Title
Open systems in classical mechanics

Journal
Journal of Mathematical Physics, 62(4)

ISSN
0022-2488

Authors
Baez, John C
Weisbart, David
Yassine, Adam M

Publication Date
2021-04-01

DOI
10.1063/5.0029885

Peer reviewed

OPEN SYSTEMS IN CLASSICAL MECHANICS

JOHN C. BAEZ[1], DAVID WEISBART[2], AND ADAM YASSINE[3]

[1,2] *Department of Mathematics*
University of California, Riverside
Riverside, California
USA

[3] *Department of Mathematics and Actuarial Science*
American University in Cairo
Cairo, Egypt

Abstract. Span categories provide a framework for formalizing mathematical models of open systems in classical mechanics. The categories appearing in classical mechanics do not have pullbacks, which requires the use of generalized span categories. We introduce categories LagSy and HamSy that respectively provide a categorical framework for the Lagrangian and Hamiltonian descriptions of open classical mechanical systems. The morphisms of LagSy and HamSy correspond to such open systems, and composition of morphisms models the construction of systems from subsystems. The Legendre transformation gives a functor from LagSy to HamSy that translates from the Lagrangian to the Hamiltonian perspective.

Contents

1. Introduction 1
2. Spans and Generalized Span Categories 5
3. Lagrangian and Hamiltonian Systems 9
4. Physical Systems as Morphisms 18
5. The Legendre Functor 24
References 30

1. Introduction

Category theory provides a formalism for unifying ideas across a wide spectrum of disciplines. The last few decades have seen the emergence of applied category theory [17, 28]. One prominent program in this subject is to describe "open" systems—that is, systems

E-mail address: [1]baez@math.ucr.edu, [2]weisbart@math.ucr.edu, [3]adam.yassine@aucegypt.edu.

that can interact with their surroundings—as morphisms in appropriate categories, where composition describes how open systems can be combined to form larger systems.

The idea of describing open systems as morphisms arose from extended topological quantum field theory, where the manifold describing space can built up by composing cobordisms, manifolds with boundary that describe smaller regions of space [3, 10, 20, 21]. It was later applied in a more down-to-earth way to electrical circuits [5, 6], Markov processes [7], and a wide variety of dynamical systems [8, 22, 26]. The morphisms in these categories are often spans or cospans with extra structure, and there are now several formalisms for constructing such categories [14].

Our goal here is to apply this idea to Lagrangian and Hamiltonian mechanics, and describe the Legendre transformation as as a functor from a category with open Lagrangian systems as morphisms to a similar category of open Hamiltonian systems. Since the study of classical systems involves solving differential equations that describe paths on general Riemannian and symplectic manifolds, it is in some ways more complicated than the examples treated earlier. The current work investigates some previously unidentified structures that appear critical to the study of open systems in classical mechanics.

The systems under consideration have a state space that is either the tangent bundle to a Riemannian manifold in the Lagrangian description or a symplectic manifold in the Hamiltonian description [2]. A path in the state space models the motion of the system. The state space of any subsystem is a quotient space of that of the entire system. For Lagrangian systems we require that the quotient maps be surjective Riemannian submersions. For Hamiltonian systems, we require that they be surjective Poisson maps between symplectic manifolds.

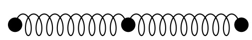

Figure 1. Three Masses

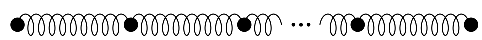

Figure 2. Many Masses

A study of the combined spring-mass system of Figure 2 serves as a simple example. Figure 1 represents a system with three point masses attached by springs, where all motion is along the same line. Figure 2 represents the more complicated system formed by attaching additional point masses and springs in series. View a pair of point masses attached by a spring as a fundamental component, or subsystem, of one of these more complicated systems. The spring-mass subsystems are open systems in the sense that both forces internal to the subsystem and external forces of the larger system govern the dynamics of the subsystems.

Figure 3 depicts the composition of subsystems to form a larger system, where two spring-mass systems combine by identifying the right mass of the system on the left with the left mass of the system on the right. Diagram 1 depicts the state spaces of these systems from a Hamiltonian perspective. Each of the maps in Diagram 1 is a canonical projection, and a Poisson map between symplectic manifolds. At the lowest level in Figure 3 are the three distinct masses. View each mass as moving along a line where the forces acting on each mass are external to the system. Each system has $T^*\mathbb{R}$, the cotangent bundle to $\mathbb{R}$, as its state space. At the middle level, view the system as two spring-mass systems, each with a state space given by $T^*\mathbb{R}^2$ and with an external force acting on one of the masses. On the top level, the total system consists of three masses interacting in series, where connecting springs mediate the interaction of the masses. The state space for this total system is a fibered product of two copies of the symplectic manifold $T^*\mathbb{R}^2$ over the manifold $T^*\mathbb{R}$.

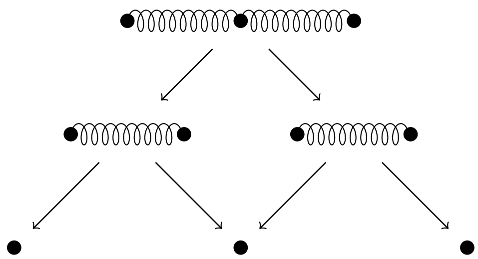

Figure 3. Three Point Masses

$$\begin{tikzcd}
 & & T^*\mathbb{R}^2 \times_{T^*\mathbb{R}} T^*\mathbb{R}^2 \arrow[dl] \arrow[dr] & & \\
 & T^*\mathbb{R}^2 \arrow[dl] \arrow[dr] & & T^*\mathbb{R}^2 \arrow[dl] \arrow[dr] & \\
T^*\mathbb{R} & & T^*\mathbb{R} & & T^*\mathbb{R}
\end{tikzcd}$$

Diagram 1. Corresponding Phase Space

The fibered product is a six dimensional symplectic manifold, whereas the cartesian product of the state spaces is an eight dimensional symplectic manifold. While the fibered product is an embedded submanifold of the product, it will not be a symplectic submanifold when endowed with the symplectic structure that it requires to be the state space of the given physical system. The Lagrangian setting is similar, but uses tangent bundles rather than cotangent bundles as the state spaces. The fibered product together with its canonical projections encapsulates the physical meaning of identifying the right mass of the left spring-mass system with the left mass of the right spring-mass system. Both Dazord in [15] and Marle in [24] had similar insights with respect to studying constrained systems, which are similar to the systems given above in the sense that the masses that connect our systems can be thought of as a geometric constraint. In fact, Dazord explicitly uses fibered products to construct the configuration and state spaces for certain constrained systems.

Suppose that X, Y, and Z are sets and f and g are functions that respectively map X and Y to the set Z. Henceforth denote by ρ_X and ρ_Y the respective canonical projections

$$\rho_X \colon X \times Y \to X \quad \text{and} \quad \rho_Y \colon X \times Y \to Y,$$

and denote by π_X and π_Y the respective restrictions of ρ_X and ρ_Y to the fibered product $X \times_Z Y$, which is the subset of $X \times Y$ consisting of all elements on which f is equal to g. The fibered product in the category Set, whose objects are sets and whose morphisms are functions, has certain universal properties recalled in Section 2. The connection between these universal properties and the construction of span categories for modeling classical mechanical systems is a central theme of the current investigation.

A span in the category Set is a pair of functions with the same source. The fibered product together with the span (π_X, π_Y) gives a prescription for composing spans in Set. Bénabou proved in [9] that if $\mathscr{C}$ is a category with pullbacks then there is a bicategory, $\mathrm{Span}(\mathscr{C})$, whose objects, morphisms, and 2-morphisms are the respective objects, spans, and maps of spans in $\mathscr{C}$. To avoid unnecessary complications we view this bicategory as a category, a *span category*, by ignoring the bicategory structure and taking isomorphism classes of spans in $\mathscr{C}$, to be defined in Section 2, as the morphisms. Fibered products define a composition of isomorphism classes of certain spans in Set that seems strikingly similar to the way in which classical mechanical systems compose.

We propose that open classical systems are morphisms in an appropriate span category, where composition of morphisms using pullbacks describes the composition of physical systems. This formalization of classical mechanics should deepen our understanding of the foundations of classical mechanics and may also offer a way to automate the modeling of classical mechanical systems. Modeling open classical mechanical systems necessitates working with spans in categories other than Set, where the fibered product lacks the universal properties that it has in Set.

It is natural to view a physical system as an isomorphism class of spans in the category of Riemannian manifolds with surjective Riemannian submersions in the Lagrangian setting, or as an isomorphism class of spans in the category of symplectic manifolds with surjective Poisson maps in the Hamiltonian setting. However, Section 2 demonstrates that neither of these categories has pullbacks. Thus, the work of Bénabou does not apply. For this same reason, it does not appear that the work of Fong [18, 19] as corrected by Courser [4, 14] can be straightforwardly modified from its cospan setting to a span setting that is useful to the present discussion. Derived geometry [11, 27] would let us use homotopy pullbacks instead of pullbacks, but in some sense this is overkill: the fibered products required for the current paper will exist and be smooth manifolds; only the universality condition of a pullback fails.

Section 2 recalls previous work required for handling this problem. Suppose that $\mathscr{C}$ and $\mathscr{C}'$ are categories and $\mathcal{F}$ is a functor from $\mathscr{C}$ to $\mathscr{C}'$. Weisbart and Yassine defined in [30] the notion of an "$\mathcal{F}$-pullback" of a cospan in $\mathscr{C}$ and the "span tightness" of the functor $\mathcal{F}$. They proved that if the functor $\mathcal{F}$ is span tight, then $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ is a category, a "generalized span category", whose objects are the objects in $\mathscr{C}$ and whose morphisms are isomorphism classes of spans in $\mathscr{C}$. Composition in this generalized span category is defined using $\mathcal{F}$-pullbacks. Generalized span categories determine the kinematical properties of open classical systems in the Hamiltonian setting and of "free" open systems in the Lagrangian setting—that is, systems where all the energy is kinetic.

In Section 3 we introduce the notion of an "augmented" span, which allows us to introduce nonzero Hamiltonians and add potentials to Lagrangians. In Section 4 we construct the augmented generalized span categories HamSy and LagSy. In the Hamiltonian setting, the augmentation determines the dynamical evolution of the system. In the Lagrangian setting, the augmentation determines the potential for the physical system, hence its dynamics as well. The categories LagSy and HamSy provide frameworks for studying open classical systems from the Lagrangian and Hamiltonian perspectives, respectively. Section 5 introduces a functor $\mathscr{L}\colon \mathsf{LagSy} \to \mathsf{HamSy}$. This functor, a version of the Legendre transformation, translates from the Lagrangian to the Hamiltonian perspective.

In future work we hope to compare the present approach to the theory of port-Hamiltonian systems, an approach to open systems in classical mechanics widely used in engineering [29].

Acknowledgements. We thank Professor Leonid Polterovich for directing us to [15] and [24], and for a conversation with Adam Yassine at a workshop at MSRI that guided us away from a fruitless direction.

2. Spans and Generalized Span Categories

2.1. **Spans and Span Categories.** Refer to [30] for further discussion of the material presented in the following subsection. We review for the reader's convenience some of the definitions and basic results from [30] that the current discussion requires.

A *span* in a category $\mathscr{C}$ is a pair of morphisms in $\mathscr{C}$ with the same source and a *cospan* in $\mathscr{C}$ is a pair of morphisms in $\mathscr{C}$ with the same target. For any span S in $\mathscr{C}$, write

$$S = (s_L, s_R)\,,$$

where S_L, S_R, and S_A are objects in $\mathscr{C}$,

$$s_L \colon S_A \to S_L, \quad \text{and} \quad s_R \colon S_A \to S_R.$$

Utilize the same notation if S is a cospan, but where s_L and s_R respectively map S_L and S_R to S_A. For any span or cospan S in $\mathscr{C}$, refer respectively to the objects S_A, S_L, and S_R in $\mathscr{C}$ as the apex, left foot, and right foot of S.

$$\begin{tikzcd}
 & S_A \arrow[dl, "s_L"'] \arrow[dr, "s_R"] & \\
C_L = S_L \arrow[dr, "c_L"'] & & C_R = S_R \arrow[dl, "c_R"] \\
 & C_A &
\end{tikzcd}$$

Diagram 2. The Pairing of S with C

$$\begin{tikzcd}
 & S_A \arrow[dl, "s_L"'] \arrow[dr, "s_R"] \arrow[dd, "\Phi"'] & \\
S_L = Q_L & & S_R = Q_R \\
 & Q_A \arrow[ul, "q_L"] \arrow[ur, "q_R"'] &
\end{tikzcd}$$

Diagram 3. A Span Morphism from S to Q

Definition 2.1. A span S in $\mathscr{C}$ is *paired with* a cospan C in $\mathscr{C}$ if

$$C_L = S_L, \quad C_R = S_R, \quad \text{and} \quad c_L \circ s_L = c_R \circ s_R.$$

The pairing of a span S with a cospan C has a diagrammatical interpretation, namely that Diagram 2 is commutative.

