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Hypersurfaces of constant curvature in asymptotically hyperbolic spaces

Author
DeConde, David Thomas

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HYPERSURFACES OF CONSTANT CURVATURE IN ASYMPTOTICALLY HYPERBOLIC SPACES

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by

David T. DeConde

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The Dissertation of David T. DeConde is approved:

Professor Richard Montgomery, Chair

Professor Debra Lewis

Professor Jie Qing

Tyrus Miller
Vice Provost and Dean of Graduate Studies

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Abstract

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by

David T. DeConde

The relationship between the geometry of a conformally compact manifold and the conformal geometry of its conformal infinity is of particular interest due to its association with the AdS/CFT correspondence of physics, a conjectured correlation between a string theory on a negatively curved Einstein manifold and a conformal field theory on its boundary at infinity. In the case of hyperbolic space $\mathbb{H}^{n+1}$ with conformal infinity the round sphere $\mathbb{S}^n$, a very precise relationship has been established between conformal invariants (the eigenvalues of the Schouten tensor) on $\mathbb{S}^n$ and Weingarten curvatures of immersed hypersurfaces [8]. This same relationship has been extended to hyperbolic Poincaré manifolds [3]. We establish a correspondence between constant scalar curvature metrics on conformal infinity and families of hypersurfaces of constant Weingarten curvature in a neighborhood of infinity. This generalizes results of Mazzeo and Pacard [31] on existence of constant curvature foliations of asymptotically hyperbolic spaces to more arbitrary Weingarten curvatures.

To Jeannette and Amy,

who were unfaltering in their support.

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Chapter 1

Introduction

Throughout history mathematics and physics have enjoyed a famously fruitful and intimate interaction with geometry being the first and most frequent point of contact. A relatively recent and remarkable case of this interplay between the two disciplines is the AdS/CFT (Anti de Sitter/Conformal Field Theory) correspondence, a conjectured equivalence between a string theory and gravity on one space, and a quantum field theory without gravity on the conformal boundary of this space. The notion of the AdS/CFT correspondence was originally proposed by Maldacena in 1997 [30], and important aspects of the theory were developed by Witten [48] and others [17].

Activity on the mathematical side of this story began in 1985 with the introduction of the ambient construction by Fefferman and Graham [10]: the goal was to find new conformal invariants by using a correspondence between asymptotically hyperbolic Einstein metrics and their conformal boundaries. This construction provides a ready mathematical manifestation of the physical model; the original statement of the correspondence provides a duality between the partition function of an n-dimensional conformal field theory (resident on a conformal manifold M) and the renormalized Einstein-Hilbert action of an Einstein metric on an $(n+1)$-dimensional manifold X with conformal boundary at infinity M.

Anti de Sitter space is a maximally symmetric pseudo-Riemannian manifold with Lorentzian metric and constant negative curvature; it is the Lorentzian analogue of hyperbolic space. This suggests hyperbolic space as a conveniently simplified model in which to study the relationship between the geometry of a bulk space and the structure of its infinity. Indeed, a very precise such connection is already well known; realizing hyperbolic space as the hyperboloid embedded in Minkowski space, and the null cone as its conformal infinity, hyperbolic symmetries of the hyperboloid and conformal transformations on the sphere at infinity are related by a natural bijection: a hyperbolic symmetry and a conformal transformation are in correspondence when they both result from restriction of the same Lorentz transformation on the ambient Minkowski space.

In fact more is known; the conformal metrics on the sphere at infinity are in correspondence with the compact immersed hypersurfaces of hyperbolic space with regular hyperbolic Gauss map, furthermore the eigenvalues of the Schouten tensor (basic conformal invariants) of metrics on the boundary are linked to the principal curvatures of these hypersurfaces by a surprisingly simple relationship [8]. These same relationships hold also for spaces which are quotients of hyperbolic space under discrete isometry groups [3]. One especially interesting use of this link in [8] is to provide a translation procedure between the Cristoffel problem [11] in hyperbolic space and the Nirenberg-Kazdan-Warner problem (there are extensive references for this problem; see the survey [29]) on the conformal boundary. This renders a geometric partial differential equation into a conformally invariant one, and similarly turns solutions to one of these problems into solutions of the other (à la Bäcklund transformations).

When the bulk space under consideration is not so homogeneous as hyperbolic space, then the recovery of such a precise connection to the conformal boundary becomes difficult. However, for asymptotically hyperbolic spaces one might hope to at least find some relationship between the bulk geometry and the conformal structure on the boundary, at least in a neighborhood of infinity. In fact, the existence of hypersurfaces in a neighborhood of the boundary with extrinsic curvature linked to the conformal structure at infinity has been demonstrated and used to produce constant mean curvature foliations of asymptotically hyperbolic manifolds [31]. It is this route we wish to explore as it seems to hold potential for eventually yielding some results analogous to those of [8] and [3] but in more general cases.

We begin with background, introducing the objects under study and the tools used to study them: chapter 2 summarizes the ambient construction and chapter 3 provides details about asymptotically hyperbolic manifolds, while chapter 4 is a brief synopsis of the Yamabe problem including just the details necessary for the sequel. In addition to their immediate relevance, it may be noted that all three of these topics hold positions of significance in the history of conformal geometry. Chapter 5 is a detailed examination of the existence of constant mean curvature hypersurfaces in asymptotically hyperbolic spaces as done by Mazzeo and Pacard in [31], though we exclude some of their results in favor of more detail on others. As will be seen in that chapter, the results regarding curvature of hypersurfaces in the bulk space are achieved via curvature prescription in the conformal structure on the boundary (cf. [8] and [3]), thus keeping with the leitmotiv of the AdS/CFT correspondence. In chapter 6 we extend these methods to hypersurfaces distinguished by other types of curvature.

The presentation here assumes familiarity with the fundamentals of Riemannian geometry and partial differential equations, which in turn require proficiency with differentiable manifolds and real and functional analysis. There are numerous good sources for this material, but we list a few recommended ones here: for analysis, [39] and [40]; for manifolds, [26] and [47]; for differential equations, [9] and [13]; for Riemannian geometry, [25] and [5].

The following conventions will be followed throughout:

1. Unless stated otherwise the Einstein summation convention will be employed, i.e. repeated indices with one raised and one lowered are summed over.

2. Tensor indices (whose position indicates valence) are raised and lowered freely via the "musical isomorphism" provided by appropriate metric. Usually the metric to be used is clear, However explicit mention is frequently provided or included in notation.

3. The definition of the Riemann curvature tensor used will be $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ for vector fields X, Y, Z, and with $[\cdot,\cdot]$ denoting the Lie bracket (this differs from the alternative definition only by sign).

4. The definition of the Laplace-Beltrami operator is simply the trace of the covariant Hessian. This operator has negative spectrum and differs from the other common convention only by sign.

Chapter 2

The Ambient Metric

In Riemannian geometry the objects of study are smooth manifolds equipped with a metric which allows the measurement of both length and angle of tangent vectors: given (M, g), $v, w \in T_pM$, we have

$$|v|^2 = g(v,v), \qquad \cos\theta = \frac{g(v,w)}{|v||w|}.$$

The maps which define equivalence in this category are isometries, diffeomorphisms that preserve the metric, and hence lengths.

In conformal geometry only the angle information is retained, there is no unambiguous length. This is the same as knowing a metric only up to scale at each point of a manifold. Hence

Definition 2.0.1 (Conformal Manifold)**.** A *conformal manifold* M is a smooth manifold equipped with an equivalence class of (pseudo-)Riemannian metric tensors $[g] =$

$\{e^{2\omega}g \ : \ \omega \in C^\infty(M)\}$, i.e. a collection of metrics on M, differing only by a factor of scale determined by some smooth function on M.

In this context the maps of relevance are those that preserve angle; conformal maps.

The Ambient Metric was originally introduced by Haantjes and Schouten in 1936-37 ([42], [43]), and independently rediscovered in modern form by Fefferman and Graham in 1985. The central idea is to apply the methods of pseudo-Riemannian geometry to conformal manifolds by realizing them as submanifolds of a (uniquely associated) pseudo-Riemannian space, with the motivating goal being to thus derive from Riemannian local invariants (e.g. curvature) conformal invariants (cf. Weyl's Theorem).

We will outline the ambient metric construction in some detail here, in particular those parts which have bearing on the asymptotic expansion of asymptotically hyperbolic metrics, and especially Poincaré and Poincaré-Einstein metrics, however extensive proofs of the technical details will be omitted and the reader is referred to [10] for the complete treatment of the subject. Note that the ambient metric construction works equally well without any modification in the pseudo-Riemannian case as there is never any use made of the positive-definite aspect of a Riemannian metric. Because the construction adds a time-like dimension to whatever sort of manifold we begin with we will simply find it convenient to assume that whenever a time-like direction is mentioned it is that one.

2.1 Motivation

Before addressing the technical details of the ambient construction it is instructive to consider the concept from the perspective of the motivating case, the conformal round sphere $(S^n, [\mathring{g}])$. S^n is (locally[1]) conformally flat, meaning that the standard sphere metric is related to the (flat) Euclidean $\mathbb{R}^n$ metric by a conformal factor $((\sigma^{-1})^*\mathring{g} = \left(\frac{4}{(1+|u|^2)^2}\right)\bar{g})$ obtained via pullback of the metric along the stereographic projection map σ Euclidean $\mathbb{R}^n$ metric by a conformal factor. This is the standard conformally flat case in the following sense.

Conformal transformations of $\mathbb{R}^n$ are generated by Euclidean motions (translations and orthogonal transformations), dilations, and inversions (in n-spheres or hyperplanes)[2]. Inversion in a sphere is not a bijection on $\mathbb{R}^n$ (as the center point is sent to infinity), so the more natural setting for studying conformal transformations is the compactification of $\mathbb{R}^n$ via the addition of $\{\infty\}$ by stereographic projection.

On the conformal sphere there is no unambiguous sense of scale at any point. Consider the following nice manner in which such a conformal sphere may be visualized. (For concreteness, picture the n-sphere as a circle; we will need two more dimensions for the ambient space.) Let $\mathcal{N}_+$ be the upper half of the standard cone in $\mathbb{R}^{n+2}$. Any horizontal plane will intersect the cone in a circle, and as the plane moves up or down the size of the circle of intersection varies. If instead of intersecting the

[1]The distinction *locally* means that there is not a global chart that is conformally flat.

[2]This is *Liouville's Theorem* for conformal mappings and applies in dimensions ≥ 3. When dim $= 2$ there is much more freedom; the *Riemann Mapping Theorem* implies that all simply-connected open domains are conformally equivalent.

cone with a plane, we use any smooth surface that might be realized as the graph of a smooth function, the result is a point-wise scaling of the circle.

In fact, we can make this precise and recapture all of the standard conformal structure on S^n from a pseudo-Riemannian metric on $\mathbb{R}^{n+2}$. Give $\mathbb{R}^{n+1} \times \mathbb{R}$ the coordinates $\chi = (\xi^1, \ldots, \xi^{n+1}, \tau)$ (the ξ^i are spacelike and τ is timelike) and the Minkowski metric $\tilde{g} = \sum_i (d\xi^i)^2 - d\tau^2$. Let $Q = \sum_i (\xi^i)^2 - \tau^2$ be the quadratic form associated to $\tilde{g}$. Then the null-cone is the quadric $\mathcal{N} = \{\chi \in \mathbb{R}^{n+2} \ : \ Q(\chi) = 0\} = \{|\xi| = \tau\}$. Projectivization yields $\mathbb{P}\mathcal{N} = \mathcal{P} = \{[\chi] \in \mathbb{R}P^{n+1} \ : \ \chi \in \mathcal{N}\} \simeq S^n$, and if $\pi : \mathcal{N} \to \mathcal{P}$ is the projection $\chi \mapsto [\chi]$, then we will use smooth sections of π to represent members of the conformal class on S^n. We want to realize the conformal structure on S^n as invariantly determined by the Minkowski metric on $\mathbb{R}^{n+2}$, but the restriction $\tilde{g}|_{T\mathcal{N}}$ is degenerate, i.e. if $X = \chi^i \partial_{\chi^i}$, (the positional vector field), then $\tilde{g}(X, V) = 0$ for all $V \in T\mathcal{N}$. Clearly for $\chi \in \mathcal{N}$ $\tilde{g}_\chi(X_\chi, X_\chi) = Q(\chi) = 0$. Then $dQ(V) = 0$ if $V \in T\mathcal{N}$, but $Q = \tilde{g}_{ij}\chi^i\chi^j$ so $dQ = 2\tilde{g}_{ij}\chi^i d\chi^j$. Thus

$$0 = dQ(V) = 2\tilde{g}_{ij}\chi^i V^j = 2\tilde{g}(X, V).$$

So, $X \perp T_\chi\mathcal{N}$ for any $\chi \in \mathcal{N}$ and hence $\tilde{g}|_{T_\chi\mathcal{N}}$ is degenerate, but it does induce an inner product g_χ on $T_{[\chi]}\mathcal{P}$ as follows: For $\pi_* : T_\chi\mathcal{N} \to T_{[\chi]}\mathcal{P}$,

$$\pi_*(X) = 0 \quad \text{and} \quad \pi_* : T_\chi\mathcal{N} / \operatorname{span} X \xrightarrow{\cong} T_{[\chi]}\mathcal{Q}$$

For $v, w \in T_{[\chi]}\mathcal{P}$ choose any $V, W \in T_\chi\mathcal{N}$ such that $\pi_* V = v, \pi_* W = w$ and define g^χ by $g^\chi(v, w) = \tilde{g}(V, W)$. This is well-defined because our choices of V and W are unique up to addition of a multiple of X, i.e. if V, V' are such that $\pi_* V = \pi_* V' = v$. Then $V' = V + \lambda X$ for $\lambda \in \mathbb{R}$ and thus the condition $X \perp T_\chi \mathcal{N}$ makes g^χ independent of such choices. For a given point $[\chi] \in \mathcal{P} \simeq S^n$ we can apply this definition for any point $\chi \in \mathbb{R}^{n+2}$ in the line $[\chi]$. Now, when $0 \neq s \in \mathbb{R}$ we wish to know how are g^χ and $g^{s\chi}$ are related.

Name the map corresponding to multiplication by s, i.e. $\chi \mapsto s\chi$, $\delta_s : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$, called "dilation by s" for obvious reasons. Then $(\delta_s)_*$ also just multiplies tangent vectors by s and hence $\tilde{g}$ is homogenous of degree two with respect to dilations, i.e. $(\delta_s)^* \tilde{g} = s^2 \tilde{g}$. Dilations correspond precisely to "scaling by s," and of course $\pi(s\chi) = \pi(\chi) = [\chi]$. Suppose $v, w \in T_{[\chi]}\mathcal{P}$ and choose $V, W \in T_\chi \mathcal{N}$ so that $\pi_* V = v, \pi_* W = w$. Then

$$\begin{aligned} g^{s\chi}(v, w) &= \tilde{g}((\delta_s)_* V, (\delta_s)_* W) = (\delta_s)^* \tilde{g}(V, W) \\ &= s^2 \tilde{g}(V, W) = s^2 g^\chi(v, w). \end{aligned}$$

Therefore $g^{s\chi} = s^2 g^\chi$; at each point $[\chi] \in \mathcal{P}$ we obtain an inner product on $T_{[\chi]}\mathcal{P}$ by pushing down $\tilde{g}_\chi$ for some $\chi \in [\chi]$ and each such selection determines the conformal scale based on the magnitude $|\chi|$. Hence the Minkowski metric on $\mathbb{R}^{n+2}$ invariantly determines the conformal structure on S^n as wished.

2.2 Coordinate Selection

Before we establish the general case definitions, we make an alternative selection of coordinates in the flat case that will prove convenient for the abstract general ambient construction. Choose coordinates:

$$x^i = \xi^i/\tau, \quad t = \tau \quad \text{and} \quad \rho = \frac{1}{2}\frac{Q}{\tau^2} = \frac{1}{2}(|x|^2 - 1)$$

corresponding to scaling up of the spacial dimensions proportional to the time component so that the upper-half of the null cone

$$\mathcal{N}_+ = \{(\xi, \tau) \in \mathbb{R}^{n+1} \times \mathbb{R} \ : \ |\xi| = \tau, \tau > 0\}$$

looks like the cylinder

$$\{|\xi| = 1, \tau > 0\}.$$

We disregard from now on points with $\tau \leq 0$; $\pi(\chi) = \pi(-\chi)$ and hence $g^{-\chi} = g^{\chi}$ and thus we lose nothing by doing so. By substitution then the Minkowski metric becomes

$$\tilde{g} = 2\rho dt^2 + 2tdtd\rho + t^2 g_\rho$$

where g_0 is the standard metric on $S^n \subset \mathbb{R}^{n+1}$, and g_ρ in general is the standard metric on the n-sphere of radius $\sqrt{2\rho + 1}$. Note that $\tilde{g}$ is homogenous of degree two in t, and $\tilde{g}|_{\rho=0,t=1} = g_0$.

2.3 The Ambient Metric Construction

The goal of the ambient construction is to produce such an ambient space and metric $\tilde{g}$ for any conformal manifold $(M, [g])$. First we construct an analog of the upper null cone $\mathcal{N}_+$.

Definition 2.3.1. For a conformal manifold $(M, [g])$ of dimension $n \geq 2$, equipped with conformal class $[g]$, of signature (p, q), the *metric bundle* $\mathcal{G} = \{(h, x) \ : \ h \in [g], x \in M\} \subset S^2T^*M$ is an $\mathbb{R}_+$-bundle over M with projection map

$$\pi : \mathcal{G} \to M, \quad (h, x) \mapsto x$$

and the maps comprising an $\mathbb{R}_+$-action on fibers, "dilation" by s for any $s \in \mathbb{R}_+$:

$$\delta_s : \mathcal{G} \to \mathcal{G}, \quad (h, x) \mapsto (s^2 h, x).$$

$\mathcal{G}$ is naturally equipped with a tautological symmetric 2-tensor $\boldsymbol{g}^0$ defined as follows. For $X, Y \in T_{(h,x)}\mathcal{G}$ let

$$\boldsymbol{g}^0(X, Y) = h(\pi_* X, \pi_* Y).$$

Then $\boldsymbol{g}^0$ is

Homogenous of degree 2 with respect to the dilations δ_s, i.e. $\delta_s^* \boldsymbol{g}^0 = s^2 \boldsymbol{g}^0$. Formally this can be seen by just unwinding the definitions, but note its geometric significance: dilations on $\mathcal{G}$ act as conformal scalings on M. Thus the metric bundle encodes the conformal structure on M.

Degenerate: If $T = \frac{d}{ds}\delta_s|_{s=1}$ is the vector field on $\mathcal{G}$ which is the infinitesimal generator of the dilations δ_s, then $\pi_* T = 0$ and $\tilde{g}(T, \cdot) = 0$; this is the generalized version of the positional vector field in the flat model.

It can be easily verified that the objects defined above, $\mathcal{G}$, π, δ_s, and $\boldsymbol{g}_0$ are all well-defined based solely on the conformal class $[g]$ and do not depend on the particular representative g, however, once a choice of representative g is fixed, we obtain a trivialization of the bundle $\mathcal{G}$; identify

$$(t, x) \in \mathbb{R}_+ \times M \text{ with } (t^2 g_x, x) \in \mathcal{G}.$$

In terms of this identification we then have

$$\pi : (t, x) \mapsto x, \qquad \delta_s : (t, x) \mapsto (st, x), \qquad T = t\partial_t, \qquad \boldsymbol{g}^0 = t^2 \pi^* g.$$

Then the chosen representative metric g can be regarded as a section of the metric bundle $\mathcal{G}$; the image of this section is the submanifold of $\mathcal{G}$ given by $t = 1$.

