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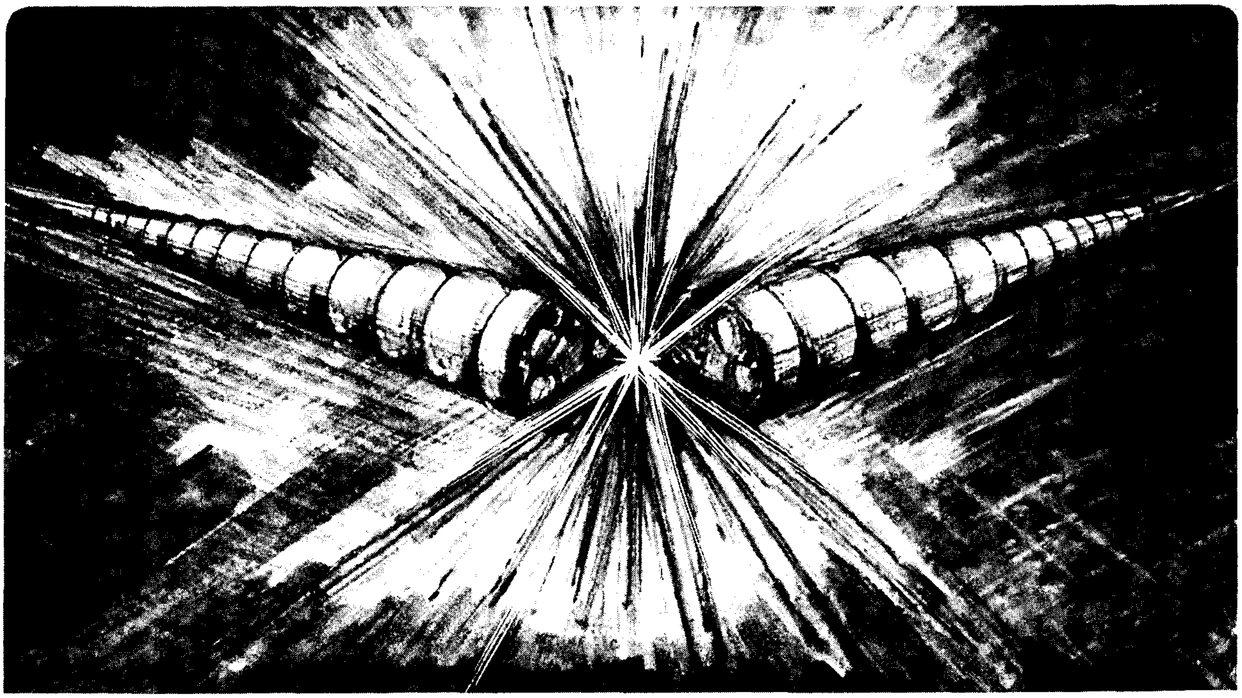

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I. Introduction

A version of the relaxation program POISSON has been produced that, for magnetostatic problems, can apply a boundary condition consistent with no external sources being present. This capability includes the treatment of axially-symmetric cases (using $A^*=\rho A$ as the working variable[†]) with a boundary whose form is that of a prolate spheroid (and hence tends toward spherical in the limit $\eta=a/\sqrt{a^2-b^2} \to \infty$). [S. Caspi, M. Helm, and L. J. Laslett, LBL-18798/UC-28 (December 1984)].

The treatment of electrostatic problems (to obtain solutions for the scalar potential V) necessarily must differ in detail from the treatment of magnetostatic problems in cases of axial symmetry. It seems desirable, therefore, first to review (§ III) the magnetostatic treatment that has been adopted for such axially-symmetric magnetostatic problems and then to suggest (§ IV) an analogous treatment that might similarly be introduced into the program to permit solution of similar electrostatic problems (again through the introduction of a prolate spheroidal boundary).

[†] The symbol A denotes the vector potential and ρ represents the radial cöordinate in a system of cylindrical cöordinates.

II. The Coördinate System

The system of prolate spheroidal coördinates to be employed is such that

$$\begin{aligned} x &= c \operatorname{Sinh} u \sin v \cos \phi = c\sqrt{\eta^2-1}\sqrt{1-\xi^2}\cos\phi \\ y &= c \operatorname{Sinh} u \sin v \sin \phi = c\sqrt{\eta^2-1}\sqrt{1-\xi^2}\sin\phi \\ z &= c \operatorname{Cosh} u \cos v = c\eta\xi \end{aligned}$$

and surfaces of constant η have semi-axes $a=c\eta$ and $b=c\sqrt{\eta^2-1}$, wherein $c=\sqrt{a^2-b^2}$ denotes the "focal distance" for the system of confocal ellipsoids. Thus, $\eta = \frac{a}{c} = \frac{a}{\sqrt{a^2-b^2}}$ and $u = \operatorname{Cosh}^{-1}\eta = \operatorname{Tanh}^{-1}(b/a)$.

III. The Axially-Symmetric Magnetostatic Case

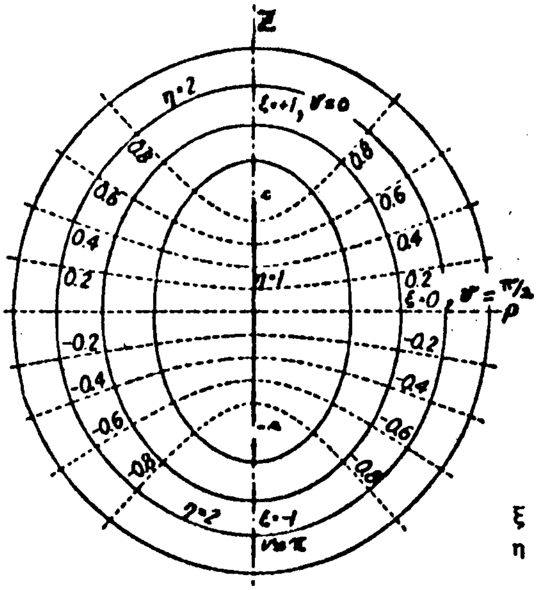
The scalar component A_ϕ of the vector potential in an axially-symmetric (no ϕ-dependence) magnetostatic problem must satisfy the differential equation

$$\sqrt{1-\xi^2}\,\frac{\partial^2}{\partial\xi^2}\left[\sqrt{1-\xi^2}\,A_\phi\right] + \sqrt{\eta^2-1}\,\frac{\partial^2}{\partial\eta^2}\left[\sqrt{\eta^2-1}\,A_\phi\right] = 0$$

in regions free of currents and magnetic material, and in a region external to all sources can be developed as a series of terms proportional to $P_n^1(\xi)$ $Q_n^1(\eta)$. For the working variable $A^*=\rho A_\phi$, one correspondingly may employ a series of terms of the form

$$\sqrt{1-\xi^2}\;P_n^1(\xi)\;\;\sqrt{\eta^2-1}\;\;Q_n^1(\eta)$$

$$= \left[\sqrt{1-\xi^2}\;\;P_n^1\,(\xi)\right] \cdot \left[\;(\eta^2-1)\,(-Q_n^{'}(\eta))\right] \;. \qquad *$$



ξ = cos v
η = Cosh u

Fig. 1

It is desirable to modify the character of such terms, through the introduction of factors that may be n-dependent, but are independent of ξ and η, so as finally to obtain forms that remain well behaved in limiting situations (such as $\eta \to \infty$) and are readily adaptable to the original operations of the program POISSON.

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*Note: Somewhat unconventionally with respect to sign, we here choose to consider $Q_n^1(\eta)$ to be defined as $\sqrt{\eta^2-1}\;[-Q_n^{'}(\eta)]$. With $Q_n^{'}(\eta) < 0$, we then conveniently have $Q_n^1(\eta) > 0$, and similarly for the functions $G_n(\eta)$ and $H_n(\eta)$ introduced on the following pages.

The form of the terms in a development of A* thus provisionally might be written

$$\left[\sqrt{1-\xi^2}\;\; P_n^1(\xi)\right]\;\frac{\eta^2-1}{\eta^{n+2}}\;G_n(\eta)\quad ,$$

wherein $$G_n(\eta) = \eta^{n+2}\left[-Q_n'(\eta)\right]$$

and for which, in particular,

$$G_0(\eta) = \eta^2\left[-Q_0'(\eta)\right] = \frac{\eta^2}{\eta^2-1} = \left(\frac{a}{b}\right)^2 .$$

It will be convenient also to introduce the n-dependent, but η-independent factor

$$G_n^\infty = \underset{\eta\to\infty}{\mathrm{L}}\quad G_n(\eta)\quad .$$

One notes that $G_0^\infty=1$. Also, from the recursion relation (Appendix to this section) $G_n^\infty = \frac{n+1}{2n+1}\;G_{n-1}^\infty$, it follows that

$$G_n^\infty = \frac{(n+1)\;n\;(n-1)\ldots}{(2n+1)(2n-1)(2n-3)\ldots}\;G_0^\infty$$

$$= \frac{(n+1)\;n\;(n-1)\ldots}{(2n+1)(2n-1)(2n-3)\ldots} = 2^n\;n!\;(n+1)!/(2n+1)!$$

One now introduces the function

$$H_n(\eta) = \frac{G_n(\eta)}{G_0(\eta)G_n^\infty}$$

and so finally, with a change only by solely n-dependent factors, the working variable A* may be developed as a series of terms proportional to

$$\frac{\sqrt{1-\xi^2}\; P_n^1(\xi)}{S(n)} \quad \eta^{-n}\; H_n(\eta)$$

$$= F_n(v) \;\bullet\; \eta^{-n}\; H_n(\eta)$$

where $$F_n(v) = \frac{\sqrt{1-\xi^2}\; P_n^1(\xi)}{S(n)}$$

$$= \frac{\sin v\; P_n^1(\cos v)}{S(n)} \quad . \qquad *$$

S(n) is the normalization factor (Appendix to § III)

$$S(n) = \left[\frac{n(n+1)}{n+\frac{1}{2}}\right]^{\frac{1}{2}} \quad .$$

We shall require subroutines able to generate values of the functions $H_n(\eta)$ at values of η associated with "inner" and "outer" boundary curves and to generate values of the functions $F_n(v)$ at the requisite v coördinates of mesh points on such boundary curves (see Appendix to this Section).

The working variable A* now may be regarded as developed on the inner boundary as

$$A^*\,(\eta_{\text{inner}},\, v) = \sum_j C_j\, F_j(v)$$

with the coefficients C_j to be considered as computable from values of A* at various locations (v_i) on the inner boundary. With such a development

---

* Recall cos v = ξ

obtainable (as shall be discussed in the following paragraph), the values of A* at points on the outer boundary then would be represented by

$$A^* (\eta_{outer}, v) = \sum_j f_j C_j F_j (v) ,$$

where

$$f_j = \frac{\eta_{outer}^{-j} H_j (\eta_{outer})}{\eta_{inner}^{-j} H_j (\eta_{inner})}$$

$$= \left(\frac{a_{inner}}{a_{outer}}\right)^j \frac{H_j (\eta_{outer})}{H_j (\eta_{inner})} .$$

The coefficients C_j, as given by a weighted least-squares procedure (e.g., with weights $W_i \propto \frac{\Delta v_i}{\Delta v_0} \frac{1}{\sin v_i}$), are such as to minimize

$$1/2 \sum_i W_i \left[\sum_j C_j F_j (v_i) - A^*(\text{inner}, v_i)\right]^2$$

and thus must satisy the simultaneous equations

$$\sum_i W_i F_\ell (v_i) \left[\sum_j C_j F_j (v_i) - A^*(\text{inner}, v_i)\right] = 0$$

or

$$\sum_{i,j} W_i F_\ell (v_i) F_j(v_i) C_j = \sum_i W_i F_\ell (v_i) A^*(\text{inner}, v_i) \quad ;$$

i.e., a set of equations equivalent to the matrix relation

$$\sum_j M_{\ell,j} C_j = V_\ell \quad ,$$

where

$$M_{\ell,j} = \sum_i W_i F_\ell (v_i) F_j (v_i)$$

and

$$V_\ell = \sum_i W_i \, F_\ell \, (v_i) \; A^*(\text{inner}, \, v_i) \quad .$$

One accordingly may write

$$C_j = \sum_\ell (M^{-1})_{j,\ell} \, V_\ell$$

$$= \sum_i W_i \sum_\ell (M^{-1})_{j,\ell} \, F_\ell \, (v_i) \; A^*(\text{inner}, \, v_i)$$

and <u>at</u> <u>points</u> $v=v_k$ <u>on</u> <u>the</u> <u>outer</u> boundary

$$A^*(n_{\text{outer}}, \, v_k) = \sum_j f_j \, C_j \, F_j$$

$$= \sum_i \left[W_i \sum_{j,\ell} f_j \, (M^{-1})_{j,\ell} \, F_\ell \, (v_i) \, F_j \, (v_k) \right] A^*(\text{inner}, \, v_i)$$

$$= \sum_i E_{k,i} \; A^*(\text{inner}, \, v_i) \quad ,$$

where $E_{k,i}$ is the matrix

$$E_{k,i} = \sum_\ell \sum_j f_j \, W_i \, (M^{-1})_{j,\ell} \, F_j \, (v_k) \, F_\ell \, (v_i)$$

that acts to revise A* at mesh points (v_k) on the outer boundary in terms of values at points (v_i) on the inner boundary.

APPENDIX TO SECTION III

The Generation of the required Functions $\frac{P_n^1(\xi)}{S(n)}$ and $H_n(\eta)$

1) Formation of the functions $F_n = \frac{\sin v \, P_n^1(\cos v)}{S(n)}$ requires evaluation of the functions $\frac{(1-\xi^2)^{1/2} \, P_n^1(\xi)}{S(n)}$ for $|\xi| \leq 1$. This may be done by iteratively executing, upward in n, the recursion relation for associated Legendre functions--adapted to contain the factor $\frac{1}{S(n)}$ where S(n) is the conventional normalization factor. Thus, having denoted by F_n the function, $\frac{(1-\xi^2)^{1/2} \, P_n^1(\xi)}{S(n)}$, the recursion relation

$$P_n^1(\xi) = \frac{(2n-1)\xi \; P_n^1(\xi) - n \; P_{n-2}^1(\xi)}{n-1}$$

leads to

$$F_n = \frac{(2n-1) \left[\frac{n(n-1)}{n-\frac{1}{2}} \right]^{1/2} \xi \; F_{n-1} - n \left[\frac{(n-1)(n-2)}{n-3/2} \right]^{1/2} F_{n-2}}{(n-1) \left[\frac{n(n+1)}{n+\frac{1}{2}} \right]^{1/2}} \; .$$

The use of the normalization factor S(n) has been introduced after noticing that the quality of the inversion of the matrix M has increased dramatically. Previously, as indicated in LBL-18063/SSC-MAG-12 (esp. p.14), we have used S(n)=n. We have since realized that using the orthogonality of Legendre polynomials in their non-approximate form yielded improved quality in the inversion of M. The orthogonality of the $P_n^1(\xi)$ is written as follows:

$$\int_{-1}^{1} P_n^1(\xi)\, P_m^1(\xi)\, d\xi = \frac{n(n+1)}{n+\frac{1}{2}}\, \delta_{n,m}$$

and the normalization factor is then defined as:

$$S(n) = \left[\frac{n(n+1)}{n+\frac{1}{2}}\right]^{\frac{1}{2}} .$$

We note that

$$\frac{P_1^1(\xi)}{S(1)} = \left(\frac{3}{4}\right)^{\frac{1}{2}} (1-\xi^2)^{\frac{1}{2}} , \qquad \frac{(1-\xi^2)^{\frac{1}{2}}\, P_1^1(\xi)}{S(1)} = \left(\frac{3}{4}\right)^{\frac{1}{2}} (1-\xi^2) ,$$

$$\frac{P_2^1(\xi)}{S(2)} = \left(\frac{15}{4}\right)^{\frac{1}{2}} (1-\xi^2)^{\frac{1}{2}}\xi , \qquad \frac{(1-\xi^2)^{\frac{1}{2}}\, P_2^1(\xi)}{S(2)} = \left(\frac{15}{4}\right)^{\frac{1}{2}} (1-\xi^2)\xi .$$

