Report No.
62-23

STRUCTURES AND MATERIALS RESEARCH
DEPARTMENT OF CIVIL ENGINEERING

LOGIC AND TOPOLOGY OF STRUCTURAL ANALYSIS

By

FRANK BARON

May 1962

STRUCTURAL ENGINEERING LABORATORY
COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY CALIFORNIA

Structures and Materials Research
Department of Civil Engineering
Division of Structural Engineering
and Structural Mechanics

LOGIC AND TOPOLOGY OF STRUCTURAL ANALYSIS

by

Frank Baron

College of Engineering
University of California
Berkeley, California
May 1962

LOGIC AND TOPOLOGY OF STRUCTURAL ANALYSIS

by

Frank Baron*

May 15, 1962

Synopsis

The paper deals with invariant forms of statements, quantities, and relationships in analysis. Group theory, set-points, and geometrical transformations are used to describe all parameters, functions, and configurations of a structure and its parts.

A generalized structure in a 3-dimensional space is transformed to a single set-member between two set-points in an N-dimensional space.

Introduction

The objectives of this paper are twofold. One objective is to state in symbolic form the invariant relationships that are applicable to any structure, or part of a structure, irrespective of the configuration of the structure. This objective is limited to the cases for which the requirements of statics and Hooke's Law hold, and for which the effects of the displacements on the forces and moments in the structure are considered negligible. The second objective is to indicate a way by means of which the configuration of a structure, and all parameters and functions of a given structure can be described. The way is by means of index notation and by sets of coordinates as employed in the study of N-dimensional spaces.

The first objective has been reported on by S. O. Asplund,(1) J. H. Argyris,(2) A. S. Hall,(3) and R. G. Woodhead(3) in terms of direct matrix notation, and by

----

* Professor of Civil Engineering, University of California, Berkeley.

G. Kron,[4] B. Langefors,[5] and S. Shore[6] in terms of index or tensor notation. The same objective is pursued herein but with a different emphasis. The emphasis is on the invariant forms that appear in symbolic statements of structural analysis. For example, invariants appear in the symbolic statements of (a) statics, (b) continuity of geometry, and (c) Hooke's Law for any structure, or sub-group of the structure. Invariants also appear in the objects that are described, the operations on the objects, and in the resulting algebraic relationships between the objects. In this sense, the emphasis herein is the same as that given in the general discipline called mathematical or symbolic logic.[7] The latter logic is a study of form. In the present application of this logic, the concepts of tensor[8,9] and matrix calculus[10] are employed. The use of tensors in this application, as in others, rests on the fact that a tensor equation is true in all coordinate systems, if true in one. This follows because tensors are invariant quantities by definition. Matrices are used to facilitate the routine work of calculations, and because the objects that are to be represented eventually must be described in sets of components.

The second objective is part of the discipline called algebraic topology.[11,12] Algebraic topology deals with continuity and with those properties of geometrical configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations, or homeomorphisms. In brief, two geometrical figures are topologically equivalent, or homeomorphic, if each can be transformed into the other by a continuous deformation; for example, like putty. This topic was illustrated for electrical applications by S. Seshu and M. B. Reed[13] and for structural applications by N. C. Lind.[14] The illustrations were in terms of a geometrical interpretation called linear graphs.

In this paper, an algebraic approach to structural topology is stressed, in which the theory of sets[15] is used to describe the coordinates of two-dimensional and three-dimensional configurations. Subsequent relationships are obtained for structural frameworks having topological equivalences. The relationships, being in algebraic form, are suited for programming of calculations to be made by means of electronic computers.

Indicial Notation

An indical notation is developed for use in the analysis of a generalized structure in space. The notation, with some modifications, is the same as that often employed in the study of tensors.[8,9] In this notation, indexes or suffixes of terms are used to identify (1) the components of terms, (2) the positions of terms (by means of sets of coordinates), and (3) arithmetical operations or transformations. Before we proceed to the analysis of structures, let us define two notational conventions that will become of real importance as labor-saving devices, and to keep the bulk of the formulae in a recognizable form.

Range Convention. When a small suffix (superscript or subscript) occurs just once in a term, it is understood to take all values 1 to N unless the contrary is specified.

Summation Convention. When a small suffix is repeated in a term (one suffix in the superscript and the other in the subscript of the term), summation with respect to that suffix is understood, the range of summation being 1 to N. Thus, the expression $\sum_{i=1}^{N} R_j^i \, p_i^{(j)}$ merely is written as $R_j^i \, p_i^{(j)}$.

The summation convention is suspended for suffixes that appear in parantheses. These suffixes are for reference in defining neighboring points that surround a given point.

The economy of the two conventions is demonstrated by the following equation:

$$P_j + R_j^i \, p_i^{(j)} = 0 \qquad (j=1,\cdot 3;\ i=1,2,\cdot 4) \qquad (1)$$

The convention informs us that there are three equations, with a sum performed over i ranging from 1 to 4 in each equation.

An index which is repeated in the same term is called a dummy index; an index which is not repeated in the same term is called a free index. Thus, j in Eq. 1 is a free index, and i is a dummy index. The following rules can be laid down for the manipulation of indices in an equation.

(1) The free index in every term of the equation must be the same, but the letter can be changed. Whatever values is given to a free index in one term, it must be given the same value in all terms, and therefore must be represented by the same letter throughout the equation.

(2) The dummy index however can be changed separately in each term, provided the same change is made within the term.

(3) The balance of free and dummy indices must be maintained on the two sides of the equation.

More freedoms than the preceding are reserved for the manipulation of indices within parantheses. This is because the latter indices are removed from the summation convention and are for identification purposes of terms. A balance of these indices is not required in an equation.

For actual calculations, the tensors in the subsequent equations are arranged for matrix multiplication; that is, row by column. This means that the multiplication of two or more tensors in a term is made in pairs along the dummy index of each adjacent pair of tensors. The dummy index of each adjacent pair heads the columns of the leading tensor and the rows of the following tensor.

The components of a tensor can quickly be determined by its valence (or order). The valence of a tensor is indicated by the suffixes of the tensor. For example, a tensor of valence one has one suffix, and a tensor of valence two has two suffixes. In matrix form, a tensor of valence one is represented by a column or row vector, and a tensor of valence two by a rectangular array of components.

Indicial notation for coordinates. In the succeeding sections, several sets of coordinates are used. These sets are described by means of suffices. For this purpose, the Latin alphabet is divided into groups such as (1, 2, ·i, ·I), (1, 2, ·j, ·J), (1, 2, ·ℓ, m, n, ·M), (1, 2, ·r, s, t, ·S), and (1, 2, ·u, v, w, ·V). Each of these groups is assigned to a specific set of coordinates. Greek suffices also are used; for example, α, β, and γ . The latter suffixes are associated with the preceding groups. Specifically, α is associated with (1, 2, ·ℓ, m, n, ·M), β with (1, 2, ·r, s, t, ·S), and γ with (1, 2, ·u, v, w, ·V).

Fundamental Relationships and Notation

A summary is given of certain fundamental relationships which occur in structural analysis. The relationships are for forces, displacements, and deformations of a structural element that lies at a point, and a member that lies between two joints. Later, it is shown that the relationships can be written in the same form for a group of members that meet at a joint, and a group of members that are connected to many joints.

Resultants. In Fig. 1, a force and moment are shown to act at point j. These are designated by F_j and M_j, respectively. The resultant at i of F_j and M_j consists of a force and moment (designated by F_i and M_i, respectively). Three orthogonal axes (x, y, and z) are shown in the Figure. A right hand rule is used for these and succeeding orthogonal axes. The x, y, z axes are used for defining the components of terms, such as F and M, along these axes.

Particular attention is called to the suffices, i and j. At this stage, these suffices merely serve as labels to identify points in space. They agree however with the conventions that are described in the preceding section. Note that a set of coordinates for i and j is not yet defined. The latter set need not be the same as that employed for defining the components of terms such as F and M. At a later stage, the suffixes such as i and j will serve as variables (for example, i = 1, 2, .. I and j = 1, 2, .. J) to identify coordinates of points in an N-dimensional space. In each of these stages, it is important to note the position of a suffix in a term; that is, as a superscript or subscript of a term. It also is important to note the order in which a suffix appears in the superscript or subscript of a term.

The x, y, z components of F_j and M_j are written in matrix form as follows:

$$F_j = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}_j \quad ; \quad M_j = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}_j \tag{2}$$

The F_j and M_j terms are assembled into a single matrix,

$$P_j = \begin{bmatrix} F \\ M \end{bmatrix}_j \tag{3}$$

P henceforth is called a load.

In Eq. 2, the indices within the brackets are for the x, y, z components of P. If desired, they can be written for the contravariant or the covariant components of P in a fundamental oblique coordinate frame (see Ref. 1a). Our interest however is principally with the indices outside of the brackets. For this reason and for simplicity herein, only the rectangular x, y, z components of terms such as P will be written.

The metric distance measured from i to j is designated by $r_i^{(j)}$. It is written in matrix form as

$$r_i^{(j)} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}_i^{(j)} \tag{4}$$

in which the positive components of r are from i to j. Consequently,

$$r_i^{(j)} = -r_j^{(i)} \tag{5}$$

Note that the upper suffixes of $r_i^{(j)}$ and $r_j^{(i)}$ are written in parantheses and are removed from the summation convention.

A 3 x 3 anti-symmetric matrix $\left[r^o\right]_j^i$ is defined in which the elements of $r_i^{(j)}$ are arranged as follows:

$$\left[r^o\right]_i^j = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}_i^j \tag{6}$$

Also, a 6 x 6 matrix R_i^j is defined

$$R_i^j = \begin{bmatrix} I & 0 \\ r^o & I \end{bmatrix}_i^j \tag{7}$$

in which I and 0 are 3 x 3 unitary and null matrices, respectively.

The resultant at point i of P_j then is given by

$$P_i = R_i^j \ P_j \tag{8}$$

that is, P_j is transformed to P_i by means of R_i^j.

