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Random Homogenization of Coercive Hamilton-Jacobi equations in 1-D

DISSERTATION

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Dedication

To my family

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Curriculum Vitae

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Abstract of the Dissertation

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By

Hongwei Gao

Doctor of Philosophy in Mathematics
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Professor Yifeng Yu, Chair, Professor Song-Ying Li, co-Chair

This dissertation considers the random homogenization of coercive Hamilton-Jacobi equations and it gives the most generalized result in 1-D. Basically, we can prove that in the stationary ergodic media, the random homogenization holds as long as the Hamiltonian is coercive. This is an extension of the result by Armstrong, Tran and Yu when the Hamiltonain is separable. We also provide some application of random homogenizaton in front propagation based on the analysis of inviscid G-equation model, it is proved that with 2-d random shear flows, the strain effect reduces the propagation of the flame front.

Introduction

0.0.1 The problem of homogenization

We study the Hamilton-Jacobi equation (see [10, 9, 8]) of the following form:

$$\begin{cases} u_t + H(Du, x, \omega) = 0 & (x,t) \in \mathbf{R}^d \times (0, \infty) \\ u(x,0) = g(x) & x \in \mathbf{R}^d \end{cases}$$

The Hamiltonian $H = H(p, x, \omega) : \mathbf{R}^d \times \mathbf{R}^d \times \Omega \to \mathbf{R}$ is coercive in p, and stationary ergodic in (x, ω) (see precise definitions in section 1.1), where ω is an element of the underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The initial condition $g(x) \in BUC(\mathbf{R}^d)$, the space of bounded uniformly continuous functions in $\mathbf{R}^d$. For each $\epsilon > 0$, $\omega \in \Omega$, let $u^\epsilon(x, t, \omega)$ be the unique solution of the equation:

$$\begin{cases} u^\epsilon_t + H\left(Du^\epsilon, \frac{x}{\epsilon}, \omega\right) = 0 & (x,t) \in \mathbf{R}^d \times (0, +\infty) \\ u^\epsilon(x,0) = g(x) & x \in \mathbf{R}^d \end{cases}$$

The central goal of stochastic homogenization is to prove that for a.e. $\omega \in \Omega$, as $\epsilon \to 0$, $u^\epsilon(x, t, \omega) \to \overline{u}(x, t)$ locally uniformly, where $\overline{u}(x, t)$ is the unique solution of the homogenized equation:

$$\begin{cases} \overline{u}_t + \overline{H}(D\overline{u}) = 0 & (x,t) \in \mathbf{R}^d \times (0,+\infty) \\ \overline{u}(x,0) = g(x) & x \in \mathbf{R}^d \end{cases}$$

0.0.2 Periodic environment

The periodic homogenization of Hamilton-Jacobi equations was first studied by Lions, Papanicolaou and Varadhan [24]. By the introduction of perturbed test function method, periodic homogenization of the general case was established by Evans [16][17], which was later extended to almost periodic environment by Ishii [20].

0.0.3 Random environment

Stationary ergodic media

If $H(p,x,\omega)$ is convex with respect to $p \in \mathbf{R}^d$, stochastic homogenization was proved independently by Souganidis [34] and by Rezakhanlou and Tarver [31]. This result was extended to time-dependent Hamiltonians by Schwab [33] when the Hamiltonian has super-linear growth in p and by Jing, Souganidis and Tran [21] for Hamiltonians with the form $a(x,t,\omega)|p|$. For those quasi-convex Hamiltonians, Siconolfi and Davini [14] established the random homogenization in 1d, and the general dimensional case was proved by Amstrong and Souganidis [4]. It remains an open problem of whether random homogenization still holds if the Hamiltonian is non-convex. The first genuinely non-convex example of stochastic homogenization was provided by Amstrong, Tran and Yu [6] for a special class of Hamiltonians with the following typical form:

$$H(p,x,\omega) = (|p|^2 - 1)^2 + V(x,\omega), \quad (p,x) \in \mathbf{R}^d \times \mathbf{R}^d$$

In the one dimensional case, the same author established in another paper [7] the random homogenization of separable Hamiltonians:

$$H(p,x,\omega) = H(p) + V(x,\omega), \quad (p,x) \in \mathbf{R} \times \mathbf{R} \text{ for any coercive } H(p)$$

The purpose of this dissertation is to extend the result of Amstrong, Tran and Yu [7] to general coercive $H(p,x,\omega)$.

Other random environments

Another typical random case is the independent and identically distributed (i.i.d.) random enviroment. Precisely speaking, it assumes that $H(p, x, \omega)$ is stationry and has a finite range of dependece or mixing condition. The homogenization of Hamilton-Jacobi equations in such kind of samplings are studied by Armstrong and Cardaliaguet [1], Armstrong and Souganidis [3], Armstrong, Cardaliaguet and Souganidis [2].

In [1], Armstrong and Cardaliaguet considered the homogenization of Hamiltonian $H(p, x, \omega)$ that is homogeneous in p and under the assumption of unit range of dependence on (x, ω) (basically, it means that $H(p, x, \omega)$ and $H(p, y, \omega)$ are independent once $|x - y| > 1$).

A counter example

When the dimension is larger than one, there indeed exists counter examples, see Ziliotto [38], for example.

0.0.4 Homogenization of second order Hamilton-Jacobi equations

The same random homogenization question can be asked to the Hamilton-Jacobi equation with viscous term, for example, see the following equation.

$$u_t^\epsilon - \epsilon \Delta u^\epsilon + H\left(Du^\epsilon, \frac{x}{\epsilon}, \omega\right) = 0$$

When the Hamitonian $H(p, x, \omega)$ is convex in the gradient variable, the homogenization has been widely studied by Lions-Souganidis [26, 27], Kosygina-Rezakhanlou-Varadhan [22], Kosygina-Varadhan [23] and Armstrong-Tran [5]. But it is open when $H(p, x, \omega)$ is nonconvex (except [1] with i.i.d. assumption for homogeneous Hamiltonian), even the 1d case as $H(p, x, \omega) = H(p) + V(x, \omega)$ with a simply "W"-shaped $H(p)$ has not been solved.

Chapter 1

Random Homogenization of coercive Hamilton-Jacobi equations in 1-D

This chapter is based mainly on the author's recent work on random homogenization of nonconvex Hamilton-Jacobi equations. The author acknowledge Springer for accepting the paper for publication. The final publication is available at Springer via http://dx.doi.org/ 10.1007/s00526-016-0968-9

1.1 Assumptions and main result

Consider the Hamiltonian $H(p, x, \omega)$ that is continuous in $(p, x) \in \mathbf{R} \times \mathbf{R}$ and measurable in $\omega \in \Omega$, assume H satisfies the following assumptions:

(**A1**) Stationary Ergodic: there exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a group $\{\tau_y\}_{y \in \mathbf{R}}$ of $\mathcal{F}$-measurable, measure-preserving transformations $\tau_y : \Omega \to \Omega$, i.e. for any x, $y \in \mathbf{R}$:

$$\tau_{x+y} = \tau_x \circ \tau_y \text{ and } \mathbf{P}[\tau_y(A)] = \mathbf{P}[A]$$

Ergodic: $A \in \mathcal{F}$, $\tau_z(A) = A$ for every $z \in \mathbf{R} \Rightarrow \mathbf{P}[A] \in \{0, 1\}$.

Stationary: $H(p, y, \tau_z\omega) = H(p, y+z, \omega)$ for any y, $z \in \mathbf{R}$ and $\omega \in \Omega$.

(**A2**) Coercive:

$$\liminf_{|p|\to+\infty} \operatorname*{ess\,inf}_{(x,\omega)\in\mathbf{R}\times\Omega} H(p, x, \omega) = +\infty$$

(**A3**) Local Uniformly Continuous: for any compact set $K \subset \mathbf{R}$,

$$|H(p, x, \omega) - H(q, y, \omega)| \le \rho_K(|p-q| + |x-y|), \quad (p, x, \omega), (q, y, \omega) \in K \times \mathbf{R} \times \Omega$$

The above ρ_K is the modulus of continuity.

Theorem 1.1.1. *Assume* (**A1**)-(**A3**) *hold and* $g(x) \in BUC(\mathbf{R})$*, for each* $\epsilon > 0$ *and* $\omega \in \Omega$*, let* $u^\epsilon(x, t, \omega)$ *be the solution of the Hamilton-Jacobi equation*

$$\begin{cases} u_t^\epsilon + H\left(Du^\epsilon, \frac{x}{\epsilon}, \omega\right) = 0 & (x, t) \in \mathbf{R} \times (0, +\infty) \\ u^\epsilon(x, 0) = g(x) & x \in \mathbf{R} \end{cases}$$

Then, there exists an effective Hamiltonian $\overline{H}(p) \in C(\mathbf{R})$ *with* $\lim_{|p|\to+\infty} \overline{H}(p) = +\infty$*, such that for a.e.* $\omega \in \Omega$*,* $\lim_{\epsilon\to0^+} u^\epsilon(x, t, \omega) = \overline{u}(x, t)$ *locally uniformly, where* $\overline{u}(x, t)$ *is the solution of the homogenized Hamilton-Jacobi equation*

$$\begin{cases} \overline{u}_t + \overline{H}(D\overline{u}) = 0 & (x, t) \in \mathbf{R} \times (0, +\infty) \\ \overline{u}(x, 0) = g(x) & x \in \mathbf{R} \end{cases}$$

1.2 Stability of homogenization

Definition 1.2.1. *(Definition 1.1 in [7])* $H(p, x, \omega)$ *is called regularly homogenizable at* $p \in \mathbf{R}$ *if there exists an* $\overline{H}(p) \in \mathbf{R}$ *such that: for any* $\lambda > 0$*, if* $v_\lambda(x, p, \omega) \in W^{1,\infty}(\mathbf{R})$ *is the unique viscosity solution of the equation*

$$\lambda v_\lambda + H(v_\lambda', x, \omega) = 0, \qquad x \in \mathbf{R}$$

Then

$$(1.1)\ \mathbf{P}\left[\omega \in \Omega : \limsup_{\lambda\to0} \max_{|x|\le\frac{R}{\lambda}} \left|\lambda v_\lambda(x, p, \omega) + \overline{H}(p)\right| = 0\right] = 1, \ \textit{for any } R > 0$$

Remark 1. *By Armstrong and Souganidis [4], with (**A1**), (1.1) is equivalent to the identity:*

$$\mathbf{P}\left[\omega \in \Omega : \lim_{\lambda \to 0} \left|\lambda v_\lambda(0, p, \omega) + \overline{H}(p)\right| = 0\right] = 1$$

Remark 2. *Homogenization with $H(p, x, \omega)$ holds if $H(p, x, \omega)$ is regularly homogenizable for each $p \in \mathbf{R}$ (see [3]). If the cell problem at p is solvable, then $H(p, x, \omega)$ is regularly homogenizable at p.*

Definition 1.2.2. *Let $G(p, x, \omega) : \mathbf{R} \times \mathbf{R} \times \Omega \to \mathbf{R}$ satisfy (**A1**), denote*

$$\begin{aligned} G_{\inf}(p) &:= \operatorname*{ess\,inf}_{(x,\omega) \in \mathbf{R} \times \Omega} G(p, x, \omega) \\ G_{\sup}(p) &:= \operatorname*{ess\,sup}_{(x,\omega) \in \mathbf{R} \times \Omega} G(p, x, \omega) \end{aligned}$$

Lemma 1.2.1. *If $G(p, x, \omega)$ satisfies $(\mathbf{A1})$ and is continuous in x, $G_{\inf}(p)$, $G_{\sup}(p) \in \mathbf{R}$, then for a.e. $\omega \in \Omega$,*

$$\begin{aligned} G_{\inf}(p) &= \operatorname*{ess\,inf}_{x \in \mathbf{R}} G(p, x, \omega) \\ G_{\sup}(p) &= \operatorname*{ess\,sup}_{x \in \mathbf{R}} G(p, x, \omega) \end{aligned}$$

Proof. Fix $p \in \mathbf{R}$, denote $g(x, \omega) := G(p, x, \omega)$. For any $\alpha \in \mathbf{R}$, define

$$A_\alpha := \{\omega \in \Omega : g(x, \omega) > \alpha, \text{ for all } x \in \mathbf{R}\} \tag{1.2}$$

Stationary implies $\tau_z A_\alpha = A_\alpha$ for any $z \in \mathbf{R}$. By ergodicity, we have $\mathbf{P}[A_\alpha] = 0$ or 1. Now let $\alpha_0 := \sup\{\alpha : \mathbf{P}[A_\alpha] = 1\}$. By (1.2), $\alpha_0 = G_{\inf}(p)$. Since $\mathbf{P}\left[A_{\alpha_0 - \frac{1}{n}}\right] = 1$, $n \in \mathbf{N}$, $\mathbf{P}\left[\bigcap_{n=1}^{\infty} A_{\alpha_0 - \frac{1}{n}}\right] = 1$. Hence

$$G_{\inf}(p) = \operatorname*{ess\,inf}_{x \in \mathbf{R}} G(p, x, \omega), \quad \omega \in \bigcap_{n=1}^{\infty} A_{\alpha_0 - \frac{1}{n}}$$

The other equality can be established similarly. □

Lemma 1.2.2. *Given uniformly coercive Hamiltonians $\{H_n(p, x, \omega)\}_{n \geq 1} \bigcup \{H(p, x, \omega)\}$ that satisfy $(\mathbf{A1})$, each $H_n(p, x, \omega)$ is regularly homogenizable for all $p \in \mathbf{R}$ and has effective Hamiltonian $\overline{H}_n(p)$. Assume for a.e. $\omega \in \Omega$ that:*

$$\lim_{n \to +\infty} ||H_n(p, x, \omega) - H(p, x, \omega)||_{L^\infty(K \times \mathbf{R})} = 0$$

for each compact set $K \subset \mathbf{R}$

Then, $H(p,x,\omega)$ is regularly homogenizable and has effective Hamiltonian $\overline{H}(p)$. Moreover,

$$\lim_{n\to+\infty}\overline{H}_n(p)=\overline{H}(p)$$

Proof. Fix $p\in\mathbf{R}$, for each $\lambda>0$, let $v_{n,\lambda}(x,p,\omega)$ and $v_\lambda(p,x,\omega)$ be solutions of the following equations:

$$\begin{aligned}\lambda v_{n,\lambda}+H_n(p+v'_{n,\lambda},x,\omega)=0,&\qquad x\in\mathbf{R}\\ \lambda v_\lambda+H(p+v'_\lambda,x,\omega)=0,&\qquad x\in\mathbf{R}\end{aligned}$$

Then,

$$-\lambda v_{n,\lambda}\in[H_{n,\inf}(p),H_{n,\sup}(p)]\quad\text{and}\quad -\lambda v_\lambda\in[H_{\inf}(p),H_{\sup}(p)$$

By uniform coercive, there exists an $r=r(p)$, such that $|v'_{n,\lambda}(x,\omega)|$, $|v'_\lambda(x,\omega)|<r$. If we denote the compact set $K:=[p-r,p+r]$, then by comparison principle,

$$\begin{aligned}|\lambda v_{n,\lambda}(0,p,\omega)-\lambda v_\lambda(0,p,\omega)| &\le \sup_{x\in\mathbf{R}}|\lambda v_{n,\lambda}(x,p,\omega)-\lambda v_\lambda(x,p,\omega)|\\ &\le ||H_n(\cdot,\cdot,\omega)-H(\cdot,\cdot,\omega)||_{L^\infty(K\times\mathbf{R})}\end{aligned}$$

Boundedness of $-\lambda v_{n,\lambda}$ implies that $\left\{\overline{H}_n(p)\right\}_{n\ge1}$ is bounded. Thus, for any subsequence $\{n_j\}_{j\ge1}$, there exists a sub-subsequence $\{n_{j_k}\}_{k\ge1}$, such that $\lim_{k\to\infty}\overline{H}_{n_{j_k}}(p_0)=h_*$. So

$$\begin{aligned}&|(-\lambda v_\lambda(0,p,\omega))-h_*|\\ \le\;&\left|(-\lambda v_\lambda(0,p,\omega))-\left(-\lambda v_{n_{j_k},\lambda}(0,p,\omega)\right)\right|+\left|\left(-\lambda v_{n_{j_k},\lambda}(0,p,\omega)\right)-\overline{H}_{n_{j_k}}(p)\right|\\ &+\left|\overline{H}_{n_{j_k}}(p)-h_*\right|\\ \le\;&||H_{n_{j_k}}(\cdot,\cdot,\omega)-H(\cdot,\cdot,\omega)||_{L^\infty\left(K\times\mathbf{R}\right)}+\left|\left(-\lambda v_{n_{j_k},\lambda}(0,p,\omega)\right)-\overline{H}_{n_{j_k}}(p)\right|\\ &+\left|\overline{H}_{n_{j_k}}(p)-h_*\right|\\ =:\;&①+②+③\end{aligned}$$

For any $\epsilon>0$, when k is large enough, $①<\frac{\epsilon}{3}$ and $③<\frac{\epsilon}{3}$. Fix such k, there exists some $\lambda_0=\lambda_0(k)$, such that, $②<\frac{\epsilon}{3}$ once $0<\lambda<\lambda_0$. Thus $\lim_{\lambda\to0}-\lambda v_\lambda(0,p,\omega)=h_*$, which means $\overline{H}(p) = h_*$. Observe that the above limit is independent of the choice of $\{n_j\}_{j\geq 1}$, then $\lim_{n\to\infty} \overline{H}_n(p) = h_*$. As a consequence,

$$\lim_{n\to\infty} \overline{H}_n(p) = \overline{H}(p)$$

□

Remark 3. *Based on this lemma, we will construct the approximation of $H(p, x, \omega)$ by constrained Hamiltonians (see Definition 1.3.1), this is the first step of reduction in this paper.*

Corollary 1.2.1. *Let $H(p, x, \omega)$ satisfy* (**A1**)-(**A3**) *and fix $p_0 \in \mathbf{R}$, then*
(1) If $H(p, x, \omega)$ is regularly homogenizable on $(-\infty, p_0)$ and $\overline{H}(p)$ is continuous, then $H(p, x, \omega)$ is also homogenizable at p_0 and $\lim_{p\to p_0^-} \overline{H}(p) = \overline{H}(p_0)$.
(2) If $H(p, x, \omega)$ is regularly homogenizable on $(p_0, +\infty)$ and $\overline{H}(p)$ is continuous, then $H(p, x, \omega)$ is also homogenizable at p_0 and $\lim_{p\to p_0^+} \overline{H}(p) = \overline{H}(p_0)$.

Proof. We only give the proof of (1), since the proof of (2) is similar. Fix any sequence $\delta_n \to 0^+$ and denote

$$H_n(p, x, \omega) := H\left(p - \delta_n, x, \omega\right)$$

By assumption, each $H_n(p, x, \omega)$ is regularly homogenizable at p_0. According to (**A3**), for each $\omega \in \Omega$ and any compact set $K \subset \mathbf{R}$, we have $\lim_{n\to\infty} ||H_n(p, x, \omega) - H(p, x, \omega)||_{L^\infty(K\times\mathbf{R})} = 0$. Thanks to Lemma 1.2.2, $H(p, x, \omega)$ is regularly homogenizable at p_0 and

$$\overline{H}(p_0) = \lim_{n\to+\infty} \overline{H_n}(p_0, x, \omega) = \lim_{n\to+\infty} \overline{H}\left(p_0 - \delta_n, x, \omega\right)$$

This is true for any sequence $\delta_n \to 0^+$, hence

$$\lim_{p\to p_0^-} \overline{H}(p, x, \omega) = \overline{H}(p_0, x, \omega)$$

□

1.2.1 Comparison Principle

Lemma 1.2.3. *Let $H(p,x,\omega)$ satisfy (**A1**)-(**A3**), for $R>0$, $1 \gg \lambda > 0$, $p \in \mathbf{R}$, let u and v both be viscosity solutions of the equation*

$$\lambda\gamma + H(p+\gamma', x, \omega) = 0, \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

If there exists a constant $M = M(p) > 0$, such that $|\lambda u| + |\lambda v| \le M(p)$.
Then, there exists a constant $C = C(p) > 0$, such that

$$|\lambda u - \lambda v| \le \frac{M(p)}{R}\sqrt{|x|^2+1} + \frac{M(p)C(p)}{R}, \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Proof. By $|\lambda u| + |\lambda v| \le M(p)$, we have $H(p+u', x, \omega) \le M(p)$ and $H(p+v', x, \omega) \le M(p)$.
By (**A2**), there exists some $r = r(p) > 0$, such that $|u'|$, $|v'| \le r(p)$.
By (**A3**), there exists some $\delta = \delta(p) > 0$, such that $|H(q_1, x, \omega) - H(q_2, x, \omega)| < 1$ if

$$q_1, q_2 \in \left[p - r(p) - \frac{M(p)}{R}, p + r(p) + \frac{M(p)}{R}\right] \quad \text{and} \quad |q_1 - q_2| < \delta$$

Now define

$$w(x) := v + \frac{M(p)}{R}\sqrt{|x|^2+1} + \frac{M(p)}{\delta(p)R\lambda}$$

Immediately, we see that $|w'| \le r(p) + \frac{M(p)}{R}$. Thus

$$H(p+w', x, \omega) \ge H(p+v', x, \omega) - \frac{M(p)}{\delta(p)R}$$

And

$$\begin{aligned}
& \lambda w + H(p+w', x, \omega) \\
= \; & \lambda v + \frac{\lambda M(p)}{R}\sqrt{|x|^2+1} + \frac{M(p)}{\delta(p)R} + H(p+w', x, \omega) \\
\ge \; & \lambda v + \frac{\lambda M(p)}{R}\sqrt{|x|^2+1} + \frac{M(p)}{\delta(p)R} + H(p+v', x, \omega) - \frac{M(p)}{\delta(p)R} \\
> \; & 0
\end{aligned}$$

Furthermore,

$$|\lambda u| + |\lambda v| \le M(p) \Rightarrow w|_{|x|=\frac{R}{\lambda}} \ge v|_{|x|=\frac{R}{\lambda}} + \frac{M(p)}{\lambda} \ge u|_{|x|=\frac{R}{\lambda}}$$

Recall the classical comparison principle, we have $w(x) \geq u(x)$ for $x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$. So

$$u - v \leq \frac{M(p)}{R}\sqrt{|x|^2+1} + \frac{M(p)}{\delta(p)R\lambda}, \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Similarly,

$$v - u \leq \frac{M(p)}{R}\sqrt{|x|^2+1} + \frac{M(p)}{\delta(p)R\lambda}, \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Thus when $\lambda \leq 1$, let $C(p) := \frac{1}{\delta(p)}$, then for $x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$, we have

$$|\lambda u - \lambda v| \leq \frac{\lambda M(p)\sqrt{|x|^2+1}}{R} + \frac{C(p)M(p)}{R} \leq \frac{M(p)\sqrt{|x|^2+1}}{R} + \frac{C(p)M(p)}{R}$$

□

1.3 Reduction by constrained Hamiltonian with index $(\widetilde{L}, L)$

1.3.1 Approximation by cluster-point-free Hamiltonians

Let $H(p,x,\omega)$ satisfy (**A1**)-(**A3**) and denote

$$h_i^{(n)}(x,\omega) := H\left(\frac{i}{n}, x, \omega\right) \quad \text{and} \quad \mathcal{E}_n := \{h_i^{(n)}(x,\omega)\}_{-n^2 \leq i \leq n^2}$$

Let $\mathcal{E}_n^+ = \{g_i^{(n)}(x,\omega)\}_{-n^2 \leq i \leq n^2}$ be another family of stationary functions and define

$$\Delta_{\mathcal{E}_n,\mathcal{E}_n^+}(p,x,\omega) = \begin{cases} g_{-n^2}^{(n)} - h_{-n^2}^{(n)} & p \in (-\infty, -n) \\ (np-i)\left[g_{i+1}^{(n)} - h_{i+1}^{(n)}\right] + (i+1-np)\left[g_i^{(n)} - h_i^{(n)}\right] & p \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \\ g_{n^2}^{(n)} - h_{n^2}^{(n)} & p \in (n, +\infty) \end{cases}$$

So $\Delta_{\mathcal{E},\mathcal{E}^+}(p,x,\omega)$ is a stationary function which is also continuous with respect to (p,x).

Lemma 1.3.1. *If $H(p,x,\omega)$ satisfies* (**A1**)-(**A3**)*, then there exists $\{H^{(n)}(p,x,\omega)\}_{n=2^j, j\in\mathbf{N}}$, such that*

*(1) $H^{(n)}(p,x,\omega)$ satisfies (**A1**)-(**A3**), $\forall n = 2^j$, $j \in \mathbf{N}$.*

(2) For $-n^2 \le i \le n^2$, $H^{(n)}\left(\frac{i}{n}, x, \omega\right)$, as functions of x, have no cluster point.