So far we have constructed the general analog of the null cone $\mathcal{N}$ of the flat case. Next we add a final dimension corresponding to the defining function of the null cone to allow us to repair the degeneracy of $\boldsymbol{g}^0$ and fill out the space on which the ambient metric $\tilde{g}$ will live. Let $\widetilde{\mathcal{G}} = \mathcal{G} \times \mathbb{R} = \mathbb{R}_+ \times M \times \mathbb{R}$. The dilations δ_s extend to $\widetilde{\mathcal{G}}$ by acting on the first factor alone, while T may similarly be extended by assuming constancy along the $\mathbb{R}$ factor. If g is a representative metric for the conformal structure with associated $\mathbb{R}_+$-fiber coordinate t, and if $(x^1, \ldots, x^n)$ are local coordinates on M, then $(t, x^1, \ldots, x^n, \rho)$ are local coordinates on $\mathcal{G} \times \mathbb{R}$. The indices 0 and ∞ will be used to refer to the t and ρ coordinates respectively, while $i, j, \ldots$ refer to the other coordinates. Capital Roman letter indices may vary over all coordinates.

The general definition is inspired by features that defined the flat model, plus the fact that while the metric bundle arises quite immediately from the conformal structure on M, the extension of the metric off of the "null cone" will require solving an ordinary differential equation in the ρ coordinate.

Definition 2.3.2 (Ambient Space)**.** An *ambient space* for a conformal manifold $(M, [g])$ is a pair $(\widetilde{\mathcal{G}}, \tilde{g})$, where:

1. $\widetilde{\mathcal{G}}$ is a dilation-invariant open neighborhood of $\mathcal{G} \times \{0\}$ in $\mathcal{G} \times \mathbb{R}$ and $\tilde{g}$ is a smooth metric on $\widetilde{\mathcal{G}}$;

2. $\tilde{g}$ is homogenous of degree 2 on $\widetilde{\mathcal{G}}$ (i.e. $\delta_s^* \tilde{g} = s^2 \tilde{g}$, for $s \in \mathbb{R}_+$);

3. The pullback $\iota^* \tilde{g}$ is the tautological tensor $\boldsymbol{g}^0$ on $\mathcal{G}$;

4. $\tilde{g}$ is Ricci-flat on $\mathcal{G} \times \{0\}$, i.e. $\mathrm{Ric}(\tilde{g}) = 0$.

When $[g]$ is a conformal class of signature (p, q), $\tilde{g}$ is a smooth metric of signature $(p + 1, q + 1)$.

The reasoning behind the axioms can be seen most easily by considering the flat model. As in the flat case, the space should be homogenous (in the geometric sense) in the dimension corresponding the conformal scaling; movement in this direction should not alter anything about the space but sense of scale. This also expresses itself as (algebraic) homogeneity of the metric with respect to dilations. Restriction of the ambient metric back to the metric bundle will be an initial condition for a second-order differential equation in terms of the metric that we get from the Ricci-flatness condition. The tautological 2-tensor on the metric bundle arises naturally, but extending this to an ambient metric on $\widetilde{\mathcal{G}}$ requires imposing additional conditions and vanishing Ricci tensor on $\mathcal{G}$ is strict enough to be useful but (mostly) solvable.

As is commonly the case in finding solutions to differential equations we do not in general expect to find global solutions for our metric in the ρ dimension so we content ourselves with a ρ-neighborhood around the metric bundle. Ambient metrics for the same conformal manifold have an appropriate sense of equivalence:

Definition 2.3.3 (Ambient-Equivalence)**.** Let $(\widetilde{\mathcal{G}}_1, \tilde{g}_1)$ and $(\widetilde{\mathcal{G}}_2, \tilde{g}_2)$ be two ambient spaces for $(M, [g])$. We say that $(\widetilde{\mathcal{G}}_1, \tilde{g}_1)$ and $(\widetilde{\mathcal{G}}_2, \tilde{g}_2)$ are *ambient equivalent* if there exist open sets $U_1 \subset \widetilde{\mathcal{G}}_1, U_2 \subset \widetilde{\mathcal{G}}_2$ and a diffeomorphism $\phi : U_1 \to U_2$ with the following properties:

1. Both U_1 and U_2 contain $G \times \{0\}$;
2. U_1 and U_2 are dilation invariant and ϕ commutes with dilations;
3. The restriction of ϕ to $G \times \{0\}$ is the identity map;
4. $\tilde{g}_1 - \phi^* \tilde{g}$ vanishes to infinite order at every point of $G \times \{0\}$.

In general when we say something like “up to diffeomorphism” about ambient metrics, it usually refers to diffeomorphisms satisfying properties like ϕ above. Having defined ambient-equivalence, we want to find a unique (up to such equivalence) ambient metric associated to each conformal manifold. So far we have

1. the second-order differential equation $\mathrm{Ric}(\tilde{g}) = 0$ and

2. the initial condition $\tilde{g}|_{T\mathcal{G}} = \boldsymbol{g}^0$.

We want to ensure that

1. this is an ordinary differential equation in terms of the ρ coordinate only, and

2. we have a first-order condition that yields a unique solution.

Ideally, we also make the actual problem easier to manage also. To this end we stipulate:

Definition 2.3.4 (Straight, Normal Form Ambient Metrics)**.** An ambient space is said to be *straight* if

1. For each point p in a dilation-invariant neighborhood $U \subset \mathcal{G}$, the parameterized dilation orbit $s \mapsto \delta_s p$ is a geodesic for $\tilde{g}$.

and in *normal form* relative to a representative metric g if the follow conditions hold:

2. For each $z \in \mathcal{G}$, the parameterized curve $(-\epsilon, \epsilon) \ni \rho \mapsto (z, \rho)$ is a geodesic for $\tilde{g}$.

3. Writing (t, x, ρ) for a point in $\mathbb{R}_+ \times M \times \mathbb{R} \sim \mathcal{G}$ under the association induced by the fixing of a representative metric g, at each point $(t, x, 0) \in \mathcal{G} \times \{0\}$, the metric tensor $\tilde{g}$ takes the form

$$\tilde{g} = \boldsymbol{g}^0 + 2t dt d\rho$$

A consequence of the existence proof of ambient metrics is that any ambient metric will be equivalent to a straight one in normal form, so it suffices to consider only such cases. Furthermore these conditions determine several of the Christoffel symbols $\widetilde{\Gamma}_{IJK}$ for $\tilde{g}$ and hence several of the components of $\tilde{g}$ itself. Hence by stipulating that $\tilde{g}$ is straight and in normal form relative to a representative metric g, it can be written nicely:

$$\tilde{\mathbf{g}}_{\mathbf{IJ}} = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_{ij} & 0 \\ t & 0 & 0 \end{pmatrix}$$

where

$$\tilde{\mathbf{g}}_{\mathbf{IJ}} = \begin{pmatrix} & & \\ & t^2 g_{ij} & \\ & & \end{pmatrix}$$

is determined by the condition $\iota^* \tilde{g} = \boldsymbol{g}_0$,

$$\tilde{\mathbf{g}}_{\mathbf{IJ}} = \begin{pmatrix} & & t \\ & & 0 \\ t & 0 & 0 \end{pmatrix}$$

is determined by $\tilde{g}$ being in normal form, and

$$\tilde{\mathbf{g}}_{\mathbf{IJ}} = \begin{pmatrix} 2\rho & 0 & \\ 0 & & \\ & & \end{pmatrix}$$

is determined by $\tilde{g}$ being straight.

The metric g_{ij} in the above expression is a function $g_{ij}(x, \rho)$ of both the point in the manifold M and the ρ coordinate. In the formula

$$\tilde{g} = 2\rho dt^2 + 2tdtd\rho + t^2 g_\rho$$

it is g_ρ, a 1-parameter family of metrics on M. In the sequel whenever a metric is referred to as "in normal form" it will also be assumed straight and expressible in the matrix form written above. This form puts our expression for the metric into a configuration that nicely mimics the flat case.

As a result of the series analysis mentioned below, both existence and uniqueness of ambient metrics is dependent on the parity of the dimension of the starting conformal manifold. If we assume our ambient metric to be straight and in normal form relative to g, then in

$$\tilde{\mathbf{g}}_{\mathbf{IJ}} = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_{ij} & 0 \\ t & 0 & 0 \end{pmatrix}$$

only the expansion of $g_{ij}(x, \rho)$ remains undetermined. The components of the Ricci tensor form a system of equations; using the expression for Ricci curvature

$$\widetilde{R}_{IJ} = \frac{1}{2}\tilde{g}^{KL}(\partial^2_{IL}\tilde{g}_{JK} + \partial^2_{JK}\tilde{g}_{IL} - \partial^2_{IJ}\tilde{g}_{KL} - \partial^2_{KL}\tilde{g}_{IJ}) + \tilde{g}^{KL}\tilde{g}^{PQ}\left(\widetilde{\Gamma}_{ILP}\widetilde{\Gamma}_{JKQ} - \widetilde{\Gamma}_{IJP}\widetilde{\Gamma}_{KLQ}\right)$$

we may compute the following components (the other two, involving the t-coordinate, will not be needed here)

$$\widetilde{R}_{ij} = g''_{ij}\rho - g^{kl}g'_{ik}g'_{jl}\rho + \frac{1}{2}g^{kl}g'_{kl}g'_{ij}\rho - \left(\frac{n}{2} - 1\right)g'_{ij} - \frac{1}{2}g^{kl}g'_{kl}g_{ij} + R_{ij}, \tag{2.3.1}$$

$$\widetilde{R}_{i\infty} = \frac{1}{2}g^{kl}(\nabla_k g'_{il} - \nabla_i g'_{kl}), \tag{2.3.2}$$

$$\widetilde{R}_{\infty\infty} = -\frac{1}{2}g^{kl}g''_{kl} + \frac{1}{4}g^{KL}g^{pq}g'_{kp}g'_{lq}. \tag{2.3.3}$$

Here $'$ denotes ∂_ρ and R_{ij} and ∇ denote the Ricci curvature and Levi-Civita connection of $g_{ij}(x,\rho)$ with ρ fixed.

The Taylor expansion of $g_{ij}(x,\rho)$ can be determined by setting these expressions equal to zero (the Ricci-flat condition) and successively differentiating and evaluating at $\rho = 0$ (in some cases tracing the equations and then substituting the solved trace part back into the original expression). In doing so one finds that

1. For n odd: Equation 2.3.1 above determines the derivatives $\partial^m_\rho g_{ij}$ for $m < n$ as well as the trace-free part of $\partial^n_\rho g_{ij}$. (The trace-part of equation 2.3.1 is automatically satisfied.) The value of the trace-part $g^{ij}\partial^n_\rho g_{ij}$ is determined by equation 2.3.3 and then all higher derivatives are determined by equation 2.3.1.

2. For n even: Equation 2.3.1 determines the derivatives $\partial^m g_{ij}$ for $m < n/2$ and also the trace part $g^{ij}\partial^m g_{ij}$. (Although calculating traces may be faster and easier using equation 2.3.3.) When trying to solve for the $m = n/2$ term the recurrence fails as one of the terms in the relation vanishes (it has a coefficient including $(n-2m)$).

The first few derivatives are

$$
\begin{aligned}
g'_{ij}|_{\rho=0} &= 2P_{ij} \\
g''_{ij}|_{\rho=0} &= -\frac{2}{(n-4)}B_{ij} + 2P_i{}^k P_{jk}, \qquad n \neq 4 \\
g'''_{ij}|_{\rho=0} &= \frac{2}{(n-4)(n-6)}B_{ij,k}{}^k - \frac{4}{(n-4)(n-6)}W_{kijl}B^{kl} - \frac{8}{(n-4}P_{k(i}B_{j)}{}^k \\
&\quad - \frac{8}{(n-4)(n-6)}P_k{}^k B_{ij} + \frac{2}{(n-2=6)}P^{kl}C_{(ij)k,l} - \frac{1}{(n-6)}C^k{}_i{}^l C_{ljk} \\
&\quad + \frac{1}{2(n-6)}C_i{}^{kl}C_{jkl} + \frac{1}{(n-6)}P^k{}_{k,l}C_{(ij)}{}^l - \frac{1}{(n-6)}W_{kijl}P^k{}_m P^{ml}, \\
&\quad n \neq 4, 6
\end{aligned}
$$

and in general, at $\rho = 0$

$$
(4-n)(6-n)\cdots(2m-n)\partial_\rho^m g_{ij} = 2(\Delta^{m-1}P_{ij} - \Delta^{m-2}P_k{}^k{}_{,ij}) + \cdots
$$

where the tensors $P_{ij}, W_{ijkl}, C_{ijk}, B_{ij}$ are defined as follows:

Schouten: $P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)}g_{ij}\right)$,

Weyl: $W_{ijkl} = R_{ijkl} - (P \oslash g)_{ijkl}$,

Cotton: $C_{ijk} = P_{ij,k} - P_{ik,j}$,

Bach: $B_{ij} = C_{ijk,}{}^{k} - P^{kl}W_{kijl}$.

Recall, the definition of *ambient-equivalence* above. Uniqueness means only up to such a diffeomorphism. Now, the summarized existence and uniqueness results for an ambient metric of a conformal manifold $(M, [g])$ of dimension n are as follows:

n **odd:** There exists an ambient metric uniquely determined to infinite order. If there is a real-analytic metric in the conformal class $[g]$ then the expansion of the ambient metric converges.

n **even:**

1. There is an ambient metric uniquely determined up to order $n/2 - 1$.
2. In general there is a local obstruction to existence. This obstruction can be realized as a conformally invariant 2-tensor on M which generalizes the Bach tensor (they coincide in dimension 4).
3. In the case of special types of conformal manifolds there are infinite order solutions for the even dimensional case. One example that will have relevance to us is when the starting manifold is conformally flat.
4. Alternative manner to handle even dimensional case: *generalized ambient metrics*. (due to Graham and Hirachi [15]) Basically, relax the smoothness requirement and instead try to construct more general formal series, potentially involving log terms. Introduce a choice of undetermined terms to allow full expansion. This yields results, but uniqueness in particular remains elusive outside of special cases.

Accordingly we must revise our definition for ambient metrics to

Definition 2.3.5 (Ambient Space)**.** An *ambient space* for a conformal manifold $(M, [g])$ is a pair $(\widetilde{\mathcal{G}}, \tilde{g})$, where:

1. $\widetilde{\mathcal{G}}$ is a dilation-invariant open neighborhood of $\mathcal{G} \times \{0\}$ in $\mathcal{G} \times \mathbb{R}$ and $\tilde{g}$ is a smooth metric on $\widetilde{\mathcal{G}}$;

2. $\tilde{g}$ is homogenous of degree 2 on $\widetilde{\mathcal{G}}$ (i.e. $\delta_s^* \tilde{g} = s^2 \tilde{g}$, for $s \in \mathbb{R}_+$);

3. The pullback $\iota^* \tilde{g}$ is the tautological tensor $\boldsymbol{g}_0$ on $\mathcal{G}$;

4. (a) If $\dim M = n$ is odd, or $n = 2$, then $\mathrm{Ric}(\tilde{g})$ vanishes to infinite order at every point of $\mathcal{G} \times \{0\}$.

   (b) If n is even, then $\mathrm{Ric}(\tilde{g}) = O(\rho^{n/2-1})$.

Briefly, in our previous definition of an ambient metric, we make the alterations:

$$\mathrm{Ric}(\tilde{g}) = 0$$

becomes

n **odd:** $\mathrm{Ric}(\tilde{g})$ vanishes to infinite order at every point of $\mathcal{G} \times \{0\}$.

n **even:** $\mathrm{Ric}(\tilde{g}) = O(\rho^{n/2-1})$

The initial terms of the ambient metric expansion are:

$$
\begin{aligned}
g_{ij}(x,\rho) = g_{ij}(x,0) + 2P_{ij}\rho + &\left(-\frac{1}{(n-4)}B_{ij} + P_i{}^k P_{jk}\right)\rho^2 + \\
&+ \frac{1}{3(n-4)(n-6)}\Bigg(B_{ij,k}{}^k - 2W_{kijl}B^{kl} - 4(n-6)P_{k(i}B_{j)}{}^k - 4P_k{}^k B_{ij} \\
&+ 4(n-4)P^{kl}C_{(ij)k,l} - 2(n-4)C^k{}_i{}^l C_{ljk} + (n-4)C_i{}^{kl}C_{jkl} \\
&+ 2(n-4)P^k{}_{k,l}C_{(ij)}{}^l - 2(n-4)W_{kijl}P^k{}_m P^{ml}\Bigg)\rho^3 + \mathcal{O}(\rho^4)
\end{aligned}
$$

Finally, though it is not (at least apparently) germane to the sequel, we remark that the obstruction to finding ambient metrics in the even dimensional case can be identified as a conformally invariant 2-tensor on M, which when $n = 4$ is the Bach tensor. Recall that when $\tilde{g}$ is a straight ambient metric in normal form $|T|_{\tilde{g}} = 2\rho t^2$, and in the case of the flat model $2\rho t^2 = |\xi|^2 - |\tau|^2 = Q(\chi)$, i.e. the standard defining function of the null cone. In the general case $Q = |T|_{\tilde{g}} = \tilde{g}(T,T)$ is a defining function for $\mathcal{G} \times \{0\} \subset \widetilde{\mathcal{G}}$ invariantly associated to $\tilde{g}$, which is homogenous of degree 2. (To verify that Q is a defining function, note that in normal form $Q = 2\rho t^2$ mod $O(\rho^{n/2+1})$). As $\mathrm{Ric}(\tilde{g}) = O(\rho^{n/2-1})$, $Q^{1-n/2}\,\mathrm{Ric}(\tilde{g})|_{T\mathcal{G}}$ is a tensor field on $\mathcal{G}$ of degree $2-n$ which annihilates T, and thus it defines a symmetric 2-tensor-density of weight $2-n$ on M which is trace-free. Evaluating this tensor-density on the image of the representative metric as a section of $\mathcal{G}$ defines a 2-tensor on M. Then the *ambient obstruction tensor* is defined by

$$\mathcal{O} = c_n(Q^{1-n/2}\,\mathrm{Ric}(\tilde{g}))|_g$$

where $c(n)$ is a constant derived from the dimension equal to

$$c_n = (-1)^{n/2-1}\frac{2^{n-2}(n/2-1)!^2}{n-2}.$$

For $\tilde{g}$ in normal form this reduces to

$$\mathcal{O}_{ij} = 2^{1-n/2}c_n(\rho^{1-n/2}\widetilde{R}_{ij})|_{\rho=0}.$$

This tensor is conformally invariant, trace-free, divergence-free, and equal to the Bach tensor when $n = 4$. For more about this tensor see [10], [14] and [15].

2.4 Conformal Curvature Tensors

Conformal curvature tensors are conformal invariants analogous to the curvature tensors of Riemannian geometry; the search for such tensors provided the original motivation for the producing the ambient construction. The general procedure for constructing these invariants is to perform the ambient construction, then calculate the Riemannian curvature of the resulting ambient metric, finally restricting this curvature tensor back to the original manifold (a submanifold of the ambient space) yields a conformally invariant tensor. Taking covariant derivatives of the ambient curvature before the restriction produces additional invariants, and in the simplest cases the well known conformally invariant tensors are recovered by this method. We will use some of the calculations of this section to realize the expansion of Poincaré-Einstein manifolds (these have conformally flat conformal infinity).

Let g be a metric on a manifold M. Then there is an ambient metric in normal form relative to g, which may be taken to be straight. Such metrics have the form:

$$\tilde{\mathbf{g}}_{\mathbf{IJ}} = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_{ij} & 0 \\ t & 0 & 0 \end{pmatrix}$$

on a neighborhood of $\mathbb{R}_+ \times M \times \{0\}$ in $\mathbb{R}_+ \times M \times \mathbb{R}$. The condition of Ricci flatness (the resulting second-order ODE in ρ) determines the 1-parameter family of metrics $g_{ij}(x, \rho)$ based on the initial metric to infinite order for n odd and modulo $O(\rho^{n/2})$ for n even, with the trace term $g^{ij}\partial_\rho^{n/2} g_{ij}|_{\rho=0}$ also determined for n even. Each of these determined (formal) Taylor coefficients is a natural invariant of the initial metric g.

