r}(t+\tau'),\ t+\tau'], \quad \text{(E-6)}$$

where $\underline{v}$ is evaluated at t. In Eq. (E-6) we interchange the order of integration and use $\underline{r}(t+\tau') = \underline{r}(t) + \underline{v}\tau'$ so that substitution into Eq. (E-5) yields

$$\underline{E}\left[\underline{r}(t+\tau),\ t+\tau\right] = \underline{E}\left[\underline{r}(t) + v\tau,\ t+\tau\right] + \frac{q_\alpha}{m_\alpha}\int_0^\tau d\tau'(\tau-\tau') \times \left\{\frac{\partial}{\partial \underline{x}}\cdot\left[\underline{E}(\underline{r}(t)+\underline{v}\tau',\ t+\tau')\,\underline{E}(\underline{x},\ t+\tau)\right]\right\}_{\underline{x}=\underline{r}(t)+\underline{v}\tau}. \quad \text{(E-7)}$$

In evaluating the ensemble average of this, the first term would vanish except that we know the particle is present "at" $\underline{r}(t) + \underline{v}\tau$, so we find $\underline{E}_{drag}(q_\alpha, \underline{v})$. As in calculating $\underset{\sim}{\mathcal{D}}_\alpha(\underline{v}, t)$, we assume the second term again yields $\underset{\sim}{C}(\underline{R}, \tau)$ so

$$\left\langle \underline{E}\left[\underline{r}(t+\tau),\ t+\tau\right]\right\rangle = \underline{E}_{drag}(q_\alpha, \underline{v}) + \frac{q_\alpha}{m_\alpha}\int_0^\tau d\tau'(\tau-\tau') \times \left\{\frac{\partial}{\partial \underline{x}}\cdot \underset{\sim}{C}\left[\underline{x} - \underline{r}(t) - \underline{v}\tau',\ \tau-\tau'\right]\right\}_{\underline{x}=\underline{r}(t)+\underline{v}\tau}. \quad \text{(E-8)}$$

The expression in braces is simply

$$\frac{1}{\tau - \tau'} \frac{\partial}{\partial \underline{v}} \cdot \underset{\sim}{C}[\underline{v}(\tau - \tau'), \tau - \tau'],$$

so, introducing $T = \tau - \tau'$, we have

$$\left\langle \underline{E}\left[\underline{r}(t+\tau), t+\tau\right]\right\rangle = \underline{E}_{drag}(q_a, \underline{v}) + \frac{q_a}{m_a}\frac{\partial}{\partial \underline{v}} \cdot \int_0^{\tau} dT\, \underset{\sim}{C}(\underline{v}T, T). \qquad \text{(E-9)}$$

On substituting this into Eq. (E-4), we see that, because Δt is chosen long compared to the correlation time, we introduce little error by extending τ in Eq. (E-9) to infinity. Then since Δt is short compared with the time scales on which the quantities in Eq. (E-9) vary, we find

$$\mathcal{Z}_a(\underline{v}, t) = \frac{q_a}{m_a} \underline{E}_{drag}(q_a, \underline{v}) + \frac{\partial}{\partial \underline{v}} \cdot \frac{q_a^2}{m_a^2} \int_0^{\infty} dT\, \underset{\sim}{C}(\underline{v}T, T). \qquad \text{(E-10)}$$

Using the symmetry $\underset{\sim}{C}(-\underline{R}, -\tau) = \underset{\sim}{C}(\underline{R}, \tau)$ and Eq. (E-3), we find Eq. (IV-17).

F. Reduction to Scalar Variables

Here certain results needed in Sec. VI. A are derived.

1. Expressions for $\partial f(v, a, t)/\partial t$ and $\underline{v} \cdot \underline{J}_e(\underline{v}, t)$

With our symmetry, the gradient in spherical coordinates yields

$$\frac{\partial f(\underline{v}, t)}{\partial \underline{v}} = \hat{\underline{v}} \frac{\partial f(v, a, t)}{\partial v} + \frac{\hat{\underline{a}}}{v} \frac{\partial f(v, a, t)}{\partial a} \qquad \text{(F-1)}$$

and the divergence in spherical polar coordinates yields

$$\frac{\partial}{\partial \underline{v}} \cdot \underline{J}_e(\underline{v}, t) = \frac{1}{v^2} \frac{\partial}{\partial v}\left[v\underline{v} \cdot \underline{J}_e\right] + \frac{1}{v \sin a} \frac{\partial}{\partial a}\left[(\sin a)\hat{\underline{a}} \cdot \underline{J}_e\right]. \qquad \text{(F-2)}$$

Since $\hat{\underline{E}} = \hat{\underline{v}} \cos\alpha - \hat{\underline{\alpha}} \sin\alpha$, we have

$$\hat{\underline{\alpha}} = \frac{1}{\sin\alpha}\left[\hat{\underline{v}} \cos\alpha - \hat{\underline{E}}\right]. \qquad \text{(F-3)}$$

Substitution of these results into Eq. (VI-7) yields Eq. (VI-9).