Suppose that S and Q are spans in $\mathscr{C}$ with S_L equal to Q_L and S_R equal to Q_R. A *span morphism in $\mathscr{C}$ from S to Q* is a morphism Φ (Diagram 3) in $\mathscr{C}$ from S_A to Q_A with

$$s_L = q_L \circ \Phi \quad \text{and} \quad s_R = q_R \circ \Phi.$$

A *span isomorphism in $\mathscr{C}$ from S to Q* is a span morphism that is additionally an isomorphism.

Proposition 2.2. *For any span isomorphism Φ, the inverse Φ^{-1} is also a span isomorphism. Furthermore, any composite of span morphisms is again a span morphism.*

Definition 2.3. A span S in $\mathscr{C}$ is a *pullback* of a cospan C in $\mathscr{C}$ if it is paired with C and if for any other span Q in $\mathscr{C}$ that is also paired with C there exists a unique span morphism Φ in $\mathscr{C}$ from Q to S (Diagram 4).

Notice that the diagram formed by pairing a span S with a cospan C, where S is a pullback of C, is a pullback diagram or a pullback square as often discussed in the literature.

Definition 2.4. A category $\mathscr{C}$ *has pullbacks* if for any cospan C in $\mathscr{C}$ there is a span S in $\mathscr{C}$ that is a pullback of C and S is unique up to a span isomorphism in $\mathscr{C}$.

Denote by Top the category whose objects are topological spaces and whose morphisms are continuous functions. The categories Set and Top are examples of categories that have pullbacks, as discussed in [30]. If C is a cospan in Set, then let ρ_L and ρ_R be the canonical projections

$$\rho_L \colon C_L \times C_R \to C_L \quad \text{and} \quad \rho_R \colon C_L \times C_R \to C_R.$$

Denote by S_A the fibered product

$$C_L \times_{C_A} C_R := \{(x, y) \in C_L \times C_R \colon (c_L \circ \rho_L)(x, y) = (c_R \circ \rho_R)(x, y)\}.$$

Take S_L and S_R to respectively equal C_L and C_R, and let s_L and s_R be the respective restrictions of ρ_L and ρ_R to the set S_A. The span (s_L, s_R) is a pullback of C. If C is a cospan in Top, then S is again a pullback of C in Top, where the topology on S_A is the subspace topology induced by the product topology on $S_L \times S_R$.

$$\begin{tikzcd}
Q_A \arrow[dr, "\exists!\Phi"] \arrow[drr, bend left, "q_R"] \arrow[ddr, bend right, "q_L"'] & & \\
& S_A \arrow[r, "s_R"] \arrow[d, "s_L"'] & C_R \arrow[d, "c_R"] \\
& C_L \arrow[r, "c_L"'] & C_A
\end{tikzcd}$$

Diagram 4. Pullback Diagram

2.2. **The Categories SympSurj and RiemSurj.** Refer to [23] for further background on Poisson geometry. A *Poisson bracket* on a smooth manifold M is an anticommutative, bilinear function from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M)$ that satisfies Leibniz's rule and the Jacobi identity. A *Poisson manifold* is the pair consisting of a smooth manifold M and a Poisson bracket on M. Suppose that $(M, \{\cdot, \cdot\}_M)$ and $(N, \{\cdot, \cdot\}_N)$ are Poisson manifolds. For each f in $C^\infty(M)$, the *Poisson vector field associated to f* is the derivation v_f given by

$$v_f(\cdot) = \{\cdot, f\}_M.$$

A smooth map Φ from M to N is a *Poisson map* if for any f and g in $C^\infty(N)$,

$$\{f, g\}_N \circ \Phi \;=\; \{f \circ \Phi, g \circ \Phi\}_M \,.$$

Symplectic manifolds are the primary objects of study in Hamiltonian mechanics. A *symplectic manifold* is a pair (M, ω_M) where M is a smooth (necessarily even dimensional) manifold and ω_M is a smooth, closed, nondegenerate 2-form on M, a *symplectic 2-form*. Suppose that the dimension of M is $2m$. For each x in M, there is a chart U containing x such that the symplectic 2-form gives rise to *Darboux coordinates* $(q_i, p_i)_{i=1}^m$ on U, coordinates such that

$$\omega_M = \sum_{i=1}^m \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

The symplectic 2-form naturally distinguishes position and momentum coordinates on M and induces an isomorphism Ω_M between the tangent and cotangent bundles. Given tangent vectors v and w in the same fiber of TM, define by $\Omega_M(v)$ the covector

$$\Omega_M(v) = \omega_M(\cdot, v)\colon w \mapsto \omega_M(w, v).$$

Since ω_M is nondegenerate, the map Ω_M is invertible. For each function f in $C^\infty(M)$, denote by D_f the *symplectic gradient of* f, which is defined by

$$D_f = \Omega_M^{-1}(\mathrm{d}f).$$

Every symplectic manifold has a Poisson structure that it inherits from its symplectic structure in the following way. For any symplectic manifold (M, ω_M), define a Poisson bracket $\{\cdot,\cdot\}_M$ on pairs (f, g) in $C^\infty(M) \times C^\infty(M)$ by

$$\{f, g\}_M = \omega_M(D_f, D_g)\,.$$

The symplectic gradient D_f is the Poisson vector field v_f associated to f, implying that

$$\{f, g\}_M = \omega_M(v_f, v_g)\,.$$

The real valued function Π_M defined by

$$\Pi_M(\mathrm{d}f, \mathrm{d}g) = \{f, g\}_M$$

is a global section of $(T^*M \wedge T^*M)^*$. The *Poisson bivector of* $(M, \{\cdot,\cdot\}_M)$ is the image of the function Π_M under the canonical isomorphism that takes $(T^*M \wedge T^*M)^*$ to $\Lambda^2 TM$. To simplify notation, denote henceforth by Π_M the Poisson bivector of $(M, \{\cdot,\cdot\}_M)$. Refer to [12, p. 30] for Proposition 2.5 and see [12, p. 44] for a proof of Proposition 2.6.

Proposition 2.5. *A smooth map Φ from $(M, \{\cdot,\cdot\}_M)$ to $(N, \{\cdot,\cdot\}_N)$ is a Poisson map if and only if*

$$\mathrm{d}\Phi(\Pi_M) = \Pi_N.$$

Proposition 2.6. *Suppose that $(M, \{\cdot,\cdot\}_M)$ is a Poisson manifold and (N, ω_N) symplectic manifold. Every Poisson map from M to N is a submersion.*

Riemannian manifolds are the primary objects of study in Lagrangian mechanics. The metric on the tangent bundle of a Riemannian manifold gives a kinetic energy associated to a particle moving in the base manifold which is the configuration space for the system, [2, p.83-84]. A *Riemannian submersion* Φ from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N) is a smooth submersion with the property that if v and w are vector fields tangent to the horizontal space $(\ker(\mathrm{d}\Phi))^{\perp}$, then

$$g_M(v, w) = g_N(\mathrm{d}\Phi(v)\,, \mathrm{d}\Phi(w))\,.$$

Table 1 specifies the categories to be henceforth denoted by Diff, $\mathsf{SurjSub}$, $\mathsf{RiemSurj}$, and $\mathsf{SympSurj}$.

Category Name	Objects	Morphisms
Diff	Smooth manifolds	Smooth maps
$\mathsf{SurjSub}$	Smooth manifolds	Surjective submersions
$\mathsf{RiemSurj}$	Riemannian manifolds	Surjective Riemannian submersions
$\mathsf{SympSurj}$	Symplectic manifolds	Surjective Poisson maps

Table 1

A example in [30] shows that the category $\mathsf{SurjSub}$ does not have pullbacks. Since this example involves manifolds that have trivial Riemannian and symplectic structures and mappings in the respective categories, the categories $\mathsf{RiemSurj}$ and $\mathsf{SympSurj}$ also do not have pullbacks.

2.3. **$\mathcal{F}$-Pullbacks and Span Tight Functors.** Assume henceforth that $\mathscr{C}$ and $\mathscr{C}'$ are categories and that $\mathcal{F}$ is a functor from $\mathscr{C}$ to $\mathscr{C}'$. For any span S in $\mathscr{C}$, denote by $\mathcal{F}(S)$ the span $(\mathcal{F}(s_L), \mathcal{F}(s_R))$. For any cospan C in $\mathscr{C}$, denote by $\mathcal{F}(C)$ the cospan $(\mathcal{F}(c_L), \mathcal{F}(c_R))$ in $\mathscr{C}'$.

Definition 2.7. The category $\mathscr{C}$ *has $\mathcal{F}$-pullbacks in $\mathscr{C}'$* if for any cospan C in $\mathscr{C}$, there is a span S in $\mathscr{C}$ that is paired with C and the span $\mathcal{F}(S)$ is a pullback of the cospan $\mathcal{F}(C)$ in $\mathscr{C}'$. In this case, the span S is an *$\mathcal{F}$-pullback* of C.

Note that if $\mathscr{C}'$ is equal to $\mathscr{C}$ and $\mathcal{F}$ is the identity functor, then an $\mathcal{F}$-pullback is simply a pullback.

Definition 2.8. Suppose that $\mathscr{C}$ has $\mathcal{F}$-pullbacks in $\mathscr{C}'$. The functor $\mathcal{F}$ is *span tight* if for any $\mathcal{F}$-pullbacks S and Q of the same cospan, the unique span isomorphism Φ from $\mathcal{F}(S)$ to $\mathcal{F}(Q)$ is $\mathcal{F}(\Psi)$ for some span isomorphism Ψ from S to Q.

Definition 2.9. For any two spans S and Q in $\mathscr{C}$ such that S_R is equal to Q_L and there is a span P in $\mathscr{C}$ that is a pullback of the cospan (s_R, q_L), denote by $S \circ_P Q$ the span in $\mathscr{C}$

given by

$$S \circ_P Q = (s_L \circ p_L, q_R \circ p_R).$$

The span $S \circ_P Q$ is the *composite of* S *and* Q *along* P. If P is an $\mathcal{F}$-pullback, then the span $S \circ_P Q$ is an $\mathcal{F}$*-pullback composite of* S *and* Q *along* P.

Identify the objects of $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ to be the objects in $\mathscr{C}$ and the isomorphism classes of spans in $\mathscr{C}$ to be the morphisms in $\mathrm{Span}(\mathscr{C}, \mathcal{F})$. If $[S]$ is an isomorphism class of spans in $\mathrm{Span}(\mathscr{C}, \mathcal{F})$, then identify S_R and S_L respectively to be the source and target of $[S]$. Define composition of isomorphism classes of spans by

$$\left[S^1\right] \circ \left[S^2\right] = \left[S^1 \circ_P S^2\right],$$

where $S^1 \circ_P S^2$ is an $\mathcal{F}$-pullback composite of S^1 and S^2. For any object X in $\mathscr{C}$, denote by Id_X the identity morphism from X to X. Define by $[(\mathrm{Id}_X, \mathrm{Id}_X)]$ the identity morphism in $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ from X to X. The following theorem is the main result of [30].

Theorem 2.10. *If* $\mathcal{F}$ *is a span tight functor from* $\mathscr{C}$ *to* $\mathscr{C}'$*, then* $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ *is a category.*

Suppose that X, Y, and Z are smooth manifolds. Suppose further that (f, g) is a cospan in $\mathsf{SurjSub}$ where f and g have respective sources X and Y and mutual target Z. Again denote by ρ_X and ρ_Y the respective projections from $X \times Y$ to X and Y and let π_X and π_Y be their respective restrictions to the embedded submanifold $X \times_Z Y$. The article [30] proves Proposition 2.11. Proposition 2.11 and Theorem 2.10 together imply that $\mathrm{Span}(\mathsf{SurjSub}, \mathcal{F})$ is a category, where $\mathcal{F}$ is the forgetful functor from $\mathsf{SurjSub}$ to Diff.

Proposition 2.11. *The span* (π_X, π_Y) *is an* $\mathcal{F}$*-pullback of* (f, g) *and so* $\mathsf{SurjSub}$ *has* $\mathcal{F}$*-pullbacks. Moreover, the functor* $\mathcal{F}$ *is span tight.*

Since we will need to work in the categories $\mathsf{SympSurj}$ and $\mathsf{RiemSurj}$, we will need to prove a similar result for these categories. The following section will provide such a result.

3. Lagrangian and Hamiltonian Systems

The description of a Lagrangian or Hamiltonian system respectively requires not only the identification of a Riemannian or Poisson span, but the additional information of a potential or a Hamiltonian, both of which are augmentations.

3.1. **Systems as Isomorphism Classes of Augmented Spans.**

We now introduce the notion of an augmentation of a span and cospan, but in the restricted settings that are significant to the current discussion. We will discuss augmentations in greater generality in an upcoming paper.

Definition 3.1. An *augmented manifold* is a pair (M, F_M), where M is a smooth manifold and F_M is a smooth real valued function defined on M. The pair (M, F_M) is an *augmented Riemannian* (*symplectic*) *manifold* if M is a Riemannian (symplectic) manifold. Refer to F_M as a *potential* (*or Hamiltonian*), denoting it by V_M (or H_M) if M is respectively a Riemannian (or symplectic) manifold.

For sake of concision, denote by $\mathfrak{M}$ any of the categories listed in Table 1.