The conformal curvature tensors will be defined in terms of the covariant derivatives of the curvature tensor of an ambient metric in normal form relative to g. Denote the curvature tensor of an ambient metric by $\widetilde{R}$ with components $\widetilde{R}_{IJKL}$. The r-th covariant derivative will be denote by $\widetilde{R}^{(r)}$ with components $\widetilde{R}^{(r)}_{IJKL,M_1\cdots M_r}$. The curvature tensor for $\tilde{g}$ can be computed directly from the explicit form of the metric given above, and the result of such computation shows that $\widetilde{R}_{IJK0} = 0$ and that

$$\widetilde{R}_{ijkl} = t^2\left[R_{ijkl} + \frac{1}{2}(g_{il}g'_{jk} + g_{jk}g'_{il} - g_{ik}g'_{jl} - g_{jl}g'_{ik}) + \frac{\rho}{2}(g'_{ik}g'_{jl} - g'_{il}g'_{jk})\right]$$

$$\widetilde{R}_{\infty jkl} = \frac{1}{2}t^2\left[\nabla_l g'_{jk} - \nabla_k g'_{jl}\right]$$

$$\widetilde{R}_{\infty jk\infty} = \frac{1}{2}t^2\left[g''_{jk} - \frac{1}{2}g^{pq}g'_{jp}g'_{kq}\right] \tag{2.4.1}$$

where $'$ denotes ∂_ρ, and ∇ and R_{ijkl} denote the Levi-Civita connection and curvature tensor respectively of the metric $g_{ij}(x, \rho)$ with ρ fixed. The symmetries of the curvature tensor imply that all other components look like one of these up to sign. Components of the ambient covariant derivatives of the ambient curvature can then be calculated by taking iterated derivatives of these formulae.

We will restrict our attention here to the case $r = 0$, as it is here that we recover the known conformally invariant curvature tensors called the Weyl, Cotton, and Bach tensors.

First recall that from the calculations involved in the formal analysis of the expansion of $\tilde{g}$ we have that

$$g'_{ij}|_{\rho=0} = 2P_{ij}, \qquad g''_{ij}|_{\rho=0} = -\frac{2}{(n-4)}B_{ij} + 2P_i{}^k P_{jk}$$

where the second derivative is only unambiguously defined when $n \neq 4$.

Now evaluating the three components of $\widetilde{R}$ listed above but restricted to the metric bundle $\rho = 0$ yields:

$$\begin{aligned} \widetilde{R}_{ijkl}|_{\rho=0} &= t^2(R_{ijkl} + g_{il}P_{jk} + g_{jk}P_{il} - g_{ik}P_{jl} - g_{jl}P_{ik}) \\ &= t^2(R_{ijkl} - (P \oslash g)_{ijkl}) \\ &= t^2 W_{ijkl} \end{aligned}$$

Where $\oslash$ denotes the Kulkarni-Nomizu product that combines two symmetric 2-tensors to produce a 4-tensor with the symmetries of a curvature tensor. At $t = 1$ of course we see the Weyl tensor of the representative metric, but also as t varies we see precisely the expected conformal transformation of the Weyl tensor (when all indices are lowered). Next we have

$$\widetilde{R}_{\infty jkl}|_{\rho=0} = t^2(P_{jk,l} - P_{jl,k})$$

$$= t^2 C_{ijk}$$

the Cotton tensor, also transforming as expected with respect to change in t. And finally

$$\begin{aligned}\widetilde{R}_{\infty jk\infty}|_{\rho=0} &= \frac{t^2}{(n-4)}\left[-B_{jk} + (n-4)P_j{}^l P_{kl}\right] - 2t^2 P_j{}^l P_{kl} \\ &= -\frac{t^2}{(n-4)} B_{jk}\end{aligned}$$

The fact that the Bach tensor is only obtained in this way when $n \neq 4$ is unfortunate, but the invariant is recovered nonetheless.

Chapter 3

Poincaré Metrics & Asymptotically Hyperbolic Manifolds

3.1 Motivation & Analogy to Flat Model

There is another construction associated to a conformal manifold that carries equivalent data to that of the ambient construction and actually arises naturally on its own, particularly in the context of contemporary physical theories. While the ambient space and associated metric fully translate conformal data into pseudo-Riemannian structure through the addition of two extra dimensions, Poincaré metrics arise through the addition of one extra (spacelike) dimension and realize a conformal manifold as the boundary of a Riemannian bulk space. Such spaces form the focus of our investigations; it is the interplay between the conformal structure of the boundary and the Riemannian geometry of the bulk space that produces the salient characteristics of these objects.

The connection between the two constructions is easily described in the flat model: restrict the Minkowski metric $\tilde{g}$ to the upper sheet of the hyperboloid $Q = -1$, this will be the standard hyperbolic metric. The hyperboloid is complete but not compact, and thus has no actual boundary, however it approaches the null-cone in every direction; the null-cone looks like a boundary at infinity for this hyperbolic space. This can be made precise by moving from the hyperboloid to the Poincaré disk model of hyperbolic space. This is via a conformal map which is very much analogous to stereographic projection. The result is a metric $\left(\frac{4}{(1+|u|^2)^2}\right)\bar{g}$ on the unit disk in $\mathbb{R}^{n+1}$, where $\bar{g}$ is the Euclidean metric there. Here the "boundary at infinity" is the unit n-sphere. This inspires the definitions that follow.

Remark 3.1.1. In the ambient construction one begins with a conformal manifold $(M, [g])$ and proceeds to construct an ambient space around it. While the construction of Poincaré metrics follows a rather similar plot, in subsequent sections our perspective will often be rather reversed, i.e. we will start with a *Riemannian* manifold (X, g) and find that its conformal compactification has "conformal infinity" $(M, [h])$. Anticipating the notation of those sections, we will use $(M, [h])$ to denote the conformal manifold that is the boundary of a Poincaré space, and (X, g) to refer to the Riemannian bulk space we construct.

Let $(M, [h])$ be a smooth conformal manifold of dimension $n \geq 3$ and X a manifold with boundary satisfying $\partial X = M$. Let $r \in C^\infty(X)$ denote a defining function for $\partial X = M$, i.e. $r > 0$ on $\operatorname{int} X$, $r = 0$ and $dr \neq 0$ on M. Most results are equally true for $n = 2$, but the plentitude of conformal mappings in two dimensions, and in particular the fact that all two dimensional manifolds are locally conformally flat, makes $n = 2$ a relatively special (frequently easy) case in most circumstances.

Definition 3.1.2 (Conformally Compact)**.** A smooth metric g on $\operatorname{int} X$ is said to be *conformally compact* if for a defining function r,

1. the conformal metric $r^2 g$ extends smoothly to a metric on X and
2. the restriction $r^2 g|_{TM}$ induces a metric on M (is nondegenerate on M).

The restriction $r^2 g|_{TM}$, which we will call $\hat{g}$, rescales by a conformal factor upon a change in the defining function r, and therefore defines a conformal structure $(M, [\hat{g}])$ on M called the *conformal infinity* of X.

The full condition of smoothness up to the boundary is frequently unnecessary, but unless otherwise indicated, we assume conformal compactifications to have at least $\mathcal{C}^{3,\alpha}$ regularity up to ∂X. Fix for convenience the notation $\bar{g} = r^2 g$.

Computing the change in curvature under a conformal transformation (see appendix B) by a factor of r^{-2} gives

$$\operatorname{Rm}(g) = -|dr|^2_{\bar{g}}(g_{ik}g_{jl} - g_{il}g_{jk}) + \operatorname{Rm}(\bar{g})r^{-2} + (\overline{\nabla}^2 r \oslash \bar{g})r^{-3} \tag{3.1.1}$$

Implying that g has asymptotically constant sectional curvature of $-|dr|^2_{\bar{g}}$ near M.

(The symbol $\oslash$ above is the Kulkarni-Nomizu product on symmetric 2-tensors:

$$(u \oslash v)_{ijkl} = u_{ik}v_{jl} + u_{jl}v_{ik} - u_{il}v_{jk} - u_{jk}v_{il}.$$

Thus $\frac{1}{2}g \oslash g$ is the curvature tensor of constant sectional curvature $+1$. Recall that the sectional curvature requires dividing $\mathrm{Rm}(g)$ by a normalization factor that will be quadratic in g.) It can be seen by taking the trace of the expression 3.1.1 above, that the Einstein condition $\mathrm{Ric}(g) = -ng$ can be satisfied (modulo $O(r^{-1})$) near the boundary of X only if $|dr|^2_{\bar{g}} = 1$ on $\partial X = M$. Such metrics are named:

Definition 3.1.3 (Asymptotically Hyperbolic)**.** A conformally compact metric g on a manifold X is called *asymptotically hyperbolic* if $|dr|^2_{r^2 g} = 1$ on ∂X.

A choice of defining function r always determines a representative metric $\hat{g} = \bar{g}|_{TM} = r^2 g|_{TM}$ in the conformal class $[g]$, however in the other direction a choice of representative metric generally only determines a defining function r modulo $O(r^2)$. (As a defining function must vanish to exactly first order at M, all higher orders of r are unseen in a member of the conformal infinity.) Stipulating that $|dr|^2_{\bar{g}} \equiv 1$ not only on M but in a neighborhood of M allows determination of a particular defining function r:

Theorem 3.1.4 (Geodesic Boundary Defining Function)**.** *Let (X, g) be an asymptotically hyperbolic manifold. Then any representative h_0 in the conformal infinity $(M, [\hat{g}])$ of (X, g) determines a unique defining function r such that $r^2 g$ extends to a metric on $\partial X = M$, $r^2 g|_{TM} = h_0$, and $|dr|^2_{\bar{g}} \equiv 1$ in a neighborhood U of M in X.*

Proof. Fix any choice of defining function ρ and let $\bar{g}_\rho = \rho^2 g$. Set $r = e^\phi \rho$ so that $\bar{g} = e^{2\phi}\bar{g}_\rho$ and $dr = e^\phi(d\rho + r d\omega)$. Then

$$|dr|^2_{\bar{g}} = |d\rho + r d\phi|^2_{\bar{g}_\rho} = |d\rho|^2_{\bar{g}_\rho} + 2\rho(\operatorname{grad}_{\bar{g}_\rho}\rho)(\phi) + \rho^2|d\phi|^2_{\bar{g}_\rho},$$

so that the condition $|dr|^2_{\bar{g}} = 1$ is equivalent to

$$2(\operatorname{grad}_{\bar{g}_\rho}\rho)(\phi) + \rho|d\phi|^2_{\bar{g}_\rho} = \frac{1 - |d\rho|^2_{\bar{g}_\rho}}{\rho}. \tag{3.1.2}$$

This is a non-characteristic first-order PDE for ϕ, so there is a solution near M with $\phi|_M = \phi_0$ arbitrarily prescribed. $\square$

For future use we will consider the solution $\phi = \phi(r, x)$ whose existence is guaranteed by the above theorem to be the result of an "extension operator" $\mathcal{E} : \mathcal{C}^{2,\alpha}(M) \to \mathcal{C}^{2,\alpha}(U)$, where U is a neighborhood of M in X. Thus, given $h_0 \in [\hat{g}]$, if $h_0 = e^{2\phi_0}(\bar{g}_\rho|_{TM})$, then $\mathcal{E}(\phi_0) = \phi$ is the extension of the conformal factor from boundary to neighborhood of the boundary that satisfies

$$r = e^\phi \rho, \qquad \bar{g} = r^2 g|_{TM} = h_0,$$

and M, $|dr|^2_{\bar{g}} \equiv 1$ in that neighborhood.

We will call such a function r the *geodesic boundary defining function* associated with h. This defining function provides an identification of $M \times [0, \epsilon)$, for some $\epsilon > 0$, with a neighborhood of M in X: the point (x, ϵ) corresponds to the point obtained by following the integral curve of $\operatorname{grad}_{\bar{g}} r$ emanating from $x \in M$ for ϵ units of time. As $|dr|^2_{\bar{g}} = 1$ the ϵ-coordinate is just r, and $\operatorname{grad}_{\bar{g}} r$ is orthogonal to the slices $M \times \{\epsilon\}$.

Identifying r with ϵ then, on $M \times [0, \delta)$ the metric takes the form $\bar{g} = dr^2 + h_r$ for a 1-parameter family of metrics h_r on M, and therefore g can be written in a manner described as:

Definition 3.1.5 (Normal Form)**.** An asymptotically hyperbolic metric g is said to be in *normal form* relative to a metric h in the conformal class of metrics on M if

$$g = r^{-2}(dr^2 + h_r),$$

where h_r is a 1-parameter family of metrics on M such that $h_0 = h$.

We will frequently make use of the asymptotic expansion (in terms of a geodesic boundary defining function r) of the boundary-tangential component of an asymptotically hyperbolic metric h in normal form, and the corresponding expansion of its inverse:

$$h_r = h_0 + h_1 r + h_2 r^2 + \mathcal{O}(r^3) \tag{3.1.3}$$

$$h_r^{-1} = h_0^{-1} - (h_0^{-1})^2 h_1 r + \left[(h_0^{-1})^3 h_1^2 - (h_0^{-1})^2 h_2\right] r^2 + \mathcal{O}(r^3). \tag{3.1.4}$$

In 3.1.4 indices have been suppressed in pursuit of simplicity but each factor of h_0^{-1} refers to the raising of an index by the metric h_0. As all tensors involved are symmetric 2-tensors the omission of specific indices should cause no confusion.

Definition 3.1.6 (Poincaré Metric)**.** A *Poincaré metric* for $(M, [h])$ is a conformally compact metric g on X°, where X is an open neighborhood of $M \times \{0\}$ in $M \times [0, \infty)$, such that:

1. g has conformal infinity $(M, [h])$.

2. If n is odd or $n = 2$, then $\mathrm{Ric}(g) + ng$ vanishes to infinite order along M, and if $n \geq 4$ is even, then $\mathrm{Ric}(g) + ng \in O(r^{n-2})$.

A Poincaré metric which is also Einstein (not just at the boundary) is called a *Poincaré-Einstein metric.* Just as in the case of ambient metrics there is an appropriate sense of equivalence up to diffeomorphisms that are the identity on the conformal infinity. Issues of existence, uniqueness, and construction can be handled directly (cf. [16]), however here the equivalence described next allows results for ambient metrics to be translated into results for Poincaré metrics (and vice versa).

Going in the other direction, that is conformally compactifying a given asymptotically hyperbolic manifold, there is also an equivalence up to diffeomorphism of conformal compactifications, ensuring that the conformal infinity of these manifolds is unique:

Theorem 3.1.7 (Cruściel-Herzlich [4]). *Let (X, g) be a conformally compactifiable asymptotically hyperbolic Riemannian manifold that admits two C^∞-conformal compactifications, $(\overline{X}_1 = X \cup \partial_\infty X_1, \bar{g}_1, \psi_1)$ and $(\overline{X}_2 = X \cup \partial_\infty X_2, \bar{g}_2, \psi_2)$, where the maps $\psi_i : X \to \overline{X}_i$, $i = 1, 2$, are the embeddings provided by the definition of compactification. Then*

$$\psi_2^{-1}\big|_{\mathrm{Int}\,\overline{X}_1} \circ \psi_1 : X_1 \to X_2$$

extends to a conformal diffeomorphism

$$\Psi : \overline{X}_1 \to \overline{X}_2$$

so that $\overline{X}_1$ and $\overline{X}_2$ are diffeomorphic as manifolds with boundary, and $\partial_\infty X_1$ and $\partial_\infty X_2$ are conformally equivalent.

3.2 Equivalence with Ambient Metrics

Recall how in the flat model the hyperboloid is identified by $\mathcal{H} = \{Q = -1, \tau > 0\}$. This quadratic form Q associated with $\tilde{g}$ provides an easy source for defining functions for both the null-cone and the hyperboloid.

Having the metric $\tilde{g}$ and the "dilation vector field" T we can construct Q invariantly by $|T|^2_{\tilde{g}} = Q$. This is verified by computation for the flat metric in normal form: $|T|^2_{\tilde{g}} = 2\rho t^2 = |\xi|^2 - |\tau|^2 = Q(\chi)$, but it should make sense intuitively since the hyperboloid is part of the "unit sphere" in a flat Lorentzian metric. (In fact, here arises the importance of straightness in this context: when $(\widetilde{\mathcal{G}}, \tilde{g})$ is a straight ambient space for $(M, [g])$, the function $|T|_{\tilde{g}}$ vanishes to precisely first-order on $\mathcal{G} \times \{0\} \subset \widetilde{G}$ and thus yields a defining function for the metric bundle.)

In the flat model then, $\mathcal{H}$ is the hyperboloid, in the general ambient space $\widetilde{\mathcal{G}}$ it is an invariantly defined hypersurface (looking like the portion of the hyperboloid very close to the null-cone); in both cases we will find a Poincaré metric associated to a given ambient metric by restriction of the ambient metric to the hypersurface. Of note here is that the construction of the ambient metric described above had positive ρ direction "outside the null-cone," while now we look at the hyperboloid sheet on the other side. This has little real effect as the equations defining an ambient metric in normal form all possess a symmetry under reflection in ρ. The only artifact of this issue will be a necessary sign-change. In the general case we define maps

projection: $\pi : \widetilde{\mathcal{G}} \subset \mathcal{G} \times \mathbb{R} \to M \times \mathbb{R}$ defined by extending the projection on $\mathcal{G}$, and

sign & scale change: $\psi : M \times \mathbb{R} \to M \times [0, \infty)$ defined by $\psi(x, \rho) = \left(x, \sqrt{2|\rho|}\right)$.

Then (shrinking $\widetilde{\mathcal{G}}$ if necessary), there is an open set $X \subset M \times [0, \infty)$ containing $M \times \{0\}$ such that $\psi \circ \pi|_{\mathcal{H}} : \mathcal{H} \to X^{\circ}$ is a diffeomorphism, and we have

Theorem 3.2.1. *If $(\widetilde{\mathcal{G}}, \tilde{g})$ is a straight ambient space for $(M, [h])$ and $\mathcal{H}$ and X are defined as above, then*

1. *the metric*

   $$g := \left((\psi \circ \pi|_{\mathcal{H}})^{-1}\right)^{*} \tilde{g}$$

   is an (even) asymptotically hyperbolic metric with conformal infinity $(M, [h])$;

2. *if $\tilde{g}$ is in normal form relative to a metric $h \in [h]$, then g is also in normal form (as a Poincaré metric) relative to h;*

3. *Every (even) asymptotically hyperbolic metric g with conformal infinity $M, [h])$ is of this form for some straight ambient metric $\tilde{g}$ for $(M, [h])$. If g is in normal*

form relative to h then $\tilde{g}$ can be taken to be in normal form relative to h, and in this case $\tilde{g}$ on $\{|T|^2_{\tilde{g}} \leq 0\}$ is uniquely determined.

Note that in general there are many different ambient metrics that may give the same Poincaré metric if the action of the diffeomorphism by which $\tilde{g}$ is pulled back scales each $\mathbb{R}_+$-fiber by an appropriate amount depending on the base point. As expected it is the insistence on both metrics being in normal form which yields the desired one-to-one association. Straightness of the ambient metric is also important; if the ambient metric is not assumed to be straight, then the Poincaré metric obtained from this association may not be asymptotically hyperbolic.

In particular, when both ambient and Poincaré metrics are in normal form then we have the following (c.f. flat model as described above): The ambient metric has the form

$$\tilde{g} = 2\rho dt^2 + 2tdtd\rho + t^2 g_\rho$$

where g_ρ is a 1-parameter family of metrics on M. Under the change of variables

$$-2\rho = r^2,$$

$$s = rt \quad \text{on} \quad \{\rho \leq 0\},$$

$\tilde{g}$ then takes the form

$$\tilde{g} = s^2 r^{-2}(dr^2 + g_\rho) - ds^2.$$

Recalling the definition of normal form for a Poincaré metric let us set

$$h_r = g_{-\frac{1}{2}r^2}$$

and then re-label g as

$$g = r^{-2}\left(dr^2 + h_r\right)$$

so that we see the ambient metric as a cone metric over g:

$$\tilde{g} = s^2 g - ds^2.$$

In these new variables, $\mathcal{H}$ is defined by $\{s = 1\}$, thus the restriction $\tilde{g}|_{T\mathcal{H}}$ is precisely the new metric g. The process may be reversed to obtain ambient metrics from Poincaré metrics, often allowing results regarding one to be translated into similar results for the other.

