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Aggregation Theory for Incomplete Systems

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Abstract

Gorman's theory of demand is extended comprehensively to incomplete systems. The incomplete systems approach dramatically increases this class of models. The separate roles of symmetry and adding up are identified in the rank and the functional form of this class of models. We show that symmetry determines rank and the maximum rank is three. We show that adding up and 0º homogeneity determines the functional form and there is no functional form restriction for an incomplete system. We prove that every full rank system and reduced rank systems with a minimal level of degeneracy can be written as a polynomial in a single function of income. A complete set of closed form solutions for the indirect objective functions of this class of models is derived. A simple method to nest rank and functional form for incomplete systems is presented.

Key Words: Aggregation, rank, functional form, integrability, incomplete systems, weak integrability

JEL Classification: D12, E21

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1. Introduction

Terence Gorman's legacy includes his seminal contributions to demand theory (Gorman 1953, 1961, 1981). Gorman (1953) first derived the necessary and sufficient conditions for the existence of a representative consumer. He then obtained the indirect preference functions for this class of demand models (Gorman 1961), since known as the *Gorman polar form*. Muellbauer (1975, 1976) extended this to include a nonlinear function of income, obtaining the *price independent generalized linear* (PIGL) and *price independent generalized logarithmic* (PIGLOG) systems. Gorman (1981) soon extended these results greatly by deriving the class of all complete demand systems that can be written as a finite sum of additive functions of nominal income, with each function multiplied by a vector of price functions. Gorman's work forms the foundation of a large and important literature in demand theory (Deaton and Muellbauer 1980; Jerison 1993; Lewbel 1987, 1988, 1989, 1990; Muellbauer 1975, 1976; Russell 1983, 1996; Russell and Farris 1993, 1998; and van Daal and Merkies 1989).

We extend this theory to incomplete demand systems. The incomplete systems approach has enormous potential to expand how we think about and model economic behavior. Here we show that it dramatically increases the set of rational economic systems without introducing *ad hoc* structure on the demands of goods that are not modeled. Our approach to extending Gorman systems also allows us to isolate the role of symmetry from the joint forces of homogeneity and adding up in determining the rank and functional form of any Gorman system of demand equations.

Many consumption and production models are among those we study. Common consumption models included are homothetic systems, the Gorman polar form, linear expenditure system (LES), quadratic expenditure system (QES), price independent generalized linear (PIGL) and price independent generalized logarithmic (PIGLOG) systems, Almost Ideal Demand System (AIDS), quadratic utility, and translog preferences. Moreover, competitive cost minimization and consumer expenditure models are formally equivalent. Common production models that are Gorman systems include the Cobb-Douglas, constant elasticity of substitution production and cost functions, quadratic production functions, normalized quadratic profit function, generalized Leontief production and cost functions, and translog direct and indirect production, cost, and profit functions. As a result, whether one is interested in aggregation or not, Gorman demand systems dominate empirical applications throughout economics.

To be consistent with the existing literature, most importantly the seminal work of Gorman, we focus our discussion on the theory of consumer choice. But this does not limit the generality of the results or the breadth of their implications. The theory developed here encompasses generalizations of all of the above models to full rank three systems, a generic class of higher order reduced rank systems, and systems with the same mathematical structure but an expansive class of functional forms beyond the small group previously known to generate coherent economic systems. The definition of an incomplete system of demand equations developed here encompasses all complete systems as a special case. Hence, we clarify several aspects of the structure of complete systems.

The rest of the paper is organized in the following way. Section two reviews, synthesizes, and extends the existing literature on complete Gorman systems. In this section, we clarify several important aspects of complete Gorman systems. We show that symmetry determines the rank of complete Gorman systems, while homogeneity and adding up determine the functional form of the income terms. We close a gap in the known solutions for the indirect preferences of Gorman systems and find a unifying expression for all of these solutions.

Section three reviews the subject of incomplete demand systems and defines an incomplete Gorman system. The definition we offer provides the greatest flexibility for the rank and functional form of the goods that are and that are not included in a demand model. We show that every full rank Gorman system or member of a generic class of reduced rank systems can be represented as a polynomial in a single function of income. We construct a complete taxonomy of closed form solutions for the indirect objective functions of every member of this class of demand models. We develop a method to nest the rank and form of many systems in this class.

The last section summarizes our results and briefly discusses some of the implications of these results. The Appendix contains a detailed set of derivations and proofs.

2. Complete Systems

This section reviews and extends the literature on complete Gorman systems. In this discussion, we exploit, combine and synthesize several very different approaches. We make use of a small number of arguments from differential geometry and the theory of Lie algebras on the real line (Hermann 1975). We also make use of Muellbauer (1975, 1976), Gorman (1981), van Daal and Merkies (1989), Lewbel (1987, 1989, 1990), and Russell and Farris (1993, 1998). We clarify several aspects of complete Gorman systems. We elucidate the different roles that Slutsky symmetry and the joint forces of 0° homogeneity and adding up play in the rank of the demand system and the functional form of the income terms in a Gorman system of demand equations. We also close an important gap in the set of solutions for the indirect preferences of complete Gorman systems and find a unifying expression for all of these solutions.

We begin with a few definitions and some notation. Let $\boldsymbol{P} \in \boldsymbol{\mathcal{P}} \subset \mathbb{R}^n_+$ be the vector of market prices for the consumption goods $\boldsymbol{q} \in \boldsymbol{\mathcal{Q}} \subset \mathbb{R}^n_+$, let $M \in \boldsymbol{\mathcal{M}} \subset \mathbb{R}_+$ be total expenditure on consumption goods, and let the consumer's utility function be $u(\boldsymbol{q})$, where $u : \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{U}} \subset \mathbb{R}$ is smooth ($u \in \boldsymbol{\mathcal{C}}^{\infty}$), increasing, and strictly quasiconcave on $\boldsymbol{\mathcal{Q}}$. We abuse language somewhat and use the sobriquet *income* to denote M throughout. Define the *nominal expenditure function* by

$$E(\boldsymbol{P}, u) \equiv \min\left\{ \boldsymbol{P}^{\intercal}\boldsymbol{q} : u(\boldsymbol{q}) \geq u \right\}. \tag{1}$$

We assume $E : \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{M}}$ is smooth ($E \in \boldsymbol{\mathcal{C}}^{\infty}$), increasing, 1° homogeneous, and concave in $\boldsymbol{P}$, and increasing in u. We also assume an interior solution for $\boldsymbol{q}$. Thus, symmetry (integrability) is the main mathematical property of interest. By duality theory, the demands for the goods $\boldsymbol{q}$ can be obtained by Hotelling's/Shephard's lemma,

$$\boldsymbol{q} = \partial E(\boldsymbol{P}, u) \big/ \partial \boldsymbol{P} = \boldsymbol{h}(\boldsymbol{P}, E(\boldsymbol{P}, u)). \tag{2}$$

An important fact is that many commonly used demand models for both production and consumption analyses, including all of the systems discussed in the introduction, can be written in the class of systems analyzed by Gorman (1981). This class of models can be expressed in terms of an additive and multiplicatively separable system of partial differential equations,

$$\boldsymbol{q} = \frac{\partial E(\boldsymbol{P}, u)}{\partial \boldsymbol{P}} = \sum_{k=1}^{K} \boldsymbol{\beta}_k(\boldsymbol{P}) H_k\left(E(\boldsymbol{P}, u)\right), \tag{3}$$

where $\boldsymbol{\beta}_k : \boldsymbol{\mathcal{P}} \to \mathbb{R}^n$ and $H_k : \boldsymbol{\mathcal{M}} \to \mathbb{R}$, $k = 1, \cdots, K$ are smooth functions of prices and income, respectively.

There are many reasons to consider demand systems in this class. Certainly, the most common one relates to the consistent aggregation across the incomes of individual consumers to market-level demands. Let the density function for the income distribution be $\varphi : \boldsymbol{\mathcal{M}} \to \mathbb{R}_+$. Then (3) implies that we only need to calculate a total of K moments of the form $\int_{\boldsymbol{\mathcal{M}}} H_k(x)\varphi(x)dx$ to obtain aggregate demands with average quantities purchased as the dependent variables.

However, there is a second reason, which is at least as important and substantially less well-understood, to seriously consider this class of demand systems. Estimating any complete system with expenditure, cost, profit, or output as an explanatory variable creates the potential for simultaneous equations bias. One natural way to address this issue in the econometric analysis of complete systems leads to Gorman's class of models.

Demand equations are not typically estimated in the form given in equation (3) because quantities, prices, and income are often not the variables of interest or the form in which the data is available. It is common for demand models to be estimated with expenditures or budget shares on the left and either log-prices and log-income, nominal prices and nominal income, or real prices and real income on the right. Moreover, in the existing theoretical work on demand systems, Gorman (1981) works with log-prices and log-income on the right and budget shares on the left, Lewbel (1987, 1989, 1990) and van Daal and Merkies work directly with (3), Howe, Pollak, and Wales (1989) write the QES in terms of expenditures on the left and nominal prices and income on the right, while Russell (1996) and Russell and Farris (1993, 1998) employ methods that are independent of the coordinate space to represent the influence of prices and income.

To consider all of these possible cases in a single framework, we require what at first blush seems to be an alternate definition of the demand system. Let $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{P})$, $\boldsymbol{g} : \mathscr{P} \to \mathscr{X} \subset \mathbb{R}^n$, $g_i \in \mathscr{C}^\infty$, $i = 1,\cdots,n$, and $\left|\partial \boldsymbol{g}(\boldsymbol{P})^\intercal \big/ \partial \boldsymbol{P}\right| \neq 0 \,\forall\, \boldsymbol{P} \in \mathscr{P}$, transform $\boldsymbol{P}$ to $\boldsymbol{x}$. Let $y = f(M)$, $f : \mathscr{M} \to \mathscr{Y} \subset \mathbb{R}$, $f \in \mathscr{C}^\infty$, and $f'(M) > 0 \,\forall\, M \in \mathscr{M}$, transform M to y. To simplify notation, denote the inverse of $\boldsymbol{g}$ as $\boldsymbol{P}(\boldsymbol{x})$ and the inverse of f as $M(y)$. Instead of (3), we can write a transformed demand system in terms of $\boldsymbol{x}$ and y as

$$\frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} \equiv f'\left(M\left(y(\boldsymbol{x},u)\right)\right)\frac{\partial \boldsymbol{P}(\boldsymbol{x})^\intercal}{\partial \boldsymbol{x}}\boldsymbol{h}\left(\boldsymbol{P}(\boldsymbol{x}), M\left(y(\boldsymbol{x},u)\right)\right) \equiv \tilde{\boldsymbol{h}}\left(\boldsymbol{x}, y\right). \tag{4}$$

Indeed, econometricians often model consumption or production decisions with the vector $\partial y \big/ \partial \boldsymbol{x}$ as functions of transformed prices $\boldsymbol{x}$ and transformed income y, rather than with quantities $\boldsymbol{q}$ as functions of nominal prices $\boldsymbol{P}$ and income M. For example, if $\boldsymbol{x} = \ln \boldsymbol{P}$ and $y = \ln M$, then $\partial y \big/ \partial x_i = P_i q_i \big/ M = w_i$ is the budget share for the i^{th} good. The right-hand-side is then naturally expressed in terms of log-prices and log-income. This is the form that the AIDS, PIGLOG, and translog systems take. On the other hand, if $y = M$ and $\boldsymbol{x} = \ln \boldsymbol{P}$, then $\partial y \big/ \partial x_i = P_i q_i = e_i$ is the expenditure on the i^{th} good. The right-hand side of (4) in this case is typically written in terms of nominal or real prices and income. This is how Bergson/Cobb-Douglas, LES, Gorman polar forms like the quadratic utility and Rotterdam models, QES, generalized Leontief, normalized quadratic, and PIGL systems are usually estimated.

It also is common practice to account for the joint determination of quantities and total expenditure by taking the conditional expectation of y and $\partial y/\partial \boldsymbol{x}$ with respect to the exogenous variables (wealth, market prices, asset returns, quasi-fixed inputs, general economy variables, demographic variables, etc.). We are led to the natural question, "Which empirical demand systems are consistent with the budget identity when expenditure, cost, or profit is endogenous or measured with error and is included as an explanatory variable in an econometric model?" Stated differently, we would like to understand the demand systems that are consistent with optimization theory and an empirical specification that matches standard empirical practice.

In this context, the primary reason to derive the variables $(\boldsymbol{x}, y)$ is to use them in place of $(\boldsymbol{q}, \boldsymbol{P}, M)$, to estimate the demands. Empirical models are usually specified as a system of $n+1$ simultaneous equations of the general form

$$\partial y/\partial \boldsymbol{x} = \tilde{\boldsymbol{h}}(\boldsymbol{x}, E(y)) + \boldsymbol{\varepsilon},\ E(\boldsymbol{\varepsilon}) = \mathbf{0}\,, \tag{5}$$

$$y = E(y) + \nu,\ E(\nu) = 0\,. \tag{6}$$

A standard interpretation is that the expectations on the right-hand-side are conditional on all available exogenous or otherwise predetermined variables. For the purposes of this discussion, it is unnecessary to impose a specific structure on the joint distribution of the random variables $(\boldsymbol{\varepsilon}, \nu)$. Nothing in (4) – (6) implies a singular distribution for $\boldsymbol{\varepsilon}$, or that $\boldsymbol{\varepsilon}$ is either stochastically independent of or jointly determined with ν. All that we require is that both are stochastic, and because they are defined as differences between observed and expected values, both have vanishing means.

However, econometric estimation of most demand systems is typically carried out using the *conditional demands*,

$$\partial y/\partial \boldsymbol{x} = \tilde{\boldsymbol{h}}(\boldsymbol{x}, y) + \tilde{\boldsymbol{\varepsilon}}\,, \tag{7}$$

because the expected or theoretical value of y is not observable. By definition, the *conditional errors*, $\tilde{\boldsymbol{\varepsilon}}$, therefore must satisfy the identity

$$\tilde{\boldsymbol{\varepsilon}} \equiv \boldsymbol{\varepsilon} + \tilde{\boldsymbol{h}}(\boldsymbol{x}, E(y)) - \tilde{\boldsymbol{h}}(\boldsymbol{x}, E(y) + \nu)\,. \tag{8}$$

Now we come to the econometric issue at hand. What role does adding up play in the coherent econometric analysis of a conditional demand system? Because we construct

$(\boldsymbol{x}, y)$ from $(\boldsymbol{q}, \boldsymbol{P}, m)$, we have

$$f'(M)\boldsymbol{q} = \frac{\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}}{\partial \boldsymbol{P}}\left[\tilde{\boldsymbol{h}}(\boldsymbol{g}(\boldsymbol{P}), E(y)) + \varepsilon\right] = \frac{\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}}{\partial \boldsymbol{P}}\left[\tilde{\boldsymbol{h}}(\boldsymbol{g}(\boldsymbol{P}), E(y)+\nu) + \tilde{\varepsilon}\right]. \tag{9}$$

Applying adding up therefore implies

$$Mf'(M) = \boldsymbol{P}^{\mathsf{T}}\frac{\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}}{\partial \boldsymbol{P}}\left[\tilde{\boldsymbol{h}}(\boldsymbol{g}(\boldsymbol{P}), E(y)) + \varepsilon\right] = \boldsymbol{P}^{\mathsf{T}}\frac{\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}}{\partial \boldsymbol{P}}\left[\tilde{\boldsymbol{h}}(\boldsymbol{g}(\boldsymbol{P}), E(y)+\nu) + \tilde{\varepsilon}\right]. \tag{10}$$

Since the variable on the left only depends on M, so must the variable on the right and in the middle. On the other hand, the variable on the right only depends on the observable value of $y = f(M)$. Therefore so must those in the middle and on the left. First, this implies the well-known fact that $\boldsymbol{P}^{\mathsf{T}}\left[\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}/\partial \boldsymbol{P}\right]\tilde{\varepsilon} = 0$ so that the joint distribution of the conditional errors is singular. Second, it implies the identity

$$\boldsymbol{P}^{\mathsf{T}}\frac{\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}}{\partial \boldsymbol{P}}\varepsilon = \boldsymbol{P}^{\mathsf{T}}\frac{\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}}{\partial \boldsymbol{P}}\tilde{\boldsymbol{h}}(\boldsymbol{g}(\boldsymbol{P}), E(y)+\nu) - \boldsymbol{P}^{\mathsf{T}}\frac{\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}}{\partial \boldsymbol{P}}\tilde{\boldsymbol{h}}(\boldsymbol{g}(\boldsymbol{P}), E(y)). \tag{11}$$

This demonstrates the fundamental issue. If $E(\varepsilon) = \boldsymbol{0}$ and $E(\nu) = 0$, both of which must be true by construction, then the only way that the conditional demands can be consistent with economic theory and adding up is when

$$Mf'(M) = \boldsymbol{P}^{\mathsf{T}}\frac{\partial \boldsymbol{g}(\boldsymbol{P})^{\mathsf{T}}}{\partial \boldsymbol{P}}\tilde{\boldsymbol{h}}(\boldsymbol{g}(\boldsymbol{P}), f(M)) = a + bf(M), \tag{12}$$

for some absolute constants a and b.

The question is, what class of conditional demand models satisfies this condition? A very large number of existing, well-known, and commonly used demand systems are members of this class, including all of those mentioned previously. However, the solution to (12) implies a restriction on the functional form of the demand system to one of the cases identified by Gorman. That is, complete Gorman systems comprise an exhaustive set of demand models in which adding up and a function of income that is endogenous, measured with error, or both leads to a tight econometric specification. This statement will be seen to follow directly from lemma 2 below. This motivates a deeper understanding of Gorman systems of demand equations beyond aggregation. Indeed, this class of models is related to the coherent specification and estimation of any demand system.

We return now to the main discussion. Rather than (3), we could write a transformed system of additive and multiplicatively separable demand equations in terms of $\boldsymbol{x}$ and y as

$$\frac{\partial y(\boldsymbol{x}, u)}{\partial \boldsymbol{x}} = \sum_{k=1}^{K}\boldsymbol{\alpha}_k(\boldsymbol{x})h_k\left(y(\boldsymbol{x}, u)\right). \tag{13}$$

where $\boldsymbol{\alpha}_k : \boldsymbol{\mathcal{X}} \to \mathbb{R}^n$ and $h_k : \boldsymbol{\mathcal{Y}} \to \mathbb{R}$, $k = 1,\cdots,K$, are smooth functions of the transformed prices and income, respectively. However, note that the two definitions are completely equivalent. The reason is that, by construction, $y(\boldsymbol{x},u) \equiv f\,(E(\boldsymbol{P}(\boldsymbol{x}),u))$, so that

$$\begin{aligned}\frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} &= f'\,(E(\boldsymbol{P}(\boldsymbol{x}),u))\frac{\partial \boldsymbol{P}(\boldsymbol{x})^{\intercal}}{\partial \boldsymbol{x}}\frac{\partial E(\boldsymbol{P}(\boldsymbol{x}),u)}{\partial \boldsymbol{P}} \\ &= \sum_{k=1}^{K}\frac{\partial \boldsymbol{P}(\boldsymbol{x})^{\intercal}}{\partial \boldsymbol{x}}\boldsymbol{\beta}_k\,(\boldsymbol{P}(\boldsymbol{x}))\frac{H_k\,(M\,(y(\boldsymbol{x},u)))}{M'\,(M\,(y(\boldsymbol{x},u)))} \\ &\equiv \sum_{k=1}^{K}\boldsymbol{\alpha}_k(\boldsymbol{x})h_k\,(y(\boldsymbol{x},u))\,. \end{aligned} \tag{14}$$

Each step is reversible. In other words, the functional separability of E from $\boldsymbol{P}$ in (3) is equivalent to functional separability of y from $\boldsymbol{x}$ in the transformed system (13). Hereafter we will call any system of partial differential equations with a finite number of additive terms, each of which is multiplicatively separable between $\boldsymbol{x}$ and y, a *Gorman system*. The upshot is that the additive and multiplicatively separable structure of all Gorman systems is independent of the coordinate space that we might choose to reflect how prices and income affect quantities.

The following simple and intuitively appealing lemma proves to be very useful in the arguments presented below. This result lets us move from one representation of $(y,\boldsymbol{x})$ to another without the need to reconsider the implication for integrability. Specifically, the lemma shows that symmetry is (trivially) independent of coordinates. In contrast, some additional structure is needed to maintain curvature after a change in coordinates.

Lemma 1. *Let* $E : \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{M}}$ *be twice differentiable on* $\boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{U}}$ *, let* $y = f(E)$, $f \in \boldsymbol{\mathcal{C}}^2$, $f' > 0\ \forall\ M \in \boldsymbol{\mathcal{M}}$ *, and let* $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{P})$ *,* $\boldsymbol{g} \in \boldsymbol{\mathcal{C}}^2\ \forall\ \boldsymbol{p} \in \boldsymbol{\mathcal{P}}$ *, with* $E(\boldsymbol{P},u) = M\,(y(\boldsymbol{g}(\boldsymbol{P}),u))$*. Then* $\partial^2 E(\boldsymbol{P},u)\big/\partial\boldsymbol{P}\partial\boldsymbol{P}^{\intercal}$ *is symmetric at* $(\boldsymbol{P},u)$ *if and only if* $\partial^2 y(\boldsymbol{g}(\boldsymbol{P}),u)\big/\partial\boldsymbol{x}\partial\boldsymbol{x}^{\intercal}$ *is symmetric at* $(\boldsymbol{g}(\boldsymbol{P}),u)$*. In addition, if* $x_i = g_i(P_i)$ *,* $g_i \in \boldsymbol{\mathcal{C}}^2$ *,* $g_i' > 0$ *,* $g_i'' \le 0, \forall\, p_i \in \boldsymbol{\mathcal{P}}_i \subset \mathbb{R}_+, \forall\, i$ *,* $M'' \le 0\ \forall\ y \in \boldsymbol{\mathcal{Y}}$ *, and y is concave in* $\boldsymbol{x}$*, then E is concave in* $\boldsymbol{P}$*.*

Gorman (1981) showed that all complete systems of the form (13) have a rank of $\boldsymbol{A}(\boldsymbol{x})$ that is at most three, and if the rank is three then the system must take one of the following three functional forms:

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{x})M + \sum_{k=1}^{K}\boldsymbol{\alpha}_k(\boldsymbol{x})M(\ln M)^k\,; \tag{15}$$

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{x})M + \sum_{\kappa \in \boldsymbol{S}} \boldsymbol{\beta}_\kappa(\boldsymbol{x})M^{1-\kappa} + \sum_{\kappa \in \boldsymbol{S}} \boldsymbol{\gamma}_\kappa(\boldsymbol{x})M^{1+\kappa}\,, \tag{16}$$

for $\boldsymbol{S}$ a set of nonzero constants; or

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{x})M + \sum_{\tau \in \boldsymbol{\mathcal{T}}} \boldsymbol{\beta}_\tau(\boldsymbol{x})M \sin(\tau \ln M) + \sum_{\tau \in \boldsymbol{\mathcal{T}}} \boldsymbol{\gamma}_\tau(\boldsymbol{x})M \cos(\tau \ln M)\,, \tag{17}$$

for $\boldsymbol{\mathcal{T}}$ a set of positive constants. This includes polynomials in M, PIGLOG models and extensions that are polynomials in $\ln M$, PIGL models and extensions of it with power functions of the form $M^{1\pm\kappa}$, and the trigonometric model (17). This identifies the possible functional forms for the income terms in any rank three complete Gorman system.

