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Locally Volume Collapsed 4-Manifolds with Respect to a Lower Sectional Curvature Bound

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by

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Committee in charge:

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Professor Ian Agol
Professor Lawrence Craig Evans
Professor Kam-Biu Luk

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Abstract

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Perelman stated without proof that a 3-dimensional compact Riemannian manifold which is locally volume collapsed, with respect to a lower curvature bound, is a graph manifold. The theorem was used to complete his Ricci flow proof of Thurston's geometrization conjecture. Kleiner and Lott gave a proof of the theorem as a part of their presentation of Perelman's proof.

In this dissertation, we generalize Kleiner and Lott's version of Perelman's theorem to 4-dimensional closed Riemannian manifolds. We show that under some regularity assumptions, if a 4-dimensional closed Riemannian manifold is locally volume collapsed then it admits an F-structure or a metric of nonnegative sectional curvature.

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Introduction

Roughly speaking, an n-dimensional Riemannian manifold M^n is said to be *collapsed* if it appears to have dimension less than n. Collapsed manifolds are studied under various curvature assumptions such as bounded sectional (or Ricci or scalar) curvature, lower sectional (or Ricci or scalar) curvature bound, or assuming that the metric is Einstein.

One way to precisely define collapsing is in terms of the injectivity radius. A Riemannian manifold M is said to be *collapsed* with bounded sectional curvature if there is a sequence of metrics $\{g_j\}$ for which the injectivity radius i_j of (M, g_j) converges uniformly to zero at all points, p, as j goes to infinity, but the sectional curvature K stays bounded (independent of p and j). Cheeger and Gromov [5, 6] first developed the theory of collapsing by showing that a Riemannian manifold is collapsed (in terms of the injectivity radius) with bounded sectional curvature if and only if it admits an *F-structure of positive rank*. An F-structure on a space X is a generalization of local torus actions where different tori (possibly not all of the same dimensions) act locally on finite covering spaces of subsets of X.

Another way to define collapsing is in terms of the volume. We say that a Riemannian manifold M is *volume collapsed* if there is a sequence of metrics $\{g_j\}$ for which the volume of (M, g_j) approaches zero as j goes to infinity. Perelman [24, Theorem 7.4] stated without proof that a 3-dimensional compact Riemannian manifold which is *locally volume collapsed*, with respect to a lower curvature bound, is a graph manifold. The theorem was used to complete his Ricci flow proof of Thurston's geometrization conjecture. As a part of their presentation [12] of Perelman's proof, Kleiner and Lott gave a proof of this theorem in [13]. Other proofs of Perelman's theorem appear in [1, 3, 16, 29].

In this dissertation, we generalize Kleiner and Lott's version ([13, Theorem 1.3]) of Perelman's theorem to closed Riemannian 4-manifolds. In short, under some regularity assumptions, if a closed Riemannian 4-manifolds is locally volume collapsed then it admits an F-structure or admits a metric of nonnegative sectional curvature. We state and discuss the result in the next section.

Other related works include Yamaguchi [34], where he studied global volume collapsed 4-manifolds with respect to a lower sectional curvature bound. Paternain and Petean [21] showed that if a compact manifold M admits an F-structure then it is volume collapsed with respect to a lower sectional curvature bound. Therefore, under some regularity assumptions, the result of this dissertation is in fact necessary and sufficient.

This dissertation is structured as follows. In the next sections, we state and discuss the result, then we give the outline of the proof, and set notations and conventions. In Chapter 2, we collect material that we will need. The rest of the dissertation, Chapter 3 to Chapter 15, is the proof of the result. See Section 1.2 for more details.

1.1 Statement of result

First, we define an intrinsic local scale function needed to define locally volume collapsed manifolds.

Definition 1.1. Let M be a complete Riemannian manifold. Given $p \in M$, the *curvature scale* R_p at p is defined as follows. If the connected component of M containing p has nonnegative sectional curvature then $R_p = \infty$. Otherwise R_p is the (unique) number $r > 0$ such that the infimum of the sectional curvatures on $B(p, r)$ equals $-\frac{1}{r^2}$.

Definition 1.2. Let c_n denote the volume of the unit ball in $\mathbb{R}^n$ and let $w \in (0, c_n)$. A complete Riemannian manifold M^n is said to be *w-locally volume collapsed with respect to a lower sectional curvature bound* if for every $p \in M^n$, $\mathrm{vol}(B(p, R_p)) \leq w R_p^n$.

Suppose that we rescale the ball $B(p, R_p)$ to have radius one. Then the resulting ball will have sectional curvature bounded from below by -1 and volume bounded above by w. As w will be small compared to the volume of the unit ball in $\mathbb{R}^n$, we can say that on the curvature scale, the manifold is locally volume collapsed with respect to a lower sectional curvature bound.

Next, we give a definition of F-structures. Roughly speaking, an F-structure on a space X is a generalization of local torus actions. Different tori (possibly not all of the same dimensions) act locally on finite covering spaces of subsets of X. These local actions satisfy compatibility conditions insuring that X is partitioned into disjoint "orbits". The concept of F-structures was introduced by Cheeger and Gromov [5, 6]. A graph manifold is an example of a manifold which admits an F-structure. The following definition of F-structures is adapted from [9]. The difference is that in this dissertation, we allow torus actions to have fixed points.

Definition 1.3. An *F-structure* on M is an open cover $\{U_i\}$ together with an action of a torus T^{n_i} on $\widetilde{U}_i$, which is a finite normal cover of U_i, with the following properties.

(1) If $U_i \cap U_j \neq \varnothing$, then there exists a covering $\pi_{ij} : \widetilde{U}_{ij} \to U_i \cap U_j$ and maps $\pi_{ij,i} : \widetilde{U}_{ij} \to \widetilde{U}_i$ and $\pi_{ij,j} : \widetilde{U}_{ij} \to \widetilde{U}_j$ so that the following diagram is commutative.

$$\begin{tikzcd}
\widetilde{U}_i \arrow[d, "\pi_i"] & \widetilde{U}_{ij} \arrow[l, "\pi_{ij,i}"'] \arrow[r, "\pi_{ij,j}"] \arrow[d, "\pi_{ij}"] & \widetilde{U}_j \arrow[d, "\pi_j"] \\
U_i & U_i \cap U_j, \arrow[l, hook'] \arrow[r, hook] & U_j
\end{tikzcd}$$

That is $\pi_i \circ \pi_{ij,i} = \pi_j \circ \pi_{ij,j} = \pi_{ij}$.

(2) There exists an action of a torus $T^{n_{ij}}$ on $\widetilde{U}_{ij}$.

(3) There exists an n_i-dimensional subtorus $T^{n_i}_{ij} \subset T^{n_{ij}}$ and a locally isomorphic group homomorphism $T^{n_i}_{ij} \to T^{n_i}$, such that $\pi_{ij,i}$ is equivariant. The same holds when we replace i by j.

The following theorem is the main result of this dissertation.

Theorem 1.4. *Let c_4 denote the volume of the unit ball in $\mathbb{R}^4$ and let $K \geq 10$ be a fixed integer. Fix a function $A : (0, \infty) \to (0, \infty)$. Then there is some $w_0 \in (0, c_4)$ such that the following holds.*

Suppose that (M, g) is a closed orientable Riemannian 4-manifold. Assume in addition that for every $p \in M$,

(1) $\mathrm{vol}(B(p, R_p)) \leq w_0 R_p^4$ *and*

(2) *For every $w' \in [w_0, c_4)$, $k \in [0, K]$, and $r \leq R_p$ such that* $\mathrm{vol}(B(p, r)) \geq w' r^4$, *the inequality*

$$|\nabla^k \mathrm{Rm}| \leq A(w') r^{-(k+2)} \tag{1.5}$$

holds in the ball $B(p, r)$.

Then M admits a metric of nonnegative sectional curvature or M admits an F-structure.

The main geometric assumption in Theorem 1.4 is the first assumption, which is a local collapsing statement. The second assumption is a technical regularity assumption. Assuming the second assumption allows us to work with a sequence of pointed Riemannian manifolds which converge in the standard C^K-topology to a C^K-smooth limit rather than having a pointed Gromov-Hausdorff convergence to an Alexandrov space limit. For the 3-dimensional analog ([24, Theorem 7.4], [13, Theorem 1.3]) of Theorem 1.4, the second assumption arises from the smoothing effect of the Ricci flow equation in its application to the geometrization conjecture. Kleiner and Lott [13] showed that the second assumption of [24, Theorem 7.4] can be removed by using the Stability Theorem of Perelman [23] instead of the standard C^K-convergence of Riemannian manifolds and using the classification of complete, noncompact, orientable, nonnegatively curved 3-dimensional Alexandrov spaces N, when N is a noncompact topological manifold by Shioya and Yamaguchi [28] instead of the classification of closed Riemannian 3-manifolds with nonnegative sectional curvature.

Paternain and Petean [21] showed that if a compact manifold M admits an F-structure then it is volume collapsed with respect to a lower sectional curvature bound. Therefore, assuming the second assumption, the statement of Theorem 1.4 is in fact necessary and sufficient. We have the following corollary.

Corollary 1.6. *With the same assumptions as in Theorem 1.4, there exists* $w_0 \in (0, c_4)$ *such that for any closed orientable Riemannian 4-manifold* M*, hypotheses (1) and (2) of Theorem 1.4 hold if and only if* M *admits a metric of nonnegative sectional curvature or* M *admits an* F*-structure.*

1.2 Outline of the proof

The proof of Theorem 1.4 is by contradiction. Assuming that the theorem is false, we get a sequence of manifolds M^α which satisfy the hypotheses of Theorem 1.4 with the parameter $\omega_0 \to 0$, but do not admit an F-structure or a metric of nonnegative sectional curvature. Using the standard C^K-convergence for Riemannian manifolds, we study the local geometry and topology of M^α, for sufficiently large α. The local geometries are based on the number of $\mathbb{R}$-factors that M^α locally approximately splits off. Next, we use these local descriptions to decompose M^α into domains which are fiber bundles. We then study all possible ways to glue fiber bundle pieces together and give an explicit configuration of M^α. Lastly, we show that M^α admits a metric of nonnegative sectional curvature or admits an F-structure. Hence, we get a contradiction.

For brevity, we will suppress the superscript α and refer to M^α by M, assuming that α is sufficiently large.

We will mainly follow and generalize the strategy and techniques developed by Kleiner and Lott in [13]. However, there are some complications that arise in proving its 4-dimensional analog. Firstly, a situation where M locally approximately splits off exactly one $\mathbb{R}$-factor is more complicated in the 4-dimensional case. This is discussed in Chapter 8 and parts of Chapter 10 to Chapter 12. Secondly, the gluing procedure and the explicit configurations of M are more involved in the 4-dimensional case. This is done in Chapter 13 to Chapter 15. Lastly, one needs to recognize a meaningful structure on M from the explicit configurations, which turned out to be F-structures.

1.2.1 Local collapsing in terms of the volume scale

In Chapter 3, we reformulate Theorem 1.4 in terms of the *volume scale* and set up the contradiction proof for the theorem.

Definition 1.7. Let c_4 denote the volume of the unit ball in $\mathbb{R}^4$. Fix $\bar{w} \in (0, c_4)$. Given $p \in M$, the *$\bar{w}$-volume scale* at p is

$$r_p(\bar{w}) = \inf\{r > 0 : \operatorname{vol}(B(p,r)) = \bar{w}r^4\}. \tag{1.8}$$

If there is no such r, then we say that the $\bar{w}$-volume scale is infinite.

In terms of the curvature scale (see Definition 1.1), Hypothesis (1) of Theorem 1.4 implies that if we rescale the ball $B(p, R_p)$ to have radius one, then the resulting ball will have sectional curvature bounded from below by -1 and volume bounded above by w_0. On the other hand, if we rescale the ball $B(p, r_p(w_0))$ to have radius one, then Hypothesis (1) implies that there is a large number $\mathcal{R}$ so that the sectional curvature on the radius $\mathcal{R}$ ball, $B'(p, \mathcal{R})$ in the rescaled manifold, is bounded below by $-\frac{1}{\mathcal{R}^2}$ while $\operatorname{vol}(B'(p,1)) = w_0$, where $B'(p,1)$ denotes a unit ball in the rescaled manifold. This means that on the volume scale, a large neighborhood of p is well approximated by a large region of a complete 4-manifold N_p, which admits a metric of nonnegatively sectional curvature. This allows us to study a local geometry of M. Moreover, if w_0 is sufficiently small, then we can say that at the volume scale, a neighborhood of p is close in a coarse sense to a space of dimension less than four. In this dissertation, we will work consistently on the volume scale.

1.2.2 Modified volume scale

Volume scales can fluctuate from point to point. This leads to difficulties in gluing local models together. In Chapter 4, we replace the *volume scale* by a *modified volume scale*, the fluctuation of which can be better controlled. We define a scale function $p \mapsto \mathfrak{r}_p$ at each point $p \in M$ such that:

(1) $\mathfrak{r}_p$ is much smaller than the curvature scale R_p.

(2) The function $p \mapsto \mathfrak{r}_p$ is smooth and has Lipschitz constant much smaller than 1.

(3) $\operatorname{vol}(B(p, \mathfrak{r}_p)) \in [w'\mathfrak{r}_p^4, \bar{w}\mathfrak{r}_p^4]$, where $w' < \bar{w}$ are suitably chosen constants lying in the interval $[w_0, c_4]$.

The existence of the modified volume scale follows from the local collapsing assumption, the Bishop-Gromov volume comparison theorem, and an argument similar to McShane's extension theorem for real-valued Lipschitz functions.

1.2.3 Implication of compactness

From Condition (1) above, we have that the rescaled manifold $\frac{1}{\mathfrak{r}_p}M$ admits a metric of nonnegative sectional curvature near p. Condition (3) implies that $\frac{1}{\mathfrak{r}_p}M$, near p, looks collapsed but not too collapsed, in the sense that the volume of the unit ball around p in the rescaled manifold $\frac{1}{\mathfrak{r}_p}M$ is small but not too small. Together with the regularity assumption (1.5) in Theorem 1.4 and the standard compactness theorems for pointed Riemannian manifolds, we can approximate a neighborhood of p in the rescaled manifold in two ways:

(1) For every $p \in M$, the rescaled pointed manifold $(\frac{1}{\mathfrak{r}_p}M, p)$ is close in the pointed C^K-topology to a pointed C^K-smooth Riemannian 4-manifold $(N_p, \star)$ which admits a metric of nonnegative sectional curvature.

(2) For every $p \in M$, the rescaled pointed manifold $(\frac{1}{\mathfrak{r}_p}M, p)$ is close in the pointed Gromov-Hausdorff topology to a pointed nonnegatively curved Alexandrov space $(X_p, \star)$ of dimension at most 3.

1.2.4 Stratification

Next in Chapter 5, we partition M into k-stratum points, for $k \in \{0, 1, 2, 3\}$, in terms of the number of $\mathbb{R}$-factors that $(\frac{1}{\mathfrak{r}_p}M, p)$ approximately splits off.

Let $0 < \beta_1 < \beta_2 < \beta_3$ be new parameters. At scale $\mathfrak{r}_p$, we partition points in M as follows:

- A point p in M is a *3-stratum* point if $(\frac{1}{\mathfrak{r}_p}M, p)$ is β_3-close to $(\mathbb{R}^3, 0)$ in the pointed Gromov-Hausdorff topology.
- A point p in M lies in the *2-stratum* if it does not lie in the 3-stratum and $(\frac{1}{\mathfrak{r}_p}M, p)$ is β_2-close to $(\mathbb{R}^2 \times Y_p, (0, \star_{Y_p}))$ in the pointed Gromov-Hausdorff topology, where Y_p is a point, a circle, an interval, or a half-line, and $\star_{Y_p}$ is a basepoint in Y_p.
- A point p in M lies in the *1-stratum* if it does not lie in the k-stratum for $k \in \{2, 3\}$ and $(\frac{1}{\mathfrak{r}_p}M, p)$ is β_1-close to $(\mathbb{R} \times Y_p, (0, \star_{Y_p}))$ in the pointed Gromov-Hausdorff topology, where Y_p is a 2-dimensional Alexandrov space.
- A point p in M lies in the *0-stratum* if it does not lie in the k-stratum for $k \in \{1, 2, 3\}$.

Furthermore, if a point $p \in M$ is in the k-stratum, then at some scale comparable to $\mathfrak{r}_p$, M is close in the pointed C^K-topology to $N_p \simeq \mathbb{R}^k \times F_p$ where F_p is given in the following table. The structures near p in the k-stratum, for $k \in \{0, 1, 2, 3\}$, are discussed in Chapter 6 to Chapter 9.

k	F_p
3	S^1
2	S^2, T^2, D^2
1	S^3/Γ, T^3/Γ, $S^2 \times S^1$, $\mathbb{R}P^3 \# \mathbb{R}P^3$, $D^3, S^2 \times_{\mathbb{Z}_2} I, S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I$
0	D^4, $S^1 \times D^3$, $S^2 \times_\omega D^2$, $(S^2 \times_\omega D^2)/\mathbb{Z}_2$, $\omega \in \mathbb{Z}$, $(\mathbb{R}P^2 \times S^1)\widetilde{\times} I$, $(S^2 \widetilde{\times} S^1)\widetilde{\times} I$, $T^2 \times D^2$, $T^2 \times_{\mathbb{Z}_2} D^2$, $\beta_k \widetilde{\times} I, k \in \{1, 2, 3, 4\}$

Additionally, we can transfer the projection map $N_p \simeq \mathbb{R}^k \times F_p \to \mathbb{R}^k$ to a map η_p defined on a large ball $B(p, C)$, at some scale comparable to $\mathfrak{r}_p$, where it defines a submersion.

1.2.5 Compatibility of the local structures

Once we have the local structure of M near each point, we investigate how the local structures fit together on their overlaps. It follows from the construction of the stratification that local structures are nearly "aligned".

For example, suppose that $p, q \in M$ are 1-stratum points with $B(p, C_p\mathfrak{r}_p) \cap B(q, C_q\mathfrak{r}_q) \neq \varnothing$ for some constants C_p and C_q. Provided that the Lipschitz constant of $p \mapsto \mathfrak{r}_p$ is small, we have that $\mathfrak{r}_p \approx \mathfrak{r}_q$. Let $z \in B(p, C_p\mathfrak{r}_p) \cap B(q, C_q\mathfrak{r}_q)$. We have two $\mathbb{R}$-factors at z, coming from the approximate splittings at p and q. If the two $\mathbb{R}$-factors do not nearly align at z, then they generate an approximate $\mathbb{R}^2$-factor at z, which then transfers to an approximate $\mathbb{R}^2$-factor at p. This contradicts to the assumption that p is in the 1-stratum. Therefore, the two $\mathbb{R}$-factors from the approximate splittings at p and q must nearly align along the overlap. It follows that the maps η_p and η_q are "almost" affine functions of each other. This will enable us to glue local structures near p and q together. Compatibilities between points in other strata follow from similar arguments.

1.2.6 Gluing the local structures together

In Chapter 10 to Chapter 12, we use the compatibility of local structures to glue them together. We mostly follow the methods in [13] in this part.

In summary, the gluing process begins with selecting a collection of points of each type in M, $\{p_i\}_{i \in I_{k\text{-stratum}}}$, for $k \in \{0, 1, 2, 3\}$, so that $\bigcup_{i \in I_{k\text{-stratum}}} B(p_i, C_i\mathfrak{r}_{p_i})$ covers the k-stratum points, for some constants C_i. Next, we try to combine the maps η_{p_i} so that we have a global fibration structure for each type of points. This is done by defining a smooth map $\mathcal{E}^0 : M \to H$ into a high-dimensional Euclidean space H. Components of $\mathcal{E}^0$ are functions of η_{p_i} and the scale function $p \mapsto \mathfrak{r}_p$, cutoff appropriately so that they define global smooth functions. See Chapter 10 for details.

It follows from pairwise compatibility of the functions η_{p_i} discussed above that the image of $\mathcal{E}^0$ of $\bigcup_{i \in I_{\text{3-stratum}}} B(p_i, C_i \mathfrak{r}_{p_i})$ is a subset $S \subset H$ which, at the right scale, is everywhere locally closed in the pointed Hausdorff sense to a 3-dimensional affine subspace. Kleiner and Lott [13] call such sets *cloudy manifolds*. They showed that a cloudy manifold of any dimension can be approximated by a core manifold W whose normal injectivity radius is controlled (see [13, Appendix B]). We use this to "upgrade" $\mathcal{E}^0$ to a new map $\mathcal{E}^1$ which is C^1-close to $\mathcal{E}^0$ and is a fibration near the 3-stratum. We repeat similar adjustments near other stratum points to obtain a map $\mathcal{E} : M \to H$ whose restriction to certain regions of M give locally trivial fibrations. For example, near the 3-stratum points, $\mathcal{E}$ yields circle fibrations. See Chapter 11 for details.

Lastly in Chapter 12, we show that the fibered regions derived from $\mathcal{E}$ have disjoint interiors and are pairwise compatible. In particular, if two fibers intersect, then one of them is contained in the other.

1.2.7 Describing the domains in terms of fiber bundle components

In Chapter 13, we describe the conclusion of Chapter 12 in terms of domains with disjoint interiors. Each domain is a compact 4-manifold with corners which is also a fiber bundle, with compatibility of fibers along the overlaps. Then, we study how fibers of different types intersect. In particular, we describe possible configurations of each fiber along the overlaps in terms of fibers of other types. We note that not all combinatorial configurations are feasible due to topological obstructions.

The following example demonstrates the decompositions of fibers.

Example 1.9. Let M_0 be a fiber bundle component $\left(\begin{tikzcd} F^4 \arrow[r] & M_0 \arrow[d] \\ & \text{pt} \end{tikzcd}\right)$ where F^4 is the unit normal bundle of a soul of a complete noncompact orientable Riemannian 4-manifold which admits a metric of nonnegative sectional curvature. For a classification, see Lemma 2.12. ∂M_0 is a closed 3-manifold.

Assume that M_0 intersects exactly two other types of fiber bundle components:

- $\left(\begin{tikzcd} T^2 \arrow[r] & M_i \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$, a T^2-bundle over a surface Σ^2, and
- $\left(\begin{tikzcd} S^1 \times D^2 \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$, an $S^1 \times D^2$-bundle over an interval I.

In Lemma 13.17, we show that $\partial M_0 = B_1 \cup A \cup B_2$ where $A \cong T^2 \times I$ is a subbundle of the boundary of $\left(\begin{array}{r} T^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, for some i, and B_1 and B_2 are $S^1 \times D^2$-fibers over the endpoints of $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_{j_1} \\ \downarrow \\ (I, \partial I) \end{array}\right)$ and $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_{j_2} \\ \downarrow \\ (I, \partial I) \end{array}\right)$, for some j_1 and j_2. It follows that $\partial M_0 \cong S^3$, $S^1 \times S^2$, or a Lens space $L(|\omega|, 1)$ and $M_0 \cong D^4$, $\pm\mathbb{C}P^2 \# D^4$, $S^1 \times D^3$, $S^2 \times D^2$, a twisted $(\mathbb{R}P^2 \times S^1)$-bundle over an interval $(\mathbb{R}P^2 \times S^1)\widetilde{\times} I$, a twisted $(S^2 \widetilde{\times} S^1)$-bundle over an interval $(S^2 \widetilde{\times} S^1)\widetilde{\times} I$ where $S^2 \widetilde{\times} S^1$ is the nonorientable S^2-bundle over S^1, or a D^2-bundle over S^2 with Euler number ω, $S^2 \times_\omega D^2$.

More generally, ∂M_0 still has the same decomposition $\partial M_0 = B_1 \cup A \cup B_2$ when we replace an occurrence of a component $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ by $\left(\begin{array}{r} T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ in the above construction. However, the topology of M_0 changes.

1.2.8 Gluing fiber bundle components into building blocks

Recall that we are trying to get a contradiction by showing that for large α, $M = M^\alpha$ admits an F-structure or a metric of nonnegative sectional curvature. In Chapter 14, we use the decompositions of fibers from Chapter 13 to glue fiber bundle components of different types together into building blocks and show that they admit F-structures. Later in Chapter 15, we finish the proof of Theorem 1.4 by describing M in terms of a configuration of building blocks and showing that M admits an F-structure.

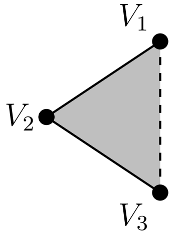
The following example illustrates the gluing process.

Example 1.10. Assume that M consists of

- three $(S^1 \times D^3)$-bundles over a point $\left(\begin{array}{r} S^1 \times D^3 \longrightarrow V_\alpha \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, $\alpha \in \{1, 2, 3\}$,
- two $(S^1 \times D^2)$-bundles over an interval $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow E_\beta \\ \downarrow \\ (\mathrm{I}, \partial\mathrm{I}) \end{array}\right)$, $\beta \in \{12, 23\}$,
- one T^2-bundle over D^2 $\left(\begin{array}{r} T^2 \longrightarrow M' \\ \downarrow \\ (\mathrm{D}^2, \partial\mathrm{D}^2) \end{array}\right)$,

– and one S^1-bundle over D^3 $\left(\begin{array}{r} S^1 \longrightarrow M'' \\ \downarrow \\ (\mathrm{D}^3, \partial \mathrm{D}^3) \end{array}\right)$.

These components are glued together according to the following diagram



where the vertices represent $\left(\begin{array}{r} S^1 \times D^3 \longrightarrow V_\alpha \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, $\alpha \in \{1, 2, 3\}$, the solid edges represent $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow E_\beta \\ \downarrow \\ (\mathrm{I}, \partial \mathrm{I}) \end{array}\right)$, $\beta \in \{12, 23\}$, and the triangular area represents $\left(\begin{array}{r} T^2 \longrightarrow M' \\ \downarrow \\ (\mathrm{D}^2, \partial \mathrm{D}^2) \end{array}\right)$. The dashed edge is a T^2-subbundle $T^2 \times I$ of $\left(\begin{array}{r} T^2 \longrightarrow \partial M' \\ \downarrow \\ \partial \mathrm{D}^2 \end{array}\right)$. For each $(S^1 \times D^2)$-fiber of $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow E_\beta \\ \downarrow \\ (\mathrm{I}, \partial \mathrm{I}) \end{array}\right)$, its T^2-boundary coincides with a T^2-fiber of $\left(\begin{array}{r} T^2 \longrightarrow \partial M' \\ \downarrow \\ \partial \mathrm{D}^2 \end{array}\right)$. The union of components represented by the above figure is glued to $\left(\begin{array}{r} S^1 \longrightarrow M'' \\ \downarrow \\ (\mathrm{D}^3, \partial \mathrm{D}^3) \end{array}\right)$ along their $(S^1 \times S^2)$-boundaries in such a way that each T^2-fiber of the dashed edge $T^2 \times I$ is a union of S^1-fibers from $\left(\begin{array}{r} S^1 \longrightarrow M'' \\ \downarrow \\ (\mathrm{D}^3, \partial \mathrm{D}^3) \end{array}\right)$.

More generally, under the same configuration, each component in Example 1.10 has more than one possible topological type. For example, in addition to $S^1 \times D^3$, each vertex can also represent a manifold diffeomorphic to $T^2 \times D^2$, D^4, or a D^2-bundle over S^2. Instead of $\left(\begin{array}{r} T^2 \longrightarrow M' \\ \downarrow \\ (\mathrm{D}^2, \partial \mathrm{D}^2) \end{array}\right)$, the triangular area can also represent $\left(\begin{array}{r} T^2 \longrightarrow M' \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ where Σ^2 is a surface with one boundary component. Instead of $\left(\begin{array}{r} S^1 \longrightarrow M'' \\ \downarrow \\ (\mathrm{D}^3, \partial \mathrm{D}^3) \end{array}\right)$, the union of the components represented by the above figure can also attach to $\left(\begin{array}{r} S^1 \longrightarrow M'' \\ \downarrow \\ (\mathrm{X}^3, \partial \mathrm{X}^3) \end{array}\right)$ where X^3 is a 3-manifold such that $\partial X^3 \cong S^2$.

1.2.9 Giving an explicit configuration and recognizing an F-structure

In Chapter 15, we finish the proof of Theorem 1.4 by describing M in terms of a configuration of building blocks and fiber bundle components then showing that M admits an F-structure or a metric of nonnegative sectional curvature.

The following example illustrates a construction of an F-structure on M.

Example 1.11. Let M be the manifold in Example 1.10. For simplicity, we assume that all fiber bundles are trivial. Let S^1 act on $\left(\begin{array}{r} S^1 \longrightarrow M'' \\ \downarrow \\ (\mathrm{D}^3, \partial \mathrm{D}^3) \end{array}\right) \cong (S^1 \times D^3)$ by rotations on S^1-fibers and act trivially on D^3. Similarly, let T^2 act on $\left(\begin{array}{r} T^2 \longrightarrow M' \\ \downarrow \\ (\mathrm{D}^2, \partial \mathrm{D}^2) \end{array}\right) \cong (T^2 \times D^2)$ by the standard T^2-action on T^2-fibers and act trivially on D^2. The T^2-action extends to $\bigcup_{\alpha,\beta}(V_\alpha \cup E_\beta) \cong (S^1 \times D^2) \times I$ in such a way that T^2 acts trivially on the I-factor and acts by an extension of the standard action on the $(S^1 \times \partial D^2)$-factor. Each $(S^1 \times D^2)$-fiber is the union of T^2-orbits over an open interval and a single orbit of dimension one. The T^2-action restricts to an S^1-action on a neighborhood of this 1-dimensional orbit. By the compatibility of fibers, the S^1 action on $\partial M''$ extends to the T^2-action on the boundary of $M' \cup \left(\bigcup_{\alpha,\beta} V_\alpha \cup E_\beta\right)$. This gives an F-structure on M.

1.3 Notation and conventions

1.3.1 Parameters and constraints

We mostly follow the notation and conventions in [13]. The proof of Theorem 1.4 involves long constructions, many steps of which generate new constants. We will refer to these constants as *parameters*. Several arguments will include a consideration of sequences of values of parameters, which one should associate with a sequence of distinct instances of the constructions.

Many arguments in this dissertation assert that certain statements hold provided that certain constraints on the parameters are satisfied. Each time we refer to such a constraint, we will assume that the inequalities in question are satisfied for the remainder of the thesis. Constraint functions will be denoted with a bar. For example, $\mu < \bar{\mu}(\beta, \sigma)$ means that $\mu \in (0, \infty)$ satisfies an upper bound which is a function of β and σ. By convention, all constraint functions take values in $(0, \infty)$.

At the end of the proof of Theorem 1.4 (Section 15.8), we will verify that it is possible to simultaneously satisfy all the constraints that appear in the proof. Since the constraints are of the form that one parameter is sufficiently large or small in terms of some other parameters, we only need to consider the order in which the parameters are consider.

We follow Perelman's convention that a condition like $a > 0$ means that a should be considered to be small, while a condition like $A < \infty$ means that A should be considered to be large. This convention is for expository purposes only.

1.3.2 Notation

We will use the following notation for cutoff functions with prescribed support. Let $\phi \in C^\infty(\mathbb{R})$ be a nonincreasing function so that $\phi|_{(-\infty,0]} = 1$, $\phi|_{[1,\infty)} = 0$, and $\phi((0,1)) \subset (0,1)$. Given $a, b \in \mathbb{R}$ with $a < b$, we define $\Phi_{a,b} \in C^\infty(\mathbb{R})$ by

$$\Phi_{a,b}(x) = \phi(a + (b-a)x), \tag{1.12}$$

so that $\Phi_{a,b}|_{(-\infty,a]} = 1$ and $\Phi_{a,b}|_{[b,\infty)} = 0$. Given $a, b, c, d \in \mathbb{R}$ with $a < b < c < d$, we define $\Phi_{a,b,c,d} \in C^\infty(\mathbb{R})$ by

$$\Phi_{a,b,c,d}(x) = \phi_{-b,-a}(-x)\phi_{c,d}(x), \tag{1.13}$$

so that $\Phi_{a,b,c,d}|_{(-\infty,a]} = 0$, $\Phi_{a,b,c,d}|_{[b,c]} = 1$ and $\Phi_{a,b,c,d}|_{[d,\infty)} = 0$.

If X is a metric space and $0 < r \leq R$, then we denote the annulus $\overline{B(x,R)} - B(x,r)$ by $A(x,r,R)$. The dimension of X refers to the Hausdorff dimension.

Let $(X, \star)$ be a pointed metric space. A *metric cone* $(C, \star)$ of $(X, \star)$ is the union of rays leaving the basepoint $\star$ such that the union of any two such rays is isometric to the union of two rays leaving the origin in $\mathbb{R}^2$. For brevity, sometimes we write C for the pointed metric space $(C, \star)$.

If Y is a subset of X and $t : Y \to (0, \infty)$ is a function, then we write $N_t(Y)$ for the neighborhood of Y with variable thickness t : $N_t(Y) = \bigcup_{y \in Y} B(y, t(y))$.

If (X, d) is a metric space and $\lambda > 0$, then we write λX for the scaled metric space $(X, \lambda d)$. That is, for any two points $x, y \in X$, the distance $d_{\lambda X}(x, y) = \lambda d_X(x, y)$. We also write $B_{\lambda X}(p, r) \subset \lambda X$ for the r-ball around p in the metric space λX.

A product metric space $X_1 \times X_2$ will always be endowed with the distance function $d_{(X_1 \times X_2)}((x_1, x_2), (y_1, y_2)) = \sqrt{d^2_{X_1}(x_1, y_1) + d^2_{X_2}(x_2, y_2)}$ for $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$.

Let X and Y be topological spaces with boundary. We denote by $X \cup_\partial Y$ the union $X \cup Y$ with the condition that $X \cap Y = \partial X \cap \partial Y$.

Let X^n be a topological space and let k be a nonnegative integer. We denote the connected sum $\underbrace{X\#X\#\ldots\#X}_{k \text{ copies}}$ by kX. When $k = 0$, we define kX to be S^n.

— 2 —

Preliminaries

In this chapter, we list the material that we will need.

We refer to [25] for basics about Riemannian geometry. We refer to [2] for basics about length spaces and Alexandrov spaces. We refer to [10, 27] for facts about 3-manifolds. We refer to [7, 8, 17] for facts about S^1 and T^2-actions on 3 and 4-manifolds.

2.1 Pointed Gromov-Hausdorff approximations

In this section, we collect definitions and basic results about the pointed Gromov-Hausdorff topology. We refer to [2, Chapter 8].

Definition 2.1. Let $(X, \star_X)$ be a pointed metric space. Given $\delta \in [0, \infty)$, two closed subspaces C_1 and C_2 are *δ-close in the pointed Hausdorff sense* if $C_1 \cap \overline{B(\star_X, \delta^{-1})}$ and $C_2 \cap \overline{B(\star_X, \delta^{-1})}$ have Hausdorff distance at most δ.

Definition 2.2. Let $(X, \star_X)$ and $(Y, \star_Y)$ be pointed metric spaces. Give $\delta \in [0, 1)$, a pointed map $f : (X, \star_X) \to (Y, \star_Y)$ is a *δ-Gromov-Hausdorff approximation* if for every $x_1, x_2 \in B(\star_X, \delta^{-1})$ and $y \in B(\star_Y, \delta^{-1} - \delta)$, we have

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \leq \delta \quad \text{and} \quad d_Y(y, f(B(\star_X, \delta^{-1}))) \leq \delta. \tag{2.3}$$

Two pointed metric spaces $(X, \star_X)$ and $(Y, \star_Y)$ are *δ-close in the pointed Gromov-Husdorff topology* if there is a δ-Gromov-Hausdorff approximation from $(X, \star_X)$ to $(Y, \star_Y)$. Although this does not define a metric space structure on the set of pointed metric spaces, it defines a topology which is metrizable.

A sequence $\{(X_i, \star_{X_i})\}_{i=1}^{\infty}$ of pointed metric spaces *Gromov-Hausdorff converges* to $(Y, \star_Y)$ if there is a sequence $\{f_i : (X_i, \star_{X_i}) \to (Y, \star_Y)\}_{i=1}^{\infty}$ of δ_i-Gromov-Hausdorff approximations, where $\delta_i \to 0$.

The pointed Gromov-Hausdorff topology is a complete metrizable topology on the set of complete proper metric spaces (up to isometry). Thus, we can talk about two metric spaces having distance at most δ from each other. In this dissertation, we only concern complete proper length spaces, which form a closed subset of the set of complete proper metric spaces under the Gromov-Hausdorff topology.

2.2 C^K-convergence

Definition 2.4. Given $K \in \mathbb{Z}^+$, let $(M_1, \star_{M_1})$ and $(M_2, \star_{M_2})$ be complete pointed C^K-smooth Riemannian manifolds. Given $\delta \in [0, \infty)$, a pointed C^{K+1}-smooth map $f : (M_1, \star_{M_1}) \to (M_2, \star_{M_2})$ is a δ-C^K *approximation* if it is a δ-Gromov-Hausdroff approximation and $\|f^* g_{M_2} - g_{M_1}\|_{C^K}$, computed on $B(\star_M, \delta^{-1})$, is bounded above by δ. Two C^K-smooth Riemannian manifolds $(M_1, \star_{M_1})$ and $(M_2, \star_{M_2})$ are said to be δ-C^K-*close* if there is a δ-C^K approximation from $(M_1, \star_{M_1})$ to $(M_2, \star_{M_2})$.

We will use the following C^K-precompactness result from [13] (see also [25, Chapter 10]).

Lemma 2.5 ([13, Lemma 3.5])**.** *Given $v, r > 0$, $n \in \mathbb{Z}^+$, and a function $A : (0, \infty) \to (0, \infty)$, the set of complete pointed C^{K+2}-smooth n-dimensional Riemannian manifolds $(M, \star_M)$ such that*

(1) $\mathrm{vol}(B(\star_M, r)) \geq v$ *and*

(2) $|\nabla^k \mathrm{Rm}| \leq A(R)$ *on* $B(\star_M, r)$*, for all* $0 \leq k \leq K$ *and* $R > 0$*,*

is precompact in the pointed C^K-topology.

The bound on the derivatives of curvature in Lemma 2.5 gives uniform C^{K+1}-bounds on the Riemannian metric in harmonic coordinates. One then obtains limit metrics which are C^K-smooth.

2.3 Alexandrov Spaces

We refer to [2] for basics about length spaces and Alexandrov spaces. In this dissertation, all Alexandrov spaces are assumed to have a finite Hausdorff dimension.

We recall the notion of a strainer (cf. [2, Definition 10.8.9]). For facts about strainers, we refer to [2, Chapter 10].

Definition 2.6. Let X be an Alexandrov space of curvature bounded below by c. Let $p \in X$. An *m-strainer at p of quality δ and scale r* is a collection $\{(a_i, b_i)\}_{i=1}^m$ of pairs of points such that $d(p, a_i) = d(p, b_i) = r$ and in terms of comparison angles,

$$\begin{aligned} \widetilde{\angle}_p(a_i, b_i) &> \pi - \delta, \\ \widetilde{\angle}_p(a_i, a_j) &> \frac{\pi}{2} - \delta, \\ \widetilde{\angle}_p(a_i, b_j) &> \frac{\pi}{2} - \delta, \\ \widetilde{\angle}_p(b_i, b_j) &> \frac{\pi}{2} - \delta, \end{aligned} \tag{2.7}$$

for all $i, j \in \{1, \ldots, m\}$, $i \neq j$. The comparison angles are defined using comparison triangles in the model space of constant curvature c.

Definition 2.8. The *strainer number* of X is the supremum of numbers m such that there exists an m-strainer of quality $\frac{1}{100m}$ at some point and some scale.

Lemma 2.9 ([2, Corollary 10.8.21]). *The Hausdorff dimension of X equals to its strainer number.*

If $(X, \star_X)$ is a pointed nonnegatively curved Alexandrov space, then there is a pointed Gromov-Hausdorff limit $C_T X = \lim_{\lambda \to \infty}(\frac{1}{\lambda}X, \star_X)$ called the *Tits cone* of X. It is a nonnegatively curved Alexandrov space which is also a metric cone.

Lemma 2.10 ([13, Lemma 3.10]). *Given $n \in \mathbb{Z}^+$, let $\{(X_i, \star_{X_i})\}_{i=1}^\infty$ be a sequence of complete pointed length spaces. Suppose that $c_i \to 0$ and $r_i \to \infty$ are positive sequences such that for each i, the ball $B(\star_{X_i}, r_i)$ has curvature bounded below by $-c_i$ and dimension bounded above by n. Then a subsequence of the $(X_i, \star_{X_i})$'s converges in the pointed Gromov-Hausdorff topology to a pointed nonnegatively curved Alexandrov space of dimension at most n.*

2.4 Topology of Riemannian 4-manifolds with nonnegative sectional curvature

Lemma 2.11 ([13, Lemma 3.11]). *Let M be a closed orientable 3-dimensional C^K-smooth Riemannian manifold with nonnegative sectional curvature. Then, M is diffeomorphic to S^3/Γ (where Γ is a finite subgroup of $\mathrm{Isom}^+(S^3) = SO^4$ which acts freely on S^3), T^3/Γ (where Γ is a finite subgroup of $\mathrm{Isom}^+(T^3)$ which acts freely on T^3), $S^1 \times S^2$, and $S^1 \times_{\mathbb{Z}_2} S^2 \cong \mathbb{R}P^3 \# \mathbb{R}P^3$.*

Wolf [33] showed that there are six orientable and four nonorientable flat closed 3-manifolds. We follow the notation in [33]. The six orientable flat 3-manifolds are denoted by $\mathcal{G}_i$, $i \in \{1, \ldots, 6\}$, where $\mathcal{G}_1 \cong T^3$. The four nonorientable flat 3-manifolds are denoted by $\mathcal{B}_i$, $i \in \{1, \ldots, 4\}$, where $\mathcal{B}_1 \cong K^2 \times S^1$. The double cover of $\mathcal{B}_1$ and $\mathcal{B}_2$ is T^3 and the double cover of $\mathcal{B}_3$ and $\mathcal{B}_4$ is $\mathcal{G}_2$.

Lemma 2.12. *Let M be a complete connected orientable 4-dimensional C^K-smooth Riemannian manifold with nonnegative sectional curvature. We have the following classification of the diffeomorphism types of M, based on the number of ends:*

- *0 ends: complete compact connected orientable 4-dimensional C^K-smooth Riemannian manifold with nonnegative sectional curvature.*
- *1 end: $\mathbb{R}^4$, $S^1 \times \mathbb{R}^3$, $T^2 \times \mathbb{R}^2$, an $\mathbb{R}^2$-bundle over K^2 $K^2 \widetilde{\times} \mathbb{R}^2 \cong T^2 \times_{\mathbb{Z}_2} \mathbb{R}^2$, $\mathbb{R}^2$-bundles over S^2 $S^2 \times_\omega R^2$ for some $\omega \in \mathbb{Z}$, $\mathbb{R}^2$-bundles over $\mathbb{R}P^2$ $(S^2 \times_\omega \mathbb{R}^2)/\mathbb{Z}_2$ for some $\omega \in \mathbb{Z}$, the twisted $\mathbb{R}$-bundle over the nonorientable S^2-bundle over S^1 $(S^2 \widetilde{\times} S^1) \widetilde{\times} \mathbb{R}$, the twisted $\mathbb{R}$-bundle over $\mathbb{R}P^2 \times S^1$ $(\mathbb{R}P^2 \times S^1) \widetilde{\times} \mathbb{R}$, and the twisted $\mathbb{R}$-bundle over $\mathcal{B}_i$ $\mathcal{B}_i \widetilde{\times} \mathbb{R}$ where $\mathcal{B}_i$ is a nonorientable compact 3-dimensional Euclidean space form for $i \in \{1, 2, 3, 4\}$.*
- *2 ends: $N \times \mathbb{R}$ where N is $S^1 \times S^2$, $S^1 \times_{\mathbb{Z}_2} S^2 \cong \mathbb{R}P^3 \# \mathbb{R}P^3$, T^3/Γ (where Γ is a finite subgroup of $\mathrm{Isom}^+(T^3)$ that acts freely on T^3), or S^3/Γ (where Γ is a finite subgroup of $\mathrm{SO}(4)$ that acts freely on S^3).*

If M has two ends then it splits off an $\mathbb{R}$-factor isometrically.

Proof. If M has no ends then it is compact. Thus, M is a complete compact connected orientable 4-dimensional C^K-smooth Riemannian manifold with nonnegative sectional curvature.

If M is noncompact, then by the Cheeger-Gromoll soul theorem, M is diffeomorphic to the total space of a vector bundle over its soul, which is a closed lower-dimensional manifold with nonnegative sectional curvature [4]. (As stated in [13], the proof in [4], which is for C^∞-metric, is also valid for C^K-smooth metrics.) The possible dimensions of soul are 0, 1, 2, and 3. The possible topologies of M are listed in the lemma.

The O'Neill formula (see [25, Chapter 3]) implies that every $\mathbb{R}^2$-bundle over S^2 admits a metric of nonnegative sectional curvature. Özaydin and Walschap [20] showed that the only $\mathbb{R}^2$-bundle over T^2 which admits a metric of nonnegative sectional curvature is the product $T^2 \times \mathbb{R}^2$.

If M has two ends then it contains a line and the Toponogov splitting theorem [32] implies that M isometrically splits off an $\mathbb{R}$-factor. The classification of closed orientable 3-dimensional C^K-smooth Riemannian manifolds with nonnegative sectional curvature was given in [13, Lemma 3.11]. □

Lemma 2.13. *For $i \in \{1, 2\}$, let W_i be diffeomorphic to $S^2 \times_{\pm 2} D^2$ and let V_i be a submanifold of $\partial W_i \cong \mathbb{R}P^3$ diffeomorphic to $S^2 \times_{\mathbb{Z}_2} I$. Let $W = W_1 \cup_\partial W_2$ by identifying V_1 and V_2. Then, $W \cong D^4 \# (S^2 \times S^2)$ or $D^4 \# (S^2 \widetilde{\times} S^2)$ where $S^2 \widetilde{\times} S^2$ is the nontrivial orientable S^2-bundle over S^2.*

Proof. Let $Z = W_1 \cup_\partial W_2$ where ∂W_1 is identify with ∂W_2 by an orientation-reversing diffeomorphism. Then, Z is diffeomorphic to the union of two copies of $S^2 \times_{\pm 2} D^2$ along their boundaries. The mapping class group of $\mathbb{R}P^3$ has one path-connected component. Hence, there is a unique way, up to isotopy, to glue two copies of $S^2 \times_{\pm 2} D^2$ to get an orientable manifold. In particular, ∂D^2-fibers from two copies of $\partial(S^2 \times_{\pm 2} D^2)$ coincide. Therefore, Z is diffeomorphic to an S^2-bundle over S^2. That is $Z \cong S^2 \times S^2$ or $S^2 \widetilde{\times} S^2$, where $S^2 \widetilde{\times} S^2$ is the nontrivial orientable S^2-bundle over S^2.

Consider that $S^2 \times_{\mathbb{Z}_2} I \cong \mathbb{R}P^3 - B^3$. For $i \in \{1,2\}$, $\partial W_i - V_i \cong \mathbb{R}P^3 - (\mathbb{R}P^3 - B^3) \cong D^3$. Z can be obtained from W by identifying $\partial W_1 - V_1$ and $\partial W_2 - V_2$. In other words, $Z \cong W \cup D^4$. Therefore, $W \cong Z - B^4 \cong D^4 \# (S^2 \times S^2)$ or $D^4 \# (S^2 \widetilde{\times} S^2)$. □

2.5 S^1 and T^2-actions

In this section, we list some S^1 and T^2-actions on the unit normal bundle of a soul of a complete noncompact orientable Riemannian 4-manifold which admits a metric of nonnegative sectional curvature. From Lemma 2.12, they are D^4, $S^1 \times D^3$, a D^2-bundle over S^2, $S^2 \times_\omega D^2$ for $\omega \in \mathbb{Z}$, a D^2-bundle over $\mathbb{R}P^2$, $(S^2 \times_\omega D^2)/\mathbb{Z}_2$ for $\omega \in \mathbb{Z}$, $(\mathbb{R}P^2 \times S^1) \widetilde{\times} I$, $(S^2 \widetilde{\times} S^1) \widetilde{\times} I$, $T^2 \times D^2$, $T^2 \times_{\mathbb{Z}_2} D^2$, or $\beta_k \widetilde{\times} I$ for $k \in \{1,2,3,4\}$.

2.5.1 D^2-bundles over S^2, $S^2 \times_\omega D^2$, $\omega \in \mathbb{Z}$

We follow the constructions in [7] for S^1 and T^2-actions on D^2-bundles over S^2.

Write S^2 as $B_1 \cup B_2$ where B_1 and B_2 are the upper and lower hemispheres respectively. For $i \in \{1,2\}$, we use the polar coordinates (r,γ) on B_i and (s,δ) on D_i^2 where D_i^2 the D^2-fiber of $S^2 \times_\omega D^2$, $r, s \in [0,1]$ and $\gamma, \delta \in [0, 2\pi)$.

For relatively prime integers u_i and v_i, we define an S^1-action on $B_i \times D_i^2$ by

$$\begin{aligned} S^1 \times (B_i \times D_i^2) &\to B_i \times D_i^2, \\ \phi \times (r,\gamma,s,\delta) &\mapsto (r, \gamma + u_i\phi, s, \delta + v_i\phi). \end{aligned} \tag{2.14}$$

If $u_2 = -u_1$ and $v_2 = -\omega u_1 + v_1$, then we obtain $Y_\omega = B_1 \times D_1^2 \cup_G B_2 \times D_2^2$ where G is an equivariant pasting $G : \partial B_1 \times D_1^2 \to \partial B_2 \times D_2^2$ so that $G(1,\gamma,s,\delta) = (1, -\gamma, s, -\omega\gamma + \delta)$. The resulting manifold Y_ω is diffeomorphic to $S^2 \times_\omega D^2$, the D^2-bundle over S^2 with Euler number ω, i.e. ω is the self-intersection number of the zero section of Y_ω.

Moreover, ∂Y_ω is obtained as the equivariant union of two solid tori $\partial Y_\omega = B_1 \times \partial D_1^2 \cup_F B_2 \times \partial D_2^2$ where $F = \begin{pmatrix} -1 & 0 \\ -\omega & 1 \end{pmatrix}$. Hence, ∂Y_ω is the Lens space $L(|\omega|, 1)$ if $\omega \neq 0$. Y_0 is the product $S^2 \times D^2$ and $\partial Y_0 = S^2 \times S^1$.

We can similarly define an effective T^2-action on $S^2 \times_\omega D^2$

$$\begin{aligned} T^2 \times (B_i \times D_i^2) &\to B_i \times D_i^2(\phi,\theta) \\ (r,\gamma,s,\delta) &\mapsto (r, \gamma + u_i\phi + w_i\theta, s, \delta + v_i\phi + t_i\theta) \end{aligned} \tag{2.15}$$

where the intergers u_i, v_i, w_i, and t_i satisfy

$$\begin{vmatrix} u_i & w_i \\ v_i & t_i \end{vmatrix} = \pm 1. \tag{2.16}$$

The pasting G defined above is T^2-equivariant if also $w_2 = -w_1$ and $t_2 = -\omega w_1 + t_1$.

We describe S^1 and T^2-actions on $S^2 \times_\omega D^2$ by the matrix

$$\begin{pmatrix} u_1 & u_2 & w_1 & w_2 \\ v_1 & v_2 & t_1 & t_2 \end{pmatrix} \tag{2.17}$$

which satisfies certain conditions.

The full list of S^1-actions on $Y_\omega \cong S^2 \times_\omega D^2$ is given in [7]. Here we present a selection that we will use. Throughout the following, $\varepsilon = \pm 1$, n is an arbitrary integer, and pairs (α_j, β_j) are relatively prime with $0 < \beta_j < \alpha_j$.

(a) Suppose

$$\varepsilon' = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{vmatrix} = \pm 1, \quad \varepsilon'' = \begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_2 & \beta_2 \end{vmatrix} = \pm 1, \quad \text{and} \quad \omega = \varepsilon'\varepsilon'' \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}. \tag{2.18}$$

Then

$$\begin{pmatrix} u_1 & u_2 & w_1 & w_2 \\ v_1 & v_2 & t_1 & t_2 \end{pmatrix} = \begin{pmatrix} \varepsilon\alpha_1 & -\varepsilon\alpha_1 & \varepsilon(\beta_1 + n\alpha_1) & -\varepsilon(\beta_1 + n\alpha_1) \\ \varepsilon\varepsilon'\alpha_2 & -\varepsilon\varepsilon''\alpha_2 & \varepsilon\varepsilon'(\beta_2 + n\alpha_2) & -\varepsilon\varepsilon''(\beta_3 + n\alpha_3) \end{pmatrix} \tag{2.19}$$

describes actions on Y_ω with the orbit space $Y_\omega^* \simeq D^3$. The restriction of the above S^1-action onto ∂Y_ω has two exceptional orbits.

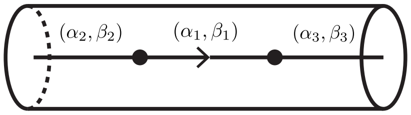


Figure 2.1: The orbit space Y_ω^* of a circle action on $S^2 \times_\omega D^2$ with two exceptional orbits on $\partial(Y_\omega^*)$

(b) Suppose

$$\varepsilon'' = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{vmatrix} = \pm 1, \quad b\alpha_1 + \beta_1 = \pm 1, \quad \varepsilon' = \begin{vmatrix} 1 & |b| \\ \alpha_1 & \beta_1 \end{vmatrix}, \quad \text{and} \quad \omega = \varepsilon'\varepsilon'' \begin{vmatrix} 1 & |b| \\ \alpha_2 & \beta_2 \end{vmatrix}. \tag{2.20}$$

Then

$$\begin{pmatrix} u_1 & u_2 & w_1 & w_2 \\ v_1 & v_2 & t_1 & t_2 \end{pmatrix} = \begin{pmatrix} \varepsilon\alpha_1 & -\varepsilon\alpha_1 & \varepsilon(\beta + n\alpha_1) & -\varepsilon(\beta + n\alpha_1) \\ \varepsilon\varepsilon' & -\varepsilon\varepsilon''\alpha_2 & \varepsilon\varepsilon'(|b| + n) & -\varepsilon\varepsilon''(\beta_2 + n\alpha_2) \end{pmatrix} \tag{2.21}$$

describes actions on Y_ω with the orbit space $Y_\omega^* \simeq D^3$. The restriction of the above S^1-action onto ∂Y_ω has one exceptional orbits.

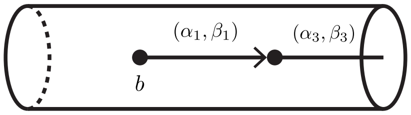


Figure 2.2: The orbit space Y_ω^* of a circle action on $S^2 \times_\omega D^2$ with one exceptional orbit on $\partial(Y_\omega^*)$

(c) Suppose

$$b'\alpha_1 + \beta_1 = \pm 1, \quad b''\alpha_1 + \beta_1 = \pm 1, \quad \varepsilon' = \begin{vmatrix} 1 & |b| \\ \alpha_1 & \beta_1 \end{vmatrix}, \tag{2.22}$$
$$\varepsilon'' = \begin{vmatrix} \alpha_1 & \beta_1 \\ 1 & |b''| \end{vmatrix}, \quad \text{and} \quad \omega = \varepsilon'\varepsilon'' \begin{vmatrix} 1 & |b| \\ 1 & |b''| \end{vmatrix}.$$

Then

$$\begin{pmatrix} u_1 & u_2 & w_1 & w_2 \\ v_1 & v_2 & t_1 & t_2 \end{pmatrix} = \begin{pmatrix} \varepsilon\alpha_1 & -\varepsilon\alpha_1 & \varepsilon(b + n\alpha_1) & -\varepsilon(b + n\alpha_1) \\ \\ \varepsilon\varepsilon' & -\varepsilon\varepsilon'' & \varepsilon\varepsilon'(|b'| + n) & -\varepsilon\varepsilon''(|b''| + n) \end{pmatrix} \tag{2.23}$$

describes actions on Y_ω with the orbit space $Y_\omega^* \simeq D^3$. The restriction of the above S^1-action onto ∂Y_ω has no exceptional orbits.

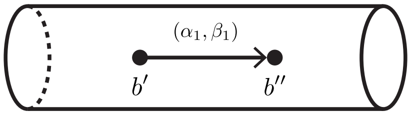


Figure 2.3: The orbit space Y_ω^* of a circle action on $S^2 \times_\omega D^2$ with no exceptional orbits on $\partial(Y_\omega^*)$

2.5.2 D^4

Consider D^4 as $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 \leq 1\}$. Let S^1 act on D^4 by $\theta \in S^1$, $\theta(z_1, z_2) = (e^{i\alpha\theta} z_1, e^{i\beta\theta} z_2)$, for some relatively prime α, β. The orbit space of the S^1-action is $(D^4)^* \cong D^3$. The restriction of the action to $S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$ gives a Seifert bundle with at most two exceptional points.

2.5.3 $S^1 \times D^3$

There is an S^1-action on $S^1 \times D^3$ by rotation on the S^1-factor and by the trivial action on the D^3-factor. An another S^1-action on $S^1 \times D^3$ is by rotation about an axis on the D^3-factor and by the trivial action on the S^1-factor.

2.5.4 $T^2 \times D^2$

There is an S^1-action on $T^2 \times D^2$ by rotation about the origin on the D^2-factor and by the trivial action on the T^2-factor. There is a T^2-action on $T^2 \times D^2$ by the standard T^2-action on the T^2-factor and by the trivial action on the D^2-factor. Note that this T^2-action also restricts to an S^1-action.

2.5.5 $(S^2 \times_\omega D^2)/\mathbb{Z}_2$, $\omega \in \mathbb{Z}$

There are S^1 and T^2-actions on the double cover $(S^2 \times_\omega D^2)$ of $(S^2 \times_\omega D^2)/\mathbb{Z}_2$ as given above.

2.5.6 $(\mathbb{R}P^2 \times S^1)\widetilde{\times}I, (S^2\widetilde{\times}S^1)\widetilde{\times}I$

There is an S^1-action on the double cover $(S^2 \times S^1) \times I$ of $(\mathbb{R}P^2 \times S^1)\widetilde{\times}I$ and $(S^2\widetilde{\times}S^1)\widetilde{\times}I$ by an S^1-action on the $S^1 \times S^2$-factor and by the trivial on the I-factor.

2.5.7 $T^2 \times_{\mathbb{Z}_2} D^2$

There are S^1 and T^2-actions on the double cover $T^2 \times D^2$ of $T^2 \times_{\mathbb{Z}_2} D^2$ as given above.

2.5.8 $\mathcal{B}_i\widetilde{\times}I$, $i \in \{1, 2, 3, 4\}$

The double cover of $\mathcal{B}_1$ and $\mathcal{B}_2$ is T^3. There are S^1 and T^2-actions on the double cover $T^3 \times I$ of $\mathcal{B}_i\widetilde{\times}I$, $i \in \{1, 2\}$, by an action on the T^3-factor and by the trivial action on the I-factor.

The double cover of $\mathcal{B}_3$ and $\mathcal{B}_4$ is $\mathcal{G}_2$. $\mathcal{G}_2$ is a T^2-bundle over S^1 which admits a T^2-action. There is a T^2-action on the double cover $\mathcal{G}^2 \times I$ of $\mathcal{B}_i\widetilde{\times}I$, $i \in \{3, 4\}$, by a T^2-action on the $\mathcal{G}_2$-factor and by the trivial action on the I-factor.

2.6 Plumbing

We refer to [7, 17] for notation and basics about plumbing.

Definition 2.24. Given two D^2-bundles $\eta_1 : D^2 \to Y_1 \to M_1$ and $\eta_2 : D^2 \to Y_2 \to M_2$ over surfaces M_1 and M_2, we define a *plumbing* $Y_1 \Box Y_2$ of Y_1 and Y_2 as follows.

Choose 2-disks $B_1 \subset M_1$ and $B_2 \subset M_2$ and the bundles over them, ξ_1 and ξ_2 respectively. Since they are trivial bundles, there are natural identifications $\mu_1 : B_1 \times D^2 \to \xi_1$ and $\mu_2 : B_2 \times D^2 \to \xi_2$. Consider the reflection $t : B_1 \times D^2 \to B_2 \times D^2$, $t(x, y) = (y, x)$ and define the homeomorphism $f : \xi_1 \to \xi_2$ by $f = \mu_2 t \mu_1^{-1}$. Pasting η_1 and η_2 together along ξ_1 and ξ_2 by the map f is called *plumbing*.

The resulting manifold of a plumbing is a 4-manifold with corners that may be smoothed. It is independent of the choices involved [17].

2.6.1 Equivariant plumbing

Fintushel [7] showed that a plumbing of D^2-bundles over S^2 can be done equivariantly with the S^1 and T^2-actions given in Subsection 2.5.1.

Lemma 2.25 ([7]). *Let Y_{ω_1} and Y_{ω_2} be D^2-bundles over S^2 constructed as in Subsection 2.5.1. Recall that we can write $Y_{\omega_1} = B_{1,1} \times D_{1,1} \cup_G B_{1,2} \times D_{1,2}$ and $Y_{\omega_2} = B_{2,1} \times D_{2,1} \cup_G B_{2,2} \times D_{2,2}$ where $B_{j,k}$ and $D_{j,k}$ are 2-disks and G is an equivariant pasting.*

Then, we may equivariantly plumb together Y_{ω_1} and Y_{ω_2} by identifying $B_{2,1} \times D_{2,1}$ with $B_{1,2} \times D_{1,2}$. The resulting manifold $Y_{\omega_1} \Box Y_{\omega_2}$ has an induced S^1-action and T^2-action.

2.7 F-structures

We refer to [5, 6, 9] for basics about F-structures.

The following definition is adapted from [9]. The difference is that we allow torus actions to have fixed points.

Definition 2.26. An *F-structure* on M is an open cover $\{U_i\}_i$ together with an action of T^{n_i} on $\widetilde{U}_i$, which is a finite normal cover of U_i, with the following properties.

(1) If $U_i \cap U_j \neq \varnothing$, then there exists a covering $\pi_{ij} : \widetilde{U}_{ij} \to U_i \cap U_j$ and maps $\pi_{ij,i} : \widetilde{U}_{ij} \to \widetilde{U}_i$ and $\pi_{ij,j} : \widetilde{U}_{ij} \to \widetilde{U}_j$ so that the following diagram is commutative.

$$\begin{tikzcd}
\widetilde{U}_i \arrow[d, "\pi_i"] & \widetilde{U}_{ij} \arrow[l, "\pi_{ij,i}"'] \arrow[r, "\pi_{ij,j}"] \arrow[d, "\pi_{ij}"] & \widetilde{U}_j \arrow[d, "\pi_j"] \\
U_i & U_i \cap U_j, \arrow[l, hook'] \arrow[r, hook] & U_j
\end{tikzcd}$$

That is $\pi_i \circ \pi_{ij,i} = \pi_j \circ \pi_{ij,j} = \pi_{ij}$.

(2) There exists an action of $T^{n_{ij}}$ on $\widetilde{U}_{ij}$.

(3) There exists an n_i-dimensional subtorus $T^{n_i}_{ij} \subset T^{n_{ij}}$ and a locally isomorphic group homomorphism $T^{n_i}_{ij} \to T^{n_i}$, such that $\pi_{ij,i}$ is equivariant. The same holds when we replace i by j.

Definition 2.27. The *orbit* of a point p in M is the minimal invariant set containing p. An F-structure is said to have *positive rank* if all orbits are of positive dimension, i.e., the action of T^{n_i} on $\widetilde{U}_i$ is fixed point free, for all $\widetilde{U}_i$.

In this dissertation, we do *not* assume that F-structures have positive rank.

Definition 2.28. If every finite normal covering $\pi_i : \widetilde{U}_i \to U_i$ in Definition 2.26 is trivial, then the F-structure is called a *T-structure*.

A graph manifold is an example of a manifold which admits a T-structure.

Example 2.29. Let $\{\Sigma_i^2\}$ be a collection of surfaces with boundary. For each i, let $U_i \cong \Sigma_i^2 \times S^1$. For each boundary component σ_i of Σ_i^2, identify the boundary component $\sigma_i \times S^1$ of U_i with a boundary component $\sigma_j \times S^1$ of U_j, for some j, where σ_j is a boundary component of Σ_j, by an element of $SL_2(\mathbb{Z})$. The resulting manifold X is a *graph manifold*.

For each i, let S^1 act on U_i by rotation on the S^1-factor and by the trivial action on Σ_i^2. On an overlap $U_i \cap U_j \cong S^1 \times S^1$, the two S^1-actions from U_i and U_j do not necessary coincide. However, if they do not coincide, then they generate a T^2-action on $U_i \cap U_j$. This T^2-action extends to a T^2-action ϕ on a neighborhood of $U_i \cap U_j$ so that $\phi|_{U_i}$ agrees with the S^1-action on U_i and $\phi|_{U_j}$ agrees with the S^1-action on U_j. In the case that the T^2-action on $U_i \cap U_j$ is not effective, i.e., the orbits are 1-dimensional, we can pass to a quotient to get an effective S^1-action on $U_i \cap U_j$. As a result, X admits a T-structure.

In this example, we have 2-dimensional torus actions on the overlaps and 1-dimensional torus actions elsewhere. The method used to construct a T-structure in this example is a typical technique for constructing an F-structure.

The following lemma says that a plumbing of two D^2-bundles over a surface admits a T-structure (see also [15] for a similar argument).

Lemma 2.30. *For $k \in \{1, 2\}$, let X_k be a D^2-bundle over a surface Σ_k^2. The plumbing $X_1 \Box X_2$ admits a T-structure. If Σ_1^2 and Σ_2^2 are closed surfaces, then $\partial(X_1 \Box X_2)$ is a graph manifold and the restriction of the T-structure to $\partial(X_1 \Box X_2)$ is a T-structure of positive rank.*

Proof. For $k \in \{1, 2\}$, the principal S^1-bundle of $D^2 \to X_k \to \Sigma_k^2$ gives local S^1-actions on X_k where S^1 acts on each D^2-fiber by rotations about the center and acts trivially on Σ_k^2. $\Sigma_k^2 \times \{0\}$ is the set of fixed orbits and the restriction of the local S^1-actions on ∂X_k are free.

Recall the plumbing construction of $X_1 \Box X_2$. Let $B_k^2 \subset \Sigma_k^2$, $k \in \{1, 2\}$, be 2-disks. Let $\phi : D_1^2 \times B_1^2 \to D_2^2 \times B_2^2$ be a map switching the fibers and the bases. That is $\phi(D_1^2, \cdot) = (\cdot, B_2^2)$ and $\phi(\cdot, B_1^2) = (D_2^2, \cdot)$. Then, $X_1 \Box X_2 = X_1 \cup_\phi X_2$ where $D_1^2 \times B_1^2 \subset X_1$ and $D_2^2 \times B_2^2 \subset X_2$ are identified by ϕ. The local S^1-actions on X_k, $k \in \{1, 2\}$, can be chosen so that they restrict to an S^1-action on $D_k^2 \times B_k^2$.

Let $Y \cong D_1^2 \times D_2^2$ be the plumbing location in $X_1 \Box X_2$. There exists a T^2-action on Y whose restriction to $(D_1^2, \cdot)$ coincides with the S^1-action on $D_1^2 \times B_1^2 \subset X_1$ and restriction to $(\cdot, D_2^2)$ coincides with the S^1-action on $D_2^2 \times B_2^2 \subset X_2$. The local S^1-actions on X_1 and X_2 and the T^2-action on a neighborhood of Y together give a T-structure on $X_1 \Box X_2$.

It follows from the plumbing construction that $\partial(X_1 \Box X_2)$ is a graph manifold in the case that Σ_1^2 and Σ_2^2 are closed surfaces. From the above construction, the T-structure on $X_1 \Box X_2$ restricts to a T-structure of positive rank on $\partial(X_1 \Box X_2)$. □

Paternain and Petean [21] showed that a connected sum of two manifolds which admit a T-structure also admits a T-structure.

Lemma 2.31 ([21, Theorem 5.9])**.** *Suppose X and Y are n-dimensional manifolds, $n > 2$, which admit a T-structure. Then $X \# Y$ also admits a T-structure.*

The proof of [21, Theorem 5.9] is by constructing new local tori actions in a neighborhood of the connected sum region. Therefore, the same proof also applies to manifolds X and Y which admit an F-structure such that at least one finite normal covering (as in Definition 2.26) is trivial.

Corollary 2.32 ([22])**.** *Suppose X and Y are n-dimensional manifolds, $n > 2$, which admit an F-structure such that at least one finite normal covering (as in Definition 2.26) is trivial. Then $X \# Y$ also admits an F-structure.*

2.8 Cloudy submanifolds

Kleiner and Lott [13] introduced the notion of a *cloudy submanifold* as a subset of a Euclidean space which looks roughly close to an affine subspace of the Euclidean space to use in their proof of [24, Theorem 7.4]. As a reference, we give the definition of a cloudy submanifold as appeared in [13, Appendix B].

Definition 2.33 ([13, Definition 20.1])**.** Suppose $C, \delta \in (0, \infty)$, $k \in \mathbb{Z}^+$, and H is a Euclidean space. A (C, δ) *cloudy k-manifold in H* is a triple $(\widetilde{S}, S, r)$, where $S \subset \widetilde{S} \subset H$ is a pair of subsets, and $r : \widetilde{S} \to (0, \infty)$ is a (possible discontinuous) function such that:

(1) For all $x, y \in \widetilde{S}$, $|r(y) - r(x)| \leq C(|x - y| + r(x))$.

(2) For all $x \in S$, the rescaled pointed subset $(\frac{1}{r(x)}\widetilde{S}, x)$ is δ-close in the pointed Hausdorff distance to $(\frac{1}{r(x)}A_x, x)$, where A_x is a k-dimensional affine subspace of H.

The following lemma says that every cloudy submanifold has a smooth "core" that comes with a smooth submersion.

Lemma 2.34 ([13, Lemma 20.2])**.** *For all $k, K \in \mathbb{Z}^+$, $\epsilon \in (0, \infty)$, and $C < \infty$, there is a $\delta = \delta(k, K, \epsilon, C) > 0$ with the following property. Suppose $(\widetilde{S}, S, r)$ is a (C, δ) cloudy k-manifold in a Euclidean space H, and for every $x \in S$, we denote by A_x an affine subspace as in Definition 2.33. Then there is a k-dimensional smooth submanifold $W \subset H$ such that*

(1) *For every point $x \in S$, the pointed Hausdorff distance from $(\frac{1}{r(x)}\widetilde{S}, x)$ to $(\frac{1}{r(x)}W, x)$ is at most ϵ.*

(2) *$W \subset N_{\epsilon r}(\widetilde{S})$.*

(3) *$W \cap N_r(S)$ is properly embedded in $N_r(S)$.*

(4) *The nearest point map $P : N_r(S) \to W$ is a well-defined smooth submersion.*

2.9 Approximate splittings and adapted coordinates

In their proof of [24, Theorem 7.4], Kleiner and Lott [13] introduced *approximate $\mathbb{R}^k$-splittings* and related concepts to capture the notion of a pointed metric space approximately splitting off an $\mathbb{R}^k$-factor. In this section, we collect basic properties of approximate splittings and related concepts that we will need in this dissertation. For more details of approximate splittings including the proofs of statements in this section, we refer the readers to [13, Section 4].

2.9.1 Splittings

We recall the notion of *splitting* of a metric space.

Definition 2.35. Let X be a metric space. A product structure on X is an isometry $\alpha : X \to X_1 \times X_2$. A *$k$-splitting* of X is a product structure $\alpha : X \to X_1 \times X_2$, where X_1 is isometric to $\mathbb{R}^k$. A *splitting* is a k-splitting for some k. Two k-splittings $\alpha : X \to X_1 \times X_2$ and $\beta : X \to Y_1 \times Y_2$ are *equivalent* if there are isometries $\phi_i : X_i \to Y_i$ such that $\beta = (\phi_1, \phi_2) \circ \alpha$.

Definition 2.36. Suppose that $j \leq k$. A j-splitting $\alpha : X \to X_1 \times X_2$ is said to be *compatible* with a k-splitting $\beta : X \to Y_1 \times Y_2$ if there is a j-splitting $\phi : Y_1 \to \mathbb{R}^j \times \mathbb{R}^{k-j}$ such that α is equivalent to the j-splitting given by the composition

$$X \xrightarrow{\beta} Y_1 \times Y_2 \xrightarrow{(\phi,\mathrm{Id})} (\mathbb{R}^j \times \mathbb{R}^{k-j}) \times Y_2 \cong \mathbb{R}^j \times (\mathbb{R}^{k-j} \times Y_2). \tag{2.37}$$

Lemma 2.38 ([13, Lemma 4.4])**.** (1) *Suppose $\alpha : X \to \mathbb{R}^k \times Y$ is a k-splitting of a metric space X, and $\beta : X \to \mathbb{R} \times Z$ is a 1-splitting. Then either β is compatible with α, or there is a 1-splitting $\gamma : Y \to \mathbb{R} \times W$ such that β is compatible with the induced splitting $X \to (\mathbb{R}^k \times \mathbb{R}) \times W$.*

(2) *Any two splittings of a metric space are compatible with a third splitting.*

2.9.2 Approximate splittings

Definition 2.39. Given a nonnegative integer k and $\delta \in [0, \infty)$, a *(k, δ)-splitting* of a pointed metric space $(X, \star_X)$ is a δ-Gromov-Hausdorff approximation $(X, \star_X) \to (X_1, \star_{X_1}) \times (X_2, \star_{X_2})$, where $(X_1, \star_{X_1})$ is isometric to $(\mathbb{R}^k, \star_{\mathbb{R}^k})$. (We allow $\mathbb{R}^k$ to have other basepoints than 0.)

The following definitions are approximate versions of equivalence and compatibility of splittings.

Definition 2.40. Suppose that $\alpha : (X, \star_X) \to (X_1, \star_{X_1}) \times (X_2, \star_{X_2})$ is a (j, δ_1)-splitting and $\beta : (X, \star_X) \to (Y_1, \star_{Y_1}) \times (Y_2, \star_{Y_2})$ is an (k, δ_2)-splitting. Then

(1) α is *ϵ-close to β* if $j = k$ and there are ϵ-Gromov-Hausdorff approximations ϕ_i : $(X_i, \star X_i) \to (Y_i, \star_{Y_i})$ such that the composition $(\phi_1, \phi_2) \circ \alpha$ is ϵ-close to β, i.e., agrees with β on $B(\star_X, \epsilon^{-1})$ up to error at most ϵ.

(2) α is *ϵ-compatible with β* if $j \leq k$ and there is a j-splitting $\gamma : (Y_1, \star_{Y_1}) \to (\mathbb{R}^j, \star_{\mathbb{R}^j}) \times (\mathbb{R}^{k-j}, \star_{\mathbb{R}^{k-j}})$ such that the (j, δ_2)-splitting defined by the composition

$$X \xrightarrow{\beta} Y_1 \times Y_2 \xrightarrow{(\gamma,\mathrm{Id})} (\mathbb{R}^j \times \mathbb{R}^{k-j}) \times Y_2 \cong \mathbb{R}^j \times (\mathbb{R}^{k-j} \times Y_2). \tag{2.41}$$

is ϵ-close to α.

Lemma 2.42 ([13, Lemma 4.10]). *Given $\delta > 0$ and $C < \infty$, there is a $\delta' = \delta'(\delta, C) > 0$ with the following property. Suppose that $(X, \star_X)$ is a complete pointed metric space with a (k, δ')-splitting α. Then for any $x \in B(\star_X, C)$, the pointed space (X, x) has a (k, δ)-splitting coming from a change of basepoint of α.*

2.9.3 Approximate splittings of Alexandrov spaces

The next lemma shows that the notions of having a good strainer and having a good approximate $\mathbb{R}^k$-splitting are essentially equivalent for Alexandrov spaces.

Lemma 2.43 ([13, Lemma 4.15]). (1) *Given $k \in \mathbb{Z}^+$ and $\delta > 0$, there is a $\delta' = (k, \delta) > 0$ with the following property. Suppose that $(X, \star_X)$ is a complete pointed nonnegatively curved Alexandrov space with a (k, δ')-splitting. Then $\star_X$ has a k-strainer of quality δ at a scale δ^{-1}.*

(2) *Given $n \in \mathbb{Z}^+$ and $\delta > 0$, there is a $\delta' = (n, \delta) > 0$ with the following property. Suppose that $(X, \star_X)$ is a complete pointed length space so that $B(\star_X, \frac{1}{\delta'})$ has curvature bounded below by $-\delta'$ and dimension bounded above by n. Suppose that for some $k \leq n$, $\star_X$ has a k-strainer $\{p^{\pm}\}_{i=1}^k$ of quality δ' at a scale $\frac{1}{\delta'}$. Then $(X, \star_X)$ has a (k, δ)-splitting $\phi : (X, \star_X) \to (\mathbb{R}^k \times X', (0, \star_{X'}))$ where the composition $\pi_{\mathbb{R}^k} \circ \phi$ has j^{th} component $d_X(p_j^+, \star_X) - d_X(p_j^+, \cdot)$.*

2.9.4 Compatibility of approximate splittings

The following lemma states that the nonexistence of an approximate $(k+1)$-splitting implies that approximate j-splittings are approximately compatible with k-splittings for $j \leq k$.

Lemma 2.44 ([13, Lemma 4.17]). *Given $j \leq k \leq n \in \mathbb{Z}^+$ and $\beta'_k, \beta_{k+1} > 0$, there are numbers $\delta = \delta(j, k, n, \beta'_k, \beta_{k+1}) > 0$, $\beta_j = \beta_j(j, k, n, \beta'_k, \beta_{k+1}) > 0$, and $\beta_k = \beta_k(j, k, n, \beta'_k, \beta_{k+1}) > 0$ with the following property. If $(X, \star_X)$ is a complete pointed length space such that*

(1) *The ball* $B(\star_X, \delta^{-1})$ *has curvature bounded below by* $-\delta$ *and dimension bounded above by* n*, and*

(2) $(X, \star_X)$ *does not admit a* $(k+1, \beta_{k+1})$*-splitting,*

then any (j, β_j)*-splitting of* $(X, \star_X)$ *is* β'_k*-compatible with any* (k, β_k)*-splitting.*

2.9.5 Overlapping cones

Recall that a pointed metric space $(X, \star_X)$ is a *metric cone* if it is a union of rays leaving the basepoint $\star$, and the union of any two rays γ_1 and γ_2 leaving $\star$ is isometric to the union of two rays $\bar{\gamma}_1$, $\bar{\gamma}_2 \subset \mathbb{R}^2$ leaving the origin $0 \in \mathbb{R}^2$.

The following lemma says that an existence of two cone points implies a 1-splitting.

Lemma 2.45 ([13, Lemma 4.19])**.** *If* $(X, \star_X)$ *is a conical nonnegatively curved Alexandrov space and there is some* $x \neq \star_X$ *so that* (X, x) *is also a conical Alexandrov space, then* X *has a 1-splitting such that the segment from* $\star_X$ *to* x *is parallel to the* $\mathbb{R}$*-factor.*

The following lemma is an approximate version of Lemma 2.45.

Lemma 2.46 ([13, Lemma 4.20])**.** *Given* $n \in \mathbb{Z}^+$ *and* $\delta > 0$*, there is a* $\delta' = \delta'(n, \delta) > 0$ *with the following property. If*

(1) $(X, \star_X)$ *is a complete pointed length space,*

(2) $x \in X$ *has* $d(\star_X, x) = 1$*, and*

(3) $(X, \star_X)$ *and* (X, x) *have pointed Gromov-Hausdorff distance less than* δ' *from conical nonnegatively curved Alexandrov spaces* CY *and* CY'*, respectively, of dimension at most* n*,*

then (X, x) *has a* $(1, \delta)$*-splitting.*

2.9.6 Adapted coordinates

Definition 2.47 ([13, Definition 4.21])**.** Suppose $0 < \delta' \leq \delta$, and let α be a (k, δ')-splitting of a complete pointed Riemannian manifold $(M, \star_M)$. Let $\Phi : B(\star_M, \delta^{-1}) \to \mathbb{R}^k$ be the composition $B(\star_M, \delta^{-1}) \to^{\alpha} \mathbb{R}^k \times X_2 \to \mathbb{R}^k$. Then a map $\phi : (B(\star_M, 1), \star_M) \to (\mathbb{R}^k, \phi(\star_M))$ defines *α-adapted coordinates of quality δ* if the following holds.

(1) ϕ is smooth and $(1+\delta)$-Lipschitz.

(2) The image of ϕ has Hausdorff distance at most δ from $B(\phi(\star_M), 1) \subset \mathbb{R}^k$.

(3) For all $m \in B(\star_M, 1)$ and $m' \in B(\star_M, \delta^{-1})$ with $d(m, m') > 1$, the unit-length initial velocity vector $v \in T_m M$ of any minimizing geodesic from m to m' satisfies

$$\left| D\phi(v) - \frac{\Phi(m') - \Phi(m)}{d(m, m')} \right| < \delta. \tag{2.48}$$

We say that a map $\phi : (B(\star_M, 1), \star_M) \to (\mathbb{R}^k, 0)$ define *adapted coordinates of quality* δ if there exists a (k, δ)-splitting α such that ϕ defines α-adapted coordinates of quality δ, as above. Likewise, $(M, \star_M)$ admits k*-dimensional adapted coordinates of quality* δ if there is a map ϕ as above which defines adapted coordinates of quality δ.

