UNIVERSITY OF CALIFORNIA

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The Combinatorics and Absoluteness of Definable Sets of Real Numbers

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ABSTRACT OF THE DISSERTATION

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This thesis divides naturally into two parts, each concerned with the extent to which the theory of $L(\mathbb{R})$ can be changed by forcing.

The first part focuses primarily on applying generic-absoluteness principles to show that definable sets of reals enjoy regularity properties. The work in Part I is joint with Itay Neeman and is adapted from our forthcoming paper [33]. This project was motivated by questions about *mad families*, maximal families of infinite subsets of ω any two of which have only finitely many members in common. We begin, in the spirit of Mathias [30], by establishing (Theorem 2.8) a strong Ramsey property for sets of reals in the Solovay model, giving a new proof of Törnquist's theorem [48] that there are no infinite mad families in the Solovay model.

In Chapter 3 we stray from the main line of inquiry to briefly study a game-theoretic characterization of filters with the Baire Property.

Neeman and Zapletal [36] showed, assuming roughly the existence of a proper class of Woodin cardinals, that the boldface theory of $L(\mathbb{R})$ cannot be changed by proper forcing. They call their result the Embedding Theorem, because they conclude that in fact there is an elementary embedding from the ground-model $L(\mathbb{R})$ to the $L(\mathbb{R})$ of any proper forcing extension that fixes every ordinal and every real. With a view toward analyzing mad families under AD^+ and in $L(\mathbb{R})$ under large-cardinal hypotheses, in Chapter 4 we establish *triangular* versions of the Embedding Theorem, Theorems 4.3 and 4.8. These are enough for us to use Mathias's methods to show (Theorem 4.5) that there are

no infinite mad families in $L(\mathbb{R})$ under large cardinals and (Theorem 4.9) that AD^+ implies that there are no infinite mad families. These are again corollaries of theorems about strong Ramsey properties under large-cardinal assumptions and AD^+, respectively. Our first theorem improves the large-cardinal assumption under which Todorcevic (see [14]) established the nonexistence of infinite mad families in $L(\mathbb{R})$.

Part I concludes with Chapter 5, a short list of open questions.

In the second part of the thesis, we conduct a finer analysis of the Embedding Theorem and its consistency strength. Schindler [41] found that the the Embedding Theorem is consistent relative to much weaker assumptions than the existence of Woodin cardinals. He defined *remarkable cardinals*, which can exist even in L, and showed that the Embedding Theorem is equiconsistent with the existence of a remarkable cardinal. His theorem resembles a theorem of Kunen from the 1980s (see [20]): the absoluteness of the theory of $L(\mathbb{R})$ to ccc forcing extensions is equiconsistent with a weakly compact cardinal. Joint with Itay Neeman [32], we improve Schindler's theorem by showing that absoluteness for σ-closed $*$ ccc posets — instead of the larger class of proper posets — implies the remarkability of $\aleph_1^V$ in L. This requires a fundamental change in the proof, since Schindler's lower-bound argument uses Jensen's reshaping forcing, which, though proper, need not be σ-closed $*$ ccc in that context. Our proof bears more resemblance to Kunen's than to Schindler's.

The proof of Theorem 6.2 splits naturally into two arguments. In Chapter 7 we extend Kunen's method of coding reals into a specializing function to allow for trees with uncountable levels that may not belong to L. This culminates in Theorem 7.4, which asserts that if there are $X \subseteq \omega_1$ and a tree $T \subseteq \omega_1$ of height ω_1 such that X is codable along T (see Definition 7.3), then $L(\mathbb{R})$-absoluteness for ccc posets must fail.

We complete the argument in Chapter 8, where we show that if in any σ-closed extension of V there is no $X \subseteq \omega_1$ codable along a tree T, then $\aleph_1^V$ must be remarkable in L.

In Chapter 9 we review Schindler's proof of generic absoluteness from a remarkable cardinal to show that the argument gives a level-by-level upper bound: a strongly λ^+-remarkable cardinal is enough to get $L(\mathbb{R})$-absoluteness for λ-linked proper posets.

Chapter 10 is devoted to partially reversing the level-by-level upper bound of Chapter 9. Adapting the methods of [31], we are able to show that $L(\mathbb{R})$-absoluteness for $\max(\mathfrak{c}, |\lambda|)$-linked posets implies that the interval $[\aleph_1^V, \lambda]$ is Σ^2_1-remarkable in L.

The dissertation of Zach Norwood is approved.