(3) $||H(p,x,\omega) - H^{(n)}(p,x,\omega)||_{L^\infty(\mathbf{R}\times\mathbf{R}\times\Omega)} \le \frac{1}{n}$.

Proof. For each $\epsilon > 0$, $-n^2 \le i \le n^2$, denote

$$H_\epsilon\left(\tfrac{i}{n}, x, \omega\right) := \tfrac{1}{\sqrt{2\pi\epsilon}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{2\epsilon}} H\left(\tfrac{i}{n}, y, \omega\right) dy$$

By (**A1**), either $H_\epsilon\left(\frac{i}{n}, x, \omega\right)$ has no cluster for a.e. $\omega \in \Omega$ or $H_\epsilon\left(\frac{i}{n}, x, \omega\right)$ has some cluster point for a.e. $\omega \in \Omega$. If $H_\epsilon\left(\frac{i}{n}, x, \omega\right)$ has a cluster point x_0, without loss of generality, we can assume $x_0 = 0$ and $H_\epsilon\left(\frac{i}{n}, 0, \omega\right) = 0$. Then $\frac{\partial^k}{\partial x^k} H_\epsilon\left(\frac{i}{n}, 0, \omega\right) = 0$, $k = 0, 1, 2, \cdots$, which means

$$\int_{\mathbf{R}} y^k e^{-\frac{y^2}{2\epsilon}} H\left(\frac{i}{n}, y, \omega\right) dy = 0$$

By Fourier analysis, we have $H\left(\frac{i}{n}, x, \omega\right) \equiv 0$. Denote

$$D_j := \left\{\frac{i}{2^j}\right\}_{-4^j \le i \le 4^j} \quad \text{and} \quad D := \bigcup_{j=1}^{\infty} D_j$$

Then D is dense in $\mathbf{R}$, if for every $d \in D$, $H_\epsilon(d, x, \omega)$ has a cluster point, then for every $d \in D$, $H(d, x, \omega)$ is independent of (x, ω). By continuity, for all $p \in \mathbf{R}$, $H(p, x, \omega)$ is independent of (x, ω), so it is already homogenized. Thus, assume for some $d_0 \in D$, $H_\epsilon(d_0, x, \omega)$ has no cluster point. Since D_j is increasing, without loss of generality, assume $d_0 = 0 \in D_j$, $j \in \mathbf{N}$. For $j \in \mathbf{N}$ and $-4^j \le i \le 4^j$, define

$$g_i^{(2^j)}(x,\omega) := \begin{cases} H\left(\frac{i}{2^j}, x, \omega\right) & \text{if } H\left(\frac{i}{2^j}, x, \omega\right) \text{ has no cluster point} \\ H\left(\frac{i}{2^j}, x, \omega\right) + \frac{H_\epsilon(0,x,\omega)}{2^j q_\epsilon} & \text{if } H\left(\frac{i}{2^j}, x, \omega\right) \text{ has a cluster point} \end{cases}$$

$$\text{Here } q_\epsilon := ||H_\epsilon(0, x, \omega)||_{L^\infty(\mathbf{R}\times\Omega)} + 1$$

Denote

$$\mathcal{E}_{2^j} := \{h_i^{(2^j)}(x,\omega)\}_{-4^j \le i \le 4^j} \quad \text{and} \quad \mathcal{E}_{2^j}^+ := \{g_i^{(2^j)}(x,\omega)\}_{-4^j \le i \le 4^j}$$

We can finish the proof by defining

$$H^{(n)}(p,x,\omega) := H(p,x,\omega) + \Delta_{\mathcal{E}_{2^j},\mathcal{E}^+_{2^j}}(p,x,\omega), \qquad n = 2^j, j \in \mathbf{N}$$

$\square$

1.3.2 Approximation by constrained Hamiltonians

In this subsection, we find a way to approximate $H(p,x,\omega)$ by $\{H_n(p,x,\omega)\}_{n\geq 1}$ in the sense of Lemma 1.2.2. Here each $H_n(p,x,\omega)$ is constrained in the following sense.

Definition 1.3.1 (Constrained Hamiltonian)**.** *A Hamiltonian $H(p,x,\omega)$ is called constrained if it satisfies the following (1)-(5).*

(1) There exists $k \in \mathbf{N}$ and $-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_{k-1} < b_{k-1} < a_k < +\infty$.

(2) For each (x,ω), $H(p,x,\omega)|_{(-\infty,a_1)}$, $H(p,x,\omega)|_{(b_1,a_2)}$, $\cdots$, $H(p,x,\omega)|_{(b_{k-1},a_k)}$ are decreasing.

(3) For each (x,ω), $H(p,x,\omega)|_{(a_k,+\infty)}$, $H(p,x,\omega)|_{(a_{k-1},b_{k-1})}$, $\cdots$, $H(p,x,\omega)|(a_1,b_1)$ are increasing.

(4) $H(p,x,\omega)$ is Lipschitz w. r. t. p (with Lipschitz constant $\mathcal{L}$), uniformly in $(x,\omega) \in \mathbf{R}\times\Omega$.

(5) Each of $H(a_i,x,\omega)$, $H(b_j,x,\omega)$, $1 \leq i \leq k$, $1 \leq j \leq k-1$ has no cluster point.

Lemma 1.3.2. *If $H(p,x,\omega)$ satisfies* (**A1**)*-*(**A3**)*, then for $n = 2^j$, $j \in \mathbf{N}$, there exists $H_n(p,x,\omega)$, such that*

(a) $\{H(p,x,\omega)\}_{n\geq 1}$ is uniformly coercive.

(b) Each $H_n(p,x,\omega)$ satisfies (**A1**)*-*(**A3**)*.*

(c) Each $H_n(p,x,\omega)$ is constrained.

(d) Fix any $\delta > 0$, then for any compact set $K \subset \mathbf{R}$, there exists an $N \in \mathbf{N}$, such that

$$||H_n(p,x,\omega) - H(p,x,\omega)||_{L^\infty(K\times\mathbf{R}\times\Omega)} < \delta, \qquad \textit{if } n > N$$

Proof. According to Lemma 1.3.1, without loss of generality, we can assume each of $H\left(\frac{i}{n},x,\omega\right)$, $-n^2 \leq i \leq n^2$ has no cluster point. We construct $H_n(p,x,\omega)$ by the following procedure.

STEP 1: For each $p \in (-\infty, n) \cup (n, \infty)$, define

$$H_n(p,x,\omega) = \begin{cases} |p+n| + H(-n,x,\omega) & p \in (-\infty,-n) \\ |p-n| + H(n,x,\omega) & p \in (n,+\infty) \end{cases}$$

STEP 2: For $k = 0, 1, 2, \cdots, 2n^2$, define

$$H_n\left(-n+\frac{k}{n},x,\omega\right) = H\left(-n+\frac{k}{n},x,\omega\right)$$

STEP 3: For $i = 0, 1, 2, \cdots, 2n^2-1$, define

$$H_n\left(-n+\frac{i}{n}+\frac{1}{2n},x,\omega\right) = \max\left\{H\left(-n+\frac{i}{n},x,\omega\right), H\left(-n+\frac{i+1}{n},x,\omega\right)\right\} + \frac{1}{n}$$

STEP 4: For $i = 0, 1, 2, \cdots, 2n^2-1$,

(1) If $p \in \left(-n+\frac{i}{n}, -n+\frac{i}{n}+\frac{1}{2n}\right)$, then there exists some $\theta \in (0,1)$, such that

$$p = \theta \times \left(-n+\frac{i}{n}\right) + (1-\theta) \times \left(-n+\frac{i}{n}+\frac{1}{2n}\right)$$

Then we define

$$H_n(p,x,\omega) = \theta H\left(-n+\frac{i}{n},x,\omega\right) + (1-\theta)H\left(-n+\frac{i}{n}+\frac{1}{2n},x,\omega\right)$$

(2) If $p \in \left(-n+\frac{i}{n}+\frac{1}{2n}, -n+\frac{i+1}{n}\right)$, then there exists some $\theta \in (0,1)$, such that

$$p = \theta \times \left(-n+\frac{i}{n}+\frac{1}{2n}\right) + (1-\theta) \times \left(-n+\frac{i+1}{n}\right)$$

Then we define

$$H_n(p,x,\omega) = \theta H\left(-n+\frac{i}{n}+\frac{1}{2n},x,\omega\right) + (1-\theta)H\left(-n+\frac{i+1}{n},x,\omega\right)$$

(a) Since $H(p,x,\omega)$ satisfies (**A2**), $\{H_n(p,x,\omega)\}_{n\geq 1}$ is uniformly coercive.

(b) By (**A1**), $H\left(-n+\frac{k}{n},x,\omega\right)$ is stationary, for $k = 0, 1, 2, \cdots, 2n^2$. So, $H_n(p,x,\omega)$, as a linear combination of these functions, is stationary and satisfies (**A1**)-(**A3**).

(c) By the above construction, such $H_n(p,x,\omega)$ is constrained with $2n^2+1$ wells. And $H_n(p,x,\omega)$ has Lipschitz constant $\mathcal{L} = 1 + n\rho_{[-n^2,n^2]}(\frac{1}{n})$ in p variable, uniformly in

$(x, \omega) \in \mathbf{R}$.

(d) By (**A3**), there exists $N \in \mathbf{N}$, such that $N > \frac{3}{\delta}$, $K \subset [-N, N]$ and

$$p, q \in K, |p - q| < \frac{1}{N} \Rightarrow |H(p, x, \omega) - H(q, x, \omega)| < \frac{\delta}{3}$$

To prove (d), it suffices to show that,

$$||H_n(p, x, \omega) - H(p, x, \omega)||_{L^\infty((K \cap (-n + \frac{k}{n}, -n + \frac{k+1}{n})) \times \mathbf{R})} < \delta$$
$$\text{for any } k \in \{0, 1, 2, \cdots, 2n^2 - 1\}$$

Denote

$$p_1 = -n + \frac{k}{n},\ p_2 = -n + \frac{k}{n} + \frac{1}{2n},\ p_3 = -n + \frac{k+1}{n}$$

Without loss of generality, assume that

$$H(p_1, x, \omega) \leq H(p_3, x, \omega)$$

Case 1: $p \in K \cap (p_1, p_2)$. Then there exists some $\theta \in (0, 1)$ such that $p = \theta p_1 + (1 - \theta) p_2$,

$$\begin{aligned}
& |H_n(p, x, \omega) - H(p, x, \omega)| \\
= & |H_n(\theta p_1 + (1-\theta)p_2, x, \omega) - H(\theta p_1 + (1-\theta)p_2, x, \omega)| \\
= & \left|\theta H(p_1, x, \omega) + (1-\theta)\left[H(p_3, x, \omega) + \frac{1}{n}\right] - H(\theta p_1 + (1-\theta)p_2, x, \omega)\right| \\
\leq & \theta|H(p_1, x, \omega) - H(\theta p_1 + (1-\theta)p_2, x, \omega)| \\
& +(1-\theta)|H(p_3, x, \omega) - H(\theta p_1 + (1-\theta)p_2, x, \omega)| + \frac{1-\theta}{n} \\
< & \delta
\end{aligned}$$

Case 2: $p \in K \cap (p_2, p_3)$. Then there exists some $\theta \in (0, 1)$ such that $p = \theta p_2 + (1 - \theta) p_3$,

$$\begin{aligned}
& |H_n(p, x, \omega) - H(p, x, \omega)| \\
= & |H_n(\theta p_2 + (1-\theta)p_3, x, \omega) - H(\theta p_2 + (1-\theta)p_3, x, \omega)| \\
= & \left|\theta\left[H(p_3, x, \omega) + \frac{1}{n}\right] + (1-\theta)H(p_3, x, \omega) - H(\theta p_1 + (1-\theta)p_2, x, \omega)\right| \\
\leq & |H(p_3, x, \omega) - H(\theta p_1 + (1-\theta)p_2, x, \omega)| + \frac{\theta}{n} \\
< & \delta
\end{aligned}$$

The above is true for all $k = 0, 1, 2, \cdots, 2n^2 - 1$, thus

$$||H_n(p, x, \omega) - H(p, x, \omega)||_{L^\infty(K\times\mathbf{R}\times\mathbf{R})} < \delta$$

□

Remark 4. *By Lemma 1.2.2 and Lemma 1.3.2, to prove Theorem 2.3.1, it suffices to consider such Hamiltonian $H(p, x, \omega)$ that is constrained (see Definition 1.3.1) and satisfies* (**A1**)-(**A3**). *So in the following sections, we only consider constrained Hamiltonians.*

1.3.3 Constrained Hamiltonian with index $(\widetilde{L}, L)$

Definition 1.3.2. *$H(p, x, \omega)$ is called constrained Hamiltonian with index $(\widetilde{L}, L)$ if*
(1) $H(p, x, \omega)$ is constrained in the sense of Definition 1.3.1.
(2) $(a_1, b_1, a_2, b_2, \cdots, a_{k-1}, b_{k-1}, a_k) = (\widetilde{p}_1, \widetilde{q}_1, \widetilde{p}_2, \widetilde{q}_2, \cdots, \widetilde{p}_{\widetilde{L}}, \widetilde{q}_{\widetilde{L}}, 0, q_L, p_L, q_{L-1}, p_{L-1}, \cdots, q_1, p_1)$.
(3) $\operatorname*{ess\,sup}_{(x,\omega)} H(\widetilde{p}_i, x, \omega) > 0, 1 \le i \le \widetilde{L}$; $\operatorname*{ess\,sup}_{(x,\omega)} H(0, x, \omega) = 0$; $\operatorname*{ess\,sup}_{(x,\omega)} H(p_j, x, \omega) > 0$, $1 \le j \le L$.
(4) Each of $H(a_i, x, \omega)$, $H(b_i, x, \omega)$, $1 \le i \le k$ has no cluster point.

Remark 5. *Apply perturbation and shift coordinates if necessary, it suffices to consider homogenization of any constrained Hamiltonian with index $(\widetilde{L}, L)$.*

Definition 1.3.3. *Let $H(p, x, \omega)$ be a constrained Hamiltonian with index $(\widetilde{L}, L)$.*
(1) For each (x, ω), denote monotone branches of $H(p, x, \omega)$ by

$$H|_{[p_1,\infty)} := \phi_{1,(x,\omega)}(p),\ H|_{[q_1,p_1]} := \phi_{2,(x,\omega)}(p),\ \cdots,\ H|_{[0,q_L]} := \phi_{2L+1,(x,\omega)}(p)$$
$$H|_{[-\infty,\widetilde{p}_1)} := \widetilde{\phi}_{1,(x,\omega)}(p),\ H|_{[\widetilde{p}_1,\widetilde{q}_1]} := \widetilde{\phi}_{2,(x,\omega)}(p),\ \cdots,\ H|_{[\widetilde{q}_{\widetilde{L}},0]} := \widetilde{\phi}_{2\widetilde{L}+1,(x,\omega)}(p)$$

(2) Denote the inverse function of each branch by

$$\left(\phi_{i,(x,\omega)}(\cdot)\right)^{-1} := \psi_{i,(x,\omega)}(\cdot), \qquad \left(\widetilde{\phi}_{i,(x,\omega)}(\cdot)\right)^{-1} := \widetilde{\psi}_{i,(x,\omega)}(\cdot)$$

(3) Denote the local extreme values by

$$m_i(x, \omega) := H(p_i, x, \omega) \quad , \quad \widetilde{m}_i(x, \omega) := H(\widetilde{p}_i, x, \omega)$$
$$M_i(x, \omega) := H(q_i, x, \omega) \quad , \quad \widetilde{M}_i(x, \omega) := H(\widetilde{q}_i, x, \omega)$$

(4) Define the two functions

$$m(x,\omega) := \min\left\{\min_{1\le i\le L} m_i(x,\omega), \min_{1\le j\le \widetilde{L}} \widetilde{m}_j(x,\omega)\right\} \tag{1.3}$$

$$M(x,\omega) := \max\left\{\max_{1\le i\le L} M_i(x,\omega), \max_{1\le j\le \widetilde{L}} \widetilde{M}_j(x,\omega)\right\} \tag{1.4}$$

1.4 Auxiliary Lemmas for Gluing Lemmas

1.4.1 Two Lemmas on the bound of gradient in the ergodic problem

Lemma 1.4.1. *Let Hamiltonian $H(p,x,\omega)$ satisfy $(\mathbf{A1})$-$(\mathbf{A3})$ and be regularly homogenizable at p_0, for each $\lambda > 0$, let $v_\lambda(x,p_0,\omega)$ be the viscosity solution of the equation:*

$$\lambda v_\lambda + H(p_0+v_\lambda', x,\omega) = 0, \qquad x\in\mathbf{R}$$

fix $P\in\mathbf{R}$, denote $\underline{P} := \operatorname*{ess\,inf}_{(x,\omega)} H(P,x,\omega)$ and $\overline{P} := \operatorname*{ess\,sup}_{(x,\omega)} H(P,x,\omega)$, then there exists an $\widetilde{\Omega}\subset\Omega$ with $\mathbf{P}[\widetilde{\Omega}]=1$, such that, for each $\omega\in\widetilde{\Omega}$, the followings are true.
(1) If $\overline{H}(p_0) < \underline{P}$, $p_0 < P$, then for any $R>0$, there exists $\lambda_0=\lambda_0(R,p_0,\omega)>0$,

$$0<\lambda<\lambda_0 \Rightarrow p_0+v_\lambda'(x,p_0,\omega)\le P \quad \text{for } x\in\left[-\tfrac{R}{\lambda},\tfrac{R}{\lambda}\right]$$

(2) If $\overline{H}(p_0) < \underline{P}$, $p_0 > P$, then for any $R>0$, there exists $\lambda_0=\lambda_0(R,p_0,\omega)>0$,

$$0<\lambda<\lambda_0 \Rightarrow p_0+v_\lambda'(x,p_0,\omega)\ge P \quad \text{for } x\in\left[-\tfrac{R}{\lambda},\tfrac{R}{\lambda}\right]$$

(3) If $\overline{H}(p_0) > \overline{P}$, $p_0 < P$, then for any $R>0$, there exists $\lambda_0=\lambda_0(R,p_0,\omega)>0$,

$$0<\lambda<\lambda_0 \Rightarrow p_0+v_\lambda'(x,p_0,\omega)\le P \quad \text{for } x\in\left[-\tfrac{R}{\lambda},\tfrac{R}{\lambda}\right]$$

(4) If $\overline{H}(p_0) > \overline{P}$, $p_0 > P$, then for any $R>0$, there exists $\lambda_0=\lambda_0(R,p_0,\omega)>0$,

$$0<\lambda<\lambda_0 \Rightarrow p_0+v_\lambda'(x,p_0,\omega)\ge P \quad \text{for } x\in\left[-\tfrac{R}{\lambda},\tfrac{R}{\lambda}\right]$$

All these inequalities are understood in the viscosity sense.

Proof. of the periodic case (1) For p_0, we have the cell problem

$$H(p_0 + v', x) = \overline{H}(p_0)$$

Suppose (1) is not true, then there exists an $x_1 \in [0,1]$, such that $p_0 + v'(x_1) > P$. On the other hand,

$$\int_1^2 p_0 + v'(x) - Pdx = p_0 - P < 0$$

So there exists some $y_1 \in [1,2]$, such that $p_0 + v'(y_1) - P < 0$. Then $\Psi(x) := p_0 + v(x) - Px$ attains local maximum at some $z_1 \in [x_1, y_1]$, which means that

$$\underline{P} \leq H(P, z_1) \leq \overline{H}(p_0) < \underline{P}$$

This is a contradiction, so we proved (1). The proofs of (2), (3) and (4) are similar.

□

Proof. of the random case (1) If it is not true, then there exists some $\Omega_1 \subset \Omega$, $\mathbf{P}[\Omega_1] > 0$, such that for any $\omega \in \Omega_1$, there are $R_1 = R_1(p_0, \omega) > 0$ and $\lambda_n \to 0$ such that

$$p_0 + v'_{\lambda_n}(x_{\lambda_n}, p_0, \omega) > P \quad \text{for some } x_{\lambda_n} \in \left[-\tfrac{R_1}{\lambda_n}, \tfrac{R_1}{\lambda_n}\right]$$

Denote $\delta := P - p_0 > 0$. For any $R > 0$, we have

$$\left| \frac{\lambda}{R} \int_{\frac{R_1}{\lambda}}^{\frac{R+R_1}{\lambda}} v'_\lambda(s, \omega) ds \right| \leq \frac{2\big(H_{\sup}(p_0) - H_{\inf}(p_0)\big)}{R}$$

Fix any $R_2 = R_2(p_0) > \frac{4(H_{\sup}(p_0) - H_{\inf}(p_0))}{\delta}$, thus for any $R \geq R_2$, we have

$$\left| \frac{\lambda}{R} \int_{\frac{R_1}{\lambda}}^{\frac{R+R_1}{\lambda}} v'_\lambda(s, p_0, \omega) ds \right| < \frac{\delta}{2} \quad \text{for any } \lambda > 0$$

So

$$\frac{\lambda_n}{R_2} \int_{\frac{R_1}{\lambda_n}}^{\frac{R_2+R_1}{\lambda_n}} p_0 + v'_{\lambda_n}(s, p_0, \omega) - Pds \leq p_0 - P + \frac{\delta}{2} < 0$$

This implies

$$p_0 + v'_{\lambda_n}(y_{\lambda_n}, \omega) - P < 0 \quad \text{for some } y_{\lambda_n} \in \left(\tfrac{R_1}{\lambda_n}, \tfrac{R_2+R_1}{\lambda_n}\right)$$

Denote $\Psi(x,\omega) = p_0 x + v_{\lambda_n}(x,\omega) - Px$, then

$$\Psi(x,\omega) \text{ is increasing (decreasing) in a neighborhood of } x_{\lambda_n}(y_{\lambda_n})$$

Since $x_{\lambda_n} < y_{\lambda_n}$, $\Psi(x,\omega)$ attains local maximum at some $z_{\lambda_n} \in (x_{\lambda_n}, y_{\lambda_n})$. So

$$\lambda_n v_{\lambda_n}(z_{\lambda_n},\omega) + H(P, z_{\lambda_n}, \omega) \leq 0 \tag{1.5}$$

Since $H(p,x,\omega)$ is regularly homogenizable at p_0, there exists $\Omega_2 \subset \Omega$, such that $\mathbf{P}[\Omega_2] = 1$,

$$\limsup_{\lambda \Rightarrow 0} \sup_{|x| \leq \frac{R_1+R_2}{\lambda}} |\lambda v_\lambda(x,\omega) + \overline{H}(p_0)| = 0 \quad \text{for each } \omega \in \Omega_2$$

Denote $\tau := \underline{P} - \overline{H}(p_0) > 0$, $\hat{\Omega} := \Omega_1 \cap \Omega_2$. So there exists some $N_1(\omega)$,

$$\sup_{|x| \leq \frac{R_1+R_2}{\lambda_n}} |\lambda_n v_{\lambda_n} + \overline{H}(p_0)| < \frac{\tau}{2} \quad \text{for any } n \geq N_1 \tag{1.6}$$

$$\mathbf{P}[\Omega_1] > 0, \mathbf{P}[\Omega_2] = 1 \Rightarrow \mathbf{P}[\hat{\Omega}] > 0 \Rightarrow \hat{\Omega} \neq \emptyset$$

Choose any $\omega \in \hat{\Omega}$ and $n \geq N_1(\omega)$, by (1.5) and (1.6),

$$\underline{P} \leq H(P, z_{\lambda_n}, \omega) \leq -\lambda_n v_{\lambda_n}(z_{\lambda_n},\omega) \leq \overline{H}(p_0) + \tfrac{\tau}{2} = \underline{P} - \tau + \tfrac{\tau}{2} = \underline{P} - \tfrac{\tau}{2}$$

This is a contradiction. Thus (1) is proved. The proofs of (2), (3) and (4) are similar. □

Lemma 1.4.2. *Let $H(p,x,\omega)$ be the Hamiltonian that satisfies* (**A1**)-(**A3**) *and be regularly homogenizable at $p_0 \in \mathbf{R}$ to $\overline{H}(p_0)$, for each λ, let $v_\lambda(x)$ be the viscosity solution of the following equation:*

$$\lambda v_\lambda(x) + H(p_0 + v'_\lambda(x), x, \omega) = 0, \qquad x \in \mathbf{R}$$

For P, $Q \in \mathbf{R}$, denote

$$\underline{P} := \operatorname*{ess\,inf}_{(x,\omega)} H(P,x,\omega) \quad , \quad \overline{P} := \operatorname*{ess\,sup}_{(x,\omega)} H(P,x,\omega)$$
$$\underline{Q} := \operatorname*{ess\,inf}_{(x,\omega)} H(Q,x,\omega) \quad , \quad \overline{Q} := \operatorname*{ess\,sup}_{(x,\omega)} H(Q,x,\omega)$$

Then, there exists an $\widetilde{\Omega} \subset \Omega$ *with* $\mathbf{P}[\tilde{\Omega}] = 1$*, such that for each* $\omega \in \widetilde{\Omega}$*, the followings are true.*

(1) If $p_0 < P$*,* $P < Q$ *and* $\overline{P} < \underline{Q}$*, then for each* $R > 0$*, there exists* $\lambda_0 = \lambda_0(R, p_0, \omega)$*,*

$$0 < \lambda < \lambda_0 \Rightarrow p_0 + v'_\lambda(x) \le Q \quad \textit{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

(2) If $p_0 < P$*,* $P < Q$ *and* $\underline{P} > \overline{Q}$*, then for each* $R > 0$*, there exists* $\lambda_0 = \lambda_0(R, p_0, \omega)$*,*

$$0 < \lambda < \lambda_0 \Rightarrow p_0 + v'_\lambda(x) \le Q \quad \textit{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

(3) If $p_0 > P$*,* $P > Q$ *and* $\overline{P} < \underline{Q}$*, then for each* $R > 0$*, there exists* $\lambda_0 = \lambda_0(R, p_0, \omega)$*,*

$$0 < \lambda < \lambda_0 \Rightarrow p_0 + v'_\lambda(x) \ge Q \quad \textit{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

(4) If $p_0 > P$*,* $P > Q$ *and* $\underline{P} > \overline{Q}$*, then for each* $R > 0$*, there exists* $\lambda_0 = \lambda_0(R, p_0, \omega)$*,*

$$0 < \lambda < \lambda_0 \Rightarrow p_0 + v'_\lambda(x) \ge Q \quad \textit{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

All these inequalities are understood in the viscosity sense.