We will refer to the following lemma as the *existence of adapted coordinates.*

Lemma 2.49 ([13, Lemma 4.23])**.** *For all $n \in \mathbb{Z}^+$ and $\delta > 0$, there is a $\delta' = \delta'(n, \delta) > 0$ with the following property. Suppose that $(M, \star_M)$ is an n-dimensional complete pointed Riemannian manifold with sectional curvature bounded below by $-(\delta')^2$ on $B(\star_M, \frac{1}{\delta'})$, which has a (k, δ')-splitting α. Then there exist α-adapted coordinates of quality δ.*

We will refer to the following lemma as the *uniqueness of adapted coordinates.*

Lemma 2.50 ([13, Lemma 4.28])**.** *Given $1 \le k \le n \in \mathbb{Z}^+$ and $\epsilon > 0$, there is an $\epsilon' = \epsilon'(n, \epsilon) > 0$ with the following properties. Suppose that*

(1) *$(M, \star_M)$ is an n-dimensional complete pointed Riemannian manifold with sectional curvature bounded below by $-(\epsilon')^2$ on $B(\star_M, \frac{1}{\epsilon'})$.*

(2) *$\alpha : (M, \star_M) \to (\mathbb{R}^k \times Z, (0, \star_Z))$ is a (k, ϵ')-splitting of $(M, \star_M)$.*

(3) *$\phi_1 : B(M, \star_M) \to (\mathbb{R}^k, 0)$ defines α-adapted coordinates of quality ϵ' on $B(\star_M, 1)$.*

(4) *Either*

    (a) *$\phi_2 : B(M, \star_M) \to (\mathbb{R}^k, 0)$ defines α-adapted coordinates of quality ϵ' on $B(\star_M, 1)$, or*

    (b) *ϕ_2 has $(1 + \epsilon')$-Lipschitz components and the following holds:*
    For every $m \in B(\star_M, 1)$ and every $j \in \{1, \ldots, k\}$, there is an $m'_j \in B(\star_M, \frac{1}{\epsilon'})$ with $d(m'_j, m) > 1$ satisfying (2.48) (with ϕ replaced by ϕ_2), such that $(\pi_{\mathbb{R}^k} \circ \alpha)(m'_j)$ lies in the ϵ'-neighborhood of the line $(\pi_{\mathbb{R}^k} \circ \alpha)(m) + \mathbb{R}e_j$, and $(\pi_Z \circ \alpha)(m'_j)$ lies in the ϵ'-ball centered at $(\pi_Z \circ \alpha)(m)$.

Then $\|\phi_1 - \phi_2\|_{C^1} < \epsilon$ on $B(\star_M, 1)$.

The next lemma shows that approximate compatibility of two approximate splittings leads to an approximate compatibility of their associated adapted coordinates.

Lemma 2.51 ([13, Lemma 4.31]). *Given* $1 \leq j \leq k \leq n \in \mathbb{Z}^+$ *and* $\epsilon > 0$*, there is an* $\epsilon' = \epsilon'(n, \epsilon) > 0$ *with the following properties. Suppose that*

(1) $(M, \star_M)$ *is an* n*-dimensional complete pointed Riemannian manifold with sectional curvature bounded below by* $-(\epsilon')^2$ *on* $B(\star_M, \frac{1}{\epsilon'})$*.*

(2) α_1 *is a* (j, ϵ')*-splitting of* $(M, \star_M)$ *and* α_2 *is a* (k, ϵ')*-splitting of* $(M, \star_M)$*.*

(3) α_1 *is* ϵ'*-compatible with* α_2*.*

(4) $\phi_1 : (M, \star_M) \to (\mathbb{R}^j, 0)$ *and* $\phi_2 : (M, \star_M) \to (\mathbb{R}^k, 0)$ *are adapted coordinates of quality* ϵ' *on* $B(\star_M, 1)$*, associated to* α_1 *and* α_2*, respectively.*

Then there exists a map $T : \mathbb{R}^k \to \mathbb{R}^j$*, which is a composition of an isometry with an orthogonal projection, such that* $\|\phi_1 - T \circ \phi_2\|_{C^1} \leq \epsilon$ *on* $B(\star_M, 1)$*.*

By rescaling, we can define adapted coordinates on a ball of any specified size, and the results of this subsection will remain valid ([13, Remark 4.35]).

— 3 —

Standing Assumptions

We now start the proof of Theorem 1.4. The proof is by contradiction.

Lemma 3.1. *If Theorem 1.4 is false then we can satisfy Standing Assumption 3.2, for an appropriate choice of* A'.

Lemma 3.1 implies that if we can get a contradiction from Standing Assumption 3.2, then we have proven Theorem 1.4. Recall from Definition 1.7 that if $w \leq w'$ then $r_p(w) \geq r_p(w')$.

Standing Assumption 3.2. *Let* $K \geq 10$ *be a fixed integer and let* $A' : (0,\infty) \times (0,\infty) \to (0,\infty)$ *be a function.*

We assume that $\{(M^\alpha, g^\alpha)\}_{\alpha=1}^\infty$ *is a sequence of connected closed Riemannian 4-manifolds such that*

(1) *For all* $p \in M^\alpha$*, the ratio* $\frac{R_p}{r_p(1/\alpha)}$ *of the curvature scale at* p *to the* $\frac{1}{\alpha}$*-volume scale at* p *is bounded below by* α.

(2) *For all* $p \in M^\alpha$*,* $\omega' \in \left[\frac{1}{\alpha}, c_4\right)$*, integer* $k \in [0, K]$*, and* $C \in (0,\alpha)$*, we have that* $|\nabla^k \mathrm{Rm}| \leq A'(C,w') r_p(w')^{-(k+2)}$ *on* $B(p, Cr_p(w'))$.

(3) *Each* M^α *does not admit a metric of nonnegative sectional curvature or an* F*-structure.*

Proof. Suppose that Theorem 1.4 is false. Then for every positive integer α, there is a manifold (M^α, g^α) which satisfies the hypothesis of Theorem 1.4 with the parameter w_0 set to $w_0^\alpha = \frac{1}{16\alpha^4}$, but M^α does not admit a metric of nonnegative sectional curvature or an F-structure.

First, we claim that for every $p^\alpha \in M^\alpha$, we have that $r_{p^\alpha}(1/\alpha) < R_{p^\alpha}$. If not, then for some $p^\alpha \in M^\alpha$, $r_{p^\alpha}(1/\alpha) \geq R_{p^\alpha}$. From the definition of $r_{p^\alpha}(1/\alpha)$,

$$\mathrm{vol}(B(p^\alpha, R_{p^\alpha})) \geq \frac{1}{\alpha} R_{p^\alpha}^4 > \frac{1}{16\alpha^4} R_{p_\alpha}^4 \tag{3.3}$$

which contradicts our choice of w_0^α.

Therefore, $r_{p^\alpha}(1/\alpha) < R_{p^\alpha}$. Then

$$\frac{1}{\alpha}(r_{p^\alpha}(1/\alpha))^4 = \mathrm{vol}(B(p^\alpha, r_{p^\alpha}(1/\alpha))) \leq \mathrm{vol}(B(p^\alpha, R_{p^\alpha})) \leq \frac{1}{16\alpha^4} R^4_{p^\alpha}. \tag{3.4}$$

Thus,

$$\frac{R_{p^\alpha}}{r_{p^\alpha}(1/\alpha)} \geq 2\alpha. \tag{3.5}$$

We then have that $\{(M^\alpha, g^\alpha)\}_{\alpha=1}^\infty$ satisfies condition (1) of Standing Assumption 3.2.

Next, we show that condition (2) of Standing Assumption 3.2 holds, for an appropriate choice of A'. First, consider that it suffices to just consider $C \in [1, \alpha)$ because a derivative bound on a larger ball implies a derivative bound on a smaller ball. For $\widetilde{w}' \in \big[\frac{1}{\alpha}, c_4\big)$, we have

$$C r_{p^\alpha}(\widetilde{w}') \leq \alpha r_{p^\alpha}(1/\alpha) \leq R_{p^\alpha}. \tag{3.6}$$

Then,

$$\mathrm{vol}(B(p^\alpha, C r_{p^\alpha}(\widetilde{w}'))) \geq \mathrm{vol}(B(p^\alpha, r_{p^\alpha}(\widetilde{w}'))) = \widetilde{w}'(r_{p^\alpha}(w'))^4 = C^{-4}\widetilde{w}'(C r_{p^\alpha}(\widetilde{w}'))^4 \tag{3.7}$$

Put $w' = C^{-4}\widetilde{w}'$. We have that

$$w_0^\alpha = \frac{1}{16\alpha^4} \leq w' < c_4. \tag{3.8}$$

Hypothesis (2) of Theorem 1.4 implies that

$$|\nabla^k \mathrm{Rm}| \leq A(w')(C r_{p^\alpha}(\widetilde{w}'))^{-(k+2)} \tag{3.9}$$

on $B(p^\alpha, C r_{p^\alpha}(\widetilde{w}'))$. Hence, condition (2) of Standing Assumption 3.2 will be satisfied, for $C \in [1, \alpha)$, if we take

$$A'(C, \widetilde{w}') = \max_{0 \leq k \leq K} A(C^{-4}\widetilde{w}')C^{-(k+2)}. \tag{3.10}$$

□

Standing Assumption 3.2 will remain in force until Chapter 15 where we will get a contradiction to the Standing Assumption.

For simplicity, we will suppress the superscript α. We will refer to M^α simply just by M. By convention, each of the statements made in the proof is to be interpreted as being valid provided that α is sufficiently large without being mentioned explicitly.

Remark 3.11. We note that for a fixed $\widehat{w} \in (0, c_4)$, conditions (1) and (2) of Standing Assumption 3.2 imply that for large α, the following holds for all $p \in M^\alpha$:

(1) $\frac{R_p}{r_p(\widehat{w})} \geq \alpha$.

(2) For each integer $k \in [0, K]$ and each $C \in (0, \alpha)$, we have $|\nabla^k \mathrm{Rm}| \leq A'(C, \widehat{w}) r_p(\widehat{w})^{-(k+2)}$ on $B(p, C r_p(\widehat{w}))$.

— 4 —

The modified volume scale $\mathfrak{r}$

In this chapter, we introduce a smooth scale function $\mathfrak{r} : M \to (0, \infty)$, which we call the *modified volume scale*. Next, we show that $\mathfrak{r}$ is like a volume scale in the sense that there is lower bounds on volume at scale $\mathfrak{r}$. This allows us to make use of C^K-paracompactness arguments at scale $\mathfrak{r}$. The advantage of $\mathfrak{r}$ over a volume scale is that the Lipschitz constant of $\mathfrak{r}$ can be made arbitrarily small. This will allow us to glue local structures together in later chapters.

Lemma 4.1 ([13, Lemma 6.1])**.** *Suppose X is a metric space, $C \in (0, \infty)$, and $l, u : X \to (0, \infty)$ are functions. Then, there is a C-Lipschitz $r : X \to (0, \infty)$ satisfying $l \leq r \leq u$ if and only if*

$$l(p) - Cd(p, q) \leq u(q) \tag{4.2}$$

for all $p, q, \in X$.

Proof. We repeat the proof in [13] here for completeness. Clearly, if such an r exists then (4.2) must hold.

Conversely, suppose that (4.2) holds and define $r : X \to (0, \infty)$ by

$$r(q) = \sup_{p \in X}\{l(p) - Cd(p, q)\}. \tag{4.3}$$

Then, $l \leq r \leq u$. For $q, q' \in X$, since $l(p) - Cd(p, q) \geq l(p) - Cd(p, q') - C(q, q')$, we obtain $r(q) \geq r(q') - Cd(q, q')$, from which it follows that r is C-Lipschitz. □

Let $\Lambda > 0$ and $\bar{w} \in (0, c_4)$ be new parameters where c_4 is the volume of the unit ball in $\mathbb{R}^4$. Put

$$w' = \frac{\bar{w}}{2(1 + 2\Lambda^{-1})^4}. \tag{4.4}$$

Corollary 4.5. *There is a smooth Λ-Lipschitz function $\mathfrak{r} : M \to (0, \infty)$ such that for every $p \in M$, we have*

$$\frac{1}{2} r_p(\bar{w}) \leq \mathfrak{r}(p) \leq 2 r_p(w') \tag{4.6}$$

Proof. Let $l : M \to (0, \infty)$ be the $\bar{w}$-volume scale and let $u : M \to (0, \infty)$ be the w'-volume scale. We first verify (4.2) with parameter $C = \frac{\Lambda}{2}$. To argue by contradiction, suppose that for some $p, q \in M$, we have $l(p) - \frac{1}{2}\Lambda d(p, q) > u(q)$. In particular, $d(p, q) < \frac{2}{\Lambda} l(p)$ and $u(q) < l(p)$. Hence, $B(p, l(p)) \subset B(q, (1 + 2\Lambda^{-1})l(p)) \subset B(p, (1 + 4\Lambda^{-1})l(p))$. It follows that

$$\mathrm{vol}(B(q, (1 + 2\Lambda^{-1})l(p)) \geq \mathrm{vol}(B(p, l(p))) = \bar{w}l^4(p) = 2w'((1 + 2\Lambda^{-1})l(p))^4. \tag{4.7}$$

For any $c > 0$, if α is sufficiently large, then the sectional curvature on $B(p, (1 + 4\Lambda^{-1})l(p))$, and hence on $B(q, (1 + 2\Lambda^{-1})l(p))$ is bounded below by $-c^2 l(p)^{-2}$. As $u(q) < l(p) < (1 + 2\Lambda^{-1})l(p)$, the Bishop-Gromov inequality implies that

$$\begin{aligned} \frac{w' u(q)^4}{\displaystyle\int_0^{\frac{u(q)}{l(p)}} \sinh^3(cr)\,\mathrm{d}r} = \frac{\mathrm{vol}(B(q, u(q)))}{\displaystyle\int_0^{\frac{u(q)}{l(p)}} \sinh^3(cr)\,\mathrm{d}r} &\geq \frac{\mathrm{vol}(B(q, (1 + 2\Lambda^{-1})l(p)))}{\displaystyle\int_0^{1+2\Lambda^{-1}} \sinh^3(cr)\,\mathrm{d}r} \\ &\geq \frac{2w'((1 + 2\Lambda^{-1})l(p))^4}{\displaystyle\int_0^{1+2\Lambda^{-1}} \sinh^3(cr)\,\mathrm{d}r}. \end{aligned} \tag{4.8}$$

Then,

$$\frac{c^3 \left(\frac{u(q)}{l(p)}\right)^4}{\displaystyle\int_0^{\frac{u(q)}{l(p)}} \sinh^3(cr)\,\mathrm{d}r} \geq \frac{2c^3(1 + 2\Lambda^{-1})^4}{\displaystyle\int_0^{1+2\Lambda^{-1}} \sinh^3(cr)\,\mathrm{d}r}. \tag{4.9}$$

Since the function $x \mapsto \dfrac{c^3}{4} \dfrac{x^4}{\int_0^x \sinh^3(cr)\,\mathrm{d}r}$ tends uniformly to 1 as $c \to 0$, for $x \in (0, 1 + 2\Lambda^{-1}]$, taking c sufficiently small gives a contradiction.

By Lemma 4.1, there is a $\frac{\Lambda}{2}$-Lipschitz function r on M satisfying $l \leq r \leq u$. The corollary now follows from [13, Corollary 3.15]. □

Notation. From now on, we will denote $\mathfrak{r}(p)$ by $\mathfrak{r}_p$.

The next lemma shows C^K-compactness at scale $\mathfrak{r}$. The following lemma is a 4-dimensional analog of [13, Lemma 6.10].

Lemma 4.10.

(1) *There is a constant* $\widehat{w} = \widehat{w}(w') > 0$ *such that* $\mathrm{vol}(B(p, \mathfrak{r}_p)) \geq \widehat{w}(\mathfrak{r}_p)^4$ *for every* $p \in M$.

(2) *For every* $p \in M$*,* $C < \infty$ *and* $k \in [0, K]$*, we have*

$$|\nabla^k \mathrm{Rm}| \leq 2^{k+2} A'(C, w') \mathfrak{r}_p^{-(k+2)} \quad \textit{on the ball } B\left(p, \frac{1}{2}C\mathfrak{r}_p\right). \tag{4.11}$$

(3) *Given $\epsilon > 0$, for sufficiently large α and for every $p \in M^\alpha$, the rescaled pointed manifold $(\frac{1}{\mathfrak{r}_p} M^\alpha, p)$ is ϵ-close in the pointed C^K-topology to a complete C^K-smooth Riemannian 4-manifold which admits a metric of nonnegative sectional curvature. Moreover, this manifold belongs to a family which is compact in the pointed C^K-topology.*

Proof. (1) As $\frac{1}{2}\mathfrak{r}_p \le r_p(w')$, the Bishop-Gromov inequality gives

$$\frac{\operatorname{vol}(B(p, \frac{1}{2}\mathfrak{r}_p))}{\displaystyle\int_0^{\frac{\mathfrak{r}_p}{2r_p(w')}} \sinh^3(r)\,\mathrm{d}r} \ge \frac{\operatorname{vol}(B(p, r_p(w')))}{\displaystyle\int_0^1 \sinh^3(r)\,\mathrm{d}r} = \frac{w'(r_p(w'))^4}{\displaystyle\int_0^1 \sinh^3(r)\,\mathrm{d}r}. \tag{4.12}$$

Then,

$$\frac{\operatorname{vol}(B(p, \frac{1}{2}\mathfrak{r}_p))}{(\frac{1}{2}\mathfrak{r}_p)^4} \ge \frac{w'}{\displaystyle\int_0^1 \sinh^3(r)\,\mathrm{d}r} \cdot \frac{\displaystyle\int_0^{\frac{\mathfrak{r}_p}{2r_p(w')}} \sinh^3(r)\,\mathrm{d}r}{\left(\frac{\mathfrak{r}_p}{2r_p(w')}\right)^4}. \tag{4.13}$$

Because $\displaystyle\int_0^A \sinh^3(r)\,\mathrm{d}r = \frac{A^4}{4} + \text{h.o.t}$ where the higher order terms are positive, we have that $\displaystyle\frac{1}{A^4}\int_0^A \sinh^3(r)\,\mathrm{d}r \ge \frac{1}{4}$. Thus,

$$\frac{\operatorname{vol}(B(p, \frac{1}{2}\mathfrak{r}_p))}{(\frac{1}{2}\mathfrak{r}_p)^4} \ge \frac{w'}{4\displaystyle\int_0^1 \sinh^3(r)\,\mathrm{d}r}. \tag{4.14}$$

Therefore,

$$\operatorname{vol}(B(p, \mathfrak{r}_p)) \ge \operatorname{vol}(B(p, \mathfrak{r}_p/2)) \ge \frac{w'}{64\displaystyle\int_0^1 \sinh^3(r)\,\mathrm{d}r}(\mathfrak{r}_p)^4, \tag{4.15}$$

which gives (1).

The proof of (2) is the same as the proof of [13, Lemma 6.10 (2)] and the proof of (3) is similar to the proof of [13, Lemma 6.10 (3)]. □

Next, we extend Lemma 4.10 to provide C^K-splitting. The next lemma is a 4-dimensional analog of [13, Lemma 6.16].

Lemma 4.16. *Given $\epsilon > 0$ and $0 \le j \le 4$, provided $\delta < \bar{\delta}(\epsilon, w')$, the following holds. If $p \in M$, and $\phi : (\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R}^j \times X, (0, \star_X))$ is a (j, δ)-splitting, then ϕ is ϵ-close to a (j, ϵ)-splitting $\widehat{\phi} : (\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R}^j \times \widehat{X}, (0, \star_{\widehat{X}}))$, where $\widehat{X}$ is a complete C^K-smooth Riemannian $(4-j)$-manifold which admits a metric of nonnegative sectional curvature, and $\widehat{\phi}$ is ϵ-close to an isometry on the ball $B(p, \epsilon^{-1}) \subset \frac{1}{\mathfrak{r}_p} M$, in the C^{K+1}-topology.*

Proof. The proof is similar to the proof of [13, Lemma 6.16]. ☐

Let $\sigma > 0$ be a new parameter. In the next lemma, we show that if the parameter $\bar{w}$ is small then the pointed 4-manifold $(\frac{1}{\mathfrak{r}_p}M, p)$ is Gromov-Hausdorff close a space of lower dimension. The next lemma is a 4-dimensional analog of [13, Lemma 6.18].

Lemma 4.17. *Under the constraint $\bar{w} < \overline{\overline{w}}(\sigma, \Lambda)$, the following holds. For every $p \in M$, the pointed space $(\frac{1}{\mathfrak{r}_p}M, p)$ is σ-close in the pointed Gromov-Hausdorff metric to a nonnegatively curved Alexandrov space of dimension at most 3.*

Proof. Suppose that the lemma is not true. Then there exist $\sigma, \Lambda > 0$ so that there is a sequence $\bar{w}_i \to 0$ and for each i, a sequence $\{M^{\alpha(i,j)}, p^{\alpha(i,j)}\}_{j=1}^{\infty}$ so that for each j, $(\frac{1}{\mathfrak{r}_{p^{\alpha(i,j)}}}M^{\alpha(i,j)}, p^{\alpha(i,j)})$ has pointed Gromov-Hausdorff distance at least σ from any nonnegatively curved Alexandrov space of dimension at most 3.

For each fixed i, $\dfrac{R_{p^{\alpha(i,j)}}}{r_{p^{\alpha(i,j)}}(w')} \to \infty$ as $j \to \infty$. Hence, $\dfrac{R_{p^{\alpha(i,j)}}}{\mathfrak{r}_{p^{\alpha(i,j)}}} \to \infty$ as $j \to \infty$. Thus, we can find some $j = j(i)$ so that $R_{p^{\alpha(i,j(i))}} \geq i\,\mathfrak{r}_{p^{\alpha(i,j(i))}}$. We relabel $M^{\alpha(i,j(i))}$ as M^i and $p^{\alpha(i,j(i))}$ as p^i. Thus, we have a sequence $\{M^i, p^i\}_{i=1}^{\infty}$ so that for each i, $(\frac{1}{\mathfrak{r}_{p^i}}M^i, p^i)$ has pointed Gromov-Hausdorff distance at least σ from any nonnegatively curved Alexandrov space of dimension at most 3, and the curvature scale at p^i is at least $i\,\mathfrak{r}_{p^i}$. In particular, a subsequence of $\{(\frac{1}{\mathfrak{r}_{p^i}}M^i, p^i)\}_{i=1}^{\infty}$ converges in the pointed Gromov-Hausdorff topology to a nonnegatively curved Alexandrov space (X, x) of dimension 4. Hence, there is a uniform lower bound on $\frac{\operatorname{vol}(B(p^i, 2\mathfrak{r}_{p^i}))}{(2\mathfrak{r}_{p^i})^4}$.

As $r_{p^i}(\bar{w}_i) \leq 2\mathfrak{r}_{p^i}$, the Bishop-Gromov inequality implies that

$$\frac{\bar{w}_i(r_{p_i}(\bar{w}_i))^4}{\displaystyle\int_0^{\frac{r_{p^i}(\bar{w}_i)}{2\mathfrak{r}_{p^i}}} \sinh^3(r)\,\mathrm{d}r} = \frac{\operatorname{vol}(B(p^i, r_{p_i}(\bar{w}_i)))}{\displaystyle\int_0^{\frac{r_{p^i}(\bar{w}_i)}{2\mathfrak{r}_{p^i}}} \sinh^3(r)\,\mathrm{d}r} \geq \frac{\operatorname{vol}(B(p^i, 2\mathfrak{r}_{p_i}))}{\displaystyle\int_0^1 \sinh^3(r)\,\mathrm{d}r}. \tag{4.18}$$

That is

$$\begin{aligned} \frac{\operatorname{vol}(B(p^i, 2\mathfrak{r}_{p^i}))}{(2\mathfrak{r}_{p^i})^4} &\leq \bar{w}_i \left(\int_0^1 \sinh^3(r)\,\mathrm{d}r\right) \frac{\left(\frac{r_{p^i}(\bar{w}_i)}{2\mathfrak{r}_{p^i}}\right)^4}{\displaystyle\int_0^{\frac{r_{p^i}(\bar{w}_i)}{2\mathfrak{r}_{p^i}}} \sinh^3(r)\,\mathrm{d}r} \\ &\leq 4\bar{w}_i \int_0^1 \sinh^3(r)\,\mathrm{d}r. \end{aligned} \tag{4.19}$$

Since $\bar{w}_i \to 0$, we get a contradiction. ☐

From now on, we will assume that the constraint

$$\bar{w} < \overline{\overline{w}}(\sigma, \Lambda) \tag{4.20}$$

is satisfied. In particular, the conclusion of Lemma 4.17 always holds.

— 5 —

Stratifications

In this chapter, we define a stratification of Riemannian 4-manifolds, based on the maximal dimension of a Euclidean factor of an approximate splitting at a point. Refer to Chapter 2 and [13, Section 4] for the definition and facts about an approximate splitting of a pointed Riemannian manifold.

Let β_i, $i \in \{0,1,2,3,4\}$ be new parameters such that $0 = \beta_0 < \beta_1 < \beta_2 < \beta_3 < \beta_4$. Recall that the parameter σ has already been introduced in Chapter 4.

Definition 5.1. A point $p \in M$ belongs to the k-stratum, $k \in \{0,1,2,3,4\}$, if $(\frac{1}{\mathfrak{r}_p}M, p)$ admits a (k, β_k)-splitting, but does not admit a (j, β_j)-splitting for any $j > k$.

We note that every point has a $(0,0)$-splitting. Thus, every point $p \in M$ belongs to the k-stratum for some $k \in \{0,1,2,3,4\}$.

Lemma 5.2. *Under the constraints $\beta_4 < \overline{\beta}_4$ and $\sigma < \overline{\sigma}$, there are no 4-stratum points.*

Proof. The proof is similar to the proof of [13, Lemma 7.2]. Let $c > 0$ be the minimal distance, in the pointed Gromov-Hausdorff metric, between $(\mathbb{R}^4, 0)$ and a nonnegatively curved Alexandrov space of dimension at most 3. Taking $\bar{\beta}_4 = \bar{\sigma} = \frac{c}{4}$, the lemma follows from Lemma 4.17. $\square$

Let $\Delta \in (\beta_3^{-1}, \infty)$ be a new parameter.

Lemma 5.3. *Under the constraint $\Delta > \overline{\Delta}(\beta_3)$, if $p \in M$ has a 3-strainer of size $\frac{\Delta}{100}\mathfrak{r}_p$ and quality $\frac{1}{\Delta}$ at p, then $(\frac{1}{\mathfrak{r}_p}M, p)$ has a $(3, \frac{1}{2}\beta_3)$-splitting $\frac{1}{\mathfrak{r}_p}M \to \mathbb{R}^3$. In particular, p is in the 3-stratum.*

Proof. The lemma follows directly from Lemma 2.43. $\square$

Definition 5.4. A 2-strainer point $p \in M$ is in the *slim 2-stratum* if there is a $(2, \beta_2)$-splitting $(\frac{1}{\mathfrak{r}_p}M, p) \to (\mathbb{R}^2 \times X, ((0,0), \star_X))$ where $\operatorname{diam}(X) \leq 10^3\Delta$.

— 6 —

The local geometry of the 3-stratum

In the next fews chapters, we study the local geometry and topology near points in different strata. We also introduce adapted coordinates, cutoff functions, and ball coverings associated with different types of strata. In this chapter, we consider the 3-stratum.

6.1 Adapted coordinates, cutoff functions, and local topology near 3-stratum points

In this chapter, let p denote a point in the 3-stratum. Let $\phi_p : (\frac{1}{\mathfrak{r}_p}M, p) \to (\mathbb{R}^3 \times X, (0, \star_X))$ be a $(3, \beta_3)$-splitting.

Lemma 6.1. *Under the constraint* $\beta_3 < \overline{\beta}_3$ *and* $\sigma < \overline{\sigma}$*,* $\operatorname{diam}(X) < 1$*.*

Proof. The proof is similar to the proof of [13, Lemma 8.1]. Suppose that the lemma is false. Then, there is a subsequence $\{M^{\alpha_j}\}$ of the sequence $\{M^\alpha\}$ and $p_j \in M^{\alpha_j}$, such that with $\beta_3 = \sigma = \frac{1}{j}$, the map $\phi_{p_j} : (\frac{1}{\mathfrak{r}_{p_j}}M^{\alpha_j}, p_j) \to (\mathbb{R}^3 \times X_j, (0, \star_{X_j}))$ violates the conclusion of the lemma, i.e. $\operatorname{diam}(X_j) \geq 1$. There is a Gromov-Hausdorff sublimit (M_∞, p_∞) of $\{M^{\alpha_j}\}$, which is a nonnegatively curved Alexandrov space of dimension at most 3, and a limiting 3-splitting $\phi_\infty : (M_\infty, p_\infty) \to (\mathbb{R}^3 \times X_\infty, (0, \star_{X_\infty}))$. The only possibility is that $\dim(M_\infty) = 3$, ϕ_∞ is an isometry, and X_∞ is a point. This contradicts the diameter assumption. □

Let $\varsigma_{\text{3-stratum}} > 0$ be a new parameter.

Lemma 6.2. *Under the constraint* $\beta_3 < \overline{\beta}_3(\varsigma_{\text{3-stratum}})$*, there is a* ϕ_p*-adapted coordinate* η_p *of quality* $\varsigma_{\text{3-stratum}}$ *on* $B(p, 200) \subset (\frac{1}{\mathfrak{r}_p}M, p)$*.*

Proof. The lemma follows from the existence of adapted coordinates (see Lemma 2.49). □

Definition 6.3. Let ζ_p be the smooth function on M which is the extension by zero of $\Phi_{8,9} \circ |\eta_p|$.

Lemma 6.4. *Under the constrains $\beta_3 < \overline{\beta}_3, \varsigma_{\text{3-stratum}} < \overline{\varsigma}_{\text{3-stratum}}$, and $\sigma < \overline{\sigma}$, $\eta_p|_{\eta_p^{-1}(B(0,100))}$ is a fibration with fiber S^1. In particular, for all $R \in (0, 100)$, $|\eta_p^{-1}|[0, R]$ is diffeomorphic to $S^1 \times \overline{B(0, R)}$.*

Proof. The proof is similar to the proof of [13, Lemma 8.4]. ☐

6.2 Selection of 3-stratum balls

Let $\mathcal{M}$ be a new parameter, which will become a bound on intersection multiplicity of balls. The corresponding bound $\overline{\mathcal{M}}$ will describe how big $\mathcal{M}$ has to be taken in order for assertions to be valid.

Let $\{p_i\}_{i \in I_{\text{3-stratum}}}$ be a maximal set of 3-stratum points of M with the property that the collection $\{B(p_i, \frac{1}{3}\mathfrak{r}_{p_i})\}_{i \in I_{\text{3-stratum}}}$ is disjoint. We write ζ_i for ζ_{p_i}.

Lemma 6.5. *Under the constraints $\mathcal{M} > \overline{\mathcal{M}}$ and $\Lambda < \overline{\Lambda}$, the following holds.*

(1) *$\bigcup_{i \in I_{\text{3-stratum}}} B(p_i, \mathfrak{r}_{p_i})$ contains all 3-stratum points.*

(2) *The intersection multiplicity of the collection $\{\operatorname{supp}(\zeta_i)\}_{i \in I_{\text{3-stratum}}}$ is bounded by $\mathcal{M}$.*

Proof. The proof is similar to the proof of [13, Lemma 8.5].

(1). Since $\{p_i\}_{i \in I_{\text{3-stratum}}}$ is a maximal set of 3-stratum points of M so that $\{B(p_i, \frac{1}{3}\mathfrak{r}_{p_i})\}_{i \in I_{\text{3-stratum}}}$ is disjoint, if p is a 3-stratum point, then $B(p, \frac{1}{3}\mathfrak{r}_p) \cap B(p_i, \frac{1}{3}\mathfrak{r}_{p_i}) \neq \varnothing$, for some $i \in I_{\text{3-stratum}}$. In particular $d(p, p_i) \leq \frac{1}{3}\mathfrak{r}_p + \frac{1}{3}\mathfrak{r}_{p_i}$. As $p \mapsto \mathfrak{r}_p$ is Λ-Lipschitz, $|\mathfrak{r}_p - \mathfrak{r}_{p_i}| \leq \Lambda d(p, p_i) \leq \Lambda\left(\frac{1}{3}\mathfrak{r}_p + \frac{1}{3}\mathfrak{r}_{p_i}\right)$. That is $\frac{1-\frac{\Lambda}{3}}{1+\frac{\Lambda}{3}} \leq \frac{\mathfrak{r}_p}{\mathfrak{r}_{p_i}} \leq \frac{1+\frac{\Lambda}{3}}{1-\frac{\Lambda}{3}}$. If we assume that $1 + \frac{2}{3}\Lambda < 1.01$, then $\frac{\mathfrak{r}_p}{\mathfrak{r}_{p_i}} \in [0.9, 1.1]$. Thus $d(p, p_i) \leq \mathfrak{r}_{p_i}$, so $p \in B(p_i, \mathfrak{r}_{p_i})$. It follows that $\bigcup_{i \in I_{\text{3-stratum}}} B(p_i, \mathfrak{r}_{p_i})$ contains all 3-stratum points.

(2). From the definition of ζ_i, if $\varsigma_{\text{3-stratum}}$ is sufficiently small, then $\operatorname{supp}(\zeta_i) \subset B(p_i, 10\mathfrak{r}_{p_i})$.

Suppose that there exists $p \in M$ such that $p \in \bigcap_{j=1}^{N} B(p_{i_j}, 10\mathfrak{r}_{p_{i_j}})$ for distinct i_j's. We relabel $\{i_j\}_{j=1}^{N}$ so that $B(p_{i_1}, \mathfrak{r}_{p_{i_1}})$ has the smallest volume among all $B(p_{i_j}, 10\mathfrak{r}_{p_{i_j}})$'s.

If 10Λ is sufficiently small, then we can assume the for all j, $\frac{1}{2} \leq \frac{\mathfrak{r}_{p_{i_j}}}{\mathfrak{r}_{p_{i_1}}} \leq 2$. Hence, the N disjoint balls $\{B(p_{i_j}, \mathfrak{r}_{p_{i_j}})\}_{j=1}^{N}$ lie in $B(p_{i_1}, 100\mathfrak{r}_{p_{i_1}})$. By Bishop-Gromov volume comparison,

$$N \leq \frac{\operatorname{vol}(B(p_{i_1}, 100\mathfrak{r}_{p_{i_1}}))}{\operatorname{vol}(B(p_{i_1}, \frac{1}{3}\mathfrak{r}_{p_{i_1}}))} \leq \frac{\displaystyle\int_0^{100} \sinh^3(r)\ \mathrm{d}r}{\displaystyle\int_0^{\frac{1}{3}} \sinh^3(r)\ \mathrm{d}r}. \tag{6.6}$$

This proves the lemma. ☐

— 7 —

The local geometry of the 2-stratum

In this chapter, we study the local geometry and topology of 2-stratum points. We also introduce adapted coordinates, cutoff functions, and ball coverings associated with the 2-stratum.

7.1 Edge 2-stratum points and associated structures

In this section we study points $p \in M$ where the pair (M, p) looks like a half 3-dimensional Euclidean space with a base point lying on the boundary. Such points define a 2-edge set E.

Recall from Definition 5.4, $p \in M$ is a slim 2-stratum point is there is a $(2, \beta_2)$-splitting $(\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R}^2 \times X, (0, \star_X))$ where $\operatorname{diam}(X) \leq 10^3 \Delta$. In this section, we show that any 2-stratum point, which is not a slim 2-stratum point, is not far from E.

We also introduce an approximate 2-edge set E', which consists of points where the edge structure is of slightly lower quality than that of E. This is a technical tool to define a smooth distance function near points in E.

Lemma 7.1. *Given $\epsilon > 0$, if $\beta_2 < \overline{\beta}_2(\epsilon)$ and $\sigma < \overline{\sigma}(\epsilon)$ then the following holds. If $(\frac{1}{\mathfrak{r}_p} M, p)$ has a $(2, \beta_2)$-splitting then there is a $(2, \epsilon)$-splitting $(\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R}^2 \times Y, (0, \star_Y))$ where Y is an Alexandrov space with $\dim(Y) \leq 1$.*

Proof. The proof is similar to the proof of [13, Lemma 9.1]. □

Let $0 < \beta_E < \beta_{E'}$ and $0 < \sigma_E < \sigma_{E'}$ be new parameters.

Definition 7.2. A point $p \in M$ is an (s, t)-*edge 2-stratum point* if there is a $(2, s)$-splitting

$$F_p : \left(\frac{1}{\mathfrak{r}_p} M, p \right) \to (\mathbb{R}^2 \times Y, (0, \star_Y)) \tag{7.3}$$

and a t-pointed-Gromov-Hausdorff approximation

$$G_p : (Y, \star_Y) \to ([0, C], 0), \tag{7.4}$$

with $C \geq 200\Delta$. Given F_p and G_p, we put

$$Q_p = (\mathrm{Id} \times G_p) \circ F_p : \left(\frac{1}{\mathfrak{r}_p} M, p\right) \to (\mathbb{R}^2 \times [0, C], (0,0)). \tag{7.5}$$

We let E denote the set of (β_E, σ_E)-edge 2-stratum points, and E' denote the set of $(\beta_{E'}, \sigma_{E'})$-edge 2-stratum points. Note that $E \subset E'$. We will refer to elements of E as *edge 2-stratum points.*

We emphasize that in the definition above, Q_p maps the basepoint $p \in M$ to $(0,0) \in \mathbb{R}^2 \times [0, C]$.

Lemma 7.6. *Under the constraints $\beta_{E'} < \overline{\beta}_{E'}$, $\sigma_{E'} < \overline{\sigma}_{E'}$, and $\beta_3 < \overline{\beta}_3$, no element $p \in E'$ can be a 3-stratum point.*

Proof. The proof is similar to the proof of [13, Lemma 9.6]. □

Lemma 7.7. *Given $\epsilon > 0$, if $\beta_{E'} < \overline{\beta}_{E'}(\epsilon, \Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\epsilon, \Delta)$, $\beta_E < \overline{\beta}_E(\beta_{E'}, \sigma_{E'})$, $\sigma_E < \overline{\sigma}_E(\beta_{E'}, \sigma_{E'})$, and $\Lambda < \overline{\Lambda}(\epsilon, \Delta)$ then the following holds.*

For $p \in E$, if Q_p is as in Definition 7.2 and $\widehat{Q}_p : (\mathbb{R}^2 \times [0, C], (0,0)) \to (\frac{1}{\mathfrak{r}_p M}, p)$ is a quasi-inverse for Q_p, then $\widehat{Q}_p(B(0, 100\Delta) \times 0)$ is $\frac{\epsilon}{2}$-Hausdorff close to $E' \cap Q_p^{-1}(B(0, 100\Delta) \times [0, 100\Delta])$.

Proof. The proof is similar to the proof of [13, Lemma 9.7]. □

Part (1) of the next lemma states that 2-stratum points are either slim 2-stratum points or lie not too far from an edge 2-stratum point. Part (2) says that E is coarsely dense in E'.

Lemma 7.8. *Under the constraints $\beta_{E'} < \overline{\beta}_{E'}(\Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\Delta)$, $\beta_E < \overline{\beta}_E(\beta_{E'}, \sigma_{E'})$, $\sigma_E < \overline{\sigma}_E(\beta_{E'}, \sigma_{E'})$, $\beta_2 < \overline{\beta}_2(\Delta, \beta_E)$, $\sigma < \overline{\sigma}(\Delta, \sigma_E)$, and $\Lambda < \overline{\Lambda}(\Delta)$, the following holds.*

1. *For every 2-stratum point p which is not in the slim 2-stratum, there is some $q \in E$ with $p \in B(q, \Delta\mathfrak{r}_q)$.*
2. *For every 2-stratum point p which is not in the slim 2-stratum and for every $p' \in E' \cap B(p, 10\Delta\mathfrak{r}_p)$, there is some $q \in E$ with $p' \in B(q, \mathfrak{r}_q)$.*

Proof. The proof is similar to the proof of [13, Lemma 9.10]. □

7.2 Regularization of the distance function $d_{E'}$

Let $d_{E'}$ be the distance function from E'. In this subsection, we will apply the smoothing results from [13, Section 3.6]. The resulting smoothing of $d_{E'}$ will define part of a good coordinate in a collar region near E.

Let $\varsigma_{E'} > 0$ be a new parameter.

Lemma 7.9. *Under the constraints* $\beta_{E'} < \overline{\beta}_{E'}(\Delta, \varsigma_{E'})$ *and* $\sigma_{E'} < \overline{\sigma}_{E'}(\Delta, \varsigma_{E'})$, *there is a function* $\rho_{E'} : M \to [0, \infty)$ *such that if* $\eta_{E'} := \frac{\rho_{E'}}{\mathfrak{r}}$ *then:*

(1) *We have*

$$\left| \frac{\rho_{E'}}{\mathfrak{r}} - \frac{d_{E'}}{\mathfrak{r}} \right| \leq \varsigma_{E'}. \tag{7.10}$$

(2) *In the set* $\eta_{E'}^{-1}\left[\frac{\Delta}{10}, 10\Delta\right] \cap \left(\frac{d_E}{\mathfrak{r}}\right)^{-1}[0, 50\Delta]$, *the function* $\rho_{E'}$ *is smooth and its gradient lies in the* $\varsigma_{E'}$*-neighborhood of the generalized gradient of* $d_{E'}$.

(3) $\rho_{E'} - d_{E'}$ *is* $\varsigma_{E'}$*-Lipschitz.*

Proof. The proof is similar to the proof of [13, Lemma 9.12]. □

7.3 Adapted coordinates tangent to the edge

In this section, $p \in E$ will denote an edge 2-stratum point and Q_p will denote a map as in Definition 7.2.

Let $\varsigma_{\text{2-edge}} > 0$ be a new parameter. Applying Lemma 2.49, we get:

Lemma 7.11. *Under the constraint* $\beta_E < \overline{\beta}_E(\Delta, \varsigma_{\text{2-edge}})$, *there is a* Q_p*-adapted coordinate*

$$\eta_p : \left(\frac{1}{\mathfrak{r}_p} M, 0\right) \supset B(p, 100\Delta) \to \mathbb{R}^2 \tag{7.12}$$

of quality $\varsigma_{\text{2-edge}}$.

We define a global function $\zeta_p : M \to [0, 1]$ by extending

$$(\Phi_{-9\Delta,-8\Delta,8\Delta,9\Delta} \circ \eta_p) \cdot (\Phi_{8\Delta,9\Delta} \circ \eta_{E'}) : B(p, 100\Delta) \to [0, 1] \tag{7.13}$$

by zero.

Lemma 7.14. *The following holds:*

(1) ζ_p *is smooth.*

(2) *Under the constraints* $\beta_3 < \overline{\beta}_3(\varsigma_{\text{3-stratum}})$, $\Lambda < \overline{\Lambda}(\varsigma_{\text{3-stratum}}, \Delta)$, $\beta_{E'} < \overline{\beta}_{E'}(\varsigma_{\text{3-stratum}}, \Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\varsigma_{\text{3-stratum}}, \Delta)$, $\beta_E < \overline{\beta}_E(\beta_3, \beta_{E'}, \sigma_{E'}, \varsigma_{\text{3-stratum}})$, $\sigma_E < \overline{\sigma}_E(\beta_3, \beta_{E'}, \sigma_{E'}, \varsigma_{\text{3-stratum}})$, $\varsigma_{E'} < \overline{\varsigma}_{E'}(\varsigma_{\text{3-stratum}})$, *and* $\varsigma_{\text{2-edge}} < \overline{\varsigma}_{\text{2-edge}}(\varsigma_{\text{3-stratum}})$, *if* $x \in (\eta_p, \eta_{E'})^{-1}(\overline{B(0, 10\Delta)} \times \left[\frac{1}{10}\Delta, 10\Delta\right])$ *then* x *is a 3-stratum point, and there is a* $(3, \beta_3)$*-splitting* $\phi : (\frac{1}{\mathfrak{r}_x} M, x) \to (\mathbb{R}^3, 0)$ *such that* $(\eta_p, \eta_{E'}) : (\frac{1}{\mathfrak{r}_x} M, x) \to (\mathbb{R}^3, \phi(x))$ *defines* ϕ*-adapted coordinates of quality* $\varsigma_{\text{3-stratum}}$ *on the ball* $B(x, 100) \subset \frac{1}{\mathfrak{r}_x} M$.

Proof. The proof is similar to the proof of [13, Lemma 9.20]. □

7.4 The topology of the edge region

In this section, we study the local topology of a suitable neighborhood of an edge 2-stratum point $p \in E$.

Lemma 7.15. *Under the constraints* $\beta_{E'} < \overline{\beta}_{E'}(\Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\Delta)$, $\beta_E < \overline{\beta}_E(\beta_{E'}, \sigma_{E'}, w')$, $\sigma_E < \overline{\sigma}_E(\beta_{E'}, \sigma_{E'})$, $\varsigma_{\text{2-edge}} < \overline{\varsigma}_{\text{2-edge}}(\Delta)$, $\varsigma_{E'} < \overline{\varsigma}_{E'}(\Delta)$, $\Lambda < \overline{\Lambda}(\Delta)$, *and* $\sigma < \overline{\sigma}(\Delta)$, *the map* η_p *restricted to* $(\eta_p, \eta_{E'})^{-1}(B(0, 4\Delta) \times (-\infty, 4\Delta]))$ *is a fibration with fiber diffeomorphic to the closed 2-disk* D^2.

Proof. The proof is similar to the proof of [13, Lemma 9.21]. □

7.5 Selection of the edge balls

Let $\{p_i\}_{i \in I_{\text{2-edge}}}$ be a maximal set of edge 2-stratum points with the property that the collection $\{B(p_i, \frac{1}{3}\Delta\mathfrak{r}_{p_i})\}_{i \in I_{\text{2-edge}}}$ is disjoint. We write ζ_i for ζ_{p_i}.

Lemma 7.16. *Under the constraints* $\mathcal{M} > \overline{\mathcal{M}}$ *and* $\Lambda < \overline{\Lambda}(\Delta)$, *the following holds.*

(1) $\bigcup_{i \in I_{\text{2-edge}}} B(p_i, \Delta\mathfrak{r}_{p_i})$ *contains* E.

(2) *The intersection multiplicity of the collection* $\{\operatorname{supp}(\zeta_i)\}_{i \in I_{\text{2-edge}}}$ *is bounded by* $\mathcal{M}$.

Proof. The proof is similar to the proof of Lemma 6.5. □

The next lemma is a useful covering of the 2-stratum points.

Lemma 7.17. *Under the constraint* $\Lambda < \overline{\Lambda}(\Delta)$, *any 2-stratum point lies in the slim 2-stratum or lies in* $\bigcup_{i \in I_{\text{2-edge}}} B(p_i, 3\Delta\mathfrak{r}_{p_i})$.

Proof. The proof is similar to the proof of [13, Lemma 9.25]. □

The next lemma will be used later for the interface between the slim 2-stratum and the edge 2-stratum.

Lemma 7.18. *Under the constraints* $\beta_E < \overline{\beta}_E(\Delta, \beta_3)$, $\varsigma_{\text{2-edge}} < \overline{\varsigma}_{\text{2-edge}}(\Delta, \beta_3)$, *and* $\Lambda < \overline{\Lambda}(\Delta)$, *the following holds. Suppose for some* $i \in I_{\text{2-edge}}$ *and* $q \in B(p_i, 10\Delta\mathfrak{r}_{p_i})$ *we have*

$$\eta_{E'}(q) < 5\Delta, \quad |\eta_{p_i}(q)| < 5\Delta. \tag{7.19}$$

Then either p_i *belongs to the slim 2-stratum, or there is a* $j \in I_{\text{2-edge}}$ *such that* $q \in B(p_j, 10\Delta\mathfrak{r}_{p_j})$ *and* $|\eta_{p_j}(q)| < 2\Delta$.

Proof. The proof is similar to the proof of [13, Lemma 9.26]. □

7.6 Additional cutoff functions

We define two additional cutoff functions for later use:

$$\zeta_{\text{2-edge}} = 1 - \Phi_{\frac{1}{2},1} \circ \left(\sum_{i \in I_{\text{2-edge}}} \zeta_i \right) \tag{7.20}$$

and

$$\zeta_{E'} = (\Phi_{0.2\Delta,0.3\Delta,8\Delta,9\Delta} \circ \eta_{E'}) \cdot \zeta_{\text{2-edge}}. \tag{7.21}$$

7.7 Adapted coordinates, cutoff functions, and local topology near slim 2-stratum points

In this section we discuss the local geometry and topology of the slim 2-stratum points.

Let p denote a point in the slim 2-stratum and let $\phi_p : (\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R}^2 \times X, (0, \star_X))$ be a $(2, \beta_2)$-splitting with $\operatorname{diam}(X) \leq 10^3 \Delta$. Let $\varsigma_{\text{2-slim}} > 0$ be a new parameter.

Lemma 7.22. *Under the constraint $\beta_2 < \overline{\beta}_2(\Delta, \varsigma_{\text{2-slim}})$, the following holds.*

1. *There is a ϕ_p-adapted coordinate η_p of quality $\varsigma_{\text{2-slim}}$ on $B(p, 10^6\Delta) \subset (\frac{1}{\mathfrak{r}_p} M, p)$.*
2. *The cutoff function*
$$(\Phi_{-9\cdot 10^5\Delta,-8\cdot 10^5\Delta,8\cdot 10^5\Delta,9\cdot 10^5\Delta}) \circ \eta_p \tag{7.23}$$
extends by zero to a smooth function ζ_p on M.

Proof. This lemma follows from the existence of adapted coordinates (see Lemma 2.49). □

Let η_p and ζ_p be as in Lemma 7.22

Lemma 7.24. *Under the constraints $\beta_2 < \overline{\beta}_2(\varsigma_{\text{2-slim}}, \Delta, w')$ and $\varsigma_{\text{2-slim}} < \overline{\varsigma}_{\text{2-slim}}(\Delta)$, $\eta_p^{-1}\{0\}$ is diffeomorphic S^2 or T^2.*

Proof. The proof is similar to the proof of [13, Lemma 10.3]. □

7.8 Selection of slim 2-stratum balls

Let $\{p_i\}_{i \in I_{\text{2-slim}}}$ be a maximal set of slim 2-stratum points with the property that the collection $\{B(p_i, \frac{1}{3}\mathfrak{r}_{p_i})\}_{i \in I_{\text{2-slim}}}$ is disjoint. We write ζ_i for ζ_{p_i}.

Lemma 7.25. *Under the constraints $\mathcal{M} > \overline{\mathcal{M}}$ and $\Lambda < \overline{\Lambda}(\Delta)$, the following holds.*

1. *$\bigcup_{i \in I_{\text{2-slim}}} B(p_i, 10^5\Delta\mathfrak{r}_{p_i})$ contains all slim 2-stratum points.*
2. *The intersection multiplicity of the collection $\{\operatorname{supp}(\zeta_i)\}_{i \in I_{\text{2-slim}}}$ is bounded by $\mathcal{M}$.*

Proof. The proof is similar to the proof of Lemma 6.5. □

— 8 —

The local geometry of the 1-stratum

In this chapter, we study the local geometry and topology near points in the 1-stratum. We start with a general lemma about 1-stratum points.

Lemma 8.1. *Given* $\epsilon > 0$*, if* $\beta_1 < \overline{\beta}_1(\epsilon)$ *and* $\sigma < \overline{\sigma}(\epsilon)$ *then the following holds. If* $(\frac{1}{\mathfrak{r}_p}M, p)$ *has a* $(1, \beta_1)$*-splitting then there is a* $(1, \epsilon)$*-splitting* $(\frac{1}{\mathfrak{r}_p}M, p) \to (\mathbb{R} \times Y, (0, \star_Y))$*, where* Y *is an Alexandrov space of dimension at most 2.*

Proof. The proof is similar to the proof of [13, Lemma 9.1]. □

8.1 The good annulus lemma for 1-stratum points

Let $\sigma_R, \delta_1 > 0, \Upsilon_1 \in (\Delta, \infty)$, and $\Upsilon_1' > 1$ be new parameters. In the next lemma, we show that at an appropriate scale larger than $\mathfrak{r}$, a neighborhood of a 1-stratum point p is well approximated by a model geometry in two different ways: firstly by $\mathbb{R} \times N_p$, where N_p is a 3-manifold with nonnegative sectional curvature, in the pointed C^K-topology, and secondly by $\mathbb{R} \times C_T N_p$ where $C_T N_p$ is the Tits cone of N_p in the pointed Gromov-Hausdorff topology.

Lemma 8.2. *Let* p *be a 1-stratum point. Under the constraints* $\Upsilon_1' > \overline{\Upsilon}_1'(\delta_1, \sigma_R, \Upsilon_1, w')$*,* $\beta_1 < \overline{\beta}_1(\delta_1, w')$*, and* $\sigma < \overline{\sigma}$*, there is* $r_p^1 \in [\Upsilon_1 \mathfrak{r}_p, \Upsilon_1' \mathfrak{r}_p]$ *so that the following holds.*

1. $(\frac{1}{r_p^1}M, p)$ *is* δ_1*-pointed Gromov-Hausdorff close to* $\mathbb{R} \times C_T N_p$ *where* $C_T N_p$ *is the Tits cone of a complete* C^K*-smooth Riemannian 3-manifold* N_p *with nonnegative sectional curvature.*
2. $(\frac{1}{r_p^1}M, p)$ *is* σ_R*-pointed Gromov-Hausdorff close to* $\mathbb{R} \times Z_p$ *where* Z_p *is a complete* C^K*-smooth Riemannian 3-manifold with nonnegative sectional curvature and* $(Z_p, \star_{Z_p})$ *is* σ_R*-pointed Gromov-Hausdorff close to* $C_T N_p$*.*
3. *The ball* $B(p, r_p^1) \subset M$ *is diffeomorphic to* $\mathbb{R} \times N_p$*.*
4. *The ball* $B(\star_{Z_p}, 1) \subset Z_p$ *is diffeomorphic to* N_p*.*

(5) *The distance function from* p *has no critical points in* $A(p, \frac{r_p^1}{100}, r_p^1) \subset M$.

(6) *The distance function from* $\star_{Z_p}$ *has no critical points in* $A(\star_{Z_p}, \frac{1}{100}, 1) \subset Z_p$.

Proof. Suppose that the conclusion (1) is not true. Then for each j, if we take $\Upsilon_1' = j\Upsilon_1$, it is not true that conclusion (1) hold for sufficiently large α. Hence, we can find a sequence $\alpha_j \to \infty$ so that for each j, $(M^{\alpha_j}, p_{\alpha_j})$ provides a counterexample with $\Upsilon_1' = j\Upsilon_1$. We can assume further that there are sequences $\sigma_j \to 0$ and $\beta_1^j \to 0$ so that for each $r_j^1 \in [\Upsilon_1 \mathfrak{r}_{p_j}, j\Upsilon_1 \mathfrak{r}_{p_j}]$, there is a $(1, \beta_1^j)$-splitting $\phi_j : (\frac{1}{\mathfrak{r}_{p_{\alpha_j}}} M_{\alpha_j}, p_{\alpha_j}) \to (\mathbb{R} \times Y_j, (0, \star_{Y_j}))$ where Y_j is an Alexandrov space of dimension at most 2 but there is no 3-dimensional Riemannian manifold N_j with nonnegative sectional curvature such that conclusion (1) holds.

Additionally, we assume that β_1^j is sufficiently small (as a function of w') so that by Lemma 4.16, there is a $(1, j^{-1})$-C^K-splitting $\widehat{\phi}_j : (\frac{1}{\mathfrak{r}_{p_{\alpha_j}}} M_{\alpha_j}, p_{\alpha_j}) \to (\mathbb{R} \times Z_j, (0, \star_{Z_j}))$ where Z_j is a complete C^K-smooth Riemannian 3-manifold with nonnegative sectional curvature and $\widehat{\phi}_j$ is j^{-1}-close to an isometry on the ball $B(p_{\alpha_j}, j) \subset \frac{1}{\mathfrak{r}_{\alpha_j}} M$ in the C^{K+1}-topology.

For notational simplicity, we relabel $(\frac{1}{\mathfrak{r}_{p_{\alpha_j}}} M_{\alpha_j}, p_{\alpha_j})$ as (M_j, p_j) and write $\mathfrak{r}_j$ for $\mathfrak{r}_{p_{\alpha_j}}$.

By Lemma 4.10, there is a subsequence of $\{(\frac{1}{\mathfrak{r}_j} M_j, p_j)\}_{j=1}^{\infty}$ converging in the pointed C^K-topology to a complete C^K-smooth 4-dimensional Riemannian manifold (M_∞, p_∞), which admits a metric of nonnegative sectional curvature, with a 1-splitting $(\mathbb{R} \times N, (0, \star_N))$ for some complete C^K-smooth Riemannian 3-manifold N with nonnegative sectional curvature. By the compatibility of approximated splittings (Lemma 2.44), $(Z_j, \star_{Z_j}) \to (N, \star_N)$ in the pointed C^K-topology.

$\mathbb{R} \times N$ is asymptotically conical. Thus, there exists $\lambda' > 0$ such that for all $\lambda > \lambda'$, $\big(\frac{1}{\lambda}(\mathbb{R} \times N), (0, \star_N)\big)$ is $\frac{\delta_1}{2}$-pointed Gromov-Hausdorff close to its asymptotic cone $C_T(\mathbb{R} \times N) \cong \mathbb{R} \times C_T N$. Because N is also asymptotically conical, λ' can also be chosen so that $(\frac{1}{\lambda} N, \star_N)$ is $\frac{\sigma_R}{2}$-pointed Gromov Hausdorff close to $C_T N$.

By critical point theory, large open balls in $\mathbb{R} \times N$ are diffeomorphic to $\mathbb{R} \times N$ itself and large open balls in N are diffeomorphic to N. Hence, there exists $\lambda' > 10^3 \max(\Upsilon_1, \lambda)$ so that for any $\lambda'' \in (\frac{1}{2}\lambda', 2\lambda')$, there are no critical points of the distance function from $(0, \star_N)$ in $A((0, \star_N), \frac{1}{10^3}\lambda'', a\lambda'') \subset \mathbb{R} \times N$ and there are no critical points of the distance function from $\star_N$ in $A(\star_N, \frac{1}{10^3}\lambda'', a\lambda'') \subset N$. Consequently, $B((0, \star_N), \lambda'')$ is diffeomorphic to $\mathbb{R} \times N$ and $B(\star_N, \lambda'')$ is diffeomorphic to N.

As $(\frac{1}{\mathfrak{r}_j} M_j, p_j) \to (\mathbb{R} \times N, (0, \star_N))$ in the pointed C^K-topology, for large j, there are no critical points of the distance function in $A(p_j, \frac{\lambda'' \mathfrak{r}_j}{100}, \lambda'' \mathfrak{r}_j) \subset M_j$ and $B(p_j, \lambda'' \mathfrak{r}_j) \subset M_j$ is diffeomorphic to $B((0, \star_N), \lambda'') \subset \mathbb{R} \times N$. Similarly, as $\big(Z_j, \star_{Z_j}\big) \to (N, \star_N)$ in the pointed C^K-topology, for large j, there are no critical points of the distance function in $A(\star_{Z_j}, \frac{\lambda''}{100}, \lambda'') \subset Z_j$ and $B(p_j, \lambda'') \subset Z_j$ is diffeomorphic to $B(\star_N, \lambda'') \subset N$. Taking $r_j^1 = \lambda'' \mathfrak{r}_j$ gives a contradiction. □

8.2 Adapted coordinates, cutoff functions, and local topology near 1-stratum points

Definition 8.3. We call a 1-stratum point p such that $C_T N_p$ is a point, i.e., N_p is compact, a *slim 1-stratum* point. Otherwise, we call p a *ridge 1-stratum* point.

For each 1-stratum point p, apply Lemma 8.2 to get β_1, σ, and a scale $r_p^1 \in [\Upsilon_1 \mathfrak{r}_p, \Upsilon_1' \mathfrak{r}_p]$ for which the conclusion of the lemma holds. Let $\widetilde{\phi}_p : (\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R} \times X_p, (0, \star_{X_p}))$ be the $(1, \beta_1)$-splitting from the conclusion (1) of the lemma and let $\phi_p : (\frac{1}{r_p^1} M, p) \to (\mathbb{R} \times Z_p, (0, \star_{Z_p}))$ be the $(1, \sigma_R)$-splitting from the conclusion (2) of the lemma.

Let $\varsigma_R > 0$ be a new parameter. Let $d_{(0,\star_{Z_p})}$ be the distance function from $(0, \star_{Z_p})$ in $\mathbb{R} \times Z_p$ and let $\pi : \mathbb{R} \times Z_p \to \mathbb{R}$ be the projection to the $\mathbb{R}$-factor.

Lemma 8.4. *Under the constraint $\sigma_R < \overline{\sigma}_R(\varsigma_R)$, there exists a function $\psi_{(0,\star_{Z_p})} : \mathbb{R} \times Z_p \to [0, \infty)$ such that:*

1. *$\psi_{(0,\star_{Z_p})}$ is C^K-smooth on $A((0, \star_{Z_p}), 0.7, 10) \subset \mathbb{R} \times Z_p$.*
2. *$||\psi_{(0,\star_{Z_p})} - d_{(0,\star_{Z_p})}||_\infty < \varsigma_R$*
3. *$\psi_{(0,\star_{Z_p})} - d_{(0,\star_{Z_p})} : \mathbb{R} \times Z_p \to [0, \infty)$ is ς_R-Lipschitz.*
4. *For every $\tau \in [-0.5, 0.5]$ and $\rho \in [1, 5]$, $(\pi, \psi_{(0,\star_{Z_p})})^{-1}(\{\tau\} \times [0, \rho])$ is diffeomorphic to the normal bundle νS where $S \subset N_p$ is a soul, if N_p is noncompact, or to N_p itself if N_p is compact.*

Proof. If σ_R is sufficiently small, $\mathbb{R} \times Z_p$ will be pointed-close to the cone $\mathbb{R} \times C_T N_p$. Hence, we can apply the same arguments as in the proof of [13, Lemma 11.3] for the proof of parts (1), (2), and (3).

(4). For any $\tau \in [-0.5, 0.5]$, we have that $\pi^{-1}(\tau) = \{\tau\} \times Z_p$ and $d^2_{\star_{Z_p}} = d^2_{(0,\star_{Z_p})} - \tau^2$ is bounded away from 0 when $A(\star_{Z_p}, 0.7, 10) \subset \{\tau\} \times Z_p$. From parts (1) to (3), if σ_R is sufficiently small, $h := \sqrt{\psi^2_{(0,\star)} - \tau^2}|_{\{\tau\} \times Z_p}$ is a smooth approximation of $d_{\star_{Z_p}}$ on $A(\star_{Z_p}, 0.8, 7) \subset \{\tau\} \times Z_p$. We apply the same arguments as in the proof of [13, Lemma 11.3 (4)] but use [13, Remark 11.4] instead of the Schoenflies theorem. We have that $h^{-1}([0, s])$ for $s \in [0.6, 6]$ is diffeomorphic to the normal bundle νS where $S \subset N_p$ is a soul, if N_p is noncompact, or N_p itself if N_p is compact. □

Let d_p be the distance function from p in $(\frac{1}{r_p^1} M, p)$. The following lemma gives a smooth approximation of d_p.

Lemma 8.5. *Under the constraint* $\delta_1 < \overline{\delta}_1(\varsigma_R)$*, there is a function* $\psi_p : \frac{1}{r_p^1}M \to [0,\infty)$ *such that*

(1) ψ_p *is smooth on* $A(p, 0.1, 10) \subset \frac{1}{r_p^1}M$*.*

(2) $||\psi_p - d_p||_\infty < \varsigma_R$

(3) $\psi_p - d_p : \frac{1}{r_p^1}M \to [0,\infty)$ *is* ς_R*-Lipschitz.*

Proof. The proof is similar to the proof of [13, Lemma 11.3]. ☐

The next two lemmas show that under appropriate constraints, η_p and $\widetilde{\eta}_p$ are compatible after a scaling.

Let $\varsigma_{\text{1-ridge}} > 0$ be a new parameter.

Lemma 8.6. *Under the constraint* $\beta_1 < \overline{\beta}_1(\varsigma_{\text{1-ridge}}, \Upsilon'_1)$*, there is a* $\widetilde{\phi}_p$*-adapted coordinate* $\widetilde{\eta}_p$ *of quality* $\varsigma_{\text{1-ridge}}$ *on* $B(p, 10\Upsilon'_1) \subset (\frac{1}{\mathfrak{r}_p}M, p)$ *where* $\widetilde{\phi}_p$ *is a* $(1, \beta_1)$*-splitting of* $(\frac{1}{\mathfrak{r}_p}M, p)$*.*

Proof. The lemma follows from the existence of adapted coordinates (Lemma 2.49). ☐

Lemma 8.7. *Under the constraints* $\beta_1 < \overline{\beta}_1(\varsigma_R, \beta_2, \Upsilon_1, \Upsilon'_1)$*,* $\Upsilon_1 > \overline{\Upsilon}_1(\beta_2)$*, and* $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\varsigma_R)$*,* $\frac{\mathfrak{r}_p}{r_p^1}\widetilde{\eta}_p$ *is a* ϕ_p*-adapted coordinate of quality* ς_R *on* $B(p, 10) \subset (\frac{1}{r_p^1}M, p)$*. Put* $\eta_p = \frac{\mathfrak{r}_p}{r_p^1}\widetilde{\eta}_p$*.*

Proof. Let $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ be parameters internal to this proof.

Consider that $\widetilde{\phi}_p : (\frac{1}{\mathfrak{r}_p}M, p) \to (\mathbb{R} \times X, (0, \star_X))$ is a $(1, \beta_1)$-splitting. Thus, $\widetilde{\phi}_p$ has distortion comparable to β_1 on $B_{\frac{1}{\mathfrak{r}_p}M}(p, \beta_1^{-1})$. Then, $\frac{\mathfrak{r}_p}{r_p^1}\widetilde{\phi}_p$ has distortion comparable to $\frac{\mathfrak{r}_p}{r_p^1}\beta_1$ on $B_{\frac{1}{r_p^1}M}(p, \frac{\mathfrak{r}_p}{r_p^1}\beta_1^{-1})$. Since $r_p^1 \in [\Upsilon_1\mathfrak{r}_p, \Upsilon'_1\mathfrak{r}_p]$, if β_1 is sufficiently small (as a function of Υ_1, Υ'_1, and ϵ_1), then $\frac{\mathfrak{r}_p}{r_p^1}\widetilde{\phi}_p$ is a $(1, \epsilon_1)$-splitting for $(\frac{1}{r_p^1}M, p)$.

Assume that $(\frac{1}{r_p^1}M, p)$ has a $(2, \epsilon_2)$-splitting. By the equivalence of a good approximate splitting and a good strainer (see Lemma 2.43) and because $\frac{r_p^1}{\mathfrak{r}_p} \geq \Upsilon_1$, if $\epsilon_2 < \overline{\epsilon}_2(\beta_2)$ and $\Upsilon_1 > \overline{\Upsilon}_1(\beta_2)$, then there is a $(2, \beta_2)$-splitting of $(\frac{1}{\mathfrak{r}_p}M, p)$. This is a contradiction. Therefore, $(\frac{1}{r_p^1}M, p)$ does not have a $(2, \epsilon_2)$-splitting.

By the compatibility of approximate splittings (see Lemma 2.44), if ϵ_1, ϵ_2, and σ_R are sufficiently small (as functions ϵ_3), then $\frac{\mathfrak{r}_p}{r_p^1}\widetilde{\phi}_p$ is ϵ_3-compatible with ϕ_p. From the uniqueness of adapted coordinates (see Lemma 2.50), if $\varsigma_{\text{1-ridge}}$ and ϵ_3 are sufficiently small (as functions of ς_R), $\frac{\mathfrak{r}_p}{r_p^1}\widetilde{\eta}_p$ is a ϕ_p-adapted coordinate of quality ς_R on $B(p, 10) \subset B(p, 10\Upsilon'_1\frac{\mathfrak{r}_p}{r_p^1}) \subset (\frac{1}{r_p^1}M, p)$. ☐

The following lemma describes the local topology of a neighborhood of a 1-stratum point.

Lemma 8.8. *Under the constraints $\varsigma_R < \overline{\varsigma}_R$, $\delta_1 < \overline{\delta}_1(\varsigma_R)$, $\sigma_R < \overline{\sigma}_R(\varsigma_R)$, if p is a ridge 1-stratum point, i.e. $C_T N_p$ is not a point, then the map η_p restricted to $(\eta_p, \psi_p)^{-1}((-0.5, 0.5)\times[0, 5])$ is a fibration with fiber diffeomorphic to D^3, $S^1 \times D^2$, $S^2 \times_{\mathbb{Z}_2} I$, or $T^2 \times_{\mathbb{Z}_2} I$.*

If p is a slim 1-stratum point, i.e. $C_T N_p$ is a point, then the map η_p restricted to $\eta_p^{-1}(-0.5, 0.5)$ is a fibration with fiber diffeomorphic to a closed orientable connected Riemannian 3-manifold with nonnegative sectional curvature.

Proof. Let $\phi_p : (\frac{1}{r_p^1}M, p) \to (\mathbb{R}\times Z_p, (0, \star_{Z_p}))$ be the $(1, \sigma_R)$-splitting that satisfies the previous lemmas. Let $\pi : \mathbb{R}\times Z_p \to \mathbb{R}$ be the projection to the $\mathbb{R}$-factor. From the definition of adapted coordinates in Definition 2.47, if σ_R and ς_R are sufficiently small, then η_p is C^1-close to $\pi \circ \phi$. Since ϕ is C^{K+1}-close to an isometry, the generalized gradients of $d_{(0,\star)} \circ \phi$ will be close to the generalized gradient of d_p in $\psi_p^{-1}((0.6, 10))$ where the gradients are taken with respect to the metric on $\frac{1}{r_p^1}M$. Hence by Lemma 8.6 and Lemma 8.7, if ς_R and δ_1 are sufficiently small, $\psi_{(0,\star)} \circ \phi$ will be C^1-close to ψ_p in the region $\psi_p^{-1}((0.6, 10))$.

For $t \in [0, 1]$, define a map $f^t : (\eta_p, \psi_p)^{-1}((-0.6, 0.6) \times [0, 10)) \to \mathbb{R}^2$ by

$$f^t = (t\eta_p + (1 - t)(\pi \circ \phi), t\psi_p + (1 - t)(\psi_{(0,\star_{Z_p})} \circ \phi)). \tag{8.9}$$

Let $F : (\eta_p, \psi_p)^{-1}((-0.6, 0.6) \times [0, 10)) \times [0, 1] \to \mathbb{R}^2$ be the map with slices $\{f^t\}_{t\in[0,1]}$. By the C^1-closeness discussed above, we can apply [13, Lemma 21.1] to conclude that $(\eta_p, \psi_p)^{-1}(\{\tau\} \times [0, 5])$ is diffeomorphic to $(\pi, \psi_{(0,\star_{Z_p})})^{-1}(\{\tau\} \times [0, 5])$ for any $\tau \in [-0.5, 0.5]$.

Finally, we claim that the restriction of η_p to $(\eta_p, \psi_p)^{-1}((-0.5, 0.5) \times [0, 5])$ is a proper submersion to $(-0.5, 0.5)$, and is therefore a fibration. The properness follows from the fact that $(\eta_p, \psi_p)^{-1}((-0.5, 0.5) \times [0, 5])$ is contained in a compact subset of the domain of (η_p, ψ_p). $D\eta_p$ is nonvanishing by the definition of adapted coordinates. If σ_R, ς_R, and δ_1 are sufficiently small, $D\eta_p$ is almost parallel to the $\mathbb{R}$-factor in the approximate splitting. Also, the angle between $D\psi_p$ and the $\mathbb{R}$-factor in the approximate splitting is contained in the interval $[\frac{\pi}{2} - 2\tan^{-1}(\frac{1}{6}), \frac{\pi}{2} + 2\tan^{-1}(\frac{1}{6})]$. In particular, $\{D\eta_p, D\psi_p\}$ is linearly independent at points on the boundary with $(\eta_p, \psi_p) \in (-0.5, 0.5) \times \{3\}$. As a result, η_p is a submersion with fiber diffeomorphic to $(\pi, \psi_{(0,\star)})^{-1}(\{\tau\} \times [0, 5])$.

$(\pi, \psi_{(0,\star)})^{-1}(\{\tau\} \times [0, 5])$ is diffeomorphic to the normal bundle νS where $S \subset N_p$ is a soul. If N_p has 2 ends, then $C_T N_p = \mathbb{R}$. If $\delta_1 < \overline{\delta}_1(\epsilon)$, then there is a 2-strainer at p of quality ϵ at scale ϵ^{-1}. By Lemma 2.43 and because $\frac{r_p^1}{\mathfrak{r}_p} \geq \Upsilon_1$, if $\epsilon < \overline{\epsilon}_1(\beta_2)$ and $\Upsilon_1 > \overline{\Upsilon}_1(\beta_2)$, then there is a $(2, \beta_2)$-splitting for $(\frac{1}{\mathfrak{r}_p}M, p)$. This is a contradiction. Hence, $C_T N_p$ has at most one end.

By the classification of complete connected orientable 3-dimensional C^K-smooth Riemannian manifolds with nonnegative sectional curvature (see [13, Lemma 3.11]), if N_p has one end, then N_p is diffeomorphic to $\mathbb{R}^3, S^1 \times \mathbb{R}^2, S^2 \times_{\mathbb{Z}_2} \mathbb{R}$, or $T^2 \times_{\mathbb{Z}_2} \mathbb{R}$. If N_p has zero ends, i.e. N_p is compact, then N_p is diffeomorphic to a closed connected orientable Riemannian

3-manifold with nonnegative sectional curvature. Therefore, $(\pi, \psi_\star)^{-1}(\{\tau\} \times [0,5])$ is diffeomorphic to D^3, $S^1 \times D^2$, $S^2 \times_{\mathbb{Z}_2} I$, or $T^2 \times_{\mathbb{Z}_2} I$ if $C_T N_p$ is not a point. Otherwise, $\pi^{-1}(\tau)$ is diffeomorphic to a closed connected orientable Riemannian 3-manifold with nonnegative sectional curvature. □

Define a smooth cutoff function $\zeta_p : M \to [0,1]$ by extending

$$\Phi_{-0.9\Delta,-0.8\Delta,0.8\Delta,0.9\Delta} \circ \widetilde{\eta}_p : (B(p, 10\Upsilon_1) \subset \frac{1}{\mathfrak{r}_p} M) \to \mathbb{R} \tag{8.10}$$

by zero. We note that $\Delta < \Upsilon_1$. Therefore, the conclusions of Lemma 8.4 to Lemma 8.8 hold for all point p where $\zeta_p > 0$.

Lemma 8.11. *Under the constraints $\sigma_R < \overline{\sigma}_R(\varsigma_{2\text{-}slim}, \beta_2)$, $\Upsilon_1 > \overline{\Upsilon}_1(\varsigma_{2\text{-}slim}, \beta_2)$, and $\Upsilon_1' > \overline{\Upsilon}_1'(\varsigma_{2\text{-}slim}, \beta_2)$, if $q \in \phi_p^{-1}([-5,5] \times A(\star_{Z_p}, \frac{1}{3}, 5))$ then q belongs to the 2-stratum or 3-stratum and there is a smooth map η_q such that $\frac{r_p^1}{\mathfrak{r}_q}\eta_q$ is an adapted coordinate for $(\frac{1}{\mathfrak{r}_q}M, q)$ of quality $\varsigma_{2\text{-}slim}$.*

Proof. Let $q \in \phi_p^{-1}([-5,5] \times A(\star_{Z_p}, \frac{1}{3}, 5))$. Let $\epsilon_i > 0$ for $i = 1, \ldots, 6$ be parameters internal to this proof.

Recall that there is a $(1, \sigma_R)$-splitting $\phi_p : (\frac{1}{r_p^1}M, p) \to (\mathbb{R} \times Z_p, (0, \star_{Z_p}))$ of $(\frac{1}{r_p^1}M, p)$. From Lemma 2.42, if $\sigma_R < \overline{\sigma}_R(\epsilon_1)$, then there is a $(1, \epsilon_1)$-splitting $\widetilde{\phi}$ of $(\frac{1}{r_p^1}M, q)$ coming from a change of basepoint of ϕ_p. If $\epsilon_1 < \overline{\epsilon}_1(\epsilon_2)$, then there is a 1-strainer $\{a_1^+, a_1^-\}$ of quality ϵ_2 and at a scale $\frac{1}{\epsilon_2}$.

Let $(t, y) = \phi_p(q) \in [-5,5] \times A(\star_{Z_p}, \frac{1}{3}, 5)$. Let $a_2^+ = \phi_p^{-1}(t, \star_{Z_p}) \in \frac{1}{r_p^1}M$ and let $a_2^+ = \phi_p^{-1}(x)$ where x is the point on the ray from $(t, \star_{Z_p})$ passing through (t, y) with the distance equal to $2d_Z(\star_{Z_p}, y)$. If σ_R is sufficiently small, then $d_{\frac{1}{r_p^1}M}(q, a_2^+) \geq \frac{1}{6}$ and $d_{\frac{1}{r_p^1}M}(q, a_2^-) \geq \frac{1}{6}$.

Since $\widetilde{\phi}$ comes from a change of basepoint of ϕ_p, we have that $\phi_p(a_1^+)$, $\phi_p(q)$, and $\phi_p(a_1^-)$ approximately aligned along the $\mathbb{R}$-factor in $\mathbb{R} \times Z_p$. Therefore, the comparison angles $\widetilde{\angle} a_1^+ q a_2^+$, $\widetilde{\angle} a_1^- q a_2^+$, $\widetilde{\angle} a_1^+ q a_2^-$, and $\widetilde{\angle} a_1^- q a_2^-$ are arbitrary close to $\frac{\pi}{2}$ if σ_R is sufficiently small. In particular, if $\sigma_R < \overline{\sigma}_R(\epsilon_3)$ and $\epsilon_2 < \overline{\epsilon}_2(\epsilon_3)$, then $\{(a_i^+, a_i^-)\}_{i=1}^2$ is a 2-strainer for $(\frac{1}{r_p^1}M, q)$ of quality ϵ_3 at a scale at least $\frac{1}{6}$. If $\epsilon_3 < \overline{\epsilon}_3(\epsilon_4)$, then there is an approximate 2-splitting of $(\frac{1}{r_p^1}M, q)$ with an adapted coordinate $\widehat{\eta}$ of quality ϵ_4. Note that the i-th component of $\widehat{\eta}$ approximates $d_{\frac{1}{r_p^1}M}(a_i^+, q) - d_{\frac{1}{r_p^1}M}(a_i^+, \cdot)$.

If $\epsilon_3 < \overline{\epsilon}_3(\epsilon_5)$, then $\{(a_i^+, a_i^-)\}_{i=1}^2$ is a 2-strainer $(\frac{1}{\mathfrak{r}_q}M, q)$ of quality ϵ_5 with scale at least $\frac{r_p^1}{6\mathfrak{r}_q}$. If $\epsilon_5 < \overline{\epsilon}_5(\beta_2, \epsilon_6)$ and $\Upsilon_1 > \overline{\Upsilon}_1(\beta_2, \epsilon_5, \epsilon_6)$, then $(\frac{1}{\mathfrak{r}_q}M, q)$ has a $(2, \beta_2)$-splitting α with an adapted coordinate γ of quality ϵ_6. Consider that the i-th component of γ approximates

$d_{\frac{1}{\mathfrak{r}_q}M}(a_i^+, q) - d_{\frac{1}{\mathfrak{r}_q}M}(a_i^+, \cdot) = \frac{r_p^1}{\mathfrak{r}_q}\left(d_{\frac{1}{r_p^1}M}(a_i^+, q) - d_{\frac{1}{r_p^1}M}(a_i^+, \cdot)\right)$. Therefore, if $\Upsilon_1' > \overline{\Upsilon}_1'(\epsilon_6)$ then $\frac{r_p^1}{\mathfrak{r}_q}\widehat{\eta}$ also approximates $d_{\frac{1}{\mathfrak{r}_q}M}(a_i^+, q) - d_{\frac{1}{\mathfrak{r}_q}M}(a_i^+, \cdot)$. From Lemma 2.50 and the proof of Lemma 2.49 (see [13, Lemma 4.23]), if $\epsilon_4 < \overline{\epsilon}_4(\varsigma_{\text{2-slim}})$ and $\epsilon_6 < \overline{\epsilon}_6(\varsigma_{\text{2-slim}})$, then $\frac{r_p^1}{\mathfrak{r}_q}\widehat{\eta}$ is an adapted coordinate for $(\frac{1}{\mathfrak{r}_q}M, q)$ of quality $\varsigma_{\text{2-slim}}$. □

8.3 Selection of ridge 1-stratum balls

Let $\{p_i\}_{i \in I_{\text{1-ridge}}}$ be a maximal set of ridge 1-stratum points with the property that the collection $B(p_i, \frac{1}{30}\Delta\mathfrak{r}_{p_i})_{i \in I_{\text{1-ridge}}}$ is disjoint. Write ζ_i for ζ_{p_i}.

