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VALUED DIFFERENCE FIELDS AND NTP$_2$

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ABSTRACT. We show that the theory of the non-standard Frobenius automorphism, acting on an algebraically closed valued field of equal characteristic 0, is NTP$_2$. More generally, in the contractive as well as in the isometric case, we prove that a σ-henselian valued difference field of equicharacteristic 0 is NTP$_2$, provided both the residue difference field and the value group (as an ordered difference group) are NTP$_2$.

1. INTRODUCTION

Model theory has proven to be a fruitful framework to study fields with extra structure. Central examples include valued fields (e.g. the work of Haskell, Hrushovski and Macpherson on the theory of algebraically closed valued fields, ACVF, see [HHM08]) and difference fields (starting with the work of Chatzidakis and Hrushovski on the theory of algebraically closed fields with a *generic* automorphism, ACFA, see [CH99]). In this paper, we are considering a combination of these, namely valued difference fields, i.e. valued fields with a distinguished automorphism (preserving the valuation ring).

Every non-principal ultraproduct of structures of the form $(\mathbb{F}_p^a, Frob_p)$ is a model of ACFA$_0$, i.e. the non-standard Frobenius is a generic automorphism. This is a deep result of Hrushovski [Hru04] which required a twisted version of the Lang-Weil estimates.

One may consider the non-standard Frobenius acting on an algebraically closed *valued* field, i.e. the limit theory of the Frobenius automorphism acting on an algebraically closed valued field of characteristic p (where p tends to infinity). Hrushovski [Hru02] gives a natural axiomatisation of this limit theory in the language of valued difference fields (denoted by VFA$_0$ in the sequel). Durhan (formerly Azgın) [Azg10] obtains an alternative axiomatisation, as well as an Ax-Kochen-Ershov principle for a certain class of valued difference fields.

The theory VFA$_0$ is interesting from an algebraic point of view. The residue field together with the induced automorphism $\overline{\sigma}$ is a model of ACFA$_0$, by the aforementioned result of Hrushovski. The induced automorphism σ_Γ on the value group Γ is ω-*increasing* (i.e. $\sigma_\Gamma(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \geq 1$; valued difference

fields satisfying this property will be called *contractive*). Thus, Γ gets the structure of a divisible torsion free ordered $\mathbb{Z}[\sigma]$-module (i.e. an ordered vector space over $\mathbb{Q}(\sigma)$, where $\sigma \gg 1$ is an indeterminate). It is sufficient to add a σ-*Hensel property* (see Definition 2.6) to obtain an axiomatisation of VFA_0.

Moreover, valued difference fields enter the study of (non-valued) difference fields by way of *transformal specialisations* (see [Hru04]). A better understanding of valued difference fields will most probably shed new light on Hrushovski's proof of the non-standard Frobenius result.

Work of Shelah on the model theoretic classification program [She90] had demonstrated the importance of understanding which combinatorial configurations a theory can encode. In the case of stable theories (i.e. theories that cannot encode linear order) he had developed a beautiful and fruitful theory of analysing types and models. Further work of Poizat, Hrushovski and other researchers, generalising the ideas of Zilber in the finite rank case, culminated in the creation of *geometric stability theory* establishing deep connections between the geometry of forking independence and properties of algebraic structures (groups and fields) definable in the theory. Later on it became clear that stability-theoretic methods can be generalised to larger contexts, and in the last twenty years there had been two main directions: simple theories [Wag00] and, more recently, NIP theories [Adl08, Sim12]. A characteristic property of these developments is that the motivation is coming both from purely model theoretic considerations and from the study of particular important algebraic structures: ACFA as a prototypical example of a simple unstable theory, and ACVF as a typical example of an (unstable) NIP theory. These lines of research had found numerous applications [Hru01, HK06, HL11].

Observe that the theory VFA_0 is neither *simple* (due to the total order in the value group) nor NIP (due to the independence property which holds in the residue field).

It turns out that in the 80's Shelah had defined another class — NTP_2 theories, or theories without the tree property of the second kind [She80]. This class generalises both simple and NIP theories, and contains new examples (e.g. any ultraproduct of p-adics is NTP_2 [Che]). Recently it had attracted attention, largely motivated by Pillay's question on equality of forking and dividing over models in NIP, and a theory of forking for NTP_2 theories had been developed [CK12, Che, BYC12].

In this paper we show the following general theorem.

Theorem (4.1). *Let $\mathcal{K} = (K, k, \Gamma, v, \mathrm{ac})$ be a valued difference fields of residue characteristic 0 (where ac is an angular component map commuting with σ). Assume that $T = \mathrm{Th}(\mathcal{K})$ eliminates $\mathbf{K}$-quantifiers, and that both the residue field k (as a difference field) and the value group Γ (as an ordered difference group) are NTP_2. Then, $\mathcal{K}$ is NTP_2.*

Our method of proof combines a new result on extending indiscernible arrays by parameters coming from NTP_2 sorts with a back-and-forth system coming from the elimination of field quantifiers. In this way, we reduce the statement to a situation where one deals with immediate extensions, and these extensions are controlled by NIP formulas.

Applying the theorem we obtain new and interesting algebraic examples of NTP_2 theories:

- VFA$_0$ is NTP$_2$. More generally, a contractive σ-henselian valued difference field of equicharacteristic 0 (where an Ax-Kochen-Ershov principle holds by [Azg10]) is NTP$_2$, provided both the theory of the value group (with the induced automorphism) and the theory of the residue field (with the induced automorphism) are NTP$_2$.
- We prove a similar result in the isometric case, where an Ax-Kochen-Ershov principle holds as well [Sca03, BMS07, AvdD11].

Similar transfer results hold in the context of valued fields (i.e. when σ is the identity), where they are derived from the usual Ax-Kochen-Ershov principle: Delon [Del81] showed this for NIP, Shelah [She09] for strongly dependent (a strengthening of NIP) and the first author [Che] for NTP$_2$ (and finiteness of burden).

A quick overview of the paper. In Section 2, we recall the basic model-theoretic results on valued difference fields and elimination of field quantifiers. Section 3 contains general results on manipulating indiscernible arrays in NTP$_2$ theories. The main theorem of the paper is then proved in Section 4.1, and applications are given in Section 4.2. We end with a list of open problems and some related observations, in particular discussing when ordered modules are NIP.

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2. Preliminaries on valued difference fields

In this section, we present the necessary material on valued difference fields which we will need for our purposes.

Moreover, we will give a survey of the results in the contractive case, in particular with respect to VFA$_0$, the theory which motivated our study. Strictly speaking, these results are not needed in the main theorem. Nevertheless, we believe this is useful for the reader, as the results in question are not so widely known and easily accessible. Most of the material in the contractive case may be found in [Azg10], but we will also need the context from [Azg07], where an angular component is used instead of a cross-section. See also [Pal12, Gia11], and the unpublished notes [Hru02] of Hrushovski, where most of the ideas in the contractive case already appear.

At the end of the section, we will briefly mention the isometric case which historically preceded the contractive case [Sca00, Sca03, BMS07, AvdD11].

2.1. **Ordered difference groups.** An *ordered difference group* is a structure of the form $\langle \Gamma, 0, +, -, <, \sigma \rangle$, where $\langle \Gamma, 0, +, -, < \rangle$ is an ordered abelian group and σ is an automorphism of $\langle \Gamma, 0, +, -, < \rangle$. The automorphism σ is called ω-*increasing* if $\sigma(\gamma) > n\gamma$ for all $\gamma \in \Gamma_{>0}$ and all natural numbers n; the corresponding difference group will also be called ω-increasing. We treat ordered difference groups as first order structures in the language $\mathcal{L}_{ODG} = \{0, +, -, <, \sigma\}$; the class of ω-increasing ordered difference groups may be axiomatised, and we denote it by IncODG.

Any $\langle \Gamma, +, <, \sigma \rangle \models$ IncODG is an ordered (and so in particular torsion free) $\mathbb{Z}[\sigma, \sigma^{-1}]$-module, where $\mathbb{Z}[\sigma]$ is the ordered ring of polynomials in the indeterminate σ with $\sigma \gg 1$. For $p = p(\sigma) = \sum z_i \sigma^i \in \mathbb{Z}[\sigma, \sigma^{-1}]$ and $\gamma \in \Gamma$, one puts $p \cdot \gamma := \sum z_i \sigma^i(\gamma)$. Conversely, any ordered $\mathbb{Z}[\sigma, \sigma^{-1}]$-module gives rise to a model of IncODG. When divisible, such modules correspond to ordered vector spaces over

the (ordered) fraction field $\mathbb{Q}(\sigma)$ of $\mathbb{Z}[\sigma]$. The theory of non-trivial divisible ordered $\mathbb{Z}[\sigma, \sigma^{-1}]$-modules will be denoted by IncDODG.

The following fact is easy (see e.g. [Pal12]).

Fact 2.1. *The theory* IncDODG *is the model-completion of* IncODG. *In particular,* IncDODG *eliminates quantifiers and is o-minimal.*

For $\gamma \in \Gamma \models$ IncODG and $\zeta = (z_0, \ldots, z_n) \in \mathbb{Z}^{n+1}$, we will sometimes denote $\sum_{i=0}^{n} z_i \sigma^i(\gamma)$ by $\sigma^\zeta(\gamma)$.

2.2. Valued difference fields.

Notation and conventions.

By a *difference field* we will always mean a field K together with a distinguished automorphism σ, i.e. what is sometimes called an *inversive* difference field.

If K is a difference field, one may form the ring of difference polynomials $K[X]_\sigma := K[X, \sigma(X), \sigma^2(X), \ldots]$. Then σ extends naturally to an endomorphism of $K[X]_\sigma$, and in this way $K[X]_\sigma$ is a difference ring extension of K.

If $K \subseteq L$ is an extension of difference fields and a is a tuple from L, then $K\langle a \rangle$ denotes the difference field generated by a over K; as a field, it is given by $K(\sigma^z(a), z \in \mathbb{Z})$. An element $a \in L$ is called σ-*algebraic* over K if $g(a) = 0$ for some non-constant $g(X) \in K[X]_\sigma$; else, it is called σ-*transcendent* over K.

Recall that a *valued field* is given by a surjective map $\mathrm{val} : K \to \Gamma_\infty$, where K is a field and $\Gamma_\infty = \Gamma \cup \{\infty\}$, with Γ an ordered abelian group and ∞ a distinct element satisfying

- $\mathrm{val}(x) = \infty \iff x = 0$;
- $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$ for all $x, y \in K$;
- $\mathrm{val}(x + y) \geq \min\{\mathrm{val}(x), \mathrm{val}(y)\}$ for all $x, y \in K$.