Now consider a set of forces and moments which act at points 1 through j as shown in Fig. 2. The resultant at i of the set $\left\{j = 1, 2, .. J\right\}$ is given by

$$P_i = R_i^j \ P_j \qquad (j = 1, 2, .. J) \tag{9}$$

in which the summation over the dummy index is from 1 to J. Thus, by means of the range and summation conventions, a single equation (Eq. 9) suffices to transform a set of P_j's to a set P_i. In matrix form, the terms of Eq. 9 are given by

$$P_i = \begin{bmatrix} F \\ M \end{bmatrix}_i \ ; \ R_i^j = \begin{bmatrix} R_i^1 \ R_i^2 \ . \ R_i^J \end{bmatrix} \tag{10}$$

and

$$P_j = \begin{bmatrix} P_1 \\ P_2 \\ . \\ P_J \end{bmatrix} \tag{11}$$

The calculation of P at point i can be transferred to a new set of axes at point k by means of the transformation

$$P_k = R_k^i \, P_i \tag{12}$$

Eqs. 9 and 12 yield

$$R_k^j = R_k^i \, R_i^j \tag{13}$$

<u>Displacement of Sections</u>: In Fig. 3, a rigid member extends from i to j. The transverse section at i is subjected to a translation $\Delta_i^{(j)}$, and to a rotation $\theta_i^{(j)}$. The subscript of each of these terms indicates the location of the transverse section to which the term is referred, and the superscript the far end of the member. The translation and rotation at i are assembled into a single term,

$$\rho_i^{(j)} = \begin{bmatrix} \Delta \\ \theta \end{bmatrix}_i^{(j)} \tag{14}$$

in which

$$\Delta_i^{(j)} = \begin{bmatrix} \Delta_x \\ \Delta_y \\ \Delta_z \end{bmatrix}_i^{(j)} ; \quad \theta_i^{(j)} = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix}_i^{(j)} \tag{15}$$

Henceforth, ρ of a section is called the displacement of the section.

For small displacements, the contribution of $\rho_i^{(j)}$ to the displacement at j is given by

$$\rho_j^{(i)} = \overline{R}_j^i \, \rho_i^{(j)} \tag{16}$$

in which

$$\overline{R}_j^i = \begin{bmatrix} I & r^o \\ 0 & I \end{bmatrix}_j^i \tag{17}$$

Note that $\overline{R}_j^i$ is the same as the transpose of R_i^j; that is,

$$\overline{R}_j^i = R_i^{j*} \tag{18}$$

(See Eq. 7 for R_i^j). The asterisk in Eq. 18 denotes transposition. Note also that the inverses of the $\overline{R}$ and R terms can be denoted merely by inverting the suffixes of these terms; that is,

$$\left[\overline{R}_j^i\right]^{-1} = \overline{R}_i^j \; ; \; \left[R_j^i\right]^{-1} = R_i^j \qquad (19)$$

in which the exponent -1 denotes inversion.

Displacements of joints: Only one suffix is required to designate the displacement of a joint, whereas, two are required to designate the displacement of a transverse section of a member. This is because a joint lies at a point, whereas, a member lies between two points. Thus, the displacement of a joint at j is designated by ρ_j.

Deformations: The influence of a discontinuity in displacement, or a concentrated deformation, on the resulting displacement of two rigid members is shown in Fig. 4. The discontinuity occurs at c and is designated by

$$\delta\rho_c = \rho_c^{(j)} - \rho_c^{(i)} \qquad (20)$$

in which

$$\rho_c^{(j)} = \overline{R}_c^j \; \rho_j^{(i)} \qquad (21)$$

and

$$\rho_c^{(i)} = \overline{R}_c^i \; \rho_i^{(j)} \qquad (22)$$

Consequently, we obtain

$$\delta\rho_c = \overline{R}_c^j \; \rho_j^{(i)} - \overline{R}_c^i \; \rho_i^{(j)} \qquad (23)$$

which when premultiplied by $\overline{R}_j^c$ yields

$$\rho_j^{(i)} = \overline{R}_j^i \; \rho_i^{(j)} + \overline{R}_j^c \; \delta\rho_c \qquad (24)$$

By a simple manipulation of indices, Eq. 24 can be rewritten as

$$\rho_k^{(i)} = \overline{R}_k^i \; \rho_i^{(k)} + \overline{R}_k^j \; \delta\rho_j \qquad (25)$$

The latter equation is for a rigid member that lies between i and k, with a discontinuity at j.

For a deformable member that lies between i and j, Eq. 24 becomes

$$\rho_j^{(i)} = \bar{R}_j^i \, \rho_i^{(j)} + \bar{R}_j^c \, d\rho_c \tag{26}$$

in which the summation over c is from i to j, and $d\rho_c$ is given by

$$d\rho_c = \begin{bmatrix} d\Delta \\ d\theta \end{bmatrix}_c \tag{27}$$

Eq. 26 henceforth is called the fundamental equation of continuity for a deformable member.

Statical Relationships. In Fig. 5, an external load is shown to act at point c of a member that lies between points i and j. The internal force and moment acting on a section of the member, say section $\binom{j}{i}$, are designated by

$$f_i^{(j)} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}_i^{(j)} ; \quad m_i^{(j)} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_i^{(j)} \tag{28}$$

These terms are gathered into a single term,

$$p_i^{(j)} = \begin{bmatrix} f \\ m \end{bmatrix}_i^{(j)} \tag{29}$$

in which the subscript of a term designates the location of the section to which the term is referred, and the superscript the opposite end of the member. The positive direction of each component of force and moment is along its respective axis. From now on, p is called an internal force.

Statics of an infinitesimally small element at c yields

$$P_c = p_c^{(j)} + p_c^{(i)} \tag{30}$$

in which

$$p_c^{(j)} = -R_c^j \, p_j^{(i)} \tag{31}$$

and

$$p_c^{(i)} = -R_c^i \, p_i^{(j)} \tag{32}$$

For member icj, statics yields

$$P_c + R_c^j \, p_j^{(i)} + R_c^i \, p_i^{(j)} = 0 \tag{33}$$

By a simple manipulation of indexes, Eq. 33 is rewritten in another form; namely,

$$P_j + R_j^i \, p_i^{(j)} = 0 \qquad (i = 1,2, \cdot I) \tag{34}$$

The latter form of equation henceforth is called the fundamental statement of statics. The statement is for a group of members that meet at j.

For other purposes, Eq. 33 is rewritten in another form; namely,

$$P_j + R_j^i \, p_i^{(j)} + R_j^k \, p_k^{(j)} = 0 \tag{35}$$

Relationships Based on Hooke's Law. A summary is given of these relationships for (a) a differential element, and (b) a finite element, when each element is loaded only at its ends. The summary is obtained from Ref. 16 with a modification in notation and an extension to include the effects of shear and axial deformations.

(a) Differential Element. Consider an element that lies at point c(see Fig. 6). The element is ds in length, is oblique to the x, y, and z axes, and is subjected to a force and moment of

$$p_c^{(i)} = \begin{bmatrix} f \\ m \end{bmatrix}_c^{(i)} \tag{36}$$

at each end. The principal axes of the cross-sections are orthogonal and are designated by 1, 2, and 3. The unit base vectors along these axes are designated by $\begin{bmatrix} q_1 \end{bmatrix}_c$, $\begin{bmatrix} q_2 \end{bmatrix}_c$, $\begin{bmatrix} q_3 \end{bmatrix}_c$, respectively, and are assembled in matrix form as follows:

$$q_c^c = \left[q_1 \; q_2 \; q_3 \right]_c^c = \begin{bmatrix} q_{1x} & q_{2x} & q_{3x} \\ q_{1y} & q_{2y} & q_{3y} \\ q_{1z} & q_{2z} & q_{3z} \end{bmatrix}_c^c \tag{37}$$

in which the q_{ij}'th element is the direction cosine of the i'th base vector of the 123 system with respect to the j'th axis of the xyz system. Note that the base vectors satisfy the condition of an orthonormal set.

The scalar components of $p_c^{(i)}$ along the new set of axes are related to the components of $p_c^{(i)}$ along the original set by the orthogonal transformation,

$$p_c^{(i)} \{1, 2, 3\} = \bar{q}_c^{c\,*} \; p_c^{(i)} \{x, y, z\} \tag{38}$$

in which

$$\bar{q}_c^c = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}_c^c ; \quad \bar{q}_c^{c\,*} = \begin{bmatrix} q^* & 0 \\ 0 & q^* \end{bmatrix}_c^c \tag{39}$$

The magnitudes of the deformations with respect to the 1, 2, and 3 axes are given by the vector scalar product,

$$d \; \bar{\rho}_c = \begin{bmatrix} d\bar{\Delta} \\ d\bar{\theta} \end{bmatrix}_c = d\bar{k}_c^{(c)c} \; \bar{q}_c^{c*} \; p_c^{(i)} \tag{40}$$

in which

$$d\bar{k}_c^{(c)\,c} = \begin{bmatrix} d\bar{j} & 0 \\ 0 & d\bar{a} \end{bmatrix}_c^{(c)c} \tag{41}$$

$$d\bar{j}_c^{(c)c} = \begin{bmatrix} dj_{11} & 0 & 0 \\ 0 & dj_{22} & 0 \\ 0 & 0 & dj_{33} \end{bmatrix}_c^{(c)c} = ds \begin{bmatrix} \frac{1}{EA} & 0 & \\ 0 & \frac{\beta_2}{GA} & \\ 0 & 0 & \frac{\beta_3}{GA} \end{bmatrix}_c^{(c)c} \tag{42}$$

and

$$d\bar{a}_c^{(c)c} = \begin{bmatrix} da_{11} & 0 & 0 \\ 0 & da_{22} & 0 \\ 0 & 0 & da_{33} \end{bmatrix}_c^{(c)c} = ds \begin{bmatrix} \frac{1}{GJ_1} & 0 & 0 \\ 0 & \frac{1}{EI_2} & 0 \\ 0 & 0 & \frac{1}{EI_3} \end{bmatrix}_c^{(c)c} \tag{43}$$

In tensor form, the deformation at c caused by $p_c^{(i)}$ is given by

$$d\rho_c = \bar{q}_c^c \, d\bar{\rho}_c \tag{44}$$

or by

$$d\rho_c = dk_c^{(c)c} \; p_c^{(i)} \tag{45}$$

in which

$$dk_c^{(c)c} = \begin{bmatrix} dj & 0 \\ 0 & da \end{bmatrix}_c^{(c)c} = \bar{q}_c^c \, d\bar{k}_c^{(c)c} \, \bar{q}_c^{(c)c*} \tag{46}$$

Eq. 45 is Hooke's Law for an element ds in length.

<u>(b) Relationships for a Finite Element</u>. Consider a member that is curved in space, lies between points i and j, and varies in cross-section along its length (see Fig. 7). In Fig. 7(a), the member is unloaded except at its ends. In Fig. 7(b), the same member is shown with displacements at both ends and with deformations along its length. It is important to note that a rotation, say $\theta_i^{(j)}$, is the rotation of the <u>transverse section</u> at point i of member $\begin{Bmatrix} j \\ i \end{Bmatrix}$, and not the slope of the displacement curve at point i. This can be noted by considering two elements, each ds in length, which meet at a point c. A difference in the shear of the two elements results in a discontinuity in slope at c. The difference in the shear however does not yield a discontinuity in the rotations of the transverse sections. For these reasons, we refer to the rotations of the transverse sections (or joints) and not to the slopes of the displacement curve. The reference permits us to have a theory of continuity which includes discontinuities in slope caused by shear.