Now the relation between the curvature of a cone metric and that of its base (as calculated from the general formula for conformal change of a metric) will elucidate the stipulation of the Einstein condition on Poincaré metrics:

Theorem 3.2.2 (Cone Metric Curvature)**.** *Let g be a metric on a manifold X of dimension $n+1$, and define a metric $\tilde{g} = s^2 g - ds^2$ on $X \times \mathbb{R}_+$. Then*

$$\begin{aligned}
\mathrm{Rm}(\tilde{g}) &= s^2\left[\mathrm{Rm}(g) + \frac{1}{2} g \oslash g\right] \\
\mathrm{Ric}(\tilde{g}) &= \mathrm{Ric}(g) + ng \\
\mathrm{R}(\tilde{g}) &= s^{-2}[\mathrm{R}(g) + n(n+1)]
\end{aligned}$$

where Rm *denotes the Riemann curvature tensor as a covariant* 4*-tensor and* $\oslash$ *is the Kulkarni-Nomizu product on symmetric* 2*-tensors:*

$$(u \oslash v)_{ijkl} = u_{ik}v_{jl} + u_{jl}v_{ik} - u_{il}v_{jk} - u_{jk}v_{il}.$$

(Thus $\frac{1}{2}g \oslash g$ is the curvature tensor of constant sectional curvature $+1$.) Here tensors on X are implicitly pulled back to $X \times \mathbb{R}_+$.

Proof. Let $s = e^y$; then $\tilde{g} = e^{2y}(g - dy^2)$ is a conformal multiple of a product metric. Under a conformal change $\tilde{g} = e^{2y}h$, the curvature tensor transforms by

$$\mathrm{Rm}(\tilde{g}) = e^{2y}[\mathrm{Rm}(h) + \Lambda \oslash h],$$

where $\Lambda = -\nabla_h^2 y + dy^2 - \frac{1}{2}|dy|_h^2 h$. When $h = g - dy^2$ we have $\mathrm{Rm}(h) = \mathrm{Rm}(g)$, $\nabla_h^2 y = 0$, and $|dy|_h^2 = -1$ so $\Lambda = \frac{1}{2}(g + dy^2)$. Thus $\Lambda \oslash h = (g + dy^2) \oslash (g - dy^2) = g \oslash g$, yielding the first equation above. The second and third equations then follow by contraction on the first. $\square$

This implies that $\tilde{g}$ is flat if and only if g has constant sectional curvature -1, $\tilde{g}$ is Ricci-flat if and only if $\mathrm{Ric}(g) = -ng$, and $\tilde{g}$ is scalar-flat if and only if g has constant scalar curvature $-n(n+1)$, making clear one significant feature of the Einstein condition on Poincaré metrics: it is precisely the Ricci-flat condition on ambient metrics.

3.3 Translation of Formal Theory

The equivalence outlined above implies that for metrics in normal form, the formal asymptotics for Poincaré metrics are precisely equivalent to those for straight ambient metrics under the change of variables $\rho = -\frac{1}{2}r^2$. In addition to this allowing us to translate several of the above theorems for ambient metrics into an equivalent form for Poincaré metrics it also implies that after making the substitution $\rho = -\frac{1}{2}r^2$ into the ambient metric expansion we get the following first few terms of the expansion for a Poincaré metric $g = r^2(dr^2 + h_r)$ in normal form:

$$(h_r)_{ij} = (h_0)_{ij} - P_{ij}r^2 + \left(-\frac{1}{4(n-4)}B_{ij} + \frac{1}{4}P_i{}^k P_{jk}\right) r^4 + \cdots$$

3.4 Poincaré-Einstein Metric Expansion

The ambient metric for the conformal infinity of a Poincaré-Einstein space can be used to quickly derive the metric expansion for the tangential component of the bulk metric in normal form. In this case the ambient metric is flat and the conformal boundary is locally conformally flat. These strong constraints ensure unique, smooth, invariant ambient and Poinaré metrics for any dimension.

When g is conformally flat we may, by definition, select a flat metric $\gamma \in [g]$. Then the expressions for the components of the ambient curvature above imply that the normal-form ambient metric

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2\gamma$$

is flat (recall that γ will be independent of ρ, as it already satisfies $\mathrm{Ric}(\gamma) = 0$ and the above ambient metric will be completely flat for every ρ). Then, knowing that $\widetilde{R} = 0$, we may use the above formulae (2.4.1) for the components of $\widetilde{R}$ to deduce

$$\widetilde{R}_{\infty jk\infty} = \frac{1}{2}t^2 \left[g''_{jk} - \frac{1}{2}g^{pq}g'_{jp}g'_{kq} \right]$$

equal to zero implies

$$0 = g''_{jk} - \frac{1}{2} g^{pq} g'_{jp} g'_{kq}.$$

Differentiating with respect to ρ and substituting the original expression for the resulting second derivatives yields $g'''_{ij} = 0$. Plugging this into the expressions for the $\mathrm{Ric}(\tilde{g})$ components used in the original formal expansion of $g_{ij}(x, \rho)$) then shows that this implies that all higher terms must consequently be zero. Thus the expansion of g_{ij} (or g_+) in the conformally flat case is only quadratic. We recall the first few terms of the expansion in the general case:

$$g_{ij}(x, \rho) = g_{ij}(x, 0) + 2P_{ij}\rho + \left(-\frac{1}{(n-4)} B_{ij} + P_i{}^k P_{jk} \right) \rho^2 + \cdots$$

Now note that in the conformally flat case the Bach tensor is zero, and hence our expansion is:

$$g_{ij}(x, \rho) = \gamma_{ij}(x) + 2P_{ij}\rho + P_i{}^k P_{jk} \rho^2$$

Theorem 3.4.1 (Ambient Metric Uniqueness for Conformally Flat Spaces)**.** *Let $n \geq 3$ and suppose that $(M, [g])$ is locally conformally flat. Then there exists an ambient metric $\tilde{g}$ for $(M, [g])$ which is flat to infinite order, and such that $\tilde{g}$ is unique to infinite order up to diffeomorphism.*

Note that when $n = 2$ of course every manifold is locally conformally flat, but there are many inequivalent ambient metrics for this dimension (these are parametrized by choice of the first-order term of their expansions: $\mathrm{tr}\,(\partial_\rho g_{ij}|_{\rho=0})$). As conformally flat is a much weaker condition on 2-dimensional manifolds this is not surprising.

Translating the above expansion to the Poincaré Metric case via the substitution $-2\rho = r^2$ and the fact that flat ambient metrics are equivalent to hyperbolic Poincaré metrics yields

Theorem 3.4.2. *Let γ be a smooth flat metric on a manifold M, and let g be a hyperbolic Poincaré metric defined on $\operatorname{int} X$, where X is a neighborhood of $M \times \{0\}$ in $M \times [0, \infty)$, with conformal infinity $(M, [\gamma])$. Then there is a neighborhood $\mathcal{U}$ of $M \times \{0\}$ in $M \times [0, \infty)$ and a diffeomorphism ϕ mapping $\mathcal{U}$ into X which restricts to the identity on $M \times \{0\}$, so that*

$$\phi^* g = r^2(dr^2 + g_r)$$

where

$$(g_r)_{ij} = \gamma_{ij} - P_{ij} r^2 + \frac{1}{4} P_{ik} P^k{}_j r^4$$

and

1. *if $n \geq 3$, the P is the Schouten tensor of γ, while*

2. *if $n = 2$, P is a symmetric 2-tensor on M satisfying*

$$P_i^{\ i} = \frac{1}{2} R \quad \text{and} \quad P_{ij,}{}^{j} = \frac{1}{2} R_{,i}.$$

As mentioned before, the 2-dimensional case requires special treatment because of the differing implications of the condition of conformal flatness.

Chapter 4

The Yamabe Problem

A critical step in all methods we describe here is the selection of a metric within a given conformal class for which the function we call κ_2 is constant. In the Poincaré-Einstein case, this means finding a metric in the conformal class of the boundary for which the scalar curvature is constant. This turns out to be a natural and well known generalization of the uniformization theorem of compact surfaces called *the Yamabe problem*. In the case of surface theory, there is only the one sectional curvature and hence a conformal adjustment of the metric can control this, in higher dimensions however, controlling the Riemannian curvature 4-tensor via choice of a single function is clearly implausible (compare degrees of freedom), and in fact even locally it does not hold, for example in dimensions $n \geq 4$ the Weyl tensor provides and obstruction to a metric being locally conformal to a Euclidean metric. Given this constraint, it seems natural to ask instead if conformal change can be used to make the scalar curvature (the completely contracted curvature tensor) constant, as this too is just a function on the manifold. The result is

Theorem 4.0.3 (The Yamabe Problem)**.** *Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$, there exists a function $\phi \in \mathcal{C}^\infty(M)$ such that the metric $e^{2\phi}g$ has constant scalar curvature.*

Yamabe claimed this result in 1960 [49], but unfortunately an error in his proof was discovered in 1968 by Trudinger [45]. The combined work of Trudinger, Aubin, and Schoen provided a complete proof as of 1984 ([45], [2], [41]).

There are a few obvious directions in which to generalize the Yamabe problem. One might ask for other curvatures to be made constant (although as we have already suggested, not all such generalizations are especially plausible). Having established the result for compact Riemannian manifolds, an immediate question is whether or not it might also hold for non-compact cases. The answer, in the most general sense, is negative, as demonstrated by counterexample in [20].

4.1 The Yamabe Invariant

Let $\hat{g} = e^{2\phi}g$ for $\phi \in \mathcal{C}^\infty(M)$ and $R_g, R_{\hat{g}}$ be the scalar curvatures of g and $\hat{g}$ respectively. Then (simply calculating from the definition of scalar curvature) these curvatures satisfy

$$R_{\hat{g}} = e^{-2\phi}\left(R_g - 2(n-1)\Delta_g\phi - (n-1)(n-2)\|\nabla^g\phi\|_g^2\right).$$

Making the substitution $e^{2\phi} = u^{\frac{4}{n-2}}$ simplifies this expression:

$$R_{\hat{g}} = u^{-\frac{n+2}{n-2}} \left(-4\frac{n-1}{n-2}\Delta_g u + R_g u \right). \tag{4.1.1}$$

Labeling

$$\overline{\mathbb{L}}_g = \left(-4\frac{n-1}{n-2}\Delta_g + R_g \right), \qquad \text{and} \qquad \lambda = R_{\hat{g}},$$

lets us write 4.1.1 as a sort of "nonlinear eigenvalue problem:"

$$\overline{\mathbb{L}}_g u = \lambda u^{\frac{n+2}{n-2}}. \tag{4.1.2}$$

Finding a positive solution $u : M \to \mathbb{R}^+$ to this nonlinear partial differential equation means finding a solution to the Yamabe problem for the manifold M; the metric $u^{\frac{4}{n-2}} g$ has constant scalar curvature λ.

Yamabe made the observation that 4.1.1 is the Euler-Lagrange equation for the normalized total scalar curvature functional:

$$Q(\hat{g}) = \frac{\int_M R_{\hat{g}} \, dV_{\hat{g}}}{\left(\int_M \, dV_{\hat{g}}\right)^{\frac{n-2}{n}}},$$

where $\hat{g}$ is allowed to vary over metrics conformal to g. This may been seen as follows:

Let

$$E(u) = \int_M R_{\hat{g}} \, dV_{\hat{g}} = \int_M u^{-\frac{n+2}{n-2}} \left(-4\frac{n-1}{n-2}\Delta_g u + R_g u \right) \, dV_{\hat{g}}.$$

If $x^1, \ldots, x^n$ are coordinates, then

$$\begin{aligned} dV_{\hat{g}} &= \sqrt{\det(\hat{g})} dx^1 \wedge \cdots \wedge dx^n \\ &= \sqrt{\det(u^{\frac{4}{n-2}} g)} dx^1 \wedge \cdots \wedge dx^n \\ &= \sqrt{u^{\frac{4n}{n-2}} \det(g)} dx^1 \wedge \cdots \wedge dx^n = u^{\frac{2n}{n-2}} dV_g \end{aligned}$$

and thus

$$E(u) = \int_M u \left(-4\frac{n-1}{n-2}\Delta_g u + R_g u \right) \, dV_g = \int_M u\mathbb{L}_g u \, dV_g, \tag{4.1.3}$$

$$\int_M dV_{\hat{g}} = \int_M u^{\frac{2n}{n-2}} \, dV_g.$$

Write $Q(\hat{g}) = Q(u^{\frac{4}{n-2}}g) = Q_g(u)$ where

$$Q_g(u) = \frac{E(u)}{\|u\|_p^2} = \|u\|_p^{-2} \int_M -4\frac{n-1}{n-2}u\Delta_g u + R_g u^2 \, dV_g, \qquad p = \frac{2n}{n-2} \tag{4.1.4}$$

As M has no boundary, integration by parts turns 4.1.4 into

$$Q_q(u) = \|u\|_p^{-2} \int_M 4\frac{n-1}{n-2}\|\nabla^g u\|_g^2 + R_g u^2 \, dV_g, \tag{4.1.5}$$

and now taking the first variation,

$$\left.\frac{d}{dt}\right|_{t=0} Q_g(u+tv) = \frac{2}{\|u\|_p^2} \int_M \left(\mathbb{L}_g u - \frac{E(u)}{\|u\|_p^p} u^{\frac{n+2}{n-2}} \right) v \, dV_g,$$

and naming $\frac{E(u)}{\|u\|_p^p} = \lambda$, it is clear that u is a critical point of Q_g if and only if it satisfies the equation 4.1.2.

Let $q = \frac{p}{p-2}$, i.e. $\frac{1}{q} + \frac{2}{p} = 1$. Then as the scalar curvature of a compact manifold is bounded, by Hölder's inequality

$$\left| \int_M R_g u^2 \, dV_g \right| \leq \left(\int_M |R_g|^q \, dV_g \right)^{\frac{1}{q}} \left(\int_M |u^2|^{\frac{p}{2}} \, dV_g \right)^{\frac{2}{p}} = C\|u\|_p^2$$

for some constant $C \geq 0$. Thus the expression in 4.1.5 is bounded below and hence

$$I(M,g) = \inf\{Q(\hat{g}) \ : \ \hat{g} \in [g]\} = \inf\{Q_g(u) \ : \ u \in \mathcal{C}^\infty(M), u > 0\}, \tag{4.1.6}$$

which is called the *Yamabe invariant* of $(M,[g])$, is a well-defined conformal invariant. The Yamabe invariant divides conformal manifolds into three classes according to its sign:

Negative: When $I(M, g) < 0$ then solutions to the Yamabe problem have constant negative scalar curvature and hence the Yamabe equation 4.1.2 is elliptic, this allows application of the maximum principle to guarantee uniqueness of the solution.

Zero: When $I(M, g) = 0$ the the manifold is confomally flat and 4.1.2 becomes linear; the solution is unique up to a constant factor.

Positive: When $I(M, g) > 0$ the structure of the solution set can be very involved. The simplest, most important and well known example is that of $(S^n, [\mathring{g}])$, the standard sphere. Obviously the round metric is a solution, but the conformal sphere is quite special because it is the only compact manifold with a noncompact group of conformal transformations. Call this group $\mathrm{Conf}(S^n)$. Then the set of all solutions to the Yamabe problem on $(S^n, [\mathring{g}])$ is given by (a result of Obata [35]) $\{\alpha(\sigma^* \mathring{g}) \ : \ \alpha \in \mathbb{R}^+, \sigma \in \mathrm{Conf}(S^n)\}$.

The Yamabe invariant played a central role in the solution of the Yamabe problem as Trudinger's modification of Yamabe's original proof worked whenever $I(M, g) \leq 0$, and Aubin then extended this to yield solution whenever $I(M, g) < I(S^n, \mathring{g})$. After this the problem shifted to showing that this condition on the Yamabe invariant is met by conformal manifolds other than the standard sphere. Finally achieving this in all cases involved the positive mass theorem of general relativity and the resulting solution is notable both for its geometric importance and for the fact that the exponent

$\frac{n+2}{n-2}$ in equation 4.1.2 is a critical exponent (where the Sobolev embedding is no longer compact) and this was the first case in which a very satisfactory existence result was obtained for a nonlinear PDE with such a critical exponent. The reader interested in complete details is recommended to consult the review [27]. Finally, we remark that the Yamabe invariant has also been used by to examine which Riemannian manifolds have a conformal compactification [23].

4.2 The Generalized Yamabe Invariant

Multiplying $\overline{\mathbb{L}}_g$ by $\frac{n-2}{4(n-1)}$ produces a scaled version of this operator called the *conformal Laplacian*:

$$\mathbb{L}_g = -\left(\Delta_g - \frac{n-2}{4(n-1)} R_g\right).$$

The name is due to the fact that (even in dimensions $n > 2$, unlike the Laplacian itself) it behaves rather nicely under conformal change:

$$\mathbb{L}_{\hat{g}} = u^{-\frac{n+2}{n-2}} \mathbb{L}_g.$$

Restricting the functional $Q_q(u)$ to functions such that $\|u\|_p = 1$ we can use 4.1.3 to express the (scaled) Yamabe invariant as follows

$$\inf \left\{ \int_M u \mathbb{L}_g u \, dV_g \ : \ \|u\|^2_{\frac{2n}{n-2}} = 1 \right\}. \tag{4.2.1}$$

By analogy,

Definition 4.2.1 (Generalized Conformal Laplacian)**.** For an asymptotically hyperbolic manifold (X, g) with conformal infinity $(M, [h_0])$, and metric in normal form $g = r^{-2}(dr^2 + h_0 + h_1 r + h_2 r^2 + \cdots)$ for a geodesic boundary defining function r, let

$$\kappa_2 = \left(\operatorname{tr}^{h_0} h_2 - \frac{1}{2}\|h_1\|_{h_0}^2\right),$$

and define the *generalized conformal Laplacian* on M to be

$$\mathbb{L}_{h_0} = -\left(\Delta_{h_0} + \frac{n-2}{2}\kappa_2\right). \tag{4.2.2}$$

Note that when X is weakly Poincaré-Einstein this is just the ordinary conformal Laplcian. Then using this, define

Definition 4.2.2 (Generalized Boundary Yamabe Invariant)**.** The *generalized boundary Yamabe invariant* of X is

$$\lambda_1 = \inf\left\{\int_M \phi_0 \mathbb{L}_{h_o}\phi_0 \; dV_{h_0} \;:\; \phi_0 \in \mathcal{C}^\infty(M), \|\phi_0\|^2_{\frac{2n}{n-2}} = 1\right\} \tag{4.2.3}$$

when for X of dimension $n \geq 3$. When $n = 2$ this becomes

$$\lambda_1 = -\int_M \kappa_2 \; dV_{h_0}.$$

By a variational principle (note that the definition is a Rayleigh quotient) this is equivalent to the first eigenvalue of $\mathbb{L}_{h_0}$.