2) Formation of the functions $H_n(\eta)$ can be obtained by the iterative execution, downward in n, of a recursion relation derived by reference to the relation satisfied by $-Q_n^{'}(\eta)$, namely

$$-Q_n^{'}(\eta) = \frac{(2n+3)\ \eta[-Q_{n+1}^{'}(\eta)] - (n+1)[-Q_{n+2}^{'}(\eta)]}{n+2} .$$

For the functions $G_n(\eta)=\eta^{n+2}\ [-Q_n^{'}(\eta)]$, then,

$$G_n(\eta) = \frac{(2n+3)\ G_{n+1}(\eta) - (n+1)\frac{1}{\eta^2}\ G_{n+2}(\eta)}{n+2} .$$

The function $G_0(\eta)=\eta^2[-Q_0^{'}(\eta)] = \frac{\eta^2}{\eta^2-1}$ and in the limit $\eta \to \infty$ yields $G_0^\infty=1$. The recursion relation in this limit then provides

$$G_n^\infty = \frac{n+1}{2n+1}\, G_{n-1}^\infty$$

$$= \frac{(n+1)\ n\ (n-1)\ldots}{(2n+1)(2n-1)(2n-3)\ldots}\, G_0^\infty = \frac{2^n n!(n+1)!}{(2n+1)!}\, G_0^\infty$$

and since, as just noted, $G_0^\infty=1$,

$$G_n^\infty = \frac{2^n\, n!\, (n+1)!}{(2n+1)!} \quad .$$

For $H_n(\eta) = \dfrac{G_n(\eta)}{G_0^\infty(\eta)\; G_n^\infty}$, then, we finally obtain the recursion relation

$$H_n(\eta) = H_{n+1}(\eta) - \frac{(n+1)(n+3)}{(2n+3)(2n+5)}\; \frac{1}{\eta^2}\, H_{n+2}(\eta) \quad , \qquad *$$

wherein $\dfrac{1}{\eta^2} = (\frac{c}{a})^2 = \dfrac{a^2-b^2}{a^2} = 1-(\frac{b}{a})^2$. The iterative execution of this recursion relation should be launched at some maximum degree N_{max} that might be twice the highest degree for which the functions H_n will be required, using provisional starting values such as

$$H_N(\eta) \cong 2^{N+½}\, \frac{(\frac{b}{a})^{½}}{(1+\frac{b}{a})^{N+½}}$$

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* Note typographical error in Eqn.(15), p.21, of LBL-18798/UC-28 (December 1984), although the relation is correctly given as the final equation [Eqn.(48)] in Appendix B (p.29) of that report.

(by reference to a large-N asymptotic form for $Q^1_n(\eta)$ as noted in Appendix C of LBL-18798/UC-28)** and, correspondingly,

$$H_{N-1}(\eta) \cong \frac{1+\frac{b}{a}}{2} \; H_N(\eta) \qquad .$$

** The suggested large-n "asymptotic" formula for Q^1_n was written somewhat carelessly at the top of p.30 [Appendix C] of LBL-18798/UC-28 (December 1984). With the unusual sign convention adopted in that report, a more proper form is

$$Q^1_n(\eta{=}\mathrm{Cosh}\ u) \cong (n\pi)^{1/2} \, \frac{e^{-(n+1/2)u}}{(2\ \mathrm{Sinh}\ u)^{1/2}} \qquad .$$

The corresponding asymptotic form for

$$G_n(\eta) \equiv \eta^{n+2} \left[-Q'_n(\eta)\right] \equiv \frac{\eta^{n+2}}{\sqrt{\eta^2-1}} \, Q^1_n(\eta) \cong (n\pi)^{1/2} \, \frac{\eta^{n+2}}{(\eta^2-1)^{1/2}} \, \frac{e^{-(n+1/2)u}}{(2\ \mathrm{Sinh}\ u)^{1/2}}$$

then becomes as indicated below as a result of the following substitutions:

** $$\eta = \frac{a}{c} = \frac{a}{\sqrt{a^2-b^2}}$$

$$\sqrt{\eta^2-1} = \frac{b}{c} = \frac{b}{\sqrt{a^2-b^2}}$$

$$e^u = \eta + \sqrt{\eta^2-1} = \frac{a+b}{c} = \frac{a+b}{\sqrt{a^2-b^2}}$$

$$e^{-u} = \eta - \sqrt{\eta^2-1} = \frac{a-b}{c} = \frac{a-b}{\sqrt{a^2-b^2}}$$

and

$$2 \text{ Sinh } u = 2\sqrt{\eta^2-1} = 2\frac{b}{c} = \frac{2b}{\sqrt{a^2-b^2}} .$$

Completion of such an iterative procedure supplies provisional values of $H_n(\eta)$ that include values for functions of degree as low as 1 and 2. Such provisional values are then used to form a correction factor

$$SF = 1./[H_1(\eta) - 0.2\ (\frac{1}{\eta^2})\ H_2(\eta)]$$

The correct values of the functions $H_n(\eta)$ of all required degrees are then found by multiplication of the provisional values by the correction factor SF (thereby assuring achievement of a normalization such that $H_0(\eta) = H_1(\eta) - 0.2\ (\frac{1}{\eta^2})H_2(\eta)$ shall equal unity, as intended).

** $$G_n(\eta) \cong (n\pi)^{1/2}\ \frac{(\frac{a}{c})^{n+2}}{\frac{b}{c}}\ \frac{(\frac{a-b}{c})^{n+1/2}}{(\frac{2b}{c})^{1/2}} = \left(\frac{n\pi}{2}\right)^{1/2} \frac{a^{n+2}}{b^{3/2}} \left(\frac{a-b}{c^2}\right)^{n+1/2}$$

$$= \left(\frac{n\pi}{2}\right)^{1/2} \frac{a^{n+2}}{b^{3/2}(a+b)^{n+1/2}}$$

$$= \left(\frac{n\pi}{2}\right)^{1/2} \frac{1}{(\frac{b}{a})^{3/2}(1+\frac{b}{a})^{n+1/2}}\ ,$$

as given by Eqn.(2c) at the bottom of the same page of the cited report. The large-n approximation for $H_n(\eta)$ suggested on p.31 of that report then follows and evidently serves to provide satisfactory starting values for the recursion relation even if $\frac{b}{a}$ is small (i.e., η not much greater than unity).

IV. The Axially-Symmetric Electrostatic Case

The scalar potential function in an axially-symmetric electrostatic problem must satisfy the differential equation

$$\frac{\partial}{\partial\xi}\left[(1-\xi^2)\ \frac{\partial V}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(\eta^2-1)\ \frac{\partial V}{\partial\eta}\right] = 0$$

in regions free of charges and dielectric material, and in a region external to all sources can be developed as a series of terms proportional to $P_n(\xi)$ $Q_n(\eta)$-i.e., in terms of ordinary Legendre functions (of the first and second kinds). We may remark that, with the customary definition of these functions, $P_o(\xi)=1$ and $Q_o(\xi) = \frac{1}{2} \ell n \frac{\eta+1}{\eta-1}$ (for $\eta>1$), so that for n=o the product written above becomes $P_o(\xi)\ Q_o(\eta) = \frac{1}{2} \ell n \frac{\eta+1}{\eta-1}$ and for large η such a term has a form $\frac{1}{\eta}$ proportional in that limit to 1/r that is characteristic of a monopole potential.

[It may be informative to examine in a similar way the nature of terms $P_n(\xi)\ Q_n(\xi)$ for a few other values of the degree n, and likewise (for comparison or contrast) examine analogous forms applicable for development of A_ϕ or of $A^*=\rho A_\phi$--see the adjoining chart wherein the entry for n=1 in the penultimate column indicates for polar coödinates a form of A_ϕ proportional to $\sin\theta/r^2$, in accordance with the form expected for a magnetic dipole (small circular loop) and cited by Wm. R. Smythe, Eqn.(1) of Section 7.04 (Ed. 1, p.266).]

TABLE 1.

	ELECTROSTATIC	MAGNETOSTATIC	
n	$P_n(\xi)\ Q_n(\eta)$, for V	$P_n^1(\xi)\ Q_n^1(\eta)$, for A_ϕ	$\sqrt{1-\xi^2}\ P_n^1(\xi)\sqrt{\eta^2-1}\ Q_n^1(\eta)$, for $A^*=\rho A_\phi$
0	$\frac{1}{2}\ \ln\frac{\eta+1}{\eta-1} \longrightarrow \frac{1}{r}$ for r large	Not defined by above form	Not defined by above form
1	$\xi\left[\frac{\eta}{2}\ \ln\frac{\eta+1}{\eta-1} - 1\right] \longrightarrow \frac{\cos\theta}{3r^2}$ for r large	$\sqrt{1-\xi^2}\ \sqrt{\eta^2-1}\ \left[\frac{1}{2}\ \ln\frac{\eta+1}{\eta-1} - \frac{\eta}{\eta^2-1}\right]$ $\longrightarrow -\frac{2}{3}\ \frac{\sin\theta}{r^2}$ for r large	$(1-\xi^2)(\eta^2-1)\ \left[\frac{1}{2}\ \ln\frac{\eta+1}{\eta-1} - \frac{\eta}{\eta^2-1}\right]$ $\longrightarrow -\frac{2}{3}\ \frac{\sin^2\theta}{r}$ for r large
2	$\frac{3\xi^2-1}{2}\left[\frac{3\eta^2-1}{4}\ \ln\frac{\eta+1}{\eta-1} - \frac{3}{2}\ \eta\right]$ $\longrightarrow \frac{2}{15}\ P_2(\cos\theta)/r^3$ for r large $= \frac{3\cos^2\theta-1}{15\ r^3}$	$3\xi\ \sqrt{1-\xi^2}\ \sqrt{\eta^2-1}\ \left[\frac{3}{2}\ \eta\ \ln\frac{\eta+1}{\eta-1} - \frac{3\eta^2-2}{\eta^2-1}\right]$ $\longrightarrow -\frac{6}{5}\ \frac{\sin\theta\cos\theta}{r^3}$ for r large	$3\xi\ (1-\xi^2)\ (\eta^2-1)\left[\frac{3}{2}\ \eta\ \ln\frac{\eta+1}{\eta-1} - \frac{3\eta^2-2}{\eta^2-1}\right]$ $\longrightarrow -\frac{6}{5}\ \frac{\sin^2\theta\cos\theta}{r^2}$ for r large

Note that in LBL-18798/UC-28 we elected to make the identification
$Q_n^1(\eta) = -(\eta^2-1)^{1/2}\ Q_n'(\eta)$ rather than the more usual convention
$Q_n^1(\eta) = (\eta^2-1)^{1/2}\ Q_n'(\eta)$ [Abramowitz & Stegun] adopted on this chart.

ETC.

The following comments refer to Table 1.:

(i) Note the asymptotic (large-η) forms

$$Q_n(\eta) \cong \frac{2^n\,(n!)^2}{(2n+1)!\;\eta^{n+1}}\;, \qquad Q_n^1(\eta) \cong -\frac{2^n\,n!\,(n+1)!}{(2n+1)!\;\eta^{n+1}}\;.$$

(ii) Note that the potential of a linear electrostatic 2^n pole is given in polar coördinates by $V = (\partial^n/\partial z^n)\,(1/r) = (-1)^n\,n!\,P_n(\cos\theta)/r^{n+1}$ [see Am. J. Phys. 26 (#6), 402 (1958)], thus being proportional to $(1/r^{n+1})$ $P_n(\cos\theta)$ in equivalence to proportionality to $\partial^n(1/r)/\partial z^n$ (holding ρ constant).

(iii) An analogous form for A*, based on proportionality to $\sin\theta$ $P_n^1(\cos\theta)/r^n$, may be expressed by proportionality to $(n+1)z\,\partial^n(1/r)/\partial z^n$ $+\,r^2\partial^{n+1}(1/r)/\partial z^{n+1}$, in which (as before) $r = \sqrt{z^2+\rho^2}$ and $z = r\cos\theta$.

It will be recalled that in the analysis of magnetostatic problems the values for the degree of the (associated) Legendre functions commenced with n=1. We have noted that for the electrostatic problems, however, we would wish also to include the degree n=0 if we wish to be able to represent a monopole contribution to the potential function (by means of the ordinary Legendre functions P_n and Q_n). It will be recognized that the presence of a net charge within the region of interest, requiring the presence of a monopole term in the potential, implies, in a sense, the presence of an equal charge of opposite sign externally (e.g., "at infinity"), and so leads to a situation that cannot be said to be strictly free of all external "sources." It nonetheless may be desirable to permit the inclusion of a monopole term (in association with n=0) in programs intended for the solution of electrostatic problems, in order to permit the solution of problems in which a net charge is present within the region of interest.

The type of terms that we have discussed, namely of the form $P_n(\xi)\ Q_n(\eta)$ with $n \geq 0$, do <u>not</u> provide for the presence of a <u>constant</u> term* in the development of the electrostatic potential. In the absence of any special provision for such a constant term, its omission will require that the specification of potential values at specific locations or on specific surfaces shall in no way be inconsistent with the potential function approaching zero at infinity.

Finally, it in any case will be recognized that, if the character of the given problem is such that there is <u>antisymmetry</u> about the equatorial plane (V <u>odd</u> with respect to the variable ξ), then only <u>odd</u> values of n need be employed in terms of the form $P_n(\xi)\ Q_n(\eta)$ noted above, while if, on the other hand, the problem is <u>symmetric</u> (V <u>even</u> with respect to ξ) only such terms with n <u>even</u> need be included (but not overlooking a term with $n=0$, if required).

In a development of the scalar potential for the axially-symmetric electrostatic problem, a sequence of terms of the form $P_n(\xi)\ Q_n(\eta)$ may conveniently be replaced by terms

$$F_m(v)\ \frac{H_m(\eta)}{\eta^m} \qquad [\text{with } m=1,2,3,\ldots],$$

wherein

$$F_m(v) = \frac{P_{m-1}(\cos v)}{S(m-1)} = \frac{P_{m-1}(\xi)}{S(m-1)}$$

and

$$H_m(\eta) = \frac{(2m-1)!}{2^{m-1}\ [(m-1)!]^2}\ \eta^m\ Q_{m-1}(\eta) \quad ,$$

$S(m-1)$ being the normalization factor (Appendix to Section IV)

$$S(m-1) = \left[\frac{1}{m-\frac{1}{2}}\right]^{\frac{1}{2}}.$$

---

*i.e., η-independent, as well as ξ-independent.

Properties of these functions, and recursion relations suitable for their formation are presented in the Appendix to this Section. The fact that $\lim_{\eta\to\infty} [H_m(\eta)] = 1$ (for all m) results in the terms $F_m(v)\ \frac{H_m(\eta)}{\eta^m}$ approaching proportionality to $P_{m-1}(\cos\theta)/r^m$ at great distances, and inclusion of a term with m=1 thus permits recognition of a monopole contribution (from an uncancelled charge) to the potential. [For problems with antisymmetry about the equatorial plane (V odd with respect to ξ), we then need use only even values of m (m=2,4,6...--corresponding to n=m-1 odd), and for problems that are symmetric about the equatorial plane only odd values of m need to be employed (and including m=1, if an uncancelled charge is present to give rise to a monopole component).]

Such a series development of the potential at one value of η (denoted η_{inner}) may then (in a source-free region) be transformed to a development at a different value of η (η_{outer}) simply through multiplication of the respective terms by the ratio

$$f_m = \frac{\eta_{outer}^{-m}\ H_m(\eta_{outer})}{\eta_{inner}^{-m}\ H_m(\eta_{inner})} = \left(\frac{a_{inner}}{a_{outer}}\right)^m \ \frac{H_m(\eta_{outer})}{H_m(\eta_{inner})}$$

--in analogy to the procedure followed in the corresponding magnetostatic case (cf. p.6).