Here, the order is extended to a total order on Γ_∞ making ∞ the maximal element, and the addition is extended so that ∞ becomes an absorbing element.

We will usually not distinguish between Γ and Γ_∞ and suppress ∞ in our paper.

The *valuation ring* is given by $\mathcal{O} = \{x \in K \mid \mathrm{val}(x) \geq 0\}$. It is a local ring, with maximal ideal $\mathfrak{m} = \{x \in K \mid \mathrm{val}(x) > 0\}$. The *residue map* is given by $\mathrm{res} : \mathcal{O} \to \mathcal{O}/\mathfrak{m} =: k$, and k is called the *residue field* of K. Sometimes, we will use $\overline{a}$ instead of $\mathrm{res}(a)$. We often write Γ_K or k_K to stress that we deal with the value group or residue field of the valued field K. An extension $K \subseteq L$ gives rise to extensions $k_K \subseteq k_L$ and $\Gamma_K \subseteq \Gamma_L$.

A *valued difference field* is a valued field K together with a distinguished automorphism σ satisfying $\sigma(\mathcal{O}) = \mathcal{O}$. Note that σ induces an automorphism $\overline{\sigma}$ of the residue field, making it a difference field. Similarly, σ induces an automorphism σ_Γ of the value group, making it an ordered difference group. Most of the time, we will drop the subscript and use σ for the automorphism on the value groups as well.

We treat valued difference fields in the three-sorted language $\mathcal{L}_{\mathbf{k}, \mathbf{\Gamma}, \sigma}$, consisting of

- the language of difference rings $\mathcal{L}_{\mathbf{K}} = \{0, 1, +, -, \times, \sigma\}$ on the valued field sort denoted by $\mathbf{K}$;
- (a copy of) the language of difference rings $\mathcal{L}_{\mathbf{k}} = \{0, 1, +, -, \times, \overline{\sigma}\}$ on the residue field sort denoted by $\mathbf{k}$;
- the language of ordered difference groups (with an additional infinite element) $\{0, +, -, <, \infty, \sigma_\Gamma\}$ on the value group sort denoted by $\mathbf{\Gamma}$, and

- the functions val : $\mathbf{K} \to \boldsymbol{\Gamma}$ and res : $\mathbf{K} \to \mathbf{k}$ between the sorts. (When considering a valued field as an $\mathcal{L}_{\mathbf{k},\boldsymbol{\Gamma},\sigma}$-structure, we may make the function res total by sending elements of negative valuation to $0 \in \mathbf{k}$.)

An ac-*valued difference field* is a valued difference field $\mathcal{K} = (K, \Gamma, k, \sigma)$ together with an *angular component map* ac : $K \to k$ satisfying the following three properties:

- $\mathrm{ac}(x) = 0$ iff $x = 0$;
- $\mathrm{ac}\restriction_{K^\times}: K^\times \to k^\times$ is a group homomorphism commuting with σ;
- for all $x \in K$ with $\mathrm{val}(x) = 0$, one has $\mathrm{ac}(x) = \mathrm{res}(x)$.

We treat ac-valued difference fields in the three-sorted language $\mathcal{L}_{\mathbf{k},\boldsymbol{\Gamma},\sigma} \cup \{\mathrm{ac}\}$. Note that the corresponding language without σ, $\overline{\sigma}$ and σ_Γ, denoted by $\mathcal{L}_{\mathbf{k},\boldsymbol{\Gamma}} \cup \{\mathrm{ac}\}$, is precisely the *language of Pas*.

If A is a substructure of $\mathcal{K} = (K, \Gamma_K, k_K)$, we write $\mathbf{K}(A)$ for the elements of A which are in sort $\mathbf{K}$. Similarly, we have $\boldsymbol{\Gamma}(A) \subseteq \Gamma_K$ and $\mathbf{k}(A) \subseteq k_K$. Note that in general $\mathrm{val}(\mathbf{K}(A))$ is a proper subset of $\boldsymbol{\Gamma}(A)$, and similarly for res and ac.

2.3. **Elimination of field quantifiers in** ac-**valued difference fields.** We gather here some useful consequences of the elimination of field quantifiers in ac-valued difference fields. We may thus treat various cases in one common framework, namely σ-henselian valued difference fields, both in the contractive and in the isometric case, and henselian valued fields (without distinguished isomorphism).

The following lemma is a consequence of compactness, taking into account that there are no function symbols in our language with arguments in $\boldsymbol{\Gamma}$ or $\mathbf{k}$ and target sort $\mathbf{K}$.

Lemma 2.2. *Let T be an $\mathcal{L}_{\mathbf{k},\boldsymbol{\Gamma},\sigma} \cup \{\mathrm{ac}\}$-theory. The following are equivalent:*

(1) *T eliminates $\mathbf{K}$-quantifiers.*
(2) *Let M and M' be models of T, with substructures $A = (\mathbf{K}(A), \boldsymbol{\Gamma}(A), \mathbf{k}(A)) \subseteq M$ and $A' = (\mathbf{K}(A'), \boldsymbol{\Gamma}(A'), \mathbf{k}(A')) \subseteq M'$. Let $f = (f_{\mathbf{K}}, f_{\boldsymbol{\Gamma}}, f_{\mathbf{k}}) : A \cong A'$ be an isomorphism such that*
  - *$f_{\boldsymbol{\Gamma}} : \boldsymbol{\Gamma}(A) \to \boldsymbol{\Gamma}(A')$ is an $\{0, +, -, <, \infty, \sigma_\Gamma\}$-elementary map, and*
  - *$f_{\mathbf{k}} : \mathbf{k}(A) \to \mathbf{k}(A')$ is an $\mathcal{L}_{\mathbf{k}}$-elementary map.*

  Then f is an elementary map.

Lemma 2.3. *Let T be an $\mathcal{L}_{\mathbf{k},\boldsymbol{\Gamma},\sigma} \cup \{\mathrm{ac}\}$-theory of ac-valued difference fields. Assume that T eliminates $\mathbf{K}$-quantifiers. Then the following holds:*

(1) *In any model $\mathcal{K} = (K, \Gamma_K, k_K) \models T$, $k_K = \mathbf{k}(\mathcal{K})$ is stably embedded, and the induced structure is that of a difference field. Similarly, $\Gamma_K = \boldsymbol{\Gamma}(\mathcal{K})$ is stably embedded and is a pure ordered difference group. Moreover, $\mathbf{k}$ and $\boldsymbol{\Gamma}$ are orthogonal, i.e. every definable subset of $\mathbf{k}^m \times \boldsymbol{\Gamma}^n$ is a finite union of rectangles.*
(2) *Let $\mathcal{K} = (K, \Gamma_K, k_K)$ and $\mathcal{L} = (L, \Gamma_L, k_L)$ be models of T.*
  (a) *One has $\mathcal{K} \equiv \mathcal{L}$ iff $k_K \equiv k_L$ (as difference fields) and $\Gamma_K \equiv \Gamma_L$ (as ordered difference groups).*
  (b) *Assume $\mathcal{K} \subseteq \mathcal{L}$. Then $\mathcal{K} \preccurlyeq \mathcal{L}$ iff $k_K \preccurlyeq k_L$ and $\Gamma_K \preccurlyeq \Gamma_L$.*
(3) *Let L/K be an immediate extension of valued difference fields, living in a model of T. Assume that $\mathrm{ac}(K) \subseteq k_K$, and let a be a tuple from L. Then $\mathrm{qftp}(a/K) \vdash \mathrm{tp}(a/K)$.*

Proof. (1) is clear by inspection of the language.

To prove (2), note that for every $b \in L$ there is $c \in K$ and $b' \in \mathcal{O}_L^\times$ such that $b = cb'$. But then $\mathrm{ac}(b) = \mathrm{ac}(c)\mathrm{res}(b')$, showing that $\mathrm{ac}(b) \in k_L = k_K$. The result follows. $\square$

Before we treat various cases of σ-henselian valued difference fields, let us mention a classical result of Pas.

Fact 2.4 ([Pas89])**.** *Let T be the theory of henselian* ac*-valued fields of residue characteristic 0, in the language of Pas $\mathcal{L}_{\mathbf{k},\mathbf{\Gamma}} \cup \{\mathrm{ac}\}$. Then T eliminates $\mathbf{K}$-quantifiers.*

2.4. σ-henselianity in contractive valued difference fields.

Definition 2.5. A valued difference field $\mathcal{K} = (K, \Gamma_K, k_K, \sigma)$ is called *contractive* if its value group Γ_K is an ω-increasing ordered difference group.

Let $\mathcal{K} = (K, \Gamma_K, k_K, \sigma)$ be a contractive valued difference field, with *fixed field* $F := \mathrm{Fix}(\sigma) := \{a \in K \mid \sigma(a) = a\}$. Then $\mathrm{res} \upharpoonright_F$ is injective. In particular, $\mathrm{char}(K) = \mathrm{char}(k_K)$. (In case $\mathcal{K}$ is σ-henselian in the sense of the definition below, res induces an isomorphism between F and $\mathrm{Fix}(\overline{\sigma})$.)

For $g(X) \in K[X]_\sigma$ non-constant, define $\mathrm{order}(g)$ to be the minimal n such that g may be written as $g(X) = G(X, \sigma(X), \sigma^2(X), \ldots, \sigma^n(X))$, for some polynomial $G \in K[X_0, \ldots, X_n]$. If $\mathrm{order}(g) = n$, we put

$$\mathrm{complexity}(g) := (n, \deg_{X_n}(G), \deg(G)) \in \mathbb{N}^3,$$

where $\deg(G)$ is the total degree of G. We say that g has *smaller complexity* than h if $\mathrm{complexity}(g) <_{lex} \mathrm{complexity}(h)$.

Recall that for any $G \in K[\overline{X}]$ there are (unique) polynomials $G_\mu \in K[\overline{X}]$ such that $G(\overline{Y} + \overline{X}) = \sum_\mu G_\mu(\overline{Y})\overline{X}^\mu$. Here, $\overline{X} = (X_0, \ldots, X_n)$, $\mu = (\mu_0, \ldots, \mu_n) \in \mathbb{N}^{n+1}$ is a multi-index, and $\overline{X}^\mu := \prod_{i=0}^n X_i^{\mu_i}$.