A Gorman system has *full rank* (Lewbel 1990) if the rank of $\boldsymbol{A}(\boldsymbol{x})$ is equal to the number of columns and therefore to the number of income functions, $h_k(y)$. We know a great deal about full rank complete Gorman systems. The results of Muellbauer (1975, 1976), Gorman (1981), Lewbel (1987, 1989, 1990), and van Daal and Merkies show that:

(a) all full rank one complete systems are homothetic,

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{x})M\,; \tag{18}$$

(b) all full rank two complete systems are either PIGL or PIGLOG,

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{x})M + \boldsymbol{\alpha}_1(\boldsymbol{x})M^{1-\kappa}\,, \tag{19}$$

for some $\kappa \neq 0$, or

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{x})M + \boldsymbol{\alpha}_1(\boldsymbol{x})M \ln M\,; \tag{20}$$

(c) every full rank three complete system is either a

a. *generalized PIGL* (including the QES with $\kappa = 1$),

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{P})M + \boldsymbol{\alpha}_1(\boldsymbol{P})M^{1-\kappa} + \boldsymbol{\alpha}_2(\boldsymbol{P})M^{1+\kappa}\,, \tag{21}$$

b. *generalized PIGLOG*,

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{P})M + \boldsymbol{\alpha}_1(\boldsymbol{P})M \ln M + \boldsymbol{\alpha}_2(\boldsymbol{P})M(\ln M)^2\,, \text{ or} \tag{22}$$

c. *trigonometric system*,

$$\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{P})M + \boldsymbol{\alpha}_1(\boldsymbol{P})M \sin(\tau \ln M) + \boldsymbol{\alpha}_2(\boldsymbol{P})M \cos(\tau \ln M)\,. \tag{23}$$

We now show that all of the full rank cases can be unified within a single framework. The easiest way to accomplish this is to generalize two types of ordinary differential equations to systems of partial differential equations. In budget share form, a full rank one complete system is a zero–order polynomial in income. A linear, first-order, ordinary differential equation is a *Bernoulli equation* if it can be written in the form,

$$\frac{d[y(x)^\kappa]}{dx} = \beta_0(x) + \beta_1(x)y(x)^\kappa\,. \tag{24}$$

We extend this to systems of partial differential equations by defining a system of first-order partial differential equations to be a *linear Bernoulli system* if it can be written as,

$$\frac{\partial[E(\boldsymbol{P},u)^{\kappa}]}{\partial\boldsymbol{P}} = \boldsymbol{\beta}_0(\boldsymbol{P}) + \boldsymbol{\beta}_1(\boldsymbol{P})E(\boldsymbol{P},u)^{\kappa} \tag{25}$$

Multiplying by $\kappa^{-1}E(\boldsymbol{P},u)^{\kappa-1}$ gives the rank two PIGL form. Similarly, a linear, first-order ordinary differential equation is a *logarithmic transformation* if it has the form

$$\frac{d[\ln y(x)]}{dx} = \beta_0(x) + \beta_1(x)\ln y(x)\,. \tag{26}$$

We also can extend this to systems. Define a system of first-order partial differential equations to be a *linear logarithmic system* if has the form

$$\frac{\partial\,[\ln E(\boldsymbol{P},u)]}{\partial\boldsymbol{P}} = \boldsymbol{\alpha}_0(\boldsymbol{P}) + \boldsymbol{\alpha}_1(\boldsymbol{P})\ln[\,E(\boldsymbol{P},u)].\tag{27}$$

Multiplying by $E(\boldsymbol{P},u)$ gives the PIGLOG form. Hence, all full rank two complete systems are first-order polynomials in a single function of income. Third, we extend these ideas to full rank three. We define a system of first-order partial differential equations to be a *quadratic Bernoulli system* if it can be written in the form[1]

$$\frac{\partial[E(\boldsymbol{P},u)^{\kappa}]}{\partial\boldsymbol{P}} = \boldsymbol{\beta}_0(\boldsymbol{P}) + \boldsymbol{\beta}_1(\boldsymbol{P})E(\boldsymbol{P},u)^{\kappa} + \boldsymbol{\beta}_2(\boldsymbol{P})E(\boldsymbol{P},u)^{2\kappa}\,. \tag{28}$$

Multiplying by $\kappa^{-1}E(\boldsymbol{P},u)^{1-\kappa}$ gives the generalized PIGL model. Similarly, a system of first–order partial differential equations is a *quadratic logarithmic system* if it can be written as

$$\frac{\partial[\ln E(\boldsymbol{P},u)]}{\partial\boldsymbol{P}} = \boldsymbol{\alpha}_0(\boldsymbol{P}) + \boldsymbol{\alpha}_1(\boldsymbol{P})\ln E(\boldsymbol{P},u) + \boldsymbol{\alpha}_2(\boldsymbol{P})[\,\ln E(\boldsymbol{P},u)]^2\,. \tag{29}$$

Multiplying by $E(\boldsymbol{P},u)$ gives the generalized PIGLOG model. Finally, define a *quadratic complex exponential system* of first–order partial differential equations by

$$\frac{\partial[E(\boldsymbol{P},u)^{\iota\tau}]}{\partial\boldsymbol{x}} = \left[\frac{\boldsymbol{\alpha}_2(\boldsymbol{P}) - \iota\boldsymbol{\alpha}_1(\boldsymbol{P})}{-2\,/\,\tau}\right] - \iota\tau\boldsymbol{\alpha}_0(\boldsymbol{P})E(\boldsymbol{P},u)^{\iota\tau} + \left[\frac{\boldsymbol{\alpha}_2(\boldsymbol{P}) + \iota\boldsymbol{\alpha}_1(\boldsymbol{P})}{2\,/\,\tau}\right]E(\boldsymbol{P},u)^{2\iota\tau}\,, \tag{30}$$

where $\iota = \sqrt{-1}$. Multiplying by $(\iota\tau)^{-1}E(\boldsymbol{P},u)^{1-\iota\tau}$ and applying de Moivre's theorem,

$$e^{\pm\iota\tau y} = \cos(\tau y) \pm \iota\sin(\tau y)\,, \tag{31}$$

with $y = \ln M$ gives the full rank three trigonometric demand system. The main trick in this case is to select the right set of complex-valued price functions to guarantee real-valued demands. In all other respects, the steps are the same as for the generalized PIGL and PIGLOG. This leads to the following conclusion.

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[1] A first-order ordinary differential equation that is quadratic in y is a *Ricatti equation.* We show here that any full rank three Gorman system is a *Ricatti system* of partial differential equations.

> ***Result:*** *All full rank complete Gorman systems can be represented as a zero-, first-, or second-order polynomial of only one function of income.*

This fact is revealed in a series of deep, and what appears to be less well-known and understood, work by Russell (1983, 1996) and Russell and Farris (1993, 1998), establishing the relationship between Gorman systems and the theory of Lie transformation groups.[2] Russell (1983) initially argued that Gorman's theorem follows from Lie's result on the maximal rank of local transformation groups on the real line. But Jerison (1993) presented a counterexample to this claim based on a polynomial demand system with more than three income functions (and therefore reduced rank) that is not a local Lie transformation group. However, Russell and Farris (1993) apply Lie's theory to show that any full rank Gorman system is a special case of the quadratic system

$$f'(M)\boldsymbol{q} = \boldsymbol{\alpha}_0(\boldsymbol{P}) + \boldsymbol{\alpha}_1(\boldsymbol{P})f(M) + \boldsymbol{\alpha}_2(\boldsymbol{P})f(M)^2\,, \tag{32}$$

for some smooth, strictly increasing function $y = f(M)$. The essence of this representation for the Gorman functional forms is captured by equations (24)–(30). However, the reduction to a polynomial representation and the restriction that this polynomial is at most a quadratic for full rank systems is purely due to symmetry. That is to say, by applying the symmetry methods developed by Lie rather than the steps followed by Gorman in his proof, we can obtain (32) without appealing to 0^{o} homogeneity or adding up. Without loss in generality, rank one has $\boldsymbol{\alpha}_0$, full rank two is linear, $\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 y$, and full rank three is the quadratic form, $\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 y + \boldsymbol{\alpha}_2 y^2$.

Russell and Farris (1998) extend their results on full rank complete systems to show that Jerison's counterexample is the only one possible in an important and generic sense (also see Russell 1996).[3] Suppose that there are $K \geq 3$ linearly independent income functions and the matrix of price functions, $\boldsymbol{A}(\boldsymbol{x})$, has the minimal possible level of linear dependence among its column vectors. This can be shown to be equivalent to the property that a maximal number of the terms, $h_k(y)h'_\ell(y) - h'_k(y)h_\ell(y)$, $k < \ell$, are spanned by the basis, $\{h_k(y)\}_{k=1}^K$,

$$h_k(y)h'_\ell(y) - h'_k(y)h_\ell(y) = c_{k\ell}^1 h_1(y) + c_{k\ell}^2 h_2(y) + \cdots c_{k\ell}^K h_K(y)\,, \tag{33}$$

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[2] We are indebted to Thomas Russell for pointing us in the direction of differential geometry as a strategy for addressing the structure of incomplete Gorman systems.

[3] Our use of the word *generic* here conveys our belief that Theorem 4 in Russell and Farris (1998) gives a precise meaning to the last paragraph and footnote in Gorman (1981).

where the $c_{k\ell}^{j}$, $j = 1,\cdots,K$, $k < \ell$, are absolute constants that do not depend on $\boldsymbol{P}$ or y. Under these conditions, a representation (definition for $y = f(M)$) exists such that the system can be written in the polynomial form

$$f'(M)\boldsymbol{q} = \boldsymbol{\alpha}_1(\boldsymbol{P}) + \boldsymbol{\alpha}_2(\boldsymbol{P})f(M) + \boldsymbol{\alpha}_3(\boldsymbol{P})f(M)^2 + \cdots + \boldsymbol{\alpha}_K(\boldsymbol{P})f(M)^{K-1}. \tag{34}$$

Russell and Farris (1998) also argue that the theory of Lie algebras on the real line implies that the rank of Gorman systems is at most three.[4] They remark further in a footnote that Gorman's (1981) theorem is a nontrivial, although specialized, extension of this area of differential geometry. It is important to note that the polynomial representation again follows purely from symmetry and that adding up plays absolutely no role in its derivation.

> ***Result:*** *For any number of income terms, the existence of a polynomial representation of a Gorman system follows from symmetry and a minimum level of linear dependence in the column vectors of the matrix of price functions,* $\boldsymbol{A}(\boldsymbol{x})$. *Homogeneity and adding up play no role in this property of Gorman systems.*

This aspect of the structure of Gorman systems plays an important role in the next section, where we analyze incomplete systems.

A common source of confusion about the structure of complete Gorman systems, including a statement by Gorman to this effect, is that the restriction on the functional form of the income terms is thought to be caused by symmetry. To the contrary, these restrictions are the consequence of 0^o homogeneity and adding up. In the full rank three case, Russell and Farris (1993) show that, *given the quadratic form implied by symmetry*,

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[4] In differential geometry, the $h_k(y)h'_\ell(y) - h'_k(y)h_\ell(y)$ are known as *Jacoby brackets* and the absolute constants $\{c_{k\ell}^{j}\}$ are the *structure constants* of the system of differential equations. If we append the differential operator, $\partial/\partial y$, on the right of the Jacoby brackets, we obtain the *Lie brackets*, $[h_k(y)h'_\ell(y) - h'_k(y)h_\ell(y)]\partial/\partial y$. The differential operators, $h_k(y)\partial/\partial y$, then are vector fields on the real line. When (33) is a complete full rank system, it is a *Lie algebra* on the finite dimensional vector space spanned by these vector fields. It is a well-known fact in differential topology that the largest Lie algebra on the real line has rank three and that the differential operators $\{\partial/\partial y \quad y\,\partial/\partial y \quad y^2\,\partial/\partial y\}$ span this vector space. Russell and Farris (1993) gives a useful introduction to these concepts and their role in Gorman systems. Guillemin and Pollack (1974), Hydon (2000), Olver (2000), and Spivak (1999) are useful references on differential geometry and the application of Lie's theory to differential equations.

adding up restricts the functional form to the cases identified by Gorman (1981). We show here that *Gorman's functional form restrictions apply to all full rank and rank three complete systems without any appeal to symmetry, only an appeal to adding up.* To see this, define the transformed expenditure function by $y(\boldsymbol{P}, u) = f\left(E(\boldsymbol{P}, u)\right)$ so that (13) with $\boldsymbol{x} = \boldsymbol{P}$ is

$$f'(M)\boldsymbol{q} = \sum_{k=1}^{K} \boldsymbol{\alpha}_k(\boldsymbol{P}) h_k\left(f(M)\right). \tag{35}$$

Adding up then implies

$$Mf'(M) = \sum_{k=1}^{K} a_k h_k\left(f(M)\right), \tag{36}$$

for a set of absolute constants $\{a_k\}_{k=1}^{K}$, since the left only depends on M while the right only depends on $y = f(M)$. Integrating this differential equation immediately implies the following result.

Lemma 2. $f : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{Y}}$, $f \in \boldsymbol{\mathcal{C}}^{\infty}$, $f' > 0$ *, satisfies the differential equation*

$$Mf'(M) = \sum_{k=1}^{K} a_k h_k\left(f(M)\right)\ \forall\ M \in \boldsymbol{\mathcal{M}},$$

if and only if $f(M) \in \left\{\ln M \quad M^{\kappa} \quad M^{\iota\tau}\right\}$*, for* κ, τ *real constants and* $\iota = \sqrt{-1}$.

A simple argument producing this result is as follows. Adding up implies that one, and without any loss of generality *only one*, income function, $h_k(f(M))$, satisfies $Mf'(M) = a_k h_k\left(f(M)\right)$. We have three cases to consider. First, if $h_k'(f(M)) = 0$, then we have the first case of the lemma. Second, if $h_k'(f(M)) = a_k$ for some constant a_k, then we either have the second or third case of the lemma. We show in the Appendix that the restriction to either purely real or purely complex roots is the consequence of adding up and the fact that the expenditure function and demands are real-valued. Hence, this restriction on the functional form of the income terms also is not due to symmetry. Third, if $h_k'(f(M)) \neq 0$ is not constant, then the change of variables to $\tilde{f}(M) = \int^{f(M)} df \big/ h_k(f)$ reproduces the first case. Note that in each case, the function $h_k(\cdot)$ either plays no role or can be eliminated by lemma 1. Also note that it is the budget identity and linear independence of the K income functions $\{h_k(y)\}_{k=1}^{K}$ that drive this result – symmetry plays absolutely no role.

In a complete nominal income system, 0° homogeneity and adding up are essentially the same, since both are consequences of a linear budget constraint. Moreover, 0°

homogeneity in all prices and income imposes almost as strong of a restriction on the functional form of the income terms in a nominal income Gorman system as does adding up. The reason is that the multiplicative separability between prices and income implies a limited number of ways that homogeneity can be attained – through addition, multiplication, or a combination of these operations.

To see this, first consider a single rank one demand function, say $q = \alpha(\boldsymbol{P})h(M)$. Euler's theorem implies $Mh'(M)\big/h(M) = a$, where $a = -\alpha_{\boldsymbol{P}}(\boldsymbol{P})^{\mathsf{T}}\boldsymbol{P}\big/\alpha(\boldsymbol{P})$ is a constant since we have separated the variables. Thus, the only functional form for $h(M)$ in which multiplication by $\alpha(\boldsymbol{P})$ produces 0° homogeneity is $h(M) = M^{a}$. Note that we have no recourse to symmetry or to adding up since there is only one demand function. Yet we obtain one of the Gorman functional forms purely from homogeneity and the multiplicative separability of prices and income. However, unlike the complete system case, it is not necessary that $a = 1$. Hence, the functional form restriction due to 0° homogeneity in a single equation is less restrictive then adding up across a system of equations.

If we consider a single demand function with two linearly independent income terms, $q = \alpha_1(\boldsymbol{P})h_1(M) + \alpha_2(\boldsymbol{P})h_2(M)$, then there are only two ways that we can attain homogeneity. In the first case, each product term can be homogeneous. This is equivalent to the previous case, but with different powers of M. In the second case, factor the demand into the product of homogeneous terms, $q = \left[\alpha_0(\boldsymbol{P}) + \tilde{\alpha}_2(\boldsymbol{P})\tilde{h}_2(M)\right]\tilde{\alpha}_1(\boldsymbol{P})h_1(M)$, where $\alpha_1(\boldsymbol{P}) = \alpha_0(\boldsymbol{P})\tilde{\alpha}_1(\boldsymbol{P})$, $\tilde{\alpha}_2(\boldsymbol{P}) = \alpha_2(\boldsymbol{P})\big/\tilde{\alpha}_1(\boldsymbol{P})$, and $\tilde{h}_2(M) = h_2(M)\big/h_1(M)$. The term on the far right again repeats the one income function case, so that $h_1(M) = M^{a_1}$, say. Euler's theorem for the other term implies that $0 = a + b\tilde{h}_2(M) + M\tilde{h}_2'(M)$, where $a = \tilde{\alpha}_2(\boldsymbol{P})^{-1}\boldsymbol{P}^{\mathsf{T}}\,\partial\alpha_0(\boldsymbol{P})\big/\partial\boldsymbol{P}$ and $b = \tilde{\alpha}_2(\boldsymbol{P})^{-1}\boldsymbol{P}^{\mathsf{T}}\,\partial\tilde{\alpha}_2(\boldsymbol{P})\big/\partial\boldsymbol{P}$. If $b \neq 0$, we again have the rank one case. Hence, assume that $b = 0$, so that $\tilde{\alpha}_2(\boldsymbol{P})$ is 0° homogeneous. Solving the resulting differential equation implies $\tilde{h}_2(M) = \ln M$. Therefore, the only functional form for which adding a function of prices generates 0° homogeneity is $\ln M$. In this case, the original income terms are $h_1(M) = M^{a_1}$ and $h_2(M) = M^{a_1}\ln M$. These resemble the Gorman functional forms, but again are not as restrictive. The reason for the weaker restriction is that we cannot appeal to adding up to require one income function to be M since there is only one demand. In the Appendix, we show that if one income function is M and another is $Mf(M)$ – the models in Muellbauer (1975, 1976) – then 0° homogeneity implies the PIGL and PIGLOG forms for a single demand. But the reason that M must be an income function in a complete system is adding up.

Rank three with three or more income functions is more involved, but the same basic principles apply. In particular, because a Gorman system is made up of a finite sum of multiplicatively separable price and income terms, we can only achieve 0^o homogeneity by addition, multiplication, or a combination of these two operations. It therefore is clear that the following result holds in any complete Gorman system.

> ***Result:*** *The Gorman functional forms for the income terms follow directly from 0^o homogeneity and adding up in a complete system. Symmetry plays no role in this property of Gorman systems.*

This result also will play an important role in the next section on incomplete Gorman systems.

We now fill an important gap in the literature on the full rank three case. By applying some minor changes to their arguments, it can be shown that van Daal and Merkies and Lewbel (1990) prove that a full rank three QES, PIGL, or PIGLOG system is integrable if and only if four functions, $\beta_1, \beta_2, \beta_3 : \mathcal{P} \to \mathbb{R}$, and $\gamma : \mathbb{R} \to \mathbb{R}$, exist such that

$$\frac{\partial}{\partial \boldsymbol{P}}\left(\frac{y(\boldsymbol{P},u) - \beta_1(\boldsymbol{P})}{\beta_3(\boldsymbol{P})}\right) = \left[\gamma(\beta_2(\boldsymbol{P})) + \left(\frac{y(\boldsymbol{P},u) - \beta_1(\boldsymbol{P})}{\beta_3(\boldsymbol{P})}\right)^2\right]\frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}}, \tag{37}$$

where $y(\boldsymbol{P},u) \equiv f(E(\boldsymbol{P},u))$ for an appropriately chosen f in each case. This can be simplified with two changes of variables. First, let $w(\boldsymbol{P},u) = [y(\boldsymbol{P},u) - \beta_1(\boldsymbol{P})]/\beta_3(\boldsymbol{P})$. Second, let $z(\boldsymbol{P},u) = -1/w(\boldsymbol{P},u)$. Then

$$\frac{\partial z(\boldsymbol{P},u)}{\partial \boldsymbol{P}} = \left[1 + \gamma(\beta_2(\boldsymbol{P}))\, z(\boldsymbol{P},u)^2\right]\frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}}, \tag{38}$$

and if $\gamma(\beta_2(\boldsymbol{P})) \equiv \lambda$ is constant, we can separate variables,

$$\frac{\partial z(\boldsymbol{P},u)/\partial P_i}{1 + \lambda z(\boldsymbol{P},u)^2} = \frac{\partial \beta_2(\boldsymbol{P})}{\partial P_i} \;\; \forall\; i = 1, \cdots, n, \tag{39}$$

so that direct integration gives

$$\phi\left(\frac{-\beta_3(\boldsymbol{P})}{y(\boldsymbol{P},u) - \beta_1(\boldsymbol{P})}\right) \equiv \int^{-\beta_3(\boldsymbol{P})/[y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})]} \frac{dz}{1 + \lambda z^2} = \beta_2(\boldsymbol{P}) + u, \tag{40}$$

where we normalize the constant of integration to be the utility index. This is the solution reported in van Daal and Merkies and Lewbel (1987, 1990). Closed form solutions for the indirect utility function in all of these cases are

$$v(\boldsymbol{P}, M) = \begin{cases} \dfrac{1}{\kappa}\tan^{-1}\left(\dfrac{-\kappa\beta_3(\boldsymbol{P})}{f(M) - \beta_1(\boldsymbol{P})}\right) - \beta_2(\boldsymbol{P}),\ \lambda > 0, \\ \\ \dfrac{f(M) - \beta_1(\boldsymbol{P}) - \kappa\beta_3(\boldsymbol{P})}{\tilde{\beta}_2(\boldsymbol{P})[f(M) - \beta_1(\boldsymbol{P}) + \kappa\beta_3(\boldsymbol{P})]},\ \lambda \leq 0, \end{cases} \tag{41}$$

where $\tilde{\beta}_2(\boldsymbol{P}) = e^{2\beta_2(\boldsymbol{P})}$, $\kappa^2 = \lambda > 0$ in the first case, and $-\kappa^2 = \lambda \leq 0$ in the second. We also can rewrite the first case in the somewhat more transparent form

$$v(\boldsymbol{P}, M) = \frac{f(M) - \tilde{\beta}_1(\boldsymbol{P})}{\tilde{\beta}_2(\boldsymbol{P})f(M) - \tilde{\beta}_3(\boldsymbol{P})}, \tag{42}$$

where $\tilde{\beta}_1 = \beta_1 + \kappa\cot(-\kappa\beta_2)$, $\tilde{\beta}_2 = \cot(-\kappa\beta_2)$, and $\tilde{\beta}_3 = \beta_1\cot(-\kappa\beta_2) - \kappa\beta_3$. Thus, the two solutions share the same fundamental structure, differing only in the specific form of their price functions.