Lemma 8.12. *Under the constraints $\mathcal{M} > \overline{\mathcal{M}}$ and $\Lambda < \overline{\Lambda}(\Delta)$, the following holds.*

(1) $\bigcup_{i \in I_{\textit{1-ridge}}} B(p_i, \frac{1}{10}\Delta\mathfrak{r}_{p_i})$ *contains all ridge 1-stratum points.*

(2) *The intersection multiplicity of the collection* $\{\operatorname{supp}(\zeta_i)\}_{i \in I_{\textit{1-ridge}}}$ *is bounded by* $\mathcal{M}$.

Proof. (1). The proof is similar to the proof of Lemma 6.5(1).

(2). From the definition of ζ_{p_i} in (8.10), if ς_R is sufficiently small then we are ensured that $\operatorname{supp}(\zeta_{p_i}) \subset B(p_i, \Delta\mathfrak{r}_{p_i})$. Suppose that for some $q \in M$, we have $q \in \bigcap_{j=1}^N B(p_{i_j}, \Delta\mathfrak{r}_{p_{i_j}})$ for distinct i_j's. We relabel so that $B(p_{i_1}, \frac{1}{30}\Delta\mathfrak{r}_{p_{i_1}})$ has the smallest volume among the $B(p_{i_j}, \mathfrak{r}_{p_{i_j}})$'s.

Note that $\{B(p_i, \frac{1}{30}\Delta\mathfrak{r}_{p_i})\}_{i \in I_{\text{1-ridge}}}$ is a disjoint collection. If $\Lambda < \overline{\Lambda}(\Delta)$, then we can assume that for all j, $\frac{1}{2} \leq \frac{\mathfrak{r}_{p_{i_j}}}{\mathfrak{r}_{p_{i_1}}} \leq 2$. Hence, the N disjoint balls $\{B(p_{i_j}, \frac{1}{30}\Delta\mathfrak{r}_{p_{i_j}})\}_{j=1}^N$ lies in $B(p_{i_1}, 100\Delta\mathfrak{r}_{p_{i_1}})$ and by Bishop-Gromov volume comparison

$$N \leq \frac{\operatorname{vol}(B(p_{i_1}, 100\Delta\mathfrak{r}_{p_{i_1}}))}{\operatorname{vol}(B(p_{i_1}, \frac{1}{30}\Delta\mathfrak{r}_{p_{i_1}}))} \leq \frac{\displaystyle\int_0^{100\Delta} \sinh^3(r)\,\mathrm{d}r}{\displaystyle\int_0^{\frac{1}{30}\Delta} \sinh^3(r)\,\mathrm{d}r}. \tag{8.13}$$

The upper bound of the right-hand side does not depend on Δ. This proves part (2) of the lemma. □

Denote $\lambda_{p_i} = \frac{r_{p_i}^1}{\mathfrak{r}_{p_i}}$. Define an additional cutoff function $\zeta_{\psi_i} : M \to [0,1]$ for each $i \in I_{\text{1-ridge}}$ by extending

$$\zeta_{\psi_{p_i}} = (\Phi_{1.1\lambda_{p_i}, 1.2\lambda_{p_i}, 4.8\lambda_{p_i}, 4.9\lambda_{p_i}} \circ \lambda_{p_i}\psi_{p_i}) \cdot \left(\Phi_{0.9\lambda_{p_i}, 1\lambda_{p_i}} \circ \widetilde{\eta}_{p_i}\right) : \frac{1}{\mathfrak{r}_{p_i}}M \to \mathbb{R} \tag{8.14}$$

by zero.

8.4 Selection of slim 1-stratum balls

Let $\{p_i\}_{i \in I_{1\text{-slim}}}$ be a maximal set of slim 1-stratum points with the property that the collection $B(p_i, \frac{1}{30}\Delta\mathfrak{r}_{p_i})_{i \in I_{1\text{-slim}}}$ is disjoint. Write ζ_i for ζ_{p_i}.

Lemma 8.15. *Under the constraints $\mathcal{M} > \overline{\mathcal{M}}$ and $\Lambda < \overline{\Lambda}(\Delta)$, the following holds.*

(1) $\bigcup_{i \in I_{1\text{-}slim}} B(p_i, \frac{1}{10}\Delta\mathfrak{r}_{p_i})$ *contains all slim 1-stratum points.*

(2) *The intersection multiplicity of the collection $\{\mathrm{supp}(\zeta_i)\}_{i \in I_{1\text{-}slim}}$ is bounded by $\mathcal{M}$.*

Proof. The proof is similar to the proof of Lemma 8.12. ☐

— 9 —

The local geometry of the 0-stratum

Points in the 0-stratum are defined by a process of elimination, i.e. they are points that are not k-stratum points for $k \in \{1, 2, 3\}$, rather than by geometric structure. In this chapter, we discuss the local geometry and topology of the 0-stratum. We show in Lemma 9.1 that M has conical structure near every point. We then use this to define radial and cutoff functions near 0-stratum points. This chapter is an analog of [13, Section 11].

Let $\delta_0 > 0$ and $\Upsilon_0, \Upsilon_0', \tau_0 > 1$ be new parameters.

9.1 The good annulus lemma

The next lemma states that for every point p in M, there is a scale at which a neighborhood of p is well approximated by a model geometry in two different ways: firstly by a nonnegatively curved 3-manifold in the pointed C^K-topology, and secondly by the Tits cone of this manifold in the pointed Gromov-Hausdorff topology.

Lemma 9.1. *Under the constraint* $\Upsilon_0' > \overline{\Upsilon}_0'(\delta_0, \Upsilon_0, \omega')$, *if* $p \in M$ *then there exists* $r_p^0 \in [\Upsilon_0 \mathfrak{r}_p, \Upsilon_0' \mathfrak{r}_p]$ *and a complete* C^K*-smooth Riemannian 4-manifold which admits a metric of nonnegative sectional curvature* N_p *such that:*

(1) $(\frac{1}{r_p^0} M, p)$ *is* δ_0*-close in the Gromov-Hausdorff topology to the Tits cone* $C_T N_p$ *of* N_p.

(2) *The ball* $B(p, r_p^0) \subset M$ *is diffeomorphic to* N_p.

(3) *The distance function from p has no critical points in the annulus* $A(p, \frac{1}{100} r_p^0, r_p^0)$.

Proof. The proof is the similar to the proof of [13, Lemma 11.1]. □

Remark 9.2. If we that the parameter σ of [13, Lemma 6.18] to be small then we can additionally conclude that $C_T N_p$ is pointed Gromov-Hausdorff close to a conical nonnegatively curved Alexandrov space of dimension at most three.

9.2 The radial function near 0-stratum point

For every $p \in M$, we apply Lemma 9.1 to get a scale $r^0_p \in [\Upsilon_0 \mathfrak{r}_p, \Upsilon'_0 \mathfrak{r}_p]$ for which the conclusion of Lemma 9.1 holds. In particular, $(\frac{1}{r^0_p} M, p)$ is δ_0-close in the pointed Gromov-Hausdorff topology to the Tits cone $C_T N_p$ of N_p, where N_p is a complete Riemannian 4-manifold which admits a metric of nonnegative sectional curvature.

Let d_p be the distance function from p in $(\frac{1}{r^0_p} M, p)$. Let $\varsigma_{\text{0-stratum}} > 0$ be a new parameter.

Lemma 9.3. *Under the constraint* $\delta_0 < \overline{\delta}_0(\varsigma_{\text{0-stratum}})$*, there is a function* $\eta_p : \frac{1}{r^0_p} M \to [0, \infty)$ *such that:*

(1) η_p *is smooth on* $A(p, 0.1, 10) \subset \frac{1}{r^0_p} M$.

(2) $\|\eta_p - d_p\|_\infty < \varsigma_{\text{0-stratum}}$.

(3) $\eta_p - d_p : \frac{1}{r^0_p} M \to [0, \infty)$ *is* $\varsigma_{\text{0-stratum}}$*-Lipschitz.*

(4) η_p *is smooth and has no critical point in* $\eta_p^{-1}([0.2, 2])$*, and for every* $p \in [0.2, 2]$*, the sublevel set* $\eta_p^{-1}([0, \rho])$ *is diffeomorphic to either the closed disk bundle in the normal bundle* νS *of the soul* $S \subset N_p$*, if* N_p *is noncompact, or to* N_p *itself if* N_p *is compact.*

(5) *The composition* $\Phi_{0.2,0.3,0.8,0.9} \circ \eta_p$ *extends by zero to a smooth cutoff function* $\zeta_p : M \to [0, 1]$.

Proof. The proof is similar to the proof of [13, Lemma 11.3] except that for (4), [13, Remark 11.4] is used instead of the Schoenflies theorem. ☐

9.3 Selecting the 0-stratum balls

The next lemma has a statement about an adapted coordinate for the radial splitting in an annular region of a 0-stratum ball. We use the parameter $\varsigma_{\text{1-ridge}}$ for the quality of this splitting. We note that there is no a priori relationship to ridge 1-stratum points. The use of this parameter will simplify the later parameter ordering.

Lemma 9.4. *Under the constraints* $\delta_0 < \overline{\delta}_0(\beta_1, \varsigma_{\text{1-ridge}})$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_1)$, $\beta_1 < \overline{\beta}_1(\varsigma_{\text{1-ridge}})$*, and* $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\varsigma_{\text{1-ridge}})$*, there is a finite collection* $\{p_i\}_{i \in I_{\text{0-stratum}}}$ *of points in* M *so that the following holds.*

(1) *The ball* $\{B(p_i, r^0_{p_i})\}_{i \in I_{\text{0-stratum}}}$ *are disjoint.*

(2) *If* $q \in B(p_i, 10 r^0_{p_i})$*, for some* $i \in I_{\text{0-stratum}}$*, then* $r^0_q \le 20 r^0_{p_i}$ *and* $\frac{r^0_{p_i}}{\mathfrak{r}_q} \ge \frac{1}{20} \Upsilon_0$.

(3) *For each* $i \in I_{\text{0-stratum}}$*, every* $q \in A(p_i, \frac{1}{10} r^0_{p_i}, r^0_{p_i})$ *belongs to the 1-stratum, 2-stratum, or 3-stratum, and there is a* $(1, \beta_1)$*-splitting of* $(\frac{1}{\mathfrak{r}_q} M, q)$*, for which* $\frac{r^0_{p_i}}{\mathfrak{r}_q} \eta_{p_i}$ *is an adapted coordinate of quality* $\varsigma_{\text{1-ridge}}$.

(4) $\bigcup_{i\in I_{0\text{-stratum}}} B(p_i, \frac{1}{10} r^0_{p_i})$ *contains all 0-stratum points.*

(5) *For each* $i \in I_{0\text{-stratum}}$*, the manifold* N_{p_i} *has at most one end.*

Proof. The proof is similar to the proof of [13, Lemma 11.5]. □

— 10 —

Mapping into Euclidean Space

In this chapter, we first construct a smooth map $\mathcal{E}^0 : M \to H$ from M into a high-dimensional Euclidean space H by using the ball collections and the geometrically defined functions from Chapter 6 to Chapter 9. We then study the behavior of $\mathcal{E}^0$ near the different strata.

10.1 Definition of the map $\mathcal{E}^0 : M \to H$

Let I_P be a copy of the index set $I_{\text{1-ridge}}$. For each $i \in I_{\text{1-ridge}}$, we denote the corresponding copy of i in I_P by i_P. Let index sets $I_{\mathfrak{r}}$ and $I_{E'}$ be singletons $I_{\mathfrak{r}} = \{\mathfrak{r}\}$ and $I_{E'} = \{E'\}$ respectively.

Let $H = \bigoplus_{i \in I} H_i$ where

- $I = I_{\mathfrak{r}} \cup I_{E'} \cup I_P \cup I_{\text{0-stratum}} \cup I_{\text{1-slim}} \cup I_{\text{1-ridge}} \cup I_{\text{2-slim}} \cup I_{\text{2-edge}} \cup I_{\text{3-stratum}}$,
- H_i is a copy of $\mathbb{R}$ when $i = \mathfrak{r}$,
- H_i is a copy of $\mathbb{R} \oplus \mathbb{R}$ when $i \in I_{\text{0-stratum}} \cup I_{\text{1-slim}} \cup I_{\text{1-ridge}} \cup I_{E'} \cup I_P$,
- H_i is a copy of $\mathbb{R}^2 \oplus \mathbb{R}$ when $i \in I_{\text{2-slim}} \cup I_{\text{2-edge}}$, and
- H_i is a copy of $\mathbb{R}^3 \oplus \mathbb{R}$ when $i \in I_{\text{3-stratum}}$.

We also define

- $H_{\text{0-stratum}} = \bigoplus_{i \in I_{\text{0-stratum}}} H_i$,
- $H_{\text{1-slim}} = \bigoplus_{i \in I_{\text{1-slim}}} H_i$,
- $H_{\text{1-ridge}} = \bigoplus_{i \in I_{\text{1-ridge}}} H_i$,
- $H_{\text{2-slim}} = \bigoplus_{i \in I_{\text{2-slim}}} H_i$,
- $H_{\text{2-edge}} = \bigoplus_{i \in I_{\text{2-edge}}} H_i$,

- $H_{\text{3-stratum}} = \bigoplus_{i \in I_{\text{3-stratum}}} H_i$,
- $Q_1 = H$
- $Q_2 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}} \oplus H_{\text{2-slim}} \oplus H_{\text{2-edge}}$
- $Q_3 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}} \oplus H_{\text{2-slim}}$
- $Q_4 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}}$
- $Q_5 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}}$
- $Q_6 = H_{\text{0-stratum}}$, and
- $\pi_{i,j} : Q_i \to Q_j$, $\pi_i = \pi_{1,i} : H \to Q_i$, $\pi_i^{\perp} : H \to Q_i^{\perp}$ are the orthogonal projections, for $1 \leq i \leq j \leq 6$.

If $x \in Q_j$, we denote the projection to a summand H_i by $\pi_{H_i}(x) = x_i$. When $i \neq \mathfrak{r}$, we write $H_i = H_i' \oplus H_i'' \simeq \mathbb{R}^{k_i} \oplus \mathbb{R}$ where $k_i \in \{1, 2, 3\}$, and we denote the decomposition of $x_i \in H_i$ into its components by $x_i = (x_i', x_i'') \in H_i' \oplus H_i''$. We denote the orthogonal projection onto H_i' and H_i'' by $\pi_{H_i'}$ and $\pi_{H_i''}$, respectively.

Recall that in Chapter 6 to Chapter 9, we define adapted coordinates η_p and cutoff functions ζ_p corresponding to points $p \in M$ of different strata types. If $\{p_i\}$ is a collection of points used to define a ball cover, as in Chapters 6 to Chapter 9, then we write η_i for η_{p_i} and ζ_i for ζ_{p_i}. For $i \in I \backslash \{\mathfrak{r}\}$, we also define a new scale parameter R_i as follows:

- If $i \in I_{\text{0-stratum}}$, then we put $R_i = r^0_{p_i}$, where $r^0_{p_i}$ is as in Lemma 9.1.
- If $i \in I_{\text{1-slim}} \cup I_{\text{1-ridge}}$, then we put $R_i = \mathfrak{r}_{p_i}$. $\eta_i := \widetilde{\eta}_{p_i}$. Recall that $\eta_{p_i} = \frac{\mathfrak{r}_{p_i}}{r^1_{p_i}} \widetilde{\eta}_{p_i}$.
- If $i \in I_{\text{2-slim}} \cup I_{\text{2-edge}} \cup I_{\text{3-stratum}}$, then we put $R_i = \mathfrak{r}_{p_i}$.
- If $i \in I_P$, then we put $R_i = \mathfrak{r}_{p_i}$, $\eta_i = \lambda_{p_i} \psi_{p_i}$, and $\zeta_i = \zeta_{\psi_{p_i}}$.
- If $i = E'$, then $R_i = \mathfrak{r}$. Note that unlike the other cases, R_i is not a constant.

The component $\mathcal{E}^0_i : M \to H_i$ of the map $\mathcal{E}^0 : M \to H$ is defined to be $\mathfrak{r}$ when $i = \mathfrak{r}$, and

$$(R_i \eta_i \zeta_i, R_i \zeta_i) \tag{10.1}$$

otherwise.

In the rest of the chapter, we study the behavior of $\mathcal{E}^0$ near the different strata. In Chapter 11, we will use these information to adjust $\mathcal{E}^0$ slightly to get a new map $\mathcal{E}$ which is a submersion in different parts of M.

10.2 The image of $\mathcal{E}^0$

Let $x = \mathcal{E}^0(p) \in H$. The components of x satisfy the following inequalities:

$$x_{\mathfrak{r}} > 0 \tag{10.2}$$

and for every $i \in I \backslash \{\mathfrak{r}\}$,

$$x_i'' \in [0, R_i] \text{ and } |x_i'| \leq c_i x_i'', \tag{10.3}$$

where

$$c_i = \begin{cases} 0.9\Delta & \text{when } i \in I_{\text{1-slim}}, \\ 0.9\Delta & \text{when } i \in I_{\text{1-ridge}}, \\ 4.9\lambda_{p_i} & \text{when } i \in I_P, \\ 9\Delta & \text{when } i \in I_{E'}, \\ 9 \cdot 10^5\Delta & \text{when } i \in I_{\text{2-slim}}, \\ 9\Delta & \text{when } i \in I_{\text{2-edge}}, \\ 9 & \text{when } i \in I_{\text{3-stratum}}. \end{cases} \tag{10.4}$$

Lemma 10.5. *Under the constraint $\Lambda \leq \overline{\Lambda}(\mathcal{M})$, there is a number $\Omega_0 = \Omega_0(\mathcal{M})$ so that for all $p \in M$, $|D\mathcal{E}^0_p| \leq \Omega_0$.*

Proof. The lemma follows from the definition of $\mathcal{E}^0$. □

10.3 Structure of $\mathcal{E}^0$ near the 3-stratum

Define

$$\widetilde{A}_1 = \bigcup_{i \in I_{\text{3-stratum}}} \{|\eta_i| \leq 8\}, \quad A_1 = \bigcup_{i \in I_{\text{3-stratum}}} \{|\eta_i| \leq 7\} \tag{10.6}$$

and

$$\widetilde{S}_1 = \mathcal{E}^0(\widetilde{A}_1), \qquad S_1 = \mathcal{E}^0(A_1). \tag{10.7}$$

In this section, we show that on a scale which is sufficiently small compared to $\mathfrak{r}$, the pair $(\widetilde{S}_1, S_1) \subset H$ is a *cloudy 2-manifold.* This is roughly because, on a scale which is small compared to $\mathfrak{r}$, near any point in A_1, the map $\mathcal{E}^0$ is well approximated in the C^1-topology by an affine function of η_i, for some $i \in I_{\text{3-stratum}}$. We refer to Section 2.8 and [13, Appendix B] for the definition and properties of cloudy manifolds.

Let $\Sigma_1, \Gamma_1 > 0$ be new parameters. Define $r_1 : \widetilde{S}_1 \to (0, \infty)$ by putting $r_1(x) = \Sigma_1 \mathfrak{r}_p$ for some $p \in (\mathcal{E}^0)^{-1}(x) \cap \widetilde{A}_1$.

Lemma 10.8. *There is a constant* $\Omega_1 = \Omega_1(\mathcal{M})$ *so that under the constraints* $\Sigma_1 < \overline{\Sigma}_1(\Gamma_1, \mathcal{M})$, $\beta_3 < \overline{\beta}_3(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M})$, $\varsigma_{\text{3-stratum}} < \overline{\varsigma}_{\text{3-stratum}}(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M})$, $\beta_2 < \overline{\beta}_2(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, $\varsigma_{\text{2-slim}} < \overline{\varsigma}_{\text{2-slim}}(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, $\varsigma_{\text{2-edge}} < \overline{\varsigma}_{\text{2-edge}}(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, $\beta_{E'} < \overline{\beta}_{E'}(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, $\varsigma_{E'} < \overline{\varsigma}_{E'}(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, $\beta_1 < \overline{\beta}_1(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M})$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, *and* $\Lambda < \overline{\Lambda}(\beta_4, \Gamma_1, \Sigma_1, \mathcal{M}, \Delta)$, *the following holds.*

(1) *The triple* $(\widetilde{S}_1, S_1, r_1)$ *is a* $(2, \Gamma_1)$*-cloudy 3-manifold.*

(2) *The affine subspaces* $\{A_x\}_{x \in S_1}$ *inherent in the definition of the cloudy 3-manifold can be chosen to have the following property. Pick* $p \in A_1$ *and put* $x = \mathcal{E}^0(p) \in S_1$*. Let* $A^0_x \subset H$ *be the linear subspace parallel to* A_x *(i.e.,* $A_x = A^0_x + x$*) and let* $\pi_{A^0_x} : H \to A^0_x$ *denote orthogonal projection onto* A^0_x*. Then*

$$\| D\mathcal{E}^0_p - \pi_{A^0_x} \circ D\mathcal{E}^0_p \| \leq \Gamma_1 \tag{10.9}$$

and

$$\Omega_1^{-1} \|v\| \leq \|(\pi_{A^0_x} \circ D\mathcal{E}^0_p)(v)\| \leq \Omega_1 \|v\| \tag{10.10}$$

for every $v \in T_pM$ *which is orthogonal to* $\ker(\pi_{A^0_x} \circ D\mathcal{E}^0_p)$*.*

(3) *Given* $i \in I_{\text{3-stratum}}$*, there is a smooth map* $\widehat{\mathcal{E}}^0_i : (\overline{B(0,8)} \subset \mathbb{R}^3) \to (H'_i)^\perp$ *such that*

$$\| D\widehat{\mathcal{E}}^0_i \| \leq \Omega_1 R_i \tag{10.11}$$

and on the subset $\{|\eta_i| \leq 8\} \subset \frac{1}{R_i} M$*, we have*

$$\left\| \frac{1}{R_i} \mathcal{E}^0 - \left(\eta_i, \frac{1}{R_i} \widehat{\mathcal{E}}^0_i \circ \eta_i \right) \right\|_{C^1} < \Gamma_1. \tag{10.12}$$

Furthermore, if $x \in S_1$*, then there are some* $i \in I_{\text{3-stratum}}$ *and* $p \in \{|\eta_i| \leq 7\}$ *such that* $x = \mathcal{E}^0(p)$ *and* $A^0_x = \operatorname{Im}(\operatorname{Id}, \frac{1}{R_i} (D\widehat{\mathcal{E}}^0_i)_{\eta_i(p)})$*.*

The parameters $\epsilon_1, \epsilon_2 > 0$ will be internal to this section, which is devoted to the proof of Lemma 10.8. Until further notice, the index i will denote a fixed element of $I_{\text{3-stratum}}$.

We put $J = \{ j \in I / \{\mathfrak{r}\} : \operatorname{supp}(\zeta_j) \cap B(p_i, 10R_i) \neq \varnothing \}$.

Sublemma 10.13. *Under the constraints* $\beta_3 < \overline{\beta}_3(\beta_4, \epsilon_1)$, $\varsigma_{\text{3-stratum}} < \overline{\varsigma}_{\text{3-stratum}}(\beta_4, \epsilon_1)$, $\beta_2 < \overline{\beta}_2(\beta_4, \epsilon_1, \Delta)$, $\varsigma_{\text{2-slim}} < \overline{\varsigma}_{\text{2-slim}}(\beta_4, \epsilon_1, \Delta)$, $\varsigma_{\text{2-edge}} < \overline{\varsigma}_{\text{2-edge}}(\beta_4, \epsilon_1, \Delta)$, $\beta_{E'} < \overline{\beta}_{E'}(\beta_4, \epsilon_1, \Delta)$, $\varsigma_{E'} < \overline{\varsigma}_{E'}(\beta_4, \epsilon_1, \Delta)$, $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\beta_4, \epsilon_1, \Delta)$, $\beta_1 < \overline{\beta}_1(\beta_4, \epsilon_1, \Delta)$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\beta_4, \Delta)$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_4, \epsilon_1, \Delta)$, *and* $\Lambda < \overline{\Lambda}(\beta_4, \epsilon_1, \Delta)$, *the following holds.*

For each $j \in J$*, there is a map* $T_{ij} : \mathbb{R}^3 \to \mathbb{R}^{k_j}$ *which is a composition of an isometry and an orthogonal projection, such that on the ball* $B(p_i, 10) \subset \frac{1}{R_i} M$*, the map* η_j *is defined and satisfies*

$$\left\| \frac{R_j}{R_i} \eta_j - (T_{ij} \circ \eta_i) \right\|_{C^1} < \epsilon_1. \tag{10.14}$$

Proof. As we are assuming the hypothesis of Lemma 5.2, there are no 4-stratum points.

If $j \in I_{\text{3-stratum}}$, then the same arguments as in the proof of [13, Lemma 12.12] apply.

Suppose that $j \in I_{\text{2-slim}} \cup I_{\text{2-edge}}$. Then, $d(p_i, p_j) \leq 10R_i + 10^5 \Delta R_j$. Since R_i and R_j are Λ-Lipschitz, $\frac{1-10\Delta\Lambda}{1+10^5\Delta\Lambda} \leq \frac{R_i}{R_j} \leq \frac{1+10^5\Delta\Lambda}{1-10\Delta\Lambda}$. In particular, if $\Lambda < \overline{\Lambda}(\Delta)$, then $\frac{R_i}{R_j}$ is arbitrary close to 1 and $d(p_i, p_j) \leq 10^6 \Delta R_j$. From Lemma 2.42, for any $\epsilon > 0$, if $\beta_2 < \overline{\beta}_2(\epsilon, \Delta)$, then the $(2, \beta_2)$-splitting of $(\frac{1}{R_j} M, p_j)$ gives a $(2, \epsilon)$-splitting of $(\frac{1}{R_j} M, p_i)$. This means that there is a map from $B_{\frac{1}{R_j} M}(p_i, \frac{1}{\epsilon})$ with the distortion comparable to ϵ. By scaling, this gives a map from $B_{\frac{1}{R_i} M}(p_i, \frac{R_j}{R_i} \frac{1}{\epsilon})$ with the distortion comparable to $\frac{R_j}{R_i}\epsilon$. Hence, for any $\epsilon' > 0$, if $\epsilon < \overline{\epsilon}(\epsilon')$ and $\Lambda < \overline{\Lambda}(\epsilon', \Delta)$, then there is a map from $B_{\frac{1}{R_i} M}(p_i, \frac{1}{\epsilon'})$ with the distortion comparable to ϵ'. In other word, there is a $(2, \epsilon')$-splitting of $(\frac{1}{R_i} M, p_i)$. Since p_i is a 3-stratum point, there is no $(4, \beta_4)$-splitting at p_i. By the compatibility of approximate splitting (Lemma 2.44), for any $\epsilon'' > 0$, if $\epsilon' < \overline{\epsilon}'(\beta_4, \epsilon'')$ and $\beta_3 < \overline{\beta}_3(\beta_4, \epsilon'')$, then the $(3, \beta_3)$-splitting of $(\frac{1}{R_i} M, p_i)$ is ϵ''-compatible with the $(2, \epsilon')$-splitting of $(\frac{1}{R_i} M, p_i)$ from the above scaling and translation. By Lemma 2.51, if $\beta_3, \beta_2, \epsilon'', \varsigma_{\text{3-stratum}}, \varsigma_{\text{2-slim}}, \varsigma_{\text{2-edge}}$ are sufficiently small (as functions of ϵ_1), then the conclusion of the sublemma holds. In summary, the sublemma holds for $j \in I_{\text{2-slim}} \cup I_{\text{2-edge}}$ if $\Lambda, \beta_2, \varsigma_{\text{2-slim}}, \varsigma_{\text{2-edge}}$ are sufficiently small (as functions of β_4, ϵ_1, and Δ) and $\beta_3, \varsigma_{\text{3-stratum}}$ are sufficiently small (as functions of β_4 and ϵ_1).

The case $j \in I_{E'}$ is the same as in the proof of [13, Lemma 12.12]. If $\beta_{E'}, \Lambda$ and $\varsigma_{E'}$ are sufficiently small (as functions of $\beta_4, \epsilon_1, \Delta$) and $\beta_3, \varsigma_{\text{3-stratum}}$ are sufficiently small (as functions of β_4 and ϵ_1), then we can apply Lemma 2.44 and Lemma 2.51 to deduce the conclusion of the sublemma.

Suppose that $j_P \in I_P$. Then, $\operatorname{supp}(\zeta_j) \cap B(p_i, 10R_i) \neq \varnothing$. There is $q \in B(p_i, 10R_i)$ such that $\zeta_{j_P}(q) = \zeta_{\psi_{p_j}}(q) > 0$ for the corresponding $j \in I_{\text{1-ridge}}$. That is $\psi_{p_j}(q) \in (1.1, 4.9)$ and $\eta_{p_j} \in (-1, 1)$. In particular, if $\Lambda < \overline{\Lambda}(\Delta, \beta_2)$, then q is in the region where there is a $(2, \beta_2)$-splitting of $(\frac{1}{\mathfrak{r}_q} M, q)$. If Λ and β_2 (as functions of Δ) are sufficiently small, then there is an approximate 2-splitting for $(\frac{1}{\mathfrak{r}_q} M, p_i)$ of arbitrary quality. If Λ is sufficiently small, $\frac{\mathfrak{r}_q}{\mathfrak{r}_{p_j}}$ is arbitrary close to 1. Hence, there is an approximate 2-splitting for $(\frac{1}{\mathfrak{r}_{p_j}} M, p_i)$ of arbitrary quality. The same arguments as in the first case imply that the conclusion of the sublemma holds if $\Lambda, \beta_2, \varsigma_{\text{2-slim}}$ are sufficiently small (as functions of Δ and ϵ_1) and $\beta_3, \varsigma_{\text{3-stratum}}$ are sufficiently small (as functions of ϵ_1).

Next, suppose that $j \in I_{\text{1-ridge}} \cup I_{\text{1-slim}}$. Note that $\operatorname{supp}(\zeta_j) \subset B(p_i, \Delta\mathfrak{r}_{p_j})$. Then, $d(p_i, p_j) \leq 10\mathfrak{r}_{p_i} + \Delta\mathfrak{r}_{p_j}$. Since $\mathfrak{r}$ is Λ-Lipschitz, $|\mathfrak{r}_{p_i} - \mathfrak{r}_{p_j}| \leq \Lambda d(p_i, p_j) \leq \Lambda(10\mathfrak{r}_{p_i} + \Delta\mathfrak{r}_{p_j})$. If $\Lambda < \overline{\Lambda}(\Delta)$, then $\frac{\mathfrak{r}_{p_i}}{\mathfrak{r}_{p_j}}$ is arbitrary close to 1 and $d(p_i, p_j) < 2\Delta\mathfrak{r}_{p_j}$. If β_1 is sufficiently small (as a function of Δ), then the $(1, \beta_1)$-splitting of $(\frac{1}{\mathfrak{r}_{p_j}} M, p_j)$ gives an approximate 1-splitting of $(\frac{1}{\mathfrak{r}_{p_j}} M, p_i)$ with arbitrary quality. By the same argument as in the first case, the conclusion of the sublemma holds if $\Lambda, \beta_1, \varsigma_{\text{1-ridge}}$ are sufficiently small (as functions of β_4, ϵ_1, and Δ) and $\beta_3, \varsigma_{\text{3-stratum}}$ are sufficiently small (as functions of β_4 and ϵ_1).

Lastly, the case $j \in I_{\text{0-stratum}}$ is the same as in the proof of [13, Lemma 12.12]. The conclusion of the sublemma holds if $\Lambda, \beta_1, \beta_3, \varsigma_{\text{0-stratum}}, \varsigma_{\text{3-stratum}}$ are sufficiently small as functions of β_4 and ϵ_1. □

We retain the hypothesis of Sublemma 10.13.

For $j \in J$, the cutoff function ζ_j is a function of the $\eta_{j'}$'s for $j' \in J$, i.e., there is a smooth function $\Phi_j \in C_c^\infty(\mathbb{R}^J)$ such that $\zeta_j(\cdot) = \Phi_j(\{\eta_{j'}(\cdot)_{j \in J}\})$. The H_j-component of $\frac{1}{R_i}\mathcal{E}^0$ can be written as

$$\frac{1}{R_i}\mathcal{E}_j^0 = \left(\frac{R_j}{R_i}\eta_j\zeta_j, \frac{R_j}{R_i}\zeta_j\right) = \left(\frac{R_j}{R_i}\eta_j \cdot (\Phi_j \circ \{\eta_{j'}\}_{j' \in J}), \frac{R_j}{R_i} \cdot (\Phi_j \circ \{\eta_{j'}\}_{j' \in J})\right). \tag{10.15}$$

Let $\mathcal{F}^0 : \mathbb{R}^3 \to H$ be the map so that the H_j-component of $\mathcal{F}^0 \circ \eta_i$, for $j \in J$, is obtained from the preceding formula by replacing each occurrence of η_j with the approximation $\frac{R_i}{R_{j'}}(T_{ij} \circ \eta_i)$. That is

$$\frac{1}{R_i}\mathcal{F}_j^0(u) = \left(T_{ij}(u) \cdot \left(\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'}(u)\right\}_{j' \in J}\right)\right), \frac{R_j}{R_i}\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'}(u)\right\}_{j' \in J}\right)\right), \tag{10.16}$$

whose $H_{\mathfrak{r}}$-component is a constant function R_i, and whose other componenets vanish. We then have

$$\frac{1}{R_i}\mathcal{F}_j^0 \circ \eta_i = \left((T_{ij} \circ \eta_i) \cdot \left(\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'} \circ \eta_i\right\}_{j' \in J}\right)\right), \frac{R_j}{R_i}\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'} \circ \eta_i\right\}_{j' \in J}\right)\right). \tag{10.17}$$

Sublemma 10.18. *Under the constraints* $\epsilon_1 < \overline{\epsilon}_1(\epsilon_2, \mathcal{M})$, $\Upsilon_0 > \overline{\Upsilon}_0(\epsilon_2, \mathcal{M})$, *and* $\Lambda < \overline{\Lambda}(\epsilon_2, \mathcal{M})$,

$$\left\|\frac{1}{R_i}\mathcal{E}^0 - \frac{1}{R_i}\mathcal{F}^0 \circ \eta_i\right\|_{C^1} < \epsilon_2 \tag{10.19}$$

on $B(p_i, 10) \subset \frac{1}{R_i}\mathcal{M}$.

Proof. The proof is the same as the proof of [13, Sublemma 12.17] except in the case $j_P \in J \cap I_P$.

Suppose that $j_P \in J \cap I_P$. The only relevant arguments of Φ_j are when $j' = j_P$ and $j' = j \in I_{\text{1-ridge}}$ corresponding to $j_P \in I_P$. Hence, from (10.15), in this case we can write

$$\frac{1}{R_i}\mathcal{E}_{j_P}^0 = \left(\frac{R_{j_P}}{R_i}\eta_{j_P} \cdot \Phi_{j_P}(\eta_j, \eta_{j_P}), \frac{R_{j_P}}{R_i} \cdot \Phi_{j_P}(\eta_j, \eta_{j_P})\right). \tag{10.20}$$

From (10.17), we also have

$$\frac{1}{R_i}\mathcal{F}^0_{j_P}\circ\eta_i=\left(T_{ij_P}(u)\cdot\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i,\frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right),\frac{R_{j_P}}{R_i}\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i,\frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right). \tag{10.21}$$

Thus, the first component of $\frac{1}{R_i}\mathcal{E}^0_{j_P}-\frac{1}{R_i}\mathcal{F}^0_{j_P}\circ\eta_i$ is

$$A:=\frac{R_{j_P}}{R_i}\eta_{j_P}\cdot\Phi_{j_P}(\eta_j,\eta_{j_P})-(T_{ij_P}\circ\eta_i)\cdot\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i,\frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right). \tag{10.22}$$

By (10.14), $\frac{R_{j_P}}{R_i}\eta_{j_P}=T_{ij_P}\circ\eta_i\pm s\epsilon_1$ for some $0\le s<1$. Then,

$$\begin{aligned}A&=(T_{ij_P}\circ\eta_i\pm s\epsilon_1)\cdot\Phi_{j_P}(\eta_j,\eta_{j_P})-(T_{ij_P}\circ\eta_i)\cdot\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i,\frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right) \qquad (10.23)\\&=(T_{ij_P}\circ\eta_i)\cdot\left(\Phi_{j_P}(\eta_j,\eta_{j_P})-\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i,\frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right)\pm s\epsilon_1\Phi_{j_P}(\eta_j,\eta_{j_P}).\end{aligned}$$

So,

$$\begin{aligned}|A|&\le\|T_{ij_P}\circ\eta_i\|\cdot\|D\Phi_{j_P}\|\left\|(\eta_j,\eta_{j_P})-\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i,\frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right\|+\epsilon_1\|\Phi_{j_P}\| \qquad (10.24)\\&\le\epsilon_1\left(\left(\frac{R_i}{R_{j_P}}\right)^2+\left(\frac{R_i}{R_j}\right)^2\right)^{\frac{1}{2}}\|T_{ij_P}\circ\eta_i\|\cdot\|D\Phi_{j_P}\|+\epsilon_1\|\Phi_{j_P}\|.\end{aligned}$$

Recall that $j\in I_{\text{1-ridge}}$ is chosen so that p_j and p_{j_P} are the same point. In particular, $\frac{R_i}{R_{j_P}}=\frac{R_i}{R_j}$. We then have

$$|A|\le\epsilon_1\frac{R_i}{R_{j_P}}\|T_{ij_P}\circ\eta_i\|\cdot\|D\Phi_{j_P}\|+\epsilon_1\|\Phi_{j_P}\|. \tag{10.25}$$

Since $j_P\in J\cap I_P$, we have that $d(p_i,p_j)<5r^1_{p_j}+10R_i\le5\Delta\mathfrak{r}_{p_j}+10\mathfrak{r}_{p_i}$. If Λ is sufficiently small (as a function of Δ), then $\frac{\mathfrak{r}_{p_i}}{\mathfrak{r}_{p_j}}$ is arbitrary close to 1. Using the fact that Φ_j has explicit bounds on its derivatives of order up to 2, if $\epsilon_1<\overline{\epsilon}_1(\epsilon_2)$ then $|A|\le\epsilon_2$.

Next, we compute a bound for $\|DA\|$. Consider that

$$\begin{aligned}DA=&\frac{R_{j_P}}{R_i}\Phi_{j_P}(\eta_j,\eta_{j_P})D\eta_{j_P}+\frac{R_{j_P}}{R_i}\eta_{j_P}D(\Phi_{j_P}(\eta_j,\eta_{j_P})) \qquad (10.26)\\&-\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i,\frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)D(T_{ij_P}\circ\eta_i)\\&-(T_{ij_P}\circ\eta_i)D\left(\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i,\frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right)\end{aligned}$$

$$\begin{aligned}
&= \Phi_{j_P}(\eta_j, \eta_{j_P})(D(T_{ij_P}\circ\eta_i) \pm s\epsilon_1) + ((T_{ij_P}\circ\eta_i) \pm s\epsilon_1) D(\Phi_{j_P}(\eta_j,\eta_{j_P}))\\
&\quad - \Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right) D(T_{ij_P}\circ\eta_i)\\
&\quad - (T_{ij_P}\circ\eta_i) D\left(\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right).
\end{aligned}$$

So,

$$\begin{aligned}
\|DA\| &\le \|D(T_{ij_P}\circ\eta_i)\| \left\|\Phi_{j_P}(\eta_j,\eta_{j_P}) - \Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right\| \qquad (10.27)\\
&\quad + \|T_{ij_P}\circ\eta_i\| \left\|D(\Phi_{j_P}(\eta_j,\eta_{j_P})) - D\left(\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right)\right\|\\
&\quad + \epsilon_1(\|\Phi_{j_P}\| + \|D\Phi_{j_P}\|)\\
&\le \epsilon_1 \frac{R_i}{R_{j_P}} \|D(T_{ij_P}\circ\eta_i)\| \|D\Phi_{j_P}\|\\
&\quad + \|T_{ij_P}\circ\eta_i\| \left\|D(\Phi_{j_P}(\eta_j,\eta_{j_P})) - D\left(\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right)\right\|\\
&\quad + \epsilon_1(\|\Phi_{j_P}\| + \|D\Phi_{j_P}\|).
\end{aligned}$$

Consider

$$\begin{aligned}
&D(\Phi_j(\eta_j,\eta_{j_P})) - D\left(\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right) \qquad (10.28)\\
&\qquad = D\Phi_{j_P}(\eta_j,\eta_{j_P}) \cdot \{D\eta_j, D\eta_{j_P}\}^T\\
&\qquad\quad - D\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right) \cdot \{\frac{R_i}{R_j}D(T_{ij}\circ\eta_i), \frac{R_i}{R_{j_P}}D(T_{ij_P}\circ\eta_i)\}^T\\
&\qquad = D\Phi_{j_P}(\eta_j,\eta_{j_P}) \cdot \{D\eta_j, D\eta_{j_P}\}^T\\
&\qquad\quad - D\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right) \cdot \{D\eta_j + s\epsilon_1\frac{R_i}{R_j}, D\eta_{j_P} + s\epsilon_1\frac{R_i}{R_{j_P}}\}^T.
\end{aligned}$$

So,

$$\begin{aligned}
&\left\|D(\Phi_{j_P}(\eta_j,\eta_{j_P})) - D\left(\Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right)\right\|\\
&\qquad \le \epsilon_1\frac{R_i}{R_{j_P}}\|D^2\Phi_{j_P}\| \|\{D\eta_j, D\eta_{j_P}\}\| + \epsilon_1\frac{R_i}{R_{j_P}}\|D\Phi_{j_P}\|. \qquad (10.29)
\end{aligned}$$

By substituting (10.29) into (10.27) and by the same argument for $|A|$, we have that if ϵ_1 is sufficiently small (as a function of ϵ_2) then $\|DA\| < \epsilon_2$. Consequently, $\|A\|_{C^1} < \epsilon_2$.

The second component of $\frac{1}{R_i}\mathcal{E}^0_{j_P} - \frac{1}{R_i}\mathcal{F}^0_{j_P}\circ\eta_i$ is

$$B := \frac{R_{j_P}}{R_i}\left(\Phi_{j_P}(\eta_j,\eta_{j_P}) - \Phi_{j_P}\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i, \frac{R_i}{R_{j_P}}T_{ij_P}\circ\eta_i\right)\right). \qquad (10.30)$$

By a similar calculation as the calculation for A,

$$|B| \leq \frac{R_{j_P}}{R_i} \cdot \epsilon_1 \frac{R_i}{R_{j_P}} \|D\Phi_{j_P}\| = \epsilon_1 \|D\Phi_{j_P}\|, \tag{10.31}$$

and

$$\begin{aligned} \|DB\| \leq \frac{R_{j_P}}{R_i} \cdot \left(\epsilon_1 \frac{R_i}{R_{j_P}} \|D^2\Phi_{j_P}\| \|\{D\eta_j, D\eta_{j_P}\}\| + \epsilon_1 \frac{R_i}{R_{j_P}} \|D\Phi_{j_P}\| \right) \\ = \epsilon_1 \|D^2\Phi_{j_P}\| \|\{D\eta_j, D\eta_{j_P}\}\| + \epsilon_1 \|D\Phi_{j_P}\|. \end{aligned} \tag{10.32}$$

If $\epsilon_1 < \overline{\epsilon}_1(\epsilon_2)$, then $\|B\|_{C^1} < \epsilon_2$. Therefore, $\left\| \frac{1}{R_i} \mathcal{E}^0_{j_P} - \frac{1}{R_i} \mathcal{F}^0_{j_P} \circ \eta_i \right\|_{C^1} < \epsilon_2$. □

Sublemma 10.33. *Given $\Sigma \in (0, \frac{1}{10})$, suppose that $|\eta_i(p)| < 8$ for some $p \in M$. Put $x = \mathcal{E}^0(p)$. For any $q \in M$, if $\mathcal{E}^0(q) \in B(x, \Sigma R_i)$, then $|\eta_i(p) - \eta_i(q)| < 20\Sigma$.*

Proof. The proof is the same as the proof of [13, Sublemma 12.21]. □

The rest of the proof of Lemma 10.8 is same as the proof of [13, Lemma 12.7].

10.4 Structure of $\mathcal{E}^0$ near the edge 2-stratum

Recall that $Q_2 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}} \oplus H_{\text{2-slim}} \oplus H_{\text{2-edge}}$, and $\pi_2 : H \to Q_2$ is the orthogonal projection.

Define

$$\widetilde{A}_2 = \bigcup_{i \in I_{\text{2-edge}}} \{|\eta_i| \leq 8\Delta, \eta_{E'} \leq 8\Delta\}, \quad A_2 = \bigcup_{i \in I_{\text{2-edge}}} \{|\eta_i| \leq 7\Delta, \eta_{E'} \leq 7\Delta\}, \tag{10.34}$$

and

$$\widetilde{S}_2 = (\pi_2 \circ \mathcal{E}^0)(\widetilde{A}_2), \quad S_2 = (\pi_2 \circ \mathcal{E}^0)(A_2). \tag{10.35}$$

Let $\Sigma_2, \Gamma_2 > 0$ be new parameters. Define $r_2 : \widetilde{S}_2 \to (0, \infty)$ by putting $r_2(x) = \Sigma_2 \mathfrak{r}_p$ for some $p \in (\pi_2 \circ \mathcal{E}^0)^{-1}(x) \cap \widetilde{A}_2$.

The analog of Lemma 10.8 for the region near edge 2-stratum points is:

Lemma 10.36. *There is a constant $\Omega_2 = \Omega_2(\mathcal{M})$ so that under the constraints $\Sigma_2 < \overline{\Sigma}_2(\Gamma_2, \mathcal{M})$, $\varsigma_{\text{2-edge}} < \overline{\varsigma}_{\text{2-edge}}(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, $\varsigma_{\text{2-slim}} < \overline{\varsigma}_{\text{2-slim}}(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, $\beta_E < \overline{\beta}_E(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, $\sigma_E < \overline{\sigma}_E(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, $\beta_2 < \overline{\beta}_2(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, $\beta_1 < \overline{\beta}_1(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$ and $\Lambda < \overline{\Lambda}(\beta_3, \Gamma_2, \Sigma_2, \mathcal{M}, \Delta)$, the following holds.*

(1) *The triple* $(\widetilde{S}_2, S_2, r_2)$ *is a* $(2, \Gamma_2)$*-cloudy 2-manifold.*

(2) *The affine subspaces* $\{A_x\}_{x \in S_2}$ *inherent in the definition of the cloudy 2-manifold can be chosen to have the following property. Pick* $p \in A_2$ *and put* $x = (\pi_2 \circ \mathcal{E}^0)(p) \in S_2$*. Let* $A^0_x \subset Q_2$ *be the linear subspace parallel to* A_x *(i.e.,* $A_x = A^0_x + x$*) and let* $\pi_{A^0_x} : H \to A^0_x$ *denote orthogonal projection onto* A^0_x*. Then*

$$\|D(\pi_2 \circ \mathcal{E}^0)_p - \pi_{A^0_x} \circ D(\pi_2 \circ \mathcal{E}^0)_p\| \leq \Gamma_2 \tag{10.37}$$

and

$$\Omega_2^{-1}\|v\| \leq \|(\pi_{A^0_x} \circ D(\pi_2 \circ \mathcal{E}^0)_p)(v)\| \leq \Omega_2\|v\| \tag{10.38}$$

for every $v \in T_pM$ *which is orthogonal to* $\ker(\pi_{A^0_x} \circ D(\pi_2 \circ \mathcal{E}^0)_p)$*.*

(3) *Given* $i \in I_{\text{2-edge}}$*, there is a smooth map* $\widehat{\mathcal{E}}^0_i : (\overline{B(0, 8\Delta)} \subset \mathbb{R}) \to (H'_i)^\perp \cap Q_2$ *such that*

$$\|D\widehat{\mathcal{E}}^0_i\| \leq \Omega_2 R_i \tag{10.39}$$

and on the subset $\{|\eta_i| \leq 8\Delta, \eta_{E'} \leq 8\Delta\}$*, we have*

$$\left\| \frac{1}{R_i}\pi_2 \circ \mathcal{E}^0 - \left(\eta_i, \frac{1}{R_i}\widehat{\mathcal{E}}^0_i \circ \eta_i\right) \right\|_{C^1} < \Gamma_2. \tag{10.40}$$

Furthermore, if $x \in S_2$*, then there are some* $i \in I_{\text{2-edge}}$ *and* $p \in \{|\eta_i| \leq 7\Delta, \eta_{E'} \leq 7\Delta\}$ *such that* $x = (\pi_2 \circ \mathcal{E}^0)(p)$ *and* $A^0_x = \operatorname{Im}(\operatorname{Id}, \frac{1}{R_i}(D\widehat{\mathcal{E}}^0_i)_{\eta_i(p)})$*.*

Proof. The proof is similar to the proof of Lemma 10.8. □

10.5 Structure of $\mathcal{E}^0$ near the slim 2-stratum

Recall that $Q_3 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}} \oplus H_{\text{2-slim}}$, and $\pi_3 : H \to Q_3$ is the orthogonal projection.

Define

$$\widetilde{A}_3 = \bigcup_{i \in I_{\text{2-slim}}} \left\{|\eta_i| \leq 8 \cdot 10^5\Delta\right\}, \quad A_3 = \bigcup_{i \in I_{\text{2-slim}}} \left\{|\eta_i| \leq 7 \cdot 10^5\Delta\right\}, \tag{10.41}$$

and

$$\widetilde{S}_3 = (\pi_3 \circ \mathcal{E}^0)(\widetilde{A}_3), \quad S_3 = (\pi_3 \circ \mathcal{E}^0)(A_3). \tag{10.42}$$

Let $\Sigma_3, \Gamma_3 > 0$ be new parameters. Define $r_3 : \widetilde{S}_3 \to (0, \infty)$ by putting $r_3(x) = \Sigma_3 \mathfrak{r}_p$ for some $p \in (\pi_3 \circ \mathcal{E}^0)^{-1}(x) \cap \widetilde{A}_3$.

The analog of Lemma 10.8 for the region near slim 2-stratum points is:

Lemma 10.43. *There is a constant $\Omega_3 = \Omega_3(\mathcal{M})$ so that under the constraints $\Sigma_3 < \overline{\Sigma}_3(\Gamma_3,\ \mathcal{M})$, $\varsigma_{\text{2-slim}} < \overline{\varsigma}_{\text{2-slim}}(\beta_3, \Gamma_3, \Sigma_3, \mathcal{M}, \Delta)$, $\beta_2 < \overline{\beta}_2(\beta_3, \Gamma_3, \Sigma_3, \mathcal{M}, \Delta)$, $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\beta_3, \Gamma_3, \Sigma_3, \mathcal{M}, \Delta)$, $\beta_1 < \overline{\beta}_1(\beta_3, \Gamma_3, \Sigma_3, \mathcal{M}, \Delta)$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\beta_3, \Gamma_3, \Sigma_3, \mathcal{M}, \Delta)$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_3, \Gamma_3, \Sigma_3, \mathcal{M}, \Delta)$, and $\Lambda < \overline{\Lambda}(\beta_3, \Gamma_3, \Sigma_3, \mathcal{M}, \Delta)$, the following holds.*

(1) *The triple $(\widetilde{S}_3, S_3, r_3)$ is a $(2, \Gamma_3)$-cloudy 2-manifold.*

(2) *The affine subspaces $\{A_x\}_{x \in S_3}$ inherent in the definition of the cloudy 3-manifold can be chosen to have the following property. Pick $p \in A_3$ and put $x = (\pi_3 \circ \mathcal{E}^0)(p) \in S_3$. Let $A^0_x \subset Q_3$ be the linear subspace parallel to A_x (i.e., $A_x = A^0_x + x$) and let $\pi_{A^0_x} : H \to A^0_x$ denote orthogonal projection onto A^0_x. Then*

$$\|D(\pi_3 \circ \mathcal{E}^0)_p - \pi_{A^0_x} \circ D(\pi_3 \circ \mathcal{E}^0)_p\| \leq \Gamma_3 \tag{10.44}$$

and

$$\Omega_3^{-1}\|v\| \leq \|(\pi_{A^0_x} \circ D(\pi_3 \circ \mathcal{E}^0)_p)(v)\| \leq \Omega_3\|v\| \tag{10.45}$$

for every $v \in T_pM$ which is orthogonal to $\ker(\pi_{A^0_x} \circ D(\pi_3 \circ \mathcal{E}^0)_p)$.

(3) *Given $i \in I_{\text{2-slim}}$, there is a smooth map $\widehat{\mathcal{E}}^0_i : (\overline{B(0, 8 \cdot 10^5\Delta)} \subset \mathbb{R}) \to (H'_i)^\perp \cap Q_3$ such that*

$$\|D\widehat{\mathcal{E}}^0_i\| \leq \Omega_3 R_i \tag{10.46}$$

and on the subset $\{|\eta_i| \leq 8 \cdot 10^5\Delta\}$, we have

$$\left\| \frac{1}{R_i}\pi_3 \circ \mathcal{E}^0 - \left(\eta_i, \frac{1}{R_i}\widehat{\mathcal{E}}^0_i \circ \eta_i\right)\right\|_{C^1} < \Gamma_3. \tag{10.47}$$

Furthermore, if $x \in S_3$, then there are some $i \in I_{\text{2-slim}}$ and $p \in \{|\eta_i| \leq 7 \cdot 10^5\Delta\}$ such that $x = (\pi_3 \circ \mathcal{E}^0)(p)$ and $A^0_x = \operatorname{Im}(\mathrm{Id}, \frac{1}{R_i}(D\widehat{\mathcal{E}}^0_i)_{\eta_i(p)})$.

Proof. The proof is similar to the proof of Lemma 10.8. □

10.6 Structure of $\mathcal{E}^0$ near the ridge 1-stratum

Recall that $Q_4 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}}$, and $\pi_4 : H \to Q_4$ is the orthogonal projection. Put

$$\widetilde{A}_4 = \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| \leq 0.8\Delta, \eta_{i_P} \leq 4.5\lambda_{p_i}\}, \quad A_4 = \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| \leq 0.7\Delta, \eta_{i_P} \leq 4\lambda_{p_i}\}, \tag{10.48}$$

where $\psi_i = \psi_{p_i}$ for the corresponding $i \in I_P$, and

$$\widetilde{S}_4 = (\pi_4 \circ \mathcal{E}^0)(\widetilde{A}_4), \quad S_4 = (\pi_4 \circ \mathcal{E}^0)(A_4). \tag{10.49}$$

Let $\Sigma_4, \Gamma_4 > 0$ be new parameters. Define $r_4 : \widetilde{S}_4 \to (0, \infty)$ by putting $r_4(x) = \Sigma_4 \mathfrak{r}_p$ for some $p \in (\pi_4 \circ \mathcal{E}^0)^{-1}(x) \cap \widetilde{A}_4$.

Lemma 10.50. *There is a constant* $\Omega_4 = \Omega_4(\mathcal{M})$ *so that under the constraints* $\Sigma_4 < \overline{\Sigma}_4(\Gamma_4, \mathcal{M})$, $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\beta_2, \Gamma_4, \Sigma_4, \mathcal{M}, \Delta)$, $\beta_1 < \overline{\beta}_1(\beta_2, \Gamma_4, \Sigma_4, \mathcal{M}, \Delta)$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\beta_2, \Gamma_4, \Sigma_4, \mathcal{M}, \Delta)$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_2, \Gamma_4, \Sigma_4, \mathcal{M}, \Delta)$, *and* $\Lambda < \overline{\Lambda}(\beta_2, \Gamma_4, \Sigma_4, \mathcal{M}, \Delta)$, *the following holds.*

(1) *The triple* $(\widetilde{S}_4, S_4, r_4)$ *is a* $(2, \Gamma_4)$*-cloudy 1-manifold.*

(2) *The affine subspaces* $\{A_x\}_{x \in S_4}$ *inherent in the definition of the cloudy 1-manifold can be chosen to have the following property. Pick* $p \in A_4$ *and put* $x = (\pi_4 \circ \mathcal{E}^0)(p) \in S_4$. *Let* $A^0_x \subset Q_4$ *be the linear subspace parallel to* A_x *(i.e.,* $A_x = A^0_x + x$*) and let* $\pi_{A^0_x} : H \to A^0_x$ *denote orthogonal projection onto* A^0_x. *Then*

$$\| D(\pi_4 \circ \mathcal{E}^0)_p - \pi_{A^0_x} \circ D(\pi_4 \circ \mathcal{E}^0)_p \| \le \Gamma_4 \tag{10.51}$$

and

$$\Omega_4^{-1} \|v\| \le \| (\pi_{A^0_x} \circ D(\pi_4 \circ \mathcal{E}^0)_p)(v) \| \le \Omega_4 \|v\| \tag{10.52}$$

for every $v \in T_pM$ *which is orthogonal to* $\ker(\pi_{A^0_x} \circ D(\pi_4 \circ \mathcal{E}^0)_p)$.

(3) *Given* $i \in I_{\text{1-ridge}}$, *there is a smooth map* $\widehat{\mathcal{E}}^0_i : (\overline{B(0, 0.8\Delta)} \subset \mathbb{R}) \to (H'_i)^\perp \cap Q_4$ *such that*

$$\| D \widehat{\mathcal{E}}^0_i \| \le \Omega_4 R_i \tag{10.53}$$

and on the subset $\{|\eta_i| \le 0.8\Delta, \eta_{i_P} \le 4.5\lambda_{p_i}\}$, *we have*

$$\left\| \frac{1}{R_i} \pi_4 \circ \mathcal{E}^0 - \left(\eta_i, \frac{1}{R_i} \widehat{\mathcal{E}}^0_i \circ \eta_i \right) \right\|_{C^1} < \Gamma_4. \tag{10.54}$$

Furthermore, if $x \in S_4$, *then there are some* $i \in I_{\text{1-ridge}}$ *and* $p \in \{|\eta_i| \le 0.7\Delta, \eta_{i_P} \le 4\lambda_{p_i}\}$ *such that* $x = (\pi_4 \circ \mathcal{E}^0)(p)$ *and* $A^0_x = \operatorname{Im}(\mathrm{Id}, \frac{1}{R_i}(D\widehat{\mathcal{E}}^0_i)_{\eta_i(p)})$.

Put $J = \{ j \in I_{\text{1-ridge}} \cup I_{\text{1-slim}} \cup I_{\text{0-stratum}} : \operatorname{supp}(\zeta_j) \cap B(p_i, \Delta R_i) \neq \varnothing \}$.

Sublemma 10.55. *Under the constraints* $\beta_1 < \overline{\beta}_1(\beta_2, \epsilon_1)$, $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\beta_2, \epsilon_1, \Delta)$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\beta_2, \epsilon_1, \Delta)$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_2, \epsilon_1, \Delta)$, *and* $\Lambda < \overline{\Lambda}(\beta_2, \epsilon_1, \Delta)$, *the following holds.*

For each $j \in J$, *there is a map* $T_{ij} : \mathbb{R} \to \mathbb{R}^{k_j}$ *which is a composition of an isometry and an orthogonal projection, such that on the ball* $B(p_i, \Upsilon_1) \subset \frac{1}{R_i} M$, *the map* η_j *is defined and satisfies*

$$\left\| \frac{R_j}{R_i} \eta_j - (T_{ij} \circ \eta_i) \right\|_{C^1} < \epsilon_1. \tag{10.56}$$

Proof. Suppose that $j \in I_{\text{1-ridge}} \cup I_{\text{1-slim}}$. Note that $\operatorname{supp}(\zeta_j) \subset B(p_j, \Delta \mathfrak{r}_{p_j})$. Then, $d(p_i, p_j) \le \Delta \mathfrak{r}_{p_i} + \Delta \mathfrak{r}_{p_j}$. If $\Lambda < \overline{\Lambda}(\Delta)$, then $\frac{\mathfrak{r}_{p_i}}{\mathfrak{r}_{p_j}}$ is arbitrary close to 1 and also $d(p_i, p_j) < 10 \Delta \mathfrak{r}_{p_j}$. If β_1 is sufficiently small (as a function of Δ and ϵ' for $\epsilon' > 0$), then the $(1, \beta_1)$-splitting of $(\frac{1}{\mathfrak{r}_{p_j}} M, p_j)$

gives a $(1, \epsilon')$-splitting of $(\frac{1}{\mathfrak{r}_{p_j}} M, p_i)$. If Λ and ϵ' is sufficiently small (as functions of Δ and ϵ'' for some $\epsilon'' > 0$), then $\frac{\mathfrak{r}_{p_i}}{\mathfrak{r}_{p_j}}$ is arbitrary close to 1 and there is a $(1, \epsilon'')$-splitting for $(\frac{1}{\mathfrak{r}_{p_i}} M, p_i)$.

By the uniqueness of approximate splittings (Lemma 2.50), if β_1 and ϵ'' are sufficiently small (as functions of $\widetilde{\epsilon} > 0$), then the $(1, \epsilon'')$-splitting is $\widetilde{\epsilon}$-compatible with the $(1, \beta_1)$-splitting for $(\frac{1}{\mathfrak{r}_{p_i}} M, p_i)$. By Lemma 2.51, the conclusion of the sublemma holds if Λ, β_1 are sufficiently small (as functions of ϵ_1, Δ) and ς_R are sufficiently small (as a function of ϵ_1).

The case $j \in I_{\text{0-stratum}}$ is the same as in the proof of [13, Sublemma 12.12]. □

We retain the hypothesis of Sublemma 10.55.

For $j \in J$, the cutoff function ζ_j is a function of the $\eta_{j'}$'s for $j' \in J$, i.e., there is a smooth function $\Phi_j \in C_c^\infty(\mathbb{R}^J)$ such that $\zeta_j(\cdot) = \Phi_j(\{\eta_{j'}(\cdot)_{j \in J}\})$. The H_j-component of $\pi_4 \circ \mathcal{E}^0$, after dividing by R_i, can be written as

$$\frac{1}{R_i}(\pi_4 \circ \mathcal{E}^0)_j = \left(\frac{R_j}{R_i}\eta_j\zeta_j, \frac{R_j}{R_i}\zeta_j\right) = \left(\frac{R_j}{R_i}\eta_j \cdot (\Phi_j \circ \{\eta_{j'}\}_{j' \in J}), \frac{R_j}{R_i} \cdot (\Phi_j \circ \{\eta_{j'}\}_{j' \in J})\right). \quad (10.57)$$

Let $\mathcal{F}^0 : \mathbb{R} \to H$ be the map so that the H_j-component of $\mathcal{F}^0 \circ \eta_i$, for $j \in J$, is obtained from the preceding formula by replacing each occurrence of η_j with the approximation $\frac{R_i}{R_{j'}}(T_{ij} \circ \eta_i)$. That is

$$\frac{1}{R_i}\mathcal{F}_j^0(u) = \left(T_{ij}(u) \cdot \left(\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'}(u)\right\}_{j' \in J}\right)\right), \frac{R_j}{R_i}\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'}(u)\right\}_{j' \in J}\right)\right). \quad (10.58)$$

We then have

$$\frac{1}{R_i}\mathcal{F}_j^0 \circ \eta_i = \left((T_{ij} \circ \eta_i) \cdot \left(\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'} \circ \eta_i\right\}_{j' \in J}\right)\right), \frac{R_j}{R_i}\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'} \circ \eta_i\right\}_{j' \in J}\right)\right). \quad (10.59)$$

Sublemma 10.60. *Under the constraints* $\epsilon_1 < \bar{\epsilon}_1(\epsilon_2, \mathcal{M})$, $\Upsilon_0 > \overline{\Upsilon}_0(\epsilon_2, \mathcal{M})$, *and* $\Lambda < \overline{\Lambda}(\epsilon_2, \mathcal{M}, \Delta)$,

$$\left\|\frac{1}{R_i}(\pi_4 \circ \mathcal{E}^0) - \frac{1}{R_i}\mathcal{F}^0 \circ \eta_i\right\|_{C^1} < \epsilon_2 \quad (10.61)$$

on $B(p_i, \Delta) \subset \frac{1}{R_i}M$.

Proof. For $j \in J \cap (I_{\text{1-slim}} \cup I_{\text{1-ridge}} \cup I_{\text{0-stratum}})$, the only relevant argument of Φ_j is when $j' = j$. Hence, we can write

$$\frac{1}{R_i}(\pi_4 \circ \mathcal{E}^0)_j = \left(\frac{R_j}{R_i}\eta_j \cdot \Phi_j(\eta_j), \frac{R_j}{R_i} \cdot \Phi_j(\eta_j)\right) \quad (10.62)$$

and

$$\frac{1}{R_i}\mathcal{F}^0_j \circ \eta_i = \left(T_{ij}(u)\cdot\Phi_j\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i\right), \frac{R_j}{R_i}\Phi_j\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i\right)\right). \tag{10.63}$$

The first component of $\frac{1}{R_i}(\pi_4\circ\mathcal{E}^0)_j - \frac{1}{R_i}\mathcal{F}^0_j\circ\eta_i$ is

$$A := \frac{R_j}{R_i}\eta_j\cdot\Phi_j(\eta_j) - T_{ij}(u)\cdot\Phi_j\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i\right). \tag{10.64}$$

By similar calculations as in the proof of Sublemma 10.18,

$$|A| \le \epsilon_1\frac{R_i}{R_j}\|T_{ij}\circ\eta_i\|\cdot\|D\Phi_j\| + \epsilon_1\|\Phi_j\| \tag{10.65}$$

and

$$\|DA\| \le \epsilon_1\frac{R_i}{R_j}\left(\|D(T_{ij}\circ\eta_i)\|\|D\Phi_j\| + \|T_{ij}\circ\eta_i\|\|D^2\Phi_j\|\|D\eta_j\| + \|D\Phi_j\|\right) + \epsilon_1(\|\Phi_j\|+\|D\Phi_j\|). \tag{10.66}$$

In the case $j \in I_{\text{1-slim}} \cup I_{\text{1-ridge}}$, if Λ is sufficiently small (as a function of Δ), then $\frac{R_i}{R_j} = \frac{\mathfrak{r}_{p_i}}{\mathfrak{r}_{p_j}}$ is arbitrary close to 1. In the case $j \in I_{\text{0-stratum}}$, from Lemma 9.4, $\frac{R_i}{R_j} \le \frac{20}{\Upsilon_0}$. Therefore, if ϵ_1 is sufficiently small and Υ_0 is sufficiently large (as functions of ϵ_2), then $\|A\|_{C^1} < \epsilon_2$.

The second component of $\frac{1}{R_i}(\pi_4\circ\mathcal{E}^0)_j - \frac{1}{R_i}\mathcal{F}^0_j\circ\eta_i$ is

$$B := \frac{R_j}{R_i}\left(\Phi_j(\eta_j) - \Phi_j\left(\frac{R_i}{R_j}T_{ij}\circ\eta_i\right)\right). \tag{10.67}$$

By similar calculations as in the proof of Sublemma 10.18,

$$|B| \le \epsilon_1\|D\Phi_j\| \tag{10.68}$$

and

$$\|DB\| \le \epsilon_1\|D^2\Phi_j\|\|\{D\eta_j, D\eta_j\}\| + \epsilon_1\|D\Phi_j\|. \tag{10.69}$$

Hence, if $\epsilon_1 < \bar{\epsilon}_1(\epsilon_2)$ then $\|B\|_{C^1} < \epsilon_2$. Therefore, $\left\|\frac{1}{R_i}\mathcal{E}^0_j - \frac{1}{R_i}\mathcal{F}^0_j\circ\eta_i\right\|_{C^1} < \epsilon_2$. □

Sublemma 10.70. *Given* $\Sigma \in (0, \frac{1}{10\Upsilon_1})$*, suppose that* $|\eta_i(p)| < 0.8\Upsilon_1$ *for some* $p \in M$*. Put* $x = \mathcal{E}^0(p)$*. For any* $q \in M$*, if* $\mathcal{E}^0(q) \in B(x, \Sigma\Upsilon_1 R_i)$ *then* $|\eta_i(p) - \eta_i(q)| < 4\Sigma\Upsilon_1^2$*.*

Proof. We know that $\zeta_i(p) = 1$. From the hypothesis of the sublemma, $|\mathcal{E}^0(p) - \mathcal{E}^0(q)| < \Sigma\Upsilon_1 R_i$. In particular, $|\zeta_i(p) - \zeta_i(q)| < \Sigma\Upsilon_1$ and $|\zeta_i(p)\eta_i(p) - \zeta_i(q)\eta_i(q)| < \Sigma\Upsilon_1$. Then

$$\begin{aligned} |\eta_i(p) - \eta_i(q)| &= \frac{1}{\zeta_i(q)}|\zeta_i(q)\eta_i(p) - \zeta_i(q)\eta_i(q)| \\ &\leq \frac{1}{\zeta_i(q)}\left(|\zeta_i(p)\eta_i(p) - \zeta_i(q)\eta_i(q)| + |\zeta_i(p) - \zeta_i(q)||\eta_i(p)|\right) \\ &\leq \frac{\Sigma\Upsilon_1 + \Sigma\Upsilon_1 \cdot 0.8\Upsilon_1}{1 - \Sigma\Upsilon_1} \leq \frac{2\Sigma\Upsilon_1^2}{1 - \Sigma\Upsilon_1} \leq 4\Sigma\Upsilon_1^2. \end{aligned} \tag{10.71}$$

Since $\Sigma < \frac{1}{2\Upsilon_1}$, this proves the sublemma. ☐

We now prove Lemma 10.50. We no longer fix $i \in I_{\text{1-ridge}}$. Given $x \in S_4$, choose $p \in A_4$ and $i \in I_{\text{1-ridge}}$ so that $\pi_4 \circ \mathcal{E}^0(p) = x$ and $|\eta_i(p)| \leq 0.8\Delta$. Put $A^0_x = \text{Im}(d\mathcal{F}^0_{\eta_i(p)})$, a 1-plane in H, and let $A_x = x + A^0_x$ be the corresponding affine subspace through x.

We first show that under the constraints $\Sigma_4 < \overline{\Sigma}_4(\Gamma_4, \mathcal{M})$, $\epsilon_2 < \overline{\epsilon}_2(\Gamma_4, \mathcal{M})$ and $\Lambda < \overline{\Lambda}(\Gamma_4, \mathcal{M})$, the triple $(\widetilde{S}_4, S_4, r_4)$ is a $(2, \Gamma_4)$-cloudy 1-manifold. First, we verify condition (1) of [13, Definition 20.2]. Pick $x, y \in \widetilde{S}_4$, and choose

$$p \in (\pi_4 \circ \mathcal{E}^0)^{-1}(x) \cap \bigcup_{i \in I_{\text{1-ridge}}} |\eta_i|^{-1}\,[0, 0.8\Delta) \cap \psi_i^{-1}[0, 4.5) \tag{10.72}$$

and

$$q \in (\pi_4 \circ \mathcal{E}^0)^{-1}(y) \cap \bigcup_{i \in I_{\text{1-ridge}}} |\eta_i|^{-1}\,[0, 0.8\Delta) \cap \psi_i^{-1}[0, 4.5) \tag{10.73}$$

satisfying $r_4(x) = \Sigma_4\mathfrak{r}_p$ and $r_4(y) = \Sigma_4\mathfrak{r}_q$.

Suppose that $d(p, q) < \frac{\mathfrak{r}_p}{\Lambda}$. Since $\mathfrak{r}$ is Λ-Lipschitz, we have that $|\mathfrak{r}_p - \mathfrak{r}_q| \leq \mathfrak{r}_p$. In this case

$$|r_4(x) - r_4(y)| = \Sigma_4|\mathfrak{r}_p - \mathfrak{r}_q| \leq \Sigma_4\mathfrak{r}_p = r_4(x). \tag{10.74}$$

Now suppose that $d(p, q) \geq 2\Delta\mathfrak{r}_p$. We claim that if Λ is sufficiently small (as a function of Δ) then this implies that $d(p, q) \geq 1.95\Delta\mathfrak{r}_q$ as well. Suppose the claim is not true. Then $2\Delta\mathfrak{r}_p \leq d(p, q) \leq 1.95\Delta\mathfrak{r}_q$, so $\frac{\mathfrak{r}_p}{\mathfrak{r}_q} \leq \frac{1.95}{2}$. On the other hand, $\mathfrak{r}_q - \mathfrak{r}_p \leq \Lambda d(p, q) \leq 1.95\Lambda\Delta\mathfrak{r}_q$. Thus, $\frac{\mathfrak{r}_p}{\mathfrak{r}_q} \geq 1 - 1.95\Lambda\Delta$. This is a contradiction if Λ is sufficiently small. Therefore, we have that $d(p, q) \geq 1.95\Delta\mathfrak{r}_q$.

Let $i, j \in I_{\text{1-ridge}}$ such that $p \in |\eta_i|^{-1}[0, 0.8\Delta) \cap \eta_{i_P}^{-1}[0, 4.5\lambda_{p_i})$ and $q \in |\eta_j|^{-1}[0, 0.8\Delta) \cap \eta_{i_P}^{-1}[0, 4.5\lambda_{p_i})$. We have $\zeta_i(p) = \zeta_j(q) = 1$. Consider $d(q, p_i) \geq d(p, q) - d(p_i, p) \geq 1.95\Delta\mathfrak{r}_q -$

$\Delta\mathfrak{r}_{p_i}$. If Λ is sufficiently small, then $\frac{\mathfrak{r}_q}{\mathfrak{r}_{p_i}}$ can be made arbitrary close to 1. Hence, $d(q, p_i) \geq 0.9\Delta\mathfrak{r}_{p_i}$. In particular, $\zeta_i(q) = 0$. Similarly, $\zeta_j(p) = 0$. Then

$$\begin{aligned} |x - y| &= |\mathcal{E}^0(p) - \mathcal{E}^0(q)| \\ &\geq \max(\mathfrak{r}_{p_i}|\zeta_i(p) - \zeta_i(q)|, \mathfrak{r}_{p_j}|\zeta_j(p) - \zeta_j(q)|) \\ &= \max(\mathfrak{r}_{p_i}, \mathfrak{r}_{p_j}) \geq \frac{1}{2}\max(\mathfrak{r}_p, \mathfrak{r}_q) = \frac{1}{2\Sigma_4}\max(r_4(x), r_4(y)). \end{aligned} \tag{10.75}$$

So $|r_4(x) - r_4(y)| \leq |x - y|$ provided $\Sigma_4 \leq \frac{1}{4}$. Therefore, condition (1) of [13, Definition 20.2] is satisfied.

We now verify condition (2) of [13, Definition 20.2]. That is, for all $x \in S_4$, the rescaled pointed subset $(\frac{1}{r_4(x)}\widetilde{S}_4, x)$ is Γ_4-close in the pointed Hausdorff distance to $(\frac{1}{r_4(x)}A^0_x, x)$. Let $x \in S_4$, $i \in I_{\text{1-ridge}}$, and $p \in M$ be such that $\pi_4 \circ \mathcal{E}^0(p) = x$ with $|\eta_i(p)| \leq 0.7\Delta$ and $\eta_{i_P}(p) \leq 4\lambda_{p_i}$. Take $\Sigma = \frac{1}{100\Delta}$ in Sublemma 10.70 and let $q \in \text{Im}(\mathcal{E}^0) \cap B(x, \Sigma\Delta R_i)$. By Sublemma 10.70, $|x - \eta_i(q)| = |\eta_i(p) - \eta_i(q)| \leq 4\Delta_1^2\Sigma = 0.04\Delta$. Thus, $\eta_i(q) \leq 0.7\Delta + 0.04\Delta = 0.74\Delta$. Moreover, since $\Sigma < 0.1$, $d(p_i, q) \leq d(q, p) + d(p, p_i) \leq \Sigma\Delta R_i + 4r^1_{p_i} \leq (\Sigma + 4)r^1_{p_i} < 4.1r^1_{p_i}$. If ς_R is sufficiently small, then $\eta_{i_P}(q) < 4.2\lambda_{p_i}$. We have that

$$\text{Im}(\pi_4 \circ \mathcal{E}^0) \cap B(x, \Sigma\Delta R_i) \subset \text{Im}\left(\pi_4 \circ \mathcal{E}^0\big|_{|\eta_i(p)|^{-1}[0,0.74\Delta)\cap\eta_{i_P}^{-1}[0,4.2\lambda_{p_i})}\right). \tag{10.76}$$

Thus, we can restrict our attention to the action of $\mathcal{E}^0$ on $|\eta_i(p)|^{-1}[0, 0.74\Delta) \cap \eta_{i_P}^{-1}[0, 4.2\lambda_{p_i})$. Consider that $\text{Im}(\mathcal{F}^0\big|_{B(0,0.74\Delta)}, x)$ is the restriction to $B(0, 0.74\Delta)$ of the graph of a function $G^0_i : H'_i \to (H'_i)^\perp$ since $T_{ii} = \text{Id}$ and $\zeta_i\big|_{B(0,0.74\Delta)} = 1$. Furthermore, in view of the universality of the functions $\{\Phi_j\}_{j\in J}$ and the bound on the cardinality of J, there are uniform C^1-estimates on G^0_i. Hence, we can find Σ_4 (as a function of Γ_4 and $\mathcal{M}$) which ensures that $\left(\frac{1}{r_4(x)}\text{Im}(\mathcal{F}^0\big|_{B(0,0.74\Delta)}), x\right)$ is $\frac{\Gamma_4}{2}$-close in the pointed Hausdorff topology to $x + \text{Im}(d\mathcal{F}^0_p)$. Finally, if the parameter ϵ_2 of Sublemma 10.60 is sufficiently small, then we can ensure that $\left(\frac{1}{r_4(x)}\text{Im}(\mathcal{E}^0), x\right)$ is Γ_4-close in the pointed Hausdorff topology to $x + \text{Im}(d\mathcal{F}^0_p)$. Thus, condition (2) of [13, Definition 20.2] is satisfied.

To finish the proof of Lemma 10.50, equation (10.51) is satisfied if the parameter ϵ_2 of Sublemma 10.60 is sufficiently small. Equation (10.52) is equivalent to upper and lower bounds on the eigenvalues of the matrix $(\pi_{A^0_x} \circ D\mathcal{E}^0_p)(\pi_{A^0_x} \circ D\mathcal{E}^0_p)^*$, which acts on the 1-dimensional space A^0_x. In view of Sublemma 10.60 and the definition of A^0_x, it is sufficient to show upper and lower bounds on the eigenvalues of $D\mathcal{F}^0_{\eta_i(p)}(D\mathcal{F}^0_{\eta_i(p)})^*$ acting on A^0_x. In terms of the function G^0_i, these are the same as the eigenvalues of $\text{Id} + ((DG^0_i)_{\eta_i(p)})^*(DG^0_i)_{\eta_i(p)}$ acting on $\mathbb{R}$. The eigenvalues are clearly bounded by 1. In view of the C^1-bound on the eigenvalues in terms of $\dim(H)$, which in turn is bounded above in terms of $\mathcal{M}$. This shows equation (10.52).

Finally, given $i \in I_{\text{1-ridge}}$, we can write $\frac{1}{R_i}\mathcal{F}^0$ on $\overline{B(0, 0.74\Delta)} \subset \mathbb{R}$ in the form $\frac{1}{R_i}\mathcal{F}^0 = (\mathrm{Id}, \frac{1}{R_i}\widehat{\mathcal{E}}_i^0)$ with respect to the orthogonal decomposition $H = H_i' \oplus (H_i')^\perp$. We use this to define $\widehat{\mathcal{E}}_i^0$. Equation (10.54) is a consequence of Sublemma 10.60. The last statement of Lemma 10.50 follows from the definition of A_x^0.

10.7 Structure of $\mathcal{E}^0$ near the slim 1-stratum

Recall that $Q_5 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}}$, and $\pi_5 : H \to Q_5$ is the orthogonal projection.

Define

$$\widetilde{A}_5 = \bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| \leq 0.8\Delta\}, \quad A_5 = \bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| \leq 0.7\Delta\}, \tag{10.77}$$

and

$$\widetilde{S}_5 = (\pi_5 \circ \mathcal{E}^0)(\widetilde{A}_5), \quad S_5 = (\pi_5 \circ \mathcal{E}^0)(A_5). \tag{10.78}$$

Let $\Sigma_5, \Gamma_5 > 0$ be new parameters. Define $r_5 : \widetilde{S}_5 \to (0, \infty)$ by putting $r_5(x) = \Sigma_5 \mathfrak{r}_p$ for some $p \in (\pi_5 \circ \mathcal{E}^0)^{-1}(x) \cap \widetilde{A}_5$.