Proof. We only give the proof of (1), since the proofs of (2), (3) and (4) are similar.
Case 1: $\overline{H}(p_0) < \underline{P}$, apply (1) of Lemma 1.4.1 to (p_0, P).
Case 2: $\overline{H}(p_0) > \overline{P}$, apply (3) of Lemma 1.4.1 to (p_0, P).
Case 3: $\overline{H}(p_0) \in [\underline{P}, \overline{P}]$, apply (1) of Lemma 1.4.1 to (p_0, Q).

□

1.4.2 Squeeze Lemma

Lemma 1.4.3. *Let* $H(p, x, \omega)$ *satisfy* (**A1**)-(**A3**) *and be constrained with index* $(\widetilde{L}, L)$*. If* $H(p, x, \omega)$ *has effective Hamiltonian* $\overline{H}(p)$ *with* $\overline{H}(q) = 0$*, then the followings are true.*

(1) If $q > 0$ *and* $\overline{H}|_{(q,+\infty)} > 0$*, then* $\overline{H}(p) \equiv 0$ *for all* $p \in [0, q]$*.*
(2) If $q < 0$ *and* $\overline{H}|_{(-\infty,q)} > 0$*, then* $\overline{H}(p) \equiv 0$ *for all* $p \in [q, 0]$*.*

Proof. (1) Recall the notation (1.3) and $H(p, x, \omega)$ is constrained with index $(\widetilde{L}, L)$, we have

$$\mathbf{E}[m(x, \omega) > 0] > 0 \tag{1.7}$$

Denote:

$$\underline{M}_i := \operatorname*{ess\,inf}_{(x,\omega)\in\mathbf{R}\times\Omega} M_i(x,\omega), \qquad \underline{M}^+ := \max_{1\le i\le L} \underline{M}_i$$
$$\underline{\widetilde{M}}_i := \operatorname*{ess\,inf}_{(x,\omega)\in\mathbf{R}\times\Omega} \widetilde{M}_i(x,\omega), \qquad \underline{M}^- := \max_{1\le i\le \widetilde{L}} \underline{\widetilde{M}}_i$$

Case 1: $\min\left\{\underline{M}^+, \underline{M}^-\right\} > 0$. Denote

$$k_+ = \max\{1 \le i \le L | \underline{M}_i > 0\}, \qquad k_- = \max\{1 \le i \le \widetilde{L} | \underline{\widetilde{M}}_i > 0\}$$

$$\widehat{H}(p,x,\omega) := \begin{cases} \mathcal{L}|p - \widetilde{q}_{k_-}| + H(\widetilde{q}_{k_-}, x, \omega) & p \in (-\infty, \widetilde{q}_{k_-}) \\ H(p,x,\omega) & p \in [\widetilde{q}_{k_-}, q_{k_+}] \\ \mathcal{L}|p - q_{k_+}| + H(q_{k_+}, x, \omega) & p \in (q_{k_+}, +\infty) \end{cases}$$

By section 1.8, $\widehat{H}(p,x,\omega)$ has a level-set convex effective Hamiltonian $\overline{\widehat{H}}(p) \ge 0$ with $\overline{\widehat{H}}(0) = 0$. For any $\lambda > 0$, let $v_\lambda(x,q,\omega)$ and $\widehat{v}_\lambda(x,q,\omega)$ be solutions of the following equations respectively,

$$\lambda v_\lambda + H(q + v'_\lambda, x, \omega) = 0,\ x \in \mathbf{R}, \qquad \lambda \widehat{v}_\lambda + \widehat{H}(q + \widehat{v}'_\lambda, x, \omega) = 0,\ x \in \mathbf{R}$$

Claim: $q_{k_-} < q < q_{k_+}$.

Proof of the **Claim**: Suppose it is not true.

(I) If $q = q_{k_+}$, then $0 = \overline{H}(q) = \overline{H}(q_{k_+}) \ge \underline{M}_{k_+} > 0$, this is a contradiction.

(II) If $q > q_{k_+}$. The arguments are divided into the following (II-1), (II-2) and (II-3).

(II-1) By Lemma 1.4.1, there exists $\Omega_1 \subset \Omega$, $\mathbf{P}[\Omega_1] = 1$ such that if $\omega \in \Omega_1$, then for any $R > 0$, there exists $\lambda_1 = \lambda_1(R,q,\omega) > 0$,

$$0 < \lambda < \lambda_1 \Rightarrow q + v'_\lambda(x,q,\omega) \ge q_{k_+} \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

(II-2) By (1.7), there are $\delta > 0$ and $\tau > 0$ such that $\mathbf{E}\left[m(x,\omega) > \delta\right] = \tau$. By ergodic theorem, there exists $\Omega_2 \subset \Omega$, $\mathbf{P}[\Omega_2] = 1$, for each $\omega \in \Omega_2$ and $R > 0$,

$$\lim_{\lambda\to 0} \frac{2\lambda}{R} \int_{-\frac{R}{\lambda}}^{\frac{R}{\lambda}} \mathbf{1}_{\{m(\cdot,\omega)>\delta\}}(x)dx = \mathbf{E}\left[m(x,\omega) > \delta\right] = \tau$$

So there exists some $\lambda_2(R,q,\omega) > 0$, such that

$$0 < \lambda < \lambda_2(R,q,\omega) \Rightarrow \frac{2\lambda}{R} \int_{-\frac{R}{\lambda}}^{\frac{R}{\lambda}} \mathbf{1}_{\{m(\cdot,\omega)>\delta\}}(x)dx > \frac{\tau}{2}$$

(II-3) Since $\overline{H}(q) = 0$, there exists $\Omega_3 \subset \Omega$, $\mathbf{P}[\Omega_3] = 1$. For each $\omega \in \Omega_3$ and $R > 0$, there exists $\lambda_3 = \lambda_3(R, q, \omega) > 0$,

$$0 < \lambda < \lambda_3 \Rightarrow |\lambda v_\lambda(x, q, \omega)| < \delta \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Denote

$$\begin{aligned} \widetilde{\lambda}(R, q, \omega) &:= \min\{\lambda_1(R, q, \omega), \lambda_2(R, q, \omega), \lambda_3(R, q, \omega)\} > 0 \\ \widetilde{\Omega} &:= \Omega_1 \bigcap \Omega_2 \bigcap \Omega_3 \end{aligned}$$

Then $\mathbf{P}[\widetilde{\Omega}] = 1$.

For each $\omega \in \widetilde{\Omega}$, when $\lambda < \widetilde{\lambda}(R, q, \omega)$, there exists $x_\lambda \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$, $m(x_\lambda, \omega) > \delta$, and then

$$\delta < m(x_\lambda, \omega) \le H(q + v'_\lambda(x_\lambda, q, \omega), x_\lambda, \omega) = -\lambda v_\lambda(x_\lambda, q, \omega) < \delta$$

This is a contradiction. (The second inequality is because we have $q + v'_\lambda(x_\lambda, q, \omega) \ge q_{k_+}$).

So, we conclude $q < q_{k_+}$. Similarly, we can prove $q_{k_-} < q$. This ends the proof of the **Claim**.

By Lemma 1.4.1, there exists $\widehat{\Omega}$, $\mathbf{P}[\widehat{\Omega}] = 1$, for $\omega \in \widehat{\Omega}$ and any $R > 0$, there exists $\widehat{\lambda}(R, q, \omega) > 0$,

$$0 < \lambda < \widehat{\lambda} \Rightarrow q_{k_+} \le q + v'_\lambda(x, q, \omega) \le q_{k_+} \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

So

$$\lambda v_\lambda(x, q, \omega) + \widehat{H}(q + v'_\lambda(x, q, \omega), x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists some constant $C > 0$, such that

$$|\lambda v_\lambda(0, q, \omega) - \lambda \widehat{v}_\lambda(0, q, \omega)| \le \frac{C}{R}$$

$R > 0$ can be chosen arbitrarily large, then

$$\overline{\widehat{H}}(q) = \lim_{\lambda \to 0} -\lambda \widehat{v}_\lambda(0, q, \omega) = \lim_{\lambda \to 0} -\lambda v_\lambda(0, q, \omega) = \overline{H}(q) = 0$$

By level-set convexity of $\overline{\widehat{H}}(p)$ and $\overline{\widehat{H}}(0) = 0$, $\overline{\widehat{H}}|_{[0,q]} \equiv 0$. Since $\widehat{H}(p, x, \omega) \geq H(p, x, \omega)$, $\overline{\widehat{H}}(p) \geq \overline{H}(p)$. On the other hand, $\overline{H}(p) \geq 0$. So $\overline{H}|_{[0,q]} \equiv 0$.

Case 2: $\min\left\{\underline{M}^+, \underline{M}^-\right\} \leq 0 < \max\left\{\underline{M}^+, \underline{M}^-\right\}$.

Construct $\widehat{H}(p, x, \omega)$ by modifying one side and similar arguments thereafter.

Case 3: $\max\left\{\underline{M}^+, \underline{M}^-\right\} \leq 0$.

By section 1.8, $\overline{H}(p)$ is level-set convex. Define $\widehat{H}(p, x, \omega) := H(p, x, \omega)$ and apply the result of **Case 1**. This finishes the proof of (1).

The proof for (2) is similar.

$\square$

1.5 Reduction by Constrained Hamiltonian with Index $(\widetilde{L}, 0)$ and $(0, L)$

Let $H(p, x, \omega)$ be a constrained Hamiltonian that satisfies (**A1**)-(**A3**). Define

$$H^+(p, x, \omega) := \begin{cases} H(p, x, \omega) & p \geq 0 \\ \mathcal{L}|p| + H(0, x, \omega) & p < 0 \end{cases}$$

$$H^-(p, x, \omega) := \begin{cases} \mathcal{L}|p| + H(0, x, \omega) & p \geq 0 \\ H(p, x, \omega) & p < 0 \end{cases}$$

Lemma 1.5.1. *If both $H^+(p, x, \omega)$ and $H^-(p, x, \omega)$ are regularly homogenizable for all $p \in \mathbf{R}$, then $H(p, x, \omega)$ is also regularly homogenizable for all $p \in \mathbf{R}$ and*

$$\overline{H}(p) = \begin{cases} \overline{H^+}(p) & p \geq 0 \\ \overline{H^-}(p) & p < 0 \end{cases}$$

Proof. Fix $p \geq 0$, $\omega \in \Omega$ and $\lambda > 0$, let $v_\lambda(x, p, \omega)$ and $v_{+,\lambda}(x, p, \omega)$ be solutions of the equations respectively,

$$\lambda v_\lambda + H(p + v_\lambda', x, \omega) = 0, x \in \mathbf{R}; \qquad \lambda v_{+,\lambda} + H^+(p + v_{+,\lambda}', x, \omega) = 0,\ x \in \mathbf{R}$$

By $H^+(p,x,\omega) \geq H(p,x,\omega)$, $\operatorname*{ess\,sup}_{(x,\omega)} H(p,x,\omega) \geq 0$ and classical comparison principle, we have

$$\liminf_{\lambda\to 0} -\lambda v_{+,\lambda}(0,p,\omega) \geq \liminf_{\lambda\to 0} -\lambda v_\lambda(0,p,\omega) \geq 0$$

Thus, if $\overline{H^+}(p) = 0$, then $\overline{H}(p) = 0$. Since $\overline{H^+}(0) = 0$, we can only consider the case: $p > 0$ and $\overline{H^+}(p) > 0$. By Lemma 1.4.1, for a.e. $\omega \in \Omega$, any $R > 0$, there exists $\lambda_0 = \lambda_0(R,p,\omega) > 0$, such that

$$0 < \lambda < \lambda_0 \Rightarrow p + v'_{+,\lambda}(x,,\omega) \geq 0, \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

So

$$\lambda v_{+,\lambda} + H(p + v'_{+,\lambda}, x, \omega) = 0, \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists a constant $C > 0$ such that

$$|\lambda v_{+,\lambda}(0,p,\omega) - \lambda v_\lambda(0,p,\omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large,

$$\lim_{\lambda\to 0} -\lambda v_\lambda(0,p,\omega) = \lim_{\lambda\to 0} -\lambda v_{+,\lambda}(0,p,\omega) = \overline{H^+}(p)$$

So $\overline{H}(p) = \overline{H^+}(p)$, $p \geq 0$. Similarly, we can prove $\overline{H}(p) = \overline{H^-}(p)$, $p \leq 0$.

□

Remark 6. *By Lemma 1.5.1, to prove the homogenization of a Hamiltonian that satisfies* (**A1**)-(**A3**) *and is constrained with index* $(\widetilde{L}, L)$*, it suffices to study those Hamiltonians that have index* $(0, L)$ *or* $(\widetilde{L}, 0)$*. Without loss of generality, in the following sections, we only consider the Hamiltonian under assumptions* (**A1**)-(**A3**) *and be constrained with index* $(0, L)$*.*

1.6 Gluing Lemmas: Reduction from Small Oscillation to Large Oscillation

In this section, $H(p,x,\omega)$ satisfies (**A1**)-(**A3**) and is constrained with index $(0, L)$. Denote

$$\underline{M} := \operatorname*{ess\,inf}_{(x,\omega)\in\mathbf{R}\times\Omega} M(x,\omega), \qquad \overline{m} := \operatorname*{ess\,sup}_{(x,\omega)\in\mathbf{R}\times\Omega} m(x,\omega)$$

There are $1 \leq \underline{k}, \overline{k} \leq L$, such that

$$\underline{M} := \operatorname*{ess\,inf}_{(x,\omega)\in\mathbf{R}\times\Omega} M_{\underline{k}}(x,\omega), \qquad \overline{m} := \operatorname*{ess\,sup}_{(x,\omega)\in\mathbf{R}\times\Omega} m_{\overline{k}}(x,\omega)$$

Definition 1.6.1 (Oscillation)**.** *Let $H(p,x,\omega)$ be constrained (see Definition 1.3.1) and satisfies* (**A1**)-(**A3**)*.*

(1) $H(p,x,\omega)$ has small oscillation if $\underline{M} \geq \overline{m}$.

(2) $H(p,x,\omega)$ has large oscillation if $\underline{M} < \overline{m}$.

Throughout this section, we assume small oscillation and denote

$$P := p_{\overline{k}} \quad \text{and} \quad Q := q_{\underline{k}}$$

1.6.1 Left Steep Side

Left steep side means $\underline{M} > \overline{m}$ and $P < Q$. Define

$$H_1(p,x,\omega) := \begin{cases} H(p,x,\omega) & p \leq Q \\ \mathcal{L}|p-Q| + H(Q,x,\omega) & p > Q \end{cases}$$

$$H_3(p,x,\omega) := \begin{cases} H(p,x,\omega) & p \geq q_{\bar{k}} \\ \mathcal{L}|p-q_{\bar{k}}| + H(q_{\bar{k}},x,\omega) & p < q_{\bar{k}} \end{cases}$$

$$H_2(p,x,\omega) := \max\{H_1(p,x,\omega), H_3(p,x,\omega)\}$$

Lemma 1.6.1. *Assume $H_i(p,x,\omega), i = 1,2,3$ are all regularly homogenizable for any $p \in \mathbf{R}$. Then $H(p,x,\omega)$ is also regularly homogenizable for any $p \in \mathbf{R}$ and*

$$\overline{H}(p) = \min\left\{\overline{H_1}(p), \overline{H_3}(p)\right\}$$

Proof. of the periodic case For any $p \in \mathbf{R}$, we have the cell problem

$$H(p + v'(x), x) = \overline{H}(p) \tag{1.8}$$

Proof by contradiction, if there are $x_1, x_2 \in [0,1]$, such that $p + v'(x_1) > Q$ and $p + v'(x_2) < P$. Then $px + v(x) - Qx$ attains local maximum at some $y_1 \in (x_1, x_2 + 1)$ and $px + v(x) - Px$ attains local minimum at some $y_2 \in (x_2, x_1 + 1)$. Thus we get a contradiction from equalities:

$$\min_{x\in[0,1]} H(Q,x) = \underline{M} \leq H(Q, y_1) \leq \overline{H}(p) \leq H(P, y_2) \leq \overline{m} = \max_{x\in[0,1]} H(P,x)$$

Thus, either $p + v'(x) \leq Q$ for all $x \in [0,1]$ or $p + v'(x) \geq P$ for all $x \in [0,1]$. By (1.8), either $\overline{H}(p) = \overline{H_1}(p)$ or $\overline{H}(p) = \overline{H_3}(p)$. On the other hand, since $H(p, x, \omega) = \min\{H_1(p, x, \omega), H_3(p, x, \omega)\}$, by comparison principle, we have $\overline{H}(p) \leq \{\overline{H_1}(p), \overline{H_3}(p)\}$. Eventually, we conclude

$$\overline{H}(p) = \{\overline{H_1}(p), \overline{H_3}(p)\}$$

□

Proof. of the random case Decompose $\mathbf{R}$ into three parts.

(**1**) If $p \in (-\infty, P)$, then $\overline{H}(p) = \overline{H_1}(p)$.

For each $\omega \in \Omega$ and $\lambda > 0$, let $v_\lambda(x, p, \omega)$ and $v_{1,\lambda}(x, p, \omega)$ be solutions of the equations respectively,

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0, \ x \in \mathbf{R}; \qquad \lambda v_{1,\lambda} + H_1(p + v'_{1,\lambda}, x, \omega) = 0, \ x \in \mathbf{R}$$

By Lemma 1.4.2, there exists $\widetilde{\Omega} \subset \Omega$, $\mathbf{P}[\widetilde{\Omega}] = 1$. For $\omega \in \widetilde{\Omega}$, any $R > 0$, there exists $\lambda_0 = \lambda_0(R, p, \omega) > 0$,

$$0 < \lambda < \lambda_0 \Rightarrow p + v'_{1,\lambda}(x, p, \omega) \leq Q \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Thus, for $0 < \lambda < \lambda_0(R, p, \omega)$,

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \text{ and } \lambda v_{1,\lambda} + H(p + v'_{1,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists $C = C(p)$, such that

$$|\lambda v_\lambda(0,p,\omega) - \lambda v_{1,\lambda}(0,p,\omega)| \le \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda\to 0^+} -\lambda v_\lambda(0,p,\omega) = \lim_{\lambda\to 0^+} -\lambda v_{1,\lambda}(0,p,\omega) = \overline{H_1}(p)$$

Thus $H(p,x,\omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H_1}(p)$, $p \in (-\infty, P)$.
(**2**) $p \in (Q,\infty)$, then $\overline{H}(p) = \overline{H_3}(p)$.
For each $\omega \in \Omega$ and $\lambda > 0$, let $v_\lambda(x,p,\omega)$ and $v_{3,\lambda}(x,p,\omega)$ be solutions of the equations respectively,

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0,\ x \in \mathbf{R}; \qquad \lambda v_{3,\lambda} + H_3(p + v'_{3,\lambda}, x, \omega) = 0,\ x \in \mathbf{R}$$

By Lemma 1.4.2, there exists $\widetilde{\Omega} \subset \Omega$, $\mathbf{P}[\widetilde{\Omega}] = 1$. For $\omega \in \widetilde{\Omega}$, any $R > 0$, there exists some $\lambda_0 = \lambda_0(R,\omega,p) > 0$,

$$0 < \lambda < \lambda_0 \Rightarrow p + v'_{3,\lambda}(x,p,\omega) \ge P, \quad x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Thus, for $0 < \lambda < \lambda_0(R,p,\omega)$, we have

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \text{ and } \lambda v_{3,\lambda} + H(p + v'_{3,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists $C = C(p)$, such that

$$|\lambda v_\lambda(0,p,\omega) - \lambda v_{3,\lambda}(0,p,\omega)| \le \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda\to 0^+} -\lambda v_\lambda(0,p,\omega) = \lim_{\lambda\to 0^+} -\lambda v_{3,\lambda}(0,p,\omega) = \overline{H_3}(p)$$

Thus $H(p,x,\omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H_3}(p)$, $p \in (Q,\infty)$.
The proof of part (**3**) is divided into the following smaller parts:
(**3.1**) Denote:

$$A := \left\{p \in (P,Q) \middle| \overline{m} < \overline{H_2}(p) < \underline{M}\right\}$$

Fix any $p \in A$, for any $\lambda > 0$, let $v_\lambda(x,p,\omega), v_{2,\lambda}(x,p,\omega)$ be solutions of the equations respectively,

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0,\ x \in \mathbf{R}; \qquad \lambda v_{2,\lambda} + H_2(p + v'_{2,\lambda}, x, \omega) = 0,\ x \in \mathbf{R}$$

By Lemma 1.4.1, for each $\omega \in \widetilde{\Omega}$, any $R > 0$, there exists $\lambda_0 = \lambda_0(R,p,\omega) > 0$, such that

$$0 < \lambda < \lambda_0 \Rightarrow P \leq p + v'_{2,\lambda}(x,\omega) \leq Q \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

So

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \text{ and } \lambda v_{2,\lambda} + H(p + v'_{2,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists $C = C(p)$, such that

$$|\lambda v_\lambda(0,p,\omega) - \lambda v_{2,\lambda}(0,p,\omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda \to 0^+} -\lambda v_\lambda(0,p,\omega) = \lim_{\lambda \to 0^+} -\lambda v_{2,\lambda}(0,p,\omega) = \overline{H_2}(p)$$

Thus $H(p,x,\omega)$ is regularly homogenizable at p and

$$\overline{H}(p) = \overline{H_2}(p) \geq \{\overline{H_1}(p), \overline{H_3}(p)\}$$

On the other hand,

$$\overline{H}(p) \leq \min\{\overline{H_1}(p), \overline{H_3}(p)\}$$

As a consequence,

$$\overline{H}(p) = \overline{H_1}(p) = \overline{H_2}(p) = \overline{H_3}(p), \quad p \in A$$

(**3.2**) For $p \in \mathbf{R}$, if $\overline{H_1}(p) < \underline{M}$, then $\overline{H}(p) = \overline{H_1}(p)$. The assumption $\overline{H_1}(p) < \underline{M}$ implies $p < Q$. By Lemma 1.4.1, for $\omega \in \widetilde{\Omega}$, any $R > 0$, there exists $\lambda_0 = \lambda_0(R,p,\omega) > 0$, such that

$$0 < \lambda < \lambda_0 \Rightarrow p + \lambda v_{1,\lambda}(x,p,\omega) \leq Q \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

So

$$\lambda v_\lambda + H(p+v'_\lambda, x, \omega) = 0 \text{ and } \lambda v_{1,\lambda} + H(p+v'_{1,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists $C = C(p)$, such that

$$|\lambda v_\lambda(0,p,\omega) - \lambda v_{1,\lambda}(0,p,\omega)| \le \frac{C}{R}$$

Since R can be chosen arbitrarily large,

$$\lim_{\lambda\to 0^+} -\lambda v_\lambda(0,p,\omega) = \lim_{\lambda\to 0^+} -\lambda v_{1,\lambda}(0,p,\omega) = \overline{H_1}(p)$$

Thus $H(p,x,\omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H_1}(p)$.