Neither Lewbel (1987, 1989, 1990) nor van Daal and Merkies succeeded in finding a closed form solution for indirect preferences if $\gamma(\beta_3(\boldsymbol{P}))$ is not a constant. We now show that only the constant case generalizes the solution for the QES obtained by Howe, Pollak, and Wales, so their quest was already over.

Lemma 3. *If* $z : \mathcal{P} \times \mathcal{U} \to \mathbb{R}$, $z \in \mathcal{C}^\infty$ *satisfies*

$$\partial z(\boldsymbol{P}, u)/\partial \boldsymbol{P} = \left[1 + \gamma(\beta_2(\boldsymbol{P}))\, z(\boldsymbol{P}, u)^2\right]\partial\beta_2(\boldsymbol{P})/\partial\boldsymbol{P},$$

then either $\gamma(\beta_2(\boldsymbol{P})) \equiv \lambda$, *a nonzero constant, or*

$$\partial z(\boldsymbol{P}, u)/\partial \boldsymbol{P} = \partial\beta_2(\boldsymbol{P})/\partial\boldsymbol{P}.$$

Remark: The original solution obtained by Howe, Pollak and Wales for the QES has the general form (using the notation in van Daal and Merkies),

$$v(\boldsymbol{P}, M) = \left(\frac{\beta_3(\boldsymbol{P})}{\beta_1(\boldsymbol{P}) - M}\right) - \beta_2(\boldsymbol{P}). \tag{43}$$

The variable z is obtained through a change of variables from w to $z = -w^{-1}$, where w is the Gorman polar form, $w(\boldsymbol{P}, u) = [y(\boldsymbol{P}, u) - \beta_1(\boldsymbol{P})]/\beta_3(\boldsymbol{P})$. As a result, the variables can be separated in the second case of the lemma, and direct integration gives

$$\frac{-\beta_3(\boldsymbol{P})}{y(\boldsymbol{P}, u) - \beta_1(\boldsymbol{P})} = \int^{-\beta_3(\boldsymbol{P})/[y(\boldsymbol{P},u) - \beta_1(\boldsymbol{P})]} dz = \beta_2(\boldsymbol{P}) + u. \tag{44}$$

Now, recalling that $f(M) = M$ in the QES reproduces the solution in Howe, Pollak, and Wales. The case that they missed is $\gamma(\beta_2(\boldsymbol{P})) = \lambda > 0$ – complex roots for the Ricatti system. We also can renormalize (41) and (42) to the form in (43) if we replace M with the appropriate $f(M)$. ■

Lewbel (1990) also solves the integrability problem for the trigonometric system, obtaining the indirect utility function in this case as

$$v(\boldsymbol{P}, M) = \beta_2(\boldsymbol{P}) + \frac{\beta_3(\boldsymbol{P})\cos\left(\tau \ln\left(M/\beta_1(\boldsymbol{P})\right)\right)}{\left[1 - \sin\left(\tau \ln\left(M/\beta_1(\boldsymbol{P})\right)\right)\right]}, \tag{45}$$

using notation consistent with previous expressions. Applying the rules for taking sums and differences of sine and cosine functions and the rules for taking products and squares of complex conjugate numbers, we can rewrite this expression in the somewhat more transparent form,

$$v(\boldsymbol{P}, M) = \beta_2(\boldsymbol{P}) - \frac{\iota\beta_3(\boldsymbol{P})(M^{\iota\tau} + \iota\beta_1(\boldsymbol{P})^{\iota\tau})}{(M^{\iota\tau} - \iota\beta_1(\boldsymbol{P})^{\iota\tau})}. \tag{46}$$

Roy's identity applied to this expression gives the same demands as in Lewbel (1990).

Hence, we have shown that all full rank three Gorman systems have the same fundamental structure whether they are defined on the complex plane or the real line. With some algebra and a renormalization of the price functions, we can rewrite (46) in the form (43) with M replaced by $M^{\iota\tau}$ and complex-valued price functions. Therefore, the functional structure of indirect preferences was correctly discovered by Howe, Pollak, and Wales once the possibility of complex solutions to the Ricatti system of partial differential equations has been taken into account.

We have limited our discussion to nominal income complete Gorman systems. However, Lewbel (1989) shows that complete Gorman systems that have been specified in terms of deflated income can attain a maximum rank of four. This plays an important role in the next section, where we analyze incomplete Gorman systems. It is precisely this rank result that guarantees that we maintain the maximum flexibility for the rank and the functional form of the goods that are not part of an incomplete Gorman system. However, we defer a complete discussion of this until later in the paper.

3. Incomplete Demand Systems

Three approaches are commonly used to address the complexity of large demand systems. One is to aggregate across commodities and estimate a complete system of demand equations using the commodity aggregates (e.g., food, clothing, housing, fuel, drink and tobacco, transportation and communication, other goods, and other services, as in Deaton and Muellbauer 1980). The second approach appeals to separability of consumer preferences and estimates a complete set of conditional demands for the goods of interest as functions of the prices for those goods and the total expenditure on that group of goods (e.g., beef, mutton, and other meat, as in Wales and Woodland 1983). The third approach specifies an incomplete system of demand equations as functions of the prices of the goods of interest, the prices of related goods, and total expenditure on all goods, or income (e.g., Burt and Brewer 1971, or Chiccetti, Fisher, and Smith 1976). This approach is commonly referred to as an *incomplete demand system*.

Incomplete information is the rule not the exception. Researchers, not wanting to, or not able to construct a complete system, often consider only a subset of demands of interest. Thus, economy and parsimony often dictate the approach that applied economists take. Another reason that incomplete systems are used is that researchers my not wish to have the strong restrictions on functional form that complete systems impose. For example, it is well known that a complete system of demand functions that is linear in quantities, prices, and income cannot be consistent with utility maximization subject to constraint (LaFrance 1985). But one or even a subset of linear demands can be consistent. Thus, one often wishes to work with an incomplete system to maintain the flexibility of model choice while also maintaining consistency with consumer theory. Whatever the reason, it is far more common for economists to model a subset of goods than the entire mix of goods that make up the consumer's expenditure decision.

In the literature reviewed in the previous section, only Russell and Farris (1993) even mention an incomplete system. They argue that (32) above completely characterizes all full rank incomplete Gorman systems for any smooth, strictly increasing function of income (page 319). However, this statement ignores the implications of relaxing adding up while maintaining homogeneity. Any group of demand equations, whether or not they form a complete system, must be 0° homogeneous in all prices and income. Only a power function can be made to be 0° homogeneous by multiplication by a function of prices. On the other hand, only the log function can be made to be 0° homogeneous by addition by a function of prices. For example, no functions $\alpha(\boldsymbol{P})$, $\beta(\boldsymbol{P})$ can make either $\alpha(\boldsymbol{P})e^{\lambda M}$ or $\alpha(\boldsymbol{P}) + \beta(\boldsymbol{P})e^{\lambda M}$ become 0° homogeneous in $(\boldsymbol{P}, M)$ for any $\lambda \neq 0$.

The rest of this section is organized in the following way. We first briefly review the properties of incomplete demand systems, explaining how they relate to and differ from complete systems, and which parts of preferences can be identified from a subset of all demands. We explain the different roles of 0° homogeneity and adding up in an incomplete system and why it is important to separately account for both. We develop a representation for an arbitrary incomplete demand system that includes all complete systems as a special case. We then extend the definition of a Gorman system of demand equations to incomplete systems in line with this representation. This extension also includes all complete Gorman systems as a special case. Our definition of an incomplete Gorman system is carefully constructed so that the only issue involved in determining the structure of this class of systems is symmetry. This leads us to our main result. The following properties follow from symmetry. The rank of an incomplete Gorman system is no more than three. A full rank system or a system with three or more income functions and a minimal level of linear dependence among the column vectors of price functions has a representation that is a polynomial in a single function of income. Because 0^o homogeneity and adding up do not apply to our representation of incomplete demand systems, there is no functional form restriction on the income terms in an incomplete Gorman system. Next we derive the closed form solutions for the indirect preferences of this entire class of models. The non-trigonometric preference functions are members of the projective transformation group (Olver 2000) and we formally develop this relationship. We give an example to illustrate the results. Finally, we derive a simple method to nest the rank and functional form of incomplete Gorman systems.

3.1 The Structure of Incomplete Demand Systems

To better understand the structure of incomplete demand systems, we first need to redefine some of our previous terms and add a couple of new definitions. We now consider $\boldsymbol{q} \in \boldsymbol{\mathcal{Q}} \subset \mathbb{R}_+^{n_q}$ to be the market goods of primary interest, with nominal market prices $\boldsymbol{P} \in \boldsymbol{\mathcal{P}} \subset \mathbb{R}_+^{n_q}$. Let $\tilde{\boldsymbol{q}} \in \tilde{\boldsymbol{\mathcal{Q}}} \subset \mathbb{R}_+^{n_{\tilde{q}}}$ be the vector of all other goods that enter the consumer's utility function, with associated nominal market prices $\tilde{\boldsymbol{P}} \in \tilde{\boldsymbol{\mathcal{P}}} \subset \mathbb{R}_+^{n_{\tilde{q}}}$. We continue to let $M \in \boldsymbol{\mathcal{M}}$ denote total expenditure on all goods (income), and define the nominal expenditure on all other goods by $\tilde{M} = \tilde{\boldsymbol{P}}^{\mathsf{T}}\tilde{\boldsymbol{q}} = M - \boldsymbol{P}^{\mathsf{T}}\boldsymbol{q} > 0$. For the remainder of the paper, we assume that $n = n_q + n_{\tilde{q}} \geq n_q + 1$, and that expenditure on other goods is strictly positive.

We extend the definition of the nominal expenditure function to

$$E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u) \equiv \min\left\{\boldsymbol{P}^{\mathsf{T}}\boldsymbol{q} + \tilde{\boldsymbol{P}}^{\mathsf{T}}\tilde{\boldsymbol{q}} : u(\boldsymbol{q}, \tilde{\boldsymbol{q}}) \geq u\right\}. \tag{47}$$

We assume that $E : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \times \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{M}}$ is analytic and has neoclassical properties in all prices and the utility index. In particular, it is increasing, $1°$ homogeneous, and concave in all prices $(\boldsymbol{P}, \tilde{\boldsymbol{P}})$, and increasing in u. Denote the Hicksian compensated demands for the goods $\boldsymbol{q}$ by

$$\boldsymbol{g}(\boldsymbol{P},\tilde{\boldsymbol{P}},u) = \partial E(\boldsymbol{P},\tilde{\boldsymbol{P}},u)\big/\partial \boldsymbol{P}\,. \tag{48}$$

We also extend our definition of the indirect utility function to

$$v(\boldsymbol{P},\tilde{\boldsymbol{P}},M) \equiv \max\left\{u(\boldsymbol{q},\tilde{\boldsymbol{q}}) : \boldsymbol{P}^{\mathsf{T}}\boldsymbol{q} + \tilde{\boldsymbol{P}}^{\mathsf{T}}\tilde{\boldsymbol{q}} \le M\right\}, \tag{49}$$

and denote the Marshallian ordinary demands for the goods $\boldsymbol{q}$ by

$$\boldsymbol{h}(\boldsymbol{P},\tilde{\boldsymbol{P}},M) = -\frac{\partial v(\boldsymbol{P},\tilde{\boldsymbol{P}},M)\big/\partial \boldsymbol{P}}{\partial v(\boldsymbol{P},\tilde{\boldsymbol{P}},M)\big/\partial M}\,. \tag{50}$$

The same relationships between the Hicksian and Marshallian demands apply for an incomplete system as for a complete system,

$$\boldsymbol{g}(\boldsymbol{P},\tilde{\boldsymbol{P}},u) = \boldsymbol{h}\left(\boldsymbol{P},\tilde{\boldsymbol{P}},E(\boldsymbol{P},\tilde{\boldsymbol{P}},u)\right), \tag{51}$$

and

$$\boldsymbol{P}^{\mathsf{T}}\boldsymbol{h}\left(\boldsymbol{P},\tilde{\boldsymbol{P}},E(\boldsymbol{P},\tilde{\boldsymbol{P}},u)\right) + \tilde{M}\left(\boldsymbol{P},\tilde{\boldsymbol{P}},E(\boldsymbol{P},\tilde{\boldsymbol{P}},u)\right) = E(\boldsymbol{P},\tilde{\boldsymbol{P}},u)\,. \tag{52}$$

LaFrance and Hanemann (1989) identify an exhaustive list of the properties implied by utility maximization for the subset of Marshallian demands for $\boldsymbol{q}$:

(d) they are 0° homogeneous in all prices and income;

(e) they are positive valued;

(f) income strictly exceeds expenditure on $\boldsymbol{q}$; and

(g) the $n_q \times n_q$ matrix of Slutsky substitution terms, $\partial \boldsymbol{h}\big/\partial \boldsymbol{P}^{\mathsf{T}} + \left(\partial \boldsymbol{h}\big/\partial M\right)\boldsymbol{h}^{\mathsf{T}}$, is symmetric and negative semidefinite.

They also show that (a)–(d) are equivalent to:

(i) the existence of a quasi-expenditure function,

$$\hat{E}\left(\boldsymbol{P},\tilde{\boldsymbol{P}},\theta(\tilde{\boldsymbol{P}},u)\right) = E(\boldsymbol{P},\tilde{\boldsymbol{P}},u),$$

which is increasing in $(\boldsymbol{P},\theta)$, concave in $\boldsymbol{P}$, and satisfies Hotelling's lemma;

(ii) the existence of a quasi-indirect utility function,

$$\theta(\tilde{\boldsymbol{P}},u) = \upsilon(\boldsymbol{P},\tilde{\boldsymbol{P}},M),$$

where υ is the inverse of $\hat{E}$ with respect to θ, which is decreasing and quasiconvex in $\boldsymbol{P}$, increasing in M, satisfies Roy's identity, and is related to the indirect utility function by

$$v(\boldsymbol{P},\tilde{\boldsymbol{P}},M) = \psi\left(\upsilon(\boldsymbol{P},\tilde{\boldsymbol{P}},M),\tilde{\boldsymbol{P}}\right)$$

where ψ is the inverse of θ with respect to u; and

(iii) the existence of a *quasi-utility function*,

$$\omega(\boldsymbol{q}, \tilde{M}, \tilde{\boldsymbol{P}}) = \min_{\boldsymbol{P}, M} \left\{ \upsilon(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M) : \boldsymbol{P}^{\mathsf{T}}\boldsymbol{q} + \tilde{M} = M \right\},$$

which is increasing and quasiconcave in $(\boldsymbol{q}, \tilde{M})$, conveys the conditional preference map for $\boldsymbol{q}$ given $\tilde{\boldsymbol{q}}$, is related to the *variable indirect utility function* (Diewert 1978, Epstein 1975) by

$$\psi\left(\omega(\boldsymbol{q}, \tilde{M}, \tilde{\boldsymbol{P}}), \tilde{\boldsymbol{P}}\right) = \max_{\tilde{\boldsymbol{q}}} \left\{ u(\boldsymbol{q}, \tilde{\boldsymbol{q}}) : \tilde{\boldsymbol{P}}^{\mathsf{T}}\tilde{\boldsymbol{q}} = \tilde{M} \right\},$$

and to the direct utility function by

$$u(\boldsymbol{q}, \tilde{\boldsymbol{q}}) = \min_{\tilde{\boldsymbol{P}}, \tilde{M}} \left\{ \psi\left(\omega(\boldsymbol{q}, \tilde{M}, \tilde{\boldsymbol{P}}), \tilde{\boldsymbol{P}}\right) : \tilde{\boldsymbol{P}}^{\mathsf{T}}\tilde{\boldsymbol{q}} = \tilde{M} \right\}.$$

LaFrance and Hanemann call (a)–(d) *weak integrability* and show that this concept of integrability for incomplete demand systems completely exhausts the implications of utility maximization. In particular, the global integrability of an incomplete demand system is not implied by utility maximization and is not based on a theoretical construct that is part of the theory of consumer choice (Epstein 1982). Finally, both the quasi-expenditure and quasi-indirect utility functions convey sufficient information about consumer preferences to admit exact welfare measurement of a change in the prices of the goods of primary interest and income. Consequently, a coherently specified incomplete demand model contains all of the necessary information required to complete any of the usual tasks of applied economic analysis.

Moreover, this theory is completely general; nothing additional has to be assumed about the functional structure of the underlying utility function, $u(\boldsymbol{q}, \tilde{\boldsymbol{q}})$, or the indirect preference functions $E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u)$, or $v(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M)$ beyond the standard conditions for utility maximization subject to a linear budget constraint. In effect, the incomplete demand system is augmented by a numeraire composite commodity, $\tilde{M}$, to obtain the budget condition, and we act as if this constitutes a complete system without any loss in generality. The demand for the "good" $\tilde{M}$ may or may not have the same functional form as the demands for the goods of interest, $\boldsymbol{q}$. This provides an additional degree of flexibility that significantly expands the class of theoretically consistent functional forms for the demands of the goods $\boldsymbol{q}$ (LaFrance 1985, 2004; von Haefen 2003).

3.2 The Role of Homogeneity

As generally is the case in a consumer choice problem, the budget set,

$$\mathcal{B}(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M) \equiv \left\{ (\boldsymbol{q}, \tilde{\boldsymbol{q}}) \in \mathcal{Q} \times \tilde{\mathcal{Q}} : \boldsymbol{P}^{\mathsf{T}}\boldsymbol{q} + \tilde{\boldsymbol{P}}^{\mathsf{T}}\tilde{\boldsymbol{q}} \leq M \right\}$$

is 0° homogeneous in all prices and income. This implies that we can divide all prices and income by any scalar and the optimal demands for $\boldsymbol{q}$ and $\tilde{\boldsymbol{q}}$ will remain unchanged. Most empirical applications employ some type of general price deflator to reflect the cost of other goods. It is convenient to use normalized prices and income in demand models to impose homogeneity. However, it is important to correctly model the way the prices $\boldsymbol{P}$ affect the demands for the goods $\boldsymbol{q}$ and not to confuse or ignore any influences, no matter how small we might assume they are, that can be masked by a general price deflator. This is particularly important when we are considering the role of symmetry, as in the present case.

As a result of these considerations, it is both flexible and convenient to normalize by a 1° homogeneous function of the other goods' prices. Therefore, let $\pi : \tilde{\boldsymbol{\mathscr{P}}} \rightarrow \mathbb{R}_+$ be a known, non-decreasing (strictly increasing in at least one element of $\tilde{\boldsymbol{P}}$), 1° homogeneous, and concave function $\tilde{\boldsymbol{P}}$. Define normalized prices and income by $\boldsymbol{p} = \boldsymbol{P}\big/\pi(\tilde{\boldsymbol{P}})$, $\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{P}}\big/\pi(\tilde{\boldsymbol{P}})$, and $m = M\big/\pi(\tilde{\boldsymbol{P}})$. Without any loss in generality, then, we can define the *normalized expenditure function* by

$$e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u) \equiv E\left(\boldsymbol{P}\big/\pi(\tilde{\boldsymbol{P}}), \tilde{\boldsymbol{P}}\big/\pi(\tilde{\boldsymbol{P}}), u\right)\Big/\pi(\tilde{\boldsymbol{P}}). \tag{53}$$

It follows that $e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u)$ is increasing in $(\boldsymbol{p},u)$, concave in $\boldsymbol{p}$, is not 1° homogeneous in $\boldsymbol{p}$, the demands for the goods $\boldsymbol{q}$ satisfy Hotelling's lemma,

$$\partial e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u)\big/\partial \boldsymbol{p} = \boldsymbol{h}\left(\boldsymbol{p}, \tilde{\boldsymbol{p}}, e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u)\right), \tag{54}$$

and the total normalized expenditure on $\boldsymbol{q}$ satisfies the inequality

$$\boldsymbol{p}^{\mathsf{T}} \boldsymbol{h}\left(\boldsymbol{p}, \tilde{\boldsymbol{p}}, e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u)\right) < e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u). \tag{55}$$

Note that since $\partial e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u)\big/\partial \boldsymbol{p} \equiv \partial E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u)\big/\partial \boldsymbol{P}$, normalizing by $\pi(\tilde{\boldsymbol{P}})$ does not change the functional relationships between E and $\boldsymbol{P}$ in any way. Adopting the convention that if $n_{\tilde{q}} = 0$ then $\pi = 1$ and (55) is an equality implies that the set of all complete systems is strictly contained in the set of all incomplete systems.

This representation of for an arbitrary incomplete demand system is quite useful for the purpose of extending complete Gorman systems to incomplete systems. There are two advantages to using this representation. First, it shows clearly that neither 0° homogeneity nor adding up restrict the functional form of income terms in the demands for $\boldsymbol{q}$. As we have seen in the previous section, these properties are what limit the functional forms of the income terms in nominal income complete Gorman systems to only three possible cases. In the next section, we show that there is no restriction on the functional form of an incomplete Gorman system. However, we also show that the rank of an incomplete Gorman system with prices and income normalized by a function of other goods' prices has a rank that is no greater than three. This is one important way that Lewbel's (1989) rank result for a deflated income Gorman system affects an incomplete system. Russell and Farris (1998) explain that the logarithmic derivative of the price index deflating income must be one of the vectors of price functions in order for the deflated income system to attain rank four. Since $\pi(\tilde{\boldsymbol{P}})$ is independent of $\boldsymbol{P}$, this cannot occur in an incomplete system when income is normalized by $\pi(\tilde{\boldsymbol{P}})$.

Second, using $\pi(\tilde{\boldsymbol{P}})$ to normalize all prices and income reserves the maximum flexibility for both the rank and the functional form of the demands for the other goods. This is a second important way that Lewbel's (1989) rank result has an affect on incomplete Gorman systems. We have almost no information on the structure of the demands for the other goods when we only include some of the demand equations in an incomplete system. A minimal set of restrictive prior assumptions on the demand equations that are not part of the formal model is therefore highly desirable. Indeed, we have the potential to achieve rank four and retain complete freedom of the functional forms for all but one income function for these goods.

Thus, for the goods that we include in the model, we trade off at most one degree of rank – from a maximum of four to a maximum of three – in exchange for complete flexibility in the functional form of the income terms. On the other hand, for the goods that are not included in the model, we reserve the potential to achieve the maximum possible rank and the greatest flexibility for the functional form.