APPENDIX TO SECTION IV

The Generation of the Required Functions $F_m(v)$ and $H_m(n)$

1) Since normalization is to be considered desirable, the function $F_m(v)$ perhaps most simply may be taken satisfactorily to be

$$F_m(v) = \frac{P_{m-1}(\cos v)}{S(m-1)} = \frac{P_{m-1}(\xi)}{S(m-1)} \quad ; \qquad m=1,2,...$$

(p.17), with "normalization". The recursion relation for the Legendre polynominal $P_n(\xi)$ is

$$P_{n+2}(\xi) = \frac{(2n+3)\xi\ P_{n+1}(\xi) - (n+1)\ P_n(\xi)}{n+2} \quad ; \qquad n=0,1,2,...$$

so that the corresponding recursion relation for the functions $F_m(v)$ becomes suitably revised

$$F_m(v) = \frac{(2m-3)\left[\frac{1}{m-3/2}\right]^{1/2} \xi\ F_{m-1}(\xi) - (m-2)\left[\frac{1}{m-5/2}\right]^{1/2} F_{m-2}(\xi)}{(m-1)\left[\frac{1}{m-1/2}\right]^{1/2}} \quad ; \ m=3,4,...$$

Note that the normalization factor S(m) is derived from the orthogonality relation

$$\int_{-1}^{1} \left[P_{m-1}(\xi)\right]^2 d\xi = \frac{1}{m-1/2} \quad ; \qquad m=1,2,...$$

$$S(m-1) = \left[\frac{1}{m-1/2}\right]^{1/2} .$$

With normalization, the functions are as follows for m=1,2, and 3:

$$F_1(v) = (\frac{1}{2})^{1/2} \quad P_0(\xi) = (\frac{1}{2})^{1/2}$$

$$F_2(v) = (\frac{3}{2})^{1/2} \quad P_1(\xi) = (\frac{3}{2})^{1/2} \xi$$

$$F_3(v) = (\frac{5}{2})^{1/2} \quad P_2(\xi) = (\frac{5}{2})^{1/2} \frac{3\xi^2-1}{2} ,$$

of which the first two can well serve to launch application of the recursion relation for generating the additional functions F_m that are required. [Recall cos $v=\xi$.]

2) The function $H_m(\eta)$ has been defined (p.17) as

$$H_m(\eta) = \frac{(2m-1)!}{2^{m-1} [(m-1)!]^2} \eta^m Q_{m-1}(\eta) \quad ,$$

wherein, in light of the asymptotic behavior of $Q_{m-1}(\eta)$, the normalization factor has been so chosen that

$$\lim_{\eta\to\infty} [H_m(\eta)] = 1 \qquad \text{(for all m)} \quad .$$

The recursion relation for $Q_n(\eta)$, written in a downward direction with respect to degree, is

$$Q_n(\eta) = \frac{(2n+3)\eta\ Q_{n+1}(\eta) - (n+2)\ Q_{n+2}(\eta)}{n+1} \quad .$$

With the functions $H_m(\eta)$ defined as above, it then follows that the corresponding recursion relation for such functions becomes

$$H_m(\eta) = H_{m+1}(\eta) - \frac{(m+1)^2}{(2m+1)(2m+3)} \frac{1}{\eta^2} H_{m+2}(\eta) \quad .$$

{It will be recalled (see p.16) that the desire to be able to accommodate the presence of a monopole term in the development of the electrostatic scalar potential motivates the suggestion that the function $H_1(\eta)$ [$=\eta\ Q_0(\eta)$] be available.}

The functions $H_m(\eta)$, as defined, are as follows for m=1,2, and 3:

$$H_1(\eta) = \eta\, Q_0(\eta) = \frac{\eta}{2} \ln \frac{\eta+1}{\eta-1}$$
$$H_2(\eta) = 3\eta^2\, Q_1(\eta) = \frac{3}{2}\eta^3 \ln \frac{\eta+1}{\eta-1} - 3\eta^2$$
$$H_3(\eta) = \frac{15}{2}\eta^3\, Q_2(\eta) = \frac{15}{8}(3\eta^5-\eta^3) \ln \frac{\eta+1}{\eta-1} - \frac{45}{4}\eta^4 \quad .$$

It can be verified that each of these functions approaches unity as $\eta\to\infty$, as we have noted is to be expected for all the functions $H_m(\eta)$. One also can verify (simply as a check) that these functions indeed satisfy the recursion relation written above. The expression, if written by way of curiosity for m=0, moreover, is found by substitution of the above forms for $H_1(\eta)$ and for $H_2(\eta)$ to indicate the identity

$$H_1(\eta) - \frac{1}{3\eta^2} H_2(\eta) = 1 \qquad \text{(for any } \eta) \quad .$$

One will wish to use the recursion relation, prepared for generation of functions $H_m(\eta)$, by commencing with convenient values for functions of high degree (m=Mmax and Mmax-1). The provisional values then so generated would next be normalized, by a common normalization factor, by taking note of the requirement

$$H_1(\eta) = \frac{\eta}{2} \ln \frac{\eta+1}{\eta-1}$$

or

$$H_1(\eta) - \frac{1}{3\eta^2} H_2(\eta) = 1 \quad .$$

Finally, we close by turning to suggestions for consistent possible starting values (no more than convenient large-m approximations) for H_{Mmax} (and for $H_{\text{Mmax}-1}$).

As a possible large-n approximation for the Legendre function of the second kind, one may suggest

$$Q_n(\eta = \text{Cosh } u) \cong \sqrt{\frac{\pi}{n}} \quad \frac{e^{-(n+½)u}}{\sqrt{2 \text{ Sinh } u}}$$

[i.e., substantially $\frac{1}{n}$ times a similar form suggested previously (p.11) for the associated function Q_n^1].

One makes use of the relations

$$\eta = \frac{a}{c} = \frac{a}{\sqrt{a^2-b^2}}$$

$$\sqrt{1-\eta^2} = \frac{b}{c} = \frac{b}{\sqrt{a^2-b^2}}$$

$$e^u = \eta + \sqrt{\eta^2-1} = \frac{a+b}{c} = \frac{a+b}{\sqrt{a^2-b^2}}$$

$$e^{-u} = \eta - \sqrt{\eta^2-1} = \frac{a-b}{c} = \frac{a-b}{\sqrt{a^2-b^2}}$$

$$2 \text{ Sinh } u = 2\sqrt{\eta^2-1} = 2\frac{b}{c} = \frac{2b}{\sqrt{a^2-b^2}}$$

$$k! \cong \sqrt{2\pi k}\,(k/e)^k \qquad \text{[Stirling]}$$

and

$$\lim_{p\to\infty} (1+\frac{z}{p})^p = e^z$$

to obtain the suggested <u>large-m</u> form

$$H_m(\eta) \overset{\text{Def.}}{\equiv} \frac{(2m-1)!}{2^{m-1}\,[(m-1)!]^2}\,\eta^m\, Q_{m-1}(\eta)$$

$$\cong 2^{m-½} \frac{1}{(\frac{b}{a})^{½}\,(1+\frac{b}{a})^{m-½}} \quad .$$

Thus, with a suitably large assignment for Mmax, reasonable starting values for the recursion relation may be suggested as

$$H_{M\max}(\eta) \doteq \frac{2^{M\max-\frac{1}{2}}}{(\frac{b}{a})^{\frac{1}{2}}\ (1+\frac{b}{a})^{M\max-\frac{1}{2}}}$$

$$H_{M\max-1}(\eta) \doteq \frac{1+\frac{b}{a}}{2} \cdot H_{M\max}(\eta) \quad .$$

Following generation of functions of lesser degree through repetitive use of the recursion relation, one then forms a correction factor -- such as

$$CF = \frac{\frac{\eta}{2}\ \ell n\ \frac{\eta+1}{\eta-1}}{H_1(\eta)}$$

or

$$CF = \frac{1}{H_1(\eta) - \frac{1}{3\eta^2} H_2(\eta)} \quad ,$$

by which all the required functions so generated are renormalized (through multiplication by CF).

A few numerical "spot checks" may be of interest with respect to the suggested "large-m" approximation

$$H_m \cong 2^{m-\frac{1}{2}} \frac{1}{(\frac{b}{a})^{\frac{1}{2}}\ (1+\frac{b}{a})^{m-\frac{1}{2}}} \quad .$$

Tabular values of $Q_n(\eta)$, for various values of the argument η are available, for example, from the W.P.A. Tables [NBS "Tables of Assoc. Legendre Functions, Columbia Univ. Press, NY, 1945]* and the corresponding values of $H_m(\eta)$, with n=m-1, may be computed therefrom. Thus, for $H_6(\eta)$ and $H_{11}(\eta)$, we obtain

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*Library Call Number QA406 M37.

Table 2. The tabulated values of $\frac{b}{a}$ (=Tanh u) are related to the entry values of η by $\frac{b}{a} = \sqrt{\eta^2-1}/\eta$. Such values are then used in the suggested "large-m" formula to form the entries for $H_{6,\text{formula}}$ and $H_{11,\text{formula}}$. Such results appear to be in rather good agreement with the "true" values that have been computed directly from published values of Q_5 and Q_{10}.

Table 2.

	n=5	m=6 H_6	m-½=5.5	n-10	m=11 H_{11}	m-½=10.5	
η	Q_5	True	Formula	Q_{10}	True	Formula	$\frac{b}{a}$
1.1	6.56414×10^{-2}	10.0734	10.3	5.28143×10^{-3}	57.0938	57.9	0.416597791
1.2	2.06130×10^{-2}	5.3318	5.41	6.75615×10^{-4}	19.0200	19.18	0.552770798
1.3	8.84960×10^{-3}	3.7002	3.74	1.48105×10^{-4}	10.0569	10.11	0.638971066
1.4	4.44631×10^{-3}	2.9001	2.92	4.27633×10^{-5}	6.5614	6.59	0.699854212
2.0	2.82977×10^{-4}	1.5688	1.573	2.86313×10^{-7}	2.2217	2.225	0.866025404
5.0	7.88950×10^{-7}	1.0679^{-}	1.0683	6.07362×10^{-12}	1.1237^{-}	1.1239	0.979795897
10.0	1.17328×10^{-8}	1.0164^{-}	1.0164^{+}	2.71639×10^{-15}	1.0292	1.0293	0.994987437

APPENDIX -- Examples

We next present several examples, to illustrate and check the computational procedures presented in this report. The cases treated are the following:

I. Magnetic, with axial symmetry:

A. Magnetically permeable ellipsoid of revolution, situated in an external applied magnetic field parallel to the axis of revolution (μ_r=250).

II. Electrostatic, with axial symmetry:

A. Electrically susceptible ellipsoid of revolution, situated in an external applied electric field parallel to the axis of revolution (ϵ_r=10).

B. Two conducting spheres, intersecting at 90 degrees, raised to a potential V_0.

C. A pair of conducting spheres of identical radii raised to potentials V_1 and V_2, with the axis of rotational symmetry lying on the line connecting their centers.

1. V2 = V1 (for even solution);
2. V2 = -V1 (for odd solution).

III. Comparison of Examples IA and IIA, with $\epsilon_r = \mu_r = 10$.

IV. Cartesian 2-D Electrostatic:

A. Two identical parallel conducting circular cylinders, at potentials $\pm V_0$.

IA. Magnetically permeable ellipsoid of revolution:

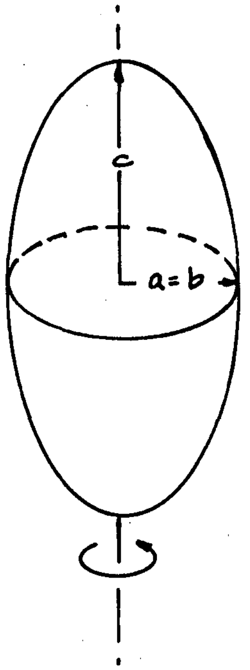
The case of a permeable ellipsoid, of constant μ_r, situated in an external applied field has been discussed in several texts [e.g., for the analogous electrostatic case see J. A. Stratton, "Electromagnetic Theory," Sect. 3.27, McGraw-Hill Book Co., Inc. (1941)]. For an external flux density that is uniform at a large distance from the ellipsoid the internal flux density also will be uniform, and will be parallel to the externally applied field if the latter is parallel to one of the principal axes of the ellipsoid. For an ellipsoid of revolution (semi-axes: a=b,c) and an applied flux density B_0 parallel to the axis of revolution, the internal flux density is given by

$$B_i = \frac{\mu_r B_0}{1+D_Z(\mu_r-1)} ,$$

where, in terms of the ratio $\kappa = c/a$, the coefficient D_Z is given by

$$D_Z = \frac{\kappa}{2} \int_0^\infty \frac{du}{(u+1)(u+\kappa^2)^{3/2}}$$

$$= \begin{cases} \dfrac{1}{1-\kappa^2} - \dfrac{\kappa}{(1-\kappa^2)^{3/2}} \cos^{-1} \kappa & \text{for } \kappa < 1 \\[2ex] \dfrac{1}{3} \quad \text{for } \kappa = 1 \\[2ex] \dfrac{\kappa \operatorname{Cosh}^{-1} \kappa}{(\kappa^2-1)^{3/2}} - \dfrac{1}{\kappa^2-1} & \text{for } \kappa > 1 . \end{cases}$$



In the solution of this axi-symmetric magnetic problem by means of POISSON we may elect to use a prolate spheroidal boundary (not necessarily confocal with the magnetic ellipsoid) and, in prolate spheroidal coördinates, represent the applied field by

$$A^*_{\mathrm{appl.}} = \rho\, A_\phi = \frac{\rho^2}{2}\, B_0$$

$$= \frac{c_0^2}{2}\, (\eta^2-1)(1-\xi^2)\, B_0$$

$$= \frac{c_0^2}{2}\, (\eta^2-1)\, \sin^2 v\;\; B_0 \quad .$$

Such an expression for $A^*_{\mathrm{appl.}}$ is employed in the computations that relate the potential on the outer boundary to that found on the nearby inner boundary [in a manner analogous to that outlined previously in Caspi <u>et</u> <u>al</u>., LBL-19050/ SSC-MAG-31 (January, 1985) for 2-D Cartesian problems]. The symmetry of this problem clearly is such that it is sufficient to recognize that A^* should be even with respect to the mid-plane $\xi=0$ and to seek such a solution solely in the region above this plane.

For $\kappa = c/a = 5/3$, we expect (from the equations cited) that

$$D_Z = \frac{45}{64}\, \ell n 3 - \frac{9}{16} \cong 0.2099618$$

and, with $\mu_r = 250$,

$$B_i/B_0 \cong 4.69215$$

for comparison with the ratio

$$B_i/B_0 = 4.6913$$

found through use of POISSON.

It should be noted that if POISSON is employed to plot lines of constant A^* in an axially symmetric problem such as that of concern here, the curves plotted are sections of <u>constant-flux</u> <u>surfaces</u> and equal intervals of A^* will not result in such lines being equally spaced in regions of constant field.