The sign of λ_1 is a conformal invariant of the class $[h_0]$:

Theorem 4.2.3. *The sign of the least eigenvalue λ_1 of $\mathbb{L}_{h_0}$ is independent of the choice of conformal representative h_0.*

Proof. Let $\hat{h}_0 = u_0^{\frac{4}{n-2}} h_0$, and $A_{ij}^k = \widehat{\Gamma}_{ij}^k - \Gamma_{ij}^k$ where $\widehat{\Gamma}_{ij}^k$ and Γ_{ij}^k are the Christoffel symbols corresponding to the metrics $\hat{h}_0$ and h_0 respectively. First we verify the identity

$$u_0^{\frac{n+2}{n-2}} \Delta_{\hat{h}_0} w = \Delta_{h_0}(u_0 w) - w \Delta_{h_0} u_0 \tag{4.2.4}$$

via direct computation. On the left-hand side we have:

$$\begin{aligned}
u_0^{\frac{n+2}{n-2}} \Delta_{\hat{h}_0} w &= u_0^{\frac{n+2}{n-2}} u_0^{\frac{4}{n-2}} h_0^{ij} \widehat{\nabla}_i \widehat{\nabla}_j w \\
&= u_0 h_0^{ij} (\nabla_i \nabla_j w - A_{ij}^k \nabla_k w) \\
&= u_0 \Delta_{h_0} w - \frac{2}{n-2} \left(h_0^{ik} \nabla_i u_0 + h_0^{kj} \nabla_j u_0 - n \nabla^k u_0 \right) \nabla_k w \\
&= u_0 \Delta_{h_0} w + 2 \nabla^k u_0 \nabla_k w
\end{aligned}$$

While on the right:

$$\begin{aligned}
\Delta_{h_0}(u_0 w) - w \Delta_{h_0} u_0 &= h_0^{ij} (\nabla_i w \nabla_j u_0 + w \nabla_i \nabla_j u_0 \\
&\quad + \nabla_i u_0 \nabla_j w + u_0 \nabla_i \nabla_j w) - w \Delta_{h_0} u_0 \\
&= \nabla^j w \nabla_j u_0 + \nabla^j u_0 \nabla_j w + u_0 \Delta w \\
&= u_0 \Delta_{h_0} w + 2 \nabla^k u_0 \nabla_k w.
\end{aligned}$$

Now we use 4.2.4 to compare $\mathbb{L}_{\hat{h}_0}$ with $\mathbb{L}_{h_0}$ and see that the generalized conformal

Laplacian shares the conformal covariance of the standard conformal Laplacian:

$$\begin{aligned}
\mathbb{L}_{\hat{h}_0} w &= -\Delta_{\hat{h}_0} w - \frac{n-2}{2}\hat{\kappa}_2 w \\
&= -u_0^{-\frac{n+2}{n-2}} \left(\Delta_{h_0}(u_0 w) - w\Delta_{h_0} u_0\right) \\
&\quad - \frac{n-2}{2}\left(u_0^{-\frac{4}{n-2}}\kappa_2 + \frac{2}{n-2} u_0^{-\frac{n+2}{n-2}} \Delta_{h_0} u_0\right) w \\
&= u_0^{-\frac{n+2}{n-2}} \left(\Delta_{h_0}(u_0 w) - \frac{n-2}{2}\kappa_2 (u_0 w)\right) \\
&= u_0^{-\frac{n+2}{n-2}} \mathbb{L}_{h_0}(u_0 w).
\end{aligned}$$

Therefore

$$\int_M w \mathbb{L}_{\hat{h}_0} w \, dV_{\hat{h}_0} = \int_M (u_0 w)\mathbb{L}_{h_0}(u_0 w)\, dV_{h_0},$$

implying that the sign of $\lambda_1(\mathbb{L}_{\hat{h}_0})$ is the same as that of $\lambda_1(\mathbb{L}_{h_0})$. $\square$

Chapter 5

Constant Mean Curvature Foliations

Spacelike foliations of Lorentzian manifolds of prescribed, and in particular constant mean curvature arise frequently in general relativity as a manner of accommodating the lack of a canonical time coordinate and finding "optimal" frames [1], [12]. In asymptotically flat manifolds constant mean curvature foliations in the exterior regions have been used to supply a definition of center of mass [19], [18], in addition to providing an intrinsic geometric structure near infinity. There are similar results for the asymptotically anti de Sitter case too [37], [33], [32].

The normal form associated to a chosen representative h_0 of the conformal infinity of an asymptotically hyperbolic manifold X automatically yields one natural foliation of X near ∂X, this is the collection of level sets of the corresponding geodesic boundary defining function. When X is sufficiently nice (e.g. hyperbolic) such levels sets may exhibit particular geometric characteristics (e.g. constant mean curvature).

This search for geometric foliations can be generalized in several directions: farther away from infinity, on more general asymptotically hyperbolic manifolds, and for more general curvature conditions. In [31] Mazzeo and Pacard concentrate on constant mean curvature foliations in a neighborhood of infinity in asymptotically (only) hyperbolic manifolds. Their first result applies to a class of manifolds less general than asymptotically hyperbolic, but more general than Poincaré-Einstein, namely manifolds that are Poincaré-Einstein "up to quadratic order:"

Definition 5.0.4. An asymptotically hyperbolic manifold (X, g) is called *weakly Poincaré-Einstein* if g has a normal form

$$g = r^{-2}(dr^2 + h_r), \qquad h_r = h_0 + h_1 r + h_2 r^2 + \cdots$$

where $h_1 = 0$ and $h_2 = -P_{h_0} = -\frac{1}{n-2}\left(\mathrm{Ric}(h_0) - \frac{R_{h_0}}{2(n-1)}h_0\right)$.

In this context the level sets of the boundary defining function r have mean curvature which is almost constant, and and in fact they can be perturbed to yield constant mean curvature foliations near the boundary. Precisely:

Theorem 5.0.5 ([31])**.** *Let (X, g) be an asymptotically hyperbolic manifold which is weakly Poincaré-Einstein with conformal infinity $(M, [\hat{g}])$. Then*

1. *If the conformal class $[\hat{g}]$ has non-positive Yamabe invariant, then there exists a unique constant mean curvature foliation of X near M.*

2. *If instead $[\hat{g}]$ has positive Yamabe invariant, then to each constant scalar curvature metric $h_0 \in [\hat{g}]$ which is non-degenerate for the linearized Yamabe equation 5.5.4, there exists an associated distinct foliation of constant mean curvature.*

In fact the weakly Poincaré-Einstein condition can be relaxed even further: $h_1 = 0$ is not needed for this to hold, only that $\operatorname{tr}^{h_0} h_1 = 0$, also while $h_2 = -P_{h_0}$ makes for a particularly easy case, there is really no restriction required on h_2. Thus an amended version of the above theorem can assert the existence of constant mean curvature foliations whenever h_1 is trace free.

N.B. The non-degeneracy assumption required in the positive Yamabe invariant case is not required in the negative case because in that case it can be shown that $D\mathcal{N} = (\Delta_{h_0} - 2\kappa_2)$ must always be invertible.

5.1 Synopsis

The strategy for proving theorem 5.0.5 can be summarized as follows:

1. Calculate the second fundamental form of a level set of a boundary defining function in terms of this defining function.

2. Take the trace of the second fundamental form to find the mean curvature of such a level set.

3. Calculate how the mean curvature of such a level set behaves with respect to

conformal change on the boundary (and hence subsequent change of defining function).

4. Employ the implicit function theorem to say that as one moves a small amount inward from the boundary (along a geodesic boundary defining function), the deviation of the mean curvature of the level sets of such a function from constant can be corrected by an appropriate conformal change on the boundary.

5.2 The Second Fundamental Form of Level Sets

Recall that $\bar{g} = r^2 g = dr^2 + h$ and let $\overline{N} = \operatorname{grad}_{\bar{g}} r$ and $N = r^{-1} \operatorname{grad}_g r$, i.e. the conformally scaled and unscaled gradients of the geodesic boundary defining function r; these are vector fields normal to the level sets of r. We begin by calculating the second fundamental form of such a hypersurface from the standard formula

$$II = -\frac{1}{2}\mathcal{L}_N g.$$

To facilitate the calculation we first note that by definition $N = r\overline{N}$, $g = r^{-2}\bar{g}$, and and $\mathcal{L}_{\overline{N}} dr^2 = 0$. Also,

Lemma 5.2.1. *Let X be a vector field on a manifold, ω a tensor field, and f a function. Then*

$$\mathcal{L}_X(f\omega) = f\mathcal{L}_X\omega + (Xf)\omega, \tag{5.2.1}$$

and if $\alpha \odot \beta = \alpha \otimes \beta + \beta \otimes \alpha$ *and* ω *is a* 2*-tensor, then*

$$\mathcal{L}_{(fX)}\omega = f\mathcal{L}_X\omega + df \odot \iota_X\omega. \tag{5.2.2}$$

Proof. Let θ be the flow of X and recall that for any $p \in X$, if t is sufficiently close to zero, then θ_t is a diffeomorphism from a neighborhood of p to a neighborhood of $\theta_t(p)$, and thus θ_t^* pulls back tensors at $\theta_t(p)$ to tensors at p.

The identity 5.2.1 is most easily seen as a particular case of the more general product rule

$$\mathcal{L}_X(\tau \otimes \omega) = \tau \otimes (\mathcal{L}_X\omega) + (\mathcal{L}_X\tau) \otimes \omega, \tag{5.2.3}$$

where by the definition of Lie derivative, in the case of a function f, i.e. a 0-tensor field, $\theta_t^* f = f \circ \theta_t$, so

$$\mathcal{L}_X f(p) = \left.\frac{d}{dt}\right|_{t=0} f(\theta_t(p)) = Xf(p).$$

The proof of 5.2.3 is as follows:

$$\begin{aligned}
(\mathcal{L}_X(\tau \otimes \omega))_p &= \lim_{t\to 0} \frac{\theta_t^*((\tau \otimes \omega)_{\theta_t(p)}) - (\tau \otimes \omega)_p}{t} \\
&= \lim_{t\to 0} \frac{\theta_t^*(\tau_{\theta_t(p)}) \otimes \theta_t^*(\omega_{\theta_t(p)}) - \tau_p \otimes \omega_p}{t} \\
&= \lim_{t\to 0} \frac{\theta_t^*(\tau_{\theta_t(p)}) \otimes \theta_t^*(\omega_{\theta_t(p)}) - \theta_t^*(\tau_{\theta_t(p)}) \otimes \omega_p}{t} \\
&+ \lim_{t\to 0} \frac{\theta_t^*(\tau_{\theta_t(p)}) \otimes \omega_p - \tau_p \otimes \omega_p}{t} \\
&= \lim_{t\to 0} \theta_t^*(\tau_{\omega_t(p)}) \otimes \frac{\omega_t^*(\omega_{\theta_t(p)}) - \omega_p}{t} \\
&+ \lim_{t\to 0} \frac{\theta_t^*(\tau_{\theta_t(p)}) - \tau_p}{t} \otimes \omega_p \\
&= \tau_p \otimes (\mathcal{L}_X\omega)_p + (\mathcal{L}_X\tau)_p \otimes \omega_p
\end{aligned}$$

Meanwhile 5.2.2 may be suitably generalized to symmetric or antisymmetric k-tensors but the second term will take an appropriately altered form. For smooth vector fields $Y_1, \ldots, Y_k$ and ω a symmetric k-tensor we have

$$
\begin{aligned}
(\mathcal{L}_{fX}\omega)(Y_1, \ldots, Y_k) &= fX(\omega(Y_1, \ldots, Y_k)) - \omega([fX, Y_1], Y_2, \ldots, Y_k) - \cdots \\
&- \omega(Y_1, \ldots, Y_{k-1}, [fX, Y_k]) \\
&= fX(\omega(Y_1, \ldots, Y_k)) - f\omega([X, Y_1], Y_2, \ldots, Y_k) - \cdots \\
&- f\omega(Y_1, \ldots, Y_{k-1}, [X, Y_k]) + (Y_1 f)\omega(X, Y_2, \ldots, Y_k) + \cdots \\
&+ (Y_k f)\omega(Y_1, \ldots, Y_{k-1}, X) \\
&= f(\mathcal{L}_X\omega) + \sum_i df(Y_i) \otimes \omega(X, Y_1, \cdots, Y_{i-1}, \widehat{Y_i}, Y_{i+1}, \ldots, Y_k),
\end{aligned}
$$

where $\widehat{Y_i}$ means to omit Y_i. When $k = 2$ then this expression is precisely the identity 5.2.2. □

The lemma makes calculation of the second fundamental form brief:

$$
\begin{aligned}
\mathcal{L}_N g = \mathcal{L}_{r\overline{N}}(r^{-2}\overline{g}) &= r\mathcal{L}_{\overline{N}}(r^{-2}\overline{g}) + dr \odot \iota_{\overline{N}}(r^{-2}\overline{g}) \\
&= r(r^{-2}\mathcal{L}_{\overline{N}}\overline{g} - 2r^3\overline{g}) + 2r^{-2}dr^2 \\
&= r^{-2}(r\mathcal{L}_{\overline{N}}\overline{g} - 2\overline{g} + 2dr^2) \\
&= r^{-2}(r\mathcal{L}_{\overline{N}}h - 2h),
\end{aligned}
$$

and for simplicity denoting $\mathcal{L}_{\overline{N}}h = \partial_r h$ from now on, we find

$$
II = \frac{1}{2}r^{-2}(2h - r\partial_r h). \tag{5.2.4}
$$

Having obtained the second fundamental form of a level set of a geodesic boundary defining function 5.2.4, the mean curvature of this hypersurface can be found by taking the trace with respect to the induced metric, i.e. $r^{-2}h$, so

$$H = \operatorname{tr}^{r^{-2}h}\left(\frac{1}{2}r^{-2}(2h - r\partial_r h)\right) = n - \frac{1}{2}\operatorname{tr}^h(r\partial_r h). \tag{5.2.5}$$

Expressing 5.2.5 in terms of the expansion 3.1.3 then gives

$$H = n - \frac{1}{2}\left(\operatorname{tr}^{h_0} h_1\right) r - \left(\operatorname{tr}^{h_0} h_2 - \frac{1}{2}\|h_1\|^2_{h_0}\right) r^2 + \mathcal{O}(r^3). \tag{5.2.6}$$

Or by defining,

$$\kappa_1 = \frac{1}{2}\operatorname{tr}^{h_0} h_1 \in \mathcal{C}^{2,\alpha}(\partial M) \qquad \text{and} \qquad \kappa_2 = \operatorname{tr}^{h_0} h_2 - \frac{1}{2}\|h_1\|^2_{h_0} \in \mathcal{C}^{1,\alpha}(\partial M)$$

we can write 5.2.6 as

$$H = n - \kappa_1 r - \kappa_2 r^2 + \mathcal{O}(r^3). \tag{5.2.7}$$

5.3 Consequences of Conformal Change

The above expresses H for a level set of r in terms of r. Next we see what effect a change of representative of the conformal class at the boundary, and hence a change of associated geodesic boundary defining function, has on H:

Let $\hat{h}_0 = e^{2\phi_0}h_0$. We then have a new geodesic boundary defining function $\hat{r} = e^{\phi}r$ associated to $\hat{h}_0$ and metric $\hat{g} = \hat{r}^2 g = e^{2\phi}\bar{g}$. Also write

$$\hat{g} = d\hat{r}^2 + \hat{h}_{\hat{r}}, \qquad \hat{h} = \hat{h}_0 + \hat{h}_1\hat{r} + \hat{h}_2\hat{r}^2 + \mathcal{O}(\hat{r}^3),$$

where

$$\hat{h}_0 = \hat{g}|_{TM} = e^{2\phi_0} h_0$$

and the expansion for $\hat{h}$ is generated as explained before for Poincaré metrics. It will be most convenient for us to parameterize this change in terms of the conformal factor on the boundary, $\phi_0 \in \mathcal{C}^{2,\alpha}(M)$, which must then be extended to to a neighborhood of M. We may use 3.1.2 to express this extension in a manner that facilitates the following calculations. Using 3.1.2 but with the assumption that both the functions ρ and r are geodesic boundary defining functions, corresponding to distinct conformal representatives, we see that

$$-2\partial_r \phi = r \|\nabla^{\bar{g}} \phi\|_{\bar{g}}^2 \tag{5.3.1}$$

and thus as $\phi_0 \in \mathcal{C}^{2,\alpha}(M)$,

$$\phi(r, x) = \phi_0(x) - \frac{1}{4} \|\nabla^{h_0} \phi_0\|_{h_0}^2 r^2 + \mathcal{O}(r^{2+\alpha}). \tag{5.3.2}$$

Having our new metric $\hat{g}$ and geodesic boundary defining function $\hat{r}$ we denote as $\widehat{N} = \hat{r}^{-1} \operatorname{grad}_g \hat{r}$ the rescaled boundary defining function gradient. As before we will employ the identities of theorem 5.2.1 but first note:

$$
\begin{aligned}
\widehat{N} &= \hat{r}^{-1}\operatorname{grad}_g \hat{r} \\
&= r^{-1}e^{-\phi}\operatorname{grad}_g(re^{\phi}) \\
&= re^{-\phi}\operatorname{grad}_{\bar{g}}(re^{\phi}) \\
&= re^{-\phi}\left((\operatorname{grad}_{\bar{g}} r)e^{\phi} + r\operatorname{grad}_{\bar{g}} e^{\phi}\right) \\
&= re^{-\phi}\left(e^{\phi}\operatorname{grad}_{\bar{g}} r + e^{\phi} r\operatorname{grad}_{\bar{g}} \phi\right) \\
&= r(\operatorname{grad}_{\bar{g}} r + \operatorname{grad}_{\bar{g}} \phi) \\
&= r(\overline{N} + r\operatorname{grad}_{\bar{g}} \phi).
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{L}_{\widehat{N}} g &= \mathcal{L}_{r(\overline{N}+r\operatorname{grad}_{\bar{g}}\phi)}(r^{-2}\bar{g}) \\
&= r\mathcal{L}_{(\overline{N}+r\operatorname{grad}_{\bar{g}}\phi)}(r^{-2}\bar{g}) + dr\odot\iota_{(\overline{N}+r\operatorname{grad}_{\bar{g}}\phi)}(r^{-2}\bar{g}) \\
&= r^{-1}\mathcal{L}_{(\overline{N}+r\operatorname{grad}_{\bar{g}}\phi)}\bar{g} - 2r^{-2}\left[(\overline{N}+r\operatorname{grad}_{\bar{g}}\phi)r\right]\bar{g} + r^{-2}dr\odot\iota_{(\overline{N}+r\operatorname{grad}_{\bar{g}}\phi)}\bar{g} \\
&= r^{-1}\mathcal{L}_{\overline{N}}\bar{g} + r^{-1}\mathcal{L}_{r\operatorname{grad}_{\bar{g}}\phi}\bar{g} - 2r^{-2}\left(1 + r\partial_r\phi\right)\bar{g} \\
&+ r^{-2}dr\odot\left(\iota_{\overline{N}}\bar{g} + r\iota_{\operatorname{grad}_{\bar{g}}\phi}\bar{g}\right) \\
&= r^{-2}\left(r\mathcal{L}_{\overline{N}}h + r^2\mathcal{L}_{(\operatorname{grad}_{\bar{g}}\phi)}\bar{g} + r\,dr\odot d\phi\right) - 2r^{-2}\bar{g} - 2r^{-1}\partial_r\phi\bar{g} \\
&+ r^{-2}dr\odot dr + r^{-1}dr\odot d\phi \\
&= r^{-2}\left(r\mathcal{L}_{\overline{N}}h - 2h\right) + \mathcal{L}_{(\operatorname{grad}_{\bar{g}}\phi)}\bar{g} + 2r^{-1}dr\odot d\phi - r^{-1}\partial_r\phi\bar{g}
\end{aligned}
$$

Using 5.3.1 and the fact that $\mathcal{L}_{(\mathrm{grad}_{\bar{g}}\,\phi)}\bar{g} = 2\,\mathrm{Hess}^{\bar{g}}(\phi)$ we find

$$II(\phi_0, \epsilon) = \left(\frac{1}{2r^2}(2h - r\partial_r h) - r^{-1}(dr \odot d\phi) - \mathrm{Hess}^{\bar{g}}(\phi) - \frac{1}{2}\|\nabla^{\bar{g}}\phi\|^2_{\bar{g}}\bar{g} \right)\Big|_{e^\phi r = \epsilon} . \tag{5.3.3}$$

As before, to obtain H we trace by the induced metric $\hat{r}^{-2}\hat{h}$ on this hypersurface (note that there is no $d\hat{r}^2$ component here so actually this trace can be just with respect to g):

$$\begin{aligned} H(\phi_0, \epsilon) &= (\mathrm{tr}^g\, II(\phi_0))|_{e^\phi r = \epsilon} = \left(\mathrm{tr}^{r^{-2}\bar{g}}\, II(\phi_0)\right)\Big|_{e^\phi r = \epsilon} \\ &= \left(\frac{1}{2}\,\mathrm{tr}^h(2h - r\partial_r h) - r^2\left(\Delta_{\bar{g}}\phi + \frac{n-1}{2}\|\nabla^{\bar{g}}\phi\|^2_{\bar{g}}\right)\right)\Big|_{e^\phi r = \epsilon} . \end{aligned}$$

So that finally we have

$$H(\phi_0, \epsilon) = n - \epsilon^2 \left(e^{-2\phi}\left(\frac{1}{2r}\,\mathrm{tr}^h\,\partial_r h + \Delta_{\bar{g}}\phi + \frac{n-1}{2}\|\nabla^{\bar{g}}\phi\|^2_{\bar{g}} \right)\right)\Big|_{e^\phi r = \epsilon} . \tag{5.3.4}$$

Using 5.3.2 we see that

$$\Delta_{\bar{g}}\phi = \Delta_{h_0}\phi_0 - \frac{1}{2}\|\nabla^{h_0}\phi_0\|^2_{h_0} + \mathcal{O}(r^\alpha), \tag{5.3.5}$$

and this along with 3.1.3 and 3.1.4 let us expand 5.3.4 to obtain

$$\begin{aligned} H(\phi_0, \epsilon) = n &- \frac{1}{2}e^{-\phi_0}\,\mathrm{tr}^{h_0}\,h_1\hat{r} \\ &+ e^{-2\phi_0}\left(\mathrm{tr}^{h_0}\,h_2 - \frac{1}{2}\|h_1\|^2_{h_0} + \Delta_{h_0}\phi_0 + \frac{n-2}{2}\|\nabla^{h_0}\phi_0\|^2_{h_0}\right)\hat{r}^2 + \mathcal{O}(\hat{r}^3). \end{aligned}$$

Comparing this expansion with the original expansion for H, 5.2.7, we can see how κ_1 and κ_2 are affected by conformal change:

$$\hat{\kappa}_1 = e^{-\phi_0}\kappa_1, \tag{5.3.6}$$

$$\hat{\kappa}_2 = e^{-2\phi_0}\left(\kappa_2 + \Delta_{h_0}\phi_0 + \frac{n-2}{2}\|\nabla^{h_0}\phi_0\|^2_{h_0}\right). \tag{5.3.7}$$

In particular, we will use 5.3.7 to reduce the general problem to the case when κ_2 is constant. That this apparently substantial restriction can be used to find constant mean curvature foliations for general κ_2 is predicated on finding solutions for equation 5.3.7 where we assume that $\hat{\kappa}_2$ is constant. For completely arbitrary smooth non-constant κ_2 this might not always be possible; sufficient conditions for reduction to the constant κ_2 case are the content of section 5.5 below. For now we assume κ_2 constant and look to frame the existence question in a form amenable to the implicit function theorem.