3.3 Incomplete Gorman Systems

Throughout the rest of the paper, we redefine $\boldsymbol{x}$ to be a vector-valued function of the normalized prices, $\boldsymbol{p}$, $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{p})$, $g_i \in \mathcal{C}^\infty \ \forall \ i$, $\left|\partial \boldsymbol{g}(\boldsymbol{p})^\intercal / \partial \boldsymbol{p}\right| \neq 0 \ \forall \ \boldsymbol{p} \in \mathcal{P} \subset \mathbb{R}_+^{n_q}$, whose inverse is denoted by $\boldsymbol{p}(\boldsymbol{x})$. We also redefine y to be a function of normalized income, $y = f(m)$, $f \in \mathcal{C}^\infty$, $f' > 0$, and denote the inverse of f by $m(y)$. We define an *incomplete Gorman system* by the following natural extension of the complete system case:

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)}{\partial \boldsymbol{x}} = \sum_{k=1}^{K} \boldsymbol{\alpha}_k(\boldsymbol{x}, \tilde{\boldsymbol{p}}) h_k\left(y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)\right). \tag{56}$$

As above, if we define $\pi = 1$ when $n_{\tilde{q}} = 0$, then the set of all complete Gorman systems is strictly contained in the set of all incomplete Gorman systems.

We are now in a position to state our main result for this class of incomplete demand systems. We focus on two important cases. The first part characterizes every full rank case. The second part extends this to reduced rank cases with a minimal degree of linear dependence between the column vectors of $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}) = \left[\boldsymbol{\alpha}_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \cdots \boldsymbol{\alpha}_K(\boldsymbol{x}, \tilde{\boldsymbol{p}})\right]$.

Proposition 1. *Every full rank weakly integrable incomplete Gorman system has* $K \leq 3$ *and a definition for* $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) \equiv f\left(e(\boldsymbol{p}(\boldsymbol{x}), \tilde{\boldsymbol{p}}, u)\right)$ *exists such that*

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)}{\partial \boldsymbol{x}} = \begin{cases} \boldsymbol{\alpha}_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}), & K = 1, \\ \boldsymbol{\alpha}_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}) y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u), & K = 2, \\ \boldsymbol{\alpha}_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}) y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) + \boldsymbol{\alpha}_3(\boldsymbol{x}, \tilde{\boldsymbol{p}}) y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)^2, & K = 3. \end{cases}$$

Conversely, if $K \geq 3$ *and a maximum number of the Jacoby brackets,*

$$h_k(y) h_\ell'(y) - h_k'(y) h_\ell(y), \; k < \ell \,,$$

are spanned by the basis $\{h_1(y) \cdots h_K(y)\}$ *in the sense that*

$$h_k(y) h_\ell'(y) - h_k'(y) h_\ell(y) = d_{k\ell}^1 h_1(y) + \cdots d_{k\ell}^K h_K(y), \; k < \ell \,,$$

where the $\left\{d_{k\ell}^j\right\}$ *are absolute constants, then* $\operatorname{rank}\left[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})\right] = 3$, *and a definition for y exists such that*

$$\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) \big/ \partial \boldsymbol{x} = \boldsymbol{\alpha}_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}) y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) + \cdots + \boldsymbol{\alpha}_K(\boldsymbol{x}, \tilde{\boldsymbol{p}}) y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)^{K-1} .$$

We offer a detailed proof of this result in the Appendix and only briefly outline a few of the steps and important ideas here. The argument hinges primarily on the invariance property in lemma 1 and a small number of results taken from the differential geometry of this class of differential equations. One difference between the approach taken here using the methods of Lie and the one followed by Gorman (1981) is the following. Gorman chose the coordinate system to be the natural logarithm of prices and income with budget shares on the left-hand-side. In many ways, this is natural for a complete system because adding up then implies that one of the income functions must be constant. In our case, we cannot use adding up in any way. But we can find a change of variables to redefine y whenever necessary so that, without loss in generality, one income function is constant. We could not find any way to extend Gorman's method of proof to incomplete systems because adding up plays an important role in many of his steps.

We proceed in the following way. Monotonicity implies that at least one of the income functions must satisfy $h_k(y) \neq 0$. Without loss in generality, let this be $h_1(y)$ and define the variable $\gamma(y) \equiv \int^y ds \big/ h_1(s)$, so that $\gamma'(y) \equiv 1 \big/ h_1(y)$ by the fundamental theorem of calculus. Then we can rewrite (56) as

$$\begin{aligned}\frac{\partial\gamma(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u))}{\partial\boldsymbol{x}} &= \gamma'(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u))\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial\boldsymbol{x}} \\ &= \frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)/\partial\boldsymbol{x}}{h_1\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right)} \\ &= \boldsymbol{\alpha}_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \sum_{k=2}^{K}\boldsymbol{\alpha}_k(\boldsymbol{x},\tilde{\boldsymbol{p}})\frac{h_k\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right)}{h_1\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right)} \\ &\equiv \boldsymbol{\alpha}_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \sum_{k=2}^{K}\boldsymbol{\alpha}_k(\boldsymbol{x},\tilde{\boldsymbol{p}})\tilde{h}_k\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right).\end{aligned} \tag{57}$$

Therefore, since lemma 1 tells us that redefining y by the composite function $\gamma(f(m))$ does not affect integrability or the multiplicative separability of $\boldsymbol{p}$ and y, we can focus on integrability of the much simpler expression[5]

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial\boldsymbol{x}} = \boldsymbol{\alpha}_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \sum_{k=2}^{K}\boldsymbol{\alpha}_k(\boldsymbol{x},\tilde{\boldsymbol{p}})h_k\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right). \tag{58}$$

When we invoke symmetry, some straightforward algebra lets us write a system of $\frac{1}{2}n_q(n_q-1)$ linear equations in the $\frac{1}{2}K(K-1)$ Jacoby brackets,

$$\sum_{k=2}^{K}\sum_{\ell=1}^{k-1}\left(\alpha_{ik}\alpha_{j\ell}-\alpha_{jk}\alpha_{i\ell}\right)\left(h_k'h_\ell - h_kh_\ell'\right) = \sum_{k=1}^{K}\left(\frac{\partial\alpha_{jk}}{\partial x_i}-\frac{\partial\alpha_{ik}}{\partial x_j}\right)h_k, \forall\, 1\le j<i=2,\cdots,n_q\,. \tag{59}$$

In contrast, Gorman (1981) constructed a basis for the vector space spanned by the elements of $\boldsymbol{h}(y)$ and as many product terms $h_k'(y)h_\ell(y)$ as necessary to express each product term $h_k'(y)h_\ell(y)$, including $h_k'(y)\,\forall\,k$, as a linear combination of that basis. This is a second difference between the approaches of Lie and Gorman. Once again, we could not construct a square (complete) system of linear, ordinary differential equations in the income functions $\{h_k(y)\}_{k=1}^{K}$ by following or extending Gorman's method of proof, because he uses both adding up and symmetry to do this.

The nature of the symmetry conditions in (59) is easier to see in matrix form, $\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}$, where $\boldsymbol{B}$ is $\frac{1}{2}n_q(n_q-1)\times\frac{1}{2}K(K-1)$, $\boldsymbol{C}$ is $\frac{1}{2}n_q(n_q-1)\times K$, $\boldsymbol{h}$ is $K\times 1$, and the vector of Jacoby brackets, $\tilde{\boldsymbol{h}}$, is $\frac{1}{2}K(K-1)\times 1$. For this to be a well-posed system we must have at least as many equations as unknowns, $n_q \ge K$, and we assume that this is so. Premultiply both sides by $\boldsymbol{B}^{\mathsf{T}}$ to obtain an equivalent square system, $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}\boldsymbol{h}$. This reveals the crux of the rank condition. $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ inherits its rank

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[5] To minimize the notation carried along in this discussion, we often omit tildes and redefine y and other functions without relabeling when it does not cause confusion.

from $\boldsymbol{A}$, which is K in the full rank case, and has dimension $\frac{1}{2}K(K-1)\times\frac{1}{2}K(K-1)$. The existence of a unique solution for $\tilde{\boldsymbol{h}}$ in terms of $\boldsymbol{h}$ therefore requires $K \leq 3$.

When $\boldsymbol{B}^{\intercal}\boldsymbol{B}$ has full rank, the well-known least squares formula gives $\tilde{\boldsymbol{h}}$ uniquely in terms of $\boldsymbol{h}$ as $\tilde{\boldsymbol{h}} = (\boldsymbol{B}^{\intercal}\boldsymbol{B})^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{C}\boldsymbol{h} \equiv \boldsymbol{D}\boldsymbol{h}$. We then note that $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ depend only on y and not on $(\boldsymbol{x}, \tilde{\boldsymbol{p}})$, while $\boldsymbol{D}$ depends only on $(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ and not on y. This implies that the elements of $\boldsymbol{D}$ are constant – they do not depend on $\boldsymbol{x}$, $\tilde{\boldsymbol{p}}$, or y. Linear independence of the income functions and existence of a unique solution also imply that not all of the elements in any row of $\boldsymbol{D}$ can vanish. Note that $\boldsymbol{D}$ has dimension $\frac{1}{2}K(K-1)\times K$. When $K = 1$, $\boldsymbol{D}$ has zero rows and there are no Jacoby brackets. When $K = 2$, $\boldsymbol{D}$ has one row and two columns. When $K = 3$, $\boldsymbol{D}$ has three rows and three columns. If $K > 3$, $\boldsymbol{D}$ has more rows than columns, so that there are more Jacoby brackets than income functions. The simple least squares formula cannot be applied to find $\tilde{\boldsymbol{h}}$ in terms of $\boldsymbol{h}$ since $\boldsymbol{B}^{\intercal}\boldsymbol{B}$ must then be singular. The main question in this case is whether there are any redundant equations in the under–identified system $\boldsymbol{B}^{\intercal}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^{\intercal}\boldsymbol{C}\boldsymbol{h}$, and if so how many.

The polynomial representation in the full rank case uses the fact that one of the income functions can be converted to the constant function to write the solution for the Jacoby brackets, $\tilde{\boldsymbol{h}} = \boldsymbol{D}\boldsymbol{h}$, as a complete system of K–1 linear, first-order, ordinary differential equations with constant coefficients subject to a set of linear side constraints that are due to symmetry. Solving the differential equations and checking for consistency with the constraints gives a polynomial representation in each full rank case.

> *This part of the proposition states that a definition for $y = f(m)$ can always be found such that every full rank weakly integrable Gorman system is at most a quadratic form. This property holds whether or not the system is complete.*

The proof of the polynomial representation and rank result when $K \geq 3$ has two parts. The first part shows that we can find a polynomial representation for the demand equations when the matrix $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ has a minimal amount of linear dependence between its column vectors. This part closely follows Russell and Farris (1998: 193-94). Beginning with a full rank three system, we add a fourth income function. By Lie's theory, at most two out of the three new Jacoby brackets can be spanned by $\{1\ y\ y^2\ h_4(y)\}$. Moreover, two of them are spanned if and only if $h_4(y) = y^3$. Otherwise symmetry is contradicted. A simple induction on K completes this part of the argument.

This part of the proposition clarifies the issues in finding a solution to the system of equations $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}$ when the $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ matrix is singular. A *redundancy* is a Jacoby bracket that is a linear combination of the elements of $\boldsymbol{h}$ and can be written as a linear combination of the other Jacoby brackets. In the polynomial case with four terms,

$$h_1(y)h_4'(y) - h_1(y)h_4'(y) = 1(3y^2) - 0(y^3) = 3y^2\,, \tag{60}$$

$$h_2(y)h_3'(y) - h_2'(y)h_3(y) = y(2y) - 1(y^2) = y^2\,. \tag{61}$$

Thus, these two equations form one redundancy. A redundancy repeats some other implication of symmetry, and therefore adds no information.

A *defect* is a Jacoby bracket that cannot be written as a linear combination of the elements of $\boldsymbol{h}$. Defects are not spanned by the elements of $\boldsymbol{h}$. Again, in the polynomial case with four terms,

$$h_3(y)h_4'(y) - h_3'(y)h_4(y) = y^2(3y^2) - (2y)y^3 = y^4 \tag{62}$$

is not spanned by $\{1\ y\ y^2\ y^3\}$. For this case, one symmetry condition is redundant and one implies a linear restriction between the columns of $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})$. Specifically, the proof of the rank restriction for this part of the proposition shows that $\boldsymbol{\alpha}_3(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \equiv \varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}})\boldsymbol{\alpha}_4(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ for some $\varphi : \mathscr{P} \times \tilde{\mathscr{P}} \to \mathbb{R}$ in this case.

In general, the system of equations we construct to reflect symmetry is linear in $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$. If symmetry is not contradicted, then only two possibilities exist. If the square, symmetric, positive semidefinite matrix $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ is nonsingular, then each Jacoby bracket can be written as a linear combination of the income functions. This is the full rank case. If $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ is singular, some equations will be redundant and others will be defects. A redundant equation adds no information. A defect implies a linear restriction between the columns of $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})$. From Lie's theory, we know that a sufficient number of defects is always added as the number of columns of $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ and elements in $\boldsymbol{h}(y)$ increase so that the rank of $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ cannot exceed three, no matter how large K becomes.

> *This part of the proposition states that polynomials produce a maximal number of redundancies and a minimal number of defects for any Gorman system. This property holds whether or not the system is complete.*

The last part of the proof is to show that any polynomial Gorman system with $K \geq 3$ has $rank[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})] \leq 3$. This part of the argument is constructive and relies only on continuity of the symmetry conditions for powers of y from K+1 to $2K$–1. This implies that if the system has a polynomial representation, $\partial y/\partial \boldsymbol{x} = \sum_{k=1}^{K} \boldsymbol{\alpha}_k y^{k-1}$, then the price vectors satisfy $\boldsymbol{\alpha}_k \equiv \varphi_k \boldsymbol{\alpha}_K\ \forall\, k \geq 3$ for some $\varphi_k : \mathscr{P} \times \tilde{\mathscr{P}} \to \mathbb{R}\ \forall\, k \geq 3$. Hence, the rank of $\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}})$ can be no greater than three. ■

3.4 Indirect Preferences

We have been able to recover a closed form solution for the indirect preferences in every full rank case, and the transformed, normalized expenditure function in each case is,

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)=\begin{cases}\beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}})+\theta(\tilde{\boldsymbol{p}},u), & K=1,\\ \beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}})+\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}},u), & K=2,\\ \beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}})-\dfrac{\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}})}{[\theta(\tilde{\boldsymbol{p}},u)-\beta_3(\boldsymbol{x},\tilde{\boldsymbol{p}})]}, & K=3,\lambda\le 0,\\ \beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}})-\dfrac{\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\tan\left(\theta(\tilde{\boldsymbol{p}},u)-\beta_3(\boldsymbol{x},\tilde{\boldsymbol{p}})\right)}, & K=3,\lambda>0,\end{cases} \tag{63}$$

where $\beta_1,\beta_2,\beta_3:\mathcal{P}\times\tilde{\mathcal{P}}\to\mathbb{R}$, $\theta:\tilde{\mathcal{P}}\times\mathcal{U}\to\mathbb{R}$, and λ is the constant term in the integral $\varphi(z)=\int^z ds\big/(1+\lambda s^2)$ in Lewbel (1990) and van Daal and Merkies. The cases $\lambda\le 0$ and $\lambda>0$ represent real and complex roots, respectively, in the Ricatti system,

$$\partial z\big/\partial\boldsymbol{x}=(1+\lambda z^2)\,\partial\gamma_3\big/\partial\boldsymbol{x}\,. \tag{64}$$

Here $z=-\gamma_2\big/(y-\gamma_1)$ and $\gamma_1,\gamma_2,\gamma_3:\mathcal{P}\times\tilde{\mathcal{P}}\to\mathbb{R}$ are price functions that extend the corresponding price functions in van Daal and Merkies (1989) to the incomplete systems case. When the roots are real, $\beta_1=\gamma_1+\kappa\gamma_3$, $\beta_2=2\kappa\gamma_3$, and $\beta_3=e^{2\gamma_2}$, with $-\lambda=\kappa^2$. In the complex roots case, we have $\beta_1=\gamma_1$, $\beta_2=\kappa\gamma_2$, and $\beta_3=\kappa\gamma_3$, with $\lambda=\kappa^2$.

We also have been able to identify the unique indirect utility function that nests the set of all non-trigonometric full rank systems and all reduced rank systems that are polynomials in y within a class of systems that are analytic in y. Let the indirect utility function be

$$v(\boldsymbol{x},\tilde{\boldsymbol{p}},y)=\upsilon\left[\left(\frac{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})-y}\right)^{\eta}-\delta(\boldsymbol{x},\tilde{\boldsymbol{p}}),\tilde{\boldsymbol{p}}\right], \tag{65}$$

where we assume $\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})>y$ for monotonicity and let η be any real number in $[1,\infty)$. By Roy's identity, the incomplete demand system for $\boldsymbol{q}$ is

$$\boldsymbol{q}=m'(y)\left(\frac{\partial\boldsymbol{p}(\boldsymbol{x})^{\intercal}}{\partial\boldsymbol{x}}\right)^{-1}\left[\frac{\partial\gamma}{\partial\boldsymbol{x}}-\frac{\partial\beta}{\partial\boldsymbol{x}}\left(\frac{\gamma-y}{\beta}\right)+\left(\frac{\beta}{\eta}\right)\frac{\partial\delta}{\partial\boldsymbol{x}}\left(\frac{\gamma-y}{\beta}\right)^{\eta+1}\right]. \tag{66}$$

For certain values of η, this takes the form of proposition 1 and illustrates the full nature of its implications. First note that there are precisely three linearly independent functions of y on the right-hand-side of (66). When $\eta=1$, we have a quadratic in y. If η is an integer greater than one, expanding the last term in square brackets with the binomial formula implies that all powers of y from 0 to $\eta + 1$ appear on the right. The demand model cannot be reduced to a quadratic for any $\eta > 1$. The first two terms in square brackets only involve the powers 0 and 1 in y and the sub-matrix of price vectors on the powers of y from 2 through $\eta + 1$ has rank one.

However, η also can be any real non-integer value in $[1,\infty)$ and preferences remain well–behaved with appropriate choices of $\{\beta,\gamma,\delta\}$. In such a case, the last term in square brackets on the right-hand-side of (66) is analytic with a convergent Taylor series expansion over the set of positive values of $\gamma - y$. The vectors of price functions for all of the powers of y greater than one are proportional and the matrix of price functions, even with an infinite number of columns, has rank no greater than three. If there are finitely many terms, then we have a polynomial in y. If the model has full rank, this polynomial is at most a quadratic. Duality implies that this is the unique functional form that nests the non-trigonometric full rank systems and the reduced rank polynomial systems within a class of analytic rank three systems. While a large set of models exists beyond quadratics, each element can be represented as an irreducible polynomial, and each has a matrix of prices coefficients with rank no greater than three.[6]

3.5 The Relationship to Local Transformation Groups

In differential geometry, the space of all real projective transformation groups is commonly associated with the *special linear group two*, $\mathfrak{sl}(2)$. This is generally defined by the set of all 2×2 real matrices,

$$\boldsymbol{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

that have a unit determinant, $\alpha\delta - \beta\gamma = 1$. The inverse matrices,

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix},$$

are members of $\mathfrak{sl}(2)$, as well as the identity $\boldsymbol{I}_2$. Any real projective transformation group can be written in the form

$$y = \frac{\alpha\theta + \beta}{\gamma\theta + \delta} \Leftrightarrow \theta = \frac{\delta y - \beta}{-\gamma y + \alpha}, \; \forall \; \alpha\delta - \beta\gamma = 1. \tag{67}$$

The set of all 2×2 matrix inverses in $\mathfrak{sl}(2)$ are one-to-one and onto the inverse functions

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[6] This argument continues to hold for complete systems by returning to the definitions of $\boldsymbol{x}$ and y in section two and applying the functional form restrictions on y.

for this group, and $\boldsymbol{I}_2$ defines the identity map in both spaces.

If we specify that $\alpha, \beta, \gamma : \boldsymbol{\mathcal{X}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$ and $\theta : \tilde{\boldsymbol{\mathcal{P}}} \times \boldsymbol{\mathcal{U}} \to \mathbb{R}$, simple algebra gives

$$\frac{\partial y}{\partial \boldsymbol{x}} = \left(\alpha \frac{\partial \beta}{\partial \boldsymbol{x}} - \beta \frac{\partial \alpha}{\partial \boldsymbol{x}}\right) + \left[\left(\beta \frac{\partial \gamma}{\partial \boldsymbol{x}} - \gamma \frac{\partial \beta}{\partial \boldsymbol{x}}\right) - \left(\alpha \frac{\partial \delta}{\partial \boldsymbol{x}} - \delta \frac{\partial \alpha}{\partial \boldsymbol{x}}\right)\right] y + \left(\gamma \frac{\partial \delta}{\partial \boldsymbol{x}} - \delta \frac{\partial \gamma}{\partial \boldsymbol{x}}\right) y^2 . \quad (68)$$

This representation defines a class of indirect utility functions in the form

$$v(\boldsymbol{p}, \tilde{\boldsymbol{p}}, m) = \upsilon\left\{\frac{\delta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}) f(m) - \beta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}})}{-\gamma(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}) f(m) + \alpha(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}})}, \tilde{\boldsymbol{p}}\right\}, \quad (69)$$

that generate incomplete Gorman systems, or equivalently, normalized and transformed expenditure functions in the form

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \frac{\alpha(\boldsymbol{x}, \tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}}, u) + \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}}, u) + \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}})}. \quad (70)$$

We immediately see the connection between the class of all non-trigonometric full rank three Gorman systems and a projective transformation group with real parameters. Note that $\gamma \neq 0$ is required for a full rank three system, so that we can rescale the price functions to obtain

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \frac{\tilde{\alpha}(\boldsymbol{x}, \tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}}, u) + \tilde{\beta}(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\theta(\tilde{\boldsymbol{p}}, u) + \tilde{\delta}(\boldsymbol{x}, \tilde{\boldsymbol{p}})} \quad (71)$$

with $\tilde{\alpha} = \alpha/\gamma$, $\tilde{\beta} = \beta/\gamma$, and $\tilde{\delta} = \delta/\gamma$. It is straightforward to convert (71) to the form in (63) by adding and subtracting $\tilde{\alpha}\tilde{\delta}$ in the numerator and rearranging terms. Also note that $\alpha = \delta = 1$ and $\gamma = 0$ gives the rank one case, while $\delta = 1$ and $\gamma = 0$ gives the full rank two case. It also can be shown that the full rank three case with complex roots produces a member of the complex projective transformation group with complex-valued price functions. Thus, the class of all full rank Gorman systems can be derived from a projective transformation group. The converse is also true – the transformed and normalized expenditure function of every full rank Gorman system is a member of this group. The group property also applies to complete Gorman systems if we define $\theta = u$, impose the functional form restrictions on y, and define α, β, γ, and δ to be functions of $\boldsymbol{P}$ rather than $(\boldsymbol{p}, \tilde{\boldsymbol{p}})$. This follows from the solutions for the full rank three indirect preferences derived at the end of section two.

The essential difference between complete and incomplete Gorman systems in this class is that (71) is valid for any $\theta(\tilde{\boldsymbol{p}}, u)$ that is smooth in $(\tilde{\boldsymbol{p}}, u)$ and monotone in u and for any smooth and monotone $y = f(m)$. The function θ is an arbitrary constant of integration obtained by integrating an incomplete system to recover the part of the expenditure function that is associated with the prices $\boldsymbol{p}$. In general, without knowledge of θ's structure − which can only be obtained from the demands for $\tilde{\boldsymbol{q}}$ − it can take any form. Moreover, it is known that (71) holds for any diffeomorphism $y = f(m)$, and there is no functional form restriction on y in an incomplete Gorman system.