(**3**.**3**) For $p > P$, if $\overline{H_3}(p) > \overline{m}$, then $\overline{H}(p) = \overline{H_3}(p)$.

By Lemma 1.4.1, for each $\omega \in \widetilde{\Omega}$, any $R > 0$, there exists $\lambda_0 = \lambda_0(R,p,\omega) > 0$, such that

$$0 < \lambda < \lambda_0 \Rightarrow p + v'_{3,\lambda}(x,p,\omega) \ge P \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

So

$$\lambda v_\lambda + H(p+v'_\lambda, x, \omega) = 0 \text{ and } \lambda v_{3,\lambda} + H(p+v'_{3,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists $C = C(p)$, such that

$$|\lambda v_\lambda(0,p,\omega) - \lambda v_{3,\lambda}(0,p,\omega)| \le \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda\to 0^+} -\lambda v_\lambda(0,p,\omega) = \lim_{\lambda\to 0^+} -\lambda v_{3,\lambda}(0,p,\omega) = \overline{H_3}(p)$$

Thus $H(p,x,\omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H_3}(p)$.

(**3**.**4**) For $p < Q$, if $\overline{H_3}(p) < \underline{M}$, then $\overline{H_2}(p) = \overline{H_3}(p) < \underline{M}$. By Lemma 1.4.1, for each $\omega \in \widetilde{\Omega}$, any $R > 0$, there exists $\lambda_0 = \lambda_0(R,p,\omega) > 0$, such that

$$0 < \lambda < \lambda_0 \Rightarrow p + v'_{3,\lambda}(x,p,\omega,) \le Q \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Here, for any $\lambda > 0$, $v_{3,\lambda}$ is the solution of the equation

$$\lambda v_{3,\lambda} + H_3(p+v'_{3,\lambda}, x, \omega) = 0, \qquad x \in \mathbf{R}$$

However, by the above upper bound,

$$\lambda v_{3,\lambda} + H_2(p + v'_{3,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Suppose for any $\lambda > 0$, $v_{2,\lambda}(x, p, \omega)$ is the solution of the equation:

$$\lambda v_{2,\lambda} + H_2(p + v'_{2,\lambda}, x, \omega) = 0, \qquad x \in \mathbf{R}$$

By Lemma 1.2.3, there exists $C = C(p)$, such that

$$|\lambda v_{2,\lambda}(0, p, \omega) - \lambda v_{3,\lambda}(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\overline{H_2}(p) = \lim_{\lambda \to 0^+} -\lambda v_{2,\lambda}(0, p, \omega) = \lim_{\lambda \to 0^+} -\lambda v_{3,\lambda}(0, p, \omega) = \overline{H_3}(p)$$

Now, we discuss the regularly homogenization of $\overline{H}(p)$ for $p \in [P, Q] \cap A^c$.

(**I**) If $p \in (P, Q)$ and $\overline{H_2}(p) \leq \overline{m}$, by the fact $\overline{m} < \underline{M}$ and

$$\max\{\overline{H_1}(p), \overline{H_3}(p)\} \leq \overline{H_2}(p)$$

we have $\overline{H_1}(p) < \underline{M}$, by (**3.2**), $\overline{H}(p) = \overline{H_1}(p)$.

(**II**) If $p \in (P, Q)$ and $\overline{H_2}(p) \geq \underline{M}$, then by (**3.4**), $\overline{H_3}(p) \geq \underline{M} > \overline{m}$. By (**3.3**), $\overline{H}(p) = \overline{H_3}(p)$.

(**III**) By Corollary 1.2.1, we have

$$\overline{H}(P) = \overline{H_1}(P) \text{ and } \overline{H}(Q) = \overline{H_3}(Q)$$

In all, for any $p \in \mathbf{R}$, either $\overline{H}(p) = \overline{H_1}(p)$ or $\overline{H}(p) = \overline{H_3}(p)$, so

$$\overline{H}(p) \geq \min\left\{\overline{H_1}(p), \overline{H_3}(p)\right\}$$

On the other hand, by classical comparison principle,

$$\overline{H}(p) \leq \min\left\{\overline{H_1}(p), \overline{H_3}(p)\right\}$$

So, we have proved:

$$\overline{H}(p) = \min\left\{\overline{H_1}(p), \overline{H_3}(p)\right\}$$

□

1.6.2 Right Steep Side:

Right steep side means $\underline{M} > \overline{m}$ and $Q \le P$. Define

$$H_1(p,x,\omega) := \begin{cases} H(p,x,\omega) & p \le Q \\ \mathcal{L}|p-Q| + H(Q,x,\omega) & p > Q \end{cases}$$

$$H_2(p,x,\omega) = \begin{cases} -\mathcal{L}|p| + H(0,x,\omega) & p < 0 \\ H(p,x,\omega) & 0 \le p \le P \\ -\mathcal{L}|p-P| + H(P,x,\omega) & p > P \end{cases}$$

$$H_3(p,x,\omega) := \begin{cases} H(p,x,\omega) & p \ge Q \\ \mathcal{L}|p-Q| + H(Q,x,\omega) & p < Q \end{cases}$$

Lemma 1.6.2. *Assume both $H_1(p,x,\omega)$ and $H_3(p,x,\omega)$ are regularly homogenizable for all $p \in \mathbf{R}$, then $H(p,x,\omega)$ is also regularly homogenizable for all p and*

$$\overline{H}(p) = \begin{cases} \overline{H_1}(p) & p \le 0 \\ \min\{\overline{H_1}(p), \overline{H_3}(p), \underline{M}\} & p \in (0,P) \\ \overline{H_3}(p) & p \ge P \end{cases}$$

Proof. of the periodic case for the middle equality For $p \in (0,P)$, we have the cell problem

$$H(p+v'(x),x) = \overline{H}(p)$$

If $p+v'(x) \le Q$, $\forall x \in [0,1]$ or $p+v'(x) \ge Q$, $\forall x \in [0,1]$, then $\overline{H}(p) = \overline{H_1}(p)$ or $\overline{H}(p) = \overline{H_3}(p)$. Otherwise, by the assumption that $\underline{M} > \overline{m}$, we have $p+v'(x) \in [0,P]$, $\forall x \in [0,1]$. There exists some $x_0 \in [0,1]$, such that $H(Q,x_0) = \min_x \max_{q \in [0,P]} H(q,x) = \underline{M}$.

So we have $\overline{H}(p) = H(p+v'(x_0),x_0) \le \underline{M}$. Therefore,

$$\overline{H}(p) \le \min\{\overline{H_1}(p), \overline{H_3}(p), \underline{M}\}$$

If $\overline{H}(p) < \underline{M}$, then by Lemma 1.4.1, we have either $p+v'(x) \le Q$, $\forall x \in [0,1]$ or $p+v'(x) \ge Q$, $\forall x \in [0,1]$ and so $\overline{H}(p) = \overline{H_1}(p)$ or $\overline{H}(p) = \overline{H_3}(p)$.

□

Proof. of the random case STEP 1: Proof of the first equality. First define

$$f(\theta) := \operatorname*{ess\,sup}_{(x,\omega)\in\mathbf{R}\times\Omega} [H(\theta Q, x, \omega)]$$

Then $f(0) = 0$, $f(1) \geq \underline{M} > \overline{m} > 0$. By the continuity of f, there exists some $\theta_0 \in (0,1)$, such that $0 < f(\theta_0) < \underline{M}$. For any $p \leq 0$, $\lambda > 0$, let $v_{1,\lambda}(x,p,\omega)$ be the solution of the equation

$$\lambda v_{1,\lambda} + H_1(p + v'_{1,\lambda}, x, \omega) = 0, \qquad x \in \mathbf{R}$$

Apply Lemma 1.4.2 to $(p, \theta_0 Q, Q)$ and $H_1(p,x,\omega)$, then for a.e. $\omega \in \Omega$, we have that for any $R > 0$, there exists $\lambda_0 = \lambda_0(R,p,\omega) > 0$,

$$0 < \lambda < \lambda_0 \Rightarrow p + v'_{1,\lambda}(x,p,\omega) \leq Q, \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Then by the definition of $H_1(p,x,\omega)$, we have

$$\lambda v_{1,\lambda} + H(p + v'_{1,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

For any $\lambda > 0$, let v_λ be the unique viscosity solution of the equation

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \quad \text{for } x \in \mathbf{R}$$

By Lemma 1.2.3, there exists $C = C(p) > 0$, such that

$$|\lambda v_\lambda(0,p,\omega) - \lambda v_{1,\lambda}(0,p,\omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large,

$$\lim_{\lambda\to 0} -\lambda v_\lambda(0,p,\omega) = \lim_{\lambda\to 0} -\lambda v_{1,\lambda}(0,p,\omega) = \overline{H_1}(p)$$

Thus, H is regularly homogenizable at p and

$$\overline{H}(p) = \overline{H_1}(p), \quad p \leq 0$$

STEP 2: Proof of the third equality. Similar as the proof of STEP 1.

STEP 3: The second equality. We divide the proofs into the following sub-STEPs

(**3.1**) **Claim:** For $p_0 \in \mathbf{R}$, if $\overline{H_1}(p_0) < \underline{M}$, then $H(p,x,\omega)$ is regularly homogenizable at p_0 and $\overline{H}(p_0) = \overline{H_1}(p_0)$. □

Proof. of ***Claim (3.1)*** By the definition of $H_1(p,x,\omega)$, $\overline{H_1}(p_0) < \underline{M}$ implies $p < Q$ (since $\overline{H_1}(p) \geq \underline{M}$ for $p \geq Q$). For each $\omega \in \Omega$ and $\lambda > 0$, let $v_\lambda(x,p_0,\omega)$ and $v_{1,\lambda}(x,p_0,\omega)$ be solutions of the equations respectively,

$$\lambda v_\lambda + H(p_0 + v'_\lambda, x, \omega) = 0,\ x \in \mathbf{R}; \qquad \lambda v_{1,\lambda} + H_1(p_0 + v'_{1,\lambda}, x, \omega) = 0,\ x \in \mathbf{R}$$

By Lemma 1.4.1, for a.e. $\omega \in \Omega$, we have the following: for each $R > 0$, there exists $\lambda_1 = \lambda_1(R, p_0, \omega) > 0$, such that

$$0 < \lambda < \lambda_0 \Rightarrow p_0 + v'_{1,\lambda} \leq Q \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

So

$$\lambda v_{1,\lambda} + H(p_0 + v'_{1,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists $C = C(p_0) > 0$, such that

$$|\lambda v_\lambda(0,p_0,\omega) - \lambda v_{1,\lambda}(0,p_0,\omega)| < \frac{C}{R}$$

Since we can choose arbitrarily large R, we have that

$$\lim_{\lambda\to 0} -\lambda v_\lambda(0,p_0,\omega) = \lim_{\lambda\to 0} -\lambda v_{1,\lambda}(0,p_0,\omega) = \overline{H_1}(p_0)$$

Thus $H(p,x,\omega)$ is regularly homogenizable at p_0 and $\overline{H}(p_0) = \overline{H_1}(p_0)$.

□

(**3.2**) **Claim:** For $p_0 \in \mathbf{R}$, if $\overline{H_3}(p_0) < \underline{M}$, then $H(p,x,\omega)$ is regularly homogenizable at p_0 and $\overline{H}(p_0) = \overline{H_3}(p_0)$.

Proof. of ***Claim (3.2)*** The proof is similar to the proof of **Claim (3.1)**. □

(**3.3**) Denote

$$q_1 = \min\left\{p \in [0,P] \middle| \overline{H_1}(p) = \underline{M}\right\} \quad \text{and} \quad q_2 = \max\left\{p \in [0,P] \middle| \overline{H_3}(p) = \underline{M}\right\}$$

(**3.1**), (**3.2**) $\Rightarrow$ $H(p,x,\omega)$ is regularly homogenizable for $p \in (0,q_1) \bigcup (q_2, P)$ and

$$\overline{H}(p) = \begin{cases} \overline{H_1}(p) & p \in (0,q_1) \\ \overline{H_3}(p) & p \in (q_2,P) \end{cases}$$

By Corollary 1.2.1, $H(p,x,\omega)$ is regularly homogenizable at q_1 and q_2 and

$$\overline{H}(q_1) = \overline{H}(q_2) = \underline{M}$$

(**3**.4) **Claim:** $H_2(p,x,\omega)$ is regularly homogenizable at q_1 and q_2, moreover,

$$\overline{H_2}(q_1) = \overline{H_2}(q_2) = \underline{M}$$

Proof. of ***Claim (3.4)*** By the definition, we have $q_1, q_2 \in (0, P_0)$. For any $\omega \in \Omega$, $\lambda > 0$, let $v_\lambda(x, q_i, \omega)$ and $v_{2,\lambda}(x, q_i, \omega)$ $(i = 1, 2)$ be solutions to the following equations respectively,

$$\lambda v_\lambda + H(q_i + v'_\lambda, x, \omega) = 0,\ x \in \mathbf{R}; \qquad \lambda v_{2,\lambda} + H_2(q_i + v'_{2,\lambda}, x, \omega) = 0,\ x \in \mathbf{R}$$

By the fact that

$$\overline{H}(q_i) = \underline{M} > \max\{\operatorname*{ess\,sup}_{(x,\omega)} H(0,x,\omega), \operatorname*{ess\,sup}_{(x,\omega)} H(P,x,\omega)\} = \overline{m}$$

By Lemma 1.4.1, then for a.e. $\omega \in \Omega$, for any $R > 0$, there exists $\lambda_2 = \lambda_2(q_i, R, \omega) > 0$,

$$0 < \lambda < \lambda_2 \Rightarrow 0 \le q_i + v'_\lambda(x, q_i, \omega) \le Q, \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

So we have

$$\lambda v_\lambda + H_2(q_i + v'_\lambda, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Apply Lemma 1.2.3, there exists some constant $C = C(q_i) > 0$, such that

$$|\lambda v_\lambda(0, q_i, \omega) - \lambda v_{2,\lambda}(0, q_i, \omega)| < \frac{C}{R}$$

We can choose arbitrarily large R, so

$$\lim_{\lambda \to 0} -\lambda v_{2,\lambda}(0, q_i, \omega) = \lim_{\lambda \to 0} -\lambda v_\lambda(0, q_i, \omega) = \overline{H}(q_i) = \underline{M}$$

Thus, $H_2(p,x,\omega)$ is regularly homogenizable at q_i and $\overline{H_2}(q_i) = \underline{M}$. □

(**3**.**5**) Denote

$$\widehat{M}(x,\omega) := \max_{\overline{k} \le j \le L, j \ne \underline{k}} M_j(x,\omega)$$

Then we have

$$\underline{\widehat{M}} := \operatorname*{ess\,inf}_{(x,\omega)\in\mathbf{R}\times\Omega} \widehat{M}(x,\omega) \le \underline{M}$$

By Lemma 1.2.2, without loss of generality, we can further assume $\underline{\widehat{M}} < \underline{M}$. This means that $\mathbf{E}[\widehat{M}(x,\omega) < \underline{M}] > 0$. Denote $\widetilde{H}(p,x,\omega) := -H_2(q_{\underline{k},0} - p, x, \omega) + \underline{M}$. If $w_\lambda(x,p,\omega)$ is a viscosity solution to

$$\lambda w_\lambda + H_2(p + w_\lambda', x, \omega) = 0, \qquad x \in \mathbf{R}$$

Then $\widetilde{w}_\lambda(x,p,\omega) := -w_\lambda(x,p,\omega)$ is a viscosity solution to

$$\lambda \widetilde{w}_\lambda + \widetilde{H}(q_{\underline{k},0} - p + \widetilde{w}_\lambda', x, \omega) + \underline{M} = 0, \qquad x \in \mathbf{R}$$

Apply Lemma 1.4.3 to $\widetilde{H}(p,x,\omega)$, we deduce that $\overline{H_2}|_{[q_1,q_2]} \equiv \underline{M}$.

(**3**.**6**) For each $p \in [q_1, q_2]$, let $v_\lambda(x,p,\omega)$ be the solution to

$$\lambda v_\lambda(x,p,\omega) + H(p + v_\lambda'(x,p,\omega), x, \omega) = 0, \qquad x \in \mathbf{R}$$

By that fact that $H(p,x,\omega) \ge H_2(p,x,\omega)$, we have

$$\mathbf{E}[\omega \in \Omega | \liminf_{\lambda\to 0} -\lambda v_\lambda(0,p,\omega) \ge \underline{M}] = 1$$

We only need to show that

$$\mathbf{E}[\omega \in \Omega | \limsup_{\lambda\to 0} -\lambda v_\lambda(0,p,\omega) \le \underline{M}] = 1$$

(**3**.**7**) For each (x,ω), define $\widehat{H}_2(p,x,\omega)$ as following:

$$\widehat{H}_2(p,x,\omega) := \begin{cases} H_2(p,x,\omega) & p \in (-\infty,0) \cup (P,\infty) \\ \text{Concave Envelope of } H_2(p,x,\omega)|_{p\in[0,P]} & p \in [0,P] \end{cases}$$

By definition, $\widehat{H}_2(p,x,\omega)$ is determined by those stationary functions: $m_i(x,\omega)$, $M_j(x,\omega)$, $1 \le i,j \le L$, so $\widehat{H}_2(p,x,\omega)$ is stationary. Then by the theory of level-set convex homogenization (see [4]), $\widehat{H}_2(p,x,\omega)$ can be homogenized to some level-set concave effective Hamiltonian $\overline{\widehat{H}}_2(p) \le \underline{M}$. Since $\widehat{H}_2(p,x,\omega) \ge H_2(p,x,\omega)$, there exists

$\widehat{q}_1 < \widehat{q}_2$ such that $[q_1, q_2] \subset [\widehat{q}_1, \widehat{q}_2]$ and $\overline{\widehat{H}}_2(\widehat{q}_1) = \overline{\widehat{H}}_2(\widehat{q}_2) = \underline{M}$. By level-set concavity, $\overline{\widehat{H}}_2|_{[\widehat{q}_1, \widehat{q}_2]} = \underline{M}$. Denote

$$\widetilde{H}_2(p, x, \omega) = \min\{\widehat{H}_2(p, x, \omega), \underline{M}\}$$

Then $\widetilde{H}_2(p, x, \omega)$ has a level-set concave effective Hamiltonian $\overline{\widetilde{H}}_2(p)$ with

$$\overline{\widetilde{H}}_2|_{[\widehat{q}_1, \widehat{q}_2]} = \underline{M}$$

For any $p_1 \in [\widehat{q}_1, \widehat{q}_2]$ and $\lambda > 0$, let $\widehat{v}_\lambda(x, p_1, \omega)$ be the solution of the equation

$$\lambda \widehat{v}_{2,\lambda} + \widehat{H}_2(p_1 + \widehat{v}'_{2,\lambda}, x, \omega) = 0, \qquad x \in \mathbf{R}$$

We will have

$$\lim_{\lambda \to 0} \inf_{|x| \leq \frac{R}{\lambda}} -\lambda \widehat{v}_{2,\lambda}(x, p_1, \omega) \geq \underline{M}$$

Since $p_1 < P$ and $0 < \overline{m} < \underline{M}$, by Lemma 1.4.1, we have that for a.e. $\omega \in \Omega$, any $R > 0$, there exists some $\lambda_0 = \lambda_0(R, p_1, \omega) > 0$,

$$0 < \lambda < \lambda_0 \Rightarrow 0 \leq p_1 + \widehat{v}'_{2,\lambda}(x, p_1, \omega) \leq P \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

Define

$$\widehat{H}(p, x, \omega) := \begin{cases} H(p, x, \omega) & p \in (-\infty, 0) \cup (P, \infty) \\ \widehat{H}_2(p, x, \omega) & p \in [0, P] \end{cases}$$

For each $\omega \in \Omega$ and $\lambda > 0$, let $\widehat{v}_\lambda(x, p_1, \omega)$ be the solution of the equation

$$\lambda \widehat{v}_\lambda + \widehat{H}(p_1 + \widehat{v}'_\lambda, x, \omega) = 0, \qquad x \in \mathbf{R}$$

Thus

$$\lambda \widehat{v}_{2,\lambda} + \widehat{H}(p_1 + \widehat{v}'_{2,\lambda}, x, \omega) = 0 \quad \text{for } x \in \left[-\tfrac{R}{\lambda}, \tfrac{R}{\lambda}\right]$$

By Lemma 1.2.3, there exists some constant $C = C(p_1) > 0$, such that

$$\left| \lambda \widehat{v}_\lambda(0, p_1, \omega) - \lambda \widehat{v}_{2,\lambda}(0, p_1, \omega) \right| \leq \frac{C}{R}$$

We can choose arbitrarily large R, so

$$\lim_{\lambda\to 0} -\lambda\widehat{v}_\lambda(0,p_1,\omega) = \lim_{\lambda\to 0} -\lambda\widehat{v}_{2,\lambda}(0,p_1,\omega) = \underline{M}$$

This means that

$$\overline{\widehat{H}}|_{[q_1,q_2]} \equiv \underline{M}$$

By the fact that $\widehat{H}(p,x,\omega) \geq H(p,x,\omega)$, we have

$$\mathbf{E}\left[\omega\in\Omega \middle| \limsup_{\lambda\to 0} -\lambda v_\lambda(0,p,\omega) \leq \underline{M}\right] = 1$$

This completes the proof. □

Lemma 1.6.3. *Let $H(p,x,\omega)$ be constrained Hamiltonian that satisfies* (**A1**)-(**A3**) *and $\underline{M} = \overline{m}$, then there exists a family of Hamiltonians $\{H_n(p,x,\omega)\}_{n\in\mathbf{N}}$, each $H_n(p,x,\omega)$ is a constrained Hamiltonian and satisfies* (**A1**)-(**A3**)*, moreover, we have $\underline{M}_n > \overline{m}_n$ and*

$$||H_n(p,x,\omega) - H(p,x,\omega)||_{L^\infty(\mathbf{R}\times\mathbf{R}\times\Omega)} \leq \frac{1}{n}$$

Proof. For each $n\in\mathbf{N}$, define the function

$$h_n(p,x,\omega) := \begin{cases} \frac{p-q_{\underline{k}}}{n(p_{\overline{k}}-q_{\underline{k}})} & p\in[q_{\underline{k}},p_{\overline{k}}] \\ -\frac{p-p_{\overline{k}}}{n(q_{\underline{k}-1}-p_{\overline{k}})}+\frac{1}{n} & p\in(p_{\overline{k}},q_{\underline{k}-1}) \\ 0 & \text{elsewhere} \end{cases}$$

And define

$$H_n(p,x,\omega) := H(p,x,\omega) - h_n(p,x,\omega)$$

Since $q_{\underline{k}}(x,\omega)$, $p_{\overline{k}}(x,\omega)$ and $q_{\underline{k}-1}(x,\omega)$ are all stationary, $H_n(p,x,\omega)$ is also stationary. By the construction, we have

$$\overline{m}_n = \overline{m} - \frac{1}{n} = \underline{M} - \frac{1}{n} < \underline{M}_n - \frac{1}{n} < \underline{M}_n$$

Moreover,

$$||H_n(p,x,\omega) - H(p,x,\omega)||_{L^\infty(\mathbf{R}\times\mathbf{R}\times\Omega)} = \frac{1}{n}$$

□

Remark 7. *In Lemma 1.6.3, if those $H_n(p,x,\omega)$ are regularly homogenizable for all $p \in \mathbf{R}$, then according to Lemma 1.2.2, $H(p,x,\omega)$ is also regularly homogenizable and $\overline{H}(p) = \lim_{n\to\infty} \overline{H_n}(p)$.*

Remark 8. *The point of Lemma 1.6.1, Lemma 1.6.2 and Lemma 1.6.3 is the following: to prove the homogenization of constrained Hamiltonian $H(p,x,\omega)$ with index $(0,L)$ and with small oscillation, it suffices to study the homogenization of constrained Hamiltonian $H(p,x,\omega)$ with index $(0,L)$ and has large oscillation.*

1.7 Auxiliary Lemmas for Large Oscillation

1.7.1 Existence Lemma

Lemma 1.7.1. *Let Hamiltonian $H(p,x,\omega)$ satisfy $(\mathbf{A1})$-$(\mathbf{A3})$ and be constrained with index $(0,L)$, then for any $\mu \geq 0, \omega \in \Omega$, there exists a Lipschitz continuous viscosity solution $u(x,\omega)$ to the equation:*

$$\begin{cases} H(u', x, \omega) = \mu, & x \in \mathbf{R} \\ u' \geq 0 \end{cases}$$

Proof. Fix $\mu \geq 0$ and $\omega \in \Omega$. By $(\mathbf{A2})$, there exists $p_0 > 0$, such that $H(p_0, x, \omega) > \mu$. Since $H(0,x,\omega) \leq \mu$, $u_+ := p_0 x$ is a super-solution and $u_- := C$ is a sub-solution for any constant C.

