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Collective Choice and the Lindahl Allocation Method

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INTRODUCTION

The cornerstone of traditional economic theory is the elegant and paradoxical result that in an economy of selfish men, resources can be allocated efficiently if consumers are allowed to pursue their own interests in competitive trading. This result is purchased at the cost of strong assumptions about the extent to which technology and preferences are compatible with individualism. It is assumed that commodities can be allocated among consumers in such a way that preferences of each consumer are concerned only with the quantity of commodities allocated to him. The individualistic consumer is not interested in the total supply of any commodity or in the quantities allocated to others.

It has been recognized, largely in the literature on "public goods," that there are commodities that do not lend themselves well to the individualistic

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formulation. This may be true because of the physical nature of commodities such as fireworks or impurities in the air, or it may be because of such psychological interactions as sympathy, envy, or emulation. In such cases it may be impossible to partition private rights or consumption in such a way that consumers are interested only in the commodities allocated to them. When commodities and preferences do not conform to the individualistic model, efficiency considerations may suggest the usefulness of collective action.

This paper suggests a formulation of the choice space that seems appropriate to a wide range of problems of joint consumption and interdependence of preference. There is a discussion of the specification and measurement of commodities and of the circumstances under which aggregation can be legitimately performed. Solutions are considered for the allocation of resources in any economy with public goods and interdependent preferences. These solutions are similar to the "Lindahl solution" suggested by Wicksell [25] and developed by Lindahl [16]. It it shown that what we call a "generalized Lindahl equilibrium" exists under very general circumstances and that there is a correspondence between the set of such equilibria and the set of Pareto optimal allocations. Some special assumptions on the nature of preferences are shown to lead to interesting simplifications of the structure of Lindahl equilibria.

Arrow [1] and Foley [14] have shown that the Arrow–Debreu proof of the existence of competitive equilibrium can be adapted to show the existence of a Lindahl equilibrium. The model presented here is somewhat more general in its assumptions about tastes and technology, and applies to a broader class of institutional arrangements. An effort is made to illuminate the special difficulties for the existence problem appearing in a nonindividualistic economy. Other discussions of Lindahl equilibrium appear in Milleron [19], Dreze and Poussin [13], Malinvaud [17], Starrett [24], Rader [22], and Bergstrom [4–6].

I.

Samuelson [23] suggests an interesting view of the public goods problem in which "public" and "private" commodities are treated as polar extremes. The analysis of efficient resource allocation then permits an elegant formal duality. Some question remains as to whether the "public goods" with which actual governments must be concerned fall neatly into the Samuelson polar case or whether such commodities are typically "mixed cases" which are neither purely private nor purely public.

The formulation that is chosen in this paper is one in which all commodities can be formally treated as purely public.[1] To avoid confusing connotations of

[1] This formulation is similar to that of Arrow [1].

the term "public goods," the basic commodities shall here be called communal commodities. Pure private commodities will appear as special cases of communal commodities. The so-called mixed cases turn out to be rather awkwardly specified aggregates of communal commodities. In order to subject the mixed cases to analysis it is necessary to disaggregate them into the communal commodities of which they are composed.

An attempt is made in the discussion to separate the formal theory from its economic interpretation. The discussion of the latter, which proceeds on an informal level is intended to illustrate that the theory when carefully applied has a reasonably "realistic" economic interpretation. The confusion that exists in the terminology of theoretical and applied public finance perhaps justifies this somewhat laborious examination.

A. Communal Commodities and Consumer Preference

There are n *consumers* (where n is a positive integer) indexed by the set $I = \{1, \ldots, n\}$. There are m *communal commodities* (where m is a positive integer) indexed by the set $J = \{1, \ldots, m\}$. An *allocation* is a point in nonnegative orthant Ω^m of the m-dimensional vector space, E^m. If $x = (x_1, \ldots, x_m)$ is an allocation, then where $j \in J$, x_j is said to be the *quantity of communal commodity* j *in allocation* x. For each consumer $i \in I$, there is a *preference relation* R_i defined on Ω^m that is complete, transitive, and reflexive. The corresponding *strict preference* and *indifference relations* P_i and I_i are defined in the usual way.

An economic interpretation of these terms is as follows. A consumer is a decision making unit, usually thought of as an individual or a family. There is a set A of conceivable states of the world. For any state of the world $\alpha \in A$, a vector $x \in \Omega^m$ can be found that lists the quantities of each of m distinct goods and services which appear in state of the world α. These vectors are called allocations. Each of the distinct goods and services is called a communal commodity. Communal commodities may be distinguished by time and location of availability as well as by to whom they are made available. The set A can be partitioned into subsets, each of which is characterized by a different vector in Ω^m. The statement "$x R_i y$" is interpreted to mean that if α and β are states of the world characterized by the vectors x and y respectively, then consumer i finds state of the world α at least as satisfactory as state β.

B. Aggregation and the Choice of Communal Commodities

When this interpretation is placed on the above definitions, the seemingly innocuous assumption "R_i is reflexive" requires that the set J of communal commodities be very carefully chosen if the economic interpretation of the theory is to be plausible. In particular, the equivalence classes that are generated by the set Ω^m of allocations must be sufficiently fine so that if any two states of the world are characterized by the same allocation, *everyone* finds them equally satisfactory.[2] For example, in the traditional theory where all consumers are "selfish," each consumer cares not how much bread is present in total but only how much bread is made available to him. Clearly, if there are k "private commodities" of the traditional sort, these must be represented in our model by nk communal commodities, a typical one of which is a particular private commodity available to a specified consumer.

In the aggregation of public goods, considerable care must be taken. It would not be reasonable to treat national defense as a communal commodity. Almost certainly, different consumers will place different relative importance on various aspects of defense expenditure. It is, therefore, most unlikely that any aggregation of armament efforts into a single scalar index would have the characteristic that for any two states of the world differing only in the aspect of armaments and for which the national defense index is identical, *all* consumers agree that the two states are equally satisfactory. Yet this is precisely what is required if R_i is to be reflexive for all $i \in I$.

These observations can be summarized in more formal terms. Consider a partition $\{A, B\}$ of the set J such that A consists of a commodities where $1 \leqslant a \leqslant m$. Allocations may be represented (x_A, x_B) where x_A and x_B are vectors of commodities in A and B respectively. Preferences of consumer i are said to be separable on A if whenever $(x_A, x_B') R_i (x_A', x_B')$ for some $x_B' \in \Omega^{m-a}$, it must be that $(x_A, x_B) R_i (x_A', x_B)$ for all $x_B \in \Omega^{m-a}$. If preferences are separable on A, one can unambiguously define the *projection* $\gtrsim_i^A$ of R_i on E^a so that $x_A \gtrsim_i^A x_A'$ if and only if $(x_A, x_B) R_i (x_A', x_B)$ for some $x_B \in \Omega^{m-a}$. One can also define the corresponding strict preference and indifference relations $>_i^A$ and $\sim_i^A$ in the natural way.

Preferences of consumers i and j are said to be *similar on A* if preferences of both consumers are separable on A and if for all x_A and x_A', $x_A \sim_i^A x_A'$ if and only if $x_A \sim_j^A x_A$. If preferences of i and j are separable on A and if $x_A \gtrsim_i^A x_A'$ if and only if $x_A \gtrsim_j^A x_A'$ then preferences of i and j are said to be *identical on A*.