The analog of Lemma 10.8 for the region near slim 1-stratum points is:

Lemma 10.79. *There is a constant $\Omega_5 = \Omega_5(\mathcal{M})$ so that under the constraints $\Sigma_5 < \overline{\Sigma}_5(\Gamma_5, \mathcal{M})$, $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\beta_2, \Gamma_5, \Sigma_5, \mathcal{M}, \Delta)$, $\beta_1 < \overline{\beta}_1(\beta_2, \Gamma_5, \Sigma_5, \mathcal{M}, \Delta)$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\beta_2, \Gamma_5, \Sigma_5, \mathcal{M}, \Delta)$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_2, \Gamma_5, \Sigma_5, \mathcal{M}, \Delta)$, and $\Lambda < \overline{\Lambda}(\beta_2, \Gamma_5, \Sigma_5, \mathcal{M}, \Delta)$, the following holds.*

(1) *The triple $(\widetilde{S}_5, S_5, r_5)$ is a $(2, \Gamma_5)$-cloudy 1-manifold.*

(2) *The affine subspaces $\{A_x\}_{x \in S_5}$ inherent in the definition of the cloudy 5-manifold can be chosen to have the following property. Pick $p \in A_5$ and put $x = (\pi_5 \circ \mathcal{E}^0)(p) \in S_5$. Let $A_x^0 \subset Q_5$ be the linear subspace parallel to A_x (i.e., $A_x = A_x^0 + x$) and let $\pi_{A_x^0} : H \to A_x^0$ denote orthogonal projection onto A_x^0. Then*

$$\|D(\pi_5 \circ \mathcal{E}^0)_p - \pi_{A_x^0} \circ D(\pi_5 \circ \mathcal{E}^0)_p\| \leq \Gamma_5 \tag{10.80}$$

and

$$\Omega_5^{-1}\|v\| \leq \|(\pi_{A_x^0} \circ D(\pi_5 \circ \mathcal{E}^0)_p)(v)\| \leq \Omega_5\|v\| \tag{10.81}$$

for every $v \in T_pM$ which is orthogonal to $\ker(\pi_{A_x^0} \circ D(\pi_5 \circ \mathcal{E}^0)_p)$.

(3) *Given $i \in I_{\text{1-slim}}$, there is a smooth map $\widehat{\mathcal{E}}_i^0 : (\overline{B(0, 0.8\Delta)} \subset \mathbb{R}) \to (H_i')^\perp \cap Q_5$ such that*

$$\|D\widehat{\mathcal{E}}_i^0\| \leq \Omega_5 R_i \tag{10.82}$$

and on the subset $\{|\eta_i| \leq 0.8\Delta\}$, we have

$$\left\| \frac{1}{R_i}\pi_5 \circ \mathcal{E}^0 - \left(\eta_i, \frac{1}{R_i}\widehat{\mathcal{E}}_i^0 \circ \eta_i\right) \right\|_{C^1} < \Gamma_5. \tag{10.83}$$

Furthermore, if $x \in S_5$, then there are some $i \in I_{\text{1-slim}}$ and $p \in \{|\eta_i| \leq 0.7\Delta\}$ such that $x = (\pi_5 \circ \mathcal{E}^0)(p)$ and $A^0_x = \operatorname{Im}(\mathrm{Id}, \frac{1}{R_i}(D\widehat{\mathcal{E}}^0_i)_{\eta_i(p)})$.

Proof. The proof is similar to the proof of Lemma 10.8. □

10.8 Structure of $\mathcal{E}^0$ near the 0-stratum

Lemma 10.84. *For $i \in I_{\text{0-stratum}}$, the only nonzero componenet of the map $\pi_6 \circ \mathcal{E}^0 : M \to Q_6 = H_{\text{0-stratum}}$ in the region $\{\eta_i \in [0.3, 0.8]\}$ is $\mathcal{E}^0_i$, where it conincides with $(R_i\eta_i, R_i)$.*

Proof. The lemma follows directly from the definitions of $\mathcal{E}^0$ and ζ_i, $i \in I_{\text{0-stratum}}$. □

— 11 —

Adjusting the map to Euclidean space

In this chapter, we modify $\mathcal{E}^0 : M \to H$ slightly to get a new map $\mathcal{E} : M \to H$ which is a submersion in different parts of M. In Chapter 12, we will use the submersion to decompose M into fibered pieces which are compatible along the intersections. The main result of this chapter is Proposition 11.1. The rest of the chapter is the proof of the proposition.

Let $c_{\text{adjust}} > 0$ be a new parameter.

Proposition 11.1. *Under the constraints imposed in this and prior chapters, there is a smooth map $\mathcal{E} : M \to H$ with the following properties:*

(1) *For every $p \in M$,*

$$\|\mathcal{E}(p) - \mathcal{E}^0(p)\| < c_{\text{adjust}}\mathfrak{r}_p \quad \textit{and} \quad \|D\mathcal{E}_p - D\mathcal{E}^0_p\| < c_{\text{adjust}}. \tag{11.2}$$

(2) *For $j \in \{1, 2, 3, 4, 5\}$, the restriction of $\pi_j \circ \mathcal{E} : M \to Q_j$ to the region $U_j \subset M$ is a submersion to a k_j-manifold $W_j \subset Q_j$, where*

$$\begin{aligned} U_1 &= \bigcup_{i \in I_{3-\text{stratum}}} \{|\eta_i| < 5\}, \qquad (11.3)\\ U_2 &= \bigcup_{i \in I_{2-\text{edge}}} \{|\eta_i| < 5\Delta, \eta_{E'} < 5\Delta\},\\ U_3 &= \bigcup_{i \in I_{2-\text{slim}}} \{|\eta_i| < 5 \cdot 10^5\Delta\},\\ U_4 &= \bigcup_{i \in I_{1-\text{ridge}}} \{|\eta_i| < 0.5\Delta, \eta_{i_P} < 3\lambda_{p_i}\},\\ U_5 &= \bigcup_{i \in I_{1-\text{slim}}} \{|\eta_i| < 0.5\Delta\}, \end{aligned}$$

and $k_1 = 3$, $k_2 = k_3 = 2$, $k_4 = k_5 = 1$.

Let $c_{\text{3-stratum}}$, $c_{\text{2-slim}}$, $c_{\text{2-edge}}$, $c_{\text{1-ridge}}$, $c_{\text{1-slim}} > 0$, and $\Xi_i > 0$ for $i \in \{1, 2, 3, 4, 5\}$ be additional new parameters. We will use these parameters in the following sections.

11.1 Overview of the proof of Proposition 11.1

The maps $\mathcal{E}^0$ and $\pi_i \circ \mathcal{E}^0$, $i \in \{1,2,3,4,5\}$, as defined in Chapter 10 behave like a "rough fibration." The goal is to promote these rough fibrations to fibrations in such a way that they are compatible on their overlap. We will do this by producing a sequence of maps $\mathcal{E}^j : M \to H$, for $j \in \{1,2,3,4,5\}$, which are successive adjustments of $\mathcal{E}^0$. This proof is an analog of the proof of [13, Proposition 13.1].

The idea for constructing $\mathcal{E}^j$ from $\mathcal{E}^{j-1}$, for $j \in \{1,2,3,4,5\}$, is as follows. First we consider the orthogonal splitting $H = Q_j \oplus Q_j^\perp$ of H. Let $\pi_j = \pi_{1,j} : H \to Q_j$ and $\pi_j^\perp : H \to Q_j^\perp$ be the orthogonal projections. In Chapter 10, we introduced a pair of subsets $(\widetilde{A}_j, A_j)$ in M whose image $(\widetilde{S}_j, S_j)$ under the composition $\pi_j \circ \mathcal{E}^{j-1}$ is a cloudy k_j-manifold in Q_j. We can think of the restriction of $\mathcal{E}^{j-1}$ to A_j as a "rough submersion" over $(\widetilde{S}_j, S_j)$. From a property of a cloudy manifold (see [13, Lemma 20.2]), there is a k_j-dimensional manifold $W_j \subset Q_j$ near $(\widetilde{S}_j, S_j)$ and a projection map P_j onto W_j, defined in a neighborhood $\widehat{W}_j$ of W_j. Thus, we have a well-defined map

$$H \supset \widehat{W}_j \times Q_j^\perp \xrightarrow{(P_j \circ \pi_j, \pi_j^\perp)} Q_j \oplus Q_j^\perp = H. \tag{11.4}$$

Then, we use a partition of unity to blend the composition $(P_j \circ \pi_j, \pi_j^\perp) \circ \mathcal{E}^{j-1}$ with $\mathcal{E}^{j-1}$ to obtain $\mathcal{E}^j : M \to H$. Under this construction, at a point $p \in M$, $|\mathcal{E}^j(p) - \mathcal{E}^{j-1}(p)| < (\text{const})\mathfrak{r}_p$ and $|D\mathcal{E}^j_p - D\mathcal{E}^{j-1}_p| <$ const, for some small constants, and $\mathcal{E}^j$ preserves the submersions defined by $\mathcal{E}^{j-1}$.

11.2 Adjusting the map near the 3-stratum

We start the adjustment process from the 3-stratum.

We take $Q_1 = H$, $Q_1^\perp = \{0\}$ and let $\widetilde{A}_1, A_1, \widetilde{S}_1, S_1$, and $r_1 : \widetilde{S}_1 \to (0,\infty)$ be as in Section 10.3.

From Lemma 10.8, $(\widetilde{S}_1, S_1, r_1)$ is a $(2, \Gamma_1)$ cloudy 3-manifold. By [13, Lemma 20.2], there is a 3-manifold $W_1^0 \subset Q_1$ so that the conclusion of [13, Lemma 20.2] holds, where the parameter ϵ in [13, Lemma 20.2] is given by $\epsilon = \Xi_1 = \Xi_1(\Gamma_1)$. In particular, there is a well-defined nearest point projection

$$P_1 : N_{r_1}(S_1) = \widehat{W}_1 \to W_1^0 \tag{11.5}$$

where N_{r_1} is a variable thickness neighborhood as defined in Section 1.3.

First, we define a cutoff function.

Lemma 11.6. *Under the constraint* $c_{\text{3-stratum}} < \bar{c}_{\text{3-stratum}}$*, there is a smooth function* $\psi_1 : H \to [0,1]$ *with the following properties:*

(1)

$$\begin{aligned} \psi_1 \circ \mathcal{E}^0 &\equiv 1 \quad \textit{in} \quad \bigcup_{i \in I_{\text{3-stratum}}} \{|\eta_i| < 6\} \ \textit{and} \\ \psi_1 \circ \mathcal{E}^0 &\equiv 0 \quad \textit{outside} \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 7\}. \end{aligned} \tag{11.7}$$

(2) $\operatorname{supp}(\psi_1) \cap \operatorname{im}(\mathcal{E}^0) \subset \widehat{W}_1$.

(3) *There is a constant* $\Omega_1' = \Omega_1'(\mathcal{M})$ *such that*

$$|(D\psi_1)_x| < \Omega_1' x_{\mathfrak{r}}^{-1} \tag{11.8}$$

for all $x \in \operatorname{im}(\mathcal{E}^3)$.

Proof. The proof is the similar to the proof of [13, Lemma 13.6] □

Define $\Psi_1 : H \to H$ by $\Psi_1(x) = x$ if $x \notin \widehat{W}_1$ and

$$\Psi_1(x) = \psi_1(x) P_1(x) + (1 - \psi_1(x))x \tag{11.9}$$

otherwise. Put $\mathcal{E}^1 = \Psi_1 \circ \mathcal{E}^0$.

Lemma 11.10. *Under the constraints* $\Sigma_1 < \overline{\Sigma}_1(\Omega_1, c_{\text{3-stratum}})$*,* $\Gamma_1 < \overline{\Gamma}_1(\Omega_1, c_{\text{3-stratum}})$*, and* $\Xi_1 < \overline{\Xi}_1(c_{\text{3-stratum}})$*, we have:*

(1) $\mathcal{E}^1$ *is smooth.*

(2) *For all* $p \in M$*,*

$$\|\mathcal{E}^1(p) - \mathcal{E}^0(p)\| < c_{\text{3-stratum}}\, \mathfrak{r}(p) \quad \textit{and} \quad \|D\mathcal{E}^1(p) - D\mathcal{E}^0(p)\| < c_{\text{3-stratum}}. \tag{11.11}$$

(3) *The restriction of* $\mathcal{E}^1$ *to* $\bigcup_{i \in I_{\text{3-stratum}}} \{|\eta_i| < 6\}$ *is a submersion to* W_1^0*.*

Proof. The proof is the similar to the proof of [13, Lemma 13.15] □

11.3 Adjusting the map near the edge 2-stratum

Recall that $Q_2 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}} \oplus H_{\text{2-slim}} \oplus H_{\text{2-edge}}$ and $\pi_2 : H \to Q_2$ is an orthogonal projection. We let $\widetilde{A}_2, A_2, \widetilde{S}_2, S_2$, and $r_2 : \widetilde{S}_2 \to (0, \infty)$ be as in Section 10.4.

Thus, by Lemma 10.36, $(\widetilde{S}_2, S_2, r_2)$ is a $(2, \Gamma_2)$ cloudy 2-manifold. By [13, Lemma 20.2], there is a 2-manifold $W_2^0 \subset Q_2$ so that the conclusion of [13, Lemma 20.2] holds, where the parameter ϵ in [13, Lemma 20.2] is given by $\Xi_2 = \Xi_2(\Gamma_2)$. In particular, there is a well-defined nearest point projection

$$P_2 : N_{r_2}(S_2) = \widehat{W}_2 \to W_2^0 \tag{11.12}$$

where N_{r_2} is a variable thickness neighborhood as defined in Section 1.3.

Lemma 11.13. *Under the constraint $c_{\text{2-edge}} < \overline{c}_{\text{2-edge}}$, there is a smooth function $\psi_2 : H \to [0,1]$ with the following properties:*

(1)

$$\psi_2 \circ \mathcal{E}^1 \equiv 1 \quad \textit{in} \quad \bigcup_{i \in I_{\text{2-edge}}} \{|\eta_i| < 6\Delta, \eta_{E'} < 6\Delta\} \textit{ and} \tag{11.14}$$
$$\psi_2 \circ \mathcal{E}^1 \equiv 0 \quad \textit{outside} \bigcup_{i \in I_{\text{2-edge}}} \{|\eta_i| < 7\Delta, \eta_{E'} < 7\Delta\}.$$

(2) $\operatorname{supp}(\psi_2) \cap \operatorname{im}(\mathcal{E}^1) \subset \widehat{W}_2 \times Q_2^\perp$.

(3) *There is a constant $\Omega_2' = \Omega_2'(\mathcal{M})$ such that*

$$|(D\psi_2)_x| < \Omega_2' x_{\mathfrak{r}}^{-1} \tag{11.15}$$

for all $x \in \operatorname{im}(\mathcal{E}^1)$.

Proof. The proof is similar to the proof of [13, Lemma 13.21] □

We can assume that $\widehat{W}_2 \subset \{x_{\mathfrak{r}} > 0\}$. Define $\Psi_2 : \{x_{\mathfrak{r}} > 0\} \to \{x_{\mathfrak{r}} > 0\}$ by $\Psi_2(x) = x$ if $\pi_2(x) \neq \widehat{W}_2$ and

$$\Psi_2(x) = (\psi_2(x) P_2(\pi_2(x)) + (1 - \psi_2(x))\pi_2(x), \pi_2^\perp(x)) \tag{11.16}$$

otherwise. Put $\mathcal{E}^2 = \Psi_2 \circ \mathcal{E}^1$.

Lemma 11.17. *Under the constraints, $\Sigma_2 < \overline{\Sigma}_2(\Omega_2, c_{\text{2-edge}})$, $\Gamma_2 < \overline{\Gamma}_2(\Omega_2, c_{\text{2-edge}})$, $\Xi_2 < \overline{\Xi}_2(c_{\text{2-edge}})$, and $c_{\text{3-stratum}} < \overline{c}_{\text{3-stratum}}(c_{\text{2-edge}})$, we have:*

(1) *$\mathcal{E}^2$ is smooth.*

(2) *For all $p \in M$,*

$$\|\mathcal{E}^2(p) - \mathcal{E}^0(p)\| < c_{\text{2-edge}}\,\mathfrak{r}(p) \quad \textit{and} \quad \|D\mathcal{E}^2(p) - D\mathcal{E}^0(p)\| < c_{\text{2-edge}}. \tag{11.18}$$

(3) *The restriction of $\pi_2 \circ \mathcal{E}^2$ to $\bigcup_{i \in I_{\text{2-edge}}} \{|\eta_i| < 6\Delta, \eta_{E'} < 6\Delta\}$ is a submersion to W_2^0.*

Proof. The proof is similar to the proof of [13, Lemma 13.34] □

11.4 Adjusting the map near the slim 2-stratum

Recall that $Q_3 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}} \oplus H_{\text{2-slim}}$ and $\pi_3 : H \to Q_3$ is an orthogonal projection. We let $\widetilde{A}_3, A_3, \widetilde{S}_3, S_3$, and $r_3 : \widetilde{S}_3 \to (0,\infty)$ be as in Section 10.5.

Thus, by Lemma 10.43, $(\widetilde{S}_3, S_3, r_3)$ is a $(2, \Gamma_3)$ cloudy 2-manifold. By [13, Lemma 20.2], there is a 2-manifold $W_3^0 \subset Q_3$ so that the conclusion of [13, Lemma 20.2] holds, where the parameter ϵ in [13, Lemma 20.2] is given by $\Xi_3 = \Xi_3(\Gamma_3)$. In particular, there is a well-defined nearest point projection

$$P_3 : N_{r_3}(S_3) = \widehat{W}_3 \to W_3^0 \tag{11.19}$$

where N_{r_3} is a variable thickness neighborhood as defined in Section 1.3.

Lemma 11.20. *Under the constraint $c_{\text{2-slim}} < \bar{c}_{\text{2-slim}}$, there is a smooth function $\psi_3 : H \to [0,1]$ with the following properties:*

(1)
$$\psi_3 \circ \mathcal{E}^2 \equiv 1 \quad \textit{in} \quad \bigcup_{i \in I_{\text{2-slim}}} \{|\eta_i| < 6 \cdot 10^5 \Delta\} \; \textit{and} \tag{11.21}$$
$$\psi_3 \circ \mathcal{E}^2 \equiv 0 \quad \textit{outside} \bigcup_{i \in I_{\text{2-slim}}} \{|\eta_i| < 7 \cdot 10^5 \Delta\}.$$

(2) $\operatorname{supp}(\psi_3) \cap \operatorname{im}(\mathcal{E}^2) \subset \widehat{W}_3 \times Q_3^\perp$.

(3) *There is a constant $\Omega_3' = \Omega_3'(\mathcal{M})$ such that*
$$|(D\psi_3)_x| < \Omega_3' x_{\mathfrak{r}}^{-1} \tag{11.22}$$
for all $x \in \operatorname{im}(\mathcal{E}^2)$.

Proof. The proof is similar to the proof of [13, Lemma 13.38] □

Define $\Psi_3 : H \to H$ by $\Psi_3(x) = x$ if $\pi_3(x) \notin \widehat{W}_3$ and

$$\Psi_3(x) = (\psi_3(x) P_3(\pi_3(x)) + (1 - \psi_3(x))\pi_3(x), \pi_3^\perp(x)) \tag{11.23}$$

otherwise. Put $\mathcal{E}^3 = \Psi_3 \circ \mathcal{E}^2$.

Lemma 11.24. *Under the constraints, $\Sigma_3 < \overline{\Sigma}_3(\Omega_3, c_{\text{2-slim}})$, $\Gamma_3 < \overline{\Gamma}_3(\Omega_3, c_{\text{2-slim}})$, $\Xi_3 < \overline{\Xi}_3(c_{\text{2-slim}})$, and $c_{\text{2-edge}} < \bar{c}_{\text{2-edge}}(c_{\text{2-slim}})$, we have:*

(1) *$\mathcal{E}^3$ is smooth.*

(2) *For all $p \in M$,*
$$\|\mathcal{E}^3(p) - \mathcal{E}^0(p)\| < c_{\text{2-slim}}\, \mathfrak{r}(p) \quad \textit{and} \quad \|D\mathcal{E}^3(p) - D\mathcal{E}^0(p)\| < c_{\text{2-slim}}. \tag{11.25}$$

(3) *The restriction of $\pi_3 \circ \mathcal{E}^3$ to $\bigcup_{i \in I_{\text{2-slim}}} \{|\eta_i| < 6 \cdot 10^5 \Delta\}$ is a submersion to W_3^0.*

Proof. The proof is similar to the proof of [13, Lemma 13.34]. □

11.5 Adjusting the map near the ridge 1-stratum

Recall that $Q_4 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}} \oplus H_{\text{1-ridge}}$ and $\pi_4 : H \to Q_4$ is an orthogonal projection. We let $\widetilde{A}_4, A_4, \widetilde{S}_4, S_4$, and $r_4 : \widetilde{S}_4 \to (0, \infty)$ be as in Section 10.6.

Thus, by Lemma 10.50, $(\widetilde{S}_4, S_4, r_4)$ is a $(2, \Gamma_4)$ cloudy 1-manifold. By [13, Lemma 20.2], there is a 1-manifold $W_4^0 \subset Q_4$ so that the conclusion of [13, Lemma 20.2] holds, where the parameter ϵ in [13, Lemma 20.2] is given by $\Xi_4 = \Xi_4(\Gamma_4)$. In particular, there is a well-defined nearest point projection

$$P_4 : N_{r_4}(S_4) = \widehat{W}_4 \to W_4^0 \tag{11.26}$$

where N_{r_4} is a variable thickness neighborhood as defined in Section 1.3.

Lemma 11.27. *Under the constraint* $c_{\text{1-ridge}} < \overline{c}_{\text{1-ridge}}$*, there is a smooth function* $\psi_4 : H \to [0, 1]$ *with the following properties:*

(1)

$$\psi_4 \circ \mathcal{E}^3 \equiv 1 \quad \textit{in} \quad \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.6\Delta, \eta_{i_P} < 3.5\lambda_{p_i}\} \textit{ and} \tag{11.28}$$

$$\psi_4 \circ \mathcal{E}^3 \equiv 0 \quad \textit{outside} \quad \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.7\Delta, \eta_{i_P} < 4\lambda_{p_i}\}.$$

(2) $\text{supp}(\psi_4) \cap \text{im}(\mathcal{E}^3) \subset \widehat{W}_4 \times Q_4^{\perp}$.

(3) *There is a constant* $\Omega_4' = \Omega_4'(\mathcal{M})$ *such that*

$$|(D\psi_4)_x| < \Omega_4' x_{\mathfrak{r}}^{-1} \tag{11.29}$$

for all $x \in \text{im}(\mathcal{E}^3)$.

Proof. If the parameter $c_{\text{2-slim}}$ is sufficiently small and Δ is sufficiently large, then by Lemma 11.24, $\|\mathcal{E}^3(p) - \mathcal{E}^0(p)\| < c_{\text{2-slim}}\mathfrak{r}(p)$. Hence, $\mathcal{E}^3(p) \in \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.6\Delta, \eta_{i_P} < 3.5\lambda_{p_i}\}$ implies that $\mathcal{E}^0(p) \in \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.61\Delta, \eta_{i_P} < 3.6\lambda_{p_i}\}$. Also, $\mathcal{E}^3(p) \notin \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.7\Delta, \eta_{i_P} < 4\lambda_{p_i}\}$ implies that $\mathcal{E}^0(p) \notin \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.69\Delta, \eta_{i_P} < 3.9\lambda_{p_i}\}$.

Define $\psi_4 : H \to [0, 1]$ by

$$\begin{aligned}\psi_4(x) = 1 - \Phi_{\frac{1}{2},1}\Bigg(\sum_{\{i \in I_{\text{1-ridge}}, x_i'' > 0\}} &\left[\Phi_{0.61\Delta, 0.65\Delta}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right)\right] \\ &\cdot \left[\Phi_{3.6\lambda_{p_i}, 3.7\lambda_{p_i}}\left(\frac{|x_{i_P}'|}{x_{i_P}''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_{i_P}''}{R_{i_P}}\right)\right)\right]\Bigg]\Bigg)\end{aligned} \tag{11.30}$$

where $i_P \in I_P$ is the index corresponding to $i \in I_{\text{1-ridge}}$. ψ_4 is clearly smooth. Also note that $x_i'' > 0$ if and only if $x_{i_P}'' > 0$.

To prove part (1), it suffices to show that

$$\begin{aligned} &\psi_4 \circ \mathcal{E}^0 \equiv 1 \quad \text{in} \quad \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.61\Delta, \eta_{i_P} < 3.6\lambda_{p_i}\}, \text{ and} \\ &\psi_4 \circ \mathcal{E}^0 \equiv 0 \quad \text{outside} \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.69\Delta, \eta_{i_P} < 3.9\lambda_{p_i}\}. \end{aligned} \tag{11.31}$$

Suppose that $i \in I_{\text{1-ridge}}$, $|\eta_i(p)| < 0.61\Delta$, and $\eta_{i_P} < 3.6\lambda_{p_i}$. Put $x = \mathcal{E}^0(p)$. Recall that $x_i'' = R_i\zeta_i(p)$ where ζ_i is given by (8.10). Hence,

$$\begin{aligned} &\frac{x_i''}{R_i} = \zeta_i(p) = 1, \\ &1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right) = 1, \text{ and} \\ &\Phi_{0.61\Delta,0.65\Delta}\left(\frac{|x_i'|}{x_i''}\right) = \Phi_{0.61\Delta,0.65\Delta}\left(|\eta_i(p)|\right) = 1. \end{aligned} \tag{11.32}$$

Also, $x_{i_P}'' = R_{i_P}\zeta_{i_P}(p)$ where $\zeta_{i_P} = \zeta_{\psi_{p_i}}$ is given by (8.14). We have

$$\begin{aligned} &\frac{x_{i_P}''}{R_{i_P}} = \zeta_{i_P}(p) = 1, \\ &1 - \Phi_{\frac{1}{2},1}\left(\frac{x_{i_P}''}{R_{i_P}}\right) = 1, \text{ and} \\ &\Phi_{3.6\lambda_{p_i},3.7\lambda_{p_i}}\left(\frac{|x_{i_P}'|}{x_{i_P}''}\right) = \Phi_{3.6\lambda_{p_i},3.7\lambda_{p_i}}\left(\eta_{i_P}\right) = 1. \end{aligned} \tag{11.33}$$

Therefore, $\psi_4(x) = 1$. Now suppose that for all $i \in I_{\text{1-ridge}}$, either

(i) $\zeta_i(p) = 0$ or $\zeta_{i_P}(p) = 0$, or

(ii) $\zeta_i(p) > 0$, $\zeta_{i_P}(p) > 0$, and $|\eta_i(p)| \geq 0.69\Delta$, or

(iii) $\zeta_i(p) > 0$, $\zeta_{i_P}(p) > 0$, $|\eta_i(p)| < 0.69\Delta$, and $\eta_{i_P} \geq 3.9\lambda_{p_i}$.

If $\zeta_i(p) = 0$ or $\zeta_{i_P}(p) = 0$, then

$$\left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_{i_P}''}{R_{i_P}}\right)\right) = 0. \tag{11.34}$$

If $\zeta_i(p) > 0$, $\zeta_{i_P}(p) > 0$, and $|\eta_i(p)| \geq 0.69\Delta$, then

$$\Phi_{0.61\Delta,0.65\Delta}\left(\frac{|x_i'|}{x_i''}\right) = 0. \tag{11.35}$$

If $\zeta_i(p) > 0$, $\zeta_{i_P}(p) > 0$, $|\eta_i(p)| < 0.69$, and $\eta_{i_P} \geq 3.9\lambda_{p_i}$, then

$$\Phi_{3.6\lambda_{p_i},3.7\lambda_{p_i}}\left(\frac{|x'_{i_P}|}{x''_{i_P}}\right) = 0. \tag{11.36}$$

In any case, $\psi_4(x) = 0$. This proves part (1) of the lemma.

To prove part (2), suppose $x = \mathcal{E}^3(p)$ and $\psi_4(x) > 0$. From part (1), $|\eta_i(p)| < 0.7\Delta$ and $\eta_{i_P} < 4\lambda_{p_i}$ for some $i \in I_{\text{1-ridge}}$. Thus, $p \in A_4$ and $x \in \mathcal{E}^3(A_4)$. Let $\widetilde{x} = (\pi_4 \circ \mathcal{E}^0)(p) \in (\pi_4 \circ \mathcal{E}^0)(A_4) = S_4$. Recall that $r_4(\widetilde{x}) = \Sigma_4 \mathfrak{r}_{\widetilde{p}}$ for some $\widetilde{p} \in (\pi_4 \circ \mathcal{E}^0)^{-1}(\widetilde{x}) \cap \widetilde{A}_4$. If Λ is sufficiently small, we have that $r_4(\widetilde{x}) \geq \frac{1}{2}\Sigma_4 \mathfrak{r}_p$. By Lemma 11.24, if $c_{\text{2-slim}} < \overline{c}_{\text{2-slim}}(\Sigma_4)$, then $\|\pi_4(x) - \widetilde{x}\| = \|\pi_4 \circ \mathcal{E}^3(p) - \pi_4 \circ \mathcal{E}^0(p)\| \leq \|\mathcal{E}^3(p) - \mathcal{E}^0(p)\| < c_{\text{2-slim}}\, \mathfrak{r}(p) < r_4(\widetilde{x})$. Hence, $\pi_4(x) \in N_{r_4}(S_4) = \widehat{W}_4 \subset Q_4$. Therefore, $x \in \widehat{W}_4 \times Q_4^{\perp}$. This proves part (2) of the lemma.

To prove part (3), suppose that $x = \mathcal{E}^3(p)$. If $x''_i > 0$, then $\zeta_i(p) > 0$. The number of indices $i \in I_{\text{1-ridge}}$ such that $x''_i > 0$ is bounded by the multiplicity of the ridge 1-stratum cover. For the remaining indices $j \in I_{\text{1-ridge}}$ such that $x''_j \leq 0$, the quantity $1 - \Phi_{\frac{1}{2},1}\left(\frac{x''_j}{R_j}\right)$ vanishes near x. Thus, by the chain rule, it suffices to bound the differentials of

$$\begin{aligned} &\Phi_{0.61\Delta,0.65\Delta}\left(\frac{|x'_i|}{x''_i}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x''_i}{R_i}\right)\right), \quad \text{and} \\ &\Phi_{3.6\lambda_{p_i},3.7\lambda_{p_i}}\left(\frac{|x'_{i_P}|}{x''_{i_P}}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x''_{i_P}}{R_{i_P}}\right)\right) \end{aligned} \tag{11.37}$$

for each $i \in I_{\text{1-ridge}}$ and its corresponding $i_P \in I_P$ for which $x''_i > 0$. Both differentials are non-zero only when $\frac{|x'_i|}{x''_i} \leq 0.65\Delta$, $\frac{x''_i}{R_i} \geq \frac{1}{2}$, $\frac{|x'_{i_P}|}{x''_{i_P}} \leq 3.7\lambda_{p_i}$, and $\frac{x''_{i_P}}{R_{i_P}} \geq \frac{1}{2}$. In this case, R_i will be comparable to $x_{\mathfrak{r}}$ and the estimate (11.29) follows. This proves part (3) of the lemma. □

Define $\Psi_4 : H \to H$ by $\Psi_4(x) = x$ if $\pi_4(x) \notin \widehat{W}_4$ and

$$\Psi_4(x) = (\psi_4(x)P_4(\pi_4(x)) + (1 - \psi_4(x))\pi_4(x), \pi_4^{\perp}(x)) \tag{11.38}$$

otherwise. Put $\mathcal{E}^4 = \Psi_4 \circ \mathcal{E}^3$.

Lemma 11.39. *Under the constraints,* $\Sigma_4 < \overline{\Sigma}_4(\Omega_4, c_{\text{1-ridge}})$*,* $\Gamma_4 < \overline{\Gamma}_4(\Omega_4, c_{\text{1-ridge}})$*,* $\Xi_4 < \overline{\Xi}_4(c_{\text{1-ridge}})$*, and* $c_{\text{2-slim}} < \overline{c}_{\text{2-slim}}(c_{\text{1-ridge}})$*, we have:*

1. $\mathcal{E}^4$ *is smooth.*
2. *For all* $p \in M$*,*

$$\|\mathcal{E}^4(p) - \mathcal{E}^0(p)\| < c_{\text{1-ridge}}\, \mathfrak{r}(p) \quad \textit{and} \quad \|D\mathcal{E}^4(p) - D\mathcal{E}^0(p)\| < c_{\text{1-ridge}}. \tag{11.40}$$

3. *The restriction of* $\pi_4 \circ \mathcal{E}^4$ *to* $\bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.6\Delta, \eta_{i_P} < 3.5\lambda_{p_i}\}$ *is a submersion to* W_4^0*.*

Proof. Part (1) of the lemma follows from part (2) of Lemma 11.27.

(2). Given $p \in M$, put $x = \mathcal{E}^3(p)$. We have

$$\begin{aligned} \|\mathcal{E}^4(p) - \mathcal{E}^3(p)\| = \|\Psi_4(x) - x\| &= \|(\psi_4(x)(P_4(\pi_4(x)) - \pi_4(x)), 0)\| \\ &\le |\psi_4(x)| \cdot |P_4(\pi_4(x)) - \pi_4(x)|. \end{aligned} \tag{11.41}$$

Now $|\psi_4(x)| \le 1$. From [13, Lemma 20.2(1)], $|P_4(\pi_4(x)) - \pi_4(x)| \le \Xi_4 r_4(x)$. From Sublemma 10.70, we can assume that $r_4(x) \le 10\mathfrak{r}_p$. If Ξ_4 is sufficiently small, then $\|\mathcal{E}^4(p) - \mathcal{E}^3(p)\| < \frac{1}{2} c_{\text{1-ridge}}\, \mathfrak{r}(p)$.

Next, consider

$$\begin{aligned} \|D\mathcal{E}^4(p) - D\mathcal{E}^3(p)\| &= \big|(D\psi_4)_x(P_4(\pi_4(x)) - \pi_4(x)) \\ &\qquad + \psi_4(x)((DP_4)_{\pi_4(x)} \circ D(\pi_4 \circ \mathcal{E}^3)_p - D(\pi_4 \circ \mathcal{E}^3)_p)\big| \\ &\le |(D\psi_4)_x| \cdot |(P_4(\pi_4(x)) - \pi_4(x))| \\ &\quad + |\psi_4(x)| \cdot |((DP_4)_{\pi_4(x)} - \pi_{A^0_x}) \circ D(\pi_4 \circ \mathcal{E}^3)_p| \\ &\quad + |\psi_4(x)| \cdot |\pi_{A^0_x} \circ D(\pi_4 \circ \mathcal{E}^3)_p - D(\pi_4 \circ \mathcal{E}^3)_p| \\ &\le |(D\psi_4)_x| \cdot |(P_4(\pi_4(x)) - \pi_4(x))| + |(DP_4)_{\pi_4(x)} - \pi_{A^0_x}| \cdot |D(\pi_4 \circ \mathcal{E}^3)_p| \\ &\quad + |\pi_{A^0_x} \circ D(\pi_4 \circ \mathcal{E}^3)_p - D(\pi_4 \circ \mathcal{E}^3)_p|. \end{aligned} \tag{11.42}$$

Equation (11.29) gives a bound on $|(D\psi_4)_x|$. [13, Lemma 20.2(1)] gives a bound on $|(P_4(\pi_4(x)) - \pi_4(x))|$. [13, Lemma 20.2(7)] gives a bound on $|(DP_4)_{\pi_4(x)} - \pi_{A^0_x}|$. Lemma 10.5 gives a bound on $|D(\pi_4 \circ \mathcal{E}^3)_p|$. Equation (10.51) gives a bound on $|\pi_{A^0_x} \circ D(\pi_4 \circ \mathcal{E}^3)_p - D(\pi_4 \circ \mathcal{E}^3)_p|$. It follows that $\|D\mathcal{E}^4(p) - D\mathcal{E}^3(p)\| < \frac{1}{2} c_{\text{1-ridge}}$.

From Lemma 11.24, if $c_{\text{2-slim}}$ is sufficiently small, then $\|\mathcal{E}^3(p) - \mathcal{E}^0(p)\| < \frac{1}{2} c_{\text{1-ridge}}\, \mathfrak{r}(p)$ and $\|D\mathcal{E}^3(p) - D\mathcal{E}^0(p)\| < \frac{1}{2} c_{\text{1-ridge}}$. It follows that $\|\mathcal{E}^4(p) - \mathcal{E}^0(p)\| < c_{\text{1-ridge}}\, \mathfrak{r}(p)$ and $\|D\mathcal{E}^4(p) - D\mathcal{E}^0(p)\| < c_{\text{1-ridge}}$. This proves part (2) of the lemma.

(3). The restriction of $\pi_4 \circ \mathcal{E}^4$ to $\bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.6\Delta, \eta_{i_P} < 3.5\lambda_{p_i}\}$ equals $P_4 \circ (\pi_4 \circ \mathcal{E}^3)$. For $p \in \bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| < 0.6\Delta, \eta_{i_P} < 3.5\lambda_{p_i}\}$, put $x = \mathcal{E}^3(p)$. Then

$$D(P_4 \circ (\pi_4 \circ \mathcal{E}^3))_p = \pi_{A^0_x} \circ D(\pi_4 \circ \mathcal{E}^3)_p + ((DP_4)_{\pi_4(x)} - \pi_{A^0_x}) \circ D(\pi_4 \circ \mathcal{E}^3)_p. \tag{11.43}$$

Using (10.52) and [13, Lemma 20.2(7)], we have that if Ξ_4 is sufficiently small, then $D(P_4 \circ (\pi_4 \circ \mathcal{E}^3))_p$ maps onto $(TW^0_4)_{P_4(\pi_4(x))}$. This proves part (3) of the lemma. □

11.6 Adjusting the map near the slim 1-stratum

Recall that $Q_5 = H_{\text{0-stratum}} \oplus H_{\text{1-slim}}$ and $\pi_5 : H \to Q_5$ is an orthogonal projection. We let $\widetilde{A}_5, A_5, \widetilde{S}_5, S_5$, and $r_5 : \widetilde{S}_5 \to (0, \infty)$ be as in Section 10.7.

Thus, by Lemma 10.79, $(\widetilde{S}_5, S_5, r_5)$ is a $(2, \Gamma_5)$ cloudy 1-manifold. By [13, Lemma 20.2], there is a 1-manifold $W^0_5 \subset Q_5$ so that the conclusion of [13, Lemma 20.2] holds, where the parameter ϵ in [13, Lemma 20.2] is given by $\Xi_5 = \Xi_5(\Gamma_5)$. In particular, there is a well-defined nearest point projection

$$P_5 : N_{r_5}(S_5) = \widehat{W}_5 \to W_5^0 \tag{11.44}$$

where N_{r_5} is a variable thickness neighborhood as defined in Section 1.3.

Lemma 11.45. *Under the constraint* $c_{\text{1-slim}} < \overline{c}_{\text{1-slim}}$*, there is a smooth function* $\psi_5 : H \to [0,1]$ *with the following properties:*

(1)

$$\begin{aligned} \psi_5 \circ \mathcal{E}^4 &\equiv 1 \quad \textit{in} \quad \bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| < 0.6\Delta\} \textit{ and} \\ \psi_5 \circ \mathcal{E}^4 &\equiv 0 \quad \textit{outside} \bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| < 0.7\Delta\}. \end{aligned} \tag{11.46}$$

(2) $\operatorname{supp}(\psi_5) \cap \operatorname{im}(\mathcal{E}^4) \subset \widehat{W}_5 \times Q_5^\perp$.

(3) *There is a constant* $\Omega_5' = \Omega_5'(\mathcal{M})$ *such that*

$$|(D\psi_5)_x| < \Omega_5' x_{\mathfrak{r}}^{-1} \tag{11.47}$$

for all $x \in \operatorname{im}(\mathcal{E}^4)$.

Proof. If the parameter $c_{\text{1-ridge}}$ is sufficiently small and Δ is sufficiently large, then by Lemma 11.39, $\|\mathcal{E}^4(p) - \mathcal{E}^0(p)\| < c_{\text{1-ridge}}\mathfrak{r}(p)$. Hence, $\mathcal{E}^4(p) \in \bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| < 0.6\Delta\}$ implies that $\mathcal{E}^0(p) \in \bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| < 0.61\Delta\}$. Also, $\mathcal{E}^4(p) \notin \bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| < 0.7\Delta\}$ implies that $\mathcal{E}^0(p) \notin \bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| < 0.69\Delta\}$.

Define $\psi_5 : H \to [0,1]$ by

$$\psi_5(x) = 1 - \Phi_{\frac{1}{2},1}\left(\sum_{\{i \in I_{\text{1-slim}},\, x_i'' > 0\}} \Phi_{0.61\Delta, 0.65\Delta}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right)\right) \tag{11.48}$$

The rest of the proof is the same as the proof of Lemma 11.27 but without the I_P case. □

Define $\Psi_5 : H \to H$ by $\Psi_5(x) = x$ if $\pi_5(x) \notin \widehat{W}_5$ and

$$\Psi_5(x) = (\psi_5(x) P_5(\pi_5(x)) + (1 - \psi_5(x))\pi_5(x), \pi_5^\perp(x)) \tag{11.49}$$

otherwise. Put $\mathcal{E}^5 = \Psi_5 \circ \mathcal{E}^4$.

Lemma 11.50. *Under the constraints,* $\Sigma_5 < \overline{\Sigma}_5(\Omega_5, c_{\text{1-slim}})$*,* $\Gamma_5 < \overline{\Gamma}_5(\Omega_5, c_{\text{1-slim}})$*,* $\Xi_5 < \overline{\Xi}_5(c_{\text{1-slim}})$*, and* $c_{\text{1-ridge}} < \overline{c}_{\text{1-ridge}}(c_{\text{1-slim}})$*, we have:*

(1) $\mathcal{E}^5$ *is smooth.*

(2) *For all* $p \in M$*,*

$$\|\mathcal{E}^5(p) - \mathcal{E}^0(p)\| < c_{\text{1-slim}}\,\mathfrak{r}(p) \quad \textit{and} \quad \|D\mathcal{E}^5(p) - D\mathcal{E}^0(p)\| < c_{\text{1-slim}}. \tag{11.51}$$

(3) *The restriction of* $\pi_5 \circ \mathcal{E}^5$ *to* $\bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| < 0.6\Delta\}$ *is a submersion to* W_5^0*.*

Proof. The proof is similar to the proof of [13, Lemma 13.34]. □

11.7 Proof of Proposition 11.1

Note from (11.9), (11.16), (11.23), (11.38), and (11.49) that Ψ_{j+1} can be factored as $\Psi_{j+1}^{Q_j} \times I_{Q_j^\perp}$ for some $\Psi_{j+1}^{Q_j} : Q_j \to Q_j$. Moreover, since $Q_j \subset Q_{j+1}$, Ψ_{j+1} can be factored as $\Psi_{j+1}^{Q_k} \times I_{Q_k^\perp}$ for some $\Psi_{j+1}^{Q_k} : Q_k \to Q_k$ for $k < j$. In particular, $\pi_k \circ \Psi_{j+1} = \Psi_{j+1}^{Q_k} \circ \pi_k$ for $j \in \{1, 2, 3, 4\}$ and $k < j$.

Put $\mathcal{E} = \mathcal{E}^5$, $c_{\text{adjust}} = c_{\text{1-slim}}$,

$$\begin{aligned}
W_1 &= (\Psi_5 \circ \Psi_4 \circ \Psi_3 \circ \Psi_2)(W_1^0) \cap \bigcup_{i \in I_{\text{3-stratum}}} \{y \in H : y_i'' > 0.9R_i, |y_i'| < 5.5R_i\}, \qquad (11.52)\\
W_2 &= (\Psi_5^{Q_2} \circ \Psi_4^{Q_2} \circ \Psi_3^{Q_2})(W_2^0) \cap \bigcup_{i \in I_{\text{2-edge}}} \left\{ y \in Q_2 : \begin{array}{c} y_i'' > 0.9R_i, |y_i'| < 5.5\Delta R_i, \\ y_{\mathfrak{r}} > 0, y_{E'} < 5.5\Delta y_{\mathfrak{r}} \end{array} \right\},\\
W_3 &= (\Psi_5^{Q_3} \circ \Psi_4^{Q_3})(W_3^0) \cap \bigcup_{i \in I_{\text{2-slim}}} \{y \in Q_3 : y_i'' > 0.9R_i, |y_i'| < 5.5 \cdot 10^5 \Delta R_i\},\\
W_4 &= \Psi_5^{Q_3}(W_4^0) \cap \bigcup_{i \in I_{\text{1-ridge}}} \left\{ y \in Q_4 : \begin{array}{c} y_i'' > 0.9R_i, |y_i'| < 0.55\Delta R_i, \\ y_{i_P}'' > 0.9R_{i_P}, y_{i_P}' < 3\lambda_{p_i} R_{i_P} \end{array} \right\}, \text{ and}\\
W_5 &= W_5^0 \cap \bigcup_{i \in I_{\text{1-slim}}} \{y \in Q_5 : y_i'' > 0.9R_i, |y_i'| < 0.55\Delta R_i\}.
\end{aligned}$$

The smoothness of $\mathcal{E}$ follows from part (1) of Lemma 11.50. Part (1) of Proposition 11.1 follows from part (2) of Lemma 11.50.

Lemma 11.53. W_i *is a* k_i*-manifold.*

Proof. The proof is similar to the proof for [13, Lemma 13.46]. □

By Lemma 11.10 (3), the restriction of $\mathcal{E}^1$ to U_1 is a submersion from U_1 to W_1^0. From Lemma 10.8 and equation (11.51), if Γ_5 and $c_{\text{1-slim}}$ are sufficiently small, then $\mathcal{E} = \Psi_5 \circ \Psi_4 \circ$

$\Psi_3 \circ \Psi_2 \circ \mathcal{E}^1$ maps U_1 to $W_1 \subset (\Psi_5 \circ \Psi_4 \circ \Psi_3 \circ \Psi_2)(W_1^0)$. To see that it is a submersion, suppose that $|\eta_i(p)| < 5$ for some $i \in I_{\text{3-stratum}}$. Put $x_0 = \mathcal{E}^0(p)$, $x_1 = \mathcal{E}^1(p)$, and $x = \mathcal{E}(p)$. Note that $x_i' = (x_1)_i' = (x_0)_i'$. From Lemma 10.8 and [13, Lemma 20.2(3)], if Ξ_1 is sufficiently small then we are ensured that $(D_{\pi_{H_i'}})_{x_0} \circ D\mathcal{E}_p^0$ maps onto $T_{(x_0)_i'}H_i' \simeq \mathbb{R}^3$. By Lemma 11.10, if $c_{\text{3-stratum}}$ is sufficiently small, then $(D_{\pi_{H_i'}})_{x_1} \circ D\mathcal{E}_p^1$ maps onto $T_{(x_1)_i'}H_i' \simeq \mathbb{R}^3$. Thus, $(D_{\pi_{H_i'}})_x \circ D\mathcal{E}_p = (D_{\pi_{H_i'}})_x \circ D(\Psi_5 \circ \Psi_4 \circ \Psi_3 \circ \Psi_2)_{x_1} \circ D\mathcal{E}_p^1 = (D_{\pi_{H_i'}})_{x_1} \circ D\mathcal{E}_p^1$ maps onto $T_{x_i'}H_i' \simeq \mathbb{R}^3$. Hence, $D\mathcal{E}_p$ must map T_pM onto T_xW_1. This shows that $\mathcal{E}$ is a submersion near p.

By Lemma 11.17 (3), the restriction of $\pi_2 \circ \mathcal{E}^2$ to U_2 is a submersion from U_2 to W_2^0. Lemma 10.36 and equation (11.51) implies that if Γ_2 and $c_{\text{1-slim}}$ are sufficiently small, then $\pi_2 \circ \mathcal{E} = \pi_2 \circ \Psi_5 \circ \Psi_4 \circ \Psi_3 \circ \mathcal{E}^2 = \Psi_5^{Q_2} \circ \Psi_4^{Q_2} \circ \Psi_3^{Q_2} \circ \pi_2 \circ \mathcal{E}^2$ maps U_2 to $W_2 \subset (\Psi_5^{Q_2} \circ \Psi_4^{Q_2} \circ \Psi_3^{Q_2})(W_2^0)$. To see that it is a submersion, suppose that $|\eta_i(p)| < 5\Delta$ for some $i \in I_{\text{2-edge}}$ and $\eta_{E'}(p) < 5\mathfrak{r}(p)$. Put $x_0 = \mathcal{E}^0(p)$, $x_2 = \mathcal{E}^2(p)$, and $x = \mathcal{E}(p)$. Note that $x_i' = (x_2)_i' = (x_0)_i'$. From Lemma 10.36 and [13, Lemma 20.2(3)], if Ξ_2 is sufficiently small, then we are ensured that $(D_{\pi_{H_i'}})_{\pi_2(x_0)} \circ D(\pi_2 \circ \mathcal{E}^0)_p$ maps onto $T_{(\pi_2(x_0))_i'}H_i' \simeq \mathbb{R}^2$. By Lemma 11.17, if $c_{\text{2-edge}}$ is sufficiently small, then $(D_{\pi_{H_i'}})_{\pi_2(x_2)} \circ D(\pi_2 \circ \mathcal{E}^2)_p$ maps onto $T_{(\pi_2(x_2))_i'}H_i' \simeq \mathbb{R}^2$. Thus, $(D_{\pi_{H_i'}})_{\pi_2(x)} \circ D(\pi_2 \circ \mathcal{E})_p = (D_{\pi_{H_i'}})_{\pi_2(x)} \circ D(\pi_2 \circ \Psi_5 \circ \Psi_4 \circ \Psi_3)_{x_2} \circ D(\mathcal{E}^2)_p = (D_{\pi_{H_i'}})_{\pi_2(x)} \circ D(\Psi_5^{Q_2} \circ \Psi_4^{Q_2} \circ \Psi_3^{Q_2})_{\pi_2(x_2)} \circ D(\pi_2 \circ \mathcal{E}^2)_p = (D_{\pi_{H_i'}})_{\pi_2(x_2)} \circ D(\pi_2 \circ \mathcal{E})_p^2$ maps onto $T_{\pi_2(x)_i'}H_i' \simeq \mathbb{R}^2$. Thus, $D(\pi_2 \circ \mathcal{E})_p$ must map T_pM onto $T_{\pi_2(x)}W_2$, showing that $\pi_2 \circ \mathcal{E}$ is a submersion near p.

By Lemma 11.24 (3), the restriction of $\pi_3 \circ \mathcal{E}^3$ to U_3 is a submersion from U_3 to W_3^0. Lemma 10.43 and equation (11.51) implies that if Γ_3 and $c_{\text{1-slim}}$ are sufficiently small then $\pi_3 \circ \mathcal{E} = \pi_3 \circ \Psi_5 \circ \Psi_4 \circ \mathcal{E}^3 = \Psi_5^{Q_3} \circ \Psi_4^{Q_3} \circ \pi_3 \circ \mathcal{E}^3$ maps U_3 to $W_3 \subset (\Psi_5^{Q_3} \circ \Psi_4^{Q_3})(W_3^0)$. By a similar argument to $\pi_2 \circ \mathcal{E}^2$ case, the restriction of $\pi_3 \circ \mathcal{E}$ to U_3 is a submersion to W_3.

By Lemma 11.39 (3), the restriction of $\pi_4 \circ \mathcal{E}^4$ to U_4 is a submersion from U_4 to W_4^0. Lemma 10.50 and equation (11.51) implies that if Γ_4 and $c_{\text{1-slim}}$ are sufficiently small then $\pi_4 \circ \mathcal{E} = \pi_4 \circ \Psi_5 \circ \mathcal{E}^4 = \Psi_5^{Q_4} \circ \pi_4 \circ \mathcal{E}^4$ maps U_4 to $W_4 \subset \Psi_5^{Q_4}(W_3^0)$. By a similar argument to $\pi_2 \circ \mathcal{E}^2$ case, the restriction of $\pi_4 \circ \mathcal{E}$ to U_4 is a submersion to W_4.

Finally, by lemma 11.50 (3), the restriction of $\pi_5 \circ \mathcal{E} = \pi_5 \circ \mathcal{E}^5$ to U_5 is a submersion to $W_5 = W_5^0$. This proves Proposition 11.1.

— 12 —

Extracting a good decomposition of M

In this chapter we will use the map $\mathcal{E}$ to decompose M into fibered domains which are compatible along the intersections. The main result of this chapter is Proposition 12.1. The rest of the chapter is the proof of the proposition.

Proposition 12.1. *There is a decomposition*

$$M = M^{\text{0-stratum}} \cup M^{\text{1-slim}} \cup M^{\text{1-ridge}} \cup M^{\text{2-slim}} \cup M^{\text{2-edge}} \cup M^{\text{3-stratum}} \tag{12.2}$$

into compact domains with disjoint interiors, where each connected component of $M^{\text{1-slim}}$, $M^{\text{1-ridge}}$, $M^{\text{2-slim}}$, $M^{\text{2-edge}}$, and $M^{\text{3-stratum}}$ may be endowed with a fibration structure, such that:

1. *$M^{\text{0-stratum}}$ and $M^{\text{1-slim}}$ are domains with smooth boundary, while $M^{\text{1-ridge}}$, $M^{\text{2-slim}}$, $M^{\text{2-edge}}$, and $M^{\text{3-stratum}}$ are smooth manifolds with corners, each point of which has a neighborhood diffeomorphic to $\mathbb{R}^{4-k} \times [0,\infty)^k$ for some $k \leq 3$.*
2. *Connected components of $M^{\text{0-stratum}}$ are diffeomorphic to a closed Riemannian 4-manifold which admits a metric of nonnegative sectional curvature or diffeomorphic to D^4, $S^1 \times D^3$, $S^2 \times_\omega D^2$, $(S^2 \times_\omega D^2)/\mathbb{Z}_2$ for $\omega \in \mathbb{Z}$, $T^2 \times D^2$, $T^2 \times_{\mathbb{Z}_2} D^2$, $(S^2 \widetilde{\times} S^1) \widetilde{\times} I$, $(\mathbb{R}P^2 \widetilde{\times} S^1) \widetilde{\times} I$, or $\mathcal{B}_i \widetilde{\times} I$ for $i \in \{1,2,3,4\}$.*
3. *The components of $M^{\text{1-slim}}$ have a fibration with fibers diffeomorphic to S^3/Γ (where Γ is a finite subgroup of $\mathrm{Isom}^+(S^3) = SO^4$ which acts freely on S^3), T^3/Γ (where Γ is a finite subgroup of $\mathrm{Isom}^+(T^3)$ which acts freely on T^3), $S^1 \times S^2$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.*
4. *The components of $M^{\text{1-ridge}}$ have a fibration with fibers diffeomorphic to D^3, $S^1 \times D^2$, $S^2 \times_{\mathbb{Z}_2} I$, or $T^2 \times_{\mathbb{Z}_2} I$.*
5. *The components of $M^{\text{2-slim}}$ have a fibration with fibers diffeomorphic to S^2 or T^2.*
6. *The components of $M^{\text{2-edge}}$ have a fibration with fibers diffeomorphic to D^2.*

(7) *$M^{\text{3-stratum}}$ is a smooth domain with corners with a smooth S^1-fibration. The S^1-fibration is compatible with any corners.*

(8) *Each fiber of the fibration $M^{\text{1-ridge}} \to B^{\text{1-ridge}}$, lying over a boundary point of the base $B^{\text{1-ridge}}$, is contained in $\partial M^{\text{0-stratum}}$ or $\partial M^{\text{1-slim}}$.*

(9) *Each fiber of the fibration $M^{\text{2-slim}} \to B^{\text{2-slim}}$ lying over a boundary point of the base $B^{\text{2-slim}}$, is contained in $\partial M^{\text{0-stratum}}$, $\partial M^{\text{1-slim}}$, a fiber of $\partial M^{\text{1-ridge}}$ induced by the fibration $M^{\text{1-ridge}} \to B^{\text{1-ridge}}$, or $M^{\text{2-edge}} \cup M^{\text{3-stratum}}$. In the $\partial M^{\text{1-ridge}}$ case, if a fiber of the fibration $M^{\text{2-slim}} \to B^{\text{2-slim}}$ is contained in a fiber over an interior point of $B^{\text{1-ridge}}$ then they coincide.*

(10) *Each fiber of the fibration $M^{\text{2-edge}} \to B^{\text{2-edge}}$, lying over a boundary point of the base $B^{\text{2-edge}}$, is contained in $\partial M^{\text{0-stratum}}$, $\partial M^{\text{1-slim}}$, a fiber of $\partial M^{\text{1-ridge}}$ induced by the fibration $M^{\text{1-ridge}} \to B^{\text{1-ridge}}$, or a fiber of $\partial M^{\text{2-slim}}$ induced by the fibration $M^{\text{2-slim}} \to B^{\text{2-slim}}$.*

(11) *The part of $\partial M^{\text{1-ridge}}$ that carries an induced 2-dimensional fibration over interior points of the base $B^{\text{1-ridge}}$ is contained in $M^{\text{2-slim}} \cup M^{\text{2-edge}} \cup M^{\text{3-stratum}}$.*

(12) *The part of $\partial M^{\text{2-edge}}$ that carries an induced S^1-fibration over interior points of the base $B^{\text{2-edge}}$ is contained in $\partial M^{\text{3-stratum}}$, and the S^1-fibration induced from $M^{\text{2-edge}}$ agrees with the one inherited from $M^{\text{3-stratum}}$.*

12.1 The definition of $M^{\text{0-stratum}}$

For each $i \in I_{\text{0-stratum}}$, put

$$M_i^{\text{0-stratum}} = B(p_i, 0.35R_i) \cup \mathcal{E}^{-1}\left\{x \in H : x_i'' \geq 0.9R_i, \frac{x_i'}{x_i''} \leq 0.4\right\}. \tag{12.3}$$

Lemma 12.4. *Under the constraints $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}$ and $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, we have that $\{M_i^{\text{0-stratum}}\}_{i \in I_{\text{0-stratum}}}$ is a disjoint collection and each $M_i^{\text{0-stratum}}$ is a compact manifold with boundary, which is diffeomorphic to one of the possibilities in Proposition 12.1 (2).*

Proof. The proof is similar to the proof of [13, Lemma 14.4]. □

We let $M^{\text{0-stratum}} = \bigcup_{i \in I_{\text{0-stratum}}} M_i^{\text{0-stratum}}$, and put $M_1 = M \setminus \operatorname{int}(M^{\text{0-stratum}})$. Thus $M^{\text{0-stratum}}$ and M_1 are smooth compact manifolds with boundary.

12.2 The definition of $M^{\text{1-slim}}$

We first truncate W_5. Put

$$W_5' = W_5 \cap \bigcup_{i \in I_{\text{1-slim}}} \left\{ x \in Q_5 : x_i'' \geq 0.9 R_i, \left| \frac{x_i'}{x_i''} \right| \leq 0.4\Delta \right\} \tag{12.5}$$

and define $U_5' = (\pi_5 \circ \mathcal{E})^{-1}(W_5')$.

Lemma 12.6. *Under the constraints $\varsigma_{\text{1-slim}} < \overline{\varsigma}_{\text{1-slim}}(\Delta)$ and $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, the following holds.*

1. $\bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| \leq 0.35\Delta\} \subset U_5' \subset U_5$, *where U_5 is as in Proposition 11.1.*
2. *The restriction of $\pi_5 \circ \mathcal{E}$ to U_5' gives a proper submersion to W_5'. In particular, it is a fibration.*
3. *The fibers of $\pi_5 \circ \mathcal{E} : U_5' \to W_5'$ are diffeomorphic to an orientable compact Riemannian 3-manifold with nonnegative sectional curvature.*
4. *M_1 intersects U_5' in a submanifold with boundary which is a union of fibers of $\pi_5 \circ \mathcal{E} : U_5' \to W_5'$.*

Proof. For a given $i \in I_{\text{1-slim}}$, suppose that $p \in M$ satisfies $|\eta_i(p)| \leq 0.35\Delta$. Putting $y = (\pi_5 \circ \mathcal{E}^0)(p) \in Q_5$, we have $y_i'' = R_i$ and $\left| \frac{y_i'}{y_i''} \right| \leq 0.35\Delta$. Put $x = (\pi_5 \circ \mathcal{E})(p) \in Q_5$. If c_{adjust} is sufficiently small, then we have that $x_i'' > 0.9R_i$ and $\left| \frac{x_i'}{x_i''} \right| < 0.4\Delta$. As $p \in U_5$, Proposition 11.1 implies that $x \in W_5$. Hence, $\bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| \leq 0.35\Delta\} \subset U_5'$.

Now suppose that $p \in U_5'$. Putting $x = (\pi_5 \circ \mathcal{E})(p)$, we have that for some $i \in I_{\text{1-slim}}$, $x_i'' > 0.9R_i$ and $\left| \frac{x_i'}{x_i''} \right| < 0.4\Delta$. Put $y = (\pi_5 \circ \mathcal{E}^0)(p)$. If c_{adjust} is sufficiently small, then $y_i'' \geq 0.8R_i$ and $\left| \frac{y_i'}{y_i''} \right| \leq 0.45\Delta$. Hence, $|\eta_i(p)| \leq 0.45\Delta$. This shows that $U_5' \subset U_5$, proving part (1) of the lemma.

By Proposition 11.1, $\pi_5 \circ \mathcal{E}$ is a submersion from U_5 to W_5. Hence it restricts to a surjective submersion on U_5'. Suppose that K is a compact subset of W_5'. Then $(\pi_5 \circ \mathcal{E})^{-1}(K)$ is a closed subset of M which is contained in $\overline{U}_5 = \{|\eta_i| \leq 0.5\Delta\}$. As $\{p_i\}_{i \in I_{\text{1-slim}}}$ are in the slim 1-stratum, it follows from the definition of adapted coordinates that $\{|\eta_i| \leq 0.5\Delta\}$ is a compact subset of M. Thus the restriction of $\pi_5 \circ \mathcal{E}$ to U_5' is a proper submersion. This proves part (2) of the lemma.

To prove part (3) of the lemma, given $x \in W_5'$, suppose that $p \in U_5'$ satisfies $(\pi_5 \circ \mathcal{E})(p) = x$. Choose $i \in I_{\text{1-slim}}$ so that $|\eta_i(p)| \leq 0.45\Delta$. If c_{adjust} is sufficiently small, then by looking at the components in H_i, one sees that for any $p' \in U_5'$ satisfying $(\pi_5 \circ \mathcal{E})(p') = x$, we have $p' \in \{|\eta_i| < 0.5\Delta\}$. Thus, to determine the topology of the fibers, it suffices to just consider the restriction of $\pi_5 \circ \mathcal{E}$ to $\{|\eta_i| < 0.5\Delta\}$.

Let $\pi_{H'_i} : Q_5 \to H'_i$ be an orthogonal projection and put $X = \pi_{H'_i}(x) \in H'_i$. Since the restriction of $\pi_{H'_i} \circ \pi_5 \circ \mathcal{E}^0$ to $\{|\eta_i| < 0.5\Delta\}$ equals η_i, we have that $\pi_{H'_i} \circ \pi_5 \circ \mathcal{E}^0$ is transverse there to X. By Lemma 8.8, $\{|\eta_i| < 0.5\Delta\} \cap (\pi_{H'_i} \circ \pi_5 \circ \mathcal{E}^0)^{-1}(X)$ is diffeomorphic to an orientable compact Riemannian 3-manifold with nonnegative sectional curvature.

Consider the restriction of $(\pi_{H'_i} \circ \pi_5 \circ \mathcal{E})$ to $\{|\eta_i| < 0.5\Delta\}$. Proposition 11.1 and [13, Lemma 21.3] imply that if c_{adjust} is sufficiently small, then the fiber $\{|\eta_i| < 0.5\Delta\} \cap (\pi_{H'_i} \circ \pi_5 \circ \mathcal{E})^{-1}(X)$ is diffeomorphic an orientable compact Riemannian 3-manifold with nonnegative sectional curvature. In particular, it is connected. Now, $(\pi_{H'_i} \circ \pi_5 \circ \mathcal{E})^{-1}(X)$ is the preimage, under $\pi_5 \circ \mathcal{E} : U'_5 \to W'_5$, of the preimage of X under $\pi_{H'_i} : W'_5 \to H'_i$. From connectedness of the fiber, the preimage of X under $\pi_{H'_i} : W'_5 \to H'_i$ must just be x. Hence $(\pi_5 \circ \mathcal{E})^{-1}(x)$ is diffeomorphic to an orientable compact Riemannian 3-manifold with nonnegative sectional curvature. This proves part (3) of the lemma.

To prove part (4) of the lemma, let $p \in M_1 \cap U'_5$. We only need to check when $p \in \partial M^{\text{0-stratum}}$. Suppose that $p \in \partial M_j^{\text{0-stratum}}$ for some $j \in I_{\text{0-stratum}}$. If $x = \mathcal{E}(p)$ then $x''_j \geq 0.9R_j$ and $x'_j = 0.4x''_j$. Let $q \in U'_5$ be a point in the same fiber of $\pi_5 \circ \mathcal{E} : U'_5 \to W'_5$ as p and put $y = \mathcal{E}(q)$. As $\pi_5(x) = \pi_5(y)$, $\pi_{H_j}(x) = \pi_{H_j}(y)$. Hence, $y''_j \geq 0.9R_j$ and $y'_j = 0.4y''_j$. In particular, $q \in \partial M_j^{\text{0-stratum}}$. Thus, the whole fiber $(\pi_5 \circ \mathcal{E})^{-1}(x)$ is in $\partial M_j^{\text{0-stratum}}$. □

Let $W''_5 \subset W'_5$ be a compact 1-dimensional manifold with boundary such that $(\pi_5 \circ \mathcal{E})^{-1}(W''_5)$ contains $\bigcup_{i \in I_{\text{1-slim}}} \{|\eta_i| < 0.35\Delta\}$ and put $M^{\text{1-slim}} = M_1 \cap (\pi_5 \circ \mathcal{E})^{-1}(W''_5)$. We endow $M^{\text{1-slim}}$ with the fibration induced by $\pi_5 \circ \mathcal{E}$.

Put $M_2 = M_1 \setminus \operatorname{int}(M^{\text{1-slim}})$.

12.3 The definition of $M^{\text{1-ridge}}$

We first truncate W_4. Put

$$W'_4 = W_4 \cap \bigcup_{i \in I_{\text{1-ridge}}} \left\{ x \in Q_4 : x''_i \geq 0.9R_i, \left| \frac{x'_i}{x''_i} \right| \leq 0.4\Delta, x''_{i_P} \geq 0.9R_{i_P}, \left| \frac{x'_{i_P}}{x''_{i_P}} \right| \leq 2.5\lambda_{p_i} \right\} \tag{12.7}$$

and define $U'_4 = (\pi_4 \circ \mathcal{E})^{-1}(W'_4)$.

Lemma 12.8. *Under the constraints $\varsigma_{\text{1-ridge}} < \overline{\varsigma}_{\text{1-ridge}}(\Delta)$ and $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, the following holds.*

(1) $\bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| \leq 0.35\Delta, |\eta_{i_P}| \leq 2\lambda_{p_i}\} \subset U'_4 \subset U_4$, *where U_4 is as in Proposition 11.1.*

(2) *The restriction of $\pi_4 \circ \mathcal{E}$ to U'_4 gives a proper submersion to W'_4. In particular, it is a fibration.*

(3) *The fibers of $\pi_4 \circ \mathcal{E} : U'_4 \to W'_4$ are diffeomorphic to D^3, $S^1 \times D^2$, $S^2 \times_{\mathbb{Z}_2} I$, or $T^2 \times_{\mathbb{Z}_2} I$.*

(4) *M_2 intersects U_4' in a submanifold with corners which is a union of fibers of $\pi_4 \circ \mathcal{E} : U_4' \to W_4'$.*

Proof. For a given $i \in I_{\text{1-ridge}}$, suppose that $p \in M$ satisfies $|\eta_i(p)| \leq 0.35\Delta$ and $|\eta_{i_P}| \leq 2\lambda_{p_i}$. Putting $y = (\pi_4 \circ \mathcal{E}^0)(p) \in Q_4$, we have $y_i'' = R_i$, $\left|\frac{y_i'}{y_i''}\right| \leq 0.35\Delta$, $y_{i_P}'' = R_{i_P}$, and $\left|\frac{y_{i_P}'}{y_{i_P}''}\right| \leq 2\lambda_{p_i}$. Put $x = (\pi_4 \circ \mathcal{E})(p) \in Q_4$. If c_{adjust} is sufficiently small, then since $\lambda_{p_i} \geq \Upsilon_1 > \Delta \gg 1$, we have that $x_i'' > 0.9R_i$, $\left|\frac{x_i'}{x_i''}\right| < 0.4\Delta$, $x_{i_P}'' > 0.9R_{i_P}$, and $\left|\frac{x_{i_P}'}{x_{i_P}''}\right| < 2.5\lambda_{p_i}$. As $p \in U_4$, Proposition 11.1 implies that $x \in W_4$. Hence, $\bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| \leq 0.35\Delta, |\eta_{i_P}| \leq 2\lambda_{p_i}\} \subset U_4'$.

Now suppose that $p \in U_4'$. Putting $x = (\pi_4 \circ \mathcal{E})(p)$, we have that for some $i \in I_{\text{1-ridge}}$, $x_i'' > 0.9R_i$, $\left|\frac{x_i'}{x_i''}\right| < 0.4\Delta$, $x_{i_P}'' > 0.9R_{i_P}$, and $\left|\frac{x_{i_P}'}{x_{i_P}''}\right| < 2.5\lambda_{p_i}$. Put $y = (\pi_4 \circ \mathcal{E}^0)(p)$. If c_{adjust} is sufficiently small, then $\left|\frac{y_i'}{y_i''}\right| \leq 0.45\Delta$, $y_{i_P}'' \geq 0.8R_{i_P}$, and $\left|\frac{y_{i_P}'}{y_{i_P}''}\right| \leq 2.7\lambda_{p_i}$. Hence, $|\eta_i(p)| \leq 0.45\Delta$ and $|\eta_{i_P}(p)| \leq 2.7\lambda_{p_i}$. This shows that $U_4' \subset U_4$, proving part (1) of the lemma.

By Proposition 11.1, $\pi_4 \circ \mathcal{E}$ is a submersion from U_4 to W_4. Hence it restricts to a surjective submersion on U_4'. Suppose that K is a compact subset of W_4'. Then $(\pi_4 \circ \mathcal{E})^{-1}(K)$ is a closed subset of M which is contained in $\overline{U}_4 = \{|\eta_i| \leq 0.5\Delta, |\eta_{i_P}| \leq 3\lambda_{p_i}\}$. As $\{p_i\}_{i \in I_{\text{1-ridge}}}$ are in the ridge 1-stratum, it follows from the definition of adapted coordinates that $\{|\eta_i| \leq 0.5\Delta\}$ is a compact subset of M. It also follows from the definition of approximated distance function ψ_{p_i} that $\{|\eta_{i_P}| \leq 3\lambda_{p_i}\}$ is a compact subset of M. Hence $\{|\eta_i| \leq 0.5\Delta, |\eta_{i_P}| \leq 3\lambda_{p_i}\}$ is a compact subset of M. Thus the restriction of $\pi_4 \circ \mathcal{E}$ to U_4' is a proper submersion. This proves part (2) of the lemma.

To prove part (3) of the lemma, given $x \in W_4'$, suppose that $p \in U_4'$ satisfies $(\pi_4 \circ \mathcal{E})(p) = x$. Choose $i \in I_{\text{1-ridge}}$ so that $|\eta_i(p)| \leq 0.45\Delta$ and $|\eta_{i_P}(p)| \leq 2.7\lambda_{p_i}$. If c_{adjust} is sufficiently small, then by looking at the components in H_i and H_{i_P}, one sees that for any $p' \in U_4'$ satisfying $(\pi_4 \circ \mathcal{E})(p') = x$, we have $p' \in \{|\eta_i| < 0.5\Delta, |\eta_{i_P}| < 3\lambda_{p_i}\}$. Thus, to determine the topology of the fiber, it suffices to just consider the restriction of $\pi_4 \circ \mathcal{E}$ to $\{|\eta_i| < 0.5\Delta, |\eta_{i_P}| < 3\lambda_{p_i}\}$.

Let $\pi_{H_i'} : Q_4 \to H_i'$ be an orthogonal projection and put $X = \pi_{H_i'}(x) \in H_i'$. Since the restriction of $\pi_{H_i'} \circ \pi_4 \circ \mathcal{E}^0$ to $\{|\eta_i| < 0.5\Delta, |\eta_{i_P}| < 3\lambda_{p_i}\}$ equals η_i, we have that $\pi_{H_i'} \circ \pi_4 \circ \mathcal{E}^0$ is transverse there to X. By Lemma 8.8, $\{|\eta_i| < 0.5\Delta, |\eta_{i_P}| < 3\lambda_{p_i}\} \cap (\pi_{H_i'} \circ \pi_4 \circ \mathcal{E}^0)^{-1}(X)$ is diffeomorphic to D^3, $S^1 \times D^2$, $S^2 \times_{\mathbb{Z}_2} I$, or $T^2 \times_{\mathbb{Z}_2} I$.

Consider the restriction of $(\pi_{H_i'} \circ \pi_4 \circ \mathcal{E})$ to $\{|\eta_i| < 0.5\Delta, |\eta_{i_P}| < 3\lambda_{p_i}\}$. Proposition 11.1 and [13, Lemma 21.3] imply that if c_{adjust} is sufficiently small then the fiber $\{|\eta_i| < 0.5\Delta, |\eta_{i_P}| < 3\lambda_{p_i}\} \cap (\pi_{H_i'} \circ \pi_4 \circ \mathcal{E})^{-1}(X)$ is diffeomorphic to D^3, $S^1 \times D^2$, $S^2 \times_{\mathbb{Z}_2} I$, or $T^2 \times_{\mathbb{Z}_2} I$. In particular, it is connected. Now, $(\pi_{H_i'} \circ \pi_4 \circ \mathcal{E})^{-1}(X)$ is the preimage, under $\pi_4 \circ \mathcal{E} : U_4' \to W_4'$, of the preimage of X under $\pi_{H_i'} : W_4' \to H_i'$. From connectedness of the fiber, the preimage of X under $\pi_{H_i'} : W_4' \to H_i'$ must just be x. Hence $(\pi_4 \circ \mathcal{E})^{-1}(x)$ is diffeomorphic to D^3, $S^1 \times D^2$, $S^2 \times_{\mathbb{Z}_2} I$, or $T^2 \times_{\mathbb{Z}_2} I$. This proves part (3) of the lemma.

To prove part (4) of the lemma, let $p \in M_2 \cap U_4'$. We only need to check when $p \in \partial M^{\text{1-slim}}$ or $p \in \partial M^{\text{0-stratum}}$.

Suppose first that $p \in \partial M_j^{\text{0-stratum}}$ for some $j \in I_{\text{0-stratum}}$. If $x = \mathcal{E}(p)$ then $x_j'' \geq 0.9R_j$ and $x_j' = 0.4x_j''$. Let $q \in U_4'$ be a point in the same fiber of $\pi_4 \circ \mathcal{E} : U_4' \to W_4'$ as p and put $y = \mathcal{E}(q)$. As $\pi_4(x) = \pi_4(y)$, $\pi_{H_j}(x) = \pi_{H_j}(y)$. Hence, $y_j'' \geq 0.9R_j$ and $y_j' = 0.4y_j''$. In particular, $q \in \partial M_j^{\text{0-stratum}}$. Thus, the whole fiber $(\pi_4 \circ \mathcal{E})^{-1}(x)$ is in $\partial M_j^{\text{0-stratum}}$.

Next, suppose that $p \in \partial M^{\text{1-slim}}$. Let $x = \mathcal{E}(p)$ and let $q \in U_4'$ be a point in the same fiber of $\pi_4 \circ \mathcal{E} : U_4' \to W_4'$ as p and put $y = \mathcal{E}(q) \in Q_4$. $\pi_4(x) = \pi_4(y)$ implies that $\pi_5(x) = \pi_5(y)$. Hence, $\pi_5(y) \in W_5''$. Since $M^{\text{1-slim}}$ is endowed with the fibration induced by $\pi_5 \circ \mathcal{E}$, q is in the same fiber of $\pi_5 \circ \mathcal{E}$ as p. Therefore, the fiber $(\pi_4 \circ \mathcal{E})^{-1}(x)$ is contained in a single fiber of the fibration of $\partial M^{\text{1-slim}}$. This proves part (4) of the lemma. ☐

Let W_4'' be a compact 1-dimensional manifold with corners such that $(\pi_4 \circ \mathcal{E})^{-1}(W_4'')$ contains $\bigcup_{i \in I_{\text{1-ridge}}} \{|\eta_i| \leq 0.35\Delta, |\eta_{i_P}| \leq 2\lambda_{p_i}\}$, and put $M^{\text{1-ridge}} = M_2 \cap (\pi_4 \circ \mathcal{E})^{-1}(W_4'')$. We endow $M^{\text{1-ridge}}$ with the fibration induced by $\pi_4 \circ \mathcal{E}$.

Put $M_3 = M_2 \setminus \operatorname{int}(M^{\text{1-ridge}})$.

12.4 The definition of $M^{\text{2-slim}}$

We first truncate W_3. Put

$$W_3' = W_3 \cap \bigcup_{i \in I_{\text{2-slim}}} \left\{ x \in Q_3 : x_i'' \geq 0.9R_i, \left| \frac{x_i'}{x_i''} \right| \leq 4 \cdot 10^5 \Delta \right\} \tag{12.9}$$

and define $U_3' = (\pi_3 \circ \mathcal{E})^{-1}(W_3')$.

Lemma 12.10. *Under the constraints $\varsigma_{\text{2-slim}} < \overline{\varsigma}_{\text{2-slim}}(\Delta)$ and $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, the following holds.*

(1) $\bigcup_{i \in I_{\text{2-slim}}} \{|\eta_i| \leq 3.5 \cdot 10^5 \Delta\} \subset U_3' \subset U_3$, *where U_3 is as in Proposition 11.1.*

(2) *The restriction of $\pi_3 \circ \mathcal{E}$ to U_3' gives a proper submersion to W_3'. In particular, it is a fibration.*

(3) *The fibers of $\pi_3 \circ \mathcal{E} : U_3' \to W_3'$ are diffeomorphic to S^2 or T^2.*

(4) *M_3 intersects U_3' in a submanifold with corners which is a union of fibers of $\pi_3 \circ \mathcal{E} : U_3' \to W_3'$.*

Proof. The proofs of parts (1), (2), and (3) of the lemma are similar to the proof of [13, Lemma 14.7].

To prove part (4) of the lemma, let $p \in M_3 \cap U_3'$. We only need to check when $p \in \partial M^{\text{1-ridge}}$, $p \in \partial M^{\text{1-slim}}$, or $p \in \partial M^{\text{0-stratum}}$. Let $x = \mathcal{E}(p)$. As in the proof of Lemma 12.8 (4), the fiber $(\pi_3 \circ \mathcal{E})^{-1}(x)$ is contained in a single fiber of the fibrations of $\partial M^{\text{1-ridge}}$, $\partial M^{\text{1-slim}}$, or $\partial M^{\text{0-stratum}}$. Additionally, consider that each fiber of an induced fibration $M^{\text{1-ridge}} \to B^{\text{1-ridge}}$ on $\partial M^{\text{1-ridge}}$, lying over an interior point of the base, is a connected compact 2-dimensional manifold. Therefore, in the case that $(\pi_3 \circ \mathcal{E})^{-1}(x)$ is contained in a fiber of the fibration of $\partial M^{\text{1-ridge}}$ over an interior point of the base $B^{\text{1-ridge}}$, the two fibers coincide. $\square$

Let W_3'' be a compact 2-dimensional manifold with corners such that $(\pi_3 \circ \mathcal{E})^{-1}(W_3'')$ contains $\bigcup_{i \in I_{\text{2-slim}}} \{|\eta_i| \leq 3.5 \cdot 10^5 \Delta\}$, and put $M^{\text{2-slim}} = M_3 \cap (\pi_3 \circ \mathcal{E})^{-1}(W_3'')$. We endow $M^{\text{2-slim}}$ with the fibration induced by $\pi_3 \circ \mathcal{E}$.

Put $M_4 = M_3 \setminus \operatorname{int}(M^{\text{2-slim}})$.

12.5 The definition of $M^{\text{2-edge}}$

We first truncate W_2. Put

$$W_2' = W_2 \cap \bigcup_{i \in I_{\text{2-edge}}} \left\{ x \in Q_2 : x_i'' \geq 0.9 R_i, \left| \frac{x_i'}{x_i''} \right| \leq 4\Delta \right\} \tag{12.11}$$

and

$$U_2' = (\pi_2 \circ \mathcal{E})^{-1}(W_2') \cap \left(\{\eta_{E'} \leq 0.35\Delta\} \cup \mathcal{E}^{-1} \left\{ x \in H : x_{\mathfrak{r}} > 0, \frac{x_{E'}}{x_{\mathfrak{r}}} \leq 4\Delta \right\} \right). \tag{12.12}$$

Lemma 12.13. *Under the constraints $\Lambda < \overline{\Lambda}(\Delta)$, $\varsigma_{\text{2-edge}} < \overline{\varsigma}_{\text{2-edge}}(\Delta)$, and $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, the following holds.*

(1) $\bigcup_{i \in I_{\text{2-edge}}} \{|\eta_i| \leq 3.5\Delta,\ |\eta_{E'}| \leq 3.5\Delta\} \subset U_2' \subset U_2$, *where U_2 is as in Proposition 11.1.*

(2) *The restriction of $\pi_2 \circ \mathcal{E}$ to U_2' gives a proper submersion to W_2'. In particular, it is a fibration.*

(3) *The fibers of $\pi_2 \circ \mathcal{E} : U_2' \to W_2'$ are diffeomorphic to D^2.*

(4) *M_4 intersects U_2' in a submanifold with corners which is a union of fibers of $\pi_2 \circ \mathcal{E} : U_2' \to W_2'$.*

Proof. The proof of the lemma is similar to the proof of [13, Lemma 14.10] and the proof of Lemma 12.8 (4). $\square$

Lemma 12.14. *Under the constraint $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, $M_4 \cap U_2'$ is compact.*

Proof. The proof of the lemma is similar to the proof of [13, Lemma 14.11]. $\square$

We put $M^{\text{2-edge}} = M_4 \cap U_2'$ and $W_2'' = (\pi_2 \circ \mathcal{E})(M^{\text{2-edge}})$. We endow $M^{\text{2-edge}}$ with the fibration induced by $\pi_2 \circ \mathcal{E}$.