STEP 1. Fix $a \in \mathbf{R}$ and let $u_- = C_a := p_0 a$, then

$$u_+(a,\omega) = u_-(a,\omega) \text{ and } u_+(x,\omega) > u_-(x,\omega), \qquad \forall x \in (a,\infty)$$

Define

$$u_a(x,\omega) \quad := \quad \sup_v \{v(x,\omega) \in C([a,\infty)) | H(v', x, \omega) \leq \mu, C_a \leq v(x,\omega) \leq p_0 x\}$$

Then

$$\begin{cases} H(u_a', x, \omega) = \mu, & x \in (a,\infty) \\ u_a(a,\omega) = p_0 a \end{cases}$$

STEP 2. Fix any $a < b$, denote

$$w(x,\omega) := u_a(x,\omega) + [u_b(b,\omega) - u_a(b,\omega)], \qquad x \geq b$$

Then

$$\begin{cases} H(w', x, \omega) = \mu, & x \in (b, \infty) \\ w(b,\omega) = p_0 b \end{cases}$$

So $u_b(x,\omega) \geq w(x,\omega)$ on $[b,\infty)$. Denote

$$\widetilde{u}_a(x,\omega) := \begin{cases} u_a(x,\omega) & x \in [a,b] \\ u_b(x,\omega) - u_b(b,\omega) + u_a(b,\omega) & x \in (b,\infty) \end{cases}$$

Then

$$p_0 x \geq \widetilde{u}_a(x,\omega) \geq u_a(x,\omega) \geq C_a, \;\; x \in [a,\infty)$$

On the other hand, by the construction, $\widetilde{u}_a(x,\omega)$ is a sub-solution, so $\widetilde{u}_a(x,\omega) \leq u_a(x,\omega)$.

Thus $\widetilde{u}_a(x,\omega) \equiv u_a(x,\omega)$, which means

$$(u_b(x,\omega) - u_a(x,\omega))\,|_{(b,\infty)} \equiv u_b(b,\omega) - u_a(b,\omega)$$

The above equality is true for any $a < b$, this also implies $u_a'(x,\omega) \geq 0$.

STEP 3. For any $n \in \mathbf{Z}$, then

$$u_n(x,\omega) - u_n(0,\omega) = u_{n+1}(x,\omega) - u_{n+1}(0,\omega), \qquad \forall x \geq n+1$$

For any $x \in \mathbf{R}$ let $m := [x]$ and define

$$u(x,\omega) := u_m(x,\omega) - u_m(0,\omega)$$

So $u(x,\omega)$ is a well-defined Lipschitz function on $\mathbf{R}$ and it is the solution of the equation

$$\begin{cases} H(u', x, \omega) = \mu & x \in \mathbf{R} \\ u' \geq 0 \end{cases}$$

□

1.7.2 Decomposition Lemma

Lemma 1.7.2. *Let $H(p,x,\omega)$ satisfy* (**A1**)-(**A3**) *and be constrained with index $(0,L)$. Fix $\mu \geq 0$ and let u be a Lipschitz continuous viscosity solution of the equation*

$$\begin{cases} H(u'(x,\omega),x,\omega) = \mu \quad x \in \mathbf{R} \\ u'(x,\omega) \geq 0 \end{cases}$$

Then there exists a sequence $\{b_i\}_{i\in\mathbf{Z}}$, such that

$$\lim_{i\to\pm\infty} b_i = \pm\infty,\ u \in C^1(I_i),\ \textit{with } I_i = (b_i, b_{i+1})$$

Moreover,

$$u'(x,\omega)|_{I_i} = \psi_{k_i,(x,\omega)}(\mu) \textit{ for some } k_i \in \{1,2,\cdots,2L+1\}$$

Proof. Fix $\omega \in \Omega$ and omit the notation ω.

STEP 1. **Claim**: for each $x \in \mathbf{R}$, there exist $\delta_x > 0$ and $l_x, r_x \in \{1,2,\cdots,2L+1\}$, such that

$$u'(y) = \begin{cases} \psi_{l_x,y}(\mu) \quad y \in (x-\delta_x, x) \\ \psi_{r_x,y}(\mu) \quad y \in (x, x+\delta_x) \end{cases}$$

We only give the proof of the first equality, since the proof for the second one is similar. Proof by contradiction, suppose this is not true at some x_0, then there exist two sequences $x_n \to x_0^-$ and $y_n \to x_0^-$, $1 \leq k_2 < k_1 \leq 2L+1$, such that

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_0, \quad u'(x_n) = \psi_{k_1,x_n}(\mu), \quad u'(y_n) = \psi_{k_2,y_n}(\mu)$$

Case 1: $k_1 \geq k_2 + 2$. Then there exists a branch between the k_1-th branch and the k_2-th branch. By (**A3**), there exist $a < b$, such that $u'(x_n) < a < b < u'(y_n)$.
Fix any $p \in [a,b]$, then $u(x) - px$ is decreasing (increasing) around $x_n(y_n)$. So, $u(x)$ attains local minimum (maximum) at $z_n^- \in (x_n, y_n)(z_n^+ \in (y_n, x_{n+1}))$, then $H(p, z_n^+) \leq \mu \leq H(p, z_n^-)$, by continuity of $H(p,x)$, there exists $z_n \in [z_n^-, z_n^+]$ with $H(p, z_n) = \mu$. By the fact that $\lim_{n\to\infty} z_n = x_0$, we have $H(p, x_0) = \mu$. This is true for any $p \in [a,b]$ and this contradicts the fact that $H(p,x,\omega)$ is constrained.
Case 2: $k_1 = k_2 + 1$, without loss of generality, let $k_1 = 2$, $k_2 = 1$.

If $m_1(x_0) < \mu$, by the similar argument used in **Case 1**, we get a contradiction.

If $m_1(x_0) > \mu$, there exists some $\delta > 0$, s.t. $m_1(\cdot)|_{(x_0-\delta,x_0)} > \mu$, let $x_n \in (x_0 - \delta, x_0)$, then $\mu = H(u'(x_n), x_n) \geq H(p_1, x_n) > \mu$, which is a contradiction.

If $m_1(x_0) = \mu$, since $m_1(x)$ has no cluster point, there exists some $\delta > 0$ such that $\mu \notin \{m_1(x)|x \in (x_0 - \delta, x_0)\}$. By the above discussion, $m_1(\cdot)|_{(x_0-\delta,x_0)} < \mu$. Let $\Phi(x) := u(x) - p_1 x$, then $\Phi'(x_n) < 0$ and $\Phi'(y_n) > 0$, so there exists some $z_n \in (x_n, y_n)$ where $\Phi(x)$ attains local minimum. So $m_1(z_n) = H(p_1, z_n) \geq \mu$, since $z_n \in (x_0 - \delta, x_0)$ when $n \gg 1$, we get the contradiction.

Thus, the **Claim** is proved.

STEP 2. Denote: $A := \{x \in \mathbf{R} | l_x \neq r_x\}$. By the above arguments, we see that A has no cluster point. Then there exists a sequence $\{b_i\}_{i \in \mathbf{Z}}$ such that $b_i < b_{i+1}$, $A \subset \{b_i\}_{i \in \mathbf{Z}}$ and $\lim_{i \to \pm\infty} b_i = \pm\infty$. We will have $r_{b_i} = l_{b_{i+1}}$. Thus $u'(x) = \psi_{r_{b_i},x}(\mu)$, $x \in (b_i, b_{i+1})$.

□

1.7.3 Homotopy between solutions

Let $H(p, x, \omega)$ be constrained with index $(0, L)$. For simplicity of notation, we omit the dependence of ω. Let $f \in L^\infty(\mathbf{R})$ and let u be a viscosity solution to $u'(x) = f(x)$ which is also a viscosity solution to

$$\begin{cases} H(u', x) = f(x), & x \in \mathbf{R} \\ u' \geq 0 \end{cases}$$

By Lemma 1.7.2, let $a_1 < a_2 < a_3$ and $f(x)|_{(a_i, a_{i+1})} = \psi_{k_i,x}(\mu)$, $k_i \in \{1, 2, \cdots, 2L + 1\}, i = 1, 2$. Denote $k = \min\{k_1, k_2\}$ and define

$$\widetilde{f}(x) := \begin{cases} f(x) & x \in (a_1, a_3)^c \\ \psi_{k,x}(\mu) & x \in (a_1, a_3) \end{cases}$$

Lemma 1.7.3. *Assume $\mu \notin \{m_i(x), M_j(x) | 1 \leq i, j \leq L, x \in (a_1, a_3)\}$. Then any solution of $u' = \widetilde{f}$ is also a viscosity solution of*

$$H(u'(x), x) = \mu, \qquad x \in \mathbf{R}$$

Proof. Since the proof is similar to that of A.3 in [7], we omit it. □

Let $I = (a, b)$, and $f_1, f_2 \in L^\infty(I)$, $f_1 \geq f_2$. Let u_1, u_2 be solutions of the following equations respectively:

$$\begin{cases} u_1' = f_1, & x \in I \\ u_1(a) = 0 \end{cases}$$

$$\begin{cases} u_2' = f_2, & x \in I \\ u_2(a) = 0 \end{cases}$$

Assume both u_1, u_2 are viscosity solutions of the equation

$$H(u', x, \omega) = \mu, \qquad x \in I \tag{1.9}$$

Then $u_2(x) \leq u_1(x) \leq u_2(x) - u_2(b) + u_1(b)$. Fix any $c \in [u_2(b), u_1(b)]$ and define

$$\begin{aligned} u_{c,*}(x) &:= \max\{u_2(x), u_1(x) - u_1(b) + c\} \\ u^{c,*}(x) &:= \min\{u_1(x), u_2(x) - u_2(b) + c\} \end{aligned}$$

Define the set

$$\mathcal{W} := \left\{ w \in W^{1,\infty}(I) | H(w', x, \omega) \leq \mu \text{ and } u_{c,*}(x) \leq w(x) \leq u^{c,*}(x) \right\}$$

And the function $w_c(x) := \sup_{w \in \mathcal{W}} w(x)$. Denote

$$\mathcal{F}_I(f_1, f_2, c)(x) := \begin{cases} w_c'(x) & \text{if } w_c \text{ is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$$

Then $u_{c,*}(x)$ ($u^{c,*}(x)$) is a viscosity sub (super) solution to equation (1.9). By Perron's method, $w_c(x)$ is a viscosity solution of the equation

$$\begin{cases} H(w_c'(x), x) = \mu, & x \in (a, b) \\ w_c(a) = 0, w_c(b) = c \end{cases}$$

Lemma 1.7.4. *Fix $a < b$, $0 < \epsilon < \frac{b-a}{2}$, let $f_1, f_2 \in L^\infty(a-\epsilon, b+\epsilon)$ such that*

$$f_1(x) \geq f_2(x),\ x \in (a-\epsilon, b+\epsilon) \quad \text{and} \quad f_1(x) = f_2(x),\ x \in (a-\epsilon, a)\bigcup(b, b+\epsilon)$$

Suppose any solution of (i=1,2)

$$\begin{cases} u_i'(x) = f_i(x) & x \in (a-\epsilon, b+\epsilon) \\ u_i(a) = 0 \end{cases}$$

is a viscosity (sub-)solution of the equation: $H(u', x) = \mu$. Fix $c \in [u_2(b), u_1(b)]$, then any solution of the equation

$$v'(x) = \begin{cases} f_1(x) = f_2(x) & x \in (a-\epsilon, a)\bigcup(b, b+\epsilon) \\ \mathcal{F}_I(f_1, f_2, c)(x) & x \in I = (a, b) \end{cases}$$

is a viscosity (sub-)solution of the equation

$$\begin{cases} H(u'(x), x) = \mu & x \in (a-\epsilon, b+\epsilon) \\ u(a) = 0, u(b) = c \end{cases}$$

Proof. See the proof of Lemma A.4 in [7]. □

1.8 Homogenization of Hamiltonian with Large Oscillation

In this section, the Hamiltonian is assumed to satisfy (**A1**)-(**A3**), be constrained (see Definition 1.3.1) with index $(0, L)$ and have large oscillation (see Definition 1.6.1).

1.8.1 Admissible decomposition and admissible functions

Recall (1.3), (1.4) and denote

$$\underline{m} := \operatorname*{ess\,inf}_{(x,\omega)} m(x,\omega), \quad \overline{M} := \operatorname*{ess\,sup}_{(x,\omega)} M(x,\omega), \quad \mathcal{P} = (\underline{m}, \overline{M})\bigcap[0,\infty)$$

Definition 1.8.1. *Fix any $\mu \in \mathcal{P}$ and $\omega \in \Omega$, a collection of disjoint finite intervals $\{I_i\}_{i\in\mathbf{Z}}$ is called a (μ, ω) admissible decomposition of* $\mathbf{R}$ *if the following (1), (2) and (3) hold.*

(1) $I_i = (a_i, a_{i+1})$, $\bigcup_{i\in\mathbf{Z}} [a_i, a_{i+1}] = \mathbf{R}$.

(2) $\mu \in \{m_j(a_i, \omega), M_j(a_i, \omega) | 1 \le j \le L\}$, *for any* $i \in \mathbf{Z}$.

(3) $\mu \notin \{m_j(x, \omega), M_j(x, \omega) | 1 \le j \le L, x \in (a_i, a_{i+1})\}$, *for any* $i \in \mathbf{Z}$.

Remark 9. *Since $H(p, x, \omega)$ is constrained and has large oscillation, such $\{I_i\}_{i\in\mathbf{Z}}$ exists and is unique. By* (**A1**)*, for any $y \in \mathbf{R}$, $\{I_i - y\}_{i\in\mathbf{Z}}$ is the $(\mu, \tau_y \omega)$ admissible decomposition of* $\mathbf{R}$.

Definition 1.8.2. *For fixed $\omega \in \Omega$ and $\mu \in \mathcal{P}$, let $\{I_i\}_{i\in\mathbf{Z}}$ be a (μ, ω) admissible decomposition of* $\mathbf{R}$*, then $f : \mathbf{R} \to \mathbf{R}$ is a (μ, ω) admissible function if the following (1), (2) and (3) hold.*

(1) $0 \le f(x) \le \max\{p \ge 0 | H(p, x, \omega) \le \overline{M}\}$.

(2) For each $i \in \mathbf{Z}$, $f(x)|_{I_i} = \psi_{j_i,x}(\mu)$, for some $j_i \in \{1, 2, \cdots, 2L + 1\}$.

(3) Any solution of $u' = f(x)$ is a viscosity solution of the equation

$$\begin{cases} H(u'(x), x, \omega) = \mu, & x \in \mathbf{R} \\ u' \ge 0 \end{cases} \tag{1.10}$$

Definition 1.8.3. *For $\mu \ge 0$ and $\omega \in \Omega$, define*

$$\mathcal{A}_\mu(\omega) := \begin{cases} \{\textit{All } (\mu, \omega) \textit{ admissible functions }\} & \mu \in \mathcal{P} \\ \psi_{2L+1,x}(\mu) & \mu \le \underline{m} \ (\textit{if } \underline{m} \ge 0) \\ \psi_{1,x}(\mu) & \mu \ge \overline{M} \end{cases}$$

Lemma 1.8.1. $\mathcal{A}_\mu(\omega) \neq \emptyset$.

Proof. Fix $\omega \in \Omega$, by Lemma 1.7.1, there exists a viscosity solution $u(x)$ of the equation (1.10). By Lemma 1.7.2, there exists a strictly increasing sequence $\{b_i\}_{i\in\mathbf{Z}}$ such that

$$\lim_{i\to\pm\infty} b_i = \pm\infty;\ u \in C^1((b_i, b_{i+1})), i \in \mathbf{Z};\ u'(x)|_{(b_i,b_{i+1})} = \psi_{k_i,x}(\mu), k_i \in \{1, 2, \cdots, 2L + 1\}$$

Let $\mu \in \mathcal{P}$ and $\{I_j\}_{j\in\mathbf{Z}}$ be the (μ, ω) admissible decomposition of $\mathbf{R}$. By refinement, we may assume that for $i \in \mathbf{Z}$, $(b_i, b_{i+1}) \subset I_{l_i}$, for some $l_i \in \mathbf{Z}$.

For each $j \in \mathbf{Z}$, denote: $s(j) = \min\{k_i | (b_i, b_{i+1}) \subset I_j\}$. And define $f(x, \omega) := \psi_{s(j),x}(\mu), x \in I_j = (a_j, a_{j+1})$. By Lemma 1.7.3, any solution to $u' = f$ is a viscosity solution of the equation (1.10). Thus $f \in \mathcal{A}_\mu(\omega)$.

If $\mu \notin \mathcal{P}$, it is clear that $\mathcal{A}_\mu(\omega) \neq \emptyset$. $\square$

Definition 1.8.4. *For each $\omega \in \Omega$ and $\mu \geq 0$, denote*

$$\overline{f}_\mu(x, \omega) := \sup\{f(x) | f \in \mathcal{A}_\mu(\omega)\}$$
$$\underline{f}_\mu(x, \omega) := \inf\{f(x) | f \in \mathcal{A}_\mu(\omega)\}$$

Lemma 1.8.2. *(1) For any $\mu \geq 0$ and $\omega \in \Omega$, $\overline{f}_\mu(x, \omega)$, $\underline{f}_\mu(x, \omega) \in \mathcal{A}_\mu(\omega)$.*
(2) $\overline{f}_\mu(x, \omega) \geq \underline{f}_\mu(x, \omega)$ and both of them are stationary.

Proof. (1) Fix any $\mu \geq 0$ and $\omega \in \Omega$. For any point $x_0 \in \mathbf{R}$, since $H(p, x, \omega)$ is constrained with index $(0, L)$, there are $f_r \in \mathcal{A}_\mu(\omega)$, $\delta_r > 0$ and $k_r \in \{1, 2, \cdots, 2L+1\}$, such that

$$\overline{f}_\mu(x, \omega)|_{(x_0, x_0+\delta_r)} = f_r(x)|_{(x_0, x_0+\delta_r)} = \psi_{k_r,x}(\mu)$$

Similarly, there are $f_l \in \mathcal{A}_\mu(\omega)$, $\delta_l > 0$ and $k_r \in \{1, 2, \cdots, 2L+1\}$, such that

$$\overline{f}_\mu(x, \omega)|_{(x_0-\delta_l, x_0)} = f_r(x)|_{(x_0-\delta_l, x_0)} = \psi_{k_l,x}(\mu)$$

(i) If $k_l = k_r = k$. Then $\psi_{k,x}(\mu)$ is continuous on $(x_0-\delta_l, x_0+\delta_r)$. Since $H(\psi_{k,x}(\mu), x, \omega) = \mu$, any solution of $u' = \psi_{k,x}(\mu)$ is the solution of the equation:

$$H(u', x, \omega) = \mu, \qquad x \in (x_0 - \delta_l, x_0 + \delta_r)$$

(ii) If $k_l < k_r$. It suffices to check any solution to $u' = \overline{f}_\mu$ is a viscosity sub-solution at x_0, which follows from the fact that

$$[\overline{f}(x_0^+), \overline{f}(x_0^-)] = [f_r(x_0^+), f_l(x_0^-)] \subset [f_l(x_0^+), f_l(x_0^-)]$$

(iii) If $k_l > k_r$. It suffices to check any solution to $u' = \overline{f}_\mu$ is a viscosity super-solution at x_0, which follows from the fact that

$$[\overline{f}(x_0^-), \overline{f}(x_0^+)] = [f_l(x_0^-), f_r(x_0^+)] \subset [f_r(x_0^-), f_r(x_0^+)]$$

So $\overline{f}_\mu(x,\omega) \in \mathcal{A}_\mu(\omega)$. Similarly, $\underline{f}_\mu(x,\omega) \in \mathcal{A}_\mu(\omega)$.

(2) It follows immediately from definition that $\overline{f}_\mu(\cdot,\omega) \geq \underline{f}_\mu(\cdot,\omega)$. By Remark 9, for any $y \in \mathbf{R}$, we have

$$\overline{f}(x,\tau_y\omega) = \sup\{f(x)|f(x) \in \mathcal{A}_\mu(\tau_y\omega)\} = \sup\{f(x)|f(x-y) \in \mathcal{A}_\mu(\omega)\} = \overline{f}(x+y,\omega)$$

Similarly, $\underline{f}(x,\tau_y\omega) = \underline{f}(x+y,\omega)$ for any $y \in \mathbf{R}$. □

1.8.2 Intermediate level set of the effective Hamiltonian

Lemma 1.8.3. *Let $H(p,x,\omega)$ satisfy* (**A1**)-(**A3**)*, assume it is constrained with index $(0,L)$ and it has large oscillation (see Definition 1.6.1). If $\mu > \underline{M}$, then for a.e. $\omega \in \Omega$, the following is true: for any $f(x) \in \mathcal{A}_\mu(\omega)$, there exists a sequence of intervals $\{J_k\}_{k\in\mathbf{Z}}$ such that*

$$J_k = (c_k, c_{k+1}), \qquad \bigcup_{k\in\mathbf{Z}}[c_k, c_{k+1}] = \mathbf{R}$$
$$\lim_{k\to\pm\infty} c_k = \pm\infty, \qquad f|_{J_{2k}} = \psi_{1,(x,\omega)}(\mu)$$

Proof. According to Lemma 1.2.1, for a.e. $\omega \in \Omega$, $\underline{M} = \operatorname*{ess\,inf}_{x\in\mathbf{R}} M(x,\omega)$. Denote $\delta := \mu - \underline{M}$ and $\epsilon := \frac{\delta}{2}$. By the Ergodic Theorem,

$$\lim_{L\to\pm\infty} \frac{1}{L}\int_0^L \mathbf{1}_{\{z,M(z,\omega)<\underline{M}+\epsilon\}}(x)dx = \mathbf{E}\left[M(0,\cdot) < \underline{M}+\epsilon\right] > 0, \qquad \text{a.e. } \omega \in \Omega$$

So, almost surely, there exists a sequence $x_i = x_i(\omega)$, such that $\lim_{i\to\pm\infty} x_i = \pm\infty$ and $M(x_i,\omega) < \underline{M}+\epsilon$. Continuity of $M(x,\omega)$ in x implies the following: for each i, there exists $\delta_i > 0$, such that $M(x,\omega) < \underline{M}+\epsilon, x \in (x_i - \delta_i,\, x_i + \delta_i)$.

Next, denote $c_{2k} := x_k - \delta_k$, $c_{2k+1} := x_k + \delta_k$ and $J_k := (c_k, c_{k+1})$. Immediately, we have $f(x)|_{J_{2k}} = \psi_{1,(x,\omega)}(\mu)$, which follows from the fact that:

$$H(f(x),x,\omega) = \mu > \underline{M}+\epsilon > M(x,\omega)|_{J_{2k}}, \text{ a.e. } \omega \in \Omega$$

□

Lemma 1.8.4. *Let $H(p,x,\omega)$ satisfy* (**A1**)-(**A3**)*, assume it is constrained with index $(0,L)$ and it has large oscillation (see Definition 1.6.1). If $0 \leq \mu < \overline{m}$, then for a.e.*

$\omega \in \Omega$, the following is true: for any $f(x) \in \mathcal{A}_\mu(\omega)$, then there exists a sequence of intervals $\{J_k\}_{k\in\mathbf{Z}}$ such that

$$J_k = (c_k, c_{k+1}), \qquad \bigcup_{k\in\mathbf{Z}} [c_k, c_{k+1}] = \mathbf{R}$$
$$\lim_{k\to\pm\infty} c_k = \pm\infty, \qquad f|_{J_{2k}} = \psi_{2L+1,x}(\mu)$$

Proof. Similar arguments as in the proof of Lemma 1.8.3. □

Lemma 1.8.5. *Let $H(p, x, \omega)$ satisfy* (**A1**)-(**A3**)*, assume it is constrained with index $(0, L)$ and it has large oscillation (see Definition 1.6.1). Fix any $\mu \geq 0$ and $p \in [\int_\Omega \underline{f}_\mu(0,\omega)d\omega, \int_\Omega \overline{f}_\mu(0,\omega)d\omega]$, there exists a stationary function $f(x,\omega) : \mathbf{R}\times\Omega \to \mathbf{R}$ such that*
(1) $p = \int_\Omega f(0,\omega)d\omega$.
(2) For a.e. $\omega \in \Omega$, any solution to $u' = f(x,\omega)$ is a solution of the equation: $H(u', x, \omega) = \mu$.