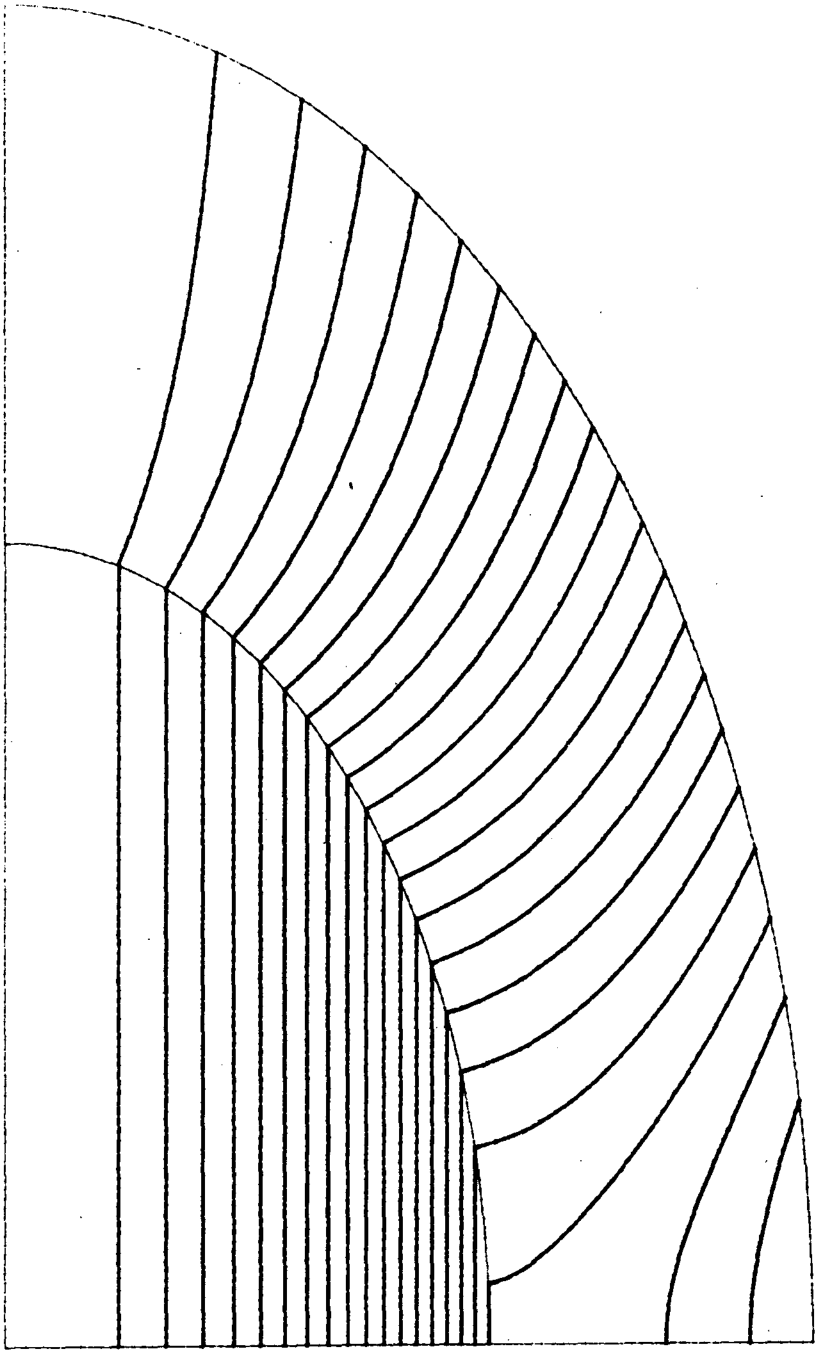


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Fig. A1. Ellipsoid of revolution in external magnetic field. Surfaces A^* = constant for $\mu_r = 250$.

IIA. Electrically susceptible ellipsoid of revolution:

For an electrostatic problem analogous to the example of the preceding section (§IA), one expects a uniform internal field given by $E_i = \frac{E_o}{1+D_Z(\epsilon_r-1)}$, with D_Z given as before in terms of the ratio $\kappa = c/a$. An electrostatically oriented POISSON solution of this problem employs a scalar potential function and the potential describing the externally applied field is written

$$V_{appl.} = - E_oZ = - E_o \, r \cos \theta = - E_oc_o \, \eta\xi.$$

One may then seek a solution for which V is odd with respect to the mid-plane $\xi = 0$.

For $\kappa = c/a = 5/3$, we again expect

$$D_Z = \frac{45}{64} \ln 3 - \frac{9}{16} \cong 0.2099618$$

and, with $\epsilon_r = 10$,

$$E_i/E_o \cong 0.34606$$

for comparison with the ratio

$$E_i/E_o = 0.346$$

found through use of POISSON.

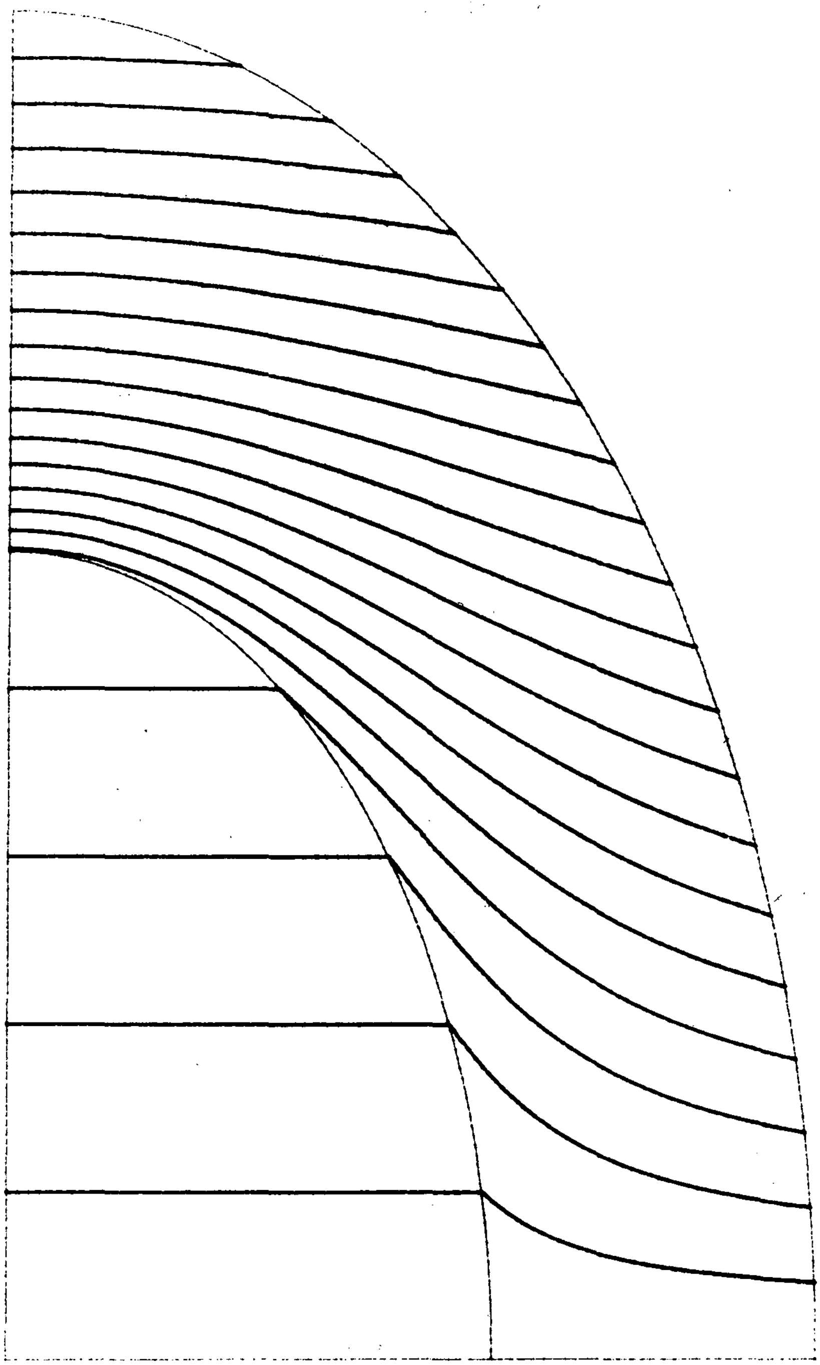

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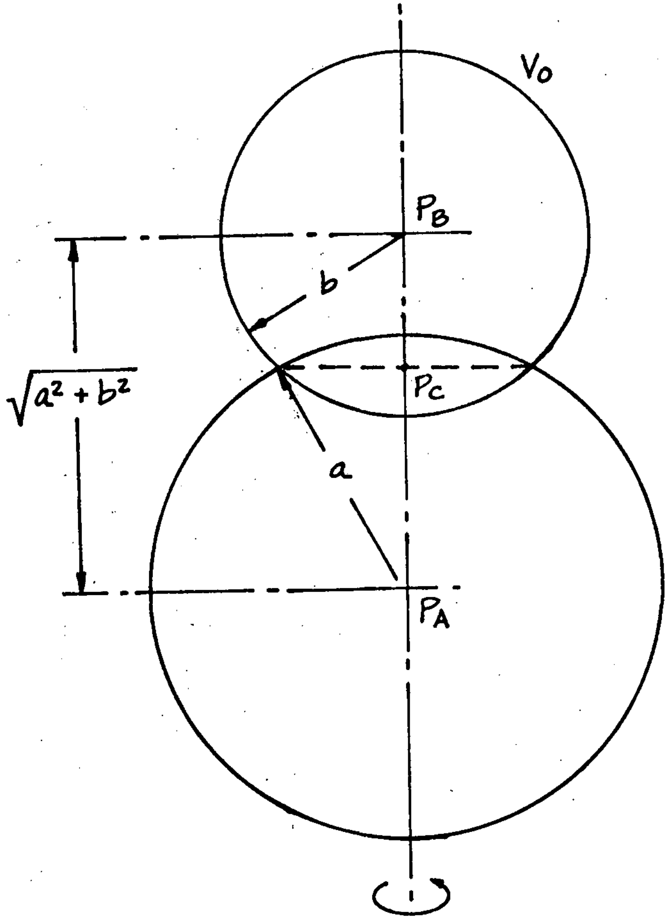
Fig. A2. Ellipsoid of revolution in external electric field. Surfaces V = constant for $\epsilon_r = 10$.

IIB. Two conducting spheres, intersecting at 90 degrees:

The electrostatic problem illustrated here is sometimes chosen to provide a text-book illustration of the technique of inversion in three dimensions [see, for example, W. R. Smythe, "Static and Dynamic Electricity," Sects. 5.09-5.103, McGraw-Hill Book Co., Inc.(1939)], but, as noted by Smythe (l. c.), the result is such that it can be described simply in terms of three suitably located image charges --viz.:



Charge at P_A: $(4\pi\epsilon_0)\, V_0 a$

Charge at P_B: $(4\pi\epsilon_0)\, V_0 b$

Charge at P_C: $-\,(4\pi\epsilon_0)\, V_0\ ab/\sqrt{a^2+b^2}$,

wherein the factors $4\pi\epsilon_0$ should be removed if one prefers to employ unrationalized cgs units.

With the radii a and b in the ratio

$$a:b = 4:3,$$

$V_0 = 10$, and the origin situated midway between the extremities of the figure, one can compute the potential at various external points for comparison with the results obtained by POISSON (using a surrounding prolate spheroidal boundary and the condition that no sources are present exterior to that boundary). See Table A1.

Table A1.

ρ	Z	V_{calc}	$V_{Poisson}$	$\frac{\Delta V}{V}$ %
0	-8.775	6.045815	6.045708	0.0017
0	-7.37864	7.529730	7.527041	0.036
0	-6.44660	9.032733	9.025843	0.07
5.07731	-1.94952	8.09189	8.09215	-0.0032
6.79910	-2.0872	6.237603	6.237634	-0.0005
6.9404	1.1423	5.967936	5.967961	-0.0004

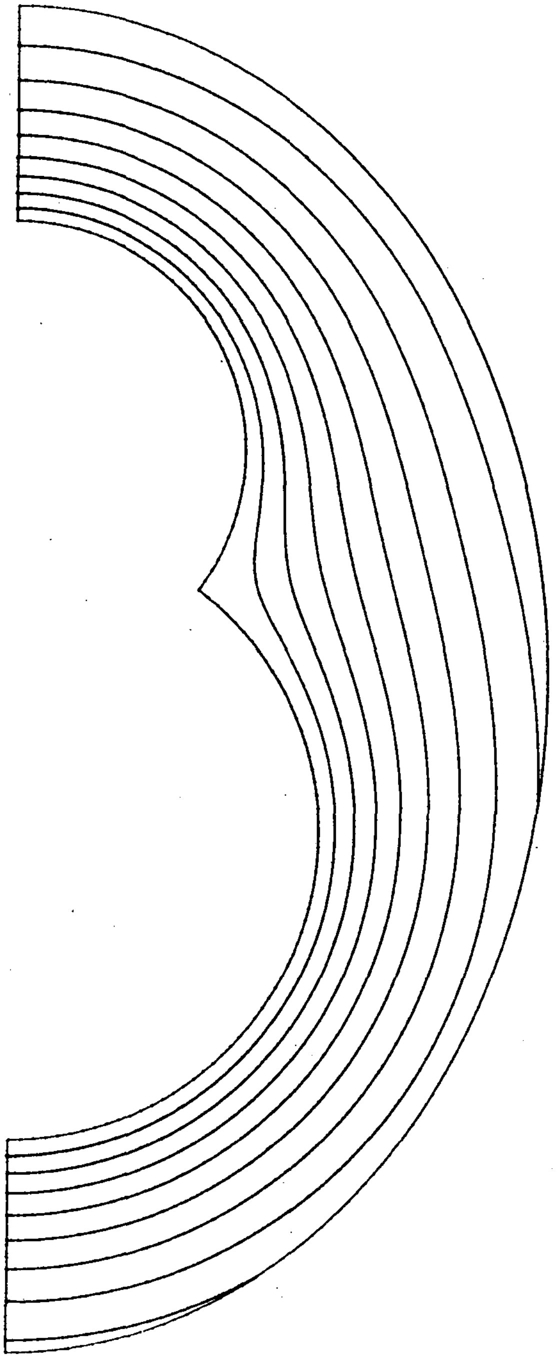


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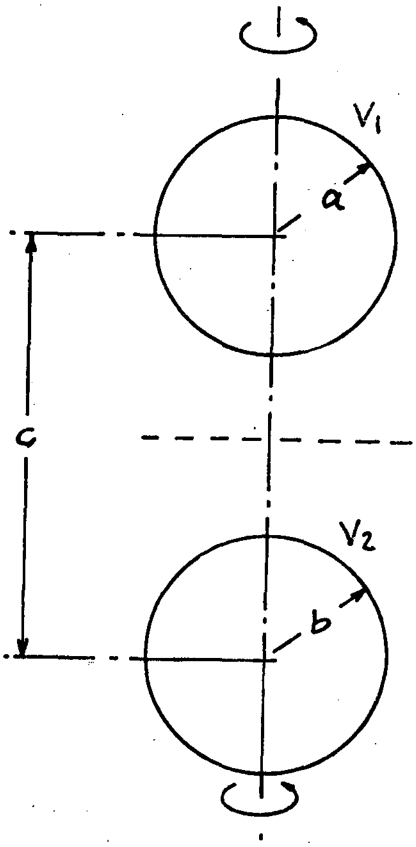
Fig. A3. Two conducting spheres, intersecting at 90 degrees and raised to a common potential. Surfaces V = constant.

IIC. A pair of conducting spheres of identical radii:

The analytic solution for the potential surrounding a pair of separated conducting spheres is frequently expressed in terms of the image potential of an infinite sequence of image charges and the results are summarized in some detail in various texts [e.g., Sir James Jeans, "The mathematical Theory of Electricity and Magnetism," Chapt. VIII, Sects. 221 ff, Cambridge Univ. Press (Ed. 5, 1948)].



Such a method of solution can be readily adopted to serve as the basis of a VAX program (TWOSP) for computing through numerical evaluations the potential external to a pair of separated spheres raised to specified potentials V1 and V2. Such a program has been constructed to provide checks of POISSON runs in which the radii of the spheres are identical, but the potentials (V1 and V2) may be such as to provide (i) symmetric solutions (V2=V1), (ii) antisymmetric solutions (V2=-V1), or, if desired, (iii) general solutions that lack symmetry or antisymmetry. The results of such comparative checks (with a relatively coarse mesh) for symmetric and antisymmetric cases are summarized in Tables A2 and A3 (a = b = 1.0, c = 4.0, |V| = 10).

Table A2.

ρ	Z	V_{calc}	$V_{Poisson}$	$\frac{\Delta V}{V}$ %
0	0	7.749476	7.772889	-0.3
1.0	0	6.989045	7.030309	-0.6
1.06175	2.00487	9.51679	9.521685	-0.05
1.46197	2.04701	7.35170	7.362267	-0.14
2.75964	2.04877	4.598179	4.532515	+1.4

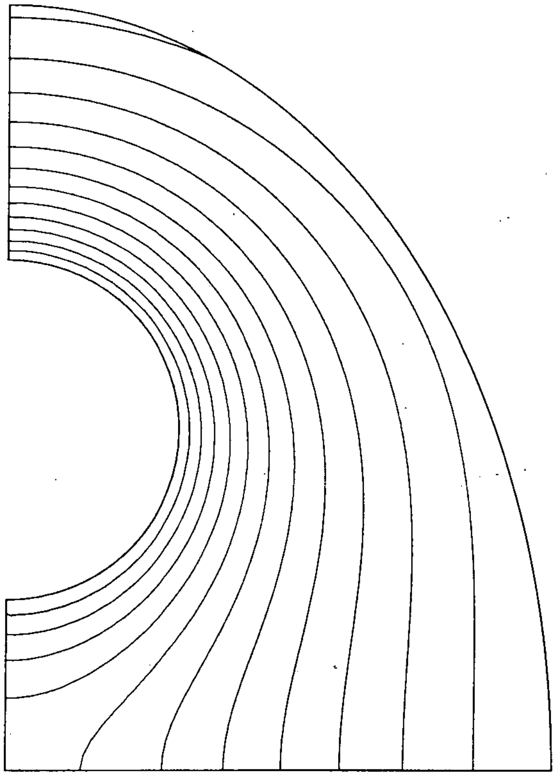


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Fig. A4. Region surrounding one of two identical charged spheres, with V1 = V2 = 10. Surfaces V = constant.