Definition 3.2. An *augmented* (*co*)*span* in $\mathfrak{M}$ is a pair (S, F_S), where S is a (co)span in $\mathfrak{M}$ and F_S is a triple $(F_{S_A}, F_{S_L}, F_{S_R})$ of smooth real valued functions defined respectively on S_A, S_L, and S_R. If $\mathfrak{M}$ is $\mathsf{RiemSurj}$ (or $\mathsf{SympSurj}$), then the given augmented span is an *augmented Riemannian* (*co*)*span* (or *augmented Poisson* (*co*)*span*). A *physical* (*co*)*span* is an augmented (co)span that is either Riemannian or Poisson. If (S, F_S) is an augmented Riemannian (Poisson) span, then refer to F_S as a *potential* (*or Hamiltonian*) and denote it by V_S (or H_S).

The apex of a Poisson span determines the kinematical properties of the system and the mapping of the apex to its feet determines the way in which the span composes with other spans and, therefore, how components of systems compose to form more complicated systems. The apex of a Riemannian span determines a free system and the augmentation will be a potential that determines the interactions in the system. The fundamental object of our study should be an isomorphism class of augmented spans rather than an augmented span because composition using $\mathcal{F}$-pullbacks is only determined up to isomorphism.

Definition 3.3. Suppose that physical spans (S, F_S) and (Q, F_Q) are either both Riemannian or both Poisson and that

$$(S_L, F_{S_L}) = (Q_L, F_{Q_L}) \quad \text{and} \quad (S_R, F_{S_R}) = (Q_R, F_{Q_R})\,.$$

A span morphism Φ from S_A to Q_A is *compatible with* F_S and F_Q if F_{S_A} is equal to $F_{Q_A} \circ \Phi$ and is, in this case, a *morphism of physical spans*. If Φ is additionally an isomorphism, then Φ is an *isomorphism of physical spans* and (S, F_S) and (Q, F_Q) are *isomorphic physical spans*.

The inverse of an isometry is again an isometry. The inverse of a Poisson diffeomorphism is again a Poisson diffeomorphism, [16, p. 10]. Proposition 3.4 follows from these facts.

Proposition 3.4. *The inverse of any Riemannian* (*or Poisson*) *span isomorphism from S to Q is a Riemannian* (*or Poisson*) *span isomorphism from Q to S.*

Denote by $[S, F_S]$ the set of all physical spans that are isomorphic to a physical span (S, F_S). Together with the fact that the composition of physical span isomorphisms is again a physical span isomorphism, Proposition 3.4 implies that isomorphism of physical spans is an equivalence relation, hence the set $[S, F_S]$ is an equivalence class.

Definition 3.5. A *Lagrangian* (or *Hamiltonian*) *system* is an isomorphism class of Riemannian (or Poisson) spans. If $[S, F_S]$ is either a Hamiltonian system or a Lagrangian system, then $[S, F_S]$ is a *physical system.* Physical systems $[S, F_S]$ and $[Q, F_Q]$ are *of the same type* if they are both Hamiltonian systems or both Lagrangian systems.

3.2. **Paths of Motion.** Suppose that S is a Riemannian span, g_{S_A} is the Riemannian metric on S_A, and V_{S_A} is a potential associated to S_A. Define by ρ_{S_A} the canonical projection from

TS_A to S_A. Define the *Lagrangian of* S on TS_A to be the function $\mathcal{L}_S$, where

$$\mathcal{L}_S(\nu) = \frac{1}{2} g_{S_A}(\nu,\nu) - V_{S_A}(\rho_{S_A}(\nu)) \quad \text{with} \quad \nu \in TS_A.$$

Definition 3.6. A path in the Riemannian manifold (S_A, g_{S_A}) is a *path of motion of* S if it minimizes the action integral of $\mathcal{L}_S$ under smooth variations with fixed endpoints.

Define on each ν in TS_A the function $\flat_{S_A}$ by

$$\flat_{S_A}(\nu) = g_{S_A}(\nu,\cdot).$$

The nondegeneracy of the metric g_{S_A} implies that the map $\flat_{S_A}$ is an invertible function from TS_A to T^*S_A. Define by $\sharp_{S_A}$ the inverse of $\flat_{S_A}$ with

$$\sharp_{S_A} \colon T^*S_A \to TS_A \quad \text{by} \quad \theta \mapsto \nu, \quad \text{where} \quad \theta = g_{S_A}(\nu,\cdot) \quad \text{and} \quad (\theta,\nu) \in T^*S_A \times TS_A.$$

Denote by $\mathrm{grad}_{S_A}(V_{S_A})$ the vector field

$$\mathrm{grad}_{S_A}(V_{S_A}) = \sharp_{S_A}(\mathrm{d}V_{S_A}).$$

Denote by ∇^{S_A} the Levi-Civita connection on the Riemannian manifold (S_A, g_{S_A}). A standard calculation shows that γ is a path of motion of the Riemannian span S if and only if it satisfies

$$\text{(EL)} \qquad \nabla^{S_A}_{\gamma'}\gamma' + \mathrm{grad}_{S_A}(V_{S_A})\big|_{\gamma} = 0,$$

the Euler–Lagrange equations. See [13] for further explanation of the details in this section.

Definition 3.7. Suppose that S is a Poisson span. Denote by $\{\cdot,\cdot\}_{S_A}$ the Poisson bracket associated to the symplectic form ω_{S_A} on the symplectic manifold S_A. A path γ in S_A is a *path of motion of* S if it is an integral curve of the the vector field v where

$$v = \{\cdot, H_{S_A}\}_{S_A}.$$

Proposition 3.8. *Suppose that* (S, F_S) *and* (Q, F_Q) *are physical spans of the same type and* Φ *is an isomorphism of physical spans taking* (S, F_S) *to* (Q, F_Q)*. If* γ *is a path of motion of* (S, F_S)*, then* $\Phi \circ \gamma$ *is a path of motion of* (Q, F_Q)*. Furthermore, every path of motion of* (Q, F_Q) *is the image of a path of motion of* (S, F_S)*.*

Proof. If S and Q are Riemannian spans and Φ is an isomorphism from S to Q, then Φ is an isometry from S_A to Q_A and V_{S_A} is equal to $V_{Q_A} \circ \Phi$. Denote by ∇^{S_A} and ∇^{Q_A} the respective Levi-Civita connections on S_A and Q_A. Suppose that p is an element of S_A and that X and Y are tangent vector fields on S_A. The map Φ is an isometry and so

$$\mathrm{d}\Phi_p\Big(\big(\nabla^{S_A}_X Y\big)(p)\Big) = \nabla^{Q_A}_{\mathrm{d}\Phi(X)}\mathrm{d}\Phi(Y)(\Phi\,(p)) \quad \text{and} \quad \mathrm{d}\Phi\Big(\mathrm{grad}_{S_A}(V_{Q_A}\circ\Phi)\Big) = \mathrm{grad}_{Q_A}(V_{Q_A}).$$

If γ is a path of motion of (S, F_S), then $\Phi \circ \gamma$ is a curve in Q_A and

$$\begin{aligned}\nabla^{Q_A}_{(\Phi\circ\gamma)'}(\Phi\circ\gamma)' + \mathrm{grad}_{Q_A}(V_{Q_A})\,\big|_{\Phi\circ\gamma} &= \nabla^{Q_A}_{\mathrm{d}\Phi(\gamma')}(\mathrm{d}\Phi(\gamma')) + \mathrm{grad}_{Q_A}(V_{Q_A})\,\big|_{\Phi\circ\gamma} \\ &= \mathrm{d}\Big(\nabla^{S_A}_{\gamma'}(\gamma') + \mathrm{grad}_{S_A}(V_{S_A})\,\big|_{\gamma}\Big)\end{aligned}$$

$$= \mathrm{d}(0) = 0,$$

where the fact that γ satisfies (EL) in S_A implies the penultimate equality. The path $\Phi \circ \gamma$ is therefore a path of motion of (Q, F_Q).

If S and Q are Poisson spans and Φ is an isomorphism from S to Q, then Φ is a Poisson diffeomorphism from S_A to Q_A and H_{S_A} is equal to $H_{Q_A} \circ \Phi$. The curve γ is path of motion of (S, F_S) if and only if it is an integral curve of the vector field $\{\cdot, H_{S_A}\}$. Suppose that α and β are smooth functions on Q_A. Since Φ is Poisson,

$$\mathrm{d}\Phi\big(\{\cdot, \alpha \circ \Phi\}_{S_A}\big)(\beta) = \{\cdot, \alpha \circ \Phi\}_{S_A}(\beta \circ \Phi) = \big(\{\beta \circ \Phi, \alpha \circ \Phi\}_{S_A}\big) = \{\beta, \alpha\}_{Q_A}$$

and so

$$\begin{aligned}(\Phi \circ \gamma)' &= \mathrm{d}\Phi\big|_\gamma\big(\{\cdot, H_{S_A}\}_{S_A}\big) \\ &= \mathrm{d}\Phi\big|_\gamma\Big(\{\cdot, H_{Q_A} \circ \Phi\}_{S_A}\Big) = \{\cdot, H_{Q_A}\}_{Q_A}\big|_{\Phi\circ\gamma}.\end{aligned}$$

The curve $\Phi \circ \gamma$ is, therefore, a path of motion of (Q, F_Q).

In both the Riemannian and Poisson settings, the map Φ^{-1} is also an isomorphism of physical spans and so every path of motion of (Q, F_Q) is the image of a path of motion of (S, F_S). □

3.3. **$\mathcal{F}$-Pullbacks of SympSurj and RiemSurj in Diff.** Proposition 3.8 implies that an isomorphism class of physical spans determines the dynamics of a physical system. Composing such isomorphism classes requires both the existence of $\mathcal{F}$-pullbacks in these categories, where $\mathcal{F}$ is an appropriate forgetful functor into Diff, as well as the span tightness of the functor $\mathcal{F}$.

Suppose X is a symplectic manifold. The Poisson bivector Π_X of X induces a map $\widetilde{\Pi}_X$ from T^*X to TX that takes any η in T^*X to the vector field $\widetilde{\Pi}_X(\eta)$ with the property that for any ν in T^*X,

$$\nu\big(\widetilde{\Pi}_X(\eta)\big) = \Pi_X(\eta, \nu).$$

Since X is symplectic, the map $\widetilde{\Pi}_X$ is an isomorphism [12, p. 17]. This isomorphism gives a way to pull back vector fields by surjective Poisson maps, a fact that, along with Proposition 2.6, is critical to the proof of Theorem 3.9. Theorem 3.9 establishes the existence of a local splitting of the tangent space of a symplectic manifold by a local foliation given by the inverse image of a surjective Poisson map.

Theorem 3.9. *Suppose that X and Z are symplectic manifolds with respective dimensions 2ℓ and $2n$ and that f is a surjective Poisson map from X to Z. Given any z in Z and a choice of Darboux coordinates $(q_i^Z, p_i^Z)_{i=1}^n$ on a chart U containing z, and given any x in X with $f(x)$ equal to z, there exist Darboux coordinates $(q_i^X, p_i^X)_{i=1}^\ell$ on a chart V containing x such that for any i in $\{1, \dots, n\}$,*

$$q_i^X = q_i^Z \circ f \quad \text{and} \quad p_i^X = p_i^Z \circ f.$$

Proof. Suppose that x_0 is in X, that U is a chart containing $f(x_0)$, and that $(q_i^Z, p_i^Z)_{i=1}^n$ is a Darboux coordinate system on U. Proposition 2.6 guarantees that f is a surjective submersion, hence it is an open map and so there is a chart V' containing x_0 with a Darboux coordinate system $(q_i^X, p_i^X)_{i=1}^\ell$ such that $f(V')$ is an open subset of U. Denote by $\mathcal{H}$ the set of all vector fields v on $f(V')$ for which there is some α in $C^\infty(f(V'))$ such that for any β in $C^\infty(f(V'))$,

$$v(\beta) = \{\beta, \alpha\}_Z.$$

Denote such a vector field by v_α. Denote by $f^*(\mathcal{H})$ the set of all vector fields w on V' for which there is an α in $C^\infty(f(V'))$ such that for any h in $C^\infty(V')$,

$$w(h) = \{h, \alpha \circ f\}_X.$$

Denote such a vector field by w_α. For any x in V' and any z in $f(V')$, denote respectively by $f^*(\mathcal{H})(x)$ and $\mathcal{H}(z)$ the set of all vector fields in $f^*(\mathcal{H})$ evaluated at x and the set of all vector fields in $\mathcal{H}$ evaluated at z. The bilinearity of the bracket implies that $\mathcal{H}(z)$ and $f^*(\mathcal{H})(x)$ are vectors spaces. Since

$$v_{-q_i^Z} = \frac{\partial}{\partial p_i^Z} \quad \text{and} \quad v_{p_i^Z} = \frac{\partial}{\partial q_i^Z},$$

for any z in $f(V')$, the vector space $\mathcal{H}(z)$ spans $T_z(U)$.