From Eqs. (F-1) and (F-3) we find

$$\hat{\underline{k}} \cdot \frac{\partial f}{\partial \underline{v}} = \frac{1}{v}\, \hat{\underline{k}} \cdot \underline{v}\ \frac{\partial f}{\partial v} - \frac{1}{v}\left(\frac{\cos\alpha}{v}\ \underline{k}\cdot\underline{v} - \cos\theta\right)\frac{\partial f}{\partial(\cos\alpha)} \qquad \text{(F-4)}$$

so, using the δ function to replace $\underline{k}\cdot\underline{v}$ with V, we have

$$\delta(V - \hat{\underline{k}}\cdot\underline{v})\hat{\underline{k}} \cdot \frac{\partial f}{\partial \underline{v}} = \delta(V - \hat{\underline{k}}\cdot\underline{v})\, \frac{1}{v}\left[V\frac{\partial f}{\partial v} - \left(\frac{V}{v}\cos\alpha - \cos\theta\right)\frac{\partial f}{\partial(\cos\alpha)}\right] \qquad \text{(F-5)}$$

which is Eq. (VI-10).

If the spherical coordinates are completed with the azimuthal angles β for $\underline{v}$ and ψ for $\underline{k}$, then we have

$$\int d^2\hat{\underline{k}}\ \delta(V - \hat{\underline{k}}\cdot\underline{v}) = \int_0^{\pi} \sin\theta\ d\theta \int_0^{2\pi} d\psi\ \delta(V - \hat{\underline{k}}\cdot\underline{v}) \qquad \text{(F-6)}$$

where

$$\hat{\underline{k}}\cdot\underline{v} = v\cos\theta\cos\alpha + v\sin\theta\sin\alpha\cos(\psi - \beta). \qquad \text{(F-7)}$$

The integrand is an even function of $\phi = \psi - \beta$, so we find

$$\int d^2\hat{\underline{k}}\ \delta(V - \hat{\underline{k}}\cdot\underline{v}) = 2\int_0^{\pi} \sin\theta\ d\theta \int_0^{\pi} d\phi\ \delta(V - v\cos\theta\cos\alpha - v\sin\theta\sin\alpha\cos\phi) \qquad \text{(F-8)}$$

which is Eq. (VI-11).

2. Expressions for $H_e(\theta, V)$, $I_e(\theta, V)$, and $R_e(\theta, V)$

With these same spherical coordinates, we have from Eq. (IV-5)

$$H_e(\theta, V) = \pi m \omega_{pe}^2 \int_0^{\infty} v^2 dv \int_0^{\pi} \sin\alpha\ d\alpha \int_0^{2\pi} d\beta\ \delta(V - \hat{\underline{k}}\cdot\underline{v})\, f(v, \alpha, t). \qquad \text{(F-9)}$$

With Eq. (F-5), Eq. (VI-4) yields

$$I_e(\theta, V) = \pi\omega_{pe}^2 \int_0^\infty v\, dv \int_0^\pi \sin\alpha\, d\alpha \int_0^{2\pi} d\beta\, \delta(V - \hat{\underline{k}} \cdot \underline{v}) \left[V \frac{\partial f}{\partial v} - \left(\frac{V}{v} \cos\alpha - \cos\theta \right) \frac{\partial f}{\partial(\cos\alpha)} \right]. \tag{F-10}$$

With Eq. (F-4) and with

$$\frac{\hat{\underline{k}} \cdot \underline{v}}{\hat{\underline{k}} \cdot \underline{v} - V} = 1 + \frac{V}{\hat{\underline{k}} \cdot \underline{v} - V}, \tag{F-11}$$