Theorem 1 *If preferences of all consumers are representable by a continuous utility function U defined on* Ω^m, *then there exists a continuous real-valued aggregator function v defined on* Ω^a *such that preferences can be represented by a function* U_i^* *defined on* Ω^{m-a+1} *where* $U_i(x) = U_i^*(v(x_A), x_B)$ *for all* $x \in \Omega^m$ *if and only if preferences of all consumers are similar on A.*

Proof It is a well-known result that preferences of consumer i can be represented in the form of $U_i^*(v_i(x_A), x_B)$ if and only if preferences of i are

[2] It is here assumed that it is possible to generate a sufficiently fine partition of states of the world from a finite-dimensional commodity space. This might not be the case if one wishes, for example, to treat time as a continuum.

separable on A (see Gorman [15]). It is easily verified that one can choose identical v_i functions for each i if and only if preferences of all consumers are similar on A. q.e.d.

If preferences of all consumers are identical on A, then U_i^* will be monotonic in the same direction with respect to v for all $i \in I$. If preferences are similar but not identical, this is not true since some consumers may like and some may dislike increases in v.

C. Survival Sets and Feasible Allocations

For each consumer $i \in I$, there is a set $C_i \subset \Omega^m$ called the *survival set of i.* There is also a set $Z \subset \Omega^m$ called the *set of feasible allocations.*

The set C_i may be interpreted as the set of all allocations that enable consumer i to survive physically for at least some short period of time. Although it is convenient to assume that consumer preference is defined on all allocations in Ω^m, it will be useful to assume stronger properties for preferences on C_i than could reasonably be assumed for preferences over the entire nonnegative orthant. For example, the economic interpretations of the assumptions of local nonsatiation and continuity of preferences at all points in Ω^m is quite implausible (see Bergstrom [4]). If these properties are assumed only at points in C_i, the economic interpretation is more reasonable. Of course, in the formal theory, C_i is simply a set that has the properties attributed to it by the axioms in which it is mentioned. The theorems in this paper do not necessarily require that C_i be coextensive with the set of allocations that enable i to survive physically although this interpretation may be useful.

The set Z can be interpreted as the set of all allocations that can be attained given the resources and technological capabilities of the economy. The institutional framework of production is deliberately left unspecified. This enables us to develop a general theorem on the existence of equilibrium that will apply under many forms of industrial organization. There may, for example, be several firms operating with or without externalities as well as several governmental activities.

D. Prices, Wealth Distribution, and Lindahl Equilibrium

In "voluntaristic" collective decision models of the Wicksell–Lindahl type, unanimous agreement is gained for a single allocation by the device of confronting different consumers with (in general) different price vectors. An equilibrium allocation maximizes the preferences of *each* consumer subject to his budget constraint at equilibrium prices.[3] Furthermore, the value of the

[3] This method presents an interesting formal contrast to the competitive pricing of "private goods" where all consumers face the same prices but may choose different quantities.

equilibrium allocation when valued at the sum of the individual prices must be at least as high as that of any other feasible allocation.

A *Lindahl price vector* is a point $p = (p^1, \ldots, p^n) \in E^{mn}$ where for all $i \in I$, $p^i = (p_1{}^i, \ldots, p_m{}^i) \in E^m$. A wealth distribution function is a function W from E^{mn} to E^n such that for $p \in E^{mn}$,

$$W(p) = (W_1(p), \ldots, W_n(p)) \qquad \text{and} \qquad \sum_{i \in I} W_i(p) = \max_{z \in Z} \sum_{i \in I} p^i z.$$

The *budget correspondence* B_i of consumer i is a mapping from E^{mn} to the set of subsets of Ω^m such that $B_i(p) = \Omega^m \cap \{x \mid p^i x \leqslant W_i(p)\}$.

An example of a wealth distribution function is that of a "private ownership economy" (e.g., Debreu [12] in which all commodities are private and in which a consumer's wealth is the value of a vector of commodities that he owns before trade occurs plus some historically predetermined fraction of the (maximized) profits of each firm). Where there are communal commodities that are not private commodities, there is a great variety of conceivable arrangements of claims to commonly enjoyed commodities. For this reason it is useful to consider a very general wealth distribution function.

A *generalized Lindahl equilibrium* under the wealth distribution function W is a point $(\bar{p}, \bar{z}) \in E^{mn} \times Z$ such that:

(i) For all $i \in I$, $\bar{z}$ maximizes R_i on $\Omega^m \cap B_i(\bar{p})$.
(ii) $\Sigma_{i \in I} \bar{p}^i \bar{z} \geqslant \Sigma_{i \in I} \bar{p}^i z$ for all $z \in Z$.

Notice that if an individual regards a communal commodity as unpleasant, we would expect his equilibrium Lindahl price for that commodity to be negative.

II. THE EXISTENCE OF LINDAHL EQUILIBRIUM

Since the requirements for an allocation to be in equilibrium are rather intricate, it is not at all obvious that they can be simultaneously met for reasonable economies. It is shown here that generalized Lindahl equilibrium does exist for a large class of economies. The assumptions employed to prove existence are similar to those used in traditional proofs of the existence of competitive equilibrium. There are, however, some interesting novelties of interpretation which are due to the nonindividualistic nature of the economy. After the statement of our existence theorem, these interpretations will be discussed.

Theorem 2 *In an economy with a set I of n consumers and a set J of m communal commodities, there exists a generalized Lindahl equilibrium if the following assumptions are satisfied:*

(A) (Individual preferences) *For each* $i \in I$, *there is a preference ordering* R_i and a survival set $C_i \subset \Omega^m$ *such that:*

A–I R_i *is a complete quasi-ordering defined on* Ω^m.

A–II (Weak convexity and regularity) *For all* $x \in \Omega^m$, *the set* $R_i(x) = \{x' \mid x' R_i x\}$ *is closed and convex.*

A–III (Continuity along straight lines in C_i) *If* $x, x' \in C_i$ *and* $x P_i x'$, *then for some* λ *such that* $0 < \lambda < 1$, $(\lambda x + (1 - \lambda)x') P_i x'$.[4]

A–IV *The set* C_i *is closed and convex.*

A–V (Local nonsatiation on C_i) *If* $x \in C_i$, *then in every open neighborhood of* x *there is an allocation* x' *such that* $x' P_i x$.

A–VI *There is a nonzero vector,* $\hat{v}^i \in \Omega^m$ *such that for* $x \in C_i$, *and all real valued* $\lambda \geq 0, x + \lambda \cdot \hat{v}^i \in C_i \cap R_i(x)$.[5]

(B) (Technology) *There is a closed, bounded, convex set* $Z \subset \Omega^m$ *of feasible allocations.*

(C) (Wealth distribution) *There is a wealth distribution function* W *such that* $W(p) = (W_1(p), \ldots, W_n(p))$ *is continuous and homogeneous of degree one, and* $\Sigma_{i \in I} W_i(p) = \max_{z \in Z} \Sigma_{i \in I} p^i z$.

(D) (Social compatibility) *The set* $\cap_{i \in I} C_i \cap Z$ *has a nonempty interior.*

(E) (Preference minimal allocation) *There exists an allocation* $w \in \cap_{i \in I} C_i \cap Z$ *called a "preference minimal allocation" such that for all* $i \in I$, $R_i(w) \subset C_i$ *and for all* $p \in E^{mn}$, $B_i(p) \cap R_i(w) \neq \phi$.

(F) (Social conflict) *For all* $h \in I$, *if* $x R_h w$ *and* $x \in Z$, *then there exists* $\hat{x} \in \cap_{i \in I} C_i \cap Z$ *such that* $\hat{x} P_i x$ *for all* $i \neq h, i \in I$.

The interpretations of assumptions A–I – A–VI are familiar from the literature on competitive equilibrium (Debreu [12]). The only novelty is the interpretation of the set C_i discussed in Section I.C. Assumption B requires that the aggregate set of technically possible transformations be convex. As mentioned above, this formulation of the technology could apply to a variety of institutional arrangements. It might be observed that assumption B does allow any commodity whether private or not to be used in the production of other commodities.