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To Mom

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In six years I have taught an astonishing number of students at UCLA. I thank those students, wherever you are, for your optimism and your curiosity, and for making me a better mathematician and educator.

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Part I

Happy and mad families

CHAPTER 1

Introduction

This part of the thesis is devoted to proving two theorems, each of which is joint with Itay Neeman and appears in [33]. The work was inspired by Mathias's celebrated 1977 paper *Happy Families* [30]. Mathias showed (Theorem 1.9) that there are no mad families in the Solovay-type model obtained by collapsing a Mahlo cardinal, and he asked whether there are mad families in the traditional Solovay model, obtained from only an inaccessible cardinal. That question remained open until Asger Törnquist answered it positively in 2015. Törnquist's proof is entirely combinatorial and makes no mention of the happy families Mathias used in his original argument. The first main theorem of this part is a strengthening of Törnquist's theorem (Theorem 2.8) that applies to Mathias's happy families. In contrast to Törnquist's proof, our proof is very similar to arguments in Mathias's original paper [30].

The second main theorem of this part concerns mad families under large-cardinal and determinacy assumptions. One of set theory's major projects is to show, often using strong assumptions, that pathological sets of reals cannot be easily defined. Theorems in this vein might assert that a specific type of pathological set cannot be Borel, or cannot be projective or ordinal-definable from reals in the presence of large cardinals, or cannot exist under AD, the Axiom of Determinacy. Mathias [30] showed that the Axiom of Determinacy for reals, $\mathsf{AD}_{\mathbb{R}}$, implies the nonexistence of infinite mad families; it is natural to ask whether this assumption can be improved to AD, as Törnquist [48] asked. Mathias argues for a strong connection between the Ramsey property and the existence of mad families, so Törnquist's question seems closely related to the longstanding open question of whether every set of reals must have the Ramsey property under AD. In Chapter 4 we give a partial answer (Theorem 4.9), confirming that there are no mad families under AD^+, a well-studied strengthening of AD. It is unknown whether AD and AD^+ are equivalent. In $L(\mathbb{R})$, AD implies AD^+ (see [8]).

We will use conventional notation and definitions throughout this thesis. In particular, we write $[\omega]^\omega$ for the set of infinite subsets of ω, and more generally $[X]^\omega$ will denote the set of all countably infinite subsets of X. For a set $X \subseteq [\omega]^\omega$, it will be convenient to write X^c for the complement $[\omega]^\omega \smallsetminus X$. We write $H(\theta)$ for the set of sets hereditarily of cardinality $< \theta$, and we use the *Boolean* forcing convention, according to which $p < q$ means that p forces more information than q.

For introductions to Gödel's constructible universe L, the basics of forcing, and other prerequisite topics, the reader is referred to the standard graduate texts [23] and [26]. In addition to those, the monograph [5] serves as a good reference for the study of definable sets of reals, with which we will be occupied in Part I. The reader of Part II looking for more information about large cardinals is directed to Kanamori's book [25].

Basic definitions

Definition 1.1. Two sets are *almost disjoint* if their intersection is finite. An *almost-disjoint family* is a family $A \subseteq [\omega]^\omega$ such that any two different members of A are almost disjoint. A *mad family* is an almost disjoint family that is maximal under inclusion. Equivalently, an almost-disjoint family A is maximal if and only if every $x \in [\omega]^\omega$ has infinite intersection with at least one member of A.

Remarks.

(1) According to our definition, any partition of ω into finitely many pieces is a mad family, but we will be interested only in infinite mad families.

(2) Zorn's lemma implies that every almost-disjoint family extends to a mad family. By applying this fact to an infinite partition of ω, one obtains an infinite mad family. The use of AC in this argument is unavoidable, by Theorem 1.9 (Mathias).

(3) A straightforward diagonalization argument shows that no infinite mad family can be countable.

Crucial to the study of mad families has been Mathias's connection between mad families and the Ramsey property. In its classical form, Ramsey's theorem states that every coloring $\chi\colon [\omega]^2 \to 2$

has an infinite monochromatic set, a set $z \in [\omega]^\omega$ such that χ is constant on $[z]^2$. It's natural to seek mild conditions on a family $H \subseteq [\omega]^\omega$ sufficient to guarantee that the monochromatic set z could always be chosen to belong to H. The problem certainly isn't made easier by requiring H to be upward-closed under $\subseteq$, and a short exercise shows that the complement of H must be closed under finite unions. Reflecting on the proof of Ramsey's theorem reveals that one more condition on H would be enough for H to have at least one monochromatic set for every 2-dimensional coloring.