Equation (VI-3) becomes

$$R_e(\theta, V) = \omega_{pe}^2 \int_0^\infty v\, dv \int_0^\pi \sin\alpha\, d\alpha \left\{ 2\pi \left[\frac{\partial f}{\partial v} - \frac{\cos\alpha}{v} \frac{\partial f}{\partial(\cos\alpha)} \right] \right.$$

$$\left. + P \int_0^{2\pi} \frac{d\beta}{\hat{\underline{k}} \cdot \underline{v} - V} \left[V \frac{\partial f}{\partial v} - \left(\frac{V}{v} \cos\alpha - \cos\theta \right) \frac{\partial f}{\partial(\cos\alpha)} \right] \right\}. \tag{F-12}$$

If we again use Eq. (F-7) and substitute $\phi = \psi - \beta$, the above integrals over β become

$$\int_0^{2\pi} d\beta\, \delta(V - \hat{\underline{k}} \cdot \underline{v}) = 2 \int_0^\pi d\phi\, \delta(V - v\cos\theta\cos\alpha - v\sin\theta\sin\alpha\cos\phi) \tag{F-13}$$

$$P \int_0^{2\pi} \frac{d\beta}{\hat{\underline{k}} \cdot \underline{v} - V} = 2P \int_0^\pi \frac{d\phi}{v\cos\theta\cos\alpha + v\sin\theta\sin\alpha\cos\phi - V}. \tag{F-14}$$

When these are used inEqs. (F-9), (F-10), and (F-12), the results are Eqs. (VI-13), (VI-14), and (VI-15).

With the change of variables $x = \cos\phi$, we have

$$\int_0^\pi d\phi\, \delta(a + b\cos\phi) = \int_{-1}^{1} dx \frac{\delta(a+bx)}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{if } |b| < |a| \\ (b^2 - a^2)^{-1/2} & \text{if } |b| > |a| \end{cases} \tag{F-15}$$

which is Eq. (VI-16).

We also must consider

$$P\int_0^{\pi} \frac{d\phi}{a + b\cos\phi} = P\int_{-\pi}^{\pi} \frac{d\phi}{2a + b(e^{i\phi} - e^{-i\phi})} . \qquad \text{(F-16)}$$

Introducing the complex variable $z = e^{i\phi}$, we find

$$P\int_0^{\pi} \frac{d\phi}{a + b\cos\phi} = \frac{1}{ib} P\oint \frac{dz}{z^2 + 2\frac{a}{b}z + 1} = \frac{1}{ib} P\oint \frac{dz}{(z - z_+)(z - z_-)} \qquad \text{(F-17)}$$

where $z_\pm = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$ and the integration is counterclockwise about the unit circle.

With $|b| > |a|$, we define $\cos\phi_0 = -\frac{a}{b}$ and find $z_\pm = \exp(\pm i\phi_0)$ so that the poles are on the unit circle. The integral over a circle slightly smaller or slightly larger than the unit circle vanishes; averaging the two cases, we find

$$P\int_0^{\pi} \frac{d\phi}{a + b\cos\phi} = 0 \qquad \text{when} \qquad |b| > |a|. \qquad \text{(F-18)}$$

With $|b| < |a|$, the principal value is not needed so the integral can be evaluated as above. If a/b is positive, the root z_+ is inside the unit circle and z_- is outside, so

$$P\int_0^{\pi} \frac{d\phi}{a + b\cos\phi} = \frac{2\pi i}{ib} \frac{1}{2\sqrt{\frac{a^2}{b^2} - 1}} \qquad \text{if} \qquad a/b > 1. \qquad \text{(F-19)}$$

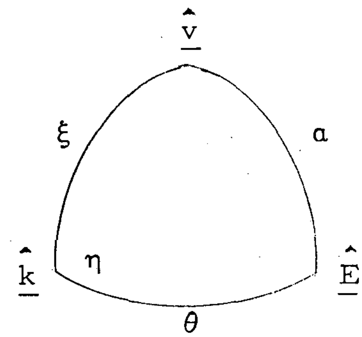
Similarly

$$P\int_0^{\pi} \frac{d\phi}{a + b\cos\phi} = \frac{2\pi i}{ib} \frac{1}{(-2)\sqrt{\frac{a^2}{b^2} - 1}} \qquad \text{if} \quad a/b < -1. \qquad \text{(F-20)}$$

These results combine to give Eq. (VI-17).