Table A3.

ρ	Z	V_{calc}	$V_{Poisson}$	$\frac{\Delta V}{V}$ %
0	0.08696	0.630816	0.6296344	+0.19
0	0.52174	4.08203	4.076604	+0.13
1.06175	2.00487	9.250107	9.261217	-0.12
1.26144	2.02583	7.337098	7.362652	-0.34
1.83801	2.06608	4.218095	4.285066	-1.58

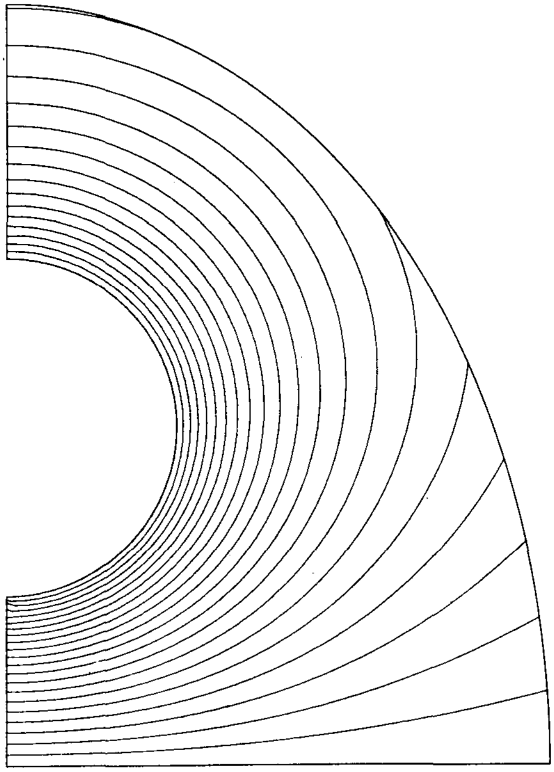

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Fig. A5. Region surrounding one of two identical charged spheres, with V1 = -V2 = 10. Surfaces V = constant.

III. Comparison of examples IA and IIA, with $\epsilon_r = \mu_r = 10$:

The reader will recognize that the examples of a magnetically or electrically permeable ellipsoid immersed in an externally applied field constitute basically the same problem -- although in the one case the problem is solved through the use of a potential $A^* = \rho A_\phi$ related to a magnetic vector potential A_ϕ and in the other through the use of a scalar electostatic potential function V. The solutions obtained accordingly may be expected to be identical in such cases provided μ_r and ϵ_r have identical values, with

$$B_i = \frac{\mu_r B_o}{1+D_Z(\mu_r-1)}$$

and

$$E_i = \frac{E_o}{1+D_Z(\epsilon_r-1)} \quad \text{or} \quad D_i = \frac{\epsilon_r D_0}{1+D_Z(\epsilon_r-1)}$$

in these respective cases.

In the magnetostatic and electrostatic solutions, plots of lines of constant A^* or plots of constant V will indicate respectively the direction of flux lines or a direction in equipotential surfaces orthogonal to flux lines. The curves resulting from such plots accordingly should be mutually orthogonal, provided the values of μ_r and ϵ_r are identical, and such a situation is illustrated by Fig. A6 in which we have superposed plots of this nature for $\mu_r = \epsilon_r = 10$. In this particular example the value of E is sufficiently small within the permeable ellipsoid that one fails to exhibit many equipotential surfaces within that volume. The curves A^* = constant and V = constant external to this ellipsoid do appear to be (as expected) mutually orthogonal. [One should recall that, as noted earlier, curves of constant A^* represent curves of constant enclosed flux and so, in cases of rotational symmetry, are not equally spaced when the flux density is constant.]

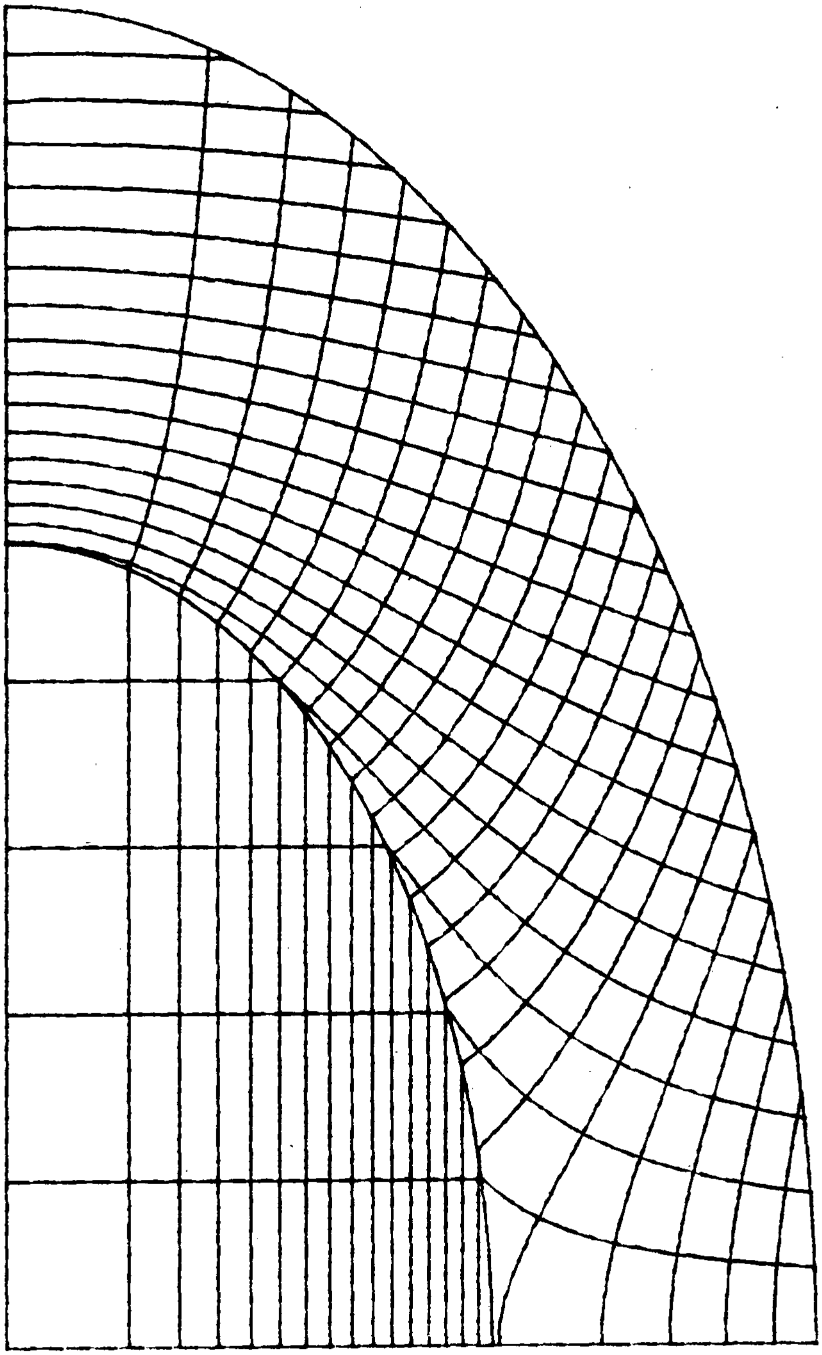

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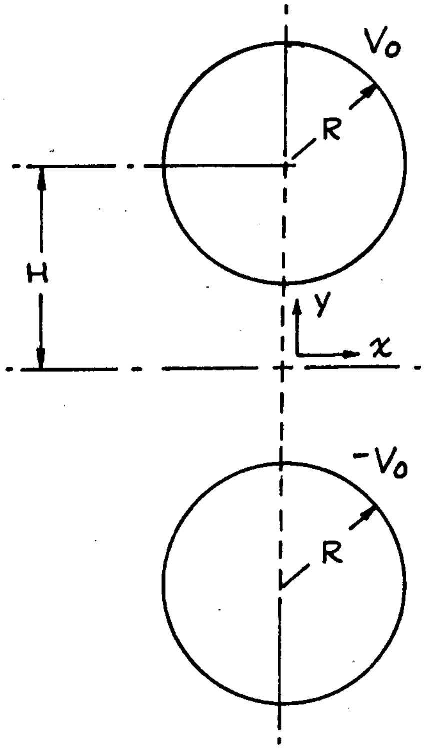
Fig. A6. Ellipsoid of revolution in external magnetic and electric field. Surfaces A^* = constant and V = constant for $\mu_r = \epsilon_r = 10$.

IV. A Cartesian 2-D electrostatic examples:

To illustrate the use of the Program POISSON for 2-D Cartesian electrostatic problems, which basically have a close similarity to 2-D magnetic problems, we present here results for the case of two identical parallel conducting circular cylinders at potentials ± V_0. For cylindrical electrodes of radius R, separation 2H (centers at 0, ±H), and potentials $\pm V_0$, the exterior potential is readily found by means of a conformal transformation [see Smythe, op. cit., Sect. 4.13]. Specifically, with



$$a = \sqrt{H^2-R^2}$$

and

$$U_0 = \ln\left[\frac{H}{R} + \sqrt{\left(\frac{H}{R}\right)^2-1}\right]$$

$$= \ln \frac{H+a}{R}$$

$$= \frac{1}{2} \ln\left(\frac{H+a}{H-a}\right) ,$$

at an external point (x,y)

$$V = \frac{V_0}{2U_0} \ln \left(\frac{(a+y)^2 + x^2}{(a-y)^2 + x^2}\right) .$$

This result has been employed to check results obtained from POISSON (in the 2-D electrostatic mode, with an elliptical boundary) in an example in which H = 10.0, R = 5.0, and V_0 = 10. See Table A4.

Table A4.

X	Y	V_{calc}	$V_{Poisson}$	$\frac{\Delta V}{V}$ %
0	1.0	1.761444	1.761872	-0.024
0	4.0	7.588682	7.592424	-0.049
5.39252	10.10902	9.506901	9.507685	-0.008
8.91380	9.97160	6.299511	6.301227	-0.027

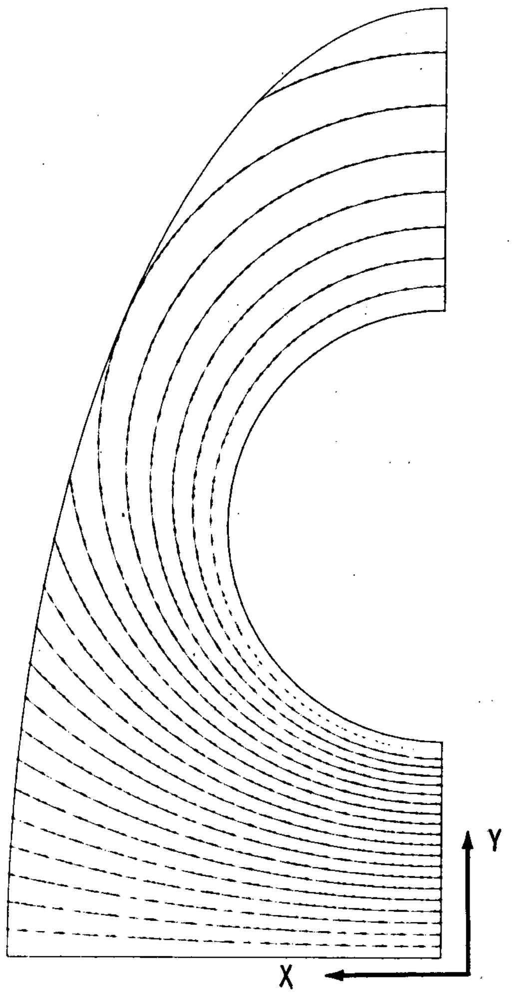


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Fig. A7. Equipotentials surrounding one of two parallel conducting circular cylinders.

NOTE

Maxima of the ξ-Dependent Functions

We add here some comments concerning the Legendre functions that have been employed to form the ξ-dependent functions introduced in the body of this Report. These comments do not, however, bear directly on the techniques we have adopted for applying our boundary condition to axially-symmetric POISSON problems, and in this sense are peripheral to the remainder of the work.

We commence with a review of the well-known normalization factors for the functions $P_n(x)$ and $P_n^1(x)$. The remainder of the Note is then concerned with the maxima of the functions $P_n(x)$, $P_n^1(x)$, and $\sqrt{1-x^2}\ P_n^1(x)$ -- as could be of concern computationally in regard to possible "exponent overflow".

In axially symmetric magnetic problems we have considered use of the function $P_n^1(\xi)$ as the ξ-dependent portion of A_ϕ, or $(1-\xi^2)^{1/2}\ P_n^1(\xi)$ as the ξ-dependent portion of $A^*=\rho A_\phi$. In axially-symmetric electrostatic problems, on the other hand, the appropriate function will be $P_n(\xi)$ to serve as the ξ-dependent portion of terms representing the scalar potential function V.

Characteristics of such functions are presented in, for example, Abramowitz & Stegun (with figures), Jahnke & Emde (with figures), and WPA Tables (NBS & Columbia Univ. Press, 1945) QA406 M37.

Well known normalization integrals are

$$\int_0^1 \left[P_n^m(x) \right]^2 dx = \frac{1}{(2n+1)} \frac{(n+m)!}{(n-m)!}$$

and, in particular,

$$\int_0^1 \left[P_n(x) \right]^2 dx = \frac{1}{2n+1} \quad , \qquad \int_0^1 \left[P_n{}^1(x) \right]^2 dx = \frac{n(n+1)}{2n+1} \quad ,$$

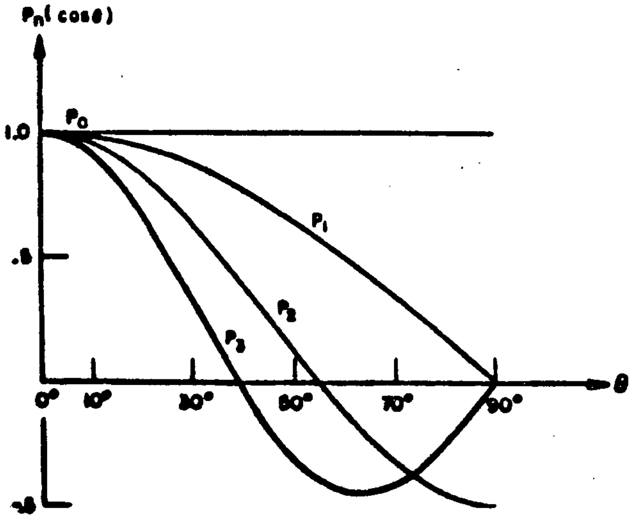
so that orthonormalized functions, for the interval 0 to 1, are respectively

$$\sqrt{2n+1}\; P_n(x) \quad \text{or} \quad \sqrt{\frac{2n+1}{n(n+1)}}\, P_n^1(x)$$

and, for the interval -1 to +1 are respectively

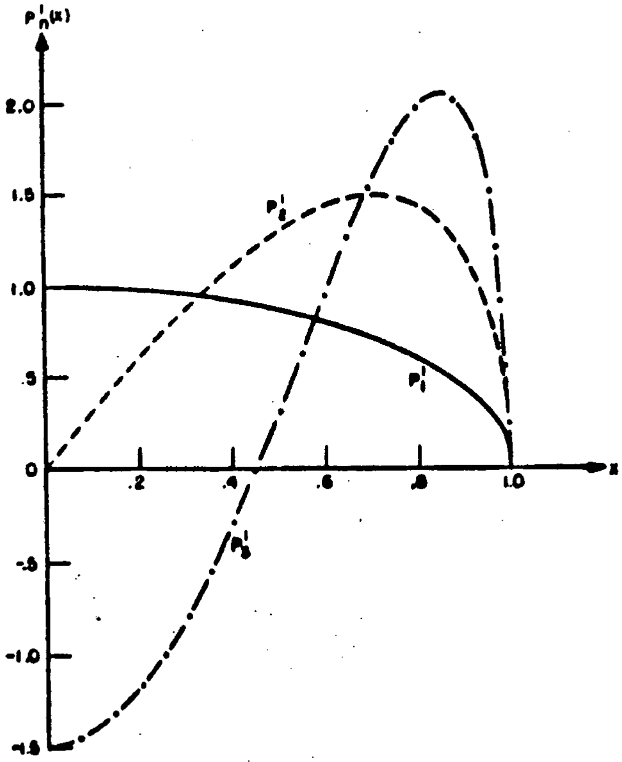
$$\overline{P_n}(x) = \sqrt{\frac{2n+1}{2}}\; P_n(x) \quad \text{or} \quad \overline{P_n^1}(x) = \sqrt{\frac{2n+1}{2n(n+1)}}\; P_n^1(x)$$

(to record the notation of Jahnke & Emde ("Funktionentafeln", Dover, NY).