From this, for $g(X) = G(X, \sigma(X), \sigma^2(X), \ldots, \sigma^n(X)) \in K[X]_\sigma$ as above, we get the following Taylor expansion of difference polynomials (in one variable)

$$g(a + X) = \sum_\mu g_\mu(a)X^\mu,$$

where $g_\mu(X) = G_\mu(X, \sigma(X), \ldots, \sigma^n(X))$ and $X^\mu := \prod_{i=0}^n (\sigma^i(X))^{\mu_i}$ for every multi-index $\mu = (\mu_0, \ldots, \mu_n)$.

Let us introduce some notation which will be used in the following definition. For $\mu \in \mathbb{N}^{n+1}$ and $\gamma \in \Gamma$, we let $|\mu| := \sum \mu_i = 1$, and $\sigma^\mu(\gamma) := \sum_{i=0}^n \mu_i \sigma^i(\gamma)$.

Definition 2.6 ([Azg10, Definition 4.5])**.**

- Let K be contractive, $g(X) \in K[X]_\sigma$, $\mathrm{order}(g) \leq n$, let $a \in K$.

  We say that (g, a) is in *σ-Hensel configuration* if $g \notin K$ and if there exists $\gamma \in \Gamma_K$ and a multi-index $\mu \in \mathbb{N}^{n+1}$ with $|\mu| = 1$ such that the following holds:

  (i) $\mathrm{val}(g(a)) = \mathrm{val}(g_\mu(a)) + \sigma^\mu(\gamma) \leq \mathrm{val}(g_\nu(a)) + \sigma^\nu(\gamma)$ for all $\nu \in \mathbb{N}^{n+1}$ with $|\nu| = 1$;

  (ii) $\mathrm{val}(g_\nu(a)) + \sigma^\nu(\gamma) < \mathrm{val}(g_{\nu+\rho}(a)) + \sigma^{\nu+\rho}(\gamma)$ for all non-zero $\nu, \rho \in \mathbb{N}^{n+1}$ with $g_\nu \neq 0$.

  Put $\gamma(g, a) :=$ the γ from above (this is uniquely determined [Azg10]).

- A contractive valued difference field K is called *σ-henselian* if for every (g, a) in σ-Hensel configuration there exists $b \in K$ with $\mathrm{val}(b - a) = \gamma(g, a)$ and $g(b) = 0$.

Remark 2.7. (1) Let (K, k, Γ, σ) be contractive and σ-henselian. Then $(k, \bar{\sigma})$ is *linearly difference closed*, i.e. for all $\alpha_0, \ldots, \alpha_n \in k$ not all 0, the equation $1 + \alpha_0 X + \alpha_1 \sigma(X) + \cdots + \alpha_n \sigma^n(X)$ has a solution in $(k, \bar{\sigma})$ [Azg10, Lemma 4.6].

(2) Conversely, let (K, k, Γ, σ) be contractive with $(k, \bar{\sigma})$ a linearly closed difference field of characteristic 0. Assume that K is a maximally complete valued field. Then (K, k, Γ, σ) is σ-henselian by [Azg10, Corollary 5.6].

(3) We now isolate a special case of (2). Let (Γ, σ) be an ω-increasing ordered difference group and $(k, \bar{\sigma})$ a linearly closed difference field of characteristic 0. Then, the Hahn field $K := k((\Gamma))$, a valued field which is naturally equipped with an automorphism σ, namely $\sigma(\sum_\gamma a_\gamma t^\gamma) := \sum_\gamma \bar{\sigma}(a_\gamma) t^{\sigma(\gamma)}$, is a contractive σ-henselian valued difference field.

For completeness, let us mention another consequence of the results in [Azg10]. (We will make no use of it in our paper.)

Remark 2.8. A σ-henselian contractive valued difference field of characteristic 0 is henselian (as a valued field).

Indeed, combining Remark 2.7(1&3) with Fact 2.10(4) below, one sees that every σ-henselian contractive valued difference field (of characteristic 0) is elementarily equivalent to a valued difference field with underlying valued field a Hahn field (which is henselian).

For $q = p^n$ a prime power, consider $\mathcal{K}_q = (K_q, \Gamma, k, \varphi_q)$, where $(K_q, \Gamma, k) \models \mathrm{ACVF}_{p,p}$ and φ_q is the Frobenius automorphism $x \mapsto x^q$.

The following follows in a straight forward way from [Azg10, Section 8] (see [Gia11, Proof of Thm 4.3.24] for a proof).

Fact 2.9. *For every non-principal ultrafilter $\mathcal{U}$ on the set Q of prime powers, $\prod_\mathcal{U} \mathcal{K}_q$ is σ-henselian, i.e. the non-standard Frobenius automorphism is σ-henselian.*

2.5. **AKE principle in the contractive case and** VFA$_0$. Denote by T_0 the theory of σ-henselian contractive valued difference fields of equal characteristic 0 (in the language $\mathcal{L}_{\mathbf{k}, \mathbf{\Gamma}, \sigma}$), and by T_0^{ac} that of σ-henselian contractive ac-valued difference fields (in the language $\mathcal{L}_{\mathbf{k}, \mathbf{\Gamma}, \sigma} \cup \{\mathrm{ac}\}$).

Fact 2.10 (Durhan).
(1) *The theory T_0^{ac} eliminates quantifiers from the field sort $\mathbf{K}$.*
(2) *In any model $\mathcal{K} = (K, \Gamma_K, k_K) \models T_0^{\mathrm{ac}}$, $k_K = \mathbf{k}(\mathcal{K})$ is stably embedded, and the induced structure is that of a difference field. Similarly, $\Gamma_K = \mathbf{\Gamma}(\mathcal{K})$ is stably embedded and is a pure ordered $\mathbb{Z}[\sigma]$-module. Moreover, $\mathbf{k}$ and $\mathbf{\Gamma}$ are orthogonal.*
(3) *Let $\mathcal{K} = (K, \Gamma_K, k_K)$ and $\mathcal{L} = (L, \Gamma_L, k_L)$ be models of T_0^{ac}.*
   (a) *$\mathcal{K} \equiv \mathcal{L}$ iff $k_K \equiv k_L$ (as difference fields) and $\Gamma_K \equiv \Gamma_L$ (as ordered $\mathbb{Z}[\sigma]$-modules).*
   (b) *Suppose $\mathcal{K} \subseteq \mathcal{L}$. Then $\mathcal{K} \preccurlyeq \mathcal{L}$ iff $k_K \preccurlyeq k_L$ and $\Gamma_K \preccurlyeq \Gamma_L$.*
(4) *Let $\mathcal{K} = (K, \Gamma_K, k_K)$ and $\mathcal{L} = (L, \Gamma_L, k_L)$ be models of T_0.*
   (a) *$\mathcal{K} \equiv \mathcal{L}$ if and only if $k_K \equiv k_L$ and $\Gamma_K \equiv \Gamma_L$.*

(b) Suppose $\mathcal{K} \subseteq \mathcal{L}$. Then $\mathcal{K} \preccurlyeq \mathcal{L}$ if and only if $k_K \preccurlyeq k_L$ and $\Gamma_K \preccurlyeq \Gamma_L$.

Proof. (1) is [Azg07, Thm 4.5.2], and (2) and (3) follow from (1) (by Lemma 2.3). (4a) is [Azg07, Thm 4.5.1]. Finally, to prove the non-trivial implication in (4b), taking an elementary extension of the pair (K, L), we may assume that both K and L are $\aleph_1$-saturated. In [Pal12, Proof of Thm 11.6], it is shown that $\mathcal{O}_M^\times$ is a pure $\mathbb{Z}[\sigma]$-submodule of $M^\times$ for every contractive valued difference field M. The saturation assumption implies that $K^\times$, $L^\times$ and Γ_K are all pure-injective $\mathbb{Z}[\sigma]$-modules (see [Hod93, Section 10.7] for facts about pure-injective modules). It follows that we get splittings for the exact sequence (of $\mathbb{Z}[\sigma]$-modules) $1 \to \mathcal{O}_K^\times \to K^\times \to \Gamma_K \to 0$ and similarly for $1 \to \mathcal{O}_L^\times \to L^\times \to \Gamma_L \to 0$, i.e. cross-sections $s_K : \Gamma_K \to K^\times$ and $s_L : \Gamma_L \to L^\times$ commuting with σ.

Moreover, as $\Gamma_K \preccurlyeq \Gamma_L$, in particular Γ_K is a pure submodule of Γ_L, and so there is a $\mathbb{Z}[\sigma]$-submodule $\Delta \leq \Gamma_L$ such that $\Gamma_L = \Gamma_K \oplus \Delta$. Now let $s_L' = s_K \oplus s_L \restriction_\Delta$: $\Gamma_L = \Gamma_K \oplus \Delta \to L^\times$. Then s_L' is a cross-section (commuting with σ) which extends s_K.

As any cross-section s gives rise to an ac-map, letting $\mathrm{ac}(x) = \mathrm{res}(x \cdot s(\mathrm{val}(x))^{-1})$, this shows that we may expand K and L so that the embedding is an embedding of models of T_0^{ac}. We then conclude by part (3a). □

Lemma 2.11. *Any model of* ACFA *is linearly difference closed.*

Proof. Let (k, σ) be a difference field, and let $\lambda(X) = 1 + \alpha_0 X + \alpha_1 \sigma(X) + \cdots + \alpha_n \sigma^n(X)$, where all α_i are from k and $\alpha_n \neq 0$.

Applying a suitable power of σ^{-1} to the equation, we may assume that $\alpha_0 \neq 0$. If $n = 0$, the statement is trivial.

Now assume $n > 0$. Consider $l := k(X_0, \ldots, X_{n-1})$, and extend σ to l, letting $\sigma(X_i) := X_{i+1}$ for $i < n-1$ and $\sigma(X_{n-1}) := -\frac{1}{\alpha_n}(1 + \sum_{i=0}^{n-1} \alpha_i X_i)$. Then $\lambda(X_0) = 0$, proving that existentially closed difference fields are linearly difference closed. □

Let IncVFA_0 be the theory of structures $\mathcal{K} = (K, k, \Gamma)$ (in the language $\mathcal{L}_{\mathbf{k},\mathbf{\Gamma},\sigma} \cup \{\mathrm{ac}\}$), where (K, k_K, Γ_K) is a contractive ac-valued difference field of characteristic 0, $(k, \overline{\sigma})$ is a difference field containing k_K, and $(\Gamma, \sigma_\Gamma) \models \mathrm{IncODG}$ contains Γ_K as a difference subgroup. (I.e. we do not require that the maps res and val are surjective.)