Put $M_5 = M_4 \setminus \operatorname{int}(M^{\text{2-edge}})$.

12.6 The definition of $M^{\text{3-stratum}}$

We first truncate W_1. Put

$$W_1' = W_1 \cap \bigcup_{i \in I_{\text{3-stratum}}} \left\{ x \in H : x_i'' \geq 0.9 R_i, \left| \frac{x_i'}{x_i''} \right| \leq 4 \right\} \tag{12.15}$$

and define $U_1' = \mathcal{E}^{-1}(W_1')$.

Lemma 12.16. *Under the constraints $\varsigma_{\text{3-stratum}} < \bar{\varsigma}_{\text{3-stratum}}$ and $c_{\text{adjust}} < \bar{c}_{\text{adjust}}$, the following holds.*

1. $\bigcup_{i \in I_{\text{3-stratum}}} \{|\eta_i| \leq 3.5\} \subset U_1' \subset U_1$, *where U_1 is as in Proposition 11.1.*
2. *The restriction of $\mathcal{E}$ to U_1' gives a proper submersion to W_1'. In particular, it is a fibration.*
3. *The fibers of $\mathcal{E} : U_1' \to W_1'$ are diffeomorphic to S^1.*
4. *M_5 intersects U_1' in a submanifold with corners which is a union of fibers of $\mathcal{E}\big|_{U_1'} : U_1' \to W_1'$.*

Proof. The proof of the lemma is similar to the proof of [13, Lemma 14.7] and the proof of Lemma 12.8 (4). ☐

We put $M^{\text{3-stratum}} = M_5$ and endow it with the fibration of $\mathcal{E}\big|_{M^{\text{3-stratum}}} : M^{\text{3-stratum}} \to \mathcal{E}(M^{\text{3-stratum}})$.

12.7 Proof of Proposition 12.1

Proposition 12.1 now follows from combining the results in this chapter.

Parts (1) to (7) of Proposition 12.1 follow directly from Lemma 12.4, Lemma 12.6, Lemma 12.8, Lemma 12.10, Lemma 12.13, and Lemma 12.16.

Suppose that part (10) is false. Then, there exists a fiber $F \cong D^2$ of $M^{\text{2-edge}}$ that is disjoint from $M^{\text{0-stratum}} \cup M^{\text{1-slim}} \cup M^{\text{1-ridge}} \cup M^{\text{2-slim}}$. From the proof of Lemma 12.16 (4), each S^1-fiber of $M^{\text{3-stratum}}$ is contained in a single fiber of $M^{\text{2-edge}}$. Therefore, F must be the total space of S^1-fibers. This is a contradiction because D^2 cannot be the total space of S^1-fibers.

Part (11) of Proposition 12.1 follows from the proof of Lemma 12.8 (4) and from the fact that $M^{\text{0-stratum}}$, $M^{\text{1-slim}}$, and $M^{\text{1-ridge}}$ have disjoint interiors. Part (9) follows from similar arguments as in the proof of Lemma 12.10 (4). Part (12) follows from similar arguments as in the proofs of Lemma 12.13 and Lemma 12.16. Part (8) follows from Lemma 12.8 (4) and from part (9).

— 13 —

Decomposing M into fiber bundle components

It follows from Proposition 12.1 that M can be decomposed into domains with disjoint interiors, where each domain is a compact 4-manifold with corners which is also a fiber bundle, with compatibility of fibers along the overlaps. In this chapter, we give a classification of the domains as fiber bundle components and describe the decompositions of fibers along the overlaps.

13.1 Fiber bundle components without boundary

If M contains a fiber bundle component without boundary, then M is a closed Riemannian 4-manifold which admits a metric of nonnegative sectional curvature or M is diffeomorphic to one of the following fiber bundles:

- $\left(\begin{array}{ccc} S^1 & \longrightarrow & M \\ & & \downarrow \\ & & X^3 \end{array}\right)$
- $\left(\begin{array}{ccc} T^2 & \longrightarrow & M \\ & & \downarrow \\ & & \Sigma^2 \end{array}\right)$
- $\left(\begin{array}{ccc} S^2 & \longrightarrow & M \\ & & \downarrow \\ & & \Sigma^2 \end{array}\right)$
- $\left(\begin{array}{ccc} S^3/\Gamma, T^3/\Gamma, S^2 \times S^1, \mathbb{R}P^3 \# \mathbb{R}P^3 & \longrightarrow & M \\ & & \downarrow \\ & & S^1 \end{array}\right)$

where X^3 is a closed 3-manifold and Σ^2 is a closed 2-manifold.

From now on in this chapter, we assume that M does not contain a fiber bundle component without boundary.

13.2 Fiber bundle components with boundary

The boundary of a fiber bundle component $\begin{pmatrix} F_i \longrightarrow & M_i \\ & \downarrow \\ & B_i \end{pmatrix}$ is $\begin{pmatrix} \partial F_i \longrightarrow & N_{i_1} \\ & \downarrow \\ & B_i \end{pmatrix} \cup \begin{pmatrix} F_i \longrightarrow & N_{i_2} \\ & \downarrow \\ & \partial B_i \end{pmatrix}$ where ∂F_i or ∂B_i may be empty.

We denote the boundary of a fiber bundle component M_i with N_i. The classification of fiber bundle components with boundary (based on the dimension of fibers) is given in the following table. See Section 2.4 for details about the topology of Riemannian 4-manifolds.

Table 13.1: Fiber bundle components with boundary

Dim	Fiber Bundle Component	Boundary
4	$\begin{pmatrix} \begin{array}{l} D^4, S^1 \times D^3, S^2 \times_\omega D^2, \\ (S^2 \times_\omega D^2)/\mathbb{Z}_2, \omega \in \mathbb{Z}, \\ (\mathbb{R}P^2 \times S^1) \widetilde{\times} I, \\ (S^2 \widetilde{\times} S^1) \widetilde{\times} I, \\ T^2 \times D^2, T^2 \times_{\mathbb{Z}_2} D^2, \\ \beta_k \widetilde{\times} I, k \in \{1,2,3,4\} \end{array} & \longrightarrow M_i \\ & \Big\downarrow \\ & \mathrm{pt} \end{pmatrix}$	$\begin{pmatrix} \begin{array}{l} S^3, S^2 \times S^1, \mathbb{R}P^3 \# \mathbb{R}P^3, \\ L(\lvert\omega\rvert, 1), L(\lvert\omega\rvert, 1)/\mathbb{Z}_2, \\ T^3, \mathcal{G}_2 \end{array} & \longrightarrow N_i \\ & \Big\downarrow \\ & \mathrm{pt} \end{pmatrix}$
3	$\begin{pmatrix} \begin{array}{l} S^3/\Gamma, T^3/\Gamma, \\ S^2 \times S^1, \mathbb{R}P^3 \# \mathbb{R}P^3 \end{array} & \longrightarrow M_i \\ & \Big\downarrow \\ & (I, \partial I) \end{pmatrix}$ $\begin{pmatrix} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow & M_i \\ & \downarrow \\ & S^1 \end{pmatrix}$ $\begin{pmatrix} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow & M_i \\ & \downarrow \\ & S^1 \end{pmatrix}$	$\begin{pmatrix} \begin{array}{l} S^3/\Gamma, T^3/\Gamma, \\ S^2 \times S^1, \mathbb{R}P^3 \# \mathbb{R}P^3 \end{array} & \longrightarrow N_i \\ & \Big\downarrow \\ & \partial I \end{pmatrix}$ $\begin{pmatrix} S^2 \longrightarrow & N_i \\ & \downarrow \\ & S^1 \end{pmatrix}$ $\begin{pmatrix} T^2 \longrightarrow & N_i \\ & \downarrow \\ & S^1 \end{pmatrix}$

Dim	Fiber Bundle Component	Boundary
3	$\left(\begin{tikzcd} D^3, S^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_i \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$	$\left(\begin{tikzcd} S^2 \arrow[r] & N_{i_1} \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right) \cup \left(\begin{tikzcd} D^3, S^2 \times_{\mathbb{Z}_2} I \arrow[r] & N_{i_2} \arrow[d] \\ & \partial I \end{tikzcd}\right)$
	$\left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_i \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$	$\left(\begin{tikzcd} T^2 \arrow[r] & N_{i_1} \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right) \cup \left(\begin{tikzcd} \begin{matrix} S^1 \times D^2, \\ T^2 \times_{\mathbb{Z}_2} I \end{matrix} \arrow[r] & N_{i_2} \arrow[d] \\ & \partial I \end{tikzcd}\right)$
2	$\left(\begin{tikzcd} S^2, T^2 \arrow[r] & M_i \arrow[d] \\ & (\Sigma^2, \partial \Sigma^2) \end{tikzcd}\right)$	$\left(\begin{tikzcd} S^2, T^2 \arrow[r] & N_i \arrow[d] \\ & \partial \Sigma^2 \end{tikzcd}\right)$
	$\left(\begin{tikzcd} D^2 \arrow[r] & M_i \arrow[d] \\ & \Sigma^2 \end{tikzcd}\right)$	$\left(\begin{tikzcd} S^1 \arrow[r] & N_i \arrow[d] \\ & \Sigma^2 \end{tikzcd}\right)$
	$\left(\begin{tikzcd} D^2 \arrow[r] & M_i \arrow[d] \\ & (\Sigma^2, \partial \Sigma^2) \end{tikzcd}\right)$	$\left(\begin{tikzcd} S^1 \arrow[r] & N_{i_1} \arrow[d] \\ & (\Sigma^2, \partial \Sigma^2) \end{tikzcd}\right) \cup \left(\begin{tikzcd} D^2 \arrow[r] & N_{i_2} \arrow[d] \\ & \partial \Sigma^2 \end{tikzcd}\right)$
1	$\left(\begin{tikzcd} S^1 \arrow[r] & M_i \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right)$	$\left(\begin{tikzcd} S^1 \arrow[r] & N_i \arrow[d] \\ & \partial X^3 \end{tikzcd}\right)$

13.3 Compatibility of fibers

It follows from Proposition 12.1 that

$$M = \bigcup_{i=1}^{N} \left(\begin{tikzcd} F_i \arrow[r] & M_i \arrow[d] \\ & B_i \end{tikzcd}\right) \tag{13.1}$$

where $\left(\begin{tikzcd} F_i \arrow[r] & M_i \arrow[d] \\ & B_i \end{tikzcd}\right)$ is a fiber bundle component given in Table 13.1. Additionally, the fiber bundle components have disjoint interiors. They intersect along the boundaries so that the fibers along the overlaps are compatible. That is, if two fibers intersect, then either one of them is contained in the other or they coincide. Consequently, a boundary fiber (F_i over a boundary point of B_i or ∂F_i over an interior point of B_i) is either contained in another boundary fiber or is the union of other boundary fibers.

In the following sections, we explicitly describe the decomposition of fibers along the overlaps as the unions of other types of fibers. This information will be used in the next chapter to glue different fiber bundle components of M into building blocks. We note that not all combinatorial configurations are feasible due to topological obstructions.

13.4 Notation

For simplicity, we define the following notation for this chapter and the following chapters.

$$\text{We will denote a fiber bundle component } \left(\begin{array}{l} D^4, S^1\times D^3, S^2\times_\omega D^2,\\ (S^2\times_\omega D^2)/\mathbb{Z}_2, \omega\in\mathbb{Z},\\ (\mathbb{R}P^2\times S^1)\widetilde{\times} I, (S^2\widetilde{\times} S^1)\widetilde{\times} I,\\ T^2\times D^2, T^2\times_{\mathbb{Z}_2} D^2,\\ \beta_k\widetilde{\times} I, k\in\{1,2,3,4\} \end{array} \longrightarrow \begin{array}{c} M_i\\ \Big\downarrow\\ \text{pt}\end{array}\right)$$

$$\text{by } \left(\begin{array}{c} D^4,\ldots\longrightarrow M_i\\ \downarrow\\ \text{pt}\end{array}\right) \text{ and denote its boundary by } \left(\begin{array}{c} S^3,\ldots\longrightarrow N_i\\ \downarrow\\ \text{pt}\end{array}\right).$$

$$\text{We will denote a fiber bundle component } \left(\begin{array}{l} S^3/\Gamma, T^3/\Gamma,\\ S^2\times S^1, \mathbb{R}P^3\#\mathbb{R}P^3 \end{array} \longrightarrow \begin{array}{c} M_i\\ \Big\downarrow\\ (I,\partial I)\end{array}\right)$$

$$\text{by } \left(\begin{array}{c} S^3/\Gamma,\ldots\longrightarrow M_i\\ \downarrow\\ (I,\partial I)\end{array}\right) \text{ and denote its boundary by } \left(\begin{array}{c} S^3/\Gamma,\ldots\longrightarrow N_i\\ \downarrow\\ \partial I\end{array}\right).$$

Let X and Y be topological spaces with boundary. We denote by $X\cup_\partial Y$ the union $X\cup Y$ with the condition that $X\cap Y=\partial X\cap\partial Y$.

We denote by $X\sqcup Y$ the union $X\cup Y$ with an emphasis that $X\cap Y=\varnothing$. Most of the time, X and Y will be subsets of M. The topology on $X\sqcup Y$ will be the topology induced from M.

13.5 Fiber bundle components with fibers S^2 or T^2

We start with the decompositions of fibers of components $\left(\begin{array}{r} S^2, T^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$.

Lemma 13.2. *Let* M_i *be a fiber bundle component* $\left(\begin{array}{r} S^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$. *Its boundary is* $\left(\begin{array}{r} S^2 \longrightarrow N_i \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$. *Let* $F \cong S^2$ *be a fiber of* $\left(\begin{array}{r} S^2 \longrightarrow N_i \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$. *Then, the following holds.*

(1) *If* $F \cap \left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right) \neq \varnothing$, *then* $F \subset \left(\begin{array}{r} S^3, \ldots \longrightarrow N_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$.

(2) *If* $F \cap \left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right) \neq \varnothing$, *then* F *is contained in a connected component of* $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$.

(3) *If* $F \cap \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right) \neq \varnothing$, *then* F *is a fiber of* $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$.

(4) *Otherwise,* $F = A_1 \cup_\partial B \cup_\partial A_2$ *where* $A_k \cong D^2$, $k \in \{1, 2\}$, *is a fiber of* $\left(\begin{array}{r} D^2 \longrightarrow N_{j_k} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, *for some* j_k, *and* $B \cong S^1 \times I$ *is a subbundle of* $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$, *for some* j. *The unions are so that each* ∂A_k, $k \in \{1, 2\}$, *is identified with a boundary component of* B.

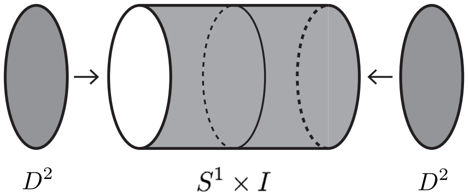


Figure 13.1: A decomposition of F in case (4). $F = A_1 \cup_\partial B \cup_\partial A_2$.

Proof. Proposition 12.1 directly implies (1), (2), and (3). Moreover, F is disjoint from $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$, and $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \end{array}\right)$ components. Hence, if F is also disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \text{pt} \end{array}\right)$, $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, and $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components, then $F \subset \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)\right] \cup_\partial \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)\right]$. We put $A = F \cap \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)\right]$ and $B = F \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)\right]$.

It follows that $A \cong \bigsqcup_j D^2$ and $B = F - A$ is the total space of S^1-fibers. Hence, the Euler characteristic $\chi(B) = 0$. Since $F \cong S^2$, $B \cong S^2 - \bigsqcup_j D^2$ must be a cylinder. Therefore, $F = A_1 \cup_\partial B \cup_\partial A_2$ where $A_k \cong D^2$, $k \in \{1, 2\}$, is a fiber of $\left(\begin{array}{r} D^2 \longrightarrow N_{j_k} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some j_k, and $B \cong S^1 \times I$ is a subbundle of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some j. ☐

Lemma 13.3. *Let M_i be a fiber bundle component $\left(\begin{array}{r} T^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$. Its boundary is $\left(\begin{array}{r} T^2 \longrightarrow N_i \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$. Let $F \cong T^2$ be a fiber of $\left(\begin{array}{r} T^2 \longrightarrow N_i \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$. Then, the following holds.*

(1) *If* $F \cap \left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right) \neq \varnothing$, *then* $F \subset \left(\begin{array}{r} S^3, \ldots \longrightarrow N_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$.

(2) *If* $F \cap \left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right) \neq \varnothing$, *then* F *is contained in a connected component of* $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$.

(3) *If* $F \cap \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \ \textit{or} \ (I, \partial I) \end{array}\right) \neq \varnothing$, *then* F *is a fiber of* $\left(\begin{array}{r} T^2 \longrightarrow N_j \\ \downarrow \\ S^1 \ \textit{or} \ (I, \partial I) \end{array}\right)$.

(4) *Otherwise,* $F \subset \left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ *for some* j. *In particular,* F *is the total space of* S^1*-fibers and* F *is disjoint from any* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *component.*

Proof. Proposition 12.1 directly implies (1), (2), and (3). Moreover, F is disjoint from $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$, and $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \end{array}\right)$ components.

Hence, if F is also disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, and $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components, then $F \subset \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)\right]$. We put $A = F \cap \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)\right]$

and $B = F \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array} \right) \right]$.

It follows that $A \cong \bigsqcup_{j=1}^m D^2$ is disjoint union of m copies of D^2 and $B = F - A$ is the total space of S^1-fibers. Hence, the Euler characteristic $\chi(B) = 0$. Since $F \cong T^2$, this is not possible unless $m = 0$. Therefore, $F = B$ is the total space of S^1-fibers and F is disjoint from any $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array} \right)$ component. By connectedness, $F \subset \left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array} \right)$, for some j. □

13.6 Fiber bundle components with fibers $D^3, S^2 \times_{\mathbb{Z}_2} I, S^1 \times D^2$, or $T^2 \times_{\mathbb{Z}_2} I$

In this section, we describe the decompositions of fibers of components $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_i \\ \downarrow \\ S^1 \end{array} \right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array} \right)$, $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_i \\ \downarrow \\ S^1 \end{array} \right)$, and $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array} \right)$.

Lemma 13.4. *Let M_i be a fiber bundle component $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_i \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array} \right)$. Its boundary is $\left(\begin{array}{r} S^2 \longrightarrow N_i \\ \downarrow \\ S^1 \end{array} \right)$ or $\left(\begin{array}{r} S^2 \longrightarrow N_{i_1} \\ \downarrow \\ (I, \partial I) \end{array} \right) \cup \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{i_2} \\ \downarrow \\ \partial I \end{array} \right)$. Let $F \cong D^3$ or $(S^2 \times_{\mathbb{Z}_2} I)$ be a fiber of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_i \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array} \right)$. Then, the following holds.*

(1) *If F is a fiber over an interior point of S^1 or I, then F is disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \text{pt} \end{array} \right)$, $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array} \right)$, $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array} \right)$, $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array} \right)$, and $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \end{array} \right)$ components.*

(2) *If F is a fiber of* $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{i_2} \\ \downarrow \\ \partial I \end{array}\right)$*, then F is disjoint from* $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$*, and* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \end{array}\right)$ *components.*

Additionally, F is contained in a boundary component of $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ *or* $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$*, for some j.*

(3) *If* $F \cap \left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right) \neq \varnothing$*, then $F \cap M_j = \partial F \cap \partial M_j$ and $\partial F \cong S^2$ is a fiber of* $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$.

(4) *Otherwise, $\partial F = A_1 \cup_\partial B \cup_\partial A_2$ where $A_k \cong D^2$, $k \in \{1, 2\}$, is a fiber of* $\left(\begin{array}{r} D^2 \longrightarrow N_{j_k} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, *for some j_k, and $B \cong S^1 \times I$ is a subbundle of* $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$*, for some j. The unions are so that ∂A_k, $k \in \{1, 2\}$, is identified with a boundary component of B.*

Proof. Proposition 12.1 directly implies (1), (2), and (3). The proof of (4) is similar to the proof of Lemma 13.2. □

Lemma 13.5. *Let M_i be a fiber bundle component* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_i \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$.

Its boundary is $\left(\begin{array}{r} T^2 \longrightarrow N_i \\ \downarrow \\ S^1 \end{array}\right)$ *or* $\left(\begin{array}{r} T^2 \longrightarrow N_{i_1} \\ \downarrow \\ (I, \partial I) \end{array}\right) \cup \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{i_2} \\ \downarrow \\ \partial I \end{array}\right)$.

Let $F \cong S^1 \times D^2$ *or* $T^2 \times_{\mathbb{Z}_2} I$ *be a fiber of* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_i \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$. *Then, the following holds.*

(1) *If* F *is a fiber over an interior point of* S^1 *or* I, *then* F *is disjoint from* $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$, *and* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \end{array}\right)$ *components.*

(2) *If* F *is a fiber of* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{i_2} \\ \downarrow \\ \partial I \end{array}\right)$, *then* F *is disjoint from* $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$, *and* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \end{array}\right)$ *components.*

Additionally, F *is contained in a boundary component of* $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ *or* $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, *for some* j.

(3) *If* $F \cap \left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right) \neq \varnothing$, *then* $F \cap M_j = \partial F \cap \partial M_j$ *and* $\partial F \cong T^2$ *is a fiber of* $\left(\begin{array}{r} T^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$.

(4) *Otherwise,* $\partial F \subset \left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ *for some* j. *In particular,* ∂F *is the total space of*

S^1-fibers and ∂F is disjoint from any $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *component.*

Proof. Proposition 12.1 directly implies (1), (2), and (3). The proof of (4) is similar to the proof of Lemma 13.3. ☐

13.7 Fiber bundle components with fibers D^2 or S^1

In this section, we describe how the fibers of components $\left(\begin{array}{r} D^2 \longrightarrow M_i \\ \downarrow \\ \Sigma^2 \end{array}\right)$, $\left(\begin{array}{r} D^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, and $\left(\begin{array}{r} S^1 \longrightarrow M_i \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ intersect with fibers of other types.

Lemma 13.6. *Let M_i be a fiber bundle component* $\left(\begin{array}{r} D^2 \longrightarrow M_i \\ \downarrow \\ \Sigma^2 \text{ or } (\Sigma^2, \partial\Sigma^2) \end{array}\right)$. *Its boundary is* $\left(\begin{array}{r} S^1 \longrightarrow N_i \\ \downarrow \\ \Sigma^2 \end{array}\right)$ *or* $\left(\begin{array}{r} S^1 \longrightarrow N_{i_1} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right) \cup \left(\begin{array}{r} D^2 \longrightarrow N_{i_2} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$. *Let $F \cong D^2$ be a fiber of* $\left(\begin{array}{r} D^2 \longrightarrow M_i \\ \downarrow \\ \Sigma^2 \text{ or } (\Sigma^2, \partial\Sigma^2) \end{array}\right)$. *Then, the following hold.*

1. *If F is a fiber over an interior point of Σ^2 or $(\Sigma^2, \partial\Sigma^2)$, then $\partial F \cong S^1$ is a fiber of* $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$, *for some j.*
2. *If F is a fiber of* $\left(\begin{array}{r} D^2 \longrightarrow N_{i_2} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, *then F is contained in*
   - *(a) the boundary of* $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$,
   - *(b) a boundary component of* $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$,

(c) *a fiber of* $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, *or*

(d) *the* S^2*-boundary of a fiber of* $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$.

Proof. The lemma follows from Proposition 12.1 and Lemma 13.2 to Lemma 13.5. □

Lemma 13.7. *Let* M_i *be a fiber bundle component* $\left(\begin{array}{r} S^1 \longrightarrow M_i \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$. *Its boundary is* $\left(\begin{array}{r} S^1 \longrightarrow N_i \\ \downarrow \\ \partial X^3 \end{array}\right)$. *Let* $F \cong S^1$ *be a fiber of* $\left(\begin{array}{r} S^1 \longrightarrow N_i \\ \downarrow \\ \partial X^3 \end{array}\right)$. *Then,* F *is contained in*

(a) *the boundary of* $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$,

(b) *a boundary component of* $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$,

(c) *a fiber of* $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$,

(d) *the* S^2*-boundary of a fiber of* $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$,

(e) *a fiber of* $\left(\begin{array}{r} T^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$,

(f) *the* T^2*-boundary of a fiber of* $\left(\begin{array}{r} S^1 \times D^3, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \text{ or } (I, \partial I) \end{array}\right)$, *or*

(g) *the* S^1*-boundary of a fiber of* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \text{ or } (\Sigma^2, \partial\Sigma^2) \end{array}\right)$.

Proof. The lemma directly follows from Proposition 12.1. ☐

13.8 Fiber bundle components $\left(\begin{array}{r} D^4, \dots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$

Let M_0 be a fiber bundle component $\left(\begin{array}{r} D^4, \dots \longrightarrow M_0 \\ \downarrow \\ \mathrm{pt} \end{array}\right)$. Its boundary is $\left(\begin{array}{r} S^3, \dots \longrightarrow \partial M_0 \\ \downarrow \\ \mathrm{pt} \end{array}\right)$. Proposition 12.1 and lemmas in the previous section imply that $\left(\begin{array}{r} S^3, \dots \longrightarrow \partial M_0 \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ is the union of fibers of other fiber bundle components from Table 13.1. In this section, we describe the decomposition of ∂M_0.

Lemma 13.8. *Let M_j be a fiber bundle component* $\left(\begin{array}{r} S^3/\Gamma, \dots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$. *If $M_0 \cap M_j \neq \varnothing$, then ∂M_0 coincides with a boundary component of M_j.*

Proof. The lemma directly follows from Proposition 12.1. ☐

Lemma 13.9. *If M_0 intersects with exactly one fiber bundle component M_j, then ∂M_0 coincides with a boundary component of M_j. Additionally, M_j is a fiber bundle component* $\left(\begin{array}{r} S^3/\Gamma, \dots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, *or* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$.

Proof. The lemma directly follows from Proposition 12.1. ☐

Lemma 13.10. *Assume that M_0 intersects with* $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *and* $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ *components, and M_0 is disjoint from fiber bundle components of* *other types, i.e.*

$$\partial M_0 \subset \left[\bigsqcup_j \left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right]. \tag{13.11}$$

Then, $\partial M_0 = B_1 \cup_\partial A \cup_\partial B_2$ *where:*

(i) $A \cong S^2 \times I$ *is a subbundle of* $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$*, for some* j*,*

(ii) $B_i \cong D^3$ *or* $S^2 \times_{\mathbb{Z}_2} I$*,* $i \in \{1, 2\}$*, is a fiber of* $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_i} \\ \downarrow \\ \partial I \end{array}\right)$*, for some* j_i*, and*

(iii) ∂B_i*,* $i \in \{1, 2\}$*, coincides with a boundary component of* A*.*

That is

$$\partial M_0 \cong \left\{ \begin{array}{c} D^3 \\ S^2 \times_{\mathbb{Z}_2} I \end{array} \right\} \cup_\partial (S^2 \times I) \cup_\partial \left\{ \begin{array}{c} D^3 \\ S^2 \times_{\mathbb{Z}_2} I \end{array} \right\}. \tag{13.12}$$

where the unions are along boundary components.

In particular,

$$\partial M_0 \cong \left\{ \begin{array}{ll} S^3 & \cong D^3 \cup D^3, \\ \mathbb{R}P^3 & \cong D^3 \cup S^2 \times_{\mathbb{Z}_2} I, \\ \mathbb{R}P^3 \# \mathbb{R}P^3 & \cong S^2 \times_{\mathbb{Z}_2} I \cup S^2 \times_{\mathbb{Z}_2} I, \end{array} \right. \tag{13.13}$$

and

$$M_0 \cong \left\{ \begin{array}{ll} D^4, \pm\mathbb{C}P^2 \# D^4 & \textit{if } \partial M_0 \cong S^3, \\ S^2 \times_{\pm 2} D^2 & \textit{if } \partial M_0 \cong \mathbb{R}P^3, \\ S^2 \times_{\mathbb{Z}_2} D^2 & \textit{if } \partial M_0 \cong \mathbb{R}P^3 \# \mathbb{R}P^3. \end{array} \right. \tag{13.14}$$

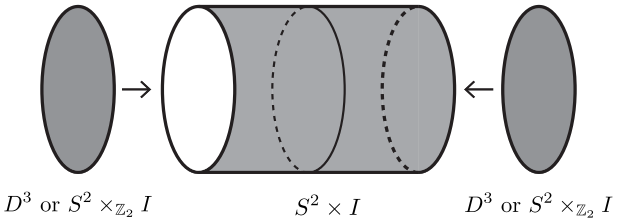


Figure 13.2: The decomposition of ∂M_0 in Lemma 13.10. $\partial M_0 = B_1 \cup_\partial A \cup_\partial B_2$.

Proof. Consider that $\left(\begin{array}{r} S^3, \ldots \longrightarrow \partial M_0 \\ \downarrow \\ \text{pt} \end{array}\right) \cap \left[\bigsqcup_j \left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)\right] \cong \bigsqcup_{j_k} (S^2 \times I)$. By Lemma 13.4, each boundary component of a copy of $S^2 \times I$ is identified with the boundary of a D^3 or

$S^2 \times_{\mathbb{Z}_2} I$-fiber from $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$, for some j. Because ∂M_0 is connected, it must contain exactly one copy of $S^2 \times I$. Consequently, ∂M_0 is the union of exactly two disjoint D^3 or $S^2 \times_{\mathbb{Z}_2} I$-fibers and one copy of $S^2 \times I \subset \left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$.

That is,

$$\partial M_0 \cong \left\{ \begin{array}{c} D^3 \\ S^2 \times_{\mathbb{Z}_2} I \end{array} \right\} \cup_\partial (S^2 \times I) \cup_\partial \left\{ \begin{array}{c} D^3 \\ S^2 \times_{\mathbb{Z}_2} I \end{array} \right\} \tag{13.15}$$

where the unions are along boundary components. Hence,

$$\partial M_0 \cong \left\{ \begin{array}{ll} D^3 \cup D^3 & \cong S^3, \\ D^3 \cup S^2 \times_{\mathbb{Z}_2} I & \cong \mathbb{R}P^3, \\ S^2 \times_{\mathbb{Z}_2} I \cup S^2 \times_{\mathbb{Z}_2} I & \cong \mathbb{R}P^3 \# \mathbb{R}P^3. \end{array} \right. \tag{13.16}$$

The classification of M_0 follows from Lemma 2.12 and Table 13.1. □

Lemma 13.17. *Assume that* M_0 *intersects with* $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *and* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ *components, and* M_0 *is disjoint from fiber bundle components of other types, i.e.*

$$\partial M_0 \subset \left[\bigsqcup_j \left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right) \right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right) \right]. \tag{13.18}$$

Then, $\partial M_0 = B_1 \cup_\partial A \cup_\partial B_2$ *where:*

(i) $A \cong T^2 \times I$ *is a subbundle of* $\left(\begin{array}{r} T^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$,

(ii) $B_i \cong S^1 \times D^2$ *or* $T^2 \times_{\mathbb{Z}_2} I$, $i \in \{1, 2\}$, *is a fiber of* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_i} \\ \downarrow \\ \partial I \end{array}\right)$.

(iii) ∂B_i, $i \in \{1, 2\}$, *coincides with a boundary component of* A.

That is,

$$\partial M_0 \cong \left\{ \begin{array}{c} S^1 \times D^2 \\ T^2 \times_{\mathbb{Z}_2} I \end{array} \right\} \cup_\partial (T^2 \times I) \cup_\partial \left\{ \begin{array}{c} S^1 \times D^2 \\ T^2 \times_{\mathbb{Z}_2} I \end{array} \right\} \tag{13.19}$$

where the unions are along boundary components.

In particular,

$$\partial M_0 \cong \left\{ \begin{array}{ll} S^1 \times D^2 \cup S^1 \times D^2 & \cong S^3, S^2 \times S^1, L(p,q), \\ S^1 \times D^2 \cup T^2 \times_{\mathbb{Z}_2} I & \cong S^2 \times S^1, \mathbb{R}P^3 \# \mathbb{R}P^3, L(p,q)/\mathbb{Z}_2, \\ T^2 \times_{\mathbb{Z}_2} I \cup T^2 \times_{\mathbb{Z}_2} I & \cong \mathcal{G}_2, \mathcal{G}_4, \mathcal{G}_5, \end{array} \right. \tag{13.20}$$

and

$$M_0 \cong \left\{ \begin{array}{ll} D^4, \pm\mathbb{C}P^2 \# D^4 & \textit{if } \partial M_0 \cong S^3, \\ S^1 \times D^3, S^2 \times D^2, (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I & \textit{if } \partial M_0 \cong S^2 \times S^1, \\ S^2 \times_\omega D^2, \omega \in \mathbb{Z} & \textit{if } \partial M_0 \cong L(|\omega|, 1), \\ S^2 \times_{\mathbb{Z}_2} D^2 & \textit{if } \partial M_0 \cong \mathbb{R}P^3 \# \mathbb{R}P^3, \\ (S^2 \times_\omega D^2)/\mathbb{Z}_2, \omega \in \mathbb{Z} & \textit{if } \partial M_0 \cong L(|\omega|, 1)/\mathbb{Z}_2, \\ T^2 \times_{\mathbb{Z}_2} D^2, \mathcal{B}_3 \widetilde{\times} I, \mathcal{B}_4 \widetilde{\times} I & \textit{if } \partial M_0 \cong \mathcal{G}_2. \end{array} \right. \tag{13.21}$$

Proof. Consider that $\begin{pmatrix} S^3, \dots \longrightarrow \partial M_0 \\ \hfill\downarrow \\ \hfill\text{pt} \end{pmatrix} \cap \left[\bigsqcup_j \begin{pmatrix} T^2 \longrightarrow N_j \\ \hfill\downarrow \\ \hfill\partial\Sigma^2 \end{pmatrix} \right] \cong \bigsqcup_{j_k} (T^2 \times I)$. By Lemma 13.5, each boundary component of a copy of $T^2 \times I$ is identified with the boundary of a $S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I$-fiber from $\begin{pmatrix} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_i} \\ \hfill\downarrow \\ \hfill\partial I \end{pmatrix}$, for some j. Because ∂M_0 is connected, it must contain exactly one copy of $T^2 \times I$. Consequently, ∂M_0 is the union of exactly two disjoint $S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I$-fibers and one copy of $T^2 \times I \subset \begin{pmatrix} T^2 \longrightarrow N_j \\ \hfill\downarrow \\ \hfill\partial\Sigma^2 \end{pmatrix}$. That is,

$$\partial M_0 \cong \left\{ \begin{array}{c} S^1 \times D^2 \\ T^2 \times_{\mathbb{Z}_2} I \end{array} \right\} \cup_\partial (T^2 \times I) \cup_\partial \left\{ \begin{array}{c} S^1 \times D^2 \\ T^2 \times_{\mathbb{Z}_2} I \end{array} \right\} \tag{13.22}$$

where the unions are along boundary components. In particular,

$$\partial M_0 \cong \left\{ \begin{array}{l} S^1 \times D^2 \cup S^1 \times D^2, \\ S^1 \times D^2 \cup T^2 \times_{\mathbb{Z}_2} I, \\ T^2 \times_{\mathbb{Z}_2} I \cup \ T^2 \times_{\mathbb{Z}_2} I, \end{array} \right. \tag{13.23}$$

where the unions are along the boundaries.

$S^1 \times D^2 \cup_\partial S^1 \times D^2$ is diffeomorphic to $S^3, S^2 \times S^1$, or a Lens space $L(p,q)$. Put $X = S^1 \times D^2 \cup_\partial T^2 \times_{\mathbb{Z}_2} I$ and let $\widehat{X}$ be a double cover of X. Then, $\widehat{X} = (S^1 \times D^2) \cup_f (T^2 \times I) \cup_f$

$(S^1 \times D^2)$ where f is an identifying map from $\partial(S^1 \times D^2)$ to a T^2-boundary component of $T^2 \times I$. We note that the identifying map must be the same for both copies of $S^1 \times D^2$. Hence, $\widehat{X}$ is diffeomorphic to $S^2 \times S^1$ or a Lens space. There are two orientable $\mathbb{Z}_2$-quotients of $S^2 \times S^1$: $S^2 \times S^1$ where $\mathbb{Z}_2$ acts on S^2 by a π-rotation around a fixed axis and acts on S^1 by a π-rotation, and $S^2 \times_{\mathbb{Z}_2} S^1 \cong \mathbb{R}P^3 \# \mathbb{R}P^3$ where $\mathbb{Z}_2$ acts on S^2 and S^1 by the antipodal map. A $\mathbb{Z}_2$-quotient of a Lens space is a Prism manifold. A double cover of $T^2 \times_{\mathbb{Z}_2} I \cup_\partial T^2 \times_{\mathbb{Z}_2} I$ is a T^2-bundle over S^1. From Table 13.1, $T^2 \times_{\mathbb{Z}_2} I \cup_\partial T^2 \times_{\mathbb{Z}_2} I$ is T^3/Γ that has a T^2-bundle over S^1 as its double cover. Hence, $T^2 \times_{\mathbb{Z}_2} I \cup_\partial T^2 \times_{\mathbb{Z}_2} I$ is $\mathcal{G}_2, \mathcal{G}_4$, or $\mathcal{G}_5$ [33].

The classification of M_0 follows from Lemma 2.12 and Table 13.1. □

Lemma 13.24. *Assume that M_0 intersects with $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ and $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ components, and M_0 is disjoint from fiber bundle components of other types, i.e.*

$$\partial M_0 \subset \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right]. \tag{13.25}$$

Then, $\partial M_0 = A \cup_\partial B_1$ or $A \cup_\partial (B_1 \sqcup B_2)$ where:

(i) A is a subbundle of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ for some j. Additionally, A is the total space of S^1-fibers over a disk or a cylinder.

(ii) $B_i \cong D^2 \times S^1$ is a component $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ for some j.

(iii) ∂B_i, $i \in \{1, 2\}$, is identified with a boundary component of A so that each ∂D^2-fiber of $\partial B_i \cong \partial D^2 \times S^1$ coincides with an S^1-fiber of ∂A.

Consequently,

$$\partial M_0 \cong \left\{ \begin{array}{ll} S^1 \times D^2 \cup S^1 \times B^2 & \cong S^3 \\ S^1 \times D^2 \cup S^1 \times (S^1 \times I) \cup S^1 \times D^2 & \cong S^2 \times S^1 \end{array} \right. \tag{13.26}$$

where $S^1 \times B^2$ and $S^1 \times (S^1 \times I)$ are contained in $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ for some j.

Hence,

$$M_0 \cong \left\{ \begin{array}{ll} D^4, \pm\mathbb{C}P^2 \# D^4 & \textit{if } \partial M_0 \cong S^3, \\ S^1 \times D^3, S^2 \times D^2, (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I & \textit{if } \partial M_0 \cong S^2 \times S^1. \end{array} \right. \tag{13.27}$$

Proof. Write $\partial M_0 = A \cup_\partial B$ where we put $A = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right]$ and $B = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)\right]$.

By Proposition 12.1, A is a disjoint union of subbundles of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$. Hence, A is the total space of S^1-fibers over a disjoint union of surfaces with boundary.

From Lemma 13.6, each fiber over an interior points of $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ is disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_0 \\ \downarrow \\ \mathrm{pt} \end{array}\right)$. Hence, B is a disjoint union of components $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial \Sigma^2 \end{array}\right) \cong D^2 \times S^1$ and D^2-subbundles of $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial \Sigma^2 \end{array}\right)$ diffeomorphic to $D^2 \times I$.

Additionally, $A \cap B = \partial A \cap \partial B$ so that the S^1-boundary of each D^2-fiber of $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial \Sigma^2 \end{array}\right)$ coincides with an S^1-fiber of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$. Since ∂M_0 is a closed 3-manifold, B does not contain connected components diffeomorphic to $D^2 \times I$. Therefore, B is a disjoint union of components $B_i = \left(\begin{array}{r} D^2 \longrightarrow N_{j_i} \\ \downarrow \\ \partial \Sigma^2 \end{array}\right) \cong D^2 \times S^1$. $\partial B_i \cong T^2$ is identified with a boundary component of A so that each ∂D^2-fiber of ∂B_i is identified with an S^1-fiber of ∂A. Because ∂M_0 is connected, A must also be connected. We have that A is a subbundle $\left(\begin{array}{r} S^1 \longrightarrow A \\ \downarrow \\ (\Sigma_A^2, \partial \Sigma_A^2) \end{array}\right)$ of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some j, where $\Sigma_A = \Sigma_A(g,n)$ is a connected surface of genus g and with $n \geq 1$ boundary components. Then,

$$\partial M_0 = A \cup_\partial \bigsqcup_{i=1}^{n} B_i \cong \left(\begin{array}{r} S^1 \longrightarrow A \\ \downarrow \\ (\widetilde{\Sigma}^2, \partial \widetilde{\Sigma}^2) \end{array}\right) \cup_\partial \bigsqcup_{i=1}^{n} (S^1 \times D^2). \tag{13.28}$$

By an explicit construction in [11, Proposition 1], $\partial M_0 \cong S^3 \# (2g+n-1)(S^2 \times S^1)$. According to the classification of $\left(\begin{array}{r} D^4, \dots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ in Table 13.1, ∂M_0 can only be diffeomorphic to S^3 or $S^2 \times S^1$ in this case. Thus, $2g+n-1 = 0$ or 1. That is $(g,n) = (0,1)$ or $(0,2)$. Therefore, Σ^2_A is a disk or a cylinder.

In summary,

$$\partial M_0 \cong \begin{cases} S^1 \times D^2 \cup S^1 \times B^2 & \cong S^3, \\ S^1 \times D^2 \cup S^1 \times (S^1 \times I) \cup S^1 \times D^2 & \cong S^2 \times S^1, \end{cases} \tag{13.29}$$

where $S^1 \times B^2$ and $S^1 \times (S^1 \times I)$ are contained in $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ for some j.

The classification of M_0 follows from Lemma 2.12 and Table 13.1. □

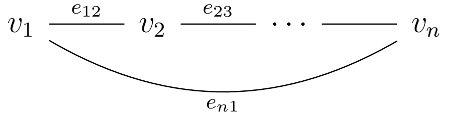
Lemma 13.30. *Assume that* M_0 *intersects with* $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components, and* M_0 *is disjoint from fiber bundle components of other types, i.e.*

$$\partial M_0 \subset \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right) \right] \cup \left[\bigsqcup_j \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right) \right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right) \right]. \tag{13.31}$$

Then, $\partial M_0 = A \cup_\partial B_1$ *or* $A \cup_\partial (B_1 \sqcup B_2)$ *where:*

(i) A *is a subbundle of* $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ *for some* j*. Additionally,* A *is the total space of* S^1*-fibers over a disk or a cylinder.*

(ii) For $i \in \{1,2\}$, B_i is a component $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some j, or a 3-manifold with boundary, which can be represented by a cycle graph



(13.32)

so that each vertex v_α represents a fiber V_α of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_\alpha} \\ \downarrow \\ \partial I \end{array}\right)$ for some j_α, and each edge $e_{\alpha(\alpha+1)}$ represents a D^2-subbundle $E_{\alpha(\alpha+1)} \cong D^2 \times I$ of $\left(\begin{array}{r} D^2 \longrightarrow N_{j_\alpha} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ for some j_α. $V_\alpha \cong D^3$ or $S^2 \times_{\mathbb{Z}_2} I$ and $V_\alpha \cap E_{\alpha(\alpha+1)} = \partial V_\alpha \cap \partial E_{\alpha(\alpha+1)}$ coincides with a connected component of $D^2 \times \partial I \subset E_{\alpha(\alpha+1)}$.

If $\partial M = A \cup_\partial B_1$, then B_1 is represented by a cyclic graph and there are at most two V_α's such that $V_\alpha \cong S^2 \times_{\mathbb{Z}_2} I$. In particular, B_1 is diffeomorphic to $S^1 \times D^2$, $(S^1 \times D^2)\#\mathbb{R}P^3$, or $(S^1 \times D^2)\#\mathbb{R}P^3\#\mathbb{R}P^3$.

If $\partial M = A \cup_\partial (B_1 \sqcup B_2)$, then all V_α's are diffeomorphic to D^3. In particular, B_1 and B_2 are diffeomorphic to $S^1 \times D^2$. At most one of B_1 and B_2 is a component $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$.

(iii) ∂B_i is identified with a boundary component of A so that each ∂D^2-fiber of $\partial B_i \cong \partial D^2 \times S^1$ coincides with an S^1-fiber of A.

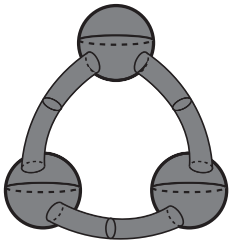
That is

$$\partial M_0 \cong \begin{cases} S^1 \times B^2 \cup S^1 \times D^2 & \cong S^3, \\ S^1 \times B^2 \cup (S^1 \times D^2)\#\mathbb{R}P^3 & \cong \mathbb{R}P^3, \\ S^1 \times B^2 \cup (S^1 \times D^2)\#(\mathbb{R}P^3\#\mathbb{R}P^3) & \cong \mathbb{R}P^3\#\mathbb{R}P^3, \\ S^1 \times D^2 \cup S^1 \times (S^1 \times I) \cup S^1 \times D^2 & \cong S^2 \times S^1, \end{cases} \tag{13.33}$$

where $S^1 \times B^2$ and $S^1 \times (S^1 \times I)$ are contained in $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ for some j.

Hence,

$$M_0 \cong \begin{cases} D^4, \pm\mathbb{C}P^2\#D^4 & \text{if } \partial M_0 \cong S^3, \\ S^2 \times_{\pm 2} D^2 & \text{if } \partial M_0 \cong \mathbb{R}P^3, \\ S^2 \times_{\mathbb{Z}_2} D^2 & \text{if } \partial M_0 \cong \mathbb{R}P^3\#\mathbb{R}P^3, \\ S^1 \times D^3, S^2 \times D^2, (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I & \text{if } \partial M_0 \cong S^2 \times S^1. \end{cases} \tag{13.34}$$



$B_1 \cong (S^1 \times D^2)\#k(\mathbb{R}P^3)$

Figure 13.3: Example of $B_1 \subset \partial M_0$ which is represented by a cycle graph with 3 vertices.

Proof. Write $\partial M_0 = A \cup_\partial E \cup_\partial V$ where we put $A = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right]$, $E = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)\right]$, and $V = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right]$.

By Proposition 12.1, A is a disjoint union of subbundles of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$. Hence, A is the total space of S^1-fibers over a disjoint union of surfaces with boundary.

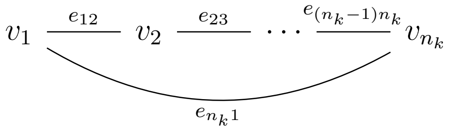
From Lemma 13.6, each fiber over an interior points of $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ is disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ and $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components. Hence, E is a disjoint union of components $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right) \cong D^2 \times S^1$ and D^2-subbundles of $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ diffeomorphic to $D^2 \times I$. The boundary of each D^2-fiber coincides with a fiber of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$.

From Lemma 13.4, fibers over an interior point of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ are disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_0 \\ \downarrow \\ \mathrm{pt} \end{array}\right)$. Hence, V is a disjoint union of fibers of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$.

Write $V = \bigsqcup_{\alpha=1}^{m} V_\alpha$, where $V_\alpha \cong D^3$ or $S^2 \times_{\mathbb{Z}_2} I$ is a connected component of V.

Write $E = \bigsqcup_i E_i$ where E_i is a connected component of E. Suppose that there is $E_i \cong S^1 \times D^2$. By Lemma 13.6, $E_i \cap V = \varnothing$. As a part of ∂M_0, ∂E_i is identified with a boundary component of A so that each ∂D^2-fiber of $\partial E_i \cong \partial D^2 \times S^1$ coincides with an S^1-fiber of ∂A.

Without loss of generality, assume that $E_i \cong D^2 \times I$ for all i. From Lemma 13.4 and Lemma 13.6, for each α, $V_\alpha \cap E = \partial V_\alpha \cap \partial E$ is exactly two copies of D^2 from $\bigsqcup_i (D^2 \times \partial I)_i \subset \bigsqcup_i \partial E_i$. Moreover, each connected component of $D^2 \times \partial I \subset E_i$ is contained in ∂V_α for some α. Therefore, each connected component of $V \cup_\partial E$ can be represented by a cycle graph



(13.35)

so that each vertex v_j represents V_α for some α, each edge $e_{j(j+1)}$ represents E_i for some i, and v_j is incident to $e_{j(j+1)}$ if and only if $V_\alpha \cap E_i \neq \varnothing$.

Put $B = V \cup_\partial E$ and write $B = \bigsqcup B_i$ where B_i is a connected component of B. ∂B_i is the total space of S^1-fibers over a circle. Each ∂D^2-fiber of ∂B_i coincides with an S^1-fiber of ∂A. B_i is diffeomorphic to $(S^1 \times D^2)\#k_i(\mathbb{R}P^3)$ where $k_i \geq 0$ is the number of E_i such that $E_i \cong S^2 \times_{\mathbb{Z}_2} I \cong D^3\#\mathbb{R}P^3$.

Because ∂M_0 is connected, A is also connected. We have that A is a subbundle $\begin{pmatrix} S^1 \longrightarrow A \\ \hfill\downarrow \\ \hfill(\Sigma_A^2, \partial\Sigma_A^2) \end{pmatrix}$ of $\begin{pmatrix} S^1 \longrightarrow N_j \\ \hfill\downarrow \\ \hfill\partial X^3 \end{pmatrix}$, for some j, where $\Sigma_A^2 = \Sigma_A^2(g,n)$ is a connected surface of genus g and with $n \geq 1$ boundary components. Then,

$$\partial M_0 = A \cup_\partial \bigsqcup_{i=1}^{n} B_i \cong \begin{pmatrix} S^1 \longrightarrow A \\ \hfill\downarrow \\ \hfill(\Sigma_A^2, \partial\Sigma_A^2) \end{pmatrix} \cup_\partial \left(\bigsqcup_{i=1}^{n} (S^1 \times D^2)\#k_i(\mathbb{R}P^3) \right). \tag{13.36}$$

By an explicit construction in [11, Proposition 1], $\partial M_0 \cong S^3\#(2g+n-1)(S^2\times S^1)\#k(\mathbb{R}P^3)$, where $k = \sum_i k_i$. According to the classification of $\begin{pmatrix} D^4, \ldots \longrightarrow M_0 \\ \hfill\downarrow \\ \hfill\text{pt} \end{pmatrix}$ in Table 13.1, ∂M_0 can only be diffeomorphic to S^3, $S^2 \times S^1$, $\mathbb{R}P^3$, or $\mathbb{R}P^3\#\mathbb{R}P^3$ in this case. Thus, $2g + n - 1 = 0$ or 1. That is $(g,n) = (0,1)$ or $(0,2)$. In other words, Σ_A^2 is a disk or a cylinder. If Σ_A^2 is a disk, then $k_1 = 0, 1$, or 2. If Σ_A^2 is a cylinder, then $k_1 = k_2 = 0$. In summary,

$$\partial M_0 \cong \begin{cases} S^1 \times B^2 \cup S^1 \times D^2 & \cong S^3, \\ S^1 \times B^2 \cup (S^1 \times D^2)\#\mathbb{R}P^3 & \cong \mathbb{R}P^3, \\ S^1 \times B^2 \cup (S^1 \times D^2)\#(\mathbb{R}P^3\#\mathbb{R}P^3) & \cong \mathbb{R}P^3\#\mathbb{R}P^3, \\ S^1 \times D^2 \cup S^1 \times (S^1 \times I) \cup S^1 \times D^2 & \cong S^2 \times S^1, \end{cases} \tag{13.37}$$

where $S^1 \times B^2$ and $S^1 \times (S^1 \times I)$ are contained in $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ for some j.

The classification of M_0 follows from Lemma 2.12 and Table 13.1. □

Lemma 13.38. *Assume that M_0 intersects with* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components, and M_0 is disjoint from fiber bundle components of other types, i.e.*

$$\partial M_0 \subset \left[\bigsqcup_j \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right]. \tag{13.39}$$

Then, $\partial M_0 = A \cup_\partial \left(\bigsqcup_i C_i\right)$ where:

(i) A is a subbundle of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ *for some j*

(ii) $C_i \cong (S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I)$ is a fiber of $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$.

(iii) $\partial C_i \cong T^2$ is identified with a boundary component of A.

It follows that ∂M_0 is a Seifert manifold.

Proof. Write $\partial M_0 = A \cup_\partial C$ where we put $A = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right]$, and $C = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right]$.

By Proposition 12.1, A is a disjoint union of subbundles of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$. Hence, A is the total space of S^1-fibers over a disjoint union of surfaces with boundary.

From Lemma 13.5, fibers over an interior point of $\left(\begin{array}{r} S^1\times D^2, T^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I,\partial I) \end{array}\right)$ are disjoint from $\left(\begin{array}{r} D^4,\ldots \longrightarrow M_0 \\ \downarrow \\ \mathrm{pt} \end{array}\right)$. Hence, C is a disjoint union of fibers of $\left(\begin{array}{lr} S^1\times D^2, & \\ T^2\times_{\mathbb{Z}_2} I & \longrightarrow N_j \\ & \Big\downarrow \\ & \partial I \end{array}\right)$. Write $C=\bigsqcup_{i=1}^m C_i$, where $C_i\cong S^1\times D^2$ or $T^2\times_{\mathbb{Z}_2} I$ is a connected component of C. Moreover, $A\cap C_i=\partial A\cap\partial C_i$ where $\partial C_i\cong T^2$ is identified with a boundary component of A.

Because ∂M_0 is connected, A is also connected. Hence,

$$\partial M_0 = A\cup_\partial \bigsqcup_i C_i \cong \left(\begin{array}{r} S^1 \longrightarrow A \\ \downarrow \\ (\Sigma_A^2,\partial\Sigma_A^2) \end{array}\right) \cup_\partial \left(\bigsqcup_i (S^1\times D^2)\right) \cup_\partial \left(\bigsqcup_i (T^2\times_{\mathbb{Z}_2} I)\right). \quad (13.40)$$

From the classification of Seifert manifolds in [17] and from the classification of M_0 in Table 13.1, ∂M_0 is a Seifert manifold. □

Lemma 13.41. *Assume that* M_0 *intersects with* $\left(\begin{array}{r} S^1\times D^2, T^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I,\partial I) \end{array}\right)$, $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2,\partial\Sigma^2) \end{array}\right)$, *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3,\partial X^3) \end{array}\right)$ *components. In addition, assume that* M_0 *may also intersect* $\left(\begin{array}{r} D^3, S^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I,\partial I) \end{array}\right)$ *components and* M_0 *is disjoint from fiber bundle components of other types, i.e.*

$$\begin{aligned} \partial M_0 \subset & \left[\bigsqcup_j \left(\begin{array}{r} D^3, S^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I,\partial I) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1\times D^2, T^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I,\partial I) \end{array}\right)\right] \\ & \cup \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2,\partial\Sigma^2) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3,\partial X^3) \end{array}\right)\right]. \quad (13.42) \end{aligned}$$

Then, $\partial M_0 = A\cup_\partial B\cup_\partial C$ *where:*

(1) *A is a subbundle of* $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$, *for some* j.

(2) *$B = B_1$ or $B = B_1 \sqcup B_2$ where B_i is a component* $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ *for some j, or a 3-manifold which can be represented by a cycle graph as in Lemma 13.30.*

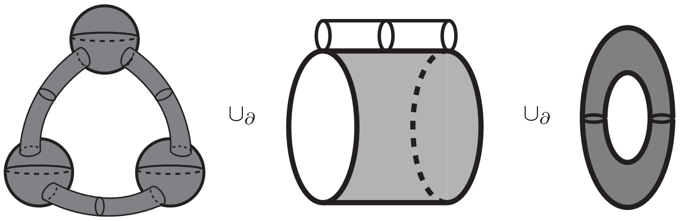
(3) *∂B_i, $i \in \{1,2\}$, is identified with a boundary component of A so that each ∂D^2-fiber of $\partial B_i \cong \partial D^2 \times S^1$ coincides with an S^1-fiber of A.*

(4) *$C = \bigsqcup_i C_i$ where $C_i \cong (S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I)$ is a fiber of* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_i} \\ \downarrow \\ \partial I \end{array}\right)$ *for some j_i.*

(5) *∂C_i is identified with a boundary component of A.*

It follows that ∂M_0 is diffeomorphic to $S^3, S^2 \times S^1, \mathbb{R}P^3, \mathbb{R}P^3\#\mathbb{R}P^3$, or a Lens space. Hence,

$$M_0 \cong \begin{cases} D^4, \pm\mathbb{C}P^2\#D^4 & \text{if } \partial M_0 \cong S^3, \\ S^2 \times_{\pm 2} D^2 & \text{if } \partial M_0 \cong \mathbb{R}P^3, \\ S^1 \times D^3, S^2 \times D^2, (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I & \text{if } \partial M_0 \cong S^2 \times S^1, \\ S^2 \times_{\mathbb{Z}_2} D^2 & \text{if } \partial M_0 \cong \mathbb{R}P^3\#\mathbb{R}P^3, \\ S^2 \times_\omega D^2 & \text{if } \partial M_0 \cong L(|\omega|, 1). \end{cases} \tag{13.43}$$



$B_1 \cong (S^1 \times D^2)\#k(\mathbb{R}P^3)$ S^1-bundle over $S^1 \times I$ $S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I$

Figure 13.4: Example of a decomposition of ∂M_0 in Lemma 13.41. $\partial M_0 = B_1 \cup_\partial A \cup_\partial C$.

Proof. Write $\partial M_0 = A \cup_\partial B \cup_\partial C$ where we put $A = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right]$, $B = \partial M_0 \cap \left(\left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right]\right)$, and $C = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right]$.

By Proposition 12.1, A is a disjoint union of subbundles of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$. Hence, A is the total space of S^1-fibers over a disjoint union of surfaces with boundary.

By the same arguments as in the proof of Lemma 13.30, we can write $B = \bigsqcup_{i=1}^{n} B_i$, where each connected component B_i is a component $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial \Sigma^2 \end{array}\right)$ for some j, or a 3-manifold which can be represented by a cycle graph such that each vertex v_α represents a fiber V_α of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_\alpha} \\ \downarrow \\ \partial I \end{array}\right)$, for some j_α, and each edge $e_{\alpha(\alpha+1)}$ represents a subbundle $E_{\alpha(\alpha+1)} \cong D^2 \times I$ of $\left(\begin{array}{r} D^2 \longrightarrow N_{j_\alpha} \\ \downarrow \\ \partial \Sigma^2 \end{array}\right)$, for some j_α. $V_\alpha \cong D^3$ or $S^2 \times_{\mathbb{Z}_2} I$ and $V_\alpha \cap E_{\alpha(\alpha+1)} = \partial V_\alpha \cap \partial E_{\alpha(\alpha+1)}$ is a connected component of $D^2 \times \partial I \subset E_{\alpha(\alpha+1)}$. Moreover, $B_i \cong (S^1 \times D^2) \# k_i (\mathbb{R}P^3)$ where $k_i \geq 0$ is the number of $E_\alpha \cong S^2 \times_{\mathbb{Z}_2} I \cong D^3 \# \mathbb{R}P^3$. ∂B_i is identified with a boundary component of A so that each ∂D^2-fiber of $\partial B_i \cong \partial D^2 \times S^1$ coincides with an S^1-fiber of A.

By the same arguments as in the proof of Lemma 13.38, we can write $C = \bigsqcup_{i=1}^{m} C_i$, where $C_i \cong S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I$ is a fiber of $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_i} \\ \downarrow \\ \partial I \end{array}\right)$ for some j_i. $\partial C_i \cong T^2$ is identified with a boundary component of A.

Because ∂M_0 is connected, A is also connected. Thus, A is a subbundle $\left(\begin{array}{r} S^1 \xrightarrow{\qquad} A \\ \downarrow \\ (\Sigma_A^2, \partial \Sigma_A^2) \end{array}\right)$ of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some j, where $\Sigma_A^2 = \Sigma_A^2(g, n+m)$ is a connected surface of genus g and with $n+m$ boundary components. Hence,

$$\begin{aligned} \partial M_0 &= A \cup_\partial \left(\bigsqcup_{i=1}^{n} B_i \right) \cup_\partial \left(\bigsqcup_{i=1}^{m} C_i \right) \\ &\cong \left(\begin{array}{r} S^1 \xrightarrow{\qquad} A \\ \downarrow \\ (\Sigma_A^2, \partial \Sigma_A^2) \end{array}\right) \cup_\partial \left(\bigsqcup_{i=1}^{n} (S^1 \times D^2) \# k_i (\mathbb{R}P^3) \right) \\ &\quad \cup_\partial \left(\bigsqcup_{i} (S^1 \times D^2) \right) \cup_\partial \left(\bigsqcup_{i} (T^2 \times_{\mathbb{Z}_2} I) \right). \end{aligned} \tag{13.44}$$

Let $\phi_i : \partial C_i \to \partial A$ be the identifying map from ∂C_i to a boundary component of A. First, assume that $C_i \cong S^1 \times D^2$. Then, $\phi_i : S^1 \times \partial D^2 \to S^1 \times \partial_i \Sigma_A^2$ for some boundary component $\partial_i\Sigma_A^2$ of Σ_A^2. Up to isotopy, $\phi_i \in SL_2(\mathbb{Z})$. Suppose further that ϕ_i does not send each $(\cdot, \partial D^2) \subset \partial C_i$ to $(S^1, \cdot) \subset S^1 \times \partial_i\Sigma_A^2$. From the classification of Seifert manifolds in [17], $A \cup_\partial C_i$ extends the S^1-fibration of A or $A \cup_\partial C_i$ is a Seifert manifold (with boundary). Next, assume that $C_i \cong T^2 \times_{\mathbb{Z}_2} I$. From the classification of Seifert manifolds in [17], $A \cup_\partial C_i$ is a Seifert manifold with two exceptional Seifert orbits.

Reindex $\{C_i\}$ so that for $i \in \{1, \dots, m'\}$, $C_i \cong S^1 \times D^2$ and ϕ_i sends $(\cdot, \partial D^2) \subset \partial C_i$ to $(S^1, \cdot) \subset S^1 \times \partial_i\Sigma_A^2$, or $C_i \cong T^2 \times_{\mathbb{Z}_2} I$. Otherwise, $C_i \cong S^1 \times D^2$ for some $i \in \{m'+1, \dots, m\}$. Put $\widetilde{C} = \bigsqcup_{i=m'+1}^{m} C_i$. Then, $A \cup_\partial \widetilde{C}$ is the total space of S^1-fibers or a Seifert manifold. Let $\widetilde{\Sigma}^2$ be the base of the S^1-bundle or the Seifert manifold $A \cup_\partial \widetilde{C}$. $\widetilde{\Sigma}^2$ has the same genus as Σ_A^2 which is equal to g. $\widetilde{\Sigma}^2$ has $n + m'$ boundary components. Therefore,

$$\begin{aligned}\partial M_0 &= (A \cup_\partial \widetilde{C}) \cup_\partial (B \sqcup (C - \widetilde{C})) \\ &\cong (\text{a Seifert manifold with base } \widetilde{\Sigma}^2(g, n+m')) \\ &\quad \cup_\partial \left(\bigsqcup_{i=1}^{n} (S^1 \times D^2)\#k_i(\mathbb{R}P^3)\right) \cup_\partial \left(\bigsqcup_{i=1}^{m'} S^1 \times D^2\right)\end{aligned} \tag{13.45}$$

where each $S^1 \times \partial D^2$ is glued to a boundary component of $A \cup_\partial \widetilde{C}$ so that each ∂D^2-fiber coincides with an S^1-fiber of $A \cup_\partial \widetilde{C}$. By [11, Proposition 2],

$$\partial M_0 \cong S^3\#(2g + (n+m') - 1)(S^2 \times S^1)\#k(\mathbb{R}P^3)\#L(p_1, q_1)\#L(p_2, q_2)\# \cdots \tag{13.46}$$

where $k = \sum_i k_i$.

According to the classification of $\left(\begin{array}{r} D^4, \dots \longrightarrow M_0 \\ \downarrow \\ \text{pt} \end{array}\right)$ in Table 13.1, ∂M_0 can only be diffeomorphic to S^3, $S^2 \times S^1$, $\mathbb{R}P^3$, $\mathbb{R}P^3\#\mathbb{R}P^3$, or $L(p, q)$ in this case. Thus, $2g + (n+m') - 1 \in \{0, 1\}$. That is $g = 0$ and $n + m' \in \{1, 2\}$. We have that (n, m') is $(1, 0), (2, 0)$, or $(1, 1)$.

Case 1: $(n, m') = (1, 0)$. Then, $B = B_1$ and $C = \widetilde{C}$. If $A \cup_\partial \widetilde{C}$ is the total space of S^1-fibers, then $A \cup_\partial \widetilde{C} \cong S^1 \times D^2$. Hence,

$$\partial M_0 = (A \cup_\partial \widetilde{C}) \cup_\partial B_1 \cong S^1 \times B^2 \cup S^1 \times D^2 \cong S^3\#k(\mathbb{R}P^3). \tag{13.47}$$

According to the classification of $\left(\begin{array}{r} D^4, \dots \longrightarrow M_0 \\ \downarrow \\ \text{pt} \end{array}\right)$ in Table 13.1, $k \leq 2$. Therefore, $\partial M_0 \cong S^3, \mathbb{R}P^3$, or $\mathbb{R}P^3\#\mathbb{R}P^3$. Otherwise, $A \cup \widetilde{C}$ is a Seifert manifold with base D^2 and with one exceptional orbit. In this case,

$$\partial M_0 = (A \cup_\partial \widetilde{C}) \cup_\partial B_1 \cong S^3\#L(p, q) \cong L(p, q). \tag{13.48}$$

Case 2: $(n, m') = (2, 0)$. Then $B = B_1 \sqcup B_2$ and $C = \widetilde{C}$. If $A \cup_\partial \widetilde{C}$ is the total space of S^1-fibers, then $A \cup_\partial \widetilde{C} \cong S^1 \times (S^1 \times I)$. Hence,

$$\begin{aligned}\partial M_0 &= (A \cup_\partial \widetilde{C}) \cup_\partial (B_1 \sqcup B_2) \\ &\cong S^1 \times (S^1 \times I) \cup (S^1 \times D^2 \sqcup S^1 \times D^2)\#k(\mathbb{R}P^3) \\ &\cong (S^1 \times S^2)\#k(\mathbb{R}P^3).\end{aligned} \tag{13.49}$$

According to the classification of $\left(\begin{array}{r} D^4, \dots \longrightarrow M_0 \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ in Table 13.1, $k = 0$ and $\partial M_0 \cong S^1 \times S^2$. By the same argument as in case 1, if $A \cup_\partial \widetilde{C}$ is a Seifert manifold, then $\partial M_0 \cong (S^1 \times S^2)\# L(p_1, q_1)\# \cdots$. This contradicts to the classification of M_0 in Table 13.1.

Case 3: $(n, m') = (1, 1)$. Then $B = B_1$ and $C - \widetilde{C} = C_1 \cong S^1 \times D^2$. By the same argument as in case 2, $A \cup_\partial \widetilde{C}$ is the total space of S^1-fibers, $k = 0$, and $\partial M_0 \cong S^1 \times S^2$. $\square$

Lemma 13.50. *Assume that M_0 intersects with* $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components, and M_0 is disjoint from fiber bundle components of other types, i.e.*

$$\begin{aligned}\partial M_0 \subset &\left[\bigsqcup_j \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right] \\ &\cup \left[\bigsqcup_j \left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right].\end{aligned} \tag{13.51}$$

Then $\partial M_0 = A \cup_\partial B_1$ or $\partial M_0 = A \cup_\partial (B_1 \sqcup B_2)$ where:

(1) $A = (\bigsqcup_i A_i)$ *where A_i is a subbundle of* $\left(\begin{array}{r} S^1 \longrightarrow N_{j_i} \\ \downarrow \\ \partial X^3 \end{array}\right)$, *for some j_i. Additionally, A_i is the total space of S^1-fibers over a disk or a cylinder. There is at most one A_i that is the total space of S^1-fibers over a cylinder.*

(2) *B_i is a component* $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ *for some j or a 3-manifold that can be represented by a connected graph G with the following properties.*

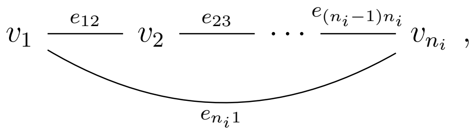
(i) Each vertex has degree at most 3.

(ii) There are two types of edges: solid edges and dashed edges. Let $E^D(G)$ be the set of dashed edges. Every vertex of odd degree is incident to exactly one dashed edge. Every vertex of degree 2 is incident to two solid edges.

(iii) Put $\mathcal{C} = G - \cup_{e \in E^D(G)} e$ and write $\mathcal{C} = \bigsqcup_i \mathcal{C}_i$ where $\mathcal{C}_i$ is a connected component of $\mathcal{C}$. Then, $\mathcal{C}_i$ is a cycle graph or a single vertex.

(iv) There exists at most one cycle subgraph of G that is not a cycle subgraph of $\mathcal{C}$.

Suppose that $\mathcal{C}_i$ is a cycle graph



(13.52)

so that the following holds. Each vertex v_j represents a 3-manifold V_j where either

(a) $V_j \cong D^3$ or $S^2 \times_{\mathbb{Z}_2} I$ is a fiber of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$, for some j, or

(b) $V_j \cong S^2$ is a boundary component of a subbundle $Z_j \cong S^2 \times I$ of $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial \Sigma^2 \end{array}\right)$, for some j.

As a vertex of G, $\deg(v_j) = 2$ in case (a) and $\deg(v_j) = 3$ in case (b).

Each edge $e_{j(j+1)}$ represents a subbundle $E_{j(j+1)} \cong D^2 \times I$ of $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial \Sigma^2 \end{array}\right)$, for some j. $V_j \cap E_{j(j+1)} = \partial V_j \cap \partial E_{j(j+1)}$ is a connected component of $D^2 \times \partial I \subset E_{\alpha(\alpha+1)}$.

If $\mathcal{C}_i$ is a single vertex v_i, then v_i represents a fiber V_i of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_i \\ \downarrow \\ \partial I \end{array}\right)$, for some i. As a vertex of G, $\deg(v_i) = 1$.

Each dashed edge e^D represents a subbundle $Z \cong S^2 \times I$ of $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial \Sigma^2 \end{array}\right)$ for some j.

For each vertex v_j, if $\deg(v_j) = 3$ and v_j is incident to e^D then $V_j \cong S^2$ is a boundary component of Z. If $\deg(v_j) = 1$ and v_j is incident to e^D, then ∂V_j coincides with a boundary component of Z.

The graph G represents $B_i \cong d_i(S^1 \times D^2) \# k_i(\mathbb{R}P^3) \# \ell_i(S^1 \times S^2)$ for some integers $d_i > 0$, $k_i \geq 0$, and $\ell_i \geq 0$.

(3) *Each boundary component of* B_i *is diffeomorphic to* $S^1 \times \partial D^2$. *It is identified with a boundary component of* A *so that each* ∂D^2*-fiber coincides with an* S^1*-fiber of* A.

If $\partial M_0 = A \cup_\partial B_1$, *then for some* $n_1 \geq 1$, $A = \sqcup_{i=1}^{n_1} A_i$ *and for every* i, A_i *is the total space of* S^1*-fibers over a disk. One of the following holds.*

(1) $B_1 \cong n_1(S^1 \times D^2)\#k_1(\mathbb{R}P^3)$ *where* $k_1 \in \{0,1,2\}$. *In this case,* $\partial M_0 \cong S^3, \mathbb{R}P^3$, *or* $\mathbb{R}P^3\#\mathbb{R}P^3$.

(2) $B_1 \cong n_1(S^1 \times D^2)\#(S^1 \times S^2)$. *In this case,* $\partial M_0 \cong S^1 \times S^2$.

If $\partial M_0 = A \cup_\partial (B_1 \sqcup B_2)$, *then* $A = \sqcup_{i=1}^{n_1+n_2-1} A_i$, *for some* $n_1, n_2 \geq 1$, *where* A_1 *is the total space of* S^1*-fibers over a cylinder and* A_i, *for all* $i \geq 2$, *is the total space of* S^1*-fibers over a disk. For* $k \in \{1,2\}$, $B_k \cong \underbrace{(S^1 \times D^2)\#\cdots\#(S^1 \times D^2)}_{n_k \text{ copies}}$ *and* $A_1 \cap B_k \neq \varnothing$. *In this case,* $\partial M_0 \cong S^1 \times S^2$.

Hence,

$$M_0 \cong \begin{cases} D^4, \pm\mathbb{C}P^2\#D^4 & \text{if } \partial M_0 \cong S^3, \\ S^2 \times_{\pm 2} D^2 & \text{if } \partial M_0 \cong \mathbb{R}P^3, \\ S^2 \times_{\mathbb{Z}_2} D^2 & \text{if } \partial M_0 \cong \mathbb{R}P^3\#\mathbb{R}P^3, \\ S^1 \times D^3, S^2 \times D^2, (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I & \text{if } \partial M_0 \cong S^2 \times S^1. \end{cases} \tag{13.53}$$

Before we prove Lemma 13.50, we give some examples of the decomposition of ∂M_0 in the lemma.

Example 13.54. Case 1 : $\partial M_0 = A_1 \cup B_1$ where $B_1 \cong (S^1 \times D^2)\#k(\mathbb{R}P^3)$ and A_1 is the total space of S^1-fibers over a disk. $\partial M_0 \cong S^3, \mathbb{R}P^3$ or $\mathbb{R}P^3\#\mathbb{R}P^3$.

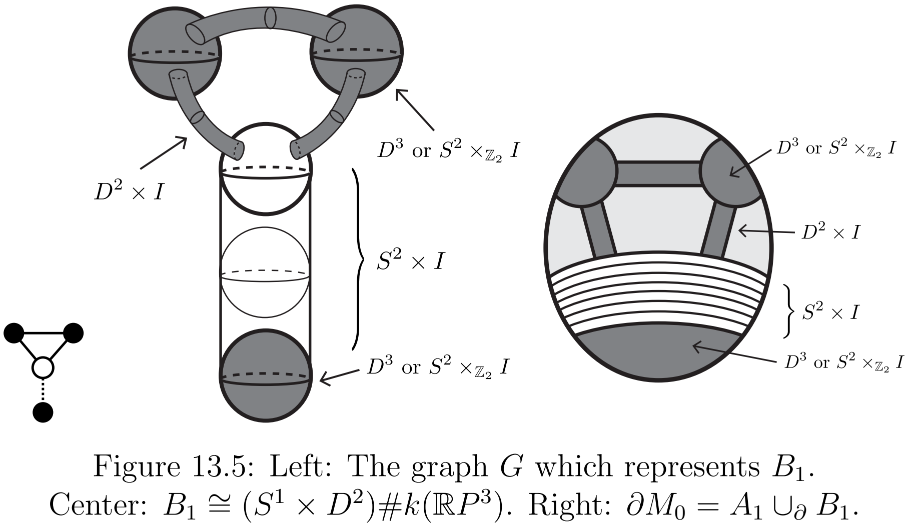


Figure 13.5: Left: The graph G which represents B_1.
Center: $B_1 \cong (S^1 \times D^2)\#k(\mathbb{R}P^3)$. Right: $\partial M_0 = A_1 \cup_\partial B_1$.

Example 13.55. Case 1 : $\partial M_0 = (A_1 \sqcup A_2) \cup_\partial B_1$ where $B_1 \cong 2(S^1 \times D^2)\#k(\mathbb{R}P^3)$, and A_1 and A_2 are the total spaces of S^1-fibers over a disk. $\partial M_0 \cong S^3, \mathbb{R}P^3$ or $\mathbb{R}P^3\#\mathbb{R}P^3$.

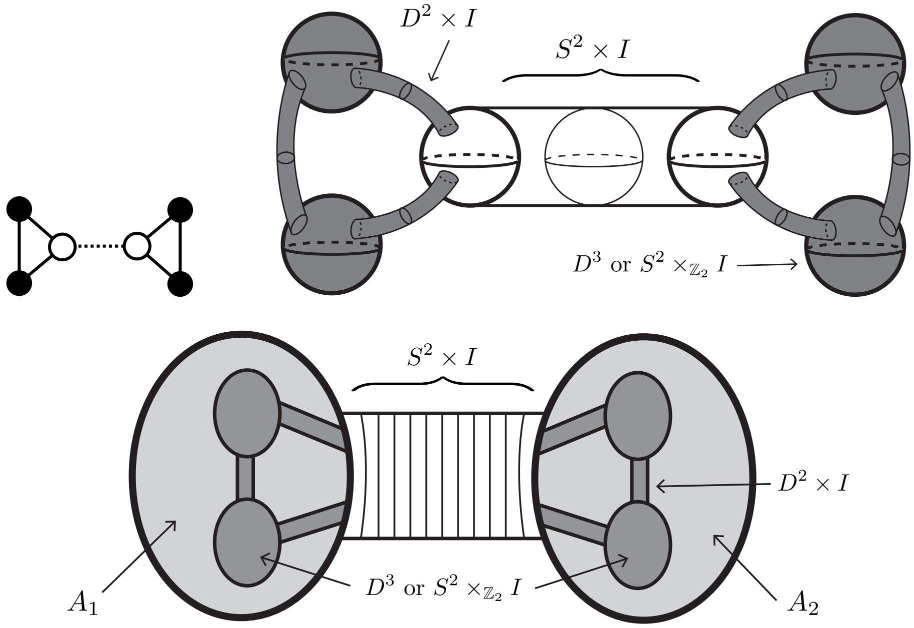


Figure 13.6: Left: The graph G which represents B_1. Right: $B_1 \cong 2(S^1 \times D^2)\#k(\mathbb{R}P^3)$. Below: $\partial M_0 = A_1 \cup_\partial B_1 \cup_\partial A_2 \cong S^3, \mathbb{R}P^3$, or $\mathbb{R}P^3\#\mathbb{R}P^3$.

Example 13.56. Case 2 : $\partial M_0 = A_1 \cup B_1$ where $B_1 \cong (S^1 \times D^2)\#(S^1 \times S^2)$ and A_1 is the total space of S^1-fibers over a cylinder. $\partial M_0 \cong S^1 \times S^2$.

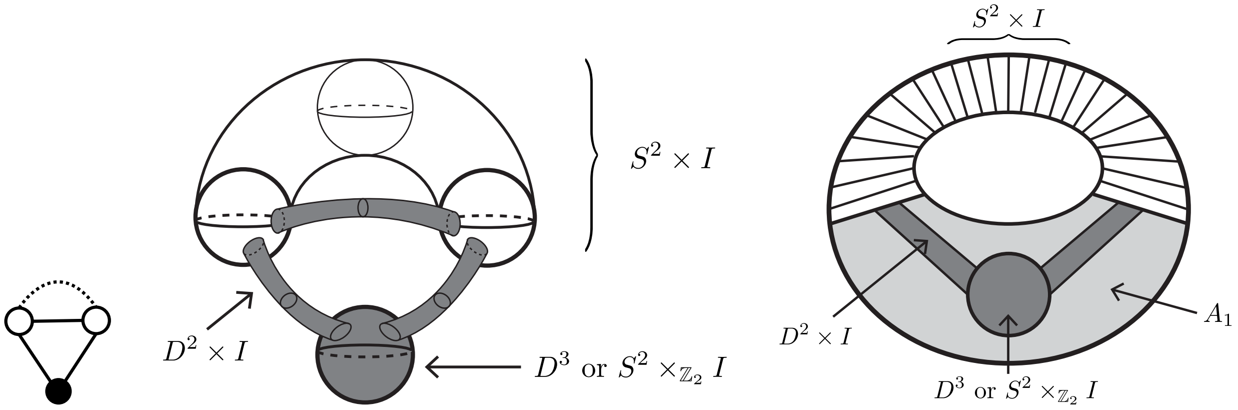


Figure 13.7: Left: The graph G which represents B_1. Center: $B_1 \cong (S^1 \times D^2)\#(S^1 \times S^2)$. Right: $\partial M_0 = A_1 \cup_\partial B_1 \cong S^1 \times S^2$.