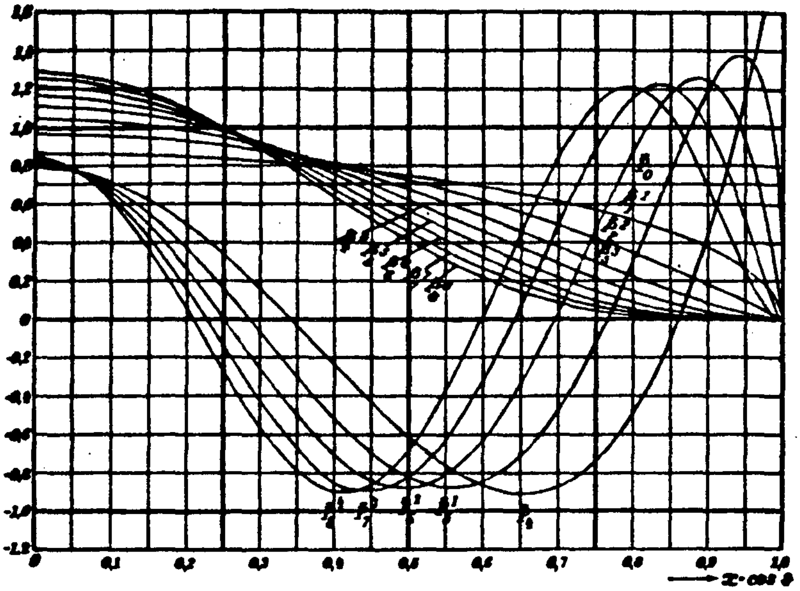
$P_n(\cos\theta)$. $n=0(1)3$.

Fig. 2



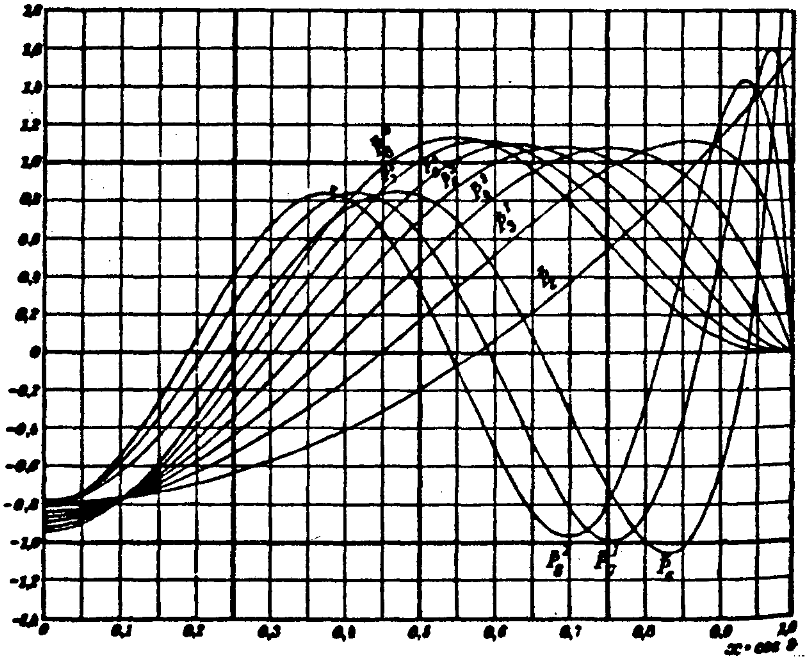
$P_n^1(x)$. $n=1(1)3$, $x \leq 1$.

Fig. 3



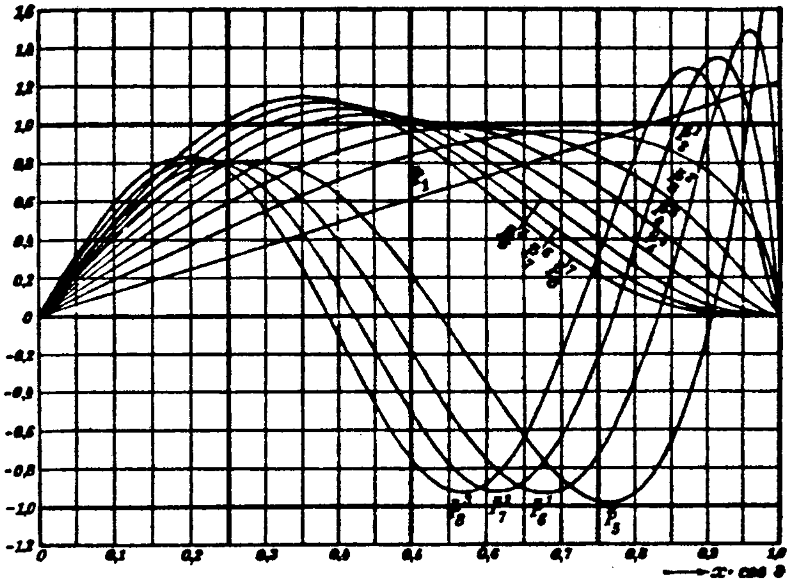
Die zugeordneten normierten Kugelfunktionen 1. Art $\overline{P_n^n}(x)$ und $\overline{P_{n+4}^n}(x)$.
The associated normalized Legendre functions of the 1st kind $\overline{P_n^n}(x)$ and $\overline{P_{n+4}^n}(x)$.

Fig. 4



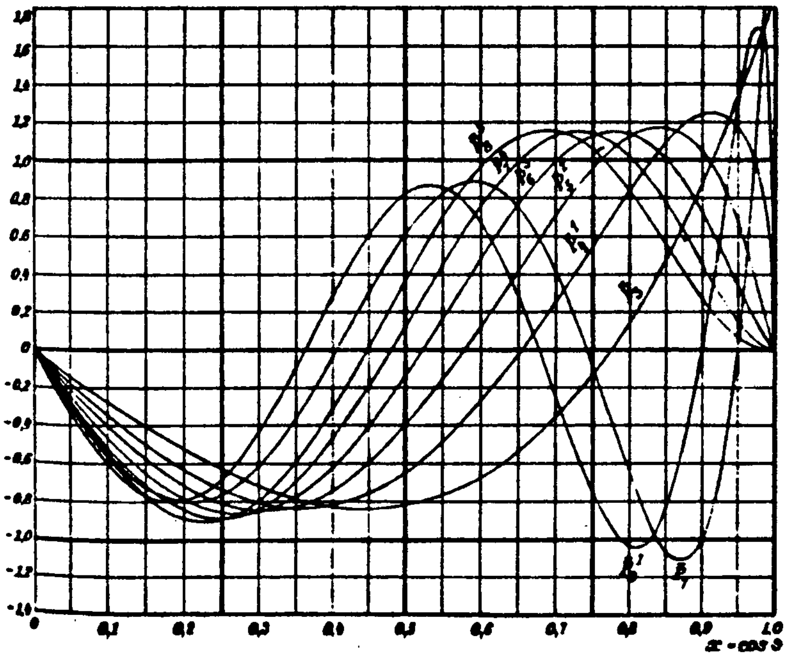
Die zugeordneten normierten Kugelfunktionen 1. Art $\overline{P}^n_{n+2}(x)$ und $\overline{P}^n_{n+6}(x)$
The associated normalized Legendre functions of the 1st kind $\overline{P}^n_{n+2}(x)$ and $\overline{P}^n_{n+6}(x)$

Fig. 5



Die zugeordneten normierten Kugelfunktionen 1. Art $\overline{P}^n_{n+1}(x)$ und $\overline{P}^n_{n+5}(x)$.
The associated normalized Legendre functions of the 1st kind $\overline{P}^n_{n+1}(x)$ and $\overline{P}^n_{n+5}(x)$.

Fig. 6



Die zugeordneten normierten Kugelfunktionen 1. Art $\overline{P}^n_{n+3}(x)$ und $\overline{P}^n_{n+7}(x)$.
The associated normalized Legendre functions of the 1st kind $\overline{P}^n_{n+3}(x)$ and $\overline{P}^n_{n+7}(x)$.

Fig. 7

In constructing the functions F(v) in terms of which we undertake to perform a development of the potential A* or V along the "inner" boundary of our mesh, we have been guided by an awareness of the normalization factors cited in the preceding paragraph and, ultimately, by the performance of our matrix-inversion routine when applied to the corresponding least-squares problem. It may also be of interest, however -- both from a mathematical viewpoint and also perhaps to provide an indication of potential difficulties that possibly could arise as a result of "exponent overflow" -- to examine the <u>maximum</u> <u>value</u> that such functions can attain (in the range $-1 \leq \xi \leq 1$) prior to "normalization". We now proceed to discuss such maximum values for the functions of concern that we mentioned earlier in this Report.

1) The ordinary Legendre function, $P_n(x)$, of integral degree and x in the interval $-1 \leq x \leq 1$, has an absolute value that attains but does not exceed unity. Thus, as is well known, $P_n(1) = 1$. Also, $P_n(-1) = (-1)^n$.

2) With respect to the associated functions $P_n^1(x)$, the graphs (Figs. 4-7, that include curves of various $P_n^1(x)$ <u>vs</u>. x) of Jahnke & Emde suggest that the functions $P_n^1(x)$ approach a greatest maximum for x near unity. More particularly, such graphs suggest that the maximum exhibits a distinct increase as n becomes larger, and that such maxima become situated closer and closer to unity as n increases. We have undertaken to estimate such behavior by an approximate analytical treatment and to check some results computationally. It is convenient in such work to introduce $y = 1-x$, since in this case interest will be focused on values of y that are small (for n large).

To anticipate, one may briefly summarize the results by stating that when n becomes large the maximum of $P_n^1(x=1-y)$ is approximately 0.582 n and occurs at $y \cong 1.695/n^2$.

The function $P_n(x=1-y)$ satisfies the Legendre differential equation (written in terms of y)

$$(y^2-2y)\,\frac{d^2P_n}{dy^2} + 2(y-1)\,\frac{dP_n}{dy} - n(n+1)\,P_n = 0$$

and one may seek a power-series solution consistent with the initial conditions

$$P_n = 1 \quad \text{for } y = 0$$

and

$$\frac{dP_n}{dy} = -\frac{1}{2}n(n+1) \qquad \text{for } y = 0$$

[cf. Smythe, § 5.157].

The result so found is

$$\begin{aligned} P_n &= a_0 + a_1y + a_2y^2 + a_3y^3 + \ldots + a_ky^k + \ldots \\ &= 1 - \frac{n(n+1)}{2}\,y \\ &\quad + \frac{n(n+1)\left[\frac{n(n+1)}{2} - 1\right]}{8}\,y^2 \\ &\quad - \frac{n(n+1)\left[\frac{n(n+1)}{2} - 1\right]\left[\frac{n(n+1)}{6} - 1\right]}{24}\,y^3 + \ldots \end{aligned}$$

(for k=1, 2,...n),

wherein

$$a_k = \frac{k(k-1) - n(n+1)}{2k^2} a_{k-1}$$

$$= - \frac{k-1}{2k} \left[\frac{n(n+1)}{k(k-1)} -1 \right] a_{k-1} \quad .$$

Because as we proceed further we shall be interested most particularly in the situation when n is <u>large</u>, it is convenient to employ the large-n approximate form

$$P_n \cong 1 - \frac{n^2}{2} y + \frac{n^4}{16} y^2 - \frac{n^6}{288} y^3 + \ldots + (-1)^k \frac{(n^2y)^k}{2^k(k!)^2} + \ldots \quad .$$

We recall that

$$P_n^1 = (1-x^2)^{1/2} \frac{dP_n}{dx}$$

$$= - (2y-y^2)^{1/2} \frac{dP_n}{dy}$$

and obtain in the large-n approximation [for which we shall be interested in values of $y=O(\frac{1}{n^2})$ and which accordingly permit replacement of $(2y-y^2)^{1/2}$ by $\frac{\sqrt{2\,n^2y}}{n}$] the result:

$$P_n^1 \cong \sqrt{\frac{n^2y}{2}} \left[1 - \frac{n^2y}{4} + \frac{(n^2y)^2}{48} - \frac{(n^2y)^3}{1152} + \frac{(n^2y)^4}{46080} \right.$$

$$\left. \ldots + (-1)^k \frac{(n^2y)^k}{2^k k!(k+1)!} + \ldots \right] n \quad .$$

A similar approximation for the derivative of this quantity leads to

$$\sqrt{2y-y^2}\,\frac{d}{dy}\,P_n^1 \cong \frac{n^2}{2}\left[1 - \frac{3}{4}(n^2y) + \frac{5}{48}(n^2y)^2 - \frac{7}{1152}(n^2y)^3 \right.$$
$$+ \frac{9}{46080}(n^2y)^4 - \frac{11}{2764800}(n^2y)^6$$
$$\left. + \ldots + (-1)^k\,\frac{2k+1}{2^k\,k!\,(k+1)!}\,(n^2y)^k + \ldots\right]$$

and the maximum value of P_n^1 in which we are interested thus may be expected to occur (in the large-n limit) at a value of y such that the square bracket vanishes in the expression written immediately above. Such a value is

$$y \cong 1.695/n^2 \quad .$$

Insertion of the approximate value $n^2y \cong 1.695$ into the large-n approximate expression for P_n^1 written at the bottom of the preceding page then leads to

$$P_n^1\Big|_{\max} \cong 0.582\ n \quad .$$

We have found the approximate results just cited (for the greatest maximum of P_n^1) to be in good agreement at large n with the results of numerical computations (Program ASSOC) -- see accompanying Table and double-logarithmic graphs. This maximum value for P_n^1, which thus appears to be essentially proportional to n, may be expected to become <u>narrower</u> as n increases (as suggested, for example, by the graphs of Jahnke and Emde).

```
      PROGRAM ASSOC
C >>> COMPUTE ASSOCIATED LEGENDRE FUNCTION FOR M=1
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION P(1000)
      NMAX = 1000
10    WRITE(*,1010)
      READ(*,*) NL
      IF (NL .GT. NMAX) GO TO 10
      NF = NL
      NL = MAX0(NL,3)
20    WRITE(*,1020)
      READ(*,*) Y
      IF (Y .LT. 0.0D0) GO TO 20
      X = 1.0D0 - Y
      IF (X .LT. 0.0D0) GO TO 20
      SQ = DSQRT((2.0D0 - Y)*Y)
      P(1) = SQ
      P(2) = 3.0D0*X*SQ
      DO 25 N=3,NL
      P(N) = ((2*N-1)*X*P(N-1) - N*P(N-2))/(N-1.0D0)
25    CONTINUE
      WRITE(*,1025) Y, NF, P(NF)
30    WRITE(*,1030)
      READ(*,*) JUMP
      IF (JUMP .EQ. 1) GO TO 10
      IF (JUMP .EQ. 2) GO TO 20
      IF (JUMP .EQ. 9) GO TO 90
      GO TO 30
90    STOP
1010  FORMAT(1H ,'TYPE NLAST')
1020  FORMAT(1H ,'TYPE Y')
1025  FORMAT(1H ,/,1H ,'Y ='1PE13.6,5X,'P('I4,') ='1PE17.10,///)
1030  FORMAT(1H ,'NEW NLAST, Y, OR QUIT? -- TYPE 1, 2, OR 9',/)
      END
```