3.6 An Illustration

The example we choose to illustrate these results is a generalized AIDS model (LaFrance 2004). Define the smooth and strictly increasing transformation of normalized income by $y = f(m) = m + ½\kappa m^2$, $\kappa > 0$ with inverse $m(y) = \left(\sqrt{1 + 2\kappa y} - 1\right)/\kappa \;\; \forall \; m > 0$. Let the normalized expenditure function be

$$e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u) = m\Big(\alpha_0(\tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p} + ½\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p} - \left(\boldsymbol{\delta}^{\top}\boldsymbol{p} + \theta(\tilde{\boldsymbol{p}}, u)e^{-\boldsymbol{\gamma}^{\top}\boldsymbol{p}}\right)^{-1}\Big). \tag{72}$$

Shephard's/Hotelling's Lemma implies that the demands for $\boldsymbol{q}$ are

$$\begin{aligned}\boldsymbol{q} = (\boldsymbol{\alpha}(\tilde{\boldsymbol{p}}) + \boldsymbol{B}\boldsymbol{p})\left(\frac{1}{1+\kappa m}\right) &+ \boldsymbol{\gamma}\frac{\left[m + ½\kappa m^2 - \left(\alpha_0(\tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p} + ½\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}\right)\right]}{(1+\kappa m)} \\ &+ (\boldsymbol{I} + \boldsymbol{\gamma}\boldsymbol{p}^{\top})\boldsymbol{\delta}\frac{\left[m + ½\kappa m^2 - \left(\alpha_0(\tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p} + ½\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}\right)\right]^2}{(1+\kappa m)}.\end{aligned} \tag{73}$$

This is a full rank three incomplete demand system with Gorman's structure in the income functions

$$\left\{1/(1+\kappa m) \quad (m + ½\kappa m^2)/(1+\kappa m) \quad (m + ½\kappa m^2)^2/(1+\kappa m)\right\}.$$

None have the form m, $m^{1\pm\kappa}$, $m(\ln m)^k$, or $m^{1\pm\iota\tau}$ required in a complete Gorman system and they cannot be reduced to this for any $\kappa > 0$. In fact, we could choose *any* sufficiently smooth and strictly increasing function $y = f(m)$ and apply it to (72) to obtain a comparable expression with three income terms, $\left\{1/f'(m) \quad f(m)/f'(m) \quad f(m)^2/f'(m)\right\}$. Each of these would be a legitimate, full rank three, incomplete Gorman system. This illustrates the functional form result.

The normalized total expenditure on other goods is

$$\begin{aligned}\tilde{m} = m - [\boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p} + \boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}]&\left(\frac{1}{1+\kappa m}\right) \\ - \boldsymbol{\gamma}^{\top}\boldsymbol{p}&\frac{\left[m + ½\kappa m^2 - \left(\alpha_0(\tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p} + ½\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}\right)\right]}{(1+\kappa m)} \\ - (1 + \boldsymbol{\gamma}^{\top}\boldsymbol{p})\boldsymbol{\delta}^{\top}\boldsymbol{p}&\frac{\left[m + ½\kappa m^2 - \left(\alpha_0(\tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p} + ½\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}\right)\right]^2}{(1+\kappa m)}.\end{aligned} \tag{74}$$

This does not have the same form as the demands for $\boldsymbol{q}$. But it is a full rank four Engel curve. Only m belongs to Gorman's class of functional forms. It is a simple exercise to show that if the Slutsky matrix for $\boldsymbol{q}$ is negative semidefinite, then the demand system for $(\boldsymbol{q}, \tilde{m})$ is globally regular. If $n_{\tilde{q}} = 1$ so that adding up *defines* the last demand equation, then not *forcing* this good to have the same functional form as the goods that are modeled relaxes the functional form restrictions on all demands and reserves full rank four for the demand that exhausts the remaining budget. Yet the entire system can be easily rationalized as globally integrable (Epstein 1982).

If, on the other hand, $n_{\tilde{q}} > 1$, then the individual demands for the goods $\tilde{\boldsymbol{q}}$ cannot be identified from the expenditure equation for these goods. It is nevertheless a simple exercise to identify sufficient conditions to globally rationalize the demands in (73). To see this, suppose that $\tilde{\boldsymbol{q}}$ is separable from $\boldsymbol{q}$ with a conditional indirect utility function that is a member of the class of rank four normalized systems in Lewbel (1990). In particular, let

$$\begin{aligned}\tilde{v}(\tilde{\boldsymbol{P}}, \tilde{M}) &= \beta_1(\tilde{\boldsymbol{P}}) - \beta_2(\tilde{\boldsymbol{P}}) \Big/ \Big[\Big(\tilde{M}/\beta_4(\tilde{\boldsymbol{P}})\Big) - \beta_3(\tilde{\boldsymbol{P}})\Big]^2 \\ &= \beta_1(\tilde{\boldsymbol{p}}) - \beta_2(\tilde{\boldsymbol{p}}) \Big/ \Big[\Big(\tilde{m}/\beta_4(\tilde{\boldsymbol{p}})\Big) - \beta_3(\tilde{\boldsymbol{p}})\Big]^2 = \tilde{v}(\tilde{\boldsymbol{p}}, \tilde{m}),\end{aligned} \tag{75}$$

where the second line follows by the 0° homogeneity of $(\beta_1, \beta_2, \beta_3)$ and the 1° homogeneity of β_4. Roy's identity implies that the conditional demands for $\tilde{\boldsymbol{q}}$ are

$$\tilde{\boldsymbol{q}} = \tilde{\boldsymbol{\alpha}}_0(\tilde{\boldsymbol{p}}) + \tilde{\boldsymbol{\alpha}}_1(\tilde{\boldsymbol{p}})\tilde{m} + \tilde{\boldsymbol{\alpha}}_2(\tilde{\boldsymbol{p}})\tilde{m}^2 + \tilde{\boldsymbol{\alpha}}_3(\tilde{\boldsymbol{p}})\tilde{m}^3, \tag{76}$$

with appropriate definitions of the price vectors, $\tilde{\boldsymbol{\alpha}}_k(\tilde{\boldsymbol{p}})$, $k = 0,1,2,3$. Note, in particular, that the deflator π *may or may not* be the same as the price index β_4, *with no effect* on the functional form in (76). Substituting the solution for $\tilde{m}$ from (74) into the conditional demands produces the unconditional demands (Gorman 1970; Blackorby, Primont, and Russell 1978). This subsystem of demands has rank four and Gorman's structure. Each own- and cross-product term of a third order polynomial defined over,

$$\Big\{m \quad 1\big/(1+\kappa m) \quad (m + \tfrac{1}{2}\kappa m^2)\big/(1+\kappa m) \quad (m + \tfrac{1}{2}\kappa m^2)^2\big/(1+\kappa m)\Big\},$$

plus the constant function appears in the demands for $\tilde{\boldsymbol{q}}$. This illustrates the flexibility that is retained for the subset of demands that are not part of an incomplete system.

However, we do not know if the goods $\tilde{\boldsymbol{q}}$ are separable from the goods $\boldsymbol{q}$, much less whether the conditional demands for these goods arise from (75). In other words, this particular set of sufficient conditions to globally rationalize (73) could be too restrictive, or they might be too loose. We simply have no way to know. There is no reason to impose these or any other set of *ad hoc* conditions on the demands that are not included in the model. However, it is clear that normalizing by $\pi(\tilde{\boldsymbol{P}})$ reserves the maximum degree of flexibility for that part of the consumer choice problem that we do not observe, model, or measure – for both functional form and rank.

3.7 Nesting Rank and Functional Form

We now present a simple method to jointly nest the rank and the functional form of incomplete Gorman systems. From our characterization of preferences, a non-trigonometric full rank Gorman system can be written as a special case of the following generalization of the QES,

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) - \frac{\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{[\theta(\tilde{\boldsymbol{p}}, u) - \beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})]}, \tag{77}$$

with $\beta_1, \beta_2, \beta_3 : \boldsymbol{\mathcal{X}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$, $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{p})$, $\left|\partial \boldsymbol{g}^{\mathsf{T}} \big/ \partial \boldsymbol{p}\right| \neq 0$, $\theta : \tilde{\boldsymbol{\mathcal{P}}} \times \boldsymbol{\mathcal{U}} \to \mathbb{R}$, and $\partial\theta \big/ \partial u > 0$. Taking partial derivatives of (77) with respect to the normalized prices $\boldsymbol{p}$ gives

$$f'(m)\boldsymbol{q} = \frac{\partial \boldsymbol{g}(\boldsymbol{p})^{\mathsf{T}}}{\partial \boldsymbol{p}} \left[\frac{\partial \beta_1}{\partial \boldsymbol{x}} + \frac{\partial \beta_2}{\partial \boldsymbol{x}} \left(\frac{f(m) - \beta_2}{\beta_3} \right) - \beta_2 \frac{\partial \beta_3}{\partial \boldsymbol{x}} \left(\frac{f(m) - \beta_2}{\beta_3} \right)^2 \right], \tag{78}$$

where all terms in the square brackets on the right are evaluated at $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{p})$.

A Box-Cox transformation on income clearly nests all non-trigonometric full rank two and three complete systems. A natural extension to incomplete Gorman systems is a Box-Cox transformation of normalized income,

$$y = (m^{\kappa} - 1) \big/ \kappa, \ \ \kappa \geq 0 . \tag{79}$$

An interesting candidate for the definition of $\boldsymbol{x}$ is a vector of Box-Cox transformations of normalized prices,

$$x_i = (p_i^{\lambda} - 1) \big/ \lambda, \ \ i = 1, \cdots, n_q, \ \ \lambda \geq 0 . \tag{80}$$

These definitions of $\boldsymbol{x}$ and y are attractive because they permit us to consider a full range of dependent variables from quantities if $\kappa = \lambda = 1$, to expenditures if $\kappa = 0$ and $\lambda = 1$, to budget shares if $\kappa = \lambda = 0$.

Writing the incomplete demand system with budget shares on the left gives,

$$\boldsymbol{w} = m^{-\kappa} \mathbf{diag}\left[p_i^{\lambda} \right] \times \left[\frac{\partial \beta_1}{\partial \boldsymbol{x}} + \frac{\partial \beta_2}{\partial \boldsymbol{x}} \left(\frac{(m^{\kappa} - 1) \big/ \kappa - \beta_2}{\beta_3} \right) - \beta_2 \frac{\partial \beta_3}{\partial \boldsymbol{x}} \left(\frac{(m^{\kappa} - 1) \big/ \kappa - \beta_2}{\beta_3} \right)^2 \right], \tag{81}$$

where $w_i = p_i q_i \big/ m$ is the budget share of the i^{th} good, and the price functions, β_1, β_2, and β_3 and their derivatives are defined in terms of the Box-Cox transformations of the normalized prices. Thus we have a natural way to nest the Gorman functional forms for all non-trigonometric systems. We also have a simple way to nest the functional form of the price variables. Rank is captured by $\left\{\partial\beta_1/\partial\boldsymbol{x} \quad \partial\beta_2/\partial\boldsymbol{x} \quad \partial\beta_3/\partial\boldsymbol{x}\right\}$, so that the price functions can be tested for dependence on $\boldsymbol{x}$ to determine the rank of the system.

To illustrate this method, again consider the generalized AIDS transformed and normalized expenditure function (LaFrance 2004),

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) \equiv \alpha_0(\tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\mathsf{T}}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{x} - \left(\boldsymbol{\delta}^{\mathsf{T}}\boldsymbol{x} + \theta(\tilde{\boldsymbol{p}},u)e^{-\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{x}}\right)^{-1}. \tag{82}$$

Applying Hotelling's lemma with the above Box-Cox definitions for $\boldsymbol{x}$ and y gives the demands in budget share form for the goods $\boldsymbol{q}$ as

$$\begin{aligned}\boldsymbol{w} = m^{-\kappa}\mathbf{diag}\left[\,p_i^{\lambda}\,\right]\Big\{&\boldsymbol{\alpha}(\tilde{\boldsymbol{p}}) + \boldsymbol{B}\boldsymbol{x} + \boldsymbol{\gamma}\Big[y - \alpha_0(\tilde{\boldsymbol{p}}) - \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\mathsf{T}}\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{x}\Big] \\ &+ (\boldsymbol{I} + \boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{x})\boldsymbol{\delta}\Big[y - \alpha_0(\tilde{\boldsymbol{p}}) - \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\mathsf{T}}\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{x}\Big]^2\Big\}.\end{aligned} \tag{83}$$

As long as $\boldsymbol{\alpha}$ and $\boldsymbol{B}$ do not vanish simultaneously, which is necessary for rank three, it follows that: (a) $\boldsymbol{\gamma} \neq \mathbf{0}$ and $\boldsymbol{\delta} \neq \mathbf{0}$ is necessary and sufficient for full rank three; (b) $\boldsymbol{\gamma} \neq \mathbf{0}$ and $\boldsymbol{\delta} = \mathbf{0}$ is necessary and sufficient for full rank two; (c) $\boldsymbol{\gamma} = \mathbf{0}$ and $\boldsymbol{\delta} \neq \mathbf{0}$ is necessary and sufficient for rank two with the linear term in the deflated and transformed superlative income variable excluded; and (d) $\boldsymbol{\gamma} = \boldsymbol{\delta} = \mathbf{0}$ is necessary and sufficient for a rank one homothetic system. Thus, we obtain a class of models that permits nesting the rank and functional form of incomplete demand systems with a generalized AIDS structure.

As a second illustration, we now construct a PIGL/PIGLOG generalization of quadratic utility. Define the functions

$$\varphi(\boldsymbol{x}) = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{x} + 2\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{x} + 1\,, \tag{84}$$

$$\eta(\boldsymbol{x},\tilde{\boldsymbol{p}}) = \alpha_0(\tilde{\boldsymbol{p}}) - \boldsymbol{\alpha}(\tilde{\boldsymbol{p}})^{\mathsf{T}}\boldsymbol{x}\,, \tag{85}$$

where $\boldsymbol{\alpha}(\tilde{\boldsymbol{p}})$ is a vector–valued function of other prices, $\alpha_0(\tilde{\boldsymbol{p}})$ is a real-valued function of other prices, $\boldsymbol{B}$ is a symmetric $n_q \times n_q$ matrix of parameters, and $\boldsymbol{\gamma}$ is a vector of parameters. The starting point for this application of the nesting procedure is the transformed normalized expenditure function,

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) = \eta(\boldsymbol{x},\tilde{\boldsymbol{p}}) - \left(\frac{\varphi(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\boldsymbol{\delta}^{\mathsf{T}}\boldsymbol{x} + \sqrt{\varphi(\boldsymbol{x},\tilde{\boldsymbol{p}})}\theta(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}\right). \tag{86}$$

Applying Hotelling's lemma with the above Box-Cox transformations gives the incomplete demand equations for the goods $\boldsymbol{q}$ in budget share form as

$$\boldsymbol{w} = m^{-\kappa}\mathbf{diag}\left[\, p_i^{\lambda}\,\right]\left\{\boldsymbol{\alpha} + \left[1 - \boldsymbol{\delta}^{\mathsf{T}}\boldsymbol{x}\left(\frac{y-\eta}{\varphi}\right)\right]\left(\frac{y-\eta}{\varphi}\right)(\boldsymbol{Bx}+\boldsymbol{\gamma}) + \frac{(y-\eta)^2}{\varphi}\boldsymbol{\delta}\right\}. \qquad (87)$$

In this case, $\kappa = \lambda = 0$ gives a rank three extension of a generalized translog type of indirect preferences, $\kappa = \lambda = 1$ gives a generalized quadratic utility type of indirect preferences, and all values of κ and λ produce a PIGL/PIGLOG model that can have rank up to three, and rank two is obtained when $\boldsymbol{\delta} = \boldsymbol{0}$. We again are able to nest both the rank and the functional form of the incomplete system of demand equations within a single unifying framework.

4. Conclusions

This paper extends Gorman's theory of demand to incomplete systems. In contrast to complete systems, there is no restriction on the functional form of the income variables. On the other hand, the maximal rank is three for both complete and incomplete Gorman systems. We close an important gap in the solutions for indirect preferences of full rank three complete systems. We also obtain closed form solutions for the indirect preferences of all full rank systems and for a large class of reduced rank systems that have a minimal degree of linear dependence in the vectors of price functions. We show the relationship between full rank Gorman systems and projective transformation groups. We develop a simple procedure to nest rank and functional form. We then use Box-Cox transformations of normalized prices and income to create two illustrations of this method.

Let us return briefly to the question of the consistent specification and estimation of econometric demand systems that include expenditure, income, cost, or profit as an explanatory variable. Every full rank and minimally degenerate reduced rank Gorman system can be expressed in a polynomial form. This implies that consistent estimation of the conditional demand equations requires precisely K–1 instruments; one each for the raw moments of y, y^2, …, y^{K-1}. An appropriate set of instruments should be readily available. Any robust IV or GMM method will produce consistent and asymptotically normal parameter estimates. Thus, a deeper understanding of Gorman systems generates a natural and straightforward solution to the endogeneity and errors in variables problems that arise in conditional demand models, both for complete and incomplete demand systems.

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Mathematical Appendix

A.1. Lemmas on Symmetry, Curvature, and Functional Form

Lemma 1. *Let* $E : \mathcal{P} \times \mathcal{U} \to \mathcal{M}$ *be twice differentiable on* $\mathcal{P} \times \mathcal{U}$, *let* $y = f(E)$, $f \in \mathcal{C}^2$, $f' > 0\ \forall\ M \in \mathcal{M}$, *and let* $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{P})$, $\boldsymbol{g} \in \mathcal{C}^2\ \forall\ \boldsymbol{p} \in \mathcal{P}$, *with* $E(\boldsymbol{P}, u) = M\left(y(\boldsymbol{g}(\boldsymbol{P}), u)\right)$. *Then* $\partial^2 E(\boldsymbol{P}, u) \big/ \partial \boldsymbol{P} \partial \boldsymbol{P}^\intercal$ *is symmetric at (P,u) if and only if* $\partial^2 y(\boldsymbol{g}(\boldsymbol{P}), u) \big/ \partial \boldsymbol{x} \partial \boldsymbol{x}^\intercal$ *is symmetric at (g(P),u). In addition, if* $x_i = g_i(P_i)$, $g_i \in \mathcal{C}^2$, $g_i' > 0$, $g_i'' \leq 0, \forall\, p_i \in \mathcal{P}_i \subset \mathbb{R}_+, \forall\, i$, $M'' \leq 0\ \forall\ y \in \mathcal{Y}$, *and y is concave in x, then E is concave in* $\boldsymbol{P}$.

Proof: We have

$$\frac{\partial E}{\partial \boldsymbol{P}} = M' \frac{\partial \boldsymbol{g}^\intercal}{\partial \boldsymbol{P}} \frac{\partial y}{\partial \boldsymbol{x}}, \tag{A.1}$$

so that

$$\frac{\partial^2 E}{\partial \boldsymbol{P} \partial \boldsymbol{P}^\intercal} = M'' \frac{\partial \boldsymbol{g}^\intercal}{\partial \boldsymbol{P}} \frac{\partial y}{\partial \boldsymbol{x}} \frac{\partial y}{\partial \boldsymbol{x}^\intercal} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{P}^\intercal} + M' \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} \frac{\partial^2 g_i}{\partial \boldsymbol{P} \partial \boldsymbol{P}^\intercal} + M' \frac{\partial \boldsymbol{g}^\intercal}{\partial \boldsymbol{P}} \frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^\intercal} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{P}^\intercal}. \tag{A.2}$$

The first two terms on the right are automatically symmetric, so that symmetry of the left-hand-side is equivalent to symmetry of the Hessian matrix on the far right-hand-side. The first two matrices on the right are negative semidefinite when $M'' \leq 0$ and $g_i'' \leq 0\ \forall\ i$, so if $\partial^2 y \big/ \partial \boldsymbol{x} \partial \boldsymbol{x}^\intercal$ is negative semidefinite then $\partial^2 E \big/ \partial \boldsymbol{P} \partial \boldsymbol{P}^\intercal$ is too. ■

Lemma 2. $f : \mathcal{M} \to \mathcal{Y}$, $f \in \mathcal{C}^\infty$, $f' > 0$, *satisfies the differential equation*

$$M f'(M) = \sum_{k=1}^{K} a_k h_k\left(f(M)\right)\ \forall\ M \in \mathcal{M},$$

if and only if $f(M) \in \left\{\ln M \quad M^\kappa \quad M^{\iota\tau}\right\}$, *for* κ, τ *real constants and* $\iota = \sqrt{-1}$.

Proof: Simply integrate the differential equation in each possible case. ■

Lemma 3. *If* $z : \mathcal{P} \times \mathcal{U} \to \mathbb{R}$, $z \in \mathcal{C}^\infty$ *satisfies the partial differential equations,*

$$\frac{\partial z(\boldsymbol{P}, u)}{\partial \boldsymbol{P}} = \left[1 + \gamma\left(\beta_2(\boldsymbol{P})\right) z(\boldsymbol{P}, u)^2\right] \frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}},$$

then either $\gamma\left(\beta_2(\boldsymbol{P})\right) \equiv \lambda$, *a nonzero constant, or*

$$\frac{\partial z(\boldsymbol{P}, u)}{\partial \boldsymbol{P}} = \frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}}.$$

Proof: Divide both sides of the system of partial differential equations by the term in square brackets on the right. This implies

$$\frac{\partial z / \partial \boldsymbol{P}}{1+\gamma z^2} = \frac{\partial \beta_2}{\partial \boldsymbol{P}}\,.$$

The matrix of second-order partial derivatives on both sides must therefore be symmetric. This in turn implies that

$$\frac{\partial^2 z / \partial \boldsymbol{P}\,\partial \boldsymbol{P}^{\mathsf{T}}}{\left[1+\gamma z^2\right]} - \frac{2\gamma z}{\left[1+\gamma z^2\right]^2}\frac{\partial z}{\partial \boldsymbol{P}}\frac{\partial z}{\partial \boldsymbol{P}^{\mathsf{T}}} - \left(\frac{z}{1+\gamma z^2}\right)^2 \gamma' \frac{\partial z}{\partial \boldsymbol{P}}\frac{\partial \beta_2}{\partial \boldsymbol{P}^{\mathsf{T}}} = \frac{\partial^2 \beta_2}{\partial \boldsymbol{P} \partial \boldsymbol{P}^{\mathsf{T}}}\,.$$

The first two matrices on the left and the matrix on the right are automatically symmetric. It follows that $\gamma' \times (\partial z / \partial \boldsymbol{P}) \times (\partial \beta_2 / \partial \boldsymbol{P})^{\mathsf{T}}$ must be symmetric. There are only two ways this can hold: either (1) $\gamma' = 0$; or (2) $z(\boldsymbol{x}, u) = \psi\left(\beta_2(\boldsymbol{x})\right) + u$ for some $\psi : \mathbb{R} \to \mathbb{R}$. In the second case, we then must have

$$\psi'(\beta_2(\boldsymbol{P})) = 1 + \gamma\left(\beta_2(\boldsymbol{P})\right)\left(\psi(\beta_2(\boldsymbol{P})) + u\right)^2,$$

identically in u. Differentiating twice with respect to u then implies $\gamma\left(\beta_2(\boldsymbol{P})\right) = 0$, which contradicts $\gamma' \neq 0$. ■

A.2. Homogeneity and PIGL and PIGLOG

Consider the quasi-linear ordinary differential equation

$$\frac{y'(x)}{y(x)} = \frac{d \ln\left(y(x)\right)}{dx} = \alpha(x) + \beta(x) f\left(y(x)\right). \tag{A.3}$$

This differential equation forms the basis for the functional form question of Muellbauer (1975, 1976). In particular, the simplest form of this question is, "What is the class of functions $f(y)$ that can satisfy (A.3) and the 0° homogeneity condition,

$$\alpha'(x)x + \beta'(x)xf(y) + \beta(x)f'(y)y \equiv 0\,?\text{"} \tag{A.4}$$

There are only two possibilities: a special case of Bernoulli's equation,

$$\frac{y'}{y} = \alpha_0 + \beta_0 \left(\frac{y}{x}\right)^{\kappa},\ \kappa \neq 0\,; \tag{A.5}$$

or a special case of the logarithmic transformation,

$$\frac{y'}{y} = \alpha_0 + \beta_0 \ln\left(\frac{y}{x}\right). \tag{A.6}$$

The reason for this can be obtained by analyzing the implications of (A.4) directly. First, consider the case where $\alpha'(x)x = 0$, so that $\alpha(x) = \alpha_0$, a constant. Then (A.4) reduces to

$$\beta'(x)xf(y) + \beta(x)f'(y)y \equiv 0\,, \tag{A.7}$$

or equivalently,

$$\frac{d\ln(f)}{d\ln(y)} = \frac{f'(y)}{f(y)}y = -\frac{\beta'(x)}{\beta(x)}x = -\frac{d\ln(\beta)}{d\ln(x)} = \kappa\,, \tag{A.8}$$

where κ is a constant because the left-hand-side is independent of x, while the right-hand-side is independent of y. Without any loss in generality, the solutions are $f(y) = y^{\kappa}$ and $\beta(x) = \beta_0 x^{-\kappa}$.

