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HOW TO DISCARD 'FREE DISPOSABILITY' – AT NO COST

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Here we demonstrate a technique for proving the existence of competitive equilibrium without assuming either free disposability or monotone preferences. Although this device is sufficiently flexible to allow generalization in several directions,[1] our discussion will be confined to a pure exchange economy constructed as fcllows. There are *m consumers* indexed by the set $I = \{1, \ldots, m\}$. For each $i \in I$, there is a *consumption set* $C_i \subset E^n$, an *initial endowment* $w_i \in E^n$, a *trade set* $X_i \equiv C_i - \{w_i\}$, a *preference relation on trades*[2] $R_i \subset X_i \times X_i$, and a *strict preference relation* $P_i \equiv \{(x, y) \in X_i \times X_i \mid (x, y) \in R_i \text{ and } (y, x) \notin R_i\}$. The following notations are equivalent: $(x, y) \in R_i$, xR_iy, $x \in R_i(y)$ and $y \in R_i^{-1}(x)$. Similar notational conventions apply for the relation P_i.

A *competitive equilibrium* is an allocation of trades $(\bar{x}_1, \ldots, \bar{x}_m) \in \prod X_i$ and a price vector $\bar{p} \in E^n$ such that $\sum_I \bar{x}_i = 0$ and for all $i \in I$, $\bar{p}\bar{x}_i = 0$ and $P_i(\bar{x}_i) \cap \{x \mid \bar{p}x \leqq 0\} = \emptyset$. A *quasi-equilibrium* is an allocation of trades $(\bar{x}_1, \ldots, \bar{x}_m) \in \prod X_i$ and a price vector $\bar{p} \in E^n$ such that $\bar{p} \neq 0$, $\sum_I \bar{x}_i = 0$, and for all $i \in I$, $P_i(\bar{x}_i) \cap \{x \mid \bar{p}x < 0\} = \emptyset$. Following a procedure devised by Debreu (1962), we first prove the existence of quasi-equilbrium and then show that (with additional assumptions) a quasi-equilibrium is a competitive equilibrium.

Theorem 1. There exists a quasi-equilibrium for an exchange economy if:

(1) *For all $i \in I$, $X_i \subset E^n$ is convex and compact and $0 \in X_i$.*

*The author is indebted to Professors Trout Rader, Robert Parks, James Little, Wayne Shafer, and Hugo Sonnenschein for instruction and encouragement. The main notion of this paper is extracted from an earlier unpublished work [Bergstrom (1973)].

[1] More general cases to which this technique has been applied or can readily be applied include Lindahl equilibrium [Bergstrom (1971)], non-convex preferences [Bergstrom (1973)] and non-transitive preferences [Bergstrom (1973 and 1975b), Gale and Mas-Collel (1975), Shafer (1976), and Shafer and Sonnenschein (1975)]. An alternative method of discarding free disposal is offered in a recent paper by Hart and Kuhn (1975).

[2] The formulation of preferences on trades is equivalent to a formulation in which preferences are defined on consumption. Thus we could define $R_i' \subset C_i \times C_i$ where $xR_i'y$ if and only if $w_i + xR_iw_i + y$. We deal with R_i rather than R_i' purely for notational economy.

(2) *For all* $i \in I$, R_i *is transitive and for all* $x \in X_i$, $R_i(x)$ *is closed and convex.*

(3) *If* $x \in \prod_I X_i$ *and* $\sum_I x_i = 0$, *then for all* $i \in I$, R_i *is locally non-satiated at* x_i.[3]

Our proof of Theorem 1 involves construction of correspondences, F_i, with domain $X_0 = \{p \in E^n \mid \|p\| \leqq 1\}$ which differ essentially from excess demand correspondences only where $\|p\| < 1$. Thus we define for each $i \in I$, the correspondences, B_i, $\mathring{B}_i$, and F_i with domain X_0 where for $p \in X_0$, $B_i(p) = \{x \in X_i \mid px \leqq (1-\|p\|)/m\}$, $\mathring{B}_i(p) = \{x \in X_i \mid px < (1-\|p\|)/m\}$, and $F_i(p) = B_i(p) \cap \{x \mid P_i(x) \cap \mathring{B}_i(p) = \emptyset\}$. The correspondence F_0 with domain $\prod_I X_i$ is defined so that $F_0(x) = \{\sum_I x_i / \|\sum_I x_i\|\}$ if $\sum_I x_i \neq 0$ and $F_0(x) = X_0$ if $\sum_I x_i = 0$.