TABLE 3.

n	y=1-x	$P_n^1(x=1-y)\|_{max.}$	n^2y	$\frac{P_n^1\|_{max.}}{n}$
1	1.0	1.0		
2	$1-\sqrt{0.5} \cong 0.2929$	1.5		
3	0.1437	2.0656		
4	0.08559	2.6401^{-}		
5	0.05687	3.2176		
6	0.04054	3.7966^{+}		
7	0.03037	$4.3765^{\bullet}$		
8	0.02360	4.9568		
9	0.01887	5.5375		
10	0.01544	6.1184		
12	0.01088	7.2807		
15	0.007068	9.0249^{-}		
20	0.004037_{5}	11.9327		
25	0.002608	$14.8412^{\bullet}$		
30	0.001823	17.7499		
40	0.0010336	23.5678		
50	0.0006647	29.3860	1.662	0.5877
60	0.004631	$35.2044^{\bullet}$	1.667	0.5867
70	0.0003411	41.0228	1.671	0.5860
80	0.0002616	46.8413	1.674	0.5855
90	0.0002070	52.6598	1.676	0.5851
100	0.0001678	58.4784^{-}	1.678	0.5848
200	0.00004216	116.664_{4}	1.686	0.5833
250	0.00002701	145.7576	1.688	0.5830
500	0.000006766	291.224	1.691_{5}	0.5824
1000	0.000001693	582.156	1.693	0.5822^{-}

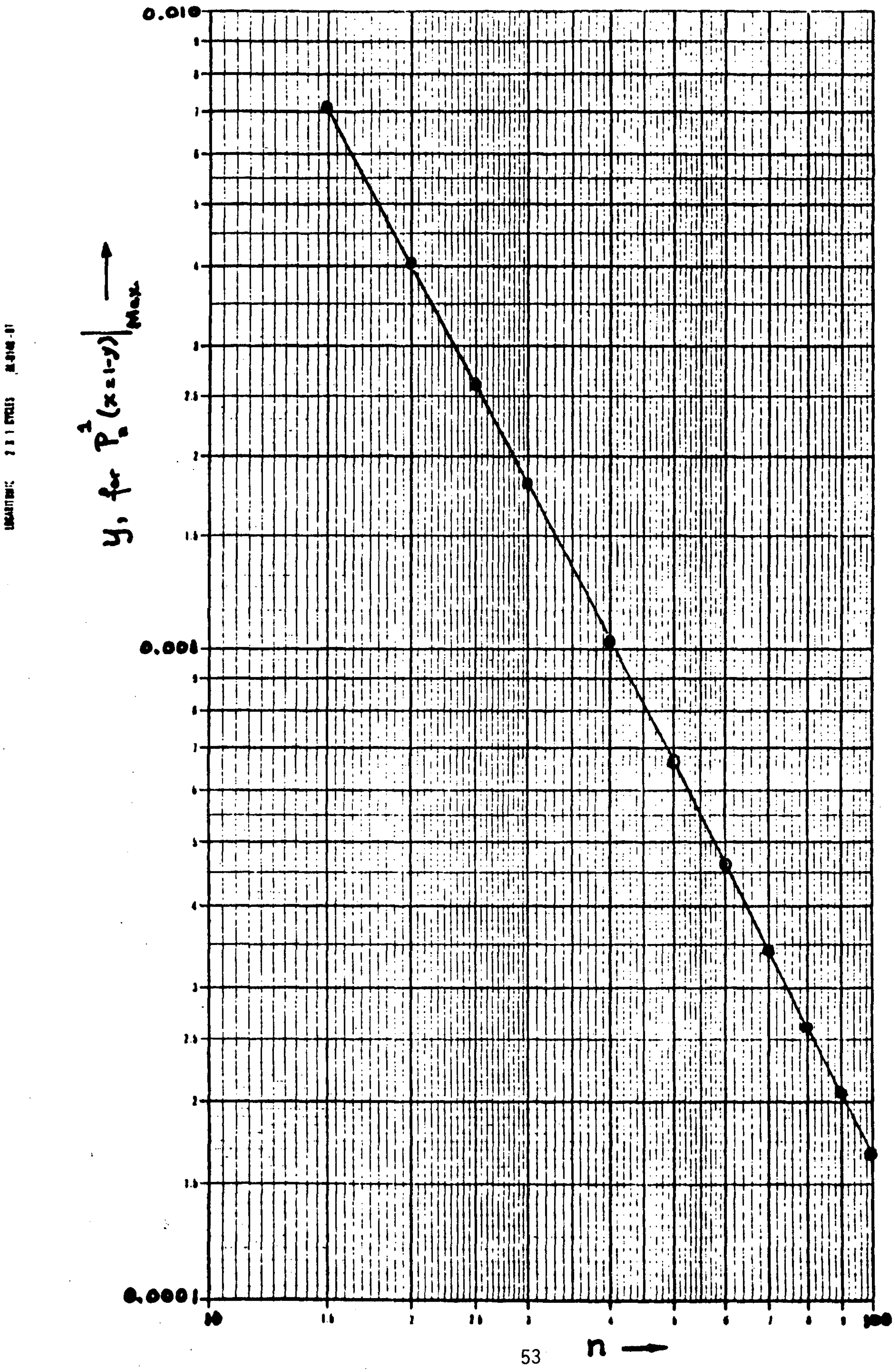

0.010
9
8
7
6
5
4
3
2.5
2
1.5
0.001
9
8
7
6
5
4
3
2.5
2
1.5
0.0001
10
1.5
2
2.5
3
4
5
6
7
8
9
100
y, for $P_n^1(x=1-y)\big|_{Max}$ →
n →

Fig. 8

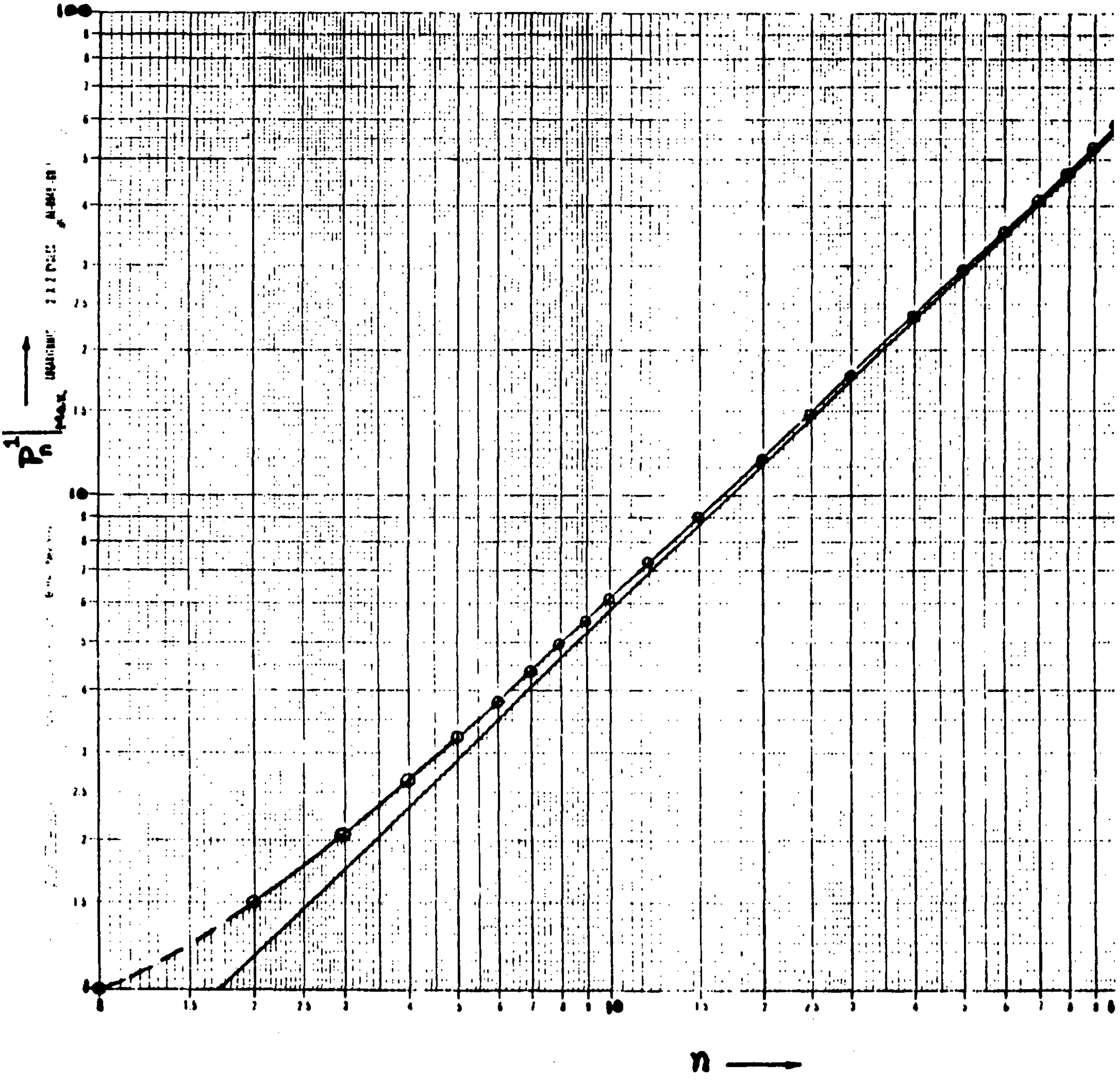
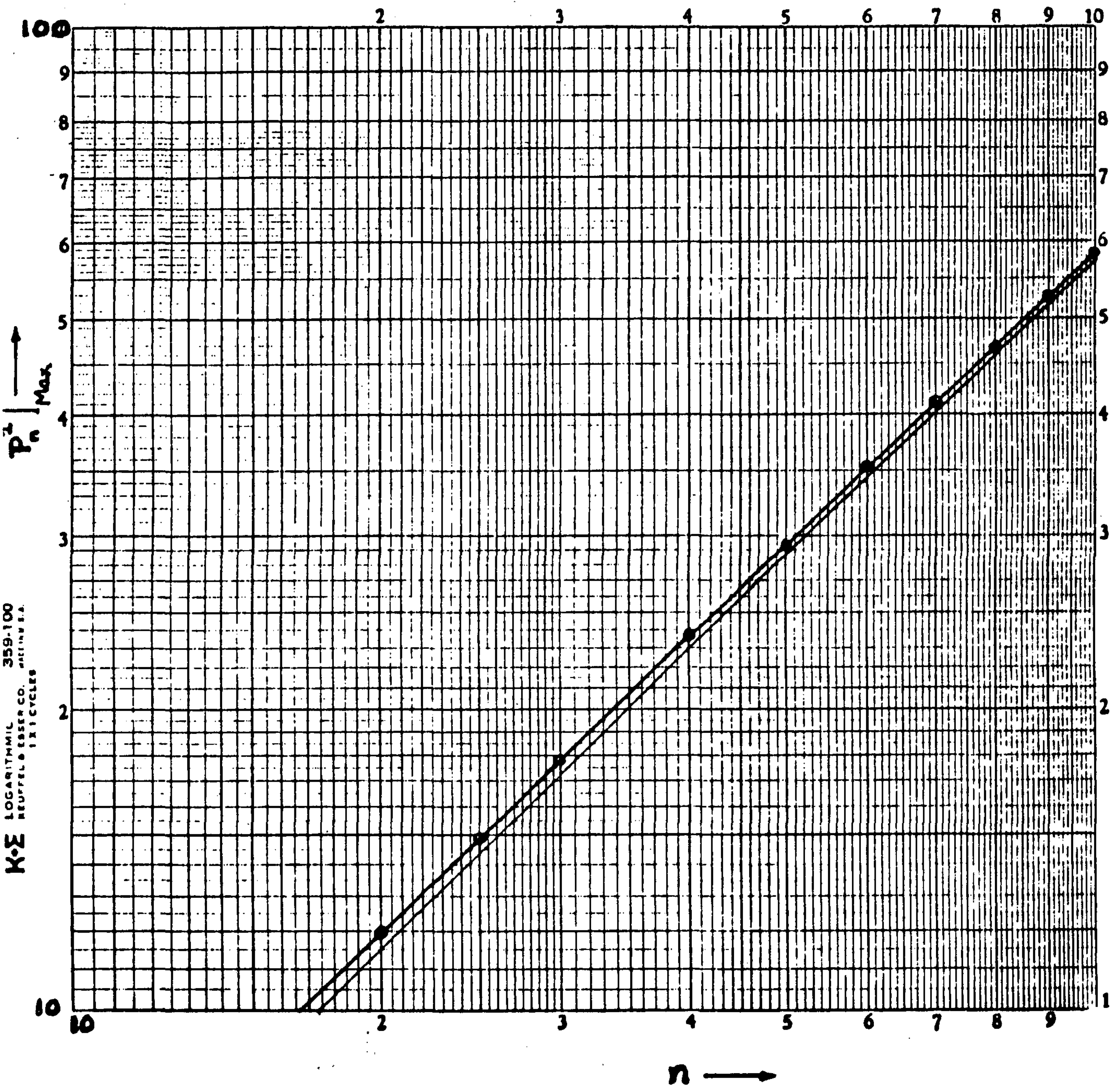


Fig. 9



100
9
8
7
6
5
4
3
2
10
10
2
3
4
5
6
7
8
9
10
$P_n^2 \big|_{Max} \longrightarrow$
$n \longrightarrow$
K·E LOGARITHMIC 359-100
KEUFFEL & ESSER CO.
1 X 1 CYCLES

Fig. 10

3) In relaxation computations pertaining to axially symmetric magnetostatic problems, however, the use of $A^*=\rho A$ as a working variable leads us to direct our attention to the characteristics of the function $(1-x^2)^{1/2}\, P_n^1(x)$ (rather than to the characteristics of $P_n^1(x)$ itself, as discussed in § 2 above).

To anticipate the results of a discussion concerned with the characteristics of the function $(1-x^2)^{1/2}\, P_n^1(x)$, the greatest maximum for this function (in absolute value) occurs either at <u>x=0</u> (y=1), for n odd, or close to that location for n even. The increase, vs. n, of such values is <u>slower</u> than that found for the maxima discussed in § 2) and asymptotically such maxima appear to approach proportionality to $n^{1/2}$.

A maximum of the function $(1-x^2)^{1/2}\, P_n^1(x)$ can be found that is analogous to that found previously, in § 2), for the function $P_n^1(x)$--although, as will be indicated subsequently, <u>more pronounced maxima can be found elsewhere</u>.

a) To discuss first the maxima found for the function $(1-x^2)^{1/2}\, P_n^1(x)$ near to y=0, we note the series development (in terms of y=1-x) valid for large-n:

$$\sqrt{1-x^2}\; P_n^1(x) \cong n^2 y - \frac{(n^2y)^2}{4} + \frac{(n^2y)^3}{48} - \frac{(n^2y)^4}{1152} + \ldots + (-1)^k \frac{(n^2y)^{k+1}}{2^k\, k!\,(k+1)!} + \ldots$$

and

$$\frac{d}{dy}\left[\sqrt{1-x^2}\;P_n^1(x)\right] \cong n^2\left[1-\frac{n^2y}{2}+\frac{(n^2y)^2}{16}-\frac{(n^2y)^3}{288}+\ldots+(-1)^k\frac{(n^2y)^k}{2^k(k!)^2}+\ldots\right] \quad .$$

The location of a stationary value for $\sqrt{1-x^2}\;P_n^1(x)$ then is suggested by setting equal to zero the square-bracket expression shown in the preceding equation--with the result

$$y \cong \frac{2.8916}{n^2} \qquad \text{(for n large)} \quad .$$

When this result is substituted into the series expression for the function $\sqrt{1-x^2}\;P_n^1(x)$ it leads to the estimate

$$\left[\sqrt{1-x^2}\;P_n^1(x)\right]\Bigg|_{\substack{\text{max.,}\\ (\underline{\text{near}}\ y=0)}} \cong 1.2485$$

for the local maximum (near y=0) when n is large. We note that this large-n estimate, <u>for this particular maximum</u>, is <u>independent</u> of n.