G. Expansion in Legendre Polynomials

If $\underline{v}$ is described in spherical polar coordinates ξ and η about the direction $\hat{\underline{k}}$ as indicated in the spherical triangle, the superposition theorem for Legendre polynomials is



$$P_\ell(\cos\alpha) = P_\ell(\cos\xi)P_\ell(\cos\theta) + 2\sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^{m}(\cos\xi)\, P_\ell^{m}(\cos\theta)\cos m\eta. \tag{G-1}$$

1. Expression for $H_e(\theta, V)$

From Eqs. (VI-5) and (VI-21), we have in the above coordinates

$$H_e(\theta, V) = \pi m \omega_{pe}^{2} \int_0^\infty v^2 dv \int_0^\pi \sin\xi \; d\xi \int_0^{2\pi} d\eta \,\delta(V - v\cos\xi) \sum_{\ell=0}^{\infty} f_\ell(v) P_\ell(\cos\alpha). \tag{G-2}$$

With $P_\ell(\cos\alpha)$ replaced with Eq. (G-1), only the term $P_\ell(\cos\xi)P_\ell(\cos\theta)$ will contribute to the integral over η. Therefore

$$H_e(\theta, V) = \sum_{\ell=0}^{\infty} H_{e\ell}(V)\; P_\ell(\cos\theta) \tag{G-3}$$

where, with $x = \cos\xi$,

$$H_{e\ell}(V) = 2\pi^2 m \omega_{pe}^{2} \int_0^\infty v^2 dv\; f_\ell(v) \int_{-1}^{1} dx\;\; \delta(V - vx) P_\ell(x) \tag{G-4}$$

or

$$H_{e\ell}(V) = 2\pi^2 m \omega_{pe}^{2} \int_V^\infty v\; dv\;\; f_\ell(v)\, P_\ell \left(\frac{V}{v}\right) . \tag{G-5}$$

2. Expression for $I_e(\theta, V)$

By the methods of Appendix F, we find

$$\hat{\underline{k}} \cdot \frac{\partial f}{\partial \underline{v}} = \cos\xi \frac{\partial f}{\partial v} + \frac{\sin^2\xi}{v} \frac{\partial f(v, a, t)}{\partial (\cos\xi)} . \tag{G-6}$$

When this is used in Eq. (VI-4), we proceed just as in Sec. G.1 and find immediately that

$$I_e(\theta, V) = \sum_{\ell=0}^{\infty} I_{e\ell}(V) P_\ell(\cos\theta) \tag{G-7}$$

with

$$I_{e\ell}(V) = 2\pi^2 \omega_{pe}^2 \int_V^\infty dv \left[V \frac{\partial f_\ell(v)}{\partial v} P_\ell\left(\frac{V}{v}\right) + \left(1 - \frac{V^2}{v^2}\right) f_\ell(v) \frac{dP_\ell(V/v)}{d(V/v)} \right] . \tag{G-8}$$

This can be written as

$$I_{e\ell}(V) = 2\pi^2 \omega_{pe}^2 \int_V^\infty dv \left\{ V \frac{\partial}{\partial v} \left[f_\ell(v) P_\ell\left(\frac{V}{v}\right) \right] + f_\ell(v) \frac{dP_\ell(V/v)}{d(V/v)} \right\} \tag{G-9}$$

and the first term can be evaluated, so we have

$$I_{e\ell}(V) = 2\pi^2 \omega_{pe}^2 \left[\int_V^\infty f_\ell(v) \frac{dP_\ell(V/v)}{d(V/v)} dv - V f_\ell(V) \right] . \tag{G-10}$$

We notice from Eqs. (G-5) and (G-10) that

$$I_{e\ell}(V) = \frac{1}{m} \frac{\partial H_{e\ell}(V)}{\partial V} . \tag{G-11}$$

More generally, it follows from the definitions of $H_e(\theta, V)$ and of $I_e(\theta, V)$ in the form of Eq. (III-22) that

$$I_a(\hat{\underline{k}}, V) = \frac{1}{m_a} \frac{\partial H_a(\hat{\underline{k}}, V)}{\partial V} . \tag{G-12}$$

3. Expression for $R_e(\theta, V)$

If we insert Eq. (G-6) in Eq. (VI-3) and reuse the same procedure, we find directly

$$R_e(\theta, V) = \sum_{\ell=0}^{\infty} R_{e\ell}(V)\, P_\ell(\cos\theta) \tag{G-13}$$

with

$$R_{e\ell}(V) = 2\pi\omega_{pe}^2 \int_0^\infty v\, dv\; P\int_{-1}^{1} \frac{dx}{x - V/v} \left[x\, \frac{\partial f_\ell(v)}{\partial v}\, P_\ell(x) + \frac{1-x^2}{v}\, f_\ell(v) \frac{dP_\ell(x)}{dx} \right] . \tag{G-14}$$

For any given ℓ, the principal-value integral over x can be evaluated by substitution of the polynomial $P_\ell(x)$ and use of the expressions

$$\frac{x^n}{x - a} = \frac{a^n}{x - a} + \sum_{p=1}^{n} a^{n-p} x^{p-1} \tag{G-15}$$

and

$$P\int_{-1}^{1} \frac{dx}{x - a} = \ln \left| \frac{1 - a}{1 + a} \right| . \tag{G-16}$$

However, no general expression valid for all ℓ is apparent.