Let VFA_0 be the theory T_0^{ac}, together with axioms expressing that $(\mathbf{k}, \overline{\sigma}) \models \mathrm{ACFA}_0$ and that $(\mathbf{\Gamma}, \sigma) \models \mathrm{IncDODG}$.

Fact 2.12. VFA_0 *is the model-companion of* IncVFA_0, *in the language of* ac-*valued difference fields. The same result holds if both* VFA_0 *and* IncVFA_0 *are restricted to the language of valued difference fields.*

Proof. For valued difference fields (without ac-map), one may show, using the proofs of [AvdD11, 2.5 & 2.6] and [Azg10, 3.3 – 3.5], that every model of IncVFA_0 embeds into some $\mathcal{K} = (K, k, \Gamma) \models \mathrm{IncVFA}_0$ with surjective res and val and such that $(k, \overline{\sigma}) \models \mathrm{ACFA}_0$ and $(\Gamma, \sigma_\Gamma) \models \mathrm{IncDODG}$. (We omit the details.) The maximal immediate extension of $\mathcal{K}$ is then σ-henselian, by Remark 2.7(2) and Lemma 2.11, so it is a model of VFA_0. As ACFA_0 and $\mathrm{IncDODG}$ are model-complete, model-completeness of VFA_0 follows from Fact 2.10(4b). (See also [Gia11, Thm 4.3.20].)

To finish the proof, it is enough to show that VFA_0 is a companion of IncVFA_0 (in the language with ac). This means that we need to show that every model of IncVFA_0 embeds into a model of VFA_0.

Let $\mathcal{K} = (K, k, \Gamma) \models \mathrm{IncVFA}_0$ be given. We first show the special case where $\Gamma = \Gamma_K$. Using only the constructions from [AvdD11, Proofs of 2.5 & 2.6], the value group does not increase, and so, similarly to the case without ac, we may embed $\mathcal{K}$ into some $\mathcal{K}' = (K', k', \Gamma) \models T_0$, with $(k', \overline{\sigma}) \models \mathrm{ACFA}_0$. Moreover, since the value group is the same, ac uniquely extends to $\mathcal{K}'$ making it a model of T_0^{ac}. By Fact 2.10, $\mathcal{K}' \equiv (k'((\Gamma)), k', \Gamma) = \mathcal{L}'$, where on $\mathcal{L}'$ we take the standard ac-map given by the first non-zero coefficient. Now $\mathcal{L}'$ embeds into a model of VFA_0, namely into $(k'((\tilde{\Gamma})), k', \tilde{\Gamma})$, where $\Gamma \subseteq \tilde{\Gamma} \models \mathrm{IncDODG}$. Since "admitting an embedding into a model of VFA_0" is a first order property, the proof of the special case is finished.

For the general case, let $\mathcal{K} = (K, k, \Gamma) \models \mathrm{IncVFA}_0$ be given. By the special case, $(K, k, \Gamma_K) \subseteq \mathcal{L} = (L, l, \Delta) \models \mathrm{VFA}_0$. We may choose $\mathcal{L}$ sufficiently saturated. Then Γ embeds into Δ over Γ_K, and we may conclude. $\square$

With a slightly different notion of σ-henselianity, Fact 2.12 had been independently obtained by Hrushovski in [Hru02], where the following consequence is also mentioned. (See also [Gia11, Thm 4.3.24].)

Fact 2.13 (Hrushovski). *Let φ be a sentence in the language of* ac-*valued difference fields. The following are equivalent:*

   (1) $\mathrm{VFA}_0 \vdash \varphi$;
   (2) $\mathcal{K}_p \models \varphi$ *for all large enough primes p.*

Proof. Every instance of the σ-Hensel scheme holds in $\mathcal{K}_p$ for $p \gg 0$, by Fact 2.9. Moreover, it is easy to see that every axiom of ordered $\mathbb{Q}(\sigma)$-vector spaces holds in $(\Gamma_{K_p}, \gamma \mapsto p\gamma)$ for $p \gg 0$. By Fact 2.10, it is thus enough to show that the limit theory of the residue difference fields coincides with ACFA_0. This is true, by a very deep result of Hrushovski [Hru04]. $\square$

Examples 2.14.   (1) Let $\mathcal{U}$ be a non-principal ultrafilter on the set of prime numbers. Then $\prod_{\mathcal{U}} \mathcal{K}_p \models \mathrm{VFA}_0$.
      (2) Let $(k, \overline{\sigma}) \models \mathrm{ACFA}_0$ (e.g. $k = \mathbb{C}$ and $\overline{\sigma}$ a sufficiently 'generic' automorphism of $\mathbb{C}$), and let Γ be a non-trivial ordered vector space over $\mathbb{Q}(\sigma)$. Then $K := k((\Gamma)) \models \mathrm{VFA}_0$.

Proof. (1) follows from Fact 2.13, and (2) is a consequence of Lemma 2.11 together with Remark 2.7(3). $\square$

2.6. **Isometric valued difference fields.** Another important class of valued difference fields is the class of valued fields with an *isometry*, where one requires that the induced automorphism σ_Γ on the value group Γ is the identity. The model theory of σ-henselian valued fields of residue characteristic 0 with an isometry is well understood, if one assumes in addition that there are *enough constants*, i.e. that every $\gamma \in \Gamma$ is of the form $\mathrm{val}(a)$ for some $a \in \mathrm{Fix}(\sigma)$.

By work of Scanlon [Sca00, Sca03], Bélair, Macintyre and Scanlon [BMS07] and then (in a slightly more general setting) Durhan and van den Dries [AvdD11], in this context one may eliminate quantifiers from the field sort **K** in the language with angular components and thus get an Ax-Kochen-Ershov principle, analogous to Fact 2.10. (In Fact 2.15 below we give a precise statement. For the definition of σ-henselianity in the isometric case, we refer to [AvdD11, Definition 4.4].) Moreover, a model-companion exists in the isometric case (see e.g., [BMS07]).

Let S_0^{ac} be the theory of σ-henselian valued fields with an isometry (having enough constants) in residue characteristic 0, considered in the language $\mathcal{L}_{\mathbf{k},\mathbf{\Gamma},\sigma} \cup \{\mathrm{ac}\}$ of ac-valued difference fields.

Fact 2.15 ([AvdD11]).   *(1) The theory S_0^{ac} eliminates $\mathbf{K}$-quantifiers.*

    (2) In any model $\mathcal{K} = (K, \Gamma_K, k_K) \models S_0^{\mathrm{ac}}$, $k_K = \mathbf{k}(\mathcal{K})$ is stably embedded, and the induced structure is that of a difference field. Similarly, $\Gamma_K = \mathbf{\Gamma}(\mathcal{K})$ is stably embedded and is a pure ordered abelian group. Moreover, $\mathbf{k}$ and $\mathbf{\Gamma}$ are orthogonal.

    (3) Let $\mathcal{K} = (K, \Gamma_K, k_K)$ and $\mathcal{L} = (L, \Gamma_L, k_L)$ be models of S_0^{ac}.

        (a) $\mathcal{K} \equiv \mathcal{L}$ if and only if $k_K \equiv k_L$ (as difference fields) and $\Gamma_K \equiv \Gamma_L$ (as ordered abelian groups).

        (b) Suppose $\mathcal{K} \subseteq \mathcal{L}$. Then $\mathcal{K} \preccurlyeq \mathcal{L}$ if and only if $k_K \preccurlyeq k_L$ and $\Gamma_K \preccurlyeq \Gamma_L$.

For p a prime number, let $W(\mathbb{F}_p^{alg})$ be the quotient field of the ring of Witt vectors with coefficients from $\mathbb{F}_p^{alg}$, with its natural valuation. On the valued field $W(\mathbb{F}_p^{alg})$, there is a natural isometry, namely the *Witt-Frobenius* automorphism which we denote by $\widetilde{Frob}_p$, sending $x = \sum_n a_n p^n \in W(\mathbb{F}_p^{alg})$ to $\sum_n a_n^p p^n$. Letting $\mathrm{ac}(x) := a_{\mathrm{val}(x)}$, we get an ac-valued difference field $\mathcal{W}_p = (W(\mathbb{F}_p^{alg}), \mathbb{Z}, \mathbb{F}_p^{alg}, \widetilde{Frob}_p)$.

The following example is discussed in [BMS07, Section 12].

Example 2.16 (Non-standard Witt-Frobenius automorphism). Let $\mathcal{U}$ be a non-principal ultrafilter on the set of prime numbers. Then $\prod_\mathcal{U} \mathcal{W}_p \models S_0^{ac}$. Moreover, $\prod_\mathcal{U} \mathcal{W}_p \equiv \prod_\mathcal{U} (\mathbb{F}_p^{alg}((t)), \sigma_p)$, where σ_p is the isometry given by $\sum a_n t^n \mapsto \sum a_n^p t^n$.

Remark 2.17. In the setting of so-called *multiplicative* valued difference fields, forming a common generalisation of the contractive and the isometric case, Pal established similar Ax-Kochen-Ershov type results (see [Pal12]), even without adding an angular component map and working in the appropriate language for the $\mathbf{RV}$ sort, where $\mathbf{RV} = K^\times / 1 + \mathfrak{m}$.

3. Indiscernible arrays and NTP$_2$

In this section we recall some facts about NTP$_2$ and prove some new lemmas. As usual, we fix a monster model $\mathbb{M}$. We don't distinguish here between finite and infinite tuples unless mentioned explicitly, and $\bar{a}$, $\bar{b}$, ... denote infinite sequences.

Definition 3.1. We say that $\varphi(x, y)$ has TP$_2$ if there are $(a_{ij})_{i,j \in \omega}$ and $k \in \omega$ such that:

    (1) $\{\varphi(x, a_{ij})\}_{j \in \omega}$ is k-inconsistent for every $i \in \omega$.

    (2) $\{\varphi(x, a_{if(i)})\}_{i \in \omega}$ is consistent for every $f : \omega \to \omega$.

A theory is called NTP$_2$ if no formula has TP$_2$.

Fact 3.2. *If a theory T is simple or NIP, then it is NTP$_2$ (see e.g. [Che, Section 2]).*

Fact 3.3 ([Che]). *If T is not NTP$_2$, then already some $\varphi(x, y)$ with $|x| = 1$ has TP$_2$.*

Definition 3.4. We say that $(c_{ij})_{i,j \in \kappa}$ is a *strongly indiscernible array* if $\bar{c}_i = (c_{ij})_{j \in \kappa}$ is indiscernible over $\bar{c}_{\neq i}$ for all i and $(\bar{c}_i)_{i \in \kappa}$ is an indiscernible sequence (of sequences).

