Title
Essays on Stochastic Bargaining and Label Informativeness

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Publication Date
2019

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Business Administration

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2019

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Abstract

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Many firms rely on salespersons to communicate with prospective customers. Such person-to-person interaction allows for two-way discovery of product fit and flexibility on price, which are particularly important for business-to-business transactions. In the first chapter, I model the sales process as a game in which a buyer and a seller discover their match sequentially while bargaining for price. The match between the product's attributes and the buyer's needs is revealed gradually over time. The seller can make price offers without commitment, and the buyer decides whether to accept or wait. Players incur flow costs and can leave at any moment. The discovery process creates a hold-up problem for the buyer that causes him to leave too early and results in inefficient no-trades. This can be alleviated by the use of a list price that puts an upper bound on the seller's offers. A lower list price encourages the buyer to stay while reducing the seller's bargaining power. But in equilibrium the players always reach agreement at a discounted price. The model thus provides a novel rationale for the pattern of "list price - discount" observed in sales. I examine whether the seller should commit to a fixed price or allow bargaining. When the seller's flow cost is high relative to the buyer's, both players are willing to participate in discovery if and only if bargaining is allowed. In such a case, bargaining leads to a Pareto improvement, which explains the prevalent use of bargaining in sales. If the buyer has private information on his outside option, the model predicts that, counter-intuitively, the buyer with a higher net value for the product pays a lower price. The chapter expands the bargaining literature by adding a discovery process that introduces a hold-up problem as well as making the product value stochastic.

The second chapter examines how counter-offers affects the hold-up problem in stochastic bargaining. Firms increasingly rely on collaboration for the development and marketing of products. The expected surplus from such collaboration can change stochastically over time due to evolving market conditions or the arrival of new information. For collaboration to happen, both firms have to agree to collaborate as well as agree on how the profit is to be split. In such cases, at what point do firms form the alliance and how do they agree on the profit split? To answer these questions, I study a model of bilateral bargaining with a surplus that

follows a Brownian motion. One firm can make repeated-offers to the other, and they switch roles after some time to allow for counteroffers. The frequency of counteroffers determines relative bargaining power, and the model captures different bargaining procedures by varying this frequency. The chapter shows that, when there is no outside option, firms collaborate after efficient delay. If there is a relevant outside option, the outcome is inefficient due to the existence of a hold-up problem faced by the weaker party. Firms form the alliance too early, taking the outside options too early, and the ex-ante probability of alliance becomes sub-optimal. Increasing the frequency of counteroffers improves social efficiency by balancing bargaining power and reducing the severity of hold-up. Furthermore, bargaining with more frequent counteroffers can lead to Pareto improvements; the proposer benefits, too, because the increased efficiency outweighs losses in bargaining power. The essay makes a step in understanding the effect of bargaining procedures on collaborative outcome, and shows how collaborators should (not) bargain.

The third chapter studies the effect of product labelling on consumer behavior empirically. Cigarettes are sold in different strengths, commonly categorized as regular, light, or ultralight. In 2009, Congress passed Tobacco Control Act (TCA) which banned tobacco companies from communicating product strengths to consumers on any marketing or packaging materials. Cigarette companies continue to sell products of different strengths by using less informative color codes, i.e., relabeling Marlboro Light to Marlboro Gold or Camel Light to Camel Blue. Brands do not use the exact same color codes, creating room for confusion. This chapter investigates the effect of such change in label informativeness on consumer choice. Using a panel of smokers from 2007 to 2012, I find a sharp decline in price sensitivity after Tobacco Control Act was passed. The finding is robust in choice models that account for preference heterogeneity, state dependence, price endogeneity, and consideration sets. This result suggests that consumers perceive products as more differentiated when strength labels change to color codes. This essay provides new evidence on the linkage between product labeling and choice behavior.

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Chapter 1

How to Make an Offer? A Stochastic Model of the Sales Process

1.1 Introduction

Sales force is an important part of the economy. The U.S. economy spends $10 billion per year on sales force (Zoltner et al. 2008). Selling through sales force is the main channel for many firms, especially those that serve other businesses. B2B firms rely on salespersons to provide product information, learn about prospective customers' needs, and persuade them to buy. Typical sales activities include discovery calls, sales pitch, demonstration, proposal, etc. These interactions between the buyer and the seller before a transaction is generally referred to as the sales process. Mantrala et al. (2010) put the sales process at the core of their framework for sales force modelling. They state that "the firm's decisions surrounding the selling process...are critical and impact response functions, operations, and, ultimately, strategies." Albeit its importance, the sales process has been an understudied topic. The details of the sales process differs greatly by industry, and can span multiple months to more than a year for enterprise buyers or complex products. In this paper, I model the sales process as a combination of two-sided information acquisition and bargaining, and study the strategic interactions between the buyer and the seller during the process. The model focuses on two key functions that person-to-person selling provides: (1) to discover the match/fit between the buyer and the seller when the market is heterogeneous; and (2) to determine the transaction price through bargaining.

Figure 1.1: Framework of Sales Process

Market Feature		Role of the Sales Process	
Heterogeneity in products and needs	→	Two-way discovery of match/fit	(Information Acquisition)
Flexibility on price	→	Negotiate discount off list price	(Bargaining)

In many industries, the value from trade can be relationship-specific due to a high degree of heterogeneity on both sides of the market. In industries such as enterprise software, industrial equipment, and professional services, sellers can differ widely in the products and services that they offer, and customers can have significant differences in the solutions they need (see, e.g., Stevens 2016 for a description of the market for data vendors). Prior to communicating with the seller, a buyer may not know how well the product or service addresses his business needs. The buyer can acquire information through the sales process to guide his purchasing decision. On the other hand, buyers have heterogeneous demand for attributes, so the seller does not know how well the product matches the buyer's needs and thus is unsure how much the potential buyer is willing to pay for it. In practice, many B2B firms have an explicit "discovery" step in their sales processes, in which the salesperson inquires about the buyer's situations and needs.[1] Industry studies find that a well-executed discovery plan plays an important role in successful sales (See, e.g., Zarges 2017 and Merkel 2017). The information about the buyer's needs helps the seller to fine-tune her selling strategy, such as whether to continue pursuing a buyer or what price to quote. For example, firms often delegate some pricing power to the salespersons, because the salespersons know more about a client's willingness-to-pay than the firm does through their communications with the client (See Mantrala et al. 2010 and Coughlan and Joseph 2011 for surveys on the literature of price delegation). The sales process allows the two parties to find out how well the product matches the customer's needs, which determines their total surplus from trading with each other. This view is also consistent with earlier work on the role of personal selling (Wernerfelt 1994a) as well as writings from practitioners (see, e.g., Nick 2017 and Mehring 2017).

Another important part of sales is bargaining. A survey of sales forces by Krafft (1999) found that 72% of sampled companies allow their salespersons to adjust price offers. The number rose to 88% for industrial-goods companies in the survey. In B2B markets, the standard practice is for the two parties to negotiate for a discount off of some list price (Mewborn et al. 2014, PwC 2013, Mukerjee 2009 p.464). Managing the list price is considered important for B2B firms even though price can be negotiated (Mewborn et al. 2014), and 85% of B2B respondents in a Bain survey believe that their pricing could improve (Kermisch and Burns 2018). Providing the buyer a discount has become the norm in many B2B markets and often represent a company's largest marketing investment (Caprio 2015, Wang 2016, and Schurmann et al. 2015). Some sellers do not publish their list prices publicly. In such a case, studies by CRM firms Hubspot (2016) and Gong (2016) show that most buyers want to discuss price in the very first sales call, forcing the seller to reveal their list price before the sales process moves on.

The common use of list price in bargaining situations brings forth many questions. For example, what roles does the list price play if most to all buyers negotiate the price? How does the choice of list price affects the sales outcome? What is the optimal choice of list price, given it is not the actual transaction price? And how does the seller makes price offers during the sales process? In this paper, I view the list price differently from a first offer.

[1]See, for example, Hubspot's sales process in Skok (2012) and Talview's sales process in Jose (2017) with more details. Both companies offer Software-as-a-Service to other businesses.

The list price serves as an upper bound on price offers during bargaining, so that players can only negotiate on discounts.[2]

This paper views the discovery and the bargaining processes as dynamic, interdependent, and simultaneous. Information on product fit arrives gradually and bargaining involves sequential offers. The revelation of product match affects bargaining strategy and vice versa. For example, intuitively, finding out that the product is a good fit is good news for the buyer. But once the seller sees that the buyer really values the product, the seller may charge a higher price (by giving a smaller discount). This reduces the buyer's incentive to discover the product fit with the seller in the first place if such activity is costly. This presents a hold-up problem, as the expectation of the future bargaining outcome affects the players' current choice of "investment" (whether to continue or leave). Another important point is that there are no natural "stages" that separate discovery and bargaining. The seller can make offers at any moment during the process. This observation motivates one to view discovery and bargaining as happening simultaneously, and allow the length of the sales process to be endogenous.

I study a game in which a buyer and a seller discover their product match sequentially while simultaneously bargaining for price. I solve the optimal selling and buying strategies in continuous time. The model provides novel insights on the role of list price and price discount in negotiations. The paper also looks at the firm's choice between allowing bargaining versus committing to a fixed price. These analyses provide implications on issues such as optimal list pricing and delegation of pricing authority. On the theoretical side, the paper expands the existing bargaining literature by adding a simultaneous discovery/matching process. This process causes product value to be stochastic, and introduces a hold-up problem to the stochastic bargaining framework.

Specifically, a buyer and a seller trade over a product that can be seen as a sum of attributes. The match between the product's attributes and the buyer's preferences is revealed to each other over time. The seller can publish a list price before the sales process that acts as a price ceiling. At each moment, players discover their match on more attributes, the seller can make price offer without future commitment, and the buyer decides whether to take that offer. Continuing the process is costly and players can choose to quit. The seller's cost can come from the salesperson's salary and product demonstration cost, while the buyer incurs cost from processing information or opportunity cost when dedicating employees to talk to the salesperson.

I show that the list price acts as an instrument to reduce the buyer's concern for hold-up. The seller uses the list price to balance the buyer's incentive to engage and the seller's bargaining power. If the seller sets the list price too high, then the buyer leaves immediately. Even if the product match will be revealed to be good, the buyer does not expect to get any surplus ex-ante because the seller can charge a high price once the good match is revealed. In

[2]Intentionally selling above the list price can be considered false advertising in countries including U.S., U.K., and Canada. B2B firms are exposed to false advertising regulations, even for promotional statements not made to the general public (Miller 2011).

order to encourage the buyer to continue gathering information for a sufficiently long time, the seller needs to set a low enough list price to limit the impact of such hold-up. On the other side, a lower list price decreases the seller's bargaining power by increasing the buyer's continuation value. The optimal list price thus has to balance these two effects. Surprisingly, I find that the parties always trade below the list price, regardless of what the list price is.[3] The reason is that it is not efficient for the seller to wait until the buyer is willing to pay the original list price. Instead, the seller prefers to make the sale earlier by offering a price discount.

This finding provides a rational explanation for the "list price - discount" pattern that is observed in sales negotiations, which has been largely ignored in the bargaining literature. There lacks sufficient explanations for why firms want to self-restrain their offers during bargaining by establishing a list price. For example, if one allows the seller to set a list price in the repeated-offers model of Fudenberg et. al. (1985), then the list price is optimal as long as it is higher than the valuation of the highest type. Providing the seller with the ability to set a list price does not affect the equilibrium outcome. Thus, "list price" and "discount" are meaningless terms in such models.

Who should leave the sales process if the product fit is revealed to be poor? Surprisingly, it is always the buyer who leaves; otherwise the seller should set a higher list price. Raising the list price has two effects: it discourages buyer engagement and improves surplus extraction. If the seller quits before the buyer does, discouraging the buyer is costless. Thus, raising the list price is strictly beneficial to the seller in such a case.

Should the seller commit to a fixed price or be open to bargaining? I find that bargaining leads to a lower final price and a higher ex-ante probability of trade than under a fixed price. Bargaining always benefits the seller and increases overall welfare, but its effect on the buyer's utility depends on the ratio of the players' costs. When the seller's cost is high relative to the buyer's, bargaining is necessary for both players to participate in the sales process.[4] The flexibility from allowing players to bargain improves overall efficiency by saving time and cost and by increasing the ex-ante success rate. In this case, bargaining is welfare-enhancing for both the buyer and the seller.

The model extends to the case in which the buyer has private information on his outside option. The seller only knows the distribution of the buyer's net valuation at each moment. This model is similar to a repeated-offers bargaining model with one-sided incomplete information (such as Fudenberg et al. 1985) but with a stochastic product value resulting from the sequential discovery of product match. In equilibrium, trade is delayed and the seller can separate different types of buyers, even though the seller can change price offers arbitrarily fast. A counter-intuitive finding is that the buyer with a higher valuation for the product pays a lower price. This is due to a combination of the efficiency gain from selling early and the high type buyer's information rent. Extensions to Bayesian learning, finite horizon, heterogeneous attributes, and time discounting are also examined.

[3]This is before considering consumer heterogeneity in costs.

[4]A player "participates" if he/she does not leave at $t = 0$.

Literature Review

The paper is related to the literature on sequential information acquisition. The papers that are closest in modelling include Roberts and Weitzman (1981), Moscarini and Smith (2001), Branco et al. (2012), and Ke et al. (2016). Similar to these studies, I use a Brownian motion to capture the effect of gradual arrival of product information on product value. Whereas previous papers focus on single-agent decision-making, this paper features two-sided learning and allows players to bargain. The addition of two-sided learning and bargaining turns the problem into a dynamic game, which greatly increases the complexity. Kruse and Strack (2015, 2017) look at a problem in which a principal tries to influence an agent's stopping decision through a transfer. The price discount in this paper can be seen as a transfer, but unlike the principal in Kruse and Strack (2015, 2017), the seller in this paper cannot commit to future transfers.

Other papers have studied bargaining with stochastic payoffs. Merlo and Wilson (1995) present a general framework of stochastic bargaining games with complete information in discrete time. Daley and Green (2017) look at a repeated-offers bargaining game with asymmetric information and gradual signals. One party knows the true quality of the product, and the other party receives noisy signals of the quality over time. In contrast, this paper features two-sided learning which creates the hold-up problem. Ortner (2017) solves a bargaining model where the seller's marginal cost changes over time. Fuchs and Skrzypacz (2010) look at bargaining with a stochastic arrival of events that can end the game. Ishii et al. (2018) studies wage bargaining with both sequential learning and stochastic arrival of competitor.

The idea that setting a list price can reduce the buyer's hold-up problem relates to the consumer search literature, which shows that sellers can use published price to encourage search. Wernerfelt (1994b) shows that, when product quality is uncertain and requires search/inspection, then the seller wants to inform the buyer about the price before search. Similarly, to encourage visits, multi-product retailers want to advertise the prices of some products to put a bound on the total price of a basket (Lal and Matutes 1994). This paper expands the concept further by allowing players to bargain after the list price is posted. Other papers have discussed the use of list price in other contexts. Xu and Duke (2017) show that the list price can be used to convince uninformed buyers of their types when the seller has superior information. Yavas and Yang (1995) and Haurin et al. (2010) discuss the signalling role of list price in the real estate market. Huang (2016) investigates why some car dealers commit to posted prices while others allow haggling. Shin (2005) shows that non-committal list prices can signal true prices when the sales process is costly.

Conceptually, the paper also relates earlier works that study the role of sales in providing information to consumers, such as Wernerfelt (1994a) and Bhardwaj et al. (2008). Shin (2007) studies a firm's decision to provide pre-sales service that reveals the match between a customer and the product, when competitor can free-ride on such service. This paper also studies a firm's decision to delegate pricing authority, but using a different approach from the principal-agent models of Lal (1986), Bhardwaj (2001), and Joseph (2001).

The paper is organized as follows. Section 2 presents the model. Section 1.3 solves the baseline case with a costless seller, and discusses the core intuitions. Section 1.4 extends to the case in which selling is costly, and shows the necessity of bargaining when selling cost is high. Section 1.5 gives the buyer private information on his outside option and discusses the results. Section 1.6 presents other extensions. Section 1.7 offers concluding remarks.

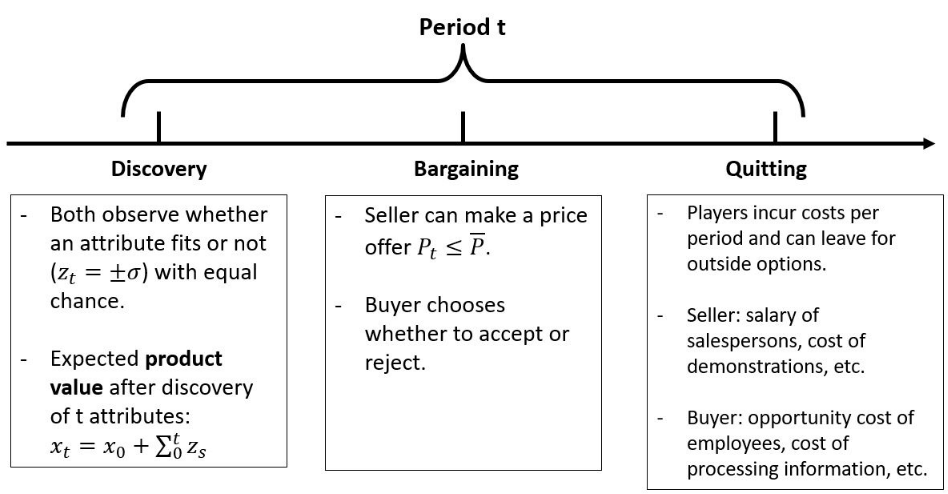
1.2 The Model

I first give an intuitive description of the discovery and the bargaining processes in discrete-time terms. Then I present the continuous-time model, which can be seen as the limit of the discrete-time game. In Section (1.10), I study the discrete-time game directly, and shows that the continuous-time solution presented in the paper represents the unique limit of discrete-time equilibrium outcomes.

Description of the Model

Consider two players, a Buyer (b) and a Seller (s). Throughout the paper, Buyer is referred to as "he" and Seller is referred to as "she". Seller owns a product that can be sold to Buyer. In each period, the two players first engage in discovery, then bargain over price, and lastly decide whether they want to continue or leave the sales process.

Figure 1.2: Illustration of the Game



Discovery as Matching The product is a combination of attributes with equal size. With equal chance[5], each attribute can either match with Buyer's need for that attribute,

[5]The model can accommodate other ex-ante probability of match. Using other probabilities increase the analytic complexity without providing additional insights.

which provides value $z_t = +\sigma\sqrt{dt}$, or does not match with Buyer's need, which gives value $-\sigma\sqrt{dt}$, where dt is the length of the each period (or size of each attribute). So $\mathbb{E}[z_t] = 0$ and $Var[z_t] = \sigma^2 dt$. Each period, players simultaneously discover whether they match on an attribute and observe z_t. The expected value of the product after observing t attributes then can be written as $x_t = x_0 + \sum_0^t z_s$, where x_0 is the expected value of the product prior to the game. As the length of each period (or size of each attribute) approaches 0, the game approaches continuous time.

As the mass of total attributes approach infinity, the product value x_t becomes a stationary process. I examine this limiting case in the main model, and allow for finite mass of attributes and heterogeneous attribute size in the extensions. In the stationary case, Buyer's expected value for the product, x_t, can be represented as a Brownian motion.

Discovery as Learning Alternatively, instead of matching on attributes, one can think of the discovery as a learning process. The product provides a true value of x_t^* to Buyer, which can change over time with a random walk of variance σ^2, due to Buyer's evolving preferences. Seller and Buyer do not know the true value x_t^*, and can only learn through sequential signals acquired during the sales process. They have a common normal prior with mean $\hat{x}_0$ and variance ρ_0. Each period, they receive a signal of x_t^* with a normal error of variance η^2, and updates the posterior mean $\hat{x}_t$ and variance $\hat{\rho}_t$ using Bayes' rule.

As the length of each period goes to 0, the signal S_t accumulates as $dS_t = x_t^* dt + \eta dW_t$, where W_t is a Wiener process. By the Kalman-Bucy filter (See Ruymgaart and Soong 1988, Ch.4), the posterior mean $\hat{x}_t$ follows $d\hat{x}_t = (\hat{\rho}_t/\eta)dB_t$ for some Wiener process B_t, and posterior variance follows $\frac{d\hat{\rho}_t}{dt} = -\hat{\rho}_t^2/\eta^2 + \sigma^2$. If $\hat{\rho}_0 = \sigma\eta$, then $\hat{\rho}_t = \hat{\rho}_0$ for all t and $\hat{x}_t$ is a stationary process, otherwise $\hat{\rho}_t$ approaches $\sigma\eta$ asymptotically over time. I examine the stationary case in the main model, and look at the non-stationary case as an extension.

Bargaining Before the game starts, Seller can set a list price $\overline{P}$, which is a commitment that Buyer can always buy the product at this price.[6] Then each period, after discovery, Seller can choose to offer a discounted price P_t with no guarantee on future offers. Buyer decides whether to accept the offer. The game ends if Buyer accepts. Notice that Seller cannot effectively offer any price above $\overline{P}$. Also, if Seller does not make an offer, it is equivalent to making an offer at $\overline{P}$, since $\overline{P}$ is always available. So there is a standing offer at each period, bounded above by $\overline{P}$.

Quitting The players incur flow costs each period. Players can choose to quit at the end of each period if the offer is not accepted. If either party quits, the game ends and the players receive their outside options.

Figure 1.3 presents a sample evolution of the product value in continuous time in the stationary baseline. Number of attributes discovered, t, is on the horizontal axis, and the expected product value, x, is on the vertical axis. The middle horizontal line represents the list price $\overline{P}$. Assume that Seller never makes a price offer, so the price stays at $\overline{P}$. The game becomes Buyer's single agent optimization problem as he decides when to buy. In the beginning, Buyer optimally chooses to wait and find out more about the product. Waiting

[6] We denote $\overline{P} = \infty$ if Seller does not set a list price.

Figure 1.3: Sample Path without Bargaining

is optimal even beyond the list price because the option value of buying a well-matched product later is greater than the utility he gets if he buys when x_t reaches exactly $\overline{P}$ (which is 0). Buyer chooses to buy when product value reaches the threshold on the top of the graph.

With bargaining, however, Seller might prefer to close the sale earlier at a discounted price. Trading earlier saves time and cost, while also eliminating the possibility of subsequently discovering a bad match that breaks down the negotiation. This point is shown in Figure 1.4.

Figure 1.4: Sample Path with Bargaining

Solving the equilibrium (defined below) in Figure 1.4 can be difficult. Because Seller cannot commit to future prices, Buyer's stopping decision has to depend on Seller's entire pricing strategy P_t in all states of the world. Conversely, Seller's optimal price offers depend on what Buyer is willing to pay in all states of the world. The game has an infinite horizon so we cannot use backward induction. An equilibrium then should be the solution to a two-sided optimal stopping problem. Buyer's strategy gives him the optimal stopping time given what Seller is willing to offer, and Seller's offering strategy gives her the optimal stopping time given what Buyer is willing to accept.

Formal Model

The game is in continuous time with an infinite horizon. There are two players: a Buyer (b) and a Seller (s). Product value x_t is observable to both players and follows a Brownian motion $dx_t = \sigma dW_t$, with initial position x_0. Let (Σ, $\mathcal{F}$, P) be the probability space that supports the Wiener process W_t, and F $= (\mathcal{F}_t)_{t\in[0,\infty)}$ be the filtration process generated by W_t satisfying the usual assumptions. As described in Section 2.1, this is the continuous-time version of a product with an infinite number of small, independent attributes, which players learn over time.

Before the game starts, Seller publishes a list price $\overline{P}$. This is a commitment that Buyer can always buy the product at $\overline{P}$. Effectively, this puts an upper bound on the price that Seller can offer to Buyer. At all times $t \geq 0$, players take actions in the following order:

1. Seller makes price offer bounded above by the list price $\overline{P}$.

2. Buyer chooses whether to buy given Seller's offer.

3. (If Buyer does not accept) Buyer and Seller simultaneously choose whether to quit or continue.

Notation Let P_t denote Seller's offer at time t, a_t be an indicator function for whether Buyer accepts the offer at time t, $q_{s,t}$ be an indicator function for whether Seller quits at time t, and $q_{b,t}$ indicates whether Buyer quits at time t. A history is denoted by $h_t \equiv (\{x_r\}_{r\leq t}, \{P_r\}_{r<t})$, which records the realizations of product value x up to time t and the past price offers.[7] A strategy profile $\theta(h_t) = (P(h_t), a(h_t, P_t), q_s(h_t), q_b(h_t))$ maps each history to Seller's price offer, $P_t = P(h_t)$, Buyer's acceptance decision given a price offer, $a_t = a(h_t, P_t)$, and each player's quitting decision, $q_{i,t} = q_i(h_t)$.

Utility The game ends if players trade or if either player quits. Let $\tau_\theta = \inf\{t \mid a_t + \sum_i q_{i,t} > 0\}$ denote the stopping time of the game given a strategy θ. Buyer (Seller) incurs flow cost c_b (c_s) during the game. Players are risk neutral and perfectly patient.[8] If Buyer

[7]In order for a history to be an decision node, all past offers must be rejected and neither player has quit. Thus in any subgame, we know that $a_r = 0$ and $q_{i,r} = 0$ for all $r < t$. So I drop them from the notation of a history. Strategies cannot depend on past acceptance and quitting decisions for the same reason.

[8]In Section 1.6, I show that using time discounting instead of flow costs does not affect results qualitatively, as long as outside options are positive so that the players have an incentive to quit. Though time discounting is more conventional in the bargaining literature, the paper uses flow cost to illustrate the "investment effort" in the hold-up problem more directly.

and Seller agree to trade at time τ, Buyer receives utility $x_\tau - P_\tau$ and Seller receives P_τ. If either player quits at time τ, then players get outside options of π_b and π_s, respectively. Thus Seller's utility at time t from a strategy θ is defined as:

$$u_t = \mathbb{E}_{\tau_\theta}\Big[- c_s(\tau_\theta - t) + P_{\tau_\theta}\mathbb{1}\{a_{\tau_\theta} = 1\} + \pi_s \mathbb{1}\{a_{\tau_\theta} = 0 \ \& \ \sum_i q_{i,t} > 0\}\Big]$$

And Buyer's utility at time t from a strategy θ is defined as:

$$v_t = \mathbb{E}_{\tau_\theta}\Big[- c_b(\tau_\theta - t) + (X_{\tau_\theta} - P_{\tau_\theta})\mathbb{1}\{a_{\tau_\theta} = 1\} + \pi_b \mathbb{1}\{a_{\tau_\theta} = 0 \ \& \ \sum_i q_{i,t} > 0\}\Big]$$

Equilibrium I look for Stationary Subgame-Perfect Equilibrium with pure strategy, henceforth referred to as "equilibrium".[9] Players' equilibrium strategies depend on state x but not on time t. I focus on stationary behavior because product value x evolves as a stationary process and time is payoff-irrelevant conditional on x. Seller's actions in equilibrium can be characterized by her price offering in each state $P(x|\overline{P}) : \mathbb{R} \mapsto [0, \overline{P}]$, and her quitting decision $q_s(x|\overline{P}) : \mathbb{R} \mapsto \{0, 1\}$. Buyer's actions can be characterized by his buying decision $a(x, P|\overline{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \{0, 1\}$, and his quitting decision $q_b(x|\overline{P}) : \mathbb{R} \mapsto \{0, 1\}$. Since everything depends on the list price $\overline{P}$, which is fixed throughout the game, I will drop $\overline{P}$ from notations moving forward. I treat the list price as an exogenous parameter, and solve the equilibrium for any arbitrary list price, then let Seller choose the list price that maximizes her ex-ante utility from bargaining.

An equilibrium of this game can be viewed as the solution to a two-sided optimal stopping problem. Buyer decides when to stop given Seller's strategy, and vice versa.

Buyer's Problem Given Seller's strategy, Buyer has to decide between three actions at each product value x. Buyer can accept Seller's offer $P(x)$, which gives utility $x - P(x)$; he can reject the offer and quit the game, which gives utility π_b; or he can reject the offer and continue the process. This is an optimal stopping problem with stopping value $W_b = \max\{x - P(x), \pi_b\}$, subjects to states in which Seller leaves. Buyer chooses the strategy that maximizes his expected payoff, $\sup_\tau \mathbb{E}[-c_b * \tau + W_b(x_\tau)]$. If Seller deviates on price, then Buyer accepts iff $x_t - P_t$ is higher than her expected utility from playing equilibrium strategies in the future.

Seller's Problem If Seller makes an offer that Buyer accepts, Seller should make the highest offer that Buyer is willing to accept. Otherwise, Seller can profitably deviate by charging a slightly higher price. Thus in equilibrium, if $a(x, P(x)) = 1$, then we should have $P(x) = \sup\{P|a(x, P) = 1\}$. Given this, Seller also decides between three actions at each moment. Seller can make the highest offer acceptable to Buyer, which gives utility

[9] As shown by Simon and Stinchcombe (1989), a strategy profile may not produce a well-fined outcome in continuous time. The utility defined above exists if and only if τ_θ is a measurable function from Σ to $\mathbb{R}^+$. Thus when considering profitable deviations, only strategies that produce a measurable stopping time is allowed. In Section (1.10), I construct an alternative equilibrium concept without restricting the strategy space.

$P(x) = \sup\{P|a(x,P) = 1\}$; she can quit the sales process, which gives utility of π_s; or she can continue (by making an unacceptable offer and do not quit). This is an optimal stopping problem with stopping value $W_s = \max\{\sup\{P|a(x,P) = 1, \pi_s\}$ subject to states in which Buyer leaves. Seller chooses the strategy that maximizes her expected payoff, $\sup_\tau \mathbb{E}[-c_s * \tau + W_s(x_\tau)]$.

Outcome An equilibrium outcome can be described by a quadruple $(A, Q, U(x), V(x))$, where $A = \{x|a(x, P(x)) = 1\}$ is the set of states such that players reach agreement, $Q = \{x|q_s(x) + q_b(x) > 0\}$ is the set of states such that some player quits, $U(x)$ is Seller's equilibrium value function, and $V(x)$ is Buyer's equilibrium value function.

For states in which players trade, Seller receives $U(x) = P(x)$ and Buyer receives $V(x) = x - P(x)$. For states in which no agreement is reached and a player quits, $U(x) = \pi_s$ and $V(x) = \pi_b$. For a state x such that players choose to continue negotiating, we can write recursively:

$$\begin{aligned} U(x,t) &= -c_s dt + e^{-r_s dt}\mathbb{E}U(x+dx, t+dt) \\ V(x,t) &= -c_b dt + e^{-r_b dt}\mathbb{E}V(x+dx, t+dt) \end{aligned} \tag{1.1}$$

Under stationarity, and by taking Taylor expansion and applying Ito's Lemma on $\mathbb{E}U$ and $\mathbb{E}V$ terms, these expressions can be reduced to the following equations:

$$\begin{aligned} r_s U(x) &= -c_s + \frac{\sigma^2}{2}U''(x) \\ r_b V(x) &= -c_b + \frac{\sigma^2}{2}V''(x) \end{aligned} \tag{1.2}$$

Given $r_s = r_b = 0$, the solutions to the equations must be of the form:

$$\begin{aligned} U(x) &= \frac{c_s}{\sigma^2}(x - \overline{P})^2 + A_s(x - \overline{P}) + B_s \\ V(x) &= \frac{c_b}{\sigma^2}(x - \overline{P})^2 + A_b(x - \overline{P}) + B_b \end{aligned} \tag{1.3}$$

for some coefficients A_s, B_s, A_b, B_b. These coefficients can be identified later by applying appropriate boundary conditions.

In this model, the product value x_t is assumed to be observable by both players. Seller observes Buyer's preference for each attribute and Buyer observes each attribute accurately. In real negotiations, potential buyers may hide their preferences in order to barter for a lower price.[10] In Section 1.5, I extend the model by giving Buyer private information on his outside option, and discuss what happens in such an environment. Also, in realistic settings, players may learn about the value with private noises. Buyer may not observe the product attributes perfectly, and Seller may not observe Buyer's needs perfectly. However, bargaining models

[10] We can motivate the truthful revelation in a simple model. Suppose that Buyer can choose whether to reveal his preference for each attribute. Buyer always chooses to reveal if he does not like the attribute. Then Seller can infer Buyer's preference when he does not reveal. Thus Buyer's preference becomes unravelled.

with two-sided private information are generally difficult, more so with stochastic arrival of information. Past works on similar topics, such as Daley and Green (2017), focused on one-sided learning in which only one player updates his belief about the value. I argue that it is important to understand what happens if both parties learn. This paper explores the special case in which information is symmetric, and illustrates the existence of a hold-up effect that helps us to understand the "list price - discount" pattern in sales. Such effect is absent if only one party learns. Future works can examine what happens if signals are private.

1.3 Costless Seller

In this section, I first assume that selling activity is costless ($c_s = 0$) and that Seller never quits on the equilibrium path. This reduces the complexity of the problem. I study the case with $c_s > 0$ in Section 4.

If Buyer and Seller reach agreement when product value is x, they split a total surplus of size x. Seller receives the price offer $P(x)$, and Buyer receives the rest of the surplus, $x - P(x)$. Alternatively, one can think of Buyer as receiving his equilibrium utility $V(x)$, which should be equal to his continuation value, otherwise Seller should raise the offer. Thus the highest price that Seller can charge at $\overline{x}$ in equilibrium corresponds to the lowest continuation value that Buyer can get at $\overline{x}$. In other words, if Seller wants to trade now, she prefers a strategy that makes continuing the game as undesirable an option as possible for Buyer.

Because the product is always available at the list price $\overline{P}$, Buyer cannot be worse off than if Seller never give a discount below $\overline{P}$. Thus, solving for Buyer's value function facing a fixed price of $\overline{P}$ provides a lower bound on Buyer's continuation value in the bargaining game. Because Seller never quits, this is Buyer's single-agent optimal stopping problem with stopping utility of $\max\{0,\ x - \overline{P}\}$. Each moment, Buyer decides between buying at $\overline{P}$, continuing, or quitting. Such problems have been studied for investment under uncertainty (eg., Dixit 1993), R&D funding (Roberts and Weitzman 1981), experimentation (Moscarini and Smith 2001), and consumer search (e.g., Branco et al. 2012).

Buyer's optimal solution is to buy when the product value reaches a threshold, denoted here as $\overline{\overline{x}}$, and quit when the product value reaches a lower threshold, denoted here as $\underline{\underline{x}}$. Let $\underline{V}(x)$ denote the Buyer's value function facing a fixed price of $\overline{P}$. Closed-form solutions of the thresholds and the value function are presented in Section (1.8).

Buyer's equilibrium payoff in the bargaining game is thus bounded below by $\underline{V}(x)$, but since Seller cannot credibly commit not to price below $\overline{P}$ in the future, it is not obvious whether this lower bound is binding. Lemma 1 states that there indeed exists an equilibrium where Buyer's continuation value is $\underline{V}(x)$, and all equilibrium in which Buyer receives $\underline{V}(x)$ must yield the same outcome. In this equilibrium outcome, Buyer is as if he is facing a fixed price of $\overline{P}$. He accepts the offer if the price makes him indifferent between continuing or stopping, and rejects the offer if the price is higher.

Lemma 1. *There exists an unique equilibrium outcome in which $V(x) = \underline{V}(x)$. In this outcome, there exist thresholds $\overline{x} \leq \overline{\overline{x}}$ and $\underline{x} = \underline{\underline{x}}$ such that*

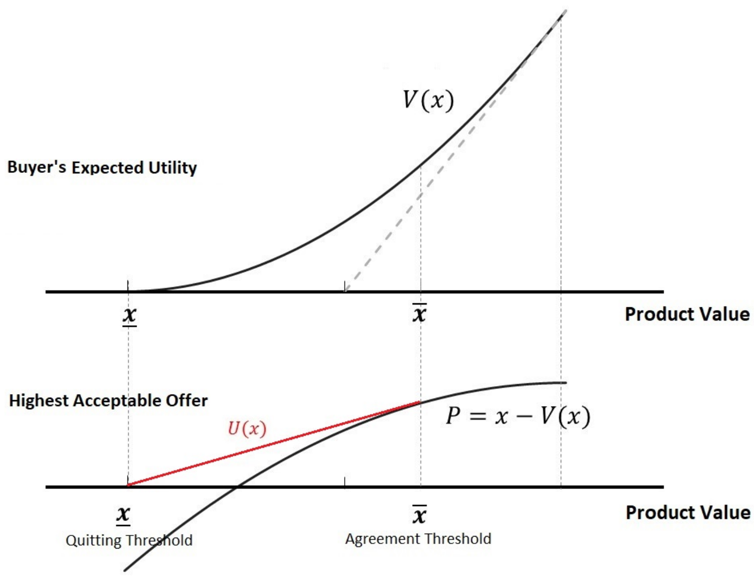
- *If $x \geq \overline{x}$, then Seller offers $P(x) = x - \underline{V}(x)$, and Buyer accepts.*
- *If $x < \overline{x}$, then Seller offers $P(x) > x - \underline{V}(x)$, and Buyer rejects.*
- *If $x \leq \underline{x}$, then Buyer quits.*

Even though Seller cannot commit to future prices, she can still enforce Buyer's continuation value to be $\underline{V}(x)$. Consider the following strategy: Seller offers price $P(x) = x - \underline{V}(x)$ if she wants to trade, and offers $P(x) > x - \underline{V}(x)$ if she does not want to trade. If Seller follows this strategy, Buyer never receives more than $\underline{V}(x)$ utility. Buyer's optimal response then is to comply with Seller. Buyer buys if $P(x) = x - \underline{V}(x)$, which is the price that makes Buyer indifferent between buying now, or continuing with a continuation value of $\underline{V}(x)$. If $P(x) > x - \underline{V}(x)$, Buyer rejects because he gets less utility than he gets from continuing, since continuation value must be at least $\underline{V}(x)$. Thus by following this pricing strategy, Seller can control when they trade. Intuitively, at every moment in time, Seller is choosing between two options. She can continue the sales process, or she can close the sale right now by offering a discount. If Seller chooses to close the sale, she offers a discounted price of $P(x) = x - \underline{V}(x)$. This transforms the game into Seller's optimal stopping problem. Solving Seller's optimal stopping problem also solves the equilibrium, since Buyer's optimal action is to comply with Seller's stopping decision. The questions become when Seller should make the final offer, and what offer Seller should make. Note that Seller can only control the stopping decision for states between thresholds $\overline{\overline{x}}$ and $\underline{\underline{x}}$, which are Buyer's stopping thresholds facing a fixed price of $\overline{P}$. At $\underline{\underline{x}}$, Buyer quits and ends the game. At $\overline{\overline{x}}$, Buyer buys the product even if it is priced at the list price $\overline{P}$, so Seller cannot delay trade beyond these points.

Figure 1.5 illustrates Seller's problem graphically. Product value x is on the horizontal axis, and shifts as the games continues. Buyer's expected value from continuing the sales process given product value x, $V(x) = \underline{V}(x)$, is shown on top half of the graph. The curve on the bottom graph shows the highest offer that Buyer is willing to accept given that the current expected product value is x, which is $P(x) = x - V(x)$. At each moment, the Seller decides between two choices: close the sale and captures $P(x)$, or wait and hope to charge a higher price in the future. Seller's value function $U(x)$ is the straight line on the bottom chart ($U(x)$ is straight due to zero flow cost). Seller's value hits π_s at $\underline{x}$ when Buyer quits, and Seller gets $P(x) = x - \underline{V}(x)$ when they trade at $\overline{x}$. To maximize $U(x)$, the agreement threshold $\overline{x}$ must make $U(x)$ and stopping value $P(x)$ tangent. Otherwise, Seller can profitably deviate by making the offer earlier or later.

Why do we care about the equilibrium outcome in which Buyer receives $\underline{V}(x)$? In Section (1.10), I show that $\underline{V}(x)$ is the unique limit of Buyer's equilibrium payoffs. If one solves the discrete-time game described in Section 2.1, excluding trivial equilibria with simultaneous quitting, then Buyer's equilibrium value functions must converge to $\underline{V}(x)$ as the game

Figure 1.5: Equilibrium Thresholds and Value Functions



approaches continuous time. As a result, the equilibrium outcome with $V(x) = \underline{V}(x)$ represents the continuous-time limit of the discrete-time equilibrium outcomes. For this reason, the equilibrium outcome with $\underline{V}(x)$ becomes our natural outcome of interest.

Proposition 1 provides the closed-form solution for this outcome on equilibrium path, under any arbitrary list price $\overline{P}$. First, I simplify the notation by normalizing outside options into x and P.[11]

Definition 1. *Define $x_n = x - \pi_b - \pi_s$, $x_{n,0} = x_0 - \pi_b - \pi_s$, $P_n = P - \pi_s$, and $\overline{P}_n = \overline{P} - \pi_s$.*

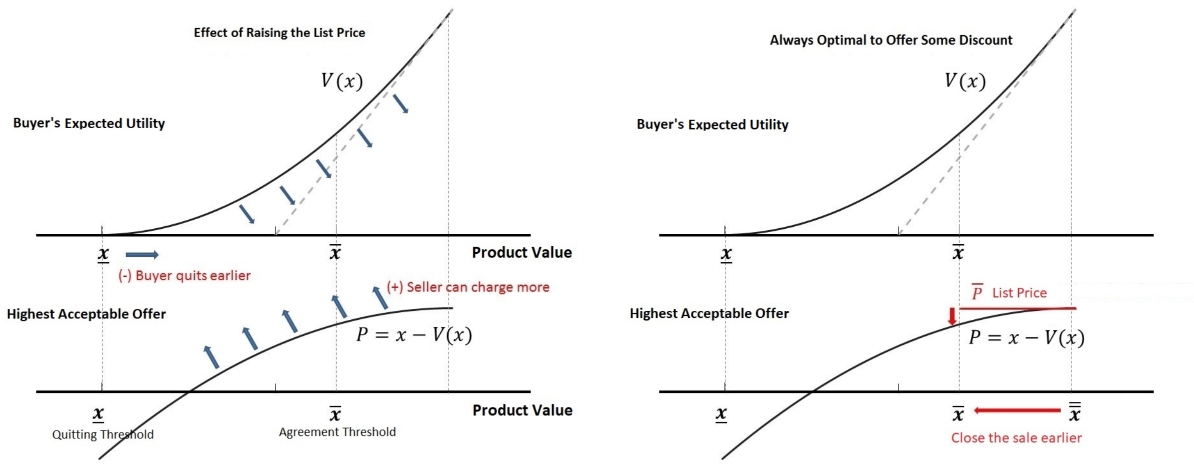
Proposition 1 (Costless Seller)**.** *Buyer and Seller trade at $\overline{x}_n = \overline{P}_n + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \overline{P}_n\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$ at price $P_n(\overline{x}_n) = \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \overline{P}_n\frac{c_b}{\sigma^2}} - 2\left(\frac{1}{4} - \overline{P}_n\frac{c_b}{\sigma^2}\right)\right]$, Buyer quits at $\underline{x}_n = \overline{P}_n - \frac{1}{4}\frac{\sigma^2}{c_b}$, and players continue for $\underline{x}_n < x_n < \overline{x}_n$. The size of the price discount is strictly positive.*

Proposition 1 shows that the list price plays a crucial role in facilitating discovery, and there exists an optimal list price for Seller. A higher list price discourages Buyer from discovering but allows Seller to extract a bigger share of the pie. Figure 1.6a illustrates this point.