Geometry of the member yields

$$\rho_i^{(j)} + \bar{R}_i^c \, d\rho_c = \bar{R}_i^j \; \rho_j^{(i)} \tag{47}$$

in which the summation over c is from i to j. Statics of the member yields

$$p^{(i)}_{c} = -R^{i}_{c}\, p^{(j)}_{i} \tag{48}$$

With Hooke's Law and statics, Eq. 47 can be rewritten as

$$\bar{R}^{c}_{i}\, dk^{(i)c}_{c}\, \bar{R}^{c*}_{i}\;\; p^{(j)}_{i} = \rho^{(j)}_{i} - \bar{R}^{j}_{i}\, \rho^{(i)}_{j} \tag{49}$$

in which c varies from i to j. Now let

$$k^{(j)i}_{i} = \bar{R}^{c}_{i}\;\; dk^{(i)c}_{c}\, \bar{R}^{c*}_{i} \tag{50}$$

and obtain

$$\rho^{(j)}_{i} = k^{(j)i}_{i}\; p^{(j)}_{i} + \bar{R}^{j}_{i}\;\; \rho^{(i)}_{j}$$

By a simple change of indices, Eq. 51 is rewritten to yield the displacements at end j; namely,

$$\rho^{(i)}_{j} = k^{(i)j}_{j}\; p^{(i)}_{j} + \bar{R}^{i}_{j}\;\; \rho^{(j)}_{i} \tag{52}$$

Eqs. 51 and 52 are the fundamental forms of Hooke's Law for a finite element which extends between two ends, and is unloaded except at the ends. The two equations can be rewritten in other forms. For example; they can be rewritten as follows:

$$\rho^{(j)}_{i} = k^{(j)j}_{i}\; p^{(i)}_{j} + \bar{R}^{j}_{i}\, \rho^{(i)}_{j} \tag{53}$$

and

$$\rho^{(i)}_{j} = k^{(i)i}_{j}\; p^{(j)}_{i} + \bar{R}^{i}_{j}\;\; \rho^{(j)}_{i} \tag{54}$$

respectively. Each form of Hooke's Law (Eqs. 51 through 54) is useful, depending on the boundary conditions that are specified at ends i and j.

In the preceding equations, $k^{(j)i}_{i}$ is the flexibility matrix for end i of member $\left\{ {(j) \atop i} \right\}$, and $k^{(i)j}_{j}$ is the flexibility matrix for end j. It can be shown that each of these terms is symmetric; that is,

$$k^{(j)i}_{i} = k^{(j)i*}_{i}\;\; ; \qquad k^{(i)j}_{j} = k^{(i)j*}_{j} \tag{55}$$

and that $k^{(i)j}_{j}$ is related to $k^{(j)i}_{i}$ by means of the transformation,

$$k_{j}^{(i)j} = \bar{R}_{j}^{i} \, k_{i}^{(j)i} \, \bar{R}_{j}^{i*} \tag{56}$$

It can also be shown that $k_{i}^{(j)j}$ and $k_{j}^{(i)i}$ are related to $k_{i}^{(j)i}$ as follows:

$$k_{j}^{(i)i} = -\bar{R}_{j}^{i} \; k_{i}^{(j)i} = k_{i}^{(j)j*} \tag{57}$$

A physical interpretation is given of the various k-terms and their indices. Each k-term designates an array of influence coefficients for the displacements at an end caused by a set of unit forces and moments at the same or opposite end of the member. The indices designate the member, the positions of the displacements, and the position of the unit forces and moments. For example, $k_{i}^{(j)j}$ designates the displacements at end i of member $\left\{ {(j) \atop i} \right\}$, caused by a set of unit forces and moments at end j.

The relationship between the forces and displacements at the ends of the member can be stated in terms of stiffness matrices. The stiffness matrix at end i of member $\left\{ {(j) \atop i} \right\}$ is related to the flexibility matrix as follows:

$$K_{i}^{(j)i} = \left[k_{i}^{(j)i} \right]^{-1} \tag{58}$$

Now, premultiply Eq. 51 by $K_{i}^{(j)i}$ and obtain

$$P_{i}^{(j)} = K_{i}^{(j)i} \rho_{i}^{(j)} - K_{i}^{(j)i} \bar{R}_{i}^{j} \; \rho_{j}^{(i)} \tag{59}$$

Further, because

$$K_{i}^{(j)j} = - K_{i}^{(j)i} \bar{R}_{i}^{(j)} \tag{60}$$

we obtain

$$P_{i}^{(j)} = K_{i}^{(j)i} \rho_{i}^{(j)} + K_{i}^{(j)j} \rho_{j}^{(i)} \tag{61}$$

By a simple change of indices, Eq. 61 is rewritten to yield the forces and moments at end j; namely,

$$P_{j}^{(i)} = K_{j}^{(i)i} \; \rho_{i}^{(j)} + K_{j}^{(i)j} \rho_{j}^{(i)} \tag{62}$$

Eqs. 61 and 62 now are grouped into a single equation as follows:

$$\begin{bmatrix} p_i^{(j)} \\ \\ p_j^{(i)} \end{bmatrix} = \begin{bmatrix} K_i^{(j)i} & K_i^{(j)j} \\ \\ K_j^{(i)i} & K_j^{(i)j} \end{bmatrix} \begin{bmatrix} \rho_i^{(j)} \\ \\ \rho_j^{(i)} \end{bmatrix} \tag{63}$$

The latter equation is Hooke's Law, or the generalized rotation-deflection equation for a general member in space.

In Eq. 63, $K_i^{(j)i}$ is the stiffness matrix for end i of member $\left\{ {(j) \atop i} \right\}$, and $K_j^{(i)j}$ is the stiffness matrix for end j. It can be shown that each of these terms is symmetric; that is,

$$K_i^{(j)i} = K_i^{(j)i*} \quad ; \quad K_j^{(i)j} = K_j^{(i)j*} \tag{64}$$

Also, $K_j^{(i)j}$ is related to $K_i^{(j)i}$ by means of the transformation,

$$K_j^{(i)j} = R_j^i \; K_i^{(j)i} \; R_j^{i*} \tag{65}$$

and $K_i^{(j)j}$ and $K_j^{(i)i}$ are related to $K_i^{(j)i}$ as follows:

$$K_i^{(j)j} = - K_i^{(j)i} \; \bar{\bar{R}}_i^j = K_j^{(i)i*} \tag{66}$$

In the same way as for the k-terms, a physical interpretation can be given to the various K-terms and their indices. Each K-term designates a set of forces at an end of a member when the same or opposite end is subjected to a set of unit displacements. In brief, $K_i^{(j)i}$ and $K_j^{(i)j}$ represent stiffness factors, and $K_j^{(i)i}$ and $K_i^{(j)j}$ represent products of stiffness factors and carry-over factors as defined in moment distribution. (See Ref. 16 for a more complete interpretation of the various K-terms.

Generalized Structure In Space

Now consider the generalized structure in space. The structure is represented in Fig. 8 and consists of deformable members which are attached to rigid joints. The merits of conceiving such joints for algebraic purposes are discussed by H. M. Westergaard in Ref. 17. The actual connections of a structure are considered to be a part of the deformable members. The external loads and reactions are assumed to act on joints only.

In Fig. 9, a sub-group of members is shown in which the members meet at a joint j. The far ends of the members are at points i and are identified by the set of coordinates x_i, $(i = 1, 2, \cdot I)$. To insure that no ambiguity exists concerning the indicial notations and the representation of the components of tensors in matrix form, all steps in the succeeding equations are shown for this sub-group.

Statics for the sub-group yields

$$P_j + R_j^i \; p_i^{(j)} = 0 \quad (j = j;\; i = 1, 2, \cdot I) \qquad (67)$$

which when summed over i becomes

$$P_j + R_j^1 \; p_1^{(j)} + \cdot + R_j^I \; p_I^{(j)} = 0 \qquad (68)$$

and, in matrix form becomes

$$P_j + \begin{bmatrix} R_j^1 & \cdot & R_j^I \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ p_2 \end{bmatrix}^{(j)} = 0 \qquad (69)$$

Hooke's Law for each member that meets at joint j is given by Eq. 61. Because $\rho_j^{(i)} = \rho_j$ for all members that meet at joint j, the suffix in the parantheses of $\rho_j^{(i)}$ is dropped and Eq. 61 is rewritten as follows:

$$p_i^{(j)} = K_i^{(j)i} \rho_i^{(j)} - K_i^{(j)i} \bar{R}_i^j \rho_j \qquad (i = 1, 2, \cdot I) \qquad (70)$$

In matrix form, the terms of Eq. 70 are given by

$$p_i^{(j)} = \begin{bmatrix} p_1 \\ \cdot \\ p_I \end{bmatrix}^{(j)} = \begin{bmatrix} K_1^{(j)1} & & \\ & \cdot & \\ & & K_I^{(j)I} \end{bmatrix} \begin{bmatrix} \rho_1 \\ \cdot \\ \rho_I \end{bmatrix}^{(j)} - \begin{bmatrix} K_1^{(j)1} & & \\ & \cdot & \\ & & K_I^{(j)I} \end{bmatrix} \begin{bmatrix} \bar{R}_1 \\ \cdot \\ \bar{R}_I \end{bmatrix}^j \rho_j \qquad (71)$$

Substituting Eq. 70 into Eq. 67 and employing the relationships that are given by

$$K_j^{(i)i} = -R_i^j K_i^{(j)i} \; ; \text{ and } \sum_{i=1}^{I} K_j^{(i)j} = R_j^i K_i^{(j)i} R_j^{i*} \qquad (i = 1, 2, \cdot I) \qquad (72)$$

yields

$$P_j = K_j^{(i)i} \rho_i^{(j)} + K_j^{(J)j} \rho_j \qquad (i = 1, 2, \cdot I) \qquad (73)$$

Note that the term $R_j^i K_i^{(j)i} R_j^{i*}$ in Eq. 72 when summed over i becomes the sum of the stiffness matrices at end j of all the members which meet at joint j. This sum is designated in Eq. 73 by the change of the suffix in the parantheses of the term $\sum_{i=1}^{I} K_j^{(i)j}$; that is, by

$$K_j^{(J)j} = \sum_{i=1}^{I} K_j^{(i)j} \qquad (74)$$

Henceforth, a capital suffix in the parantheses of a term designates a sum of the term that is denoted by the principal letter and the other suffixes.