We start with some auxiliary results on finding indiscernible sequences and arrays.

Lemma 3.5. *(1) Let C be a small set, $\bar{a} = (a_i)_{i \in \omega}$ be a C-indiscernible sequence, b given, and let $p(x, a_0) = \mathrm{tp}(b/a_0 C)$. Assume that $\bigcup_{i \in \omega} p(x, a_i)$ is consistent. Then there is some $\bar{a}'$ indiscernible over bC and such that $\bar{a}' \equiv_{a_0 C} \bar{a}$.*

(2) Let C be a small set and $(a_{\alpha i})_{\alpha < n, i < \omega}$ be an array with $n < \omega$. Then for any finite $\Delta \in L(C)$ and $N < \omega$ we can find Δ-mutually indiscernible sequences $(a_{\alpha, i_{\alpha 0}}, ..., a_{\alpha, i_{\alpha N}}) \subset \bar{a}_\alpha$, $i_{\alpha 0} < ... < i_{\alpha N} \in \omega$, $\alpha < n$.

(3) Assume that we are given $(\bar{a}_i)_{i \in \kappa}$ and a small set C such that $\bar{a}_i$ is indiscernible over $\bar{a}_{<i} (a_{j0})_{j > i} C$ for all $i \in \kappa$. Then there exists an array $(\bar{a}'_i)_{i \in \kappa}$ such that $\bar{a}'_i \equiv_{a_{i0} C} \bar{a}_i$ and $\bar{a}'_i$ is indiscernible over $\bar{a}'_{\neq i} C$ for all i.

Proof. (1) By applying an automorphism it is enough to find $b' \equiv_{a_0 C} b$ such that $\bar{a}$ is indiscernible over $b'C$. Let Δ be an arbitrary finite set of formulas over C. Let $b^+ \models \bigcup_{i \in \omega} p(x, a_i)$. By Ramsey there is an infinite subsequence $\bar{a}^+$ of $\bar{a}$ which is Δ-indiscernible over b^+. Let σ be a C-automorphism sending $\bar{a}^+$ to $\bar{a}$. Then $\bar{a}$ is Δ-indiscernible over $\sigma(b^+)$ and $\sigma(b^+) \equiv_{a_0 C} b$. We find b' by compactness.

(2) By the finitary Ramsey theorem there are natural numbers $(N_\alpha)_{\alpha < n}$ such that for every $\alpha < n$ and every set A of size $\sum_{\beta < \alpha} N_\alpha + (n - 1 - \alpha) \times N$, every sequence of elements of length N_α contains a subsequence of length N which is Δ-indiscernible over A.

Let $\bar{a}_\alpha^+ = (a_{\alpha i})_{i < N_\alpha}$. By reverse induction and the choice of N_α we can find $\bar{a}'_\alpha$ such that:

- $\bar{a}'_\alpha$ is a subsequence of $\bar{a}_\alpha^+$,
- $|\bar{a}'_\alpha| = N$,
- $\bar{a}'_\alpha$ is Δ-indiscernible over $\bar{a}_{<\alpha}^+ \bar{a}'_{>\alpha}$.

But then $\bar{a}'_0, \ldots, \bar{a}'_{n-1}$ are as wanted.

(3) By compactness, it is enough to prove the statement for finite κ. Let $\Delta \in L(C)$ finite and $N \in \omega$ be arbitrary. By (2) we can find Δ-mutually indiscernible sequences $\bar{a}_\alpha^+ = (a_{\alpha, i_{\alpha 0}}, ..., a_{\alpha, i_{\alpha N}}) \subset \bar{a}_\alpha$ for $\alpha \in \kappa$. It follows from the assumption that $a_{0, i_{00}} a_{1, i_{10}} \ldots a_{\kappa - 1, i_{(\kappa - 1) 0}} \equiv_C a_{00} a_{10} \ldots a_{(\kappa - 1) 0}$. Let σ be a C-automorphism sending the former to the latter. Then we have:

- $\sigma(\bar{a}_0^+), \ldots, \sigma(\bar{a}_{\kappa - 1}^+)$ are mutually Δ-indiscernible,
- $a_{\alpha, i_{\alpha 0}}, ..., a_{\alpha, i_{\alpha N}} \equiv_C a_{\alpha 0} \ldots a_{\alpha N}$ by indiscernibility, so $\sigma(\bar{a}_\alpha^+) \equiv_C (a_{\alpha i})_{i \leq N}$, which together with $\sigma(a_{\alpha, i_{\alpha 0}}) = a_{\alpha 0}$ implies that $\sigma(\bar{a}_\alpha^+) \equiv_{a_{\alpha 0} C} (a_{\alpha i})_{i \leq N}$, for each α.

By compactness we find $\bar{a}'_0, \ldots, \bar{a}'_{\kappa - 1}$ as wanted. $\square$

Lemma 3.6. *Let $(c_{ij})_{i, j \in \omega}$ be a strongly indiscernible array such that $(\bar{c}_i)_{i \in \omega}$ is indiscernible over a, and let b be given. Then we can find b_{ij} such that $(b_{ij} c_{ij})_{i, j \in \omega}$ is a strongly indiscernible array, $bc_{00} \equiv b_{ij} c_{ij}$ for all i, j and $(\bar{b}_i \bar{c}_i)_{i \in \omega}$ is indiscernible over a.*

Proof. Set $b_{00} = b$ and let b_{ij} be such that $b_{ij} c_{ij} \equiv b_{00} c_{00}$. By Lemma 3.5(2) and compactness, applying an automorphism we may assume that $(\bar{b}_i \bar{c}_i)_{i \in \omega}$ are

mutually indiscernible. By Ramsey, compactness and applying automorphisms over a, we may assume in addition that $(\bar{b}_i \bar{c}_i)_{i \in \omega}$ is indiscernible over a. $\square$

Given a definable set D, by D_{ind} we mean the full induced structure on it. The next lemma is a generalisation of a lemma from [Che, Section 1].

Lemma 3.7. *Let an $\emptyset$-definable set D be stably embedded and assume that D_{ind} is* NTP$_2$*. Let $\bar{b} \subset D$ with $|\bar{b}| \leq \lambda$ be given.*

Assume that $(\bar{c}_i)_{i \in \kappa}$ is an array with mutually indiscernible rows over C, and $\bar{c}_i = (c_{ij})_{j \in \omega}$. If $\kappa \geq (\lambda + |T|)^+$, then there is $i \in \kappa$ and $\bar{c}'$ such that:

- *$\bar{c}' \equiv_{c_{i0} C} \bar{c}_i$*
- *$\bar{c}'$ is indiscernible over $C\bar{b}$.*

Proof. Let $p_i(\bar{x}, c_{i0}) = \operatorname{tp}(\bar{b}/c_{i0}C)$.

We claim that $\bigcup_{j \in \omega} p_i(\bar{x}, c_{ij})$ is consistent for some $i \in \kappa$.

Assume not, then by compactness and indiscernibility, for every $i \in \kappa$ we have some $\varphi_i(x_i, c_{i0}d_i) \in p_i$ (with finite $x_i \subset \bar{x}$), $d_i \in C$, and $k_i \in \omega$ such that $\{\varphi_i(x_i, c_{ij}d_i)\}_{j \in \omega}$ is k_i-inconsistent. As D is stably embedded, for each i there is some $\psi_i(x_i, e_{i0})$ with $e_{i0} \in D$ such that $\psi_i(x_i, e_{i0}) \cap D = \varphi_i(x_i, c_{i0}d_i) \cap D$. As the type of $c_{i0}d_i$ says that there is an element e_{i0} with this property, by the indiscernibility of the rows over C we can find $e_{ij} \in D$ such that $\psi_i(x_i, e_{ij}) \cap D = \varphi_i(x_i, c_{ij}d_i) \cap D$ for all i, j. As κ was chosen large enough, by throwing some rows away we may assume that $\psi_i = \psi$, $x_i = x$ and $k_i = k$.

But then we have:

- $\{\psi(x, e_{ij}) \wedge D(x)\}_{j \in \omega}$ is k-inconsistent for every i (as $\{\varphi_i(x_i, c_{ij}d_j)\}_{j \in \omega}$ is k-inconsistent),
- $\{\psi(x, e_{if(i)}) \wedge D(x)\}_{i \in \kappa}$ is consistent for every $f : \kappa \to \omega$ (it is witnessed by $\bar{b}$ for $f(i) = 0$, and follows for an arbitrary f by the mutual indiscernibility of the rows).

This is a contradiction to D_{ind} being NTP$_2$.

So let i be as given by the claim. But then by Lemma 3.5(2) we can find $\bar{c}' \equiv_{c_{i0} C} \bar{c}_i$ such that $\bar{c}'$ is indiscernible over $\bar{b}C$. $\square$

Lemma 3.8 (The Array Extension Lemma). *Let D be a stably embedded $\emptyset$-definable set and assume that D_{ind} is* NTP$_2$*.*

Assume that

- *$(\bar{c}_i)_{i \in \omega}$ is indiscernible over a,*
- *$(c_{ij})_{i,j \in \omega}$ is a strongly indiscernible array.*

Let a small $b \subseteq D$ be given. Then we can find $(c_{ij}^*)_{i,j \in \omega}$ and $(b_{ij}^*)_{i,j \in \omega}$ such that:

(1) *$(\bar{b}_i^* \bar{c}_i^*)_{i \in \omega}$ is indiscernible over a,*
(2) *$(\bar{b}_i^* \bar{c}_i^*)_{i \in \omega}$ are mutually indiscernible,*
(3) *$\bar{c}_i^* \equiv_{c_{i0}} \bar{c}_i$ for all $i \in \omega$ (so in particular $c_{i0}^* = c_{i0}$),*
(4) *$abc_{00} \equiv ab_{00}^* c_{00}^*$.*

In particular $(b_{ij}^* c_{ij}^*)_{i,j \in \omega}$ is a strongly indiscernible array.

Proof. By compactness, it suffices to prove the result for finite b.

First, by indiscernibility, Ramsey and applying automorphisms over a, we can find b_i such that $ab_i \bar{c}_i \equiv ab \bar{c}_0$ and $(b_i \bar{c}_i)_{i \in \omega}$ is indiscernible over a.