As $\overline{P}$ increases, $\underline{x} = \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$ increases, which means that Buyer leaves the sales process earlier when he receives unfavorable information about product match. In the extreme

[11] The notation x_n represents the net surplus from trade, and the notation P_n captures Seller's net gain from trade. This normalization gets rid of outside options but does not affect value functions. It's easy to check that $\underline{V}(x|\overline{P}, \pi_b) = \underline{V}(x_n|\overline{P}_n, 0)$. One can solve the equilibrium outcome using x_n and P_n, then back out the original solution. The rest of the section will simply work with x_n and P_n.

Figure 1.6: The Roles of List Price and Discount



(a) (b)

case where list price is too high, Buyer leaves immediately, because the expected gain from trade is not enough to justify the cost of staying in the sales process. This is conceptually similar to the hold-up problem in Wernerfelt (1994b). It is costly for Buyer to find out about product match (or quality), and Seller can hold Buyer up by charging a price equal to the product value after Buyer incur the cost. This gives Buyer negative utility ex-ante, and as a result, Buyer chooses to not spend any effort in the first place. Thus in order to encourage Buyer to participate, Seller has to impose a low enough list price to raise the option value of discovery for Buyer. If the product fit is bad, Buyer does not have to buy the product, but if the product fit is good, Buyer is guaranteed to pay no more than the list price. On the downside, a lower list price restricts Seller's ability to bargain. Buyer's higher continuation value means that Seller has to offer lower price in order to close the sale, because $P(x) = x - \underline{V}(x, \overline{P})$ decreases as $\overline{P}$ increases. The list price can be viewed as an instrument that balances Buyer's incentives to engage and Seller's bargaining power, and the optimal choice must balance these two effects.

The second finding is that the final trading price is always lower than the list price, regardless of what the list price is. Thus, Proposition 1 predicts that the sale must come at a discount.[12]

Figure 1.6b illustrates why it is never optimal for Seller to sell at the list price. If Seller never offers a price discount, Buyer will continue to learn about the product until product value reaches $\overline{\overline{x}} = \overline{P} + \frac{\sigma^2}{4c_b}$ (the vertical threshold on the right). As Buyer gets close to this threshold, Buyer's value function $\underline{V}(x)$ becomes increasingly tangent to $x - \overline{P}$. As a result,

[12]In reality, some customers may buy at the list price. Note that the model so far only allows for a single type of buyer. If there are different types of buyers with different costs and starting positions, then some buyers could buy at the list price in equilibrium.

$P(x)$ becomes increasingly tangent to $\overline{P}$. So, Buyer is willing to accept offers with discounts that approach 0. Seller can close the deal earlier by sacrificing very little on price. Closing the sale earlier is beneficial because it increases the ex-ante success rate and saves cost (in this case, only Buyer's cost is saved, but Seller can extract Buyer's savings through price). The amount of discount required to close the sale increases at a faster rate as product value decreases, so it is not optimal to close the deal too early. In other words, having some discovery of product match is valuable to Seller, because discovery can increase total surplus by resolving uncertainty. Seller can charge a higher price if the product match is revealed to be good. The optimal timing and size of the price offer in Proposition 1 balances the risk of losing the sale with the potential of selling at a higher price. Proposition 2 solves the optimal list price.

Proposition 2. *(Optimal List Price and Outcome at $t = 0$)*

- *For intermediate values of $x_{n,0}$ ($-\frac{1}{4}\frac{\sigma^2}{c_b} < x_{n,0} < \frac{1}{16}\frac{\sigma^2}{c_b}$), the optimal list price is $\frac{1}{3}x_{n,0} - \frac{1}{18}\frac{\sigma^2}{c_b}\sqrt{1 - 12x_{n,0}\frac{c_b}{\sigma^2}} + \frac{7}{36}\frac{\sigma^2}{c_b}$, and the game continues beyond $t = 0$.*
- *For higher $x_{n,0}$, any list price above $x_{n,0}+\frac{\sigma^2}{4c_b}$ is optimal. Parties trade at price $P_n = x_{n,0}$ at $t = 0$.*
- *For lower $x_{n,0}$, any non-negative list price is optimal. Buyer quits at $t = 0$.*

Proof. Given Proposition 1, maximizing Seller's ex-ante utility over $\overline{P}$ yields the results. □

Proposition 2 shows that the discovery/bargaining only takes place if the initial surplus from trade, $x_{n,0} = x_0 - \pi_b - \pi_s$, is not too big or too small. If the initial surplus is big enough ($x_{n,0} \geq \frac{1}{16}\frac{\sigma^2}{c_b}$), then Seller does not want Buyer to learn anything about the product. Seller publishes a list price high enough to deter Buyer from discovering, and offers a monopoly spot price that takes all existing surplus. On the other hand, if the initial surplus is too low ($x_{n,0} \leq -\frac{1}{4}\frac{\sigma^2}{c_b}$), matching is socially inefficient. Buyer will leave the game even if the list price is set to 0.[13]

Given that some firms allow salespersons to bargain with customers while others do not (Kraft 1994), it is important to understand the effect of allowing players to bargain. When coming to the market, should Seller commit to a fixed price or should she allow price to be negotiated downward? The following corollary compares the equilibrium outcome under the optimal list price to the outcome if Seller commits to a fixed price $\overline{P}$.

Corollary 3 (Comparison to Fixed Price for Costless Seller)**.** *The optimal list price under bargaining is higher than the optimal fixed price. The final price under bargaining is lower than the optimal fixed price. Expected length of the game is shorter under bargaining. Ex-ante social welfare is higher but Buyer's utility is lower.*

[13]Since only Buyer has cost, Buyer's action is socially optimal when the list price is 0.

Proof. Let T denote the total length of the game. Buyer's ex-ante utility must satisfy $V(x_{n,0}) = \frac{x_{n,0}-\underline{x}_n}{\overline{x}_n-\underline{x}_n}(\overline{x}_n - P(x_n)) - c_b\mathbb{E}[T]$, where $\frac{x_{n,0}-\underline{x}_n}{\overline{x}_n-\underline{x}_n}$ is the ex-ante probability that players reach agreement. Expected length of the sales process is thus calculated as $\mathbb{E}[T] = \frac{1}{c_b}\left[\frac{x_{n,0}-\underline{x}_n}{\overline{x}_n-\underline{x}_n}(\overline{x}_n - P(x_n)) - V(x_{n,0})\right]$. The comparisons are straightforward. □

Figures 1.4 simulate two sample paths and illustrate the value of bargaining to the Seller. Trade happens at time τ, with price $P(x_\tau)$ slightly below the list price. However, this small discount significantly decreases Buyer's buying threshold from $\overline{P}+\frac{\sigma^2}{4c_b}$ to $\overline{x}$. The dotted paths in Figure 2.2 simulates what happens if Seller commits to the list price. Buyer continues to discover the match for a period of time. If the subsequent match is good, then Seller's ability to bargain significantly decreases the length of the process. The cost savings are captured by Seller through price and increases her ex-ante utility. If players subsequently discover that the product match is bad, then the game ends without a trade. Thus the lower trading threshold from bargaining increases the ex-ante success rate. Note that the graph does not show the full price path. There are infinite price paths leading up to time τ that produce the same equilibrium outcome. Price strategies before time τ only need to be high enough so that Buyer does not want to buy. Keeping the price at the list price before time τ, for example, would work.

Corollary 4. *(Comparative Statics w.r.t c_b, π_s, π_b)*
For $-\frac{1}{4}\frac{\sigma^2}{c_b} < x_{n,0} < \frac{1}{16}\frac{\sigma^2}{c_b}$:

- *Optimal list price and final price decrease in Buyer's cost, c_b; increase in Seller's outside option, π_s; and decrease in Buyer's outside option, π_b.*
- *Size of price discount decreases in Buyer's cost, c_b, and outside options, π_s and π_b.*
- *Ex-ante probability of trade is unaffected by Buyer's cost, c_b, and decreases in outside options, π_s and π_b.*
- *Expected length of the game decreases in Buyer's cost, c_b, and has inverse U-shape in outside options, π_s and π_b.*

Interestingly, the expected length of the game is non-monotonic in outside options. Intuitively, one would expect that worse outside options make players more interested in matching with each other. However, as outside options become increasingly poor, surplus from trade gets higher (after normalization). As a result, Seller is more inclined to close the deal early. Thus the length of the sales process is short for both very good and very bad outside options.

Another finding is that the ex-ante probability of trade is unaffected by Buyer's cost. For a given list price, the probability of trade decreases in c_b. However, a higher cost for Buyer makes the hold-up problem more severe, which then pushes Seller to set a lower list price. In equilibrium, these two effects negate each other. Note that this is not true if bargaining is not allowed. If players cannot bargain, one can show that the ex-ante probability of trade is monotonically decreasing in Buyer's cost, even if Seller set the price optimally.

Furthermore, when Buyer's cost of continuing the sales process increases, the price discount he receives actually decreases. The reason is that with a higher cost for Buyer, Seller has to set a lower list price. Buyer still receives a lower final price even though the size of the discount is smaller.

1.4 Costly Selling

In this section, selling is costly so that Seller may want to quit before Buyer does. Following notation from the last section, let $\underline{x} = \sup\{x| \sum_i q_i(x) > 0\}$ denote the threshold where the "earliest" quitting happens. We can normalize outside options to $\pi_b = \pi_s = 0$ WLOG as in Definition 1.

When both players can choose to quit, we have trivial equilibria in which both players quit simultaneously. If the opponent quits now, then a player is indifferent between quitting and not quitting, so quitting is weakly optimal. As a result, quitting at any state x can be supported in an equilibrium, by having both players quit simultaneously.[14] To avoid this triviality, I restrict attention to equilibrium outcomes that satisfy the following condition.

Condition 1. *Either* $U(\underline{x}) = U'(\underline{x}) = 0$ *or* $V(\underline{x}) = V'(\underline{x}) = 0$.

The condition implies that the quitting decision is optimal for at least one of the player, even if the other player never quits. In a single agent optimal stopping problem, the quitting threshold must satisfy the value-matching condition, $u_i(\underline{x}) = 0$, and smooth-pasting condition, $u_i'(\underline{x}) = 0$. The value-matching condition ensures that the player does not quit if continuation value is positive, and the smooth-pasting condition ensures that the timing of quitting is optimal (Dixit 1993, pg.34-37). This condition is satisfied for Buyer's quitting threshold in Section 2, for example. In Section (1.10), I show that if one requires the players to quit if and only if quitting is strictly preferred, then the limit of discrete-time equilibrium outcomes satisfies Condition 1. Thus Condition 1 eliminates the simultaneous quitting triviality.

It is unclear which player quits first in equilibrium.[15] Intuitively, the player with a higher flow cost is more likely to quit earlier. However, Proposition 5 below shows that this is not the case. If Seller chooses the list price optimally, then Buyer always quits before Seller does.

As in Section (1.3), I first derive the lower bound on Buyer's equilibrium value function. One can solve for Buyer's value function facing a fixed price of $\overline{P}$, subject to a game-ending state at $\underline{x}$. This gives a lower bound on the Buyer's equilibrium value function if the quitting

[14] Due to the nature of continuous time, a strategy profile can achieve the effect of simultaneous quitting without having players quitting at the same x. For example, suppose Seller quits in a set B except a single point x'. Then on x', Buyer cannot extend the game whether he quits or not. The game ends immediately even if Buyer does not quit, making him indifferent between quitting at x' or not. This example illustrates that it it not sufficient to simply restricting players to not quit at the same x.

[15] That is, whether $\underline{x}$ is the optimal quitting threshold for Buyer or for Seller. If $\underline{x}$ is optimal for Buyer (Seller), then we must have $V(\underline{x}) = V'(\underline{x}) = 0$ ($U(\underline{x}) = U'(\underline{x}) = 0$), due to the discussion above.

threshold is $\underline{x}$. Denote this lower bound as $\underline{V}(x, \underline{x})$. The closed-form solution for this lower bound is in Section (1.9).

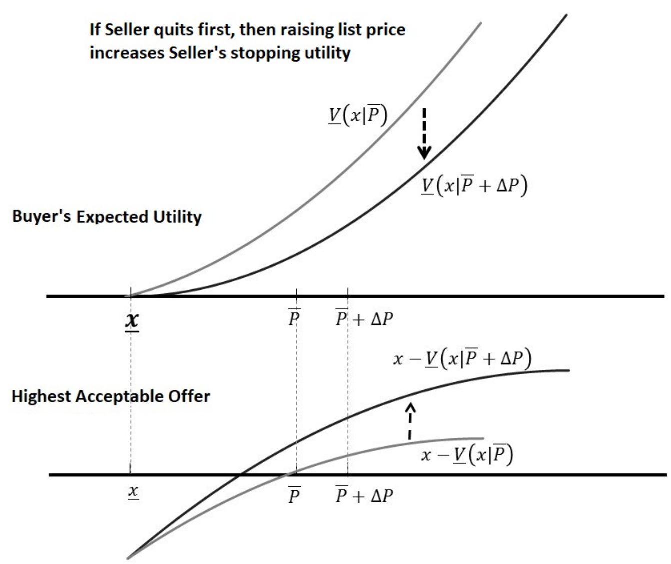
Then one can construct equilibrium strategies that give such payoff to Buyer and satisfy Condition 1. As before, this lower bound payoff is the limit of Buyer's equilibrium payoff in the discrete-time game, so this is the outcome of interest in this paper.

Lemma 2. *There exists an unique equilibrium outcome in which* $V(x) = \underline{V}(x, \underline{x})$ *and satisfies Condition 1.*

Proposition 5. *If Seller quits earlier than Buyer does, i.e.,* $\sup\{x|q_s(x) = 1\} > \sup\{x|q_b(x) = 1\}$*, then the list price* $\overline{P}$ *is sub-optimal (too low) for the equilibrium outcome in Lemma 2.*

Figure 1.7 provides a sketch of the proof. If Seller quits first at $\underline{x}$, then increasing list price by $\Delta P = \underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b} - \overline{P}$ is strictly better for the Seller. This implies that the original list price was sub-optimal. Setting list price to $\overline{P} + \Delta P$ leads to a new stopping problem for Seller with the same quitting threshold but a higher stopping payoff in every state. Thus, Seller's ex-ante utility must be strictly higher under the new list price. Note that this new list price $\overline{P} + \Delta P$ is **not** the optimal list price. The optimal list price must be even higher.

Figure 1.7: Raising List Price If Seller Quits First



The intuition of Proposition 5 can be found in Figure 1.6b. Suppose under a list price Seller quits first. Consider the effect of raising the list price. As Figure 1.6b shows, increasing the list price has both positive and negative effects for Seller. The positive effect is that a higher list price lowers Buyer's continuation value, which increases the surplus that Seller can extract in every state. The negative effect is that Buyer's lower continuation value pushes him to quit earlier. However, if Seller is quitting earlier than Buyer does anyway, then pushing Buyer to quit earlier has no effect. Thus, raising the list price is strictly beneficial to the Seller. The original list price must be sub-optimal.

Proposition 5 implies that, to find the outcome under optimal list price, one can solve the equilibrium outcome by assuming that $\underline{x} = \underline{x_b} = \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$, then maximizing $U(x_0)$ over $\overline{P}$, and finally verifying that indeed $\underline{x_b}(\overline{P}^*) \geq \underline{x_s}(\overline{P}^*)$.

For tractability, I restrict attention to the case of $x_0 = 0$ for the next Proposition.

Proposition 6 (Costly Selling with $x_0 = 0$)**.** *Let* $k = \frac{c_s}{c_b}$. *The optimal list price is* $\overline{P} = \frac{\sigma^2}{c_b}(\frac{1}{4} - \frac{27+10k-6\sqrt{9+10k+k^2}}{16k^2})$. *Buyer and Seller trade at* $\overline{x} = \overline{P} + \frac{\sigma^2}{c_b}\left[\frac{3-\sqrt{\frac{9+k}{1+k}}}{4k} - \frac{1}{4}\right]$ *at price* $P(\overline{x}) = \overline{P} - \frac{\sigma^2}{c_b}\left(\frac{3-\sqrt{\frac{9+k}{1+k}}}{4k} - \frac{1}{2}\right)^2$. *Buyer quits at* $\underline{x} = \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$.

If Seller has the option to commit to selling at a fixed price, should Seller commit or be open to bargaining? Corollary 7 shows that bargaining is necessary for discovery to take place when Seller's cost is high relative to Buyer's. If the ratio of c_s to c_b exceeds a certain threshold, and if price is fixed, one of the players must quit immediately at $t = 0$, regardless of the level of the price. Thus there does not exist a fixed price such that both are willing to "sit down" at time 0. However, this problem is avoided if bargaining is allowed. The length of the sales process and the ex-ante chance of a trade are always positive under bargaining. The ability to bargain lowers both player's expected costs by giving Seller the flexibility to close the sale early. This flexibility is particularly beneficial when Seller's flow cost is high.

Suppose Seller charges a fixed price, when Seller' cost is very high, Seller needs to charge a high price to make up for her expected cost of selling. But a higher price pushes Buyer to wait longer before buying, which makes the process even more costly for Seller, who in turn has to charge an even higher price. If Seller charges too high a price, Buyer quits immediately. Thus, there does not exist any fixed price such that both players are willing to participate in the sales process beyond $t = 0$. If Seller is open to bargaining, however, then Seller can reduce her cost of selling by offering a discount to close the sale earlier. This flexibility allows Seller to post a lower list price initially. Also, Seller always benefits from Buyer engaging in the sales process for at least some positive amount of time. If the product match is good in that "period", Seller can close the sale at a positive price. If the product match is bad, Buyer will quit and Seller gets 0. Thus the option value of discovery is always positive at time 0, regardless of Seller's cost. This effect guarantees that Seller does not charge a list price too high. When Seller's cost is low, Buyer prefers to face a fixed price over bargaining, but when the ratio $k = \frac{c_s}{c_b}$ exceeds a certain threshold, both Buyer and Seller gain from bargaining.

Corollary 7 (Importance of Bargaining)**.** *If* $k(= \frac{c_s}{c_b}) \geq x_0\frac{4c_b}{\sigma^2} + 1$, *then the game ends at* $t = 0$ *if price is fixed, but ends at* $t > 0$ *with positive ex-ante probability of trade if price can be bargained. Seller's ex-ante utility is higher under bargaining for all* k. *Buyer's ex-ante utility is higher under bargaining if* $k > \underline{k}$, *for some* $\underline{k} \leq x_0\frac{4c_b}{\sigma^2} + 1$.

This result helps to explain the prevalence of bargaining in many B2B industries. Firms have to incur significant cost in employing and training salespersons, and salespersons often spend a significant amount of resources on each client. Corollary 7 shows that, if this cost is high relative to the customer's cost of participating in the sales process, then the firm must be open to bargaining. Otherwise, trade cannot take place.

In practice, a firm can choose whether to allow bargaining by choosing whether to delegate the pricing authority to its salespersons. This question has been studied under principal-agent models. See, e.g., Lal (1986), Bhardwaj (2001), and Joseph (2001). The principal-agent models used in these papers highlight the disadvantage of delegating pricing authority when selling cost is high. Salespersons have the incentive to give customers too much discount so they can shirk on selling efforts. This problem is intensified when the selling efforts are more costly. On the other hand, the information acquisition approach of this paper emphasizes the advantage of delegating pricing authority when selling cost is high. Using a survey of 270 companies from different industries, Hansen et al. (2008) find that firms that need more calls to close a sale are more likely to delegate pricing authority to salespersons. Corollary 7 provides one explanation for this empirical observation.

Corollary 8 (Comparative Statics w.r.t c_s and c_b for $x_0 = 0$)**.**

- *Optimal list price decreases in Buyer's cost, c_b, and increases in Seller's cost, c_s.*
- *Final price decreases in both players' costs, c_b and c_s.*
- *Size of price discount decreases in Buyer's cost, c_b, and increases in Seller's cost, c_s.*
- *Expected length of the game decreases in both player's costs, c_b and c_s.*

Interestingly, a higher c_s leads to a higher $\overline{P}$ but a lower $P(\overline{x})$. This means that, when Seller's cost increases, she posts a higher list price, but gives a bigger discount and sells at a lower price than before. The lower final price helps to reduce the length of the game and saves cost. Note that, if Seller is not able to give a discount (as under a fixed price), Seller would want to charge a higher price when cost increases. This eventually leads to the no-discovery result from Corollary 7.

1.5 Private Outside Option

The previous sections assume that Buyer's expected value for the product, x_t, is fully observable to Seller. In many settings, though, the buyer may have private information regarding his willingness-to-pay. In particular, even if the seller observes the buyer's preference for each attribute, the seller may not know what the buyer's outside option is. This section expands the model to incorporate this scenario.

There are two types of Buyer with different outside options: a (H)igh type and a (L)ow type. The type is Buyer's private information. L type is the Buyer with a **better** outside option, and H type is the Buyer with a **worse** outside option. For the same attributes and preferences, the buyer with a better outside option has a lower willingness-to-pay. Seller learns Buyer's preference for each attribute during the sales process, but she is uncertain about Buyer's willingness-to-pay for the product due to Buyer's private information. This gives rise to information asymmetry. If we normalize Buyer's outside option into his product value, then at each moment, Seller is facing a distribution of Buyers with different levels of net product value. Seller observes how this distribution shifts over time, but does not observe where Buyer falls within this distribution.

Formally, let $i \in \{H, L\}$ denote Buyer's type. Nature draws H type with probability λ and L type with probability $1-\lambda$. Define ϵ as the different in outside options between the two types. Let x_H and X_L denote the H type and L type's product values net of their respective outside options. Denote $x = \frac{1}{2}(x_H + x_L)$ as the common state variable. Then we can write $x_H = x + \frac{\epsilon}{2}$ and $x_L = x - \frac{\epsilon}{2}$. Both H type and L type incur flow cost $c_b > 0$. Seller is assumed to have no cost and never quits. Positive selling cost does not impact the outcome qualitatively.

I look for Stationary Sequential Equilibrium with pure strategies. Equilibrium utilities and strategies now depend on two state variables, product value x and Seller's belief μ, where $\mu \in [0, 1]$ denote Seller's belief that Buyer is type H. The players' equilibrium strategies are represented by $P(x, \mu)$, $a_i(x, \mu, P)$, and $q_i(\mu)$. Type i's value function is $V_i(x, \mu)$. Note that with pure strategies, Seller's belief μ can only take 3 values: 0, 1, or λ. Denote μ_L for the belief that Buyer is L type, μ_H for the belief that Buyer is H type, and μ_{HL} for the belief that the Buyer can be either. Note that if $\mu = \mu_L$ or $\mu = \mu_H$, then there is no private information. This happens on the path of a separating equilibrium. I assume that Seller does not update her belief off the equilibrium path.

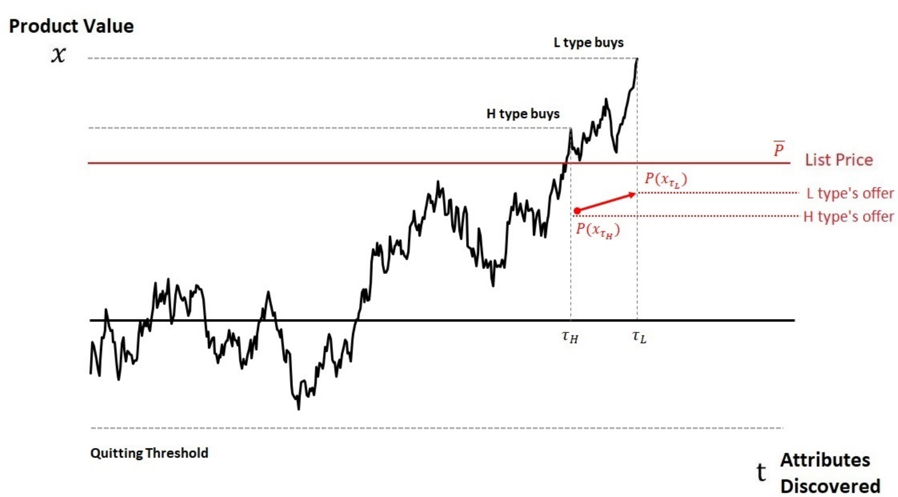
Similar to earlier sections, I first look for the lower bound on Buyer's equilibrium utility. Let $\underline{V}_i(x, \mu)$ denote the lower bound on type i's equilibrium value function in state x and under belief μ. If only one type of Buyer remains ($\mu = \mu_H$ or $\mu = \mu_L$), then the lower bound is the same as in Section 3. If Buyer receives this lower bound when only one type remains, then one can show that L type must have the same lower bound even when both types are still in the game. The proof is roughly structured as follows. First, Lemma 3 in Section (1.8) proves that, if Buyer receives the lower bound from Section 3 when $\mu = \mu_H$ or $\mu = \mu_L$, then H type must buy (weakly) before L type in any equilibrium, because H type suffers from revealing his type. If L type is about to accept the offer, H type is always better off taking L type's offer. If H type waits, he becomes the only remaining type, and receives the Buyer's payoff from Section 3, which is strictly inferior to taking L type's offer. Second, Lemma 4 in Section (1.8) shows that, if H type buys earlier than L type, then L type's utility has the same lower bound regardless of μ. Since H type will not wait beyond L type's offer, Seller cannot use $\overline{P}$ as a threat to H type. Seller's offer to the L type then implies a new lower bound on H type's value function. H type cannot be worse than if Seller never offers a discount until Seller makes an offer for the L type. This gives both types of Buyer's lower bounds when $\mu = \mu_{HL}$. The closed-form solutions for $\underline{V}_i(x, \mu)$ is presented in Section (1.8).

Seller then faces the new optimal stopping problem regarding when to sell to H type and at what price. Proposition 9 describes the equilibrium outcome in which each type of Buyer receives the lower bound of his equilibrium value functions. We only need to solve for the case of $\overline{P} < x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$, otherwise the types unravel immediately.[16]

Proposition 9 (For $\overline{P} < x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$)**.** *There exists an unique equilibrium outcome in which* $V_i(x,\mu) = \underline{V}_i(x,\mu)$. *Buyer with type* $i \in \{H, L\}$ *quits at* $\underline{x}_i = \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$, *and buys at* $\overline{x}_i = \overline{P} + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$. *H type buys earlier and pays less than L type.*

Figure (1.8) presents a sample equilibrium path. H type buys earlier, and pays a lower price. Trading price increases from time τ_H to τ_L. The right panel on Figure (1.9) shows x_H and x_L separately. Both types of Buyer buy when their net product value reach $\overline{x}$. H type reaches the threshold earlier, because his worse outside option translates to a higher net value from buying the product.

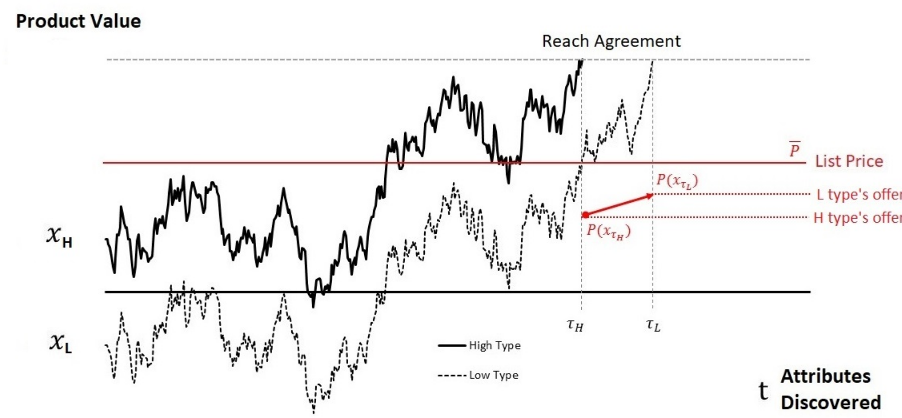
Figure 1.8: Sample Path with Private Types



Surprisingly, having a private outside option does not affect Buyer's buying and quitting thresholds in the equilibrium. Both types of Buyers buy and quit at the same thresholds on their respective net product values, and these thresholds are the same as Buyer's thresholds without private information. The finding that H type pays less than L type runs counter to many existing bargaining models, in which the seller makes declining offers to screen through buyers' reservation prices, so types that buy earlier pay higher prices.

[16]By Lemma 4, L type should act the same way as in Proposition 1. Thus, when $\overline{P} \geq x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$, L type buys immediately if $x_0 - \frac{\epsilon}{2} > 0$ or quits immediately if $x_0 - \frac{\epsilon}{2} \leq 0$. By Lemma 3, H type should buy immediately if L type buys immediately. If L type quits immediately and H type stays, then there is only a single type of Buyer remaining, which is again solved in Proposition 1. So we do not need to solve for the case of $\overline{P} \geq x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$.

Figure 1.9: Private Types with Normalized Outside Option



Why does L type not take H type's offer, if waiting is costly and price is rising? The reason is that, when Seller makes an offer to H type, L type's net product value is less by ϵ, and this difference makes L type strictly prefer to wait and discover more. On the flip side, Seller has to offer H type a lower price because H type has the option to pretend to be L type. H type can wait until x_L reaches the buying threshold, and then pay the same price as L type. However, Seller prefers to close the sale with H type earlier, which saves cost as well as locking in the surplus. Intuitively, if Buyer already comes into the negotiation with a good intention to buy, then the subsequent discovery is less valuable, and closing the sale earlier is more efficient. But in order for H type Buyer to voluntarily reveal his type, Seller needs to make a more generous offer. This is analogous to the problem of product line design with different types of buyers. H type buyer can take L type's contract, but the seller may not want H type to do so. This creates a binding incentive compatibility constraint that forces the seller to concede more utility to H type. In this game, Seller wants Buyers to self-sort into processes of different lengths: a shorter process for H type and a longer process for L type. The price cut for H type is needed to satisfy the incentive compatibility constraint so that H type does not undertake L type's process. The price difference represents H type's information rent. The size of this rent is $P(\overline{x}+\frac{\epsilon}{2},\mu_L)-P(\overline{x}-\frac{\epsilon}{2},\mu_{HL}) = \underline{V}(\overline{x})-\frac{c_b}{\sigma^2}(\overline{x}-\frac{\epsilon}{2}-\overline{P})^2-\alpha_H(\overline{x}-\frac{\epsilon}{2}-\overline{P})-\beta_H$, where α_H and β_H are in equations (1.21) in Section (1.8).

This model with private types can be seen as an extension to repeated-offers bargaining models with one-sided incomplete information, such as Fudenberg et al. (1985). The novelty here is that, with the additional discovery of product match, the product value follows a public diffusion process. Fudenberg et al.(1985) show that, if Seller can change price arbitrarily fast, then all types buy immediately at a price equal to the value of the lowest type. If players incur flow cost, the bargaining game is akin to an one-shot monopoly pricing, because no Buyer will wait beyond $t = 0$. In either case, trade is immediate, and Seller cannot separate different types. With the stochastic value (and the existence of list price), now trade happens with delay, and Seller can separate Buyers of different types. Buyer with the higher ex-ante value are offered a lower price due to information rent.

One feature of this model is that the relationship between length of the sale process and trading price is probabilistic and price can increase over time. This does not mean, however, that empirically one should expect to observe prices to rise with the length of negotiation. Many factors ignored in this model can cause a downward trend in price. For example, if Seller can commit to each price offer for a short period of time and if players have time discounting, then Seller can use declining price to screen different types. This is common in the repeated-offers bargaining literature. Section 1.6 shows a few extensions in which price can decrease over time - for example, if the product has a finite mass of attributes, or if players observe signals of the true match value. However, the core intuition should still remain. The fact that H type has a worse outside option makes the product more appealing ex-ante and decreases the value of information acquisition. This encourages Seller to trade with him earlier, which puts downward pressure on H type's price in order for the offer to be incentive compatible.

1.6 Other Extensions

Finite Mass of Attributes

The baselin model assumes that players match on an infinite mass of attributes during the discovery process. This assumption allows one to study stationary equilibria, as players never exhaust the information that they can learn. In reality, a product may have only a limited number of features, or consumers may care about only a finite subset of a product's features. As a result, players exhaust all the information they can learn from the other party if the game continues long enough.

In this section, I extend the model from Section 3 to incorporate such a case. Suppose that the product has a finite mass of attributes, T. Players can only discover their product match up to time T. The product value x_t follows a Brownian motion between time 0 and time T. Markov Perfect Equilibria under a finite mass of attributes can no longer be stationary. Players' strategies and value functions now depend on both state x and time t. Equilibrium value functions satisfy the following partial differential equations:

$$\begin{aligned} V_{xx} &= \frac{2}{\sigma^2}(c_b - V_t) \\ U_{xx} &= \frac{2}{\sigma^2}(c_b - U_t) \end{aligned} \tag{1.4}$$

where V_{xx} and U_{xx} are second-order derivatives w.r.t x and V_t and U_t are first-order derivatives w.r.t t. These are the same as Equations (1.2), except that V_t and U_t are no longer 0 due to the non-stationarity of the problem.

The analysis is similar to Section 1.3. One first solves for the lower bound on Buyer's equilibrium value function, $\underline{V}(x, t|\overline{P})$, by solving for Buyer's optimal stopping problem facing a fixed price of $\overline{P}$. Buyer's stopping thresholds, $\overline{\overline{x}}(t)$ and $\underline{\underline{x}}(t)$, now change with time t. The optimal $\overline{\overline{x}}(t)$ and $\underline{\underline{x}}(t)$ must satisfy the value-matching and smooth-pasting conditions:

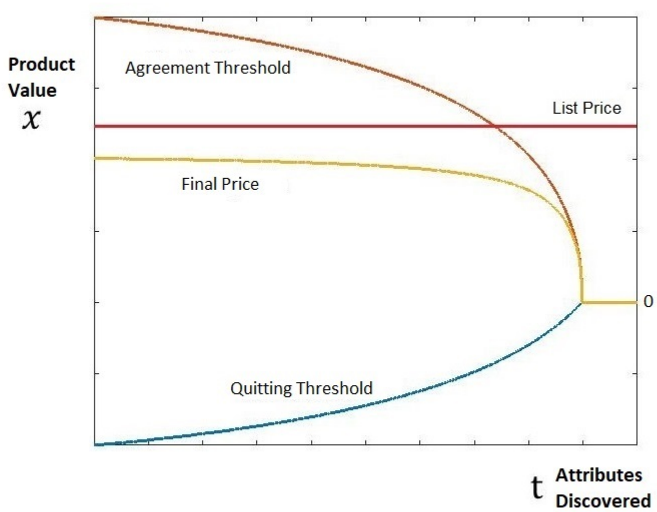
$$\begin{cases} V(\overline{\overline{x}}(t), t) = \overline{\overline{x}}(t) - \overline{P} & \text{and} \quad V(\underline{\underline{x}}(t), t) = 0 \\ V_x(\overline{\overline{x}}(t), t) = 1 & \text{and} \quad V_x(\underline{\underline{x}}(t), t) = 0 \end{cases} \tag{1.5}$$

The solution to these boundary conditions can be solved numerically. By the same reasoning as in Lemma 1, one can construct an equilibrium where Buyer gets this lower bound as his utility, $V(x,t) = \underline{V}(x,t)$. At each moment, Seller decides whether to stop by offering $P(x,t) = x_t - \underline{V}(x,t)$ to Buyer, or to continue the game. Buyer cannot do better than complying with Seller's strategy, so he buys when Seller offers $P(x,t) = x_t - \underline{V}(x,t)$. Trade happens when product value reaches Seller's stopping threshold $\overline{x}(t)$, and Buyer quits when product value reaches $\underline{x}(t) = \underline{\underline{x}}(t)$. Seller's choice of trading threshold $\overline{x}(t)$ must satisfy:

$$\begin{cases} U(\overline{x}(t), t) & = \overline{x}(t) - \underline{V}(\overline{x}(t), t) \\ U_x(\overline{x}(t), t) & = 1 - \underline{V}_x(\overline{x}(t), t) \end{cases} \tag{1.6}$$

These two conditions pin down the equilibrium trading threshold $\overline{x}(t)$ as well as the equilibrium price $P(\overline{x}(t))$. Figure 10 presents the solution for the case of $T = 1$, $c_b = 0.25$, $\sigma^2 = 1$, $x_0 = 0$, at a list price of 0.5.

Figure 1.10: Finite Mass of Attributes



One can show that both the trading threshold and trading price decline over time towards 0, and Buyer's quitting threshold increases over time towards 0. Also, players never wait until they exhaust all attributes. Because Buyer makes his quitting decision as if he is facing a fixed price, his quitting threshold must approach $\overline{P}$ as time approaches T. Thus, the quitting threshold crosses $x = 0$ strictly before time T. Suppose the quitting threshold is above 0, and the product value x is between 0 and the quitting threshold; then, instead of letting Buyer quit, Seller wants to sell to him at the price of $P(x, t) = x$. Seller must also want to sell to all types above x, because the stopping value $x - V(x, t)$ is concave in x and $U(x, t)$ is convex in x. Because this argument holds for all positive x, Seller must immediately sell if $x > 0$, and Buyer quits immediately for $x \leq 0$. Thus, the game must end before Buyer's quitting threshold crosses 0, which happens strictly before players exhaust all attributes at time T.

Signals on True Product Value

Instead of matching on attribute, an alternative way to model the discovery process is to have players observe signals on the true value of the product and update their beliefs using Bayes' rule. In Section 2.1, I describe how the model can be interpreted as a stationary case of learning from sequential signals. In this section, I explore a non-stationary case. In particular, I study the case in which the true value of the product does not change over time, so that players always get "closer and closer" to the truth by receiving more signals.

Suppose that, before the negotiation begins, Buyer and Seller have a common prior on the true value x^* that follows a normal distribution with mean ν and variance e^2. Each moment during the game, players observe a common signal that equals true value plus a normal error term that is independent across time. Players update their beliefs about the product value as bargaining continues.

Specifically, let S_t denote the cumulative signal up to time t. The signal is assumed to follow the process $dS_t = x^* dt + \eta dW$. An incremental signal of size dt is thus the true value $x^* dt$ plus a normal error with variance $\eta^2 dt$. Bayesian updating on the normal prior implies that the expected value of the product after observing t signals, x_t, can be written as

$$x_t = \frac{\nu/e^2 + S_t/\eta^2}{1/e^2 + t/\eta^2} \tag{1.7}$$

The change in expected product value x_t can be decomposed as

$$dx_t = \frac{x^* - x_t}{\eta^2(1/e^2 + t/\eta^2)} dt + \frac{1}{\eta(1/e^2 + t/\eta^2)} dW \tag{1.8}$$

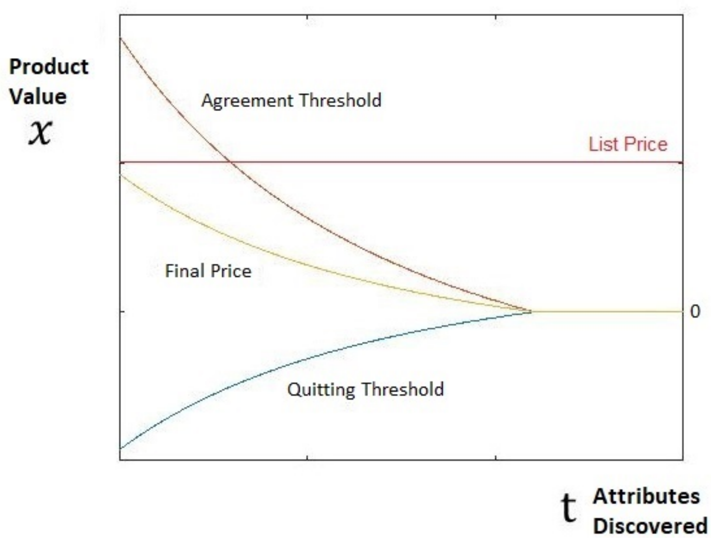
The expected change in the expected product value, conditional on observing S_t, $\mathbb{E}[dx_t|S_t]$, must be 0, because $\mathbb{E}[x^*|S_t] = x_t$ by definition of x_t. The variance of the change in expected product value with dt signals is $\sigma_t^2 dt = \frac{1}{\eta^2(1/e^2+t/\eta^2)^2} dt$. Thus, the process x_t can be described as a Brownian motion with a variance that decreases with time t.

The rest of the analysis is carried out similarly to earlier sections. Equilibrium value functions satisfy the following partial differential equations:

$$
\begin{aligned}
V_{xx} &= \frac{2}{\sigma_t^2}(c_b - V_t) \\
U_{xx} &= \frac{2}{\sigma_t^2}(c_b - U_t)
\end{aligned}
\tag{1.9}
$$

One first solves for Buyer's optimal stopping problem facing a fixed price of $\overline{P}$. This gives the lower bound on Buyer's equilibrium value function, $\underline{V}(x,t|\overline{P})$, and Buyer's quitting threshold $\underline{\underline{x}}(t)$. One can then construct an equilibrium where Buyer gets $V(x,t) = \underline{V}(x,t)$ as his utility. At each moment, Seller decides whether to stop and receive $P(x,t) = x_t - \underline{V}(x,t)$, or to continue. Buyer optimally chooses to comply with Seller's strategy. Trade happens when product value reaches Seller's stopping threshold $\overline{x}(t)$, and Buyer quits when product value reaches $\underline{x}(t) = \underline{\underline{x}}(t)$. Solving Seller's optimal stopping problem gives the equilibrium outcome. Figure 1.10 illustrate the case of $c_b = 0.25$, $\nu = 0$, $e^2 = 1$, $\eta^2 = 1$, at a list price of 0.5.

Figure 1.11: Bayesian Updating



Similar to the previous extension to finite attributes, one can show that both the trading threshold and trading price decline towards 0, and Buyer's quitting threshold increases towards 0. Furthermore, there exists a time by which the players must trade or quit, even though the game does not impose an explicit deadline. This implicit deadline happens when Buyer's quitting threshold hits 0. Because Buyer makes his quitting decision as if he is facing a fixed price, his quitting threshold must approach $\overline{P}$ over time. Thus, the quitting threshold must reach 0 in finite length. Similar to the finite attributes case, when the quitting threshold is above 0, Seller immediately sells to Buyer for all $x > 0$, and Buyer quits immediately for all $x \leq 0$. Thus, the game must end by the time Buyer's quitting threshold hits 0, which is finite.