Let F be the function

$$F \colon \mathcal{H} \to f^*(\mathcal{H}) \quad \text{by} \quad F(v_\alpha) = w_\alpha.$$

The fact that f is Poisson implies that

$$\begin{aligned} \mathrm{d}f(w_\alpha)(\beta) &= w_\alpha(\beta \circ f) \\ &= \{\beta \circ f, \alpha \circ f\}_X \\ &= \{\beta, \alpha\}_Z = v_\alpha(\beta), \end{aligned}$$

and so

$$\mathrm{d}f(F(v_\alpha)) = v_\alpha.$$

Similarly, for any w_α in $f^*(\mathcal{H})$,

$$F(\mathrm{d}f(w_\alpha)) = F(v_\alpha) = w_\alpha.$$

The maps F and $\mathrm{d}f|_{\mathcal{H}}$ are therefore inverses of each other and so for each x in V', the vector spaces $\mathcal{H}(f(x))$ and $f^*(\mathcal{H})(x)$ are isomorphic. Both of these vector spaces are of the same dimension as Z.

For any w_α and $w_{\alpha'}$ in $f^*(\mathcal{H})$, the Jacobi identity implies that

$$\begin{aligned} [w_\alpha, w_{\alpha'}]_{TX} &= w_\alpha(w_{\alpha'}(\beta)) - w_{\alpha'}(w_\alpha(\beta)) \\ &= \{w_{\alpha'}(\beta), \alpha \circ f\}_X - \{w_\alpha(\beta), \alpha' \circ f\}_X \\ &= \{\{\beta \circ f, \alpha' \circ f\}_X, \alpha \circ f\}_X - \{\{\beta \circ f, \alpha \circ f\}_X, \alpha' \circ f\}_X \\ &= \{\beta, \{\alpha' \circ f, \alpha \circ f\}_X\}_X = w_{\{\alpha,\alpha'\}}(\beta), \end{aligned}$$

and so the space of vector fields $f^*(\mathcal{H})$ is closed under the bracket $[\cdot,\cdot]_{TX}$ on TX. Frobenius' Theorem for involutive distributions implies that for any x in V' there is a submanifold W of V' such that $f^*(\mathcal{H})(x)$ is the tangent space T_xW. Since

$$f^*(\mathcal{H})(x) \cap \ker(\mathrm{d}f\big|_x) = \{0\},$$

the rank-nullity theorem implies that

$$T_xV' = f^*(\mathcal{H})(x) \oplus \ker(\mathrm{d}f\big|_x).$$

Define the function g from W to Z to be the restriction of f to the submanifold W. The form $g^*(\omega_Z)$ is a closed 2-form on W as the pullback of the closed 2-form ω_Z restricted to $f(V')$. Suppose that there is a v in TW such that for all w in TW, $g^*(\omega_Z)(v,w)$ is equal to zero. In this case,

$$0 = g^*(\omega_Z)(v,w) = \omega_Z(\mathrm{d}g(v), \mathrm{d}g(w)),$$

and so

$$\omega_Z(\mathrm{d}g(v), \cdot) = 0$$

since $\mathrm{d}g\big|_x$ is surjective at each point x of W. Nondegeneracy of ω_Z implies that $\mathrm{d}g(v)$ is equal to zero and the injectivity of $\mathrm{d}g$ further implies that v is equal to zero. The form $g^*(\omega_Z)$ is, therefore, a symplectic form on W.

For any (η,ζ) in $C^\infty(V') \times C^\infty(V')$,

$$\begin{aligned} f^*(\omega_Z)\big|_x(w_\eta, w_\zeta) &= \omega_Z(\mathrm{d}f(w_\eta), \mathrm{d}f(w_\zeta))|_{f(x)} \\ &= \omega_Z(v_\eta, v_\zeta)\big|_{f(x)} \\ &= \{\eta,\zeta\}_Z\big|_{f(x)} \\ &= \{\eta\circ f, \zeta\circ f\}_X\big|_x = \omega_X(w_\eta, w_\zeta)\big|_x, \end{aligned} \tag{1}$$

where the assumption that f is Poisson implies the penultimate equality. The pullback $f^*(\omega_Z)$ is therefore the restriction of ω_X to $TW \times TW$. The manifold W is an embedded symplectic submanifold of V' and so [25, p.124, Exercise 3.38] implies that there is an open set V of V' that contains x_0 and a Darboux coordinate system $(q_i^X, p_i^X)_{i=1}^{\ell}$ on V such that for any x in V and i strictly larger than n,

$$q_i^X(x) = p_i^X(x) = 0.$$

Define

$$\omega_A = \sum_{i=1}^{n} \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X \quad \text{and} \quad \omega_B = \sum_{i=n+1}^{\ell} \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X,$$

so that in the open set V, ω_X is equal to the sum of ω_A and ω_B. The form ω_B is the restriction of ω_X to $(TW \times TW) \cap (TV \times TV)$ and so (1) implies that ω_B is equal to $f^*(\omega_X)$. Furthermore, for any θ in $C^\infty(U)$,

$$\begin{aligned} (f^*(\mathrm{d}q_i^Z))(w_\theta)|_x &= \mathrm{d}q_i^Z(\mathrm{d}f(w_\theta))\big|_x \\ &= \mathrm{d}q_i^Z(v_\theta)\big|_{f(x)} \end{aligned}$$

$$
\begin{aligned}
&= v_\theta(q_i^Z)\big|_{f(x)} \\
&= \{q_i^Z, \theta\}_Z\big|_{f(x)} \\
&= \{q_i^Z \circ f, \theta \circ f\}_X\big|_x = \mathrm{d}(q_i^Z \circ f) w_\theta\big|_x.
\end{aligned}
$$

Every element of TW is of the form w_θ for some θ in $C^\infty(U)$, implying that

$$
f^*(\mathrm{d}q_i^Z) = \mathrm{d}(q_i^Z \circ f) \quad \text{and} \quad f^*(\mathrm{d}p_i^Z) = \mathrm{d}(p_i^Z \circ f). \tag{2}
$$

Use (2) together with the coordinate representation of ω_Z to obtain the equality

$$
f^*(\omega_Z) = \sum_{i=1}^{n} \mathrm{d}(q_i^Z \circ f) \wedge \mathrm{d}(p_i^Z \circ f),
$$

that implies that in the chart V,

$$
\omega_X = \sum_{i=1}^{n} \mathrm{d}(q_i^Z \circ f) \wedge \mathrm{d}(p_i^Z \circ f) + \sum_{i=n+1}^{\ell} \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X.
$$

The coordinate system ϕ on V given by

$$
\phi = (q_1^Z \circ f, p_1^Z \circ f, \ldots, q_n^Z \circ f, p_n^Z \circ f, q_{n+1}^X, p_{n+1}^X, \ldots, q_\ell^X, p_\ell^X)
$$

is, therefore, a Darboux coordinate system on V. □

Denote by π_Z the map

$$
\pi_Z = f \circ \pi_X = g \circ \pi_Y,
$$

where π_X and π_Y are the projections from $X \times_Z Y$ to Z. More generally, for any span Q that is paired with a cospan (f, g), define by q_M the map

$$
q_M = f \circ q_L = g \circ q_R.
$$

Theorem 3.10. *Suppose that (f, g) is a cospan in SympSurj with*

$$
f\colon X \to Z \quad \text{and} \quad g\colon Y \to Z,
$$

with 2ℓ, $2m$, $2n$ the respective dimensions of X, Y, and Z, and suppose that ω_X, ω_Y, and ω_Z are the respective symplectic forms on X, Y, and Z. Suppose that Q is a span in SympSurj that is paired with (f, g) and suppose that Q_A has dimension $2(\ell + m - n)$. The 2-form ω_{Q_A}, given by

$$
\omega_{Q_A} = q_L^*(\omega_X) + q_R^*(\omega_Y) - q_M^*(\omega_Z)\,,
$$

is the symplectic form on Q_A. Moreover, the 2-form ω, given by

$$
\omega = \pi_X^*(\omega_X) + \pi_Y^*(\omega_Y) - \pi_Z^*(\omega_Z)
$$

is the unique symplectic form on $X \times_Z Y$ with the property that (π_X, π_Y) is paired with (f, g).

Proof. Suppose that a is in Q_A. Since Z is a symplectic manifold, there is on some chart U_Z containing $q_M(a)$ a Darboux coordinate system Ψ^Z with

$$\Psi^Z = \left(q_k^Z, p_k^Z\right)_{k\in\{1,\ldots,n\}} : U_Z \to \mathbb{R}^{2n}.$$

Since $q_M(a)$ is equal to $f(q_L(a))$, Theorem 3.9 implies that there is a chart U_X containing $q_L(a)$ and a Darboux coordinate system Ψ^X on U_X with

$$\Psi^X = \left(q_i^X, p_i^X, q_k^Z \circ f, p_k^Z \circ f\right)_{\substack{i\in\{1,\ldots,\ell-n\}\\ k\in\{1,\ldots,n\}}} : U_X \to \mathbb{R}^{2\ell}.$$

Similarly, there is a chart U_Y containing $q_R(a)$ and a Darboux coordinate system Ψ^Y on U_Y with

$$\Psi^Y = \left(q_j^Y, p_j^Y, q_k^Z \circ g, p_k^Z \circ g\right)_{\substack{j\in\{1,\ldots,m-n\}\\ k\in\{1,\ldots,n\}}} : U_Y \to \mathbb{R}^{2m}.$$

For each k in $\{1,\ldots,n\}$, the equality of $f\circ q_L$ and $g \circ q_R$ implies that

$$q_k^Z \circ f \circ q_L = q_k^Z \circ g \circ q_R = q_k^Z \circ q_M \quad \text{and} \quad p_k^Z \circ f \circ q_L = p_k^Z \circ g \circ q_R = p_k^Z \circ q_M.$$

Furthermore, there is a chart U containing a with the property that $q_L(U)$ and $q_R(U)$ are, respectively, subsets of U_X and U_Y. Denote respectively by $\tilde{q}_i^X$, $\tilde{p}_i^X$, $\tilde{q}_j^Y$, $\tilde{p}_j^Y$, $\tilde{q}_k^Z$, $\tilde{p}_k^Z$ the functions $q_i^X \circ q_L$, $p_i^X \circ q_L$, $q_j^Y \circ q_R$, $p_j^Y \circ q_R$, $q_k^Z \circ q_M$, and $p_k^Z \circ q_M$ acting on Q_A. The map Ψ given by

$$\Psi = \left(\tilde{q}_i^X, \tilde{p}_i^X, \tilde{q}_j^Y, \tilde{p}_j^Y, \tilde{q}_k^Z, \tilde{p}_k^Z\right)_{\substack{i\in\{1,\ldots,\ell-n\}\\ j\in\{1,\ldots,m-n\}\\ k\in\{1,\ldots,n\}}} : U \to \mathbb{R}^{2(\ell+m-n)}$$

is a homeomorphism from U to an open subset of $\mathbb{R}^{2(\ell+m-n)}$ and hence a coordinate system on U that is a Darboux coordinate system. The 2-form ω_{Q_A} is therefore the form

$$\omega_{Q_A} = \sum_{i=1}^{\ell-n} \mathrm{d}\tilde{q}_i^X \wedge \mathrm{d}\tilde{p}_i^X + \sum_{j=1}^{m-n} \mathrm{d}\tilde{q}_j^Y \wedge \mathrm{d}\tilde{p}_j^Y + \sum_{k=1}^{n} \mathrm{d}\tilde{q}_k^Z \wedge \mathrm{d}\tilde{p}_k^Z,$$

proving that if there is a span Q with the given properties, then the symplectic form on Q_A is determined by the cospan (f,g). It does not, however, prove that there is such a span.

Proposition 3.6 of [30] implies that $X \times_Z Y$ is a smooth manifold of dimension $2(\ell+m-n)$. Suppose v is in $T_a(X\times_Z Y)$ and for any w in $T_a(X \times_Z Y)$, $\omega(v,w)$ is zero. There are coefficients $a^i, b^i, c^j, e^j, s^k, t^k$ such that, using Einstein summation convention,

$$v = a^i \partial\tilde{q}_i^X + b^i \partial\tilde{p}_i^X + c^j \partial\tilde{q}_j^Y + e^j \partial\tilde{p}_j^Y + s^k \partial\tilde{q}_k^Z + t^k \partial\tilde{p}_k^Z.$$

For a fixed i,

$$-\omega(v, \partial\tilde{q}_i^X) = b^i = 0.$$

A similar calculation shows that all of the given coefficients are zero, implying that v is equal to zero and so ω is nondegenerate. The form ω is the sum of pullbacks of smooth closed forms, and so smooth and closed itself, hence symplectic. The construction of ω ensures that the smooth surjections π_X and π_Y are Poisson maps on the symplectic manifold $(X\times_Z Y, \omega)$, hence (π_X, π_Y) is paired with (f,g). $\square$

Theorem 3.11. *Suppose that* (f, g) *is a cospan in* RiemSurj *with*

$$f\colon X \to Z \quad \text{and} \quad g\colon Y \to Z$$

and that g_X*,* g_Y*, and* g_Z *are the metric tensors on* X*,* Y*, and* Z*, respectively. The tensor* $g_{X\times_Z Y}$*, given by*

$$g_{X\times_Z Y} = \pi_X^*(g_X) + \pi_Y^*(g_Y) - \pi_Z^*(g_Z)\,,$$

is the unique metric tensor on $X \times_Z Y$ *such that the span* (π_X, π_Y) *is paired with* (f, g)*.*

Proof. Since every surjective Riemannian submersion is a surjective submersion, the fibered product $X \times_Z Y$ is a smooth manifold. If $g_{X\times_Z Y}$ is positive definite, then $(X \times_Z Y, g_{X\times_Z Y})$ is a Riemannian manifold since $g_{X\times_Z Y}$ is a symmetric tensor as a sum of pullbacks of symmetric tensors. It suffices to show that $g_{X\times_Z Y}$ is nondegenerate.