Again by compactness, indiscernibility and applying automorphisms over a, it is enough to find $\bar{c}^*_{<n}, \bar{b}^*_{<n}$ satisfying (2), (3) and (4′) for every $n \in \omega$, where

(4′) $abc_{00} \equiv ab^*_{k0} c^*_{k0}$ for all $k < n$.

So fix $n \in \omega$ and let $I = I_0 + I_1 + \ldots + I_{n-1} = |T|^+ + \ldots + |T|^{+n}$ (where for a cardinal κ we let κ^{+n} denote the nth successor of κ). By compactness we may expand our sequence to $(b_i \bar{c}_i)_{i \in I}$ with the same Ehrenfeucht-Mostowski type over a.

By reverse induction on $k < n$ we find i_k, $\bar{c}^+_k$, $\bar{b}^+_k$ such that:

(a) $i_k \in I_k$,
(b) $\bar{c}^+_k \equiv_{c_{i_k 0}} \bar{c}_{i_k}$ (so in particular $c^+_{k0} = c_{i_k 0}$),
(c) $\bar{c}_{\in I_{<k}} \bar{c}^+_k \bar{c}^+_{k+1} \ldots \bar{c}^+_{n-1} \equiv \bar{c}_{\in I_{<k}} \bar{c}_{i_k} \bar{c}_{i_{k+1}} \ldots \bar{c}_{i_{n-1}}$,
(d) $b^+_{k0} = b_{i_k}$ and $\bar{b}^+_k \subseteq D$,
(e) $\left(b^+_{kj} c^+_{kj} \right)_{j \in \omega}$ is indiscernible over $\bar{b}^+_{>k} \bar{c}^+_{>k} b_{\in I_{<k}} \bar{c}_{\in I_{<k}}$.

In step k, let $C = \bar{c}^+_{>k} \bar{c}_{\in I_{<k}}$ and $\bar{b} = \bar{b}^+_{>k} b_{\in I_{<k}}$. Then $\bar{b} \subseteq D$, $|\bar{b}| \leq |T|^{+k}$ and $(\bar{c}_i)_{i \in I_k}$ are mutually indiscernible over C (by (c) for $k+1$ and the assumption on $(\bar{c}_i)_{i \in I}$). As $I_k = |T|^{+(k+1)}$, it follows by Lemma 3.7 that there is some $i_k \in I_k$ and $\bar{c}'_k$ indiscernible over $\bar{b}C$ and such that $\bar{c}'_k \equiv_{c_{i_k 0} C} \bar{c}_{i_k}$. Let $b'_{k0} = b_{i_k}$ and b'_{kj} be such that $b'_{k0} c'_{k0} \equiv_{\bar{b}C} b'_{kj} c'_{kj}$. By Ramsey, compactness and $\bar{b}C$-automorphisms we can find a $\bar{b}C$-indiscernible sequence $\left(b^+_{kj} c^+_{kj} \right)_{j \in \omega}$ such that $b^+_{k0} c^+_{k0} = b'_{k0} c'_{k0}$ and $\bar{c}^+_k \equiv_{\bar{b}C} \bar{c}'_k$. Now (b) and (c) follow from (c) for $k+1$, and $\bar{c}^+_k \equiv_C \bar{c}'_k \equiv_C \bar{c}_{i_k}$ and $c^+_{k0} = c_{i_k 0}$. Parts (a), (d) and (e) are clearly satisfied by construction.

By Lemma 3.5(3) and (e) we find sequences $\left(b^{++}_{kj} c^{++}_{kj} \right)_{j \in \omega}$ for $k < n$ which are mutually indiscernible and such that $\bar{c}^{++}_k \bar{b}^{++}_k \equiv_{b^+_{k0} c^+_{k0}} \bar{c}^+_k \bar{b}^+_k$.

Finally, let σ be an a-automorphism sending $b_{i<n} \bar{c}_{i<n}$ to $b_{<n} \bar{c}_{<n}$, and let $\bar{b}^*_k = \sigma \left(\bar{b}^{++}_k \right)$ and $\bar{c}^*_k = \sigma \left(\bar{c}^{++}_k \right)$ for $k < n$.

We have:

- $\left(\bar{b}^*_k \bar{c}^*_k \right)_{k<n}$ are mutually indiscernible (as $\left(\bar{b}^{++}_k \bar{c}^{++}_k \right)_{k<n}$ are),
- $\bar{c}^*_k \equiv_{c_{k0}} \bar{c}_k$ (as $\bar{c}^{++}_k \equiv_{c^+_{k0}} \bar{c}^+_k$, $\bar{c}^+_k \equiv_{c_{i_k 0}} \bar{c}_{i_k}$, $c^+_{k0} = c_{i_k 0}$ and σ is an automorphism),
- $abc_{00} \equiv ab^*_{k0} c^*_{k0}$ for all $k < n$, as $b^*_{k0} c^*_{k0} = b_k c_{k0}$ by the construction and $ab_k c_{k0} \equiv abc_{00}$.

$\square$

Lemma 3.9. *In any theory, $\varphi(x,y)$ has* TP$_2$ *if and only if there is a strongly indiscernible array $(a_{ij})_{i,j \in \omega}$ witnessing it (as in Definition 3.1) and $c \models \{\varphi(x, a_{i0})\}_{i \in \omega}$ such that the sequence of rows $(\bar{a}_i)_{i \in \omega}$ is indiscernible over c.*

Proof. By Lemma 3.5(2), Ramsey and compactness. $\square$

Definition 3.10. We say that a (partial) type $p(x)$ over A is NTP$_2$-*determined* if there is $\Phi \subseteq p$ closed under conjunction, such that $\Phi(x) \vdash p(x)$ and such that for every $\varphi(x, a) \in \Phi$, $\varphi(x, y)$ is NTP$_2$.

Lemma 3.11. *Let $(a_{ij})_{i,j \in \omega}$ be a strongly indiscernible array, $\varphi(x,y)$ a formula and let $c \models \{\varphi(x, a_{i0})\}_{i \in \omega}$, moreover assume that the sequence of rows $(\bar{a}_i)_{i \in \omega}$ is*

indiscernible over c. Assume that $p(x, a_{00}) = \mathrm{tp}(c/a_{00})$ is NTP_2-determined. Then $\{\varphi(x, a_{0j})\}_{j \in \omega}$ is consistent.

Proof. By the choice of c we have $\varphi(x, a_{00}) \in p(x, a_{00})$, then by compactness (and as Φ is closed under conjunctions) there is some $\psi(x, a_{00}) \in \Phi(x)$ such that $\psi(x, a_{00}) \vdash \varphi(x, a_{00})$. By strong indiscernibility it follows that $\psi(x, a_{ij}) \vdash \varphi(x, a_{ij})$ for all $i, j \in \omega$. Note also that $c \models \bigcup_{i \in \omega}\{p(x, a_{i0})\}$, so in particular $c \models \{\psi(x, a_{i0})\}_{i \in \omega}$. As $\psi(x, z)$ is NTP_2, it follows that for some $i \in \omega$ the set $\{\psi(x, a_{ij})\}_{j \in \omega}$ is consistent, so by strong indiscernibility the set $\{\psi(x, a_{0j})\}_{j \in \omega}$ is consistent. But this implies that $\{\varphi(x, a_{0j})\}_{j \in \omega}$ is consistent. $\qquad\square$

4. Preservation of NTP_2

In this section we prove the main results of the paper, concerning the preservation of NTP_2 in various σ-henselian valued difference fields. We first prove a general preservation result and then apply this in various contexts.

4.1. **A general preservation result.**

Theorem 4.1. *Let $\mathcal{K} = (K, k, \Gamma)$ be an ac-valued difference fields of residue characteristic 0. Assume that $T = \mathrm{Th}(\mathcal{K})$ eliminates $\mathbf{K}$-quantifiers, and that both the residue field k (as a difference field) and the value group Γ (as an ordered difference group) are NTP_2. Then, $\mathcal{K}$ is NTP_2.*

Lemma 4.2. *Let $\varphi(x, y)$ be a quantifier-free formula (in the language $\mathcal{L}_{k,\Gamma,\sigma} \cup \{\mathrm{ac}\}$). Then it is NIP in every ac-valued difference field of residue characteristic 0.*

Proof. We may write $\varphi(x, y)$ as $\psi(x, \sigma(x), \ldots, \sigma^n(x), y, \sigma(y), \ldots, \sigma^n(y))$, where the formula $\psi(x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n)$ is quantifier-free in the language of ac-valued fields $\mathcal{L}_{\mathbf{k},\mathbf{\Gamma}} \cup \{\mathrm{ac}\}$.

Claim. *Every ac-valued field in residue characteristic 0 embeds into an algebraically closed ac-valued field.*

Proof of the claim. This should be well known. We could not find a reference, and so we give a proof. Let $\mathcal{K} = (K, k, \Gamma)$ be an ac-valued field of residue characteristic 0. The ac-map (uniquely) extends to the henselisation of $\mathcal{K}$, so we may assume that $\mathcal{K}$ is henselian. We may now argue as in the proof of Fact 2.12. It follows from Pas' theorem (Fact 2.4) that $\mathcal{K} \equiv (k((\Gamma)), k, \Gamma)$, where the latter is endowed with the standard ac-map. Any Hahn series field embeds (as an ac-valued field) in an algebraically closed Hahn series field. The result follows. $\qquad\square$

By a result of Delon [Del81] mentioned in the introduction, the theory $\mathrm{ACVF}_{0,0}$, in the language with ac, is NIP. It thus follows from the claim that $\psi(\bar{x}, \bar{y})$ is NIP in every ac-valued field of residue characteristic 0. Now, assume that $(a_i)_{i \in \omega}$ and $(b_s)_{s \subseteq \omega}$ from some ac-valued difference field $\mathcal{K}$ are such that $\mathcal{K} \models \varphi(a_i, b_s) \Leftrightarrow i \in s$. But then, letting $\bar{a}_i = (a_i, \sigma(a_i), \ldots, \sigma^n(a_i)), \bar{b}_s = (b_s, \sigma(b_s), \ldots, \sigma^n(b_s))$, we get $\mathcal{K} \models \psi(\bar{a}_i, \bar{b}_s) \Leftrightarrow i \in s$. This contradicts the fact that $\psi(\bar{x}, \bar{y})$ is NIP. $\qquad\square$

Proof of Theorem 4.1. We fix some monster model $\mathbb{M} \models T$. Suppose there is a formula $\varphi(x, y)$ with TP_2. By Fact 3.3, we may assume that $|x| = 1$. As the induced structures on $\mathbf{\Gamma}$ and on $\mathbf{k}$ are NTP_2, combining Lemma 3.9 and Lemma 3.7 we may assume that x is a variable from the valued field sort $\mathbf{K}$.