Declining Importance of Attributes

In the baseline model, all attributes are assumed to be equally important. In reality, some attributes may be more important for a product's value, and the two parties can match on these attributes first. As players discover, later attributes have less impact on the expected value of the product. This differs from the base model because the discovery process is now represented by $dx_t = \sigma_t dW_t$, where σ_t decreases in t. Note that this is closely related to the Bayesian updating extension above, where $\sigma_t^2 dt = \frac{1}{\eta^2(1/e^2+t/\eta^2)^2} dt$. Once σ_t is given, the rest of the analysis can be carried out in the same way. In the equilibrium, both the trading threshold and trading price decline over time and Buyer's quitting threshold increases over time. The game must end before Buyer's quitting threshold hits 0, which happens in finite time.

Time Discounting

The previous sections assume that players incur flow costs and do no have time discounting. The existence of flow costs creates the hold-up problem, because Buyer quits if he does not expect his payoff from future bargaining to compensate for his costly discovery effort today. Alternatively, players can have time discounting instead of flow costs. In this section, I show that the modelling choice between flow cost and time discounting does not affect the main results of the paper, as long as players have positive outside options so that they have incentives to quit.

Let r denote the common discount rate for both players. For non-terminating states, the players' value functions must satisfy the ODE's in Equation (1.2), and with $c_s = c_b = 0$, the solutions to these ODE's are of the form:

$$\begin{aligned} V(x) &= A_b e^{\sqrt{\frac{2r}{\sigma^2}}x} + B_b e^{-\sqrt{\frac{2r}{\sigma^2}}x} \\ U(x) &= A_s e^{\sqrt{\frac{2r}{\sigma^2}}x} + B_s e^{-\sqrt{\frac{2r}{\sigma^2}}x} \end{aligned} \tag{1.10}$$

for some constants A_b, A_s, B_b, and B_s.

For simplicity, assume that $\pi_b > 0$ and $\pi_s = 0$. This mimics the baseline model in Section 1.3, where only Buyer has an incentive to quit. Given a list price $\overline{P}$, one can calculate the lower bound on Buyer's utility, $\underline{V}(x|\overline{P})$, by solving for Buyer's optimal stopping problem facing a fixed price of $\overline{P}$. The value-matching and smooth-pasting boundary conditions are:

$$\begin{cases} A_b e^{\sqrt{\frac{2r}{\sigma^2}}\underline{x}} + B_b e^{-\sqrt{\frac{2r}{\sigma^2}}\underline{x}} = \pi_b & \text{and} \quad A_b e^{\sqrt{\frac{2r}{\sigma^2}}\underline{x}} - B_b e^{-\sqrt{\frac{2r}{\sigma^2}}\underline{x}} = 0 \\ A_b e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} + B_b e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} = \overline{x} - \overline{P} & \text{and} \quad A_b e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} - B_b e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} = \sqrt{\frac{\sigma^2}{2r}} \end{cases} \tag{1.11}$$

Solving this system of equations, one can find that

$$\underline{V}(x|\overline{P}) = \frac{1}{2}(\sqrt{\frac{\sigma^2}{2r} + \pi_b{}^2} + \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{P}-\sqrt{\frac{\sigma^2}{2r}+\pi_b{}^2})} + \frac{1}{2}(\sqrt{\frac{\sigma^2}{2r} + \pi_b{}^2} - \sqrt{\frac{\sigma^2}{2r}})e^{-\sqrt{\frac{2r}{\sigma^2}}(x-\overline{P}-\sqrt{\frac{\sigma^2}{2r}+\pi_b{}^2})} \tag{1.12}$$

Buyer's buying threshold is $\overline{\overline{x}} = \overline{P} + \sqrt{\frac{\sigma^2}{2r} + \pi_b{}^2}$, and Buyer's quitting threshold is $\underline{\underline{x}} = \overline{P} + \sqrt{\frac{\sigma^2}{2r} + \pi_b{}^2} - \sqrt{\frac{\sigma^2}{2r}} \log(\sqrt{\frac{\sigma^2}{2r\pi_b{}^2}} + \sqrt{\frac{\sigma^2}{2r\pi_b{}^2} + 1})$.

The equilibrium with $V(x) = \underline{V}(x)$ can be constructed as before. Seller offers prices $P(x) = x - \underline{V}(x)$ in states where Seller wants to trade, and offers higher prices in other states. The problem transforms into Seller's optimal stopping problem with stopping value $P(x) = x - \underline{V}(x)$, subject to Buyer's quitting threshold $\underline{x} = \underline{\underline{x}}$. The solution of Seller's stopping problem is also an equilibrium outcome, because Buyer's optimal strategy is to comply with Seller's trading threshold. Buyer receives $\underline{V}(x)$ in this equilibrium.

The qualitative results are the same as with flow costs. One can see that $\underline{x}$ increases with $\overline{P}$, and $P(x)$ decreases with $\overline{P}$, thus the optimal list price reduces Buyer's hold-up concern at the cost of Seller's bargaining power. Also, the derivative of $P(x)$ approaches 0 as x approaches $\overline{\overline{x}}$, which implies that Seller's optimal stopping point must be lower than $\overline{\overline{x}}$. Thus, players always trade at a price below the list price, regardless of the level of the list price.

If Seller's outside option, π_s, is also positive, then Seller may choose to quit when the state is bad enough. One can again prove that, on the equilibrium path, Buyer must quit before Seller does. The intuition is the same as in Section 4. If Seller's quitting threshold is higher than Buyer's, then the hold-up problem is not binding; thus, raising the list price increases Seller's stopping payoff $x - \underline{V}(x)$ for all x. This is strictly beneficial to Seller. Furthermore, if Seller commits to a fixed price, then when $\frac{\pi_s}{\pi_b}$ is above a certain threshold, the game must end at $T = 0$. If Seller allows bargaining, then the players must stay and discover beyond $t = 0$. Thus, bargaining is necessary for "conversation" when selling is more costly than buying; here the cost comes from delaying the realization of the outside option.

The results for private outside options can also be replicated. One can show that H type must buy before L type does, following the logic of Lemma 3. Then L type must receive the same payoff as in the full information game, per Lemma 4. If x reaches L type's buying threshold, then H type takes L type's offer, and if x reaches L type's quitting threshold, then L type quits and H type is revealed. These boundaries pin down H type's equilibrium value function, and Seller chooses the optimal time and price for the H type. Seller offers H type a lower price than she offers to L type in order to incentivize H type to buy earlier.

1.7 Conclusion

This paper presents the sales process as a combination of two-sided information acquisition and price bargaining. A seller and a buyer discovers the match between the product's attributes and the buyer's needs sequentially while the seller can make repeated offers bounded above by a self-imposed list price.

The combination of two-sided discovery with bargaining creates a hold-up problem, and the paper illustrates the role of the list price in solving this hold-up problem. Because the seller makes the offers, the buyer may be concerned that his future gain from bargaining cannot compensate for his discovery effort today. As a result, the buyer does not participate in the sales process without a list price or if the list price is too high. A lower list price incentivizes the buyer to stay but reduces the seller's bargaining power. The optimal list price balances these two effects. However, the final price is always below the list price. The seller can encourage the buyer to trade earlier by offering a spot discount, and some level of discount is always optimal regardless of the list price. This model provides a rationalization for the common "list price - discount" pattern that one observes in sales negotiations, which is novel in the literature.

Should the seller commit to a fixed price or be open to bargaining? The model shows that allowing bargaining is always beneficial to the seller, but can hurt the buyer if the buyer's cost is high. When the seller's flow cost is relatively high, bargaining is necessary for both players to participate in the discovery process. This result highlights the importance of allowing salespersons and customers to bargain, especially in B2B industries where selling activities can be resource-intensive.

The model features information asymmetry when the buyer has private information on his outside option. The stochastic nature of the product value and the existence of list price enable the seller to separate different types of buyers, even though the seller has no commitment power on each offer. Surprisingly, the buyer with a higher net valuation from the product (or a lower outside option) pays a lower price due to information rent and the seller's incentive to trade earlier. The discovery process thus benefits higher types by putting downward pressure on the prices they receive.

This paper is limited in several ways. First, the model assumes that only the seller can make price offers. Future research can look at what happens if counter-offers are allowed. Second, the model only includes one potential buyer with one potential seller. Competition on either side of the market can impact the outcome. The model also ignores many aspects of a real-world sales process, such as prospecting and service delivery. Last but not least, the model assumes that all communication to be honest. The seller truthfully reveals product information, and the buyer truthfully reveals his preference for attributes. This is done to avoid the complexity of modelling two-sided learning with private signals. Future research can consider what happens if discovery is asymmetric, or if players can choose what messages to send.

1.8 Lower Bound on Buyer's Value Function

Costless Seller

Let $\underline{V}(x)$ denote the Buyer's value function facing a fixed price of $\overline{P}$. Let $\overline{\overline{x}}$ denote the Buyer's buying threshold, and let $\underline{\underline{x}}$ denote the Buyer's quitting threshold. By Equation (1.3), Buyer's value function must be of the form $\underline{V}(x) = \frac{c_b}{\sigma^2}(x-\overline{P})^2 + \alpha(x-\overline{P}) + \beta$ for some coefficients α and β. The function has to satisfy the following boundary conditions:

$$\begin{cases} \underline{V}(\overline{\overline{x}}) = \overline{\overline{x}} - \overline{P} & \text{and} \quad \underline{V}(\underline{\underline{x}}) = \pi_b \\ V'(\overline{\overline{x}}) = 1 & \text{and} \quad V'(\underline{\underline{x}}) = 0 \end{cases} \tag{1.13}$$

The first two conditions ensure that the value function must match the stopping value when Buyer buys or quits. The last two conditions, often referred to as "smooth-pasting" conditions, ensure that the stopping time is optimal. See Dixit (1993, pg.34-37) for more details.

Solving this system of equations shows that Buyer buys at

$$\overline{\overline{x}} = \overline{P} + \frac{\sigma^2}{4c_b} + \pi_b \tag{1.14}$$

and quits at

$$\underline{\underline{x}} = \overline{P} - \frac{\sigma^2}{4c_b} - \pi_b \tag{1.15}$$

Buyer's value function is

$$\underline{V}(x|\overline{P},\pi_b) = \begin{cases} \pi_b, & x-\pi_b \le \overline{P} - \frac{\sigma^2}{4c_b} \\ \frac{c_b}{\sigma^2}(x-\pi_b-\overline{P})^2 + \frac{1}{2}(x-\pi_b-\overline{P}) + \frac{\sigma^2}{16c_b} + \pi_b, & x-\pi_b \in \left(\overline{P} - \frac{\sigma^2}{4c_b}, \overline{P} + \frac{\sigma^2}{4c_b}\right) \\ x-\overline{P}, & x-\pi_b \ge \overline{P} + \frac{\sigma^2}{4c_b} \end{cases} \tag{1.16}$$

Costly Seller

Since $\overline{P}$ is the upper bound on price, Buyer cannot be worse off than if price is fixed at $\overline{P}$ forever. However, the quitting threshold is not necessarily Buyer's choice any more, since Seller now has cost and may choose to quit. We can calculate the lower bound on Buyer's value function for a given list price $\overline{P}$ and a given quitting threshold $\underline{x} = \sup\{x|\sum_i q_i(x) > 0\}$.

We claim that for any x, there must exist $x' < x$ such that some player quits at x'. Suppose not, then players' utilities must approach $-\infty$ as $x \to -\infty$, and players should quit, a contradiction. Thus $\underline{x} = \sup\{x|\sum_i q_i(x) > 0\}$ exists.

First consider what happens for $x \le \underline{x}$. We have $V(\underline{x}) = \max\{x-\overline{P}, 0\}$. If no offer is accepted at $x \in Q$, then game ends immediately and Buyer receives 0 (since outside option is normalized to 0). Then by subgame-perfection, Seller offers price equal to $\min\{x, \overline{P}\}$. Then the lower bound on $V(x)$ for $x \leq \underline{x}$ is $\max\{x - \overline{P}, 0\}$, because the state cannot jump to above $\underline{x}$ without crossing $\underline{x}$.

Next we consider what happens for $x > \underline{x}$. Treating $\underline{x}$ as an exogenous stopping point, let $\underline{V}(x, \underline{x}|\overline{P},)$ denote the lower bound. By Ito's Lemma, $\underline{V}(x, \underline{x}|\overline{P}) = -c_b + \frac{\sigma^2}{2}\frac{d}{dx^2}\underline{V}(x, \underline{x}|\overline{P})$, which implies $\underline{V}(x, \underline{x}|\overline{P}) = \frac{c_b}{\sigma^2}(x - \overline{P})^2 + \alpha(x - \overline{P}) + \beta$ for some α and β. Let $\overline{\overline{x}}$ denote the point at which Buyer chooses to buy if price is fixed at $\overline{P}$. Then we have following value-matching and smooth-pasting conditions.:

$$\begin{cases} \frac{c_b}{\sigma^2}(\overline{\overline{x}} - \overline{P})^2 + \alpha(\overline{\overline{x}} - \overline{P}) + \beta & = \overline{\overline{x}} - \overline{P} \\ \frac{2c_b}{\sigma^2}(\overline{\overline{x}} - \overline{P}) + \alpha & = 1 \\ \frac{c_b}{\sigma^2}(\underline{x} - \overline{P})^2 + \alpha(\underline{x} - \overline{P}) + \beta & = 0 \end{cases} \tag{1.17}$$

Solving these 3 conditions and combining with the fact that $\underline{V}(x) = 0 \quad \forall x \leq \underline{x}$, we get:

$$\underline{V}(x, \underline{x}|\overline{P}) = \begin{cases} 0, & x \leq \underline{x} \\ \frac{c_b}{\sigma^2}(x + \eta - \overline{P})^2 + \frac{1}{2}(x + \eta - \overline{P}) + \frac{\sigma^2}{16c_b}, & x \in \left(\underline{x}, \overline{\overline{x}}\right) \\ x - \overline{P}, & x \geq \overline{\overline{x}} \end{cases} \tag{1.18}$$

where $\eta = \frac{1}{4}\frac{\sigma^2}{c_b} - (\underline{x} - \overline{P}) - \sqrt{-\frac{\sigma^2}{c_b}(\underline{x} - \overline{P})}$ and $\overline{\overline{x}} = \overline{P} + 1 + \sqrt{1 - (\underline{x} - \overline{P})^2 - \frac{\sigma^2}{c_b}(\underline{x} - \overline{P})}$.

Note that the Costless Seller lower bound in equation (1.16) is a special case of this by adding the condition $V'(\underline{x}) = 0$. If $\underline{x}$ is Buyer's optimal quitting threshold, then $\eta = 0$ and his value function does not depend on Seller's cost. Buyer's quitting threshold $\underline{x}_b$ is smaller than $\overline{P}$, so $V(x, \underline{x}) = 0$ for $x \leq \underline{x}$. If Seller quits first, then Buyer's value function is shifted down and left from a positive η.

Private Outside Option

Let $\underline{V}_i(x, \mu)$ denote type i's lower bound in state x with Seller's belief μ.

First, if $\mu = \mu_H$ or $\mu = \mu_L$, then Buyer's type is revealed. Only one type of Buyer remains and the problem is identical to the costless seller model in Section 3. Thus the lower bound is the same as in equation (1.16). Thus we must have $\underline{V}_i(x, \mu_i) = \underline{V}(x_i)$ from equation (1.16).

It remains to solve for $\underline{V}_i(x, \mu_{HL})$. The following two Lemmas prove that, if type L receives $\underline{V}(x_L)$ in equilibrium when his type is reveled, then H type must buy (weakly) earlier than L type when the type has not been revealed, and the existence of H type does not change L type's lower bound.

Lemma 3. *In any equilibrium such that $V_L(x, \mu_L) = \underline{V}(x_L)$ from equation (1.16), H type must buy (weakly) earlier than L type. That is, if $a_L(x, \mu_{HL}, P(x, \mu_{HL})) = 1$, then $a_H(x, \mu_{HL}, P(x, \mu_{HL})) = 1$.*

Proof. Suppose $\mu = \mu_{HL}$ so that both types are still in the game. Since L type cannot be worse than if price is fixed at $\overline{P}$, we must have $V_L(x, \mu_{HL}) \geq \underline{V}(x_L)$, where $\underline{V}$ is from Equation (1.16). If L type buys at state x, then we must have $P(x, \mu_{HL}) \leq x_L - \underline{V}(x_L)$. If H types also buy at x, he gets utility of at least $x_H - (x_L - \underline{V}(x_L)) = \underline{V}(x_L) + \epsilon$. If H type does not buy at x, then he will be the only type remaining after L type buys, and will get continuation value $V_H(x, \mu_H) = \underline{V}(x_H) = \underline{V}(x_L + \epsilon)$. It is straightforward to show that $\underline{V}(x_L) + \epsilon \geq \underline{V}(x_L + \epsilon)$, hence H type must buy no later than L type does. □

Lemma 4. *In any equilibrium, if* $V_L(x, \mu_L) = \underline{V}(x_L)$ *from equation (1.16), then there exists an equilibrium outcome in which* $V_L(x, \mu_{HL}) = \underline{V}(x_L)$*. If* $V_L(x, \mu_{HL}) = \underline{V}(x_L)$*, then L type quits at* $\underline{x}_L = \underline{x}_L = \overline{P} - \frac{\sigma^2}{4c_b}$*, and buys at* $\overline{x}_L = \overline{x}_L = \overline{P} + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$ *at price* $P_L = P(x_L = \overline{x}_L) = \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}} - 2\left(\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}\right)\right]$.

Proof. First we see that, if $\mu = L$, the game is the same as in Section 1.3. Thus, L type buys at $x_L \geq \overline{P} + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$ and quits at $x_L \leq \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$. Denote these two thresholds $\overline{x}^o$ and $\underline{x}^o$, respectively.

Now for $\mu = \mu_{HL}$, by Lemma 3, H type must buy weakly earlier than L type in equilibrium. Suppose that it is strictly earlier ($\overline{x}_H < \overline{x}_L + \epsilon$), then we have a separating equilibrium. After H type buys, L type buys immediately if $x_L \geq \overline{x}^o$. Thus H type must buy strictly before x_H reaches $\overline{x}^o + \epsilon$, otherwise there cannot be separation. As a result, on equilibrium path, L type buys at $x_L = \overline{x}^o$ without uncertainty on types. The existence of H type does not affect L type's equilibrium payoff. Thus $V_L(x, \mu_{HL}) = V_L(x, \mu_L) = \underline{V}(x_L)$ as in Lemma 1, and he buys at $\overline{x}_L = \overline{x}^o$ and quits at $\underline{x}_L = \underline{x}^o$ as in Proposition 1.

Suppose instead we have a pooling equilibrium where H and L types buy together ($\overline{x}_H = \overline{x}_L + \epsilon$). If Seller cannot separate the two types, then it is as if she's only dealing with L type. Using arguments in Lemma 1, Seller can charge up to $P(x, \mu) = P(x_L + \frac{\epsilon}{2}, \mu) = x_L - \underline{V}(x_L)$ at the point of trade. In equilibrium, Seller charges this price if she wants to trade and above this price if she does not want to trade. L type behaves as if there's no high type. Then we must have $\overline{x}_L = \overline{x}^o$; otherwise Seller can profitably deviate by charging $P(x, \mu) = P(x_L + \frac{\epsilon}{2}, \mu) = x_L - \underline{V}(x_L)$ for $x_L \geq \overline{x}^o$ and charging $P(x, \mu) > P(x_L + \frac{\epsilon}{2}, \mu) = x_L - \underline{V}(x_L)$ for $x_L < \overline{x}^o$.

If $\overline{P} \geq x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$, then $x_{L|t=0} < \underline{x}_L$; thus L type wants to quit immediately. For $\overline{P} < x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$, we have shown above that L type behaves the same as in Proposition 1. Thus the existence of H type does not affect L type's equilibrium utility or strategy. □

By Lemma 3 and Lemma 4, L type buys and quits at same thresholds as in the case with just a single type. The existence of H type does not affect L type's lower bound value function, nor does it affect L type's equilibrium strategies if he receives this lower bound.

Thus we must have:

$$\begin{aligned}\underline{V}_L(x,\mu_{HL}) = \underline{V}_L(x,\mu_L) &= \underline{V}(x-\frac{\epsilon}{2})\\ &= \underline{V}(x_L)\\ &= \frac{c_b}{\sigma^2}(x_L-\overline{P})^2+\frac{1}{2}(x_L-\overline{P})+\frac{\sigma^2}{16c_b}, \qquad x_L \in \big(\overline{P}-\frac{\sigma^2}{4c_b},\overline{P}+\frac{\sigma^2}{4c_b}\big)\end{aligned} \tag{1.19}$$

The lower bound on H type's value function when $\mu = \mu_{HL}$ must be higher than $\underline{V}(x)$ from equations (1.16). Seller cannot "threaten" to never make discount, because H type rationally expects Seller to make L type an offer when x_L reaches $\overline{x}_L$.

When x drops to $\underline{x}_L + \frac{\epsilon}{2}$, L type quits and H type is revealed with product value x_H. When x increases to $\overline{x}_L + \frac{\epsilon}{2}$, L type buys, and by Lemma 3, H type buys too. When L quits at $x_L = \underline{x}_L$, H's utility becomes $\underline{V}(\underline{x}_L+\epsilon)$. When L buys, H's utility becomes $\underline{V}(\overline{x}_L)+\epsilon$. These two conditions bound H type's value function from below. By Taylor's expansion and Ito's Lemma, we have

$$V_H(x,\mu_{HL}) = V_H(x_H-\frac{\epsilon}{2},\mu_{HL}) = \frac{c_b}{\sigma^2}(x_H-\frac{\epsilon}{2}-\overline{P})^2+\alpha_H(x_H-\frac{\epsilon}{2}-\overline{P})+\beta_H \qquad \forall x_H \in [\underline{x}_L+\epsilon,\overline{x}_L+\epsilon] \tag{1.20}$$

The two conditions translate to:

$$\begin{cases}\frac{c_b}{\sigma^2}(\frac{\epsilon}{2}-\frac{\sigma^2}{4c_b})^2+\alpha_H(\frac{\epsilon}{2}-\frac{\sigma^2}{4c_b})+\beta_H = \frac{c_b}{\sigma^2}(\epsilon-\frac{\sigma^2}{4c_b})^2+\frac{1}{2}(\epsilon-\frac{\sigma^2}{4c_b})+\frac{\sigma^2}{16c_b}\\ \frac{c_b}{\sigma^2}(\overline{x}_L+\frac{\epsilon}{2}-\overline{P})^2+\alpha_H(\overline{x}_L+\frac{\epsilon}{2}-\overline{P})+\beta_H = \frac{c_b}{\sigma^2}(\overline{x}_L-\overline{P})^2+\frac{1}{2}(\overline{x}_L-\overline{P})+\frac{\sigma^2}{16c_b}+\epsilon\end{cases} \tag{1.21}$$

Solving these two equations produces α_H, β_H, and consequently $V_H(x,\mu_{HL})$. The solution is:

$$\begin{aligned}\alpha_H &= \frac{1}{2}+\frac{\frac{\epsilon}{2}-\frac{c_b}{\sigma^2}\epsilon^2-\epsilon\sqrt{\frac{1}{4}-\overline{P}\frac{c_b}{\sigma^2}}}{\frac{\sigma^2}{c_b}\sqrt{\frac{1}{4}-\overline{P}\frac{c_b}{\sigma^2}}}\\ \beta_H &= \frac{3c_b}{4\sigma^2}\epsilon^2+\frac{\sigma^2}{16c_b}-\frac{\frac{\epsilon}{2}-\frac{c_b}{\sigma^2}\epsilon^2-\epsilon\sqrt{\frac{1}{4}-\overline{P}\frac{c_b}{\sigma^2}}}{\frac{\sigma^2}{c_b}\sqrt{\frac{1}{4}-\overline{P}\frac{c_b}{\sigma^2}}}(\frac{\epsilon}{2}-\frac{\sigma^2}{4c_b})\end{aligned} \tag{1.22}$$

1.9 Proofs

Section 3

Proof of Lemma 1

First we assume the existence of an equilibrium with $V(x) = \underline{V}(x)$, and show that if $V(x) = \underline{V}(x)$, then there's an unique outcome and it has the stated properties.

Suppose $V(x) = \underline{V}(x)$. Because $V(x) = \underline{V}(x)$, Buyer's quitting strategies must also be the same as if price is fixed at $\overline{P}$. So Buyer quits for all $x < \underline{x} = \underline{\underline{x}}$. Given the continuation value, Buyer buys at x if and only if $P(x) \leq x - \underline{V}(x)$, otherwise Buyer strictly prefers to wait. So the maximum price that Seller can extract is $x - \underline{V}(x)$. This transforms Seller's problem into an optimal stopping problem. Seller can stop and receive a payoff of $x - \underline{V}(x)$, or continue. If Seller wants to continue, she must charge $P(x) > x - \underline{V}(x)$.

The solution to Seller's stopping problem is a threshold $\overline{x}$, such that Seller stops for all $x > \overline{x}$. Suppose there exists $\tilde{x} > \overline{x}$ such that $a(x, P(x)) = 0$ for an open neighborhood around $\tilde{x}$. Let $x_{left} = sup\{x|x \in A \ \& \ x < \tilde{x}\}$, and $x_{right} = inf\{x|x \in A \ \& \ x > \tilde{x}\}$. Because there's no trade between x_{left} and x_{right}, $U(\tilde{x_n})$ is on a convex function connecting $U(x_{left})$ and $U(x_{right})$. But Buyer is willing to accept $P(x) = x - \underline{V}(x)$ which is a concave function connecting $U(x_{left})$ and $U(x_{right})$, so we must have $U(\tilde{x}) < \tilde{x} - \underline{V}(\tilde{x}, P(\tilde{x}))$. Thus Seller should stop at $\tilde{x}$. If there's no open neighborhood around $\tilde{x}$, then when $x_t = \tilde{x}$, the games stops at $\tau = 0$ and pays utility of 0 by definition of utilities in Section 2. Thus Seller should stop at $\tilde{x}$, a contradiction. The threshold must be lower than $\overline{\overline{x}}$, because for $x > \overline{\overline{x}}$, Seller cannot charge price higher than $x - \underline{V}(x)$, since $\overline{\overline{x}} - \underline{V}(\overline{\overline{x}}) = \overline{P}$, which is an upper bound on price.

Next we prove that the strategy we propose above is indeed an equilibrium.

Let $A = [\overline{x}, \infty]$. If $P(x) = x - \underline{V}(x)$ for $x \in A$, and $P(x) > x - \underline{V}(x)$ for $x \notin A$, then Buyer receives $V(x) = \underline{V}(x)$ for $x \in A$. But this implies that $V(x) = \underline{V}(x)$ for all x, because $V(x)$ and $\underline{V}$ share the same diffusion process with the same flow cost. Buyer's continuation value is then $\underline{V}(x)$ for all x. Given the continuation value, Buyer's optimal strategy is to buy at x if and only if $P(x) \leq x - \underline{V}(x)$. Seller does not have a profitable deviation either, since A is the solution to the optimal stopping problem with stopping payoff of $x - \underline{V}(x)$.

Proof of Proposition 1

Take any $\overline{P}_n \leq x_{n,0} + \frac{1}{4}\frac{\sigma^2}{c_b}$. Given Lemma 1, Buyer has the same utility as if she's facing a fixed price at $\overline{P}$. So we know that $V(x) = \underline{V}(x)$ from equation (1.16) in Section (1.8).

Since Seller controls when to trade, the trading threshold $\overline{x}_n$ must solve the optimal stopping problem with stopping value $x - \underline{V}(x_n, \overline{P}_n)$. By Taylor expansion and Ito's Lemma, Seller's value function satisfies $rU(x_n) = -c_s + \frac{\sigma^2}{2}U''(x_n)$. Having $r = 0$ and $c_s = 0$ imply a linear value function for the Seller $U(x_n) = \alpha_s(x_n - \overline{P}_n) + \beta_s$. We have three boundary conditions. The first two conditions match values at quitting and trading points. The third condition is a first-order "smooth-pasting" condition ensuring the stopping time is optimal

(Dixit 1993).

$$\begin{cases} \alpha_s\Big(-\frac{1}{4}\frac{\sigma^2}{c_b}\Big)+\beta_s & =0 \\ \alpha_s(\overline{x}_n-\overline{P}_n)+\beta_s & =\overline{x}_n-\frac{c_b}{\sigma^2}(\overline{x}_n-\overline{P}_n)^2-\frac{1}{2}(\overline{x}_n-\overline{P}_n)-\frac{\sigma^2}{16c_b} \\ \alpha_s & =1-\frac{2c_b}{\sigma^2}(\overline{x}_n-\overline{P}_n)+1/2 \end{cases} \tag{1.23}$$

Solving the system of equations, we get the trading threshold

$$\overline{x}_n=\overline{P}_n+\frac{\sigma^2}{c_b}\Big[\sqrt{\frac{1}{4}-\overline{P}_n\frac{c_b}{\sigma^2}}-\frac{1}{4}\Big]$$

and trading price

$$P_n(\overline{x}_n)=\overline{x}_n-\underline{V}(\overline{x}_n,\overline{P}_n)=\frac{\sigma^2}{c_b}\Big[\sqrt{\frac{1}{4}-\overline{P}_n\frac{c_b}{\sigma^2}}-2\Big(\frac{1}{4}-\overline{P}_n\frac{c_b}{\sigma^2}\Big)\Big]$$

The optimal list price is found by

$$\underset{\overline{P}_n}{\operatorname{argmax}}\, U(0)=\underset{\overline{P}_n}{\operatorname{argmax}}\, \alpha_b(-\overline{P}_n)+\beta_b=\underset{\overline{P}_n}{\operatorname{argmax}}\,\big(\frac{1}{4}\frac{\sigma^2}{c_b}-\overline{P}_n\big)\big(1-2\sqrt{\frac{1}{4}-\overline{P}_n\frac{c_b}{\sigma^2}}\big)$$

If $\overline{P}>\frac{1}{4}\frac{\sigma^2}{c_b}$, then $\underline{x}_n=\overline{P}-\frac{1}{4}\frac{\sigma^2}{c_b}>0$. At $\underline{x}_n$, Seller would offer $P(\underline{x}_n)=\underline{x}_n$ and Buyer buys. The only difference now is that Seller receives utility of $\underline{x}_n$ instead of 0 at Buyer's quitting threshold. We can re-solve the above system of equations by swapping out the first condition of Equations (1.23) with

$$\alpha_b\Big(-\frac{1}{4}\frac{\sigma^2}{c_b}\Big)+\beta_b=\overline{P}_n-\frac{1}{4}\frac{\sigma^2}{c_b} \tag{1.24}$$

and we get $\overline{x}_n=\underline{x}_n=\overline{P}_n-\frac{1}{4}\frac{\sigma^2}{c_b}$. Thus players trade immediately for $x_n\geq\underline{x}_n$. Furthermore, for $0<x_n\leq\underline{x}_n$, Seller would offer $P_n(x_n)=x_n$ and Buyer accepts, otherwise Buyer would quit. For $x_n\leq 0$, Buyer quits immediately. Thus the game must end immediately for all $x_n\in\mathbb{R}$.

Section 4

Proof of Lemma 2 In A.2 we have shown that $\underline{V}(x+\eta,\overline{P})-\eta$ is the lower bound on $V(x)$.

Suppose $V(\underline{x})=V'(\underline{x})=0$, then the case is identical to the case of a costless seller. So we can use the proof directly from Lemma 1. The lower bound $\underline{V}(x)$ is from equations (1.16). Buyer quits at $\underline{x}=\overline{P}-\frac{1}{4}\frac{\sigma^2}{c_b}$. Seller solves an optimal stopping problem with stopping value $x-\underline{v}(x)$ and an exogenous stopping point at $\underline{x}$, which yields some unique stopping point $\overline{x}$. Player trade for $x\geq\overline{x}$.

Suppose $U(\underline{x}) = U'(\underline{x}) = 0$ and $V'(\underline{x}) \neq 0$, first we see that in this case, we must have $\underline{x} > \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$, otherwise, $\underline{V}(x, \underline{x}) < 0$ for $x < \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$, and Buyer would quit at $\overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$, a contradiction. An equilibrium outcome should be Seller's optimal stopping problem, with stopping payoff of $x - \underline{V}(x, \underline{x})$.

Let $\overline{\overline{x}}$ denote the point where Buyer would buy if price is fixed at $\overline{P}$, and let $\overline{x}$ denote the point where trade happens in equilibrium. From earlier analysis, we know that Buyer's value function is in the form of $V(x) = \frac{c_b}{\sigma^2}(x - \overline{P})^2 + \alpha_b(x - \overline{P}) + \beta_b$ and Seller's value function is in the form of $U(x) = \frac{c_s}{\sigma^2}(x - \overline{P})^2 + \alpha_s(x - \overline{P}) + \beta_s$. Then $\overline{\overline{x}}$, $\overline{x}$, and $\underline{x}$ must satisfy the following seven value-matching and smooth-pasting boundary conditions - the first 3 for the Buyer and the last 4 for the Seller:

$$\begin{cases} \frac{c_b}{\sigma^2}(\overline{\overline{x}} - \overline{P})^2 + \alpha_b(\overline{\overline{x}} - \overline{P}) + \beta_b & = \overline{\overline{x}} - \overline{P} \\ 2\frac{c_b}{\sigma^2}(\overline{\overline{x}} - \overline{P}) + \alpha_b & = 1 \\ \frac{c_b}{\sigma^2}(\underline{x} - \overline{P})^2 + \alpha_b(\underline{x} - \overline{P}) + \beta_b & = 0 \\ \frac{c_s}{\sigma^2}(\underline{x} - \overline{P})^2 + \alpha_s(\underline{x} - \overline{P}) + \beta_s & = 0 \\ 2\frac{c_s}{\sigma^2}(\underline{x} - \overline{P}) + \alpha_s & = 0 \\ \frac{c_s}{\sigma^2}(\overline{x} - \overline{P})^2 + \alpha_s(\overline{x} - \overline{P}) + \beta_s & = \overline{x} - V(x) \\ 2\frac{c_s}{\sigma^2}(\overline{x} - \overline{P}) + \alpha_s & = 1 - \frac{d}{dx}V(x) \end{cases} \tag{1.25}$$

Solving for $\overline{x}$, $\underline{x}$, α_b, β_b, α_s, and β_s then generates the equilibrium outcome.

Lastly, we show that the game cannot simultaneously have an equilibrium outcome with $V'(\underline{x}) = 0$ and an outcome with $U'(\underline{x}) = 0$ and $V'(\underline{x}) \neq 0$. Using the last six conditions from the system of equations above we can derive $\underline{x} = -\frac{c_b}{c_s}\overline{P}$. One can then compare this threshold to the quitting threshold for $V'(\underline{x}) = 0$, which is $\underline{x} = \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$. Only the higher of the two thresholds can be supported in an equilibrium. If the earliest quitting happens at the lower threshold, then in the region between the two thresholds, one of the player must have a negative continuation value. Such player can profitably deviate by quitting at those states.

Proof of Proposition 5

We prove by showing that, if Seller quits earlier than Buyer, then there's a higher $\overline{P}$ such that the equilibrium outcome in Lemma 2 generates a higher ex-ante utility for Seller. Suppose in equilibrium, $\underline{x} = \max\{\underline{x}_b, \underline{x}_s\} = \underline{x}_s > \underline{x}_b$. Given Lemma 1 and Proposition 1, we know that $\overline{P} < \underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b}$; otherwise Buyer's quitting threshold must be higher than $\underline{x}$, which is a contradiction. We will show that increasing $\overline{P}$ to $\underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b}$ is strictly better for the Seller ex-ante.

As an interim step, we note that, if $\underline{x}_s > \underline{x}_b$, then $\underline{x}_s$ decreases as $\overline{P}$ increases. Since $\underline{x}_s > \underline{x}_b$, we can safely assume $\underline{x}_b = -\infty$, since shifting Buyer's quitting threshold downward has no impact on the equilibrium payoff. In the proof of Lemma 2, we showed that $\underline{x}_s = -\frac{c_b}{c_s}\overline{P}$, Thus $\underline{x}_s = \underline{x}$ decreases as $\overline{P}$ increases.

Now we prove that increasing $\overline{P}$ to $\underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b}$ is strictly better for the Seller ex-ante. Fixing an $\underline{x}$ in equilibrium, we can think of Seller as facing an optimal stopping problem regarding when to sell, with a lower boundary at $\underline{x}$. Her utility from stopping is $P(x) = x - V(x) = x - \underline{V}(x+\eta, \overline{P}) + \eta$. Now suppose the list price is increased from $\overline{P}$ to $\overline{P} + \Delta P$, where $\Delta P = \underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b} - \overline{P}$. Seller's quitting threshold is decreased as argued above, and Buyer's quitting threshold increases to $\overline{P} + \Delta P - \frac{1}{4}\frac{\sigma^2}{c_b} = \underline{x}$, so the lower boundary of the game, $\underline{x} = \max\{\underline{x}_b, \underline{x}_s\}$, is unchanged. With the new list price $\overline{P} + \Delta P$, η becomes 0, and Seller's payoff from trading is $x - \underline{V}(x, \overline{P} + \Delta P)$, which is strictly higher than her selling price under $\overline{P}$: $x - \underline{V}(x+\eta, \overline{P}) + \eta$, for all $x > \underline{x}$. Thus Seller is facing the same lower boundary under the two list prices, and a stopping utility under $\overline{P} + \Delta P$ that strictly dominates $\overline{P}$. Thus regardless of where she wants to trade, Seller's ex-ante utility is strictly higher under $\overline{P} + \Delta P$ than under $\overline{P}$. Thus $\overline{P}$ is sub-optimal.

Proof of Proposition 6 From Proposition 5, we know that Buyer quits first in equilibrium. Thus given $\overline{P}$, Buyer's value function must be

$$V(x) = \underline{V}(x) = \frac{c_b}{\sigma^2}(x - \overline{P})^2 + \frac{1}{2}(x - \overline{P}) + \frac{\sigma^2}{16c_b}$$

and Buyer's quitting threshold is $\underline{x} = \overline{P} - \frac{\sigma^2}{4c_b}$.

Because $c_s > 0$, by $U''(x) = \frac{2c_s}{\sigma^2}$, we know that $U(x) = \frac{c_s}{\sigma^2}(x - \overline{P})^2 + A_s(x - \overline{P}) + B_s$ for some coefficients A_s and B_s. At $\underline{x}$, we must have (1) $U(\underline{x}) = 0$. Since $\overline{x}$ is Seller's optimal stopping points, $\overline{x}$ must satisfy: (2) $U(\overline{x}) = P(\overline{x}) = \overline{x} - V(\overline{x})$, and (3) $U'(\overline{x}) = 1 - V'(\overline{x})$. From these three conditions, we can solve $\overline{x}$, A_s, and B_s by solving the following system of equations:

$$\begin{cases} \frac{c_s}{\sigma^2}(-\frac{\sigma^2}{4c_b})^2 + A_s(-\frac{\sigma^2}{4c_b}) + B_s & = 0 \\ \frac{c_s}{\sigma^2}(\overline{x} - \overline{P})^2 + A_s(\overline{x} - \overline{P}) + B_s & = \overline{x} - \frac{c_b}{\sigma^2}(\overline{x} - \overline{P})^2 - \frac{1}{2}(\overline{x} - \overline{P}) - \frac{\sigma^2}{16c_b} \\ 2\frac{c_s}{\sigma^2}(\overline{x} - \overline{P}) + A_s & = 1 - \frac{2c_b}{\sigma^2}(\overline{x} - \overline{P}) - \frac{1}{2} \end{cases} \tag{1.26}$$

From which we get

$$\begin{cases} \overline{x} & = \overline{P} - \frac{\sigma^2}{4c_b} + \frac{\sigma^2}{c_b}\frac{1}{\sqrt{1+k}}\sqrt{\frac{1}{4} - r} \\ A_s & = \frac{2+k}{2} - 2\sqrt{1+k}\sqrt{\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}} \\ B_s & = (\frac{1}{4} + \frac{k}{16})\frac{\sigma^2}{c_b} - \frac{\sqrt{1+k}}{2}\frac{\sigma^2}{c_b}\sqrt{\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}}. \end{cases} \tag{1.27}$$

Plugging A_s and B_s into $U(x_0)$, and letting $r = \overline{P}\frac{c_b}{\sigma^2}$, we get

$$U(x_0) = kr^2(\frac{\sigma^2}{c_b})^2 - (1 + \frac{k}{2} - 2\sqrt{1+k}\sqrt{\frac{1}{4} - r})r(\frac{\sigma^2}{c_b})^2 + (\frac{1}{4} + \frac{3k}{16})\frac{\sigma^2}{c_b} - \frac{\sqrt{1+k}}{2}\frac{\sigma^2}{c_b}\sqrt{\frac{1}{4} - r}$$
$$+ k\frac{\sigma^2}{c_b}{x_0}^2 - 2krx_0 + (1 + \frac{k}{2})x_0 - 2x_0\sqrt{1+k}\sqrt{\frac{1}{4} - r}.$$

Maximize $U(0)$ with respect to r to get the optimal r^* and $\overline{P}^* = r^*\frac{\sigma^2}{c_b}$. For $x_0 = 0$, we get $r^* = \frac{1}{4} - \frac{27+10k-6\sqrt{9+10k+k^2}}{16k^2}$. Plugging $\overline{P}$ into $\overline{x} = \overline{P} - \frac{\sigma^2}{4c_b} + \frac{\sigma^2}{c_b}\frac{1}{\sqrt{1+k}}\sqrt{\frac{1}{4} - r}$ produces $\overline{x}$.

Plugging $\overline{P}$ into $P(\overline{x}) = \overline{x} - V(\overline{x})$ produces $P(\overline{x})$. It's easy to check that $\lim_{x \to \underline{x}^+} U'(x) > 0$ and $U(x) > 0 \; \forall x > \underline{x}$, which confirms that Seller does not want to quit before Buyer does.

Proof of Corollary 7 and 8

To make the comparison, we need to first solve for the equilibrium under a fixed price. Let P^* be the optimal fixed price. We solve separately for the cases of (1) Buyer quits first and (2) Seller quits first.