Proof. Fix $\omega \in \Omega$. Suppose $\underline{u}'(x,\omega) = \underline{f}_\mu(x,\omega)$ and $\overline{u}'(x,\omega) = \overline{f}_\mu(x,\omega)$, Lemma 1.8.2 implies $H(\underline{u}', x, \omega) = \mu$, $H(\overline{u}', x, \omega) = \mu$.
According to Lemma 1.8.3 and Lemma 1.8.4, there exists a sequence of intervals $\{I_k\}_{k\in\mathbf{Z}}$, where $I_k = (a_k, a_{k+1})$, such that $\lim_{k\to\pm\infty} a_k = \pm\infty$ and

$$\underline{f}_\mu(x,\omega) = \overline{f}_\mu(x,\omega),\ x \in I_{2k} \quad \text{and} \quad \underline{f}_\mu(x,\omega) \leq \overline{f}_\mu(x,\omega),\ x \in I_{2k+1}$$

Denote

$$\underline{d}_i = \int_{a_i}^{a_{i+1}} \underline{f}_\mu(s,\omega)ds \quad \text{and} \quad \overline{d}_i = \int_{a_i}^{a_{i+1}} \overline{f}_\mu(s,\omega)ds$$

For each $t \in [0,1]$, accordingly define $f_t : \mathbf{R}\times\Omega \to \mathbf{R}$ as

$$f_t(x,\omega) := \begin{cases} \underline{f}_\mu(x,\omega) = \overline{f}_\mu(x,\omega) & x \in I_{2i} \\ \mathcal{F}_{I_{2i+1}}(\overline{f}_\mu, \underline{f}_\mu, t\overline{d}_i + (1-t)\underline{d}_i) & x \in I_{2i+1} \end{cases}$$

Therefore $f_t(x,\omega)$ is stationary and

$$\int_{a_0}^{a_i} f_t(x,\omega)dx = \int_{a_0}^{a_i} t\overline{f}_\mu(x,\omega) + (1-t)\underline{f}_\mu(x,\omega)dx$$

By (**A2**), $\underline{f}_\mu$ and $\overline{f}_\mu$ are bounded. Then there exists some constant $C > 0$, such that

$$\begin{aligned}
& \frac{1}{|a_i - a_0|}\left|\int_{a_0}^{a_i} f_t(x,\omega)dx - \int_{a_0}^{a_i} f_s(x,\omega)dx\right| \\
= \;& |t-s|\left|\int_{a_0}^{a_i}\left(\overline{f}_\mu(s,\omega) - \underline{f}_\mu(s,\omega)\right)ds\right| \\
\leq \;& C\,|t-s|
\end{aligned}$$

Hence

$$\lim_{L\to\infty}\frac{1}{L}\left|\int_0^L f_t(x,\omega)dx - \int_0^L f_s(x,\omega)dx\right| \leq C\,|t-s|$$

Apply the Ergodic Theorem, we have for a.e. $\omega \in \Omega$ that

$$\lim_{L\to\infty}\frac{1}{L}\int_0^L f_t(x,\omega)dx = \mathbf{E}\left[f_t(0,\omega)\right] \quad \text{and} \quad \lim_{L\to\infty}\frac{1}{L}\int_0^L f_s(x,\omega)dx = \mathbf{E}\left[f_s(0,\omega)\right]$$

Thus $|\mathbf{E}\left[f_t(0,\omega)\right] - \mathbf{E}\left[f_s(0,\omega)\right]| \leq C\,|t-s|$, which means $\mathbf{E}\left[f_t(0,\omega)\right]$ is a continuous function of t. As a consequence of this observation,

$$\bigcup_{t\in[0,1]}\mathbf{E}\left[f_t(0,\omega)\right] = \left[\int_\Omega \underline{f}_\mu(0,\omega)d\omega, \int_\Omega \overline{f}_\mu(0,\omega)d\omega\right]$$

Finally, if we fix any $p \in [\int_\Omega \underline{f}_\mu(0,\omega)d\omega, \int_\Omega \overline{f}_\mu(0,\omega)d\omega]$, there must be some $t_0 = t_0(p) \in [0,1]$, such that $\mathbf{E}[f_{t_0}(0,\omega)] = p$. Just let u be the solution of $u' = f_{t_0}(x,\omega)$, Lemma 1.7.4 states that u is also a solution of $H(u',x,\omega) = \mu$.

□

Lemma 1.8.6. *Let $H(p,x,\omega)$ satisfy (**A1**)-(**A3**), assume it is constrained with index $(0,L)$ and it has large oscillation (see Definition 1.6.1). Fix $\omega \in \Omega$, if $\mu_m \to \mu$ and $f_m(x) \in \mathcal{A}_{\mu_m}(\omega)$, then the following (1), (2) and (3) hold.*

(1) If $\mu \in \mathcal{P}$, then $\limsup_{m\to\infty} f_m(x) \in \mathcal{A}_\mu(\omega)$ and $\liminf_{m\to\infty} f_m(x) \in \mathcal{A}_\mu(\omega)$.

(2) If $\underline{m} \geq 0$ and $\mu \leq \underline{m}$, then except a countable set,

$$\limsup_{m\to\infty} f_m(x) = \liminf_{m\to\infty} f_m(x) = \psi_{2L+1,(x,\omega)}(\mu)$$

(3) If $\mu \geq \overline{M}$, then except a countable set,

$$\limsup_{m\to\infty} f_m(x) = \liminf_{m\to\infty} f_m(x) = \psi_{1,(x,\omega)}(\mu)$$

Proof. Prove $f(x) = \limsup_{m\to\infty} f_m(x) \in \mathcal{A}_\mu(\omega)$ only since the proof for lim inf is similar.
(1) Let $\{I_i\}_{i\in\mathbf{Z}}$ be the (μ,ω) admissible decomposition of $\mathbf{R}$. Fix $k \in \mathbf{Z}$ and $0 < \epsilon \ll 1$, then, there exists some $N = N(\epsilon) \in \mathbf{N}$, such that if $m > N$,

$$\mu_m \notin \{m_i(x,\omega), M_j(x,\omega) | 1 \le i, j \le L, x \in (a_k+\epsilon, a_{k+1}-\epsilon) \cup (a_{k+1}+\epsilon, a_{k+2}-\epsilon)\}$$

So, there are l, $\widetilde{l}$, q, $\widetilde{q} \in \{1, 2, \cdots, 2L+1\}$, $\{f_{l_n}\}_{n\ge1}$ and $\{f_{q_n}\}_{n\ge1}$, such that

$$f_{l_n}(x) = \begin{cases} \psi_{l,(x,\omega)}(\mu) & x \in (a_k + \frac{1}{n}, a_{k+1} - \frac{1}{n}) \\ \psi_{\widetilde{l},(x,\omega)}(\mu) & x \in (a_{k+1} + \frac{1}{n}, a_{k+2} - \frac{1}{n}) \end{cases}$$

$$f_{q_n}(x) = \begin{cases} \psi_{\widetilde{q},(x,\omega)}(\mu) & x \in (a_k + \frac{1}{n}, a_{k+1} - \frac{1}{n}) \\ \psi_{q,(x,\omega)}(\mu) & x \in (a_{k+1} + \frac{1}{n}, a_{k+2} - \frac{1}{n}) \end{cases}$$

Moreover,

$$f(x)|_{I_k} = \psi_{l,(x,\omega)}(\mu) \quad \text{and} \quad f(x)|_{I_{k+1}} = \psi_{q,(x,\omega)}(\mu)$$

To prove (1), we only need to show that the solution of $u' = f$ is a viscosity solution of (1.10) at a_{k+1}. Define $u_l \in W^{1,\infty}(a_k, a_{k+2})$ and $u_q \in W^{1,\infty}(a_k, a_{k+2})$ respectively be solutions of

$$u_l'(x) = \begin{cases} \psi_{l,(x,\omega)}(\mu) & x \in I_k \\ \psi_{\widetilde{l},(x,\omega)}(\mu) & x \in I_{k+1} \end{cases}$$

and

$$u_q'(x) = \begin{cases} \psi_{\widetilde{q},(x,\omega)}(\mu) & x \in I_k \\ \psi_{q,(x,\omega)}(\mu) & x \in I_{k+1} \end{cases}$$

By stability of viscosity solutions, both of u_l and u_q are also viscosity solutions to

$$H(v'(x), x, \omega) = \mu, \qquad x \in (a_k, a_{k+2})$$

It is easy to see that the jump of f at a_{k+1} is contained in the jump of u_l' or the jump of u_q' at a_{k+1}, therefore the solution of $u' = f$ is a viscosity solution of (1.10).
(2) Denote $A = \{x \in \mathbf{R} | \mu = m_i(x) \text{ for some } 1 \le i \le L\}$. Since each of $m_i(x,\omega)$ has

no cluster point, A is countable. Since $\underline{m} \geq 0$ and $\mu \leq \underline{m}$, if $x \notin A$, then we must have

$$\limsup_{m\to\infty} f_m(x) = \liminf_{m\to\infty} f_m(x) = \psi_{2L+1,(x,\omega)}(\mu)$$

(3) Denote $B = \{x \in \mathbf{R} | \mu = M_j(x) \text{ for some } 1 \leq j \leq L\}$. Since each $M_j(x,\omega)$ has no cluster point, B is countable. Since $\mu \geq \overline{M}$, if $x \notin B$, then we must have

$$\limsup_{m\to\infty} f_m(x) = \liminf_{m\to\infty} f_m(x) = \psi_{1,(x,\omega)}(\mu)$$

□

Definition 1.8.5. *For each $\mu \geq 0$, denote $\mathcal{I}_\mu = \left[\int_\Omega \underline{f}_\mu(0,\omega)d\omega, \int_\Omega \overline{f}_\mu(0,\omega)d\omega\right]$.*

Remark 10. *Recalling Lemma 1.8.5, if $\mu \neq \nu$, then $\mathcal{I}_\mu \bigcap \mathcal{I}_\nu = \emptyset$.*

Lemma 1.8.7. *If $\lim_{m\to\infty} \mu_m = \mu$, then*

$$\begin{aligned} \int_\Omega \overline{f}_\mu(0,\omega)d\omega &\geq \limsup_{m\to\infty} \int_\Omega \overline{f}_{\mu_m}(0,\omega)d\omega \\ \int_\Omega \underline{f}_\mu(0,\omega)d\omega &\leq \liminf_{m\to\infty} \int_\Omega \underline{f}_{\mu_m}(0,\omega)d\omega \end{aligned}$$

Moreover, $\bigcup_{\mu\geq 0} \mathcal{I}_\mu = [q_0, \infty)$ with $q_0 = \int_\Omega \underline{f}_0(0,\omega)d\omega$.

Proof. See the proof of Lemma 3.8 in [7]. □

Remark 11. *So far, we have established that for those $p \in [q_0, \infty)$, the cell problem is always solvable (hence the Hamiltonian is regularly homogenizable for those p).*

1.8.3 Extreme level set of effective Hamiltonian

In this subsection, we will study the minimum piece of $\overline{H}(p)$, which turns out to be 0. Denote

$$z_l(x,\omega) := \min\left\{p \leq 0 : H(q,x,\omega) \leq 0 \text{ on } [p,0]\right\}$$

Lemma 1.8.8. *Let $H(p,x,\omega)$ satisfy* (**A1**)-(**A3**)*, assume it is constrained with index $(0,L)$ and it has large oscillation (see Definition 1.6.1). For any p between $\mathbf{E}[z_l(0,\omega)]$ and $\mathbf{E}[\underline{f}_0(0,\omega)]$, there exists a stationary function $f(x,\omega)$ such that $p = \mathbf{E}[f(0,\omega)]$ and any solution to $u' = f$ is a viscosity sub-solution of $H(u',x,\omega) = 0$, $x \in \mathbf{R}$.*

Proof. Since $H(p,x,\omega)$ is constrained with index $(0,L)$, $\overline{m} = \operatorname*{ess\,sup}_{(x,\omega)\in\mathbf{R}\times\Omega} m(x,\omega) > 0$. By similar arguments in the proof of Lemma 1.8.3, then: for a.e. $\omega\in\Omega$, there exists $\{b_i\}_{i\in\mathbf{Z}}$ such that

$$\lim_{i\to\pm\infty} b_i = \pm\infty, \quad m(x,\omega)|_{(b_{2i},b_{2i+1})} \in \left[\tfrac{3\overline{m}}{4}, \overline{m}\right], \quad m(x,\omega)|_{(b_{2i+1},b_{2i+2})} \le \frac{3}{4}\overline{m}$$

Fix any such ω and for each $i\in\mathbf{Z}$, we denote

$$\underline{r}_i = \int_{b_i}^{b_{i+1}} z_l(x,\omega)dx \quad \text{and} \quad \overline{r}_i = \int_{b_i}^{b_{i+1}} \underline{f}_0(x,\omega)dx$$

Fix $t\in(0,1)$, the following part of proof is devoted to define a stationary function $f_t(x,\omega)$.

STEP 1: Modification on (b_{2i},b_{2i+1}). First denote

$$f_{l,t}(x,\omega) = \begin{cases} (1-t)\underline{f}_0(x,\omega) + tz_l(x,\omega) & x\in\bigcup_{i\in\mathbf{Z}}(b_{2i},b_{2i+1}) \\ z_l(x,\omega) & x\in\bigcup_{i\in\mathbf{Z}}[b_{2i+1},b_{2i+2}] \end{cases}$$

$$f_{r,t}(x,\omega) = \begin{cases} (1-t)\underline{f}_0(x,\omega) + tz_l(x,\omega) & x\in\bigcup_{i\in\mathbf{Z}}(b_{2i},b_{2i+1}) \\ \underline{f}_0(x,\omega) & x\in\bigcup_{i\in\mathbf{Z}}[b_{2i+1},b_{2i+2}] \end{cases}$$

Suppose u is a solution of the equation $u' = f_{l,t}$ or the equation $u' = f_{r,t}$, then in viscosity sense, we have $H(u'(x,\omega),x,\omega)\le 0, x\in\mathbf{R}$ (This is because $H(p,x,\omega)$ is convex in p on $(z_l(x,\omega),\underline{f}_0(x,\omega))$ for any $x\in(b_{2i},b_{2i+1})$).

STEP 2: Modification on $[b_{2i+1},b_{2i+2}]$. Define

$$f_t := \begin{cases} \mathcal{F}_{I_{2i+1}}(\underline{f}_0, z_l(x,\omega), (1-t)\overline{r}_i + t\underline{r}_i) & x\in[b_{2i+1},b_{2i+2}] \\ f_{l,t}(x,\omega) = f_{r,t}(x,\omega) & x\in(b_{2i},b_{2i+1}) \end{cases}$$

By Lemma 1.7.4, if $u' = f_t$, then in viscosity sense, we have $H(u'(x,\omega),x,\omega)\le 0$, $x\in\mathbf{R}$. By similar arguments as in the proof of Lemma 1.8.5, there exists some constant $C>0$, such that

$$\frac{1}{|b_i - b_0|}\left|\int_{b_0}^{b_i} f_t(x,\omega)dx - \int_{b_0}^{b_i} f_s(x,\omega)dx\right| \le C|t-s|$$

$$\lim_{L\to+\infty}\frac{1}{L}\left|\int_0^L f_t(x,\omega)dx - \int_0^L f_s(x,\omega)dx\right| \le C|t-s|$$

Recall the Ergodic theorem, the above means

$$|\mathbf{E}\left[f_t(0,\omega)\right] - \mathbf{E}[f_s(0,\omega)]| \le C|t-s|$$

Hence $\mathbf{E}[f_t(0,\omega)]$ is a continuous function with respect to t.

Since $\mathbf{E}\left[f_0(0,\omega)\right] = \mathbf{E}[\underline{f}_0(0,\omega)]$ and $\mathbf{E}[f_1(0,\omega)] = \mathbf{E}[z_l(0,\omega)]$, we conclude that

$$\bigcup_{t\in[0,1]} \mathbf{E}[f_t(0,\omega)] = \Big[\mathbf{E}[z_l(0,\omega)], \mathbf{E}[\underline{f}_0(0,\omega)]\Big]$$

So for any $p \in \Big[\mathbf{E}[z_l(0,\omega)], \mathbf{E}[\underline{f}_0(0,\omega)]\Big]$, there exists $t = t(p)$, such that $p = \mathbf{E}[f_t(0,\omega)]$, therefore, any solution of $u' = f_t(x,\omega)$ is a viscosity sub-solution of $H(v', x, \omega) = 0$, $x \in \mathbf{R}$. □

Corollary 1.8.1. *Let $H(p,x,\omega)$ satisfy* (**A1**)-(**A3**)*, assume it is constrained with index $(0, L)$ and it has large oscillation (see Definition 1.6.1). Fix any p between $\mathbf{E}[z_l(0,\omega)]$ and $\mathbf{E}[\underline{f}_0(0,\omega)]$, for any $\lambda > 0$, assume $v_\lambda(x,p,\omega)$ be the viscosity solution of the equation*

$$\lambda v_\lambda + H(p + v_\lambda', x, \omega) = 0, \qquad x \in \mathbf{R}$$

Then for a.e. $\omega \in \Omega$, we have

$$\limsup_{\lambda\to 0} -\lambda v_\lambda(0,p,\omega) \le 0$$

Proof. From Lemma 1.8.8, we see that for any $p \in \Big[\mathbf{E}[z_l(0,\omega)], \mathbf{E}[\underline{f}_0(0,\omega)]\Big]$, there exists a Lipschitz continuous function $w(x,p,\omega)$, which is the solution of the equation

$$\begin{cases} H(p + w', x, \omega) \le 0 \\ \mathbf{E}[w'(x,\omega)] = 0 \\ w'(x,\omega) \text{ is stationary} \end{cases}$$

Without loss of generality, assume that $w(0,p,\omega) = 0$, apply the Ergodic theorem, we have for a.e. $\omega \in \Omega$ that $w(x,p,\omega)$ is sub-linear, which follows from the following observations:

$$\lim_{L\to\infty}\frac{w(L,p,\omega)}{L} = \lim_{L\to\infty}\frac{1}{L}\int_0^L w'(s,p,\omega)ds = \mathbf{E}[w'(0,p,\omega)] = 0$$

$$\lim_{L\to\infty}\frac{w(-L,p,\omega)}{L}=\lim_{L\to\infty}\frac{1}{L}\int_0^{-L}w'(s,p,\omega)ds=\mathbf{E}[w'(0,p,\omega)]=0$$

These imply that, for each fixed $R>0$, we have

$$\lim_{\lambda\to 0}\max\left\{\lambda w\left(\frac{R}{\lambda},p,\omega\right),\lambda w\left(-\frac{R}{\lambda},p,\omega\right)\right\}=0$$

As a result of above statement, we also have

$$\lim_{\lambda\to 0}\max_{x\in\left[-\frac{R}{\lambda},\frac{R}{\lambda}\right]}\{|\lambda w\left(x,p,\omega\right)|,|\lambda w\left(x,p,\omega\right)|\}=0$$

Denote $A(\lambda,R):=\max_{x\in\left[-\frac{R}{\lambda},\frac{R}{\lambda}\right]}\{|\lambda w\left(x,p,\omega\right)|,|\lambda w\left(x,p,\omega\right)|\}$, then for any $\delta>0$, there exists some $\lambda_0=\lambda_0(R,\delta)>0$, when $\lambda<\lambda_0$, $A(\lambda,R)<\delta$. Note that on $\left[-\frac{R}{\lambda},\frac{R}{\lambda}\right]$, we have

$$\lambda w+H(p+w',x,\omega)\le\lambda w\le\delta=\lambda\left(v_\lambda+\frac{\delta}{\lambda}\right)+H(p+v'_\lambda,x,\omega)$$

Recall Lemma 1.2.3, which gives

$$\lambda w(0,p,\omega)-(\lambda v_\lambda(0,p,\omega)+\delta)\le\frac{C(p)}{R}\text{ for some constant }C(p)>0$$

Choose $R>\frac{C(p)}{\delta}$, then $-\lambda v_\lambda(0,p,\omega)<2\delta$. The above argument is true for arbitrary $\delta>0$.

Therefore, for a.e. $\omega\in\Omega$,

$$\limsup_{\lambda\to 0}-\lambda v_\lambda(0,p,\omega)\le 0$$

□

Lemma 1.8.9. *Let $H(p,x,\omega)$ satisfy* (**A1**)-(**A3**)*, assume it is constrained with index $(0,L)$ and it has large oscillation (see Definition 1.6.1). Then for a.e. $\omega\in\Omega$, we have that*

$$\liminf_{\lambda\to 0}-\lambda v_\lambda(x,p,\omega)\ge 0\quad\textit{for any }x\in\mathbf{R}$$

where $p\in\mathbf{R}$ and where $v_\lambda(\cdot,p,\omega)\in W^{1,\infty}(\mathbf{R})$ is the unique viscosity solution of the equation:

$$\lambda v_\lambda+H(p+v'_\lambda,x,\omega)=0,\qquad x\in\mathbf{R}$$

Proof. By assumption, $\operatorname*{ess\,inf}_{(x,\omega)\in\mathbf{R}\times\Omega} H(0,x,\omega) < 0$, for each $(x,\omega)\in\mathbf{R}\times\Omega$. Denote

$$V(x,\omega) := \min\{H(0,x,\omega), m(x,\omega)\}$$

Then $V(x,\omega)\le 0$ and it is a bounded continuous stationary function. Then

$$H_+(p,x,\omega) := H(p,x,\omega) - V(x,\omega) \ge 0$$

For a.e. $\omega\in\Omega$ and any $\delta>0$, there are (by similar arguments as in the proof of Lemma 1.8.3):

$$I_i = (a_i, a_{i+1}),\quad \lim_{i\to\pm\infty} a_i = \pm\infty \quad \text{and } -\delta \le V(x,\omega)\le 0,\ x\in(a_{2i}, a_{2i+1})$$

Then

$$\liminf_{\lambda\to 0,\ x\in(a_{2i},a_{2i+1})} -\lambda v_\lambda(x,\omega) \ge -\delta$$

On the other hand, for each $\omega\in\Omega$, there exists a sequence $\lambda_n\to 0$ and a constant $C\in\mathbf{R}$, such that

$$-\lambda_n v_{\lambda_n}(x,\omega)\to C \qquad \text{locally uniformly in } \mathbf{R}$$

Then, $C\ge -\delta$. Because $\delta>0$ can be arbitrarily close to 0, $C\ge 0$. Thus $\liminf_{\lambda\to 0} -\lambda v_\lambda(x,\omega)\ge 0$.

□

Remark 12. *Corollary 1.8.1 and Lemma 1.8.9 demonstrate that for any p between $\mathbf{E}[z_l(0,\omega)]$ and $\mathbf{E}[\underline{f}_0(0,\omega)]]$, $H(p,x,\omega)$ is regularly homogenizable (see Definition 1.2.1) and $\overline{H}(p)=0$.*

Definition 1.8.6. *If $H(p,x,\omega)$ is constrained with index $(0,L)$, denote by $\Psi_{(x,\omega)}(\cdot)$ the inverse function of $H(\cdot,x,\omega)|_{(-\infty,0)}$.*

Lemma 1.8.10. *For $p\in(-\infty,\mathbf{E}[z_l(0,\omega)])$, $H(p,x,\omega)$ is regularly homogenizable.*

Proof. For each $\mu\ge 0$, denote $p_\mu=\mathbf{E}[\Psi_{(0,\omega)}(\mu)]$, let $v(x,\omega)$ be the solution of the equation

$$v'(x,\omega) = \Psi_{(x,\omega)}(\mu) - p_\mu$$

Then v is a sub-linear (by the Ergodic theorem) solution of $H(p+v', x, \omega) = \mu$, $x \in \mathbf{R}$. The lemma follows from the fact that

$$(-\infty, \mathbf{E}[z_l(0,\omega)]) = \bigcup_{\mu>0} \{p_\mu\}$$

□

Remark 13. *From the construction of the effective Hamiltonian* $\overline{H}(p)$*, in the case of large oscillation,* $\overline{H}(p)$ *is coercive, continuous and level-set convex.*

Proof. of Theorem 2.3.1 It follows from Remark 4, Remark 5, Remark 6, Remark 8, Remark 11, Remark 12 and Lemma 1.8.10. □

1.9 Future problem: Random homogenization of nonconvex Hamilton-Jacobi equations in high dimensional cases

It is natural to ask if we can extend the argument in 1D to general high dimension. As the counter-example [38] indicates, the random homogenization of general nonconvex Hamilton-Jacobi equations may not be true in high dimensional case. However, we can still ask if the homogenization still can be proved for some particular type of Hamilton-Jacobi equations.