Lemma 1. Assumptions 1 and 2 of Theorem 1 imply that for $i = 0, 1, \ldots, m$, *the correspondence* F_i *has a closed graph with non-empty, convex image sets.*

Proof. It is straightforward to verify that F_0 has the desired properties. For $i \in I$, let $(p^n, x^n) \to (p, x)$ where $x^n \in F_i(p^n)$ for all n. Then for all n, $p^n x^n \leqq (1-\|p^n\|)/m$. Therefore $px \leqq (1-\|p\|)/m$ and since X_i is closed, $x \in B_i(p)$. If $x' \in \mathring{B}_i(p)$, then for all n sufficiently large, $p^n x' < (1-\|p^n\|)/m$ and since $x^n \in F_i(p^n)$, $x' \notin P_i(x^n)$. Therefore $x^n \in R_i(x')$ for all sufficiently large n, and since $R_i(x')$ is closed, $x \in R_i(x')$. It then follows that $x \in F_i(p)$ and hence that F_i has a closed graph. Convexity of the image sets, $F_i(p)$, follows from convexity of the sets $B_i(p)$ and $R_i(x)$. Since for all $i \in I$, X_i is compact and $0 \in X_i$, the sets, $B_i(p)$ are non-empty and compact. Non-emptiness of the sets, $F_i(p)$, then follows from Assumption 2. Q.E.D.

Proof of Theorem 1. Define the correspondence F with domain, $\prod_{i=0}^m X_i$ so that $F(p, x_1, \ldots, x_m) = F_0(\sum_I x_i) \times F_1(p) \times \ldots \times F_m(p)$. According to Lemma 1 the correspondences F_i have closed graphs and non-empty, compact image sets. These properties are inherited by F which maps the convex, compact set $\prod_{i=0}^m X_i$ into its subsets. Kakutani's fixed point theorem therefore establishes the existence of $(\bar{p}, \bar{x}) \in F(\bar{p}, \bar{x})$. We show that $\bar{p}$ and $\bar{x}$ constitute a quasi-equilibrium.

If $\|\bar{p}\| < 1$, then since $\bar{p} \in F_0(\sum_I \bar{x}_i)$, it must be that $\sum_I \bar{x}_i = 0$. Then, by Assumption 3, R_i is locally non-satiated at $\bar{x}_i$ for each $i \in I$. Since $\bar{x}_i \in F_i(\bar{p})$, it follows that $\bar{p}\bar{x}_i = (1-\|\bar{p}\|)/m$ for all $i \in I$ and that $\bar{p} \sum_I \bar{x}_i = 1 - \|\bar{p}\| > 0$.[4] But this is impossible since $\sum_I \bar{x}_i = 0$. Therefore it cannot be that $\|\bar{p}\| < 1$. We must conclude that $\|\bar{p}\| = 1$.

If $\sum_I \bar{x}_i \neq 0$, then since $\bar{p} \in F_0(\sum_I \bar{x}_i)$, it must be that $\bar{p} \sum_I \bar{x}_i = \|\sum_I \bar{x}_i\| > 0$.

[3]Preferences are said to be locally non-satiated at x_i if every neighborhood of x_i contains a point x_i' such that $x_i' P_i x_i$. Notice that we could not consistently assume that preferences are locally non-satiated everywhere on X_i. Assumptions 1 and 2 imply that for each i, there exists $\bar{x}_i \in X_i$ such that $P_i(\bar{x}_i) = \emptyset$.

[4]Suppose $\bar{p}\bar{x}_i < (1-\|\bar{p}\|)/m$. Then since P_i is locally non-satiated as $\bar{x}_i$, there exists $\bar{x}_i' \in X_i$ such that $x_i' P_i \bar{x}_i$ and $\bar{p} x_i' < (1-\|\bar{p}\|)/m$. But this is impossible if $\bar{x}_i \in F_i(\bar{p})$.

But since $\bar{x}_i \in F_i(\bar{p})$ and $\|\bar{p}\| = 1$, $\bar{p}\bar{x}_i \leqq 0$ for all $i \in I$ and thus $\bar{p} \sum_I \bar{x}_i \leqq 0$. This contradicts our previous assertion. Therefore it must be that $\sum_I \bar{x}_i = 0$. From the results that $\|\bar{p}\| = 1, \sum_I \bar{x}_i = 0$, and $\bar{x}_i \in F_i(\bar{p})$ for all $i \in I$, it is immediate that $\bar{x}$ and $\bar{p}$ constitute a quasi-equilibrium. Q.E.D.