Suppose Buyer quits first. We know that Buyer's value function is $V(x) = \frac{c_b}{\sigma^2}(x - P^*)^2 + \frac{1}{2}(x - P^*) + \frac{\sigma^2}{16c_b}$, and Seller's value function is $U(x) = \frac{c_s}{\sigma^2}(x - P^*)^2 + A_s(x - P^*) + B_s$ for some coefficients A_s and B_s. Let $\overline{x}$ and $\underline{x}$ denote Buyer's buying and quitting thresholds, respectively. Since both buying and quitting are Buyer's decision, we get $\overline{x} = P^* + \frac{\sigma^2}{4c_b}$ and $\underline{x} = P^* - \frac{\sigma^2}{4c_b}$ from Section 3. Solving $U(\overline{x}) = P^*$ and $U(\underline{x}) = 0$ simultaneously give us $A_s = P^* \frac{2c_b}{\sigma^2}$ and $B_s = -k\frac{\sigma^2}{16c_b} + \frac{1}{2}P^*$. Thus

$$U(x_0) = \frac{kc_b}{\sigma^2}(x_0 - P^*)^2 + P^* \frac{2c_b}{\sigma^2}(x_0 - P^*) + -k\frac{\sigma^2}{16c_b} + \frac{1}{2}P^* \tag{1.28}$$

Taking derivative with respect to P^* and setting to zero, we get $P^* = \frac{1}{2-k}\frac{\sigma^2}{4c_b} + \frac{1-k}{2-k}x_0$. To make sure $\underline{x} < x_0$, we need $k < x_0\frac{4c_b}{\sigma^2} + 1$, otherwise Buyer quits immediately.

Now suppose Seller quits first. Since now quitting is Seller's decision, we do not know coefficients A_b and B_b in Buyer's value function $V(x) = \frac{c_b}{\sigma^2}(x - P^*)^2 + A_b(x - P^*) + B_b$. We still have $U(x) = \frac{c_s}{\sigma^2}(x - P^*)^2 + A_s(x - P^*) + B_s$. To solve A_b, B_b, A_s, B_s, $\overline{x}$, and $\underline{x}$, we solve the following system of equations:

$$\begin{cases} U(\overline{x}) = P^* & \text{and} \quad U(\underline{x}) = 0 \\ V(\overline{x}) = \overline{x} - P^* & \text{and} \quad V(\underline{x}) = 0 \\ U'(\underline{x}) = 0 & \text{and} \quad V'(\overline{x}) = 1 \end{cases} \tag{1.29}$$

This gives $\underline{x} = (1 - \frac{1}{k})P^*$, and $U(x_0) = \frac{kc_b}{\sigma^2}(x_0 - P^*)^2 + \frac{2c_b}{\sigma^2}P^*(x_0 - P^*) + \frac{kc_b}{\sigma^2}\left(\frac{P^*}{k}\right)^2$. Then maximize $U(x_0)$ with respect to P^*. If $k \leq x_0\frac{4c_b}{\sigma^2} + 1$, then $\frac{d}{dP}U > 0$ for all P. This implies Seller will raise the price till Buyer quits first, i.e., $P^* - \frac{\sigma^2}{4c_b} \geq (1 - \frac{1}{k})P^*$. If $k > x_0\frac{4c_b}{\sigma^2} + 1$, we get $P^* = \frac{k}{k-1}x_0 = \underline{x}$. Thus Seller quits immediately. Thus there does not exist an equilibrium with positive length such that Seller quits strictly before Buyer. For $k < x_0\frac{4c_b}{\sigma^2} + 1$, the length fo the game is positive and Buyer quits first. For $k >= x_0\frac{4c_b}{\sigma^2} + 1$, one player stops immediately regardless of the price.

Now we can compare the equilibrium outcome under bargaining to the outcome under a fixed price. Particularly, for $x_0 = 0$, we can compare $\overline{P} = \frac{\sigma^2}{c_b}\left[\frac{1}{4} - \frac{27+10k-6\sqrt{9+10k+k^2}}{16k^2}\right]$ with $P^* = \frac{1}{2-k}\frac{\sigma^*}{4c_b}$. There exists a $\underline{k} < 1$ such that $\overline{P} > P^*$ for $k < \underline{k}$ and $\overline{P} < P^*$ for $k > \underline{k}$. I did not find a closed-form solution for $\underline{k}$.

Section 5

Proof of Proposition 9

If $\mu = \mu_L$ or $\mu = \mu_H$, then we have proved the result in Lemma 1 and Proposition 1.

From Lemma 3 and Lemma 4, we also know the equilibrium outcome if Buyer is L type and belief is μ_{HL}. The outcome is exactly the same is in Proposition 1. L type quits at $\underline{x}_L$ and buys at $\overline{x}_H$.

So the only case we need to prove is when Buyer is H type and Seller's belief is $\mu = \mu_{HL}$. As before, Seller does not need to offer H type more utility than the lower bound of H type's continuation value. When L quits at $x_L = \underline{x}_L$, H's utility becomes $\underline{V}(\underline{x}_L + \epsilon)$. When L buys, H's utility becomes $\underline{V}(\overline{x}_L) + \epsilon$. These two conditions bound H type's value function from below. Given H type's lower bound $\underline{V}_H(x, \mu_{HL})$, It's easy to check that $\underline{V}_H(x, \mu_{HL}) > \underline{V}_L(x+\epsilon, \mu_{HL})$ for $\underline{x}_L + \frac{\epsilon}{2} < x < \overline{x}_L + \frac{\epsilon}{2}$. Thus H type has a higher utility than L type in every state, even after compensating L type for the difference in outside options, ϵ. It's also easy to check that $\underline{V}_H(x, \mu_{HL}) - \epsilon < \underline{V}_L(x + \epsilon, \mu_{HL})$ for $\underline{x}_L + \frac{\epsilon}{2} < x < \overline{x}_L + \frac{\epsilon}{2}$; thus L type would not buy when Seller makes an offer to the H type, $P(x, \mu_{HL}) = x_H - \underline{V}_H(x, \mu_{HL})$. Now, given $V_H(x, \mu_{HL}) = \underline{V}_H(x, \mu_{HL})$, Seller receives $P(x, \mu_{HL}) = x_H - V_H(x, \mu_{HL})$ if she trades with H type. Seller's decision of when to trade with H type must solve the following optimal stopping problem, with two value matching boundary conditions and one smooth-pasting boundary condition:

$$\begin{cases} \alpha_s(\overline{x}_H - \frac{\epsilon}{2}) + \beta_s = -(\overline{x}_H - \frac{\epsilon}{2} - \overline{P})^2 + (1 - \alpha_H)(\overline{x}_H - \frac{\epsilon}{2} - \overline{P}) + \overline{P} - \beta_H \\ \alpha_s = -2(\overline{x}_H - \frac{\epsilon}{2} - \overline{P}) + (1 - \alpha_H) \\ \alpha_s(\frac{\epsilon}{2} - \frac{\sigma^2}{4c_b}) + \beta_s = (1 - 2\sqrt{\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}})\epsilon \end{cases} \tag{1.30}$$

where α_H and β_H are solutions to equations 1.21. Using the system of 6 equations, we can derive the follwoing condition

$$(\overline{x}_H - \overline{P} - \frac{\sigma^2}{4c_b})^2 + \overline{P} - \frac{\sigma^2}{4c_b} = 0$$

which implies that

$$\overline{x}_H = \overline{x}_L = \overline{P} + \frac{\sigma^2}{c_b}\Big[\sqrt{\frac{1}{4} - \overline{P}\frac{c_b}{\sigma^2}} - \frac{1}{4}\Big]$$

Note that even though they buy at the same threshold, H type arrives at the threshold earlier than L type does, since $x_H = x_L + \epsilon$ at all times. Also, this common threshold is the same trading threshold as in Proposition 1, when there is only a single type.

When L type quits, H type's value function becomes $\underline{V}(\underline{x}_L + \epsilon) > 0$, thus H type does not quit as long as L is present. After L quits, H has the same quitting threshold as in Proposition 1. So

$$\underline{x}_i = \underline{x} = \overline{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$$

At their respective times of trade, L type pays price

$$P_L = P(\overline{x} + \frac{\epsilon}{2}, \mu_L) = \overline{x} - V_L(\overline{x} + \frac{\epsilon}{2})$$

and H type pays price

$$P_H = P(\overline{x} - \frac{\epsilon}{2}, \mu_{HL}) = \overline{x} - V_H(\overline{x} - \frac{\epsilon}{2})$$

Thus $P_L - P_H = V_H(\overline{x} - \frac{\epsilon}{2}) - V_L(\overline{x} + \frac{\epsilon}{2})$. Since we showed above that $V_H(x) > V_L(x + \epsilon)$, this proves that $P_L > P_H$.

1.10 Discrete-time Analog

In this section, I present the discrete-time analog of the continuous-time model. I show that the equilibrium outcomes presented in paper represent the limit of discrete-time equilibrium outcomes. Specifically, because the equilibrium outcomes presented in the paper is identified through Buyer's value function, we want to prove that, as the period length in the discrete-time game approaches 0, Buyer's equilibrium value function must approach $\underline{V}(x)$ from equation (1.16) or $\underline{V}(x, \underline{x})$ from equation (1.18) and satisfies Condition 1.

I first formally state the discrete-time game, then introduce a refinement that eliminates trivial equilibria in which players quit simultaneously, then prove the uniqueness of the limit as the game approaches continuous time.

The discrete-time game follows the model described in Section 2.1. Let $\mathbb{G}(\Delta t)$ denote the game with period length of Δt. Players act at time $t \in \{0, \Delta t, 2\Delta t, \cdots\}$. There are two players, a Seller (s) and a Buyer (b). Product value is denoted as x, with $x = x_0$ at time $t = 0$. The state variable x_t evolves as a Markov chain, with $x_{t+\Delta t} = x_t + \sigma\sqrt{\Delta t}$ with probability $\frac{1}{2}$ and $x_{t+\Delta t} = x_t - \sigma\sqrt{\Delta t}$ with probability $\frac{1}{2}$. Let $\mathbf{X}(\Delta t) = \{x_0 + \alpha\sigma\sqrt{\Delta t} \mid \alpha \in \mathbb{N}\}$ denote the grid on x spanned by $\sigma\sqrt{\Delta t}$ from x_0. We must have $x_t \in \mathbf{X}(\Delta t)\ \forall t$. State x_t is observable to both players.

Before the game starts,a list price $\overline{P} \in \mathbb{R}^+ +$ is chosen. Then at $t \geq 0$:

1. Seller chooses price P_t subject to $P_t \leq \overline{P}$.

2. Buyer chooses whether to buy with $a_t \in \{0, 1\}$.

3. If $a_t = 0$, then both players choose whether to quit with $q_{i,t} \in \{0, 1\}$. If no player quits, then Buyer incurs cost and the game moves to time $t + \Delta t$. If either player quits, the game ends and players receive their outside options.

I look for stationary SPE with pure strategy, henceforth referred as equilibrium. Given a list price, Seller's equilibrium strategy is characterized by $(P(x), q_s(x))$, with price offer $P(x) : \mathbb{R} \mapsto [0, \overline{P}]$ and quitting decision $q_s(x) : \mathbb{R} \mapsto \{0, 1\}$. Buyer's equilibrium strategy is characterized by $(a(x, P), q_b(x))$, with buying decision $a(x, P) : \mathbb{R}^2 \mapsto \{0, 1\}$ and quitting decision $q_b(x) : \mathbb{R} \mapsto \{0, 1\}$. Since everything depends on list price $\overline{P}$, which is fixed throughout the game, I do not include $\overline{P}$ from notations.

Buyer incurs continuation cost of $c_b * \Delta t$ and Seller incurs continuation cost $c_s * \Delta t$ at the end of each period if $a(x_t) = 0$ and $q_s(x_t) = q_b(x_t) = 0$. If $a(x_t) = 1$, Buyer receives utility

$x_t - P_t$ and Seller receives P_t. If either player quits, the game ends and players get outside options of π_b and π_s, which is normalized to 0 (see Definition 1 in Section 3).

Given a strategy profile ϕ, let $u(x, \phi)$ denote the seller's expected utility in state x, and $v(x, \phi)$ denote buyer's expected utility in state x. Let $V_1(x)$ denote Buyer's value function in equilibrium, and $U_1(x)$ denote Seller's equilibrium value function. Let $V_2(x)$ denote Buyer's expected value from continuing to the next period. That is, $V_2(x)$ represents Buyer's expected utility of rejecting the trade and if both players choose not to quit this period.

We can write $V_2(x)$ recursively as:

$$V_2(x) = -c_b \Delta t + \frac{1}{2} V_1(x + \sigma\sqrt{\Delta t}) + \frac{1}{2} V_1(x - \sigma\sqrt{\Delta t}) \tag{1.31}$$

When players trade, Seller must offer the highest price that Buyer is willing to accept. If players quit at x, then Seller can charge $P(x) = \max\{x, \overline{P}\}$. If players don't quit at x, and if $V_2(x) \geq 0$, Seller can charge up to $x - V_2(x)$. If players don't quit at x and $V_2(x) < 0$, Seller can charge up to x. Also Seller cannot charge above the list price. This implies:

$$V_1(x) = \begin{cases} \max\{0, x - \overline{P}\} & \forall x \text{ s.t. } q(x) = 1 \\ \max\{0, V_2(x)\} & \forall x \leq \overline{P} \text{ s.t. } q(x) = 0 \\ \max\{V_2(x), x - \overline{P}\} & \forall x > \overline{P} \text{ s.t. } q(x) = 0 \end{cases} \tag{1.32}$$

Before we proceed, note that there is an infinite number of trivial equilibria in which players quit simultaneously. If the opponent is quitting in a given state, then the player is indifferent between quitting and not quitting, thus quitting is always optimal. By the same reasoning, the opponent's quitting decision is optimal too. As a result, we can construct equilibria with arbitrary quitting rules, as long as both players quit at the same time. These equilibria exist due to the simultaneity of players' quitting decisions, and do not offer any insight. One extreme example is that $q_b(x) = q_s(x) = 1 \quad \forall x$, $P(x) = max\{0,\ min\{x, \overline{P}\}\ \}$, and $a(x, \overline{P}, P) = \mathbb{I}\{x \geq 0\}$ constitutes an equilibrium. In this equilibrium, players always quit no matter what the state x is; Seller charges a price equal to the value of the product (bounded between 0 and $\overline{P}$); Buyer buys immediately if product value is positive because he's indifferent between buying and quitting. In this equilibrium, the game ends immediately, no matter what the initial position is.

To remove these simultaneous quitting equilibria, I introduce the following refinement, which is equivalent to applying trembling hand only to players' quitting strategies.

Refinement. *Player quits if and only if quitting is strictly preferred.*

Under this refinement, players cannot quit simultaneously. If the opponent quits, then the player is indifferent and will choose to stay. This refinement is not restrictive. Suppose in an equilibrium, Seller quits for $x \leq \underline{x_s}$ and Buyer quits for $x \leq \underline{x_b}$. Now keep $\overline{P}$, $P(\cdot)$, and $a(\cdot)$ as before, and see whether $\underline{x_s} - \epsilon$ and $\underline{x_b}$ constitute a equilibrium, and whether $\underline{x_b} - \epsilon$ and $\underline{x_s}$ constitute a equilibrium. If neither is an equilibrium, that means that players quit only because the other player is quitting. If either perturbation on $\underline{x_s}$ or on $\underline{x_b}$ constitutes an equilibrium, then the new equilibrium has the same equilibrium outcome as before, and the payoffs are not affected by the restriction that $\underline{x_s} \neq \underline{x_b}$.

Now we are ready to prove the following result. Let $V_{\Delta t}$ denote an index family of Buyer's equilibrium value functions for the index Δt. Let $U_{\Delta t}$ denote an index family of Seller's equilibrium value functions for the index Δt. Let $q(x) = \max\{q_b(x), q_s(x)\}$ and $\underline{x} = \sup\{x|q(s) = 1\}$. Let $U(x) = \lim_{\Delta t \to 0} U_{\Delta t}$ and $V(x) = \lim_{\Delta t \to 0} V_{\Delta t}$ denote the limit of value functions as the game approaches continuous time.

Theorem 5. *As $\Delta t \to 0$, $V_{\Delta t} \to \underline{V}(x, \underline{x})$ from equation (1.18). Condition 1 is satisfied in the limit.*

Step 1. *There exists an $\overline{\overline{x}}$ such that $a(x) = 1$ and $P(x) = \overline{P}$ iff $x \geq \overline{\overline{x}}$.*

Here we claim that there exists a threshold such that, when x is above the threshold, Seller charges $\overline{P}$ and Buyer buys immediately.

First we see that, if $P(x) = \overline{P}$ and $a(x) = 1$, then $x \geq \overline{P}$. If $x \geq \overline{P}$, then $V_1(x) = \max\{V_2(x), x - \overline{P}\}$ if $q(x) = 0$ and $V_1(x) = x - \overline{P}$ if $q(x) = 1$. Seller must charge $P(x) = \overline{P}$ if $V_2(x) < x - \overline{P}$. Let $\widetilde{P}(x)$ denote the highest price that Buyer is willing to pay at $x > \overline{P}$. Buyer buys at $\overline{P}$ if and only if $\widetilde{P} = \overline{P}$.

Now we prove that $\widetilde{P}$ must be non-decreasing for $x > \overline{P}$. Suppose $\widetilde{P}$ decreases for some $x < \overline{P}$. Then there exists $x' > \overline{P}$ such that $\widetilde{P}(x' - \sigma\sqrt{\Delta t}) > \widetilde{P}(x')$. This implies that $\widetilde{P}(x') < \overline{P}$, so players do not quit at x. Then $\widetilde{P}(x') = x' - V_2(x') = \frac{1}{2}(wide\widetilde{P}(x' - \sigma\sqrt{\Delta t}) + \widetilde{P}(x' + \sigma\sqrt{\Delta t}) + c_b\Delta t$. By rearranging the terms, we get: $\widetilde{P}(x' + \sigma\sqrt{\Delta t}) - \widetilde{P}(x') = \widetilde{P}(x') - \widetilde{P}(x' - \sigma\sqrt{\Delta t}) - 2c_b\Delta t < 0$. Thus $\widetilde{P}(x' + \sigma\sqrt{\Delta t}) < \tilde{P}(x') - c_b\Delta t$. By induction, $\widetilde{P} \to -\infty$ as $x \to \infty$. When $\widetilde{P} < 0$, there cannot be trade since Seller can't charge a negative price. Players must not quit if $\widetilde{P} < \overline{P}$. Thus neither trading or quitting can occur after x passes some threshold. However, this implies that players' value functions must approach $-\infty$, so players should quit, which is a contradiction.

Given that $\widetilde{P}$ must be non-decreasing, we need to show that $\widetilde{P}$ must reach $\overline{P}$ at some point. Suppose $\widetilde{P}(x)$ never reaches $\overline{P}$. Previously we showed that if $x > \overline{P}$ and $\widetilde{P}(x) < \overline{P}$, then $\widetilde{P}(x+\sigma\sqrt{\Delta t}) - \widetilde{P}(x) = \widetilde{P}(x) - \widetilde{P}(x-\sigma\sqrt{\Delta t}) - 2c_b\Delta t < 0$. Because $\widetilde{P}$ is non-decreasing, $\widetilde{P}(x) - \widetilde{P}(x - \sigma\sqrt{\Delta t}) - 2c_b\Delta t > 0$, thus $\widetilde{P}(x) - \widetilde{P}(x - \sigma\sqrt{\Delta t}) > 2c_b\Delta t$ for all $x > \overline{P}$. Then $\widetilde{P}$ must reach $\overline{P}$ for some large x. This concludes the proof. We can find $\overline{\overline{x}}$ by taking the $\min\{x \mid \widetilde{P} = \overline{P}\}$.

Step 2. *For Δt small enough, there exists an $\underline{x} < \overline{P}$ such that $q(x) = 1$ iff $x \leq \underline{x}$.*

There exists a threshold $\underline{x}$ such that, players quit if and only if the current state is below the threshold. First, we can see that Buyer must quit for some states that are low enough. When x is negative, there cannot be trade, and players pay costs to continue the game. As x approaches $-\infty$, continuation value also goes $-\infty$ and so players must quit.

Next, for $x \le \overline{P}$, we can show that if $q(x) = 1$, then $q(x - \sigma\sqrt{\Delta t}) = 1$. If $q(x) = 1$, then we must have $V_1(x) = 0$. Suppose that $q_b(x - \sigma\sqrt{\Delta t}) = 0$; this implies that

$$V_2(x - \sigma\sqrt{\Delta t}) = -c_b\Delta t + \frac{1}{2}V_1(x) + \frac{1}{2}V_1(x - 2\sigma\sqrt{\Delta t}) \ge 0 \tag{1.33}$$

which means that $V_1(x - 2\sigma\sqrt{\Delta t}) > V_2(x - \sigma\sqrt{\Delta t} \ge 0$. By induction, we have $V_1(x') > 0$ for all $x' < x - 2\sigma\sqrt{\Delta t}$. Thus players never quits for $x' < x$, which is a contradiction.

Lastly, I show that, for Δt small enough, we must have $q(x) = 0$ if $x > \overline{P}$. Suppose not, then we have some $x' > \overline{P}$ such that $q(x') = 1$. Then we must have $P(x') = \overline{P}$ and $a(x') = 1$, which implies that $x' \ge \overline{\overline{x}}$. Then the continuation value for Buyer $V_2(x') = -c_b\Delta t + \frac{1}{2}V_1(x' - \sigma\sqrt{\Delta t}) + \frac{1}{2}V_1(x' + \sigma\sqrt{\Delta t}) \ge -c_b\Delta t + \frac{1}{2}(x' + \sigma\sqrt{\Delta t} - \overline{P}) > -c_b\Delta t + \frac{1}{2}\sigma\sqrt{\Delta t}$, which is strictly positive for Δt small enough, so Buyer does not quit. Seller's continuation value $U_2(x') = -c_s\Delta t + \frac{1}{2}U_1(x' - \sigma\sqrt{\Delta t}) + \frac{1}{2}U_1(x' + \sigma\sqrt{\Delta t}) \ge -c_s\Delta t + \frac{1}{2}\overline{P}$, which is strictly positive for Δt small enough, so Seller does not quit. Thus $q(x') = 0$, a contradiction. This concludes the proof.

Step 3. *Let $\{V_{\Delta t}(x) : \mathbf{X}(\Delta t) \mapsto \mathbb{R}\}_{\Delta t}$ be an index family of Buyer's value functions. Then $V_{\Delta t}(x) \to \underline{V}(x)$ as $\Delta t \to 0$.*

Given the existence of $\overline{\overline{x}}$ and $\underline{\underline{x}}$, we can now specify $V_1(x)$ in three cases:

$$V_1(x) = \begin{cases} 0, & \forall x \le \underline{\underline{x}} \\ -c_b\Delta t + \frac{1}{2}V_1(x + \sigma\sqrt{\Delta t}) + \frac{1}{2}V_1(x - \sigma\sqrt{\Delta t}), & \forall \underline{\underline{x}} < x < \overline{\overline{x}} \\ x - \overline{P}, & \forall x \ge \overline{\overline{x}} \end{cases} \tag{1.34}$$

Rearranging the terms for $\underline{x} < x < \overline{\overline{x}}$, we get

$$\frac{2c}{\sigma^2} = \frac{\frac{V_1(x+\sigma\sqrt{\Delta t}) - V_1(x)}{\sigma\sqrt{\Delta t}} + \frac{V_1(x) - V_1(x-\sigma\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}}}{\sigma\sqrt{\Delta t}} \qquad \text{for } \underline{x} < x < \overline{\overline{x}}$$

Also, at $\overline{\overline{x}}$, we must have:

$$V_1(\overline{\overline{x}}) \ge -c_b\Delta t + \frac{1}{2}V_1(\overline{\overline{x} + \sigma\sqrt{\Delta t}}) + \frac{1}{2}V_1(\overline{\overline{x} - \sigma\sqrt{\Delta t}})$$

which rearranges to:

$$\frac{V_1(\overline{\overline{x}}) - V_1(\overline{\overline{x}} - \sigma\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}} \ge 1 - \frac{2c}{\sigma}\sqrt{\Delta t} \tag{1.35}$$

Let $V(x)$ denote the limit of $V_1(x) \to V(x)$ as $\Delta t \to 0$. In the limit, the above 5 conditions converge to:

$$\begin{cases} V(x) = 0 \quad \forall x \le \underline{x} \\ V''(x) = \frac{2c}{\sigma^2} \quad \text{for } \underline{x} < x < \overline{\overline{x}} \\ V(x) = x - \overline{P} \quad \forall x \ge \overline{\overline{x}} \\ V'_{-}(\overline{\overline{x}}) \ge 1 \end{cases}$$

Fixing an $\underline{x}$, one can solve for $V(x)$ using the above 4 conditions. The solution is:

$$\underline{V}(x|\overline{P}) = \begin{cases} 0, & x \le \underline{x} \\ \frac{c_b}{\sigma^2}(x+\eta-\overline{P})^2 + \frac{1}{2}(x+\eta-\overline{P}) + \frac{\sigma^2}{16c_b}, & x \in \left(\underline{x}, \overline{\overline{x}}\right) \\ x-\overline{P}, & x \ge \overline{\overline{x}} \end{cases} \tag{1.36}$$

where $\eta = \frac{1}{4}\frac{\sigma^2}{c_b} - (\underline{x}-\overline{P}) - \sqrt{-\frac{\sigma^2}{c_b}(\underline{x}-\overline{P})}$ and $\overline{\overline{x}} = \overline{P} + 1 + \sqrt{1-(\underline{x}-\overline{P})^2 - \frac{\sigma^2}{c_b}(\underline{x}-\overline{P})}$. This is $\underline{V}(x,\underline{x})$ from equation (1.18) for $\underline{x} < \overline{P}$.

Step 4. *Either $U(\underline{x}) = U'(\underline{x}) = 0$, or $V(\underline{x}) = V'(\underline{x}) = 0$.*

Our refinement implies that only one player can quit at a given state. So either $q_b(\underline{x}) = 1$ or $q_s(\underline{x}) = 1$.

Suppose $q_b(\underline{x}) = 1$. That means the continuation value must be less than the outside option for Buyer:

$$0 \le -c_b\Delta t + \frac{1}{2}V_1(\underline{x} - \sigma\sqrt{\Delta t}) + \frac{1}{2}V_1(\underline{x} + \sigma\sqrt{\Delta t})$$

Thus $\frac{V_1(\underline{x}+\sigma\sqrt{\Delta t}) - V_1(\underline{x}-\sigma\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}} \le c_b\sqrt{\Delta t}$.

Similarly, suppose $q_s(\underline{x}) = 1$. That means the continuation value must be less than the outside option for Seller:

$$0 \le -c_s\Delta t + \frac{1}{2}U_1(\underline{x} - \sigma\sqrt{\Delta t}) + \frac{1}{2}U_1(\underline{x} + \sigma\sqrt{\Delta t})$$

Thus $\frac{U_1(\underline{x}+\sigma\sqrt{\Delta t}) - U_1(\underline{x}-\sigma\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}} \le c_s\sqrt{\Delta t}$.

So for any Δt, we must have:

$$\min\left\{\frac{V_1(\underline{x}+\sigma\sqrt{\Delta t}) - V_1(\underline{x}-\sigma\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}} - c_b\sqrt{\Delta t}, \frac{U_1(\underline{x}+\sigma\sqrt{\Delta t}) - U_1(\underline{x}-\sigma\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}} - c_s\sqrt{\Delta t}\right\} \le 0$$

Thus in the limit as $\Delta t \to 0$, we have $\min\{U'(\underline{x}), V'(\underline{x})\} \le 0$.

Suppose $V'(\underline{x}) \le 0$. Since we know $V(\underline{x}) = 0$, and equilibrium value function cannot be negative, then $V'(\underline{x}) = 0$. Suppose $U'(\underline{x}) \le 0$. If $\underline{x} > 0$, then $U(x) = x$ for $x \le \underline{x}$ because Buyer is wiling to accept price up to x. This implies $U'_-(\underline{x}) = 1$, a contradiction. Thus $\underline{x} \le 0$, so we must have $U(\underline{x}) = 0$.

This concludes the proof. In the limit, either $U(\underline{x}) = U'(\underline{x}) = 0$ or $V(\underline{x}) = V'(\underline{x}) = 0$ must hold.

1.11 Alternative Equilibrium Concept

Standard equilibrium notions such as subgame-perfection may not work well in continuous-time games. A strategy profile can produce no well-defined outcome in continuous time, or produce equilibrium outcomes that do not correspond to anything in the discrete-time game, because SPE does not impose the same restrictions on strategies in continuous time as in discrete time. See Simon and Stinchcombe (1989) for more discussions. In the main text of the paper, I constrain players to strategies such that the resulting stopping time τ_θ is a measurable function. One undesirable feature from doing so is that each player's available deviation strategies now depend on the opponent's strategy. In this section, I describe an alternative equilibrium concept of the game without such joint restriction on the strategy space. I first describe each component of the equilibrium concept, before formally defining it at the end.

Stationarity I focus on stationary behavior with pure strategies. Players' equilibrium strategies depend on state x but not on time t. Seller's actions in equilibrium can be characterized by the list price $\overline{P} \in \mathbb{R}^+ \cup +\infty$, her price offering in each state $P(x, \overline{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto (-\infty, \overline{P})$, and her quitting decision $q_s(x, \overline{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \{0, 1\}$, where $q_s = 1$ means quitting. Buyer's actions can be characterized by his buying decision $a(x, P, \overline{P}) : \mathbb{R}^2 \times \mathbb{R}^+ \mapsto \{0, 1\}$ where $a = 1$ means accepting the offer, and his quitting decision $q_b(x, \overline{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \{0, 1\}$. Since everything depends on the list price $\overline{P}$, which is fixed throughout the game, I will drop $\overline{P}$ from notations going forward. We will treat the list price as an exogenous parameter, and solve the equilibrium for any arbitrary list price, then let Seller chooses the list price that maximizes her ex-ante utility from bargaining under such list price.

Buyer's Problem Given Seller's strategy, Buyer has to decide between three actions for each product value x. Buyer can accept Seller's offer $P(x)$, which gives utility $x - P(x)$; he can reject the offer and quit the game, which gives utility π_b; or he can reject the offer and continue to wait. This is an optimal stopping problem, with stopping value $W_b = \max\{x - P(x), \pi_b\}$, subjects to states in which Seller leaves. Buyer chooses the strategy that maximizes his expected payoff, $\sup_\tau \mathbb{E}[-c * \tau + W_b(x_\tau)]$, where τ is the stopping time. Buyer's equilibrium strategy $a(x, p)$ and $q_b(x)$ has to be a solution to this problem.

Highest Acceptable Offer If Seller makes an offer that Buyer accepts, Seller should make the highest offer that Buyer is willing to accept. Otherwise, Seller can profitably deviate by charging a slightly higher price. Thus in equilibrium, if $a(x, P(x)) = 1$, then we should have $P(x) = \sup\{P | a(x, P) = 1\}$.

Seller's Problem Given this, Seller also decides between three actions at each moment. Seller can make a sale, which gives utility $P(x) = \sup\{P | a(x, P) = 1\}$; she can quit the game, which gives utility of π_s; or she can continue to wait (by making an unacceptable and do not quit). This is an optimal stopping problem, with stopping value $W_s = \max\{\sup\{P | a(x, P) = 1, \pi_s\}$, subject to states in which Buyer leaves. Buyer chooses the strategy that maximizes his expected payoff, $\sup_\tau \mathbb{E}[-c * \tau + W_b(x_\tau)]$, where τ is the stopping time. Seller's equilibrium strategy $\overline{P}$, $P(x)$, and $q_b(x)$ has to be a solution to this problem.

Outcome Given a strategy profile described above, the games ends at time $\tau = \inf\{t \mid a(x_t, P(x_t) =$

1 or $q_s(x) + q_b(x) > 0\}$, which is the earliest time by which an offer is accepted or one of the players quits. Given a starting value x_0, the equilibrium outcome can be described by a quadruple $\{A, Q, U(x), V(x)\}$, where $A = \{x|a(x, P(x)) = 1\}$ is the set of states such that players reach agreement, $Q = \{x|q_s(x) + q_b(x) > 1\}$ is the set of states such that some player quits, $U(x)$ is Seller's equilibrium value function, and $V(x)$ is Buyer's equilibrium value function.

For states in which players trade, then Seller receives $U(x) = P(x)$ and Buyer receives $V(x) = x - P(x)$. For states in which no agreement is reached and a player quits, $U(x) = \pi_s$ and $V(x) = \pi_b$. For state x such that players choose to continue negotiating, we can write recursively:

$$\begin{aligned} U(x,t) &= -c_s dt + e^{-r_s dt}\mathbb{E}U(x + dx, t + dt) \\ V(x,t) &= -c_b dt + e^{-r_b dt}\mathbb{E}V(x + dx, t + dt) \end{aligned} \tag{1.37}$$

Under stationarity, and by taking Taylor expansion and applying Ito's Lemma on $\mathbb{E}U$ and $\mathbb{E}V$ terms, these expressions can be reduced to the following equations:

$$\begin{aligned} r_s U(x) &= -c_s + \frac{\sigma^2}{2}U''(x) \\ r_b V(x) &= -c_b + \frac{\sigma^2}{2}V''(x) \end{aligned} \tag{1.38}$$

Given $r_s = r_b = 0$, the solutions to the equations must be of the form:

$$\begin{aligned} U(x) &= \frac{c_s}{\sigma^2}(x - \overline{P})^2 + A_s(x - \overline{P}) + B_s \\ V(x) &= \frac{c_b}{\sigma^2}(x - \overline{P})^2 + A_b(x - \overline{P}) + B_b \end{aligned} \tag{1.39}$$

for some coefficients A_s, B_s, A_b, B_b. These coefficients can be identified later by applying appropriate boundary conditions.

Definition 2. *A strategy profile is an equilibrium if it can be describe by* $\{P(x),\ q_s(x),\ a(x,P),\ q_b(x)\}$ *such that:*

1. *(Buyer's Problem) Given* $\{P(x),\ q_s(x)\}$, $\{a(x,P),\ q_b(x)\}$ *solves* $\sup_\tau \mathbb{E}[-c*\tau + W_b(x_\tau)]$.
2. *(Highest Acceptable Price)* $P(x) = \sup\{P|a(x,P) = 1\}$ *if* $a(x, P(x)) = 1$.
3. *(Seller's Problem) Given* $\{a(x,P),\ q_b(x)\}$, $\{P(x),\ q_s(x)\}$ *solves* $\sup_\tau \mathbb{E}[-c*\tau + W_s(x_\tau)]$.
4. *(Trivial Multiplicity) A and Q are closed.* [17]

[17] The game can have trivial equilibrium outcomes that differ only on sets of measure 0 with no relevance to payoffs. I restrict A and Q to be closed sets to rule out this type of multiplicity. See Ortner 2017 for a similar restriction.

Chapter 2

Bargaining Between Collaborators of a Stochastic Project

2.1 Introduction

Firms increasingly rely on collaboration for development and marketing of products and services (Sivadas and Dwyer 2000). Each year, more than 2,000 alliances are formed, and this number has continued to rise rapidly (Steinhilber 2013). Such alliances can be formed for a variety of purposes, ranging from joint product development, to co-marketing campaigns, to global distribution partnerships. Such collaborations allow firms to combine complementary resources (dAspremont and Jacquemin 1988), access technology and markets for new business opportunities (Hamel 1991), and achieve economies of scale (Gomes-Casseres 1997). While past papers (e.g., Bucklin and Sengupta 1993) have studied firm behaviorand determinants of success *after* firms enter into alliances, this paper is interested in firms' strategies *before* they form an alliance. The paper asks the following questions: when do firms form alliances and how do they split the profits if (1) the profitability of the project evolves stochastically and (2) firms have to reach an agreement through bargaining.

The expected surplus generated from a collaboration project can evolve over time due to changing market conditions, arrival of information, or matching between prospective collaborators. For example, in 2017, Amazon explored a partnership with DISH Networks to create their own wireless networks. A proprietary wireless network became attractive to Amazon at that time because more Americans had shifted to mobile Internet and smart appliances (Fung 2017). The shift in consumer preferences opened up business opportunities that would not have been profitable in earlier years. The appeal of a project can also evolve with the arrival of new information. For example, companies often conduct multiple rounds of market research or trials before committing to a major decision. The data collected from research updates a firm's belief about the attractiveness of a market. In 2016, after the successes of many smaller scale collaborations, Red Bull and GoPro entered into a global content partnership. When asked about the partnership, the CEO of GoPro described the two brands as "extremely compatible and collaborative", and stated that "the feedback [GoPro gets] from [Red Bull] is phenomenal." (Beer 2016). The lapse of time also reveals whether a project is successful. For example, after failing to reach an agreement in their previous negotiation, Starbucks breached its exclusive contract with Kraft in order to enter the coffee pods market in 2010. The contract dispute lasted three years, during which time Starbucks enjoyed great success in the pods market. The settlement resulted in Starbucks paying $2.75 million to Kraft, which would have been different had the market performed differently (Shonk 2018). The expected gains from collaboration can also change as partners discover how well they match with each other. In 2014, Apple and IBM formed a partnership to create enterprise apps, but their conversation started a couple years prior. The two firms dedicated teams to work with each other and find complementarities before finally establishing the partnership (Cook and Rometty, 2014). Last but not least, a common tactic in negotiation is to work with the opponent in finding new "win-win" solutions (see, e.g., Bazerman et al. 1985 and De Dreu et al. 2000). Such efforts have uncertain impacts on the total surplus to be bargained. Sovereign debt restructuring is another common example in which the total surplus changes stochastically over time.

If a project's expected return evolves stochastically, then the timing of implementation becomes endogenous. If the project only requires a single firm to implement, the previous literature has shown that the firm will want to delay implementing it until the expected surplus reaches a certain threshold (see, e.g., Dixit 1993). However, the decision is not so clear if an alliance between two firms is required to implement it. In many alliances, one can argue that neither firm has pricing power over the other. Prospective partners must both agree to implement the project and agree on how to split the profit, in order for the alliance to be formed. This naturally leads to a bargaining situation. Whether two firms form an alliance, and when they form it, can be studied as the outcome of a sequential bargaining game.

If two firms can collaborate to implement a project with a stochastically evolving return, when do they reach an agreement to collaborate? How do they agree to split the profit from the alliance? Is the outcome efficient? How does bargaining power (as determined by the bargaining procedure) affect their decisions to form the alliance? To answer these questions, I present a model of bilateral bargaining with a stochastic surplus. At each moment, one firm is the proposer and the other firm is the responder. If the proposer wants to form an alliance at that time, it can propose an offer to the responder without future commitment. The responder then decides whether to accept the offer, wait, or quit and take the outside option. Firms switch their roles as proposer and responder upon the arrival of a Poisson process, so the previous responder becomes the new proposer and can make counteroffers.[1]

A common feature of sequential bargaining games is that the bargaining power is determined by who gets to make the offer and when. In this paper, bargaining power is governed by the arrival rate of the switching between the proposer and the responder. I refer to this

[1]This model of bargaining can also be thought of as an alternating-offers model, with no commitment between when an offer is made and when the next counter-offer arrives.

arrival rate as the frequency of counteroffers. If counteroffers are more frequent, the two parties are more balanced in their power. If counteroffers arrive less frequently, the bargaining power favors the current proposer more because the responder has to wait longer before it can make offers.

One concern in modelling bargaining as a sequential game is that the outcome of the game is determined by the choice of the extensive form, but real negotiations could be undertaken in a variety of ways. This paper relaxes this concern by examining how the outcome varies with the frequency of counteroffers, which has the effect of varying the procedures by which firms bargain. As the frequency becomes infinitely low, the game approaches the repeated-offers paradigms of Fudenberg et al. (1985) and Gul et al. (1986), where one firm makes all the offers. As the frequency becomes infinitely high, the game approaches the continuous-time limit of the alternating-offers game by Rubinstein (1982). Intermediate levels of frequency capture a class of symmetric bargaining procedures between the two extremes.

The core result of this paper is that one bargaining procedure can be "better" than another in fostering collaboration. Changing the bargaining procedure (by changing the frequency of counteroffers) can be mutually beneficial by improving the ex-ante chance of alliance and making the timing of such collaboration more efficient. Particularly, bargaining with a higher frequency of counteroffers can Pareto dominate lower frequencies, suggesting that both parties and any social planner may prefer more balanced bargaining power. Bucklin and Sengupta (1993) find that reducing power imbalances between partners can improve the effectiveness of a co-marketing alliance. This paper extends their conclusion by suggesting that reducing power imbalances may also improve the efficiency of the formation of such alliances in the first place.

The model produces a unique equilibrium outcome under symmetric and stationary strategies. When the outside option is non-positive, the quitting decision becomes irrelevant and the alliance must be formed at some point. Firms delay agreement until it is socially efficient to implement the project. Upon agreement, the responder receives a larger share of the surplus than the share it receives under a constant surplus scenario. The stochastic nature of the project thus increases the responder's bargaining power due to the responder's ability unilaterally delay collaboration.

When the outside option is strictly positive, quitting becomes a relevant option. The responder quits when the project surplus drops to a sufficiently low point. In that case, the formation of alliance is no longer efficient. The ex-ante probability of alliance is sub-optimal. Firms implement the project too early and negotiation breaks down too early compared to the socially optimal levels. Such inefficiency is a result of a hold-up problem faced by the responder. Waiting can be seen as a relationship-specific investment. The cost of waiting is the discounted value lost in not taking the outside option. The return on that investment is determined by future bargaining. Thus, the responder, who has less bargaining power, under-invests and quits too early. In order to avoid the higher chance of breakdown, firms then implement the project "prematurely". This hold-up problem is mitigated by having more frequent counteroffers. A higher frequency of counteroffers balances bargaining power, which leads to agreement and quitting decisions that are more socially efficient. Increasing the frequency of counteroffers not only redistributes but also expands total welfare.

The proposer benefits from a more balanced bargaining procedure if the expansion of total welfare dominates the redistribution of welfare. Particularly, when the initial size of the surplus is not too big or too small, a low frequency of counteroffers can be Pareto dominated by a high frequency of counteroffers. This phenomenon does not happen if the return from collaboration is constant or if quitting is irrelevant. Those cases do not exhibit the hold-up problem, so varying the frequency of counteroffers does not affect total utility, only the split of that utility.

The paper is organized as follows. After literature review, Section 2 present the model. Section 3 presents the equilibrium outcome and its efficiency when quitting is irrelevant. Section 4 solves the case with quitting and discusses the effect of counteroffers and bargaining power on social and Pareto efficiency. Section 5 concludes the paper.