H. Evaluation of $I_g(\theta)$

With V/v set equal to zero, Eqs. (VI-16) and (VI-18) yield

$$\int_0^\pi d\phi\, \delta(a + b\cos\phi) = (\sin^2\theta - \cos^2 a)^{-1/2}/v \tag{H-1}$$

when $\sin^2\theta > \cos^2 a$ and zero otherwise. With the same approximation in Eq. (VI-14), we have

$$I_e(\theta, V) = 2\pi\omega_{pe}^2 \int_0^\infty dv \int_{-\sin\theta}^{\sin\theta} \frac{d(\cos a)}{\sqrt{\sin^2\theta - \cos^2 a}}\; \cos\theta\, \frac{\partial f(v, a)}{\partial (\cos a)} . \tag{H-2}$$

Introducing the dimensionless variables and functions of Sec. VI. C and including only the contribution of g(x, y), we have

$$I_g(\theta) = -\pi\cos\theta \int_{-\sin\theta}^{\sin\theta} \frac{dy}{\sqrt{\sin^2\theta - y^2}}\,\rho(y) \qquad \text{(H-3)}$$

where

$$\rho(y) = \frac{\partial}{\partial y}\int_0^\infty g(x, y)\,dx\,. \qquad \text{(H-4)}$$

We may rewrite Eq. (H-3) by changing variables to ϕ where

$$\begin{aligned} y &= \sin\theta\,\cos\phi \quad \text{when} \quad y > 0 \\ y &= -\sin\theta\,\cos\phi \quad \text{when} \quad y < 0\,. \end{aligned} \qquad \text{(H-5)}$$

This yields

$$I_g(\theta) = -\pi\cos\theta \int_0^{\pi/2} d\phi \left[\rho(\sin\theta\,\cos\phi) + \rho(-\sin\theta\,\cos\phi)\right]. \qquad \text{(H-6)}$$

I. Evaluation of $J_1(x)$ and $J_2(x)$

From Eqs. (VI-81) and (VI-59) we find

$$J_1(x) = \frac{-9}{2\pi^{3/2}x}\int_0^x du\,u\,R(u)\int_{-1}^1 y\,dy\int_{r_-}^{r_+} d(\cos\theta)\,D^{-1/2}\left\{\pi^{-3/2}e^{-x^2} + f(x)y\right.$$

$$\left. + \frac{H(\theta, u)}{I(\theta, u)}\,\frac{u}{x}\left[\frac{-2x}{\pi^{3/2}}\,e^{-x^2} + y\,\frac{\partial f}{\partial x}\right] + \frac{H(\theta, u)}{I(\theta, u)}\,\frac{1}{x}\left(\cos\theta - \frac{uy}{x}\right) f(x)\right\} \qquad \text{(I-1)}$$

where

$$r_\pm = uy/x \pm \sqrt{1 - y^2}\,\sqrt{1 - u^2/x^2} \qquad \text{(I-2)}$$

and

$$D = (r_+ - \cos\theta)(\cos\theta - r_-) = 1 - u^2/x^2 - \cos^2\theta - y^2 + 2ux(\cos\theta)/y\,. \qquad \text{(I-3)}$$

Notice that D is unchanged by the interchange of u/x, $\cos\theta$, and y. The integral over y and $\cos\theta$ is actually over the region where D is positive, which is the interior of an ellipse. Therefore, we may trivially interchange the order of integration as

$$\int_{-1}^{1} dy \int_{r_-}^{r_+} d(\cos\theta) \left[\quad \right] = \int_{-1}^{1} d(\cos\theta) \int_{r_-'}^{r_+'} dy \left[\quad \right] \qquad \text{(I-4)}$$

where

$$r_\pm' = u(\cos\theta)/x \pm \sin\theta \sqrt{1 - u^2/x^2} \equiv c \pm d \qquad \text{(I-5)}$$

and

$$D = (r_+' - y)(y - r_-') \,. \qquad \text{(I-6)}$$

The integrals over y are now easily evaluated with the substitution $y = c + d\cos\phi$, so

$$\int_{r_-'}^{r_+'} y^n D^{-1/2}\, dy = \int_0^\pi d\phi (c + d\cos\phi)^n \qquad \text{(I-7)}$$

$$= \begin{cases} \pi & \text{with } n = 0 \\ \pi u(\cos\theta)/x & \text{with } n = 1 \\ \pi u^2(\cos\theta)^2/x^2 + \pi(1-\cos^2\theta)(1-u^2/x^2)/2 & \text{with } n = 2. \end{cases}$$