Let $(a_{ij})_{i,j \in \omega}$ be an array witnessing that $\varphi(x, y)$ has TP$_2$, and let a be a realisation of the first column, i.e. $a \models \bigwedge_{i \in \omega} \varphi(x, a_{i0})$. By Lemma 3.9 we may assume that:

- $(a_{ij})_{i,j \in \omega}$ is a strongly indiscernible array;
- the sequence of rows $(\overline{a}_i)_{i \in \omega}$ is a-indiscernible.

In our proof, we will successively construct arrays (a_{ij}^α), where α is an ordinal $\leq \omega$ and a_{ij}^α is a countable tuple from $\mathbb{M}$, starting with $a_{ij}^0 = a_{ij}$, and such that, for any $\beta > \alpha$, there is a decomposition $(a_{ij}^\beta) = (a_{ij}^* b_{ij}^*)$ satisfying

(1) $(a_{ij}^\beta)_{i,j \in \omega}$ is a strongly indiscernible array;
(2) the sequence of rows $(\overline{a}_i^\beta)_{i \in \omega}$ is a-indiscernible, and
(3) $\overline{a}_i^* \equiv_{a_{i0}^\alpha} \overline{a}_i^\alpha$ for all $i \in \omega$.

It follows from (3) that the the first column is just an extension of the original one, and that in particular we still have $a \models \bigwedge_{i \in \omega} \varphi(x, a_{i0}^\beta)$. Also by (3), the rows $\{\varphi(x, a_{ij}^\beta)\}_{j \in \omega}$ are still inconsistent. As a consequence, for any β, the array (a_{ij}^β) witnesses that φ has TP$_2$.

Even though only the first column $(a_{i0}^\alpha)_{i \in \omega}$ has to be a subtuple of (a_{i0}^β), we will say, somewhat inaccurately, that we *extend* the array, when we pass from (a_{ij}^α) to (a_{ij}^β). An extension of arrays will be called *good* if it satisfies the properties (1)–(3) above.

The construction will be done in steps, following the back-and-forth system one may infer from (2) in Lemma 2.2. There will be two kinds of successor steps: *auxiliary steps*, where Lemma 3.6 is used to extend the array (a_{ij}^α) carelessly, to add new parameters; *treating steps*, where the Array Extension Lemma (Lemma 3.8) is used to extend the array (a_{ij}^α) carefully, respecting partial information from $\mathrm{tp}(a/a_{00}^\alpha)$ coming from the NTP$_2$ sorts $\mathbf{\Gamma}$ and $\mathbf{k}$. The final step dealing with an immediate extension will follow from Lemma 3.11.

If (a_{ij}^α) have been constructed for all $\alpha < \omega$ such that (a_{ij}^β) is a good extension of (a_{ij}^α) for all $\alpha < \beta < \omega$, then we may find an array (a_{ij}^ω) with $a_{i0}^\omega = \bigcup_{\alpha < \omega} a_{i0}^\alpha$ for all $i \in \omega$ and such that (a_{ij}^ω) is a good extension of (a_{ij}^α), for all $\alpha < \omega$. Indeed, this follows from compactness, as properties (1)–(3) are type-definable in the variables $(x_{ij}^\omega)_{i,j \in \omega}$.

(I) Given (a_{ij}^α), there is a good extension $(a_{ij}^{\alpha+1})$ such that $(a_{00}^{\alpha+1})$ enumerates a substructure $\mathcal{K}^{\alpha+1} = (K^{\alpha+1}, k^{\alpha+1}, \Gamma^{\alpha+1})$ where both $K^{\alpha+1}$ and $k^{\alpha+1}$ are difference fields and $\Gamma^{\alpha+1}$ is an ordered $\mathbb{Z}[\sigma]$-module.
    [By Lemma 3.6.]

    In what follows, we may always assume that a_{00}^α is as in the conclusion of step I. To ease the notation, we write $a_{00}^\alpha = (K, k, \Gamma)$ and $a_{00}^{\alpha+1} = (K', k', \Gamma')$. Let $L := K\langle a \rangle$, and $L' := K'\langle a \rangle$. Recall that $a \models \bigwedge_{i \in \omega} \varphi(x, a_{i0})$.

(II) Given (K, k, Γ), there is a good extension such that $k_{K'} \supseteq k$.
    [By Lemma 3.6.]

(III) Given (K, k, Γ), there is a good extension such that $\Gamma_{K'} \supseteq \Gamma$.
    [By Lemma 3.6.]

(IV) Given (K, k, Γ), there is a good extension such that $\mathrm{ac}(L) \subseteq k'$.
       [By Lemma 3.8 with $b = \mathrm{ac}(L)$, as $\mathbf{k}$ is stably embedded (Lemma 2.3(1)), and NTP$_2$ by assumption.]

 (V) Given (K, k, Γ), there is a good extension such that $\Gamma_L \subseteq \Gamma'$.
       [By Lemma 3.8, as $\mathbf{\Gamma}$ is stably embedded (Lemma 2.3(1)), and NTP$_2$ by assumption.]

       Iterating steps I-V and passing to the limit, we may thus construct a good extension (a_{ij}^ω) such that (K, k, Γ) is coming from an ac-valued difference field (i.e. $k = k_K$ and $\Gamma = \Gamma_K$) and such that L/K is an immediate extension.

(VI) As $\mathrm{ac}(K) \subseteq k_K$, we may apply Lemma 2.3(2), and so $\mathrm{tp}(a/K)$ is determined by its quantifier-free part. By Lemma 4.2 every quantifier-free formula is NIP, so in particular is NTP$_2$. Thus, $\mathrm{tp}(a/K)$ is NTP$_2$-determined. From Lemma 3.11 it then follows that $\{\varphi(x, a_{0j}^\omega)\}_{j \in \omega}$ is consistent. But $\{\varphi(x, a_{ij}^\omega)\}_{i,j < \omega}$ is a witness that $\varphi(x, y)$ is TP$_2$ — a contradiction. $\qquad\square$

In fact, it is easy to see that the previous proof gives the following stronger result.

Remark 4.3. With the notations and assumptions of Theorem 4.1, suppose that $T_r' \supseteq \mathrm{Th}(k, \overline\sigma)$ and $T_v' \supseteq \mathrm{Th}(\Gamma, \sigma)$ are expansions which are both NTP$_2$. Then, $T' := T \cup T_r' \cup T_v'$ eliminates $\mathbf{K}$-quantifiers and is NTP$_2$. $\qquad\square$

4.2. **Applications to σ-henselian valued difference fields.** We start with the contractive case. We fix a completion T of T_0^{ac}, so T is of the form $T_0^{\mathrm{ac}} \cup T_r \cup T_v$, where T_r is a complete theory of difference fields of characteristic 0 (which has to be assumed to be linearly difference closed by Remark 2.7) and T_v is a complete theory of ordered $\mathbb{Z}[\sigma, \sigma^{-1}]$-modules.

Theorem 4.4. *Assume that both T_r and T_v are NTP$_2$. Then, T is NTP$_2$.*

Proof. Combine Theorem 4.1 with Fact 2.10(1). $\qquad\square$

Corollary 4.5. *Every completion of VFA$_0$ is NTP$_2$.*

Proof. Every completion of ACFA$_0$ is simple by [CH99], so in particular it is NTP$_2$ by Fact 3.2. The theory IncDODG is o-minimal by Fact 2.1, so in particular it is NIP and thus NTP$_2$, by Fact 3.2. We may thus conclude by Theorem 4.4. $\qquad\square$

Next, we consider the isometric case.

Theorem 4.6. *Let $\mathcal{K} = (K, k, \Gamma, \sigma, \mathrm{val}, \mathrm{ac}) \models S_0^{ac}$, i.e., $\mathcal{K}$ is a σ-henselian valued difference field of residue characteristic 0, σ is an isometry and there are enough constants. Then $\mathrm{Th}(\mathcal{K})$ is NTP$_2$ if and only if $\mathrm{Th}(k, \overline\sigma)$ is NTP$_2$.*

Proof. Combine Theorem 4.1 with Fact 2.15. Note that in the isometric case, as well as in the case of henselian valued fields (Fact 4.7), there is no condition on Γ, since $\sigma_\Gamma = \mathrm{id}$ in these cases, and so the induced structure is that of an ordered abelian group. By a result of Gurevich and Schmidt [GS84], any ordered abelian group is NIP. $\qquad\square$

Finally, our result applies to valued fields without an automorphism in the language as well.

Corollary 4.7 ([Che])**.** *Let* $\mathcal{K} = (K, \Gamma, k)$ *be a henselian valued field of residue characteristic* 0 *in the Denef-Pas language. Assume that the theory of* k *is* NTP$_2$. *Then the theory of* $\mathcal{K}$ *is* NTP$_2$.

Proof. Any ac-valued field may be considered as an ac-valued difference field, with $\sigma = \mathrm{id}$. As (K, Γ, k) eliminates field quantifiers (Fact 2.4), Theorem 4.1 applies. □

We remark that the proof from [Che] also shows that strength is preserved, see Section 5.1.

Remark 4.8. One may show in the same way that in the multiplicative case from [Pal12] (see Remark 2.17), the valued difference field is NTP$_2$, provided **RV** (with the induced structure) is NTP$_2$.

5. Open problems

In the last section we discuss some open problems, pose several questions and consider possible research directions around model-theoretic properties of valued difference fields.

5.1. **Further model theoretic properties of** VFA$_0$.

Definition 5.1. A theory is called *strong* if there are no $(\varphi_i(x, y_i), \bar{a}_i, k_i)$ with $\bar{a}_i = (a_{ij})_{j \in \omega}$ and $k_i \in \omega$ such that:

- $\{\varphi_i(x, a_{ij})\}_{j \in \omega}$ is k_i-inconsistent, for all $i \in \omega$,
- $\{\varphi_i(x, a_{if(i)})\}_{i \in \omega}$ is consistent for every $f : \omega \to \omega$.

Strong theories were defined by Adler in [Adl07]. They form a subclass of NTP$_2$ theories which can be viewed as 'super NTP$_2$'. For more on strong theories and the related notion of *burden* see [Che].

Question 5.2. Is VFA$_0$ strong?