Assumption C specifies the class of wealth distribution functions for which the theorem applies. As the discussion of Section I.D suggests, it is useful to consider a broader class of wealth distribution functions than simply those appropriate for a private ownership economy.

Assumption D demands a minimal amount of social compatibility in the sense that there must be some feasible allocation such that every individual can survive

[4] This assumption is slightly weaker than the familiar assumption that lower contour sets are closed. See Rader [21].

[5] Assumption A–VI involves no substantial loss of generality since an artificial commodity could be introduced which is both harmless and useless.

at that allocation or any allocation sufficiently close to it. A similar assumption is contained in the traditional models of private competitive equilibrium (see e.g. McKenzie [18, Assumption 5]). Where there are nonprivate commodities, assumption D requires also that there be no "public good" that is one man's necessity and another man's poison.

Assumption E places an additional restriction on the wealth distribution function. It is assumed that at any price vector, each individual's budget constraint allows him the choice of some allocation that he likes at least as well as the preference minimal allocation. The preference minimal allocation in the private ownership economy of the traditional models is the "initial" allocation. There the equivalent of assumption E is obtained by assuming that each consumer could survive with his initial holdings. Here the term *preference minimal* rather than *initial* is used because there is nothing in the formal structure requiring that the allocation w would persist in the absence of social intercourse. In fact it may be useful to interpret w as an allocation in which some communal commodities that provide at least a minimal social infrastructure are present by mutual agreement.

Assumption F requires conflict of interest in the economy in the following sense. If any consumer likes a feasible allocation x as well as the preference minimal allocation, then there exists another feasible allocation that is preferred by all other consumers to x. Thus it would be in the interest of all other consumers to "enslave" him if they could. In the individualistic private ownership economy, assumption F would follow either from the assumption that all consumers have initial holdings interior to their consumption sets and have locally nonsatiated preferences (Arrow and Debreu [2]) or the assumption that everyone can surrender some commodity vector which can be transformed into something desirable for each other consumer (McKenzie [18] or Bergstrom [13]). Since our model allows the possibility of benevolence or more complicated interactions between consumers, it might be that even if a consumer could surrender a quantity of a good desirable to all others, the others would not wish him to do so because of benevolent feelings for him. An example that illustrates this difficulty is presented in Bergstrom [4, p. 394].

The proof of Theorem 2 will be aided by the following lemmas. Lemma 1 is proved by Debreu [10].

Lemma 1 *Let P be the intersection of the unit sphere $\{p \mid p \in E^{mn}$ and $\Sigma_{i=1}^{mn} p_i^2 = 1\}$ with a convex cone in E^{mn} that is not a linear manifold. Let E be an upper semicontinuous mapping from P to the set of subsets of E^{mn} such that $E(p) = E(\lambda p)$ for all $\lambda > 0$ and all $p \in P$ and such that $px \leqslant 0$ whenever $x \in E(p)$. Then there exists a $\bar{p} \in P$ such that $E(\bar{p}) \cap -P^* \neq \phi$, where $P^* = \{x \mid px \geqslant 0$ for all $p \in P\}$.*

Lemma 2 *The assumptions of Theorem 2 excluding* A–III, D, *and* F *imply* *that there exists a feasible allocation* $\bar{z} \in Z$ *and a nonzero Lindahl price vector* $\bar{p} \in E^{mn}$ *such that:*

(i) *For all* $i \in I$, $\bar{z} \in C_i \cap \{x \mid \bar{p}^i x \leqslant W_i(\bar{p})\}$ *and if* $x \in \Omega^m \cap \{x \mid \bar{p}^i x < W_i(\bar{p})\}$, *then* $\bar{z} R_i x$.

(ii) $\Sigma_{i \in I} \bar{p}^i \bar{z} \geqslant \Sigma_{i \in I} \bar{p}^i z$ *for all* $z \in Z$.

If in part (i) of the conclusion $\leqslant$ were substituted for $<$, then the conclusion would state that $(\bar{p}, \bar{z})$ is a generalized Lindahl equilibrium. An allocation and a Lindahl price vector satisfying Lemma 2 is exactly analogous to what Debreu calls a "quasi-equilibrium" for a private economy (Debreu [11]).

Following the procedure of Debreu [11], we first restrict attention to choice on a bounded subset of Ω^m. Assumption B allows one to choose a closed bounded cube $\tilde{Z}$ in Ω^m that contains $Z \cap \Omega^m$ in its interior. Assumption A–VI allows us to restrict attention to a set of price vectors of the kind required for Debreu's lemma. For each $i \in I$, let $\hat{v}^i$ be the vector mentioned in A–VI. Define the set $V \equiv \{v \mid v = (\lambda_1 \hat{v}^1, \ldots, \lambda_n \hat{v}^n)$ and $\lambda_i \geqslant 0$ for all $i \in I\}$. Clearly V is a closed convex cone in Ω^{mn} that contains points other than the origin. Define the set $V^* \equiv \{p \mid p \in E^{mn}$ and $pv \geqslant 0$ for all $v \in V\}$. Then V^* is a closed convex cone that is not a linear manifold and if $p = (p^1, \ldots, p^n) \in V^*$, then $p^i v^i \geqslant 0$ for all $i \in I$. Define the set $P \equiv V^* \cap \{p \mid |p| = 1\}$.

Define the mappings, M, F_i, and E with domain P so that:

$$M(p) \equiv Z \cap \{z \mid \sum_{i \in I} p^i z \geqslant \sum_{i \in I} p^i z' \text{ if } z' \in Z\}.$$

$$F_i(p) \equiv \tilde{Z} \cap R_i(w) \cap B_i(p) \cap \{x \mid x R_i x' \text{ if } p^i x' < W_i(p) \text{ and } x' \in \tilde{Z}\}.$$

$$E(p) \equiv \{(z^1 - z, \ldots, z^n - z) \mid z^i \in F_i(p) \text{ for all } i \in I \text{ and } z \in M(p)\}.$$

It is not difficult to show that the assumptions of Lemma 2 imply that each mapping is upper semicontinuous and has nonempty convex image sets.

For all $i \in I$, if $z^i \in F_i(p)$, then $p^i z^i \leqslant W_i(p)$. Hence

$$\sum_{i \in I} p^i z^i \leqslant \sum_{i \in I} W_i(p) = \max_{\hat{z} \in Z} \sum_{i \in I} p^i \hat{z}.$$

Therefore, if $z \in M(p)$ and $x = (z^1 - z, \ldots, z^n - z) \in E(p)$, it must be that

$$px = \sum_{i \in I} p^i z^i - \max_{\hat{z} \in Z} \sum_{i \in I} p^i \hat{z} \leqslant 0.$$

Since W is homogeneous of degree 1, $E(\lambda p) = E(p)$ for all $\lambda > 0$.

The set P and the correspondence E thus satisfy the hypothesis of Lemma 1. Therefore for some $\bar{p} \in P$, $E(\bar{p}) \cap -P^* \neq \phi$. Since V is a closed convex cone, it follows from the duality theorem for closed convex cones that $-P^* = -V$. Therefore there exist $\bar{p} \in P$, $\bar{z} \in M(\bar{p})$, $(\bar{z}^1, \ldots, \bar{z}^n) \in \Pi_{i \in I} F_i(\bar{p})$, and $(\bar{v}_1, \ldots, \bar{v}_n) \in V$ such that for all $i \in I$, $\bar{z}^i + \bar{v}^i = \bar{z}$.