Definition 1.2. If $y_0 \supseteq y_1 \supseteq y_2 \supseteq \cdots$ is a decreasing sequence of subsets of ω, then we call a set $y_\infty \in [\omega]^\omega$ a *diagonalization* of the sequence $\langle y_n : n < \omega \rangle$ iff $f(n+1) \in y_{f(n)}$ for every $n < \omega$, where $f : \omega \to \omega$ is the increasing enumeration of y_∞.

A set $H \subseteq [\omega]^\omega$ is called a *happy family* if it contains every cofinite set and satisfies the following three conditions:

(i) (upward-closure) If $x \in H$ and $y \supseteq x$, then $y \in H$ too.

(ii) (pigeonhole) If $x_0 \cup x_1 \cup \cdots \cup x_n \in H$, then $x_k \in H$ for some k.

(iii) (selectivity) Every decreasing sequence $y_0 \supseteq y_1 \supseteq \cdots$ of members of H has a diagonalization y_∞ in H.

Remarks.

(1) A brief meditation on this definition reveals that if y_∞ diagonalizes $\vec{y}$, then so does every infinite subset of y_∞.

(2) Clauses (i) & (ii) simply assert that the complement of H, together with all finite subsets of ω, forms an ideal on ω, so a family satisfying (i) & (ii) is called a *coideal*. Accordingly, happy families are often called *selective coideals* in the literature.

Noting that happy families have monochromatic sets for all two-dimensional colorings, Mathias began to investigate when they have monochromatic sets for infinite-dimensional colorings.

Definition 1.3. If H is a coideal (typically a happy family) and $X \subseteq [\omega]^\omega$ is a set of reals, then we say X is *H-Ramsey* if there is in H a monochromatic set for the coloring associated to X, that is, if there is $z \in H$ such that $[z]^\omega \subseteq X$ or $[z]^\omega \subseteq X^c$.

The following fact explains the connection and the terminology.

Proposition 1.4 (Mathias [30])**.** If $A \subseteq [\omega]^\omega$ is an infinite almost-disjoint family and $I(A)$ is the ideal generated by A, then $[\omega]^\omega \setminus I(A)$ is a happy family.

If A is an infinite mad family, then a set belongs to $[\omega]^\omega \setminus I(A)$ if and only if it has infinite intersection with infinitely many members of A.

Mathias provides two other prototypical examples of happy families.

(i) $[\omega]^\omega$ is a happy family.

(ii) Every Ramsey ultrafilter is a happy family.

We can take item (ii) as a definition: an ultrafilter F on ω is a *Ramsey ultrafilter* if and only if it is a happy family.

Notice that a set $X \subseteq [\omega]^\omega$ has the classical Ramsey property if and only if X is $[\omega]^\omega$-Ramsey, in our terminology. See [18] and the references therein for an account of classical infinitary Ramsey Theory, including some expansions of Mathias's ideas.

This observation follows immediately from the definitions, but it deserves special emphasis.

Lemma 1.5. An infinite almost-disjoint family A is a mad family if and only if $I(A)^c$ is *not* $I(A)^c$-Ramsey.

Mathias used Lemma 1.5 to prove several results about the non-existence of definable mad families. His strategy, which we will emulate, is first to prove that definable (suitably understood) sets are H-Ramsey for every happy family H. Then, if a mad family A were definable, then its corresponding happy family $I(A)$ would be too. These two facts are incompatible, by Lemma 1.5.

Mathias's analysis

To establish the H-Ramsey property, Mathias used what is now known as *Mathias forcing guided by* H, and which we denote $\mathbb{M}_H$. Conditions are pairs $\langle s, x\rangle$, where s is a finite subset of ω, and $x \in H$.

The ordering is defined as follows: $\langle s, x\rangle \leq \langle t, y\rangle$ iff t is an initial segment of s and $x \subseteq y$. The first component s of a condition is called its *stem*, and the second component is called its *commitment*. If G is $\mathbb{M}_H$-generic over V, then the union of the stems of conditions in G naturally gives a real $g \subseteq \omega$ that can be shown to generate a filter $F \subseteq H$ that is, roughly speaking, a Ramsey ultrafilter from the perspective of the ground model. In the case that $H = F$ is already a Ramsey ultrafilter in V, this property characterizes $\mathbb{M}_F$-generic reals: g is $\mathbb{M}_F$-generic over V if and only if g is almost-included in every member of F.