5.4 Linearization

When $\phi_0 = 0$ and $r = \epsilon$, the assumption $\operatorname{tr}^{h_0} h_1 \equiv 0$ makes 5.2.7 into

$$H(0, \epsilon) = n - \kappa_2 \epsilon^2 + \mathcal{O}(\epsilon^3),$$

where $\kappa_2 \in \mathcal{C}^{1,\alpha}(M)$. Assume κ_2 to be constant, we wish to find for each $\epsilon > 0$ a function $\phi_0(\epsilon)$ on M such that

$$H(\phi_0(\epsilon), \epsilon) = n - \kappa_2 \epsilon^2 \tag{5.4.1}$$

By 5.3.4 this is equivalent to

$$\epsilon^2 \left(e^{-2\phi} \left(\frac{1}{2r} \operatorname{tr}^h \partial_r h + \Delta_{\bar{g}} \phi + \frac{n-1}{2} \|\nabla^{\bar{g}} \phi\|_{\bar{g}}^2 \right) \right) \Big|_{e^\phi r = \epsilon} = \kappa_2 \epsilon^2$$

Or letting

$$\mathcal{N}(\phi_0, \epsilon) = \left(e^{-2\phi} \left(\frac{1}{2r} \operatorname{tr}^h \partial_r h + \Delta_{\bar{g}} \phi + \frac{n-1}{2} \|\nabla^{\bar{g}} \phi\|_{\bar{g}}^2 \right) \right) \Big|_{e^\phi r = \epsilon},$$

we are looking to solve

$$\mathcal{N}(\phi_0, \epsilon) - \kappa_2 = 0 \tag{5.4.2}$$

for $\phi_0(\epsilon)$. Note that when $\kappa_1 = 0$, $\mathcal{N}(0,0) = \kappa_2$, and because in this case we see from 5.2.5 and 5.2.6 that $\operatorname{tr}^h \partial_r h = 2\kappa_2 r + \mathcal{O}(r^2)$, it follows that $\mathcal{N}$ is $\mathcal{C}^1$ up to the boundary ($\epsilon = 0$). We are well poised to apply the implicit function theorem, provided that the linearization of $\mathcal{N}$ is invertible at $\phi_0 = 0, \epsilon = 0$. To this end, we compute the linearization of $\mathcal{N}$ with respect to ϕ_0, which we will denote $D_{\phi_0}\mathcal{N}$. First, express $\mathcal{N}$ as a composition $\mathcal{N} = \mathcal{R} \circ \mathcal{L} \circ \mathcal{E}$, where $\mathcal{E}$ is the extension operator defined previously, $\mathcal{R}(\phi, \epsilon)$ is restriction of a function ϕ on X to the hypersurface corresponding to $\{e^\phi r = \epsilon\}$, and

$$\mathcal{L} : \phi \mapsto \left(e^{-2\phi} \left(\frac{1}{2r} \operatorname{tr}^h \partial_r h + \Delta_{\bar{g}} \phi + \frac{n-1}{2} \|\nabla^{\bar{g}} \phi\|^2_{\bar{g}} \right) \right). \tag{5.4.3}$$

By the chain-rule we have $D_{\phi_0}\mathcal{N}(0,0) = D_{\phi_0}\mathcal{R}(0,0) \circ D_{\phi_0}\mathcal{L}(0,0) \circ D_{\phi_0}\mathcal{E}(0,0)$. Let us consider each piece:

$D_{\phi_0}\mathcal{R}(0,0)$ This is simply restriction to M.

$D_{\phi_0}\mathcal{E}(0,0)$ To see how this operator acts on functions $\psi_0 \in \mathcal{C}^{2,\alpha}(M)$ we linearize the equation used to derive $\mathcal{E}$, 5.3.1 to get

$$\partial_r \psi = 0, \qquad \text{with} \qquad \psi(0,x) = \psi_0(x),$$

and thus

$$\psi(r,x) = \psi_0(x),$$

i.e. constant extension.

$D_{\phi_0}\mathcal{L}(0,0)$ We linearize $\mathcal{L}$ at $\phi_0 = 0$:

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\mathcal{L}(t\psi) &= \left.\frac{d}{dt}\right|_{t=0} e^{-2t\psi}\left(\frac{1}{2r}\operatorname{tr}^h \partial_r h + \Delta_{\bar{g}}(t\psi) + \frac{n-1}{2}\|\nabla^{\bar{g}} t\psi\|^2_{\bar{g}}\right) \\ &= \left(\Delta_{\bar{g}} - \frac{1}{r}\operatorname{tr}^h \partial_r h\right).\end{aligned}$$

We have already seen above that $\operatorname{tr}^h \partial_r h = 2\kappa_2 r + \mathcal{O}(r^2)$ and combining this with our knowledge that $D_{\phi_0}\mathcal{E}(0,0)$ is constant extension plus 5.3.2 and 5.3.5 yields:

$$D_{\phi_0}\mathcal{N}(0,0) = (\Delta_{h_0} - 2\kappa_2)\,. \tag{5.4.4}$$

Now provided 5.4.4 is invertible the implicit function theorem (A.0.7) asserts existence of a unique smooth function $\epsilon \mapsto \phi_0$ such that $\phi_0(0) = 0$ and $\mathcal{N}(\phi_0(\epsilon),\epsilon) - \kappa_2 = 0$ for all sufficiently small $0 \leq \epsilon$. Thus ϵ parameterizes collections of constant mean curvature hypersurfaces near M.

When $D_{\phi_0}\mathcal{N}(0,0)$ is non-degenerate the implicit function theorem may be employed. This provokes the questions: When does this non-degeneracy condition fail to hold? Can the existence result of the original theorem be in any sense salvaged in such an event?

So long as we assume κ_2 to be constant, the first of these questions is in effect the eigenvalue problem:

$$\Delta\phi + \lambda\phi = 0$$

but with the eigenvalue $\lambda = -2\kappa_2$ specified. The question becomes: Is $-2\kappa_2$ in the spectrum of M? Perhaps in some circumstances we may exclude this possibility via some relations between the geometry of M and the bounds of $\mathrm{Spec}(M)$ without needing to actually calculate the spectrum itself. Unfortunately, we already know a natural and quite common example of when this degeneracy occurs: when the conformal boundary is the round sphere, then our linearized operator has eigenvalue $-2\kappa_2 = n$ with associated eigenspace comprised by the first degree spherical harmonics. This defect is an obvious target for future improvement.

5.5 Reduction to a Yamabe-type Problem

The method of proof employed above assumed κ_2 to be constant. In the case of weakly Poincaré-Einstein metrics, $h_1 = 0$ and $h_2 = -P_{h_0}$, so that

$$\kappa_2 = \mathrm{tr}^{h_0} h_2 = -\frac{R_{h_0}}{2(n-1)}$$

and thus we will see below that equation 5.5.1 can be written as the Yamabe equation. This suggests that the κ_2 constant condition can be ensured via solution of a Yamabe-type problem.

Recalling 5.3.7 we are seeking non-degenerate solutions to

$$e^{-2\phi_0}\left(\kappa_2 + \Delta_{h_0}\phi_0 + \frac{n-2}{2}\|\nabla^{h_0}\phi_0\|^2_{h_0}\right) - \hat{\kappa}_2 = 0 \tag{5.5.1}$$

where $\hat{\kappa}_2$, corresponding to κ_2 after a conformal change of metric $\hat{h}_0 = e^{2\phi_0}h_0$, is constant.

Let $n \geq 3$. Recalling the generalized conformal Laplacian

$$\mathbb{L}_{h_0} = -\left(\Delta_{h_0} + \frac{n-2}{2}\kappa_2\right),$$

we can rewrite 5.5.1 so as to make it more clearly a Yamabe-type equation. To this end set $\phi_0 = \frac{2}{n-2}\log u_0$ which transforms 5.5.1 into

$$\mathbb{L}_{h_0} u_0 + \frac{n-2}{2}\hat{\kappa}_2 u_0^{\frac{n+2}{n-2}} = 0 \tag{5.5.2}$$

N.B. that in the weakly Poincaré-Einstein case, as just mentioned, this is precisely the Yamabe equation, as then $\kappa_2 = -\frac{R_{h_0}}{2(n-2)}$. This illustrates the opposite sign relationship between κ_2 and the generalized Yamabe invariant.

Lemma 5.5.1. *The sign of $\lambda_1(\mathbb{L}_{h_0})$ is opposite to the sign of $\hat{\kappa}_2$. That is,*

1. *if $\lambda_1(\mathbb{L}_{h_0}) < 0$ then $\hat{\kappa}_2 > 0$,*

2. *if $\lambda_1(\mathbb{L}_{h_0}) > 0$ then $\hat{\kappa}_2 < 0$, and*

3. *if $\lambda_1(\mathbb{L}_{h_0}) = 0$ then $\hat{\kappa}_2 = 0$.*

Proof. Suppose that ψ_1 is the eigenfunction corresponding to $\lambda_1(\mathbb{L}_{h_0})$ and that $u_0 > 0$ is a solution to 5.5.2. Multiplying 5.5.2 by ψ_1 and integrating yields

$$\int_M \psi_1 \mathbb{L}_{h_0} u_0 \, dV_{h_0} + \frac{n-2}{2}\hat{\kappa}_2 \int_M \psi_1 u_0^{\frac{n+2}{n-2}} \, dV_{h_0} = 0.$$

Integration by parts (M is compact and without boundary) then produces

$$\lambda_1 \int_M \psi_1 u_0 \, dV_{h_0} + \frac{n-2}{2}\hat{\kappa}_2 \int_M \psi_1 u_0^{\frac{n+2}{n-2}} \, dV_{h_0} = 0.$$

As $\psi_1 > 0$ and $u_0 > 0$, λ_1 and $\hat{\kappa}_2$ must have opposite signs or both be zero. $\square$

Theorem 5.5.2. *Let (X, g) be an asymptotically hyperbolic manifold with $\kappa_1 = 0$, and the operator $(\Delta_{h_0} - 2\kappa_2)$ invertible. Then if the generalized boundary Yamabe invariant (equivalently, the least eigenvalue of the generalized conformal conformal Laplacian) is non-positive there exists a constant mean curvature foliation of X within a neighborhood of conformal infinity. If the generalized boundary Yamabe invariant is negative then there is a unique monotone constant mean curvature foliation near $\partial_\infty X$.*

Proof. Suppose that $\lambda_1(\mathbb{L}_{h_0}) < 0$ and hence $\hat{\kappa}_2 > 0$. For each $1 < p < \frac{n+2}{n-2}$ minimize the functional

$$E_p(u) = \frac{1}{2}\int_M \left(|\nabla^{h_0} u|^2_{h_0} - \frac{n-2}{2}\kappa_2 u^2\right) dV_{h_0} + \frac{n-2}{2(p+1)}\hat{\kappa}_2 \int_M |u|^{p+1}\, dV_{h_0}. \quad (5.5.3)$$

Existence of such minimizers u_p is a consequence of the compactness of the Sobolev embedding $H^1(M) \hookrightarrow L^p(M)$ when $p < \frac{n+2}{n-2}$ and they satisfy the Euler-Lagrange equation of 5.5.3:

$$\Delta_{h_0} u_p + \frac{n-2}{2}\kappa_2 u_p - \frac{n-2}{2}\hat{\kappa}_2 u_p^p = 0.$$

Smoothness of the u_p is ensured by elliptic regularity theory. To find a smooth positive solution to 5.5.2 we wish to take a limit of a subsequence of $\{u_p\}$ as $p \nearrow \frac{n+2}{n-2}$. To ensure uniform convergence we use the Arzela-Ascoli theorem after verifying its hypotheses are met. Recalling that M is compact and without boundary, it will suffice to find a uniform bound for $\{u_p\}$, as Schauder estimates will then yield a uniform bound of the Hölder norms. Let $x_p \in M$ be a point where u_p attains its maximum, then

$$\kappa_2(x_p)u_p(x_p) \geq \hat{\kappa}_2 u_p^p(x_p),$$

and thus

$$\hat{\kappa}_2 \|u_p\|_{L^\infty}^{p-1} \leq \|\kappa_2\|_{L^\infty}$$

yielding the required uniform bound.

To demonstrate uniqueness, assume u and v are both positive solutions of 5.5.2 (with the same value of $\hat{\kappa}_2$), and let $\psi = \frac{v}{u}$. Using the calculations in theorem 4.2.3 we compute

$$\Delta_{\hat{h}_0}\psi_0 + \frac{n-2}{2}\hat{\kappa}_2\left(\psi - \psi^{\frac{n+2}{n-2}}\right)$$

where $\hat{h}_0 = u^{\frac{4}{n-2}}h_0$. Where ψ achieves its supremum $\Delta_{\hat{h}_0}\psi \leq 0$, and so we have $\psi \leq 1$ everywhere. Similarly, considering the point at which ψ reaches its infimum we find $\psi \geq 1$. Thus $\psi \equiv 1$, proving uniqueness.

This encompasses the case $n \geq 3$. When $n = 2$ a similar argument produces the result but the definition of the generalized boundary Yamabe invariant in that case means that the appropriate assumption is $\int_M \kappa_2 \, dV_{h_0} > 0$. When $\lambda_1(\mathbb{L}_{h_0}) = 0$, equation 5.5.2 becomes linear elliptic and thus a solution exists. $\square$

Recall that in theorem 5.0.5 when the generalized boundary Yamabe invariant is positive, the invertibility of the linearized Yamabe-type equation is sufficient for existence of constant mean curvature foliations. It should be remarked that this is because applying the extension and restriction operators to equation 5.4.3 to find the explicit form of $\mathcal{N}$ yields

$$\phi_0 \mapsto e^{-2\phi_0}\left(\kappa_2 + \Delta_{h_0}\phi_0 + \frac{n-2}{2}\|\nabla^{h_0}\phi_0\|^2_{h_0}\right),$$

i.e. precisely the expression for the conformal transformation of κ_2 (5.3.7). Thus it seems that the conformal adjustment of the boundary on the boundary to produce constant mean curvature level sets of the boundary defining function is quite directly linked to the Yamabe-type problem for finding constant κ_2. This has implications for the handling of the degenerate situations, as it means that conformal adjustments cannot be used to circumvent the degeneracy of the linearization of $\mathcal{N}$.

The linearization of 5.5.2 at u is

$$\Delta_{h_o} + \frac{n-2}{2}\kappa_2 - \frac{n+2}{2}\hat{\kappa}_2 u^{\frac{4}{n-2}} = 0 \tag{5.5.4}$$

and hence combining the calculations of theorem 4.2.3 with equation 5.5.2 yields

$$\left(\Delta_{h_0} + \frac{n-2}{2}\kappa_2 - \frac{n+2}{2}\hat{\kappa}_2 u^{\frac{4}{n-2}}\right)(uw) = u^{\frac{n+2}{n-2}}(\Delta_{\hat{h}_0} - 2\hat{\kappa}_2)w.$$

Then the assumption that $\hat{\kappa}_2 > 0$ implies that $(\Delta_{\hat{h}_0} - 2\hat{\kappa}_2)$ and hence 5.5.4 must be invertible.

Chapter 6

Foliations of Constant Weingarten Curvature

Finally, we extend the methods described in chapter 5 to functions of the principal curvatures other than the mean curvature.

Let $\lambda_1, \ldots, \lambda_n$ be the principal curvatures (eigenvalues of the second fundamental form) of a hypersurface Σ and $f(\lambda_1, \ldots, \lambda_n)$ a symmetric function of these curvatures. Then f is called a *Weingarten curvature* of Σ. The most commonly studied of Weingarten conditions is $f(\lambda_1, \ldots, \lambda_n) = \sum_i \lambda_i = c$ for some constant c, i.e. hypersurfaces of constant mean curvature. Prescriptions of Weingarten curvature have been used in sphere theorems for hypersurfaces in Euclidean spaces in [38], Euclidean and hyperbolic spaces in [38], and in spaces of constant curvature in [46] and [6]. In all but the last of these cases, investigation is restricted to $f = \sigma_k$, the elementary symmetric polynomials $\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum \lambda_{i_1} \cdots \lambda_{i_k}$. In [31] foliations near infinity of constant σ_k-curvature of asymptotically hyperbolic spaces are briefly considered and we will continue that consideration here.

A similar sort of hypersurface curvature prescription in hyperbolic spaces is a topic of [8]. In this case however the Weingarten functionals are applied to the principal curvature *radii*, i.e. the reciprocals of the principal curvatures, and the existence of hypersurfaces of specified curvature is sought. When $f = \sigma_1$ this is a formulation of the classical Christoffel problem but in hyperbolic space. Exploiting the duality of hyperbolic $(n+1)$-space with de Sitter $(n+1)$-space provided by embedding both in $(n+2)$-dimensional Minkowski space, they exhibit a very precise relationship between the scalar curvature of metrics in the conformal structure at infinity and the mean curvature of immersed hypersurfaces in $\mathbb{H}^{n+1}$. In [3] this same relationship is obtained through the methods and perspective of asymptotically hyperbolic geometry, while the results are also extended to quotients of hyperbolic space by discrete groups of isometries (even so the same duality between hyperbolic and de Sitter space is still required).