Literature Review

Much of the marketing and management literature on alliances has focused on investment decisions and their effectiveness *after* firms form an alliance. Amaldoss et al. (2000), Amaldoss and Staelin (2010), Bhaskaran and Krishnan (2009), and Amaldoss and Rapaport (2005) study factors that affect resource-commitment decisions in an alliance, and explore structural solutions that mitigate free-riding problems among partners in an alliance. See also, Harrigan (1988) and Bucklin and Sengupta (1993). In comparison, less focus has been placed on factors that lead to the formation of alliances. Robertson and Gatignon (1998) study factors that predict whether firms join technological alliances. Cai and Raju (2016) study market entry as an alliance. They show that market size and market competitiveness can determine when entry as an alliance is more profitable than entering independently. This paper studies the investment decisions *before* firms form an alliance and how bargaining procedure affects the probability and timing of the alliance. A conceptually related paper is Frankel (1998), in which players can exert effort to expand the size of the "pie", but sometimes under-invest due to the hold-up problem.

This paper features bargaining with a stochastic payoff and a stochastic proposing order. Merlo and Wilson (1995, 1998) present a discrete-time model with a stochastic pie and prove the uniqueness of a stationary payoff when there is no outside option. Cripps (1997) allows the pie to follow a geometric Brownian motion. Furusawa and Wen (2003) study the case of stochastic disagreement payoffs and allow the proposer to delay proposing. In Hanazono and Watanabe (2016), firms receive private signals on an i.i.d stochastic pie. Another stream of literature expands the alternating-offers model of Rubinstein (1982) to allow for a random order of proposers determined by a homogeneous Markov process, see e.g., Binmore (1987), Muthoo (1999), and Houba (2008). In Yildiz (2004) and Simsek and Yildiz (2016), players have uncommon priors on the recognition process and update their beliefs over time. Daley and Green (2018), Ishii et al. (2018), and Ning (2018) study stochastic bargaining in continuous time but do not allow for counteroffers. In Ortner (2019), the right to propose is stochastic, but the total surplus is fixed. In contrast, this paper allows for both stochastic surplus and a random order of proposing.

The paper is also related to optimal stopping problems with stochastic payoffs. Earlier works have examined the cases of R&D funding (e.g., Roberts and Weitzman 1981), options (e.g., Dixit and Pindyck 1994), and consumer search (e.g., Branco et al. 2010). But these papers only feature a single decision-maker. Wilson (2001), Compte and Jehiel (2010), Cho and Matsui (2013), and Kamada and Muto (2015) consider multi-agent search problems in which unanimous agreement is required to stop. They show that the limiting case of the search model relates to the Nash bargaining solution, but do not allow explicit bargaining between the players. In this paper, stopping requires an agreement between the two firms, but they are allowed to make offers to facilitate such agreement.

2.2 The Model

The game is in continuous time with an infinite horizon. Two risk-neutral firms, i and j, can form an alliance to collaborate on a project. They bargain over whether to collaborate and how to split the returns from the alliance.

Stochastic Surplus

The expected surplus generated from the collaboration project, x_t, is observable to both firms and is assumed to follow a Brownian motion $dx_t = \sigma dW_t$ with volatility $\sigma \geq 0$ and initial position x_0, where W_t is a Wiener process. Below I present micro motivations for a stochastic surplus in which a Brownian motion can be derived as a limiting case.

A hypothetical firm is considering whether to enter a new market. To operate in the new market, the firm has to collaborate with a partner. The alliance may be needed to co-develop the product, engage in co-marketing campaign, or provide distribution and localized service.

Evolving Consumer Preference The potential profit from the alliance depends on consumer preference in the market, which evolves over time. For example, consider a Hotelling line of length l. A mass of consumers is located at $z_t \in [0, l]$ at time t, and z_t takes a random walk with reflecting boundaries at 0 and l. There exists a competitive fringe at location 0 with price normalized to 0, and the alliance can enter at location l. If the collaborators enter at time t, the highest price they can charge to make a sale is $p_t = 2z_t - l$, which is a random walk with reflecting boundaries at l and $-l$. As $l \to \infty$, the price approaches a Brownian motion.

Arrival of Information Suppose that the new product provides a value of v_t to consumers, where v_t follows a random walk with variance σ^2. However, the firm does not observe the true value of v_t, but can obtain signals by conducting market research. Firms have a normal prior with mean $\hat{v}_0$ and variance $\hat{\rho}_0$. At each moment, the two firms receive a signal of v_t with a normal error of variance η^2, and updates the posterior mean $\hat{v}_t$ and variance $\hat{\rho}_t$ using Bayes' rule. The signal S_t accumulates as $dS_t = v_t dt + \eta dW_t$, where W_t is a Wiener process. By the Kalman-Bucy filter (see Ruymgaart and Soong 1988, ch.4), the posterior mean $\hat{v}_t$ follows $d\hat{v}_t = (\hat{\rho}_t/\eta)dB_t$ for some Wiener process B_t, and posterior variance follows $\frac{d\hat{\rho}_t}{dt} = -\hat{\rho}_t^2/\eta^2 + \sigma^2$. The posterior variance $\hat{\rho}_t$ approaches $\sigma\eta$ asymptotically over time. If $\hat{\rho}_0 = \sigma\eta$, then $\hat{v}_t$ is a Brownian motion with variance $\sigma^2\eta^2$.[2]

Matching The success of the alliance depends on how well the hypothetical firm matches with its partner. The two companies discover their match over time as they build relationship and explore the potential alliance. There is a mass of attributes important for the success of the project. The end product provides more value to consumers if the two collaborators match on more attributes. With equal chance, the collaborators match on a particular attribute, which delivers $z_t = +\sigma\sqrt{dt}$ value to consumers, or do not match, delievering $-\sigma\sqrt{dt}$ instead, where dt is the size of each attribute, so that we have $\mathbb{E}[z_t] = 0$ and $Var[z_t] = \sigma^2 dt$. The expected value of the product after observing t attributes can be written as $x_t = x_0 + \sum_0^t z_s$, where x_0 is the prior. As the mass of total attributes approaches infinity and dt approaches 0, the expected product value x_t becomes a Brownian motion.

Bargaining

Firms cannot write contingency contracts on when they will collaborate and how they will split the profit from the alliance. Instead, at the moment of its formation, both firms must agree to implement the project and agree on how they will split the surplus. The split and the timing of agreement is determined by sequential bargaining.

Extensive Form The order of movement at time t is determined by a recognition process f_t. We call firm i the *Proposer* at time t if $f_t = i$ and the *Responder* at time t if $f_t \neq i$. At each moment, the Proposer can propose a split of x_t and the Responder decides whether to accept the proposal if presented. The roles are switched upon the arrival of a Poisson process with rate λ.[3] If there is no switching in a time interval, then the roles remain the same and the same Proposer can make repeated offers. Thus, λ captures the speed at which counteroffers arrive, and $N(t)$ captures the number of times that offers have been countered. Let (Σ, $\mathcal{F}$, P) be the probability space that supports the Wiener process W_t and the Poisson counting process N_t, and F $= (\mathcal{F}_t)_{t\in[0,\infty)}$ be the filtration process satisfying the usual assumptions.

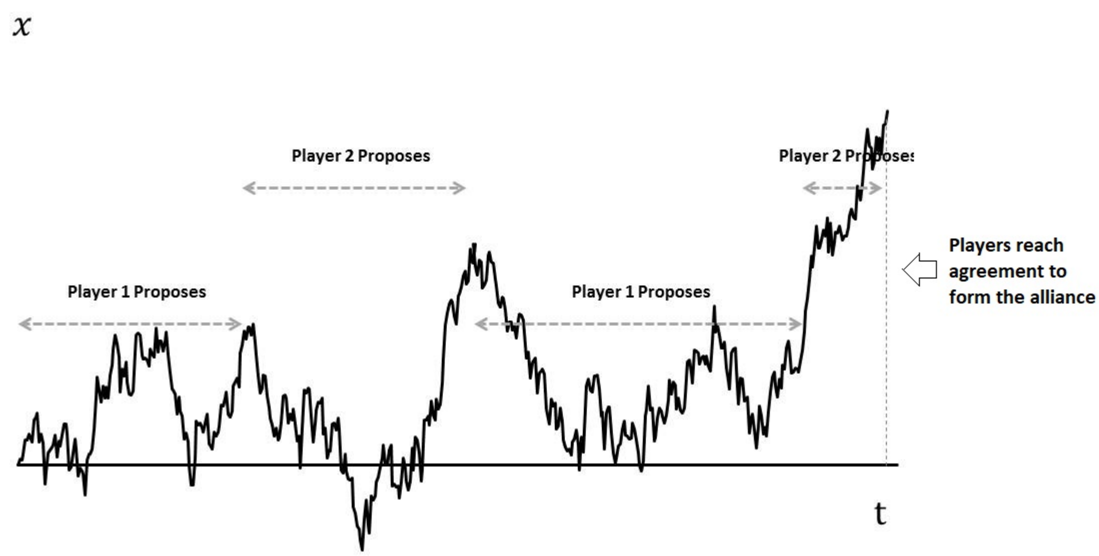
The game is played as follows. At time t, the surplus x_t and the identity of the Proposer f_t are realized. Upon the realization, the Proposer can choose to proposes a split $s_t \in R_+^2$ such that $s_1 + s_2 = x_t$ (so no waste is allowed). Let p_t denote the amount offered to the Responder, and denote $p_t = -\infty$ if the Proposer does not make an offer, since making an

[2]Note that one can model with fixed consumer preference as well. Suppose consumers values the product at either v_h or v_l, and firms receive signals of the true state and update their belief $\hat{\pi}_t$ that the state is v_h. Then one can show that the log-likelihood of the posterior, $\log(\frac{\hat{\pi}_t}{1-\hat{\pi}_t})$, follows a Brownian motion. The surplus from forming the alliance at time t is $\hat{\pi}_t(v_h - v_l)$.

[3]Assuming WLOG that $f_0 = 1$, then $f_t = f_0 + N(t) \bmod 2$, where $\{N(t), t \geq 0\}$ is a Poisson counting process with rate λ.

unacceptable offer is equivalent to not making one.[4] Given an offer p_t, the Responder chooses whether to accept or reject. If the offer is accepted, firms implement the project and split the surplus as agreed. If the offer is rejected, the Responder chooses whether to continue or to quit. If the Responder chooses to quit, the game ends and both firms receive their outside options. The game continues with new realizations of x_t and f_t until an offer is accepted or a firm quits. Figure 1 illustrates the game graphically.

Figure 2.1: Illustration of the Game



Utility Firms are symmetric with discount rate r. Upon reaching an agreement, the Proposer receives utility of $x_t - p_t$ and the Responder receives utility of p_t. If the Responder quits, then each firm receive its outside option, ω.

Let a_t be an indicator function for whether the Responder agrees to the Proposer's offer at time t, and q_t be an indicator function for whether Responder quits at time t. Then the game ends at $\tau = \inf\{t|a_t = 1 \text{ or } q_t = 1\}$. If firms reach agreement at time τ, i.e., $a_\tau = 1$, then the expected utility of firm i at time t is defined as:

$$u_{i,t} = e^{-r(\tau-t)}[\overbrace{\mathbb{1}\{f_\tau = i\}(x_\tau - p_\tau)}^{\text{Proposer utility at } \tau} + \overbrace{\mathbb{1}\{f_\tau \neq i\}p_\tau}^{\text{Responder utility at } \tau}]$$

If negotiation breaks down at time τ, i.e., $a_\tau = 0$ and $q_\tau = 1$, then the expected utility of firm i at time t is:

$$u_{i,t} = e^{-r(\tau-t)}\omega$$

[4]Note that the Proposer may not make an offer if it does not want to form the alliance at time t.

Firms receive zero utility if they neither reach an agreement nor quit, i.e., $\tau = \infty$ or $\max(a_\tau, q_\tau) = 0$. I refer to the outside option, ω, as *irrelevant* if $\omega \leq 0$, since it can never be taken in equilibrium. I refer to the outside option as *relevant* if ω is strictly positive.

Equilibrium Because the firms are symmetric and the stochastic processes are stationary, the paper focuses on symmetric and stationary MPE, henceforth referred to as *equilibrium*. Equilibrium strategies depend on the current size of the surplus and the bargaining role of each firm, but not on time or identity of the firms. Thus, a strategy profile is an equilibrium if it satisfies subgame-perfection[5] and can be described by:

1. A proposing strategy $p_t = p(x_t) : \mathcal{R} \mapsto \mathcal{R}$ for the Proposer at time t.
2. An agreement strategy $a_t = a(x_t, p_t) : \mathcal{R}^2 \mapsto \{0, 1\}$ for the Responder at time t.
3. A quitting strategy $q_t = q(x_t) : \mathcal{R} \mapsto \{0, 1\}$ for the Responder at time t.

Outcome An equilibrium outcome can be described by $\{U(x), V(x), A, Q\}$, where $U(x)$ is the value function of the Proposer in state x, $V(x)$ is the value function of the Responder in state x, $A = \{x \mid a(x, p(x)) = 1\}$ is the set of states in which firms reach agreement, and $Q = \{x \mid q(x) = 1\}$ is the set of states in which the negotiation breaks down.[6]

If $x_t \in A$, then the Proposer offers the Responder $p(x_t)$ and the Responder accepts, so $U(x_t) = 1 - p(x_t)$ and $V(x_t) = p(x_t)$. If $x_t \in Q$, then the Responder quits, so $U(x_t) = V(x_t) = \omega$. If x_t is in neither $\mathcal{A}$ or $\mathcal{Q}$, then the game continues. In this case, one can show that the value functions must satisfy:

$$\begin{aligned}(r+\lambda)U(x) &= \frac{\sigma^2}{2}U''(x) + \lambda V(x)\\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x)\end{aligned} \tag{2.1}$$

or

$$\begin{aligned}(U+V)(x) &= \frac{\sigma^2}{2r}(U+V)''(x)\\ (U-V)(x) &= \frac{\sigma^2}{2r+4\lambda}(U-V)''(x)\end{aligned} \tag{2.2}$$

The function $(U+V)(x)$ is the sum of both firms' expected utilities in state x and captures social value. Through $(U+V)(x)$ one can examine whether an outcome is socially efficient.

[5]In continuous-time games, a strategy profile may not produce a well-defined outcome. See Simon and Stinchcombe (1989). When checking for profitable deviations, only strategy profiles such that the stopping time τ is a measurable function from Σ to R^+ is considered.

[6]By the definition of utility, firms receive payoffs of 0 if they neither reach agreement nor quit at τ. This effectively restrict A and Q to be closed sets. Without the restriction to closed sets, one can create alternative equilibrium outcomes with agreement region $A \setminus Z_1$ and quitting region $Q \setminus Z_2$, where Z_1 and Z_2 are sets of measure 0. See Ortner (2019) for a similar restriction.

The function $(U-V)(x)$ is the difference in the utilities and captures the advantage of being the Proposer. The full derivation of these equations is found in Section (2.6).

In equilibrium, the sum of the firms' value functions have to exceed the current surplus size. Intuitively, if the sum of their utilities by following their equilibrium strategies is less than what is available right now, then they can profitably deviate by splitting the current surplus or taking their respective outside options.

Lemma 6. *In equilibrium, $(U+V)(x) \geq x$ for all $x \in \mathcal{R}$.*

Only the Responder is allowed to quit in this game. I argue that this is innocuous. Intuitively, the Responder has a weaker position in the negotiation and thus a stronger incentive to quit. In Section 3 and 4, I show that the Proposer always has a (weakly) higher continuation value in equilibrium. Thus, the Proposer never wants to quit if the Responder stays in the game, so any equilibrium of this game must also be an equilibrium if the Proposer is allowed to quit. Allowing the Proposer to quit adds trivial equilibria in which firms quit at the same time.[7] Only allowing the Responder to quit eliminates this trivial multiplicity.

Frequency of Counteroffers and Bargaining Power

The arrival rate λ captures the frequency by which offers are countered. When the Poisson event arrives, the previous Responder becomes the new Proposer and can make counteroffers. The firms remain in their new roles until the next Poisson event arrives. Varying the parameter λ is analogous to varying the extensive-form by which the firms bargain. A game with a higher λ features more frequent counteroffers. If $\lambda \to \infty$, the game approaches the continuous-time limit of the alternating-offers paradigm of Rubinstein (1982). On the other hand, as $\lambda \to 0$, all offers are made by the Proposer at $t = 0$ (in probability). Thus the game approaches the repeated-offers paradigm of Fudenberg et al. (1985) and Gul et al. (1986) (but without asymmetric information).

As in other sequential bargaining games, bargaining power between two symmetric agents is determined by who has the ability to propose; hence, in this game, the choice of λ. This point can be illustrated by examining the static case. Assume $\sigma = 0$ so that the surplus generated from the project is fixed. Also assume $x_0 > \omega = 0$, so the outside option is irrelevant. The game with a static surplus has a unique equilibrium outcome. Plugging $\sigma = 0$ into equations (2.1) shows that the expected utility for the Responder if it rejects the first offer is $\frac{\lambda}{r+\lambda}U(x_0)$. Thus, the Responder accepts if and only if $p(x_0) \geq \frac{\lambda}{r+\lambda}U(x_0)$. So the Proposer offers $p_0 = \frac{\lambda}{r+\lambda}U(x_0)$. This implies that the Proposer gets $\frac{r+\lambda}{r+2\lambda}x_0$, and the

[7]If the quitting decisions are made simultaneously, quitting is always (weakly) optimal if the opponent is quitting at the same time. So, quitting at any state can be supported in an equilibrium. Sequential order does not eliminate this problem. For example, take any random threshold $\underline{x} < \omega$, and let both firms quit for all $x < \underline{x}$. Then there's no profitable deviation in these states regardless the order of quitting. Because the state evolves continuously, the second mover is indifferent because the opponent will quit in the "next instant" regardless. The state x_t cannot jump out of the quitting region. Thus any choice of $\underline{x} < \omega$ can be supported in an equilibrium in this fashion.

Responder gets $\frac{\lambda}{r+2\lambda}x_0$. As λ decreases, the Proposer gets a larger share of the pie. Less frequent counteroffers translates to higher bargaining power for the Proposer. Conversely, as λ increases towards infinity, allowing more frequent counteroffers, the Proposer loses its advantage and the profit split approaches being even. It is worth noting that this equilibrium is analogous to the symmetric equilibrium from Rubinstein (1982). Particularly, if we define $\delta = \frac{\lambda}{r+\lambda}$, then the Proposer's share is $\frac{1}{1+\delta}$ and the Responder's share is $\frac{\delta}{1+\delta}$. This equivalence no longer holds when the size of the surplus is stochastic as in Section 3 and 4.

To model bargaining sequentially, one faces the problem of choosing the "right" game. Fudenberg et al. (1985) point out two issues: "[f]irst, because the results depend on the extensive form, one needs to argue that the chosen specification is [...] a good approximation to the extensive forms actually played. Second, even if one particular extensive form were used in almost all bargaining, the analysis is incomplete because it has not [...] begun to address the questions of why that extensive form is used." This view is echoed by Sutton (1986). This paper makes an attempt to address these two issues. First, by varying the frequency of counteroffers, λ, one can examine how the equilibrium outcome is affected by the choice of bargaining procedures within a class. Second, it provides a meaningful comparison between different procedures. Section 4 shows that, when the outside option is relevant, the bargaining outcome under a higher λ can dominate the bargaining outcome under a lower λ in both social and Pareto efficiency. In such cases, both firms and any social planner should prefer to bargain with more frequent counteroffers. This finding helps to explain why a particular bargaining procedure is (not) used.

2.3 Irrelevant Outside Options

This section studies the case where quitting is not a feasible action. Assume that the outside option is irrelevant, or $\omega \leq 0$. Then firms must remain in the game until they reach an agreement, because the expected utility from continuing is always better than the outside option. The quitting decision thus can be ignored in the analysis. The two firms will form alliance with certainty, but what is unknown is the timing of their collaboration and the split they agree to.

Socially Efficient Outcome

As a benchmark, I first show what the socially efficient outcome is. It is socially optimal to delay the project when the surplus is small, because of the option value of waiting for a higher surplus in the future. This is an optimal stopping problem with a discount rate r (see, e.g., Dixit 1993). The optimal decision features a threshold $\overline{x_s}$. The social planner implements the project if the surplus is equal or above the threshold, and waits otherwise.

One can show that the socially efficient threshold is $\overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$ and the value function of the social planner is $W_s(x) = \sqrt{\frac{\sigma^2}{2r}}e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} = \sqrt{\frac{\sigma^2}{2r}}e^{\sqrt{\frac{2r}{\sigma^2}}x-1}$. See Section (2.6) for details.

Equilibrium Outcome

The following Lemma shows that in equilibrium, firms reach agreement once the surplus exceeds some threshold. First, it is easy to see that firms must reach an agreement at some point, because the utility from bargaining indefinitely is 0. If they reach an agreement at some state x_l, one can show that they must also reach agreement for any state $x_h > x_l$. Conditional on implementing the project at state x_l, the socially optimal decision for state x_h is to implement the project immediately. Thus, x_h is both an upper bound (because it is socially optimal) and a lower bound (per Lemma 1) on the sum of the firms' value functions at x_h. Intuitively, if firms are willing to implement the project at x_l, then at least one of the firms must also want to implement the project at x_h. Because firms can transfer utility through bargaining, the firm that wants to implement the project can offer the other firm enough so that an agreement is reached immediately.

Lemma 7. *In equilibrium, $\exists\, \overline{x} \geq 0$ such that firms reach an agreement if x_t is in $A = [\overline{x}, \infty)$.*

Given an agreement threshold $\overline{x}$, firms form the alliance once the surplus reaches that threshold. To characterize an equilibrium outcome, one only needs to solve for the agreement threshold, $\overline{x}$, and the value functions for the Proposer, $U(x)$, and for the Responder, $V(x)$.

Proposition 1 shows that the agreement threshold in equilibrium must be equal to the socially efficient threshold. Thus, the sum of firms' equilibrium value functions must also equal to the social planner's value function. The intuition is that, if the outcome is inefficient, then the two firms can "coordinate" a profitable deviation. If the agreement threshold is not socially optimal, then there exist a different threshold that improves total welfare, so at least one firm must benefit from such deviation. Bargaining then allows that firm to transfer some of the efficiency gain to the other firm, so that both firms can benefit. The only agreement threshold that firms cannot mutually benefit from deviating is the socially optimal one. Thus the alliance is formed as a social planner would: the Responder rejects the Proposer's offer when the return of the project is smaller than $\overline{x_s}$, and accepts the offer when the return is larger.

Knowing the agreement threshold, one can find proposal rules that implements such threshold. Both firms have to be willing to stop at the socially efficient threshold, and at least one firm does not want to stop below it. The proposal rule is unique in the agreement region, which implies that the equilibrium outcome is unique.[8] Section 2.3 has shown that, if the surplus x_t is constant over time, then the Responder receives $\frac{\lambda}{r+2\lambda}x_\tau$ upon agreement. I refer to this as the *static share* for the Responder.

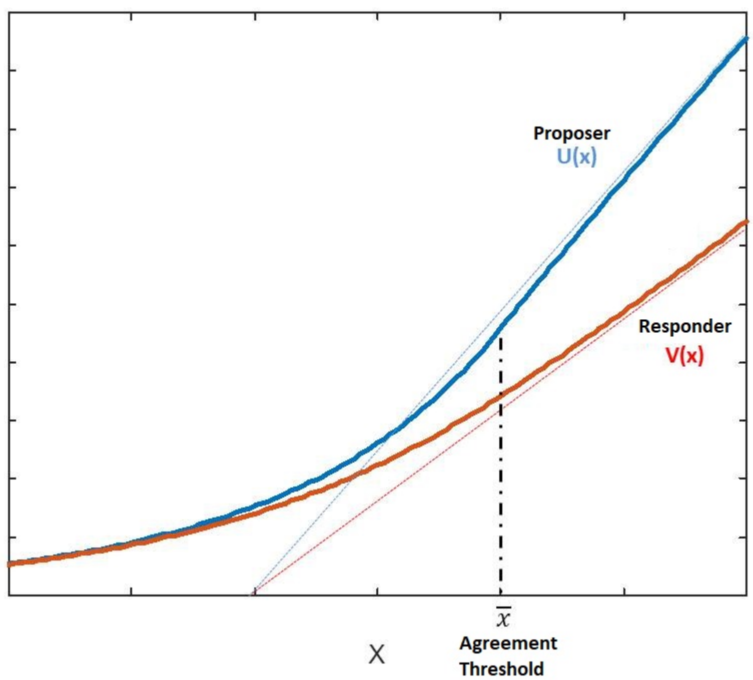
Proposition 10 (Irrelevant Outside Option with $\omega \leq 0$)**.** *There exists a unique equilibrium outcome. The agreement threshold is $\overline{x} = \overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$. The Proposer has a higher expected utility than the Responder for all x. Upon agreement, the Responder receives strictly more than $\frac{\lambda}{r+2\lambda}x_\tau$, the static share.*

[8] The proposal rule can be different for the non-agreement region, but such multiplicity in strategy do not affect the equilibrium outcome. If an offer is rejected, the size of that offer becomes irrelevant.

Proposition 1 says that, when two firms do not have any option other than forming an alliance, their timing of collaboration will be efficient. They wait until the project looks sufficiently profitable before implementing it. The bargaining protocol only determines how the return from the alliance is split. This social efficiency of bargaining with a stochastic pie has been shown by Merlo and Wilson (1995, 1998). Proposition 1 can be seen as a continuous-time analogy to their results.

Figure 2.2 depicts the value functions and the trading threshold graphically. The Proposer's value function is always strictly above the Responder's, illustrating the advantage of being the Proposer. Closed-form solutions of the value functions $U(x)$ and $V(x)$ are presented in Section (2.6).

Figure 2.2: Equilibrium Outcome



Another result from Proposition 1 is that, when the alliance is formed, the Responder gets a bigger share of the surplus than it would receive if the return of the project were static. Thus, the stochastic nature of the project reduces the Proposer's advantage. In Figure 2.2, the two dotted lines represent the static shares that firms would receive. In equilibrium, the Responder's utility strictly above the static share in all states. Also, the equilibrium split approaches the static split as $x \to \infty$.

The evolving surplus provides more bargaining power to the Responder because the Responder has the power to unilaterally delay the project. Because the Responder gets a smaller share of the surplus from collaborating, it incurs less cost from delaying and so has a higher incentive to wait. To see this, suppose that the only available split is their static share, which is $\frac{r+\lambda}{r+2\lambda}x_t$ for the Proposer and $\frac{\lambda}{r+2\lambda}x_t$ for the Responder. Firms then decide individually at each moment, whether they agree to form the alliance. One can show that the Responder must have a higher agreement threshold. Since both firms have to agree, the Responder effectively controls when the agreement is reached. The Proposer thus has to offer more than the static share to encourage the Responder to agree earlier.

Figure 2.3: Responder Delay Agreement Under Static Share

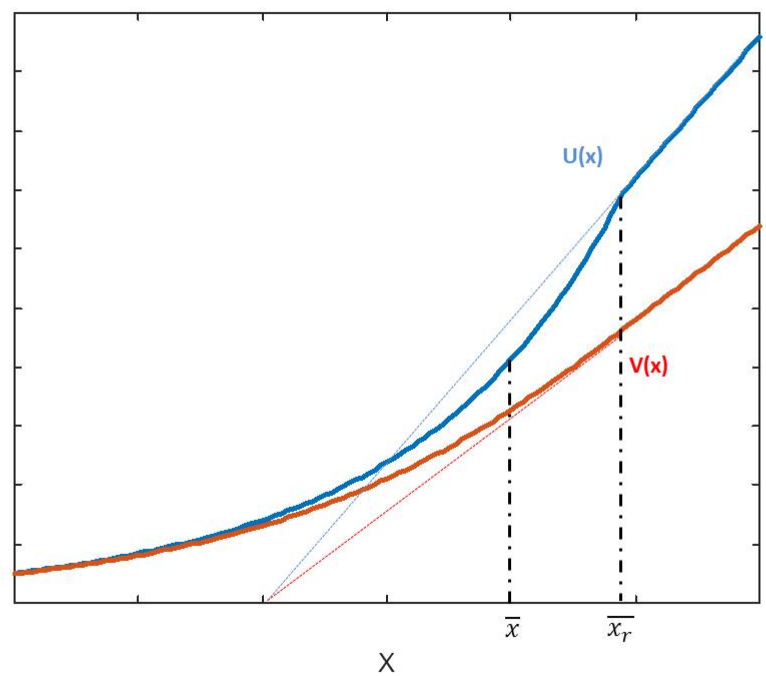


Figure 2.3 illustrates this example. The threshold $\overline{x}$ is the equilibrium threshold under bargaining, and the threshold $\overline{x_r}$ is the Responder's stopping threshold if it can only get $\frac{r+\lambda}{r+2\lambda}x_t$. For states between $\overline{x}$ and $\overline{x_r}$, the sum of firms' value functions $U(x_t) + V(x_t)$ is smaller than x_t. Thus firms can mutually benefit by collaborating immediately with a bigger share offered to the Responder.

Note that the agreement threshold in equilibrium does not depend on λ. The choice of λ does not affect total welfare but only how it is distributed. One can easily verify that $U(x)$ decreases with λ and $V(x)$ increases with λ for all x_0. A lower frequency of counteroffers benefits the (current) Proposer, and a higher frequency makes the bargaining power more balanced, as expected. As $\lambda \to \infty$, each firm receives half of the social value. As $\lambda \to 0$, the Proposer receives the entire profit from the alliance.

Corollary 11 (Comparative Statics w.r.t λ)**.** *The equilibrium outcome is socially efficient for all λ. The Proposer's ex-ante utility strictly increases with λ, and the Responder's ex-ante* *utility strictly decreases with* λ.

2.4 Relevant Outside Options and the Hold-Up Problem

Suppose now that the outside option is relevant, or $\omega > 0$. The two firms do not have to form an alliance with each other. In this case, the outcome is no longer socially efficient. More importantly, the level of efficiency is determined by the frequency of counteroffers, λ. A higher λ makes firms act more patiently and leads to a more efficient timing of the alliance. Furthermore, the equilibrium outcome under a higher λ can Pareto dominate the outcome under a lower λ, suggesting that there can be mutual gain by allowing for more counteroffers (and, hence, more balanced bargaining power).

Socially Efficient Outcome

A social planner with discount rate r and outside option 2ω chooses at each moment whether to implement the project, wait, or take the outside option. The social planner implements the project if the surplus exceeds the threshold $\overline{x_s}$, and takes the outside option if the surplus falls below the threshold $\underline{x_s}$. Solving the social planner's optimal stopping problem shows that the socially efficient thresholds are

$$\overline{x_s} = \sqrt{\frac{\sigma^2}{2r} + 4\omega^2} \quad \text{and} \quad \underline{x_s} = \overline{x_s} - \sqrt{\frac{\sigma^2}{2r}} \log\Big(\frac{\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}}{2\omega}\Big)$$

The social value function for continuation states $\underline{x_s} < x < \overline{x_s}$ is:

$$W_s(x) = \frac{1}{2}\Big(\overline{x_s} + \sqrt{\frac{\sigma^2}{2r}}\Big)e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} + \frac{1}{2}\Big(\overline{x_s} - \sqrt{\frac{\sigma^2}{2r}}\Big)e^{-\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} \tag{2.3}$$

See Section (2.6) for more details.

Equilibrium Outcome

Similarly, there exists an agreement threshold $\overline{x}$. Firms form alliance immediately if the surplus is larger than the threshold. The agreement threshold cannot be higher than the socially efficient threshold $\overline{x_s}$, otherwise there exists a mutually beneficial offer between $\overline{x_s}$ and $\overline{x}$. Also, the threshold has be higher than the sum of outside options for the alliance to be profitable.

Lemma 8. *In equilibrium,* $\exists$ *an agreement threshold* $\overline{x}$ *such that* $\overline{x_s} \geq \overline{x} \geq 2\omega$. *The alliance is formed if* $x_t \in A = [\overline{x}, \infty)$.

Because quitting is an option now, to characterize an equilibrium outcome we have to specify the states in which the Responder quits. The following result says those states can be described by a quitting threshold $\underline{x}$. The negotiation breaks down when the surplus drops below that a threshold. The quitting threshold has to be higher than the socially efficient threshold $\underline{x_s}$.

Lemma 9. *In equilibrium, $\exists$ a quitting threshold $\underline{x}$ such that $\underline{x} \geq \underline{x_s}$. The Responder quits if $x_t \in Q = (-\infty, \underline{x}]$.*

An equilibrium outcome can be described by the thresholds, $\overline{x}$ and $\underline{x}$, and the value functions, $U(x)$ and $V(x)$. When quitting is not an option, firms form alliance with certainty and their timing of collaboration is efficient, as shown in Proposition 1. This is no longer true when the outside option becomes relevant. In equilibrium, firms form alliance too early, negotiation breaks down too early, and the ex-ante probability of alliance is sub-optimal. Proposition 3 described the unique equilibrium outcome and how $\overline{x}$, $\underline{x}$, and the ex-ante probability of alliance changes with λ.

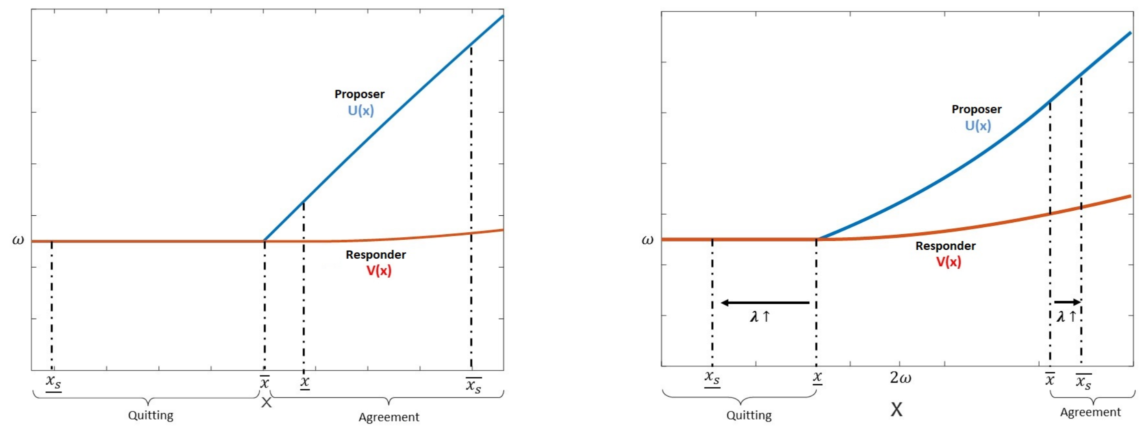
Proposition 12 (Relevant Outside Option with $\omega > 0$)**.**

1. *There exists a unique equilibrium outcome with agreement threshold $\overline{x} < \overline{x_s}$ and quitting threshold $\underline{x} > \underline{x_s}$. The Proposer has a higher expected utility than the Responder for all $x > \underline{x}$.*
2. *Let $\underline{\lambda} = \frac{4r^2\omega^2 + \sqrt{16r^4\omega^4 + 8\sigma^2 r^3\omega^2}}{2\sigma^2}$:*
   (a) *If $\lambda \leq \underline{\lambda}$, then $\underline{x} \geq \overline{x} = 2\omega$. Firms form alliance or take outside options at $t = 0$.*
   (b) *If $\lambda > \underline{\lambda}$, then $\overline{x} > \underline{x}$. Negotiation continues for $x_t \in (\underline{x}, \overline{x})$. The agreement threshold $\overline{x}$ strictly increases in λ and the quitting threshold $\underline{x}$ strictly decreases in λ.*
   (c) *There exists a cutoff $\tilde{x}(\lambda)$ such that the ex-ante probability of alliance increases in λ for $x_0 < \tilde{x}(\lambda)$ and decreases in λ for $x_0 > \tilde{x}(\lambda)$.*

The closed-form expression of firms' value functions can be found in Section (2.6). Figure 2.4 graphically depicts the unique equilibrium outcome, and compares it to the socially efficient outcome. Figure 4(a) shows the case of $\lambda \leq \underline{\lambda}$. In this case, $\underline{x} \geq \overline{x}$, so that all x falls into either the agreement region or the quitting region. The alliance is formed immediately if $x_0 \geq 2\omega$, and broken off otherwise. Figure 4(b) shows the case of $\lambda > \underline{\lambda}$. Firms continue to wait if $x_t \in (\underline{x}, \overline{x})$, but still stop too early compared to the socially efficient thresholds. Both $\overline{x}$ and $\underline{x}$ move towards the socially efficient thresholds as λ increases. The ex-ante probability of alliance can either increase or decrease in λ depending on the initial position, x_0.

The Responder quits too early due to a hold-up problem resulting from its lack of bargaining power. When deciding between quitting and waiting, the Responder has to weigh its cost and benefit from continuing. The Responder's cost of continuation is the discounting of

Figure 2.4: Equilibrium Outcome with Relevant Outside Option



(a) $\lambda \leq \underline{\lambda}$ (b) $\lambda > \underline{\lambda}$

its outside option, and its benefit of continuation is the potential of forming an alliance with high surplus. However, its benefit from the alliance is determined through bargaining, and its current lack of bargaining power means an expected disadvantage in future bargaining. If we view the decision of continuing as a relationship-specific investment, then the Responder under-invests because it incurs half of the social cost but captures less than half of the social gain. Given that the Responder quits earlier, the Proposer wants to implement the project earlier to avoid the higher risk of breakdown. Note that a lower agreement threshold is efficient conditional on having a higher quitting threshold, so the inefficiency comes from the early quitting decision.

The severity of the hold-up problem depends on the bargaining power between the two firms. One can show that, if the only split allowed is an even split, and both firms only decide whether to accept that split, wait, or quit, then the outcome must be socially efficient. The uneven split from bargaining causes the weaker party, the Responder, to "under-invest" in the relationship and act too impatiently. Thus, a higher frequency of counteroffers, which balances the bargaining power, improves social efficiency. As λ approaches ∞, the collaboration outcome approaches the socially efficient outcome.

Corollary 13 (Effect of λ on Social Efficiency)**.** *Ex-ante welfare $U(x_0)+V(x_0)$ increases in λ. As $\lambda \to \infty$, the agreement and quitting thresholds approach the socially efficient thresholds, and the utility of each firm approaches one-half of the socially efficient total welfare.*

Increasing the frequency of counteroffers does two things. It redistributes total welfare between the two firms, as in other sequential bargaining games. But in this case it also expands the total welfare. This fact implies that a higher frequency of counteroffers may not necessarily be detrimental to the Proposer. With less bargaining power, the Proposer gets a smaller share of the pie, but the total size of the pie becomes larger. If the increased efficiency outweighs the loss of bargaining power for the Proposer, then both firms prefer to bargain under the higher λ. Proposition 5 outlines the conditions under which λ affects Pareto efficiency.

For simpler language, we say that λ_1 *is Pareto Dominated by* λ_2 if the equilibrium outcome under frequency λ_1 is Pareto dominated by the equilibrium outcome under frequency λ_2.

Proposition 14 (Effect of λ on Pareto efficiency)**.**

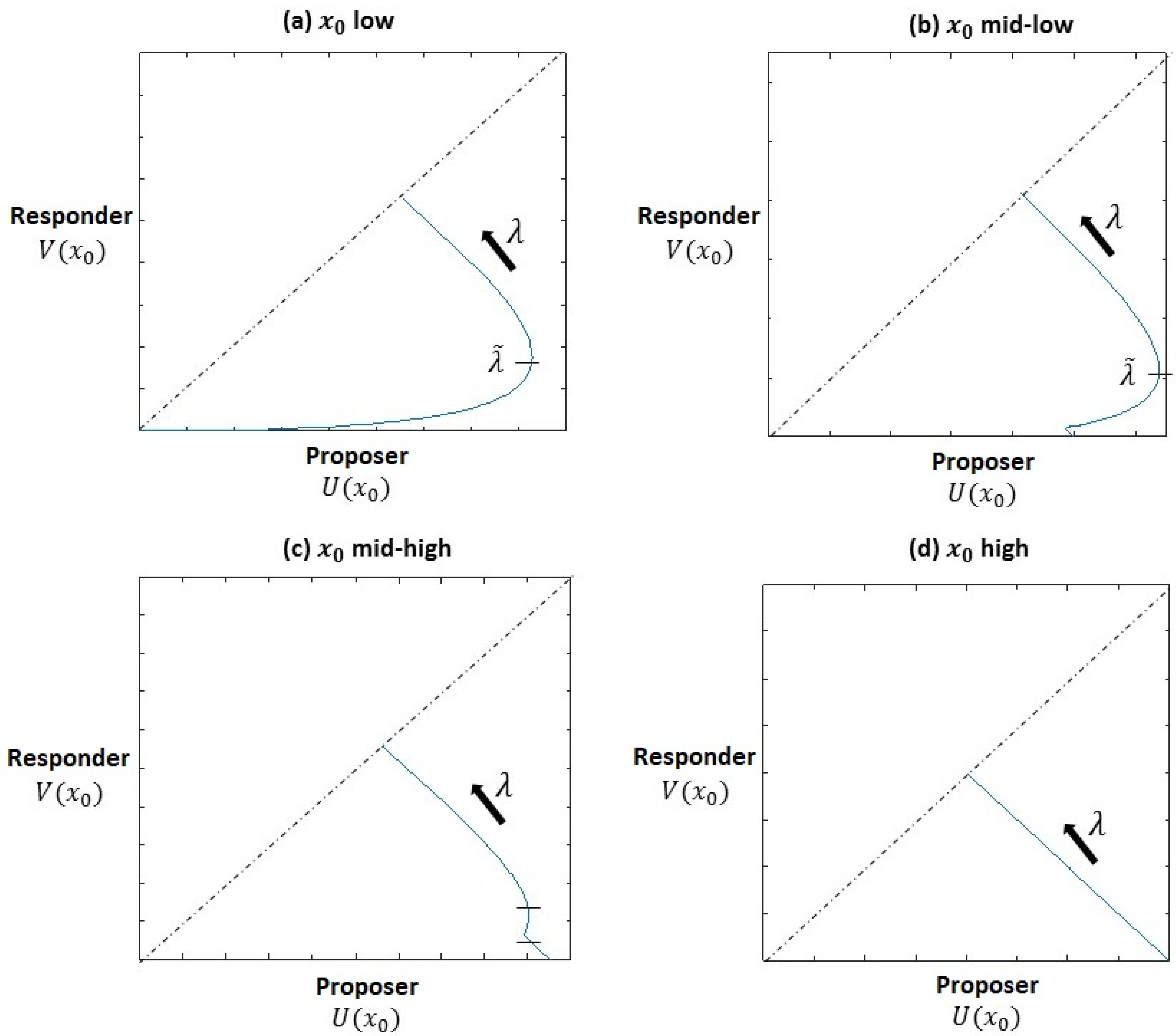
1. *For an intermediate range of* x_0*, there exists* $\tilde{\lambda}$ *such that any* $\lambda \leq \tilde{\lambda}$ *is Pareto dominated by some* $\lambda' > \tilde{\lambda}$*.*
2. *For all* λ*, there exists a range of* x_0 *such that* λ *is Pareto dominated by some* $\lambda' > \lambda$*.*
3. *If* $x_0 < \underline{x_s}$ *or* $x_0 > \overline{x_s}$*, then no* λ *is Pareto dominated.*

Proposition 5(1) states that there is an intermediate range of initial values such that all low λ are Pareto dominated. Both firms benefit from more frequent counteroffers if the frequency of counter-offers is below some threshold. The Proposer benefits from a loss of bargaining power because a more efficient timing of collaboration outweighs the negative effect of giving up a bigger share to its opponent. Proposition 5(2) states that no choice of λ is immune to Pareto improvement. Regardless of the level of λ, there is always some initial value such that firms can mutually benefit from an even higher frequency of counteroffers. Proposition 5(3) states that, if the surplus of the project at the beginning of the negotiation is too low ($x_0 < \underline{x_s}$) or too high ($x_0 > \overline{x_s}$), then the frequency of counteroffers does not affect Pareto efficiency. In these cases, the outcome is socially optimal regardless of λ, per Corollary 4. So total welfare is not impacted by λ, only the distribution of it.