Since $\bar{z}^i \in F_i(\bar{p})$ and $\bar{z} = \bar{z}^i + \bar{v}^i R_i \bar{z}^i$, we can be assured for all $i \in I$, that

$\bar{z} \in F_i(\bar{p})$ if $\bar{p}^i \bar{z} \leqslant W_i(\bar{p})$. This is demonstrated as follows. Since $\bar{z} \in Z \subset \operatorname{int} \tilde{Z}$ and since $\bar{z}^i \in F_i(\bar{p})$, and $\bar{z} R_i \bar{z}^i$, it follows from the assumption of local nonsatiation that $\bar{p}^i \bar{z} \geqslant W_i(\bar{p})$ for all $i \in I$. But

$$\sum_{i \in I} \bar{p}^i \bar{z} = \max_{z \in Z} \sum_{i \in I} \bar{p}^i \bar{z} = \sum_{i \in I} W_i(\bar{p}).$$

Therefore $\bar{p}^i \bar{z} = W_i(\bar{p})$ for all $i \in I$. Hence $\bar{z} \in F_i(\bar{p})$ for all $i \in I$. Since, also $\bar{z} \in M(\bar{p})$, conditions (i) and (ii) of the lemma are satisfied except for the restriction of choice to $\tilde{Z}$. To remove the artificial bounding cube $\tilde{Z}$, simply follow the procedure of Debreu [11]. q.e.d.

We can now complete the proof of Theorem 2. It remains to show that an allocation z that satisfies Lemma 2 actually maximizes preferences on $B_i(\bar{p})$ for all $i \in I$. If for any consumer $i \in I, \bar{z}$ does not maximize R_i on $B_i(\bar{p})$, then it follows by a well-known argument of Debreu [12] that $\min_{x \in C_i} \bar{p}^i x = W_i(\bar{p})$. Suppose that for some $h \in I, \min_{x \in C_h} \bar{p}^h x = W_h(\bar{p})$.
Since $z R_h w$, there is an allocation $\hat{z} \in \bigcap_{i \in I} C_i \cap Z$ such that $\hat{z} P_i \bar{z}$ for all $i \in I$, $i \neq h$. Since $\hat{z} \in C_h$, $\bar{p}^h \hat{z} \geqslant W_h(\bar{p})$. Since $\hat{z} P_i \bar{z}$ for all $i \in I, i \neq h$, it must be that $\bar{p}^i \hat{z} \geqslant W_i(\bar{p})$ for all $i \in I$ and that strict inequality obtains for every $i \in I$, such that $\min_{x \in C_i} \bar{p}^i x < W_i(\bar{p})$. Assumption D implies that for some $i \in I$, $\min_{x \in C_i} \bar{p}^i x < W_i(\bar{p})$.[6] Therefore

$$\sum_{i \in I} \bar{p}^i z > \sum_{i \in I} W_i(\bar{p}) = \max_{z \in Z} \sum_{i \in I} \bar{p}^i z.$$

But since $\hat{z} \in Z$, this is a contradiction. Therefore we must conclude that for all $i \in I$, $\min_{x \in C_i} \bar{p}^i x < W_i(\bar{p})$. It then follows that for all $i \in I$, $\bar{z}$ maximizes R_i on $B_i(\bar{p})$. q.e.d.

The next result is a stronger version of Theorem 2, which will be useful for showing that when preferences and/or production possibility sets have special structure, there can be found Lindahl equilibrium prices that have special form. For example, we shall wish to show that where there are classical private goods, these will have "private" prices and where there is crowding of an impersonal sort, the corresponding Lindahl prices can take the form of uniform "tolls" for users.

Theorem 3 *For each $i \in I$, let S_i be a linear subspace of E^m such that for all $z^i \in C_i$, $(z^i + S_i) \cap C_i \subset R_i(z_i)$. Let Y be a linear subspace in E^m such that $(Z + Y) \cap C_I \subset Z$ (where $C_I = \bigcap_{i \in I} C_i$). If $\bigcap_{i \in I} (C_i + S_i) \subset C_I$, and if the assumptions of Theorem 2 hold, then there exists a Lindahl equilibrium $(\bar{p}, \bar{z})$*

[6] Suppose that for all $i \in I$, $\bar{p}^i x \geqslant W_i(\bar{p})$ for all $x \in C_i$. Then if $x \in \bigcap_{i \in I} C_i$, $\sum_{i \in I} \bar{p}^i x \geqslant \sum_{i \in I} W_i(\bar{p}) = \max_{z \in Z} \sum_{i \in I} \bar{p}^i z$. But this means that $\bigcap_{i \in I} C_i$ and Z are separated by $\sum_{i \in I} \bar{p}^i$ which is impossible if $\bigcap_{i \in I} C_i \cap Z$ has a nonempty interior.

such that for all $i \in I$, $\bar{p}^i x = 0$ *for all* $x \in S_i$ *and such that* $\Sigma_{i \in I} \bar{p}^i y = 0$ *for all* $y \in Y$.

Proof Let $S = \Pi_{i \in I} S_i$ and let $T = \{(y, \ldots, y) \mid y \in Y\}$. Let $Q = \{p \in E^{mn} \mid |p| = 1\} \cap V^* \cap S^* \cap T^*$.[7] Since S and T are linear subspaces, if $p \in Q$, then $p^i x^i = 0$ for all $x^i \in S_i$, and $\Sigma_{i \in I} p^i y = 0$ for all $y \in Y$. Thus Theorem 3 will be established if we demonstrate the existence of Lindahl equilibrium $(\bar{p}, \bar{z})$ where $\bar{p} \in Q$. To do this we need only to show that there exists $(\bar{p}, \bar{z})$ where $\bar{p} \in Q$ which satisfies conditions (i) and (ii) of Lemma 2. The same argument used to prove Theorem 2 from Lemma 2 can then be applied to show that $(\bar{p}, \bar{z})$ is a Lindahl equilibrium.

Replacing the set P by the set Q in the arguments of the first part of the proof of Lemma 2 leads us to conclude that there exists $\bar{p} \in Q$ such that $E(\bar{p}) \cap -Q^* \neq \phi$. From the theory of convex cones, it is known that

$$Q^* = (V^* \cap S^* \cap T^*)^* = V^{**} + S^{**} + T^{**} = V + S + T.$$

Thus for some $\bar{p} \in Q$, $E(\bar{p}) \cap - V - S - T \neq \phi$.

Therefore there exists $(\bar{z}^1, \ldots, \bar{z}^n) \in \Pi_{i \in I} F_i(\bar{p})$, $(\bar{x}^1, \ldots, \bar{x}^n) \in S$, $(\bar{v}^1, \ldots, \bar{v}^n) \in V$, $\bar{z} \in M(\bar{p})$, and $\bar{y} \in Y$ such that for all $i \in I$, $\bar{z}^i - \bar{z} = -\bar{v}^i - \bar{x}^i - \bar{y}$, or equivalently $\bar{z} - \bar{y} = \bar{z}^i + \bar{v}^i + \bar{x}^i$. Since $\bar{z}^i + \bar{v}^i \in C_i$ for all $i \in I$, $\bar{z} - \bar{y} \in \cap_I (C_i + S_i) \subset C_i$. Therefore $\bar{z} - \bar{y} \in (Z + Y) \cap C_I \subset Z \cap C_I$. Let $\bar{\bar{z}} = \bar{z} - \bar{y}$. Since $\Sigma_{i \in I} \bar{p}^i \bar{y} = 0$, $\bar{\bar{z}} \in M(\bar{p})$. For all $i \in I$, $\bar{\bar{z}} R_i \bar{z}^i$ since $\bar{\bar{z}} \in C_i \cap \bar{z}^i + V_i + S_i$. Also $\bar{z}^i \in F_i(\bar{p})$ and $\bar{\bar{z}} \in Z \subset -\text{Int}\, \bar{Z}$. From the assumption of local nonsatiation it follows that $\bar{p}^i \bar{\bar{z}} \geqslant W_i(\bar{p})$ for all $i \in I$. But since $\bar{\bar{z}} \in M(\bar{p})$, it must be that $\Sigma_{i \in I} \bar{p}^i \bar{\bar{z}} = \Sigma_{i \in I} W_i(\bar{p})$. Therefore for all $i \in I$, $\bar{p}^i \bar{\bar{z}} = W_i(\bar{p})$ and hence $\bar{\bar{z}} \in \cap_{i \in I} F_i(\bar{p}) \cap M(\bar{p})$. Thus $(\bar{p}, \bar{\bar{z}})$ satisfies the conditions of Lemma 2. q.e.d.