The simplest such Hamiltonian (except the case in [6]) is the above (d = 2) one with $H(p, x, \omega) = H(p) + AV(x, \omega)$, where $H(p)$ is a rotation of the curve $\beta(s), s \geq 0$ and $A \geq 0$. In this example, the function $H(p)$ has local oscillation 1, we fix some $V(x) : \mathbf{R}^2 \to \mathbf{R}$, which is $[0, 1]^2$-periodic (a special case of stationary ergodic), the picture on the right-hand side are effective Hamiltonians corresponding to different scale A, the simulation result is from Yu-Yu Liu (National Cheng Kung University, Taiwan). As we know in one dimensional case, when $A \geq 1$, the effective Hamiltonian $\overline{H}(p)$ becomes level-set convex. However, from the above simulations, we see that evern when $A = 1$ (the red curve), $\overline{H}(p)$ still shows some nonconvexity. This indicates that the 2D case is more complicated than that of 1D. In fact, the homotopy argument in the proof of large oscillation case cannot extend easily to general dimension situation.

Chapter 2

Applications to the study of G-equations

This chapter is based mainly on the author's previous work on the front propagation by analyzing G-equaton model. The author thanks the American Mathematical Society for the publication, the final publication is available at Proceedings of The American Mathematical Society via http://dx.doi.org/10.1090/proc/12930

2.1 Introduction

2.1.1 G-equations and strain effect

The G-equation is a well known model in the study of turbulent combustion. It is the level set formulation of interface motion laws in the thin interface regime. In the simplest model of the G-equation, the normal velocity of the interface equals a positive constant s_L (which is called the laminar speed) plus the normal projection of the fluid velocity $V(x)$, which gives the inviscid G-equation (in the general dimensional situation, we use x as spatial variable and t as time variable):

$$\begin{cases} G_t + V(x) \cdot DG + s_L|DG| = 0 & (x,t) \in \mathbf{R}^d \times (0,\infty) \\ G(x,0) = G_0(x) & x \in \mathbf{R}^d \end{cases} \tag{2.1}$$

In reality, inter-facial fluctuations appear in front propagation. So, there is a family of G-equations with different oscillation scales.

$$
\begin{cases} G^\epsilon_t + V(\frac{x}{\epsilon}) \cdot DG^\epsilon + s_L|DG^\epsilon| = 0 & (x,t) \in \mathbf{R}^d \times (0,\infty) \\ G^\epsilon(x,0) = G_0(x) & x \in \mathbf{R}^d \end{cases} \tag{2.2}
$$

When $V(x)$ is $\mathbb{Z}^d$-periodic and nearly compressible, Xin-Yu [35] and Cardaliaguet-Nolen-Souganidis [11] independently proved that $G^\epsilon(x,t) \to \overline{G}(x,t)$ locally uniformly and $\overline{G}$ solves the homogenized Hamilton-Jacobi equation:

$$
\begin{cases} \overline{G}_t + \overline{H}(D\overline{G}) = 0 & (x,t) \in \mathbf{R}^d \times (0,\infty) \\ \overline{G}(x,0) = G_0(x) & x \in \mathbf{R}^d \end{cases} \tag{2.3}
$$

The effective Hamiltonian $\overline{H}$ is called the turbulent flame speed or the turbulent burning velocity in combustion literature. For the stationary ergodic divergence-free flow $V(x,\omega)$ which is the gradient of a stream function that satisfies some integrability condition, Nolen-Novikov [29] first proved homogenization for the 2-dimensional case. Then for the general dimensional case, if $V(x,\omega)$ is divergence-free and has appropriately small mean, Cardaliaguet-Souganidis [13] proved the homogenization.

Since the flow will stretch or compress the front flame surface, the reaction over the flame front will be affected. Thus the laminar speed s_L depends on the flame stretch and therefore can not be constant. To model the strain effect in the G-equation, people extend s_L to $s_L + c\vec{n} \cdot DV \cdot \vec{n}$, where $\vec{n}$ represents the normal direction. Here the Markstein length c is proportional to the flame thickness. Hence the induced strain G-equation is [30][28][12]

$$
G_t + V(x,\omega) \cdot DG + s_L|DG| + c\frac{DG}{|DG|} \cdot S \cdot DG = 0 \quad S := \tfrac{DV + (DV)^\top}{2} \tag{2.4}
$$

Some interesting questions are:

(1) Can this strain G-equation be homogenized?

(2) If yes, how does the strain term affect the turbulent flame speed $\overline{H}(p)$?

Remark 14. *The strain G-equation (2.4) is highly non-coercive and non-convex, which increase the difficulty of homogenization.*

When V is a 2-d periodic Cellular Flow, Xin-Yu [36] showed that due to the existence of a strain term, when the flow intensity (magnitude of V) is large enough, the effective Hamiltonian becomes zero. This means that under the effect of strain, the flame is quenched when the flow is too strong.

In this short article, we investigate those questions for 2-d random Shear Flows $V(x,\omega)$ in the stationary ergodic setting.

2.1.2 2-d random Shear Flows

For the 2-d problem, we denote the space variable by $(x,y) \in \mathbf{R}^2$. Without loss of generality, we assume $s_L = 1$. We study the problem under a random Shear Flow $V = (v(y,\omega), 0)$ and assume $v(y,\omega)$ is stationary ergodic (See section 2 for precise definitions).

Let $p = (m,n)$, the cell problem, if it exists, becomes:

$$\sqrt{(m+G_x)^2+(n+G_y)^2}+v(y,\omega)\cdot(m+G_x)+c\frac{(m+G_x)(n+G_y)v'}{\sqrt{(m+G_x)^2+(n+G_y)^2}} = \overline{H}(m,n,c)$$

This can be reduced to a 1-d problem:

$$\sqrt{m^2+(n+G_y)^2}+v(y,\omega)\cdot m+c\frac{m(n+G_y)v'}{\sqrt{m^2+(n+G_y)^2}} = \overline{H}(m,n,c) \tag{2.5}$$

So, it suffices to study the following 1-d Hamiltonian. Since the case $m = 0$ is trivial, let's fix $m \neq 0$, and we will denote $\overline{H}(m,n,c)$ by $\overline{H}(n,c)$. As a function of p, the following Hamiltonian is not convex but it is level-set convex.

$$H(p,x,\omega,c) = \sqrt{m^2+p^2}+v(x,\omega)\cdot m+c\frac{mpv'(x,\omega)}{\sqrt{m^2+p^2}} \tag{2.6}$$

In fact,

$$\frac{\partial H(p,x,\omega,c)}{\partial p} = \frac{p^3+m^2p+cv'm^3}{(m^2+p^2)^{\frac{3}{2}}} \tag{2.7}$$

For fixed x,ω,c, $\frac{\partial H}{\partial p} = 0$ has a unique real root and $\lim\limits_{p\to-\infty}\frac{\partial H}{\partial p} = -1$, $\lim\limits_{p\to+\infty}\frac{\partial H}{\partial p} = 1$. Thus the Hamiltonian is level-set convex(see **(A2)** in section 2). Actually, by the above facts, H is strictly level-set convex. This means that for each fixed x,ω,c, for any $\mu \in \mathbf{R}$, $\{p : H(p,x,\omega,c) = \mu\}$ has no interior point.

2.1.3 Random homogenization of Hamilton-Jacobi equations with level-set convex Hamiltonians

Armstrong-Souganidis [4] proved random homogenization of the Hamilton-Jacobi equations with level-set convex Hamiltonians. In addition to level-set convexity they require more assumptions. Their proofs depend on the existence of a family of auxiliary functions $\Lambda_\lambda \in C(\mathbf{R} \times \mathbf{R})$ that are nondecreasing in both of the arguments and satisfying

$$\text{For all } \mu \neq \nu,\ \Lambda_\lambda(\mu, \nu) < \max\{\mu, \nu\} \tag{2.8}$$

$$H(\lambda p + (1-\lambda)q, y, \omega) \leq \Lambda_\lambda(H(p, y, \omega), H(q, y, \omega)) \tag{2.9}$$

where $p, q, y \in \mathbf{R}^d, \omega \in \Omega$, for each $0 < \lambda_1 \leq \lambda_2 \leq \frac{1}{2}$, $\Lambda_{\lambda_1} \geq \Lambda_{\lambda_2}$. However, the existence of Λ_λ is not straightforward.

So, it is not obvious if the Hamiltonian (2.6) satisfies (2.8), (2.9). However, based on a very simple modification of the method in [4], we will show in section 2.2 that (2.8), (2.9) are not necessary and random homogenization holds for any level-set convex Hamiltonian with general dimension. Actually, to prove the homogenization of 1-d Hamiltonian in section 2.3, we do not need the result of section 2.2. In fact, random homogenization of 1-d coercive Hamiltonian has been established by Armstrong-Tran-Yu [7] in separable case and extended by the author [18] to general coercive case. The following is our main result which will be proved in section 2.3. Throughout this paper, all solutions of PDEs are interpreted in the viscosity sense [15].

Theorem 2.1.1. *For the 2-dimensional case in the the stationary ergodic setting with a shear flow $V = (v(y,\omega), 0)$ such that $v(\cdot, \omega) \in C^\infty(\mathbf{R})$ and $v(x,\omega), v'(x,\omega) \in L^\infty(\mathbf{R} \times \Omega)$. Then*

1. *The G-equation with strain term (2.4) can be homogenized.*

2. *For any unit vector $p = (m, n) \in \mathbf{R}^2$ and $c > 0$,*

$$\overline{H}(p) = \overline{H}(m,n) \geq \overline{H}(p,c) = \overline{H}(m,n,c) \geq |m| + \sup_{(y,\omega)\in\mathbf{R}\times\Omega} mv(y,\omega)$$

3. *If* $\overline{H}(p) > |m| + \sup_{(y,\omega)\in\mathbf{R}\times\Omega} mv(y,\omega)$ *and* $\mathbf{E}[v] = 0$*, then* $\overline{H}(p) = \overline{H}(p,c)$ *if and only if* $mv \equiv 0$.

Remark 15. *Statement (2) means that the strain term reduces the turbulent flame speed. Since* v' *changes sign, this is not obvious at all. This result is consistent with concensus in combustion literature that the strain rate plays an important role in slowing down or even quenching flame propagation [32]. This fact has been observed by Xin-Yu [37] in the periodic setting.*

2.2 A remark on homogenization of level-set convex Hamiltonians

In this section, we claim that random Hamilton-Jacobi equations with Hamiltonians that are merely level-set convex can be homogenized. Here $(x,t) \in \mathbf{R}^d \times \mathbf{R}$ is the space-time variable.

2.2.1 Assumptions

Consider the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(Du, x, \omega) = 0 & (x,t) \in \mathbf{R}^d \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \mathbf{R}^d \end{cases}$$

For H, we assume:

(A1) Stationary Ergodicity: There exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a group $\{\tau_y\}_{y\in\mathbf{R}^d}$ of $\mathcal{F}$-measurable, measure-preserving transformations $\tau_y : \Omega \to \Omega$, i.e. for any $x, y \in \mathbf{R}^d$:

$$\tau_{x+y} = \tau_x \circ \tau_y \text{ and } \mathbf{P}[\tau_y(A)] = \mathbf{P}[A]$$

Ergodicity:

$$A \in \mathcal{F},\ \tau_z(A) = A \text{ for every } z \in \mathbf{R}^d \Rightarrow \mathbf{P}[A] \in \{0,1\}$$

Stationary:

$$H(p, y, \tau_z\omega) = H(p, y+z, \omega)$$

(A2) Level-Set Convexity: For every $(y, \omega) \in \mathbf{R}^d \times \Omega$ and $p, q \in \mathbf{R}^d$.

$$H\left(\frac{p+q}{2}, y, \omega\right) \leq \max\{H(p, y, \omega), H(q, y, \omega)\}$$

(A3) Coercivity:

$$\lim_{|p|\to\infty} \operatorname*{ess\,inf}_{(y,\omega)\in\mathbf{R}^d\times\Omega} H(p, y, \omega) = +\infty$$

(A4) Boundedness and Uniform Continuity:

$$\{H(\cdot, \cdot, \omega) : \omega \in \Omega\} \text{ is bounded and equicontinuous on } B_R \times \mathbf{R}^d \text{ for any } R > 0.$$

2.2.2 Comparison Principle for Metric Problem

We adopt the same notations as in [4]; by stationary ergodicity, these are independent of random variable ω. So, we suppress the random variable.

Notation 2.2.1.

$$\mathcal{L} := \{\textit{Real-valued global Lipschitz functions in } \mathbf{R}^d\}$$

$$\overline{H}_* := \inf_{w\in\mathcal{L}} \operatorname*{ess\,sup}_{y\in\mathbf{R}^d} H(Dw, y)$$

$$\mathcal{S} := \left\{w \in \mathcal{L} : \lim_{|y|\to\infty} \frac{w(y)}{|y|} = 0\right\}$$

$$\widehat{H}(p) := \inf_{w\in\mathcal{S}} \operatorname*{ess\,sup}_{y\in\mathbf{R}^d} H(p + Dw, y)$$

For fixed $x \in \mathbf{R}^d$ and $\mu \geq \overline{H}_*$, we consider the metric problem

$$\begin{cases} H(p + Dv, y) = \mu \quad y \in \mathbf{R}^d \diagdown \{x\} \\ v(x) = 0 \end{cases} \tag{2.10}$$

The idea in [4] to determine $\overline{H}(p)$ is to homogenize each level set of H. The main tool is a comparison principle (Proposition 3.1 of [4]) of the metric problem, and the proof of the comparison principle depends on the additional assumptions (2.8) and (2.9). For general level-set convex Hamiltonians, we cannot prove the same comparison principle.

Since homogenization is closed under uniform limits, the following question arises: can we add a small perturbation to the level-set convex Hamiltonian such that the perturbed Hamiltonian satisfy (2.8) and (2.9), and then take the limit? This may work for a carefully constructed perturbation, but it does not work if we simply perturb $H(p,x,\omega)$ by $H_\epsilon(p,x,\omega) := \epsilon|p|^2 + H(p,x,\omega)$ as the following simple example shows.

Example 2.2.1. *Consider* $d = 1$ *and* $H(p,x,\omega) = H(p)$ *defined by*

$$H(p) := \begin{cases} -p & p \in (-\infty, 0] \\ 0 & p \in (0,1] \\ -p+1 & p \in (1,2] \\ p-3 & p \in (2,+\infty) \end{cases}$$

$$H_\epsilon(p) := \epsilon|p|^2 + H(p)$$

Then for $0 < \epsilon \ll 1$, $H_\epsilon(0) = 0, H_\epsilon(1) = \epsilon > 0$, $H_\epsilon(2) = 4\epsilon - 1 < 0$ *violates the level-set convexity. So the perturbation may destroy the structure of level-set convexity.*

Fortunately, we observe that by the method introduced in [4], wherever we need the comparison principle we actually only need the following weak version comparison principle. To prove the weak version comparison principle, level-set convexity is sufficient.

Lemma 2.2.1 (Weak Comparison Principle)**.** *Assume* $\widehat{H}(p) < \mu < \nu < +\infty$ *and* u, $-v \in USC(\mathbf{R}^d)$ *solve the following equation.*

$$H(p+Du,y) \le \mu < \nu \le H(p+Dv,y) \text{ in } \mathbf{R}^d \setminus K,\ K \subset \mathbf{R}^d \text{ compact} \tag{2.11}$$

with

$$\liminf_{|y|\to\infty} \frac{v(y)}{|y|} \ge 0 \tag{2.12}$$

Then

$$\sup_{\mathbf{R}^d}(u-v) = \max_K (u-v) \tag{2.13}$$

Remark 16. *Without loss of generality, we only consider the case $p = 0$.*

(A4) *and $\mu < \nu \Rightarrow$ by adding a function with arbitrary small gradient to v, we can assume without of loss of generality that:*

$$A := \liminf_{|y|\to\infty} \frac{v(y)}{|y|} > 0 \tag{2.14}$$

(A3) *$\Rightarrow u$ is Lipschitz $\Rightarrow \exists\ a := \limsup_{|y|\to\infty} \frac{u(y)}{|y|} < \infty$*

Claim 1. *$A \geq a$.*

Proof of the Claim. It suffices to prove $I := \{0 \leq \lambda \leq 1 : A \geq \lambda a\} = [0, 1]$. If $a \leq 0$, there is nothing to prove. Assume $a > 0$ and let $s := \sup I$. Since I is closed, $I = [0, s]$. It suffices to show that for any $\lambda \in I \cap [0, 1)$, $\exists\ 0 < \delta \ll 1$ such that $\lambda + \delta \in I$.

By the assumption $\widehat{H}(0) < \mu$, we can choose $w \in \mathcal{S}$, such that

$$H(Dw, y) < \mu$$

Fix R, $\epsilon > 0$, let $0 < \delta < \min\{\frac{\epsilon}{2a}, 1 - \lambda\}$, and define

$$\phi_R(y) := \sqrt{R^2 + |y|^2} - R$$

Define

$$\widetilde{u} := (\lambda + \delta)u + (1 - \lambda - \delta)w \text{ and } \widetilde{v} := v + \epsilon\phi_R$$

Then by level-set convexity,

$$\begin{aligned} H(D\widetilde{u}, y) &= H((\lambda + \delta)Du + (1 - \lambda - \delta)Dw, y) \\ &\leq \max\{H(Du, y), H(Dw, y)\} \\ &\leq \mu \end{aligned}$$

By **(A4)**, if we choose $\epsilon \ll 1$, we have

$$\mu < H(D\widetilde{v}, y)$$

$$
\begin{aligned}
\liminf_{|y|\to\infty} \frac{\widetilde{v}-\widetilde{u}}{|y|} &= \liminf_{|y|\to\infty} \frac{(v+\epsilon\phi_R) - \big[(\lambda+\delta)u + (1-\lambda-\delta)w\big]}{|y|} \\
&= \liminf_{|y|\to\infty} \left[\frac{v-\lambda u}{|y|} + \epsilon\frac{\phi_R}{|y|} - \delta\frac{u}{|y|} - \frac{(1-\lambda-\delta)w}{|y|}\right] \\
&\geq 0 + \epsilon - \delta\cdot a - 0 \\
&\geq \epsilon - \frac{\epsilon}{2a}\cdot a \\
&= \frac{\epsilon}{2}
\end{aligned}
$$

So, $\widetilde{u}-\widetilde{v}$ attains its maximum in a bounded domain. And by the comparison principle in bounded domain.

$$
\widetilde{u} - \widetilde{v} \leq \max_K(\widetilde{u} - \widetilde{v})
$$

Let $R \to \infty$

$$
\widetilde{u} - v \leq \max_K(\widetilde{u} - v)
$$

Which leads to

$$
(\lambda+\delta)u + (1-\lambda-\delta)w - v \leq \max_K \big((\lambda+\delta)u + (1-\lambda-\delta)w - v\big) \tag{2.15}
$$

So,

$$
\begin{aligned}
\liminf_{|y|\to\infty} \frac{v-(\lambda+\delta)u}{|y|} &= \liminf_{|y|\to\infty} \frac{v - [(\lambda+\delta)u + (1-\lambda-\delta)w]}{|y|} \\
&= \liminf_{|y|\to\infty} \frac{v-\widetilde{u}}{|y|} \\
&\geq \liminf_{|y|\to\infty} \frac{-\min_K(v-\widetilde{u})}{|y|} \\
&= 0
\end{aligned}
$$

So, $I = [0,1]$, i.e. $A \geq a$.

□

Proof of Lemma 2.3. Since $I = [0,1]$, by argument similar to the one above, for any $\lambda \in [0,1)$

$$
\lambda u + (1-\lambda)w - v \leq \max_K \big(\lambda u + (1-\lambda)w - v\big) \tag{2.16}
$$

Letting $\lambda \to 1$, we get the lemma. □

Remark 17. *The following statements give the homogenization of general level-set convex Hamiltonians under **(A1)-(A4)**.*

(I) *In [4], the assumptions (2.8), (2.9) are only used to derive the comparison principle (Proposition 3.1 of [4]).*

(II) *The above proof of the weak comparison principle does not require (2.8), (2.9).*

(III) *The weak comparison principle is sufficient to obtain the homogenization. Actually, in [4], the comparison principle is used mainly in two places. One is the construction of a maximal solution of the metric problem, the other is the proof of homogenization where the comparison principle is used to control the convergence in the Ergodic problem. Wherever we need the comparison principle, the above weak version comparison principle is sufficient.*

2.3 The effect of strain term

We will study the following more general Hamiltonians with **(B1)-(B4)**.

$$H(p,x,\omega,c) = \sqrt{m^2+p^2} + cs(x,\omega)\cdot\frac{p}{\sqrt{m^2+p^2}} + k(x,\omega) \tag{2.17}$$

(B1) Fix $\omega \in \Omega$, $k(x,\omega)$, $s(x,\omega) \in C^\infty(\mathbf{R})$ and $k(x,\omega)$, $s(x,\omega) \in L^\infty(\mathbf{R}\times\Omega)$.

(B2) Fix $\omega \in \Omega$, if $k(x,\omega)$ achieves its local maximum value, then $s(x,\omega) = 0$.

(B3) The event $\{\omega \in \Omega : k(x,\omega) \text{ or } s(x,\omega) \text{ is constant}\}$ is not of probability 1.

(B4) $k(x,\omega)$ and $s(x,\omega)$ are stationary. For any $x \in \mathbf{R}$, $\mathbf{E}[s(x,\omega)] = 0$.

Remark 18. *(1) The strain G-equation is a special case with $k = mv$ and $s = mv'$.*

*(2) We keep the existence of $\{\tau_z\}_{z\in\mathbf{R}}$, which is ergodic. By this fact and **(B1)-(B4)**, without loss of generality, we can assume there are $\underline{k} < \overline{k}$, $\underline{s} < 0 < \overline{s}$, such that for all $\omega \in \Omega$,*

$$\inf_{x\in\mathbf{R}} k(x,\omega) = \underline{k},\ \sup_{x\in\mathbf{R}} k(x,\omega) = \overline{k},\ \inf_{x\in\mathbf{R}} s(x,\omega) = \underline{s},\ \sup_{x\in\mathbf{R}} s(x,\omega) = \overline{s}$$

(3) Since $H(\cdot,x,\omega,c)$ is level-set convex and $\min_{p\in\mathbf{R}} H(p,x,\omega,c) \le |m| + \overline{k}$, there exist $p_-(x,\omega,c) \le p_+(x,\omega,c)$ and p_-, p_+ that are continuous functions of x with

$$\{p : H(p,x,\omega,c) > |m| + \overline{k}\} = (-\infty, p_-(x,\omega,c)) \cup (p_+(x,\omega,c), \infty)$$

(4) By the homogenization result in [4], the following is true: for a.e. $\omega \in \Omega$ and any $\delta > 0$, if u^δ is the unique viscosity solution of

$$\delta u^\delta + H\left(p + (u^\delta)', x, \omega, c\right) = 0,\ \ x \in \mathbf{R}$$

then we have

$$\lim_{\delta\to 0} -\delta u^\delta(0,\omega) = \overline{H}(p,c). \tag{2.18}$$

Without loss of generality, we can assume this statement is true for every $\omega \in \Omega$.

(5) It is easy to see $\overline{H}_* = \min_{p\in\mathbf{R}} \overline{H}(p,c) = |m| + \overline{k}$. By level-set convexity of $\overline{H}(p,c)$, there exist $\overline{p}_-(c) \le \overline{p}_+(c)$ with

$$\left\{p : \overline{H}(p,c) > |m| + \overline{k}\right\} = (-\infty, \overline{p}_-(c)) \cup (\overline{p}_+(c), \infty)$$

(6) We will show (from Lemma 2.3.1 to Lemma 2.3.3) that if $\overline{H}(p,c) > \overline{H}_*$, for fixed ω, c, the following cell problem has a sub-linear solution $\gamma(x,\omega,c)$.

$$H(p + \gamma', x, \omega, c) = \overline{H}(p,c),\ \ x \in \mathbf{R} \tag{2.19}$$

(7) Cell problems (2.19) do not have solutions (see [25]) in general. Here in the 1-dimensional level-set convex setting, the above remark (6) says for those $\overline{H}(p,c) > \overline{H}_*$, cell problems do have solutions. More generally, in the 1-dimensional coercive situation, if $\overline{H}(p)$ is not a local extreme value, the solution of the cell problem at $p \in \mathbf{R}$

always exists (see [7]). As for those $\overline{H}(p,c) = \overline{H}_*$, the identity (2.18) can be obtained by using comparison principle (see [7]).