Figure 2.5(a) to 5(d) trace the ex-ante utilities of the two firms as functions of λ for different initial surplus x_0. In figure 2.5(a) and 2.5(b) with low (but still higher than $\underline{x_s}$) initial surplus, all λ smaller than some $\tilde{\lambda}$ are Pareto dominated, and all $\lambda \geq \tilde{\lambda}$ are on the Pareto frontier. In figure 2.5(c) with a higher initial surplus, some middle levels of λ are dominated, but both high and low λ's are on the frontier. In Figure 2.5(d), the initial surplus is above $\overline{x}$ so that all choices of λ are efficient, since the alliance is formed immediately.

Under what procedures, then, should firms bargain? Or alternatively, if we observe firms bargaining in a certain way, why was such a procedure selected? Past literature has been mostly silent on these questions. If the bargaining procedure only affects the distribution of welfare, then it is unclear why one procedure would be "better" than another. Proposition 5 shows that, when the surplus is stochastic and there are outside options, the choice of bargaining procedure impacts the total welfare as well as the distribution of such welfare. Under certain conditions, bargaining with more counteroffers can benefit both firms by improving the efficiency of their collaboration.

Figure 2.5: Ex-ante Utilities as Functions of λ for Various Initial Values



2.5 Conclusion

In this paper, two firms can form an alliance to collaborate on a project with a stochastically evolving return. They bargain over whether to form the alliance and how to split the surplus from collaboration. The paper investigates the effect of bargaining procedure and bargaining power on the timing and efficiency of alliance formation. Two symmetric firms with time discounting and outside options bargain over a surplus that follows a random walk. One firm makes repeated-offers to its opponent, and they switch roles following a Poisson process to allow for counteroffers. The frequency of counteroffers controls their relative bargaining power, as a lower frequency favors the current proposer. The paper presents the unique symmetric and stationary equilibrium outcome. When the outside option is non-positive, the project is implemented with a socially efficient delay, and the final split of the surplus is more balanced than under a static surplus. However, when the outside option is strictly positive, the outcome is no longer socially efficient. The alliance is formed too early and negotiation breaks down too early compared to what is socially optimal. The inefficiency is caused by a hold-up problem faced by the weaker party. A higher frequency of counteroffers evens out the bargaining power and reduces the severity of the hold-up problem. The increase in social efficiency can outweigh the loss of bargaining power for the stronger party. As a result, bargaining with more frequent counteroffers (and, hence, a more balanced procedure) can produce a Pareto improvement.

This paper provides theoretical insights on how bargaining procedure affects collaboration outcomes, and how potential collaborators should or should not bargain. However, this paper remains agnostic about how the bargaining procedure is selected. I assume that a frequency of counteroffers is exogenously set. Future research might explore what happens if the selection is done endogenously by the parties in negotiation, or what happens if the bargaining procedure is asymmetric.

2.6 Derivation of Value Functions

If $x \in A$, then the Proposer offers the Responder $p(x)$ and the Responder accepts immediately, so $U(x) = 1 - p(x)$ and $V(x) = p(x)$. If $x \in Q$, then the Responder quits immediately, so $U(x) = V(x) = \omega$. If $x \in \mathcal{R} \setminus (\mathcal{A} \cup \mathcal{Q})$, then the Responder rejects the offer but continues to wait. In this case, the value functions can be written recursively as:

$$\begin{aligned} U(x) &= e^{-rdt}\ \mathbb{E}[\mathbb{1}\{f_{t+dt} = f_t\}U(x+dx) + \mathbb{1}\{f_{t+dt} \neq f_t\}V(x+dx)] + o(dt) \\ V(x) &= e^{-rdt}\ \mathbb{E}[\mathbb{1}\{f_{t+dt} = f_t\}V(x+dx) + \mathbb{1}\{f_{t+dt} \neq f_t\}U(x+dx)] + o(dt) \end{aligned} \tag{2.4}$$

Since $f_t = 1 + N(t) \bmod 2$, where $N(t)$ follows a Poisson counting process, the probability that the counter-offer event arrives once in dt is $\lambda dt + o(dt)$, and the probability that it arrives more than once is of $o(dt)$. Applying Ito's Lemma to $\mathbb{E}[V(x+dx)]$ and $\mathbb{E}[U(x+dx)]$, one can get:

$$\begin{aligned} U(x) &= e^{-rdt} \left\{(1-\lambda dt)\left[U(x) + \frac{\sigma^2}{2}U''(x)\right] + \lambda dt \left[V(x) + \frac{\sigma^2}{2}V''(x)\right]\right\} + o(dt) \\ &= (1-rdt) \left\{(1-\lambda dt)\left[U(x) + \frac{\sigma^2}{2}U''(x)\right] + \lambda dt \left[V(x) + \frac{\sigma^2}{2}V''(x)\right]\right\} + o(dt) \\ V(x) &= e^{-rdt} \left\{(1-\lambda dt)\left[V(x) + \frac{\sigma^2}{2}V''(x)\right] + \lambda dt \left[U(x) + \frac{\sigma^2}{2}U''(x)\right]\right\} + o(dt) \\ &= (1-rdt) \left\{(1-\lambda dt)\left[V(x) + \frac{\sigma^2}{2}V''(x)\right] + \lambda dt \left[U(x) + \frac{\sigma^2}{2}U''(x)\right]\right\} + o(dt) \end{aligned} \tag{2.5}$$

which after simplification and taking $dt \to 0$ becomes:

$$\begin{aligned} (r+\lambda)U(x) &= \frac{\sigma^2}{2}U''(x) + \lambda V(x) \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{2.6}$$

Adding and subtracting the two equations produces:

$$\begin{aligned} (U+V)(x) &= \frac{\sigma^2}{2r}(U+V)''(x) \\ (U-V)(x) &= \frac{\sigma^2}{2r+4\lambda}(U-V)''(x) \end{aligned} \tag{2.7}$$

The solution to these two differential equations is of the form:

$$\begin{aligned} (U+V)(x) &= \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} \\ (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \end{aligned} \tag{2.8}$$

for some coefficients α_+, α_-, β_+, and β_-.

Irrelevant Outside Option

Socially Efficient Outcome The social value function, denoted as $W_s(x)$, for $x \leq \overline{x_s}$ is of the form:

$$W_s(x) = C_1 e^{\sqrt{\frac{2r}{\sigma^2}}x} + C_2 e^{-\sqrt{\frac{2r}{\sigma^2}}x} \tag{2.9}$$

We must have $C_2 = 0$, because the value function has to approach 0 as $x \to -\infty$.

The threshold $\overline{x_s}$ has to satisfy $W_s(\overline{x_s}) = \overline{x_s}$ and $W_s'(\overline{x_s}) = \frac{dx}{dx} = 1$. The second condition is referred to as smooth-pasting and guarantees the optimal timing of the stoppage (See, e.g., Dixit 1993). Solving these two conditions gives the socially efficient threshold

$$\overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$$

and the social value function

$$W_s(x) = \sqrt{\frac{\sigma^2}{2r}} e^{\sqrt{\frac{2r}{\sigma^2}}(x - \overline{x_s})} = \sqrt{\frac{\sigma^2}{2r}} e^{\sqrt{\frac{2r}{\sigma^2}}x - 1}$$

.

Equilibrium Outcome Given an agreement threshold $\overline{x}$, firms implement the project immediately when the surplus reaches that threshold. Thus for $x \geq \overline{x}$, the sum of their value function equals to the surplus. Also, the Responder must be indifferent between accepting the offer and rejecting the offer in equilibrium, as the Proposer must not offer anything more than necessary. The Responder's value function from rejecting the offer is characterized in equation (2.6). These imply that the firms' value functions for $x \geq \overline{x}$ must satisfy:

$$\begin{aligned} U(x) + V(x) &= x \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{2.10}$$

Combining the two, one gets:

$$V(x) = \frac{\lambda}{r + 2\lambda}x + \frac{\sigma^2}{2r + 4\lambda}V''(x)$$

which has solution in the form:

$$V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_1 e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x \geq \overline{x} \tag{2.11}$$

for some coefficients γ_1 and γ_2. The Responder can never get more than the full surplus or agree to accept a negative amount, so $0 < V(x) \leq x \; \forall x$; thus $\gamma_1 = 0$, otherwise this is violated for x large enough. So $V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x}$ for some γ_2.

For states below the threshold, the Responder rejects the offer and the project is delayed. Firms' value functions must follow the recursive formulations in equations (2.8). Note that

$(U+V)(x)$ captures the social value function, and $(U-V)(x)$ captures the advantage of being the Proposer. As $x \to -\infty$, the social value must approaches zero, which implies $\beta_+ = 0$. Similarly, as social value approaches zero, the difference between the two firms has to approach zero, which implies $\beta_- = 0$. Thus we have a simpler version:

$$\begin{aligned} (U+V)(x) &= \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} \qquad \forall x < \overline{x} \\ (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x < \overline{x} \end{aligned} \tag{2.12}$$

Following the proof of Proposition 1, we get $\alpha_+ = \sqrt{\frac{\sigma^2}{2r}} e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x_s})} = \sqrt{\frac{\sigma^2}{2r}} e^{-1}$, $\alpha_- = \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{-\sqrt{\frac{2r+4\lambda}{2r}}}$, and $\gamma_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{2r}}}$. This pins down the closed-form solutions of the equilibrium value functions:

$$U(x) = \begin{cases} \frac{1}{2}\left(e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} + \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}\right) & \forall x < \sqrt{\frac{\sigma^2}{2r}} \\ \frac{r+\lambda}{r+2\lambda}x - \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\overline{x}-x)} & \forall x \geq \sqrt{\frac{\sigma^2}{2r}} \end{cases} \tag{2.13}$$

$$V(x) = \begin{cases} \frac{1}{2}\left(e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} - \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}\right) & \forall x < \sqrt{\frac{\sigma^2}{2r}} \\ \frac{\lambda}{r+2\lambda}x + \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\overline{x}-x)} & \forall x \geq \sqrt{\frac{\sigma^2}{2r}} \end{cases} \tag{2.14}$$

where $\overline{x} = \sqrt{\frac{\sigma^2}{2r}}$.

Relevant Outside Option

Socially Efficient Outcome A social planner with discount rate r and outside option 2ω choose at each moment whether to implement the project, wait, or take the outside option. The social planner implements the project if the surplus reaches above the threshold $\overline{x_s}$, but takes the outside option if the surplus reaches below the threshold $\underline{x_s}$. The social planner's value function has the same form as equations (2.9):

$$W_s(x) = \gamma_1 e^{\sqrt{\frac{2r}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r}{\sigma^2}}x}$$

The social efficient thresholds have to satisfy:

$$\begin{cases} W_s(\overline{x}) = \overline{x} & W_s(\underline{x}) = 2\omega \\ W_s'(\overline{x}) = 1 & W_s'(\underline{x}) = 0 \end{cases} \tag{2.15}$$

where the first two are value-matching conditions and the last two are smooth-pasting conditions. Together they imply that the socially efficient thresholds have to be

$$\overline{x_s} = \sqrt{\frac{\sigma^2}{2r} + 4\omega^2} \quad \text{and} \quad \underline{x_s} = \overline{x_s} - \sqrt{\frac{\sigma^2}{2r}} \log\left(\frac{\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}}{2\omega}\right)$$

The social value function for continuation states $\underline{x_s} < x < \overline{x_s}$ is:

$$W_s(x) = \frac{1}{2}\Big(\overline{x_s} + \sqrt{\frac{\sigma^2}{2r}}\Big)e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} + \frac{1}{2}\Big(\overline{x_s} - \sqrt{\frac{\sigma^2}{2r}}\Big)e^{-\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} \tag{2.16}$$

Equilibrium Outcome An equilibrium outcome can be described by $\overline{x}$, $\underline{x}$, $U(x)$ and $V(x)$. Given an agreement threshold $\overline{x}$, firm implement the project immediately when the surplus reaches that threshold. Thus for $x \geq \overline{x}$, the sum of their value function equals to the surplus. Also, the Responder must be indifferent between accepting the offer and rejecting the offer in equilibrium, as the Proposer must not offer anything more than necessary. The Responder's value function from rejecting the offer is characterized in equation (2.6). These imply that the firms' value functions for $x \geq \overline{x}$ must satisfy:

$$\begin{aligned} U(x) + V(x) &= x \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{2.17}$$

Combining the two, one gets:

$$V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_1 e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x \geq \overline{x} \tag{2.18}$$

for some coefficients γ_1 and γ_2. The Responder can never get more than the full surplus or agree to accept negative amount, so $0 < V(x) \leq x \; \forall x$; thus $\gamma_1 = 0$, otherwise this is violated for x large enough. So $V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x}$ for some γ_2.

In the case of $\underline{x} \geq \overline{x}$, γ_2 can be solved using the conditions $V(\underline{x}) = \omega$ and $V'(\underline{x}) = 0$. See proof of Proposition 3 for details. The Responder's utility is

$$V(x) = \frac{\lambda}{r+2\lambda}x + \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\underline{x}-x)} \quad \text{for} \quad x \geq \underline{x} \quad \text{and} \quad V(x) = \omega \quad \text{for} \quad x < \underline{x}$$

The Proposer's utility is

$$U(x) = x - V(x) \text{ for } x \geq 2\omega, \quad \text{and } U(x) = \omega \text{ for } x \leq 2\omega$$

In the case of $\underline{x} < \overline{x}$. There is a region of waiting between $\underline{x}$ and $\overline{x}$. For $\underline{x} < x < \overline{x}$, the value functions follow equations (2.8):

$$\begin{aligned} (U+V)(x) &= \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} \\ (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \end{aligned} \tag{2.19}$$

For $x < \underline{x}$, the Responder quits and both firms receive $U(x) = V(x) = \omega$.

So the closed-form expressions of equilibrium value functions are:

$$U(x) = \begin{cases} \frac{r+\lambda}{r+2\lambda}x - \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } x \geq \overline{x} \\ \frac{1}{2}\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \frac{1}{2}\beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } \underline{x} < x < \overline{x} \\ \omega & \text{for } x \leq \underline{x} \end{cases} \tag{2.20}$$

and

$$V(x) = \begin{cases} \frac{\lambda}{r+2\lambda}x + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } x \geq \overline{x} \\ \frac{1}{2}\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} - \frac{1}{2}\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} - \frac{1}{2}\beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } \underline{x} < x < \overline{x} \\ \omega & \text{for } x \leq \underline{x} \end{cases} \tag{2.21}$$

with coefficients

$$\begin{cases} \alpha_+ = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \beta_+ = \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} \\ \gamma_2 = \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} + \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} - \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases} \tag{2.22}$$

from equations (2.23) in the proof of Proposition 3, and $\overline{x}$ is the solution to the implicit function.

$$F(\overline{x}, \lambda) = \sqrt{\frac{r}{r+2\lambda}}\Big(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\Big)\Big(\frac{2\omega + \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} + \sqrt{\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2} = 0$$

2.7 Proofs

Proof of Lemma 1 We must have $U(x)+V(x) \geq x\ \forall x$, otherwise the equilibrium does not satisfy sub-game perfection. If the sum of their expected utilities are smaller than the current cake size, then there must be a profitable deviation. To see that, assume there is a state x such that $U(x)+V(x) < x$, then the Responder must accept all offers $p \geq V(x)$. Then the Proposer can propose a $p = V(x)$, and receives immediate payment of $x - V(x)$. Thus $U(x)+V(x) = x$, a contradiction.

Proof of Lemma 2 First, one can show that there must $\exists x$ s.t. $a(x, p(x)) = 1$ in intervals $(\underline{m}, \infty)$ for all $\underline{m}$. Thus firms must reach agreement in some state, and must also reach agreement again for some larger cake size. Suppose they never reach agreement, then $U(x) + V(x) = 0 < x$ for $x > 0$, a contradiction to Lemma 1. Suppose there exists a highest cake size such that firms reach agreement, denote that state as $\overline{m}$. Firms do not reach agreement for all $x > \overline{m}$. Since firms do not quit, then their value functions must be decreasing as x increases beyond $\overline{m}$. Thus for $x > \overline{m}$, we have $U(x)+V(x) < U(\overline{m})+V(\overline{m}) = \overline{m} < x$, again a contradiction to Lemma 1.

Now define $\overline{x} = inf\{x \mid a(x, p(x)) = 1\}$. By definition, we establish that $a(x, p(x)) = 0$ $\forall x < \overline{x}$.

Second, one can show that, if $a(x, p(x')) = 1$ and $a(x, p(x'')) = 1$ for some $x' < x''$, then $\nexists$ a open set $Z \subset (x', x'')$ s.t. $a(x, p(x)) = 0\ \forall x \in Z$. This means that, firms cannot disagree on an open set between two agreement states. Suppose not, then take $x_l = \sup\{x < Z \mid a(x, p(x)) = 1\}$ as the first state smaller than Z in which firms trade, and take $x_r = \inf\{x > Z \mid a(x, p(x)) = 1\}$ to be the first state bigger than Z in which firms trade. Then for any $x \in Z$, firms delay until the state reaches x_l or x_r. The probability of reaching x_l first is $\frac{x_r - x}{x_r - x_l}$. Thus without time discounting ($r = 0$), the sum of firms value functions must be $U(x) + V(x) = \frac{x_r - x}{x_r - x_l}[U(x_l) + V(x_l)] + \frac{x - x_l}{x_r - x_l}[U(x_r) + V(x_r)] = \frac{x_r - x}{x_r - x_l}x_l + \frac{x - x_l}{x_r - x_l}x_r = x$. If $r > 0$, then we must have $U(x) + V(x) < x$, a contradiction to Lemma 1.

By the utility definition, $A = \{x \mid a(x, p(x)) = 1\}$ is a closed set. Thus A must be $[\overline{x}, \infty)$.

Proof of Proposition 1 We know that $(U + V)(\overline{x}) = \overline{x}$ as firms split a cake of size $\overline{x}$. To prove $\overline{x} = \overline{x_s}$ in equilibrium, we only need to show that $(U + V)'(x) = 1$ hold, because $\overline{x_x}$ is the unique value that satisfies both conditions.

First, we know that $\lim_{x \to \overline{x}^+}(U + V)'(x) = 1$, because $(U + V)(x) = x$ for $x \geq \overline{x}$. If $\lim_{x \to \overline{x}^-}(U + V)'(x) > 1$, then $\exists x < \overline{x}$ such that $(U + V)(x) < x$. Then there's a profitable deviation at that state, a contradiction to Lemma 1. If $\lim_{x \to \overline{x}^-}(U + V)'(x) < 1$, then we must have either $\lim_{x \to \overline{x}^-} U'(x) < \lim_{x \to \overline{x}^+} U'(x)$ or $\lim_{x \to \overline{x}^-} V'(x) < \lim_{x \to \overline{x}^+}$. In either case, the firm with the convex kink at $\overline{x}$ can profitably deviate by delaying agreement for an infinitesimal dt. This follows from standard proof of the smooth-pasting condition in optimal stopping problem (see Dixit 1993 for details). This implies that $\lim_{x \to \overline{x}^-}(U + V)'(x) = 1$, and thus $(U + V)'(\overline{x}) = 1$. Combining with $(U + V)(\overline{x}) = \overline{x}$, we prove that $\overline{x} = \overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$.

Knowing the threshold, now we need to solve for how the cake is split. An accepted offer $p(x)$ must equal to $V(x)$, thus we want to solve for $V(x)$ for $x \geq \overline{x}$. Given $\lim_{x \to \overline{x}^-}(U +$

$V)'(x) = \lim_{x\to\overline{x}^+}(U+V)'(x)$ and that neither firm's value function can have a convex kink at $\overline{x}$, we can conclude that:

$$\begin{cases} \lim_{x\to\overline{x}^+} V(x) = \lim_{x\to\overline{x}^-} V(x) \\ \lim_{x\to\overline{x}^+} V'(x) = \lim_{x\to\overline{x}^-} V'(x) \end{cases}$$

Plugging in equations (2.11), (2.12), and $\alpha_+ = \sqrt{\frac{\sigma^2}{2r}}e^{-1}$, we get:

$$\begin{cases} \frac{1}{2}\left(\sqrt{\frac{\sigma^2}{2r}}e^{-1}e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}}\right) - \frac{\alpha_-}{2}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} & = \frac{\lambda}{r+2\lambda}\overline{x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \frac{1}{2} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\frac{\alpha_-}{2}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} & = \frac{\lambda}{r+2\lambda} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

Solving the system of equations gives:

$$\begin{cases} \alpha_- & = \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{-\sqrt{\frac{2r+4\lambda}{2r}}} \\ \gamma_2 & = \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{\sqrt{\frac{2r+4\lambda}{2r}}} \end{cases}$$

Plugging in coefficients α_+, α_-, and γ_2 into equations (2.11) and (2.12) gives value functions $U(x)$ and $V(x)$. Particularly, for $x \geq \overline{x}$, $V(x) = \frac{\lambda}{r+2\lambda}x + \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{\sqrt{\frac{2r+4\lambda}{2r}}(\overline{x}-x)}$, which shows that the Responder receives strictly more than $\frac{\lambda}{r+2\lambda}x$.

Also $\alpha_- = \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{-\sqrt{\frac{2r+4\lambda}{2r}}} > 0$, so $(U-V)(x) = \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}}$ must also be positive, which concludes the proof.

Proof of Corollary 2 Because $\overline{x} = \overline{x_s}$, the equilibrium outcome must be socially efficient. Taking derivatives of $U(x)$ and $V(x)$ from equations (2.13) and (2.14) with respect to λ proves the rest. Particularly, for $x < \overline{x}$, the terms $\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)$, $\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)$, and $e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}$ all decrease in λ, thus $\frac{dU(x)}{d\lambda} < 0$ and $\frac{dV(x)}{d\lambda} > 0$.

Proof of Lemma 3 The proof is similar to the proof of Lemma 2.

First, one can show that there must $\exists x$ s.t. $a(x, p(x)) = 1$ in intervals $(\underline{m}, \infty)$ for all $\underline{m}$. Thus firms must reach agreement in some state, and must also reach agreement again for some larger state. Suppose they never reach agreement, then $U(x) + V(x) = 0 < x$ for $x > 0$, a contradiction to Lemma 1. Suppose there exists a highest cake size such that firms reach agreement, denote that state as $\overline{m}$. Firms do not reach agreement for all $x > \overline{m}$. Then their value functions must be decreasing as x increases beyond $\overline{m}$. Then we have $U(x) + V(x) \leq \overline{m} < x$ for some $x > \overline{m}$, again a contradiction to Lemma 1.

Now define $\overline{x} = inf\{x \mid a(x, p(x)) = 1\}$. By definition, we establish that $a(x, p(x)) = 0$ $\forall x < \overline{x}$.

Second, one can show that, if $a(x, p(x')) = 1$ and $a(x, p(x'')) = 1$ for some $x' < x''$, then $\nexists$ a open set $Z \subset (x', x'')$ s.t. $a(x, p(x)) = 0$ $\forall x \in Z$. This means that, firms cannot disagree on an open set between two agreement states. Suppose not, then take $x_l = \sup\{x < Z \mid a(x, p(x)) = 1\}$ as the first state smaller than Z in which firms trade, and take $x_r = \inf\{x > Z \mid a(x, p(x)) = 1\}$ to be the first state bigger than Z in which firms trade. Because firms trade at x_l, we must have $x_l \geq 2\omega$. Then $Z > 2\omega$. The Responder must not quit in any state in Z. Otherwise, the Proposer can offer ω to the Responder at that state and the Responder accepts, a contradiction to the assumption that firms do not trade in Z. Then for any $x \in Z$, firms delay until the state reaches x_l or x_r. The probability of reaching x_l first is $\frac{x_r - x}{x_r - x_l}$. Thus without time discounting ($r = 0$), the sum of firms value functions must be

$$U(x)+V(x) = \frac{x_r - x}{x_r - x_l}[U(x_l)+V(x_l)] + \frac{x - x_l}{x_r - x_l}[U(x_r)+V(x_r)] = \frac{x_r - x}{x_r - x_l}x_l + \frac{x - x_l}{x_r - x_l}x_r = x$$

If $r > 0$, then we must have $U(x) + V(x) < x$, a contradiction to Lemma 1.

By the utility definition, $A = \{x \mid a(x, p(x)) = 1\}$ is a closed set. Thus A must be $[\overline{x}, \infty)$.

Next we need to prove that $\overline{x} \leq \overline{x}_s$. Suppose instead $\overline{x} > \overline{x_s}$, then for $x \in (\overline{x_s}, \overline{x})$, we have $U(x) + V(x) \geq x$ by Lemma 1. Also, the total payoff in equilibrium cannot exceed the socially efficient payoff, which is x for $x > \overline{x_s}$. Thus we must have $U(x) + V(x) = x$ for $x \in (\overline{x_s}, \overline{x})$. However, given the form of $U(x)$ and $V(x)$ in equations (2.8), no parameters can satisfy $U(x) + V(x) = x$ in an open interval, a contradiction.

Proof of Lemma 4 First, one can show that there must $\exists x$ s.t. $q(x) = 1$ in intervals $(-\infty, \overline{m})$ for all $\overline{m}$. Thus the Responder must quit in some state, and must also quit too for some even lower state. Suppose the Responder never quits, then $V(x)$ approaches 0 as $x \to -\infty$. Then the Responder can profitably deviate by quitting. Suppose there exists a lowest state such that the Responder quits. Denote that state as $\underline{m}$, so the Responder does not quit for all $x < \underline{m}$. Then again $V(x)$ approaches 0 as $x \to -\infty$, for which the Responder can deviate by quitting.

Now define $\underline{x} = sup\{x \mid q(x) = 1\}$. By definition, we establish that $q(x) = 0\ \forall x > \underline{x}$.

Second, one can show that, if $q(x') = 1$ and $q(x'') = 1$ for some $x' < x''$, then $\nexists$ a open set $Z \subset (x', x'')$ s.t. $q(x) = 0\ \forall x \in Z$. This means that, the Responder cannot choose to continue on any open set between two quitting states. Suppose not, then take $x_l = \sup\{x < Z \mid q(x) = 1\}$ as the first state smaller than Z in which the Responder quits, and take $x_r = \inf\{x > Z \mid q(x) = 1\}$ to be the first state bigger than Z in which the Responder quits. The Responder gets utility of ω at x_l and x_r. If an agreement is reached in this interval, then the Proposer must make an offer such that the Repsonder is indifferent between agreeing or not. Thus for $x_l < x < x_r$, the Responder must gets utility smaller than ω due to discounting. The Responder can profitably deviate by quitting in this interval.

By the utility definition, $Q = \{x \mid q(x)) = 1\}$ is a closed set. Thus Q must be of the form $[-\infty, \underline{x})$.

Third, we need to prove that $\underline{x} \geq \underline{x}_s$ from Section 4.1 and Section (2.6). Suppose instead $\underline{x} < \underline{x_s}$. For states $x < \underline{x}$, the socially efficient outcome is quitting, and the equilibrium outcome cannot have a higher payoff than the socially efficient payoff. Thus $U(x) + V(x) \leq 2\omega$. Because the Responder does not quit in the region $(\underline{x}, \underline{x_s})$, we must have $V(x) \geq \omega$

in this region. This implies $U(x) \leq \omega$ in $(\underline{x}, \underline{x_s})$. By the form of $U(x)$ and $V(x)$ from equations (2.2), they cannot be flat functions. Then there must exists some $x' > \underline{x}$ such that $U(x') < V(x')$ and $U'(x') < V'(x')$. This implies that $(U-V)(x') < 0$ and $(U-V)'(x') < 0$.

Again by equations (2.2), if $(U-V)(x') < 0$ then $(U-V)''(x') < 0$, which implies that $(U-V)'(x) < 0$ and $(U-V)(x) < 0$ for $\underline{x} < x < \overline{x}$. Thus we must have $U'_-(\overline{x}) < V'_-(\overline{x})$ and $U(\overline{x}) < V(\overline{x})$. Because $(U+V)(x) = x$ for $x \geq \overline{x}$, we must have $(U+V)'_-(\overline{x}) \leq 1$, otherwise there exists some $x < \overline{x}$ such that $(U+V)(x) < x$, a contradiction to Lemma 1. Thus $U'_-(\overline{x}) < \frac{1}{2}$.

Because $U(\overline{x}) < V(\overline{x})$, γ_2 from equation (2.18) must be positive. Then by equations (2.17) and (2.18), we have $U'_+(\overline{x}) > V'_+(\overline{x})$. Because $(U+V)(x) = x$ for $x \geq \overline{x}$, we can conclude that $U'_+(x) > \frac{1}{2}$.

Because $U'_-(\overline{x}) < \frac{1}{2}$ and $U'_+(x) > \frac{1}{2}$, there is a convex kink on $U(x)$ at $\overline{x}$. Then the Proposer can profitably deviate by delaying the trade for a small dt. Thus we cannot have $\underline{x} < \underline{x_s}$ in equilibrium, concluding the proof.

Proof of Proposition 3 We solve for the equilibrium outcome in two cases.

Case 1: $\underline{x} \geq \overline{x}$**:**

In this case, the game always end immediately. So the firms must trade for $x_0 > 2\omega$. Thus $\overline{x} = 2\omega$. On equilibrium path, firms trade immediately for $x_0 \geq 2\omega$, and the Responder quit immediately for $x_0 < 2\omega$. We know that $V(x) = \omega$ for $x \leq \underline{x}$ is the Responder's equilibrium payoff when it quits. Then we have $V'_-(\underline{x}) = 0$. In order for the quitting threshold to be optimal, we need to have $V'_+(\underline{x}) = 0$, otherwise the Responder can profitably deviate by delaying quitting for time dt. Thus we have

$$V(\underline{x}) = \omega \quad \textbf{and} \quad V'(\underline{x}) = 0$$

The form of $V(x)$ is given in equation (2.18) in Section (2.6). Plugging in equation (2.18) with $\gamma_1 = 0$, we get:

$$\begin{cases} \frac{\lambda}{r+2\lambda}\underline{x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = \omega \\ \frac{\lambda}{r+2\lambda} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = 0 \end{cases}$$

Solving the two equations, we find that the solution exists if and only if $\lambda \leq \underline{\lambda} = \frac{4r^2\omega^2 + \sqrt{16r^4\omega^4 + 8\sigma^2 r^3\omega^2}}{2\sigma^2}$. The solutions are:

$$\begin{cases} \underline{x} = \frac{r+2\lambda}{\lambda}\omega - \sqrt{\frac{\sigma^2}{2r+4\lambda}} \\ \gamma_2 = \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} \end{cases}$$

The Responder's utility is $V(x) = \frac{\lambda}{r+2\lambda}x + \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\underline{x}-x)}$ for $x \geq \underline{x}$ and $V(x) = \omega$ for $x < \underline{x}$. The Proposer's utility is $U(x) = x - V(x)$ for $x \geq \underline{x}$, $U(x) = x - \omega$ for $2\omega < x < \underline{x}$, and $U(x) = \omega$ for $x \leq 2\omega$.

Case 2: $\underline{x} < \overline{x}$**:**

An equilibrium outcome has to satisfy the following four value-matching conditions:

$$\begin{cases} (U+V)(\underline{x}) = 2\omega \\ (U-V)(\underline{x}) = 0 \\ (U+V)(\overline{x}) = \overline{x} \\ (U-V)(\overline{x}) = \overline{x} - 2V(\overline{x}) = \frac{r}{r+2\lambda}\underline{x} - 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

where $V(\overline{x})$ comes from equation (2.18). An equilibrium outcome also has to satisfy the following three conditions:

1. $V'(\underline{x}) = 0$ or $(U+V)'(\underline{x}) = (U-V)'(\underline{x})$ for optimality of the quitting threshold.
2. $(U+V)'(\overline{x}) = 1$. We know that $(U+V)'_+(\overline{x}) = 1$ since $(U+V)(x) = x$ for $x > \overline{x}$. If $(U+V)'_-(\overline{x}) > 1$, then one of the firm has a convex kink on his/her value function at $\overline{x}$, and can profitably deviate and delaying trade for dt. If $(U+V)'_-(\overline{x}) < 1$, then $(U+V)(x) < x$ for some $x < \overline{x}$, a contradiction to Lemma 1.
3. $(U-V)'_-(\overline{x}) = (U-V)'_+(\overline{x})$. We know that $(U+V)'_-(\overline{x}) = (U+V)'_+(\overline{x}) = 1$. Also, there cannot be convex kink for either $U(x)$ or $V(x)$ at $\overline{x}$, otherwise a profitable deviation exists. Thus $U'_-(\overline{x}) = U'_+(\overline{x})$ and $V'_-(\overline{x}) = V'_+(\overline{x})$.

Using the forms of $(U+V)(x)$ and $(U-V)(x)$ from equations (2.8) for the region $\underline{x} < x < \overline{x}$, the above seven conditions constitute the following system of equations:

$$\begin{cases} \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\underline{x}} = 2\omega \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = 0 \\ \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} = \overline{x} \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} = \frac{r}{r+2\lambda}\overline{x} - 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \sqrt{\frac{2r}{\sigma^2}}(\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\underline{x}}) = \sqrt{\frac{2r+4\lambda}{\sigma^2}}(\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}}) \\ \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} - \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} = \sqrt{\frac{\sigma^2}{2r}} \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} - \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} = \frac{r}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} + 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

Solving the system of equations, we get the following implicit equation of $\overline{x}$ and λ:

$$F(\overline{x},\lambda) = \sqrt{\frac{r}{r+2\lambda}}\Big(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\Big)\Big(\frac{2\omega + \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} + \sqrt{\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2} = 0$$

By Lemma 3, $2\omega \le \overline{x} \le \overline{x_s}$. Here I use a Lemma that I prove at the end of this Section.

Lemma 10. *The equation $F(\overline{x},\lambda)$ has a unique solution $\overline{x}(\lambda)$ in the range of $2\omega \le \overline{x} \le \overline{x_s} = \sqrt{\frac{\sigma^2}{2r}+4\omega^2}$. The solution $\overline{x}(\lambda)$ is increasing in λ for $\lambda \ge \underline{\lambda}$, $\overline{x}(\underline{\lambda}) = 2\omega$, and $\overline{x} \to \overline{x_s}$ as $\lambda \to \infty$.*

By Lemma 5, the equilibrium outcome is unique. If $\lambda \le \underline{\lambda}$, then the equilibrium outcome is in case 1. In equilibrium, $\underline{x} \ge \overline{x}$, and the game ends immediately. If $\lambda > \underline{\lambda}$, then the equilibrium outcome is in case 2. There exists a unique $\overline{x}$ as a function of λ in the range of $(2\omega, \overline{x_s})$. We can solve the rest of the parameters as functions of λ and $\overline{x}$. The solutions are:

$$\begin{cases} \alpha_+ = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \beta_+ = \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \underline{x} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}} \log \frac{2\omega - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} - \sqrt{\frac{\sigma^2}{2r}}} \\ \alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} \\ \gamma_2 = \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} + \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} - \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases} \tag{2.23}$$

To prove Proposition 3(1): Note that case 1 only exists for $\lambda \le \underline{\lambda}$, and case 2 only exists for $\lambda > \underline{\lambda}$. So for each λ, there is a unique equilibrium outcome. From Lemma 5 we get $\frac{d\overline{x}}{d\lambda} > 0$ and $\overline{x} \to \overline{x_s}$ as $\lambda \to \infty$.

Next we need to prove that $(U-V)(x) \ge 0$ for all x. Suppose $\lambda \le \underline{\lambda}$. Then $(U-V)(x) = 0$ for $x \le 2\omega$, $(U-V)(x) = x - 2\omega$ for $x \in (\omega, \underline{x})$, and $(U-V)(x) = \frac{r}{r+2\lambda}x - \frac{2\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\underline{x}-x)}$ for $x \ge \underline{x}$. Then $\frac{d(U-V)}{dx} \ge 0$, so $U(x) \ge V(x)$ for all x.

Suppose $\lambda > \underline{\lambda}$. Then $(U-V)(x) = 0$ for $x \le \underline{x}$. For $x \in (\underline{x}, \overline{x})$,

$$\begin{aligned} (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \beta_- \sqrt{\frac{2r+4\lambda}{\sigma^2}}x \\ &= \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})} - \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-x-\overline{x})} \\ &= \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})\left[e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})} - e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-x-\overline{x})}\right] \\ &\ge 0 \end{aligned}$$

because $x-\overline{x} > 2\underline{x}-x-\overline{x}$. For $x \geq \overline{x}$,

$$\begin{aligned}(U-V)(x) &= \frac{r}{r+2\lambda}x - 2\gamma_2 e^{\sqrt{-\frac{2r+4\lambda}{\sigma^2}}x} \\ &= \frac{r}{r+2\lambda}x - \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-x-\overline{x})} - \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}-\sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\overline{x}-x)} \\ &\geq \frac{r}{r+2\lambda}x - \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}) - \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}-\sqrt{\frac{\sigma^2}{2r+4\lambda}}) \\ &= 0\end{aligned}$$

Thus the Proposer has a weakly higher utility in all states x and for all λ.

For Proposition 3(2)(b): The only thing left to prove is that $\frac{d\underline{x}}{d\lambda} < 0$ for $\lambda > \underline{\lambda}$. Because $\frac{d\overline{x}}{d\lambda} > 0$, we just need to prove $\frac{d\underline{x}}{d\overline{x}} < 0$. From equations (2.23), we have: $\underline{x} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}} \log \frac{2\omega - \sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}-\sqrt{\frac{\sigma^2}{2r}}} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}} \log \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}$. Then taking derivative of $\underline{x}$ with respect to $\overline{x}$, we get:

$$\frac{d\overline{x}}{d\underline{x}} = \frac{\overline{x}\big(2\omega\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}} - \overline{x}(\sqrt{\frac{\sigma^2}{2r}}+\overline{x}) + 4\omega^2\big)}{(\sqrt{\frac{\sigma^2}{2r}}+\overline{x})\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}\big(2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}\big)}$$

Given that the denominator is strictly positive, the sign of $\frac{d\overline{x}}{d\underline{x}}$ depends on the sign of $2\omega\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}} - \overline{x}(\sqrt{\frac{\sigma^2}{2r}}+\overline{x}) + 4\omega^2 = 2\omega\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}} - \overline{x}\sqrt{\frac{\sigma^2}{2r}} + 4\omega^2 - \overline{x}^2$. Given that $\overline{x} > 2\omega$, we have $\sqrt{\frac{\sigma^2}{2r}} > \sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}$, thus $2\omega\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}} - \overline{x}\sqrt{\frac{\sigma^2}{2r}} + 4\omega^2 - \overline{x}^2 < 0$.

One can check that $\underline{x} = \underline{x_s}$ when $\overline{x} = \overline{x_s}$. So $\underline{x}$ approaches $\underline{x_s}$ as $\lambda \to \infty$.

For Proposition 3(2)(c), note that the ex-ante probability of alliance is only non-trivial for $x_0 \in (\underline{x}, \overline{x})$, so we only need to focus on this case. Because x_t follows a Brownian motion, the ex-ante probability of reaching $\overline{x}$ before $\underline{x}$ can be written as

$$\mu = \frac{x_0 - \underline{x}}{\overline{x}_0\underline{x}}$$

. Then

$$\frac{d\mu}{d\lambda} = \frac{(\overline{x}-\underline{x})(-\frac{d\underline{x}}{d\lambda}) - (x_0-\underline{x})(\frac{d\overline{x}}{d\lambda}-\frac{d\underline{x}}{d\lambda})}{(\overline{x}-\underline{x})^2} = \frac{(\overline{x}-x_0)(-\frac{d\underline{x}}{d\lambda}) - (x_0-\underline{x})\frac{d\overline{x}}{d\lambda}}{(\overline{x}-\underline{x})^2}$$

Let $\tilde{x}(\lambda) = \frac{\overline{x}(-\frac{d\underline{x}}{d\lambda})+\underline{x}\frac{d\overline{x}}{d\lambda}}{\frac{d\overline{x}}{d\lambda}-\frac{d\underline{x}}{d\lambda})-(x_0-\underline{x})}$. Since $\frac{d\overline{x}}{d\lambda} > 0$ and $-\frac{d\underline{x}}{d\lambda} > 0$, then $\frac{d\mu}{d\lambda} > 0$ for $x_0 < \tilde{x}(\lambda)$ and $\frac{d\mu}{d\lambda} < 0$ for $x_0 > \tilde{x}(\lambda)$.

Proof of Corollary 4 If $\lambda \leq \underline{\lambda}$, then game ends immediately, so $(U+V)(x) = \max\{x_0, 2\omega\}$. Ex-ante welfare is not affected by λ.

Suppose $\lambda > \underline{\lambda}$. Plugging coefficients from equations (2.23) into $(U{+}V)(x)$ from equations (2.8), we get:

$$(U+V)(x) = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} + \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)}$$

for $\underline{x} < x < \overline{x}$. The derivative with respect to λ is:

$$\frac{d(U+V)(x)}{d\lambda} = \frac{d(U+V)(x)}{d\overline{x}}\frac{d\overline{x}}{d\lambda} = \frac{1}{2}\sqrt{\frac{2r}{\sigma^2}}\overline{x}(e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)} - e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)})\frac{d\overline{x}}{d\lambda}$$

Because $\frac{1}{2}\sqrt{\frac{2r}{\sigma^2}}\overline{x}(e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)} - e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)}) > 0$ and $\frac{d\overline{x}}{d\lambda} > 0$ from Lemma 5, we can conclude that $\frac{d(U+V)(x)}{d\lambda} > 0$. Thus if $x_0 \in (\underline{x}, \overline{x})$, then the ex-ante welfare is strictly increasing in λ for $\lambda < \underline{\lambda}$. If $x_0 \notin (\underline{x}, \overline{x})$, then the game ends immediately and the ex-ante welfare is not affected by λ.