Follow the proof of Theorem 3.10, using the splitting of the tangent spaces

$$TX = (\ker(\mathrm{d}f))^\perp \oplus (\ker(\mathrm{d}f)) \quad \text{and} \quad TY = (\ker(\mathrm{d}g))^\perp \oplus (\ker(\mathrm{d}g))$$

rather than the previous appeal to Theorem 3.9 to obtain an expression for $g_{X\times_Z Y}$ in local coordinates. Together with this local coordinate representation of $g_{X\times_Z Y}$, the fact that the maps π_X, π_Y and π_Z are surjective Riemannian submersions implies that $g_{X\times_Z Y}$ is nondegenerate. The proof is similar to the proof of Theorem 3.10 and so the details are left to the reader to verify. □

Note that the symplectic form on $X \times_Z Y$ in Theorem 3.10 is not the pullback by the inclusion map of the symplectic form on $X \times Y$ to the manifold $X \times_Z Y$. While the pullback form is symplectic, the span (π_X, π_Y) will no longer be a span in SympSurj when $X \times_Z Y$ is endowed instead with the pullback form. The analogous statements about the potential choices for the metric tensor are true in the Riemannian setting.

Theorem 3.12. *The forgetful functors from* SympSurj *to* Diff *and from* RiemSurj *to* Diff *are span tight.*

Proof. Suppose that $\mathcal{F}$ is the forgetful functor from SympSurj to Diff. Since every morphism in SympSurj is a surjective submersion, the functor $\mathcal{F}$ maps SympSurj to the subcategory SurjSub of Diff. If (f, g) is a cospan in SympSurj, and π_X and π_Y are, as defined above, the respective projections from $X \times_Z Y$ to X and Y, then Proposition 2.11 implies that $(\mathcal{F}(\pi_X)\,, \mathcal{F}(\pi_Y))$ is a span in Diff that is a pullback of the cospan $(\mathcal{F}(f)\,, \mathcal{F}(g))$. Therefore, SympSurj has $\mathcal{F}$-pullbacks in Diff. Suppose now that Q is a span in SympSurj that is also an $\mathcal{F}$-pullback of (f, g). In this case, the span $\mathcal{F}(Q)$ is a span in Diff that is a pullback of $(\mathcal{F}(f)\,, \mathcal{F}(g))$ and so there is a span diffeomorphism Φ from $\mathcal{F}(Q)$ to $\mathcal{F}(X \times_Z Y)$. Since Φ is a span morphism,

$$\mathcal{F}(q_L) \circ \Phi^{-1} = \mathcal{F}(\pi_X), \quad \mathcal{F}(q_R) \circ \Phi^{-1} = \mathcal{F}(\pi_Y), \quad \text{and} \quad \mathcal{F}(f) \circ \mathcal{F}(q_L) \circ \Phi^{-1} = \mathcal{F}(\pi_Z). \tag{3}$$

Denote respectively by ω, ω_X, ω_Y, and ω_Z the symplectic forms on $X \times_Z Y$, X, Y, and Z. The equalities of (3) imply that

$$\omega = \mathcal{F}(\pi_X)^*(\omega_X) + \mathcal{F}(\pi_Y)^*(\omega_Y) - \mathcal{F}(\pi_Z)^*(\omega_Z)$$

$$
\begin{aligned}
&= \left(\mathcal{F}(q_L)\circ\Phi^{-1}\right)^*(\omega_X) + \left(\mathcal{F}(q_R)\circ\Phi^{-1}\right)^*(\omega_Y) - \left(\mathcal{F}(f)\circ\mathcal{F}(q_L)\circ\Phi^{-1}\right)^*(\omega_Z)\\
&= \left(\Phi^{-1}\right)^*\Big(\mathcal{F}(q_L)^*(\omega_X) + \mathcal{F}(q_R)^*(\omega_Y) - (\mathcal{F}(f)\circ\mathcal{F}(q_L))^*(\omega_Z)\Big)\\
&= \left(\Phi^{-1}\right)^*(\omega_{Q_A}),
\end{aligned}
$$

where ω_{Q_A} is the unique 2-form on Q_A such that Q is paired with (f,g). Let Ψ be the map from (Q_A,ω_{Q_A}) to $(X\times_Z Y,\omega)$ that acts as Φ on the underlying manifolds. The map Ψ is, therefore, a diffeomorphism and Ψ^{-1} is a symplectic map, hence Ψ is a symplectomorphism. Since every symplectomorphism is a Poisson diffeomorphism, Ψ isomorphism in the category SympSurj with $\mathcal{F}(\Psi)$ equal to Φ, [1, p. 195].

A similar argument proves the theorem in the case of RiemSurj. □

Corollary. *If $\mathcal{F}$ is the forgetful functor from SympSurj to Diff (resp. RiemSurj to Diff), then* $\mathrm{Span}(\textit{SympSurj},\mathcal{F})$ *(resp.* $\mathrm{Span}(\textit{RiemSurj},\mathcal{F})$*) is a category.*

While Theorems 2.10 and 3.12 imply that $\mathrm{Span}(\mathsf{SympSurj},\mathcal{F})$ and $\mathrm{Span}(\mathsf{RiemSurj},\mathcal{F})$ are categories, where $\mathcal{F}$ is the appropriate forgetful functor into Diff, to show that physical systems are morphisms of a category requires additional verifications. The next section provides the necessary verifications.

4. Physical Systems as Morphisms

This section constructs the categories LagSy and HamSy, whose objects are respectively augmented Riemannian manifolds or augmented symplectic manifolds and whose morphisms are isomorphism classes of the physical spans appropriate to the given category.

4.1. The Categories HamSy and LagSy.

Definition 4.1. The physical system $[S,F_S]$ is *composable* with the physical system $[Q,F_Q]$ if:

(1) both are physical systems of the same type;
(2) if (S,F_S) and (Q,F_Q) are respective representatives of the equivalence classes $[S,F_S]$ and $[Q,F_Q]$, then (S_R,F_{S_R}) is equal to (Q_L,F_{Q_L}).

Assume below that the physical system $[S,F_S]$ is composable with $[Q,F_Q]$, and (S,F_S) and (Q,F_Q) are, respectively, representatives of $[S,F_S]$ and $[Q,F_Q]$. To simplify notation, let

$$S_A = X,\ S_L = V,\ S_R = Q_L = Z,\ Q_A = Y,\ \text{and } Q_R = W.$$

Again denote by $X\times_Z Y$ the fibered product and by π_X, π_Y, and π_Z the respective projections to X, Y, and Z. Define by $[S,F_S]\circ[Q,F_Q]$ the augmented span given by

$$[S,F_S]\circ[Q,F_Q] = [(s_L\circ\pi_X, q_R\circ\pi_Y), F_{S\circ Q}],$$

where

$$F_{S\circ Q} = (F_X\circ\pi_X + F_Y\circ\pi_Y - F_Z\circ\pi_Z, F_V, F_W).$$

Theorem 4.2. *The Hamiltonian systems are the morphisms in a category, HamSy, whose objects are augmented symplectic manifolds. The Lagrangian systems are the morphisms in a category, LagSy, whose objects are augmented Riemannian manifolds.*

Proof. To prove the theorem, it suffices to show that: (1) composition of morphisms in HamSy and in LagSy is well defined; (2) both HamSy and LagSy have left and right unit laws; and (3) composition of morphisms in HamSy and in LagSy is associative. Since $\mathrm{Span}(\mathsf{RiemSurj}, \mathcal{F})$ and $\mathrm{Span}(\mathsf{SympSurj}, \mathcal{F})$ are categories, to show that HamSy and LagSy are categories, it suffices to show that the augmentations are compatible with the various span isomorphisms that arise in defining the categories $\mathrm{Span}(\mathsf{RiemSurj}, \mathcal{F})$ and $\mathrm{Span}(\mathsf{SympSurj}, \mathcal{F})$. Suppose that $[S, F_S]$ and $[Q, F_Q]$ are both morphisms in HamSy and denote by $\mathcal{F}$ the forgetful functor from $\mathsf{SympSurj}$ to Diff.

(1) Suppose that $[S', F_{S'}]$ is equal to $[S, F_S]$ and that α is an isomorphism of augmented spans with

$$\alpha\colon X = S_A \to S'_A.$$

Suppose that $[Q', F_{Q'}]$ is equal to $[Q, F_Q]$ and that β is an isomorphism of augmented spans with

$$\beta\colon Y = Q_A \to Q'_A.$$

Since (Z, F_Z) is the right foot of (S, F_S) and the left foot of (Q, F_Q),

$$\left(S'_R, F_{S'_R}\right) = \left(Q'_L, F_{Q'_L}\right) = (Z, F_Z)\,.$$

If P is an $\mathcal{F}$-pullback of (s'_R, q'_L), then there is a span isomorphism Φ in $\mathsf{SympSurj}$ with

$$\Phi\colon X \times_Z Y \to P_A.$$

The augmented span $(S', F_{S'}) \circ_P (Q', F_{Q'})$ is given by

$$(S', F_{S'}) \circ_P (Q', F_{Q'}) = \left(\left(s'_L \circ p_L, q'_R \circ p_R\right), F_{S' \circ_P Q'}\right),$$

where

$$F_{S' \circ_P Q'} = \left(F_{S'_A} \circ p_L + F_{Q'_A} \circ p_R - F_Z \circ s'_R \circ p_L, F_V, F_W\right).$$

Since α and β are isomorphisms of augmented spans,

$$F_{S'_A} \circ \alpha = F_X \quad \text{and} \quad F_{Q'_A} \circ \beta = F_Y.$$

The function Φ is a span isomorphism and so

$$p_L \circ \Phi = \alpha \circ \pi_X \quad \text{and} \quad p_R \circ \Phi = \beta \circ \pi_Y,$$

hence

$$F_{S'_A} \circ p_L \circ \Phi = F_{S'_A} \circ \alpha \circ \pi_X = F_X \circ \pi_X.$$

Similar arguments show that

$$F_{Q'_A} \circ p_R \circ \Phi = F_Y \circ \pi_Y \quad \text{and} \quad F_Z \circ s'_R \circ p_L \circ \Phi = F_Z \circ \pi_Z,$$

and so

$$F_{S \circ Q} = (F_{S' \circ_P Q'}) \circ \Phi. \tag{4}$$

Equality (4) implies that Φ is an augmented span isomorphism, hence the composition of $[S, F_S]$ and $[Q, F_Q]$ is independent of representative. The composite $[S, F_S] \circ [Q, F_Q]$ is, therefore, a well defined morphism from (Q_R, F_{Q_R}) to (S_L, F_{S_L}).

$$\begin{tikzcd}
 & & & P_A^3 \arrow[dl, "p_L^3"'] \arrow[dr, "p_R^3"] \arrow[dd, "m^3"] & & & P_A^4 \arrow[lll, "\Phi"'] \arrow[dllll, dashed, bend right, "p_L^4"'] \arrow[dddll, dashed, "m^4"'] \arrow[ddl, dashed, bend left, "p_R^4"] \\
 & & P_A^1 \arrow[dl, "p_L^1"'] \arrow[dr, "p_R^1"] \arrow[dd, "m^1"] & & P_A^2 \arrow[dl, "p_L^2"'] \arrow[dr, "p_R^2"] \arrow[dd, "m^2"] & & \\
 & S_A \arrow[dl, "s_L"'] \arrow[dr, "s_R"] & & Q_A \arrow[dl, "q_L"'] \arrow[dr, "q_R"] & & T_A \arrow[dl, "t_L"'] \arrow[dr, "t_R"] & \\
S_L & & S_R = Q_L & & Q_R = T_L & & T_R
\end{tikzcd}$$

Diagram 5. Associativity of Augmented Span Composition

(2) Let $[S, F_S]$ be a morphism with source (S_R, F_{S_R}) and target (S_L, F_{S_L}). Let $\left(\mathrm{I}_{S_R}, F_{\mathrm{I}_{S_R}}\right)$ be a representative of the identity augmented span with source (S_R, F_{S_R}) and target (S_R, F_{S_R}). The equality

$$[S] \circ [\mathrm{I}_{S_R}] = [S]$$

follows from the fact that Span(SympSurj, $\mathcal{F}$) is a category. Let the span P be an $\mathcal{F}$-pullback of (s_R, I_{S_R}), where

$$P_L = P_A = S_A,\ P_R = S_R,\ p_L = \mathrm{Id}_{S_A},\ \text{and } p_R = s_R.$$

The equalities

$$\begin{aligned}
F_{P_A} &= F_{S_L} \circ p_L + F_{S_R} \circ s_R - F_{S_R} \circ s_R \circ p_L \\
&= F_{S_L} \circ \mathrm{Id}_{\mathrm{S_A}} + F_{S_R} \circ s_R - F_{S_R} \circ s_R \circ \mathrm{Id}_{S_A} = F_{S_L}
\end{aligned}$$

imply that there is an augmented span isomorphism from $(S, F_S) \circ (\mathrm{I}_{S_R}, F_{S_R})$ to (S, F_S), and so

$$[S, F_S] \circ [\mathrm{I}_{S_R}, F_{S_R}] = [S, F_S]\,.$$

A similar argument shows that

$$[\mathrm{I}_{S_L}, F_{S_L}] \circ [S, F_S] = [S, F_S]\,.$$

Therefore, HamSy has left and right unit laws.