III. SPECIAL LINDAHL EQUILIBRIA

A. Private Commodities

Private commodities will be treated as special cases of communal commodities, and it will be shown that there exist Lindahl equilibria with a corresponding special structure.

In the classical model of a private goods economy, one man's bread can be costlessly transformed into another man's bread. Furthermore each individual is interested in his own bread consumption but not in the bread consumption of others. These notions motivate the following definitions.

Consumer i is said to be *not concerned with commodity* j if whenever $x \in C_i$, and $x' \in \Omega^m$ such that $x_k' = x_k$ for all $k \neq j$, it follows that $x' \in C_i$ and $x' I_i x$.

[7] For any set X, X^* denotes the polar cone of X.

Otherwise consumer j is said to be *concerned with commodity j*. A *private commodity specific to consumer i* is a communal commodity j such that i is concerned with commodity j but no consumer other than i is concerned with j.

A set $k = \{k_1, \ldots, k_s\}$ of communal commodities is said to be *freely exchangeable* if $(Z + T_k) \cap \Omega^m \subset Z$ where $T_k = \{x \in E^m \mid x_j = 0 \text{ for } j \notin k \text{ and } \Sigma_{k_i \in k} x_{k_i} = 0\}$. A *nonspecific exchangeable private commodity* is a freely exchangeable set $k = \{k_1, \ldots, k_n\}$ of communal commodities such that for each $i \in I$, k_i is a private commodity specific to consumer i. Let K be the set of all nonspecific exchangeable private commodities. Then where $k \in K$ and $i \in I$, the communal commodity k_i is an *exchangeable private commodity specific to consumer i.*

These definitions simply define in formal language the properties of private commodities that are implicitly assumed in the classical theory. Thus if k is the nonspecific, exchangeable private commodity "bread," then k_i is the specific exchangeable private commodity "bread for consumer i."

It will be seen that the pricing of specific exchangeable private commodities in Lindahl equilibrium is similar to that of private commodities in a competitive economy. A *Lindahl–Samuelson* equilibrium is a generalized Lindahl equilibrium $(\bar{p}, \bar{z})$ such that for every nonspecific exchangeable commodity $k \in K$ and for all $i, j \in I$, $\bar{p}^i_{k_i} = \bar{p}^j_{k_j} \equiv \bar{p}_k$ and if $i \neq j$, $p^i_{k_j} = 0$. In a Lindahl–Samuelson equilibrium, then Mr. i pays the same price for the commodity, "bread for i," as Mr. j pays for the commodity "bread for j." Also Mr. i pays nothing for the commodity "bread for j." In the special case where all communal commodities are specific exchangeable private commodities, a Lindahl–Samuelson equilibrium is a competitive equilibrium.

Theorem 4 *If the assumptions of Theorem 2 hold, there exists a Lindahl–Samuelson equilibrium.*

Proof We apply Theorem 3. For each $i \in I$, let $S_i = \{x^i \in E^m \mid x^i_j = 0 \text{ if } j \neq k_h$ for some $k \in K$ and some $h \neq i\}$. Let $Y = \Sigma_{k \in K} T_k$. Clearly the sets S_i and Y are linear subspaces in E^m. Since for all $i \in I$, $C_i \subset \Omega^m$, the definition of exchangeable private commodities implies that for all $z^i \in C_i$, $(z^i + S_i) \cap C_i \subset R_i(z^i)$ and that $(Z + Y) \cap C_I \subset Z$. If $z \in \bigcap_{i \in I}(C_i + S_i)$, then $z \in \Omega^m$. Therefore

$$\bigcap_{i \in I} (C_i + S_i) = \bigcap_{i \in I} ((C_i + S_i) \cap \Omega^m) = \bigcap_{i \in I} C_i.$$

Thus the hypothesis of Theorem 3 is true. It follows that there exists a generalized Lindahl equilibrium $(\bar{p}, \bar{z})$ such that for all $i \in I, \bar{p}^i x = 0$ for all $x \in S_i$ and such that $\Sigma_{i \in I} \bar{p}^i y = 0$ for all $y \in Y$. If $\bar{p}^i x = 0$ for all $x \in S_i$, then evidently $\bar{p}^i_{k_h} = 0$ for all $k \in K$ and all $h \neq i$. If $\Sigma_{i \in I} \bar{p}^i y = 0$ for all $y \in Y$, then for all $k \in K$ and all $h, h' \in I$, $\Sigma_{i \in I} \bar{p}^i_{k_h} = \Sigma_{i \in I} \bar{p}^i_{k_{h'}}$. But since $\bar{p}^i_{k_h} = 0$ if $i \neq h$, it must be that for all $k \in K$ and all h, $i \in I$, $\bar{p}^h_{k_h} = \bar{p}^i_{k_i} \equiv \bar{p}_k$. Thus $(\bar{p}, \bar{z})$ is a Lindahl–Samuelson equilibrium. q.e.d.

B. Congestion and Impersonal Externalities

It is of some interest to examine the way in which congested public facilities can be treated within our model. Where a public facility such as a highway or park is subject to crowding, the commodity space must be sufficiently disaggregated so that allocations describe not only the size of the facility but also the amount of crowding. If, for example, each consumer's enjoyment of a park depends on the size of the park, the extent to which he uses it, and the extent to which each other consumer uses it, we would wish to include as distinct communal commodities the utilization of the park by each consumer (possibly treating use at different times as different commodities) as well as the size of the park.

Often it would seem reasonable to suppose that individuals do not care *who* is crowding them but are interested only in the size of the facility, the *total* use of the facility by others, and in the extent to which they themselves use it. If this is the case, the system of Lindahl prices can be somewhat simplified.

An *impersonal externality* is a set $m = \{m_1, \ldots, m_n\} \subset J$ such that for all $i \in I$, if z, $z' \in C_i$ and if $z_j = z_j'$ for all $j \notin m$, $z_{m_i} = z'_{m_i}$, and $\Sigma_{h \in I} z'_{m_h} = \Sigma_{h \in I} z'_{m_h}$, then $z \, I_i \, z'$. Thus the impersonal externality m might be use of a park. The communal commodity m_i is use of the park by consumer i. Consumer i is indifferent between any two allocations that differ in the distribution of park use among others, but which have the same use by himself, the same *total* use by others, and the same quantities of all other communal commodities.

A further simplification of the system of Lindahl prices can be made if we are willing to assume that use of a particular public facility by any individual does not directly affect aggregate production possibilities for other commodities. This assumption is made explicit as follows. A communal commodity j is *independently produced* if $Z + Y_j \cap C_I \subset Z$ where Y_j is the linear subspace $\{y \in E^m \mid y_k = 0 \text{ if } k \neq j\}$. In effect, all restrictions on the feasible quantities of commodity j are expressed as restrictions on the sets C_i. As an example, suppose that commodity j is use of a specified park by a certain individual. Of course this individual cannot possibly spend more than 24 hours a day in the park. Furthermore, time he spends in the park cannot be spent using other public facilities, nor can this time be spent working. These constraints can be incorporated in the model by obvious restrictions on the sets C_i. Once this is done, there is no reason to suppose that use of the park, as such, has any direct effect on production possibilities for other commodities. Thus we may reasonably assume that $Z + Y_j \cap C_I \subset Z$.