Now suppose that $\alpha'(x)x \neq 0$, so that

$$\beta'(x)xf(y) + \beta(x)f'(y)y = -\alpha'(x)x\,. \tag{A.9}$$

Since the right-hand-side is again independent of y, at least one of the terms on the left also must be independent of y. If $f'(y) = 0$, so that $f(y) = f_0$ is constant, we obtain the degenerate case where the functions of y on the right-hand-side of (A.3) are not linearly independent. Hence, it must be that $\beta'(x)x = 0$, i.e., $\beta(x) = \beta$, a constant, and

$$f'(y)y = \frac{df(y)}{d\ln(y)} = -\frac{\alpha'(x)x}{\beta} = \lambda\,, \tag{A.10}$$

where λ is a constant again because the left-hand-side is independent of $\boldsymbol{x}$ and the right-hand-side is independent of y. Solving the left side gives

$$f(y) = \lambda\ln(y) + \gamma\,, \tag{A.11}$$

while the right-hand-side can be rewritten as

$$\frac{d\alpha(x)}{d\ln(x)} = -\lambda\beta\,, \tag{A.12}$$

which has solution

$$\alpha(x) = \alpha - \lambda\beta\ln(x)\,. \tag{A.13}$$

Combining (A.11) and (A.13), we obtain (A.6), with $\alpha_0 = \alpha + \beta\gamma$ and $\beta_0 = \beta\lambda$.

The implication is that, for ranks one and two demand models in this class, the admissible forms of $f(y)$ are completely determined by homogeneity.

When we consider incomplete demand systems, we do not have homogeneity in the prices of interest or adding up to restrict the functional form. For Bernoulli's differential equation,

$$y' = \alpha(x)y + \beta(x)y^{1-\kappa},\ \kappa \neq 0\,, \tag{A.14}$$

if we note that $d\left(y^{\kappa}/\kappa\right)\big/dx = y^{\kappa-1}y'$, we can rewrite this as the linear ordinary differential equation in $f(y) = y^{\kappa}/\kappa$,

$$\frac{d}{dx}\left(y^{\kappa}/\kappa\right) = y^{\kappa-1}y' = \left(\kappa\alpha(x)\right)\left(y^{\kappa}/\kappa\right) + \beta(x)\,, \tag{A.15}$$

with complete solution

$$y(x)=\left[\kappa e^{\int^x \kappa\alpha(s)ds}\left(\int^x e^{-\int^s \kappa\alpha(t)dt}\beta(s)ds+c\right)\right]^{1/\kappa}. \tag{A.16}$$

Similarly, the logarithmic first-order linear differential equation is

$$\frac{d\ln(y)}{dx}=\frac{y'}{y}=\alpha(x)+\beta(x)\ln(y), \tag{A.17}$$

with complete solution

$$y(x)=\exp\left\{e^{\int^x \beta(s)ds}\left(\int^x e^{-\int^s \beta(t)dt}\alpha(s)ds+c\right)\right\}. \tag{A.18}$$

The generic nature of both of these differential equations is that they can be written as simple linear first-order ordinary differential equations,

$$\frac{df(y(x))}{dx}=\alpha(x)+\beta(x)f(y(x)). \tag{A.19}$$

When y is normalized income and the demands do not absorb all of the budget, homogeneity and adding up do not impose any restriction on the class of functions $f(y)$ that can solve this differential equation, and the complete class of solutions is

$$y(x)=f^{-1}\left[e^{\int^x \kappa\alpha(s)ds}\left(\int^x e^{-\int^s \kappa\alpha(t)dt}\beta(s)ds+c\right)\right]. \tag{A.20}$$

The upshot is that if one of the income functions is M, which must be the case in any complete system, and the demands are zero degree homogeneous in prices and (nominal) income, then we are restricted to the PIGL and PIGLOG functional forms. Conversely, when the system is incomplete and income has been normalized by a function of other goods' prices, then neither of these restrictions must apply, and there is no functional form restriction in a full rank two incomplete Gorman system.

A.3. Unique Representations of Gorman Systems

We need two conditions on the number of goods relative to the number of income functions and the relationship between and among the price and income functions to ensure that the demand system has a unique representation. Let the $n\times K$ matrix of price functions be denoted by $\boldsymbol{A}(\boldsymbol{x})=[\boldsymbol{\alpha}_1(\boldsymbol{x})\cdots\boldsymbol{\alpha}_K(\boldsymbol{x})]$ and let the $K\times 1$ vector of income functions be denoted by $\boldsymbol{h}(y)$. The first condition we need is that the $\{h_k(y)\}_{k=1}^K$ are linearly independent with respect to the constants in K–dimensional space. That is, there can exist no $\boldsymbol{c}\in\mathbb{R}^K$ satisfying $\boldsymbol{c}\neq\boldsymbol{0}$ and $\boldsymbol{c}^\intercal\boldsymbol{h}(y^1)=0\ \ \forall\ y^1\in\boldsymbol{\mathcal{N}}(y)\subset\mathbb{R}$, where $\boldsymbol{\mathcal{N}}(y)$ is an open neighborhood of an arbitrary point in the interior of $\boldsymbol{\mathcal{Y}}\subset\mathbb{R}$, the domain of definition for y. The reason we need this condition is that if it is not satisfied, then $\forall\ \boldsymbol{d}\in\mathbb{R}^K$, adding the n–vector $\boldsymbol{A}(\boldsymbol{x})\boldsymbol{d}\boldsymbol{c}^{\mathsf{T}}\boldsymbol{h}(y) \equiv \boldsymbol{0}$ to the system of demands does not change it,

$$\begin{aligned}\frac{\partial y}{\partial \boldsymbol{x}} &= \sum_{k=1}^{K} \boldsymbol{\alpha}_k(\boldsymbol{x}) h_k(y) + \sum_{k=1}^{K} \boldsymbol{\alpha}_k(\boldsymbol{x}) d_k \left(\sum_{\ell=1}^{K} c_\ell h_\ell(y) \right) \\ &= \sum_{k=1}^{K} \boldsymbol{\alpha}_k(\boldsymbol{x}) \Big[h_k(y) + d_k \textstyle\sum_{\ell=1}^{K} c_\ell h_\ell(y) \Big] \\ &= \boldsymbol{A}(\boldsymbol{x}) \big(\boldsymbol{I} + \boldsymbol{d}\boldsymbol{c}^{\mathsf{T}} \big) \boldsymbol{h}(y)\,. \end{aligned} \tag{A.21}$$

We could therefore choose different $\boldsymbol{d}$ vectors to make the matrix $\boldsymbol{A}(\boldsymbol{x})$ become anything at all, and the demand model would be meaningless.

Similarly, we require that the columns of $\boldsymbol{A}(\boldsymbol{x})$ are linearly independent with respect to the K–dimensional constants. For this to hold, there can be no $\boldsymbol{c} \in \mathbb{R}^K$ that satisfies $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{A}(\boldsymbol{x}^1)\boldsymbol{c} = \boldsymbol{0} \;\; \forall \;\boldsymbol{x}^1 \in \boldsymbol{\mathcal{N}}(\boldsymbol{x})$, where here $\boldsymbol{\mathcal{N}}(\boldsymbol{x})$ is an open neighborhood of any point in the interior of $\boldsymbol{\mathcal{X}} \subset \mathbb{R}^n$, the domain of definition for $\boldsymbol{x}$. As before, if this did not hold, then $\forall \;\boldsymbol{d} \in \mathbb{R}^K$, adding $\boldsymbol{A}(\boldsymbol{x})\boldsymbol{c}\boldsymbol{d}^{\mathsf{T}}\boldsymbol{h}(y) \equiv \boldsymbol{0}$ to the system does not change it,

$$\frac{\partial y}{\partial \boldsymbol{x}} = \sum_{k=1}^{K} \Big[\boldsymbol{\alpha}_k(\boldsymbol{x}) + \Big(\textstyle\sum_{\ell=1}^{K} \boldsymbol{\alpha}_\ell(\boldsymbol{x}) c_\ell \Big) d_k \Big] h_k(y) = \boldsymbol{A}(\boldsymbol{x}) \big(\boldsymbol{I} + \boldsymbol{c}\boldsymbol{d}^{\mathsf{T}} \big) \boldsymbol{h}(y)\,. \tag{A.22}$$

We could again choose different vectors $\boldsymbol{d}$ to make the matrix $\boldsymbol{A}(\boldsymbol{x})$ become anything, and the demand model makes no sense. We assume throughout that the dimensions of $\boldsymbol{A}$ and $\boldsymbol{h}$ have been reduced as necessary to guarantee a unique representation (see Gorman 1981: 358-59; or Russell and Farris 1998: 201-202).

It is important to keep in mind that linear independence across the set of K–dimensional constants is not equivalent to $\boldsymbol{A}(\boldsymbol{x})$ having full column rank. In particular, if the vector of coefficients in a linear combination of the columns of $\boldsymbol{A}(\boldsymbol{x})$ is a function of $\boldsymbol{x}$ and/or y, then a vector $\boldsymbol{c}(\boldsymbol{x},y)$ might exist such that $\boldsymbol{A}(\boldsymbol{x})\boldsymbol{c}(\boldsymbol{x},y) = 0$ even if both properties discussed above are satisfied if $\boldsymbol{A}(\boldsymbol{x})$ does not have full column rank.

A.4. Purely Real or Purely Complex Roots in Rank 3

In the rank three case for Gorman systems, one case that warrants further consideration is when the adding up condition implies $Mf'(M) = bf(M)$ and the constant b may be a real or a complex number, say $b = b_1 + b_1\iota$. The question of interest in this case is why must b be either purely real or purely complex? Here we show that this restriction follows from the fact that the expenditure function and the demand functions are real-valued, and is not due to Slutsky symmetry. To see this, suppose that b is complex and integrate the linear, first-order ordinary differential equation to get $f(M) = M^{b_0 + b_1\iota}$ in a similar manner as for the rank two case. It is sufficient to consider the full rank three case since complex roots must appear as pairs of complex conjugates. We can write the demands for $\boldsymbol{q}$ in the form

$$f'(M)\boldsymbol{q} = (\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\iota) + (\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1\iota)f(M) + (\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1\iota)f(M)^2 \tag{A.23}$$

for real-valued some functions $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1 : \mathcal{P} \rightarrow \mathbb{R}^n$. Substituting for $f(M)$ and $f'(M)$ into this equation, and solving for $\boldsymbol{q}$, we have

$$\boldsymbol{q} = M\left\{\left(\frac{\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\iota}{b_0 + b_1\iota}\right)M^{-(b_0+b_1\iota)} + \left(\frac{\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1\iota}{b_0 + b_1\iota}\right) + \left(\frac{\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1\iota}{b_0 + b_1\iota}\right)M^{(b_0+b_1\iota)}\right\}. \tag{A.24}$$

The terms in the braces must be real-valued, and applying de Moivre's theorem, this implies that each individual vector of terms inside the braces of

$$\begin{aligned}\boldsymbol{q} = M\Bigg\{&\left[\left(\frac{\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\iota}{b_0 + b_1\iota}\right)M^{-b_0} + \left(\frac{\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1\iota}{b_0 + b_1\iota}\right)M^{b_0}\right]\cos\left(b_1 \ln M\right) + \left(\frac{\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1\iota}{b_0 + b_1\iota}\right) \\ &+\iota\left[\left(\frac{\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1\iota}{b_0 + b_1\iota}\right)M^{b_0} - \left(\frac{\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\iota}{b_0 + b_1\iota}\right)M^{-b_0}\right]\sin\left(b_1 \ln M\right)\Bigg\}\end{aligned} \tag{A.25}$$

must be real-valued. First, we must have $\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1\iota = (b_0 + b_1\iota)\tilde{\boldsymbol{\beta}}$ for some $\tilde{\boldsymbol{\beta}} : \mathcal{P} \rightarrow \mathbb{R}^n$. Second, it can not be the case that $\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\iota = (b_0 + b_1\iota)\tilde{\boldsymbol{\alpha}}$ and $\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1\iota = (b_0 + b_1\iota)\tilde{\boldsymbol{\gamma}}$ for any $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\gamma}} : \mathcal{P} \rightarrow \mathbb{R}^n$, since the vector of coefficients on $\sin\left(b_1 \ln M\right)$ will be complex-valued. Hence, it follows that if $b_1 \neq 0$ then $b_0 = 0$. Conversely, if $b_0 \neq 0$ then $b_1 = 0$ since the two terms M^{-b_0} and M^{+b_0} are linearly independent for all $b_0 \neq 0$. Finally, in the purely complex case, we must have $-\iota(\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\iota) - \iota(\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1\iota)$ and $-(\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\iota) + (\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1\iota)$ both as real-valued functions. This, in turn, implies that $\boldsymbol{\gamma}_0 = -\boldsymbol{\alpha}_0$ and $\boldsymbol{\gamma}_1 = \boldsymbol{\alpha}_1$, and the demands take the form

$$\boldsymbol{q} = M\left[\tilde{\boldsymbol{\alpha}}_0 + \tilde{\boldsymbol{\alpha}}_1 \cos\left(b_1 \ln M\right) + \tilde{\boldsymbol{\alpha}}_2 \sin\left(b_1 \ln M\right)\right], \tag{A.26}$$

where $\tilde{\boldsymbol{\alpha}}_0 = \tilde{\boldsymbol{\beta}}$, $\tilde{\boldsymbol{\alpha}}_1 = 2\boldsymbol{\alpha}_1/b_1$, and $\tilde{\boldsymbol{\alpha}}_2 = -2\boldsymbol{\alpha}_2/b_1$.

A.5. Proof of Proposition 1.

Proposition 1. *Every full rank weakly integrable incomplete Gorman system has* $K \leq 3$ *and a definition for* $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) \equiv f\left(e(\boldsymbol{p}(\boldsymbol{x}), \tilde{\boldsymbol{p}}, u)\right)$ *exists such that*

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)}{\partial \boldsymbol{x}} = \begin{cases} \boldsymbol{\alpha}_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}), & K = 1, \\ \boldsymbol{\alpha}_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u), & K = 2, \\ \boldsymbol{\alpha}_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) + \boldsymbol{\alpha}_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)^2, & K = 3. \end{cases}$$

Conversely, if $K \geq 3$ *and a maximum number of the Jacoby brackets,*

$$h_k(y)h_\ell'(y) - h_k'(y)h_\ell(y),\ k < \ell\ ,$$

are spanned by the basis $\{h_1(y)\cdots h_K(y)\}$ *in the sense that*

$$h_k(y)h_\ell'(y) - h_k'(y)h_\ell(y) = d_{k\ell}^1 h_1(y) + \cdots d_{k\ell}^K h_K(y),\ k < \ell\ ,$$

where the $\{d_{k\ell}^j\}$ *are absolute constants, then* $rank\left[\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}})\right] = 3$ *, and a definition for* y *exists such that*

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \boldsymbol{\alpha}_2(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) + \cdots + \boldsymbol{\alpha}_K(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)^{K-1}.$$

Proof: By Young's theorem, the second-order cross partial derivatives of y with respect to $\boldsymbol{x}$ must be symmetric for integrability,

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \sum_{k=1}^{K}\left(\frac{\partial \alpha_{ik}}{\partial x_j} h_k + \alpha_{ik}h_k' \sum_{\ell=1}^{K}\alpha_{j\ell}h_\ell\right)$$
$$= \sum_{k=1}^{K}\left(\frac{\partial \alpha_{jk}}{\partial x_i} h_k + \alpha_{jk}h_k' \sum_{\ell=1}^{K}\alpha_{i\ell}h_\ell\right) = \frac{\partial^2 y}{\partial x_j \partial x_i} \quad \forall \quad i \neq j\,. \tag{A.27}$$

These can be re-expressed in terms of $\frac{1}{2}n_q(n_q-1)$ vanishing differences,

$$0 = \sum_{k=1}^{K}\left(\frac{\partial \alpha_{ik}}{\partial x_j} - \frac{\partial \alpha_{jk}}{\partial x_i}\right)h_k + \sum_{k=1}^{K}\sum_{\ell=1}^{K}\alpha_{ik}\alpha_{j\ell}\left(h_k'h_\ell - h_kh_\ell'\right), \forall\ j < i = 2,\cdots,n_q\,. \tag{A.28}$$

In the double sum on the right-hand-side if (A.28), when $k = \ell$, the term $\alpha_{ik}\alpha_{jk}$ is multiplied by the Jacoby bracket, $h_k'h_k - h_kh_k' = 0$. On the other hand, when $k \neq \ell$, the Jacoby bracket $h_k'h_\ell - h_kh_\ell'$ appears twice, once multiplied by $\alpha_{ik}\alpha_{j\ell}$ and once multiplied by $-\alpha_{i\ell}\alpha_{jk}$. Therefore, rewrite (A.28) as

$$0 = \sum_{k=1}^{K}\left(\frac{\partial \alpha_{ik}}{\partial x_j} - \frac{\partial \alpha_{jk}}{\partial x_i}\right)h_k + \sum_{k=2}^{K}\sum_{\ell=1}^{k-1}\left(\alpha_{ik}\alpha_{j\ell} - \alpha_{jk}\alpha_{i\ell}\right)\left(h_k'h_\ell - h_kh_\ell'\right), j < i = 2\cdots n_q\,, \tag{A.29}$$

a linear system of $\frac{1}{2}n_q(n_q-1)$ equations in the $\frac{1}{2}K(K-1)$ Jacoby brackets $h_k'h_\ell - h_kh_\ell'$.

The first step in the proof of the proposition is to restate (A.29) in matrix form. Define

$$\boldsymbol{B} = \begin{bmatrix} \alpha_{22}\alpha_{11} - \alpha_{12}\alpha_{21} & \cdots & \alpha_{2k}\alpha_{1\ell} - \alpha_{1k}\alpha_{2\ell} & \cdots & \alpha_{2K}\alpha_{1K-1} - \alpha_{1K}\alpha_{2K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{i2}\alpha_{j1} - \alpha_{j2}\alpha_{i1} & \cdots & \alpha_{ik}\alpha_{j\ell} - \alpha_{ik}\alpha_{j\ell} & \cdots & \alpha_{2K}\alpha_{1K-1} - \alpha_{1K}\alpha_{2K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n_q2}\alpha_{n_q-1,1} - \alpha_{n_q-1,2}\alpha_{n_q1} & \cdots & \alpha_{n_qk}\alpha_{n_q-1,\ell} - \alpha_{n_q-1,k}\alpha_{n_q\ell} & \cdots & \alpha_{n_qK}\alpha_{n_q-1,K-1} - \alpha_{n_q-1,K}\alpha_{n_qK-1} \end{bmatrix},$$

$$\boldsymbol{C} = -\begin{bmatrix} \dfrac{\partial \alpha_{11}}{\partial x_2} - \dfrac{\partial \alpha_{21}}{\partial x_1} & \cdots & \dfrac{\partial \alpha_{1K}}{\partial x_2} - \dfrac{\partial \alpha_{2K}}{\partial x_1} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial \alpha_{i1}}{\partial x_j} - \dfrac{\partial \alpha_{j1}}{\partial x_i} & \cdots & \dfrac{\partial \alpha_{iK}}{\partial x_j} - \dfrac{\partial \alpha_{jK}}{\partial x_i} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial \alpha_{n_q 1}}{\partial x_{n_q-1}} - \dfrac{\partial \alpha_{n_q-1,1}}{\partial x_{n_q}} & \cdots & \dfrac{\partial \alpha_{n_q K}}{\partial x_{n_q-1}} - \dfrac{\partial \alpha_{n_q-1K}}{\partial x_{n_q}} \end{bmatrix},$$

$$\boldsymbol{h} = \begin{bmatrix} h_1 & \cdots & h_K \end{bmatrix}^{\mathsf{T}},$$

and
$$\tilde{\boldsymbol{h}} = \begin{bmatrix} h_2' h_1 - h_2 h_1' & \cdots & h_k' h_\ell - h_k h_\ell' & \cdots & h_K' h_{K-1} - h_K h_{K-1}' \end{bmatrix}^{\mathsf{T}}.$$

$\boldsymbol{B}$ is $½n_q(n_q-1)\times ½K(K-1)$, $\boldsymbol{C}$ is $½n_q(n_q-1)\times K$, $\boldsymbol{h}$ is $K\times 1$, and the vector of Jacoby brackets $\tilde{\boldsymbol{h}}$ is $½K(K-1)\times 1$.