Using these, we find

$$J_1(x) = \frac{-9}{2\sqrt{\pi}\, x} \int_0^x du\, u\, R(u) \int_{-1}^{1} d(\cos\theta) \left\{ \frac{u}{x} \cos\theta \left[\pi^{-3/2} e^{-x^2} \left(1 - 2u \frac{H(\theta, u)}{I(\theta, u)}\right) + \frac{H(\theta, u)}{I(\theta, u)} \frac{\cos\theta}{x} f(x) \right] + \left[\frac{u^2}{x^2} \cos^2\theta + \frac{1}{2} (1 - \cos^2\theta) \left(1 - \frac{u^2}{x^2}\right) \right] \right.$$

$$\left. \times \left[f(x) \left(1 - \frac{u}{x^2} \frac{H(\theta, u)}{I(\theta, u)}\right) + \frac{H(\theta, u)}{I(\theta, u)} \frac{u}{x} \frac{\partial f}{\partial x} \right] \right\}, \qquad \text{(I-8)}$$

and similarly

$$J_2(x) = \frac{-9}{2\sqrt{\pi}\,x} \int_0^x du\, R(u) \int_{-1}^{1} d(\cos\theta) \cos\theta \left\{ \pi^{-3/2} e^{-x^2} \left[1 - 2u \frac{H(\theta, u)}{I(\theta, u)} \right] \right.$$

$$\left. + \frac{H(\theta, u)}{I(\theta, u)} \frac{\cos\theta}{x} f(x) + \frac{u}{x} \cos\theta \left[f(x) \left(1 - \frac{u}{x^2} \frac{H(\theta, u)}{I(\theta, u)} \right) + \frac{H(\theta, u)}{I(\theta, u)} \frac{u}{x} \frac{\partial f}{\partial x} \right] \right\} . \tag{I-9}$$

The integrals over θ are simply the $V_n(u)$ defined by Eq. (VI-84) along with the simple results

$$\int_{-1}^{1} d(\cos\theta) \cos^n\theta = \begin{cases} 2 & \text{with} \quad n = 0 \\ 0 & \text{with} \quad n = 1 \\ 2/3 & \text{with} \quad n = 2 \, . \end{cases} \tag{I-10}$$

When these results are used in Eqs. (I-8) and (I-9), we find Eqs. (VI-82) and (VI-83).

J. Evaluation of $W_n(u, y)$

We define

$$\Delta I(u, \cos\theta) = I_1(u) \cos\theta + I_g(\theta) \tag{J-1}$$

so that

$$I(u, \theta) = I_0(u) + \Delta I(u, \cos\theta) \tag{J-2}$$

where ΔI is an odd function of $\cos\theta$. Then from Eq. (VI-106)

$$W_n(u, y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} d(\cos\theta) \frac{\cos^n\theta}{\sqrt{1-y^2-\cos^2\theta}} \frac{1}{I_0(u) + \Delta I(u, \cos\theta)} \, . \tag{J-3}$$

We now change variables to ϕ, where

$$\begin{aligned} \cos\theta &= \sqrt{1-y^2}\cos\phi && \text{when} \quad \cos\theta > 0 \\ \cos\theta &= -\sqrt{1-y^2}\cos\phi && \text{when} \quad \cos\theta < 0 . \end{aligned} \tag{J-4}$$

and find

$$W_n(u, y) = \int_0^{\pi/2} d\phi \left[\frac{(\sqrt{1-y^2}\cos\phi)^n}{I_0(u)+\Delta I(u,\sqrt{1-y^2}\cos\phi)} + \frac{(-\sqrt{1-y^2}\cos\phi)^n}{I_0(u)-\Delta I(u,\sqrt{1-y^2}\cos\phi)} \right]. \tag{J-5}$$