Eq. 73 is the fundamental equation of elasticity for a sub-group j connected to a neighborhood i. One such equation can be written for each joint j, including the joints at the reactions. When the conditions at the reactions are specified[6] and statics of the external loads is considered, the following equation can be written:

$$P_\eta = K_\eta^{(\eta)\eta} \; \rho_\eta \qquad (\eta = 1, 2, \cdot\cdot J) \qquad (75)$$

which yields

$$\rho_\eta = \left[K_\eta^{(\eta)\eta} \right]^{-1} P_\eta \qquad (\eta = 1, 2, \cdot\cdot J) \qquad (76)$$

In Eq. 75, the prime unknowns are displacements. If desired, equations can be formed in which the unknowns are the internal forces, or a mixture of internal forces and displacements. The latter equations can be formed by means of the fundamental relationships that are given in the preceding section.

Special Cases of Sub-Group. Several cases of the sub-group shown in Fig. 9 are considered. The cases serve as bases for determining constants for use in distribution procedures, iteration methods, and partitioning of matrices. Eq. 73 is recalled for ready reference:

$$P_j = K_j^{(i)i} \; \rho_i^{(j)} + K_j^{(J)j} \; \rho_j \qquad (i = 1, 2, \cdot I) \qquad (73)$$

Case 1. (see Fig. 9). For this case, the ends at i are considered fixed and P_j is an unbalanced load at the joint. Consequently, $\rho_i^{(j)}$ is null and Eq. 73. yields

$$\rho_j = \left[K_j^{(J)j} \right]^{-1} P_j \qquad (77)$$

in which $K_j^{(J)j}$ is the stiffness matrix for the entire joint j. From Hooke's Law (see Eqs. 61 and 62), the forces at end j of each sub-member are given by

$$p_j^{(i)} = K_j^{(i)j} \left[K_j^{(J)j} \right]^{-1} P_j \quad (i = 1, 2, \cdot I) \quad (78)$$

and at end i by

$$p_i^{(j)} = K_i^{(j)j} \left[K_j^{(J)j} \right]^{-1} P_j \quad (i = 1, 2, \cdot I) \quad (79)$$

Eq. 78 is interpreted to read as follows: An unbalanced load P_j is distributed to ends j of the sub-members that meet at j in proportion to the relative stiffnesses of the sub-members. Further, the load at j is carried over (or transferred) to ends i as given by Eq. 79. As in moment distribution, forces carried-over from ends i to joint j are summed up at j to give new unbalanced loads at j. The latter loads follow from statics (see Eq. 67).

Case 1, together with the concepts of line and block operators (or sub-sets of a set), have served as a basis for programming the electronic calculations of a generalized distribution procedure for the general structure in space.

Case 2a (see Fig. 10a). For this case, a sub-group lies along a curve ijk, with a load at j. By a simple change of a dummy index in Eq. 73, we obtain

$$P_j = K_j^{(i)i} \rho_i + K_j^{(J)j} \rho_j + K_j^{(k)k} \rho_k \tag{80}$$

The latter equation can be interpreted as a central difference equation for sub-group ijk. It can be modified to include a branch-member that extends from j to j'.

Case 2b (see Fig. 10b). For this case, a set of sub-members lie along a line defined by the set of points

$$x_\psi = (x_1, x_2, .. x_i, x_j, x_k, .. x_J) \tag{81}$$

in which

$$\psi = (1, \cdot i, j, k, \cdot J) \tag{82}$$

$$i = j - 1$$

and

$$k = j + 1 \tag{84}$$

By a simple change of the indexes in Eq. 73 and application of the range convention, Eq. 73 can be rewritten as

$$P_\psi = K_\psi^{(\psi)\psi} \rho_\psi \tag{85}$$

in which

$$P_\psi = \begin{bmatrix} P_1 \\ \cdot \\ P_j \\ \cdot \\ P_J \end{bmatrix} ; \quad \rho_\psi = \begin{bmatrix} \rho_1 \\ \cdot \\ \rho_j \\ \cdot \\ \rho_J \end{bmatrix} \qquad (86)$$

and $K_\psi^{(\psi)\psi}$ is the symmetric band matrix

$$K_\psi^{(\psi)\psi} = \begin{bmatrix} K_1^{(1)1} K_1^{(2)2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & K_j^{(i)i} K_j^{(J)j} K_j^{(k)k} & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & K_J^{(i)I} K_J^{(J)J} \end{bmatrix} \qquad (87)$$

See Ref. 18 for obtaining the inverse of a band matrix.

It can be shown that the inverse of $K_\psi^{(\psi)\psi}$ is related to the flexibility matrix at end 1 of the member that lies between 1 and J. Note that the flexibility matrix has already been obtained for a finite element that lies between two points and is unloaded except at its ends (see Eq. 50).

Cases 1, 2a, and 2b are reexamined. In Case 1, the members meet at a joint and are arranged in parallel (parallel being used as for electrical circuits). For this kind of group, the stiffness matrix is the sum of the stiffness matrices for all members that meet at the joint. Before the sum is performed, the stiffness matrix for each member is transformed to the end that lies at the joint. (See Eq. 65 for the transformation of a stiffness matrix).

In Cases 2a and 2b, the members (or segments) lie along a curve and are arranged in series (series being used as for electrical circuits). For this kind of group, the flexibility matrix for an end of the group is the sum of the flexibilities of all segments. Before this sum is performed, the flexibility of each segment is transformed to the end that is considered. (See Eq. 56 for the transformation of a flexibility matrix).

Cases 1, 2a, and 2b can serve as a basis for the development of a computational technique in which the concepts of stiffness paths and flexibility paths are employed. A description of this technique is beyond the scope of this paper. The technique is similar to that described by C. E. Pearson[19] and that employed in the analysis of electrical circuits.

Sets of Coordinates

In the preceding sections, no restrictions are placed on the selection of the coordinates to be used in identifying points on a structure; for example, points j and i. Various systems of coordinates can be used. A few that are particularly useful in structural applications are illustrated.

<u>Coordinate Set A.</u> In this set (see Fig. 11), the joints j of a structure are identified by the set of values,

$$x_{\psi} = \left\{ x_1, \; x_2, \; \cdot \; x_j, \; \cdot \; x_J \right\} \tag{88}$$

in which ψ takes all values from 1 to J (J being the number of joints of the structure). The element $\left\{ x_j \right\}$ of set x_{ψ} denotes a particular joint j.

The far ends i of the members that meet at a joint j are identified by the sub-set

$$x^{\phi}_{(j)} = \left\{ x^1, \; x^2, \; \cdot \; x^i, \; \cdot \; x^I \right\}_{(j)} \tag{89}$$

in which ϕ takes all values from 1 to I (I being the number of members that intersect at joint j). The element $\left\{ x^i_{(j)} \right\}$ of set $x^{\phi}_{(j)}$ denotes a particular end $\left\{ \begin{matrix} i \\ (j) \end{matrix} \right\}$.

This system of coordinates already has been illustrated in the preceding section.

<u>Coordinate Set B.</u> Consider the sets of coordinates that are represented in Figs. 12(a) and 12 (b). In each case, the set is composed of two sub-sets of

curves; sub-set

$$x_\alpha = \left\{x_1, \cdot x_\ell, x_m, x_n, \cdot M\right\} \tag{90}$$

and sub-set

$$x_\beta = \left\{x_1, \cdot x_r, x_s, x_t, \cdot S\right\} \tag{91}$$

in which α is associated with m, and β with s. Note that the suffixes ℓ, n and r, s are dependent variables; that is, their values are given by

$$\ell = m - 1; \; n = m + 1 \tag{92}$$

and

$$r = s - 1; \; t = s + 1 \tag{93}$$

In Fig. 12, the intersections of the curves lie on a surface in Euclidean space. All points on this surface are represented by the set $x_{\alpha\beta}$ of which x_α and x_β are sub-sets, or curves.

Let us assume that the curves x_α and x_β coincide with the axes of the members of a given structure and are the same as the coordinates of a space, V_2. In this way, the position of each joint (or application of load) can be specified. (When necessary, non-existent members can be introduced to define positions of joints. The non-existent members are purely imaginary and offer no resistance to loads.) Thus, a set of values,

$$\left\{x_m, x_s\right\}$$

define the coordinates of a joint, say the joint labeled j in Fig. 12. By means of index notation, the load at j is designated by

$$P_{ms}$$

and the displacement at j by

$$\rho_{ms}$$

in which the subscripts indicate that the joint lies at the intersection of members (or coordinates) x_m and x_s. In the same way, the load at the joint labeled a (see Fig. 12) is designated by P_{mt}.

Note that the valence (or order) of P_{ms} is one. The two suffixes ms simply identify suffix j in the term P_j for the coordinate system that is used. The same note is made concerning any subsequent substitution of coordinates for j and i. This follows because a change in coordinates does not change a tensor quantity.

In Fig. 12, ja is a sub-member of x_m and jb is a sub-member of x_s. Several different index notations can be used to designate terms referred to a specific end-section of a given sub-member. For example, the term p at end j of sub-member ja can be designated by

$$p_{ms}^{(mt)}$$

and ρ at end j of sub-member jb by

$$\rho_{ms}^{(\ell s)}$$

In this example, the subscripts of a term designate the end-section to which the term is referred, and the superscripts the opposite end of the sub-member.

In the same way, the stiffness matrix for end j of sub-member ja is designated by

$$K_{ms}^{(mt)mt}$$

The distance vector measured from j to a is denoted by

$$r_{ms}^{(mt)}$$

and the transformation matrix R that is associated with $\left[r^{c} \right]_{ms}^{mt}$ is denoted by

$$R_{ms}^{mt}$$

Note that in each case one of the upper suffixes of a four index term is identical in value and position to a lower suffix. The suffix that is common in a term indicates the coordinate of the sub-member, and the member of which the sub-member is a part. If desired, the suffix that is common in a term can be eliminated from the upper (or lower) suffixes, provided that the remaining suffixes, are retained in value and position. The latter notation is not followed herein. Instead, all suffices of a term are retained.

For another example of a possible index notation, the term p at ends j of a sub-group as shown in Fig. 13 can be designated by

$$p_{ms}^{(i)} \qquad (i = 1, 2, \cdot I)$$

and ρ at ends j by

$$\rho_{ms}^{(i)} \qquad (i = 1, 2, \cdot I)$$

In this example, two sets of coordinates are used; namely, sets A and B, of which A is a sub-set of B. Set B is used to define the positions of ends j (see the subscripts), and set A is used to define the opposite ends of the members that meet at j (see the superscripts). Note that the elements of set A are the same as those of a rotating vector which sweeps, with its origin at j, through points i. The latter example of combining coordinate sets is useful in describing the configurations of structures composed of ribs, lattices, or rosettes, See Refs 20, 21 and 22 for numerous illustrations of such configurations. It can be shown that many of these configurations are homeomorphic.