Example 13.57. Case 3 : $\partial M_0 = A_1 \cup B_1$ where $B_1 \cong (S^1 \times D^2)\#(S^1 \times D^2)$. A_1 is the total space of S^1-fibers over a cylinder. $\partial M_0 \cong S^1 \times S^2$. We note that the graph G which represents B_1 is the same as in Example 13.55.

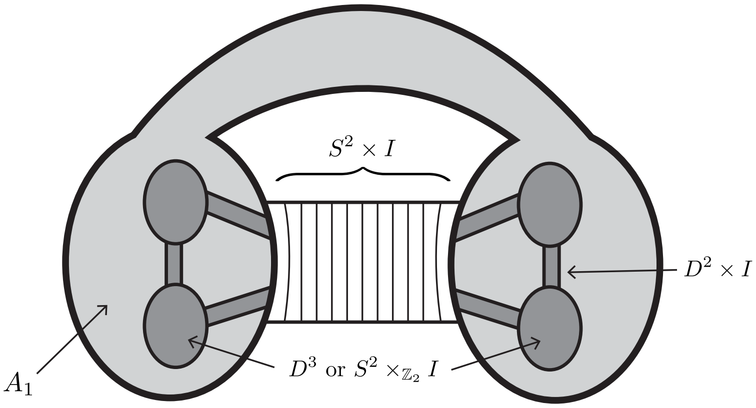


Figure 13.8: $\partial M_0 = A_1 \cup_\partial B_1 \cong S^1 \times S^2$ with B_1 as in Example 13.55

Proof. Write $\partial M_0 = A \cup_\partial E \cup_\partial W \cup_\partial Z$ where we put $A = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{ccc} S^1 & \longrightarrow & M_j \\ & & \downarrow \\ & & (X^3, \partial X^3) \end{array}\right)\right]$,

$E = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{ccc} D^2 & \longrightarrow & M_j \\ & & \downarrow \\ & & (\Sigma^2, \partial \Sigma^2) \end{array}\right)\right]$, $W = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{ccc} D^3, S^2 \times_{\mathbb{Z}_2} I & \longrightarrow & M_j \\ & & \downarrow \\ & & (I, \partial I) \end{array}\right)\right]$, and $Z = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{ccc} S^2 & \longrightarrow & M_j \\ & & \downarrow \\ & & (\Sigma^2, \partial \Sigma^2) \end{array}\right)\right]$.

By Proposition 12.1, A is a disjoint union of subbundles of $\left(\begin{array}{ccc} S^1 & \longrightarrow & N_j \\ & & \downarrow \\ & & \partial X^3 \end{array}\right)$. Hence, A is diffeomorphic the total space of S^1-fibers over a disjoint union of surfaces with boundary.

From Lemma 13.6, each fiber over an interior points of $\left(\begin{array}{ccc} D^2 & \longrightarrow & M_j \\ & & \downarrow \\ & & (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ is disjoint from $\left(\begin{array}{ccc} D^4, \ldots & \longrightarrow & M_i \\ & & \downarrow \\ & & \mathrm{pt} \end{array}\right)$, $\left(\begin{array}{ccc} D^3, S^2 \times_{\mathbb{Z}_2} I & \longrightarrow & M_j \\ & & \downarrow \\ & & (I, \partial I) \end{array}\right)$, and $\left(\begin{array}{ccc} S^2 & \longrightarrow & M_j \\ & & \downarrow \\ & & (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ components.

Hence, E is a disjoint union of components $\left(\begin{array}{ccc} D^2 & \longrightarrow & N_j \\ & & \downarrow \\ & & \partial \Sigma^2 \end{array}\right) \cong D^2 \times S^1$ and subbundles of

$\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ diffeomorphic to $D^2 \times I$. The boundary of each D^2-fiber coincides with an S^1-fiber of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$.

From Lemma 13.4, fibers over an interior point of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ are disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_0 \\ \downarrow \\ \text{pt} \end{array}\right)$. Hence, W is a disjoint union of fibers of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$. Write $W = \bigsqcup_s W_s$, where $W_s \cong D^3$ or $S^2 \times_{\mathbb{Z}_2} I$ is a connected component of W.

From Lemma 13.2, Z is a disjoint union of subbundles of $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ diffeomorphic to $S^2 \times I$. Write $Z = \bigsqcup_t Z_t$, where Z_t is a connected component of Z. If $Z_t \cap W_s \neq \varnothing$, then ∂W_s coincides with a boundary component of Z_t.

Write $E = \bigsqcup_{i=1}^n E_i$ where E_i is a connected component of E. If $E_i \cong S^1 \times D^2$, then by Lemma 13.6, $E_i \cap W = \varnothing$ and $E_i \cap Z = \varnothing$. Hence, as a part of ∂M_0, ∂E_i is identified with a boundary component of A so that each ∂D^2-fiber of $\partial B_i \cong \partial D^2 \times S^1$ coincides with an S^1-fiber of ∂A. Without loss of generality, assume that $E_i \cong D^2 \times I$ for all i.

If there exist t, s_1, and s_2 such that $Z_t \cap W_{s_1} \neq \varnothing$ and $Z_\beta \cap W_{s_2} \neq \varnothing$, then $W_{s_1} \cup_\partial Z_\beta \cup_\partial W_{s_2}$ is a closed manifold. Consequently, $\partial M_0 = W_{s_1} \cup_\partial Z_\beta \cup_\partial W_{s_2}$. This is a contradiction since $A \neq \varnothing$. Therefore, every Z_t intersects with at most one connected component of W.

We construct a graph G to represent $E \cup_\partial W \cup_\partial Z$ as follows.

(1) For every connected component W_s of W, construct a vertex w_s to represent $W_s \cong D^3$ or $S^2 \times_{\mathbb{Z}_2} I$.

(2) For every connected component Z_t of Z such that $Z_t \cap W_s \neq \varnothing$, construct a vertex z_{t_1} to represent the boundary component of Z_t that is disjoint from W_s. Connect vertices z_{t_1} and w_s with a dashed edge.

(3) For every connected component Z_t of Z such that $Z_t \cap W = \varnothing$, let $\partial_1 Z_t$ and $\partial_2 Z_t$ denote its two boundary components and construct vertices z_{t_1} and z_{t_2} to represent $\partial_1 Z_t$ and $\partial_2 Z_t$ respectively. Connect z_{t_1} and z_{t_2} with a dashed edge.

(4) Let V_1 and V_2 be connected components of W represented by vertices v_1 and v_2 where $v_1 \neq v_2$. Connect vertices v_1 and v_2 with a solid edge if there is a connected component $E_i \cong D^2 \times I$ of E such that $V_1 \cap E_i \neq \varnothing$ and $V_2 \cap E_i \neq \varnothing$.

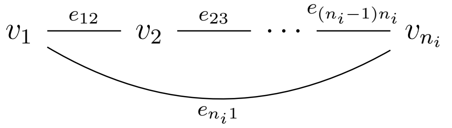
(5) Let V_1 be a connected component of W represented by the vertex v_1. If there exists a connected component $E_i \cong D^2 \times I$ of E such that $V_1 \cap E_i$ is diffeomorphic to two copies of D^2, then we construct a loop of solid edge incident to v_1.

Next, we show that all vertices have degree at most 3. From Lemma 13.4 and Lemma 13.6, if $W_s \cap E \neq \varnothing$, then $W_s \cap E = \partial W_s \cap \partial E$ is two D^2-fibers of $\left(\begin{array}{r} D^2 \longrightarrow \partial E_{i_k} \\ \downarrow \\ \partial I \end{array}\right)$, for some i_k, $k \in \{1, 2\}$. Similarly, from Lemma 13.2 and Lemma 13.6, if $Z_t \cap E_i \neq \varnothing$, then $Z_t \cap E = \partial Z_t \cap \partial E$ is two D^2-fibers of $\left(\begin{array}{r} D^2 \longrightarrow \partial E_{i_k} \\ \downarrow \\ \partial I \end{array}\right)$, for some i_k, $k \in \{1, 2\}$. Every connected component of $D^2 \times \partial I \subset \partial E_i$ is contained in ∂W_s or ∂Z_t for some s and t. In other words, every vertex is incident to either zero or two solid edges. From the construction of G, each vertex has at most one dashed edge. Thus, each vertex has degree at most 3. Every vertex of odd degree is incident to exactly one dashed edge and all other edges are solid edges.

Let $E^D(G)$ be the set of all dashed edges and define $\mathcal{C} = G - \cup_{e \in E^D(G)} e$. We have that all vertices in $\mathcal{C}$ have degree zero or degree two. Write $\mathcal{C} = \bigsqcup_i \mathcal{C}_i$ where $\mathcal{C}_i$ is a connected component of $\mathcal{C}$. Then, each $\mathcal{C}_i$ is a cycle graph or a single vertex.

First assume that $\mathcal{C}_i$ is a single vertex w_s, for some s. As a vertex of G, $\deg(w_s) = 1$ and w_s is adjacent to a vertex z_{t_1} for some t. w_s represents a connected component $W_s \cong D^3$ or $S^2 \times_{\mathbb{Z}_2} I$ and ∂W_s is identified with a boundary component of $Z_t \cong S^2 \times I$. In particular, $W_s \cup Z_t \cong W_s$. Let G' be the graph obtained from G by replacing the subgraph z_{t_1} ------ w_s with the vertex w_s. We have that the 3-manifold represented by G' is diffeomorphic to the 3-manifold represented by G. Therefore, we can assume without loss of generality that all vertices have degree 2 or 3. In other words, every $\mathcal{C}_i$ is a cycle graph.

Let $\mathcal{C}_i$ be a cycle graph



(13.58)

where each vertex v_ℓ represents a 3-manifold V_ℓ so that

(1) $V_\ell = W_s \cong D^3$ or $S^2 \times_{\mathbb{Z}_2} I$, for some s, or

(2) $V_\ell \cong S^2$ is a boundary component of $Z_t \cong S^2 \times I$, for some t.

Each edge $e_{\ell(\ell+1)}$ represents a component $E_\ell \cong D^2 \times I$ intersecting both V_ℓ and $V_{\ell+1}$.

Let B_i be the 3-manifold represented by $\mathcal{C}_i$. It follows that $B_i \cong (S^1 \times D^2)\#k_i(\mathbb{R}P^3) - \bigsqcup_{u=1}^{p_i} B^3$ where k_i is the number of $V_\ell \cong S^2 \times_{\mathbb{Z}_2} I$ and p_i is the number of $V_\ell \cong S^2$. Every

S^2-boundary component of B_i is a boundary component of $Z_t \cong S^2 \times I$, for some t. Thus, a dashed edge between vertices z_{t_1} and z_{t_2} corresponds to identifying two S^2-boundary components of B_{i_1} and B_{i_2}, for some i_1 and i_2.

We can construct $E \cup_\partial W \cup_\partial Z$ from the graph G inductively as follows.

(1) First, we put $\mathcal{B}_0 = \bigsqcup_i B_i$.

(2) Put $\mathcal{B}_1 = \mathcal{B}_0 \cup_\partial Z_1 = (\bigsqcup_i B_i) \cup_\partial Z_1$.

(3) We have that $\mathcal{B}_1$ is diffeomorphic to $\mathcal{B}_0$ with two S^2-boundary components of B_{i_1} and B_{i_2}, for some i_1, i_2, identified.

(4) If $i_1 = i_2$, then $B_{i_1} \cup_\partial Z_1 \cong (S^1 \times D^2)\#k_{i_1}(\mathbb{R}P^3)\#(S^1 \times S^2) - \bigcup_{u=1}^{p_{i_1}-2} B^3$. If $i_1 \neq i_2$, then $B_{i_1} \cup_\partial Z_1 \cup_\partial B_{i_2} \cong B_{i_1}\#B_{i_2} \cong 2(S^1 \times D^2)\#(k_{i_1} + k_{i_2})(\mathbb{R}P^3) - \bigcup_{u=1}^{p_{i_1}+p_{i_2}-2} B^3$.

(5) Redefine $B_{i_1} = B_{i_1}\#B_{i_2}$ and reindex $\{B_i\}$. After reindexing, we have that $\mathcal{B}_1 = \bigsqcup_i B_i$.

(6) In the general case, put $\mathcal{B}_{j+1} = \mathcal{B}_j \cup_\partial Z_j$ where $\mathcal{B}_j = \bigsqcup_i B_i$ and $B_i \cong n_i(S^1 \times D^2) \# k_i(\mathbb{R}P^3) \# \ell_i(S^1 \times S^2) - \bigcup_{u=1}^{p_i} B^3$ for some integers $n_i > 0$ and $k_i, \ell_i, p_i \geq 0$.

(7) $\mathcal{B}_{j+1}$ is diffeomorphic to $\mathcal{B}_j$ with two S^2-boundary components of B_{i_1} and B_{i_2}, for some i_1, i_2, identified.

(8) If $i_1 = i_2$, then $B_{i_1} \cup_\partial Z_j \cong n_{i_1}(S^1 \times D^2)\#k_{i_1}(\mathbb{R}P^3)\#(\ell_{i_1} + 1)(S^1 \times S^2) - \bigcup_{u=1}^{p_{i_1}-2} B^3$. If $i_1 \neq i_2$, then $B_{i_1} \cup_\partial Z_j \cup_\partial B_{i_2} \cong B_{i_1}\#B_{i_2} \cong (n_{i_1} + n_{i_2})(S^1 \times D^2)\#(k_{i_1} + k_{i_2})(\mathbb{R}P^3)\# (\ell_{i_1} + \ell_{i_2})(S^1 \times S^2) - \bigcup_{u=1}^{p_{i_1}+p_{i_2}-2} B^3$.

(9) Repeat the process for all connected components Z_t of Z. Finally, we get the manifold $E \cup_\partial W \cup_\partial Z = \bigsqcup_i B_i \cong \bigsqcup_i n_i(S^1 \times D^2)\#k_i(\mathbb{R}P^3)\#\ell_i(S^1 \times S^2)$, for some integers $n_i > 0$, $k_i, \ell_i \geq 0$.

Write $A = \bigsqcup_i A_i$ where A_i is a connected component of A. Each A_i is the total space of S^1-fiber over a surface $\Sigma^2_{A_i} = \Sigma^2_{A_i}(g_i, d_i)$ of genus g_i and with d_i boundary components. Then,

$$\partial M_0 = A \cup_\partial (E \cup_\partial W \cup_\partial Z) \cong \left(\bigsqcup_i A_i\right) \cup_\partial \left(\bigsqcup_i n_i(S^1 \times D^2)\right) \#k(\mathbb{R}P^3)\#\ell(S^1 \times S^2) \quad (13.59)$$

where $k = \sum_i k_i$ and $\ell = \sum_i \ell_i$. We note that ℓ is the number of cycle subgraphs of G that are not subgraphs of $\mathcal{C}$, and k is the number of $W_s \cong S^2 \times_{\mathbb{Z}_2} I$. Each copy of $S^1 \times D^2$ is glued to a boundary component of A so that ∂D^2-fibers of $S^1 \times \partial D^2$ coincide with S^1-fibers of A.

By an explicit construction in [11, Proposition 1],

$$\left(\bigsqcup_i A_i\right) \cup_\partial \left(\bigsqcup_i n_i(S^1 \times D^2)\right) \cong S^3\#q(S^1 \times S^2) \quad (13.60)$$

where $q = \sum_i (g_i + d_i - 1) + r$ and r is the number of pairs of boundary components of the same B_i that are glued to the same connected component of A. Therefore,

$$\partial M_0 \cong S^3 \# k(\mathbb{R}P^3) \# (\ell + q)(S^1 \times S^2). \tag{13.61}$$

According to the classification of $$\left(\begin{tikzcd} D^4, \dots \arrow[r] & M_0 \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right)$$ in Table 13.1, $\ell = q = 0$ and $k \in \{0, 1, 2\}$, or $\ell + q = 1$ and $k = 0$.

Case 1: $\ell = q = 0$. Then, $r = 0$, $g_i = 0$, and $d_i = 1$ for all i. Thus, A_i is the total space of S^1-fibers over D^2, for all i. Because ∂M_0 is a closed manifold, $E \cup_\partial W \cup_\partial Z = \bigsqcup_i B_i$ is connected. Hence $\bigsqcup_j B_j = B_1 \cong n_1(S^1 \times D^2) \# k(\mathbb{R}P^3)$ where n_1 is the number of connected components of A and $k \in \{0, 1, 2\}$. Therefore, $\partial M_0 \cong S^3, \mathbb{R}P^3$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Case 2: $q = 0$ and $\ell = 1$. By the same arguments as in case 1, A_i is the total space of S^1-fibers over D^2, for all i, and $E \cup_\partial W \cup_\partial Z$ has exactly one connected component $B_1 \cong n_1(S^1 \times D^2) \# (S^1 \times S^2)$. In this case, $\partial M_0 \cong S^1 \times S^2$.

Case 3: $q = 1$ and $\ell = 0$. If $r = 1$, then $\sum_i (g_i + d_i - 1) = 0$. By the same arguments as in case 1, A_i is the total space of S^1-fibers over D^2, for all i. However, because $r = 1$, there exists a connected component of A that has at least two boundary components. This is a contradiction. Therefore, $r = 0$ and $\sum_i (g_i + d_i - 1) = 1$. It follows that $g_i = 0$ for all i, and without loss of generality $d_1 = 2$ and $d_i = 1$ for all $i \geq 2$. A_1 is the total space of S^1-fibers over $S^2 \times I$, and A_i is the total space of S^1-fiber over D^2, for all $i \geq 2$. $(E \cup_\partial W \cup_\partial Z) = \bigsqcup_i B_i = B_1 \sqcup B_2$ where $B_k \cong n_k(S^1 \times D^2)$, for some $n_k \geq 1$, $k \in \{1, 2\}$. $A_1 \cap B_1 \neq \varnothing$ and $A_1 \cap B_2 \neq \varnothing$. A has $n_1 + n_2 - 1$ connected components. Alternatively, B_2 is a component $$\left(\begin{tikzcd} D^2 \arrow[r] & N_j \arrow[d] \\ & \partial\Sigma^2 \end{tikzcd}\right)$$ for some j. In this case, $\partial M_0 \cong S^1 \times S^2$.

The classification of M_0 follows from Lemma 2.12 and Table 13.1. □

Combining Lemma 13.41 and Lemma 13.50, we get the following lemma.

Lemma 13.62. *Assume that M_0 intersects with* $$\left(\begin{tikzcd} D^3, S^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right),$$

$$\left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right), \left(\begin{tikzcd} D^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right), \left(\begin{tikzcd} S^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right),$$

and $$\left(\begin{tikzcd} S^1 \arrow[r] & M_j \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right)$$ *components, and M_0 is disjoint from fiber bundle components of* *other types, i.e.*

$$\begin{aligned}\partial M_0 \subset & \left[\bigsqcup_j \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right] \cup \\ & \left[\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)\right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right].\end{aligned} \tag{13.63}$$

Then, the conclusion of Lemma 13.50 is still valid but with every occurrence of A_i *in the statement replaced by* $A_i \cup_\partial \bigsqcup_{i_k} C_{i_k}$ *where* $C_{i_k} \cong S^1 \times D^2$ *or* $T^2 \times_{\mathbb{Z}_2} I$ *is a fiber of* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_{i_k}} \\ \downarrow \\ \partial I \end{array}\right)$*, for some* j_{i_k}*.* ∂M_0 *is diffeomorphic to* S^3*,* $S^1 \times S^2$*,* $\mathbb{R}P^3$*,* $\mathbb{R}P^3 \# \mathbb{R}P^3$*, or a Lens space.*

Proof. The proof of Lemma 13.41 is still valid when every occurrence of A_i is replaced by $A_i \cup_\partial \bigsqcup_{i_k} C_{i_k}$. □

The following two lemmas describe small adjustments to Lemma 13.30, Lemma 13.38, Lemma 13.41, Lemma 13.50, and Lemma 13.62 in the case that M_0 also intersects $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ components in addition to $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ components.

Lemma 13.64. *In the assumption of Lemma 13.38, assume that* M_0 *also intersects* $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ *components. That is,* M_0 *intersects with* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$*,* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$*, and* $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ *components and* M_0 *is disjoint from fiber bundle components of other types, i.e.*

$$\partial M_0 \subset \left[\bigsqcup_j \begin{pmatrix} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow & M_j \\ & \downarrow \\ & (I, \partial I) \end{pmatrix}\right] \cup \left[\bigsqcup_j \begin{pmatrix} S^1 \longrightarrow & M_j \\ & \downarrow \\ & (X^3, \partial X^3) \end{pmatrix}\right]$$
$$\cup \left[\bigsqcup_j \begin{pmatrix} T^2 \longrightarrow & M_j \\ & \downarrow \\ & (\Sigma^2, \partial \Sigma^2) \end{pmatrix}\right]. \quad (13.65)$$

Then, $\partial M_0 = A \cup_\partial C \cup_\partial T$ *where:*

- *(i)* $A = \bigsqcup_i A_i$ *where* A_i *is a subbundle of* $\begin{pmatrix} S^1 \longrightarrow & N_{j_i} \\ & \downarrow \\ & \partial X^3 \end{pmatrix}$ *for some* j_i *and* A_i *is connected.*
- *(ii)* $C = \bigsqcup_i C_i$ *where* $C_i \cong (S^1 \times D^2$ *or* $T^2 \times_{\mathbb{Z}_2} I)$ *is a fiber of* $\begin{pmatrix} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow & N_{j_i} \\ & \downarrow \\ & \partial I \end{pmatrix}$, *for some* j_i.
- *(iii)* $T = \bigsqcup_i T_i$ *where* $T_i \cong T^2 \times I$ *is a subbundle of* $\begin{pmatrix} T^2 \longrightarrow & N_{j_i} \\ & \downarrow \\ & \partial \Sigma^2 \end{pmatrix}$, *for some* j_i.
- *(iv)* $\partial C_i \cong T^2$ *is identified with a boundary component of* A *or a boundary component of* T_j, *for some* j.
- *(v) Each boundary component of* T_i *is identified with a boundary component of* A *or a boundary component of* C_j, *for some* j.
- *(vi) For every* i, $T_i \cap A \neq \varnothing$.

It follows that ∂M_0 *is a Seifert manifold.*

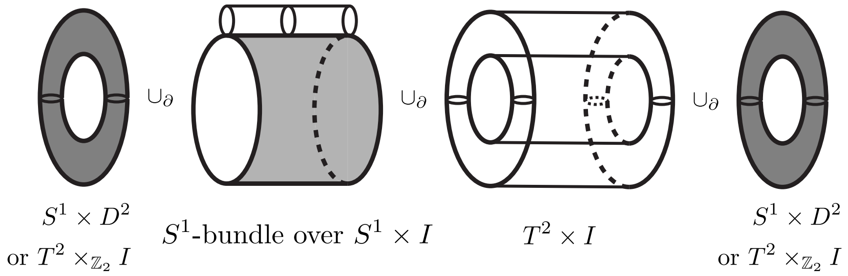


Figure 13.9: Example of a decomposition of ∂M_0 in Lemma 13.64.

Proof. Write $\partial M_0 = A \cup_\partial C \cup_\partial T$ where we put $A = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3)\end{array}\right)\right]$, $C = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I)\end{array}\right)\right]$, and $T = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2)\end{array}\right)\right]$.

By Proposition 12.1, A is a disjoint union of subbundles of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3\end{array}\right)$. Hence, A is the total space of S^1-fibers over a disjoint union of surfaces with boundary. Write $A = \bigsqcup_i A_i$ where A_i is a connected component of A.

From Lemma 13.5, fibers over an interior point of $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I)\end{array}\right)$ are disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_0 \\ \downarrow \\ \mathrm{pt}\end{array}\right)$. Hence, C is a disjoint union of fibers of $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I\end{array}\right)$. Write $C = \bigsqcup_{i=1} C_i$, where $C_i \cong S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I$ is a connected component of C. If $C_i \cap A \neq \varnothing$, then ∂C_i is identified with a boundary component of A.

From Lemma 13.3, T is a disjoint union of subbundles of $\left(\begin{array}{r} T^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2\end{array}\right)$ components which are diffeomorphic to $T^2 \times I$. Write $T = \bigsqcup_i T_i$ where $T_i \cong T^2 \times I$ is a connected component of T.

From Lemma 13.5, if $T_i \cap C_j \neq \varnothing$, then ∂C_j is identified with a boundary component of T_i. Suppose that for some i, j_1, and j_2, $T_i \cap C_{j_1} \neq \varnothing$ and $T_i \cap C_{j_2} \neq \varnothing$. Then, $C_{j_1} \cup_\partial T_i \cup_\partial C_{j_2}$ is a closed manifold. Consequently, $\partial M_0 = C_{j_1} \cup_\partial T_i \cup_\partial C_{j_2}$. This is a contradiction since $A \neq \varnothing$. Therefore, every T_i intersects with at most one connected component of C. In other words, $T_i \cap A \neq \varnothing$. From Lemma 13.3, it follows that for every i, a boundary component of T_i coincides with a boundary component of A.

In summary, each boundary component $\partial C_i \cong T^2$ of C is identified with a boundary component of $A \cup_\partial T$. Because ∂M_0 is connected, $A \cup_\partial T$ must be connected. Since A_i is the total spaces of S^1-fibers over a surface and $T_j \cong T^2 \times I$, for every i and j, we have that

$A \cup_\partial T$ is a connected graph manifold with boundary. Hence,

$$\begin{aligned}\partial M_0 &= (A \cup_\partial T) \cup_\partial \left(\bigsqcup_i C_i\right) \\ &\cong Q^3 \cup_\partial \left(\bigsqcup_i (S^1 \times D^2)\right) \cup_\partial \left(\bigsqcup_i (T^2 \times_{\mathbb{Z}_2} I)\right)\end{aligned} \tag{13.66}$$

where Q^3 is a graph manifold.

According to the classification of

$$\left(\begin{tikzcd} D^4, \ldots \arrow[r] & M_0 \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right)$$

in Table 13.1 and the classification of Seifert manifolds in [17], ∂M_0 is a Seifert manifold. □

Lemma 13.67. *In the assumptions of Lemma 13.30, Lemma 13.41, Lemma 13.50, and Lemma 13.62, assume that M_0 also intersects*

$$\left(\begin{tikzcd} T^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$$

components.

The conclusion of the lemmas are still valid when an occurrence of A_j, an S^1-subbundle of

$$\left(\begin{tikzcd} S^1 \arrow[r] & \partial M_j \arrow[d] \\ & \partial X^3 \end{tikzcd}\right),$$

is replaced by $A'_j = S \cup_\partial T$ where:

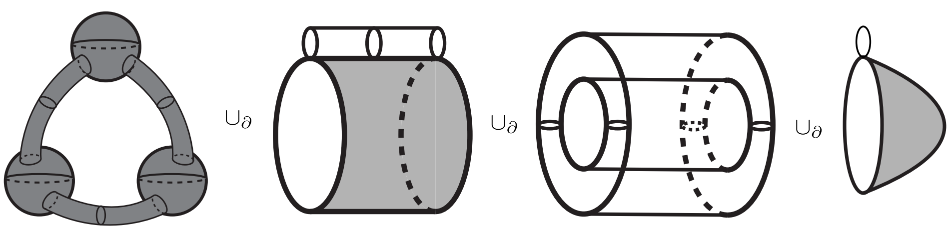
(1) *A'_j is connected.*

(2) *$S = \bigsqcup_i S_i$ where S_i is an S^1-subbundle of*
$$\left(\begin{tikzcd} S^1 \arrow[r] & N_{j_i} \arrow[d] \\ & \partial X^3 \end{tikzcd}\right),$$
for some j_i, and S_i is connected.

(3) *$T = \bigsqcup_i T_i$ where $T_i \cong T^2 \times I$ is a T^2-subbundle of*
$$\left(\begin{tikzcd} T^2 \arrow[r] & N_{j_i} \arrow[d] \\ & \partial\Sigma^2 \end{tikzcd}\right),$$
for some j_i.

(4) *For every i, each boundary component of T_i is identified with a boundary component of S. In particular, $\partial A'_j \subset \partial S$.*

For Lemma 13.30 and Lemma 13.50, in addition to $S^3, S^2 \times S^1, \mathbb{R}P^3$, and $\mathbb{R}P^3 \# \mathbb{R}P^3$, ∂M_0 can also be diffeomorphic to a Lens space.



$B_1 \cong (S^1 \times D^2)\#k(\mathbb{R}P^3)$ S^1-bundle over $S^1 \times I$ $T^2 \times I$ S^1-bundle over D^2

Figure 13.10: Example of a decomposition of ∂M_0 in Lemma 13.67.

Proof. Put $S = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right]$ and $T = \partial M_0 \cap \left[\bigsqcup_j \left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)\right]$. Write $S = \bigsqcup_i S_i$ where S_i is a connected component of S and $T = \bigsqcup_i T_i$ where T_i is a connected component of T. Put $A = S \cup_\partial T$ and write $A = \bigsqcup_j A'_j$ where A'_j is a connected component of A.

From Lemma 13.3, $T_i \cong T^2 \times I$ is a T^2-subbundle of $\left(\begin{array}{r} T^2 \longrightarrow N_{j_i} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some j_i. Moreover, T is disjoint from any $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ and $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ components. Therefore, each boundary component of T_i is identified with a boundary component of S. Since $T \neq \varnothing$, we must have that $\partial A'_j \subset \partial S$.

In the statements and the proofs of Lemma 13.30, Lemma 13.41, and Lemma 13.50, a connected component A_j of A is the total space of S^1-fibers over a connected surface. The arguments about a decomposition of ∂M_0 in the proofs of above lemmas only use the information on the boundary components of A_j. Therefore, arguments and conclusions of the lemmas are still valid when an occurrence of A_j is replaced by $A'_j = S \cup_\partial T$.

According to the classification of $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_0 \\ \downarrow \\ \text{pt} \end{array}\right)$ in Table 13.1 and from explicit constructions in [11], $\partial M_0 \cong S^3, S^2 \times S^1, \mathbb{R}P^3$, and $\mathbb{R}P^3\#\mathbb{R}P^3$, or a Lens space. □

13.9 Fiber bundle components $\left(\begin{tikzcd} S^3/\Gamma, \dots \arrow[r] & M_i \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$

Every statement about $\partial M_0 = \left(\begin{tikzcd} S^3, \dots \arrow[r] & \partial M_0 \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right)$ in the previous sections also applies to a boundary component of $\left(\begin{tikzcd} \begin{array}{l} S^3/\Gamma, T^3/\Gamma, \\ S^2 \times S^1, \mathbb{R}P^3 \# \mathbb{R}P^3 \end{array} \arrow[r] & M_i \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$. This is because the proofs in the previous sections only use Proposition 12.1 and the fact that ∂M_0 is a closed Seifert manifold. Therefore, we have the following lemma.

Lemma 13.68. *Lemma 13.9 to Lemma 13.50 are still valid when an occurrence of ∂M_0 is replaced with a boundary component of* $\left(\begin{tikzcd} \begin{array}{l} S^3/\Gamma, T^3/\Gamma, \\ S^2 \times S^1, \mathbb{R}P^3 \# \mathbb{R}P^3 \end{array} \arrow[r] & M_i \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$.

— 14 —

Gluing fiber bundle components into building blocks

Recall that we are trying to get a contradiction to Standing Assumption 3.2. At this point, we have a sequence of connected closed Riemannian 4-manifolds $\{M^\alpha\}_{\alpha=1}^\infty$ such that for large α, M^α satisfies the conclusions of Proposition 12.1 and all lemmas in Chapter 13. As mentioned in Chapter 3, we refer to M^α just by M. To get a contradiction to Standing Assumption 3.2, we need to show that M admits an F-structure or a metric of nonnegative sectional curvature.

As a result of Proposition 12.1 and Chapter 13, M can be decomposed into fiber bundle components (see Table 13.1) which have disjoint interiors and are compatible along the overlaps. The next step is to find all possible ways to glue these components together.

In this chapter, we start the gluing process by gluing the fiber bundle components into *elementary building blocks*. Then we construct more complicated building blocks from different types of elementary building blocks. In addition to the gluing process, we also show that the building blocks admit an F-structure.

In the next chapter, we will finish the proof of Theorem 1.4 by describing M in terms of a configuration of building blocks and showing that M admits an F-structure or a metric of nonnegative sectional curvature.

14.1 Gluing $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ and $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components

Let M_i be a component $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$. If there exists a component $M_j = \left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ such that $M_i \cap M_j \neq \varnothing$, then by Lemma 13.8, ∂M_j is a boundary component of M_i.

If M_i intersects $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_{j_\ell} \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ for $\ell \in \{1, 2\}$, where $M_{j_1} \neq M_{j_2}$, then $M = M_{j_1} \cup_\partial M_i \cup_\partial M_{j_2} \cong M_{j_1} \cup_\partial M_{j_2}$.

If M_i intersects exactly one $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ component, then $M_i \cup_\partial M_j \cong M_j$. From Lemma 13.68, the decomposition of a boundary component of $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ is the same as the decomposition of $\left(\begin{array}{r} S^3, \ldots \longrightarrow N_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$.

Therefore, we can assume without loss of generality that $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ is disjoint from any $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ components.

14.2 Overview of the gluing strategy

To simplify the gluing process, in the following sections we assume that M does not contain any $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ component. In the next chapter, we will show that constructions and results in this chapter are still valid when an occurrence of a component $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ is replaced by a boundary component of $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$.

We start by gluing the fiber bundle components in Table 13.1 into elementary building blocks, then we construct general building blocks by combining different types of elementary building blocks.

There are four types of elementary building blocks: $(2, S^2)$, $(2, T^2)$, $(2, D^2)$, $(1, S^1 \times D^2)$. In the following sections, we will use graphs and polyhedrons to represent building blocks and to provide gluing instructions. The number n of an elementary building block of type (n, F) corresponds to the dimension of its representation while the manifold F corresponds to its representative fiber type. The following table shows fiber bundle components that are involved in a construction of each type of elementary building block.

Table 14.1: Elementary building blocks

Type	Fiber bundle components involved
$(2, S^2)$	$\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right), \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right), \left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$
$(2, T^2)$	$\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right), \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right), \left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$
$(2, D^2)$	$\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right), \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right), \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right),$ $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$
$(1, S^1 \times D^2)$	$\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right), \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right), \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$

14.3 Elementary building block of type $(2, S^2)$

In this section, we construct an elementary building block of type $(2, S^2)$. The fiber bundle components that are involved in the constructions are $\left(\begin{array}{r} D^4, \dots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, and $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$.

Definition 14.1. Let M_k be a component $\left(\begin{array}{r} S^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$. Assume that every boundary components of M_k is attached to a manifold W_j where W_j is the manifold W in the conclusion of Lemma 14.3. We call the union $M_k \cup \left(\bigsqcup_j W_j\right)$ an *elementary building block of type* $(2, S^2)$.

We can represent an elementary building block of type $(2, S^2)$ by a disjoint union of solid polygons where the boundary of each polygon is the cycle graph representing the manifold W in Lemma 14.3 and the interior represents attaching W to a boundary component of M_k.

Example 14.2. The following is a model example of an elementary of building block of type $(2, S^2)$.

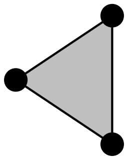


Figure 14.1: A representation of an elementary of building block of type $(2, S^2)$

In this example, there are three $\left(\begin{array}{r} D^4 \longrightarrow M_{i_\ell} \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ components, $\ell \in \{1, 2, 3\}$, which are represented by vertices, and three $\left(\begin{array}{r} D^3 \longrightarrow M_{j_\ell} \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components, $\ell \in \{1, 2, 3\}$, which are represented by edges. Denote the union of all M_{i_ℓ} and M_{j_ℓ} by W. We have that $W \cong S^1 \times D^3$. There is one $\left(\begin{array}{r} S^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ component, where $\Sigma^2 \cong D^2$. The interior of the triangle represents attaching ∂W to a boundary component of M_k. In this example, $M = W \cup_\partial M_k \cong (S^1 \times D^3) \cup (S^2 \times D^2) \cong S^4$.

From Lemma 13.10, if $M_0 = \begin{pmatrix} D^4, \dots \longrightarrow & M_0 \\ & \downarrow \\ & \mathrm{pt} \end{pmatrix}$ only intersects $\begin{pmatrix} S^2 \longrightarrow & M_j \\ & \downarrow \\ & (\Sigma^2, \partial\Sigma^2) \end{pmatrix}$ and $\begin{pmatrix} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow & M_j \\ & \downarrow \\ & (I, \partial I) \end{pmatrix}$ components, then $M_0 \cong D^4$, $\pm\mathbb{C}P^2 \# D^4$, $S^2 \times_{\pm 2} D^2$, or $S^2 \times_{\mathbb{Z}_2} D^2$.

Lemma 14.3. *Let* $\{M_i\}_{i \in \mathcal{A}_0}$ *be a collection of* $\begin{pmatrix} D^4, \pm\mathbb{C}P^2\#D^4, & \\ S^2 \times_{\pm 2} D^2, S^2 \times_{\mathbb{Z}_2} D^2 & \longrightarrow M_i \\ & \Big\downarrow \\ & \mathrm{pt} \end{pmatrix}$ *components such that* M_i *only intersects* $\begin{pmatrix} S^2 \longrightarrow & M_j \\ & \downarrow \\ & (\Sigma^2, \partial\Sigma^2) \end{pmatrix}$ *and* $\begin{pmatrix} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow & M_j \\ & \downarrow \\ & (I, \partial I) \end{pmatrix}$ *components (as in Lemma 13.10).*

Let $\{M_j\}_{j \in \mathcal{A}_1}$ *be a collection of* $\begin{pmatrix} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow & M_j \\ & \downarrow \\ & (I, \partial I) \end{pmatrix}$ *components such that both fibers of* $\begin{pmatrix} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow & N_j \\ & \downarrow \\ & \partial I \end{pmatrix}$ *are contained in* $\bigsqcup_{i \in \mathcal{A}_0} \partial M_i$.

Let W *be a connected component of* $M - \bigsqcup_p \begin{pmatrix} S^2 \longrightarrow & M_p \\ & \downarrow \\ & (\Sigma^2, \partial\Sigma^2) \end{pmatrix}$ *such that* W *only intersects components* M_i, $i \in \mathcal{A}_0$, *and* M_j, $j \in \mathcal{A}_1$, *i.e.* $W \subset \left(\bigsqcup_{i \in \mathcal{A}_0} M_i\right) \cup \left(\bigsqcup_{j \in \mathcal{A}_1} M_j\right)$. *Then, the following holds.*

1. *W can be represented by a cycle graph so that each vertex represents a component M_i, for some $i \in \mathcal{A}_0$, and each edge represents a component M_j, for some $j \in \mathcal{A}_1$.*
2. *$W \cong (S^1 \times D^3) \# n_1(\mathbb{C}P^2) \# n_2(-\mathbb{C}P^2) \# n_3(S^2 \times S^2)$ or $S^1 \times (S^2 \times_{\mathbb{Z}_2} I) \cong S^1 \times (\mathbb{R}P^3 \# D^3)$ for some integers $n_1, n_2, n_3 \geq 0$. In particular, $\partial W \cong S^1 \times \partial D^3$.*
3. *As a part of M, ∂W is identified with a boundary component of $\begin{pmatrix} S^2 \longrightarrow & M_k \\ & \downarrow \\ & (\Sigma^2, \partial\Sigma^2) \end{pmatrix}$, for some k, so that each ∂D^3-fiber of ∂W coincides with an S^2-fiber of ∂M_k.*

Proof. (1). We construct a graph to represent W as follows. Let G be a graph such that each vertex v_i represents a connected component M_i, for some $i \in \mathcal{A}_0$, and each vertex e_j

represents a connected component M_j, for some $j \in \mathcal{A}_1$. A vertex v_i is incident to an edge e_j if and only if $M_i \cap M_j \neq \varnothing$.

From Lemma 13.10, for each $i \in \mathcal{A}_0$, $\partial M_i = \left(\begin{array}{lr} S^3, \mathbb{R}P^3 \# \mathbb{R}P^3, & \\ \mathbb{R}P^3, S^1 \times S^2 & \longrightarrow N_i \\ & \Big\downarrow \\ & \mathrm{pt} \end{array}\right)$ contains exactly two fibers B_{j_1} and B_{j_2} from $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_1} \\ \downarrow \\ \partial I \end{array}\right)$ and $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_{j_2} \\ \downarrow \\ \partial I \end{array}\right)$, for some $j_1, j_2 \in \mathcal{A}_1$ respectively. It follows that every vertex v of G has degree two. Therefore, G is a cycle graph.

(2). Let v be a vertex and let e_1 and e_2 be edges incident to v.

$$\overset{e_1}{\rule{2.5em}{0.4pt}}\; v \;\overset{e_2}{\rule{2.5em}{0.4pt}} \tag{14.4}$$

We note that if G has only one vertex, e_1 and e_2 are the same edge. Let $M_i = \left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ be the component represented by v and let $M_{j_k} = \left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j_k} \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $k \in \{1, 2\}$, be the component represented by e_k. We have the following cases.

(a) Case $M_{j_k} = \left(\begin{array}{r} D^3 \longrightarrow M_{j_k} \\ \downarrow \\ (I, \partial I) \end{array}\right) \cong D^3 \times I$ for $k \in \{1, 2\}$.

By Lemma 13.10, $M_i \cong D^4$ or $\pm\mathbb{C}P^2 \# D^4$ and $M_{j_k} \cup_\partial M_i \cong D^3$, for $k \in \{1, 2\}$. If $M_i \cong D^4$, then $M_{j_1} \cup_\partial M_i \cup_\partial M_{j_2} \cong (D^3 \times I) \cup_\partial D^4 \cup_\partial (D^3 \times I) \cong D^3 \times I$. If $M_i \cong \pm\mathbb{C}P^2 \# D^4$, then $M_{j_1} \cup_\partial M_i \cup_\partial M_{j_2} \cong (D^3 \times I) \cup_\partial (\pm\mathbb{C}P^2 \# D^4) \cup_\partial (D^3 \times I) \cong (D^3 \times I) \# (\pm\mathbb{C}P^2)$.

(b) Case $M_{j_k} = \left(\begin{array}{r} S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j_k} \\ \downarrow \\ (I, \partial I) \end{array}\right) \cong (S^2 \times_{\mathbb{Z}_2} I) \times I$ for $k \in \{1, 2\}$.

By Lemma 13.10, $M_i \cong S^2 \times_{\mathbb{Z}_2} D^2$ and $M_{j_k} \cup_\partial M_i \cong S^2 \times_{\mathbb{Z}_2} I$, for $k \in \{1, 2\}$. Thus, $M_{j_1} \cup_\partial M_i \cup_\partial M_{j_2} \cong ((S^2 \times_{\mathbb{Z}_2} I) \times I) \cup_\partial (S^2 \times_{\mathbb{Z}_2} D^2) \cup_\partial ((S^2 \times_{\mathbb{Z}_2} I) \times I) \cong (S^2 \times_{\mathbb{Z}_2} I) \times I$. In other words,

$$\overset{M_{j_1}}{\rule{2.5em}{0.4pt}}\; M_i \;\overset{M_{j_2}}{\rule{2.5em}{0.4pt}} \quad \cong \quad \overset{M_{j'}}{\rule{2.5em}{0.4pt}} \tag{14.5}$$

where $\overset{M_{j_1}}{\overline{\qquad}}\ M_i\ \overset{M_{j_2}}{\overline{\qquad}}$ denotes $M_{j_1} \cup_\partial M_i \cup_\partial M_{j_2}$. Hence, by considering $M_{j_1} \cup_\partial M_i \cup_\partial M_{j_2}$ as a single component $M_{j'} = \begin{pmatrix} S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j'} \\ \hfill\downarrow \\ \hfill(I, \partial I) \end{pmatrix}$, we can assume that there is no vertex of this type, unless G has only one vertex.

In the case that G has exactly one vertex, W is the total space of $(S^2 \times_{\mathbb{Z}_2} I)$-fibers over S^1. The mapping class group of orientation preserving homeomorphism of $S^2 \times_{\mathbb{Z}_2} I$ is trivial [34, Lemma 9.12]. Therefore, $W \cong S^1 \times (S^2 \times_{\mathbb{Z}_2} I) \cong S^1 \times (\mathbb{R}P^3 \# D^3)$.

(c) Case $M_{j_1} = \begin{pmatrix} D^3 \longrightarrow M_{j_1} \\ \hfill\downarrow \\ \hfill(I, \partial I) \end{pmatrix} \cong D^3 \times I$ and $M_{j_2} = \begin{pmatrix} S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j_2} \\ \hfill\downarrow \\ \hfill(I, \partial I) \end{pmatrix}$ $\cong (S^2 \times_{\mathbb{Z}_2} I) \times I$.

By Lemma 13.10, $M_i \cong S^2 \times_{\pm 2} D^2$, $M_{j_1} \cap M_i \cong D^3$, and $M_{j_2} \cap M_i \cong S^2 \times_{\mathbb{Z}_2} I$.

Let v' be the vertex adjacent to v via e_2. By the assumption in the previous case, we have that v' represents a component $M_{i'} = \begin{pmatrix} S^2 \times_{\pm 2} D^2 \longrightarrow M_{i'} \\ \hfill\downarrow \\ \hfill\text{pt} \end{pmatrix}$, for some $i' \in \mathcal{A}_0$.

Additionally, v' is incident to an edge e_3 representing a component $\begin{pmatrix} D^3 \longrightarrow M_{j_3} \\ \hfill\downarrow \\ \hfill(I, \partial I) \end{pmatrix} \cong$ $D^3 \times I$, for some $j_3 \in \mathcal{A}_1$.

$$\overset{e_1}{\overline{\qquad}}\ v\ \overset{e_2}{\overline{\qquad}}\ v'\ \overset{e_3}{\overline{\qquad}} \tag{14.6}$$

$M_i \cup_\partial M_{j_2} \cup_\partial M_{i'}$ is diffeomorphic to $(S^2 \times_{\pm 2} D^2) \cup_\partial (S^2 \times_{\pm 2} D^2)$ where two copies of $(S^2 \times_{\mathbb{Z}_2} I) \cong \mathbb{R}P^3 \# D^3$ on their $\mathbb{R}P^3$-boundaries are identified. Therefore, $M_i \cup_\partial M_{j_2} \cup_\partial M_{i'}$ is diffeomorphic to $(D^3 \times I)\#(S^2 \times S^2)$ or $(D^3 \times I)\#(S^2 \widetilde{\times} S^2)$ where $S^2 \widetilde{\times} S^2 \cong \mathbb{C}P^2 \# (-\mathbb{C}P^2)$ is the nontrivial orientable S^2-bundle over S^2 (see Lemma 2.13).

From all cases, we have that if W is not diffeomorphic to $S^1 \times (\mathbb{R}P^3 \# D^3)$, then

$$W \cong \left(\bigcup_{i=1}^{m} (D^3 \times [0,1])_i \right) \# n_1(\mathbb{C}P^2) \# n_2(-\mathbb{C}P^2) \# n_3(S^2 \times S^2) \tag{14.7}$$

where the union is so that $(D^3 \times \{1\})_i$ is identified with $(D^3 \times \{0\})_{i+1}$, $i \in \mathbb{Z}/m\mathbb{Z}$, for some integers $n_1, n_2, n_3 \geq 0$. Because the mapping class group of orientation preserving homeomorphism of D^3 is trivial,

$$W \cong (S^1 \times D^3) \# n_1(\mathbb{C}P^2) \# n_2(-\mathbb{C}P^2) \# n_3(S^2 \times S^2) \tag{14.8}$$

for some integers $n_1, n_2, n_3 \geq 0$. In particular, $\partial W \cong S^1 \times \partial D^3$.

(3) From Lemma 13.2 and Lemma 13.4, the boundary of each fiber of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $j \in \mathcal{A}_1$, coincides with an S^2-fiber of of $\left(\begin{array}{r} S^2 \longrightarrow N_k \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some k. From Lemma 13.10, $\partial M_0 - (B_{j_1} \cup_\partial B_{j_2}) \cong S^2 \times I$ is a subbundle of $\left(\begin{array}{r} S^2 \longrightarrow N_k \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some k. By connectedness, ∂W is identified with a boundary component of $\left(\begin{array}{r} S^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, for some k, so that each ∂D^3-fiber of ∂W coincides with an S^2-fiber of ∂M_k. □

In the following lemma, we construct an F-structure on an elementary building block of type $(2, S^2)$.

Lemma 14.9. *Let Y be an elementary building block of type $(2, S^2)$. That is, $Y = M_k \cup \left(\bigsqcup_j W_j\right)$ for some component $M_k = \left(\begin{array}{r} S^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ and W_j is a manifold represented by a cycle graph in Lemma 14.3. Then, Y admits an F-structure.*

Proof. From Lemma 14.3,

$$Y \cong \left[M_k \cup_\partial \left(\bigsqcup_i S^1 \times D^3 \right) \cup_\partial \left(\bigsqcup_j S^1 \times (\mathbb{R}P^3 \# D^3) \right) \right] \# n(\mathbb{C}P^2) \# m(-\mathbb{C}P^2) \# p(S^2 \times S^2) \tag{14.10}$$

where the union is so that each ∂D^3-fiber of ∂W_j is identified with an S^2-fiber of M_k, for some integers $n, m, p \geq 0$.

An orientable S^2-bundle over a compact surface with boundary is trivial [14]. Let S^1 act on M_k by rotations (with two fixed points) on each S^2-fiber and act trivially on the base $(\Sigma^2, \partial\Sigma^2)$. Let S^1 act on each copy of $S^1 \times D^3$ by extending the S^1-action on S^2-fibers of ∂M_k to rotations on D^3-fibers about an axis. For each copy of $S^1 \times (\mathbb{R}P^3 \# D^3) \cong S^1 \times (S^2 \times_{\mathbb{Z}_2} I)$, we consider the double covering $S^1 \times (S^2 \times I) \xrightarrow{\pi} S^1 \times (S^2 \times_{\mathbb{Z}_2} I)$. Let S^1 act on $S^1 \times (S^2 \times I)$ by rotations on each S^2-fiber and act trivially on the S^1 and I-factors. The action can be made compatible with π and the S^1-action on M_k.

The above construction gives an F-structure on $M_k \cup_\partial \bigsqcup_i (S^1 \times D^3) \cup_\partial \bigsqcup_j (S^1 \times (\mathbb{R}P^3 \# D^3))$. Paternain and Petean [21, Theorem 5.9] showed that the connected sum of two manifolds X and Y with F-structure admits an F-structure, provided that X and Y have at least one open set with a trivial normal covering (as in Definition 2.26). Since $\mathbb{C}P^2, -\mathbb{C}P^2$, and $S^2 \times S^2$ admit a T-structure [21], Y admits an F-structure. ☐

14.4 Elementary building block of type $(2, T^2)$

In this section, we construct an elementary building block of type $(2, T^2)$. The fiber bundle components that are involved in the constructions are

$$\left(\begin{tikzcd} D^4, \dots \arrow[r] & M_j \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right),$$

$$\left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right), \text{ and } \left(\begin{tikzcd} T^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right).$$

Definition 14.11. Let M_k be a component $\left(\begin{tikzcd} T^2 \arrow[r] & M_k \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$. Assume that every boundary component of M_k is attached to a manifold W_j where W_j is the manifold W in the conclusion of Lemma 14.14. We call the union $M_k \cup \left(\bigsqcup_j W_j\right)$ an *elementary building block of type* $(2, T^2)$.

We represent an elementary building block of type $(2, T^2)$ by a disjoint union of solid polygons where the boundary of each polygon is the cycle graph representing the manifold W in Lemma 14.14 and the interior represents attaching W to a boundary component of M_k.

Example 14.12. The following is a model of an elementary building block of type $(2, T^2)$.

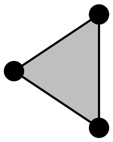


Figure 14.2: A representation of an elementary building block of type $(2, T^2)$

In this example, there are three $\left(\begin{tikzcd} S^1 \times D^3 \arrow[r] & M_{i_\ell} \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right)$ components, $\ell \in \{1, 2, 3\}$, which are represented by vertices, and three $\left(\begin{tikzcd} S^1 \times D^2 \arrow[r] & M_{j_\ell} \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$ components, $\ell \in \{1, 2, 3\}$, which are represented by edges. Denote the union of all M_{i_ℓ} and M_{j_ℓ} by W. We have that

$W \cong S^1 \times (S^1 \times D^2)$. There is one $\begin{pmatrix}\begin{array}{r} T^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2)\end{array}\end{pmatrix}$ component, where $\Sigma^2 \cong D^2$. The interior of the triangle represents attaching ∂W to a boundary component of M_k In this case, $M = M_k \cup_\partial W \cong S^1 \times (S^1 \times D^2) \cup D^2 \times T^2 \cong S^1 \times S^3$.

From Lemma 13.17, if $M_0 = \begin{pmatrix}\begin{array}{r} D^4, \dots \longrightarrow M_0 \\ \downarrow \\ \mathrm{pt}\end{array}\end{pmatrix}$ only intersects $\begin{pmatrix}\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2)\end{array}\end{pmatrix}$ and $\begin{pmatrix}\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I)\end{array}\end{pmatrix}$ components, then

$$M_0 \cong \begin{cases} D^4, \pm\mathbb{C}P^2 \# D^4 & \text{if } \partial M_0 \cong S^3, \\ S^1 \times D^3, S^2 \times D^2, (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I & \text{if } \partial M_0 \cong S^2 \times S^1, \\ S^2 \times_\omega D^2, \omega \in \mathbb{Z} & \text{if } \partial M_0 \cong L(|\omega|, 1), \\ S^2 \times_{\mathbb{Z}_2} D^2 & \text{if } \partial M_0 \cong \mathbb{R}P^3 \# \mathbb{R}P^3, \\ (S^2 \times_\omega D^2)/\mathbb{Z}_2, \omega \in \mathbb{Z} & \text{if } \partial M_0 \cong L(|\omega|, 1)/\mathbb{Z}_2, \\ T^2 \times_{\mathbb{Z}_2} D^2, \mathcal{B}_3 \widetilde{\times} I, \mathcal{B}_4 \widetilde{\times} I & \text{if } \partial M_0 \cong \mathcal{G}_2. \end{cases} \tag{14.13}$$

Lemma 14.14. *Let* $\{M_i\}_{i \in \mathcal{A}_0}$ *be a collection of* $\begin{pmatrix}\begin{array}{r} D^4, \dots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt}\end{array}\end{pmatrix}$ *components such that* M_i *only intersects* $\begin{pmatrix}\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2)\end{array}\end{pmatrix}$ *and* $\begin{pmatrix}\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I)\end{array}\end{pmatrix}$ *components (as in Lemma 13.17).*

Let $\{M_j\}_{j \in \mathcal{A}_1}$ *be a collection of* $\begin{pmatrix}\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I)\end{array}\end{pmatrix}$ *components such that both fibers of* $\begin{pmatrix}\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I\end{array}\end{pmatrix}$ *are contained in* $\bigsqcup_{i \in \mathcal{A}_0} \partial M_i$.

Let W *be a connected component of* $M - \bigsqcup_p \begin{pmatrix}\begin{array}{r} T^2 \longrightarrow M_p \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2)\end{array}\end{pmatrix}$ *such that* W *only intersects components* M_i, $i \in \mathcal{A}_0$, *and* M_j, $j \in \mathcal{A}_1$, *i.e.* $W \subset \left(\bigsqcup_{i \in \mathcal{A}_0} M_i\right) \cup \left(\bigsqcup_{j \in \mathcal{A}_1} M_j\right)$. *Then, the following holds.*

(1) *W can be represented by a cycle graph so that each vertex represents M_i, for some $i \in \mathcal{A}_0$ and each edge represents M_j, for some $j \in \mathcal{A}_1$.*

(2) *∂W is a T^2-bundle over S^1. W admits an F-structure which restricts to local T^2-actions on ∂W. The local T^2-actions on ∂W are free and their orbits coincide with T^2-fibers of ∂W.*

(3) *As a part of M, ∂W is identified with a boundary component of*

$$\left(\begin{tikzcd} T^2 \arrow[r] & M_k \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right),$$

for some k, so that T^2-fibers of ∂W coincide with T^2-fibers of M_k.

Proof. Here, we prove part (1) and part (3). In the next subsection, we give a proof of part (2).

(1). We construct a graph to represent W as follows. Let G be a graph such that each vertex v_i represents a connected component M_i, for some $i \in \mathcal{A}_0$, and each vertex e_j represents a connected component M_j, for some $j \in \mathcal{A}_1$. A vertex v_i is incident to an edge e_j if and only if $M_i \cap M_j \neq \varnothing$.

From Lemma 13.17, for each $i \in \mathcal{A}_0$, ∂M_i contains exactly two fibers F_{j_1} and F_{j_2} from

$$\left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & N_{j_1} \arrow[d] \\ & \partial I \end{tikzcd}\right) \text{ and } \left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & N_{j_2} \arrow[d] \\ & \partial I \end{tikzcd}\right),$$

for some $j_1, j_2 \in \mathcal{A}_1$ respectively. It follows that every vertex v of G has degree two. Therefore, G is a cycle graph.

(3) From Lemma 13.3 and Lemma 13.5, the boundary of each fiber of

$$\left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right) \text{ coincides with a } T^2\text{-fiber of of } \left(\begin{tikzcd} T^2 \arrow[r] & N_k \arrow[d] \\ & \partial\Sigma^2 \end{tikzcd}\right), \text{ for some } k.$$

From Lemma 13.17, $\partial M_0 - (F_{j_1} \cup_\partial F_{j_2}) \cong T^2 \times I$ is a subbundle of

$$\left(\begin{tikzcd} T^2 \arrow[r] & N_k \arrow[d] \\ & \partial\Sigma^2 \end{tikzcd}\right),$$

for some k. By connectedness, ∂W is identified with a boundary component of

$$\left(\begin{tikzcd} T^2 \arrow[r] & M_k \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right),$$

for some k so that each T^2-fiber of ∂W coincides with a T^2-fiber of ∂M_k. □

14.4.1 Proof of Lemma 14.14 (2)

In this subsection, we prove part (2) of Lemma 14.14. Let v be a vertex and let e_1 and e_2 be edges incident to v.

$$\xrightarrow{e_1}\!\!\!\!\!\!\quad v \quad \overset{e_2}{\text{———}} \qquad (14.15)$$

We note that if G has only one vertex, e_1 and e_2 are the same edge. Let

$$M_i^v = \left(\begin{tikzcd} D^4, \dots \arrow[r] & M_i^v \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right)$$

be the component represented by v and let

$$M_{j_k}^e = \left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_{j_k}^e \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right), k \in \{1,2\},$$

be the component represented by e_k. We denote the fiber of

$$\left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & N_{j_k}^e \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$$

that is contained in ∂M_i^v by F_{j_k}, for $k \in \{1,2\}$. We have the following cases.

(a) Case $M_{j_k}^e = \left(\begin{tikzcd} S^1 \times D^2 \arrow[r] & M_{j_k}^e \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right) \cong (S^1 \times D^2) \times I$ for $k \in \{1,2\}$.

By Lemma 13.17, $\partial M_i^v = B_1 \cup_\partial A \cup_\partial B_2$ where $A \cong T^2 \times I$ and $B_k \cong S^1 \times D^2$, $k \in \{1,2\}$. $F_{j_k} \subset M_{j_k}^e$ attaches to $B_k \subset M_i^v$. Consider that $\partial M_i^v \cong B_1 \cup_\partial B_2 \cong (S^1 \times D^2) \cup_\partial (S^1 \times D^2)$ via the identifying map $\varphi : \underbrace{(S^1 \times \partial D^2)}_{\partial B_1} \to \underbrace{(S^1 \times \partial D^2)}_{\partial B_2}$. Up to isotopy, $\varphi \in SL_2(\mathbb{Z})$. $\partial M_i^v \cong L(p,q)$ if and only if φ sends a meridian $\{x\} \times \partial D^2$ to a circle of slope $\frac{q}{p}$. We adapt the convention from [10] that a meridian has slope ∞, a longitude $S^1 \times \{y\}$ has slope 0, $L(1,0) \cong S^3$, and $L(0,1) \cong S^1 \times S^2$. In particular,

$$M_i^v \cong \begin{cases} D^4, \pm\mathbb{C}P^2 \# D^4 & \text{if } \partial M_i^v \cong S^3, \\ S^1 \times D^3, S^2 \times D^2, (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I & \text{if } \partial M_i^v \cong S^1 \times S^2, \\ S^2 \times_\omega D^2 & \text{if } \partial M_i^v \cong L(|\omega|, 1). \end{cases} \tag{14.16}$$

We have the following cases.

(i) $M_i^v \cong S^1 \times D^3$.

In this case, $\partial M_i^v \cong S^1 \times \partial D^3 \cong S^1 \times S^2$ and φ sends a meridian to a meridian. Then, we can consider $M_i^v \cong S^1 \times (D^2 \times [0,1])$ where $F_{j_1} \to B_1$ sends $\{x\} \times D^2$ to $\{x\} \times (D^2 \times \{0\})$ and $F_{j_2} \to B_2$ sends $\{x\} \times D^2$ to $\{x\} \times (D^2 \times \{1\})$.

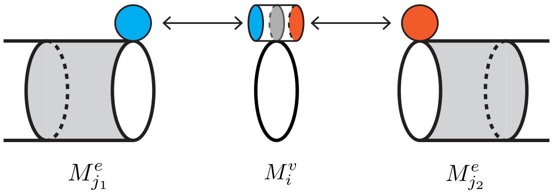


Figure 14.3: $M_{j_1}^e \cup_\partial M_i^v \cup_\partial M_{j_2}^e$ in case (a) when $M_i^v \cong S^1 \times D^3$. In this figure, only one D^2-fiber of $M_{j_k}^e$, $k \in \{1,2\}$ and one D^3-fiber of M_i^v are showed.

Hence,

$$M^e_{j_1} \cup_\partial M^v_i \cup_\partial M^e_{j_2} \cong ((S^1 \times D^2) \times I) \cup_\partial (S^1 \times D^3) \cup_\partial ((S^1 \times D^2) \times I) \cong (S^1 \times D^2) \times I. \tag{14.17}$$

In other words, the manifold represented by the graph

$$v' \text{ ——— } v \text{ ——— } v'' \tag{14.18}$$

is diffeomorphic to the manifold represented by the graph

$$v' \text{ ————— } v''. \tag{14.19}$$

Therefore, we can assume that there are no vertices of this type unless the graph G has only one vertex. In the case that G has one vertex, $W \cong (S^1 \times D^2) \times S^1$ and $\partial W \cong \partial(S^1 \times D^2) \times S^1 \cong T^2 \times S^1$.

(ii) $M^v_i \cong S^2 \times D^2$.

In this case, $\partial M^v_i \cong S^2 \times \partial D^2 \cong S^2 \times S^1$ and φ sends a meridian to a meridian. Consider $\partial M^v_i \cong S^2 \times D^2$ as a D^2-bundle over S^2. Then, $F_k \cong D^2_k \times \partial D^2$ where D^2_k is a 2-disk subset of the base S^2. Thus, we have that $\{x\} \times D^2 \mapsto D^2_k \times \{y\}$.

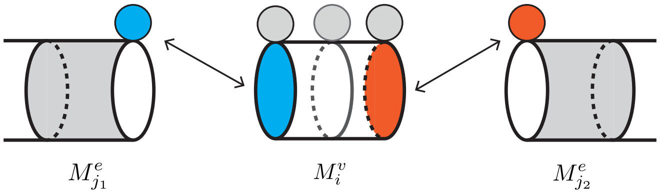


Figure 14.4: $M^e_{j_1} \cup_\partial M^v_i \cup_\partial M^e_{j_2}$ in case (a) when $M^v_i \cong S^2 \times D^2$. In this figure, only one D^2-fiber of $M^e_{j_k}$, $k \in \{1, 2\}$ and one D^2-fiber of M^v_i are showed.

Let $(M^e)'_{j_k} \cong M^e_{j_k} \cup_\partial (D^2 \times S^1)$ along the identity attaching map $\partial F_{j_k} \cong S^1 \times \partial D^2 \to \partial D^2 \times S^1$. It follows that $(M^e)'_{j_k}$ is a trivial D^2-bundle over a surface $\Sigma^2_k \cong (S^1 \times I) \cup_\partial D^2 \cong D^2$. Then, we have that $M^e_{j_k} \cup_\partial M^v_i \cong (M^e)'_{j_k} \Box M^v_i \cong (\Sigma^2_k \times D^2) \Box (S^2 \times D^2)$ where $\Box$ denotes a plumbing (see Section 2.6). Therefore,

$$\overset{M^e_{j_1}}{\text{———}} \; M^v_i \; \overset{M^e_{j_2}}{\text{———}} \quad \cong \underbrace{(\Sigma^2_1 \times D^2)}_{(M^e)'_{j_1}} \Box \underbrace{(S^2 \times D^2)}_{M^v_i} \Box \underbrace{(\Sigma^2_2 \times D^2)}_{(M^e)'_{j_2}} \tag{14.20}$$

where $\overset{M^e_{j_1}}{\text{———}} \; M^v_i \; \overset{M^e_{j_2}}{\text{———}}$ denotes $M^e_{j_1} \cup_\partial M^v_i \cup_\partial M^e_{j_2}$.

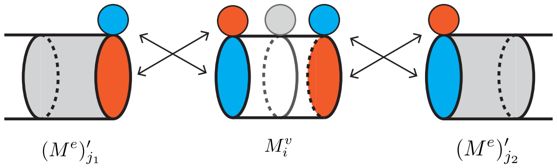


Figure 14.5: $(M^e)'_{j_1} \cup_\partial M_i^v \cup_\partial (M^e)'_{j_2}$ in case (a) when $M_i^v \cong S^2 \times D^2$ in terms of plumbing

Suppose that there exists a component $M^v_{i_2} = \left(\begin{array}{r} S^2 \times D^2 \longrightarrow M^v_{i_2} \\ \downarrow \\ \text{pt} \end{array}\right)$, $i_2 \in \mathcal{A}_0$, such that $M^v_{i_2} \cap M^e_{j_2} \neq \varnothing$. In other words, there exists a vertex v_2 (representing $M^v_{i_2}$) adjacent to v via the edge e_2. Let $(M^e)''_{j_2} \cong (M^e)'_{j_2} \cup_\partial (D^2 \times S^1)$ along the identity attaching map $S^1 \times \partial\Sigma_2^2 \to \partial D^2 \times S^1$. It follows that $(M^e)''_{j_2}$ is a trivial D^2-bundle over $\Sigma_2^2 \cup_\partial D^2 \cong S^2$. Therefore,

$$\stackrel{M^e_{j_1}}{\rule{2em}{0.4pt}} \; M_i^v \; \stackrel{M^e_{j_2}}{\rule{2em}{0.4pt}} \; M^v_{i_2} \;\cong \underbrace{(\Sigma_1^2 \times D^2)}_{(M^e)'_{j_1}} \square \underbrace{(S^2 \times D^2)}_{M_i^v} \square \underbrace{(S^2 \times D^2)}_{(M^e)''_{j_2}} \square \underbrace{(S^2 \times D^2)}_{M^v_{i_2}}. \tag{14.21}$$

If the graph G has one vertex, then W is a cyclic plumbing of two copies of $S^2 \times D^2$, i.e. W is a 4-manifold with plumbing diagram

$$\begin{tikzcd} W \cong \; S^2 \times D^2 \arrow[r, bend left, dash, "\square"] \arrow[r, bend right, dash, "\square"'] & S^2 \times D^2 \end{tikzcd} \; . \tag{14.22}$$

In particular, ∂W is a T^2-bundle over S^1. From Lemma 2.30, W admits a T-structure which restricts to local T^2-actions on ∂W. The local T^2-actions on ∂W are free and their orbits coincide with T^2-fibers of ∂W.

(iii) $M_i^v \cong S^2 \times_\omega D^2$, $\omega \in \mathbb{Z}$, $\omega \neq 0$.

We note that $S^2 \times_{\mp 1} D^2 \cong \pm\mathbb{C}P^2 \# D^4$ ([26, Section 2.4]). In this case, $\partial M_i^v \cong L(|\omega|, 1)$ and $\varphi : \underbrace{(S^1 \times \partial D^2)}_{\partial B_1} \to \underbrace{(S^1 \times \partial D^2)}_{\partial B_2}$ is isotopic to the linear map $\begin{pmatrix} -1 & 0 \\ \omega & 1 \end{pmatrix}$.

Consider $\partial M_i^v \cong S^2 \times_\omega D^2$ as a D^2-bundle over S^2. Write $S^2 = \widehat{D}_1^2 \cup_\partial \widehat{D}_2^2$ where $\widehat{D}_k^2$, $k \in \{1, 2\}$, is a 2-disk and consider $B_k \cong \widehat{D}_k^2 \times \partial D^2$.

Then, we can consider $\partial M_i^v \cong \widehat{D}_1^2 \times D_1^2 \cup_\psi \widehat{D}_2^2 \times D_2^2$ where the attaching map $\psi : \partial\widehat{D}_1^2 \times D_1^2 \to \partial\widehat{D}_2^2 \times D_2^2$ is a linear extension of φ. Using the polar coordinates $(s, \theta) \times (r, \phi)$, $r, s \in [0, 1]$, $\theta, \phi \in [0, 2\pi)$ on $\widehat{D}_k^2 \times D_k^2$, we have that $\psi((1, \theta) \times (r, \phi)) = (1, -\theta) \times (r, \omega\theta + \phi)$.

Let $(M^e)'_{j_k} \cong M^e_{j_k} \cup_\partial (D^2 \times S^1)$ along the identity attaching map $\partial F_{j_k} \cong S^1 \times \partial D^2 \to \partial D^2 \times S^1$. By the same argument as in the previous case,

$$\overset{M^e_{j_1}}{\overline{}}\; M^v_i \;\overset{M^e_{j_2}}{\overline{}} \quad \cong \underbrace{(\Sigma^2_1 \times D^2)}_{(M^e)'_{j_1}} \square \underbrace{(S^2 \times_\omega D^2)}_{M^v_i} \square \underbrace{(\Sigma^2_2 \times D^2)}_{(M^e)'_{j_2}}. \tag{14.23}$$

Similarly, if there exists a component $M^v_{i_2} = \begin{pmatrix} S^2 \times_\omega D^2 \longrightarrow & M^v_{i_2} \\ & \downarrow \\ & \text{pt} \end{pmatrix}$, $i_2 \in \mathcal{A}_0$, such that $M^v_{i_2} \cap M^e_{j_2} \neq \varnothing$, then

$$\overset{M^e_{j_1}}{\overline{}}\; M^v_i \;\overset{M^e_{j_2}}{\overline{}}\; M^v_{i_2} \;\cong \underbrace{(\Sigma^2_1 \times D^2)}_{(M^e)'_{j_1}} \square \underbrace{(S^2 \times_\omega D^2)}_{M^v_i} \square \underbrace{(S^2 \times D^2)}_{(M^e)''_{j_2}} \square \underbrace{(S^2 \times_\omega D^2)}_{M^v_{i_2}}. \tag{14.24}$$

If the graph G has one vertex, then W is a 4-manifold with plumbing diagram

$$W \cong \begin{tikzcd} S^2 \times D^2 \arrow[r, bend left, dash, "\square"] \arrow[r, bend right, dash, "\square"'] & S^2 \times_\omega D^2 \end{tikzcd}. \tag{14.25}$$

In particular, ∂W is a T^2-bundle over S^1. From Lemma 2.30, W admits a T-structure which restricts to local T^2-actions on ∂W. The local T^2-actions on ∂W are free and their orbits coincide with T^2-fibers of ∂W.

(iv) $M^v_i \cong D^4$.

In this case, $\partial M^v_i \cong S^3$ and φ sends a meridian to a longitude. Consider $M^v_i \cong D^4 \cong D^2_1 \times D^2_2$ where D^2_1 and D^2_2 are 2-disks. Then, $\partial M^v_i \cong (D^2_1 \times \partial D^2_2) \cup (\partial D^2_1 \times D^2_2)$. Let $\phi_1 : F_{j_1} \cong S^1 \times D^2 \to D^2_1 \times \partial D^2_2$ and $\phi_2 : F_{j_2} \cong S^1 \times D^2 \to \partial D^2_1 \times D^2_2$ be the attaching maps from $M^e_{j_1}$ and $M^e_{j_2}$ to M^v_i respectively. We have that $(\phi_2|_{\partial F_{j_2}})_2^{-1} \circ (\phi_1|_{\partial F_{j_1}})$ must be isotopic to $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $(M^e)'_{j_k} \cong M^e_{j_k} \cup_\partial (D^2 \times S^1)$ along attaching map $\phi_1|_{\partial F_{j_1}} : \partial F_{j_k} \cong S^1 \times \partial D^2 \to \partial D^2 \times S^1$. It follows that $(M^e)'_{j_k}$ is a D^2-bundle over $\Sigma^2_k \cong (S^1 \times I) \cup_\partial D^2 \cong D^2$ and $F_{j_k} \cup_\partial (D^2 \times S^1) \cong \widehat{D}^2_1 \times D^2$ for some 2-disk $\widehat{D}^2_k \subset \Sigma^2_k$. Hence,

$$\overset{M^e_{j_1}}{\overline{}}\; M^v_i \;\overset{M^e_{j_2}}{\overline{}} \quad \cong (M^e)'_{j_1} \cup_\psi (M^e)'_{j_2} \tag{14.26}$$

where $\psi : \widehat{D}^2_1 \times D^2 \to \widehat{D}^2_2 \times D^2$ is an extension of $(\phi_2^{-1} \circ \phi_1) : F_{j_1} \to F_{j_2}$. ψ sends $\widehat{D}^2_1 \times \{y\}$ to $\{x\} \times D^2$ and sends $\{x\} \times D^2$ to $\widehat{D}^2_2 \times \{y\}$. In other words,

$$(M^e)'_{j_1} \cup_\psi (M^e)'_{j_2} \cong (M^e)'_{j_1} \square (M^e)'_{j_2}. \tag{14.27}$$

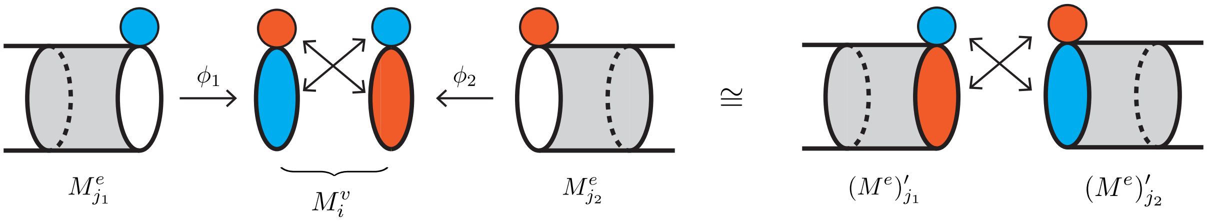


Figure 14.6: $M^e_{j_1} \cup_\partial M^v_i \cup_\partial M^e_{j_2}$ in case (a) when $M^v_i \cong D^4$ in terms of plumbing

Suppose that there exists a component $M^v_{i_2} = \left(\begin{array}{cc} S^2 \times_\omega D^2 \longrightarrow & M^v_{i_2} \\ & \downarrow \\ & \text{pt} \end{array}\right)$, $\omega \in \mathbb{Z}$, $i_2 \in \mathcal{A}_0$, such that $M^v_{i_2} \cap M^e_{j_2} \neq \varnothing$. In other words, there exists a vertex v' (representing $M^v_{i_2}$) adjacent to v via the edge e_2. Let $(M^e)''_{j_2} \cong (M^e)'_{j_2} \cup_\partial (D^2 \times S^1)$ along the identity attaching map $S^1 \times \partial\Sigma^2_2 \to \partial D^2 \times S^1$. It follows that $(M^e)''_{j_2}$ is a D^2-bundle over S^2. In contrast to the case $M^v_i \cong S^2 \times_\omega D^2$, $(M^e)''_{j_2}$ is not necessary a trivial bundle. Therefore,

$$\overset{M^e_{j_1}}{\text{———}} \; M^v_i \; \overset{M^e_{j_2}}{\text{———}} \; M^v_{i_2} \;\cong \underbrace{(\Sigma^2_1 \times D^2)}_{(M^e)'_{j_1}} \Box \underbrace{(S^2 \times_{\omega'} D^2)}_{(M^e)''_{j_2}} \Box \underbrace{(S^2 \times_\omega D^2)}_{M^v_{i_2}}. \qquad (14.28)$$

If the graph G has one vertex, then W can be obtained from $(M^e)''_{j_1}$ by identifying $\widehat{D}^2_1 \times D^2_1 \subset (M^e)''_{j_1}$ and $\widehat{D}^2_2 \times D^2_2 \subset (M^e)''_{j_1}$ with ψ. Consider that $(M^e)''_{j_1} \cong \widehat{D}^2_1 \times D^2_1 \cup \widehat{D}^2_2 \times D^2_2$ where ψ^{-1} is the attaching map. Thus, $(M^e)''_{j_1}$ is a trivial D^2-bundle over S^2. Therefore, W is diffeomorphic to the resulting manifold of a self-plumbing of $S^2 \times D^2$. In particular, ∂W is a T^2-bundle over S^1. From Lemma 2.30, W admits a T-structure which restricts to local T^2-actions on ∂W. The local T^2-actions on ∂W are free and their orbits coincide with T^2-fibers of ∂W.

(v) $M^v_i \cong (S^2 \widetilde{\times} S^1) \widetilde{\times} I$ or $(\mathbb{R}P^2 \times S^1) \widetilde{\times} I$.

In this case, $\partial M^v_i \cong S^1 \times S^2$. Let U_i be an open neighborhood of M^v_i in $M^e_{j_1} \cup_\partial M^v_i \cup_\partial M^e_{j_2}$ such that $V_{j_k} = U_i \cap M^e_{j_k}$ is a subbundle of $M^e_{j_k}$ and $V_{j_k} \cap M^e_{j_k} = F_{j_k}$, $k \in \{1, 2\}$.

Let $\widetilde{U}_i \xrightarrow{\pi} U_i$ be a double covering constructed as follows. $\widetilde{U}_i = \widetilde{V}_{j_1} \cup_\partial \widetilde{M}_i \cup_\partial \widetilde{V}_{j_2}$ where we put $\widetilde{V}_{j_k} = \pi^{-1}(V_{j_k})$, $k \in \{1, 2\}$, and $\widetilde{M}_i \cong (S^2 \times S^1) \times [0, 1]$ is a double cover of M^v_i. We denote an $(S^2 \times S^1)$-fiber of $\widetilde{M}_i$ by $(S^2 \times S^1) \times \{t\}$, $t \in [0, 1]$. Write $\widetilde{V}_{j_k} = V_{j_k}(0) \sqcup V_{j_k}(1)$ where $\pi(V_{j_k}(0)) = \pi(V_{j_k}(1)) = V_{j_k}$. For $k \in \{1, 2\}$, put $F_{j_k}(0) = V_{j_k}(0) \cap (S^2 \times S^1) \times \{0\}$ and $F_{j_k}(1) = V_{j_k}(1) \cap (S^2 \times S^1) \times \{1\}$. Then, $\pi(F_{j_k}(0)) = \pi(F_{j_k}(1)) = F_{j_k}$.

For $k \in \{1, 2\}$, let $\mathcal{F}_{j_k}$ be a subset of $\widetilde{M}_i$ so that $\mathcal{F}_{j_k} \cong (S^1 \times D^2) \times [0, 1]$ and $(S^1 \times D^2) \times \{t\} \subset (S^2 \times S^1) \times \{t\}$ for all $t \in [0, 1]$. We also require that $(S^1 \times$

$D^2) \times \{0\}$ coincides with $F_{j_k}(0)$ and $(S^1 \times D^2) \times \{1\}$ coincides with $F_{j_k}(1)$. It follows that $V_{j_k}(0) \cup \mathcal{F}_{j_k} \cup V_{j_k}(1) \cong (S^1 \times D^2) \times I$. Moreover, $\widetilde{M_i} - (\mathcal{F}_{j_1} \cup \mathcal{F}_{j_2}) \cong ((S^1 \times I) \times S^1) \times [0,1] \cong (T^2 \times I) \times [0,1]$. The two boundary components of $(T^2 \times \partial I) \times \{t\}$ coincide with the boundary of an $(S^1 \times D^2)$-fiber of $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$ respectively.