K. Effect of Ion Waves

We neglect the effects of $\underline{E}_0$ and of "ordinary" collisions for the moment. Then Eqs. (VI-100) and (VI-105) yield

$$\frac{\partial h(x, y, \tau)}{\partial \tau} = \frac{1}{x^3} \frac{\partial}{\partial y} \left[S(y) \frac{\partial h(x, y, \tau)}{\partial y} \right]. \tag{K-1}$$

According to this equation, we have

$$\int_{-1}^{1} dy h(x, y, \tau) = C(x) \tag{K-2}$$

where $C(x)$ is independent of τ. That is, the ion waves do not alter the number of electrons with speed v (in our approximation). We now define

$$\mathcal{H}(x, \tau) = \int_{-1}^{1} dy \left[h(x, y, \tau) \right]^2. \tag{K-3}$$

If we write

$$h(x, y, \tau) = \frac{1}{2} C(x) + \delta h(x, y, \tau) \tag{K-4}$$

we find that

$$\mathcal{H}(x, \tau) = \frac{1}{2} \left[C(x) \right]^2 + \int_{-1}^{1} dy \left[\delta h(x, y, \tau) \right]^2 \tag{K-5}$$

since

$$\int_{-1}^{1} dy\, \delta h(x, y, \tau) = 0. \tag{K-6}$$

We thus see that the minimum value $\mathcal{H}(x, \tau)$ could have is $C^2(x)/2$ will occur if, and only if, $\delta h(x, y, \tau)$ vanishes, so $h(x, y, \tau)$ is isotropic. We next consider

$$\frac{\partial \mathcal{H}(x, \tau)}{\partial \tau} = 2 \int_{-1}^{1} h \frac{\partial h}{\partial \tau} dy = \frac{2}{x^3} \int_{-1}^{1} h \frac{\partial}{\partial y} \left[S(y) \frac{\partial h}{\partial y} \right] dy. \qquad \text{(K-7)}$$

Since $S(y)$ vanishes at $y = \pm 1$, partial integration yields

$$\frac{\partial \mathcal{H}(x, \tau)}{\partial \tau} = \frac{-2}{x^3} \int_{-1}^{1} S(y) \left[\frac{\partial h(x, y, \tau)}{\partial y} \right]^2 dy. \qquad \text{(K-8)}$$

But since $S(y)$ is positive except at $y = \pm 1$, this implies

$$\frac{\partial \mathcal{H}(x, \tau)}{\partial \tau} \leqslant 0 \qquad \text{(K-9)}$$

and the equality holds if, and only if, $\partial h(x, y, \tau)/\partial y$ vanishes for all y.

We have thus proven that (a) within our approximations the ion waves always tend to make the electron velocity distribution more isotropic and that (b) their effect vanishes only when the electron velocity distribution is isotropic.

A similar "H-theorem" is well known for one-dimensional problems in quasi-linear theory.[30]

One may verify that the above discussion depends only upon the assumption $u \ll x$, which of course will fail at small x. Although we have derived this theorem only for the restricted anisotropy of our problem, it can probably be easily derived for arbitrary anisotropy. It is probably true in unstable plasmas as well as in stable plasmas.

L. Partial List of Functions and Symbols

A plain letter like v denotes a scalar variable, a letter underlined like $\underset{\sim}{C}$ denotes a second-rank tensor quantity, and a letter underlined like $\underline{k}$ represents a vector quantity with magnitude k and direction $\hat{\underline{k}}$ so $\underline{k} = k\hat{\underline{k}}$.

The symbol $\langle\ \rangle$ indicates an average, often an ensemble average, of the enclosed quantity.

The meaning of subscripts is usually self-explanatory. The subscripts e and i in most cases label electron and ion quantities respectively. Numerical subscripts 0, 1, 2, etc. in many cases label coefficients in an expansion in Legendre polynomials.

Below we list primarily the variables and functions that appear in more than one section.

$a = a_e$	electron thermal speed
$A = a_i$	ion thermal speed
$b_{min} \approx e^2/\theta_e$	minimum impact parameter in Landau equation
$b_{max} \approx D_e$	maximum impact parameter in Landau equation
$\underset{\sim}{C}(\underline{R}, \tau)$	autocorrelation function for electric-field fluctuations
D_e	electron Debye length
$\underset{\sim}{\mathcal{D}}_\alpha(\underline{v}, t)$	Lenard-Balescu form of Fokker-Planck coefficient for species α
e	magnitude of electron charge
$\underline{E}(\underline{r}, t)$	electric field
$\underline{E}_0(\underline{r}, t)$	electric field not produced by the volume of plasma under consideration
E_{crit}	critical value of E_0 at which the plasma in the Spitzer-Härm problem becomes unstable to ion waves
$E_{run} \equiv \dfrac{n_e e a_e}{\sigma_{SH}}$	value of E_0 above which the electrons would quickly run away, according to the Landau equation
$E = E_0/E_{run}$	dimensionless E_0
$\underline{E}_{drag}(q, \underline{v}')$	drag on a test particle with velocity $\underline{v}'$ and charge q