The case $H = [\omega]^\omega$ gives the classical *Mathias forcing*. The chief properties of $\mathbb{M}_H$ are these:

Proposition 1.6 (Mathias)**.** Suppose that H is a happy family.

(Prikry Property): if $\langle s, x\rangle \in \mathbb{M}_H$ is a condition and σ is a sentence in the forcing language, then there is a condition $\langle s, x^*\rangle \leq \langle s, x\rangle$ that forces either σ or $\neg\sigma$.

(Characterization of genericity): Suppose that g is a real in an outer model of V. Then g is $\mathbb{M}_H$-generic over V iff for every mad family $A \subseteq H$ in V, g is almost-included in some member of A.

(Mathias Property): if g is $\mathbb{M}_H$-generic over V and $g^* \in [g]^\omega$, then g^* is $\mathbb{M}_H$-generic over V also.

Remarks.

(1) It is worth emphasizing that the Prikry Property allows the extension $\langle s, x^*\rangle$ to be chosen to have the same stem as $\langle s, x\rangle$.

(2) The Mathias Property follows easily from the characterization of genericity.

(3) In the case that $H = F$ is a Ramsey ultrafilter, the characterization of genericity can be simplified: g is $\mathbb{M}_F$-generic iff g is almost-included in every member of F.

Mathias gives several proofs that analytic sets are H-Ramsey for every happy family H. We carefully review one here ([30, paragraphs following Prop 2.9]), both to illustrate the techniques and to motivate later arguments.

Theorem 1.7 (Mathias)**.** Suppose that $X \subseteq [\omega]^\omega$ is analytic and suppose that H is a happy family. Then X is H-Ramsey.

Proof. Let φ be Σ^1_1 and r a real such that $X = \{x \in [\omega]^\omega : \varphi[x, r]\}$. We can find a countable transitive model M with $r \in M$ and an elementary embedding $\pi\colon M \prec V_\theta$ for some sufficiently large initial segment V_θ of V. By insisting that $H \in \pi'' M$, we can ensure that $\overline{H} := H \cap M \in M$, and $M \models$ "$\overline{H}$ is a happy family.".

Let $\dot{g}$ be the standard name for the $\mathbb{M}_{\overline{H}}$-generic real. Apply the Prikry Property of $\mathbb{M}_{\overline{H}}$ in M to find $x \in \overline{H}$ such that $\langle 0, x\rangle$ forces (over M) either $\varphi[\dot{g}, r]$ or its negation. Since M is countable, we can use the characterization of $\mathbb{M}_{\overline{H}}$-genericity to find a real $g \in H$ that is $\mathbb{M}_{\overline{H}}$-generic over M below $\langle 0, x\rangle$ (meaning that $g \subseteq^* x$). By the Mathias Property, any real $g' \in [g]^\omega$ is also $\mathbb{M}_{\overline{H}}$-generic over M, so we have

$$
\begin{aligned}
g' \in X &\iff \varphi[g', r] && \text{(definition of } X\text{)} \\
&\iff M[g'] \models \varphi[g', r] && (\Sigma^1_1 \text{ absoluteness}) \\
&\iff \langle \emptyset, x\rangle \Vdash^M_{\mathbb{M}_{\overline{H}}} \varphi[\dot{g}, r] && \text{(genericity of } g'\text{)}
\end{aligned}
$$

We conclude that $[g]^\omega \subseteq X$ or $[g]^\omega \subseteq X^c$, according to whether $\langle 0, x\rangle$ forces $\varphi[\dot{g}, r]$ or $\neg\varphi[\dot{g}, r]$. ∎

The proof of Theorem 1.7 depends on the Mathias and Prikry properties of $\mathbb{M}_{\overline{H}}$, which in turn depend on H's being a happy family in M. In Chapter 2, we will define a poset that, when guided by a non-Ramsey ultrafilter, still has the Mathias and Prikry properties, and use this to adapt the argument.

The proof also depends on the absoluteness of the Σ^1_1 formula between V and $M[g']$. We will show in Chapter 4 how to repeat the argument by establishing stronger absoluteness theorems under large-cardinal or determinacy assumptions. In Chapter 2, we can use the homogeneity of the collapse poset that gives the Solovay model instead of appealing to an absoluteness result.

In his study of happy families in the Solovay model, Mathias established two theorems to support his conjecture that the Solovay model has no infinite mad families.

Theorem 1.8 (Mathias [30])**.** In the Solovay model, every set $x \in [\omega]^\omega$ has the Ramsey property, i.e., is $[\omega]^\omega$-Ramsey.