The following remark implies that VFA$_0$ is not of finite burden, as in a simple theory burden of a type equals the supremum of the weights of its completions [Adl07].

Remark 5.3.   (1) Let T be a simple theory. Assume that in T there is a (type-)definable infinite field F and a (type-)definable F-vector space V of dimension $\geq n$. Then, there is a type $p(x) \vdash x \in V$ such that $w(p) \geq n$.
   (2) Every completion of ACFA has a 1-type of weight $\geq n$, for any $n \in \omega$.

Proof. To prove (1), choose $v_1, \ldots, v_n \in V$ which are F-linearly independent. Choose any non-algebraic type $q(x)$ such that $q(x) \vdash x \in F$. Let $\bar{b} = (b_1, \ldots, b_n)$ be a sequence of independent realisations of q (over some model M containing the v_i's). Let $b := \sum_{i=1}^{n} b_i v_i$. Since $\bar{b}$ and b are interdefinable over M, we compute (see e.g. [Wag00, Lemma 5.2.4]): $w(\mathrm{tp}(b/M)) = w(\mathrm{tp}(\bar{b}/M)) = n\, w(q) \geq n$.
   The second part follows, considering $F := \mathrm{Fix}(\sigma)$ and $V := K$ and noting that the dimension of V over F is infinite. □

Definition 5.4. We say that a formula $\varphi(x, y)$ is *resilient* if it satisfies the following property:

- For any indiscernible sequences $\bar{a} = (a_i)_{i\in\mathbb{Z}}$ and $\bar{b} = (b_i)_{i\in\mathbb{Z}}$ such that $a_0 = b_0$ and $\bar{b}$ is indiscernible over $a_{\neq 0}$, if $\{\varphi(x, a_i)\}_{i\in\mathbb{Z}}$ is consistent, then $\{\varphi(x, b_i)\}_{i\in\mathbb{Z}}$ is consistent.

A theory is resilient if it implies that every formula is resilient.

Resilient theories were introduced in [BYC12] where it was observed that:

Remark 5.5.   (1) Every formula in a simple theory is resilient.
  (2) Every NIP formula is resilient.
  (3) Every resilient theory is NTP_2.

It is not known if there are NTP_2 theories which are not resilient.

Conjecture 5.6. *An analog of Theorem 4.1 holds for resilience.*

An earlier version of the article contained a purported proof of this following the strategy of the proof for NTP_2, but a flaw was pointed out by the referee.

Some further model theoretic properties of VFA_0 are of interest, both for sets in the real sort and in M^{eq}, and most importantly in the *geometric sorts* from [HHM08]:

Question 5.7.   (1) Is VFA_0 extensible? I.e. is it true that for every small set A, every type $p(x) \in S(A)$ has a global extension which does not fork over A? Note that it is enough to check this property for 1-types.
  (2) Is VFA_0 low? I.e., is it true that for every formula $\varphi(x, y)$ there is $k \in \omega$ such that for every indiscernible sequence $(a_i)_{i\in\omega}$, the set $\{\varphi(x, a_i)\}_{i\in\omega}$ is consistent if and only if it is k-consistent?
  (3) Does VFA_0 eliminate $\exists^\infty$?

It seems tempting to try to develop a theory of *simple* domination in VFA_0 (parallel to stable domination from [HHM08]). Some elements of the theory of simple types in NTP_2 theories are developed in [Che].

Question 5.8. Is it true in VFA_0 that a union of two stably embedded sets is stably embedded? Is it at least true for simple stably embedded sets?

5.2. **Ordered modules.** With a view on our main results, it would be interesting to know which (ω-increasing) ordered difference groups are NTP_2, or even NIP. We will put this issue in a larger context. Let R be an ordered ring. Recall that an *ordered R-module* is an ordered abelian group $\langle M, 0, +, < \rangle$ together with an action of R by endomorphisms which is compatible with the orderings, i.e. such that $r \cdot m > 0$ for all $r > 0$ from R and all $m > 0$ from M. We consider ordered R-modules in the language $\mathcal{L}_{R-mod,<} = \{0, +, <\} \cup \{\lambda_r \mid r \in R\}$, where λ_r is a unary function which is interpreted by the scalar multiplication by r.

Question 5.9.   (1) Are all ordered R-modules NIP (for all ordered rings R)?
  (2) More specifically, are all ω-increasing ordered difference groups NIP? (This corresponds to the case where $R = \mathbb{Z}[\sigma, \sigma^{-1}]$, see Section 2.1.)

Recall that every module is stable (see e.g. [Hod93]), and that every ordered abelian group is NIP, by a result of Gurevich and Schmidt [GS84]. Therefore, one might suspect a positive answer even to the first question. It seems that the answer to this question is unknown.

We now give a result which covers some easy cases. There are some similarities with work of Robinson and Zakon on (archimedean) ordered abelian groups [RZ60].

Proposition 5.10. *Let R be an ordered ring. Assume the following conditions:*

(i) *R is a principal ideal domain;*
(ii) *R is densely ordered;*
(iii) *for every prime $\pi \in R$, the ideal πR is dense in R, and*
(iv) *for every prime $\pi \in R$, $R/\pi R$ is infinite.*

Let T be the $\mathcal{L}_{R-mod,<}$-theory of R, considered as an ordered module over itself. Then the following holds:

(1) *A non-zero ordered R-module M is a model of T iff, for every prime $\pi \in R$,*
    (a) *πM is dense in M, and*
    (b) *$M/\pi M$ is infinite.*
(2) *T eliminates quantifiers in the language $\mathcal{L}_{R-mod,<} \cup \{P_r, r \in R\}$, where $P_r(x) :\Leftrightarrow \exists y\, r \cdot y = x$.*
(3) *T is NIP.*

Proof. It is a classical result (see, e.g. [Pre88]) that in the class of R-modules (without the order), for a ring R satisfying (i) and (iv), a non-zero R-module M is elementarily equivalent to R (as an R-module) iff M is torsion free and $M/\pi M$ is infinite for every prime $\pi \in R$. Moreover, $T \restriction_{\mathcal{L}_{R-mod}}$ eliminates quantifiers in the language $\mathcal{L}_{R-mod} \cup \{P_r, r \in R\}$.

Now put $\mathcal{L} = \mathcal{L}_{R-mod,<} \cup \{P_r, r \in R\}$, and let T' be the $\mathcal{L}$-theory of non-zero ordered R-modules satisfying (a) and (b). We will show that T' eliminates quantifiers in $\mathcal{L}$. This will prove (1) and (2). We use a standard back-and-forth argument. Let M and N be two models of T', with N sufficiently saturated. Assume that $f : A \cong B$ is an $\mathcal{L}$-isomorphism between finitely generated substructures $A \subseteq M$ and $B \subseteq N$.

Now let $\tilde{a} \in M$. If $\tilde{a}$ is in the divisible hull of A, then f extends (even uniquely) to an $\mathcal{L}$-isomorphism $g : A + R\tilde{a} \to B + R\tilde{b}$ for some $\tilde{b} \in N$.

We now assume that $\tilde{a}$ is not in the divisible hull of A. By the elimination of quantifiers down to $\mathcal{L}_{R-mod} \cup \{P_r, r \in R\}$ mentioned in the first paragraph, there is $\tilde{b}' \in N$ such that $\tilde{a} \mapsto \tilde{b}'$ defines an extension of f to an $\mathcal{L}_{R-mod} \cup \{P_r, r \in R\}$-isomorphism $g' : A + R\tilde{a} \cong B + R\tilde{b}'$. Of course, g' might not preserve the order. We will show that there is $d \in N$ such that d is divisible by every non-zero $r \in R$ and such that $\tilde{a} \mapsto \tilde{b} = \tilde{b}' + d$ defines an extension of f to an $\mathcal{L}$-isomorphism $g : A + R\tilde{a} \cong B + R\tilde{b}$.

Let $Q(R)$ be the field of fractions of R. Recall that if C is an ordered R-module, the order extends uniquely to $Q(C) = C \otimes_R Q(R)$ so that $Q(C)$ is an ordered R-module extending C. Now by assumption we have $R\tilde{a} \cap A = (0)$, so $\tilde{a}$ determines a cut (L, R) in $Q(A)$. Let $(f(L), f(R))$ be the cut over $Q(B)$ induced by f. Over $Q(B + R\tilde{b}')$, we may look at the (consistent) partial type $\pi(x)$ given by $f(L) - \tilde{b}' < x < f(R) - \tilde{b}'$. Note that the density assumption (a) implies that rN is dense in N for any non-zero r. Thus, by saturation of N, we may find $d \in N$ such that d is divisible by any non-zero $r \in R$ and such that $d \models \pi$. By construction, $\tilde{b} = \tilde{b}' + d$ is as we want. (In particular $\tilde{b}$ is not in the divisible hull of B.)

(3) follows from (2), taking into account that it is enough to show that any formula $\varphi(x, y)$ with x a singleton is NIP, and that NIP formulas are closed under Boolean combinations. $\qquad\square$

We note that condition (ii) in the proposition actually follows from (iii) and (i).

Corollary 5.11. *Consider the ordered field* $\mathbb{Q}(\sigma)$, *with* $\sigma \gg 1$. *Then every ordered subring of* $\mathbb{Q}(\sigma)$ *containing* $\mathbb{Q}[\sigma, \sigma^{-1}]$ *is* NIP, *considered as an ordered module over itself. In particular, the ordered difference group* $\mathbb{Q}[\sigma, \sigma^{-1}]$ *is* NIP.

Proof. We need to show that the hypotheses of Proposition 5.10 hold. So let R be an ordered ring with $\mathbb{Q}[\sigma, \sigma^{-1}] \subsetneq \mathbb{Q}(\sigma)$.

If A is a PID with field of fractions K, then every ring B with $A \subseteq B \subseteq K$ is a PID, as B is necessarily a localisation of R. This shows that R is a PID.

Property (ii) holds since $(R, +)$ is a divisible ordered abelian group.

For (iii), note that $\mathbb{Q}[\sigma, \sigma^{-1}]$ is dense in $\mathbb{Q}((\sigma))$. More generally, for any $0 \neq s \in \mathbb{Q}((\sigma))$, the set $s\mathbb{Q}[\sigma, \sigma^{-1}]$ is dense in $\mathbb{Q}((\sigma))$. In particular, for any prime π of R, πR is dense in $\mathbb{Q}((\sigma))$, and so in R as well.

(iv) is clear, as πR is a proper $\mathbb{Q}$-vector subspace of R. $\square$

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