(3) Refer to Diagram 5 for the naming of the maps below, where all spans paired with a given cospan are augmented $\mathcal{F}$-pullbacks of the given cospan and the diagram is commutative. Let (P^3, F_{P^3}) be an $\mathcal{F}$-pullback of (p_R^1, p_L^2) and let (P^4, F_{P^4}) be an $\mathcal{F}$-pullback of $(q_R \circ p_R^1, t_L)$.

To prove (3), show first that there is an augmented span isomorphism from the augmented span $\big((S, F_S) \circ_{(P^1, F_{P^1})} (Q, F_Q)\big) \circ_{(P^4, F_{P^4})} (T, F_T)$ to the augmented span (P, F_P) that is given by the composite $\big((S, F_S) \circ_{(P^1, F_{P1})} (Q, F_Q)\big) \circ_{(P^3, F_{P3})} \big((Q, F_Q) \circ_{(P^2, F_{P2})} (T, F_T)\big)$. A similar argument will show that there is an augmented span isomorphism from the augmented span $(S, F_S) \circ \big((Q, F_Q) \circ (T, F_T)\big)$ to (P, F_P) and the result follows by the fact that inverses and compositions of augmented span isomorphisms are augmented span isomorphisms. Since Lemma 5.3 of [30] proves the existence of a span isomorphism between the non-augmented spans, it suffices to show that this span isomorphism is compatible with the augmentations for the two composite spans.

The commutativity of Diagram 5 and the definition of the composition of augmented spans together imply that

$$\begin{aligned}
F_{P^4_A} &= F_{P^1_A} \circ p_L^4 + F_{T_A} \circ p_R^4 - F_{Q_R} \circ m^4 \\
&= F_{P^1_A} \circ p_L^3 \circ \Phi + F_{T_A} \circ p_R^2 \circ p_R^3 \circ \Phi - F_{Q_R} \circ m^2 \circ p_R^3 \circ \Phi. \\
&= \Big(F_{P^1_A} \circ p_L^3 + F_{T_A} \circ p_R^2 \circ p_R^3 - F_{Q_R} \circ m^2 \circ p_R^3\Big) \circ \Phi \\
&= \Big(F_{P^1_A} \circ p_L^3 + \big(F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\big) \circ p_R^3\Big) \circ \Phi \\
&= \Big(F_{P^1_A} \circ p_L^3 + \big(F_{Q_A} \circ p_L^2 - F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\big) \circ p_R^3\Big) \circ \Phi \\
&= \Big(F_{P^1_A} \circ p_L^3 + \big(F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\big) \circ p_R^3 - F_{Q_A} \circ p_L^2 \circ p_R^3\Big) \circ \Phi \\
&= \Big(F_{P^1_A} \circ p_L^3 + \big(F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\big) \circ p_R^3 - F_{Q_A} \circ m^3\Big) \circ \Phi \\
&= \Big(F_{P^1_A} \circ p_L^3 + F_{P^2} \circ p_R^3 - F_{Q_A} \circ m^3\Big) \circ \Phi \\
&= F_{P^3_A} \circ \Phi.
\end{aligned}$$

Therefore, the span isomorphism Φ is compatible with the augmentations F_{P^4} and F_{P^3}.

The above arguments are independent of the morphisms being in HamSy. Repeat the arguments above in the setting of LagSy to complete the proof of the theorem. $\square$

4.2. **Motivating Example.** Suppose that the spring-mass system with three masses given in Figure 3 has masses m_1, m_2, and m_3 respectively as the left, middle, and right masses of the system. Suppose further that the spring constants of the left and right springs are respectively k_1 and k_2. The spring-mass system with three masses is a composite of two spring-mass systems with two masses each. We now discuss a category theoretic construction of a model for the composite system with its subsystems.

Let $[S, V_S]$ be a Lagrangian system describing the left-spring mass system and $[Q, V_Q]$ be a Lagrangian systems describing the right spring-mass system. Denote both S_R and Q_L by Z, since S_R is equal to Q_L, and by V_Z the augmentation on Z. Take a representative (S, V_S) of the Langrangian system $[S, V_S]$ to be the augmented Riemannian span with the manifold S_A equal to $\mathbb{R}^2$ and the manifolds S_L and Z equal to $\mathbb{R}$. Let g_1 be the standard Riemannian metric on $\mathbb{R}$. Let ρ_L and ρ_R be the canonical projections on $\mathbb{R}^2$ with

$$\rho_L(q_1, q_2) = q_1 \quad \text{and} \quad \rho_R(q_1, q_2) = q_2.$$

Denote by g_2 the standard Riemannian metric on $\mathbb{R}^2$. Endow S_L with the Riemannian metric g_{S_L} and Z with the Riemannian metric g_Z, where g_{S_L} and g_Z are given by

$$g_{S_L} = m_1 g_1 \quad \text{and} \quad g_Z = m_2 g_1.$$

Define by g_{S_A} the metric on $\mathbb{R}^2$ given for all v and w in $T_{(q_1,q_2)}\mathbb{R}^2$ by

$$g_{S_A}(v, w) = g_{S_L}(\mathrm{d}\rho_L(v), \mathrm{d}\rho_L(w)) + g_Z(\mathrm{d}\rho_R(v), \mathrm{d}\rho_R(w)).$$

Denote respectively by s_L and s_R the functions from S_A to S_L and from S_A to Z that act on underlying manifolds as the projections ρ_L and ρ_R. The augmentation V_S is the triple of maps

$$V_S = (V_{S_A}, V_{S_L}, V_Z) \quad \text{with} \quad V_{S_A}(q_1, q_2) = \frac{k_1}{2}(q_1 - q_2)^2, \ V_{S_L} \equiv 0, \text{ and } V_Z \equiv 0.$$

Define similarly the Riemannian span (Q, V_Q), but with the Riemannian metric g_{Q_R} on Q_R and the augmentations V_{Q_A} and V_{Q_R} given by

$$g_{Q_R} = m_3 g_1, \ V_{Q_A}(q_2, q_3) = \frac{k_2}{2}(q_2 - q_3)^2, \text{ and } V_{Q_R} \equiv 0.$$

Define by g_{Q_A} the metric on $\mathbb{R}^2$ given for all v and w in $T_{(q_2,q_3)}\mathbb{R}^2$ by

$$g_{Q_A}(v, w) = g_Z(\mathrm{d}\rho_L(v), \mathrm{d}\rho_L(w)) + g_{Q_R}(\mathrm{d}\rho_R(v), \mathrm{d}\rho_R(w)).$$

$$\begin{tikzcd}
 & & \mathbb{R}^3 \arrow[dl] \arrow[dr] & & \\
 & \mathbb{R}^2 \arrow[dl] \arrow[dr] & & \mathbb{R}^2 \arrow[dl] \arrow[dr] & \\
\mathbb{R} & & \mathbb{R} & & \mathbb{R}
\end{tikzcd}$$

Diagram 6. Configuration Spaces for Three Point Masses

Denote by π_L and π_R the respective projections from $S_A \times_Z Q_A$ to S_A and to Q_A and by π_M the map $s_R \circ \pi_L$, which is also the map $q_R \circ \pi_R$. Denote by $g_{S_A \times_Z Q_A}$ the Riemannian metric on $S_A \times_Z Q_A$ given by

$$g_{S_A \times_Z Q_A} = \pi_L^*(g_{S_A}) + \pi_R^*(g_{Q_A}) - \pi_Z^*(g_Z).$$

The augmentation $V_{S_A \times_Z Q_A}$ is then given by

$$V_{S_A \times_Z Q_A} = \pi_L^*(V_{S_A}) + \pi_R^*(V_{Q_A}) - \pi_M^*(V_Z).$$

Let Φ be the diffeomorphism from $S_A \times_Z Q_A$ to $\mathbb{R}^3$ given by

$$\Phi(q_1, q_2, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_2, \dot{q}_3) = (q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3).$$

Denote by P_A the Riemannian manifold $\mathbb{R}^3$, and by p_L and p_R the maps

$$p_L = s_L \circ \pi_L \circ \Phi^{-1} \quad \text{and} \quad p_R = s_R \circ \pi_R \circ \Phi^{-1}.$$

Denote similarly by V_{P_A} the potential

$$V_{P_A} = V_{S_A \times_Z Q_A} \circ \Phi^{-1}.$$

Define a Riemannian metric g_{P_A} on P_A by

$$g_{P_A} = (\Phi^{-1})^*(g_{S_A \times_Z Q_A}),$$

making Φ an isometry. The Lagrangian for the composite system is $\mathcal{L}_{P_A}$ where for every ν in TP_A,

$$\mathcal{L}_{P_A}(\nu) = \frac{1}{2} g_{P_A}(\nu, \nu) - V_{P_A}(\rho_{P_A}(\nu)) \,.$$

The Lagrangian $\mathcal{L}$ of the system with configuration space given by $\mathbb{R}^3$ is given with respect to coordinate system (q_1, q_2, q_3) by

$$\begin{aligned}
\mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) &= \frac{m_1}{2}(\dot{q}_1)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_3}{2}(\dot{q}_3)^2 - \frac{m_2}{2}(\dot{q}_2)^2 \\
&\quad - \frac{k_1}{2}(q_1 - q_2)^2 - \frac{k_1}{2}(q_2 - q_3)^2 + 0 \quad (\text{since } V_Z \equiv 0) \\
&= \frac{m_1}{2}(\dot{q}_1)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_3}{2}(\dot{q}_3)^2 - \frac{k_1}{2}(q_1 - q_2)^2 - \frac{k_1}{2}(q_2 - q_3)^2.
\end{aligned}$$

The Riemannian span (P, F_P) is a representative of the Lagrangian system $[S, F_S] \circ [Q, F_Q]$. The Lagrangian $\mathcal{L}$ on P_A is the Lagrangian for the given system of three masses and two springs with configuration space equal to $\mathbb{R}^3$. We leave the determination of the Hamiltonian system to the reader as it is a straightforward exercise given the previous discussion and the result of the next section.

In general, a description of a composite system requires a prior description of the subsystems. The subsystems need not themselves have descriptions as composite systems and it remains an open problem to determine the simplest subsystems that are required to construct from them any other system as a composite. If two subsystems that share a common component form a complicated system, and if we know how to map the subsystems into two pieces, one of which is the common component, then we can view the complicated system as a composite system in our formalism. We carefully work through a selection of examples in an upcoming paper where we more carefully develop computational tools.

5. The Legendre Functor

This section constructs a functor $\mathscr{L}$ from LagSy to HamSy that preserves the paths of motion.

Suppose that (M, g_M) is a Riemannian manifold of dimension m. The canonical 2-form, ω_{T^*M}, is the exterior derivative of the tautological 1-form and is a symplectic form on T^*M, [2, p. 202]. Denote respectively by π_M and ρ_M the canonical projections from T^*M to M and from TM to M. Suppose a is a point of M. There is a chart U of M containing a that is the domain of coordinates $(x_i)_{i\in\{1,\ldots,m\}}$. The set of 1-forms $\{\mathrm{d}x_i \colon i \in \{1,\ldots,m\}\}$ trivializes the subbundle T^*U. Define for each i the real valued functions p_i^M on T^*U with the property that for all θ in T^*M,

$$\theta = \sum_{i=1}^{m} p_i^M(\theta) \frac{\partial}{\partial x_i}\bigg|_{\pi_M(\theta)}.$$

The p_i^M are the *momenta* associated with the x_i coordinates. For each i, the function p_i^M is the evaluation map $\mathrm{ev}_{\frac{\partial}{\partial x_i}\big|_{\pi_M(\theta)}}$ that is defined by the equality

$$\mathrm{ev}_{\frac{\partial}{\partial x_i}\big|_{\pi(\theta)}}(\theta) = \theta\left(\frac{\partial}{\partial x_i}\bigg|_{\pi_M(\theta)}\right).$$

For each i, define q_i^M by

$$q_i^M = x_i \circ \pi_M.$$

The function given by $\left(q_i^M, p_i^M\right)_{i\in\{1,\ldots,m\}}$ on $\pi_M^{-1}(U)$ is a Darboux coordinate system, that is

$$\omega_{T^*M} = \sum_{i=1}^{m} \mathrm{d}q_i^M \wedge \mathrm{d}p_i^M.$$

Define for each i the real valued function $\hat{q}_i^M$ on TM with the property that if v is in $\rho_M^{-1}(U)$, then

$$v = \sum_{i=1}^{m} \hat{q}_i^M(v)\, \frac{\partial}{\partial x_i}\bigg|_{\rho_M(v)}.$$

Note that $\hat{q}_i^M$ is the function defined for each v in TU by

$$\hat{q}_i^M(v) = \mathrm{d}x_i\big|_{\rho_M(v)}(v).$$

Denote ambiguously by q_i^M the function

$$q_i^M = x_i \circ \rho_M$$

on TU. The coordinate system $\left(q_i^M, \hat{q}_i^M\right)$ is a coordinate system on $\rho_M^{-1}(\pi_M(U))$.