This gives the symmetry conditions in matrix form as

$$\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}. \tag{A.30}$$

For this to be a well-posed system of equations we must have at least as many equations as unknowns, which is equivalent to $n_q \geq K$. Assume this is so. Premultiply both sides of (A.30) by $\boldsymbol{B}^{\mathsf{T}}$ to get the square system, $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}\boldsymbol{h}$. The rank result of Lie (1880) when $\boldsymbol{B}$ has full column rank is that $½K(K-1) \leq K$, equivalently, $K \leq 3$ (Hermann 1975: 143-146). The reason is a direct result of linear algebra. The rank of $\boldsymbol{B}$ is inherited from the rank of a (Hermann 1975: 141). Since $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ is of order $½K(K-1)\times ½K(K-1)$ and has rank no greater than K (the rank of $\boldsymbol{A}$), it follows that $K \leq 3$, completing the proof of the first part of the proposition for the full rank case.

The next step is to obtain the representation result for the full rank case. Assume that $\boldsymbol{B}$ has full column rank. The least squares formula for $\tilde{\boldsymbol{h}}$ as a function of $\boldsymbol{h}$ is

$$\tilde{\boldsymbol{h}} = (\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}\boldsymbol{h} \equiv \boldsymbol{D}\boldsymbol{h}. \tag{A.31}$$

The vectors $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ depend only on y and not on $(\boldsymbol{x}, \tilde{\boldsymbol{p}})$, while the matrix $\boldsymbol{D}$ depends only on $(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ and not on y. It follows that the elements of $\boldsymbol{D}$ are absolute constants; a fundamental property of this type of system. $\boldsymbol{B}$ is of order $½n_q(n_q-1)\times ½K(K-1)$ and $\boldsymbol{C}$ is of order $½n_q(n_q-1)\times K$, so that $\boldsymbol{D}$ is of order $½K(K-1)\times K$. When $K = 1$, $\boldsymbol{D}$ has zero rows (there are no Jacoby brackets), when $K = 2$, $\boldsymbol{D}$ has one row and two columns), and when $K = 3$, $\boldsymbol{D}$ has three rows and three columns. If $K > 3$, then $\boldsymbol{D}$ would have more rows than columns (i.e., more Jacoby brackets than income functions), and the full rank condition cannot be satisfied. We address each full rank case in turn.

Rank 1:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}_1 h_1(y)\,. \tag{A.32}$$

Integrability implies that

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^\mathsf{T}} = \frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^\mathsf{T}} h_1(y) + h_1'(y) \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1^\mathsf{T} \tag{A.33}$$

is symmetric. Hence, $\partial \boldsymbol{\alpha}_1 \big/ \partial \boldsymbol{x}^\mathsf{T}$ must be symmetric, which is necessary and sufficient for the existence of a function, $\beta : \boldsymbol{\mathscr{P}} \times \tilde{\boldsymbol{\mathscr{P}}} \to \mathbb{R}$, such that $\partial \beta \big/ \partial \boldsymbol{x} = \boldsymbol{\alpha}_1$. Rewrite the system as

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial \beta}{\partial \boldsymbol{x}} h_1(y)\,, \tag{A.34}$$

and separate the variables (recall that $h_1(y) \neq 0$ is required for $\partial y \big/ \partial \boldsymbol{x} \gg \mathbf{0}$) to obtain

$$\gamma(y) \equiv \int^y h_1(s)^{-1} ds = \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}}, u)\,. \tag{A.35}$$

From this we have

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \gamma'(y) \frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)} \frac{\partial y}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}_1\,. \tag{A.36}$$

Therefore, a representation for y exists (by composing γ and f), such that $\partial y \big/ \partial \boldsymbol{x} = \boldsymbol{\alpha}_1$ and $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}}, u)$, with $\boldsymbol{\alpha}_1 = \partial \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \big/ \partial \boldsymbol{x}$.

Rank 2:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}_1 h_1(y) + \boldsymbol{\alpha}_2 h_2(y)\,. \tag{A.37}$$

Integrability implies that

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^\mathsf{T}} = \frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^\mathsf{T}} h_1 + \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{x}^\mathsf{T}} h_2 + \left(\boldsymbol{\alpha}_1 h_1' + \boldsymbol{\alpha}_2 h_2'\right)\left(\boldsymbol{\alpha}_1 h_1 + \boldsymbol{\alpha}_2 h_2\right)^\mathsf{T} \tag{A.38}$$

is symmetric. Expanding gives

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^\mathsf{T}} = \frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^\mathsf{T}} h_1 + \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{x}^\mathsf{T}} h_2 + \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1^\mathsf{T} h_1' h_1 + \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2^\mathsf{T} h_1' h_2 + \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_1^\mathsf{T} h_2' h_1 + \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_2^\mathsf{T} h_2' h_2\,, \tag{A.39}$$

and the terms $\boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1^\mathsf{T} h_1' h_1$ and $\boldsymbol{\alpha}_2 \boldsymbol{\alpha}_2^\mathsf{T} h_2' h_2$ are automatically symmetric. Since $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are linearly independent, $\boldsymbol{\alpha}_2 \neq c \boldsymbol{\alpha}_1$ for any $c \in \mathbb{R}$. Otherwise, $rank\left(\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})\right) = [\boldsymbol{\alpha}_1\ \boldsymbol{\alpha}_2]$ is only 1, not 2. Hence, $\boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2^\mathsf{T}$ is not symmetric. Since h_1 and h_2 are functionally independent (equivalently, are locally linearly independent), $h_1' h_2 \neq h_1 h_2'$. Otherwise, $h_2 = c h_1$ for some constant $c \in \mathbb{R}$; a contradiction. Hence, we can premultiply the reduced symmetry conditions by $\boldsymbol{\alpha}_1^\mathsf{T}$ and postmultiply by $\boldsymbol{\alpha}_2$ to obtain

$$
\begin{aligned}
&\boldsymbol{\alpha}_1^{\intercal}\left(\frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^{\intercal}} h_1 + \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{x}^{\intercal}} h_2 + \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2^{\intercal} h_1' h_2 + \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_1^{\intercal} h_2' h_1\right)\boldsymbol{\alpha}_2 = \\
&\boldsymbol{\alpha}_1^{\intercal} \frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^{\intercal}} \boldsymbol{\alpha}_2 h_1 + \boldsymbol{\alpha}_1^{\intercal} \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{x}^{\intercal}} \boldsymbol{\alpha}_2 h_2 + \boldsymbol{\alpha}_1^{\intercal} \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2^{\intercal} \boldsymbol{\alpha}_2 h_1' h_2 + (\boldsymbol{\alpha}_1^{\intercal} \boldsymbol{\alpha}_2)^2 h_2' h_1 = \\
&\boldsymbol{\alpha}_i^{\intercal} \boldsymbol{\alpha}_j \; \boldsymbol{\alpha}_1^{\intercal} \frac{\partial \boldsymbol{\alpha}_1^{\intercal}}{\partial \boldsymbol{x}} \boldsymbol{\alpha}_2 h_1 + \boldsymbol{\alpha}_1^{\intercal} \frac{\partial \boldsymbol{\alpha}_2^{\intercal}}{\partial \boldsymbol{x}} \boldsymbol{\alpha}_2 h_2 + (\boldsymbol{\alpha}_1^{\intercal} \boldsymbol{\alpha}_2)^2 h_1' h_2 + \boldsymbol{\alpha}_1^{\intercal} \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2^{\intercal} \boldsymbol{\alpha}_2 h_2' h_1 = \\
&\boldsymbol{\alpha}_1^{\intercal}\left(\frac{\partial \boldsymbol{\alpha}_1^{\intercal}}{\partial \boldsymbol{x}} h_1 + \frac{\partial \boldsymbol{\alpha}_2^{\intercal}}{\partial \boldsymbol{x}} h_2 + \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_1^{\intercal} h_1' h_2 + \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2^{\intercal} h_2' h_1\right)\boldsymbol{\alpha}_2 \,.
\end{aligned} \tag{A.40}
$$

Group common terms in the $h_k h_\ell'$ and the h_k and rearrange to write

$$
\left[\boldsymbol{\alpha}_1^{\intercal}\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2^{\intercal}\boldsymbol{\alpha}_2 - (\boldsymbol{\alpha}_1^{\intercal}\boldsymbol{\alpha}_2)^2\right](h_1 h_2' - h_1' h_2) = \boldsymbol{\alpha}_1^{\intercal}\left(\frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^{\intercal}} - \frac{\partial \boldsymbol{\alpha}_1^{\intercal}}{\partial \boldsymbol{x}}\right)\boldsymbol{\alpha}_2 h_1 + \boldsymbol{\alpha}_1^{\intercal}\left(\frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{x}^{\intercal}} - \frac{\partial \boldsymbol{\alpha}_2^{\intercal}}{\partial \boldsymbol{x}}\right)\boldsymbol{\alpha}_2 h_2 \,. \tag{A.41}
$$

Solving for the Jacoby bracket, $h_1 h_2' - h_1' h_2$, we have

$$
\begin{aligned}
h_1 h_2' - h_1' h_2 &= \left[\frac{\boldsymbol{\alpha}_1^{\intercal}\left(\partial \boldsymbol{\alpha}_1 / \partial \boldsymbol{x}^{\intercal} - \partial \boldsymbol{\alpha}_1^{\intercal} / \partial \boldsymbol{x}\right)\boldsymbol{\alpha}_2}{\boldsymbol{\alpha}_1^{\intercal}\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2^{\intercal}\boldsymbol{\alpha}_2 - (\boldsymbol{\alpha}_1^{\intercal}\boldsymbol{\alpha}_2)^2}\right] h_1 + \left[\frac{\boldsymbol{\alpha}_1^{\intercal}\left(\partial \boldsymbol{\alpha}_2 / \partial \boldsymbol{x}^{\intercal} - \partial \boldsymbol{\alpha}_2^{\intercal} / \partial \boldsymbol{x}\right)\boldsymbol{\alpha}_2}{\boldsymbol{\alpha}_1^{\intercal}\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2^{\intercal}\boldsymbol{\alpha}_2 - (\boldsymbol{\alpha}_1^{\intercal}\boldsymbol{\alpha}_2)^2}\right] h_2 \\
&\equiv c_1 h_1 + c_1 h_2 \,,
\end{aligned} \tag{A.42}
$$

with c_1 and c_2 constants, both of which cannot vanish. Without any loss in generality, let $h_1 \neq 0$ (both h_i cannot vanish simultaneously and neither can vanish over an open set). Dividing both sides of (A.42) by h_1 and solving for h_2' gives

$$
h_2'(y) = c_1 - \frac{h_1'(y)}{h_1(y)} + \frac{c_2}{h_1(y)} h_2(y) \,. \tag{A.43}
$$

Let $c_1 \neq 0$ (reverse the roles of h_1 and h_2, if necessary) and make a change of variables to $\tilde{h}_1 = c_1 h_1 + c_1 h_2$, with $\tilde{h}_1' = c_1 h_1' + c_1 h_2'$, and to $\tilde{h}_2 = h_2 / c_1$, with $\tilde{h}_2' = h_2' / c_1$. Then

$$
\tilde{h}_1 \tilde{h}_2' - \tilde{h}_1' \tilde{h}_2 = (c_1 h_1 + c_2 h_2)(h_2'/c_1) - (c_1 h_1' + c_2 h_2')(h_2/c_1) = h_1 h_2' - h_1' h_2 \,. \tag{A.44}
$$

We now have

$$
\tilde{h}_1 \tilde{h}_2' - \tilde{h}_1' \tilde{h}_2 = h_1 h_2' - h_1' h_2 = c_1 h_1 + c_2 h_2 = \tilde{h}_1 \,. \tag{A.45}
$$

In other words (abusing notation by dropping the ~'s), we form particular linear combinations of the h_i such that

$$
h_1 h_2' - h_1' h_2 = h_1 \neq 0 \,, \tag{A.46}
$$

equivalently,

$$
h_2' - \frac{h_1'}{h_1} h_2 = 1 \,. \tag{A.47}
$$

Since

$$\frac{d}{dy}\left(\frac{h_2}{h_1}\right) = \frac{h_2'}{h_1} - \frac{h_1'}{h_1^2} h_2 = \frac{1}{h_1}, \tag{A.48}$$

direct integration gives

$$\int \frac{d}{dy}\left(\frac{h_2(y)}{h_1(y)}\right) dy = \int \frac{dy}{h_1(y)}, \tag{A.49}$$

equivalently,

$$h_2(y) = h_1(y) \int^y \frac{ds}{h_1(s)}. \tag{A.50}$$

Define $\gamma(y) = \int^y h_1(s)^{-1} ds$ and rewrite (A.37) as

$$\frac{\partial y}{\partial \boldsymbol{x}} = \left[\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \gamma(y)\right] h_1(y). \tag{A.51}$$

Since $\gamma'(y) = \frac{d}{dy} \int^y h_1(s)^{-1} ds = h_1(y)^{-1}$, this is equivalent to

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \gamma'(y) \frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)} \frac{\partial y}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \gamma(y). \tag{A.52}$$

We thus can change the definition of $y = f(m)$ to incorporate $\gamma(y)$ through composition, and any full rank 2 system has a representation for y such that $\partial y / \partial \boldsymbol{x} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 y$.

Rank 3:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}_1 h_1(y) + \boldsymbol{\alpha}_2 h_2(y) + \boldsymbol{\alpha}_3 h_3(y) \tag{A.53}$$

The derivations in this case are considerably more involved. We make use of several previous results and techniques from the theory of differential equations to simplify and reduce the calculations. Let $h_1(y) \neq 0$, define $\gamma(y) = \int^y h_1(s)^{-1} ds$, and rewrite (A.53) as

$$\begin{aligned} \frac{\partial \gamma}{\partial \boldsymbol{x}} = \gamma'(y) \frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)} \frac{\partial y}{\partial \boldsymbol{x}} &= \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \left(\frac{h_2(y)}{h_1(y)}\right) + \boldsymbol{\alpha}_3 \left(\frac{h_3(y)}{h_1(y)}\right) \\ &\equiv \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \tilde{h}_2(y) + \boldsymbol{\alpha}_3 \tilde{h}_3(y) \end{aligned} \tag{A.54}$$

By Lemma 1 symmetry is coordinate free. Therefore, consider the representation (again dropping the ~'s and redefining y if necessary)

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 h_2(y) + \boldsymbol{\alpha}_3 h_3(y). \tag{A.55}$$

The least squares conversion of the symmetry conditions gives

$$\begin{aligned} h_2'(y) &= c_{12}^1 + c_{12}^2 h_2(y) + c_{12}^3 h_3(y), \\ h_3'(y) &= c_{13}^1 + c_{13}^2 h_2(y) + c_{13}^3 h_3(y), \\ h_2(y) h_3'(y) - h_2'(y) h_3(y) &= c_{23}^1 + c_{23}^2 h_2(y) + c_{23}^3 h_3(y), \end{aligned} \tag{A.56}$$

where the $\{c_{ij}^k\}$ are constants and cannot all be zero in any given equation. The first two equations form a complete system of linear, ordinary differential equations with constant coefficients. These would be straightforward to solve if the system were not constrained by the third equation (the Jacoby bracket for $h_2(y)$ and $h_3(y)$).

Our plan of attack is to calculate the complete solution to the two-equation system of differential equations and then check for consistency with the third equation. This second step restricts the set of values that the c_{ij}^k can assume in an integrable system. Differentiate the first equation with respect to y and substitute out $h_3'(y)$ and $h_3(y)$,

$$\begin{aligned} h_2''(y) &= c_{12}^2 h_2'(y) + c_{12}^3 h_3'(y) \\ &= c_{12}^2 h_2'(y) + c_{12}^3 \left[c_{13}^1 + c_{13}^2 h_2(y) + c_{13}^3 h_3(y)\right] \\ &= c_{12}^3 c_{13}^1 + c_{12}^2 h_2'(y) + c_{12}^3 c_{13}^2 h_2(y) + c_{12}^3 c_{13}^3 h_3(y) \\ &= c_{12}^3 c_{13}^1 + c_{12}^2 h_2'(y) + c_{12}^3 c_{13}^2 h_2(y) + c_{13}^3 \left[h_2'(y) - c_{12}^1 + c_{12}^2 h_2(y)\right] \\ &= \left(c_{12}^2 + c_{13}^3\right) h_2'(y) + \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right) h_2(y) + \left(c_{13}^1 c_{12}^3 - c_{12}^1 c_{13}^3\right). \end{aligned} \tag{A.57}$$

The homogeneous part of this second-order differential equation is

$$h_2''(y) - \left(c_{12}^2 + c_{13}^3\right) h_2'(y) - \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right) h_2(y) = 0\,. \tag{A.58}$$

Trying $h_2(y) = e^{\lambda y}$ produces the characteristic equation

$$\lambda^2 - \left(c_{12}^2 + c_{13}^3\right)\lambda - \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right) = 0\,, \tag{A.59}$$

with characteristic roots

$$\lambda = \tfrac{1}{2}\left[c_{12}^2 + c_{13}^3 \pm \sqrt{\left(c_{12}^2 + c_{13}^3\right)^2 + 4\left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right)}\right]. \tag{A.60}$$

If $c_{12}^2 = c_{12}^3 = c_{13}^3 = 0$, then $\lambda = 0$ is the only root, and the complete solution has the form

$$\begin{aligned} h_2(y) &= a_2 + b_2 y + c_2 y^2\,, \\ h_3(y) &= a_3 + b_3 y + c_3 y^2\,, \end{aligned} \tag{A.61}$$

where the $\{a_k, b_k, c_k\}_{k=2}^3$ are constants. Define $\tilde{\boldsymbol{\alpha}}_1 = \boldsymbol{\alpha}_1 + a_2\boldsymbol{\alpha}_2 + a_3\boldsymbol{\alpha}_3$, $\tilde{\boldsymbol{\alpha}}_2 = b_2\boldsymbol{\alpha}_2 + b_3\boldsymbol{\alpha}_3$, and $\tilde{\boldsymbol{\alpha}}_3 = c_2\boldsymbol{\alpha}_2 + c_3\boldsymbol{\alpha}_3$. Then we have

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\boldsymbol{\alpha}}_1 + \tilde{\boldsymbol{\alpha}}_2 y + \tilde{\boldsymbol{\alpha}}_3 y^2\,. \tag{A.62}$$

The last step in this part of the proof is to show that this is the only possibility in the full rank 3 case. If any of $c_{12}^2 \neq 0$, $c_{12}^3 \neq 0$, or $c_{13}^3 \neq 0$, then we need to consider distinct and repeated roots separately. With distinct roots, the complete solution to the two ordinary differential equations takes the general form

$$h_2(y) = a_2 + b_2 e^{\lambda_1 y} + c_2 e^{\lambda_2 y}\,,$$

$$h_2(y) = a_3 + b_3 e^{\lambda_1 y} + c_3 e^{\lambda_2 y}, \tag{A.63}$$

where the $\{a_k, b_k, c_k\}_{k=2}^{3}$ again are constants. As before, define $\tilde{\boldsymbol{\alpha}}_1 = \boldsymbol{\alpha}_1 + a_2\boldsymbol{\alpha}_2 + a_3\boldsymbol{\alpha}_3$, $\tilde{\boldsymbol{\alpha}}_2 = b_2\boldsymbol{\alpha}_2 + b_3\boldsymbol{\alpha}_3$ and $\tilde{\boldsymbol{\alpha}}_3 = c_2\boldsymbol{\alpha}_2 + c_3\boldsymbol{\alpha}_3$, and rewrite (A.55) as

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\boldsymbol{\alpha}}_1 + \tilde{\boldsymbol{\alpha}}_2 e^{\lambda_1 y} + \tilde{\boldsymbol{\alpha}}_3 e^{\lambda_2 y}. \tag{A.64}$$

The equation for the Jacoby bracket $h_2 h_3' - h_2' h_3$ now takes the form

$$(\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)y} = c_{23}^1 + c_{23}^2 e^{\lambda_1 y} + c_{23}^3 e^{\lambda_2 y}, \tag{A.65}$$

where $\lambda_2 - \lambda_1 = \sqrt{\left(c_{12}^2 + c_{13}^3\right)^2 + 4\left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right)}$ and $\lambda_2 + \lambda_1 = c_{12}^2 + c_{13}^3$; a contradiction for all $(\lambda_1, \lambda_2) \neq (0,0)$, for either real or complex roots. Therefore, the roots must be real and equal, $\lambda = ½\left(c_{12}^2 + c_{13}^3\right)$. Once again form the above linear combinations of the $\boldsymbol{\alpha}_k$'s, let $h_2(y) = e^{\lambda y}$ and $h_3(y) = ye^{\lambda y}$, and rewrite (A.55) as

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\boldsymbol{\alpha}}_1 + \tilde{\boldsymbol{\alpha}}_2 e^{\lambda y} + \tilde{\boldsymbol{\alpha}}_3 y e^{\lambda y}. \tag{A.66}$$

In this case, the equation for the Jacoby bracket, $h_2 h_3' - h_2' h_3$, takes the form

$$e^{2\lambda y} = c_{23}^1 + c_{23}^2 e^{\lambda y} + c_{23}^3 y e^{\lambda y}, \tag{A.67}$$

a contradiction for all $\lambda \neq 0$. Hence, only a repeated vanishing root is possible and a representation for y exists in any full rank 3 system such that

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 y + \boldsymbol{\alpha}_3 y^2. \tag{A.68}$$

This completes the proof of the full rank representation part of the proposition.

The next step in the proof of the proposition is to show that polynomials constitute the class of minimal deficit demand systems when $K > 3$. This is accomplished by an inductive argument, and we proceed with the induction beginning with $K = 4$. When $K = 4$ there are a total of six Jacoby brackets, but the dimension of the vector space spanned by the basis $\{h_1\, h_2 \cdots h_4\}$ is only four. We know from the theory of Lie algebras on the real line that at least one of the Jacoby brackets must lie outside of this space. We have shown that by redefining y and modifying the $\boldsymbol{\alpha}_k$'s to accommodate the change in y, $\{1\, y\, y^2\}$ is the largest Lie algebra on the real line. The structure of this vector space is

$$\begin{bmatrix} h_2' \\ h_3' \\ h_2 h_3' - h_2' h_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}. \tag{A.69}$$

If we add a fourth income function to this system, the above derivations apply with minor modifications. Therefore, add $h_4(y)$ to $\{1\, y\, y^2\}$. At most two of the new equations for the Jacoby bracket can be consistent. Without loss in generality, consider the fourth and fifth symmetry conditions to be

$$\begin{aligned} h_4'(y) &= c_{14}^1 + c_{14}^2 y + c_{14}^3 y^2 + c_{14}^4 h_4(y)\,, \\ y h_4'(y) - h_4(y) &= c_{24}^1 + c_{24}^2 y + c_{24}^3 y^2 + c_{24}^4 h_4(y)\,. \end{aligned} \tag{A.70}$$

The Jacoby bracket conditions then are

$$\begin{bmatrix} h_2' \\ h_3' \\ h_2 h_3' - h_2' h_3 \\ h_4' \\ h_2 h_4' - h_2' h_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ y^2 \\ h_4' \\ y h_4' - h_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c_{14}^1 & c_{14}^2 & c_{14}^2 & c_{14}^3 \\ c_{24}^1 & c_{24}^2 & c_{24}^3 & c_{24}^4 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ h_4 \end{bmatrix}. \tag{A.71}$$

(A.70) is two linear, first-order ordinary differential equations in $h_4(y)$. The most direct route is to solve the first and check the second for consistency. If $c_{14}^4 = 0$, then integrating the first equation gives

$$h_4(y) = c + c_{14}^1 y + ½ c_{14}^2 y^2 + ⅓ c_{14}^3 y^3\,, \tag{A.72}$$

where c is a constant of integration. Applying similar modifications to the $\boldsymbol{\alpha}_k$'s as before, we have $h_k(y) = y^{k-1},\ k = 1, 2, 3, 4$. The second equation becomes

$$y h_4'(y) - h_4(y) = 3y^3 - y^3 = 2y^3 = 0 \cdot 1 + 0 \cdot y + 0 \cdot y^2 + 2y^3\,, \tag{A.73}$$

which lies in the space that is spanned by $\{1\, y\, y^2\, y^3\}$. Of course, the last Jacoby bracket, $h_3 h_4' - h_3' h_4 = 3y^4 - 2y^4 = y^4$, falls outside of this space, as it must.