Coordinate Set C. Now consider the set that is represented in Fig. 14. The set is composed of three sub-sets of surfaces; namely;

$$x_{\alpha}^{(\beta\gamma)} = \{x_1, \cdot x_{\ell}, x_m, x_n, \cdot M\}^{(\beta\gamma)} \qquad (94)$$

$$x_{\beta}^{(\gamma\alpha)} = \{x_1, \cdot x_r, x_s, x_t, \cdot S\}^{(\gamma\alpha)} \qquad (95)$$

and

$$x_{\gamma}^{(\alpha\beta)} = \{x_1, \cdot x_u, x_v, x_w, \cdot V\}^{(\alpha\beta)} \qquad (96)$$

in which α is associated with m, β with s, and γ with v. Note that the suffixes ℓ, n; r, t; and u, w are dependent variables; that is, their values are given by

$$\ell = m - 1; \; n = m + 1 \qquad (97)$$

$$r = s - 1; \; t = s + 1 \qquad (98)$$

and

$$u = v - 1; \; w = v + 1 \qquad (99)$$

Also, note that the intersection of any two surfaces (one element from each sub-set) is a curve. The intersection of any three surfaces (one element from each sub-set) is a point.

Let us assume that the various curves coincide with the axes of the members of a given structure, and that the points coincide with the joints of the structure. In this way, the position of each joint can be specified. Thus, a set of values

$$\left\{x_m, x_s, x_v\right\}$$

define the coordinate of a joint, say the joint labeled j in Fig. 14.

In the same way as in the preceding section, the load at j is designated by P_{msv}, (2) the displacement at end j of the sub-member labeled ja is designated by $\rho_{msv}^{(mtv)}$, and (3) the force at end j of the sub-member labeled jb is designated by $p_{msv}^{(msw)}$. (See Fig. 14).

Again, another index notation can be used to designate terms referred to a specific end-section of a given sub-member. For example, the terms p referred to ends j of the sub-group shown in Fig. 14 can be designated by

$$p_{msv}^{(i)} \qquad (i = 1, 2, \cdot\cdot I)$$

and ρ at ends j by

$$\rho_{msv}^{(i)} \qquad (i = 1, 2, \cdot\cdot I)$$

In this example, coordinate sets A and C are used to define the configuration of a structure, and all parameters and variables of the structure. Again, A is a sub-set of C.

Examples of Groups and Set-Points

Several ways that terms can be grouped to describe the behavior of a structure are given. They are illustrated by examples that do not represent complete solutions to problems, but rather ways of grouping the various terms of a structure into sub-groups. The sub-groups are similar to partitions employed in tear methods[23] of matrix analysis, and to line and block operators[22] employed in relaxation procedures.

The examples also illustrate the use of set-points to describe the configurations of structures.

For these purposes, Eqs. 73 and 80 are recalled.

$$P_j = K_j^{(i)i} \rho_i^{(j)} + K_j^{(J)j} \rho_j \tag{73}$$

$$P_j = K_j^{(i)i} \rho_i^{(j)} + K_j^{(J)j} \rho_j + K_j^{(k)k} \rho_k^{(j)} \tag{80}$$

<u>Example 1</u>. Consider a structure which has a configuration that is of the same homeomorphic class as shown in Fig. 12. The configuration can be described by coordinate set B and sub-set A, that is, set B is used to identify a joint j, and sub-set A is used to identify joints i that are connected to j. We now inspect various sub-groups of the structure.

Sub-group 1a (see Fig. 15a): For this sub-group, Eq. 73 becomes

$$P_{ms} = K_{ms}^{(i)i} \rho_i^{(ms)} + K_{ms}^{(MS)ms} \rho_{ms} \tag{100}$$

in which the summation over i is for values of

$$i = mt,\ \ell s,\ mr,\ ns \tag{101}$$

Eq. 100 can be interpreted as a mapping operation in which the sub-group shown in Fig. 15a is mapped into a V_N space (see Fig. 16a). The sub-group in the V_N space consists of a single set-member that lies between two set-points; namely, set-points j and i. The set-points are given by

$$j = \{ms\} \tag{102}$$

$$i = \{mt,\ \ell s,\ mr,\ ns\} \tag{103}$$

For the special case that $\rho_i^{(j)} = 0$; that is, ends i are fixed, we obtain

$$P_{ms} = K_{ms}^{(MS)ms} \rho_{ms} \tag{104}$$

which yields

$$\rho_{ms} = \left[K_{ms}^{(MS)ms} \right]^{-1} P_{ms} \tag{105}$$

For another special case; namely, that for which ends i = mt and i = mr are fixed, we obtain

$$P_{ms} = K_{ms}^{(\ell s)\ell s} \rho_{\ell s} + K_{ms}^{(MS)ms} \rho_{ms} + K_{ms}^{(ns)ns} \rho_{ns} \tag{106}$$

which is of the same form as Eq. 80. Eq. 106 can also be interpreted as a mapping operation in which the special sub-group is mapped into a V_N space (see Fig. 16b). The sub-group in the V_N space consists of two members which lie along a curve s equals a constant. The curve goes through three points; ℓs, ms, and ns.

Sub-group 1b (see Fig. 15b). This sub-group consists of M joints that lie along a defining line s equals a constant. For this group, Eq. 73 becomes

$$P_{\alpha s} = K_{\alpha s}^{(\alpha i)\alpha i} \rho_{\alpha i}^{(\alpha s)} + K_{\alpha s}^{(\alpha s)\alpha s} \rho_{\alpha s} \tag{107}$$

in which

$$s = \text{a constant} \tag{108}$$

$$r = s - 1; \; t = s + 1 \tag{109}$$

$$i = r \text{ and } t \tag{110}$$

and

$$\alpha = \left\{1,\ 2,\ \cdot\cdot\ m\ \cdot\ M\right\} \tag{111}$$

The Greek letter α is introduced for ease in distinguishing terms that are associated with individual joints on the structure from those that are associated with all joints $\alpha = \left\{1,\ 2,\ \cdot\cdot\ m\ \cdot\ M\right\}$ that lie along lines r, s, and t equal constants.

By means of the range convention, $P_{\alpha s}$ and $\rho_{\alpha s}$ are written in matrix form as follows:

way, $\rho_{mr}^{(ms)}$ in Eq. 114 is associated with point mr of member $\left\{\begin{matrix}(ms)\\ mr\end{matrix}\right\}$; whereas, $\rho_{\alpha r}^{(\alpha s)}$ is associated with all points $\alpha = \left\{1, 2, \cdot m, \cdot M\right\}$ which lie along the line r equals a constant, (r = s - 1).

Again, the concept of a mapping operation or a geometrical transformation is used. Eq. 107 is a statement of Hooke's Law for a single set-member that lies between two set-points in a V_N space (see Fig. 16c). The two points are αs and $\underline{\alpha i}$.

Eq. 107 when written in the form of Eq. 80 becomes

$$P_{\alpha s} = K_{\alpha s}^{(\alpha r)\alpha r} \rho_{\alpha r}^{(\alpha s)} + K_{\alpha s}^{(\alpha s)\alpha s} \rho_{\alpha s} + K_{\alpha s}^{(\alpha t)\alpha t} \rho_{\alpha t}^{(\alpha s)} \tag{117}$$

The given sub-group now is mapped into two set-members which lie along a curve in a V_N space (see Fig. 16d). The curve goes through three points; αr, αs, and αt.

Now consider a special case of sub-group 1b. For this case, $\rho_{\alpha r}^{(\alpha s)}$ and $\rho_{\alpha t}^{(\alpha s)}$ are null; that is, the ends that lie along the defining lines αr and αt equal constants are fixed. Eq. 117 then becomes

$$P_{\alpha s} = K_{\alpha s}^{(\alpha s)\alpha s} \rho_{\alpha s} \tag{118}$$

which yields

$$\rho_{\alpha s} = \left[K_{\alpha s}^{(\alpha s)\alpha s}\right]^{-1} P_{\alpha s} \tag{119}$$

The forces and moments at all ends of the members can now be obtained by means of Eqs. 61 and 62.

The special case can be interpreted to yield line-sets (or line operators) of stiffness factors and products of stiffness factors and carry-over factors for use in relaxation procedures. The relaxations then are performed for entire sets of joints that lie along a line, or for sub-sets of joints that lie along segments of a line.

<u>Sub-group 1c</u> (see Fig. 12a) This sub-group consists of M x S joints that lie along a surface, as on a shell (γ equals a constant). For this group, we recall Eq. 117 for ready reference;

$$P_{\alpha s} = K_{\alpha s}^{(\alpha r)\alpha r} \rho_{\alpha r}^{(\alpha s)} + K_{\alpha s}^{(\alpha s)\alpha s} \rho_{\alpha s} + K_{\alpha s}^{(\alpha t)\alpha t} \rho_{\alpha t}^{(\alpha s)} \tag{117}$$

The latter equation is valid for all joints $\alpha = \{1, 2, \cdot m, \cdot M\}$ which lie along a curve s equals a constant. One such equation can be written for each set-point α that lies along a curve

$$\beta = \{1, 2, \cdot s, \cdot S\} \tag{120}$$

Conceptually, this is the same as mapping all members and joints of the structure that is in a Euclidean space onto a plane of a V_N space. The submembers in the V_N space lie along a curve that is defined by the set of points, $\beta = \{1, 2, \cdot s, \cdot S\}$. When all such equations are gathered into a single equation, we obtain

$$P_{\alpha\beta} = K_{\alpha\beta}^{(\alpha\beta)\alpha\beta} \rho_{\alpha\beta} \tag{121}$$

in which

$$P_{\alpha\beta} = \begin{bmatrix} P_{\alpha 1} \\ \cdot \\ P_{\alpha s} \\ \cdot \\ P_{\alpha S} \end{bmatrix}; \qquad \rho_{\alpha\beta} = \begin{bmatrix} \rho_{\alpha 1} \\ \cdot \\ \rho_{\alpha s} \\ \cdot \\ \rho_{\alpha S} \end{bmatrix} \tag{122}$$

and

$$K_{\alpha\beta}^{(\alpha\beta)\alpha\beta} = \begin{bmatrix} K_{\alpha 1}^{(\alpha 1)\alpha 1} & K_{\alpha 1}^{(\alpha 2)\alpha 2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & K_{\alpha s}^{(\alpha r)\alpha r} & K_{\alpha s}^{(\alpha s)\alpha s} & K_{\alpha s}^{(\alpha t)\alpha t} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & K^{(\alpha, S-1)\alpha, S-1} & K_{\alpha S}^{(\alpha S)\alpha S} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \tag{123}$$

The latter operation is conceived as the intersection of a curve with a surface in a V_N space; the intersection being a set-point, $\alpha\beta$. Thus, all joints and members of the structure are "contracted" into a single set-point in an N-dimensional space (see Fig. 16e). Eq. 121 is a statement of proportionality (Hooke's Law) for the entire structure. It is of the same form as Eq. 45 for a differential element that lies at a single point in a 3-dimensional space.