$f_\alpha(\underline{r}, \underline{v}, t)$	distribution function for species α
$f(\underline{v}, t) = f_e(\underline{v}, t)$	velocity distribution of electrons
$F(\underline{v}, t) = f_i(\underline{v}, t)$	velocity distribution of ions
$F(V; \hat{\underline{k}})$	a function from which $R(\hat{\underline{k}}, V)$ and $I(\hat{\underline{k}}, V)$ can be calculated [see Eq. (III-20)]
$\mathcal{F}_\alpha(\underline{v}, t)$	Lenard-Balescu form of dynamic friction for species α
$f(x)$	dimensionless $f_1(v)$
$g(x, y)$	dimensionless nonlinear correction in $f(v, \alpha)$
$h(x, y)$	dimensionless $f(v, \alpha)$
$H(\hat{\underline{k}}, V)$	a function involved in $C(R, \tau)$ and in the Lenard-Balescu equations [see Eq. (IV-10)]
$I(\hat{\underline{k}}, V)$	imaginary part of $k^2 - k^2\epsilon(\underline{k}, kV)$ with V real
$\underline{j}(t)$	electrical-current density
$\underline{J}_\alpha(\underline{v}, t)$	current of species α in velocity space
$\underline{k}$	variable of Fourier transform in space; wave number
$k_m \sim [b_{min}]^{-1}$	the cutoff in the Lenard-Balescu equations [see Eq. (V-1)]
$K(\hat{\underline{k}}, V)$	a function in the Lenard-Balescu equations [see Eq. (V-1)]
$m_\alpha = M_\alpha$	mass of a particle of species α
$m = m_e$	mass of an electron
$M = m_i$	mass of an ion
n_α	number density of species α
$n = n_e$	number density of electrons
P	a principal-value integration
$P_\ell(y)$	Legendre polynomials
$q_\alpha = Z_\alpha e$	charge of a particle of species α
$\underline{r}$	position variable
$R(\hat{\underline{k}}, V)$	real part of $k^2 - k^2\epsilon(\underline{k}, kV)$ with V real
$S(y)$	function that determines the effect of ion waves upon the electron velocity distribution [see Eq. (VI-105]
t	time

$u = V/a_e$	dimensionless phase speed
$\underline{U}$	relative velocity in displaced Maxwellian electron-proton plasma
U_{crit}	value of U beyond which certain ion waves grow
$\underline{v}$	velocity variable for particles
V	real part of phase speed ω/k
V_{min}, V_{max}	in a Maxwellian electron-proton plasma, ion waves can exist only within $V_{min} < V < V_{max}$
$W(\theta, u)$	distribution of energy in fluctuations associated with ion waves [see Eq. (VII-9)]
$x = v/a_e$	dimensionless particle speed
$y = \cos\alpha$	angular variable in electron velocity distribution
$Z_\alpha = q_\alpha/e$	charge of particle of species α in units e
α	index for labeling species
α	angle between $\underline{v}$ and $\underline{E}_0$
β	thermoelectric coefficient
$\gamma = \theta_e/\theta_i$	ratio of temperatures (in Sec. VI. C only)
γ	imaginary part of frequency ω; growth rate
$\delta = \sqrt{m/M}$	square root of mass ratio
$\delta(\ )$	Dirac δ function of variable in parentheses
$\epsilon = A/a$	ratio of thermal speeds (ion to electron)
$\epsilon(\underline{k}, \omega)$	Vlasov dielectric function
θ	angle between $\underline{k}$ and $\underline{E}_0$
θ_α	temperature of species α in energy units
$\rho_0(\underline{r}, t)$	a charge density not part of the plasma under consideration
$\Lambda = 4\pi n_e D_e^3$	the plasma parameter
ν_c	a collision frequency [see Eq. (VI-31)]
σ	electrical conductivity (appears with various subscripts)
$\tau = \nu_c t$	dimensionless time

$\omega = kV + i\gamma$	variable of Laplace transforms in time; angular frequency
$\omega_{p\alpha}$	plasma frequency of species α

FOOTNOTES AND REFERENCES

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