A salient point of comparison between the results in [8] and [3], and those of this section (besides the use of principal radii instead of curvatures) is that the curvature interplay between hypersurfaces and conformal infinity seem to differ in a potentially interesting way. In both cases a constant curvature metric on the conformal boundary is tied to a constant curvature hypersurface in the bulk space. The equivalence provided in [8] and [3] relates differing Weingarten curvatures on hypersurfaces to differing curvatures on metrics on the conformal boundary, in the case of constant mean radial curvature hypersurfaces the boundary curvature in question is the scalar curvature. Curvature conditions involving elementary symmetric functions of higher order produce more complicated relationships, e.g. the hypersurfaces of constant $\sum c_i \sigma_i$-curvature correspond to boundary metrics with constant σ_k of the eigenvalues of the Schouten tensor. In contrast, the methods of this section produce constant f-curvature hypersurfaces for most Weingarten functionals f, predicated on finding only constant scalar curvature metrics on the conformal boundary of a Poincaré-Einstein manifold.

6.1 Spectral Functions of the Second Fundamental Form

In this section we will implicitly exploit the isomorphism between smooth sections of the $(1,1)$-tensor bundle $T_1^1(X)$ and smooth maps from a manifold M^n to the set of $(n \times n)$-matrices to refer interchangeably to $(1,1)$-tensors and matrices.

Definition 6.1.1. A real-valued function F which takes as argument real-valued symmetric matrices is called *spectral* if $F(A) = F(UAU^T)$ for all A in the domain of F and all orthogonal matrices U. Equivalently, F is a function on symmetric matrices that depends only on the eigenvalues of its argument.

The restriction of a spectral function F to the subspace of diagonal $n \times n$ matrices defines a function on $\mathbb{R}^n$, $f(x) = F(\mathrm{Diag}(x))$ where $\mathrm{Diag}(x)$ is the diagonal matrix with entries the components of $x \in \mathbb{R}^n$. As permutation matrices are orthogonal, f is by construction a *symmetric* function in the sense that $f(Px) = f(x)$ for any permutation P. Furthermore, we can write $F(A) = (f \circ \lambda)(A)$ where $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ is the *eigenvalue* map which sends a symmetric matrix to the vector of it's eigenvalues (suppose the components of this vector to be in non-increasing order to make this a single-valued function).

The question of precisely what degree of smoothness the function F inherits from the map f is actually a delicate one in general because while the eigenvalue map λ is always continuous, it is not necessarily differentiable with respect to A. A thorough examination of the regularity of λ can be found in [22] and some implications for F are explored in [44] and [28] for example. Since the classical statements of problems of prescribed curvature (e.g. Christoffel, Minkowski) are made in terms of a symmetric function of principal curvatures, i.e. in terms of a function like f, we use F primarily for conceptual and notational purposes and might simply assume that F retains "enough" smoothness. In fact, when the argument A of λ depends on a single real parameter t and all $A(t)$ are symmetric, a result of Rellich [36] (outlined more succinctly in [22]) ensures that λ will be as smooth as $A(t)$, and this is precisely the context of concern here.

As before we calculate the second fundamental form $II(\phi_0)$ of level sets of the defining function $\hat{r}$ associated to the boundary conformal representative $\hat{h}_0 = e^{2\phi} h_0$. Instead of taking the trace of II to find the mean curvature of level sets we instead apply any function F with the following properties:

1. F is a spectral function,

2. F is $\mathcal{C}^2$.

Equivalently, we require that $F(A) = (f \circ \lambda)(A)$ for λ the eigenvalue map and f a function on the n eigenvalues such that

1. f is symmetric,

2. f is $\mathcal{C}^2$.

Finally we will make one further assumption on f, the importance of which will become clear in the course of our investigation:

3. f has non-vanishing first-derivative at $(1, \ldots, 1)$.

The σ_k provide simple and concrete examples of this class of function; σ_1-curvature is mean curvature and σ_n-curvature is Gaussian curvature.

As F is a function on symmetric matrices we first use the metric g to raise one index of II to obtain a $(1,1)$-tensor denoted $g^{-1}II$. (N.B. that in this section occurrences of g^{-1}, $\bar{g}^{-1}$ and h^{-1} denote raised indices, not traces.) For convenience recall the expression for $II(\phi_0)$, 5.3.3:

$$II(\phi_0, \epsilon) = \left(\frac{1}{2r^2}(2h - r\partial_r h) - r^{-1}(dr \odot d\phi) - \mathrm{Hess}^{\bar{g}}(\phi) - \frac{1}{2}\|\nabla^{\bar{g}}\phi\|^2_{\bar{g}}\bar{g} \right)\bigg|_{e^{\phi}r=\epsilon}.$$

Then $g^{-1}II = (r^{-2}\bar{g})^{-1}II = \bar{g}^{-1}(r^2 II)$ where

$$\bar{g}^{-1}(r^2 II)(\phi_0, \epsilon) = \bar{g}^{-1}\Bigg((h - \frac{1}{2} r\partial_r h) \\ - r(dr \odot d\phi) - r^2 \left(\mathrm{Hess}^{\bar{g}}(\phi) - \frac{1}{2}\|\nabla^{\bar{g}}\phi\|^2_{\bar{g}}\bar{g} \right) \Bigg)\Bigg|_{e^\phi r = \epsilon},$$

and in particular

$$g^{-1}II(0) = h^{-1}(h - \frac{1}{2} r\partial_r h) \tag{6.1.1}$$

$$= \delta_i{}^j - \frac{1}{2}(h_1)_i{}^j r - \left((h_2)_i{}^j - \frac{1}{2}(h_1 \cdot h_1)_i{}^j \right) r^2 + \mathcal{O}(r^3), \tag{6.1.2}$$

where the h_i refer to the terms in our original expansion of h and $(h_1 \cdot h_1)_i{}^j$ denotes composition (natural pairing) of the $(1,1)$-tensor $(h_1)_i{}^j$ with itself.

Let $A_i{}^j = g^{-1}II(0)$ and $\lambda_0 = (1, \ldots, 1)$ and note that the symmetry of f means that taking partial derivatives and then evaluating at λ_0 yields a value dependent only on the order of the derivative; denote these values by

$$c_0 = f(\lambda_0), \qquad c_1 = \frac{\partial f}{\partial \lambda_i}(\lambda_0), \qquad c_2 = \frac{\partial^2 f}{\partial \lambda_j \partial \lambda_i}(\lambda_0).$$

Finally note

Lemma 6.1.2. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a symmetric set, $g : \mathcal{S} \to \mathbb{R}$, $\mathcal{M}_n$ the set of symmetric real $n \times n$ matrices A with n-tuple of eigenvalues $\lambda(A)$ lying within $\mathcal{S}$. For $A \in \mathcal{M}_n$ let $G(A) = g(\lambda(A))$. Let A_0 be diagonal, hence with diagonal entries $\lambda(A_0) = \lambda_0$. Then*

$$\frac{\partial G}{\partial a_i^j}(A_0) = \delta_{ij} \frac{\partial g}{\partial \lambda_i}(\lambda_0) \tag{6.1.3}$$

Proof. Note that G is spectral, i.e. $G(A) = G(UAU^T)$ for U orthogonal, so it suffices to consider diagonal A. The lemma is then an immediate consequence of the chain-rule. $\square$

We calculate the expansion of $F(A)$ in terms of r by taking derivatives:

$$
\begin{aligned}
\frac{d}{dr}F(A)\bigg|_{r=0} &= \sum_{i,j} \frac{\partial F}{\partial a_i^j}(A_0)\frac{d}{dr}[A_i{}^j]_{r=0} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}(\lambda_0)\frac{d}{dr}[A_i{}^i]_{r=0} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}(\lambda_0)(-\frac{1}{2}(h_1)_i{}^i) \\
&= -\frac{c_1}{2}\operatorname{tr}^{h_0} h_1 \\
&= -c_1\kappa_1
\end{aligned}
$$

Where the second line follows from lemma 6.1.3 and the third from 6.1.2. Similarly,

$$\begin{aligned}
\frac{d^2}{dr^2}F(A)\bigg|_{r=0} &= \sum_{i,j,k,l} \frac{\partial F}{\partial a_k^l \partial a_i^j}(A_0)\left(\frac{d}{dr}[A_k{}^l]_{r=0}\right)\left(\frac{d}{dr}[A_i{}^j]_{r=0}\right) \\
&+ \sum_{i,j} \frac{\partial F}{\partial a_i^j}(A_0)\frac{d^2}{dr^2}[A_i{}^j]_{r=0} \\
&= \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_k}(\lambda_0)\left(\frac{d}{dr}[A_k{}^k]_{r=0}\right)\left(\frac{d}{dr}[A_i{}^i]_{r=0}\right) \\
&+ \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}(A_0)\frac{d^2}{dr^2}[A_i{}^i]_{r=0} \\
&= \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_k}(\lambda_0)\left[\left(-\frac{1}{2}\right)^2 (h_1)_i{}^i (h_1)_k{}^k\right] \\
&- 2\sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}\left((h_2)_i{}^i - \frac{1}{2}(h_1 \cdot h_1)_i{}^i\right) \\
&= \frac{c_2}{4}\sum_{i,k=1}^{n}(h_1)_i{}^i (h_1)_k{}^k - 2c_1 \sum_{i=1}^{n}\left((h_2)_i{}^i - \frac{1}{2}(h_1 \cdot h_1)_i{}^i\right) \\
&= -2c_1\left(\operatorname{tr}^{h_0} h_2 - \frac{1}{2}\|h_1\|_{h_0}^2\right) + \frac{c_2}{4}\sigma_2(\lambda((h_0)^{-1}h_1)) \\
&= -2c_1\kappa_2 + \frac{c_2}{4}\sigma_2(\lambda(h_1)).
\end{aligned}$$

Where $\sigma_2(\lambda(h_1)) = \sum_{j=1}^{n}(\lambda_j((h_0)^{-1}h_1))^2$ denotes the second order elementary symmetric polynomial in the eigenvalues of h_1. Thus

$$F(0,\epsilon) = F(g^{-1}II(0)) = c_0 - c_1\kappa_1\epsilon - \left(c_1\kappa_2 - \frac{c_2}{8}\sigma_2(\lambda(h_1))\right)\epsilon^2 + \mathcal{O}(\epsilon^3),$$

and just as before we seek $\epsilon \mapsto \phi_0$ such that

$$F(\phi_0(\epsilon),\epsilon) = c_0 - c_1\kappa_2\epsilon^2,$$

or equivalently

$$\frac{1}{\epsilon^2}\left(F(\phi_0,\epsilon) - c_0\right) + c_1\kappa_2 = 0. \tag{6.1.4}$$

6.2 Linearization & Result

Recalling that κ_2 is assumed constant, we linearize

$$\mathcal{N}(\phi_0, \epsilon) = \frac{1}{\epsilon^2}\left(F(\phi_0, \epsilon) - c_0\right) \tag{6.2.1}$$

with respect to ϕ_0 at $(0,0)$. Just as in the mean curvature case we factor $\mathcal{N} = \mathcal{R} \circ \mathcal{L} \circ \mathcal{E}$ with $\mathcal{R}$ and $\mathcal{E}$ being extension and restriction as before, but this time with $\mathcal{L}$ being

$$\begin{aligned}\phi \mapsto r^{-2}e^{-2\phi} F\Bigg(&\bar{g}^{-1}(h - \frac{1}{2}r\partial_r h) - \bar{g}^{-1}r(dr \odot d\phi)\\ &- \bar{g}^{-1}r^2\left(\mathrm{Hess}^{\bar{g}}(\phi) - \frac{1}{2}\|\nabla^{\bar{g}}\phi\|^2_{\bar{g}}\bar{g}\right)\Bigg) - r^{-2}e^{-2\phi}c_0,\end{aligned}$$

so that again $D_{\phi_0}\mathcal{N}(0,0) = D_{\phi_0}\mathcal{R}(0,0) \circ D_{\phi_0}\mathcal{L}(0,0) \circ D_{\phi_0}\mathcal{E}(0,0)$. Once again we have that $D_{\phi_0}\mathcal{R}(0,0)$ is restriction to M, and $D_{\phi_0}\mathcal{E}(0,0)$ extends functions by $\psi(r,x) = \psi_0(x)$ (i.e. $\partial_r\psi = 0$). The linearization of our new $\mathcal{L}$ at $\phi = 0$, $\epsilon = 0$ is:

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(t\psi) &= \left.\frac{d}{dt}\right|_{t=0} r^{-2}e^{-2t\psi}F\Big(g^{-1}II(0) - \bar{g}^{-1}r(dr \odot d(t\psi))\\ &\quad - \bar{g}^{-1}r^2\nabla^{\bar{g}}_i(\nabla^{\bar{g}})^j(t\psi) - \frac{r^2}{2}\|\nabla^{\bar{g}}t\psi\|^2_{\bar{g}}\delta_i{}^j\Big) - r^{-2}e^{-2t\psi}c_0\\ &= -2r^{-2}\psi F(g^{-1}II(0)) + \sum_{i=1}^n \frac{\partial f}{\partial\lambda_i}(\lambda_0)\frac{d}{dt}[g^{-1}II(t\psi)_i{}^i]_{t=0} + 2r^{-2}c_0\\ &= -2r^{-2}\psi F(g^{-1}II(0)) + c_1(-2r^{-1}\partial_r\psi - \Delta_{\bar{g}}\psi) + 2r^{-2}c_0\psi\\ &= -2r^{-2}\left(c_0 - c_1\kappa_2 r^2 + \mathcal{O}(r^3)\right)\psi - c_1\Delta_{\bar{g}}\psi + 2r^{-2}c_0\psi.\end{aligned}$$

Using 6.1.2 and 5.3.5, we have:

$$D_{\phi_0}\mathcal{N}(0,0) = -c_1\left(\Delta_{h_0} - 2\kappa_2\right). \tag{6.2.2}$$

Now provided 6.2.2 is invertible the implicit function theorem yields a unique smooth function $\epsilon \mapsto \phi_0$ satisfying 6.1.4 and thus ϵ parameterizes collections of constant f-curvature hypersurfaces near M.

Comparing 6.2.2 with 5.4.4 it is clear that the only additional hypothesis needed to make this generalization to the case of any f-curvature is that $c_1 \neq 0$. This is expressed in

Theorem 6.2.1. *Let (X, g) be an asymptotically hyperbolic manifold which is weakly Poincaré-Einstein with conformal infinity $(M, [\hat{g}])$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with the following properties:*

1. *f is symmetric,*
2. *f is $\mathcal{C}^2$,*
3. *f has non-vanishing first-derivative at $(1, \ldots, 1)$.*

Then

1. *If the conformal class $[\hat{g}]$ has non-positive Yamabe invariant, then there exists a unique constant f-curvature foliation of X near M.*
2. *If instead $[\hat{g}]$ has positive Yamabe invariant, then to each constant scalar curvature metric $h_0 \in [\hat{g}]$ for which the linear operator $(\Delta_{h_0} - 2\kappa_2)$ is invertible there exists an associated distinct foliation of constant f-curvature.*

Examining the weakly Poincaré-Einstein case, this linearization is

$$D_{\phi_0}\mathcal{N}(0,0) = -c_1\left(\Delta_{h_0} + \frac{R_{h_0}}{n-1}\right).$$

A minor issue to note is that while in the constant mean curvature case the condition $h_1 = 0$ is stronger than required (h_1 being traceless is enough to use these methods, and in fact with much more trouble even this can be made unnecessary [31]), it becomes necessary (at least via these methods) for more arbitrary f-curvatures.

6.3 A Corollary

Let us briefly examine what may be said when the expansion h_r is neither arbitrary, nor necessarily so well characterized as weakly Poincaré-Einstein. Suppose that $h_1 = 0$, but $h_p \neq 0$ for some $p > 1$ (clearly $h_0 \neq 0$ is by definition always true as it is a Riemannian metric). The methods of chapter 5 require only $h_1 = 0$, not the full strength of the prescribed quadratic term of the weakly Poincaré-Einstein condition. Then we have the following corollary of theorem 6.2.1:

Theorem 6.3.1. *Let (X, g) be an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{g}])$ and geodesic boundary defining function r and metric g such that*

$$g = r^{-2}(dr^2 + h_0 + h_1 r + \cdots + h_p r^p + \cdots)$$

where $h_i = 0$ for $i = 1, \ldots p-1$, but $h_p \neq 0$, and where $p > 1$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with the following properties:

1. *f is symmetric,*

2. *f is $\mathcal{C}^p$,*

3. *f has non-vanishing first-derivative at $(1, \ldots, 1)$.*

Then when $p = 2$:

1. *If the conformal class $[\hat{g}]$ has non-positive generalized boundary Yamabe invariant, then there exists a unique constant f-curvature foliation of X near M.*

2. *If instead $[\hat{g}]$ has positive generalized boundary Yamabe invariant, then to each constant scalar curvature metric $h_0 \in [\hat{g}]$ for which the operator $(\Delta_{h_0} - p\kappa_p)$ is invertible there exists an associated distinct foliation of constant f-curvature.*

When $p > 2$, if the operator $(\Delta_{h_0} - p\kappa_p)$ is invertible, there exists a unique constant f-curvature foliation of X near M.

Proof. To begin, consider the case of $p = 3$ in detail, as the results of chapter 5 already cover $p = 2$. We can continue the expansion 6.1.2 of $g^{-1}II$ by one more term:

$$
\begin{aligned}
g^{-1}II = {} & \delta_i{}^j - \frac{1}{2}(h_1)_i{}^j r - \left((h_2)_i{}^j - \frac{1}{2}(h_1 \cdot h_1)_i{}^j\right) r^2 \\
& - \frac{3}{2}\left((h_3)_i{}^i - (h_1 \cdot h_2)_i{}^i - (h_1 \cdot h_1 \cdot h_1)_i{}^i\right) r^3 + \mathcal{O}(r^4),
\end{aligned}
$$

and then use this to extend the mean curvature expansion, finding κ_3,

$$\begin{aligned}
H &= \operatorname{tr}^h \left(h - \frac{1}{2} r \partial_r h \right)\bigg|_{r=\epsilon} \\
&= n - \left(\frac{1}{2} \operatorname{tr}^{h_0} h_1 \right) \epsilon - \left(\operatorname{tr}^{h_0} h_2 - \|h_1\|^2_{h_0} \right) \epsilon^2 \\
&- \frac{3}{2} \left(\operatorname{tr}^{h_0} h_3 - \langle h_1, h_2 \rangle_{h_0} - p_3(\lambda(h_1)) \right) \epsilon^3 + \mathcal{O}(\epsilon^4) \\
&= n - \kappa_1 \epsilon - \kappa_2 \epsilon^2 - \kappa_3 \epsilon^3 + \mathcal{O}(\epsilon^4).
\end{aligned}$$

Then using this, the f-curvature expansion may also be continued:

$$\begin{aligned}
\frac{d^3}{dr^3} F(A)\bigg|_{r=0} &= \sum_{i,j,k=1}^{n} \frac{\partial^3 f}{\partial \lambda_i \partial \lambda_j \partial \lambda_k}(\lambda_0) \left[\left(-\frac{1}{2} \right)^3 (h_1)_i{}^i (h_1)_j{}^j (h_1)_k{}^k \right] \\
&+ 3 \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}(\lambda_0)(h_1)_i{}^i \left((h_2)_j{}^j - \frac{1}{2}(h_1 \cdot h_1)_j{}^j \right) \\
&- \frac{9}{2} \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i} \left((h_3)_i{}^i - (h_1 \cdot h_2)_i{}^i - (h_1 \cdot h_1 \cdot h_1)_i{}^i \right) \\
&= \frac{c_3}{8} p_3(\lambda(h_1)) + 3 c_2 \nu(h_1, h_2) - 3 c_1 \kappa_3
\end{aligned}$$

where we have labeled the mixed terms $\nu(h_1, h_2)$, and $p_3(\lambda(h_1))$ refers to the degree-3 power sum in the eigenvalues of h_1, and thus

$$\begin{aligned}
F(0, \epsilon) = F(g^{-1} II(0)) &= c_0 - c_1 \kappa_1 \epsilon - \left(c_1 \kappa_2 - \frac{c_2}{8} \sigma_2(\lambda(h_1)) \right) \epsilon^2 \\
&- \left(c_1 \kappa_3 - c_2 \nu(h_1, h_2) - \frac{c_3}{24} p_3(\lambda(h_1)) \right) \epsilon^3 + \mathcal{O}(\epsilon^4),
\end{aligned}$$