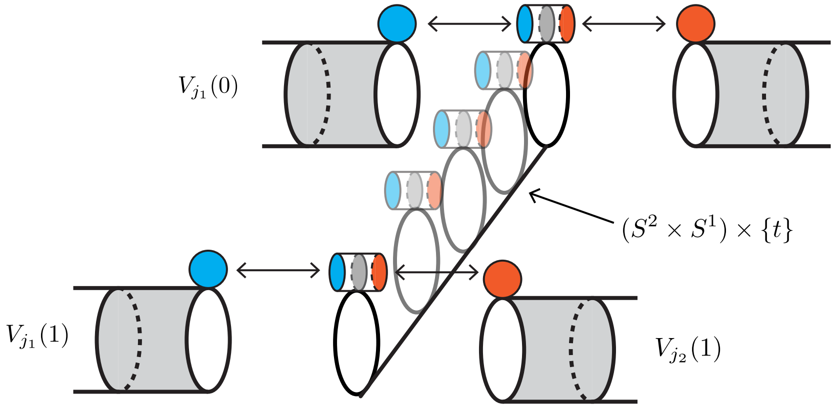


Figure 14.7: $\widetilde{U}_i = \widetilde{V}_{j_1} \cup_\partial \widetilde{M_i} \cup_\partial \widetilde{V}_{j_2}$ when $M_i^v \cong (S^2 \widetilde{\times} S^1) \widetilde{\times} I$

Let T^2 act on $\widetilde{M_i} - (\mathcal{F}_{j_1} \cup \mathcal{F}_{j_2}) \cong (T^2 \times I) \times [0,1]$ by the standard action on the T^2-factor and act trivially on the other factors. Let ϕ be a T^2-action on $V_{j_k}(0) \cup \mathcal{F}_{j_k} \cup V_{j_k}(1) \cong (S^1 \times D^2) \times I$ by any action that restricts to a free T^2-action on $\partial(S^1 \times D^2) \times I$ and such that $\pi \circ \phi = \pi$. As a result, we get local T^2-actions on $\widetilde{U}_i$ that are compatible with π.

To show that W admits an F-structure which restricts to local T^2-actions on ∂W, it suffices to construct local T^2-actions on 4-manifolds represented by the graphs

$$G_1 = \quad \overset{e_1}{\text{———}} \; v_1 \; \overset{e_2}{\text{———}} \; \cdots \; \overset{e_m}{\text{———}} \; v_m \; \overset{e_{m+1}}{\text{———}} \tag{14.29}$$

and

$$G_2 = \begin{tikzcd} v_1 \arrow[r, dash, "e_1"] \arrow[rrr, dash, bend right, "e_m"'] & v_2 \arrow[r, dash, "e_2"] & \cdots \arrow[r, dash] & v_m \end{tikzcd} \tag{14.30}$$

where all vertices are from cases (i) to (iv). In other words, G_1 and G_2 represents the resulting manifold of a linear or a cyclic plumbing of D^2-bundles over S^2. Fintushel [7] constructed local T^2-actions on such manifolds.

Therefore, there exists local T^2-actions on W and from the constructions in [7], their restrictions to the boundary are free and their orbits coincide with T^2-fibers of ∂W.

(b) Case $M^e_{j_1} = \begin{pmatrix} S^1 \times D^2 \longrightarrow M^e_{j_1} \\ \hfill \downarrow \\ \hfill (I, \partial I) \end{pmatrix} = (S^1 \times D^2) \times I$ and $M^e_{j_2} = \begin{pmatrix} T^2 \times_{\mathbb{Z}_2} I \longrightarrow M^e_{j_2} \\ \hfill \downarrow \\ \hfill (I, \partial I) \end{pmatrix} =$ $(T^2 \times_{\mathbb{Z}_2} I) \times I$.

By Lemma 13.17, $M^v_i \cong S^1 \times D^3, S^2 \times D^2$, $(\mathbb{R}P^2 \times S^1)\widetilde{\times} I$, $(S^2 \widetilde{\times} S^1)\widetilde{\times} I$, $S^2 \times_{\mathbb{Z}_2} D^2$, or $(S^2 \times_\omega D^2)/\mathbb{Z}_2$.

Let v' be the vertex joined with v by e_2 and let e_3 be an edge incident to v'. v' represents a component $M^v_{i'} = \begin{pmatrix} D^4, \dots \longrightarrow M^v_{i'} \\ \hfill\downarrow \\ \hfill\text{pt} \end{pmatrix}$, for some $i' \in \mathcal{A}_0$. In this case, we assume that e_3 represents $M^e_{j_3} \cong (S^1 \times D^2) \times I$. The case that e_3 represents $M^e_{j_3} \cong (T^2 \times_{\mathbb{Z}_2} I) \times I$ will be considered later in case (c). We have the following subgraph of G

$$\cdots \;\overset{e_1}{\text{———}}\; v \;\overset{e_2}{\text{------}}\; v' \;\overset{e_3}{\text{———}}\; \cdots \tag{14.31}$$

where dashed edges represent $\begin{pmatrix} T^2 \times_{\mathbb{Z}_2} I \longrightarrow M^e_j \\ \hfill\downarrow \\ \hfill(I, \partial I) \end{pmatrix}$ components and solid edges represent $\begin{pmatrix} S^1 \times D^2 \longrightarrow M^e_j \\ \hfill\downarrow \\ \hfill(I, \partial I) \end{pmatrix}$ components. We have the following cases.

(i) $M^v_i, M^v_{i'} \cong S^1 \times D^3$ or $S^2 \times D^2$

Let $\widetilde{M}^e_{j_2} \xrightarrow{\pi} M^e_{j_2}$ be a double covering. $\widetilde{M}^e_{j_2} \cong (T^2 \times I) \times [0,1]$. $\pi((T^2 \times I) \times \{0\})$ coincides with T^2-fibers of $A_i \subset \partial M^v_i$ and $\pi((T^2 \times I) \times \{1\})$ coincides with T^2-fibers of $A_{i'} \subset \partial M^v_{i'}$.

Let T^2 act on $\widetilde{M}^e_{j_2}$ by the standard action on the T^2-factor and act trivially on the I-factors. By the same constructions as in case (a), there are local T^2-actions on M^v_i and $M^v_{i'}$ so that their restrictions on ∂M^v_i and $\partial M^v_{i'}$ are free. The T^2-action on $\widetilde{M}^e_{j_2}$ is compatible with π and with local T^2-actions on M^v_i and $M^v_{i'}$.

(ii) $M^v_i \cong (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I, S^2 \times_{\mathbb{Z}_2} D^2$, or $(S^2 \times_\omega D^2)/\mathbb{Z}_2$ and $M^v_{i'} \cong S^1 \times D^3$ or $S^2 \times D^2$.

Let U be an open neighborhood of $M^v_i \cup M^e_{j_2}$ in W such that $U \cap M^e_{j_1}$ is a subbundle of $M^e_{j_1}$. Put $V_{j_1} = U \cap M^e_{j_1}$ and $F_{j_1} = V_{j_1} \cap M^v_i$. We have that $F_{j_1} \cong S^1 \times D^2$.

Let $\widetilde{U} \xrightarrow{\pi} U$ be a double covering. $\widetilde{U} = \widetilde{V}_{j_1} \cup \widetilde{M}^v_i \cup \widetilde{M}^e_{j_2}$ where $\widetilde{V}_{j_1}$ is a double cover of V_{j_1}, $\widetilde{M}^e_{j_2}$ is a double cover of $M^e_{j_2}$ and $\widetilde{M}^v_i$ is a double cover of M^v_i. We have that $\widetilde{V}_{j_1}$ is the union of two copies of $S^1 \times D^2$'s, which we denote by $V_{j_1}(0)$ and $V_{j_1}(1)$. Then, $\pi(V_{j_1}(0)) = \pi(V_{j_1}(0)) = V_{j_1}$. Next, we have that $\widetilde{M}^e_{j_2} \cong (T^2 \times [0,1]) \times [0,1]$. We denote a T^2-fiber of $\widetilde{M}^e_{j_2}$ by $T^2 \times (s,t)$ where $(s,t) \in [0,1] \times [0,1]$. Lastly, $\widetilde{M}^v_i \cong (S^2 \times S^1) \times I$ or $S^2 \times_\omega D^2$, $\omega \in \mathbb{Z}$.

From Lemma 13.17, $\partial M^v_i = B_i \cup_\partial A_i \cup_\partial C_i$ where $B_i \cong S^1 \times D^2$, $C_i \cong T^2 \times_{\mathbb{Z}_2} I$, and $A_i \cong T^2 \times I$. Then, $\partial \widetilde{M}^v_i = B_{i,1} \cup_\partial A_{i_1} \cup_\partial \widetilde{C}_i \cup_\partial A_{i_2} \cup_\partial B_{i,2}$ where $B_{i,1} \cong B_{i,2} \cong$

$S^1 \times D^2$, $A_{i_1} \cong A_{i_2} \cong T^2 \times I$, and $\widetilde{C}_i \cong T^2 \times I$. We have that $B_{i_1} = V_{j_1}(0) \cap \partial\widetilde{M}^v_i$ and $B_{i_2} = V_{j_1}(1) \cap \partial\widetilde{M}^v_i$. Additionally, each $T^2 \times (s, 0)$ coincides with a T^2-fiber of $\widetilde{C}_i$. Similar arguments apply for each $T^2 \times (s, 1)$, $s \in [0, 1]$.

Let T^2 act on $\widetilde{M}^e_{j_2}$ by the standard action on the T^2-factor and act trivially on the $[0, 1]$-factors. By the same constructions as in case (a), there is a T^2-action on $\widetilde{V}_{j_1} \cup \widetilde{M}^v_i$ whose orbits coincide with the orbits of the T^2-action on $\widetilde{M}^e_{j_2}$ along the overlaps.

By the same arguments as in case (a), there are local T^2-actions on M^v_i and $M^v_{i'}$ so that their restrictions on $\partial M^v_{i'}$ are free and their orbits coincide with T^2-fibers of A_i and $A_{i'}$. In particular, the T^2-action on $\widetilde{M}^e_{j_2}$ is compatible with π and with local T^2-actions on M^v_i and $M^v_{i'}$. Therefore, there are local T^2-actions on $\widetilde{U}$ that are compatible with π and local T^2-actions on $W - U$.

(iii) $M^v_i, M^v_{i'} \cong (\mathbb{R}P^2 \times S^1)\widetilde{\times} I, (S^2 \widetilde{\times} S^1)\widetilde{\times} I, S^2 \times_{\mathbb{Z}_2} D^2$, or $(S^2 \times_\omega D^2)/\mathbb{Z}_2$.

Let U be an open neighborhood of $M^v_i \cup_\partial M^e_{j_2} \cup_\partial M^v_{i'}$ in W such that $V_{j_1} = U \cap M^e_{j_1}$ is a subbundle of $M^e_{j_1}$ and $V_{j_3} = U \cap M^e_{j_3}$ is a subbundle of $M^e_{j_3}$.

Let $\widetilde{U} \xrightarrow{\pi} U$ be a double covering constructed as follows. $\widetilde{U} = \widetilde{V}_{j_0} \cup_\partial \widetilde{M}^v_i \cup_\partial \widetilde{M}^e_{j_2} \cup_\partial \widetilde{M}^v_{i'} \cup_\partial \widetilde{V}_{j_3}$ where $\widetilde{V}_{j_k}$ is a double cover of V_{j_k}, $k \in \{1, 3\}$, $\widetilde{M}^v_i$ is a double cover of M^v_i, $\widetilde{M}^v_{i'}$ is a double cover of $M^v_{i'}$, and $\widetilde{M}^e_{j_2}$ is a double cover of $M^e_{j_2}$.

By similar arguments as in case (ii), there are local T^2-actions on $\widetilde{U}$ that are compatible with π and local T^2-actions on $W - U$.

(c) Case $M^e_{j_k} = \left(\begin{array}{r} T^2 \times I \longrightarrow M^e_{j_k} \\ \downarrow \\ (I, \partial I) \end{array}\right) \cong (T^2 \times_{\mathbb{Z}_2} I) \times I$ for $k \in \{1, 2\}$.

By Lemma 13.17, $M^v_i \cong T^2 \times_{\mathbb{Z}_2} D^2, \mathcal{B}_3 \widetilde{\times} I$, or $\mathcal{B}_4 \widetilde{\times} I$. For $k \in \{1, 2\}$, let v_k be the vertex adjacent to v via the edge e_k. Let e_0 be the edge incident to v_1 and e_3 be the edge incident to v_2. First we assume that $M^e_{j_0} \cong M^e_{j_3} \cong (S^1 \times D^2) \times I$. That is $M^v_{i_1}$ and $M^v_{i_2}$ are from case (ii). We have the following subgraph of G

$$\cdots \overset{e_0}{\text{———}} \; v_1 \; \overset{e_1}{\text{------}} \; v \; \overset{e_2}{\text{------}} \; v_2 \; \overset{e_3}{\text{———}} \cdots \tag{14.32}$$

where dashed edges represent $\left(\begin{array}{r} T^2 \times_{\mathbb{Z}_2} I \longrightarrow M^e_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components and solid edges represent $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M^e_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components.

Let $\widetilde{M_i^v}$ be a double cover of M_i^v. From Lemma 13.17, $\partial M_i^v = C_1 \cup_\partial A_1 \cup_\partial C_2$ where $C_1 \cong C_2 \cong T^2 \times_{\mathbb{Z}_2} I$, and $A \cong T^2 \times I$. Then,

$$\partial \widetilde{M_i^v} = \underbrace{\widetilde{C}_1 \quad \cup_\partial \quad A_{i_1} \quad \cup_\partial \quad \widetilde{C}_2 \quad \cup_\partial \quad A_{i_2}}_{\cup_\partial} \tag{14.33}$$

where $A_{i_1} \cong A_{i_2} \cong T^2 \times I$, and $\widetilde{C}_1 \cong \widetilde{C}_2 \cong T^2 \times I$ are double covers of C_1 and C_2 respectively. Thus, $\partial \widetilde{M_i^v}$ is a T^2-bundle over S^1.

Let $Z = M_{j_1}^e \cup_\partial M_i^v \cup_\partial M_{j_2}^e$. Let $\widetilde{Z} \xrightarrow{\pi} Z$ be a double covering. We have $\widetilde{Z} = \widetilde{M_{j_1}^e} \cup_\partial \widetilde{M_i^v} \cup_\partial \widetilde{M_{j_2}^e}$ where $\widetilde{M_{j_k}^e} \cong (T^2 \times [0,1]) \times [0,1]$ is a double cover of $M_{j_k}^e$, $k \in \{1,2\}$. We denote each T^2-fiber of $\widetilde{M_{j_k}^e}$ by $T^2 \times (s,t)$ for $(s,t) \in [0,1] \times [0,1]$. Then, $\widetilde{M_{j_1}^e} \cap \partial \widetilde{M_i^v} = T^2 \times [0,1] \times \{1\}$ coincides with $\widetilde{C}_1$ fiberwise. Similarly, $\widetilde{M_{j_2}^e} \cap \partial \widetilde{M_i^v} = T^2 \times [0,1] \times \{0\}$ coincides with $\widetilde{C}_2$ fiberwise.

By similar arguments as in case (ii), for each $k \in \{1,2\}$, there exists an F-structure on $M_{i_k}^v \cup M_{j_k}^e$ which restricts to the standard T^2-action on the T^2-factors and to the trivial action on the I-factors of $\widetilde{M_{j_k}^e} \cong (T^2 \times I) \times I$.

$\widetilde{M_i^v} \cong T^2 \times D^2$ or $\mathcal{G}_2 \times I$. In both cases, there exists a T^2-action on $\widetilde{M_i^v}$ whose orbits coincides with T^2-fibers of $\widetilde{C}_k \subset \partial \widetilde{M_i^v}$, $k \in \{1,2\}$. Therefore, there exists an F-structure on $M_{i_1}^v \cup M_{j_1}^e \cup M_i^v \cup M_{j_2}^e \cup M_{i_2}^v$.

More generally, we have the following subgraph of G,

$$\cdots \overset{e_0}{\rule[0.5ex]{3em}{0.4pt}} v_1 \overset{e_1}{\text{------}} v_2 \text{ ------ } \cdots \text{ ------ } v_{m-1} \overset{e_{m-1}}{\text{------}} v_m \overset{e_m}{\rule[0.5ex]{3em}{0.4pt}} \cdots \tag{14.34}$$

By repeating the above argument on $M_{j_2}^v, M_{j_3}^v, \ldots, M_{j_{m-1}}^v$, we get an F-structure on $Z = M_{i_1}^v \cup M_{j_1}^e \cup \ldots \cup M_{j_{m-1}}^e \cup M_{i_m}^v$.

In the case that G has only one vertex, $W = M_i^v \cup M_{j_1}^e$ where $M_i^v \cap M_{j_1}^e$ is the union of two copies of $T^2 \times_{\mathbb{Z}_2} I$. Let $\widetilde{W}$ be a double cover of W. $\widetilde{W} = \widetilde{M_i^v} \cup \widetilde{M_{j_1}^e}$ where $\widetilde{M_{j_1}^e} \cap \widetilde{M_i^v} = \widetilde{C}_1 \cup \widetilde{C}_2$. Let T^2 act by the standard T^2-action on the T^2-factor and act trivially on the $(I \times I)$-factor of $\widetilde{M_{j_1}^e}$. There exists a T^2-action on $\widetilde{M_i^v}$ whose orbits coincides with T^2-fibers of $\widetilde{C}_k \subset \partial \widetilde{M_i^v}$, $k \in \{1,2\}$. Therefore, there exists an F-structure on W which restricts to local free T^2-actions on ∂W.

It follows from case (a) to case (c) that there is a collection of an open sets $\{U_i\}$ which covers W so that there exists a T^2-action on U_i or its double cover $\widetilde{U}_i$ and the actions are compatible with the covering maps and with each other along the intersections. Additionally, their restrictions on ∂W are free. Therefore, W admits an F-structure which restricts to local free T^2-actions on ∂W. This completes the proof of Lemma 14.14 (2).

14.4.2 An F-structure on an elementary building block of type $(2, T^2)$

In the following lemma, we show that an elementary building block of type $(2, T^2)$ admits an F-structure by extending the F-structure on the manifold W constructed in Lemma 14.14 to the component $\left(\begin{array}{r} T^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ it attaches to.

Lemma 14.35. *Let M_k be a component $\left(\begin{array}{r} T^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ such that every boundary component of M_k attaches to a manifold W_j, for some j, where W_j is a manifold W in Lemma 14.14. Let Y be the union $M_k \cup \left(\bigsqcup_{j=1}^{n_k} W_j\right)$. Then, Y admits an F-structure.*

Proof. Since M_k is the total space of T^2-fibers over a surface, there are local T^2-actions on M_k which are free and whose orbits coincide with T^2-fibers.

From Lemma 14.14, each W_j admits an F-structure which restricts to local T^2-actions on ∂W_j. The local T^2-actions on ∂W_j are free and their orbits coincide with T^2-fibers of ∂W_j. Also, ∂W_j is identified with a boundary component of M_k so that T^2-fibers coincide. In particular, the F-structure on W_j is compatible with local T^2-actions on M_k. Therefore, $Y = M_k \cup \left(\bigsqcup_j W_j\right)$ admits an F-structure. □

14.5 Elementary building block of type $(2, D^2)$

In this section, we consider a connected component of $M - \bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ that only contains $\left(\begin{array}{r} D^4, \dots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, and $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components.

Definition 14.36. We call a component Y in Lemma 14.38 an *elementary building block of type* $(2, D^2)$.

We represent an elementary building block of type $(2, D^2)$ by a polyhedron.

Example 14.37. The following is a model example of an elementary building block of type $(2, D^2)$.

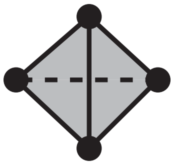


Figure 14.8: A representation of an elementary building block of type $(2, D^2)$

In this example, there are four $\left(\begin{array}{r} D^4 \longrightarrow M_{i_\ell} \\ \downarrow \\ \text{pt} \end{array}\right)$ components, $\ell \in \{1, \ldots, 4\}$, which are represented by vertices, and six $\left(\begin{array}{r} D^3 \longrightarrow M_{j_\ell} \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components, $\ell \in \{1, \ldots, 6\}$, which are represented by edges. Denote the union of all M_{i_ℓ} and M_{j_ℓ} by W. W is represented by the 1-skeleton of the tetrahedron. There are four $\left(\begin{array}{r} D^2 \longrightarrow M_{k_\ell} \\ \downarrow \\ (\Sigma^2_{k_\ell}, \partial\Sigma^2_{k_\ell}) \end{array}\right)$ components, $\ell \in \{1, \ldots, 4\}$, where $\Sigma^2_{k_\ell} \cong D^2$. Each face of the tetrahedron represents attaching M_{k_ℓ} to W along the D^2-bundle over $\partial\Sigma^2_{k_\ell}$. Let Y be the union of all components. Y is represented by the tetrahedron. We have that Y is a D^2-bundle over S^2. ∂Y is identified with a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some p, so that ∂D^2-fibers of ∂Y coincide with S^1-fibers of M_p.

From Lemma if 13.30, $M_0 = \left(\begin{array}{r} D^4, \ldots \longrightarrow M_0 \\ \downarrow \\ \text{pt} \end{array}\right)$ only intersects $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, and $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ components, then $M_0 \cong D^4$, $\pm\mathbb{C}P^2 \# D^4$, $S^2 \times_{\pm 2} D^2$, or $S^2 \times_{\mathbb{Z}_2} D^2$.

Lemma 14.38. *Let* $\{M_i\}_{i\in\mathcal{A}_0}$ *be a collection of* $\left(\begin{array}{ll} D^4, \pm\mathbb{C}P^2\# D^4, & \\ S^2\times_{\pm 2} D^2, S^2\times_{\mathbb{Z}_2} D^2 & \longrightarrow M_i \\ & \qquad\Big\downarrow \\ & \qquad\mathrm{pt} \end{array}\right)$ *components such that* M_i *only intersects* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components (as in Lemma 13.30).*

Let $\{M_j\}_{j\in\mathcal{A}_1}$ *be a collection of* $\left(\begin{array}{r} D^3, S^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ *components such that both fibers of* $\left(\begin{array}{r} D^3, S^2\times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$ *are contained in* $\bigsqcup_{i\in\mathcal{A}_0}\partial M_i$.

Let $\{M_k\}_{k\in\mathcal{A}_2}$ *be a collection of* $\left(\begin{array}{r} D^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *components such that* $\left(\begin{array}{r} D^2 \longrightarrow N_{k_1} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ *are contained in* $\left(\bigsqcup_{i\in\mathcal{A}_0}\partial M_i\right)\cup_{\partial}\left(\bigsqcup_{i\in\mathcal{A}_1}\partial M_j\right)$. *Note that* $\partial M_k = \left(\begin{array}{r} D^2 \longrightarrow N_{k_1} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right) \cup \left(\begin{array}{r} S^1 \longrightarrow N_{k_2} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$.

Let Y *be a connected component of* $M - \bigsqcup_p \left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *such that* Y *only intersects components* M_i, $i\in\mathcal{A}_0$, M_j, $j\in\mathcal{A}_1$, *and* M_k, $k\in\mathcal{A}_2$, *i.e.* $Y\subset\left(\bigsqcup_{i\in\mathcal{A}_0}M_i\right)\cup\left(\bigsqcup_{j\in\mathcal{A}_1}M_j\right)\cup\left(\bigsqcup_{k\in\mathcal{A}_k}M_k\right)$.

Put $W = Y - \left(\bigsqcup_{k\in\mathcal{A}_2}M_k\right)$ *and write* $W = \bigsqcup_\ell W_\ell$ *where* W_ℓ *is a connected component of* W.

Then, the following holds.

(1) *W_ℓ can be represented by the 1-skeleton of a polyhedron so that each vertex represents a component M_i, for some $i \in \mathcal{A}_0$, and each edge represents a component M_j, for some $j \in \mathcal{A}_1$.*

(2) *Y can be represented by the disjoint union of polyhedrons so that each connected component of the 1-skeletons represents W_ℓ, for some ℓ, and each face represents attaching a connected component of $\left(\begin{array}{r} D^2 \longrightarrow N_{k_1} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some $k \in \mathcal{A}_2$, to W.*

(3) *Y admits an F-structure whose restriction to ∂M_0 has positive rank.*

(4) *∂Y is the total space of S^1-fibers over a closed surface. As a part of M, ∂Y is identified with a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some p, so that S^1-fibers coincide.*

Proof. Here we prove parts (1), (2), and (4). The proof of part (3) is given in the next subsection.

(1), (2). For simplicity, we assume that Y is the only connected component of $M - \bigsqcup_p \left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ that only intersects components M_i, $i \in \mathcal{A}_0$, M_j, $j \in \mathcal{A}_1$, and M_k, $k \in \mathcal{A}_2$.

From Lemma 13.30, for each $i \in \mathcal{A}_0$, $\partial M_i = A_i \cup_\partial B_{i,1}$ or $A \cup_\partial (B_{i,1} \sqcup B_{i,2})$ where A_i is a subbundle of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some j, and $B_{i,k}$, $k \in \{1, 2\}$, is a component $\left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some j, or a 3-manifold which is represented by a cycle graph G_i so that each vertex represents a fiber of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$, for some $j \in \mathcal{A}_1$, and each edge represents a D^2-subbundle $E^i \cong D^2 \times I$ of $\left(\begin{array}{r} D^2 \longrightarrow N_k \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some $k \in \mathcal{A}_2$.

By Lemma 13.4, each fiber of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ contains exactly two D^2-fibers of $\left(\begin{array}{r} D^2 \longrightarrow M_{k_1} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ and $\left(\begin{array}{r} D^2 \longrightarrow M_{k_2} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, for some k_1 and k_2. By connectedness, each M_j, $j \in \mathcal{A}_1$, contains two copies of $D^2 \times I$ so that each D^2-fiber is contained in a D^3 or $S^2 \times_{\mathbb{Z}_2} I$-fiber of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$. The two copies of $D^2 \times I$ coincide with $M_j \cap \left(\begin{array}{r} D^2 \longrightarrow M_{k_1} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ and $M_j \cap \left(\begin{array}{r} D^2 \longrightarrow M_{k_2} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, for some $k_1, k_2 \in \mathcal{A}_2$. Put $\widehat{E}_{j_1} = M_j \cap \left(\begin{array}{r} D^2 \longrightarrow M_{k_1} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ and $\widehat{E}_{j_2} = M_j \cap \left(\begin{array}{r} D^2 \longrightarrow M_{k_1} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$.

Let $\{M_i\}_{i \in \mathcal{A}_0^\ell}$ be the collection of all M_i, $i \in \mathcal{A}_0$, such that $M_i \subset W_\ell$ and $\{M_j\}_{j \in \mathcal{A}_1^\ell}$ be the collection of all M_j, $j \in \mathcal{A}_1$, such that $M_j \subset W_\ell$. Let $X \subset W_\ell$ be the union of all D^2-subbundle $E^i \cong D^2 \times I$ contained in ∂M_i, $i \in \mathcal{A}_0^\ell$, and all $\widehat{E}_{j_1} \cong D^2 \times I$ and $\widehat{E}_{j_2} \cong D^2 \times I$ contained in ∂M_j, $j \in \mathcal{A}_1^\ell$. Then, X is a disjoint union of copies of $D^2 \times S^1$. As a part of M, each connected component of X is identified with a connected component of $\left(\begin{array}{r} D^2 \longrightarrow N_k \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some $k \in \mathcal{A}_2$, so that D^2-fibers of X coincide with D^2-fibers of N_k.

We represent each M_i, $i \in \mathcal{A}_0^\ell$, by a vertex and each M_j, $j \in \mathcal{A}_1^\ell$, by an edge. A vertex v_i connects to an edge e_j if and only if $M_i \cap M_j \neq \varnothing$. From the above construction, the union of all vertices and edges is the 1-skeleton of a polyhedron so that each face corresponds to identifying a connected component of X with a boundary component of $\left(\begin{array}{r} D^2 \longrightarrow N_k \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some $k \in \mathcal{A}_2$.

(4). It follows from the compatibility of fibers in Lemma 13.6, Lemma 13.24, and Lemma 13.30, and from the construction in the proof of part (1) and part (2) that ∂Y is the total space of S^1-fibers. By connectedness, ∂Y is identified with exactly one boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some ℓ, so that their S^1-fibers coincide. □

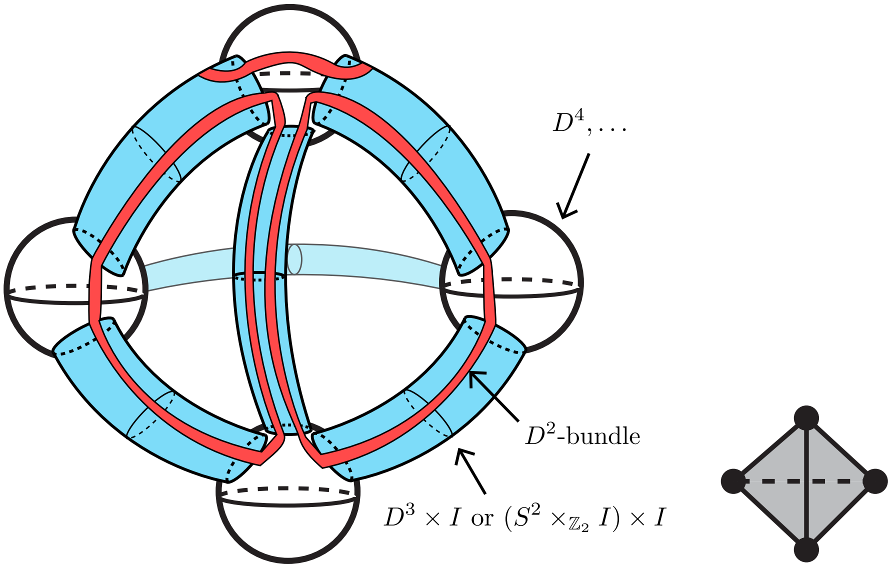


Figure 14.9: Example of the configuration of an elementary building block of type $(2, D^2)$ and its representation. Each D^2-bundle over a circle (a connected component of X) is identified with $\left(\begin{array}{rr} D^2 \longrightarrow & N_k \\ & \downarrow \\ & \partial\Sigma^2 \end{array}\right)$, for some k. There are four D^2-bundles over a circle, which are represented by the four faces of the tetrahedron.

14.5.1 Proof of Lemma 14.38 (3)

In this subsection, we give a proof of Lemma 14.38 (3).

Put $Z = Y - \left(\bigsqcup_{i \in \mathcal{A}_0} M_i\right)$. First, we assume that for all $j \in \mathcal{A}_1$, $M_j = \left(\begin{array}{rr} D^3 \longrightarrow & M_j \\ & \downarrow \\ & (I, \partial I) \end{array}\right)$ $\cong D^3 \times I$. We will show that Z is the total space of D^2-fibers over a surface. Z is not necessary connected.

Case $M_j \cong D^3 \times I$, for all $j \in \mathcal{A}_1$

By the same arguments as in the proof of Lemma 14.38 (1) and (2), each M_j, $j \in \mathcal{A}_1$, contains

$$\widehat{E}_{j_1} = M_j \cap \left(\begin{array}{rr} D^2 \longrightarrow & M_{k_1} \\ & \downarrow \\ & (\Sigma^2, \partial\Sigma^2) \end{array}\right) \cong D^2 \times I \text{ and } \widehat{E}_{j_2} = M_j \cap \left(\begin{array}{rr} D^2 \longrightarrow & M_{k_2} \\ & \downarrow \\ & (\Sigma^2, \partial\Sigma^2) \end{array}\right) \cong D^2 \times I,$$

for some $k_1, k_2 \in \mathcal{A}_2$, so that each D^2-fiber is contained in the boundary of a D^3-fiber of M_j. Therefore, we can consider $M_j \cong D^3 \times I$ as $(D^2 \times [0,1]) \times I$ so that $\widehat{E}_{j_1}$ coincides with $(D^2 \times \{0\}) \times I$ and $\widehat{E}_{j_2}$ coincides with $(D^2 \times \{1\}) \times I$. In particular, M_j is the total space of D^2-fibers whose fibers coincide with fibers of M_{k_1} and M_{k_2} along the overlaps. It follows that $M_{k_1} \cup M_j \cup M_{k_2}$ is the total space of D^2-fibers.

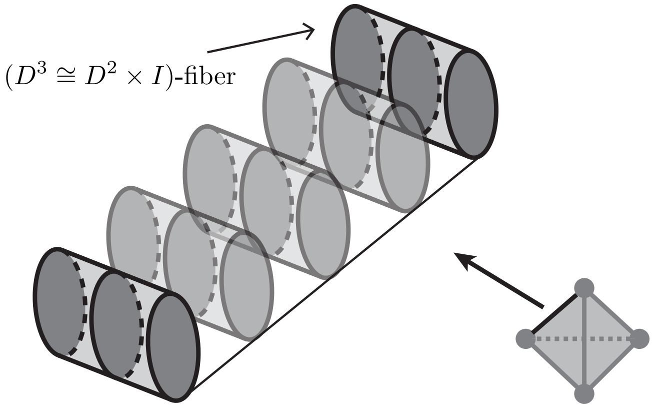


Figure 14.10: $M_j \cong D^3 \times I$ as $(D^2 \times [0,1]) \times I$

By applying the above argument to every M_j, $j \in \mathcal{A}_1$, such that $M_j \cap Z \neq \varnothing$, we have that Z is the total space of D^2-fibers over a surface $(\Sigma^2_Z, \partial\Sigma^2_Z)$, for some surface Σ^2_Z. Moreover, the total space of D^2-fibers over $\partial\Sigma^2_Z$ is contained in $\bigsqcup_{i \in \mathcal{A}_0} M_i$.

Without loss of generality, assume that Y has one connected component. In particular, $M_i \cap Z \neq \varnothing$, for all $i \in \mathcal{A}_0$. From Lemma 13.30, $\partial M_i \cong S^3$ or $S^2 \times S^1$. Next, we study $Z \cup_\partial M_i$. We have the following cases.

(i) $M_i \cong D^4$.

From Lemma 13.30, $\partial M_i = A \cup_\partial B_1$ where $A \cong S^1 \times D^2$ is a subbundle of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some j, and $B_1 \cong S^1 \times D^2$ is identified with a D^2-bundle over a boundary component of Σ^2_Z. The union is so that $(S^1, \cdot) \subset \partial A$ is identified with $(\cdot, \partial D^2) \subset \partial B_1$. By extending this attaching map to M_i, we can consider M_i as $\Sigma^2_i \times D^2$ where Σ^2_i is a 2-disk and B_1 is identified with $\partial\Sigma^2_i \times D^2$. In particular, D^2-fibers of M_i and Z coincide. Then, $Z \cup_\partial M_i$ is the total space of D^2-fibers over the surface $\Sigma^2_i \cup_\partial \Sigma^2_Z$ where the 2-disk Σ^2_i attaches to a boundary component of Σ^2_Z. Hence, $Z \cup_\partial M_i$ is the total space of D^2-fibers over a surface.

(ii) $M_i \cong \pm\mathbb{C}P^2 \# D^4$.

Since $M_i \cap Z = \partial M_i \cap \partial Z$, the same construction as in the case $M_i \cong D^4$ applies. We have that $Z \cup M_i \cong (Z \cup D^4)\#(\pm\mathbb{C}P^2)$.

(iii) $M_i \cong S^1 \times D^3$.

In this case, $\partial M_i \cong S^1 \times S^2$. From Lemma 13.30, $\partial M_i = B_1 \cup_\partial A \cup_\partial B_2$ where $A \cong S^1 \times (S^1 \times [0,1])$ is a subbundle of $\left(\begin{tikzcd} S^1 \arrow[r] & N_j \arrow[d] \\ & \partial X^3 \end{tikzcd}\right)$, for some j, and $B_1, B_2 \cong S^1 \times D^2$ are identified with two D^2-bundles over a boundary component of Σ^2_Z. The unions are so that an S^1-fiber $S^1 \times (\{\cdot\} \times \{0\}) \subset \partial A$ is identified with $(\cdot, \partial D^2) \subset \partial B_1$ and a fiber $S^1 \times (\{\cdot\} \times \{1\}) \subset \partial A$ is identified with $(\cdot, \partial D^2) \subset \partial B_2$. By extending this attaching map to M_i, we can consider M_i as $\Sigma^2_i \times D^2$ where Σ^2_i is a cylinder and B_k, $k \in \{1,2\}$ is identified with a connected component of $\partial \Sigma^2_i \times D^2$. In particular, D^2-fibers of Z extend to M_i. Then, $Z \cup_\partial M_i$ is the total space of D^2-fibers over the surface $\Sigma^2_i \cup_\partial \Sigma^2_Z$ where $\Sigma^2_i \cong S^1 \times I$ connects two boundary components of Σ^2_Z. Hence, $Z \cup_\partial M_i$ is the total space of D^2-fibers over a surface.

(iv) $M_i \cong S^2 \times D^2$.

In this case, $\partial M_i \cong S^1 \times S^2$. From Lemma 13.30, $\partial M_i = B_1 \cup_\partial A \cup_\partial B_2$ where $A \cong S^1 \times (S^1 \times I)$ is a subbundle of $\left(\begin{tikzcd} S^1 \arrow[r] & N_j \arrow[d] \\ & \partial X^3 \end{tikzcd}\right)$, for some j, and $B_1, B_2 \cong S^1 \times D^2$ are identified with two D^2-bundles over a boundary components of Σ^2_Z. Denote the two boundary components of Σ^2_Z by σ_1 and σ_2 respectively. Then, B_k, $k \in \{1,2\}$, is identified with $\sigma_k \times D^2 \subset Z$. We note that the D^2-factor of B_k, $k \in \{1,2\}$, is contained in the S^2-factor of M_i while the S^1-factor of B_k coincides with the ∂D^2-factor of ∂M_i.

For $k \in \{1,2\}$, let U_k be a neighborhood of $\sigma_k \times D^2$ in Z so that $U_k \cong (\sigma_k \times D^2) \times [0,1)$. Then, U_k is a D^2-bundle over a cylinder $\sigma_k \times [0,1)$ where $\sigma_k \times \{0\}$ is identified with B_k. By the same plumbing construction as in the proof of Lemma 14.14, we have that $U_1 \cup M_i \cup U_2 \cong (\Sigma^2_1 \times D^2) \square (S^2 \times D^2) \square (\Sigma^2_2 \times D^2)$ where $\Sigma^2_k = (\sigma_k \times [0,1)) \cup D^2$ and the union by gluing $\sigma_k \times \{0\}$ to ∂D^2. That is $\Sigma^2_k \cong D^2$.

(v) $M_i \cong (S^2 \widetilde{\times} I) \widetilde{\times} I$ or $(\mathbb{R}P^3 \times S^1) \widetilde{\times} I$.

In this case, $\partial M_i \cong S^1 \times S^2$. From Lemma 13.30, $\partial M_i = B_1 \cup_\partial A \cup_\partial B_2$ where $A \cong S^1 \times (S^1 \times I)$ is a subbundle of $\left(\begin{tikzcd} S^1 \arrow[r] & N_j \arrow[d] \\ & \partial X^3 \end{tikzcd}\right)$, for some j, and $B_1, B_2 \cong S^1 \times D^2$ are identified with two D^2-bundles over a boundary components of Σ^2_Z. Denote the two boundary components by σ_1 and σ_2 so that B_k, $k \in \{1,2\}$, is identified with $\sigma_k \times D^2 \subset Z$.

For $k \in \{1,2\}$, let V_k be a subbundle of $Z = \left(\begin{tikzcd} D^2 \arrow[r] & Z \arrow[d] \\ & (\Sigma^2_Z, \partial \Sigma^2_Z) \end{tikzcd}\right)$ so that $V_k \cong D^2 \times (\sigma_k \times [0, \epsilon))$ where $\sigma_k \times [0,\epsilon)$ denotes a neighborhood of σ_k in Σ^2_Z. Let $U = V_1 \cup_\partial M_i \cup_\partial V_2$.

We have that U is an open neighborhood of M_i in $Z \cup_\partial M_i$, $V_1 \cap M_i = B_1$, and $V_2 \cap M_i = B_2$.

Let $\widetilde{U} \xrightarrow{\pi} U$ be a double covering. Then, $\widetilde{U} = \widetilde{V}_1 \cup \widetilde{M_i} \cup \widetilde{V}_2$ where $\widetilde{M_i}$ is a double cover of M_i and $\widetilde{V}_k$, $k \in \{1, 2\}$, is a double cover of V_k. We will show that there are local S^1-actions on $\widetilde{U}$ that are compatible with π.

Consider that $\widetilde{M_i} \cong (S^2 \times S^1) \times I$. We will refer to each $(S^2 \times S^1)$-fiber of $\widetilde{M_i}$ by $(S^2 \times S^1) \times \{t\}$, for some $t \in [0, 1]$. For each $k \in \{1, 2\}$, $\widetilde{V}_k$ is the union of two copies of $D^2 \times (\sigma_k \times [0, \epsilon))$, which we denote by $V_k(0)$ and $V_k(1)$. Then, $\pi(V_k(0)) = \pi(V_k(1)) = V_k$. $V_k(0)$ attaches to $\widetilde{M_i}$ along $(S^2 \times S^1) \times \{0\}$ so that each D^2-fiber of $V_k(0)$ is contained in an S^2-fiber of $(S^2 \times S^1) \times \{0\} \subset \partial\widetilde{M_i}$. $V_k(1)$ attaches to $\widetilde{M_i}$ along $(S^2 \times S^1) \times \{1\}$ similarly.

Let $L_1 \cong (D^2 \times S^1) \times [0, 1]$ be a submanifold of $\widetilde{M_i}$ so that for each $s \in S^1$ and $t \in [0, 1]$, $(D^2 \times \{s\}) \times \{t\} \subset L_1$ is contained in $(S^2 \times \{s\}) \times \{t\} \subset \widetilde{M_i}$. Additionally, $(D^2 \times S^1) \times \{0\}$ coincides with $V_1(0) \cap \widetilde{M_i}$ and $(D^2 \times S^1) \times \{1\}$ coincides with $V_1(1) \cap \widetilde{M_i}$. In particular, $\widetilde{V}_1 \cup L_1 = V_1(0) \cup L_1 \cup V_1(1)$ is the total space of D^2-fibers over a cylinder. In other words, D^2-fibers of $\widetilde{V}_1$ extend to L_1. Let L_2 be constructed in the same manner so that $(D^2 \times S^1) \times \{0\} \subset L_2$ coincides with $V_2(0) \cap \widetilde{M_i}$ and $(D^2 \times S^1) \times \{1\} \subset L_2$ coincides with $V_2(1) \cap \widetilde{M_i}$. Then, $\widetilde{V}_2 \cup L_2 = V_2(0) \cup L_2 \cup V_2(1)$ is the total space of D^2-fibers over a cylinder.

Put $Q = U - [(\widetilde{V}_1 \cup L_1) \cup (\widetilde{V}_2 \cup L_2)]$. Then, $Q \cong ((S^2 - 2D^2) \times S^1) \times I \cong ((S^1 \times I) \times S^1) \times I$. We have that $U = L_1 \cup_\partial Q \cup_\partial L_2$ where the unions are so that (∂D^2)-fibers of ∂L_1 and ∂L_2 coincides with $(S^1, \cdot, \cdot, \cdot)$-fibers of Q.

Let S^1 act on Q by rotations on the first S^1-factor and act trivially on other factors. Let S^1 act on L_k, $k \in \{1, 2\}$, by rotations about the center on the D^2-factor and act trivially on other factors. Consequently, we get local S^1-actions on $U = L_1 \cup_\partial Q \cup_\partial L_2$ that are compatible with π and with local S^1-actions on Z. In particular, the images under π of the orbits of the local S^1-actions on $\partial\widetilde{U}$ coincides with S^1-fibers of ∂U.

Case $M_j \cong (S^2 \times_{\mathbb{Z}_2} I) \times I$, for some $j \in \mathcal{A}_1$

Next, we assume that there exists M_j, $j \in \mathcal{A}_1$, such that

$$M_j = \left(\begin{tikzcd} S^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd} \right) \cong$$

$(S^2 \times_{\mathbb{Z}_2} I) \times I$. From Lemma 13.30, if a component M_i, $i \in \mathcal{A}_1$ intersects M_j, then $M_i \cong S^2 \times_{\pm 2} D^2$ or $S^2 \times_{\mathbb{Z}_2} D^2$. $M_i \cong S^2 \times_{\pm 2} D^2$ if M_i intersects with exactly one $M_j \cong (S^2 \times_{\mathbb{Z}_2} I) \times I$, for some $j \in \mathcal{A}_1$. $M_i \cong S^2 \times_{\mathbb{Z}_2} D^2$ if M_i intersects with exactly two $M_{j_k} \cong (S^2 \times_{\mathbb{Z}_2} I) \times I$, for some $j_k \in \mathcal{A}_1$, $k \in \{1, 2\}$.

Let $\mathcal{A}_0'$ be the collection of $i \in \mathcal{A}_0$ so that $M_i \cong S^2 \times_{\pm 2} D^2$ or $S^2 \times_{\mathbb{Z}_2} D^2$ and let $\mathcal{A}_1'$ be the collection of $j \in \mathcal{A}_1$ so that $M_j \cong (S^2 \times_{\mathbb{Z}_2} I) \times I$. Put $X = \left(\bigsqcup_{i\in\mathcal{A}_0'} M_i\right) \cup \left(\bigsqcup_{j\in\mathcal{A}_1'} M_j\right)$ and $\widetilde{Z} = Y - X$. Write $X = \bigsqcup_q X_q$ where X_q is a connected component of X. Then, X_q can be written as

$$X_q = \begin{tikzcd} M_{i_1} \arrow[r, dash, "M_{j_1}"] & M_{i_2} \arrow[r, dash, "M_{j_2}"] & \cdots \arrow[r, dash, "M_{j_{d-1}}"] & M_{i_d} \end{tikzcd} \tag{14.39}$$

or

$$X_q = \begin{tikzcd} M_{i_1} \arrow[r, dash, "M_{j_1}"] \arrow[rrr, dash, bend right, "M_{j_d}"'] & M_{i_2} \arrow[r, dash, "M_{j_2}"] & \cdots \arrow[r, dash, "M_{j_{d-1}}"] & M_{i_d} \end{tikzcd} \tag{14.40}$$

where $\begin{tikzcd} M_{i_k} \arrow[r, dash, "M_{j_k}"] & M_{i_{k+1}} \end{tikzcd}$ denotes the union $M_{i_k} \cup_\partial M_{j_k} \cup_\partial M_{i_{k+1}}$ so that $M_{i_k} \cap M_{j_k}$ and $M_{j_k} \cap M_{i_{k+1}}$ are the two fibers of $(S^2 \times_{\mathbb{Z}_2} I) \times \partial I \subset \partial M_j$.

(i) Case $X_q = \begin{tikzcd} M_{i_1} \arrow[r, dash, "M_{j_1}"] & M_{i_2} \arrow[r, dash, "M_{j_2}"] & \cdots \arrow[r, dash, "M_{j_{d-1}}"] & M_{i_d} \end{tikzcd}$.

In this case, $M_{i_1}, M_{i_d} \cong S^2 \times_{\pm 2} D^2$ and $M_{i_k} \cong S^2 \times_{\mathbb{Z}_2} D^2$ for $k \notin \{1, d\}$. As in the proof of Lemma 14.3, if $M_{i_k} \cong M_{i_{k+1}} \cong S^2 \times_{\mathbb{Z}_2} D^2$, then $M_{i_k} \cup_\partial M_{j_k} \cup_\partial M_{i_{k+1}} \cong (S^2 \times_{\mathbb{Z}_2} I) \times I$. Therefore, $X_q \cong (S^2 \times_{\pm 2} D^2) \cup_\partial ((S^2 \times_{\mathbb{Z}_2} I) \times I) \cup_\partial (S^2 \times_{\pm 2} D^2)$. By the same arguments as in the proof of Lemma 14.3 and from Lemma 2.13, $X_q \cong (S^2 \times S^2) \# D^4$ or $(S^2 \widetilde{\times} S^2) \# D^4$. It follows that $\widetilde{Z} \cup X \cong (\widetilde{Z} \cup D^4) \# (S^2 \times S^2)$ or $(\widetilde{Z} \cup D^4) \# (S^2 \widetilde{\times} S^2)$.

(ii) Case $X_q = \begin{tikzcd} M_{i_1} \arrow[r, dash, "M_{j_1}"] \arrow[rrr, dash, bend right, "M_{j_m}"'] & M_{i_2} \arrow[r, dash, "M_{j_2}"] & \cdots \arrow[r, dash, "M_{j_{d-1}}"] & M_{i_d} \end{tikzcd}$.

In this case $M_{i_k} \cong S^2 \times_{\mathbb{Z}_2} D^2$ for all k. As in the proof of Lemma 14.3, $M_{i_k} \cup_\partial M_{j_k} \cup_\partial M_{i_{k+1}} \cong (S^2 \times_{\mathbb{Z}_2} I) \times I$. Thus, X_q is diffeomorphic to an $(S^2 \times_{\mathbb{Z}_2} I)$-bundle over S^1. By the same arguments as in the case $M_i \cong (S^2 \widetilde{\times} S^1) \widetilde{\times} I$, there is a double cover $\widetilde{U}$ of a neighborhood U of X_q and there are local S^1-actions on U which are compatible with the covering map and local S^1-actions on $\widetilde{Z}$.

Assemble all components

Lastly, we assemble all components. Let $\mathcal{A}_0''$ be the collection of $i \in \mathcal{A}_0$ such that $M_i \cong D^4$, $\pm\mathcal{C}P^2 \# D^4$, $S^2 \times D^2$, or $S^1 \times D^3$. Let $\mathcal{A}_1''$ be the collection of $j \in \mathcal{A}_1$ such that $M_j \cong D^3 \times I$.

Put $Y_1 = \left(\bigsqcup_{i\in\mathcal{A}_0''} M_i\right) \cup \left(\bigsqcup_{j\in\mathcal{A}_1''} M_j\right) \cup \left(\bigsqcup_{k\in\mathcal{A}_2} M_k\right)$ and $Y_2 = Y - Y_1$. From all cases above, we have that

$$Y_1 \cong (\text{plumbings of } D^2\text{-bundles over}\Sigma^2_{Y_1}) \# n_1(\mathbb{C}P^2) \# n_2(-\mathbb{C}P^2) \# n_3(S^2 \times S^2) \tag{14.41}$$

where $\Sigma^2_{Y_1}$ is a surface and for some integers $n_1, n_2, n_3 \geq 0$. Additionally, $Y_2 \cap Y_1$ is the disjoint union of copies of $S^1 \times D^2$ and Y_2 admits an F-structure which is compatible to the standard local S^1-actions on D^2-fibers of Y_1.

From Lemma 2.30, plumbings of D^2-bundles over a surface admits a T-structure. Away from the plumbing locations, the T-structure restricts to local S^1-actions by rotations about the center on each D^2-fiber. Paternain and Petean [21, Theorem 5.9] showed that the connected sum of two manifolds which admit a T-structure also admits a T-structure. Therefore, Y_1 admits a T-structure which restricts to the standard local S^1-actions on a D^2-bundle over a neighborhood of $\partial\Sigma^2_{Y_1}$. Therefore, $Y = Y_1 \cup_\partial Y_2$ admits an F-structure whose restriction to ∂Y has positive rank. This completes the proof of Lemma 14.38 (3).

Example 14.42. In this example, $Y \cong (S^2 \times_{\omega_1} D^2)\Box(S^2 \times D^2)\Box(S^2 \times_{\omega_2} D^2)$, for some $\omega_1, \omega_2 \in \mathbb{Z}$. All vertices represent a manifold diffeomorphic to D^4 except one vertex which represents a manifold diffeomorphic to $S^2 \times D^2$.

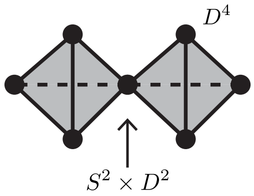


Figure 14.11: A representation of an elementary building block of type $(2, D^2)$

14.6 Elementary building block of type $(1, S^1 \times D^2)$

In this section, we consider a connected component of $M - \bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ that only contains $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ and $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components.

Definition 14.43. We call a component W in Lemma 14.46 an *elementary building block of type* $(1, S^1 \times D^2)$.

We represent an elementary building block of type $(1, S^1 \times D^2)$ by a graph.

Example 14.44. The following is an example of an elementary building block of type $(1, S^1 \times D^2)$.

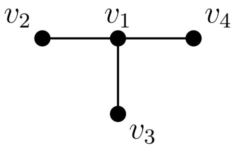


Figure 14.12: A representation of an elementary building block of type $(1, S^1 \times D^2)$

In this example, there are three $\left(\begin{array}{r} D^4 \longrightarrow M_{i_\ell} \\ \downarrow \\ \text{pt} \end{array}\right)$ components, $\ell \in \{2, 3, 4\}$, which are represented by the vertices v_2, v_3, and v_4, respectively. There is one $\left(\begin{array}{r} T^2 \times D^2 \longrightarrow M_{i_1} \\ \downarrow \\ \text{pt} \end{array}\right)$ component, which is represented by the vertex v_1. There are three $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_{j_\ell} \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components, $\ell \in \{1, 2, 3\}$, which are represented by edges. Let W be the union of all components. W is represented by the graph above.

The topology of W depends on how each $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_{j_\ell} \\ \downarrow \\ (I, \partial I) \end{array}\right)$ component attaches to $\left(\begin{array}{r} D^4, T^2 \times D^2 \longrightarrow M_{i_\ell} \\ \downarrow \\ \text{pt} \end{array}\right)$ components. For example, W can be diffeomorphic to a 4-manifold with plumbing diagram

$$\begin{tikzcd} D^4 \arrow[r, dash] & S^2 \times D^2 \arrow[r, dash] & T^2 \times D^2 \arrow[r, dash] \arrow[d, dash] & S^2 \times D^2 \arrow[r, dash] & D^4 \\ & & S^2 \times D^2 \arrow[r, dash] & D^4 & \end{tikzcd} \quad . \tag{14.45}$$

Moreover, ∂W is the total space of S^1-fibers and ∂W attaches to a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_k \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ so that S^1-fibers coincide.

Lemma 14.46. *Let* $\{M_i\}_{i\in\mathcal{A}_0}$ *be a collection of* $\left(\begin{array}{r} D^4, \dots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ *components such that* M_i *only intersects* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components (as in Lemma 13.38).*

Let $\{M_j\}_{j\in\mathcal{A}_1}$ *be a collection of* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ *components such that both fibers of* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$ *are contained in* $\bigsqcup_{i\in\mathcal{A}_0} \partial M_i$.

Let W *be a connected component of* $M - \bigsqcup_p \left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *such that* W *only intersects components* M_i*,* $i \in \mathcal{A}_0$ *and* M_j*,* $j \in \mathcal{A}_1$*, i.e.* $W \subset \left(\bigsqcup_{i\in\mathcal{A}_0} M_i\right) \cup \left(\bigsqcup_{j\in\mathcal{A}_1} M_j\right)$*. Then, the following holds.*

1. *W can be represented by a graph so that each vertex represents M_i, for some $i \in \mathcal{A}_0$ and each edge represents M_j, for some $j \in \mathcal{A}_1$.*
2. *W admits an F-structure whose restriction to ∂W has positive rank.*
3. *∂W is the total space of S^1-fibers. As a part of M, ∂W is identified with a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_k \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some k, so that S^1-fibers of ∂W coincide with S^1-fibers of M_k.*

Proof. Here we proof parts (1) and (3) of the lemma. The proof of part (2) will be given in the next subsection.

(1). We construct a graph to represent W as follows. Let G be a graph such that each vertex v_i represents a connected component M_i, for some $i \in \mathcal{A}_0$, and each vertex e_j represents a connected component M_j, for some $j \in \mathcal{A}_1$. A vertex v_i is incident to an edge e_j if and only if $M_i \cap M_j \neq \varnothing$.

(3). It follows from compatibility of fibers in Lemma 13.5 and Lemma 13.38, and from the configuration in part (1) that ∂W is the total space of S^1-fibers. By connectedness, ∂W

is identified with exactly one boundary component of $\left(\begin{array}{ccc} S^1 & \longrightarrow & M_k \\ & & \downarrow \\ & & (X^3, \partial X^3) \end{array}\right)$, for some k, so that their S^1-fibers coincide. $\square$

14.6.1 Proof of Lemma 14.46 (2)

In this subsection, we prove Lemma 14.46 (2). For simplicity, we assume that W is the only connected component of $M - \bigsqcup_p \left(\begin{array}{ccc} S^1 & \longrightarrow & M_p \\ & & \downarrow \\ & & (X^3, \partial X^3) \end{array}\right)$ that only intersects components M_i, $i \in \mathcal{A}_0$ and M_j, $j \in \mathcal{A}_1$.

First, we assume that for all $j \in \mathcal{A}_1$, $M_j = \left(\begin{array}{ccc} S^1 \times D^2 & \longrightarrow & M_j \\ & & \downarrow \\ & & (I, \partial I) \end{array}\right) \cong (S^1 \times D^2) \times I$.

Case $M_j \cong (S^1 \times D^2) \times I$ **for all** $j \in \mathcal{A}_1$

Let $M_i = \left(\begin{array}{ccc} D^4, \ldots & \longrightarrow & M_i \\ & & \downarrow \\ & & \mathrm{pt} \end{array}\right)$, $i \in \mathcal{A}_0$. From Lemma 13.38, $\partial M_i = A_i \cup_\partial \left(\bigsqcup_s C_{i,s}\right)$ where:

(i) A_i is a subbundle of $\left(\begin{array}{ccc} S^1 & \longrightarrow & N_k \\ & & \downarrow \\ & & \partial X^3 \end{array}\right)$, for some k.

(ii) $C_{i,s} \cong S^1 \times D^2$ is a fiber of $\left(\begin{array}{ccc} S^1 \times D^2 & \longrightarrow & M_j \\ & & \downarrow \\ & & \partial I \end{array}\right)$, for some $j \in \mathcal{A}_1$.

(iii) $\partial C_{i,s} \cong T^2$ is identified with a boundary component of A_i.

Because ∂M_i is connected, A_i is also connected. Thus, A_i is a subbundle $\left(\begin{array}{ccc} S^1 & \longrightarrow & A_i \\ & & \downarrow \\ & & (\Sigma^2_{A_i}, \partial \Sigma^2_{A_i}) \end{array}\right)$ of $\left(\begin{array}{ccc} S^1 & \longrightarrow & N_k \\ & & \downarrow \\ & & \partial X^3 \end{array}\right)$ for some k. We will refer to a boundary component of A_i which attaches to $C_{i,s}$ by $S^1 \times \sigma_{i,s}$, where $\sigma_{i,s}$ is a boundary component of $\Sigma^2_{A_i}$. Let $\phi_{i,s} : \underbrace{S^1 \times \partial D^2}_{\cong \partial C_{i,s}} \to S^1 \times \sigma_{i,s}$ be the attaching map. Up to isotopy, there are three possibilities:

(a) $\phi_{i,s}$ maps $(\cdot, \partial D^2) \subset \partial C_{i,s}$ to $(S^1, \cdot) \subset S^1 \times \sigma_{i,s}$.

(b) $\phi_{i,s}$ maps $(\cdot, \partial D^2) \subset \partial C_{i,s}$ to $(\cdot, \partial \Sigma^2_{A_i}) \subset S^1 \times \sigma_{i,s}$.

(c) $\phi_{i,s}$ maps $(\cdot, \partial D^2)$ to a circle of slope $\dfrac{q}{p}$, where $p, q \neq 0$, in $S^1 \times \sigma_{i,s}$.

For each $M_j = \left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $j \in \mathcal{A}_1$, the two fibers of $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$ are attached to M_{i_1} and M_{i_2}, for some $i_1, i_2 \in \mathcal{A}_0$, respectively. From the compatibility of circle fibers in Lemma 13.5 and Lemma 13.38, the two attaching maps ϕ_{i_1,s_1} and ϕ_{i_2,s_2} from M_{i_1} and M_{i_2} to M_j must be the same type.

Attaching map of type (a): Suppose that there exists $C_{i,s}$ so that $\phi_{i,s}$ is an attaching map of type (a), i.e., $\phi_{i,s}$ sends $(\cdot, \partial D^2) \subset \partial C_{i,s}$ to $(S^1, \cdot) \subset S^1 \times \sigma_{i,s}$. From explicit constructions in [11] and from the classification of $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \text{pt} \end{array}\right)$ in Table 13.1, M_i has at most two attaching maps of type (a). In particular, if M_i has exactly one $\phi_{i,s}$ of type (a) then $\partial M_i \cong S^3, \mathbb{R}P^3, \mathbb{R}P^3 \# \mathbb{R}P^3$, or $L(|\omega|, 1)$, $\omega \in \mathbb{Z}$. If M_i has exactly two ϕ_{i,s_1} and ϕ_{i,s_2} of type (a) then $\partial M_i \cong S^2 \times S^1$.

Let X be the union of all M_j, $j \in \mathcal{A}_1$, such that there exists M_i, $i \in \mathcal{A}_0$, where the attaching map from M_i to M_j is of type (a). Write $X = \bigsqcup_m X_m$ where X_m is a connected component of X. Let G_m be a graph representing X_m (as constructed in Lemma 14.46 (1)). Then, G_m must have one of the following forms.

(i)

$$G_m = v_1 \;\text{——}\; v_2 \;\text{——}\; \cdots \;\text{——}\; v_d \tag{14.47}$$

where v_1 and v_s represents M_{i_1} and M_{i_s} so that $\partial M_{i_1}, \partial M_{i_d} \cong S^3, \mathbb{R}P^3, \mathbb{R}P^3 \# \mathbb{R}P^3$, or $L(|\omega|, 1)$, $\omega \in \mathbb{Z}$, and v_k, $k \notin \{1, s\}$, represents M_{i_k} so that $\partial M_{i_k} \cong S^2 \times S^1$.

(ii)

$$G_m = \begin{tikzcd} v_1 \arrow[r, dash] \arrow[rrr, dash, bend right] & v_2 \arrow[r, dash] & \cdots \arrow[r, dash] & v_d \end{tikzcd} \tag{14.48}$$

where and v_k represents M_{i_k} so that $\partial M_{i_k} \cong S^2 \times S^1$ for all k. If for all k, $M_{i_k} \cong S^1 \times D^3$, then $X_m \cong T^2 \times D^2$.

If $X_m \cong T^2 \times D^2 \cong S^1 \times (S^1 \times D^2)$, then there exists a T^2-action on X_m by the standard T^2-action on the $(S^1 \times D^2)$-factor and by the trivial action on the S^1-factor. Otherwise, by the same constructions as in the proof of Lemma 14.14, X_m admits an F-structure whose restriction to ∂X_m is of positive rank.

Consequently, in order to construct an F-structure on W, we can assume without loss of generality that for all attaching map is of type (b) or (c).

Attaching map of type (b) or (c): Let M_i, $i \in \mathcal{A}_0$, and M_j, $j \in \mathcal{A}_1$, be such that $M_i \cap M_j \neq \varnothing$ and the attaching map from M_i to M_j is of type (b) or (c). In each of the the following cases, we will show that $M_i \cup_\partial M_j$ admits an F-structure which is compatible with fibration structures on other components it intersects with.

Each M_j, $j \in \mathcal{A}_1$, intersects with two components M_{i_1} and M_{i_2}, $i_1, i_2 \in \mathcal{A}_0$. From the compatibility of circle fibers in Lemma 13.5 and Lemma 13.38, and the following constructions, we will have that the F-structures on $M_{i_1} \cup_\partial M_j$ and $M_{i_2} \cup_\partial M_j$ can be combined.

As a result from all cases, we will have that W admits an F-structure.

(i) $M_i \cong S^2 \times_\omega D^2$, $\omega \in \mathbb{Z}$.

A list of S^1-actions on $S^2 \times_\omega D^2$ is given in Section 2.5. The orbits of the S^1-actions on $\partial M_i \cong L(|\omega|, 1)$ coincide with Seifert orbits on $L(|\omega|, 1)$. In particular, there are at most two attaching maps of type (c).

Let M_j, $j \in \mathcal{A}_1$, be such that $M_j \cap M_i = C_{i,1}$. We consider M_j as $(S^1 \times D^2) \times [0,1]$ so that $(S^1 \times D^2) \times \{0\}$ coincides with $C_{i,1}$. We use the coordinates $((e^{i\gamma_1}, re^{i\gamma_2}), t)$, $\gamma_1, \gamma_2 \in [0, 2\pi)$, $t \in [0,1]$, on M_j.

If the attaching map $\phi_{i,1}$ is of type (c), then it introduces an exceptional orbit with orbit invariant (u, v) on ∂M_i. In this case, let $\psi_j : S^1 \times [(S^1 \times D^2) \times [0,1]] \to (S^1 \times D^2) \times [0,1]$ be an S^1-action on M_j so that

$$\psi_j : \{\theta\} \times ((e^{i\gamma_1}, re^{i\gamma_2}), t) \mapsto ((e^{i(\gamma_1 + u\theta)}, re^{i(\gamma_2 + v\theta)}), t). \tag{14.49}$$

If the attaching map $\phi_{i,1}$ is of type (b), then let $\psi_j : S^1 \times [(S^1 \times D^2) \times [0,1]] \to (S^1 \times D^2) \times [0,1]$ be an S^1-action on M_j so that

$$\psi_j : \{\theta\} \times ((e^{i\gamma_1}, re^{i\gamma_2}), t) \mapsto ((e^{i(\gamma_1 + \theta)}, re^{i\gamma_2}), t). \tag{14.50}$$

In both cases, the orbits of ψ_j on M_j and the orbits of the S^1-action on M_i coincide along $M_j \cap M_i = \partial M_j \cap \partial M_i$. If the S^1-actions do not agree, then they generate a T^2-action with orbits of dimension one on a neighborhood U of $\partial M_j \cap \partial M_i$ which is compatible with both S^1-actions. If the T^2-action is not effective, then we pass to a quotient to get an effective S^1-action on U.

(ii) $M_i \cong D^4$.

Consider $M_i \cong D^4$ as $D^2 \times D^2$ with coordinates $(re^{i\gamma_1}, se^{i\gamma_2})$, $\gamma_1, \gamma_2 \in [0, 2\pi)$. Let $\psi_{u,v}$ be an S^1-action on M_i defined by

$$\psi_{u,v} : \{\theta\} \times (re^{i\gamma_1}, se^{i\gamma_2}) \mapsto (re^{i(\gamma_1+u\theta)}, re^{i(\gamma_2+v\theta)}) \tag{14.51}$$

for some $(u, v) \in \mathbb{Z}^2$. The restriction of $\psi_{u,v}$ to $\partial D^4 \cong S^3$ gives a Seifert fibration $S^3 \to S^3/\psi_{u,v} \cong S^2$. The orbit space is $(D^4)^* \cong D^3$ with one interior fixed point and possibly at most two exceptional segments whose endpoints are the fixed point and an exceptional orbit on the boundary $\partial(D^4)^* \cong S^2$.

We construct an S^1-action on M_j, $j \in \mathcal{A}_1$, that connects to M_i as in case (i).

(iii) $M_i \cong S^1 \times D^3$.

In this case, $\partial M_i \cong S^1 \times S^2$. The Seifert orbit of $S^1 \times S^2$ is either S^2 with no exceptional orbits or S^2 with two exceptional Seifert orbits of the same order. In the first case, there is an S^1-action by rotations on the S^1-factor and by the trivially action on the S^2-factor. This action extends to $S^1 \times D^3$ so that S^1 acts by rotations on the S^1-factor and acts trivially on the D^3-factor.

In the second case, there is an S^1-action on M_i which is obtained from a quotient of an S^1-action on $S^2 \times \mathbb{R}$ where S^1 acts on the S^2-factor by a screw motion of finite order. Extend the screw motion on S^2 to D^3 to get an S^1-action on $S^1 \times D^3$. The orbit space of this action is D^3 with one exceptional segment with two endpoints on the boundary $\partial D^3 \cong S^2$.

We construct S^1-actions on M_j, $j \in \mathcal{A}_1$, that connects to M_i as in case (i).

(iv) $M_i \cong T^2 \times D^2$.

In this case, $\partial M_i \cong T^3$ whose Seifert orbit space is T^2 with no exceptional orbits. Hence, all attaching maps are of type (b). From Lemma 13.38, $\partial M_i = A \cup \left(\bigsqcup_j C_j\right)$ where A is the total space of S^1-fibers over a surface Σ^2_A whose fibers coincide with fibers of

$$\left(\begin{tikzcd} S^1 \arrow[r] & M_k \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right)$$

, for some k, and $C_j \cong S^1 \times D^2$. We consider $\partial M_i \cong T^3$ as the total space of S^1 fibers over T^2. Denote the base T^2 by $S^1_b \times S^1_b$ and denote the fibers by S^1_f. Then, $\partial M_i \cong S^1_b \times S^1_b \times S^1_f$ where $\Sigma^2_A \subset S^1_b \times S^1_b$ and the S^1_f-factor coincides with fibers of

$$\left(\begin{tikzcd} S^1 \arrow[r] & M_k \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right)$$

, for some k.

$\partial M_i \cong T^2 \times \partial D^2$. Up to isotropy, the (∂D^2)-factor either coincides with the S^1_b-factor or with the S^1_f-factor. If the ∂D^2-factor coincides with the S^1_b-factor, then the T^2-factor coincides with $S^1_b \times S^1_f$. Let S^1 act on $M_i \cong T^2 \times D^2$ by acting trivially on the D^2-factor and acting on T^2 by rotations along the S^1_f-factor and trivially along the S^1_b-factor. On $M_j \cong (S^1 \times D^2) \times I$, $j \in \mathcal{A}_1$, that connects to M_i, we let S^1 act by rotations on the S^1-factor and act trivially on other factors. In particular, the orbits of S^1-actions on M_i and on M_j coincide along $M_i \cap M_j = \partial M_i \cap \partial M_j = C_j$. If the S^1-actions do not agree, then they generate a T^2-action with orbits of dimension one on a neighborhood of $\partial M_i \cap \partial M_j$ that is compatible with the two S^1-actions. We pass to a quotient to get an effective S^1-action if the T^2-action is not effective.

If the ∂D^2-factor coincides with the S^1_f-factor, then the T^2-factor coincides with $S^1_b \times S^1_b$. We consider $M_i \cong T^2 \times D^2$ as a D^2-bundle over T^2. By the same plumbing construction as in the proof of Lemma 14.14, we have that $M_i \cup M_j \cong (T^2 \times D^2)\Box(\Sigma^2_j \times D^2)$ where $\Sigma^2_j \cong D^2$. $(\partial\Sigma^2_j, \cdot)$ is identified with $(S^1, \cdot) \times \{0\} \subset M_j$. Let $\mathcal{A}^i_1$ be the collection of all $j \in \mathcal{A}_1$ such that $M_i \cap M_j \neq \varnothing$. Then, $M_i \cup \left(\bigsqcup_{j \in \mathcal{A}^i_1} M_j\right)$ is a plumbing of $T^2 \times D^2$ with copies of D^2-bundles over D^2. From Lemma 2.30, $M_i \cup \left(\bigsqcup_{j \in \mathcal{A}^i_1} M_j\right)$ admits a T-structure.

(v) $M_i \cong S^2 \times_{\mathbb{Z}_2} D^2$, $(S^2 \times_\omega D^2)/\mathbb{Z}_2$, or $T^2 \times_{\mathbb{Z}_2} D^2$.

For each M_j, $j \in \mathcal{A}_1$, such that $M_j \cap M_i = \partial M_i \cap \partial M_j \neq \varnothing$, we consider M_j as $(S^1 \times D^2) \times [0,1]$ where $(S^1 \times D^2) \times \{0\}$ coincides with $\partial M_i \cap \partial M_j$. Let U_i be an open neighborhood of M_i so that $V_{i,j} = U_i \cap M_j$ is an $(S^1 \times D^2)$-subbundle of M_j.

Let $\widetilde{U}_i \xrightarrow{\pi} U_i$ be a double covering. Then, $\widetilde{U}_i = \widetilde{M_i} \cup \left(\bigsqcup_j \widetilde{V}_{i,j}\right)$ where $\widetilde{V}_{i,j}$ is a double cover of $V_{i,j}$ and $\widetilde{M_i}$ is a double cover of M_i. $\widetilde{U}_i \cong \widetilde{M_i} \cong S^2 \times D^2, S^2 \times_\omega D^2$, or $T^2 \times D^2$. By the same arguments as in cases (i) and (iv), $\widetilde{U}_i$ admits a T-structure which is compatible with π. Therefore, U_i admits an F-structure.

(vi) $M_i \cong (\mathbb{R}P^3 \times S^1)\widetilde{\times} I$, $(S^2 \widetilde{\times} S^1) \times I$, $\mathcal{B}_k \widetilde{\times} I$, $k \in \{1,2,3,4\}$.

We have that

$$\partial M_i \cong \begin{cases} S^2 \times S^1 & \text{if } M_i \cong (\mathbb{R}P^3 \times S^1)\widetilde{\times} I \text{ or } (S^2 \widetilde{\times} S^1) \times I, \\ T^3 & \text{if } M_i \cong \mathcal{B}_k \widetilde{\times} I, k \in \{1,2\}, \\ \mathcal{G}_2 & \text{if } M_i \cong \mathcal{B}_k \widetilde{\times} I, k \in \{3,4\}. \end{cases} \tag{14.52}$$

Let $\mathcal{A}^i_1$ be a collection of $j \in \mathcal{A}_1$ such that $M_j \cap M_i = \partial M_i \cap \partial M_j \neq \varnothing$. We will consider M_j, $j \in \mathcal{A}_1$, as $(S^1 \times D^2)_j \times [0,1]$ where $(S^1 \times D^2)_j \times \{0\}$ coincides with $\partial M_i \cap \partial M_j$. From Lemma 13.38, $\partial M_i = A \cup \left(\bigsqcup_{j \in \mathcal{A}^i_1} C_j\right)$ where A is the total space of S^1-fibers over a surface Σ^2_A whose fibers coincide with fibers of

$$\left(\begin{tikzcd} S^1 \arrow[r] & M_k \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right),$$

for some k, and $C_j = \partial M_i \cap \partial M_j = (S^1 \times D^2)_j \times \{0\}$.

Let U_i be an open neighborhood of M_i in W so that $V_j = U_i \cap M_j$ is an $(S^1 \times D^2)$-subbundle of M_j. Let $\widetilde{U}_i \xrightarrow{\pi} U_i$ be a double covering. Then, $\widetilde{U} = \widetilde{M}_i \cup \left(\bigsqcup_j \widetilde{V}_j\right)$ where $\widetilde{V}_j$ is a double cover of V_j and $\widetilde{M}_i$ is a double cover of M_i. We have that $\widetilde{M}_i \cong \partial M_i \times I$. That is $\widetilde{M}_i \cong (S^2 \times S^1) \times I, T^3 \times I$, or $\mathcal{G}_2 \times I$. We will refer to a (∂M_i)-fiber of $\widetilde{M}_i$ by $(\partial M_i) \times \{t\}$ for some $t \in [0,1]$. Write $\widetilde{V}_j = V_j(0) \sqcup V_j(1)$ where $\pi(V_j(0)) = \pi(V_j(1)) = V_j$. Put $C_j(s) = V_j(s) \cap \partial M_i \times \{s\}$ for $s \in \{0,1\}$. Then, $\pi(C_j(0)) = \pi(C_j(1)) = C_j$.

For each $j \in \mathcal{A}_1^i$, let $\mathcal{C}_j$ be a subset of $\widetilde{M}_i$ so that $\mathcal{C}_j \cong (S^1 \times D^2) \times [0,1]$ and for all $t \in [0,1]$, $(S^1 \times D^2) \times \{t\} \subset \mathcal{C}_j$ is contained in $\partial M_i \times \{t\}$. We also require that for each $s \in \{0,1\}$, $(S^1 \times D^2) \times \{s\}$ coincides with $C_j(s)$. It follows that $V_j(0) \cup \mathcal{C}_j \cup V_j(1) \cong (S^1 \times D^2) \times I$. Moreover, $\widetilde{M}_i - (\bigsqcup_{j \in \mathcal{A}_1^i} \mathcal{C}_j) \cong \left(\begin{array}{r} S^1 \longrightarrow A \\ \downarrow \\ (\Sigma^2_A, \partial\Sigma^2_A) \end{array}\right) \times [0,1]$.

Each connected component of $\left(\begin{array}{r} S^1 \longrightarrow \partial A \\ \downarrow \\ \partial\Sigma^2_A \end{array}\right) \times \{t\}$ coincides with the boundary of an $(S^1 \times D^2)$-fiber of $\mathcal{C}_j$, for some $j \in \mathcal{A}_1^i$.

There are local S^1-actions on $\widetilde{M}_i - (\bigsqcup_{j \in \mathcal{A}_1^i} \mathcal{C}_j)$ whose orbits coincide with S^1-fibers of $\left(\begin{array}{r} S^1 \longrightarrow A \\ \downarrow \\ (\Sigma^2_A, \partial\Sigma^2_A) \end{array}\right) \times [0,1]$. Let ϕ_j be any S^1-action on $V_j(0) \cup \mathcal{C}_j \cup V_j(1) \cong (S^1 \times D^2) \times I$ which is compatible with π and whose restriction to $\partial(S^1 \times D^2) \times I$ is free. As a result, we get local S^1-actions on $\widetilde{U}_i$ that is compatible with π.

From all cases above, we have that if $M_j \cong (S^1 \times D^2) \times I$ for all $j \in \mathcal{A}_1$, then W admits an F-structure whose restriction on ∂W has positive rank.

Case $M_j \cong (T^2 \times_{\mathbb{Z}_2} I) \times I$ **for some** $j \in \mathcal{A}_1$

Next, we assume that there exists $j \in \mathcal{A}_1$ so that $M_j = \left(\begin{array}{r} T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right) \cong (T^2 \times_{\mathbb{Z}_2} I) \times I$. Let $\mathcal{A}_1'$ be the collection of $j \in \mathcal{A}_1$ such that $M_j \cong (T^2 \times_{\mathbb{Z}_2} I) \times I$. Let $\mathcal{A}_0'$ be the collection of $i \in \mathcal{A}_0$ such that $M_i \cap M_j \neq \varnothing$, for some $j \in \mathcal{A}_1'$.

Put $Z = \left(\bigsqcup_{j \in \mathcal{A}_1'} M_j\right) \cup_\partial \left(\bigsqcup_{i \in \mathcal{A}_0'} M_i\right)$. From Lemma 13.38, for all $i \in \mathcal{A}_0'$, $\partial M_i \cong S^2 \times S^1$, $\mathbb{R}P^3 \# \mathbb{R}P^3$, $L(|\omega|, 1)/\mathbb{Z}_2$, or $\mathcal{G}_2$. From the classification of Seifert manifolds ([17, 27]), a Seifert orbit space of $S^3/\Gamma, T^3/\Gamma$, or $(S^2 \times S^1)/\Gamma$ contains at most four exceptional orbits.

Since $T^2 \times_{\mathbb{Z}_2} I \subset \partial M_{i_k}$ introduces two exceptional orbits of the same order, each M_i, $i \in \mathcal{A}_0$, connects to at most two M_j, $j \in \mathcal{A}_1'$.

Write $Z = \bigsqcup_m Z_m$ where Z_m is a connected component of Z. It follows that either

$$\begin{tikzcd} Z_m = M_{i_1} \arrow[r, dash, "M_{j_1}"] & M_{i_2} \arrow[r, dash, "M_{j_2}"] & \cdots \arrow[r, dash, "M_{j_{d-1}}"] & M_{i_d} \end{tikzcd} \tag{14.53}$$

or

$$\begin{tikzcd} Z_m = M_{i_1} \arrow[r, dash, "M_{j_1}"] \arrow[rrr, dash, bend right, "M_{j_d}"'] & M_{i_2} \arrow[r, dash, "M_{j_2}"] & \cdots \arrow[r, dash, "M_{j_{d-1}}"] & M_{i_d} \end{tikzcd} \tag{14.54}$$

where $\begin{tikzcd} M_{i_k} \arrow[r, dash, "M_{j_k}"] & M_{i_{k+1}} \end{tikzcd}$ denotes the union $M_{i_k} \cup_\partial M_{j_k} \cup_\partial M_{i_{k+1}}$ so that $M_{i_k} \cap M_{j_k}$ and $M_{j_k} \cap M_{i_{k+1}}$ are the two fibers of $(T^2 \times_{\mathbb{Z}_2} I) \times \partial I \subset \partial M_j$. In the first case, $\partial M_{i_k} \cong \mathbb{R}P^3 \# \mathbb{R}P^3, L(|\omega|, 1)/\mathbb{Z}_2$, or $\mathcal{G}_2$ for $k \notin \{1, d\}$ and $\partial M_{i_1}, \partial M_{i_d} \cong S^2 \times S^1, \mathbb{R}P^3 \# \mathbb{R}P^3, L(|\omega|, 1)/\mathbb{Z}_2$, or $\mathcal{G}_2$. In the second case, $\partial M_{i_k} \cong \mathbb{R}P^3 \# \mathbb{R}P^3, L(|\omega|, 1)/\mathbb{Z}_2$, or $\mathcal{G}_2$ for all k. We construct an F-structure on Z_m as follows.

For each j_k, let $\widetilde{M}_{j_k} \xrightarrow{\pi_{j_k}} M_{j_k}$ be a double covering. Then, $\widetilde{M}_{j_k} \cong (T^2 \times I) \times I$. Let T^2 act on $\widetilde{M}_{j_k}$ by the standard action on the T^2-factor and by the trivial action on the I-factors. By the compatibility of fibers from Lemma 13.5, each $((T^2 \times \partial I), \cdot)$-fiber of $(T^2 \times I) \times I$ is the total space of S^1-fibers from $\left(\begin{tikzcd} S^1 \arrow[r] & M_k \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right)$, for some k.