(8) From Lemma 2.3.1 to Lemma 2.3.3, we always fix c. In Theorem 2.3.1, we fix $p = (m,n)$ and study how $\overline{H}(p,c)$ depends on c.

Lemma 2.3.1. *Fix $c \in [0,\infty)$. For any $\mu \in (\overline{H}_*, \infty)$, there exists a unique $P_+(\mu,c)$, such that for each ω, the equation*

$$\begin{cases} H(P_+(\mu,c) + \gamma'(x,\omega,c), x, \omega, c) = \mu, \ \ x \in \mathbf{R} \\ P_+(\mu,c) + \gamma'(x,\omega,c) > p_+(x,\omega,c) \end{cases}$$

admits a viscosity solution $\gamma(x,\omega,c)$ and for a.e. $\omega \in \Omega$, $\gamma(x,\omega,c)$ is sub-linear.

Proof. For each $\mu > \overline{H}_*$, consider the equation

$$H(u'(x,\omega,c), x, \omega, c) = \mu, \ \ x \in \mathbf{R}$$

By the fact that $\min_{p \in \mathbf{R}} H(p,x,\omega,c) \le \overline{H}_*$ and $H(p,x,\omega,c)$ is strictly level-set convex. There are exactly two solutions of $u'(x,\omega,c)$, one is less than $p_-(x,\omega,c)$, the other is greater than $p_+(x,\omega,c)$. We choose the latter one, by stationary of H, $u'(x,\omega,c)$ is stationary. By smoothness of $H(\cdot,\cdot,\omega,c)$ and $\mu > \overline{H}_*$, $u'(x,\omega,c)$ is smooth with respect to x, so by continuity, we always have that $u'(x,\omega,c) > p_+(x,\omega,c)$.

Since $H(p,x,\omega,c)$ is coercive with respect to p, uniformly with respect to $x \in \mathbf{R}$, $u'(x,\omega,c)$ is bounded. We can define

$$P_+(\mu,c) := \mathbf{E}\left[u'(x,\omega,c)\right]$$

Due to the stationary of u', the expectation is independent of x and is uniquely defined for each $c \ge 0$ and $\mu > \overline{H}_*$.

Then we define the function

$$\gamma(x,\omega,c) := u(x,\omega,c) - P_+(\mu,c) \cdot x$$

Then $\mathbf{E}[\gamma'(x,\omega,c)] = 0$ and by sub-additive Ergodic Theorem, for a.e. $\omega \in \Omega$,

$$\lim_{|x|\to\infty} \frac{\gamma(x,\omega,c) - \gamma(0,\omega,c)}{|x|} = \lim_{|x|\to\infty} \frac{1}{|x|} \int_0^x \gamma'(s,\omega,c)ds = \mathbf{E}[\gamma'(0,\omega,c)] = 0$$

Hence for a.e. $\omega \in \Omega$, $\gamma(x,\omega,c)$ is sub-linear and this is the desired solution. □

By the same argument, we have:

Lemma 2.3.2. *Fix $c \in [0, \infty)$. For any $\mu \in (\overline{H}_*, \infty)$, there exists a unique $P_-(\mu, c)$, such that for each ω, the equation*

$$\begin{cases} H(P_-(\mu, c) + \gamma'(x, \omega, c), x, \omega, c) = \mu, \ \ x \in \mathbf{R} \\ P_-(\mu, c) + \gamma'(x, \omega, c) < p_-(x, \omega, c) \end{cases}$$

admits a viscosity solution $\gamma(x, \omega, c)$ and for a.e. $\omega \in \Omega$, $\gamma(x, \omega, c)$ is sub-linear.

Proposition 2.3.1. *Fix $c \in [0, +\infty)$. The function $P_+(\mu, c) : (\overline{H}_*, +\infty) \longrightarrow \mathbf{R}$ has the following properties:*

1. *$P_+(\mu, c)$ is strictly increasing.*
2. *$P_+(\mu, c)$ is continuous.*
3. *$\lim_{\mu \to +\infty} P_+(\mu, c) = +\infty$.*

Proof. (1) Since $H(p, x, \omega, c)$, as a function of p, is strictly increasing on $(p_+(x, \omega, c), +\infty)$, and it's uniformly continuous. So

$$\overline{H}_* < \mu_1 < \mu_2 < \infty \implies P_+(\mu_1, c) < P_+(\mu_2, c)$$

(2) Suppose μ_n, $\mu \in (\overline{H}_*, +\infty)$ and $\mu_n \to \mu$ as $n \to \infty$. Accordingly, we can solve u'_n and u' by Lemma 2.3.1.

$$H(u'_n(x, \omega, c), x, \omega, c) = \mu_n, \quad x \in \mathbf{R}$$
$$H(u'(x, \omega, c), x, \omega, c) = \mu, \quad x \in \mathbf{R}$$

For each fixed $(x, \omega, c) \in \mathbf{R} \times \Omega \times \mathbf{R}^+$, by the fact that $H(\cdot, x, \omega, c)$ is smooth and strictly increasing on $(p_+(x, \omega, c), +\infty)$, we have $\lim_{n \to \infty} u'_n(x, \omega, c) = u'(x, \omega, c)$, by bounded convergence theorem, $\lim_{n \to \infty} \mathbf{E}[u'_n(x, \omega, c)] = \mathbf{E}[u'(x, \omega, c)]$.

Thus, $\lim_{n \to \infty} P_+(\mu_n, c) = P_+(\mu, c)$.

(3) If $P_+(\mu, c)$ is bounded, since $k(x, \omega)$, $s(x, \omega)$ are uniformly bounded and then $H(P_+(\mu, c), x, \omega, c)$ is uniformly bounded. Let $\mathbf{E}[\gamma'(x, \omega)] = 0$ and $\gamma(x, \omega)$ solves

$$H(P_+(\mu, c) + \gamma', x, \omega, c) = \mu, \ \ x \in \mathbf{R}$$

Since $\gamma(x,\omega)$ is sub-linear and smooth. For any $\epsilon > 0$, there is some interval $(a(\mu), b(\mu))$ on which $|\gamma'| < \epsilon$ (Otherwise, by continuity, γ will be at least linear growth at infinity). So $H(P_+(\mu,c) + \gamma', x, \omega, c)$ is uniformly bounded on $(a(\mu), b(\mu))$, this gives a contradiction when $\mu \to +\infty$. □

Similarly, we can prove:

Proposition 2.3.2. *Fix $c \in [0, +\infty)$. The function $P_-(\mu, c) : (\overline{H}_*, +\infty) \longrightarrow \mathbf{R}$ has the following properties:*

1. *$P_-(\mu, c)$ is strictly decreasing.*
2. *$P_-(\mu, c)$ is continuous.*
3. *$\lim_{\mu \to +\infty} P_-(\mu, c) = -\infty$.*

Definition 2.3.1. *By the above propositions of $P_+(\mu, c)$ and $P_-(\mu, c)$, we denote their inverse functions by $\mu_+(p, c)$ and $\mu_-(p, c)$.*

$$\mu_+(p,c) : \left(\inf_\mu P_+(\mu,c), +\infty\right) \longrightarrow (\overline{H}_*, +\infty)$$

$$\mu_-(p,c) : \left(-\infty, \sup_\mu P_-(\mu,c)\right) \longrightarrow (\overline{H}_*, +\infty)$$

And then we can define the continuous level-set convex function $\mu(p, c)$.

$$\mu(p,c) := \begin{cases} \mu_-(p,c) & \text{if } p \in \left(-\infty, \sup_\mu P_-(\mu,c)\right) \\ \overline{H}_* & \text{if } p \in \left[\sup_\mu P_-(\mu,c), \inf_\mu P_+(\mu,c)\right] \\ \mu_+(p,c) & \text{if } p \in \left(\inf_\mu P_+(\mu,c), +\infty\right) \end{cases}$$

Lemma 2.3.3. $\mu(p,c) = \overline{H}(p,c)$

Proof. By the existence of cell problem,

$$\mu(p,c) = \overline{H}(p,c), \ \forall p \in \left(-\infty, \sup_\mu P_-(\mu,c)\right) \bigcup \left(\inf_\mu P_+(\mu,c), +\infty\right)$$

By level-set convexity of $\overline{H}(p,c)$ and $\overline{H}(p,c) \geq \overline{H}_*$, we have

$$\overline{H}(p,c) = \overline{H}_*, \ \forall p \in \left[\sup_{\mu} P_-(\mu,c), \inf_{\mu} P_+(\mu,c)\right]$$

So $\mu(p,c) = \overline{H}(p,c)$. □

The next theorem is aimed to study the dependence of $\overline{H}(n,c)$ on c. As mentioned under (2.5), $\overline{H}(n,c)$ is equal to $\overline{H}(m,n,c)$ in the original 2-d problem. We will fix a unit vector $(m,n) \in \mathbf{R}^2$ and denote $h(c) := \overline{H}(n,c) = \overline{H}(m,n,c)$ in the following theorem.

Theorem 2.3.1. *Under **(B1)-(B4)**, fix a unit vector $(m,n) \in \mathbf{R}^2$ with $mn \neq 0$.*

1. *$h(c) \in C^{0,1}(\mathbf{R}^+)$ and $\|s\| := \|s(x,\omega)\|_{L^\infty(\mathbf{R}\times\Omega)}$ is the Lipschitz constant.*
2. *$h'(c) \leq 0$ for a.e. $c \in (0,\infty)$. If $h(c) > \overline{H}_*$, $h'(c) < 0$.*
3. *There exists $\overline{c} > 0$, when $c > \overline{c}$, $h(c) = \overline{H}_*$.*

Proof. (1) Fix c_1, $c_2 \in (0,\infty)$, then $\overline{H}_* \leq h(c_1)$, $h(c_2) < \infty$. For each $0 < \delta \ll 1$, let u^δ, v^δ be the unique solutions of the following two equations respectively.

$$\delta u^\delta + \sqrt{m^2 + (n + (u^\delta)')^2} + \frac{c_1(n + (u^\delta)')s(x,\omega)}{\sqrt{m^2 + (n + (u^\delta)')^2}} + k(x,\omega) = 0, \ x \in \mathbf{R} \tag{2.20}$$

$$\delta v^\delta + \sqrt{m^2 + (n + (v^\delta)')^2} + \frac{c_2(n + (v^\delta)')s(x,\omega)}{\sqrt{m^2 + (n + (v^\delta)')^2}} + k(x,\omega) = 0, \ x \in \mathbf{R} \tag{2.21}$$

By Remark 18.

$$\lim_{\delta\to 0} |\delta u^\delta(0,\omega) + h(c_1)| = \lim_{\delta\to 0} |\delta v^\delta(0,\omega) + h(c_2)| = 0$$

Since $w(x,\omega) := v + \frac{1}{\delta}\|s\||c_2 - c_1|$ is a super solution of (2.20). By comparison principle, $\delta u^\delta \leq \delta v^\delta + \|s\||c_2 - c_1|$, similarly, $\delta v^\delta \leq \delta u^\delta + \|s\||c_2 - c_1|$, thus

$$|\delta u^\delta(0,\omega) - \delta v^\delta(0,\omega)| \leq \|s\||c_2 - c_1|$$

Let $\delta \to 0$, we get

$$|h(c_2) - h(c_1)| \leq \|s\||c_2 - c_1|$$

(2) Fix $c_0 > 0$. Without of loss of generality, we assume $n > 0$. Lipschitz function is differentiable a.e., so if $h(c_0) = \overline{H}_*$ and $h(c_0)$ is differentiable at c_0, then $h'(c_0) = 0$.

Now assume $h(c_0) > \overline{H}_*$ and denote $f(t) := \sqrt{m^2 + t^2}$.

We will focus on the cell problem $H(n + u', x, \omega, c) = h(c)$.

By continuity, there is some $\epsilon > 0$, such that for $c \in I_\epsilon = (c_0 - \epsilon, c_0 + \epsilon) \cap \mathbf{R}^+$, $h(c) - \overline{H}_*$ has a positive lower bound. Since $u(x, \omega, c)$ is smooth, $n + u'(x, \omega, c) > 0$ has a positive lower bound.

To show $h'(c_0) < 0$, we first show:

Claim 1: For $c \in I_\epsilon$ $\Big($here $t = n + u'$ in $f(t)\Big)$.

$$f' + cs(x,\omega)f'' = \frac{(n+u')^3 + m^2(n+u') + cs(x,\omega)m^2}{(m^2 + (n+u')^2)^{\frac{3}{2}}} > 0 \tag{2.22}$$

To prove **Claim 1**, it suffices to show $(n + u') + cs(x, \omega) > 0$.

By the fact that $\overline{H}_* = |m| + \overline{k}$ and

$$\sqrt{m^2 + (n+u')^2} + \frac{c(n+u')s(x,\omega)}{\sqrt{m^2 + (n+u')^2}} + k(x,\omega) = h(c) > \overline{H}_*$$

We have

$$\sqrt{m^2 + (n+u')^2} - |m| + \frac{c(n+u')s(x,\omega)}{\sqrt{m^2 + (n+u')^2}} > 0$$

This is equivalent to

$$\frac{(n+u')^2}{\sqrt{m^2 + (n+u')^2} + |m|} + \frac{c(n+u')s(x,\omega)}{\sqrt{m^2 + (n+u')^2}} > 0$$

So

$$\frac{(n+u')^2 + c(n+u')s(x)}{\sqrt{m^2 + (n+u')^2}} = \frac{(n+u')^2}{\sqrt{m^2 + (n+u')^2}} + \frac{c(n+u')s(x,\omega)}{\sqrt{m^2 + (n+u')^2}} > 0$$

Which means

$$(n + u')(n + u' + cs(x, \omega)) > 0$$

The fact that $n + u' > 0$ implies $n + u' + cs(x, \omega) > 0$. Thus **Claim 1** is proved.

Immediately, we have:

$$\mathbf{E}\left[\frac{1}{f' + cs(x,\omega)f''}\right] > 0 \tag{2.23}$$

The fact that

$$f' + cs(x,\omega)f'' = \frac{(n+u')^3 + m^2(n+u') + cs(x,\omega)m^2}{(m^2+(n+u')^2)^{\frac{3}{2}}} > \frac{(n+u')^3}{(m^2+(n+u')^2)^{\frac{3}{2}}}$$

implies $f' + cs(x,\omega)f''$ has a positive lower bound. And the fact that

$$f'(n+u') = \frac{n+u'}{\sqrt{m^2+\left(n+u'\right)^2}}$$

implies $f'(n+u')$ has a positive lower bound.

If we denote

$$a(x,\omega,c) := \frac{cf''}{f'} = \frac{m^2 c}{(n+u')(m^2+(n+u')^2)} > 0$$

Then by dividing f' in (2.22).

$$1 + a(x,\omega,c)s(x,\omega) > 0 \text{ has a positive lower bound}$$

Now, the cell problem can be rewritten as (2.24). Since $F(t) := f(t) + csf'(t) + k$ is smooth and increasing with respect to $t = n + u'(x,\omega,c)$ and $h(c) \in C^{0,1}(\mathbf{R}^+)$, $u' = F^{-1}(h(c)) - n$ is differentiable a.e. with respect to c.

$$f(n+u'(x,\omega,c)) + cs(x,\omega)f'(n+u'(x,\omega,c)) + k(x,\omega) = h(c) \tag{2.24}$$

Differentiate it w.r.t. c gives: $\left(\text{here } \frac{\partial}{\partial c}(u') := \frac{\partial}{\partial c}(\frac{\partial}{\partial x}u(x,\omega,c))\right)$

$$h'(c) \cdot \frac{1}{f' + cs(x,\omega)f''} = \frac{s(x,\omega)}{1+a(x,\omega,c)s(x,\omega)} + \frac{\partial}{\partial c}(u') \tag{2.25}$$

The above positive lower bounds as well as the boundedness of h' and $s(x,\omega)$ implies that $\frac{\partial}{\partial c}(u')$ is bounded uniformly for $(c,x,\omega) \in I_\epsilon \times \mathbf{R} \times \Omega$. This will allow us to apply bounded convergence theorem in (2.27).

Taking expectation in (2.25) gives:

$$h'(c_0) \cdot \mathbf{E}\left[\frac{1}{f' + c_0 s(x,\omega)f''}\right] = \mathbf{E}\left[\frac{s(x,\omega)}{1+a(x,\omega,c_0)s(x,\omega)}\right] + \mathbf{E}\left[\frac{\partial}{\partial c}(u')(x,\omega,c_0)\right]$$

Choose $I_\epsilon \ni c_k \to c_0$, by bounded convergence theorem and the fact $\mathbf{E}[u'] = 0$.

$$\mathbf{E}\left[\frac{\partial}{\partial c}(u')(x,\omega,c_0)\right] = \mathbf{E}\left[\lim_{I_\epsilon \ni c_k \to c_0} \frac{u'(x,\omega,c_k) - u'(x,\omega,c_0)}{c_k - c_0}\right] \tag{2.26}$$

$$= \lim_{I_\epsilon \ni c_k \to c_0} \mathbf{E}\left[\frac{u'(x,\omega,c_k) - u'(x,\omega,c_0)}{c_k - c_0}\right] \tag{2.27}$$

$$= 0 \tag{2.28}$$

Recall that $a(x,\omega,c_0) > 0$, $1 + a(x,\omega,c_0)s(x,\omega) > 0$ and $s(x,\omega)$ is not a constant function, we have

$$\begin{aligned}
\mathbf{E}\left[\frac{s(x,\omega)}{1+a(x,\omega,c_0)s(x,\omega)}\right] &= \mathbf{E}\left[\frac{s(x,\omega)}{1+a(x,\omega,c_0)s(x,\omega)}\chi_{\{\omega:s(x,\omega)>0\}}\right] \\
&+ \mathbf{E}\left[\frac{s(x,\omega)}{1+a(x,\omega,c_0)s(x,\omega)}\chi_{\{\omega:s(x,\omega)\le 0\}}\right] \\
&< \mathbf{E}\left[s(x,\omega)\chi_{\{\omega:s(x,\omega)>0\}}\right] + \mathbf{E}\left[s(x,\omega)\chi_{\{\omega:s(x,\omega)\le 0\}}\right] \\
&= \mathbf{E}[s(x,\omega)] \\
&= 0
\end{aligned}$$

Combine these with (2.23), we can conclude:

$$h'(c_0) < 0$$

(3) Without loss of generality, let $n > 0$ and $\tau := |\underline{s}| = -\underline{s} > 0$.

For each $\omega \in \Omega$, there are countable disjoint intervals $\{(l_i(\omega), r_i(\omega)) : r_{i-1} \le l_i, i \in \mathbb{Z}\}$ such that

$$A(\omega) := \left\{x : s(x,\omega) < -\frac{\tau}{2}\right\} = \bigcup_{i\in\mathbb{Z}} (l_i(\omega), r_i(\omega))$$

Denote

$$B(\omega) := \mathbf{R} \diagdown A(\omega) = \bigcup_{i\in\mathbb{Z}} [r_i(\omega), l_{i+1}(\omega)]$$

Since $s(x,\omega) < -\frac{\tau}{2}$ on $A(\omega)$, by the stationary of $\chi_{A(\omega)}(x)$ and $\chi_{B(\omega)}(x)$ we have for a.e. $\omega \in \Omega$:

$$\alpha := \lim_{L\to+\infty} \frac{1}{2L}\int_{-L}^{L} \chi_{A(\omega)}(x)dx = \mathbf{P}\left[\omega \in \Omega : s(0,\omega) < -\frac{\tau}{2}\right]$$

By **(B4)** and Remark 18, $\alpha \in (0,1)$. Now we can construct a smooth stationary function $\psi(x,\omega)$ with $\psi(x,\omega) = 0$ on $B(\omega)$ and

$$\frac{1}{r_i - l_i}\int_{l_i}^{r_i} \psi(x,\omega)dx = \frac{n}{\alpha} \text{ and } 0 \le \psi(x,\omega) \le \frac{2n}{\alpha}$$

Then we will have

$$\lim_{L\to+\infty} \frac{1}{2L}\int_{-L}^{L} \psi(x,\omega)dx = \lim_{L\to+\infty} \frac{1}{L}\int_{0}^{L} \psi(x,\omega)dx = \lim_{L\to+\infty} \frac{1}{L}\int_{-L}^{0} \psi(x,\omega)dx = n$$

Let $\phi'(x,\omega) := \psi(x,\omega) - n$, then

$$\lim_{|x|\to\infty} \frac{\phi(x,\omega)-\phi(0,\omega)}{|x|} = \lim_{|x|\to\infty} \frac{1}{|x|}\int_0^x \phi'(s,\omega)ds = 0$$

Which means that for a.e. $\omega \in \Omega$, $\phi(x,\omega)$ is sub-linear.

The derivative of $g(t) := \sqrt{m^2+t^2} + \frac{cts(x,\omega)}{\sqrt{m^2+t^2}}$ with respect to t is $\frac{t^3+m^2t+m^2cs(x,\omega)}{(m^2+t^2)^{\frac{3}{2}}}$, Let $\overline{c} := \frac{2}{\tau m^2}\left[(\frac{2n}{\alpha})^3 + m^2(\frac{2n}{\alpha})\right] > 0$. For all $x \in A(\omega)$, if $c > \overline{c}$, then $g'(t) < 0$ for $t \in [0, \frac{2n}{\alpha}]$. By the construction of ϕ,

$$0 \le n + \phi'(x,\omega) = \psi(x,\omega) \le \frac{2n}{\alpha}$$

And recall that $\text{supp}(n+\phi'(x,\omega)) \subset A(\omega)$, then

$$\begin{aligned}
\max_{x\in\mathbf{R}}\left\{\sqrt{m^2+(n+\phi')^2} + \frac{c(n+\phi')s}{\sqrt{m^2+(n+\phi')^2}} + k\right\} &\le \max_{x\in\mathbf{R}}\{|m| + k(x,\omega)\} \\
&= |m| + \max_{x\in\mathbf{R}} k(x,\omega) \\
&= \overline{H}_*
\end{aligned}$$

If $h(c) > \overline{H}_*$, by Lemma 2.3.3, the cell problem has solution $u(x,\omega)$ which is sub-linear for a.e. $\omega \in \Omega$. By above construction, ϕ is also sub-linear for a.e. $\omega \in \Omega$. Fix such ω that both of $\phi(x,\omega)$ and $u(x,\omega)$ are sub-linear. So for any $\delta > 0$, $u(x,\omega) - \phi(x,\omega) + \delta\sqrt{x^2+1}$ can achieve minimum at some point x_δ, so

$$h(c) \le H\left(n + \phi'(x_\delta,\omega) - \delta\frac{x_\delta}{\sqrt{x_\delta^2+1}}, x_\delta, \omega\right)$$

$\delta \to 0 \implies h(c) \le \max_{x\in\mathbf{R}} H(n+\phi'(x,\omega),x,\omega) = \overline{H}_*$, this is a contradiction.
Thus $h(c) = \overline{H}_*$ when $c > \overline{c}$.

□

Proof of theorem 2.1.1. (1) comes from section 2.

(2) If $mn \neq 0$, by Theorem 2.3.1 with $k(x,\omega) = mv(x,\omega)$, $s(x,\omega) = mv'(x,\omega)$.

If $m = 0$, $\overline{H}(p) = \overline{H}(p,c) = |n| = 1 > 0 = |m| + \sup_{(x,\omega)\in\mathbf{R}\times\Omega} mv(x,\omega)$.

If $n = 0$, $\overline{H}(p) = \overline{H}(p, c) = |m| + \sup_{(x,\omega)\in\mathbf{R}\times\Omega} mv(x, \omega)$.

(3) If $mv \equiv 0$, then $\overline{H}(p) \equiv \overline{H}(p, c)$.

Suppose $\overline{H}(m, n) = \overline{H}(m, n, c) > \overline{H}_*$.

If $mn \neq 0$ we must have v is constant, otherwise by Theorem 2.3.1, $\overline{H}(m, n, c) > \overline{H}(m, n)$ which gives a contradiction. By $\mathbf{E}[v] = 0$, we must have $mv = 0$.

If $m = 0$ then $mv \equiv 0$.

If $n = 0$, this is impossible since $\overline{H}(m, n) = \overline{H}(m, n, c) \equiv \overline{H}_*$.

Thus $mv \equiv 0$. □

2.4 Future problem

Same question can be asked regarding the strain effect with more generak imcompressible flow. For example, we can study the problem with t-dependent shear flow. More challengely, we can investigate the question under cellular flow (see figure **??**) with unsteady or random center.

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