It turns out that if m is an impersonal externality such that m_i is independently produced for each $i \in I$, then there exists a Lindahl equilibrium $(\bar{p}, \bar{z})$ such that for each $i \in I$, there is some scalar $\bar{p}_m{}^i$ such that $\bar{p}^i_{m_k} = \bar{p}_m{}^i$ for all $k \neq i$, and such that for all $i \in I$, $\Sigma_{j \in I} \bar{p}^i_{m_j} = 0$. When this is the case, the portion of consumer i's budget constraint that relates to the impersonal externality is

$$\begin{aligned} \bar{p}^i_{m_i} x_{m_i} + \sum_{j \neq i} \bar{p}_j{}^i x_{m_j} &= \bar{p}^i_{m_i} x_{m_i} + \bar{p}_{\bar{m}}{}^i \sum_{j \neq i} x_{m_j} \\ &= -\sum_{j \in I} \bar{p}_{\bar{m}}{}^j x_{m_i} + \bar{p}_m{}^i \sum_{j \in I} x_{m_j} \\ &= \bar{q}_m x_{m_i} - t_m{}^i \bar{q}_m \sum_{j \in I} x_{m_j} \end{aligned}$$

where $\bar{q}_m = -\Sigma_{j \in I} \bar{p}_{\bar{m}}{}^j$ and $t_m{}^i = \bar{p}_{\bar{m}}{}^i / \Sigma_{j \in I} \bar{p}_{\bar{m}}{}^j$. Suppose, for example, the impersonal externality m is use of a public facility and that each individual dislikes being crowded by others. Then the $\bar{p}_{\bar{m}}{}^i$ will all be negative. The Lindahl prices reduce to a system of uniform tolls for use of the facility where each consumer pays at the rate $\bar{q}_m = -\Sigma_{j \in I} \bar{p}_{\bar{m}}{}^j$ for the use of the facility. Revenue from the toll is disbursed in such a way that consumer i receives the fraction $t_m{}^i$ of the total revenue from tolls for each i.

It is of some interest to look at the very special case where preferences are identical so that there is a Lindahl equilibrium in which for some $\bar{p}_m, p_{\bar{m}}{}^i = \bar{p}_m$ for all $i \in I$. Where there are n consumers, $\bar{q} = n\bar{p}_m$ and $t_m{}^i = 1/n$. Each consumer i in equilibrium chooses the same value $x^i_{m_j} = x_m$ for all $j \in I$. Thus he pays tolls amounting to $\bar{q}x_m = n\bar{p}_m x_m$. He receives $n^{-1}\bar{q}nx_m = n\bar{p}_m x_m$ from the total toll revenue.

Thus his receipts from tolls precisely equal his payments. Of course his behavior is by no means the same as in a system without tolls. The cost to him of using the road by an extra amount Δ is

$$\bar{q}\Delta - t_m{}^i \bar{q}\Delta = \bar{p}_m n \left(1 - \frac{1}{n}\right)\Delta = \bar{p}_m(n-1)\Delta.$$

Thus the marginal private cost is $\bar{p}_m(n-1)$, which is in turn equal to marginal social cost.

Theorem 5 proves the assertions we have made about the special structure of Lindahl prices for impersonal externalities and independently produced commodities.

Theorem 5 *If the assumptions of Theorem 2 are true and if m is an impersonal externality, then there exists a generalized Lindahl equilibrium $(\bar{p}, \bar{z})$ in which for all $i \in I$ there is some $\bar{p}_{\bar{m}}{}^i$ such that $\bar{p}^i_{m_j} = \bar{p}_{\bar{m}}{}^i$ for all $j \neq i$. Furthermore if the communal commodity m_l is independently produced, then $\Sigma_{j \in I} \bar{p}^i_{m_j} = 0$.*

Proof Let $S_i = \{x^i \in E^m \mid x^i_j = 0$ if $j \neq m_k$ for some $k \neq i\} \cap$

$\{x_i \in E^m \mid \Sigma_{h \in I} x^i_{m_h} = 0\}$. Let N be the set $\{m_j \in m \mid m_j$ is independently produced$\}$ and let $Y = \Sigma_{m_j \in N} Y_{m_j}$. The sets S_i and Y are linear subspaces. Also $Z + Y \cap C_I \subset Z$ and $\bigcap_{i \in I}(C_i + S_i) \subset C_I$. We can therefore apply Theorem 3. There exists a generalized Lindahl equilibrium $(\bar{p}, \bar{z})$ such that for all $i \in I$ and all $x \in S_i, \bar{p}^i x = 0$ and such that for all $y \in Y, \Sigma_{i \in I} \bar{p}^i y = 0$. But this implies that for all $i \in I$ and for all $h, k \in I$ such that $h \neq i, k \neq i, \bar{p}^i_{m_k} = \bar{p}^i_{m_h} \equiv \bar{p}_{\bar{m}}{}^i$ and that for all $m_j \in N$, $\Sigma_{i \in I} \bar{p}^i = 0$. q.e.d.

C. Aggregation and Decentralized Lindahl Equilibrium

In Section I, conditions were stated under which it is possible to aggregate commodities in some subset A of the commodity space. It seems reasonable that one might be able to find a simplified structure for the Lindahl equilibrium price vectors allowing some decentralization of decision making on subsets of the commodity space that admit aggregation. Theorem 6 below states a result of this kind.

A *decentralized Lindahl equilibrium* for the partition $\{A_1, \ldots, A_r, B\}$ of J is a generalized Lindahl equilibrium $(\bar{p}, \bar{x})$ such that for each A_k, $k = 1, \ldots, r$, there exists a nonnegative vector $(\bar{\lambda}_k{}^1, \ldots, \bar{\lambda}_k{}^n)$ where $\Sigma_{i \in I} \bar{\lambda}_k{}^i = 1$ and such that for all $i \in I$ and all $j \in A_k$, $\bar{p}_j{}^i = \bar{\lambda}_k{}^i \bar{p}_j$ where $\bar{p}_j \equiv \Sigma_{i \in I} \bar{p}_j{}^i$.

Theorem 6 *If the assumptions of Theorem 2 are true, if $W(p) = W(\hat{p})$ whenever $\Sigma_{i \in I} p^i = \Sigma_{i \in I} \hat{p}^i$, and if for the partition $\{A_1, \ldots, A_r, B\}$ of J, preferences of all consumers are identical on A_k for each $k = 1, \ldots, r$, then there exists a decentralized Lindahl equilibrium for the partition $\{A_1, \ldots, A_r, B\}$.*

Proof It will be shown that Theorem 3 holds in the case where $r = 1$. A simple induction argument extends the result to $r \geqslant 1$. Suppose that J is partitioned into sets A and B such that preferences of all consumers are identical on A. Let $(\bar{p}, \bar{x})$ be a generalized Lindahl equilibrium. Define $\bar{p}_A \equiv \Sigma_{i \in I} \bar{p}_A{}^i$. Let

$$V \equiv \{(z_1, \ldots, z_n, y_1, \ldots, y_n) \mid \text{for all } i \in I, z_i = \bar{p}_A(x_A{}^i - \bar{x}_A) \text{ and } y_i = \bar{p}_B{}^i(x_B{}^i - \bar{x}_B) \text{ where } (x_A{}^i, x_B{}^i) R_i \bar{x}\}.$$

Let

$$H = \{(\underbrace{z, \ldots, z}_{n \text{ times}}, y_1, \ldots, y_n) \mid z + \sum_{i \in I} y_i \leqslant 0\}.$$