By assumption $\kappa_1 = \kappa_2 = 0$. Employing the same methods as before, in this case we are looking to solve

$$F(\phi_0(\epsilon), \epsilon) = c_0 - c_1 \kappa_3 \epsilon^3,$$

or

$$\frac{1}{\epsilon^3}\left(F(\phi_0(\epsilon)) - c_0\right) + c_1\kappa_3 = 0.$$

As κ_3 can be assume constant by lemma 6.3.2, we linearize

$$\overline{\mathcal{N}}(\phi_0, \epsilon) = \frac{1}{\epsilon^3}\left(F(\phi_0(\epsilon)) - c_0\right)$$

to obtain

$$\begin{aligned} D_{\phi_0}\overline{\mathcal{N}}(0,0)(\psi_0) &= \left(-3r^{-3}\left(c_0 - c_1\kappa_3 r^3 + \mathcal{O}(r^4)\right)\psi - c_1\Delta_{\bar{g}}\psi + 3r^{-3}c_0\psi\right)\Big|_{r=0} \\ &= -c_1\left(\Delta_{h_0} - 3\kappa_3\right)\psi_0. \end{aligned}$$

An appeal to *Faà di Bruno's* formula ([7], [21]),

$$\frac{d^n}{dr^n}f(g(r)) = \sum \frac{n!}{m_1!1!^{m_1}m_2!2!^{m_2}\cdots m_n!n!^{m_n}} f^{(m_1+\cdots+m_n)}(g(r)) \prod_{j=1}^{n}(g^{(j)}(r))^{m_j},$$

(the sum is over all n-tuples of non-negative integers satisfying $1m_1+2m_2+\cdots+nm_n = n$) demonstrates that whenever $h_i = 0$ for $i = 1, \ldots p-1$, while $h_p \neq 0$, we will be solving

$$\frac{1}{\epsilon^p}\left(F(\phi_0(\epsilon)) - c_0\right) + c_1\kappa_p = 0$$

for $\epsilon \mapsto \phi_0$. Linearizing as in theorem 6.2.1 and employing the implicit function theorem produces constant f-curvature foliations whenever the linear operator $(\Delta_{h_0} - p\kappa_p)$ is invertible. Clearly in all cases the condition $c_1 = \frac{\partial f}{\partial \lambda_i}(\lambda_0) \neq 0$ is required. $\square$

Finally, as we have always employed the assumption that κ_p can be made constant by a conformal change, let us verify that this assumption is a significant one only when

$p = 2$ (the case already addressed in chapter 5). The manner in which conformal change affects H is quite simple away from order-2. Recall

$$H(\phi_0, \epsilon) = \left(\frac{1}{2}\operatorname{tr}^h(2h - r\partial_r h) - r^2\left(\Delta_{\bar{g}}\phi + \frac{n-1}{2}\|\nabla^{\bar{g}}\phi\|_{\bar{g}}^2\right)\right)\Bigg|_{e^\phi r = \epsilon}. \tag{6.3.1}$$

It is clear from this expression that all of the impact of conformal change is seen in the r^2-coefficient, i.e. κ_2. Indeed, expanding 6.3.1 we see

$$\begin{aligned}
H(\phi_0, \epsilon) = n &- \frac{1}{2}e^{-\phi_0}\operatorname{tr}^{h_0} h_1 \hat{r} \\
&- e^{-2\phi_0}\left(\operatorname{tr}^{h_0} h_2 - \frac{1}{2}\|h_1\|_{h_0}^2 + \Delta_{h_0}\phi_0 + \frac{n-2}{2}\|\nabla^{h_0}\phi_0\|_{h_0}^2\right)\hat{r}^2 \\
&- e^{-3\phi_0}\left(\operatorname{tr}^{h_0} h_3 - \langle h_1, h_2\rangle_{h_0} - p_3(\lambda(h_1))\right)\hat{r}^3 + \mathcal{O}(\hat{r}^4),
\end{aligned}$$

implying that

$$\begin{aligned}
\hat{\kappa}_1 &= e^{-\phi_0}\kappa_1, \\
\hat{\kappa}_2 &= e^{-2\phi_0}\left(\kappa_2 + \Delta_{h_0}\phi_0 + \frac{n-2}{2}\|\nabla^{h_0}\phi_0\|_{h_0}^2\right), \\
\hat{\kappa}_3 &= e^{-3\phi_0}\kappa_3, \\
&\vdots \\
\hat{\kappa}_p &= e^{-p\phi_0}\kappa_p \\
&\vdots
\end{aligned}$$

Therefore only when $p = 2$ does the stipulation that κ_p be made constant require solving a Yamabe-type equation; even in that case, as we have seen (see chapter 5), it may be done. This may be expressed as the

Lemma 6.3.2. *Let (X, g) be an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{g}])$ and r a geodesic boundary defining function. For any fixed r let*

$$H(r) = n - \kappa_1 r - \kappa_2 r^2 - \cdots - \kappa_p r^p - \cdots$$

be the mean curvature of the hypersurface defined by a level set of r. For any p there exists some metric $h_0 \in [\hat{g}]$ such that κ_p is constant. Furthermore, the sign of κ_p is an invariant under conformal change.

6.4 Conclusion

The results of this section exhibit the fullest scope of Weingarten curvature functionals to which this method of implicit function theorem applied to linearized Yamabe-type equation appears to be applicable. Precisely, any Weingarten curvature functional will by definition be a symmetric function of the principal curvatures, and it is clear from the proofs above that the condition of non-zero first derivative is always necessary in employing this method.

These results link the existence of hypersurfaces of constant f-curvature for a very extensive class of Weingarten functionals f to constant scalar curvature metrics in the conformal class at infinity. An intriguing application of this would be to use an appropriate translation of the results in [8] and [3] to then tie such hypersurfaces back to prescribed curvatures of other types on the boundary. Until those results can be extended to more general spaces (beyond completely hyperbolic) this naturally restricts the setting for such applications, however, it provides the potential to link Yamabe-like semilinear elliptic equations on the boundary at infinity with a potential variety of other sorts of differential equations on this same manifold.

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Appendix A

Calculus in Banach Spaces

We collect here some essential elements of calculus on Banach spaces. Only needed definitions and theorems will be included; the presentation here largely follows that of [34] with the exception of the use of the uniform contraction principle. The primary purpose of this appendix is to provide a precise statement and proof of the implicit function theorem for Banach spaces. For a stunningly thorough examination of the history, underlying concepts, and the many generalizations and applications of this powerful tool, the reader is recommended to see [24].

Definition A.0.1 (Fréchet Derivative)**.** Let X and Y be Banach spaces and $\mathcal{L}(X, Y) \subset \mathcal{C}(X, Y)$ denote the set of bounded linear functions from X to Y (a subset of continuous functions from X to Y). Let U be an open subset of X. Then a function $F : U \to Y$ is called (Fréchet) differentiable at $x \in U$ if there exists a linear function

$DF(x) \in \mathcal{L}(X, Y)$ such that

$$F(x+u) = F(x) + DF(x)u + \mathfrak{o}(u).$$

The map $DF(x)$ is called the (Fréchet) derivative of F at x. If F is differentiable at all $x \in U$ then it is called (Fréchet) differentiable and

$$DF : U \to \mathcal{L}(X, Y), \qquad x \mapsto DF(x)$$

If DF is continuous we write $F \in \mathcal{C}^1(U, Y)$

(Note that the Landau symbols $\mathfrak{o}$ and $\mathcal{O}$ will always be taken to include the application of appropriate norms.) As with all derivatives there are many equivalent notations. For convenience, we will often write the Fréchet derivative of the map F, at x, applied to u, as $D_x F(u)$ instead of $DF(x) \cdot u$. Establishing the implicit function theorem in Banach spaces will require, as expected from the finite dimensional case, a contraction mapping principle, but first we verify that the familiar chain rule still holds in the infinite dimensional case. This wholly norm-based proof requires no special ingredients different from those of introductory calculus.

Theorem A.0.2 (Chain Rule)**.** *Let $F \in \mathcal{C}^p(X, Y)$ and $G \in \mathcal{C}^p(Y, Z)$ for Banach spaces X, Y and Z, $p \geq 1$. Then $G \circ F \in \mathcal{C}^p(X, Z)$ and*

$$D(G \circ F) = DG(F(x)) \circ DF(x), \qquad \forall x \in X.$$

Proof. Let $x, u \in X$ and $y = F(x) \in Y$. Then

$$\begin{aligned}
(G \circ F)(x+u) - (G \circ F)(x) &= G(F(x+u)) - G(F(x)) \\
&= G(F(x+u) - F(x) + F(x)) - G(y) \\
&= G(v+y) - G(y),
\end{aligned}$$

where $v = F(x+u) - F(x)$. Therefore

$$\|(G \circ F)(x+u) - (G \circ F)(x) - DG(y) \cdot v\| = \mathfrak{o}(\|v\|).$$

As $\|v - DF(x)u\| = \mathfrak{o}(\|u\|)$, we have

$$\begin{aligned}
&\|(G \circ F)(x+u) - (G \circ F)(x) - DG(y) \cdot DF(x) \cdot u\| \\
&= \|(G \circ F)(x+u) - (G \circ F)(x) - DG(y) \cdot v + DG(y) \cdot v - DG(y) \cdot DF(x) \cdot u\| \\
&= \mathfrak{o}(\|v\|) + \mathfrak{o}(\|u\|).
\end{aligned}$$

Then as F is continuous at x, $\|v\| = \mathfrak{o}(\|u\|)$, so

$$D(G \circ F) \cdot u = DG(F(x)) \cdot DF(x) \cdot u$$

as desired. □

Definition A.0.3 (Contraction)**.** A fixed point of a mapping $F : C \subset X \to C$ is an elements $x \in C$ such that $F(x) = x$. The mapping F is called a *contraction* if there exists a (contraction) constant $\theta \in [0, 1)$ such that

$$|F(x) - F(\bar{x})| \leq \theta |x - \bar{x}|, \qquad \forall x, \bar{x} \in C.$$

Clearly a contraction is continuous. We denote repeated applications of the mapping F by: $F^n(x) = F(F^{n-1}(x)), F^0(x) = x$.

Theorem A.0.4 (Contraction Principle). *Let X be a Banach space, $C \subset X$ a closed subset, and $F : C \to C$ be a contraction, then F has a unique fixed point $\bar{x} \in C$ such that*

$$|F^n(x) - \bar{x}| \leq \frac{\theta^n}{1-\theta}|F(x) - x|, \qquad x \in C. \tag{A.0.1}$$

Proof. Fix $x_0 \in C$ and let $x_n = F^n(x_0)$. Then

$$|x_{n+1} - n_n| \leq \theta|x_n - x_{n-1}| \leq \cdots \leq \theta^n|x_1 - x_0|,$$

and thus by the triangle inequality (for $n > m$)

$$|x_n - x_m| \leq \sum_{i=M+1}^{n} |x_i - X_{i-1}| \leq \theta^m \sum_{i=0}^{n-m-1} \theta^i|x_1 - x_0| \leq \frac{\theta^m}{1-\theta}|x_1 - x_0|. \tag{A.0.2}$$

Therefore x_n is Cauchy and has limit $\bar{x}$. Furthermore,

$$|F(\bar{x}) - \bar{x}| = \lim_{n\to\infty} |x_{x+1} - x_n| = 0$$

implies that $\bar{x}$ is a fixed point of F and A.0.1 follows after letting $n \to \infty$ in A.0.2.

The uniqueness argument goes as follows: if $x = F(x)$ and $\bar{x} = F(\bar{x})$, then $|x - \bar{x}| = |F(x) - F(\bar{x})| \leq \theta|x - \bar{x}|$, but as $\theta < 1$, $|x - \bar{x}| = 0$. □

To apply the contraction principle to proving an implicit function theorem for Banach spaces we will need to verify that fixed points of parameterized contractions vary with regularity. To this end we define

Definition A.0.5 (Uniform Contraction)**.** If $U \subset X$ and $V \subset Y$ are open subsets of the Banach spaces X and Y and $F : \overline{U} \times V \to U$ a mapping such that

$$|F(x,y) - F(\bar{x},y)| \leq \theta |x - \bar{x}|, \qquad \forall x, \bar{x} \in \overline{U}, y \in V \tag{A.0.3}$$

for some $\theta \in [0,1)$, then F is called a *uniform contraction.*

Then we must extend the contraction principle to such maps:

Theorem A.0.6 (Uniform Contraction Principle)**.** *Let X, Y be Banach spaces, $U \subset X$, $V \subset Y$ be open subsets, and $F : \overline{U} \times V \to U$ be a uniform contraction with unique fixed point $\bar{x}(y) \in U$ of the mapping $F(\cdot, y)$. If $F \in \mathcal{C}^p(U \times V, U)$, $p \geq 1$, then $\bar{x}(\cdot) \in \mathcal{C}^p(V, U)$.*

Proof. We begin by showing that $\bar{x}(\cdot)$ is continuous. Using A.0.3 we have

$$\begin{aligned}
|\bar{x}(y+v) - \bar{x}(y)| &= |F(\bar{x}(y+v), y+v) - F(\bar{x}(y), y+v) \\
&\quad + F(\bar{x}(y), y+v) - F(\bar{X}(y), y)| \\
&\leq \theta |\bar{x}(y+v) - \bar{x}(y)| + |F(\bar{x}(y), y+v) - F(\bar{x}(y), y)|
\end{aligned}$$

and thus

$$|\bar{x}(y+v) - \bar{x}(y)| \leq \frac{1}{1-\theta} |F(\bar{x}(y), y+v) - F(\bar{x}(y), y)|. \tag{A.0.4}$$

Therefore $\bar{x}(\cdot) \in \mathcal{C}(V, U)$.

Now suppose that $p = 1$ and formally differentiate the fixed point equation $\bar{x}(y) = F(\bar{x}(y), y)$ with respect to y:

$$D\bar{x}(y) = D_x F(\bar{x}(y), y) D\bar{x} + D_y F(\bar{x}(y), y). \tag{A.0.5}$$

Then looking at A.0.5 as a fixed point equation of the form $x' = G(x', y)$, with

$$G(\cdot, y) : \mathcal{L}(Y, X) \to \mathcal{L}(Y, X),$$

$$x' \to D_x F(\bar{x}(y), y)x' + D_y F(\bar{x}(y), y),$$

we see that G is a uniform contraction because $|D_x F(\bar{x}(y), y)| \leq \theta$ by Theorem **??**. Therefore we have a unique continuous solution $\bar{x}'(y)$.

We still must verify that $\bar{x}'$ is actually a Fréchet derivative, i.e.

$$\bar{x}(y + v) - \bar{x}(y) - \bar{x}'(y)v = \mathfrak{o}(v).$$

Call $u = \bar{x}(y + v) - \bar{x}(y)$. Then using A.0.5 and the fact that $\bar{x}(y)$ is a fixed point we get

$$\begin{aligned}
&(1 - D_x F(\bar{x}(y), y))(u - \bar{x}'(y)) \\
&= F(\bar{x}(y) + u, y + v) - F(\bar{x}(y), y) - D_x F(\bar{x}(y), y)u - D_y F(\bar{x}(y), y)v \\
&= \mathfrak{o}(u) + \mathfrak{o}(v)
\end{aligned}$$

as $F \in \mathcal{C}^1(U \times V, U)$ by assumption. As we have just remarked, $|D_x F(\bar{x}(y), y)| \leq \theta$, thus

$$\left|\frac{1}{1 - D_x F(\bar{x}(y), y)}\right| \leq \left|\frac{1}{1 - \theta}\right|$$

and by A.0.4 $u = \mathcal{O}(v)$. Therefore $u - \bar{x}'(y)v = \mathfrak{o}(v)$, so $\bar{x}'$ is the derivative as desired.

Higher degrees of regularity result from induction: Suppose that $D\bar{x}(y) \in \mathcal{C}^{p-1}$ holds for $p - 1 \geq 1$ when $F \in \mathcal{C}^p$, then $\bar{x}(y) \in \mathcal{C}^{p-1}$ and the fact that $D\bar{x}$ satisfies A.0.5 means that actually $\bar{x}(y) \in \mathcal{C}^p(V, U)$. $\square$

Finally we can state and prove our goal, the

Theorem A.0.7 (Implicit Function Theorem)**.** *Let X, Y, Z be Banach spaces and $U \subset X$, $V \subset Y$ be open. Let $F \in \mathcal{C}^p(U \times V, Z)$, $p \geq 1$, and fix $(x_0, y_0) \in U \times V$. Suppose $D_x F(x_0, y_0) \in \mathcal{L}(X, Z)$ is an isomorphism. Then there exists an open neighborhood $U_0 \times V_0 \subset U \times V$ of (x_0, y_0) such that for each $y \in V_0$ there exists a unique point $(u(y), y) \in U_0 \times V_0$ satisfying $F(u(y), y) = F(x_0, y_0)$, and the map u is in $\mathcal{C}^p(V_0, Z)$. Furthermore, u fulfills*

$$Du(y) = -(D_x F(u(y), y))^{-1} \circ D_y F(u(y), y). \tag{A.0.6}$$

Proof. First note that the translation $F - F(x_0, y_0) \to F$ means we may assume that $F(x_0, y_0) = 0$. We construct a function with fixed points corresponding to solutions of $F(x, y) = 0$; let

$$G(x, y) = x - (D_x F(x_0, y_0))^{-1} \circ F(x, y).$$

Now G has the same smoothness as F, and as $\|D_x G(x_0, y_0)\| = 0$, we can find neighborhoods U_0 and V_0 around x_0 and y_0 such that for $(x, y) \in U_0 \times V_0$ we have $\|D_x G(x, y)\| \leq \theta < 1$, that is $G : U_0 \times V_0 \to U_0$ is a uniform contraction. The result then follows from the uniform contraction principle A.0.6.

The formula A.0.6 follows from the chain rule by differentiating $F(u(y), y) = 0$. $\square$

Appendix B

Formulas Associated with Conformal Change

Let g be a Riemannian metric on a smooth manifold M and ϕ a smooth real-valued function on M. Then

$$\hat{g} = e^{2\phi} g$$

is also a Riemannian metric on M. We say that $\hat{g}$ is *conformal* to g, and as conformality is evidently an equivalence relation on metrics, a conformal structure on a smooth manifold is an equivalence class of smooth metrics on this manifold $[g] = \{e^{2\phi} g \ : \ \phi \in \mathcal{C}^{\infty}(M)\}$. For convenience we list here equalities expressing the manner in which several tensors associated with a metric transform under conformal change, all are simply the result of calculation from the definitions. Quantities associated to the metric $\hat{g}$ are denoted by a $\widehat{\ }$ above their symbol.

FIrst, the metric and its Christoffel symbols (N.B. the Christoffel symbols are not actually proper tensors.):

$$\hat{g}_{ij} = e^{2\phi} g_{ij}$$

$$\widehat{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \partial_j \phi + \delta^k_j \partial_i \phi - g_{ij} \nabla^k \phi$$

The volume element and Laplacian:

$$d\widehat{V} = e^{n\phi} dV$$

$$\Delta f = \nabla^i \partial_i f$$

Note the sign of the Laplacian here.

$$\widehat{\Delta} f = e^{-2\phi} \left(\Delta f + (n-2) \nabla^k \phi \nabla_k f \right)$$

The curvature tensors (Riemann, Ricci, and scalar):

$$\widehat{R}_{ijkl} = e^{2\phi} \left(R_{ijkl} - \left[g \oslash \left(\nabla \partial \phi - \partial \phi \partial \phi + \frac{1}{2} \|\nabla \phi\|^2 g \right) \right]_{ijkl} \right)$$

$$\widehat{R}_{ij} = R_{ij} - (n-2) \left[\nabla_i \partial_j \phi - (\partial_i \phi)(\partial_j \phi) \right] + \left(\Delta \phi - (n-2) \|\nabla \phi\|^2 \right) g_{ij}$$

$$\widehat{R} = e^{-2\phi} \left(R + 2(n-1) \Delta \phi - (n-2)(n-1) \|\nabla \phi\|^2 \right)$$