For each i_k, if $M_{i_k} \cong S^1 \times D^3$ or $S^2 \times D^2$, then let $\widetilde{M}_{i_k} = M_{i_k}$. By the same construction as in the proof of the case $M_j \cong (S^1 \times D^2) \times I$, there is an S^1-action on $\widetilde{M}_{i_k}$ that is compatible with the T^2-action on $\widetilde{M}_{j_k}$. Otherwise, we let $\widetilde{M}_{i_k} \xrightarrow{\pi_{i_k}} M_{i_k}$ be a double covering. Then, $\widetilde{M}_{i_k} \cong S^2 \times_\omega D^2, \omega \in \mathbb{Z}, T^2 \times D^2, (S^2 \times S^1) \times I$, or $\mathcal{G}_2 \times I$.

If $\widetilde{M}_{i_k} \cong S^2 \times_\omega D^2$, or $T^2 \times D^2$, then a similar construction as in the case $M_j \cong (S^1 \times D^2) \times I$ gives an S^1-action on $\widetilde{M}_{i_k}$. Since $S^2 \times S^1$ and $\mathcal{G}_2$ are the total space of S^1-fibers, if $\widetilde{M}_{i_k} \cong (S^2 \times S^1) \times I$ or $\mathcal{G}_2 \times I$, then there is an S^1-action on $\widetilde{M}_{i_k}$ whose orbits coincide with the S^1-fibers on the $S^2 \times S^1$ or $\mathcal{G}_2$-factor. From Lemma 13.38, $\widetilde{M}_{j_k} \cap \widetilde{M}_{i_k} = \partial \widetilde{M}_{j_k} \cap \partial \widetilde{M}_{i_k} \cong T^2 \times I$ where each T^2-fiber is the total space of S^1-fibers of $\partial \widetilde{M}_{i_k}$. Therefore, the S^1-action on $\widetilde{M}_{i_k}$ is compatible with the T^2-action on $\widetilde{M}_{j_k}$ along their overlap.

As a result, Z_m admits an F-structure. By the same arguments as in the case $M_j \cong (S^1 \times D^2) \times I$, the F-structure is compatible with S^1-actions on all $M_j \cong (S^1 \times D^2) \times I$ such that $Z_m \cap M_j \neq \varnothing$.

From all cases, we have that W admits an F-structure whose restriction on ∂W has positive rank. This completes the proof of Lemma 14.46 (2).

14.7 Combining elementary building blocks of type $(2, D^2)$ and $(1, S^1 \times D^2)$

In this section, we show that a connected component of $M - \bigsqcup_\ell \left(\begin{tikzcd} S^1 \arrow[r] & M_\ell \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right)$ that only contains $\left(\begin{tikzcd} D^4, \dots \arrow[r] & M_j \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right)$, $\left(\begin{tikzcd} D^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$, $\left(\begin{tikzcd} D^3, S^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$, and $\left(\begin{tikzcd} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$ components is the union of elementary building blocks of type $(2, D^2)$ and elementary building blocks of type $(1, S^1 \times D^2)$. We call such components building blocks of type $(2, D^2) + (1, S^1 \times D^2)$.

Definition 14.55. We call the manifold W in Lemma 14.57 a *building block of type* $(2, D^2) +$ $(1, S^1 \times D^2)$.

We represent a building block of type $(2, D^2) + (1, S^1 \times D^2)$ by a join of the polyhedrons representing elementary building blocks of type $(2, D^2)$ (see Lemma 14.38) and the graphs representing elementary building blocks of type $(1, S^1 \times D^2)$ (see Lemma 14.46) by identifying some vertices of the polyhedrons with vertices of the graphs.

Example 14.56. The following is a model example of a building block of type $(2, D^2) +$ $(1, S^1 \times D^2)$.

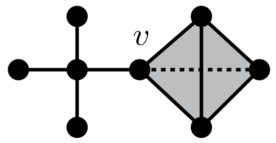
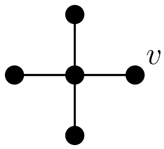


Figure 14.13: A representation of a building block of type $(2, D^2) + (1, S^1 \times D^2)$

In this example, we have an elementary building block of type $(2, D^2)$ represented by a tetrahedron and an elementary building block of type $(1, S^1 \times D^2)$ represented by a graph with four vertices. The two elementary building blocks are joined via a component $\left(\begin{tikzcd} S^1 \times D^2 \arrow[r] & M_j \arrow[d] \\ & (I, \partial I) \end{tikzcd}\right)$. In Lemma 14.57, we show that the resulting manifold W is the plumbing of an elementary building block of type $(2, D^2)$ represented by the tetrahedron and an an elementary building block of type $(1, S^1 \times D^2)$ represented by the graph



where the vertex v represents the component $\left(\begin{array}{r} D^4,\ldots\longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ connecting an elementary building block of type $(2, D^2)$ to an elementary building block of type $(1, S^1\times D^2)$. ∂W is the total space of S^1-fibers. ∂W is identified with a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3,\partial X^3) \end{array}\right)$, for some p, so that S^1-fibers coincide.

From Lemma 13.41, if $M_0 = \left(\begin{array}{r} D^4,\ldots\longrightarrow M_0 \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ only intersects $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2,\partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I,\partial I) \end{array}\right)$, $\left(\begin{array}{r} S^1\times D^2, T^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I,\partial I) \end{array}\right)$, and $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3,\partial X^3) \end{array}\right)$ components, then $M_0 \cong D^4, \pm\mathbb{C}P^2\# D^4, S^2\times_{\pm 2} D^2, S^2\times_{\mathbb{Z}_2} D^2, S^1\times D^3, (\mathbb{R}P^3\times S^1)\widetilde{\times} I, (S^2\widetilde{\times} S^1)\widetilde{\times} I$, or $S^2\times_\omega D^2$, $\omega\in\mathbb{Z}$.

Lemma 14.57. *Let* $\{M_i\}_{i\in\mathcal{A}_0}$ *be a collection of* $\left(\begin{array}{r} D^4,\ldots\longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ *components such that* M_i *only intersects* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2,\partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2\times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I,\partial I) \end{array}\right)$, $\left(\begin{array}{lr} S^1\times D^2, & \\ T^2\times_{\mathbb{Z}_2} I & \longrightarrow M_j \\ & \Big\downarrow \\ & (I,\partial I) \end{array}\right)$, *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3,\partial X^3) \end{array}\right)$ *components (as in Lemma 13.41).*

Let W *be a connected component of* $M - \bigsqcup_p \left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3,\partial X^3) \end{array}\right)$ *such that* W *contains a component* M_i, $i\in\mathcal{A}_0$, *and* W *is disjoint from any* $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2,\partial\Sigma^2) \end{array}\right)$ *or* $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2,\partial\Sigma^2) \end{array}\right)$

components. Then, the following holds.

1. *W can be represented a join of the polyhedrons representing elementary building blocks of type $(2, D^2)$ (see Lemma 14.38) and the graphs representing elementary building blocks of type $(1, S^1 \times D^2)$ (see Lemma 14.46) by identifying some vertices of the polyhedrons with vertices of the graphs. These vertices correspond to M_i, for some $i \in \mathcal{A}_0$.*
2. *W admits an F-structure whose restriction to ∂W has positive rank.*
3. *∂W is the total space of S^1-fibers. As a part of M, ∂W is identified with a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_k \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some k, so that S^1-fibers of ∂W coincide with S^1-fibers of M_k.*

Proof. Here we prove part (1) and part (3) of the lemma. In the next subsection, we give a proof of part (2) of the lemma.

(1). Let $M_i = \left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, for some $i \in \mathcal{A}_0$. From Lemma 13.41, $\partial M_i = B \cup_\partial A \cup_\partial C$ where $A \subset \left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ is the total space of S^1-fibers over a surface Σ^2_A, $B \cong S^1 \times D^2$, $(S^1 \times D^2)\#\mathbb{R}P^2$, or $(S^1 \times D^2)\#(\mathbb{R}P^2\#\mathbb{R}P^2)$, is the union of D^2-subbundles of $\bigsqcup_j \left(\begin{array}{r} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ and fibers of $\bigsqcup_j \left(\begin{array}{r} D^3, S^1 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$, and C is a fiber of $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$. In particular, M_i intersects both an elementary building block of type $(2, D^2)$ and an elementary building block of type $(1, S^1 \times D^2)$.

It follows from the constructions in Lemma 14.38 (1) and Lemma 14.46 (1) that W can be represented a join of the polyhedrons representing elementary building blocks of type $(2, D^2)$ and the graphs representing elementary building blocks of type $(1, S^1 \times D^2)$ by identifying some vertices of the polyhedrons with vertices (of degree one) of the graphs. These vertices correspond to M_i, $i \in \mathcal{A}_0$.

(3). It follows from the compatibility of fibers in Lemma 13.41, the conclusions of Lemma 14.38 and Lemma 14.46, and from part (1), that ∂W is the total space of S^1-fibers. By connectedness, ∂W is identified with exactly one boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_k \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some k, so that their S^1-fibers coincide. □

14.7.1 Proof of Lemma 14.57 (2)

In this section, we prove Lemma 14.57 (2). Without loss of generality, we assume that W contains only one M_i, $i \in \mathcal{A}_0$. Then, $W - M_i$ has exactly two connected components. One connected component corresponds to an elementary building block of type $(2, D^2)$ (see Lemma 14.38) and the other connected component corresponds to an elementary building block of type $(1, S^1 \times D^2)$ (see Lemma 14.46). Denote the connected component of $W - M_i$ corresponding to an elementary building block of type $(2, D^2)$ by $X^{(2,D^2)}$ and the connected component corresponding to an elementary building block of type $(1, S^1 \times D^2)$ by $X^{(1,S^1\times D^2)}$. Hence, W can be represented as in the following figure.

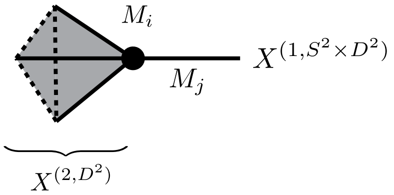


Figure 14.14: A representation of a building block of type $(2, D^2) + (1, S^1 \times D^2)$

We have that

$$W = X^{(2,D^2)} \cup_\partial M_i \xrightarrow{\ M_j\ } X^{(1,S^1\times D^2)} \tag{14.58}$$

where $M_j = \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array} \right)$ and $M_i \xrightarrow{\ M_j\ } X^{(1,S^1\times D^2)}$ denotes $M_i \cup_\partial M_j \cup_\partial X^{(1,S^1\times D^2)}$. From part (1), $\partial M_i = B \cup_\partial A \cup_\partial C$ where B coincides with $X^{(2,D^2)} \cap M_i = \partial X^{(2,D^2)} \cap \partial M_i$ and C coincides with $M_j \cap X^{(1,S^1\times D^2)} = \partial M_j \cap \partial X^{(1,S^1\times D^2)}$.

Case $M_j \cong (S^1 \times D^2) \times I$

From Lemma 14.38, we can assume that $X^{(2,D^2)}$ is a D^2-bundle over a cylinder near $\partial X^{(2,D^2)} \cap \partial M_i$. Hence, we can apply the same plumbing construction as in the proof of Lemma 14.14. Thus,

$$W = X^{(2,D^2)} \cup_\partial M_i \xrightarrow{\ M_j\ } X^{(1,S^1\times D^2)} \cong (X^{(2,D^2)} \cup_\partial D^4) \square (M_i \xrightarrow{\ M_j\ } X^{(1,S^1\times D^2)}) \tag{14.59}$$

where the plumbing locations are contained in D^4 and M_i.

From Lemma 14.38 and Lemma 14.46, $X^{(2,D^2)} \cup_\partial D^4$ is an elementary building block of type $(2, D^2)$ and $M_i \xrightarrow{\ M_j\ } X^{(1,S^1\times D^2)}$ is an elementary building block of type $(1, S^1 \times D^2)$. Additionally, they admit an F-structure whose restriction to a neighborhood of D^4 and M_i is a T-structure. In other words, the normal covers of open neighborhoods of D^4 and M_i

associated with the F-structures are trivial. Since the plumbing locations are contained in D^4 and M_i, Lemma 2.30 implies that W admits an F-structure.

Case $M_j \cong (T^2 \times_{\mathbb{Z}_2} I) \times I$

From Lemma 13.41, $\partial M_i \cong \mathbb{R}P^3 \# \mathbb{R}P^3$ so $M_i \cong S^2 \times_{\mathbb{Z}_2} D^2$. Let U be a neighborhood of $M_j \cup_\partial M_i$ in W so that $V = U \cap X^{(2,D^2)}$ is a D^2-bundle over a cylinder. Let $\widetilde{U}_i \xrightarrow{\pi} U_i$ be a double covering. Then, $\widetilde{U} = \widetilde{M_j} \cup_\partial \widetilde{M_i} \cup_\partial \widetilde{V}$ where $\widetilde{M_j} \cong (T^2 \times I) \times I$ is a double cover of M_j, $\widetilde{M_i} \cong S^2 \times D^2$ is a double cover of M_i, and $\widetilde{V}$ is a double cover of V. $\widetilde{V} = V_1 \sqcup V_2$ where V_1 and V_2 are D^2-bundles over a cylinder and $\pi(V_1) = \pi(V_2) = V$. $V_k \cap \widetilde{M_i} \cong S^1 \times D^2$, $k \in \{1, 2\}$.

By similar constructions as in the proofs of Lemma 14.38 and Lemma 14.46, $\widetilde{U}$ admits a T-structure which is compatible with π, local S^1-actions on $X^{(2,D^2)}$ near $\partial X^{(2,D^2)} \cap \partial M_i$, and the F-structure on $X^{(1,S^1 \times D^2)}$. Therefore, U admits an F-structure which is compatible with the F-structures on $X^{(1,S^1 \times D^2)}$ and $X^{(2,D^2)}$. Hence, W admits an F-structure whose restriction to ∂W has positive rank. This proves Lemma 14.57 (2).

14.8 Combining elementary building blocks of type $(2, S^2)$ and $(2, D^2)$

In this section, we consider a connected component of $M - \bigsqcup_\ell \begin{pmatrix} \begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array} \end{pmatrix}$ that only contains $\begin{pmatrix} \begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array} \end{pmatrix}$, $\begin{pmatrix} \begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array} \end{pmatrix}$, $\begin{pmatrix} \begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array} \end{pmatrix}$, and $\begin{pmatrix} \begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array} \end{pmatrix}$ components.

Definition 14.60. We call the manifold W in Lemma 14.62 a *building block of type* $(2, S^2) + (2, D^2)$.

We represent a building block of type $(2, S^2) + (2, D^2)$ by a union of the polyhedrons representing elementary building blocks of type $(2, D^2)$ (see Lemma 14.38) and the solid polygons representing elementary building blocks of type $(2, S^2)$ (see Lemma 14.3) by identifying some edges.

Example 14.61. The following is a model example of a building block of type $(2, S^2) + (2, D^2)$.

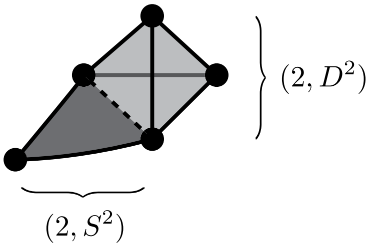


Figure 14.15: A representation of a building block of type $(2, S^2) + (2, D^2)$

In this example, we have an elementary building block of type $(2, D^2)$ represented by a tetrahedron and an elementary building block of type $(2, S^2)$ represented by a triangle. An edge of the triangle and an edge of the tetrahedron are identified. In Lemma 14.62, we show that the resulting manifold Y is the connected sum of an elementary building block of type $(2, D^2)$ represented by the tetrahedron and an an elementary building block of type $(2, S^2)$ represented by the triangle.

∂Y is the total space of ∂D^2-fibers. ∂Y is identified with a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some p, so that S^1-fibers coincide with ∂D^2-fibers of ∂Y.

Lemma 14.62. *Let* $\{M_i\}_{i \in \mathcal{A}_0}$ *be a collection of* $\left(\begin{array}{r} D^4, \dots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ *components such that* M_i *only intersects* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components (as in Lemma 13.50) or* M_i *only intersects* $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, *and* $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components (as in Lemma 13.62).*

Let W *be a connected component of* $M - \bigsqcup_j \left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *such that* W *contains a component* M_i*,* $i \in \mathcal{A}_0$*, and* W *is disjoint from any* $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *component. Then, the following holds.*

(1)

$$W = \left(Y - \bigsqcup_{\ell=1}^{m} B^4\right) \cup \left(Z - \bigsqcup_{\ell=1}^{m} B^4\right). \tag{14.63}$$

where $Y = \bigsqcup_j Y_j$ *so that each connected component* Y_j *is an elementary building block of type* $(2, S^2)$ *and* $Z = \bigsqcup_j Z_j$ *so that each connected component* Z_j *is an elementary building block of type* $(2, D^2)$*. The union is by identifying* S^3*-boundary components of* $\bigsqcup_{\ell=1}^{m} B^4$*.*

W *can be represented by a union of the polyhedrons representing elementary building blocks of type* $(2, D^2)$ *(see Lemma 14.38) and the solid polygons representing elementary building blocks of type* $(2, S^2)$ *(see Lemma 14.3) by identifying some edges.*

(2) $\partial W = \partial Z$*. As a part of* M*, each connected component of* ∂W *is identified with a boundary component of* $\left(\begin{array}{r} S^1 \longrightarrow M_k \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$*, for some* k*.*

(3) W *admits an* F*-structure whose restriction to* ∂W *has positive rank.*

Proof. (1). Consider a component M_i for some $i \in \mathcal{A}_0$. From the decomposition of ∂M_i in Lemma 13.50 and Lemma 13.62, there exists a collection $\{E_j\}_{j=1}^{n_i}$ of S^2-subbundles of $\left(\begin{array}{r} S^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ so that $E_j \cong S^2 \times I$ and $E_j \cap \partial M_i \neq \varnothing$. We consider E_j as $S^2 \times [0, 3]$ and denote the subbundle $S^2 \times [0, 1] \subset E_j$ by $E_j^{(1)}$, the subbundle $S^2 \times [1, 2]$ by $E_j^{(2)}$, and the subbundle $S^2 \times [2, 3]$ by $E_j^{(3)}$. Then, $E_j \cap \partial M_i$ coincides with $E_j^{(1)}$. Additionally, we denote the two boundary components $S^2 \times \{0\}$ and $S^2 \times \{1\}$ of $E_j^{(1)}$ by $V_{j,1}$ and $V_{j,2}$ respectively. From Lemma 13.50 and Lemma 13.62, one of the following holds.

(i) $V_{j,1}$ coincides with a boundary component of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow N_k \\ \downarrow \\ \partial I \end{array}\right)$, for some k. $V_{j,2} =$

$A \cup_\partial (B_1 \sqcup B_2)$ where $B_s \cong D^2$, $s \in \{1, 2\}$, is a fiber of $\left(\begin{array}{r} D^2 \longrightarrow N_{j_s} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some j_s, and A is a subbundle of $\left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some ℓ.

(ii) Both $V_{j,1}$ and $V_{j,2}$ can be decomposed into $A \cup_\partial (B_1 \sqcup B_2)$ as in (i).

From Lemma 13.2 and by connectedness, for each $k \in \{1, 2\}$, there is a subbundle $F_{j_k} \cong D^2 \times I$ of $\left(\begin{array}{r} D^2 \longrightarrow N_{j_k} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$ so that each D^2-fiber of F_{j_k} is contained in an S^2-fiber of $E_j^{(2)} = S^2 \times [1, 2]$. D^2-fibers of $F_{j_1} \sqcup F_{j_2}$ on $E_j^{(1)} \cap E_j^{(2)} \cong S^2$ coincide with D^2-fibers $B_1 \sqcup B_2$ on $V_{j,2}$. $E_j^{(2)} - (F_{j_1} \sqcup F_{j_2}) \cong (S^1 \times I) \times I$ is contained in $\left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some ℓ, so that S^1-fibers coincide. Moreover, from Lemma 13.50 and by connectedness, there exists $M_{i'}$, $i' \in \mathcal{A}_0$, so that $M_{i'} \cap E_j = E_j^{(3)}$. Similarly, $E_j^{(2)} \cap E_j^{(3)} \cong S^2$ can be decomposed into $A' \cup_\partial (B'_1 \sqcup B'_2)$ in the same way as $V_{j,2} = E_j^{(1)} \cap E_j^{(2)}$.

As a result, $E_j^{(2)}$ has the same decomposition as the boundary of the component $\left(\begin{array}{r} D^3 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ represented by an edge of a polyhedron in the construction of an elementary building block of type $(2, D^2)$ in Lemma 14.38.

Therefore, the construction of an elementary building block of type $(2, D^2)$ in Lemma 14.38 is still valid when we replace an occurrence of $M_j \cong D^3 \times I$ by $E_j \cong S^2 \times I$. This process is equivalent (up to diffeomorphism) to removing B^4 from the interior of an elementary building block of type $(2, D^2)$, removing B^4 from the interior of an elementary building block of type $(2, S^2)$, then identifying their S^3-boundaries. Hence,

$$W = \left(Y - \bigsqcup_{\ell=1}^{m} B^4 \right) \cup \left(Z - \bigsqcup_{\ell=1}^{m} B^4 \right) \tag{14.64}$$

where $Y = \bigsqcup_j Y_j$ so that each connected component Y_j is an elementary building block of type $(2, S^2)$ and $Z = \bigsqcup_j Z_j$ so that each connected component Z_j is an elementary building block of type $(2, D^2)$. The union is by identifying S^3-boundary components of $\bigsqcup_{\ell=1}^m B^4$.

(2). From Lemma 14.3, Y is a closed manifold. Hence, $\partial W = \partial Z$. Part (2) of the lemma then follows directly from Lemma 14.38.

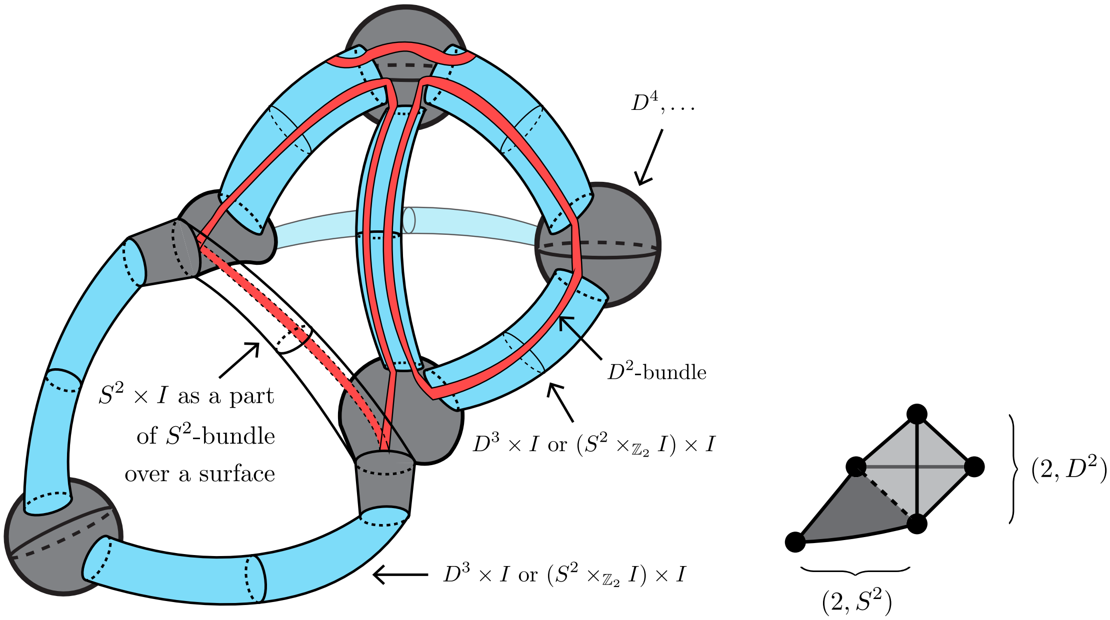


Figure 14.16: Example of the configuration of a building block of type $(2, S^2) + (2, D^2)$ and its representation. Each D^2-bundle over a circle is identified with $\begin{pmatrix} D^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{pmatrix}$.

(3). From Lemma 14.38, every Z_j admits an F-structure and there exists a component $\begin{pmatrix} D^2 \longrightarrow M_k \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{pmatrix}$ such that $M_k \subset Z_j$. The F-structure on Z_j restricts to local S^1-actions on M_k. From Lemma 14.9, every Y_j admits an F-structure with at least one open set with a trivial normal covering (in the sense of Definition 2.26).

Paternain and Petean [21, Theorem 5.9] showed that the connected sum of two manifolds with F-structure admits an F-structure, provided that the manifolds have at least one open set with a trivial normal covering (in the sense of Definition 2.26). The new F-structure is constructed by finding appropriate S^1-actions in a neighborhood of connected sum locations. Hence, [21, Theorem 5.9] also applies to (14.64). Therefore, W admits an F-structure. □

The constructions in the proof of Lemma 14.62 only concern components $\begin{pmatrix} D^3 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{pmatrix}$ as a part of an elementary building block of type $(2, D^2)$. Therefore, we get the following corollary.

Corollary 14.65. *The conclusion of Lemma 14.62 is still valid when an occurrence of an elementary building block Z_j of type $(2, D^2)$ is replaced by a building block of type $(2, D^2) + (1, S^1 \times D^2)$.*

Proof. The proof of the lemma is similar to the proof of Lemma 14.62. Lemma 14.57 is used in the proof instead of Lemma 14.38 and Lemma 13.62 is used instead of Lemma 13.50. ☐

Definition 14.66. We call the resulting manifold in Corollary 14.65 a *building block of type* $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$.

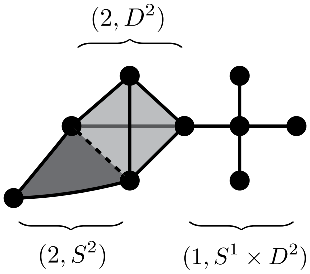


Figure 14.17: Example of a representation of a building block of type $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$

14.9 Connected components of $M - \bigsqcup_\ell \left(\begin{array}{ccc} S^1 & \longrightarrow & M_\ell \\ & & \downarrow \\ & & (X^3, \partial X^3) \end{array}\right)$ that contains $\left(\begin{array}{ccc} T^2 & \longrightarrow & M_i \\ & & \downarrow \\ & & (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ components

In this section, we consider connected components of $M - \bigsqcup_\ell \left(\begin{array}{ccc} S^1 & \longrightarrow & M_\ell \\ & & \downarrow \\ & & (X^3, \partial X^3) \end{array}\right)$ from the previous sections that also contain $\left(\begin{array}{ccc} T^2 & \longrightarrow & M_i \\ & & \downarrow \\ & & (\Sigma^2, \partial \Sigma^2) \end{array}\right)$ components. There are two possibilities which will be described in Lemma 14.67 and Lemma 14.75.

14.9.1 Combining an elementary building block of type $(2, T^2)$ with other building blocks

Lemma 14.67 shows that we can combine an elementary building blocks of type $(2, T^2)$ with an elementary building block of type $(2, D^2)$, an elementary building block of type

$(1, S^1 \times D^2)$, a building block of type $(2, S^2) + (1, D^2)$, or a building block of type $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$.

Example 14.67. The following example combines an elementary building blocks of type $(2, T^2)$ with an elementary building block of type $(1, S^1 \times D^2)$.

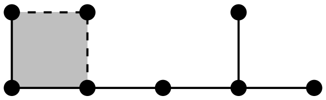


Figure 14.18: A representation of a building block of type $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$

The rectangle represents an elementary building block of type $(2, T^2)$. Dashed edges represent removing $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components in the construction of an elementary building block of type $(2, T^2)$ (as in Lemma 14.14). The rest of the figure is a graph which represents an elementary building block of type $(1, S^1 \times D^2)$ (as in Lemma 14.46).

In Lemma 14.68, we show that the resulting manifold Y admits an F-structure. ∂Y is a graph manifold and the F-structure restricts to a T-structure with positive rank on ∂Y. ∂Y is identified with a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_p \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some p.

Lemma 14.68. *Let* M_2 *be a component* $\left(\begin{array}{r} T^2 \longrightarrow M_2 \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$. *Suppose that there exists a boundary component* N_2' *of* M_2 *such that* N_2' *only intersects* $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, *and* $\left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components, i.e.*

$$N_2' \subset \left[\bigsqcup_i \left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right) \right] \cup \left[\bigsqcup_j \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right) \right] \cup \left[\bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right) \right]. \tag{14.69}$$

Let $\{M_i\}_{i\in\mathcal{A}_0}$ *be a collection of* $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ *components such that* $M_i \cap N_2' \neq \varnothing$*. Also assume that* $\partial M_i \cap \bigsqcup_p \left(\begin{array}{r} T^2 \longrightarrow M_p \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *is connected. That is* $\partial M_i \cap \bigsqcup_p \left(\begin{array}{r} T^2 \longrightarrow M_p \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ $= \partial M_i \cap N_2' \cong T^2 \times I$*.*

Let $\{M_j\}_{j\in\mathcal{A}_1}$ *be a collection of* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ *components such that* $M_j \cap N_2' \neq \varnothing$*.*

Let $W = \left(\bigsqcup_{i\in\mathcal{A}_0} M_i\right) \cup \left(\bigsqcup_{j\in\mathcal{A}_1} M_j\right)$*. Then, the following holds.*

(1) *We can represent* W *together with its attaching data to* N_2' *by a cycle graph* $\mathcal{C}$ *as follows. Each vertex* v_i *represents a component* M_i*,* $i \in \mathcal{A}_0$*. There are two types of edges: solid edges and dashed edges. A solid edge represents a component* M_j*,* $j \in \mathcal{A}_1$*, and a dashed edge represents a connected component of* $N_2' - \partial W$ *diffeomorphic to* $(T^2 \times I)$*.*

We can consider W *as an elementary building block of type* $(2, T^2)$ *but with the interior of some* $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ *components removed. The cycle graph* $\mathcal{C}$ *is obtained by labeling some edges of a cycle graph in Lemma 14.14 as dashed edges.*

(2) *Let* v_i *be a vertex representing a component* M_i*,* $i \in \mathcal{A}_0$*. If* v_i *is incident to exactly one dashed edge, then either*

a) there exists a component $M_{j'} = \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j'} \\ \downarrow \\ (I, \partial I) \end{array}\right)$*, for some* j'*, so that* $M_{j'}$ *is not contained in* W *and* $M_{j'} \cap M_i \neq \varnothing$ *(*M_i *is as in in Lemma 13.64), or*

b) $\partial M_i = A \cup_\partial B \cup_\partial C$ *where* A *is the total space of* S^1*-fibers over a surface with one boundary component,* $B \cong T^2 \times I$ *is a subbundle of* N_2'*, and* $C \cong S^1 \times D^2$ *or* $T^2 \times_{\mathbb{Z}_2} I$ *is a fiber of a component* $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow N_j \\ \downarrow \\ \partial I \end{array}\right)$*, for some* j *(*M_i *is as in Lemma 13.67).*

If v_i is incident to two dashed edges, then either

a) there exists a component $M_{j'} = \begin{pmatrix} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j'} \\ \hfill\downarrow\ \ \\ \hfill (I, \partial I) \end{pmatrix}$*, for some j', so that $M_{j'}$ is not contained in W and $M_{j'} \cap M_i \neq \varnothing$ (M_i as in Lemma 13.64), or*

b) $\partial M_i = A \cup_\partial B$ where A is the total space of S^1-fibers over a surface (not necessary connected) with two boundary components and $B \cong T^2 \times I$ is a subbundle of N_2' (M_i is as in Lemma 13.67).

(3) *Let Z be a connected component of* $M - \bigsqcup_\ell \begin{pmatrix} S^1 \longrightarrow M_\ell \\ \hfill\downarrow\ \ \\ \hfill (X^3, \partial X^3) \end{pmatrix}$ *such that $M_2 \subset Z$.*

Assume that Z contains exactly one $\begin{pmatrix} T^2 \longrightarrow M_p \\ \hfill\downarrow\ \ \\ \hfill (\Sigma^2, \partial\Sigma^2) \end{pmatrix}$ *component. Let Y be a connected component of $Z - M_2$ such that $W \subset Y$.*

Suppose that there exists exactly one component $M_{j'} \cong (S^1 \times D^2) \times I$ or $(T^2 \times_{\mathbb{Z}_2} I) \times I$ such that $M_{j'}$ is not contained in W and $M_{j'} \cap M_i \neq \varnothing$, for some $i \in \mathcal{A}_0$, as in (2). Then,

$$Y = \ W \ \overset{M_{j'}}{\rule{2em}{0.4pt}}\ X = W \cup_\partial M_{j'} \cup_\partial X \tag{14.70}$$

where X is a 4-manifold with boundary which admits an F-structure, $M_{j'} \cap W = \partial M_{j'} \cap \partial M_i$, $M_{i'} \subset X$, and $M_{j'} \cap X = \partial M_{j'} \cap \partial M_{i'}$. (In particular, X is an elementary building block of type $(2, D^2)$, an elementary building block of type $(1, S^1 \times D^2)$, a building block of type $(2, S^2) + (2, D^2)$, a building block of type $(2, D^2) + (1, S^1 \times D^2)$, or a building block of type $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$)

W admits an F-structure that is compatible with local T^2-actions on M_2 and the F-structure on $M_{j'} \cup X$.

(4) *In general, when there are more than one* $\begin{pmatrix} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j'} \\ \hfill\downarrow\ \ \\ \hfill (I, \partial I) \end{pmatrix}$ *components as in (2), Y admits an F-structure that is compatible with local T^2-actions on* $\bigsqcup_\ell \begin{pmatrix} T^2 \longrightarrow M_\ell \\ \hfill\downarrow\ \ \\ \hfill (\Sigma^2, \partial\Sigma^2) \end{pmatrix}$ *and with local S^1-actions on* $\bigsqcup_\ell \begin{pmatrix} S^1 \longrightarrow M_\ell \\ \hfill\downarrow\ \ \\ \hfill (X^3, \partial X^3) \end{pmatrix}$*. ∂Y is a graph manifold and the F-structure on Y restricts to a T-structure with positive rank on ∂Y.*

Proof. (1). We construct a graph $\mathcal{C}$ to represent W together with its attaching data to N_2' as follows. We construct a vertex v_i for each component M_i, $i \in \mathcal{A}_0$. Vertices v_{i_1} and v_{i_2} are adjacent via a solid edge e_j if and only if there is a component M_j, $j \in \mathcal{A}_1$, so that $M_{i_1} \cap M_j \neq \varnothing$ and $M_{i_2} \cap M_j \neq \varnothing$. From Lemma 13.3, $N_2' - \partial W = \bigsqcup_j L_j$ where $L_j \cong T^2 \times I$ is a subbundle of N_2'. We construct a dashed edge e_j^D connecting vertices v_{i_1} and v_{i_2} if and only there is L_j such that $L_j \cap M_{i_1} \neq \varnothing$ and $L_j \cap M_{i_2} \neq \varnothing$.

The decomposition of M_i, $i \in \mathcal{A}_0$, such that v_i is incident only to solid edges is given in Lemma 13.17. The decomposition of M_i such that v_i is incident to both solid edges and dashed edges is given in Lemma 13.64. The decomposition of M_i such that v_i is incident only to dashed edges is given in Lemma 13.67. It follows from the lemmas that $\deg(v_i) = 2$ for all i. Therefore, $\mathcal{C}$ is a cycle graph.

By a similar construction as in the proof of Lemma 14.14, the cycle graph $\mathcal{C}$ can be obtained by labeling some edges of a cycle graph in Lemma 14.14 as dashed edges. In other words, we can consider W as an elementary building block of type $(2, T^2)$ with the interior of some $\left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components removed.

(2). This follows directly from the decomposition of M_i in Lemma 13.64 and Lemma 13.67.

(3). Let Z be a connected component of $M - \bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ such that $W \cup M_2 \subset Z$. Assume that Z contains exactly one $\left(\begin{array}{r} T^2 \longrightarrow M_p \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ component, i.e. $Z \cap \bigsqcup_p \left(\begin{array}{r} T^2 \longrightarrow M_p \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right) = Z \cap M_2$.

Put $Y = Z - M_2$. Suppose that there exists exactly one component $M_{j'} = \left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j'} \\ \downarrow \\ (I, \partial I) \end{array}\right)$ such that $M_{j'}$ is not contained in W and $M_{j'} \cap M_i \neq \varnothing$, for some $i \in \mathcal{A}_0$, as in (2). Put $X = Y - (W \cup M_{j'})$. Then, $M_{i'} \subset X$ and

$$Y = \ W \overset{M_{j'}}{\overline{\qquad\quad}} X = W \cup_\partial M_{j'} \cup_\partial X \tag{14.71}$$

where $M_{j'} \cap W = \partial M_{j'} \cap \partial M_i$, $M_{i'} \subset X$, and $M_{j'} \cap X = \partial M_{j'} \cap \partial M_{i'}$.

From Lemma 13.5, the boundary of each $(S^1 \times D^2)$ or $(T^2 \times_{\mathbb{Z}_2} I)$-fiber of $M_{j'}$ is the total space of S^1-fibers from $\left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some ℓ. By connectedness, $\partial M_{i'} \cap \left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$ $\neq \varnothing$. The decompositions of $\partial M_{i'}$ in these cases are given in Lemma 13.38, Lemma 13.41, Lemma 13.62, Lemma 13.64, and Lemma 13.67. It follows that X is an elementary building block of type $(2, D^2)$, an elementary building block of type $(1, S^1 \times D^2)$, a building block of type $(2, S^2) + (1, D^2)$, a building block of type $(2, D^2) + (1, S^1 \times D^2)$, or a building block of type $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$.

Let W_p be a connected component of W. By the same argument as in the proof of Lemma 14.14,

$$W_p = \; M_{i_1} \overset{M_{j_1}}{\rule[0.5ex]{3em}{0.4pt}} M_{i_2} \rule[0.5ex]{3em}{0.4pt} \cdots \rule[0.5ex]{3em}{0.4pt} M_{i_m} \tag{14.72}$$

where $M_{i_k} \overset{M_{j_k}}{\rule[0.5ex]{3em}{0.4pt}} M_{i_{k+1}}$ denotes $M_{i_k} \cup_\partial M_{j_k} \cup_\partial M_{i_{k+1}}$ so that $M_{i_k} \cap M_{i_{k+1}} = \varnothing$. For all $k \in \{2, 3, \ldots, m-1\}$,

$$\partial M_{i_k} \cong \left\{ \begin{array}{c} S^1 \times D^2 \\ T^2 \times_{\mathbb{Z}_2} I \end{array} \right\} \cup_\partial (T^2 \times I) \cup_\partial \left\{ \begin{array}{c} S^1 \times D^2 \\ T^2 \times_{\mathbb{Z}_2} I \end{array} \right\}. \tag{14.73}$$

In particular, the decomposition of ∂M_{i_k} is the same as the decomposition of $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ in Lemma 14.14. Hence, the construction of an F-structure in the proof of Lemma 14.14 also applies to this case. We have that $W_p - M_{i_1} - M_{i_m}$ admits an F-structure whose restriction to $\partial W_p - M_{i_1} - M_{i_m}$ is local T^2-actions. Additionally, the F-structure restricts to T^2-actions on M_{j_1} and $M_{j_{m-1}}$ (or their double covers).

Without loss of generality, we assume that $M_{i_1} \cap M_{j'} \neq \varnothing$. Then, $\partial M_{i_m} \cap M_{j'} = \varnothing$. From Lemma 13.64, $\partial M_{i_m} = A \cup_\partial B \cup_\partial C$ where A is the total space of S^1-fibers over a surface with one boundary component, $B \cong T^2 \times I$ is a subbundle of N_2', and $C \cong S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I$. ∂M_{i_1} and ∂M_{i_m} are Seifert manifolds.

If $M_{i_m} \cong D^4, S^2 \times_\omega D^2, \omega \in \mathbb{Z}$, or $S^1 \times D^3$, then $M_{j_{m-1}} \cong (S^1 \times D^2) \times I$. By a similar argument as in the proof of Lemma 14.14, there is an S^1-action on M_{i_m} so that the orbits are compatible with the orbits of the T^2-action on $M_{j_{m-1}}$. The S^1-action on M_{i_m} and the T^2-action on $M_{j_{m-1}}$ generate a T^3-action of degree one or two on a neighborhood of $M_{i_m} \cap M_{j_{m-1}}$. By passing to a quotient, we get an effective T^2 or S^1-action on a neighborhood of $M_{i_m} \cap M_{j_{m-1}}$ which is compatible with the S^1-action on M_{i_m} and the T^2-action on $M_{j_{m-1}}$.

If $M_{i_m} \cong (S^2 \times_\omega D^2)/\mathbb{Z}_2, \omega \in \mathbb{Z}$, $(\mathbb{R}P^3 \times S^1) \widetilde{\times} I$, or $(S^2 \widetilde{\times} S^1) \widetilde{\times} I$, then $M_{j_{m-1}} \cong S^1 \times D^2$ or $T^2 \times_{\mathbb{Z}_2} I$. If $M_{j_{m-1}} \cong S^1 \times D^2$ then we let $\widetilde{M}_{j_{m-1}} = M_{j_{m-1}}$. If $M_{j_{m-1}} \cong (T^2 \times_{\mathbb{Z}_2} I) \times I$, then we let $\widetilde{M}_{j_{m-1}} \cong (T^2 \times I) \times I$ be the double cover of $M_{j_{m-1}}$. Let $\widetilde{M}_{i_m}$ be a double cover of M_{i_m}. By a similar argument as in the proof of Lemma 14.14 and in above cases, there is an effective T^2 or S^1-action on a neighborhood of $\widetilde{M}_{i_m} \cap \widetilde{M}_{j_{m-1}}$ which is compatible with the S^1-action on $\widetilde{M}_{i_m}$ and the T^2-action on $\widetilde{M}_{j_{m-1}}$.

If $M_{i_m} \cong T^2 \times D^2$, then by a similar argument as in the proof of Lemma 14.14, $M_{j_{m-1}} \cup_\partial M_{i_m}$ admits a T-structure which restricts to T^2-actions on M_{i_m} and $M_{j_{m-1}}$. If $M_{i_m} \cong T^2 \times_{\mathbb{Z}_2} D^2$ or $\mathcal{B}_k \widetilde{\times} I$, then the same argument applies on its double covers.

From all cases, we can extend the F-structure on $W_p - M_{i_1} - M_{i_m}$ to M_{i_m}. Similarly, we can extend the F-structure on $W_p - M_{i_1} - M_{i_m}$ to M_{i_1}. Therefore, W admits an F-structure that is compatible with local T^2-actions on M_2.

(4). From the proofs of Lemma 14.38, Lemma 14.46, Lemma 14.57, Lemma 14.62, Lemma 14.68, and the proof of part (3), the F-structures on X and W are compatible with the S^1 or T^2-action on $M_{j'}$ (or its double cover). Therefore, $Y = W \cup_\partial M_{j'} \cup_\partial X$ admits an F-structure. It follows from part (3) that ∂Y is a graph manifold and the F-structure restricts to a T-structure with positive rank on ∂Y. The F-structure is also compatible with local T^2-actions on $\bigsqcup_\ell \left(\begin{array}{ccc} T^2 & \longrightarrow & M_\ell \\ & & \downarrow \\ & & (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ and local S^1-actions on $\bigsqcup_\ell \left(\begin{array}{ccc} S^1 & \longrightarrow & M_\ell \\ & & \downarrow \\ & & (X^3, \partial X^3) \end{array}\right)$.

The above construction is done on a neighborhood of $M_{i_1} \cap M_{j'}$. Thus, the same construction applies for the general case when there are more than one $\left(\begin{array}{ccc} \begin{array}{l} S^1 \times D^2, \\ T^2 \times_{\mathbb{Z}_2} I \end{array} & \longrightarrow & M_{j'} \\ & & \Big\downarrow \\ & & (I, \partial I) \end{array}\right)$ components such that $M_{j'} \cap M_i \neq \varnothing$, for some $i \in \mathcal{A}_0$, and $M_{j'} \cap N_2' = \varnothing$. Hence, Y admits an F-structure. ∂Y is a graph manifold and the F-structure restricts to a T-structure with positive rank on ∂Y. Moreover, the F-structure is compatible with local T^2-actions on $\bigsqcup_\ell \left(\begin{array}{ccc} T^2 & \longrightarrow & M_\ell \\ & & \downarrow \\ & & (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ and local S^1-actions on $\bigsqcup_\ell \left(\begin{array}{ccc} S^1 & \longrightarrow & M_\ell \\ & & \downarrow \\ & & (X^3, \partial X^3) \end{array}\right)$. ☐

14.9.2 Attaching $\left(\begin{tikzcd} T^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$ to any $\left(\begin{tikzcd} D^4, \dots \arrow[r] & M_i \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right)$ component

The next lemma shows that we can attach a component $\left(\begin{tikzcd} T^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$ to any $\left(\begin{tikzcd} D^4, \dots \arrow[r] & M_i \arrow[d] \\ & \mathrm{pt} \end{tikzcd}\right)$ component such that $M_i \cap \left(\begin{tikzcd} S^1 \arrow[r] & M_\ell \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right) \neq \varnothing$, for some ℓ.

Example 14.74. The following example demonstrates attaching a component $\left(\begin{tikzcd} T^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$ to elementary building blocks of type $(2, D^2)$ and $(1, S^1 \times D^2)$.

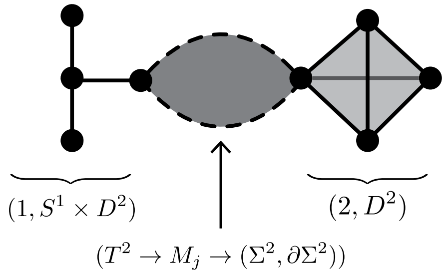


Figure 14.19: Attaching a component $\left(\begin{tikzcd} T^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$ to elementary building blocks of type $(2, D^2)$ and $(1, S^1 \times D^2)$

In this figure, the tetrahedron represents an elementary building block of type $(2, D^2)$ and the graph represents an elementary building block of type $(1, S^1 \times D^2)$. The region bounded by dashed curves represents a $\left(\begin{tikzcd} T^2 \arrow[r] & M_j \arrow[d] \\ & (\Sigma^2, \partial\Sigma^2) \end{tikzcd}\right)$ component. In Lemma 14.75, we show that the resulting manifold Y admits an F-structure. ∂Y is identified with a boundary component of $\left(\begin{tikzcd} S^1 \arrow[r] & M_p \arrow[d] \\ & (X^3, \partial X^3) \end{tikzcd}\right)$, for some p.

Lemma 14.75. *Let M_2 be a component* $\left(\begin{array}{r} T^2 \longrightarrow M_2 \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$. *Suppose that there exists a boundary component N_2' of M_2 such that N_2' only intersects* $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ *and* $\left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *components, i.e.*

$$N_2' \subset \left[\bigsqcup_i \left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)\right] \cup \left[\bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)\right]. \tag{14.76}$$

Let U_2 be a neighborhood of N_2' in M_2 so that $U_2 \cong N_2' \times [0, \epsilon)$, for some $\epsilon > 0$. U_2 is a T^2-subbundle of M_2.

Let $\{M_i\}_{i \in \mathcal{A}_0}$ be a collection of $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_i \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ *components such that $M_i \cap N_2' \neq \varnothing$. Also assume that* $\partial M_i \cap \bigsqcup_p \left(\begin{array}{r} T^2 \longrightarrow M_p \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *is connected. That is* $\partial M_i \cap \bigsqcup_p \left(\begin{array}{r} T^2 \longrightarrow M_p \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ $= \partial M_i \cap N_2' \cong T^2 \times I$.

Let Z be a connected component of $M - \bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ *such that Z contains M_2. Assume that there Z contains exactly one* $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$ *component.*

Let $\{Y_j\}_{j \in \mathcal{J}}$ be a collection of connected components of $Z - M_2$ such that $Y_j \cap N_2' \neq \varnothing$. Then, each Y_j, $j \in \mathcal{J}$, admits an F-structure which is compatible with local T^2-actions on M_2 and local S^1-actions on $\bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$. *In particular, Y_j is an elementary building block of type $(2, D^2)$, an elementary building block of type $(1, S^1 \times D^2)$, a building block of type $(2, D^2) + (1, S^1 \times D^2)$, a building block of type $(2, S^2) + (2, D^2)$, a building block of type $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$, or the manifold Y in the conclusion of Lemma 14.68.*

Let X be the connected component of $\partial\left(U_2 \cup \bigsqcup_{j\in\mathcal{J}} Y_j\right)$ such that $X \cap N_2' \neq \varnothing$. As a part of M, X attaches to a component $\left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some ℓ. Each T^2-fiber of $X \cap N_2'$ is the total space of S^1-fibers from N_ℓ. $U_2 \cup \bigsqcup_{j\in\mathcal{J}} Y_j$ admits an F-structure which is compatible with local T^2-actions on M_2 and local S^1-actions on M_ℓ.

Proof. It follows from Lemma 13.67 that every argument in the proofs of Lemma 14.38, Lemma 14.46, Lemma 14.57, Lemma 14.62, and Lemma 14.68 that involves $\left(\begin{array}{r} D^4 \longrightarrow M_q \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ components such that $M_j \cap \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right) \neq \varnothing$, for some ℓ, is still valid when an occurrence of M_q is replaced by M_i, for some $i \in \mathcal{A}_0$. Therefore, Y_j admits an F-structure which is compatible with local S^1-actions on $\bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$. By similar arguments as in the proof of Lemma 14.62, the F-structure is also compatible with local T^2-actions on M_2. In particular, each Y_j is an elementary building block of type $(2, D^2)$, an elementary building block of type $(1, S^1 \times D^2)$, a building block of type $(2, D^2) + (1, S^1 \times D^2)$, a building block of type $(2, S^2) + (2, D^2)$, a building block of type $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$, or the manifold Y in the conclusion of Lemma 14.68.

For each $i \in \mathcal{A}_0$, put $S_i = N_2' \cap \partial M_i \cong T^2 \times I$. From Lemma 13.67, we have that

$$\begin{tikzcd} N_2' = S_1 \arrow[r, dash, "E_1"] \arrow[rrr, dash, bend right, "E_m"'] & S_2 \arrow[r, dash, "E_2"] & \cdots \arrow[r, dash, "E_{m-1}"] & S_m \end{tikzcd} \tag{14.77}$$

where $E_i \cong T^2 \times I$ is a connected component of $N_2' - \bigsqcup_i M_i$ and $S_i \xrightarrow{E_i} S_{i+1}$ represents $S_i \cup_\partial E_i \cup_\partial S_{i+1} \cong (T^2 \times I) \cup_\partial (T^2 \times I) \cup_\partial (T^2 \times I) \cong T^2 \times I$. By Lemma 13.3, each T^2-fiber of E_i is the total space of S^1-fibers from $\left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$. By Lemma 13.67, $E_i \cap \partial M_i$ is the total space of S^1-fibers which coincide to both S^1-fibers of E_i and ∂M_i.

Let X be a connected component of $\partial\left(U_2 \cup \bigsqcup_{j\in\mathcal{J}} Y_j\right)$ such that $X \cap N_2' \neq \varnothing$. For simplicity, we first assume that each Y_j, $j \in \mathcal{J}$, contains exactly one M_i, $i \in \mathcal{A}_0$. Then,

$$\begin{tikzcd} X = \partial Y_1 \arrow[r, dash, "E_1"] \arrow[rrr, dash, bend right, "E_m"'] & \partial Y_2 \arrow[r, dash, "E_2"] & \cdots \arrow[r, dash] & \partial Y_m \end{tikzcd} \tag{14.78}$$

where the two boundary components of $\partial Y_i - S_i$ are identified with a boundary component of E_i and E_{i+1} respectively. From the compatibility of S^1-fibers and from Lemma 14.38, Lemma 14.46, Lemma 14.57, Lemma 14.62, and Lemma 14.68, X is the total space of S^1-fibers. By connectedness, X attaches to a boundary component of $\left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some ℓ so that S^1-fibers coincide. If there exists Y_j, $j \in \mathcal{J}$, such that Y_j contains M_{i_1} and M_{i_2}, $i_1, i_2 \in \mathcal{A}_0$, then the same argument applies.

Let T^2 act freely on U_2 so that the orbits coincide with T^2-fibers of U_2. From Lemma 14.38, Lemma 14.46, Lemma 14.57, Lemma 14.62, and Lemma 14.68, each Y_j, $j \in \mathcal{J}$, admits an F-structure whose restriction to M_i, $i \in \mathcal{A}_0$, such that $M_i \subset Y_j$ and $M_i \cap N_2' \neq \varnothing$, is compatible with T^2-fibers on $N_2' \cap \partial M_i$ and S^1-fibers on $\partial M_i \cap \left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$. Hence, $U_2 \cup \bigsqcup_j Y_j$ admits an F-structure which is compatible with local T^2-actions on M_2 and local S^1-actions on M_ℓ. $\square$

— 15 —

Proof of Theorem 1.4

In this chapter, we finish the proof of Theorem 1.4. We describe M in terms of a configuration of building blocks and fiber bundle components then show that M admits an F-structure or a metric of nonnegative sectional curvature.

15.1 M contains a fiber bundle component without boundary

From Section 13.1, if M contains a fiber bundle component without boundary, then M is a closed Riemannian 4-manifold which admits a metric of nonnegative sectional curvature or M is $\left(\begin{array}{r} S^1 \longrightarrow M \\ \downarrow \\ X^3 \end{array}\right)$, $\left(\begin{array}{r} T^2 \longrightarrow M \\ \downarrow \\ \Sigma^2 \end{array}\right)$, $\left(\begin{array}{r} S^2 \longrightarrow M \\ \downarrow \\ \Sigma^2 \end{array}\right)$, or $\left(\begin{array}{r} S^3/\Gamma, T^3/\Gamma, S^2 \times S^1, \mathbb{R}P^3\#\mathbb{R}P^3 \longrightarrow M \\ \downarrow \\ S^1 \end{array}\right)$. In the later case, M admits local S^1 or T^2-actions. In particular, M admits an F-structure.

In the following sections, we assume that M does not contain a fiber bundle component without boundary.

15.2 M does not contain a component with 2 or 3-dimensional base

First, we assume that M does not contain a component with 2 or 3-dimensional base. From Section 2.5, $M = M_{i_1} \cup_\partial M_1 \cup_\partial M_{i_2} \cong M_{i_1} \cup_\partial M_{i_2}$ where M_{i_k}, $k \in \{1,2\}$, is a component $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_{i_k} \\ \downarrow \\ \text{pt} \end{array}\right)$, and M_1 is a component $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_1 \\ \downarrow \\ (I, \partial I) \end{array}\right)$. The classification of M_{i_k}, $k \in \{1,2\}$, is given in Table 13.1. From Section 2.5, M_{i_k} (or its double cover $\widetilde{M_{i_k}}$) admits an S^1 or T^2-action whose restriction to ∂M_{i_k} (or $\partial\widetilde{M_{i_k}}$) is free.

Consider M_1 as $F \times [1,2]$ where $F \cong S^3/\Gamma, T^3/\Gamma, S^2 \times S^1$, or $\mathbb{R}P^3\#\mathbb{R}P^3$. We must have that $F \cong \partial M_{i_k}$, $k \in \{1,2\}$. Additionally, consider $M_{i_1} \cap M_1$ as $F \times \{1\}$ and $M_{i_2} \cap M_1$ as $F \times \{2\}$. Extend the S^1 or T^2-action on M_{i_1} (or $\widetilde{M_{i_1}}$) to $F \times [1,2]$ (or its double cover

$\widetilde{F} \times [1,2]$) so that it restricts to the same action on each fiber $F \times \{t\}$ (or $\widetilde{F} \times \{t\}$), $t \in [1,2]$, and restricts to the trivial action on the $[1,2]$-factor. The action on $F \times [1,2]$ (or $\widetilde{F} \times [1,2]$) and the action on M_{i_2} (or $\widetilde{M_{i_2}}$) together generate a higher dimensional torus action on $M_1 \cap M_{i_2} = F \times \{2\}$ (or $\widetilde{F} \times \{2\}$). If the action is not effective, then we pass to a quotient to get an effective lower dimensional torus action. Therefore, M admits an F-structure.

From Section 14.1, it suffices to assume that every $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ component is disjoint from $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ components. For simplicity, in the following sections, we assume that M does not contain any $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ component. Later in Section 15.6, we will show that the conclusions of the following sections are still valid when we replace an occurrence of a $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ component by a boundary component of $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$.

15.3 Boundary components of $\left(\begin{array}{r} S^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$

Let M_i be a component $\left(\begin{array}{r} S^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$. Let N_i' be a boundary component of M_i. From the results in previous two chapters, $N_i' \cong S^2 \times S^1$ is identified with one of the following.

(i) A component $\left(\begin{array}{r} S^1 \times D^3, S^2 \times D^2 \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, for some j, along the boundary $\left(\begin{array}{r} S^1 \times S^2 \longrightarrow \partial M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$.

(ii) A component $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, for some j, along the boundary $\left(\begin{array}{r} S^2 \longrightarrow \partial M_j \\ \downarrow \\ S^1 \end{array}\right)$ so that S^2-fibers coincide.

(iii) A 4-manifold W represented by a cycle graph where each vertex represents a component $\left(\begin{array}{lr} D^4, \pm\mathbb{C}P^2\#D^4 & \\ S^2 \times_{\pm 2} D^2, S^2 \times_{\mathbb{Z}_2} D^2 & \longrightarrow M_{j_k} \\ & \Big\downarrow \\ & \text{pt} \end{array}\right)$, for some j_k, and each edge represents a component $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j_\ell} \\ \downarrow \\ (I, \partial I) \end{array}\right)$, for some j_ℓ, as in Lemma 14.3. $W \cong (S^1 \times D^3)\# n_1(\mathbb{C}P^2)\# \; n_2(-\mathbb{C}P^2)\# \; n_3(S^2 \times S^2)$, for some integers $n_1, n_2, n_3 \geq 0$, or $W \cong S^1 \times (\mathbb{R}P^3\#D^3)$. In particular, $\partial W \cong S^1 \times \partial D^3$. ∂D^3-fibers of ∂W coincide with S^2-fibers of N_i'.

(iv) A 4-manifold as in (iii) but with some occurrences of $\left(\begin{array}{r} D^3, S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$ components replaced by the union $V_{j_1} \cup_\partial E_\ell \cup_\partial V_{j_2} \cong S^2 \times I$ where $V_{j_k} \cong D^2 \times I$, $k \in \{1, 2\}$, is a subbundle of $\left(\begin{array}{r} D^2 \longrightarrow N_{j_k} \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, for some j_k, and $E_\ell \cong (S^1 \times I) \times I$ is a subbundle of $\left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some ℓ. $V_{j_k} \cap E_\ell \cong S^1 \times I$ so that the boundary of each D^2-fiber of V_{j_k} coincides with a boundary component of an $(S^1 \times I)$-fiber of E_ℓ. (See Lemma 14.62).

(v) The union $V_{j_1} \cup_\partial E_\ell \cup_\partial V_{j_2}$ where V_{j_k}, $k \in \{1, 2\}$, is a boundary component of $\left(\begin{array}{r} D^2 \longrightarrow M_{j_k} \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$, for some j_k, and $E_\ell \cong (S^1 \times I) \times S^1$ is a subbundle of $\left(\begin{array}{r} S^1 \longrightarrow N_\ell \\ \downarrow \\ \partial X^3 \end{array}\right)$, for some ℓ. $V_{j_k} \cap E_\ell \cong S^2 \times S^1$ so that the boundary of each D^2-fiber of V_{j_k} coincides with a boundary component of an $(S^1 \times I)$-fiber of E_ℓ.

Lemma 15.1. *There are local S^1-actions on M_i which are compatible with an F-structure on the manifold attaching to N_i'.*

Proof. Let U_i be a neighborhood of N_i' in M_i so that $U_i \cong S^2 \times [0, \epsilon) \times S^1$ where $[0, \epsilon) \times S^1$ is a neighborhood of a boundary component of the base Σ^2 and so that N_i' is identified with

$S^2 \times \{0\} \times S^1$. Since M_i is an S^2-bundle over a surface Σ^2, there is the associated principal bundle $S^1 \longrightarrow F \longrightarrow \Sigma^2$. Therefore, there are local S^1-actions on M_i where S^1 acts trivially on Σ^2 and acts by rotations (with two fixed points) on each S^2-fiber. The local S^1-actions can be chosen so that they restrict to an S^1-action on U_i.

In cases (i) and (ii), if $M_j = \left(\begin{array}{r} S^1 \times D^3 \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ or $M_j = \left(\begin{array}{r} D^3 \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, then the S^1-action on U_i extends to $M_j \cup_\partial U_i$ so that S^1 acts trivially on the S^1-factor and acts by rotations on each D^3-fiber. If $M_j = \left(\begin{array}{r} S^2 \times D^2 \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ then the S^1-action on U_i extends to $M_j \cup_\partial U_i$ so that S^1 acts trivially on the D^2-factor and acts by rotations on the S^2-factor. If $M_j = \left(\begin{array}{r} S^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, then consider its double cover $\widetilde{M_j} \cong (S^2 \times I) \times S^1$. There is an S^1-action on $\widetilde{M_j}$ so that S^1 acts trivially on the $(I \times S^1)$-factor and acts by rotations on the S^2-factor so that it is compatible with the S^1-action on U_i. Hence, there are local S^1-actions on U_i which are compatible with an F-structure on the manifold attaching to N_i' in cases (i) and (ii).

For cases (iii) and (iv), the result follows from Lemma 14.3, Lemma 14.9, and Lemma 14.62.The proof of case (v) is similar to the proof of Lemma 14.62. $\square$

15.4 Boundary components of $\left(\begin{array}{r} T^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$

Let M_i be a component $\left(\begin{array}{r} T^2 \longrightarrow M_i \\ \downarrow \\ (\Sigma^2, \partial\Sigma^2) \end{array}\right)$. Let N_i' be a boundary component of M_i. N_i' is the total space of T^2-fibers over S^1. From the results in previous two chapters, N_i' is identified with one of the following.

(i) A component $\left(\begin{array}{lr} T^2 \times D^2, T^2 \times_{\mathbb{Z}_2} D^2, & \\ \mathcal{B}_k \widetilde{\times} I, k \in \{1,2,3,4\} & \longrightarrow M_j \\ & \Big\downarrow \\ & \mathrm{pt} \end{array}\right)$, for some j, along the boundary $\left(\begin{array}{r} T^3, \mathcal{G}_2 \longrightarrow N_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$.

(ii) A component $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, for some j, along the boundary $\left(\begin{array}{r} T^2 \longrightarrow N_j \\ \downarrow \\ S^1 \end{array}\right)$ so that T^2-fibers coincide.

(iii) A boundary component of a component $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, for some j, which is also a T^2-bundle over S^1.

(iv) A 4-manifold represented by a cycle graph as in Lemma 14.46 where each vertex represents a component $\left(\begin{array}{r} F \longrightarrow M_{j_k} \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, for some j_k, where $F \cong D^4$, $\pm\mathbb{C}P^2\#D^4$, $S^1\times D^3$, $S^2\times D^2$, $(\mathbb{R}P^2\times S^1)\widetilde{\times}I$, $(S^2\widetilde{\times}S^1)\widetilde{\times}I$, $S^2\times_\omega D^2$, $S^2\times_{\mathbb{Z}_2} D^2$, $(S^2\times_\omega D^2)/\mathbb{Z}_2$, $T^2\times_{\mathbb{Z}_2} D^2$, $\mathcal{B}_3\widetilde{\times}I$, or $\mathcal{B}_4\widetilde{\times}I$, and each edge represents a component $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j_\ell} \\ \downarrow \\ (I, \partial I) \end{array}\right)$, for some j_ℓ, as in Lemma 14.14.

(v) A 4-manifold as in (iv) but with some occurrences of $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_{j_\ell} \\ \downarrow \\ (I, \partial I) \end{array}\right)$ components replaced by S^1-subbundles of $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \partial X^3 \end{array}\right)$ diffeomorphic to $(S^1\times S^1)\times I$. (See Lemma 14.68.)

(vi) The union of S^1-subbundles of $\left(\begin{array}{r} S^1 \longrightarrow N_{j_k} \\ \downarrow \\ \partial X^3 \end{array}\right)$ components and copies of $(T^2 \times I)$-subsets of $\left(\begin{array}{r} S^3, \ldots \longrightarrow N_{j_\ell} \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ components. (See Lemma 14.75.)

Lemma 15.2. *There are local T^2-actions on M_i which are compatible with an F-structure on the manifold attaching to N_i'.*

Proof. N_i' is the total space of T^2-fibers over S^1. Let U_i be a neighborhood of N_i' in M_i so that $U_i \cong N_i' \times [0, \epsilon)$. There are local T^2-actions on M_i where T^2 acts trivially on the base and acts by the standard T^2-action on each T^2-fiber.

In case (i), $M_j \cong T^2 \times D^2$, $T^2 \times_{\mathbb{Z}_2} D^2$ or $\mathcal{B}_k\widetilde{\times}I$, $k \in \{1, 2, 3, 4\}$.

(a) $M_j \cong T^2 \times D^2$.

Consider that $\partial M_j \cong T^2 \times \partial D^2 = S^1 \times S^1 \times \partial D^2$. Up to isotopy, either T^2-fibers of $\partial M_j \cong T^2 \times \partial D^2$ coincide with T^2-fibers of U_i or $(S^1 \times \partial D^2)$-fibers of ∂M_j coincide with T^2-fibers of U_i. In the first case, we let T^2 act on M_j by the rotations

$$\begin{aligned} T^2 \times T^2 &\to T^2 \\ (\phi, \theta) \times (u, v) &\mapsto (u + \phi, v + \theta) \end{aligned} \tag{15.3}$$

where we use the coordinates $u, v, \phi, \theta \in [0, 2\pi)$, on the T^2-factor and act trivially on the D^2-factor. Consequently, we get local free T^2-actions on $M_j \cup U_i$. In the second case, we let S^1 act by rotations on the D^2-factor of $M_j \cong T^2 \times D^2$ and act trivially on the T^2-factor. We have that the T^2-action on U_i restricts to the S^1-action on M_j. Hence, $M_j \cup U_i$ admits a T-structure.

(b) $M_j \cong T^2 \times_{\mathbb{Z}_2} D^2$.

In this case, we apply the same argument as in the case $M_j \cong T^2 \times D^2$ to its double cover $\widetilde{M_j} \cong T^2 \times D^2$. As a result, $M_j \cup U_i$ admits an F-structure which restricts to a T^2-action on U_i.

(c) $M_j \cong \mathcal{B}_k \widetilde{\times} I$.

Consider a double cover $\widetilde{V} = U_{i,1} \cup_\partial \widetilde{M_j} \cup_\partial U_{i,2}$ of $V = M_j \cup U_i$ where $\widetilde{M_j} \cong T^3 \times I$ or $\mathcal{G}_2 \times I$ is a double cover of M_j and $U_{i,1}$ and $U_{i,2}$ are copies of U_i. Consider that $\widetilde{V}$ is a T^2-bundle over a surface. Hence, there are local T^2-actions on $\widetilde{V}$ that are compatible with the double covering $\widetilde{V} \longrightarrow V$. Therefore, $M_j \cup U_i$ admits an F-structure which restricts to local T^2-actions on U_i.

In case (ii), if $M_j = \left(\begin{array}{r} S^1 \times D^2 \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, then the local T^2-actions on U_i extend to $M_j \cup_\partial U_i$ so that T^2 acts trivially on the base S^1 and acts on each $S^1 \times D^2$-fiber by the action

$$\begin{aligned} T^2 \times (S^1 \times D^2) &\to S^1 \times D^2 \\ (\phi, \theta) \times (u, r, v) &\mapsto (u + \phi, r, v + \theta) \end{aligned} \tag{15.4}$$

where we use the coordinates $\phi, \theta, u, v \in [0, 2\pi)$, and $r \in [0, 1]$. This extends the T^2-action on each T^2-fiber of U_i. Hence, there are local T^2-actions on $M_j \cup U_i$. If $M_j = \left(\begin{array}{r} T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, then the same argument as in case (i) : $M_j \cong \mathcal{B}_k \widetilde{\times} I$ applies. From both cases, we have that $M_j \cup U_i$ admits an F-structure which restricts to a T^2-action on U_i.

In case (iii), there are local S^1-actions on $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$. From Lemma 13.3, each T^2-fiber of N_i' is the total space of S^1-fibers from M_j. Hence, local S^1-fibers from M_j and local T^2-fibers from M_i are compatible in the sense of T-structure. As a result, there is a T-structure on $M_j \cup M_i$ that restricts to local T^2-actions on M_i.

For cases (iv), (v), and (vi), the result follows from Lemma 14.14, Lemma 14.9, Lemma 14.68, and Lemma 14.75. □

15.5 Boundary components of $\left(\begin{array}{r} S^1 \longrightarrow M_i \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$

Let M_i be a component $\left(\begin{array}{r} S^1 \longrightarrow M_i \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$. Let N_i' be a boundary component of M_i. N_i' is the total space of S^1-fibers over a closed surface. From the results in previous two chapters, N_i' is identified with one of the following.

(i) A component $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$, for some j, along the boundary $\left(\begin{array}{r} S^3, \ldots \longrightarrow N_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$.

(ii) A component $\left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \end{array}\right)$, for some j, along the boundary $\left(\begin{array}{r} S^1 \longrightarrow N_j \\ \downarrow \\ \Sigma^2 \end{array}\right)$ so that S^1-fibers coincide.

(iii) A component $\left(\begin{array}{r} S^1 \times D^2, T^2 \times_{\mathbb{Z}_2} I \longrightarrow M_j \\ \downarrow \\ S^1 \end{array}\right)$, for some j, along the boundary $\left(\begin{array}{r} T^2 \longrightarrow N_j \\ \downarrow \\ S^1 \end{array}\right)$ so that each T^2-fiber of N_j is the total space of S^1-fibers from N_i'.

(iv) A boundary component of a component $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$, for some j, so that along the overlap, each T^2-fiber is the total space of S^1-fibers from N_i'.

(v) An elementary building block of type $(2, D^2)$ (see Lemma 14.38).

(vi) An elementary building block of type $(1, S^1 \times D^2)$ (see Lemma 14.46).

(vii) A building block of type $(2, D^2)+(1, S^1 \times D^2)$, a building block of type $(2, S^2)+(2, D^2)$, or a building block of type $(2, S^2) + (2, D^2) + (1, S^1 \times D^2)$ (see Lemma 14.57, Lemma 14.62, and Corollary 14.65).

(viii) The union of T^2-subbundles of $\left(\begin{array}{r} T^2 \longrightarrow N_j \\ \downarrow \\ \partial\Sigma^2 \end{array}\right)$, which are diffeomorphic to $T^2 \times I$, and other building blocks as described in Lemma 14.68 and Lemma 14.75.

Lemma 15.5. *There are local S^1-actions on M_i which are compatible with an F-structure on the manifold attaching to N_i'.*

Proof. Let U_i be a neighborhood of N_i' in M_i so that $U_i \cong N_i' \times [0, \epsilon)$. Since M_i is an S^1-bundle over X^3, there are local S^1-actions on M_i where S^1 acts trivially on the base and acts by rotations on each S^1-fiber. The local S^1-actions can be chosen so that it restricts to an S^1-action on U_i.

For case (i), there is an S^1 or T^2-action on M_j (or its double cover $\widetilde{M_j}$) whose restriction to ∂M_j (or $\partial\widetilde{M_j}$) is free (see Section 2.5). The action on ∂M_j (or $\widetilde{M_j}$) and the local S^1-actions on U_i together generate higher dimensional local torus actions on a neighborhood of ∂M_j in $M_j \cup U_i$ (or $\partial\widetilde{M_j}$ in $\widetilde{M_j} \cup \widetilde{U_i}$). If they are not effective, then we can pass to a quotient to get effective local actions. Therefore, $M_j \cup U_i$ admits an F-structure.

In case (ii), $M_j = \left(\begin{array}{r} D^2 \longrightarrow M_j \\ \downarrow \\ \Sigma^2 \end{array}\right)$. We can extend the local S^1-actions on U_i to M_j so that S^1 acts trivially on the base Σ^2 and acts by rotations about the center on each D^2-fiber.

For cases (iii) and (iv), the result follows from the same arguments as in the proof of cases (ii) and (iii) of Lemma 15.2. For cases (v), (vi), (vii), and (viii), the result follows from Lemma 14.38, 14.46, Lemma 14.57, Lemma 14.68, and Lemma 14.75. ☐

15.6 Replacing $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_0 \\ \downarrow \\ \text{pt} \end{array}\right)$ with a boundary component of $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$

In this section, we show that the conclusions of Lemma 15.1, Lemma 15.2, and Lemma 15.5 are still valid when we replace an occurrence of a component $\left(\begin{array}{r} D^4, \ldots \longrightarrow M_j \\ \downarrow \\ \text{pt} \end{array}\right)$ by a boundary component of $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$.

If an occurrence of a $\left(\begin{array}{r} D^4, \ldots \rightarrow M_j \\ \downarrow \\ \text{pt} \end{array}\right)$ component in case (i) of Section 15.3, Section 15.4, and Section 15.5 is replaced by a boundary component of $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_j \\ \downarrow \\ (I, \partial I) \end{array}\right)$, then by the same argument as in Section 15.2, the conclusions of Lemma 15.1, Lemma 15.2, and Lemma 15.5 are still valid. Therefore, from now on we can assume that there are no boundary components of $\left(\begin{array}{r} S^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$, $\left(\begin{array}{r} T^2 \longrightarrow M_j \\ \downarrow \\ (\Sigma^2, \partial \Sigma^2) \end{array}\right)$, and $\left(\begin{array}{r} S^1 \longrightarrow M_j \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$, that only intersect a component $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$.

From Lemma 13.68, the decomposition of a boundary component of $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ is the same as the decomposition of $\left(\begin{array}{r} S^3, \ldots \rightarrow \partial M_0 \\ \downarrow \\ \text{pt} \end{array}\right)$. It follows that the gluing description of building blocks in the previous chapter (Lemma 14.38, Lemma 14.46, Lemma 14.57, Lemma 14.62, Corollary 14.65, Lemma 14.68, and Lemma 14.75) does not change when an occurrence of a component $\left(\begin{array}{r} D^4, \ldots \rightarrow M_0 \\ \downarrow \\ \text{pt} \end{array}\right)$ is replaced by a boundary component of $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$.

Since we will construct an F-structure near $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$, it suffices to assume that M contains only one $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ component, which we denote by M_1. Then, we can write M as

$$M = \; W_0 \overset{M_1}{\rule[0.5ex]{2.5em}{0.4pt}} W_1 \tag{15.6}$$

or

$$M = \begin{tikzcd} W_0 \arrow[loop above, no head, "M_1"] \end{tikzcd} \tag{15.7}$$

where $W_0 \overset{M_1}{\rule[0.5ex]{2.5em}{0.4pt}} W_1$ denotes $W_0 \cup_\partial M_1 \cup_\partial W_1$ and W_0 and W_1 are the resulting manifolds from Section 15.3, Section 15.4, and Section 15.5 but with a component $\left(\begin{array}{r} D^4, \dots \longrightarrow M_j \\ \downarrow \\ \mathrm{pt} \end{array}\right)$ replaced by a boundary component of M_1. Because we will construct an F-structure in a neighborhood of M_1, it suffices to assume that M is as in (15.6).

Lemma 15.8. *Let* $M = \; W_0 \overset{M_1}{\rule[0.5ex]{2.5em}{0.4pt}} W_1$ *be defined as in (15.6). Then,* M *admits an* F*-structure.*

Proof. Consider $M_1 = \left(\begin{array}{r} S^3/\Gamma, \dots \longrightarrow M_1 \\ \downarrow \\ (I, \partial I) \end{array}\right)$ as $F \times [0,1]$ where $F \cong S^3/\Gamma, T^3/\Gamma, S^2 \times S^1$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Let U be a neighborhood of M_1 in M so that $U \cong F \times (-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$. Put $V_0 = F \times (-\epsilon, 0] \subset W_0$ and $V_1 = F \times [1+\epsilon) \subset W_1$. Then, $U = V_0 \cup M_1 \cup V_1$. Let $\widetilde{U}$ be a finite cover of U so that $\widetilde{U}/\Gamma \cong U$. Then, $\widetilde{U} = \widetilde{V}_0 \cup (\widetilde{F} \times [0,1]) \cup \widetilde{V}_1$ where $\widetilde{F}$ is a finite cover of F so that $\widetilde{F}/\Gamma \cong F$, $\widetilde{V}_0 = \widetilde{F} \times (-\epsilon, 0]$ and $\widetilde{V}_1 = \widetilde{F} \times [1, 1+\epsilon)$. We have that, $\widetilde{F} \cong S^3, T^3$, or $S^2 \times S^1$.

From the constructions of F-structures on building blocks and associated components in Lemma 14.38, Lemma 14.46, Lemma 14.57, Lemma 14.62, Lemma 14.68, and Lemma 14.75, W_0 and W_1 admit an F-structure whose restriction to ∂M_1 is an F-structure. In particular, V_0 and V_1 admit an F-structure whose restriction to each fiber is an F-structure. Moreover, from our choices of F-structures in the lemmas, $\widetilde{V}_0$ and $\widetilde{V}_1$ admit a T-structure whose restriction to each fiber is a T-structure.

Let S^1 act on $\widetilde{F} \times [0,1]$ by a free action on the $\widetilde{F}$-factor and act trivially on the $[0,1]$-factor. On $\widetilde{V}_0 \cap \widetilde{F} \times [0,1]$, if the S^1-action and torus actions from the T-structure on $\widetilde{V}_0$ do not coincide, then they generate higher dimensional torus actions. By passing to quotients, we get a T-structure on a neighborhood of $\widetilde{V}_0 \cap \widetilde{F} \times [0,1]$, which is compatible to the T-structure on $\widetilde{V}_0$ and the S^1-action on $\widetilde{F} \times [0,1]$. Consequently, $\widetilde{V}_0 \cup \widetilde{F} \times [0,1]$ admits a T-structure. By repeating the same argument, $\widetilde{U} = \widetilde{V}_0 \cup (\widetilde{F} \times [0,1]) \cup \widetilde{V}_1$ admits a T-structure. Therefore, U admits an F-structure. ☐

15.7 The gluing instruction

By Section 15.2, Section 15.6, and Lemma 15.8, it suffices to assume that M does not contain any $\left(\begin{array}{r} S^3/\Gamma, \ldots \longrightarrow M_i \\ \downarrow \\ (I, \partial I) \end{array}\right)$ component. First, we glue all fiber bundle components contained in $Y = M - \bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ as described in Section 15.3 and Section 15.4. By the lemmas in Chapter 14, boundary components of Y are S^1-bundles over surfaces. By Lemma 15.1 and Lemma 15.2, Y admits an F-structure which is compatible with S^1-fibers of ∂Y. Finally, we glue $\bigsqcup_\ell \left(\begin{array}{r} S^1 \longrightarrow M_\ell \\ \downarrow \\ (X^3, \partial X^3) \end{array}\right)$ with Y as described in Section 15.5. By Lemma 15.5, M admits an F-structure.

15.8 Satisfying the constraints

We have now shown that M is a 4-dimensional closed C^K-smooth Riemannian manifold which admits an F-structure or a metric of nonnegative sectional curvature. Hence, we have shown:

Proposition 15.9. *Under the constraints imposed in earlier chapters, M admits an F-structure or a metric of nonnegative sectional curvature.*

We now verify that it is possible to simultaneously satisfy all the constraints that appeared in the construction. We indicate a partial ordering of the parameters which is respected by all the constraints appearing in this dissertation. We denote $A \prec B$ if and only if $A < \overline{A}(B)$ or $A > \overline{A}(B)$. This means that every constraint on a given parameter is an upper or lower bound given as a function of other parameters which are strictly smaller in the partial order. It follows that all constraints can be satisfied simultaneously.

$$\begin{gathered} \{\mathcal{M}, \beta_4\} \prec \{c_{\text{1-slim}}, \Omega_i, \Omega_i'\} \prec \Gamma_5 \prec \{\Sigma_5, \Xi_5\} \prec c_{\text{1-ridge}} \prec \Gamma_4 \prec \{\Sigma_4, \Xi_4\} \prec c_{\text{2-slim}} \prec \Gamma_3 \prec \\ \{\Sigma_3, \Xi_3\} \prec c_{\text{2-edge}} \prec \Gamma_2 \prec \{\Sigma_2, \Xi_2\} \prec c_{\text{3-stratum}} \prec \Gamma_1 \prec \{\Sigma_1, \Xi_1\} \prec \varsigma_{\text{3-stratum}} \prec \beta_3 \prec \\ \Delta \prec \varsigma_{E'} \prec \{\varsigma_{\text{2-slim}}, \varsigma_{\text{2-edge}}, \beta_{E'}, \sigma_{E'}\} \prec \sigma_E \prec \{\sigma, \Lambda\} \prec \bar{w} \prec w' \prec \beta_E, \prec \beta_2 \prec \\ \Upsilon_1 \prec \varsigma_R \prec \{\varsigma_{\text{1-ridge}}, \delta_1, \sigma_R\} \prec \Upsilon_1' \prec \beta_1 \prec \{\Upsilon_0, \delta_0\} \prec \varsigma_{\text{0-stratum}} \prec \Upsilon_0'. \end{gathered} \tag{15.10}$$

This proves Theorem 1.4.

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