It is easily verified that H and V are convex sets in E^{2n} containing the origin. We now show that V does not intersect the interior of H. Suppose $(z, \ldots, z, y_1, \ldots, y_n) \in V$. Then for each $i \in I$, there exists an $(\hat{x}_A{}^i, \hat{x}_B{}^i)$ such that $(\hat{x}_A{}^i, \hat{x}_B{}^i) R_i \bar{x}$ and such that $\bar{p}_A(\hat{x}_A{}^i - \bar{x}_A) = z$ and $p_B{}^i(\hat{x}_B{}^i - \bar{x}_B) = y_i$. Since preferences of all consumers are identical on A, the projection of each preference ordering R_i on A gives the same ordering $\gtrsim_A$, on A for each $i \in I$. Choose a bundle $\hat{x}_A$ that maximizes $\gtrsim_A$ subject to $\bar{p}_A(x_A - \bar{x}_A) < z$. Clearly

$(\hat{x}_A, \hat{x}_B) R_i \bar{x}$ for all $i \in I$. But this implies that for all $i \in I$,

$$\bar{p}_A{}^i(\hat{x}_A - \bar{x}_A) + \bar{p}_B{}^i(\hat{x}_B - \bar{x}_B) \geqslant 0.$$

Hence

$$\left(\sum_{i\in I} \bar{p}_A{}^i\right)(\hat{x}_A - \bar{x}_A) + \sum_{i\in I} \bar{p}_B{}^i(\hat{x}_B{}^i - \bar{x}_B) \geqslant 0.$$

But

$$z = \left(\sum_{i\in I} \bar{p}_A{}^i\right)(\hat{x}_A - \bar{x}_A).$$

Therefore $z + \sum_{i\in I} y_i \geqslant 0$. It follows that $(z, \ldots, z, y_1, \ldots, y_n)$ cannot be in the interior of H. By the separation theorem for convex sets, there is a vector $(\bar{\lambda}^1, \ldots, \bar{\lambda}^n, \bar{\mu}^1, \ldots, \bar{\mu}^n)$ such that:

(i) $(\sum_{i\in I}\bar{\lambda}^i)z + \sum_{i\in I}\bar{\mu}^i y_i \leqslant 0$ if $(z, \ldots, z, y_1, \ldots, y_n) \in H$; and
(ii) $\sum_{i\in I}\bar{\lambda}^i z_i + \sum_{i\in I}\bar{\mu}^i y_i \geqslant 0$ if $(z_1, \ldots, z_n, y_1, \ldots, y_n) \in V$.

But (i) implies that for some such vector, $\sum_{i\in I}\bar{\lambda}^i = 1$ and $\bar{\mu}^i = 1$ for all $i \in I$. It is easily shown that (ii) implies that for all $i \in I$, $\bar{\lambda}^i \geqslant 0$ and that if $(x_A, x_B) R_i \bar{x}$ then $\bar{\lambda}^i \bar{p}_A x_A + \bar{p}_B{}^i x_B \geqslant \bar{\lambda}^i \bar{p}_A \bar{x}_A + \bar{p}_B{}^i x_B$. It is not hard to show that $\bar{\lambda}^i \bar{p}_A \bar{x}_A + \bar{p}_B{}^i \bar{x}_B = W_i(\bar{p})$ for all $i \in I$. Applying the same procedure used in the final step of Theorem 2 one can show that $\bar{x}$ maximizes R_i subject to $\bar{\lambda}^i \bar{p}_A x_A + \bar{p}_B{}^i x_B \leqslant W_i(\bar{p})$ for each $i \in I$. This completes the proof. q.e.d.

This theorem could be given the following interpretation for decentralized governmental administration. If preferences of all consumers are identical on A_k for $k = 1, \ldots, r$, one could delegate the responsibility for providing commodities in each A_k to a separate bureau. Each bureau would be given a budget b_k and instructed to maximize the commonly held projection $\succsim A_k$, of preferences on A_k subject to $\sum_{j\in A_k} \bar{p}_j x_j \leqslant b_k$. A share $\lambda_k{}^i$ of the budget b_k is assigned to be paid by each consumer i. Equilibrium cost shares $\bar{\lambda}_k{}^i$ and budgets $\bar{b}_k$ are found so that, knowing how each bureau will spend its budget, each consumer agrees that so long as his cost share is $\bar{\lambda}_k{}^i$, the proper size for the budget of the bureau is $\bar{b}_k$.

Another application of Theorem 3 concerns the problem of Pareto efficient income redistribution among benevolent consumers. This problem is discussed in Bergstrom [4]. Theorem 5 provides an alternative proof of the main theorem of that paper.

IV.

A. Generalized Lindahl Equilibrium is Pareto Optimal

Theorem 7 *If $(\bar{p}, \bar{z})$ is a generalized Lindahl equilibrium and if preferences of all consumers are locally nonsatiated at $\bar{z}$, then the allocation $\bar{z}$ is Pareto optimal.*

Proof As in the traditional proofs of the optimality of competitive equilibrium it can be shown that for all $i \in I$, if $z P_i \bar{z}$, then $\bar{p}^i z > \bar{p}^i \bar{z} = W_i(\bar{p})$ and if $z R_i \bar{z}$, then $\bar{p}^i z \geqslant \bar{p}^i \bar{z} = W_i(\bar{p})$. Therefore if the allocation z' is Pareto superior to $\bar{z}$, it must be that $\Sigma_{i \in I} \bar{p}^i z' > \Sigma_{i \in I} W_i(\bar{p})$. But by assumption, $\Sigma_{i \in I} W_i(\bar{p}) \geqslant \Sigma_{i \in I} \bar{p}^i z$ for all $z \in Z$. Therefore $z' \notin Z$. It follows that $\bar{z}$ is Pareto optimal. q.e.d.

B. Pareto Optimal Allocations Can Be Sustained as Generalized Lindahl Equilibria

Theorem 8 *If assumptions* A, B, D, *and* E *of Theorem 2 are true and if* $\bar{z}$ *is a Pareto optimal allocation such that:*

(F′) *For all* $h \in I$, *there exists a* $\hat{z} \in \bigcap_{i \in I} C_i \cap Z$ *such that* $\hat{z} P_i \bar{z}$ *for all* $i \neq h$, $i \in I$, *then there exists a wealth distribution vector* $W = (W_1, \ldots, W_n)$ *and a Lindahl price vector* $\bar{p}$ *such that* $(\bar{p}, \bar{z})$ *is a generalized Lindahl equilibrium under the income distribution function* $W(p) = \overline{W}$.

Proof Consider the sets R^* and Z^* where $R^* = \Pi_{i \in I} R_i(\bar{z})$ and $Z^* = \{(z, \ldots, z) \mid z \in Z\}$. Assumptions A–II and B ensure that R^* and Z^* are convex sets. Local nonsatiation implies that R^* does not intersect the interior Z. According to the separation theorem for convex sets, there exists a vector $\bar{p} = (\bar{p}^1, \ldots, \bar{p}^n) \in E^{mn}$ that separates R^* and Z^*. Since $\bar{z} \in R^* \cap Z^*$, the following results follow:

(i) For all $i \in I$, if $z \in R_i(\bar{z})$, then $\bar{p}^i z \geqslant \bar{p}^i \bar{z}$.
(ii) For all $z \in Z$, $\Sigma_{i \in I} \bar{p}^i z \leqslant \Sigma_{i \in I} \bar{p}^i \bar{z}$.