The Riemannian metric g_M on TM induces a Riemannian metric on the cotangent bundle T^*M, to be denoted g_M^* and for each a in U defined on the pair (θ_1, θ_2) in $T_a^*M \times T_a^*M$ by

$$g_M^*(\theta_1, \theta_2) = g_M(\sharp_M(\theta_1), \sharp_M(\theta_2)) = \sum_{i,j=1}^{m} g_M^{ij}(a) p_i^M(\theta_1) p_j^M(\theta_2),$$

where g_M^{ij} denotes the (i, j) entry of the inverse of the matrix given by g_M in the $(q_i^M, \hat{q}_i^M)$ coordinates. For all v in TM and θ in T^*M, denote respectively by $g_M(\cdot)$ and $g_M^*(\cdot)$ the quadratic forms

$$g_M(v) = g_M(v, v) \quad \text{and} \quad g_M^*(\theta) = g_M^*(\theta, \theta). \tag{5}$$

Define $\mathscr{K}$ as a map from Riemannian manifolds to symplectic manifolds by

$$\mathscr{K}(M, g_M) = (T^*M, \omega_{T^*M}).$$

For any surjective Riemannian submersion f from M to N, define (see Diagram 7) $\mathscr{K}(f)$ by

$$\mathscr{K}(f) = \flat_N \circ \mathrm{d}f \circ \sharp_M.$$

To simplify the notation, denote by F the function $\mathscr{K}(f)$.

Suppose that M and N are smooth manifolds of respective dimensions m and n and suppose further that f is a surjective Riemannian submersion from M to N. For any point p in M there is a coordinate system $(x_1, \ldots, x_m)$ of $\mathcal{A}_M$ on a chart containing p and a coordinate system $(y_1, \ldots, y_n)$ of $\mathcal{A}_N$ on a chart containing $f(p)$ such that for all i in $\{1, \ldots, n\}$ and k in $\{n + 1, \ldots, m\}$,

$$x_i = y_i \circ f \quad \text{and} \quad \frac{\partial}{\partial x_k} \in \ker(\mathrm{d}f).$$

Let j be an index varying in the set $\{1, \ldots, n\}$. For each i and each j, denote respectively by q_i^M and q_j^N the functions $x_i \circ \pi_M$ and $y_j \circ \pi_N$ and denote by p_i^M and p_j^N the momenta associated with the coordinate functions x_i and y_j. Use the above notation for the following lemma, as well as for the rest of the section.

$$\begin{tikzcd}
M \arrow[r, "f"] & N \\
TM \arrow[u, "\rho_M"] \arrow[r, "\mathrm{d}f"] \arrow[d, dashed, shift left, "\flat_M"] & TN \arrow[u, "\rho_N"'] \arrow[d, shift left, "\flat_N"] \\
T^*M \arrow[u, shift left, "\sharp_M"] \arrow[r, "F = \mathscr{K}(f)"'] \arrow[uu, bend left=60, "\pi_M"] & T^*N \arrow[u, dashed, shift left, "\sharp_N"] \arrow[uu, bend right=60, "\pi_N"']
\end{tikzcd}$$

Diagram 7. Composition of $\mathrm{d}f$ with the Musical Isomorphisms

Lemma 5.1. *For all p_j^M, p_j^N, and F defined as above,*

$$p_j^M = p_j^N \circ F.$$

Proof. For all j in $\{1, \ldots, n\}$,

$$\mathrm{d}f\left(\left.\frac{\partial}{\partial x_j}\right|_a\right) = \mathrm{d}f\left(\left.\frac{\partial}{\partial (y_j \circ f)}\right|_a\right) = \left.\frac{\partial}{\partial y_j}\right|_{f(a)}.$$

For all θ in T^*U, there is an element X of TU with θ equal to $g_M(X, \cdot)$. In this case, the form $F(\theta)$ is equal to $g_N(\mathrm{d}f(X), \cdot)$, and so

$$p_j^M(\theta) = \mathrm{ev}_{\left.\frac{\partial}{\partial x_j}\right|_{\pi_M(\theta)}}(\theta) = g_M\left(X, \left.\frac{\partial}{\partial x_j}\right|_{\pi_M(\theta)}\right).$$

The function f is Riemannian, implying that

$$g_M\left(X, \left.\frac{\partial}{\partial (y_j \circ f)}\right|_{\pi_M(\theta)}\right) = g_N\left(\mathrm{d}f(X), \mathrm{d}f\left(\left.\frac{\partial}{\partial (y_j \circ f)}\right|_{\pi_M(\theta)}\right)\right)$$

and so

$$\begin{aligned} p_j^M(\theta) &= g_N\left(\mathrm{d}f(X), \left.\frac{\partial}{\partial y_j}\right|_{f(\pi_M(\theta))}\right) \\ &= g_N\left(\mathrm{d}f(X), \left.\frac{\partial}{\partial y_j}\right|_{\pi_N(F(\theta))}\right) \\ &= F(\theta)\left(\left.\frac{\partial}{\partial y_j}\right|_{\pi_N(F(\theta))}\right) \\ &= \mathrm{ev}_{\left.\frac{\partial}{\partial y_j}\right|_{\pi_M(\theta)}}(F(\theta)) = (\pi_N \circ F)(\theta), \end{aligned}$$

which proves the desired equality. □

Proposition 5.2. *For any surjective Riemannian submersion f from a Riemannian manifold M to a Riemannian manifold N, the function $\mathscr{K}(f)$ is a surjective Poisson map.*

Proof. Suppose M and N have respective dimensions m and n. The map $\mathscr{K}$ maps Riemannian manifolds to symplectic manifolds. Once again denote by F the map $\mathscr{K}(f)$. Suppose that Π_{T^*M} and Π_{T^*N} respectively denote the Poisson bivectors for T^*M and T^*N. For any α and β in $C^\infty(N)$ and any a in M,

$$\begin{aligned} \mathrm{d}F_a(\Pi_{T^*M})(\alpha, \beta) &= \Pi_{T^*M}(\alpha \circ F, \beta \circ F)\Big|_a \\ &= \sum_{i=1}^m \left(\frac{\partial(\alpha \circ F)}{\partial q_i^M}\frac{\partial(\beta \circ F)}{\partial p_i^M} - \frac{\partial(\beta \circ F)}{\partial q_i^M}\frac{\partial(\alpha \circ F)}{\partial p_i^M}\right)\Bigg|_a \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \left(\frac{\partial(\alpha\circ F)}{\partial q_i^M}\frac{\partial(\beta\circ F)}{\partial p_i^M} - \frac{\partial(\beta\circ F)}{\partial q_i^M}\frac{\partial(\alpha\circ F)}{\partial p_i^M} \right)\Bigg|_a \\
&= \sum_{i=1}^{n} \left(\frac{\partial(\alpha\circ F)}{\partial(q_i^N\circ F)}\frac{\partial(\beta\circ F)}{\partial(p_i^N\circ F)} - \frac{\partial(\beta\circ F)}{\partial(q_i^N\circ F)}\frac{\partial(\alpha\circ F)}{\partial(p_i^N\circ F)} \right)\Bigg|_a \qquad (6) \\
&= \sum_{i=1}^{n} \left(\frac{\partial(\alpha)}{\partial q_i^N}\frac{\partial(\beta)}{\partial p_i^N} - \frac{\partial(\beta)}{\partial q_i^N}\frac{\partial(\alpha)}{\partial p_i^N} \right)\Bigg|_{F(a)} = \Pi_{T^*N}(\alpha,\beta)\Big|_{F(a)},
\end{aligned}
$$

where Lemma 5.1 implies the equality in (6). Therefore, $\mathrm{d}F(\Pi_{T^*M})$ is equal to Π_{T^*N}, which implies that F is a Poisson map. The map f is a surjective submersion, therefore $\mathrm{d}f$ is surjective. The nondegeneracy of g implies that F is also surjective and so $\mathscr{K}$ maps the morphisms in $\mathsf{RiemSurj}$ to morphisms in $\mathsf{SympSurj}$. □

Lemma 5.3. *For any Riemannian spans S and Q and any span isomorphism Φ from S to Q, the function $\mathscr{K}(\Phi)$ is a span isomorphism from $\mathscr{K}(S)$ to $\mathscr{K}(Q)$.*

Proof. Suppose that Φ is a span isomorphism from S and Q. In this case, $\mathscr{K}(\Phi)$ is Poisson. Since $\mathscr{K}(\Phi)$ is a diffeomorphism and Poisson, it is an isomorphism in the category $\mathsf{SympSurj}$. Recall that the isomorphisms in $\mathsf{SympSurj}$ are Poisson diffeomorphisms, which are symplectomorphisms since the objects in $\mathsf{SympSurj}$ are symplectic manifolds, [1, p. 195]. Since Φ is a span morphism,

$$
s_L = q_L \circ \Phi \quad \text{and} \quad s_R = q_R \circ \Phi,
$$

implying that

$$
\begin{aligned}
\mathscr{K}(s_L) &= \mathscr{K}(q_L \circ \Phi) \\
&= \flat_{Q_L} \circ \mathrm{d}(q_L \circ \Phi) \circ \sharp_{S_A} \\
&= \flat_{Q_L} \circ \mathrm{d}q_L \circ \mathrm{d}\Phi \circ \sharp_{S_A} \\
&= \flat_{Q_L} \circ \mathrm{d}q_L \circ (\sharp_{Q_A} \circ \flat_{Q_A}) \circ \mathrm{d}\Phi \circ \sharp_{S_A} \\
&= (\flat_{Q_L}\mathrm{d}q_L \circ \sharp_{Q_A}) \circ (\flat_{Q_L} \circ \mathrm{d}\Phi \circ \sharp_{S_A}) = \mathscr{K}(q_L) \circ \mathscr{K}(\Phi).
\end{aligned}
$$

A similar argument shows that

$$
\mathscr{K}(s_R) = \mathscr{K}(q_R) \circ \mathscr{K}(\Phi),
$$

proving that $\mathscr{K}(\Phi)$ is a span morphism. Therefore, for any spans S and Q in $\mathsf{RiemSurj}$ that are span isomorphic, the spans $\mathscr{K}(S)$ and $\mathscr{K}(Q)$ are also span isomorphic. □

Lemma 5.4. *For any Riemannian submersion f that is compatible with a Riemannian augmentation, the function $\mathscr{K}(f)$ is a Poisson map that is compatible with the Hamiltonian augmentation that is the image under $\mathscr{K}$ of the Riemannian augmentation.*

Proof. For any span isomorphism Φ from S to Q that is compatible with F_S and F_Q,

$$
V_{S_A} = V_{Q_A} \circ \Phi.
$$

The isomorphism Φ is Riemannian, hence an isometry. Therefore,

$$g^*_{S_A} = g^*_{Q_A} \circ \mathscr{K}(\Phi),$$

and so

$$\begin{aligned} H_{S_A} &= \frac{1}{2} g^*_{S_A} + V_{S_A} \circ \pi_{S_A} \\ &= \frac{1}{2} g^*_{Q_A} \circ \mathscr{K}(\Phi) + V_{Q_A} \circ \pi_{Q_A} \circ \mathscr{K}(\Phi) = H_{Q_A} \circ \mathscr{K}(\Phi). \end{aligned}$$

□

Suppose that S is a Riemannian span and let $\star$ denote either of the letters A, L, or R. Define $\mathscr{K}(S_\star, V_\star)$ by

$$\mathscr{K}(S_\star, V_\star) = (\mathscr{K}(S_\star), H_\star)$$

where for all η in $S_\star$,

$$H_{S_\star}(\eta) = \frac{1}{2} g^*_{S_\star}(\eta) + (V_\star \circ \pi_{S_\star})(\eta).$$

Each object of LagSy is an augmented Riemannian manifold and so $\mathscr{K}$ maps the objects of LagSy to the objects of HamSy. Define $\mathscr{L}$ to be $\mathscr{K}$ on the objects of LagSy and for each morphism $[S]$ in LagSy, define $\mathscr{L}([S])$ by

$$\mathscr{L}([S]) = [\mathscr{K}(S)].$$

Theorem 5.5. *The map $\mathscr{L}$ is a functor from LagSy to HamSy. Suppose that π_{S_A} is the canonical projection from T^*S_A to S_A. Suppose that the Lagrangian system $[S]$ has a path of motion γ on the manifold S_A that is specified by the representative S of $[S]$ and suppose that γ intersects a point x of S_A at time zero. In this case, the path $\mathscr{K} \circ \gamma$ is a path determined by $\mathscr{L}([S])$, valued in the symplectic manifold $\mathscr{K}(S_A)$, and $\pi_{S_A} \circ \mathscr{K} \circ \gamma$ also intersects x at time zero.*

Proof. The map $\mathscr{L}$ maps Riemannian manifolds to symplectic manifolds and potentials to Hamiltonians, and therefore maps the objects of LagSy to the objects of HamSy. Proposition 5.2 implies that $\mathscr{L}$ maps surjective Riemannian submersions to surjective Poisson maps, and so if S is a Riemannian span, then $\mathscr{K}(S)$ is a Poisson span. Lemma 5.4 implies that if (S, F_S) and (Q, F_Q) are isomorphic as augmented Riemannian spans, then $\mathscr{K}(S, F_S)$ and $\mathscr{K}(Q, F_Q)$ are also isomorphic as augmented Poisson spans and so $\mathscr{L}$ is well defined on Lagrangian systems, mapping them to Hamiltonian systems.