Such "small-y maxima" have been sought numerically (aided by Program ASTAR), with results shown on the accompanying Table. The expected characteristics of such maxima appear to be confirmed by the tabulation.

TABLE 4.

Small-y Maxima of $\sqrt{1-x^2}\;P_n^1(x)$

n	y=1-x	Local Maximum	n^2y
1	-	-	
2	$1-1/\sqrt{3} = 0.42265$	$\frac{2}{3}\sqrt{3} = 1.15470$	
3	$1-\sqrt{3/5} = 0.2254$	1.2	
4	0.1389	$1.21899^{\bullet}$	
5	0.09382	$1.22868^{\bullet}$	
6	0.06753	1.23427	
7	0.05089	1.23779^{+}	
8	0.03971	1.24015	
9	0.03184	1.24180_{5-}	
10	0.02609	1.24301	2.609
12	0.01844	1.24461	2.655
15	0.01201	1.24596^{-}	2.702
20	0.006871	1.24703	2.748
25	0.004443	1.24753	2.777
30	0.003107	1.24781	2.796
40	0.001762	1.24809	2.819
50	0.001134	1.24822	2.835
60	0.0007899	$1.248295^{\bullet}$	2.844^{-}
70	0.0005817	1.24834	2.850
80	0.0004462	1.24837^{-}	2.856^{-}
90	0.0003530	1.24838_{6-}	2.859
100	0.0002863	$1.24840^{\bullet}$	2.863
200	0.00007193	1.2484442	2.877
250	0.00004608	1.2484496	2.880
500	0.00001154	1.248457^{-}	2.885
1000	0.000002889	1.248459^{-}	2.889

b) The associated Legendre function $P_n^1(x)$ typically will exhibit several locations in the range $0 \leq x \leq 1$ at which the absolute value of this function passes through a maximum and this same feature remains present for the function $\sqrt{1-x^2}\, P_n^1(x)$. Specifically,

<u>For n even</u>, the absolute value of the function exhibits $\frac{n}{2}$ maxima (in the range $0 \leq x \leq 1$), and

<u>For n odd</u>, the absolute value exhibits $\frac{n+1}{2}$ maxima, of which one occurs at $x=0$ $(y=1)$.

For <u>n odd</u>, the maximum that occurs at $x=0$ is

$$\left| \sqrt{1-x^2}\, P_n^1(x) \right| \underset{x=o}{=} \left| P_n^1(o) \right|$$

$$= \left| (-1)^{\frac{n-1}{2}} \frac{n!}{2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2} \right|$$

$$= \frac{n!}{2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2} .$$

For <u>n even</u> and $x=o$, we have $P_n^1(o)=0$, but for a value of x somewhat greater than zero, the function $\sqrt{1-x^2}\, P_n^1(x)$ then exhibits a magnitude maximum that is not markedly different from the result cited above for a nearby odd value of n.

The particular maxima just cited (for n odd, or for n even) will be the most prominent of all the many magnitude maxima that may be present in the interval $0 \leq x \leq 1$ for any particular value of n. This feature is illustrated by the following tabulations, for $n=8$ and $n=9$ (numerical results):

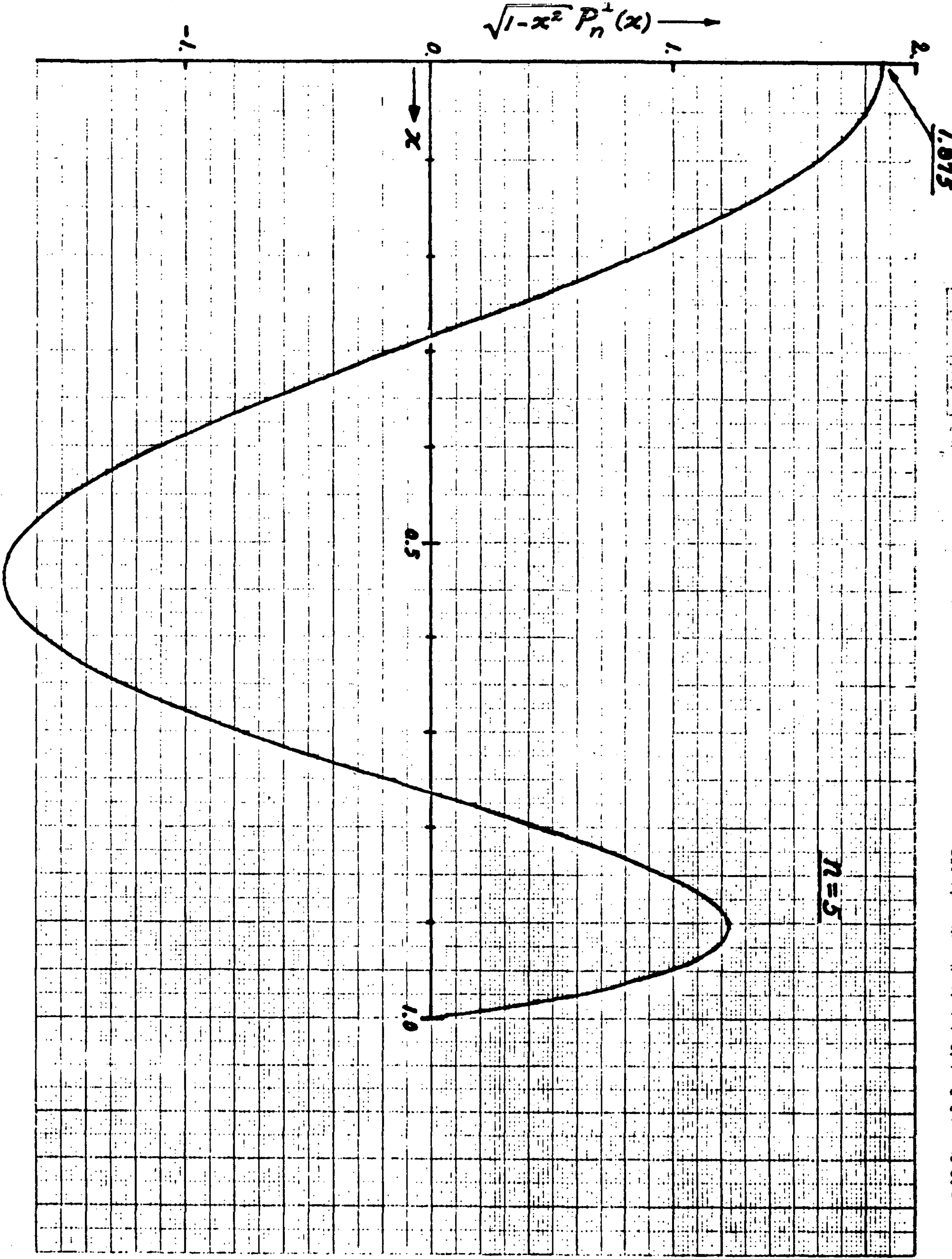


Fig. 11

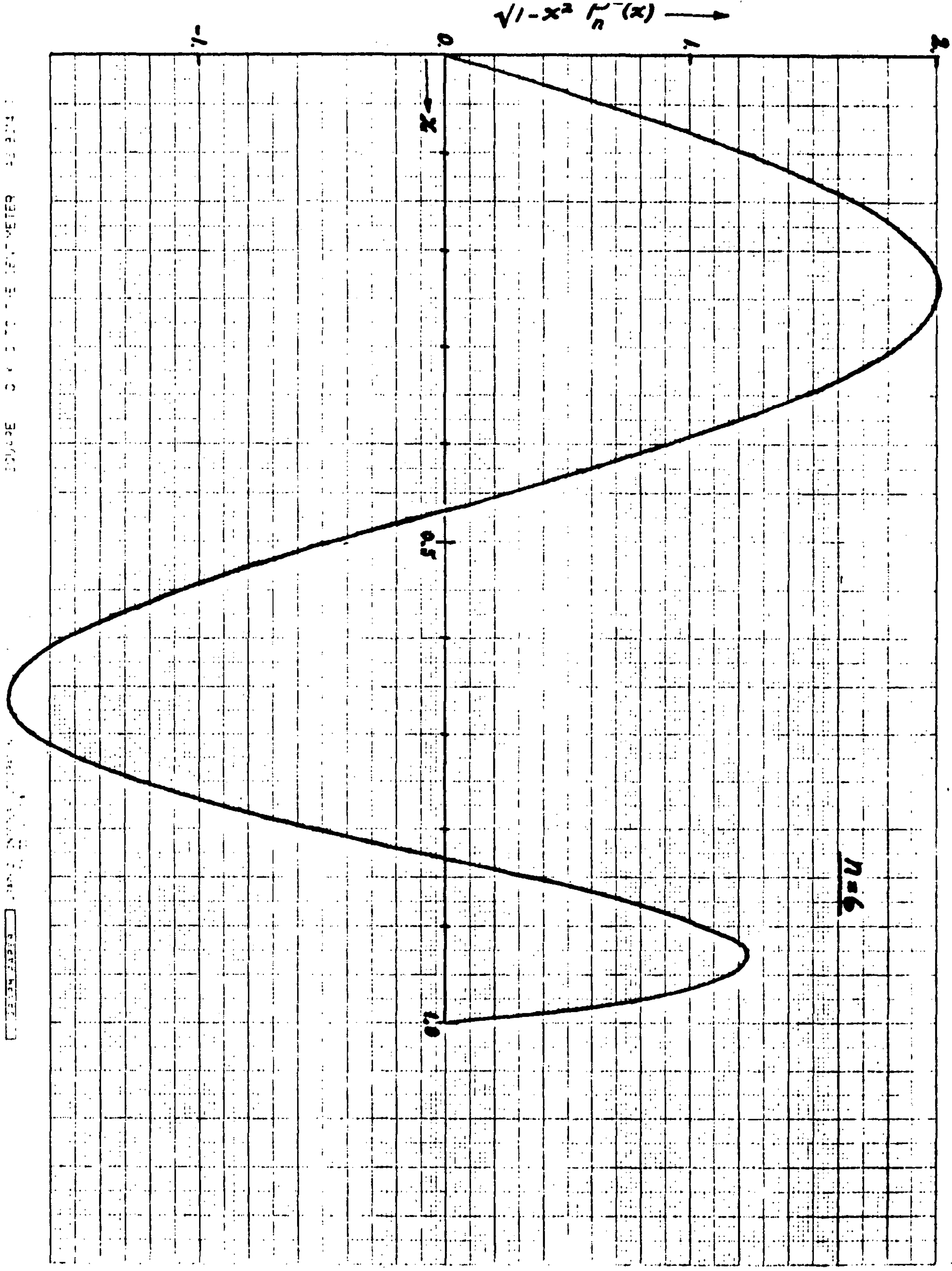


Fig. 12

```
1 THRU 40
              PROGRAM ASTAR
              IMPLICIT REAL*8(A-H,O-Z)
              DIMENSION P(1000)
              NMAX = 1000
      10      WRITE(*,1010)
              READ(*,*) NL
              IF (NL .GT. NMAX) GO TO 10
              NF = NL
              NL = MAX0(NL,3)
      20      WRITE(*,1020)
              READ(*,*) Y
              IF (Y .LT. 0.0D0) GO TO 20
              X = 1.0D0 - Y
              IF (X .LT. 0.0D0) GO TO 20
              SQ = DSQRT((2.0D0 - Y)*Y)
              P(1) = SQ
              P(2) = 3.0D0*X*SQ
              DO 25 N=3,NL
              P(N) = ((2*N-1)*X*P(N-1) - N*P(N-2))/(N-1.0D0)
      25      CONTINUE
              F = SQ*P(NF)
              WRITE(*,1025) Y, NF, F
      30      WRITE(*,1030)
              READ(*,*) JUMP
              IF (JUMP .EQ. 1) GO TO 10
              IF (JUMP .EQ. 2) GO TO 20
              IF (JUMP .EQ. 9) GO TO 90
              GO TO 30
      90      STOP
      1010    FORMAT(1H ,'TYPE NLAST')
      1020    FORMAT(1H ,'TYPE Y')
      1025    FORMAT(1H ,/,1H ,'Y ='1PE13.6,5X,'F('I4,') ='1PE17.10,///)
      1030    FORMAT(1H ,'NEW NLAST, Y, OR QUIT? -- TYPE 1, 2, OR 9',/)
              END
*
```

TABLE 5.

Magnitude Maxima for $\sqrt{1-x^2}\ P_n^1(x)$

n=8

y=1-x	x	Function	
1.0	0	0	
0.8166	0.1834	-2.30844	
0.4745	0.5255	2.14816	4 "maxima"
0.2033	0.7967	-1.81261	
0.03970	0.96030	1.24015	
0	1.0	0	

n=9

y=1-x	x	Function	
1.0	0	2.46094	
0.6757	0.3243	-2.39372	
0.3866	0.6134	2.18793	5 "maxima"
0.1640	0.8360	-1.82566	
0.03184	0.96816	1.24180	
0	1.0	0	

We have remarked that the greatest magnitude attained by $\sqrt{1-x^2}\;P_n^1(x)$, for x in the range $0 \leq x \leq 1$, is either

$$\frac{n!}{2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2} \qquad \text{(for n odd),}$$

or a value close to this (using a neighboring value of n) for n even. For large n this expression may be replaced conveniently by the Stirling approximation:

$$\left|\sqrt{1-x^2}\;P_n^1(x)\right|_{\text{Max.}} \approx \left|\sqrt{\frac{2(n+1)}{\pi}}\right| .$$

The accompanying Table illustrates such values.

TABLE 6.

Values of $\sqrt{1-x^2}\ P_n^1$ for Maximum Magnitude

n	y=1-x	x	Function	$\sqrt{\frac{2(n+1)}{\pi}}$
8	0.8166	0.1834	-2.30844	
9	1.0	0	+2.4609375	
10	0.8511	0.1489	$+2.57247_7$	
11	1.0	0	-2.70703_{125}	
15	1.0	0	-3.14209	
16	0.9050	0.0950	-3.23443	
25	1.0	0	$+4.02951^-$	4.0684
26	0.9408	0.0592	$+4.10411^+$	4.1459
35	1.0	0	-4.75418	4.7873
36	0.9570	0.0430	-4.81843	4.8533
45	1.0	0	+5.38219	5.4115
46	0.96623	0.03377	+5.43945	5.4700
55	1.0	0	-5.94423	5.9708
56	0.97220	0.02780	-5.99637_6	6.0239
65	1.0	0	+6.45754	6.4820
66	0.97638	0.02362	$+6.50573_6$	6.5310
75	1.0	0	-6.93295^+	6.9558
76	0.97947	0.02053	-6.97798	7.0014
85	1.0	0	+7.37780	7.3993
86	0.98184	0.01816	+7.42021	7.4422
95	1.0	0	-7.79731	7.8176
96	0.98372	0.01628	-7.837505	7.8583
99	1.0	0	-7.95892	7.9788
100	0.98437	0.01563	-7.998329	8.0186
101	1.0	0	+8.03851	8.0582

TABLE 6.

(continued)

Values of $\sqrt{1-x^2}\, P_n^1$ for Maximum Magnitude

n	y=1-x	x	Function	$\sqrt{\frac{2(n+1)}{\pi}}$
115	1.0	0	-8.57498	8.5935
125	1.0	0	+8.93848	8.9562
135	1.0	0	-9.28776^{+}	9.3049^{-}
145	1.0	0	+9.62438	9.6409
155	1.0	0	-9.94962	9.9656
195	1.0	0	-11.1561$_{45}$	11.1704
205	1.0	0	+11.43791	11.4518
295	1.0	0	-13.7157	13.7273
305	1.0	0	+13.94588	13.9573
395	1.0	0	-15.86768	15.8777
405	1.0	0	+16.06703	16.0769
495	1.0	0	-17.76078	17.7697
505	1.0	0	+17.93910$_{4}$	17.9480
695	1.0	0	-21.04208	21.0496
705	1.0	0	+21.19281	21.2003
795	1.0	0	-22.50402$_{5-}$	22.5111
805	1.0	0	+22.64503	22.6521^{-}
895	1.0	0	-23.87662	23.8833
905	1.0	0	+24.00957	24.0162
998	0.998427	0.001573	+25.21238	25.2187
999	1.0	0	-25.22502	25.2313
1000	0.998430	0.001570	-25.23762	25.2439

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