Let us reexamine $K_{\alpha\beta}^{(\alpha\beta)\alpha\beta}$ in Eq. 123. The latter term is a symmetric band matrix and is associated with all joints that lie along the surface defined by the coordinate set $\alpha\beta$. Note that the K-terms in the brackets of Eq. 123 are associated with all joints that lie along a curve β equals a constant. Further, inspect the s'th row of $K_{\alpha\beta}^{(\alpha\beta)\alpha\beta}$ and note that the terms, $K_{\alpha s}^{(\alpha r)\alpha r}$ and $K_{\alpha s}^{(\alpha t)\alpha t}$, are of the same kind as those given by Eq. 115. Also, $K_{\alpha s}^{(\alpha s)\alpha s}$ is of the same kind as that given by Eq. 116.

In this example, no distinctions are made between the joints that lie at the intersections of members and those that lie at the supports. If desired, the joints that lie at the supports can be represented by the single set-point $\phi = \{1, 2, \cdot\, i \,\cdot\, I\}$ and all other joints of the structure by another set-point $\alpha\beta$. The members of the structure then are "mapped" onto a single set-member that lies between points $\alpha\beta$ and ϕ of a V_N space. The resulting expression for Hooke's Law is of the form given by Eq. 73.

For use in a relaxation procedure, this example can be interpreted to yield surface-sets (or block-operators) of stiffness factors, and products of stiffness factors and carry-over factors, for segmental areas of the entire "surface". In this procedure, the entire surface of the structure is segmented into sub-surfaces which intersect along curves. The curves of intersection are selected members of the structure. Fixed-edge forces then are obtained for each segmental area; also, blocks of stiffness factors and products of stiffness factors and carry-over factors. A sequence of relaxation then is performed for the unbalanced sets of forces along the curves of intersection. In this way, difficulties that at times arise with the sensitivity of computations or slowness of convergence can be minimized. It should be mentioned that the concepts of set-points and mapping operations are helpful in defining this sequence of computations.

<u>Example 2.</u> Consider a structure which has a configuration that is of the same homeomorphic class as shown in Fig. 17 (the structure is 3-dimensional). The configuration can be described by coordinate set C and sub-set A, that is; set C is used to identify a joint j, and sub-set A is used to identify joints i that are connected to j.

This example can be inspected by means of sub-groups in the same way as Example 1. A summary only is given of the resulting expressions for this example.

Sub-groups 2a, 2b, and 2c of this example (see Fig. 18) are of the same kind as those of the preceding example, except that more terms now appear in the resulting expressions. For example, Eq. 73 for sub-group 2a now becomes

$$P_{msv} = K_{msv}^{(i)i} \rho_{i}^{(msv)} + K_{msv}^{(MSV)msv} \rho_{msv} \tag{124}$$

in which the summation over i is for values of

$$i = \{\ell sv,\ nsv,\ mrv,\ mtv,\ msu,\ msw\} \tag{125}$$

Also, Eq. 73 for sub-group 2b now becomes

$$P_{\alpha sv} = K_{\alpha sv}^{(\alpha i)\alpha i} \rho_{\alpha i}^{(\alpha sv)} + K_{\alpha sv}^{(\alpha sv)\alpha sv} \rho_{\alpha sv} \tag{126}$$

in which

$$s = \text{a constant};\ r = s - 1;\ t = s + 1 \tag{127}$$

$$v = \text{a constant};\ u = v - 1;\ w = v + 1 \tag{128}$$

$$i = \{rv,\ tv,\ su,\ sw\} \tag{129}$$

and

$$\alpha = \{1,\ 2,\ \cdot\ m,\ \cdot\ M\} \tag{130}$$

For sub-group 2c, we obtain

$$P_{\alpha\beta v} = K_{\alpha\beta v}^{(\alpha\beta i)\alpha\beta i} \rho_{\alpha\beta i}^{(\alpha\beta v)} + K_{\alpha\beta v}^{(\alpha\beta v)\alpha\beta v} \rho_{\alpha\beta v} \tag{131}$$

in which

$$v = \text{a constant};\ u = v - 1;\ w = v + 1 \tag{132}$$

$$\alpha = \left\{1,\ 2,\ \cdot\ m\ \cdot\ M\right\} \tag{133}$$

$$i = u,\ w \tag{134}$$

and

$$\beta = \left\{1,\ 2,\ \cdot\ s\ \cdot\ S\right\} \tag{135}$$

Finally, for the sub-group that consists of all joints of the structure (see Fig. 17), we obtain

$$P_{\alpha\beta\gamma} = K^{(\alpha\beta\gamma)\alpha\beta\gamma}_{\alpha\beta\gamma}\ \rho_{\alpha\beta\gamma} \tag{136}$$

in which

$$\alpha = \left\{1,\ 2,\ \cdot\ m,\ \cdot\ M\right\} \tag{137}$$

$$\beta = \left\{1,\ 2,\ \cdot\ s,\ \cdot\ S\right\} \tag{138}$$

and

$$\gamma = \left\{1,\ 2,\ \cdot\ v,\ \cdot\ V\right\} \tag{139}$$

By means of the range and summation conventions, the term $K^{(\alpha\beta\gamma)\alpha\beta\gamma}_{\alpha\beta\gamma}$ becomes

$$K^{(\alpha\beta\gamma)\alpha\beta\gamma}_{\alpha\beta\gamma} = \begin{bmatrix} K^{(\alpha\beta1)\alpha\beta1}_{\alpha\beta1} & K^{(\alpha\beta2)\alpha\beta2}_{\alpha\beta1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & \cdot & K^{(\alpha\beta u)\alpha\beta u}_{\alpha\beta v} & K^{(\alpha\beta v)\alpha\beta v}_{\alpha\beta v} & K^{(\alpha\beta w)\alpha\beta w}_{\alpha\beta v} & \cdot & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & K^{(\alpha\beta,V-1)\alpha\beta,V-1}_{\alpha\beta V} & K^{(\alpha\beta V)\alpha\beta V}_{\alpha\beta V} \end{bmatrix} \tag{140}$$

Again, the same concepts of set-points, mapping operations, and block operators can be employed. It merely is mentioned that for each sub-group 2a, 2b, and 2c the members and joints of the structure are mapped into a single set-member that lies between two set-points in an N-dimensional space. For the group that consists of all members and joints, the structure is mapped (or contracted) into a single set-point.

If desired, the form of Eq. 80 can be used to group the various relationships of the structure. In this case, each successive grouping of members and joints can be interpreted as a mapping operation in which the members and joints of the sub-group are mapped into two set-members that lie along a curve. The curve for each sub-group goes through three set-points.

The preceding examples are for particular configurations of structures. Other configurations, such as those that have more than four or six members meeting at a joint, can be considered.

Concluding Remarks

For practical purposes, the abstract concepts of set-points and geometrical transformations are useful in structural analysis.

It has been shown that the symbolic statements of

(a) statics; $$P_j - R_j^i \, p_i^{(j)} = 0 \qquad (34)$$

(b) continuity of geometry; $$\rho_j = \overline{R}_j^i \, \rho_i^{(j)} + \overline{R}_j^c \, d\rho_c \qquad (26)$$

and (c), Hooke's Law; $$P_j = K_j^{(i)j} \, \rho_j^{(i)} + K_j^{(i)i} \, \rho_i^{(j)} \qquad (73)$$

for a single member which lies between two points, are of the same forms, respectively, as those for (1) an entire set of members which are connected to N-number of joints in a 3-dimensional space, and (2) a single set-member which lies between two set-points in an N-dimensional space. Further, the statement of proportionality

$$d\rho_c = dk_c^{(c)c} \, p_c \qquad (45)$$

between forces and deformations of a differential element which lies at a single point in a 3-dimensional space is of the same form as

$$\rho_\eta = K_\eta^{(\eta)\eta} \quad P_\eta \qquad (75)$$

for an entire structure which lies at a single set-point in an N-dimensional space. Thus, it is clearer conceptually to consider the fundamental relationships

4. "Tensorial Analysis and Equivalent Circuits of Elastic Structures," by G. Kron, J. Franklin Inst., Vol. 238, No. 6, 1944. See also, "Tensors for Circuits," by G. Kron, Dover Publications, Inc., New York, 2nd edition, 1959.

5. "Structural Analysis of Sweptback Wings by Matrix Transformation" by B. Langefors, SAAB Airc. Co., Linkoping, Tech. Note 3, 1951.

6. "The Elements of Matrix Structural Analysis" by S. Shore, 2nd Conference on Electronic Computation, ASCE, Pittsburgh, 1960, p. 145.

7. "Introduction to Symbolic Logic and Its Applications," by R. Carnap, Dover Publications, Inc., New York, 1958.

8. "Tensor Calculus" by J. L. Synge and A. Schild, Univ. of Toronto Press, Toronto, 1949.

9. "Tensor Calculus" by B. Spain, Oliver and Boyd, Edinburgh, 1953.

10. "Determinants and Matrices" by A. C. Aitken, Oliver and Boyd, Edinburgh; 1958. See also, "Elementary Matrices" by R. A. Frazer, W. J. Duncan, and A. R. Collar, Cambridge, Univ. Press, Cambridge, 1957.

11. "Topology" by E. M. Patterson, Oliver and Boyd, Edinburgh, 1956.

12. "Elements of the Topology of Plane Sets of Points" by M. H. A. Newman, Cambridge Univ. Press, Cambridge, 1961.

13. "Linear Graphs and Electrical Networks," by S. Seshu and M. B. Reed, Addison-Wesley Pub. Co., Reading, Mass., 1961.

14. "Analysis of Structures by System Theory" by N. C. Lind, Journal of the Structural Div., ASCE., Vol. 88, No. ST2, April, 1962, p. 1.

15. "Theory of Sets" by E. Kamke, Dover Publications Inc., New York, 1950.

16. "Matrix Analysis of Structures Curved in Space" by Frank Baron, Journal of the Structural Division, ASCE, March, 1961, p. 17.