As $\lambda \to \infty$, $\overline{x} \to \overline{x_s}$ by Lemma 5. As $\overline{x} \to \overline{x_s}$, we have $\underline{x} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}} \log \frac{2\omega - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} - \sqrt{\frac{\sigma^2}{2r}}} \to \underline{x_s} = \overline{x_s} - \sqrt{\frac{\sigma^2}{2r}} \log\left(\frac{\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}}{2\omega}\right)$. Also $(U+V)(x) \to (U+V)_s(x)$ from equation (2.16) in Section (2.6). So the total welfare approaches the socially efficient welfare. Also, both $\alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}}$ and $\beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})}$ approach 0 as $\lambda \to \infty$, so $(U-V)(x) \to 0$. Thus $(U+V)(x_0) = (U+V)_s(x_0)$ and $(U-V)(x_0) = 0$ in the limit, so each firm approaches half of the socially efficient total welfare.

Proof of Proposition 5 First we prove Proposition 5(1). Let $I = (\underline{x_s}, 2\omega)$, and $\tilde{\lambda} = \underline{\lambda}$. If $\lambda \leq \underline{\lambda}$, then $\underline{x} \geq \overline{x}$ by Proposition 3, so the game ends immediately. If $x_0 \in I$, then the Responder quits at time 0 and both firms get outside option of ω. Again by Proposition 3, as $\lambda \to \infty$, $\underline{x} \to \underline{x_s}$ and $(U+V)(x_0) \to (U+V)_s(x_0)$. So there exists a $\lambda' > \underline{\lambda}$ such that $\underline{x}(\lambda') < x_0$ and $(U+V)(x_0) > 2\omega$, because $(U+V)_s(x_0) > 2\omega$. So the total ex-ante welfare is higher under λ'. The Responder can never be worse off than ω, and the Proposer has weakly higher utility than the Responder. Thus λ' Pareto dominates λ.

For Proposition 5(2): For any λ, $\underline{x}(\lambda) > \underline{x_s}$. Take any x_0 in the interval $(\underline{x_s}, \underline{x})$. For such x_0, the ex-ante utilities for both firms under λ is ω. Then by Proposition 3, as $\lambda \to \infty$, $\underline{x} \to \underline{x_s}$ and $(U+V)(x_0) \to (U+V)_s(x_0)$. So there exists a $\lambda' > \lambda$ such that $\underline{x}(\lambda') < x_0$ and $(U+V)(x_0) > 2\omega$. This must be a Pareto improvement, similarly to the argument above.

For Proposition 5(3): If $x_0 < \underline{x_s}$ or $x_0 > \overline{x_s}$, then it is socially optimal to stop immediately. The equilibrium outcome is socially optimal regardless of λ, so total welfare does change. So no λ can Pareto dominates another.

Lemma 5

$$F(\overline{x},\lambda)=\sqrt{\frac{r}{r+2\lambda}}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})(\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}})^{\sqrt{\frac{r+2\lambda}{r}}}-\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}=0$$

has a unique solution $\overline{x}(\lambda)$ in the range of $2\omega \leq \overline{x} \leq \overline{x_s} = \sqrt{\frac{\sigma^2}{2r}+4\omega^2}$. The solution $\overline{x}(\lambda)$ is increasing in λ for $\lambda \geq \underline{\lambda}$, $\overline{x}(\underline{\lambda}) = 2\omega$, and $\overline{x} \to \overline{x_s}$ as $\lambda \to \infty$.

Proof. First, $\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}} \leq 1$ because $\overline{x} \geq 2\omega$ by Lemma 3. Thus $\frac{\partial F}{\partial \lambda} < 0$ because every term decreases in λ.

By implicit function theorem, to prove $\frac{\partial \overline{x}}{\partial \lambda} > 0$, we need to prove that $\frac{\partial F}{\partial \overline{x}} > 0$ for $2\omega \leq \overline{x} \leq \overline{x_s}$. We prove by the following steps. We show that $\frac{\partial F}{\partial \overline{x}} = M(\overline{x}) * E(\overline{x})$ for some M and E. The term $M(\overline{x})$ is always positive. The term $E(\overline{x})$ is positive at $\overline{x} = 2\omega$, and $\frac{\partial E}{\partial \overline{x}} > 0$ for $\overline{x} > 2\omega$, thus concluding that $\frac{\partial F}{\partial \overline{x}} > 0$.

Let $M = \left(\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\frac{\sigma^2}{2r}}\right)^{\sqrt{\frac{r+2\lambda}{r}}}$. Then $M > 0$ for $\overline{x} < \overline{x_s} = \sqrt{4\omega^2+\frac{\sigma^2}{2r}}$, then:

$$\begin{aligned}
\frac{\partial F}{\partial \overline{x}} &= \frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}} - \frac{\overline{x}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}\frac{1}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}M - \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}M + M \\
&= M * \Big[\frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}(\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}})^{\sqrt{\frac{r+2\lambda}{r}}} \\
&\quad - \frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}} - \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}+1\Big] \\
&= M * E(\overline{x})
\end{aligned} \tag{2.24}$$

At $\overline{x} = 2\omega$, $E(\overline{x}) = \left(\frac{2\omega}{\sqrt{\frac{\sigma^2}{2r}+1}}\right)\left(\frac{2\omega+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{\frac{\sigma^2}{2r}}}\right) > 0$. We can write $E(\overline{x})$ as $E = \frac{\overline{x}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} * B -$

$\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}+1$, where

$$
\begin{aligned}
B &= \Big(\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} \\
&> \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} - \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} \\
&= \frac{\sqrt{\frac{\sigma^2}{2r}}-\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} > 0
\end{aligned} \tag{2.25}
$$

with $\frac{dB}{d\overline{x}} > 0$. Then taking derivative of E with respect to $\overline{x}$, we get:

$$
\frac{dE}{d\overline{x}} = \frac{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}+\overline{x}^2/\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}^2} * B + \frac{\overline{x}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}\frac{dB}{d\overline{x}} - \frac{\sqrt{\frac{\sigma^2}{2r}}-\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{(\overline{x}+\sqrt{\frac{\sigma^2}{2r}})^2} \tag{2.26}
$$

Then $\frac{dE}{d\overline{x}}$ is positive because each of the terms is positive for $\overline{x} \in (2\omega, \overline{x_s})$. Because $E(\overline{x} = 2\omega) > 0$, this implies $E(x) > 0$ for all $\overline{x} \geq 2\omega$. Thus $\frac{dF}{d\overline{x}} = M * E > 0$ for $\overline{x} \in (2\omega, \overline{x_s})$. Combining with the fact that $\frac{dF}{d\lambda} < 0$, by implicit function theorem, there exists a function $\overline{x}(\lambda)$ in that range and $\frac{d\overline{x}}{d\lambda} > 0$.

□

Chapter 3

Label Informativeness and Price Sensitivity in the Cigarettes Market

3.1 Introduction

[1] Cigarette products differentiate in their strengths. Most cigarettes sold on the market can be categorized as regular, light, or ultralight. Most of the brands offer all 3 strengths. In 2009, Congress passed the Tobacco Control Act (henceforth referred to as "TCA"). The bill bans tobacco companies from using strength descriptors on any marketing material or packaging, effectively blocking any explicit communication of the strength of a product. However, firms are still allowed to sell cigarettes of different strengths. Companies adapted by changing product labels to less informative color codes. While the original goal of the law is to reduce consumer misconception about health risks of "light" cigarettes, an unintended consequence is that it obfuscates information on product attribute. This paper seeks to address the effect of such change in label informativeness on consumer choice.

One area that cigarettes of different strengths differ is their ventilation level. A ventilated cigarette mixes smoke with air as smokers inhale, reducing the amount of tar and nicotine intake in each puff. Cigarettes with no or little ventilation are referred as "regular" and those with more ventilation are referred to as "light/low/mild" or "ultra-light" depending on the extant of ventilation. Exactly which category a product falls into is determined through machine measurement defined by CDC (CDC Fact Sheets Low Yield Cigarettes). For years, tobacco companies marketed these products as healthier substitutes to regular cigarettes,

[1]Researcher(s) own analyses calculated (or derived) based in part on data from The Nielsen Company (US), LLC and marketing databases provided through the Nielsen Datasets at the Kilts Center for Marketing Data Center at The University of Chicago Booth School of Business. The conclusions drawn from the Nielsen data are those of the researcher(s) and do not reflect the views of Nielsen. Nielsen is not responsible for, had no role in, and was not involved in analyzing and preparing the results reported herein. The conclusions drawn from the Nielsen data are those of the researchers and do not reflect the views of Nielsen. Nielsen is not responsible for, had no role in, and was not involved in analyzing and preparing the results reported herein.

and research confirmed that a large portion of consumers indeed believed light cigarettes to be less harmful (Borland et al., 2008; Yong et al. 2011). However, medical research find that smokers who switched from regular cigarettes to light cigarettes tend to compensate the lower nicotine intake by smoking more frequently or inhaling deeper, and as a result, there is no evidence that switching to light cigarettes actually reduce health risk (Scherer, 1999).

In 2009, congress passed Tobacco Control Act. Among other measures, it bans the usage of strength descriptions such as "regular", "light", or "ultra-light" on any marketing or packaging materials. However, the law does not actually prohibit tobacco companies from manufacturing and selling these cigarettes. Tobacco companies responded by re-labeling product lines with color codes. For example, "Camel Light" was re-branded to "Camel Blue", while "Marlboro Light" became "Marlboro Gold". Medical studies show that Tobacco Control Act achieved very little success in its original goal of correcting consumers beliefs on the health benefits of light cigarettes. But as a side effect, the law obfuscates product information, as consumers are less able to identify product characteristics from its name and packaging. This can prevent consumers from switching to less familiar brands due to additional costs and risks, making consumers less responsive to price changes in competing brands. Given that price discounts account for over 90% of marketing expenditure in cigarette industry, this change could impact competitive nature of the industry and firms marketing strategy. This paper seeks to answer the following questions. How do consumers shopping behaviors change following the Tobacco Control Act? What is the most likely mechanism that caused such change in behavior? Do tobacco companies change their pricing and promotion strategies as a result of such change? Does the effect of obfuscation disappear over time?

Nielsen's retailer scanner data and household panel data from 2007 to 2012 are used to address these questions. Data shows an increase in cigarette price and an increase in price dispersion among strengths after the law is passed and after the compliance deadline. Overall consumption trends down for the time period, but the trend started well before passing of TCA. I see no noticeable change in market shares or overall cigarette consumption as a result of the law. This suggests that Tobacco Control Act had minimal impact in discouraging smokers from smoking, resonating previous findings from medical researchers on the health effect of TCA. However, I find a sharp reduction in cross-purchasing frequency after implementation of descriptor ban. This that consumers were less willing to buy brands outside of their most favorite choice after the label change.

To further explore the effect of TCA on brand choice, discrete-choice models are run for the four year period from 2008-2011, which includes 1.5 years before passing of the law, 1 year from passing of the law to deadline of compliance, and 1.5 years after compliance. In order to focus solely on the laws effect on brand choice, I exclude smokers who start or quit during the time period. Firms could comply anytime from June 2009 to June 2010, and the exact time that each firm complied is unknown. This prohibits one from conducting a regression discontinuity study. Instead, relying on the size of Nielsens data, I take a less parametric approach in estimating the effect of TCA. The data is divided by quarter, and regressions are run separately for each quarter. Then I examine the time series of estimates to see the effect of Tobacco Control Act.

I find that the effect of price on consumer choice dropped significantly as Tobacco Control Act went into effect, and remained low for the remaining time period. On face value, a decrease in the size of price coefficient suggests that either consumers care less about money or care more about the differences between products. Assuming that the underlying preference for money did not change, the result implies that consumers perceive products to be more differentiated after the labelling change. However, before such conclusion can be reached, I consider four alternative mechanisms that could cause this change in behavior. An observed decrease in price sensitivity can be due by an increase in preference heterogeneity, an increase in inertia or switching cost, an increase in the level of price endogeneity, or a decrease in the average size of consideration sets. These effects, if not specified in the choice model, can bias the price coefficient in the observed direction. I find that the drop price sensitivity is robust in models that account for these four factors.

This paper has three main contributions. First, it provides empirical evidence on how product packaging or labelling affect consumers choice. The paper shows that, decreasing the informativeness of product labels can increase the perceived level of differentiation among products. Second, previous literature on obfuscation focus mostly of the obfuscation of price information, while this paper study the effect of obfuscating product characteristics. Finally, this paper provides policy implications by documenting the unintended consequence of Tobacco Control Act on consumer behavior. Many countries are considering adopting a similar ban on strength descriptors. Policy makers may want to its effect on consumer choice when drafting regulations.

The remaining sections are organized as follows. After literature review, Section (3.2) provides background on the industry and Tobacco Control Act. Section (3.3) describes the data and presents descriptive evidence. Section (3.4) estimates choice models and examines the effect of TCA on consumer behavior. Section (3.5) concludes the paper.

Literature Review

Coefficients in a discrete-choice model can be biased when the model is mis-specified. Thus a change in price elasticity can be due to a change in the level of bias rather than an actual change to consumer's price sensitivity. Villas-Boas and Winer (1999) show that endogeneity bias price coefficient towards zero. Chiang, Chib, and Narasimhan (1999) show that both price and state dependence can be biased if we ignore consideration set effect. Chintagunta, Kyriazidou, and Perktold (1998) shows that price coefficient can be biased if one does not account for preference heterogeneity. Abramson et. al. (2000) use simulation to test how each of the above mentioned source bias price coefficients, and find that responsiveness to price is underestimated if state dependence is ignored. This paper accounts for the above factors.

This paper also relates to the literature on consideration set. Mehta, Rajiv, and Srinivasan (2003) structurally estimated consumer search cost and consideration set. However, their methods suffer from curse of dimensionality when the number of alternative products increases. Van Nierop, et. al. (2010) proposed a different model that reduced the dimensionality of the problem, and used experimental data to confirm that consideration set can be reliably estimated. They model consumer choice as a two-stage process. In the first stage, consumers choose to consider a product if its consideration utility crosses a threshold. This is conceptually similar to the reduced form model of Bronnenberg and VanHonacker (1996). Abaluck and Adams (2018) show that consideration probabilities can be identified without auxiliary data or exclusion of instruments from first or second stage. This paper follows Abaluck and Adams (2018) in estimating consideration set.

Research on obfuscation and search cost has shown that firms can increase market power when product information becomes harder to comprehend. Stahl (1989) shows that existence of search cost for some consumers lead to higher price, more promotions, and higher profit for the firm. Wilson (2010) and Ellison and Wolitzky (2011) shows that under different competitive structures, there can be equilibrium in which firms endogenously choose to increase search cost for consumers. However, there are two major differences between our paper and the papers above. Whereas prior literature focuses on obfuscation on price information, this paper analyzes the effect of obfuscation on product characteristics. Also the source of obfuscation in this paper is regulation instead of endogenous decision. Ellison and Ellison (2009) provide rare empirical evidence on effect of product information obfuscation. They find that internet retailers sometimes do not display all product information, and such obfuscation leads to lower aggregate price sensitivity.

An increases in differentiation can be either perceptual or informative. Ampuero and Vila (2006) find that packaging color influence consumers' perception of product positioning. Wanke, Herrmann, and Schaffner (2006) find that consumers' perceptions of hypothetical hotels are influenced by the hotels' names, even when they are informed that the hotels are otherwise the same. The context of this paper also resembles informational differentiation of experience goods. Products can appear more differentiated in the eyes of consumers "due to consumers imperfect knowledge of the products quality or fit with their preferences." (Tirole 1988) While a Marlboro Light user can easily transition to Marlboro Gold given its same look, a substitute such as Camel Light becomes harder to recognize given its label change to Camel Blue. This asymmetry in a consumer's knowledge of brands can make consumers less responsive to price promotions.

3.2 Industry Background and Tobacco Control Act

The cigarette industry in the U.S. is large but declining. Cigarette consumption has declined steadily from over 600 billion units in early 1980s to 268 billion units in 2012. Most cigarette brands sold in the U.S. are owned by one of five tobacco companies: Philip Morris (a subsidiary of Altria), Reynolds, Lorillard, Commonwealth, and Vector. Among them, Philip Morris and Reynolds together own nine of top 10 brands, which accounted for over 80% of market share in 2010. The two most popular brands today, Marlboro and Pall Mall, are owned by Philip Morris and Reynolds respectively (FTC cigarette report 2012).

Cigarettes are offered in different strengths, which can be roughly categorized as "regular", "light", or "ultralight". These cigarettes taste different, with "regular" cigarettes providing a stronger, fuller taste of tobacco. The formal definition of strength depends on tar yield or ventilation level. CDC measures tar yield on standardized smoking machines, and defines cigarettes with tar yield of more than 15mg as regular cigarettes, those with tar yield between 6mg and 15mg as light cigarettes, and those with less than 6mg as ultralight (CDC Fact Sheet Low-Yield Cigarette). Light cigarettes are very popular. In 2011, FDA estimated that light cigarettes account for 52% of market share (FTC Cigarette Report 2011).

Part of the appeal of light cigarettes is their perceived health benefits over regular cigarettes. Research shows that consumers believe light cigarettes are healthier than regular cigarettes (Scherer 1999, Shiffman et. al. 2001). However, clinical studies show no evidence of health benefit when switching from regular to light cigarettes (National Cancer Institute 2001; Thun and Burns 2001). This disparity between clinical evidence and consumer perception creates concern for public health as well as false advertising.

Other than brand and strength, price is the most important marketing mix variable that influences smokers choices. Tauras, Peck, and Chaloupka (2006) find that price variation including price promotion account for most of the change in market share change in the cigarette industry. According to FTC Cigarette Report for 2012, price discounts paid to retailers or wholesalers in order to lower price for consumers account for 85% of cigarette industrys total marketing expenditures, and other price-related expenditures including promotional allowances and coupons account for another 10% of the marketing expenditure. On the other hand, advertising of cigarettes had been heavily regulated since 1970s. Advertising on TV and radio is completely banned, and advertising on printed media is prohibited if there are readers who are under the age of 21. The single largest category of advertising expenditure in the 2000s is POS advertising, which only account for less than 2% of total marketing expenditure. Additionally, previous research find the effect of POS advertising on cigarette sales to be inconclusive (Baltagi and Levin 1986; Wakefield et al. 2002).

In 2009, Congress passed Family Smoking Prevention and Tobacco Control Act (hereafter referred to as "Tobacco Control Act" or "TCA") to further tighten regulation on marketing of tobacco products. Tobacco Control Act introduced a variety of new regulations on tobacco marketing activities. It banned the sale of flavored cigarettes (except menthol), banned sports and entertainment event sponsorship, and added restrictions on placement of advertising in publications or in stores that are targeted at youth. Most notably, the new law banned the use of terms such as "light", "ultra-light", "mild" or any other similar descriptors on any marketing or packaging materials. Companies were mandated to remove strength descriptors, with a deadline to comply of June 2010. However, the law did not prohibit tobacco companies from manufacturing and selling these light cigarettes. Tobacco companies responded to the new law by re-labeling their product lines with color codes. For example, "Marlboro Light" is renamed as "Marlboro Gold", and "Marlboro Regular" is renamed as "Marlboro Red". Given that these products were already packaged in different color boxes before TCA, the change was mostly only on the product label. However, the color

Figure 3.1: Color Schemes of Marlboro, Pall Mall, Winston, and Camel

	Regular	**Light**	**Ultralight**
Marlboro	Red	Gold	Silver
Pall Mall	Red	Blue	Orange
Winston	Red	Blue	Silver
Camel	Yellow	Blue	Silver

scheme is not uniform across brands. This creates potential room for consumer confusion, as the inconsistent labelling across firms can make product comparison more difficult. Figure (3.1) shows the color schemes for four of the top brands at the time: Marlboro, Pall Mall, Winston, and Camel. As the table shows, the color schemes overlap but still differ across the four brands. Most brands kept their entire product line with minimal change to the actual product, simply substituting strength descriptors with color codes on the packaging.[2]

FDA stated in Tobacco Control Act that the goal of banning strength descriptors is to eliminate misconception about light cigarettes' health benefits. On its website, FDA explicitly expresses concerns that such incorrect belief prevents smokers from quitting and attracts new smokers. Unfortunately, studies show that TCA is rather unsuccessful in its mission. Medical researchers find that simply removing "light" descriptors from the labels have little intended effect. Borland et al. (2008) and Yong et al. (2011) show that theres little effect of banning light descriptors in correcting beliefs in U.S. as well as other countries that implemented similar measures. Cohen et al. (2014) find that removing "light" descriptors did not increase cessation rate, and most previous light smokers continue to buy light cigarettes under new product labels. In Connolly and Albert (2013), participants report that it is easy to recognize their usual brand, suggesting little effect of labelling change on the purchase of their favorite brand. They also found that companies actively educate retailers and existing consumers about the name change prior to effective date of the law. However, there is evidence of consumer confusion. Bansal-Travers et al. (2011) found that smokers incorrectly infer strengths from colors more than 30% of the time and the ability to correctly identify color code by ventilation level differ by age and by brand. Specifically, subjects are more able to identify Marlboro than an unfamiliar brand.

There has been relatively few research on cigarette brand choices. Most smokers are loyal and tend to purchase the same brand repeatedly. Cornelius et al. (2015) find that between 2007 and 2011, 16% to 29% of surveyed smokers switch brand and 29% to 33% of

[2]Two other potential measures were being discussed but not implemented, including banning menthol flavored cigarettes and adding graphic warning signs on cigarette packaging. No other major changes such as M&A or regulation happened in the U.S. market between 2008-2012 (Bialous and Peeters 2012).

Figure 3.2: Whether Brands Switched to Color Names

Date	07/09	12/09	02/10	03/10	03/10	03/10	04/10	05/10	05/10	06/10	07/10
STATE	KY	MASS	MD	KY	OR	AL	AL	KY	MASS	MASS	AL
Brand											
MARLBORO	NO	NO	NO	NO	NO	NO	YES	YES			YES
PALL MALL	YES	YES	BOTH	YES	YES	BOTH	BOTH	YES		YES	YES
WINSTON	NO	NO	NO	NO	YES	YES	YES	YES		YES	YES
DORAL	NO	NO	NO	NO	YES	YES	YES	YES		YES	YES
BASIC	NO	NO	NO	NO	NO	NO	YES	YES			YES
PYRAMID	NO	NO	NO	NO	NO	NO	NO	BOTH	YES	YES	BOTH
MISTY	YES	YES	NO	YES	YES	BOTH	BOTH	YES		YES	YES
USA GOLD	NO	NO	NO	NO	NO	NO	NO	YES			YES
SONOMA	NO	NO	NO	NO	NO	NO	NO	YES			YES
CAMEL	NO	NO	NO	NO	YES	BOTH	BOTH	YES		YES	BOTH

surveyed smokers switch products within the same brand. Dawes (2014) uses IRI panel data and confirms that a large portion of smokers do not change their favorite brands overtime, but even these “loyal” customers buy competing brands close to half of the time, suggesting significant level of cross-purchasing.

One issue in studying the effect of TCA is that we do not know exactly when firms changed their product labels. Firms were given a year to comply after TCA is passed, and the process was likely not instant. Also, sources suggest that firms complied at different times. Philip Morris marketing brochures suggests that PM did not adopt name changes for its brands until 2010. On the other hand, Reynolds changed Pall Mall to color codes immediately in the summer of 2009. State tax documents, which record a product’s registered name, provide some qualitative evidence on when each brand adopted the new labels. I reviewed available archived state tax documents and report the result for top 10 brands in Figure (3.2). The table confirms that there is significant heterogeneity across brands.

3.3 Descriptive Evidence

Data

The data is provided by Nielsen. Both Retail Scanner Data and Consumer Panel Data are used. The data covers observations from 2007 to 2012.

Retail Scanner Data (RMS) collects weekly pricing, volume and store merchandising conditions from around 35,000 grocery, drug, and other stores in the U.S. The identity of the stores are hidden and replaced by an ID number. The data covers year 2006 and after. The data is at UPC level, and encompass a wide variety of product categories, including cigarettes. The scanner data is useful in detecting aggregate market response after removal of strength descriptors from packaging.

Consumer Panel Data (CPD) follows a panel of approximately 60,000 U.S. households who report their daily purchases. Panelists are spread out geographically and a projection factor for each household is used to adjust the panel demographics to match general demographics. Panelists use in-home scanner to report what they buy and how much they pay from each shopping trip. The store they shop at is also recorded if such store is a participant in the scanner data. The panel data allows us to estimate effect of TCA on a granular, trip level. This is the primary data source used for choice modelling in Section (3.4).

Both the scanner data and the panel data have their problems. A problem with the scanner data is that it does not capture the majority of stores in which cigarettes are bought. According to a Euromonitor study in 2012, 60% of cigarettes sales occur in convenience stores such as 7-Eleven, private brick-and-mortar, and gas stations. Nielsens data only cover 2% of total sales from the convenience store channel. However, cigarettes sales captured in the scanner data should still provide abundant data points for our purpose as it captures over half of sales from grocery and drug stores in the U.S. Although there can be selection bias if consumer who buy at grocery and drug stores behave differently from consumer who buy at other stores, I do not have evidence or convincing argument supporting this view. This channel problem does not exist in the panel data since it captures all purchases regardless of store. However, the panel data is recorded on the household level. I cannot distinguish which member in the household buys the cigarette, nor can I know how many people in the household are smokers. In the sample, 21% of households are single, 44% are double, 28% have 3-4 members, and 7% have 5 or more members. In Section 4 I show that single households do not behave very differently from households with multiple members when purchasing cigarettes.

Feature and display of tobacco products are highly restricted during the time period studied. This is confirmed in the data, as very few observations in the scanner data have either feature or display. As a result, I ignore feature and display from my analyses. According to FDA Cigarette Report, over 95% of point-of-sale marketing expenditures are in the form of price discounts or related activities. Additionally, previous research find the effect of POS advertising on cigarette sales to be inconclusive.(Cameron, 1998; Baltagi and Levin, 1986; Wakefield et al., 2002)

The paper focuses on the time period between 2008 to 2011. I use purchase records in 2007 to generate loyalty measure, and use 2012 data to confirm that smokers did not quit in 2011. This window provides one and a half years of observation before passing of TCA, one year during the implementation, and one and a half years after the compliance deadline. The data is recorded on UPC level with many UPCs for the same product. I consolidate UPCs by strength and by brand. In this paper, a product is defined as a brand-strength combination. Package size is normalized to 20 cigarette per pack. To further cut down

Figure 3.3: Market Share of Top 10 Brands among Panelists

Brand	2007	2008	2009	2010	2011	2012
MARLBORO	36.66	37.78	37.48	37.08	37.88	39.34
PALL MALL	5.35	6.27	10.78	15.10	15.49	15.18
WINSTON	6.39	6.30	5.89	5.03	4.32	4.36
DORAL	6.81	5.93	4.74	3.58	2.23	1.50
BASIC	5.78	5.77	5.00	2.98	2.07	1.13
CAMEL	4.40	3.78	3.24	2.49	1.62	2.27
MISTY	2.45	3.03	2.27	2.22	2.19	2.13
USA GOLD	3.06	2.51	2.42	2.37	2.20	1.63
SONOMA	2.00	2.22	2.02	1.69	1.62	0.81
PYRAMID	0.00	0.01	0.39	2.06	3.51	4.09

irrelevant product characteristics, I dropped unfiltered cigarettes and flavored cigarettes from my data. Unfiltered cigarettes only represent about 2% of sales and should not have any impact. Non-menthol flavored cigarettes have very little sales and are banned altogether in September 2009. Menthol-flavored cigarettes account for slightly above 26% of sales in 2009 and 2010 and 27.5% in 2011. Since TCA affects all cigarettes packaging uniformly regardless of flavor, dropping menthol-flavored cigarettes should have no effects. I later do robustness check with menthol cigarettes included to confirm that.

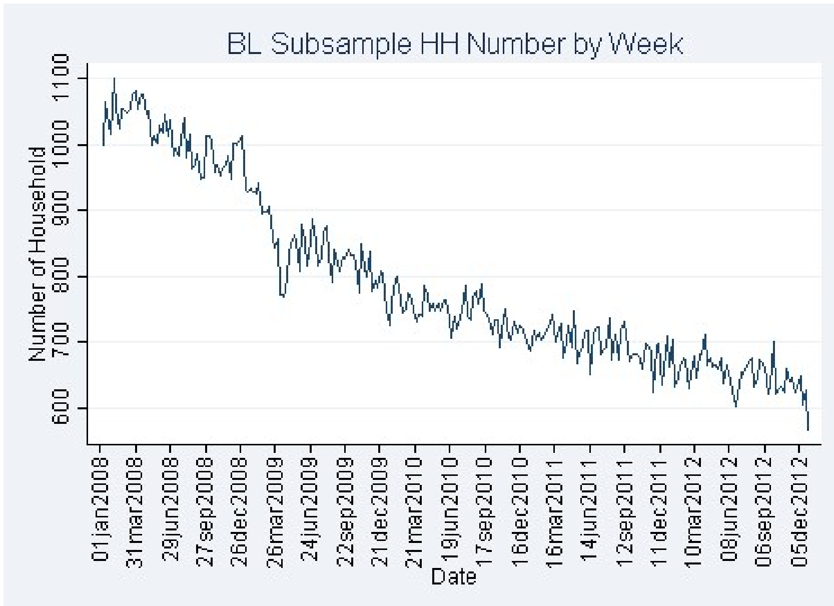
Summary Statistics

Nielsen's consumer panel data has around 60,000 households per year. Nielsen retains about 80% of panelists from year to year. About 26,000 households remain in the panel from 2007 to 2012. Between 6000 and 9000 households purchase cigarettes in a given year, but that number declines steadily. Out of the 26,000 households who remained in the panel in the six year period, 5700 households purchase cigarettes, and 1300 households purchase cigarettes in every year from 2007 to 2012.

Brands and Strengths

There are over 100 brands in the market. Figure (3.3) shows market shares of top 10 brands from 2008-2011 estimated from the panel data. As the table shows, Marlboro alone captures over half of cigarette sales, and 9 of top 10 brands are owned by two firms, Philip Morris and R.J. Reynolds. Thus the industry is very concentrated, despite the large number of brands.

Figure 3.4: Number of Households with Cigarette Purchases by Week



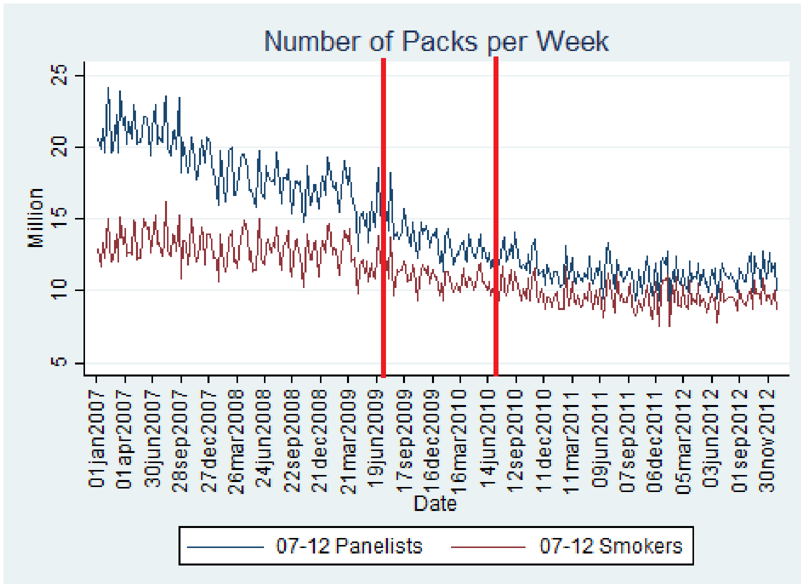
Most brands offer cigarettes in different strength. Out of 124 brands In 2010, 77% offer regular cigarettes, 68% offer light, 48% offer ultralight. 40% of these brands offer only one strength, 26% offer two strengths, and 34% offer all three strengths. In particular, all top 10 brands except Misty offer all three strength in the time period. The number one product in 2010 is Marlboro Light, followed by Marlboro Regular, Marlboro Ultralight, Pall Mall Regular, and Camel Light.

Consumption by Panelists

Total cigarette consumption declines over the period, as more households quit smoking. Figure (3.4) shows the number of households that purchase cigarettes in a given week, for panelists who remain in the panel from 2008 to 2012. The first bar in the graph denotes the date that TCA was passed, and the second bar denotes the date the compliance deadline. As noted in Section (3.2), firms changed labels at different points of time during the one-year window. The number goes from 1100 in early 2008 to 600 in 2012. The large drop in early 2009 correspond to the biggest federal tax increase on cigarettes, which went into effect roughly 2 months before TCA is passed. As a result of decrease in number of smokers, total cigarette consumption by panelists decreases as well.

While the number of smokers decreases steadily, the consumption for the remaining smokers remain relatively stable. Figure (3.5) shows the total number of packs of cigarettes bought per week for 07-12 panelists versus 07-12 smokers, weighted by each households projection factor provided by Nielsen, which account for the demographics difference between the panel and general population. A household is defined as a smoker in year Y if it purchase

Figure 3.5: Total Number of Packs by Week



cigarette in that year, and 07-12 smokers are those households that purchase cigarettes every year in the sample period. Although consumption declines in this period, there is no visible difference in the trend before and after the passing of Tobacco Control Act. Figure (3.6) shows the number of cigarette trips per week for 07-12 panelists versus 07-12 smokers, ignoring the quantity purchased. As one can see, for households who continue smoking, their weekly cigarette consumption was relatively stable. Most of the decline in cigarette consumption can be attributed to people who quit smoking. The above findings coincide with earlier research that shows that TCA was ineffective in lowering consumption or increasing cessation. In Section (3.4), to study brand choice behaviors, I focus only on household who remain as smokers, to maintain a consistent composition of households over time.

Retail Market Share and Price

The Retail Scanner Data records about \$4000 million and 800 million packs of cigarettes sales per year in its participating stores. Average cigarette price increased and quantity decreased in the time period, resulting in a flat trend in sales. There is seasonality in cigarette sales, with peak in summer and trough in winter. An increase on federal cigarette tax from \$0.39 to \$1.01 was put into effect in April 2009, causing prices of all brands to elevate in late first quarter and early second quarter.

There is no recognizable pattern in market share movement from the implementation of TCA in 2009 and 2010. Figure (3.7) shows the market shares of Marlboro, Pall Mall, Winston, and Camel. There is no visible kink or jump between 2009 and 2010 that is consistent across brands.

Figure 3.6: Number of Cigarette Trips by Week

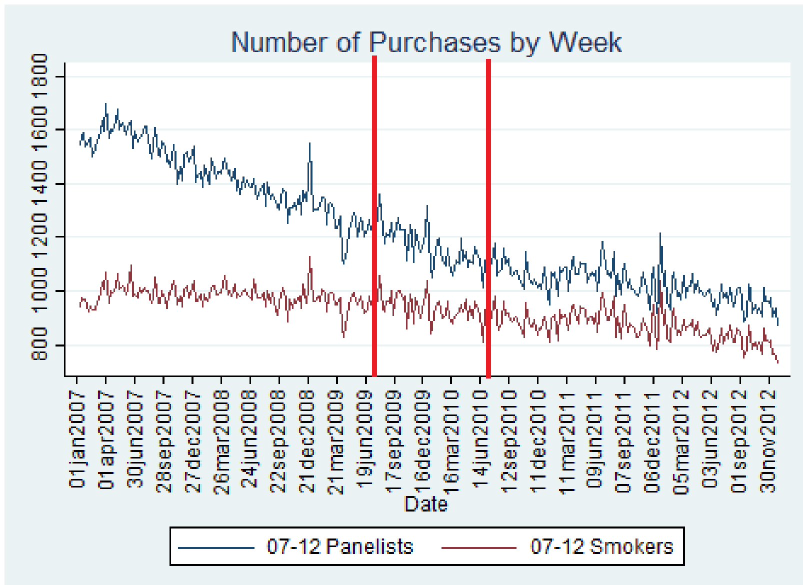
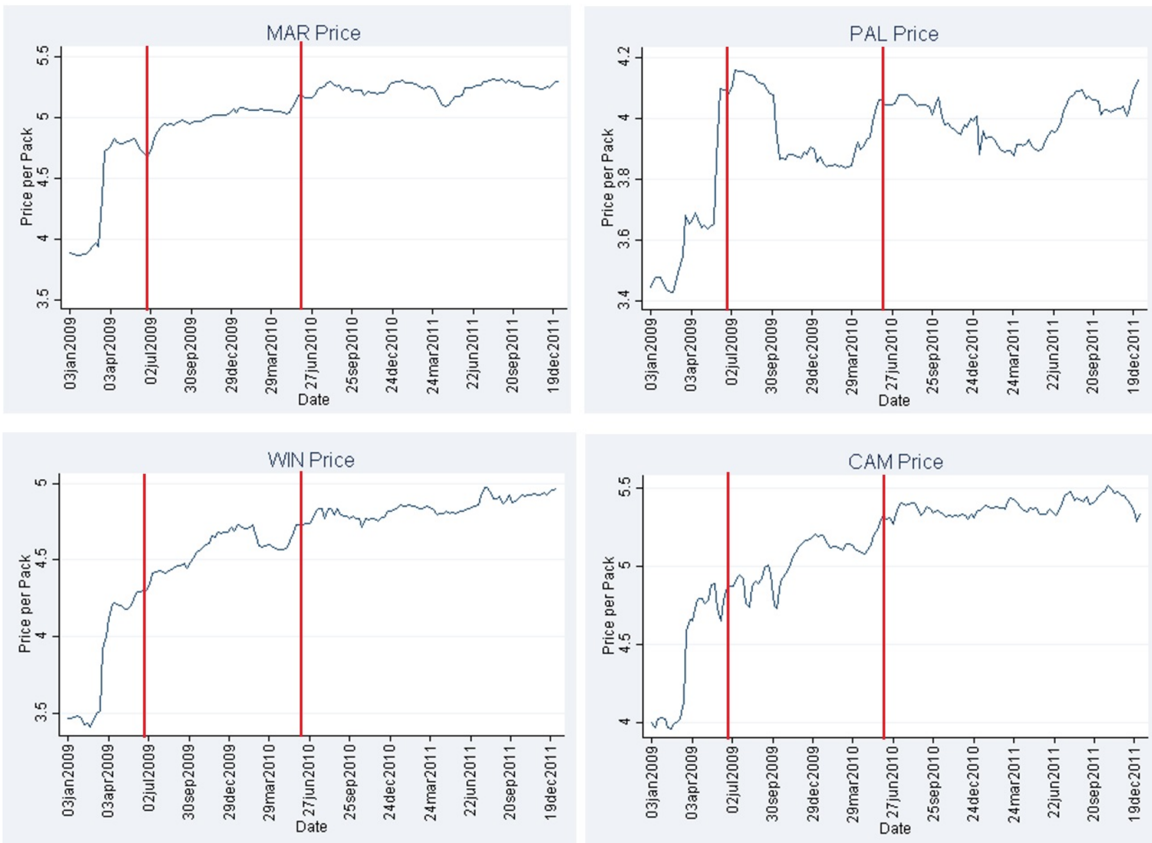


Figure 3.7: Weekly Market Share of Marlboro, Pall Mall, Winston, and Camel

Figure 3.8: Weekly Prices of Marlboro, Pall Mall, Winston, and Camel



Price show interesting movements, besides the jump in early 2009 due to federal tax increase. As shown in Figure (3.8) below, average cigarette price for all three strengths jumped both at the passing of TCA in the summer 2009 and the compliance deadline in the summer of 2010. Also, prices for different strengths start to diverge in late 2009 and early 2010. Regular cigarette price remains flat while light and ultralight became more expensive over the next two years. This divergence is graphed in Figure (3.9). The increase in cigarette price can be partly attributed to seasonality. Cigarette price also increased in the summer of 2011, but in a smaller amount and over a longer period of time. The evidence is not conclusive, but a price increase around the passing and deadline of TCA compliance may suggest that firms are less concerned about switching or cross-purchasing after strength descriptors are removed. Also, divergence in prices among different strengths could be a result of less substituting across strengths.

Cross-Purchasing

Monitoring brand switching for an individual is difficult. It is hard to identify when consumers switch from one product to another, because consumers can alternate among

Figure 3.9: Weekly Prices of Marlboro, Pall Mall, Winston, and Camel

multiple choices. Instead, I look at their cross-purchasing behavior. Cross-purchase happen when consumers buy outside of their favorite brands, possibly due to price promotion, variety-seeking, or simply idiosyncratic taste shock. In the cigarettes market, switching of favorite brand is infrequent. However, even loyal customers buy competing brands often (Dawes 2014), creating a significant amount of cross-purchasing. If the removal of strength descriptor affects consumers choice of brands or strengths, it should be able observable in their cross-purchasing patterns.

First I confirm that cross-purchasing exists in the data. Figure (3.10) shows the distribution of the number of brands each smoking household purchased in 2009. About 40% of households bought more than one brands in 2009. The same pattern holds for single households, proving that the distribution was not a result of having multiple smokers in a household. For households that bought more than one brands, Figure (3.11) shows the distribution of the share of purchase for a household's favorite brand, for households that purchased more than 1 brand. Again there exits a lot of heterogeneity. Many smokers routinely buy multiple brands, even if they do not switch their favorite brand.

Figure (3.12) graphs average number of brands per household per week. The graphs only include households who smoked continually from 2007 to 2012 and bought cigarettes in that week. Note that this measures cross-purchasing at a very small window, mostly focused on high frequency smokers. If a household buys two different brands in the same week, then the number of brands is two. If a household buys brand A in one week, then buys brand B in second week, then it is counted as only one brand in each week. Figure (3.13) shows average number of brands per household per month. This gives a bigger window for capturing cross-purchase and tells a similar story. Both graphs show a noticeable drop in cross-purchasing after TCA was passed and put into effect.