Suppose that M is a Riemannian manifold. Denote by $\mathcal{L}_M$ the Lagrangian on TM, where for each ν in TM,

$$\mathcal{L}_M(\nu) = \frac{1}{2} g_M(\nu, \nu) - V_M(\rho_M(\nu)).$$

Denote by H_M the Hamiltonian associated to V_M and by $\{\cdot, \cdot\}_{T^*M}$ the Poisson bracket as given above in the construction of $\mathscr{L}$. It is a standard result in classical mechanics that a path γ on M is a solution to (EL) if and only if it is an integral curve of $\{\cdot, H_M\}_M$, [13, p.25, Theorem 3.13]. This proves the last two statements of the theorem. To prove that $\mathscr{L}$

is a functor, it suffices to show further that: (1) $\mathscr{L}$ preserves composition and (2) $\mathscr{L}$ maps identity morphisms to identity morphisms.

To show (1), suppose that $[S, F_S]$ and $[Q, F_Q]$ are augmented Riemannian spans and that $[S, F_S]$ is composable with $[Q, F_Q]$. Suppose that P is an $\mathcal{F}$-pullback of (s_R, q_L), where P_A is the fibered product $S_A \times_{S_R} Q_A$ and p_R and p_L are the respective restrictions of the projections on $S_A \times Q_A$ to S_A and Q_A. The map $\mathscr{K}$ maps $S_A \times_{S_R} Q_A$ to its cotangent bundle $T^*(S_A \times_{S_R} Q_A)$, which is isomorphic in $\mathsf{SympSurj}$ to the manifold $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$. The symplectic form on $T^*(S_A \times_{S_R} Q_A)$ is given by the canonical 2-form and the symplectic form ω on $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$ is given by

$$\omega = \mathscr{K}(p_L)^*(\omega_{T^*S_A}) + \mathscr{K}(p_R)^*(\omega_{T^*Q_A}) - \mathscr{K}(p_L)^*(\mathscr{K}(s_R)^*(\omega_{T^*S_R})).$$

The symplectomorphism Φ from $T^*(S_A \times_{S_R} Q_A)$ to $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$ is consistent with the augmentations. Lemma 5.4 implies that

$$\begin{aligned}
\mathscr{L}([S, F_S] \circ [Q, F_Q]) &= \mathscr{L}([(S, F_S) \circ_P (Q, F_Q)]) \\
&= [\mathscr{K}((S, F_S) \circ_P (Q, F_Q))] \\
&= [\mathscr{K}(S, F_S) \circ_{\mathscr{K}(P)} \mathscr{K}(Q, F_Q)] \\
&= [\mathscr{K}(S, F_S)] \circ [\mathscr{K}(Q, F_Q)] = \mathscr{L}([S, F_S]) \circ \mathscr{L}([Q, F_Q]),
\end{aligned}$$

where the penultimate equality holds because $\mathscr{K}(P)$ is an $\mathcal{F}$-pullback.

To show (2), suppose that (X, V_X) is an augmented Riemannian manifold and that Id_X is the identity map from X to X. Denote by I_X the span $(\mathrm{Id}_X, \mathrm{Id}_X)$. The span $\mathscr{K}(\mathrm{I}_X)$ is the pair $(\mathscr{K}(\mathrm{Id}_X), \mathscr{K}(\mathrm{Id}_X))$ where $\mathscr{K}(\mathrm{Id}_X)$ is the identity map Id_{T^*X} from T^*X to T^*X. Furthermore, $\mathscr{K}$ maps the augmentation V_X to the augmentation H_{T^*X} where

$$H_{T^*X} = \frac{1}{2} g_X^* + V_X \circ \pi_X.$$

Suppose that S is an augmented Hamiltonian span with (S_L, H_{S_L}) equal to (T^*X, H_{T^*X}). Let Q be the $\mathcal{F}$-pullback of the cospan $(\mathscr{K}(\mathrm{Id}_X), s_L)$ with the property that Q_A is the symplectic manifold $T^*X \times_{T^*X} S_A$. The maps q_L and q_R are the respective restrictions to the manifold $T^*X \times_{T^*X} S_A$ of the canonical projections of the manifold $T^*X \times S_A$ to T^*X and S_A and are symplectomorphisms. The definition of the augmentation on a pullback implies that

$$\begin{aligned}
H_{Q_A} = &\left(\frac{1}{2} g_X^* + V_X \circ \pi_X\right) \circ q_L + \left(\frac{1}{2} g_{S_A}^* + V_{S_A} \circ \pi_{S_A}\right) \circ q_R \\
&- \left(\frac{1}{2} g_X^* + V_X \circ \pi_X\right) \circ q_L \circ \mathrm{Id}_{T^*X} \\
= &\left(\frac{1}{2} g_X^* + V_X \circ \pi_X\right) \circ q_L + \left(\frac{1}{2} g_{S_A}^* + V_{S_A} \circ \pi_{S_A}\right) \circ q_R - \left(\frac{1}{2} g_X^* + V_X \circ \pi_X\right) \circ q_L \\
= &\left(\frac{1}{2} g_{S_A}^* + V_{S_A} \circ \pi_{S_A}\right) \circ q_R = H_{S_A} \circ q_R,
\end{aligned}$$

hence

$$H_{Q_A} = H_{S_A} \circ q_R.$$

The map q_R is, therefore, compatible with the augmentations. Since Q is paired with $(\mathscr{K}(\mathrm{Id}_X), s_L)$,

$$s_L \circ q_R = \mathrm{Id}_X \circ q_L = q_L,$$

and so q_R is a span isomorphism mapping the composite $(\mathscr{K}(\mathrm{I}_X) \circ q_L, s_R \circ q_R)$ to the span S that is compatible with the augmentations. This compatibility implies that

$$\mathscr{L}([\mathrm{I}_X, V_{\mathrm{I}_X}]) \circ [S, H_S] = [\mathscr{K}(\mathrm{I}_X, V_X) \circ (S, H_S)] = [S, H_S].$$

Similar arguments show that for any augmented Hamiltonian span $(S', H_{S'})$ such that $(S'_R, H_{S'_R})$ is equal to (T^*X, H_{T^*X}),

$$[S', H_{S'}] \circ \mathscr{L}([\mathrm{I}_X, V_X]) = [S', H_{S'}],$$

and so $\mathscr{L}([\mathrm{I}_X, V_X])$ is the identity map with source and target (T^*X, H_{T^*X}).

□

We call the functor $\mathscr{L}$ from LagSy to HamSy the *Legendre functor*. It is a generalization of the Legendre transformation which translates from the Lagrangian to the Hamiltonian description of an open system in classical mechanics.

References

[1] Abraham, R., Marsden, J.: *Foundations of Mechanics*. Addison-Wesley Publishing Company, Boston (1987). (Referred to on page 18, 27.)

[2] Arnol'd, V. I.: *Mathematical Methods of Classical Mechanics*. Springer, Berlin (1989). (Referred to on page 2, 8, 24.)

[3] Baez, J. C., Dolan, J.: *Higher-dimensional algebra and topological quantum field theory*. J. Math. Phys. 36 (1995), 6073–6105. Available as arXiv:q-alg/9503002. (Referred to on page 2.)

[4] Baez, J. C., Courser, K.: *Structured cospans*. Available as arXiv:1911.04630. (Referred to on page 4.)

[5] Baez, J. C., Coya, B., Rebro, F.: *Props in network theory*. Theor. Appl. Categ. 33 (2018), 727–783. Available as arXiv:1707.08321. (Referred to on page 2.)

[6] Baez, J. C., Fong, B.: *A compositional framework for passive linear networks*. Theor. Appl. Categ. 33 (2018), 1158–1222. Available as arXiv:1504.05625. (Referred to on page 2.)

[7] Baez, J. C., Fong, B., Pollard, B.: *A compositional framework for Markov processes*. J. Math. Phys. 57 (2016), 033301. Available as arXiv:1508.06448. (Referred to on page 2.)

[8] Baez, J. C., Pollard, B.: *A compositional framework for reaction networks*. Rev. Math. Phys. **29** (2017). Available as https://arxiv.org/abs/1704.02051. (Referred to on page 2.)

[9] Bénabou, J.: *Introduction to bicategories*. *Reports of the Midwest Category Seminar*, Springer Lecture Notes in Mathematics **47**, Springer, Berlin (1967). (Referred to on page 3.)

[10] Brunetti, R., Fredenhagen, K., Verch, R.: *The generally covariant locality principle—a new paradigm for local quantum field theory*. Commun. Math. Phys. 237 (2003), 31–68. Available as arXiv:math-ph/0112041. (Referred to on page 2.)

[11] Calaque, D.: *Derived stacks in symplectic geometry*. Available as arXiv:1802.09643. (Referred to on page 4.)

[12] Cannas da Silva, A., Weinstein, A.: *Geometric Models for Noncommutative Algebras*. Berkeley Mathematics Lecture Notes (Book 10), American Mathematical Society, Providence (1999). (Referred to on page 7, 12.)

[13] Cortés, V., Haupt, A. S.: *Mathematical Methods of Classical Physics*. Springer, Berlin (2017). (Referred to on page 11, 28.)

[14] Courser, K.: *Open Systems: a Double Categorical Perspective*, Ph.D. thesis, U. C. Riverside, 2020. Available at http://math.ucr.edu/home/baez/thesis_courser.pdf. (Referred to on page 2, 4.)

[15] Dazord, P.: *Mécanique hamiltonienne en presence de contraintes*. Illinois J. Math. 38 (1994), 148–175. (Referred to on page 3, 5.)

[16] Dufour, J-P., Zung, N. T.: *Poisson structures and their normal forms*. Birkhäuser Basel (2005). (Referred to on page 10.)

[17] Fong, B., Spivak, D.: *An Invitation to Applied Category Theory: Seven Sketches in Compositionality*. Cambridge U. Press Cambrige (2019). (Referred to on page 1.)

[18] Fong, B.: *Decorated cospans*. Theor. Appl. Categ. 30 (2015),1096–1120. Available as arXiv:1502.00872. (Referred to on page 4.)

[19] Fong, B: *The Algebra of Open and Interconnected Systems*, Ph.D. thesis, University of Oxford, 2016. Available as arXiv:1609.05382. (Referred to on page 4.)

[20] Freed, D.: *Higher algebraic structures and quantization*. Commun. Math. Phys. 159 (1994), 343–398. (Referred to on page 2.)

[21] Haugseng, R.: *Iterated spans and classical topological field theories*. Math. Z. 289 (2018), 1427–1488. Available as https://arxiv.org/abs/1409.0837. (Referred to on page 2.)

[22] Lerman, E., Spivak, D.: An algebra of open continuous time dynamical systems and networks. Available as arXiv:1602.01017. (Referred to on page 2.)

[23] Libermann, P., Marle, C.-M.: *Symplectic Geometry and Analytical Mechanics*. D. Reidel, Dordrecht (1987). (Referred to on page 6.)

[24] Marle, C.-M.: *Reduction of constrained mechanical systems and stability of relative equilibria*. Commun. Math. Phys. 174 (1995), 295–318. Available at https://projecteuclid.org/euclid.cmp/1104275295. (Referred to on page 3, 5.)

[25] McDuff, D., Salamon, D.: *Introduction to Symplectic Topology*. Oxford University Press, Oxford (2017). (Referred to on page 14.)

[26] Schultz, P., Spivak, D., Vasilakopoulou, C.: Dynamical systems and sheaves. Available as arXiv:1609.08086. (Referred to on page 2.)

[27] Spivak, D.: *Derived smooth manifolds*. Duke Math. J. 153 (2010), 55–128. (Referred to on page 4.)

[28] Spivak, D.: *Category Theory for Scientists*. The MIT Press Cambridge 2014. Available as arXiv:1302.6946. (Referred to on page 1.)

[29] Van Der Schaft, A., Jeltsema, D.: *Port-hamiltonian systems theory: an introductory overview*. Found. Trends Syst. Control 1 (2014), 173–378. (Referred to on page 4.)

[30] Weisbart, D., Yassine, A.: *Constructing span categories from categories without pullbacks*. Available as arXiv:2007.07752. (Referred to on page 4, 5, 6, 8, 9, 16, 21.)