17. "On the Method of Complementary Energy" by H. M. Westergaard, Transactions, ASCE, Vol. 107, 1942, p. 765.

18. "Inversion of Band Matrices" by S. O. Asplund, 2nd Conference on Electronic Computation, ASCE, Pittsburgh, 1960, p. 513.

19. "A Computational Technique for Three-Dimensional Pin-Jointed Structures" by C. E. Pearson, 2nd Conference on Electronic Computation, ASCE, Pittsburgh, 1960, p. 327.

20. "Numerical Methods of Analysis in Engineering" by L. E. Grinter, Ed., Macmillan Co., 1949.

21. "Relaxation Methods in Engineering Science" by R. V. Southwell, Oxford University Press, Oxford, 1940.

22. "Relaxation Methods" by F. S. Shaw, Dover Publications, Inc., 1953.

23. "Solving Highly Complex Elastic Structures in Easy Stages" by G. Kron, Journal of Applied Mechanics, ASME, Vol. 22, No. 2, June, 1955, pp. 235-244.

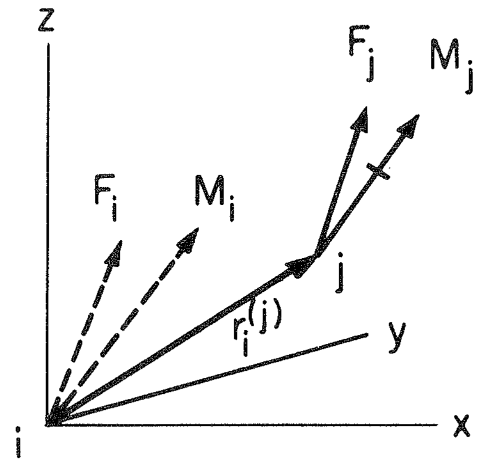