Households also decreased the number of strengths they buy, indicating less shopping across strengths as well. Figure (3.14) and (3.15) show weekly and monthly number of strengths per household, for households who smoked continually from 2007 to 2012 and bought cigarettes in that week. Again the number begins to decrease after the TCA is

Figure 3.10: Number of Brands per Household

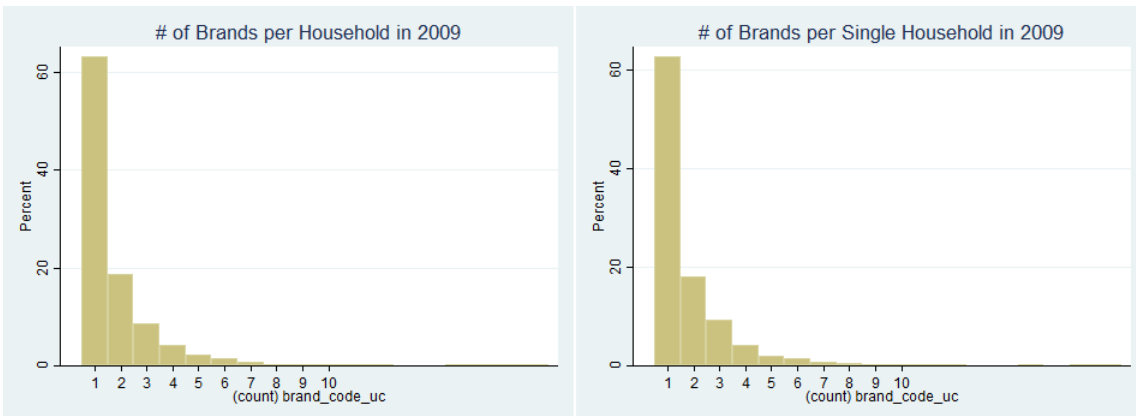
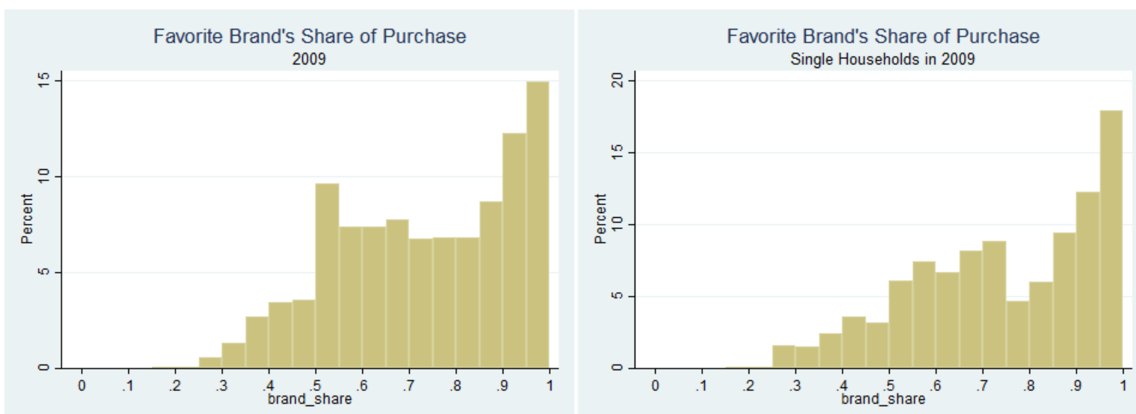


Figure 3.11: Favorite Brand's Share of Purchase



passed, and continue to decline through 2012. The number does not rise back to pre-TCA level in the two years following the implementation or the new regulation.

The evidence on cross-purchasing suggest that the removal of strength descriptors discourages consumers from shopping less familiar brands and strengths. However, a few problems remain to be addressed. First, the phenomenon of cross-purchasing does not capture more general switching behaviors. Second, the movements of cross-purchases are confounded with price movements at the same time. For example, in March and April of 2009, theres a spike in both number of brands and number of strengths purchased. This was due to a large, upcoming federal tax on cigarette sales. As brands increase their prices in anticipation of tax increase, consumers are motivated to shop for cheaper, alternative brands. Given that price is the most important marketing mix variable in the cigarettes market, it is important to make sure that the observed change in consumer behavior is not simply a response to supply-side changes.

Figure 3.12: Weekly Average Number of Brands

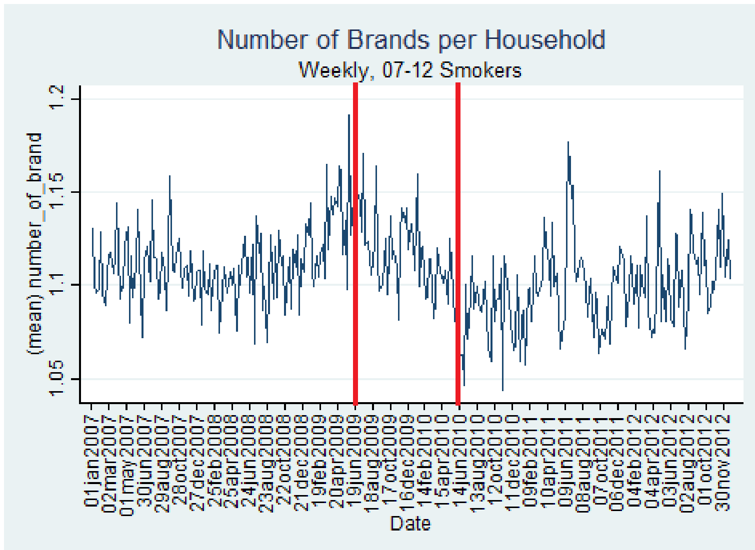


Figure 3.13: Monthly Average Number of Brands

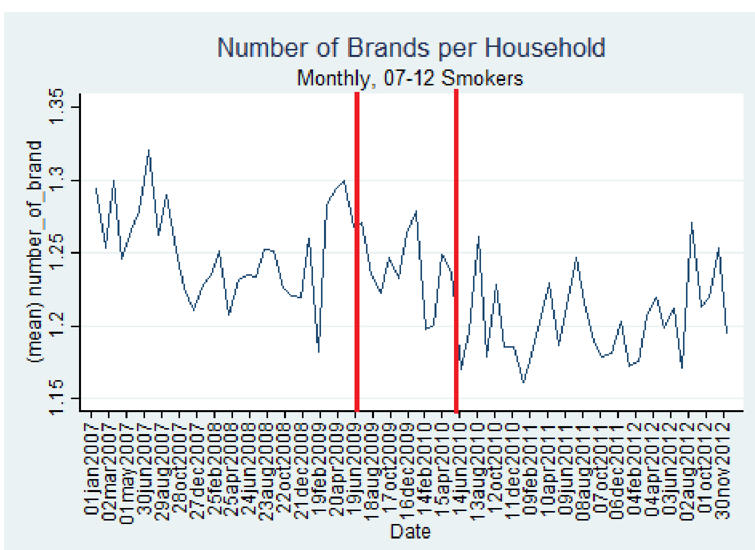


Figure 3.14: Weekly Average Number of Strengths

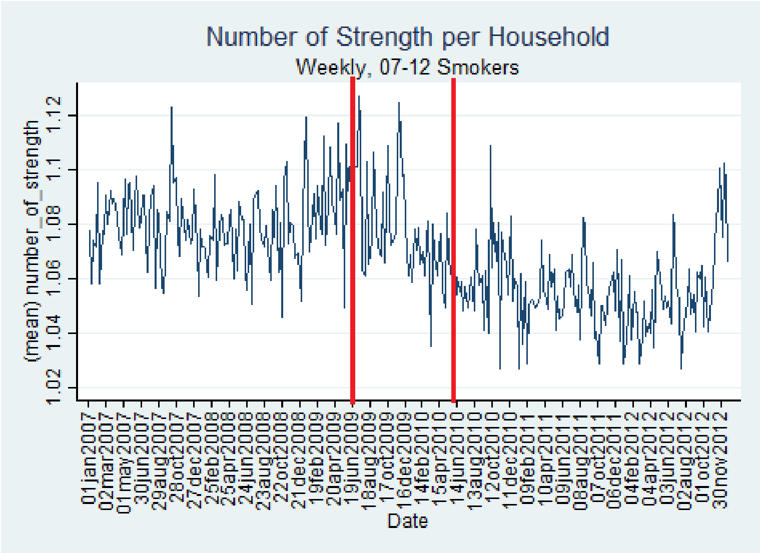
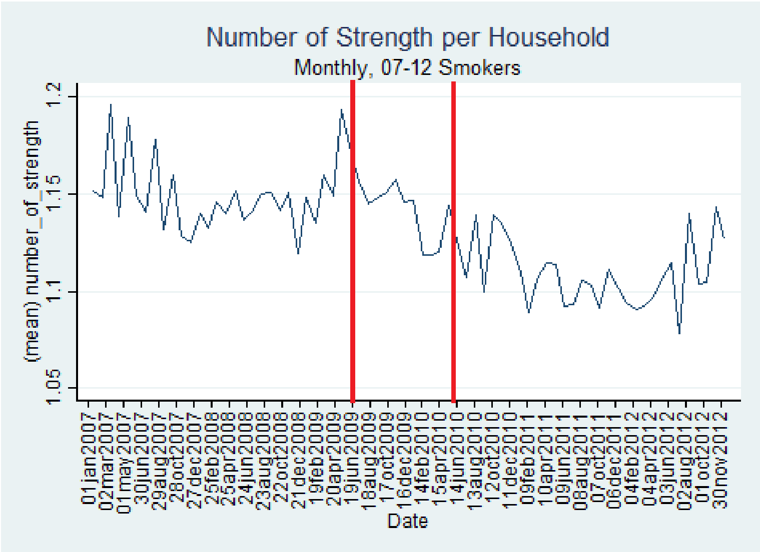


Figure 3.15: Monthly Average Number of Strengths



3.4 Effect on Price Sensitivity

In this section, I estimate discrete-choice models to detect change in consumer behavior following Tobacco Control Act. As a first step, a simple logit model with only intercepts and price is estimated. For household i, product j, and trip t, the utility of buying product j is:

$$U_{ijt} = \mu_j + \beta p_{ijt} + \epsilon_{ijt} \tag{3.1}$$

where ϵ_{ijt} is an i.i.d type I extreme value error term.

A product is defined as a combination of brand and strength. Size of the package is normalized to a 20 cigarettes pack, and price is normalized to price per 20 cigarettes. I include the top 7 brands by market share: Marlboro, Pall Mall, Winston, Camel, Basic, Doral, Misty. These 7 brands account for 69% of purchases in the sample. With the exception of Misty, each brand sells all three strengths, totaling up to 20 products.

I only include households who: (1) remained in Nielsens panel in each year from 2007 to 2012; (2) had at least 100 trips buying cigarettes (habitual smokers); and (3) purchased cigarettes in 2007 and 2012. Condition (3) made sure that households in the estimation sample did not start or quit smoking between 2008 and 2011. These three conditions ensure that the sample consist of a stable population of habitual smokers throughout the sample time period. From Figure (3.6) in Section (3.3), one can see that households who continued to smoke did not show any sign of slowing down their consumption post Tobacco Control Act. This is crucial because there is no outside option in the choice set. Due to the nature of panel data, an observation exists only when a household buys cigarettes. Thus I do not consider "not smoking" as a choice and solely focus on consumers choice among different brands and strengths. It is then important to ensure that the sample consists of the same composition of smokers who maintained their smoking frequencies over time.

The panel data only records the price of the chosen product of each trip. Prices and availability of a household's alternatives are approximated using Retail Scanner Data. If a product has no recorded sale in retailer scanner data in a zip code in a given week, I assume such product to be unavailable for consumers living in that zip code. This approximation method is consistent with observed choices. Only 0.25% of trips in the sample end up choosing products not available. Also, availability for the 7 brands included in the analysis do not fluctuate much over time. Given availability, I approximate the prices that a household see by using the average price of each alternative product in that week in the households zip code. Some price variations across stores and days in a week are not captured. This could bias price coefficient away from zero, if consumers are more likely to buy products that are cheaper than the regional, weekly average. This should not affect our analysis though, because we observe a movement of price coefficients towards zero and because the same problem exist for the entire sample period.

As noted in Section (3.2), one issue in estimating the effect of TCA is that we do not know exactly when firms changed their product labels. Instead, the model is estimated separately for each quarter from 2008 to 2011, and the results are compared over time. Doing so gives

Figure 3.16: Marginal Effect of Price on Choice Probability in Equation (3.1)

16 regressions for each specification. I also run regressions separately for each month as a robustness check and confirm that results are similar.

Figure (3.16) shows the marginal effect of price on choice probability over time. As the figure shows, consumers price sensitivity dropped after Tobacco Control Act was passed and implemented. The marginal effect of price remained low for the remaining of the sample period, showing now sign of learning. This observation confirms the descriptive evidence that consumers are less likely to switch between products. Consumers respond less to price movements after the strength descriptors are removed.

If the choice model is correctly specified, then the price coefficient represent how money compensates for the difference in utility between protects. Assuming that consumers maintain the same underlying preference for money, then a smaller marginal effect of price implies a larger difference in the utility between products. In another word, consumers perceive the products to be more differentiated than before. The increase in differentiation could be perceptual, as researchers have shown that both name and color affect consumers' perception of a product (Ampuero and Vila 2006; Plasschaert 1995; Wanke, Herrmann, and Schaffner 2006), or informative, as in Tirole (1988). However, the simple logit model in equation (3.1) is likely mis-specified. Past research has shown that the price coefficient can be biased if the model fails to account for a number of factors. The introduction of Tobacco Control Act could have other effects that cause a change in the level of bias in the observed direction. In particular, Villas-Boas and Winer (1999) show that endogeneity bias price coefficient towards zero. Chintagunta, Kyriazidou, and Perktold (1998) shows that price coefficient can be biased toward zero if one does not account for preference heterogeneity. Abramson et. al. (2000) find that price responsiveness is underestimated if state dependence is ignored. Chiang, Chib, and Narasimhan (1999) show that price coefficient is biased toward zero is consideration set is ignored. If Tobacco Control Act affects preference heterogeneity, price endogeneity, or state dependence, then one could observe a shift in price coefficients even though underlying preferences have not changed.

In the following sections, I test for the possibility that the change in price sensitivity was due to additional bias. First, a choice model is specified to include preference heterogeneity, state dependence, and price endogeneity. Then a two-stage model is used to to account for change in the average size of consideration set. In both cases, consumers continued to be less price-sensitive following the change of product labels. Even though in this setting, a change in perceived differentiation cannot be directly identified, the results suggest that this explanation is more credible than the alternatives discussed.

Accounting for Preference Heterogeneity, State Dependence, and Price Endogeneity

The utility of household i from purchasing product j on trip t is defined as:

$$U_{ijt} = \mu_{ij} + \beta p_{jt} + \theta^{br} l^{br}_{ijt}(\lambda^{br}) + \theta^{str} l^{str}_{ijt}(\lambda^{str}) + \epsilon^{a}_{ijt} + \epsilon^{b}_{ijt} \tag{3.2}$$

where ϵ^{a}_{ijt} is an error term with i.i.d type I extreme value. The second error term ϵ^{b}_{ijt} represents market or brand variables that are not observed by the researcher by affect prices, and is assumed to be normally distributed with mean zero and variance σ^2_b.

The term $l^{br}_{ijt}(\lambda^{br})$ represents household i's loyalty level towards product j's brand, and $l^{str}_{ijt}(\lambda^{str})$ represents household i's loyalty level towards product j's strength. The loyalty levels are functions of the smoothing parameter λ's, as in Guadagni and Little (1993):

$$l^{br}_{ijt}(\lambda^{br}) = \lambda^{br} l^{br}_{ij,t-1} + (1-\lambda^{br}) y^{br}_{ij,t-1} \tag{3.3}$$

$$l^{str}_{ijt}(\lambda^{str}) = \lambda^{str} l^{str}_{ij,t-1} + (1-\lambda^{str}) y^{str}_{ij,t-1} \tag{3.4}$$

where $y^{br}_{ij,t-1}$ is an indicator function equal to 1 if household i bought the brand of j on the last trip, and $y^{str}_{ij,t-1}$ is an indicator function equal to 1 if household i bought the strength of j on the last trip. The exponential smoothing parameters

Due to the high numbers of products included in the estimation, it is difficult to estimate mixed logit model with an individual intercept for each product. Instead, I defined product intercept as sum of brand intercept and strength intercept, similar to the treatment of Fader and Hardie (1996). The product intercept μ_{ij} becomes:

$$\mu_{ij} = \mu_{i,br(j)} + \mu_{i,str(j)}$$

where $br(j)$ and $str(j)$ represent the brand and strength of product j, respectively, with $\mu_{i,br(j)} \sim N(\mu_{br(j)}, \sigma_{br(j)})$, and $\mu_{i,str(j)} \sim N(\mu_{str(j)}, \sigma_{str(j)})$. The mean and standard deviation of brand and strength intercepts $\mu_{br(j)}$, $\sigma_{br(j)}$, $\mu_{str(j)}$, and $\sigma_{str(j)}$ will be estimated. For simplicity, I assume no correlation in brand or strength coefficients across products. That is, $\mu_{i,br(j)} \perp \mu_{i,br(k)}$ and $\mu_{i,str(j)} \perp \mu_{i,str(k)}$ for all $j \neq k$.

Consumption tax on cigarettes account for a large share of the price and vary significantly across states and cities and across time. The price of product j on household i's trip t is parametrically defined as a function of cigarettes tax and unobservable factor:

$$p_{ijt} = \alpha_0 + \alpha_1 s_{it} + \omega_{ijt} \tag{3.5}$$

The term s_{it} denotes the cigarettes tax in the region where household i made their t's trip. The term ω_{ijt} is correlated to the error term ϵ^b_{ijt} in equation (3.2). This captures the idea that firms set prices endogenously based on demand factors unobserved to the econometrician. I assume that ω_{ijt} and ϵ^b_{ijt} are jointly normal and independent across j.

Estimation

The specification is estimated separately for each quarter from 2008 to 2011. Thus there are 16 regressions for each specification. A household's 2017 purchases are use to initilize loyalty variables. If a household has less than 20 purchases in 2017, then the first 20 trips are set aside to initialize loyalty variables. On a households first trip in 2017, I assume the household has equal loyalties over all brands and strengths that add up to one. Then loyalty variables for the remaining trips are calculated according to equation (3.3).

Brand and Strength dummies are normalized to the most frequent choice in each quarter. Thus there are 6 brand dummies and 2 strength dummies to be estimated. All models are estimated using maximum likelihood. The loyalty smoothing parameters enters the utility function non-linearly, and thus poses difficulty in computation. I first estimate the smoothing parameters using a fixed coefficient model following the method developed by Fader, Lattin and Little (1992), which uses Taylor expansion to transform the model into a linear one, and estimate recursively to retrieve maximum likelihood estimators. The estimated parameters from fixed coefficient model are all close 85%. Then for simplicity, I fix smoothing parameters at 85% for both brand and strength and for all periods in the random coefficient model. I use alternative numbers ranging from 80% to 90% as robustness check and find results to be stable.

I use control function approach to account for price endogeneity as in Petrin and Train (2010). Given that ω_{ijt} and ϵ^b_{ijt} are jointly normal and independent across j, one can write:

$$\mathbb{E}[\epsilon^b_{ijt} | \omega_{ijt}] = \gamma \omega_{ijt} \tag{3.6}$$

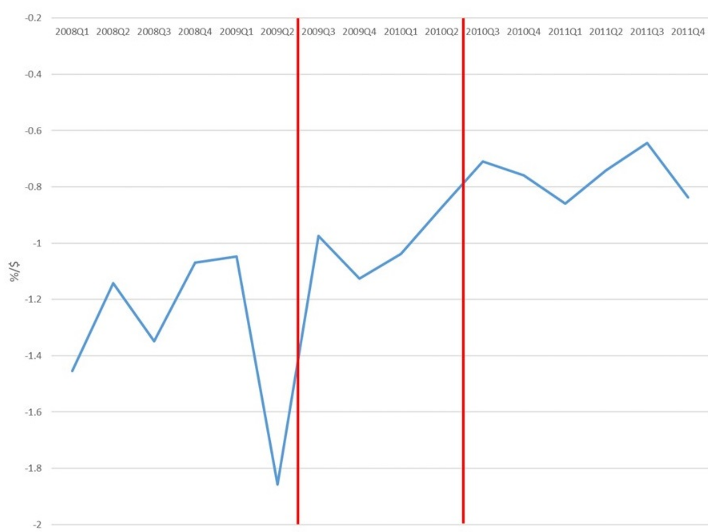
$$\epsilon^b_{ijt} = \gamma \omega_{ijt} + \sigma \eta_{ijt} \tag{3.7}$$

where η_{ijt} is standard normal and σ captures the standard deviation of ϵ^b_{ijt} conditional on ω_{ijt}. Substituting equation (3.7) into equation (3.2) gives us:

$$U_{ijt} = \mu_{ij} + \beta p_{jt} + \theta^{br} l^{br}_{ijt}(\lambda^{br}) + \theta^{str} l^{str}_{ijt}(\lambda^{str}) + \gamma \omega_{ijt} + \sigma \eta_{ijt} + \epsilon^a_{ijt} \tag{3.8}$$

The estimation has three steps for each quarter. First, I obtain a product's average price from each week for each zip code from the Retail Scanner Data. Then, the average price is regressed on the sum of federal, state, and local tax to obtain the residual $\hat{\omega}_{ijt}$. For the last step, equation (3.8) is estimated as a mixed logit using $\hat{\omega}_{ijt}$, with mixing over μ_{ij} and η_{ijt}.

Figure 3.17: Marginal Effect of Price on Choice Probability in Equation (3.2)



Results and Robustness

The appendix shows the complete regression results for all 16 quarters in the sample. Coefficients other than price show no visible shift in value after the new regulation. The size of the price coefficient jumps to a different level with the passage of TCA. Figure () graphs the average marginal effect of price on choice probability over time. The effect of TCA on price sensitivity is visible and significant. Note that there is a high magnitude of price effect in 2019 Q2. This jump is the result of a large increase in federal cigarette tax in that quarter. Consumers were more responsive to the price differences as firms raise their prices in responses to the new tax. However, even if we exclude this data point, the average marginal effect for the six quarters after the compliance deadline is still only 63% of the marginal effect in the period before TCA was passed. This shows that the observed change in choice behavior cannot be explained by changes in consumer heterogeneity, price endogeneity, or state dependence.

For robustness, I ran variations of the model with only consumer heterogeneity, price endogeneity, or state dependence, and all combinations of the three. I also run the model with a longer time window from 2007 to 2012. Results show that marginal effects in 2007 were stable and similar to 2008 levels. This confirms that the change in price sensitivity was not caused by the Great Recession. Additionally, I run a variation where all other brands are combined by strengths into three additional choices: other regular, other light, and other ultralight. The result is repeated in each case. Another specification includes an interaction term of loyalty and price. As expected, consumers act less price-sensitive toward products that they have a higher loyalty for. In another specification, I replace the G&L loyalty variable with two variables (a static loyalty variable and last brand purchased) to separate cross-sectional and longitudinal heterogeneity. This approach was previously used by Bucklin and Gupta (1992) and Bell and Lattin (2000). It intends to separate control for state dependence and static brand preference. Lastly, I set a month instead of a quarter as period to get "finer" estimates. The result resembles quarterly estimates, but with more fluctuations from month to month.

Other coefficients in my estimates can also be biased. However, they should not affect our finding. Keane (1997) and Heckman (1981) show that state dependence is upward biased if we ignore heterogeneity. Chiang, Chib, and Narasimhan (1999) show that state dependence is upward biased if we ignore consideration set. However, Abramson et al. (2000) shows that state dependence should be robust against price endogeneity, preference heterogeneity, and choice set effect. Use of loyalty model itself could contaminate price coefficient, if past price and promotions are not counted. Though this is theoretically true, studies found such effect to be small (Fader and Hardie 1996 and Abramson et al. 2000).

Consideration Set

In this section, I test if the change in price coefficient can be explained by a change in the consideration set. Ignoring choice set effect can significantly downward bias price coefficient (Chiang, Chib, and Narasimhan 1999; Abramson et al. 2000). The removal of strengths information from product packaging and labelling can increase consumers search cost. Besides search for price information, consumers now also needs to figure out which colored products are comparable. Such mental burden can leads to a shrink in the size of his/her consideration set.

Various papers have modeled consumer choice in two stages. In the first stage, a subset of products form the consideration set based on their attributes. In the second stage, consumers make choice only within the consideration set formed in the first stage. The treatment of consideration sets in this paper closely follows Abaluck and Adams (2018) and Goeree (2008). Other papers with a consideration stage followed by a choice stage include Andrews and Srinivasan (1995), Mehta, Rajiv, and Srinivasan (2003), and van Nierop et al. (2010).

For ease of computing, the model does not allow random coefficients. That is, explicit preference heterogeneity and price endogeneity are not included. Loyalty variables are still included and should capture some of the preference heterogeneity. For household i on trip t, the utility of buying product j in the second stage of the model is:

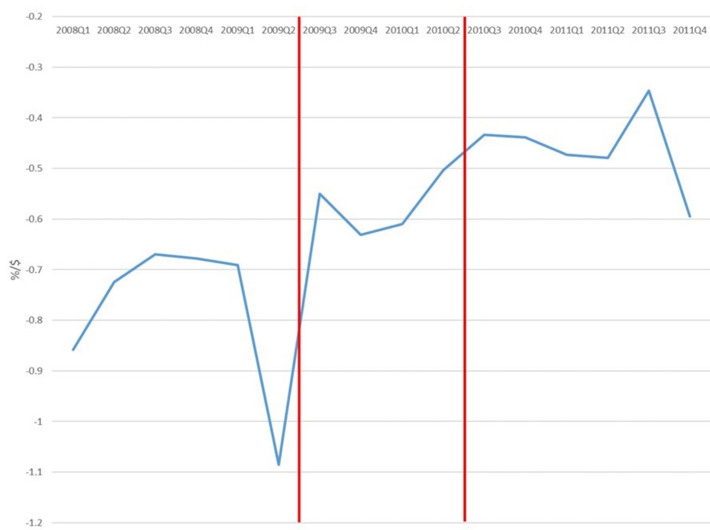
$$U_{ijt} = \mu_{br(j)} + \mu_{str(j)} + \beta_2 p_{ijt} + \theta^{br} l_{ijt}^{br}(\lambda^{br}) + \theta^{str} l_{ijt}^{str}(\lambda^{str}) + \epsilon_{ijt} \tag{3.9}$$

where ϵ_{ijt} is i.i.d type I extreme value.

For the first stage, I consider two different specifications from Abaluck and Adams (2018). In the alternative-specific consideration (ASC) model, the probability of each product being in the consideration set is independent from other products. In the first stage, a product is considered if

$$V_{ijt} = \alpha_{br(j)} + \alpha_{str(j)} + \beta_1 p_{ijt} + \theta^{br} l_{ijt}^{br}(\lambda^{br}) + \theta^{str} l_{ijt}^{str}(\lambda^{str}) > \xi_{ijt} \tag{3.10}$$

Figure 3.18: Marginal Effect of Price on Choice Probability for ASC



where ξ_{ijt} is i.i.d type I extreme value. The probability for consumer i to consider product j then is

$$\phi_{ijt} = \frac{exp\big(\alpha_{br(j)} + \alpha_{str(j)} + \beta_1 p_{ijt} + \theta^{br} l_{ijt}^{br}(\lambda^{br}) + \theta^{str} l_{ijt}^{str}(\lambda^{str})\big)}{1 + exp\big(\alpha_{br(j)} + \alpha_{str(j)} + \beta_1 p_{ijt} + \theta^{br} l_{ijt}^{br}(\lambda^{br}) + \theta^{str} l_{ijt}^{str}(\lambda^{str})\big)}$$

In the default-specific consideration (DSC) model, a household either only considers the default product, or considers all products. The default product for household i is defined as the most frequently purchased product by that household in the past. Let $j = 0$ denote the default product. The probability that a household only considers the default product is:

$$\pi_{it} = \frac{exp\big(\alpha_{br(0)} + \alpha_{str(0)} + \beta_1 p_{i0t} + \theta^{br} l_{i0t}^{br}(\lambda^{br}) + \theta^{str} l_{i0t}^{str}(\lambda^{str})\big)}{1 + exp\big(\alpha_{br(0)} + \alpha_{str(0)} + \beta_1 p_{i0t} + \theta^{br} l_{i0t}^{br}(\lambda^{br}) + \theta^{str} l_{i0t}^{str}(\lambda^{str})\big)}$$

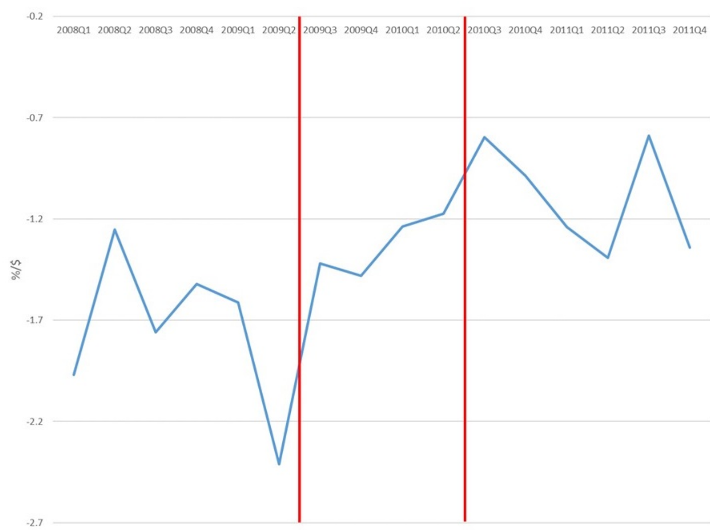
And the probability that a household considers all products on trip t is thus:

$$1 - \pi_{it} = 1 - \frac{exp\big(\alpha_{br(0)} + \alpha_{str(0)} + \beta_1 p_{i0t} + \theta^{br} l_{i0t}^{br}(\lambda^{br}) + \theta^{str} l_{i0t}^{str}(\lambda^{str})\big)}{1 + exp\big(\alpha_{br(0)} + \alpha_{str(0)} + \beta_1 p_{i0t} + \theta^{br} l_{i0t}^{br}(\lambda^{br}) + \theta^{str} l_{i0t}^{str}(\lambda^{str})\big)}$$

The models are estimated using simulated maximum likelihood. The estimation is done using the alogit STATA package described in Abaluck, Adams, and Caceres (2017).

Figure (3.18) and (3.19) show the unconditional marginal effect of price on choice probability in ASC and DSC models. The graphs tell a similar story as before. Consumers became less price-sensitive after strength descriptors were removed. Change in the size of consideration sets (plus loyalty) cannot explain this change in behavior.

Figure 3.19: Marginal Effect of Price on Choice Probability for DSC



3.5 Conclusion

In 2009, Congress passed Tobacco Control Act, which bans tobacco companies from communicating product strengths on any marketing or packaging materials. Since then, tobacco companies rely on color codes to continue selling cigarettes of different strengths, i.e., relabeling Marlboro Light to Marlboro Gold and Camel Light to Camel Blue. Different brands do not use the exact same color schemes, further creating confusion. This paper investigates the effect of such change in label informativeness on consumer choices.

Using Nielsen's data, I find an increase in cigarette price and an increase in price dispersion between different strengths after TCA went into effect. No change in overall cigarette consumption trend is observed, which resonates with previous findings that suggest Tobacco Control Act had minimal impact on cessation. However, the data shows a reduction in cross-purchasing after strength descriptors were banned. Consumers became less willing to buy brands or strengths outside of their most frequent choice.

To further explore the effect of removing strength descriptors, I fit discrete-choice models to data from a panel of smokers from 2008 to 2011. Firms had a one-year window to comply to the new regulation, and the specific timings of compliance are unknown and likely differ across firms. To address this issue, I divide the data into 16 quarter, separately estimate every model for each quarter, and track changes in behavior over time. The estimates show a decrease of price sensitivity after Tobacco Control Act was passed and implemented. The effect persisted in the 6 quarter after the compliance deadline, showing no sign of dynamic learning. The observed change in price sensitivity is robust to specifications that account for preference heterogeneity, state dependence, price endogeneity, and limited attention/consideration set. Thus it cannot be explained by changes to these four factors. One remaining explanation, assuming preference for money was stable, is that consumers perceived products to be more differentiated after labels changed from strengths to color codes. This paper provides evidence on the effect of product label informativeness on consumer choice and documents an unintended effect brought by Tobacco Control Act.

3.6 Estimation from Section 3.4

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
VARIABLES	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
price	-1.254***		-1.091***		-1.141***		-0.884***		-0.813***	
	(0.0556)		(0.0558)		(0.0547)		(0.0606)		(0.0541)	
Brand Loyalty	6.459***		6.513***		6.354***		6.285***		6.066***	
	(0.130)		(0.130)		(0.126)		(0.125)		(0.120)	
Strength Loyalty	4.943***		5.003***		4.702***		4.761***		4.837***	
	(0.118)		(0.113)		(0.108)		(0.115)		(0.111)	
omega	0.675**		0.742***		1.037***		0.362		-0.153	
	(0.309)		(0.272)		(0.295)		(0.245)		(0.229)	
Basic	-1.573***	1.207***	-1.420***	1.079***	-1.943***	1.328***	-1.470***	1.381***	-1.209***	0.748***
	(0.198)	(0.175)	(0.211)	(0.174)	(0.209)	(0.173)	(0.213)	(0.177)	(0.145)	(0.140)
Camel	-1.099***	1.084***	-1.093***	1.217***	-1.629***	1.272***	-1.502***	1.479***	-1.673***	1.376***
	(0.196)	(0.206)	(0.238)	(0.244)	(0.248)	(0.207)	(0.290)	(0.266)	(0.269)	(0.251)
Doral	-1.744***	-0.540***	-1.501***	0.436**	-1.470***	0.757***	-1.398***	0.904***	-1.879***	0.927***
	(0.185)	(0.165)	(0.189)	(0.188)	(0.168)	(0.141)	(0.190)	(0.193)	(0.181)	(0.198)
Misty	-2.308***	0.879***	-2.010***	0.921***	-2.114***	-0.823**	-1.789***	0.672**	-2.087***	0.972***
	(0.327)	(0.301)	(0.325)	(0.305)	(0.321)	(0.344)	(0.279)	(0.324)	(0.349)	(0.320)
Pall Mall	-1.304***	0.594***	-1.417***	1.136***	-1.836***	1.312***	-1.269***	0.761***	-1.208***	0.979***
	(0.140)	(0.126)	(0.167)	(0.156)	(0.188)	(0.151)	(0.143)	(0.164)	(0.136)	(0.126)
Winston	-1.319***	1.144***	-1.133***	1.014***	-1.410***	0.763***	-1.660***	0.896***	-1.595***	1.500***
	(0.196)	(0.169)	(0.183)	(0.173)	(0.173)	(0.149)	(0.184)	(0.168)	(0.244)	(0.213)
Ultra-light	-0.943***	1.723***	-0.984***	1.137***	-1.059***	1.492***	-0.836***	0.811***	-0.739***	-0.752***
	(0.166)	(0.172)	(0.141)	(0.152)	(0.132)	(0.168)	(0.117)	(0.133)	(0.119)	(0.140)
Light	-0.173**	0.568***	-0.254***	0.690***	-0.212***	0.696***	-0.206**	0.981***	-0.171*	1.258***
	(0.0813)	(0.112)	(0.0798)	(0.110)	(0.0755)	(0.127)	(0.0810)	(0.126)	(0.102)	(0.118)
eta	0	2.138***	0	1.263***	0	1.336***	0	1.835***	0	1.404***
	(0)	(0.208)	(0)	(0.323)	(0)	(0.194)	(0)	(0.257)	(0)	(0.226)
Observations	141,399	141,399	148,499	148,499	148,239	148,239	144,987	144,987	144,064	144,064
Quarter	2008Q1	2008Q1	2008Q2	2008Q2	2008Q3	2008Q3	2008Q4	2008Q4	2009Q1	2009Q1

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
VARIABLES	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
price	-1.286***		-0.747***		-0.911***		-0.943***		-0.883***	
	(0.0571)		(0.0436)		(0.0478)		(0.0402)		(0.0428)	
Brand Loyalty	6.040***		6.214***		6.418***		6.632***		6.958***	
	(0.116)		(0.115)		(0.126)		(0.141)		(0.157)	
Strength Loyalty	4.806***		4.546***		4.760***		4.993***		5.398***	
	(0.109)		(0.102)		(0.101)		(0.121)		(0.140)	
omega	0.813***		-0.141		0.762***		0.804***		0.423	
	(0.255)		(0.265)		(0.275)		(0.292)		(0.287)	
Basic	-1.661***	1.065***	-1.636***	1.035***	-1.524***	0.989***	-1.322***	0.735***	-1.191***	1.235***
	(0.177)	(0.120)	(0.214)	(0.181)	(0.196)	(0.162)	(0.182)	(0.189)	(0.246)	(0.238)
Camel	-1.804***	1.436***	-1.439***	1.607***	-1.046***	1.064***	-1.448***	1.190***	-1.375***	1.225***
	(0.297)	(0.218)	(0.234)	(0.202)	(0.200)	(0.192)	(0.315)	(0.285)	(0.296)	(0.231)
Doral	-2.040***	0.779***	-1.667***	0.587***	-1.991***	1.277***	-1.861***	0.751***	-2.258***	0.588**
	(0.212)	(0.247)	(0.190)	(0.226)	(0.238)	(0.198)	(0.220)	(0.227)	(0.207)	(0.234)
Misty	-1.825***	-0.397	-1.687***	0.815***	-1.913***	0.775**	-1.943***	1.180***	-2.115***	0.824***
	(0.208)	(0.274)	(0.283)	(0.306)	(0.288)	(0.305)	(0.383)	(0.344)	(0.308)	(0.293)
Pall Mall	-1.628***	1.184***	-0.997***	0.767***	-1.098***	1.128***	-1.395***	1.475***	-1.231***	1.007***
	(0.134)	(0.133)	(0.121)	(0.117)	(0.131)	(0.125)	(0.153)	(0.157)	(0.149)	(0.143)
Winston	-1.833***	1.250***	-1.468***	0.754***	-1.643***	-0.973***	-1.617***	0.866***	-1.649***	1.102***
	(0.213)	(0.171)	(0.193)	(0.149)	(0.192)	(0.184)	(0.230)	(0.229)	(0.219)	(0.226)
Ultra-light	-0.716***	0.492***	-0.822***	1.194***	-0.797***	1.332***	-0.604***	0.675***	-0.863***	-0.882***
	(0.101)	(0.118)	(0.123)	(0.169)	(0.139)	(0.151)	(0.113)	(0.172)	(0.135)	(0.150)
Light	-0.174*	1.396***	-0.192**	0.733***	0.0681	0.644***	-0.358***	1.029***	-0.349***	1.511***
	(0.0889)	(0.116)	(0.0780)	(0.110)	(0.0773)	(0.101)	(0.0976)	(0.149)	(0.117)	(0.128)
eta	0	1.599***	0	1.629***	0	1.208***	0	0.808***	0	0.750**
	(0)	(0.352)	(0)	(0.221)	(0)	(0.390)	(0)	(0.269)	(0)	(0.363)
Observations	142,151	142,151	146,196	146,196	146,088	146,088	138,423	138,423	137,950	137,950
Quarter	2009Q2	2009Q2	2009Q3	2009Q3	2009Q4	2009Q4	2010Q1	2010Q1	2010Q2	2010Q2

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)
VARIABLES	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
price	-0.757***		-0.790***		-0.835***		-0.751***		-0.653***		-0.843***	
	(0.0472)		(0.0492)		(0.0521)		(0.0562)		(0.0597)		(0.0653)	
Brand Loyalty	6.726***		6.869***		6.633***		6.681***		6.717***		6.450***	
	(0.154)		(0.166)		(0.160)		(0.161)		(0.165)		(0.152)	
Strength Loya	5.080***		5.262***		5.124***		5.234***		4.987***		5.084***	
	(0.120)		(0.139)		(0.130)		(0.138)		(0.129)		(0.135)	
omega	0.339		0.421		0.442		0.689**		0.774**		0.159	
	(0.284)		(0.296)		(0.292)		(0.320)		(0.312)		(0.291)	
Basic	-2.268***	1.751***	-1.513***	0.992***	-1.628***	1.196***	-1.837***	1.061***	-1.930***	1.506***	-1.607***	1.189***
	(0.362)	(0.250)	(0.318)	(0.231)	(0.301)	(0.258)	(0.313)	(0.221)	(0.414)	(0.356)	(0.319)	(0.328)
Camel	-1.245***	0.889**	-1.882***	1.672***	-2.338***	1.847***	-2.102***	1.577***	-1.755***	1.345***	-1.596***	1.318***
	(0.317)	(0.359)	(0.388)	(0.217)	(0.468)	(0.347)	(0.370)	(0.217)	(0.425)	(0.399)	(0.390)	(0.334)
Doral	-1.750***	0.644*	-2.381***	0.925***	-2.857***	1.273***	-2.085***	1.121***	-1.890***	0.956***	-1.791***	0.529
	(0.239)	(0.356)	(0.341)	(0.317)	(0.321)	(0.244)	(0.331)	(0.274)	(0.339)	(0.298)	(0.301)	(0.393)
Misty	-1.350***	0.643**	-1.773***	0.933***	-2.162***	1.112***	-1.747***	1.306***	-1.317***	1.049***	-1.455***	0.746**
	(0.261)	(0.269)	(0.258)	(0.249)	(0.416)	(0.317)	(0.377)	(0.397)	(0.348)	(0.331)	(0.297)	(0.350)
Pall Mall	-1.175***	1.264***	-1.012***	1.187***	-1.752***	0.947***	-1.048***	1.020***	-0.801***	0.882***	-1.070***	0.867***
	(0.152)	(0.144)	(0.149)	(0.142)	(0.150)	(0.123)	(0.137)	(0.179)	(0.154)	(0.160)	(0.151)	(0.152)
Winston	-1.388***	0.855***	-1.339***	0.825***	-1.976***	1.315***	-1.582***	0.584***	-1.577***	0.756***	-1.509***	0.825***
	(0.219)	(0.223)	(0.241)	(0.266)	(0.255)	(0.186)	(0.220)	(0.201)	(0.235)	(0.212)	(0.223)	(0.169)
Ultra-light	-0.911***	0.708***	-0.430***	0.746***	-0.832***	1.296***	-0.760***	1.034***	-0.754***	0.944***	-0.703***	0.829***
	(0.138)	(0.196)	(0.122)	(0.157)	(0.161)	(0.185)	(0.125)	(0.144)	(0.162)	(0.297)	(0.144)	(0.184)
Light	-0.399***	0.963***	-0.397***	1.398***	-0.335***	1.007***	-0.492***	1.482***	-0.232*	1.391***	0.148	1.327***
	(0.102)	(0.113)	(0.114)	(0.144)	(0.0995)	(0.130)	(0.122)	(0.135)	(0.124)	(0.156)	(0.113)	(0.145)
eta	0	0.916**	0	1.032***	0	1.152***	0	0.959**	0	0.511	0	-0.0499
	(0)	(0.438)	(0)	(0.390)	(0)	(0.446)	(0)	(0.410)	(0)	(0.587)	(0)	(0.687)
Observations	135,345	135,345	126,330	126,330	127,200	127,200	128,278	128,278	126,475	126,475	118,563	118,563
Quarter	2010Q3	2010Q3	2010Q4	2010Q4	2011Q1	2011Q1	2011Q2	2011Q2	2011Q3	2011Q3	2011Q4	2011Q4

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

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