Remark 1. The assumption that R_i is transitive is used only to show that $F_i(p)$ is non-empty. For this purpose it would be sufficient to make the weaker assumption that R_i takes maximal elements on convex, compact sets. Assumptions weaker than transitivity which guarantee this property are studied in Sonnenschein (1971), and Bergstrom (1973, 1975a and 1975b).

Remark 2. The assumption that each X_i is compact can be replaced by the weaker assumption that each X_i is closed and bounded from below. This is accomplished by a straightforward application of the truncation technique exploited by Debreu (1962).

Remark 3. Theorem 1 can be slightly strengthened to assert the existence of a quasi-equilibrium $(\bar{p}, \bar{x})$ in which $\bar{x}_i R_i 0$ for all $i \in I$. This is accomplished by simply replacing the correspondence $F_i(p)$ by $F_i'(p) = R_i(0) \cap F_i(p)$ at each point in the argument.

There are several known sets of conditions which guarantee that a quasi-equilibrium is a competitive equilibrium. We conclude by presenting one useful variant.

Lemma 2. Let $(\bar{p}, \bar{x})$ be a quasi-equilibrium such that $\bar{x}_i R_i 0$ for all $i \in I$. Then $(\bar{p}, \bar{x})$ is a competitive equilibrium if:

(i) $0 \in \operatorname{Int} \sum_I X_i$.

(ii) *For all $i \in I$, if xP_iy and $z \in X_i$, there exists $\lambda > 0$ such that $\lambda x + (1-\lambda) zP_iy$.*[5]

(iii) *If $x \in \prod_I R_i(0)$ and $\sum_I x_i = 0$, then for each $i \in I$, there exists $\hat{x} \in \prod_I X_i$ such that $\hat{x}_j P_j x_j$ for all $j \neq i$ and $\sum_{j \neq i} \hat{x}_j + \Theta \hat{x}_i = 0$ for some scalar $\Theta > 0$.*[6]

Proof. Let $K = \{i \in I \mid \bar{p}\mathring{x}_i < 0$ for some $\mathring{x}_i \in X_i\}$. Let $i \in K$ and suppose $\bar{p}x_i \leqq 0$ and $x_i P_i \bar{x}_i$. Where $\mathring{x}_i \in X_i$ and $\bar{p}\mathring{x}_i < 0$, let $x_i(\lambda) = \lambda \mathring{x}_i + (1-\lambda)x_i$. Then for all $\lambda > 0$, $\bar{p}x_i(\lambda) < 0$ and by Assumption (ii), $x_i(\lambda)P_i\bar{x}_i$ for some

[5]This assumption is weaker than the assumption that $P_i(x)$ is open for all $x \in X_i$. Rader (1974) shows that where there is a household production, the weaker assumption is significantly more plausible than the stronger.

[6]If there always exists such an allocation with $\Theta = 1$, then for every feasible allocation which each consumer likes as well as no trade, it would be possible for the other traders to 'exploit' any individual by forcing him to make a trade more favorable to everyone except himself. The possibility that Θ may exceed one weakens the assumption further.

$\lambda > 0$. But this is impossible since $(\bar{p}, \bar{x})$ is a quasi-equilibrium. Therefore for all $i \in K$, $P_i(\bar{x}_i) \cap \{x \mid \bar{p}x \leqq 0\} = \emptyset$. Thus if $K = I$, then $(\bar{p}, \bar{x})$ must be a competitive equilibrium.

Suppose $i \in I$ and $i \notin K$. By Assumption 3, there exists $\hat{x} \in \prod_I X_i$ such that $\hat{x}_j P_j \bar{x}_j$ for all $j \neq i$ and $\sum_{j \neq i} \hat{x}_j = -\Theta \hat{x}_i$ for some $\Theta > 0$. Since $(\bar{p}, \bar{x})$ is a quasi-equilibrium, and K is non-empty, it must be that $\bar{p}\hat{x}_j \geqq 0$ for all $j \neq i$ with strict inequality for some $j \neq i$. Therefore $\bar{p} \sum_{j \neq i} \hat{x}_j = -\Theta \bar{p}\hat{x}_i > 0$. It follows that $\bar{p}\hat{x}_i < 0$. But this cannot be if $i \notin K$. Therefore $K = I$ and hence $(\bar{p}, \bar{x})$ is a competitive equilibrium. Q.E.D.

As an immediate consequence of Theorem 1, Remark 3, and Lemma 2 we have the following:

Theorem 2. If an exchange economy satisfies the conditions of Theorem 1 and Lemma 3, then a competitive equilibrium exists.

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