FIG. 1

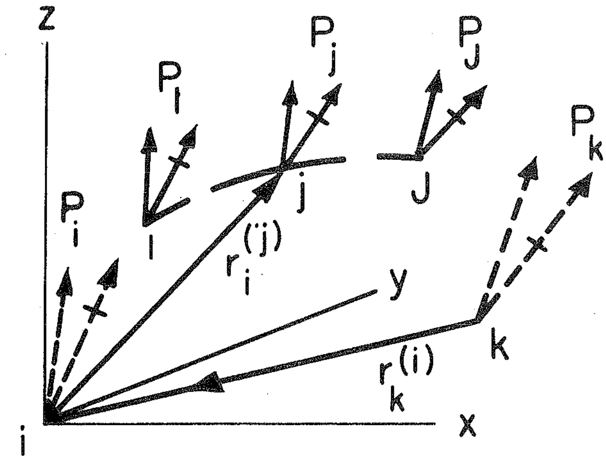

FIG. 2

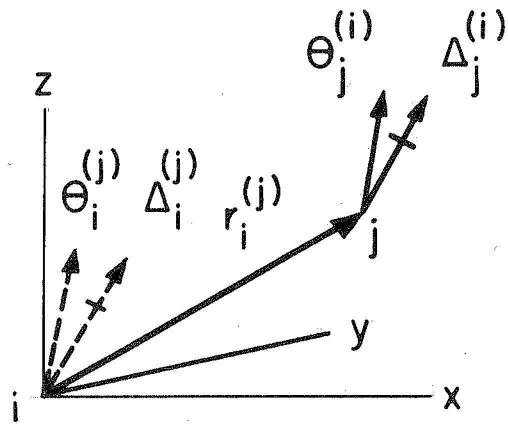

FIG. 3

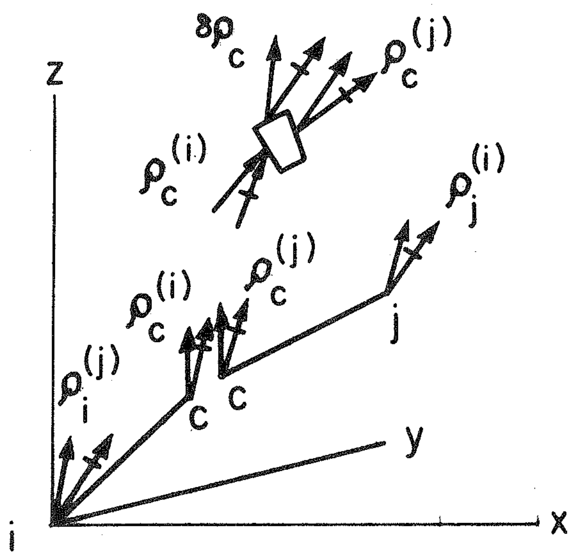

FIG. 4

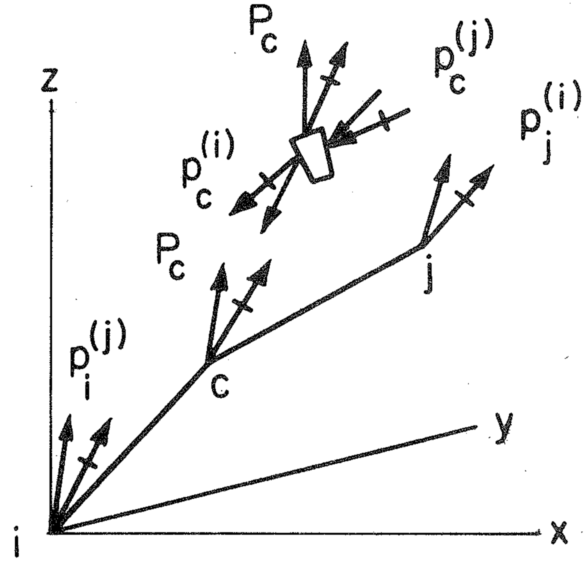

FIG. 5

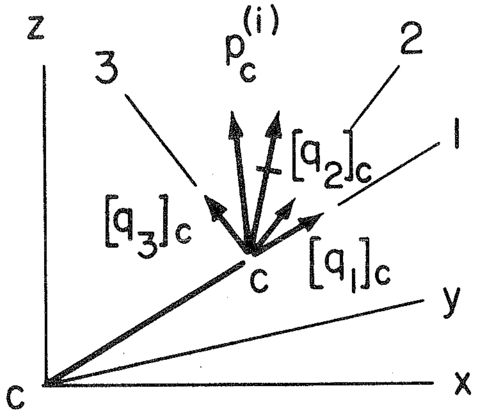

FIG. 6

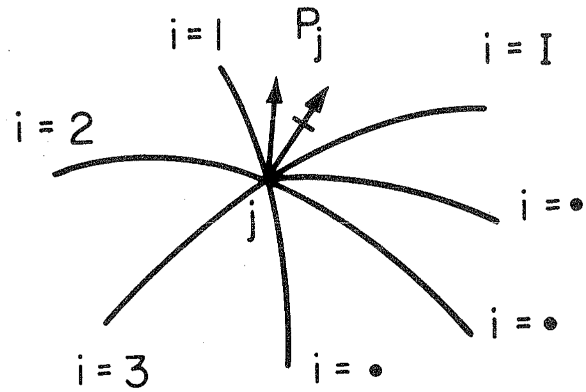

FIG. 9

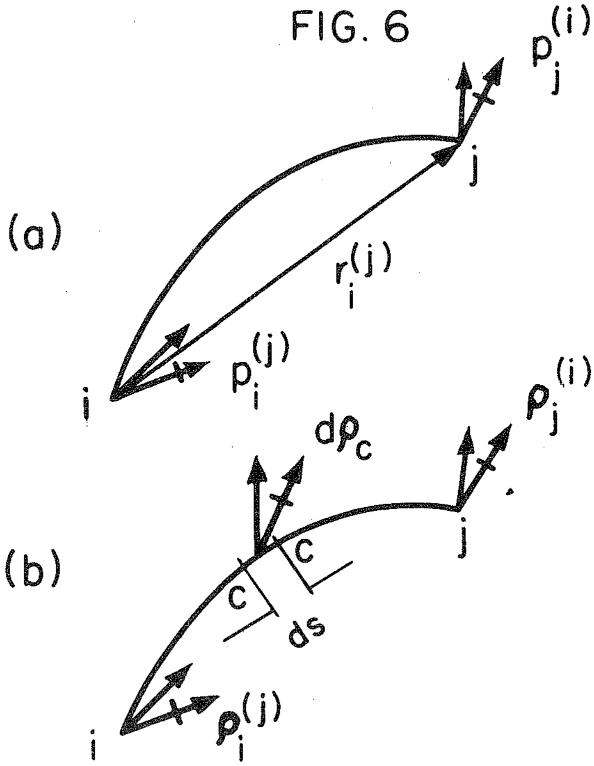

FIG. 7

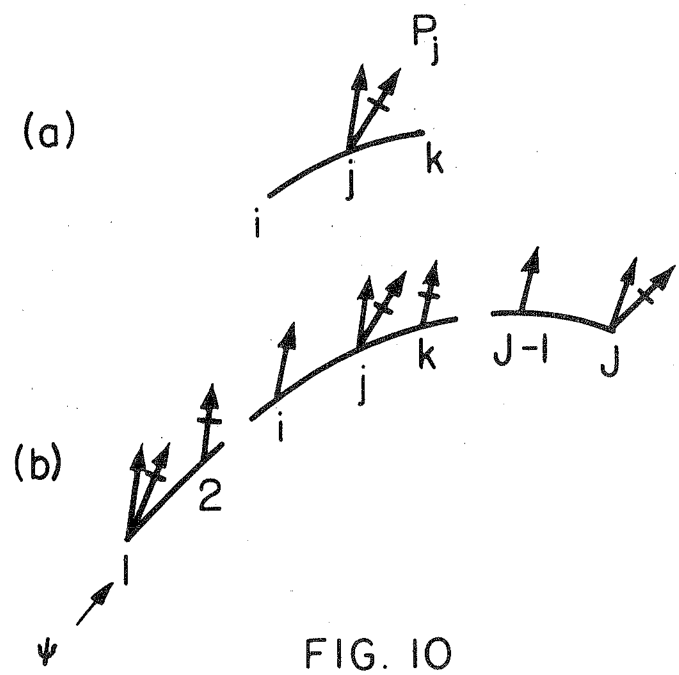

FIG. 10

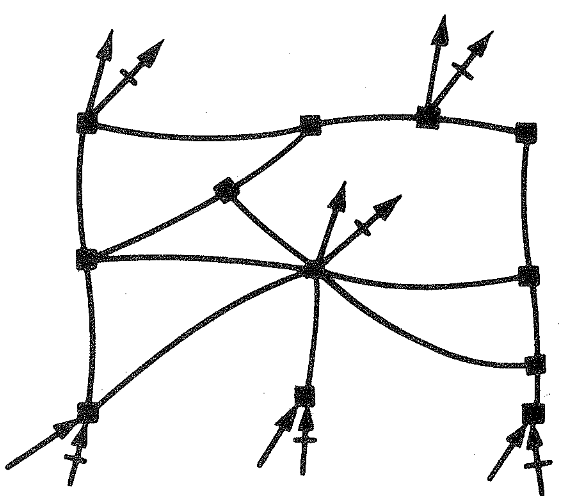

FIG. 8

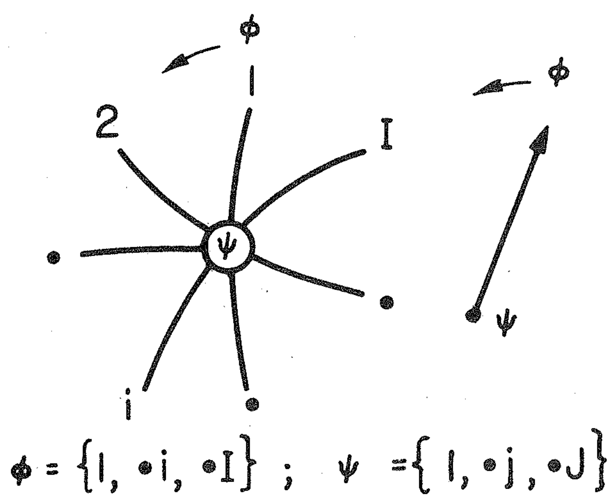

FIG. 11

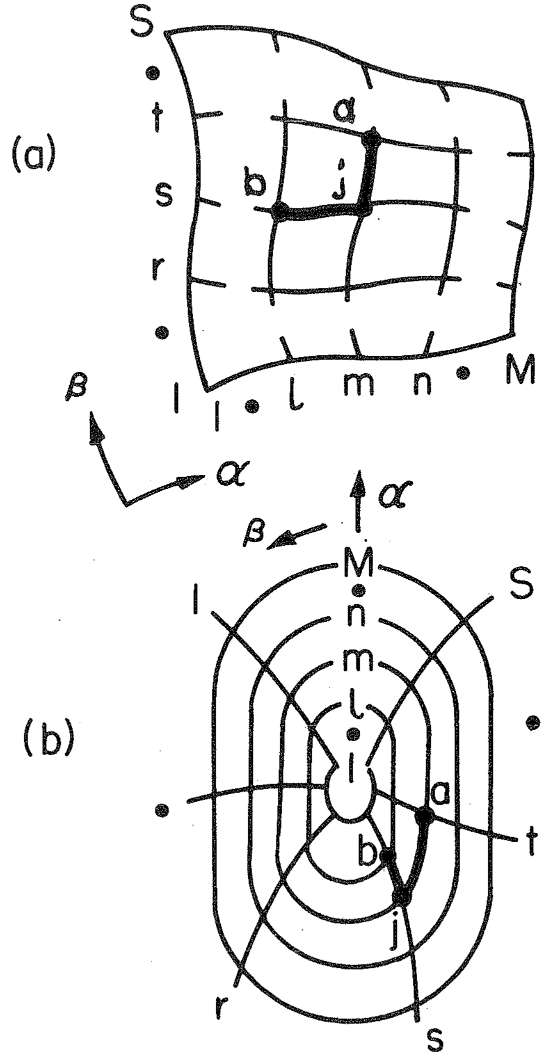

FIG. 12

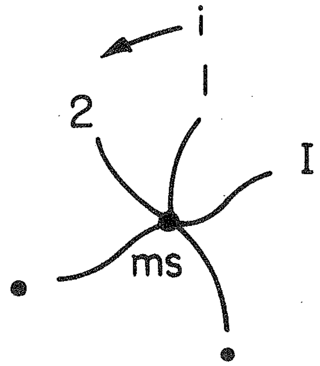

FIG. 13

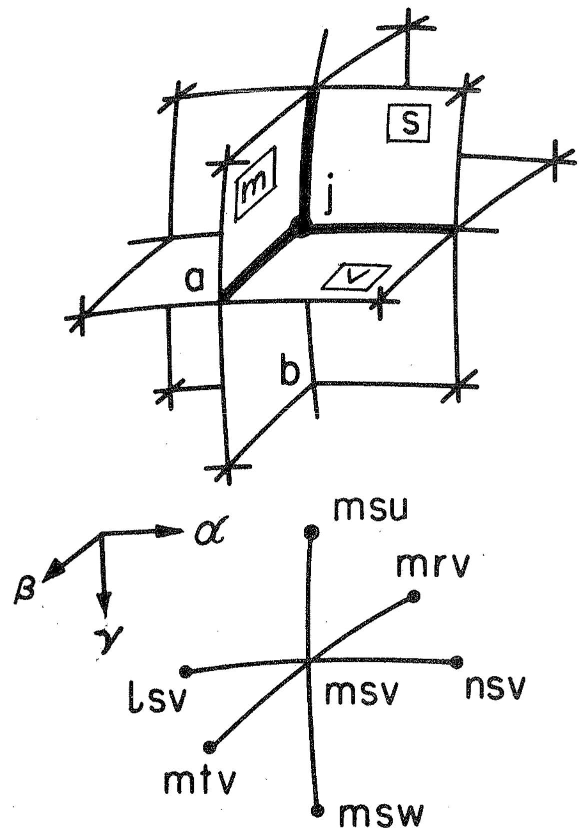

FIG. 14

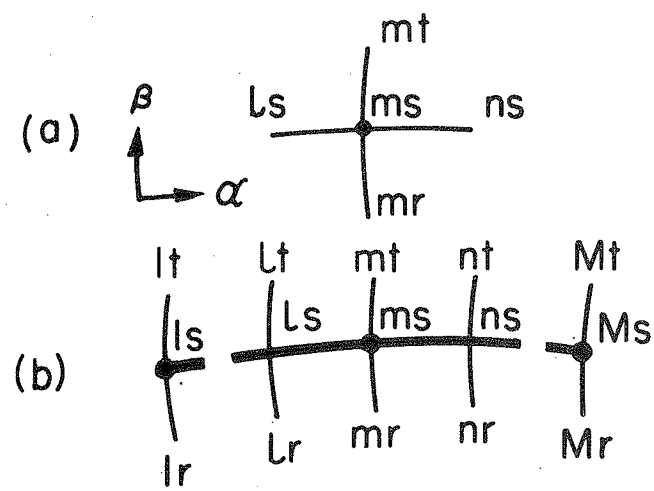

FIG. 15

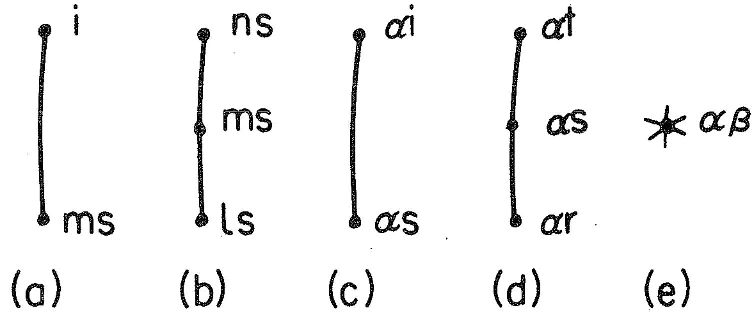

FIG. 16

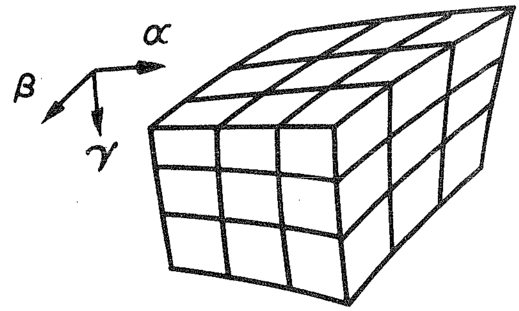

FIG. 17

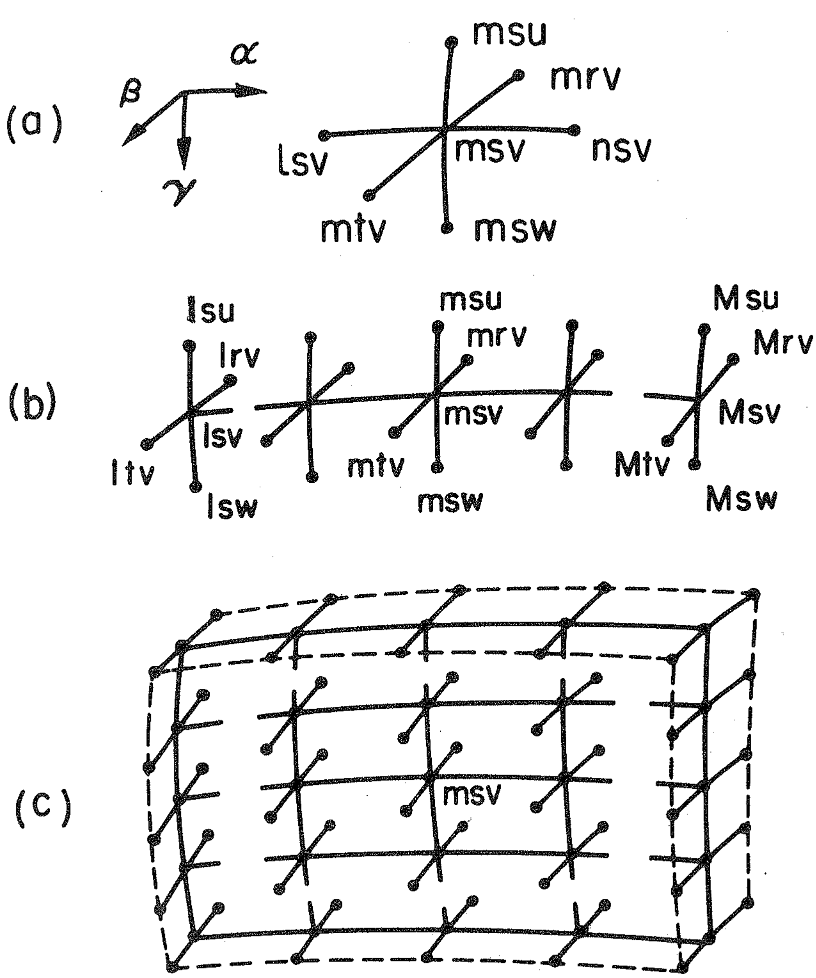

FIG. 18