It remains to be shown for all $i \in I$ that if $z P_i \bar{z}$, then $\bar{p}^i z > \bar{p}^i \bar{z}$. Assumption A–III and result (i) of the previous paragraph can be used to show that this is the case for i such that $\min_{z \in C_i} \bar{p}^i z < \bar{p}^i \bar{z}$. It will be demonstrated that in fact $\min_{z \in C_i} \bar{p}^i z < \bar{p}^i z$ for all $i \in I$. Suppose that for some $h \in I$, $\min_{z \in C_i} \bar{p}^i z = \bar{p}^i \bar{z}$. By assumption F′ there is an allocation $\hat{z} \in \bigcap_{i \in I} C_i$ such that $\hat{z} P_i \bar{z}$ for all $i \in I$ where $i \neq h$. Result (i) above implies that $\bar{p}^i \hat{z} \geqslant \bar{p}^i \bar{z}$ for all $i \in I$. As demonstrated in footnote 6, assumption D implies that $\min_{z \in C_i} \bar{p}^k z < \bar{p}^k \bar{z}$ for some $k \in I$. Therefore $\bar{p}^k \hat{z} > \bar{p}^k \bar{z}$. Hence $\Sigma_{i \in I} \bar{p}^i \hat{z} > \Sigma_{i \in I} \bar{p}^k \bar{z}$. But this contradicts result (ii) of the previous paragraph. Therefore for all $i \in I$, $\min_{z \in C_i} \bar{p}^i z < \bar{p}^i \bar{z}$. It follows that $\bar{z}$ maximizes R_i on $\{z \mid \bar{p}^i z \leqslant \bar{p}^i \bar{z}\}$ for all $i \in I$. Therefore the Lindahl price vector $\bar{p}$ and the wealth distribution vector $(\overline{W}_1, \ldots, \overline{W}_n) = (\bar{p}^1 \bar{z}, \ldots, p^n \bar{z})$ satisfy the conditions of Theorem 4. q.e.d.

V.

A. Units of Measurement and Convexity

The assumption that specific private commodities are freely exchangeable puts a restriction on the choice of units of measurement. In particular, suppose that a nonspecific private commodity k can be measured in two different choices of units such that $y_k = f(x_k)$ where x_k is the quantity of k measured in one choice of unit and y_k in the other. If specific private commodities in k are to be freely exchangeable under either choice of unit, then it must be that $f(\Sigma_{i \in I} x_{ki}) = \Sigma_{i \in I} f(x_{ki})$. But this implies that $f(x) = cx$ for some scalar x. Thus the choice of units of measurement is determined up to a multiplicative constant if the property of free exchange ability is to hold.

One has more freedom of choice of units in measuring quantities of communal commodities that are not freely exchangeable. If some method of measurement provides a sufficiently fine partition of states of the world as discussed in Section 1.A, then any choice of units that are strictly monotonic functions of the initially chosen units will also allow a sufficient amount of disaggregation. If, however, one wishes to satisfy the assumptions of Theorem 2, care must be taken to choose units so that preferences are weakly convex *and* the set of feasible allocations is convex.

The following example illustrates this principle. Consider a village in which there is one nonspecific private commodity, bread, and one nonprivate communal commodity, a church tower. Preferences of consumers are represented by $u_i = x_i + a_i h^2$ where x_i is bread consumed by i and h is the height of the church tower measured in feet. Where $a_i \geqslant 0$, preferences are nonconvex in terms of these units. The community initially holds a fixed stock w of bread. Bread may be converted into church tower by means of a production function of the form $h = \sqrt{x_h}$ where x_h is the quantity of bread used to build the tower. The set of feasible allocations is then $\Omega^{n+1} \cap \{(x_1, \ldots, x_n, h) \mid \Sigma_{i=1}^n x_i + h^2 = w\}$. If one chooses to measure units of church tower by the square root of height, one can write the utility functions as $u_i = x_i + a_i z$ and the set of feasible allocations as $\Omega^{n+1} \cap \{(x_1, \ldots, x_n, z) \mid \Sigma_{i=1}^n x_i + z = w\}$ where z is the square of the height of the tower measured in feet. Under this choice of units, preferences are weakly convex and the set of feasible allocations is convex.

B. Further Implications of Convexity

An argument due to Starrett [24] suggests that the aggregate production possibility set is unlikely to be convex in the presence of external diseconomies in production. Starrett's results suggest that some care must be taken in employing negative Lindahl prices to guide production decisions. Other discussions of nonconvex externalities in production are found in Bradford and Baumol [9] and Bergstrom [7].

The assumption of convex preferences also limits the admissable kinds of disagreement about the utility of communal commodities. For example, the following remark is true.

Remark *Suppose that preferences of two consumers are similar on $A \subset J$ and* *are represented by $U_1(v(x_A), x_B)$ and $U_2(v(x_A), x_B)$ where U_1 is an increasing and U_2 a decreasing function of v. Then either (a) preferences of at least one consumer are nonconvex, or (b) all isoquants of v are linear.*

Proof Consider any x_A and x_A' such that $v(x_A) = v(x_A',)$. If $v(\lambda x_A + (1-\lambda) x_A') < v(x_A)$, then $(x_A, x_B) I_1 (x_A', x_B)$ but $(x_A, x_B) P_1 (\lambda x_A + (1-\lambda)x_A', x_B)$. In this case, preferences of consumer 1 are nonconvex. But if $v(\lambda x_A + (1-\lambda)x_A') > v(x_A)$, then $(x_A, x_B) P_2 (\lambda x_A + (1-\lambda)x_A', x_B)$ so that preferences of consumer 2 are nonconvex. The only case in which preferences of both consumers are convex is where $v(\lambda x_A + (1-\lambda)x_A') = v(x_A)$. q.e.d.

Consider an economy in which some communal commodity is liked by some consumers and disliked by others. In Lindahl equilibrium consumers who like the commodity must pay those who dislike the commodity to induce them to accept it. Suppose the controversial commodity is produced according to a concave production function. If one wished to further disaggregate the commodity space, he could define preferences on the inputs used to produce the controversial commodity instead of on the commodity itself. But if this is done, preferences of the consumers who dislike the commodity will be nonconvex in terms of the inputs. In general, no Lindahl equilibrium will exist on the disaggregated commodity space. At any Lindahl prices on the factors, it would always be in the interest of those who liked the commodity to maximize output for any given factor cost while those who dislike the commodity would wish to sabotage production by minimizing output given the total factor cost.

VI. CONCLUSION

Theorems on the existence of competitive equilibrium make no explicit claims for the possibility or stability of institutions that would enforce competitive behavior. The tatônnement paradigm and the result that the core of an economy shrinks to the set of competitive equilibria as the economy becomes large each suggest some plausibility for the notion that nearly competitive behavior might actually occur in a world relatively innocent of nonconvexity and coercion.

All that has been demonstrated in this paper is that for seemingly reasonable economies, the conditions required for Lindahl equilibrium are not in themselves logically contradictory. The question remains open whether there are workable institutional arrangements that would sustain Lindahl equilibrium.

One approach to this problem would be to consider a central authority that acquired information about tastes and technology and used this information to compute a system of taxes and subsidies that approximate a Lindahl solution. Aside from computational difficulties, there remains the so-called "free rider problem." If an individual's stated preferences influence the prices that he pays for communal commodities, it may be in his interest to misrepresent his preferences. A discussion of the free rider problem and references to the literature are found in Bergstrom [8].

A second line of approach is the study of the relation of the Lindahl equilibrium to the core. As demonstrated by Foley [14], if all communal commodities are desirable for all consumers, a Lindahl equilibrium is in the core. The relation between Lindahl equilibrium and the core when some commodities are disliked by some consumers is studied by Bergstrom [5]. If benefits from some communal commodities are diffused to all consumers, the core remains larger than the set of Lindahl equilibria even if the economy becomes large (see Muench [20]). Where the nonprivate communal commodities are (as police, parks, or streets) of a spatially localized nature, consumers may be interested only in the quantities of services made available in their home communities. Suppose blocking coalitions can be formed as alternative communities with different local services and tax structures. There may then be some hope that when the number of possible communities becomes large, the core approaches the set of Lindahl equilibria. As yet this remains an unsolved problem. An interesting attempt along these lines is made by Rader [22].

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