If $c_{14}^4 \neq 0$, integrating by parts twice gives the complete solution as

$$h_4(y) = -\left[\frac{c_{14}^1}{c_{14}^4} + \frac{c_{14}^2}{\left(c_{14}^4\right)^2} + \frac{2c_{14}^3}{\left(c_{14}^4\right)^3}\right] - \left[\frac{c_{14}^2}{c_{14}^4} + \frac{c_{14}^3}{\left(c_{14}^4\right)^2}\right] y - \frac{c_{14}^3}{c_{14}^4} y^2 + c e^{c_{14}^4 y}\,, \tag{A.74}$$

where c is again a constant of integration. Once more using the above device to adjust the $\boldsymbol{\alpha}_k$'s, we have $h_4(y) = e^{c_{14}^4 y}$, and the second equation becomes

$$y h_4'(y) - h_4(y) = \left(c_{14}^4 y - 1\right) e^{c_{14}^4 y} = c_{24}^1 + c_{24}^2 y + c_{24}^3 y^2 + c_{24}^4 e^{c_{14}^4 y}\,, \tag{A.75}$$

a contradiction. Hence, the structure with four income functions and a maximum number of Jacoby brackets spanned by the income functions $\{1\, y\, y^2\, y^3\}$ is

$$\begin{bmatrix} h_2' \\ h_3' \\ h_4' \\ h_2h_3' - h_2'h_3 \\ h_2h_4' - h_2'h_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ 3y^2 \\ y^2 \\ 2y^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \end{bmatrix}. \tag{A.76}$$

Only one (the minimal possible number) Jacoby bracket, $h_3h_4' - h_3'h_4$, out of the total of six, falls outside of this space.

The induction is completed by identical steps to show that if a basis with K functions is $\{1\, y\, y^2 \cdots y^{K-1}\}$ and we add a K+1[st] function, $h_{K+1}(y)$, then the maximal increase in the number of spanned Jacoby brackets occurs when $h_{K+1}(y) = y^K$.

The final step in the proof of this proposition is to show that for the polynomial class of Gorman Engel curve systems, $rank[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})] \leq 3$. We proceed constructively by showing that if the system of demands has the polynomial representation,[1]

$$\frac{\partial y}{\partial \boldsymbol{x}} = \sum_{k=0}^{K} \boldsymbol{\alpha}_k y^k \,, \tag{A.77}$$

and is weakly integrable, then there exist $\varphi_k : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R},\ k = 2, \ldots, K$ such that

$$\boldsymbol{\alpha}_k \equiv \varphi_k \boldsymbol{\alpha}_K \ \ \forall \ k \geq 2 \,. \tag{A.78}$$

Integrability is equivalent to symmetry of the matrix

$$\sum_{k=0}^{K} \frac{\partial \boldsymbol{\alpha}_i}{\partial \boldsymbol{x}^{\intercal}} y^k + \sum_{k=1}^{K} \sum_{\ell=0}^{K} k \boldsymbol{\alpha}_k \boldsymbol{\alpha}_\ell^{\intercal} y^{k+\ell-1} \,. \tag{A.79}$$

By continuity, symmetry requires that each like power of y has a symmetric coefficient matrix, and all of the matrices for powers K+1 through $2K$–2 involve nontrivial symmetry conditions without involving any of the $\partial \boldsymbol{\alpha}_k \big/ \partial \boldsymbol{x}^{\intercal}$ terms. The matrix on y^{2K-1} only involves $\boldsymbol{\alpha}_K \boldsymbol{\alpha}_K^{\intercal}$ and is symmetric. Combine terms in like powers of y and apply a backward recursion beginning with the matrix on y^{2K-2}, so that

$$(K-1)\boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_K^{\intercal} + K\boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-\mathbf{1}}^{\intercal} \tag{A.80}$$

is symmetric if and only if $\boldsymbol{\alpha}_{K-1} \equiv \varphi_{K-1}\boldsymbol{\alpha}_K$ for some $\varphi_{K-1} : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$. Similarly,

$$(K-2)\boldsymbol{\alpha}_{K-2}\boldsymbol{\alpha}_K^{\intercal} + (K-1)\boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_{K-1}^{\intercal} + K\boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-2}^{\intercal} \tag{A.81}$$

---

[1] Switching indexes from {1,…,K} to {0,…,K} greatly simplifies the algebra and notation in this part of the proof without affecting the structure of the underlying problem in any way.

is symmetric if and only if $\boldsymbol{\alpha}_{K-2} \equiv \varphi_{K-2}\boldsymbol{\alpha}_K$ for some $\varphi_{K-2} : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$. Applying the recursive argument, consider the matrix on y^{2K-4},

$$(K-3)\boldsymbol{\alpha}_{K-3}\boldsymbol{\alpha}_K^{\mathsf{T}} + (K-2)\boldsymbol{\alpha}_{K-2}\boldsymbol{\alpha}_{K-1}^{\mathsf{T}} + (K-1)\boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_{K-2}^{\mathsf{T}} + K\boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-3}^{\mathsf{T}} . \quad \text{(A.82)}$$

The middle pair of terms are both symmetric, since

$$\boldsymbol{\alpha}_{K-2}\boldsymbol{\alpha}_{K-1}^{\mathsf{T}} = \varphi_{K-2}\varphi_{K-1}\boldsymbol{\alpha}_K\boldsymbol{\alpha}_K^{\mathsf{T}} = \boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_{K-2}^{\mathsf{T}} .$$

The matrix $\left(\boldsymbol{\alpha}_{K-3}\boldsymbol{\alpha}_K^{\mathsf{T}} + \boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-3}^{\mathsf{T}}\right)$ is automatically symmetric. Therefore, the matrix on y^{2K-4} is symmetric if and only if $\boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-3}^{\mathsf{T}}$ is symmetric, if and only if $\boldsymbol{\alpha}_{K-3} \equiv \varphi_{K-3}\boldsymbol{\alpha}_K$, for some $\varphi_{K-3} : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$. This completes the argument when $3 \le K \le 5$.

If $K > 5$, for each j satisfying $4 \le j \le K-1$, group like terms, substitute $\boldsymbol{\alpha}_{K-i} \equiv \varphi_{K-i}\boldsymbol{\alpha}_K$ for each $i < j$, and appeal to symmetry of the matrix $\boldsymbol{\alpha}_{K+1-j}\boldsymbol{\alpha}_K^{\mathsf{T}} + \boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K+1-j}^{\mathsf{T}}$. Then symmetry sequentially requires that each matrix of the following form is symmetric:

$$(j-1)\boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K+1-j}^{\mathsf{T}} + \sum_{i=1}^{j-2}(K-i)\varphi_{K-i}\varphi_{K+1+i-j}\boldsymbol{\alpha}_K\boldsymbol{\alpha}_K^{\mathsf{T}} . \quad \text{(A.83)}$$

This holds if and only if $\boldsymbol{\alpha}_{K+1-j} \equiv \varphi_{K+1-j}\boldsymbol{\alpha}_K$ for $\varphi_{K+1-j} : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$. When $j = 4$ we have the result for $\boldsymbol{\alpha}_{K-3}$; when $j = K-1$ we have it for $\boldsymbol{\alpha}_2$; and $\forall\ K > 2$, we have $\boldsymbol{\alpha}_k \equiv \varphi_k\boldsymbol{\alpha}_K\ \ \forall\ k = 2,\ldots,K$ so that $rank[\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}})] \le 3$. ■

A.6. Characterizing Indirect Preferences

In this section, we characterize the class of indirect preferences for each of the full rank cases and present and discuss an example of indirect preferences that gives rise to a rank three demand model with more than three income terms.

Rank 1:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial \beta}{\partial \boldsymbol{x}} . \quad \text{(A.84)}$$

Simply integrating gives

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) = \beta(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}},u) . \quad \text{(A.85)}$$

This is the translation group representation of indirect preferences for the rank one case. Solving for the normalized expenditure function gives

$$e(\boldsymbol{p},\tilde{\boldsymbol{p}},u) = m\left(\beta(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}},u)\right). \quad \text{(A.86)}$$

Equivalently, the indirect utility function has the form

$$v(\boldsymbol{p},\tilde{\boldsymbol{p}},m) = \psi\left(f(m) - \beta(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}}),\tilde{\boldsymbol{p}}\right), \quad \text{(A.87)}$$

where ψ is the inverse of θ with respect to u. Since $\boldsymbol{q} = \boldsymbol{diag}[g_i'] \times \left(\partial\beta/\partial\boldsymbol{x}\right)\Big/ f'$, the demands for **q** in the rank 1 case are homothetic with income elasticity $-mf''(m)\big/f'(m)$. If

$f(m)=m^{\kappa}$, the common income elasticity $1-\kappa$ is constant, but only equals one in the limiting case $f(m)=\ln(m)$. More general transformations do not result in a constant income elasticity, although it must be independent of prices in this class of demands.

Rank 2:

$$\frac{\partial y}{\partial \boldsymbol{x}}=\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2 y\,. \tag{A.88}$$

Symmetry in this case implies

$$\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^{\mathsf{T}}}=\frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^{\mathsf{T}}}+\frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{x}^{\mathsf{T}}}y+\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1^{\mathsf{T}}+\boldsymbol{\alpha}_2\boldsymbol{\alpha}_2^{\mathsf{T}}y=\frac{\partial \boldsymbol{\alpha}_1^{\mathsf{T}}}{\partial \boldsymbol{x}}+\frac{\partial \boldsymbol{\alpha}_2^{\mathsf{T}}}{\partial \boldsymbol{x}}y+\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2^{\mathsf{T}}+\boldsymbol{\alpha}_2\boldsymbol{\alpha}_2^{\mathsf{T}}y\,. \tag{A.89}$$

Eliminating the symmetric matrix $\boldsymbol{\alpha}_2\boldsymbol{\alpha}_2^{\mathsf{T}}y$ from both sides and equating the matrices that multiply like powers of y implies

$$\frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^{\mathsf{T}}}+\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1^{\mathsf{T}}=\frac{\partial \boldsymbol{\alpha}_1^{\mathsf{T}}}{\partial \boldsymbol{x}}+\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2^{\mathsf{T}}\,, \tag{A.90}$$

and that $\partial\boldsymbol{\alpha}_2/\partial\boldsymbol{x}^{\mathsf{T}}$ is symmetric. The latter property implies the existence of a function $\beta:\boldsymbol{\mathscr{P}}\times\tilde{\boldsymbol{\mathscr{P}}}\to\mathbb{R}$ such that $\partial\beta/\partial\boldsymbol{x}=\boldsymbol{\alpha}_2$. It follows that $\partial\boldsymbol{\alpha}_1/\partial\boldsymbol{x}^{\mathsf{T}}+\left(\partial\beta/\partial\boldsymbol{x}\right)\boldsymbol{\alpha}_1^{\mathsf{T}}$ is symmetric. Equivalently, we can rewrite (A.88) in the form

$$\frac{\partial y}{\partial \boldsymbol{x}}=\boldsymbol{\alpha}_1+\frac{\partial\beta}{\partial \boldsymbol{x}}y\,, \tag{A.91}$$

with

$$\frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^{\mathsf{T}}}-\boldsymbol{\alpha}_1\frac{\partial\beta}{\partial \boldsymbol{x}^{\mathsf{T}}}=\frac{\partial \boldsymbol{\alpha}_1^{\mathsf{T}}}{\partial \boldsymbol{x}}-\frac{\partial\beta}{\partial \boldsymbol{x}}\boldsymbol{\alpha}_1^{\mathsf{T}}\,,$$

symmetric. We can apply the integrating factor $e^{-\beta}$ by noting that

$$\frac{\partial}{\partial \boldsymbol{x}}\left(ye^{-\beta}\right)=\left(\frac{\partial y}{\partial \boldsymbol{x}}-\frac{\partial\beta}{\partial \boldsymbol{x}}y\right)e^{-\beta}\,, \tag{A.92}$$

and

$$\frac{\partial}{\partial \boldsymbol{x}^{\mathsf{T}}}\left(\boldsymbol{\alpha}_1e^{-\beta}\right)=\left(\frac{\partial \boldsymbol{\alpha}_1}{\partial \boldsymbol{x}^{\mathsf{T}}}-\boldsymbol{\alpha}_1\frac{\partial\beta}{\partial \boldsymbol{x}^{\mathsf{T}}}\right)e^{-\beta} \tag{A.93}$$

is symmetric. This is equivalent to the existence of a function $\gamma:\boldsymbol{\mathscr{P}}\times\tilde{\boldsymbol{\mathscr{P}}}\to\mathbb{R}$ such that $\partial\gamma/\partial\boldsymbol{x}=\boldsymbol{\alpha}_1e^{-\beta}$, and integrating gives the transformed deflated expenditure function as

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)=e^{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}})}\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})+e^{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}})}\theta(\tilde{\boldsymbol{p}},u)\,. \tag{A.94}$$

Let $e^{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}})}\equiv\delta(\boldsymbol{x},\tilde{\boldsymbol{p}})$, abuse notation and relabel $e^{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}})}\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})$ as $\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})$, and rewrite (A.94) in the form

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)=\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})+\delta(\boldsymbol{x},\tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}},u)\,. \tag{A.95}$$

This quasi-linear form is the translation and scaling group representation of indirect preferences in the full rank two case. We can write the normalized expenditure function as

$$e(\boldsymbol{x},\tilde{\boldsymbol{p}},u)=m\left(\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})+\delta(\boldsymbol{x},\tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}},u)\right), \tag{A.96}$$

and the indirect utility function as

$$v(\boldsymbol{p}, \tilde{\boldsymbol{p}}, m) = \psi\left(\frac{f(m) - \gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\delta(\boldsymbol{x}, \tilde{\boldsymbol{p}})}, \tilde{\boldsymbol{p}}\right). \tag{A.97}$$

Rank 3:

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 y + \boldsymbol{\alpha}_3 y^2 . \tag{A.98}$$

First, we note that the methods of van Daal and Merkies (1989) for solving integrability of the complete quadratic expenditure system apply without change to our problem. The only difference is that the homogeneity properties they identified do not apply here. Thus, there is no need to reproduce their steps. They show that (A.98) is integrable if and only if there exist functions, $\beta_1, \beta_2, \beta_3 : \mathcal{P} \times \tilde{\mathcal{P}} \to \mathbb{R}$, and $\gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial \beta_1}{\partial \boldsymbol{x}} + \gamma_2(\beta_2)\beta_3 \frac{\partial \beta_2}{\partial \boldsymbol{x}} + \frac{\partial \beta_3}{\partial \boldsymbol{x}} \frac{(y - \beta_1)}{\beta_3} + \frac{\partial \beta_2}{\partial \boldsymbol{x}} \frac{(y - \beta_1)^2}{\beta_3} . \tag{A.99}$$

This can be rewritten in the form

$$\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{y - \beta_1}{\beta_3}\right) = \frac{1}{\beta_3}\left(\frac{\partial y}{\partial \boldsymbol{x}} - \frac{\partial \beta_1}{\partial \boldsymbol{x}}\right) - \frac{(y - \beta_1)}{\beta_3^2} \frac{\partial \beta_3}{\partial \boldsymbol{x}} = \left[\gamma_2(\beta_2) + \frac{(y - \beta_1)^2}{\beta_3^2}\right] \frac{\partial \beta_2}{\partial \boldsymbol{x}} . \tag{A.100}$$

We can simplify this even further by making two simple changes of variables. First, let $w = (y - \beta_1)/\beta_3$, so that

$$\frac{\partial w}{\partial \boldsymbol{x}} = \left[\gamma_2(\beta_2) + w^2\right] \frac{\partial \beta_2}{\partial \boldsymbol{x}} \tag{A.101}$$

Second, let $z = -1/w$, so that

$$\frac{\partial z}{\partial \boldsymbol{x}} = \left[1 + \gamma_2(\beta_2) z^2\right] \frac{\partial \beta_2}{\partial \boldsymbol{x}} . \tag{A.102}$$

Since $\gamma(\beta_2) \equiv \lambda$ must be a constant by lemma 3, we can separate the variables so that

$$\frac{dz}{1 + \lambda z^2} = \frac{\partial \beta_2}{\partial x_i} \;\; \forall \;\; i = 1, \cdots, n_q . \tag{A.103}$$

This is an exact system of partial differential equations and the solution is found by direct integration,

$$\phi\left(\frac{-\beta_3}{y - \beta_1}\right) \equiv \int^{-\beta_3/(y - \beta_1)} \frac{dz}{(1 + \lambda z^2)} = \beta_2 + \theta , \tag{A.104}$$

where $\theta(\tilde{\boldsymbol{p}}, u)$ is the "constant of integration." This is readily recognized as the solution obtained by van Daal and Merkies (1989) and applied by Lewbel (1990) to full rank three QPIGL and QPIGLOG complete systems.

If $\lambda > 0$, we can make a third change of variables to $s = \kappa z$, where $\lambda = \kappa^2 > 0$, so that

$$\int^{-\beta_3/(y-\beta_1)} \frac{dz}{1+(\kappa z)^2} = \int^{-\kappa\beta_3/(y-\beta_1)} \frac{ds}{\kappa(1+s^2)} = \frac{1}{\kappa}\tan^{-1}\left(\frac{-\kappa\beta_3}{y-\beta_1}\right) = \beta_2 + \theta\,. \qquad \text{(A.105)}$$

Solving for y, the normalized and transformed expenditure function is

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) - \frac{\kappa\beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\tan\left\{\kappa\left[\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}}, u)\right]\right\}}\,. \qquad \text{(A.106)}$$

If $\lambda < 0$, set $-\lambda = \kappa^2$, with $1 + \lambda z^2 = (1+\kappa z)(1-\kappa z)$. Partial fractions imply

$$\frac{1}{1+\lambda z^2} = \frac{1}{1-(\kappa z)^2} = \frac{½}{(1-\kappa z)} + \frac{½}{(1+\kappa z)}\,. \qquad \text{(A.107)}$$

Integrating now gives

$$\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\lambda z^2)} = ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1-\kappa z)} + ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\kappa z)}$$

$$= \frac{1}{2\kappa}\ln\left[\frac{y-\beta_1-\kappa\beta_3}{y-\beta_1+\kappa\beta_3}\right] = \beta_2 + \theta\,. \qquad \text{(A.108)}$$

Solving for y, the normalized and transformed expenditure function is

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \frac{\beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \kappa\beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}}) - \tilde{\beta}_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})\big(\beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) - \kappa\beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})\big)\tilde{\theta}(\tilde{\boldsymbol{p}}, u)}{1 - \tilde{\beta}_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})\tilde{\theta}(\tilde{\boldsymbol{p}}, u)}\,, \qquad \text{(A.109)}$$

where $\tilde{\beta}_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}) = \mathrm{e}^{2\kappa\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})}$ and $\tilde{\theta}(\tilde{\boldsymbol{p}}, u) = \mathrm{e}^{2\kappa\theta(\tilde{\boldsymbol{p}}, u)}$.

In the real roots case, the space of all projective transformation groups with real parameters is referred to in differential topology as special linear group two and is denoted by $\mathfrak{sl}(2)$. It is standard practice in Lie group theory to identify the space $\mathfrak{sl}(2)$ with the set of 2×2 real matrices

$$\boldsymbol{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with unit determinant, $\alpha\delta - \beta\gamma = 1$. Indeed, we have

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

as a member of this group, and if we write

$$y = \frac{\alpha\theta + \beta}{\gamma\theta + \delta} \;\Leftrightarrow\; \theta = \frac{\delta y - \beta}{-\gamma y + \alpha}\,, \qquad \text{(A.110)}$$

we can see immediately that 2×2 matrix inverses in this set are one-to-one and onto with the inverse functions of the projective transformation group, while $\boldsymbol{I}_2$ defines the identity map in both spaces. Simple algebra then shows that

$$\frac{\partial y}{\partial \boldsymbol{x}} = \left(\alpha \frac{\partial \beta}{\partial \boldsymbol{x}} - \beta \frac{\partial \alpha}{\partial \boldsymbol{x}}\right) + \left[\left(\beta \frac{\partial \gamma}{\partial \boldsymbol{x}} - \gamma \frac{\partial \beta}{\partial \boldsymbol{x}}\right) - \left(\alpha \frac{\partial \delta}{\partial \boldsymbol{x}} - \delta \frac{\partial \alpha}{\partial \boldsymbol{x}}\right)\right] y + \left(\gamma \frac{\partial \delta}{\partial \boldsymbol{x}} - \delta \frac{\partial \gamma}{\partial \boldsymbol{x}}\right) y^2 . \quad \text{(A.111)}$$

Integrablity is represented by the four Jacoby brackets between the $\{\alpha,\beta,\gamma,\delta\}$ functions with respect to $\boldsymbol{x}$. A class of indirect utility functions generating Gorman systems is

$$v(\boldsymbol{p}, \tilde{\boldsymbol{p}}, m) = \upsilon \left\{ \frac{\delta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}) f(m) - \beta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}})}{-\gamma(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}) f(m) + \alpha(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}})}, \tilde{\boldsymbol{p}} \right\}, \;\; \alpha\delta - \beta\gamma \equiv 1 , \quad \text{(A.112)}$$

the normalized and transformed expenditure function is

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \frac{\alpha(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \theta(\tilde{\boldsymbol{p}}, u) + \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \theta(\tilde{\boldsymbol{p}}, u) + \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}})}, \;\; \alpha\delta - \beta\gamma \equiv 1 . \quad \text{(A.113)}$$

Note that $\gamma \neq 0$ is required for a full rank three system, and we can define this class of preferences in terms of Lie's (1880) rank three transformation group,

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \frac{\tilde{\alpha}(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \theta(\tilde{\boldsymbol{p}}, u) + \tilde{\beta}(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\theta(\tilde{\boldsymbol{p}}, u) + \tilde{\delta}(\boldsymbol{x}, \tilde{\boldsymbol{p}})} \quad \text{(A.114)}$$

where $\tilde{\alpha} = \alpha/\gamma$, $\tilde{\beta} = \beta/\gamma$, and $\tilde{\delta} = \delta/\gamma$. Adding and subtracting $\tilde{\alpha}\tilde{\delta}$ from the numerator and rearranging terms reproduces the form given in proposition 1. ■