Theorem 1.9 (Mathias [30])**.** Suppose that G is generic over V for the Levy collapse of a Mahlo cardinal. In $V[G]$, every set $x \subseteq [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every happy family $H \in V[G]$. Consequently, there are no infinite mad families in $L(\mathbb{R})^{V[G]}$.

There are other strong forms of the Ramsey property known to hold in the Solovay model. For example, Bagaria and Di Prisco [3] use generic-absoluteness results to establish parametrized partition properties, some of which generalize the Ramsey property. Some regularity properties of this kind (the Perfect Set Property, for instance) hold even in $L(\mathbb{R})[U]$, the extension of the Solovay model by a generic ultrafilter (see [12]), though in that model there are sets without the Ramsey property, non-Lebesgue-measurable sets, and sets without the Baire property.

To prove that there are no infinite mad families in the Solovay model (obtained by collapsing an inaccessible cardinal), one might hope to imitate Mathias's proof of Theorem 1.9, but Eisworth showed that the conclusion of Theorem 1.9 is too strong for only an inaccessible.

Theorem 1.10 (Eisworth [13])**.** Suppose that CH holds and that every set $X \in [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every happy family H. Then $\aleph_1$ is Mahlo in L.

Eisworth's theorem doesn't seem to leave any room for proving a version of Theorem 1.9 that uses only an inaccessible, and Törnquist's purely combinatorial proof seems to confirm this suspicion. But a closer examination of Eisworth's proof suggests an alternative approach. Assuming that $\aleph_1$ is not Mahlo in L, Eisworth builds Ramsey ultrafilters U such that not every set in $L(\mathbb{R})$ can be U-Ramsey. These ultrafilters certainly aren't definable, in the sense that they won't belong to $L(\mathbb{R})$. This suggests that the Solovay model from an inaccessible could still satisfy the restriction of the conclusion of Theorem 1.9 to definable happy families. And indeed, in Chapter 2 we will prove:

Theorem. If G is generic over V for the Levy collapse of an inaccessible cardinal, then in $V[G]$ every set $X \subseteq [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every happy family $H \in L(\mathbb{R})$.

Using Lemma 1.5, we obtain as a corollary Törnquist's theorem [48] that there are no infinite mad families in $L(\mathbb{R})^{V[G]}$, the Solovay model. Törnquist was the first to show that the nonexistence of infinite mad families is consistent relative to an inaccessible cardinal. Recently, Horowitz and Shelah [21] have shown that this inaccessible is unnecessary, that is, that the theory ZF + "there is no infinite mad family" is in fact equiconsistent with ZFC.

CHAPTER 2

The Solovay model

The main difficulty in adapting Mathias's arguments to our needs is that, while a coideal H may be a happy family in $V[G]$, its intersection with an intermediate extension $V[G \restriction \alpha] \subseteq V[G]$ is unlikely to be closed under diagonalizations. For this reason, there may not be (in $V[G \restriction \alpha]$) any Ramsey ultrafilters $U \subseteq H$ to guide Mathias forcing over M. Mathias forcing guided by an ultrafilter $U \subseteq H$ is the primary tool Mathias uses to prove that a set is H-Ramsey, but it has the Mathias and Prikry properties if and only if U is Ramsey.[1] Consequently, we will need a different poset to serve the same purpose in the absence of Ramsey ultrafilters.

Diagonalization forcing

We use standard notation for finite sequences, so, for instance, $\mathrm{lh}(s)$ denotes the length of s (which, for us, will always be the same as its domain). It will be convenient to refer to the last term of s by $\mathrm{last}(s)$, i.e., $\mathrm{last}(s) = s(\mathrm{lh}(s) - 1)$.

A set y_∞ *almost diagonalizes* a $\subseteq$-decreasing sequence $\vec{y} = \langle y_0, y_1, \ldots \rangle$ if there is some $n < \omega$ such that $y_\infty \setminus n$ diagonalizes $\vec{y}$. Just as Mathias forcing guided by an ultrafilter generically adds a set almost-included in every measure-one set, we need a poset that generically adds a set almost-diagonalizing every decreasing sequence of measure-one sets.

Fix a nonprincipal ultrafilter U on ω. (Crucially, we do not assume that U is a Ramsey ultrafilter.)

Definition 2.1. Conditions in $\mathbb{P}_U$ are pairs $\langle s, \vec{y} \rangle$ such that

(i) $s \in \omega^{<\omega}$ is a finite, strictly increasing sequence,

[1]This follows from [14, Theorems 2.4, 4.1]. See [30, Theorem 2.10] for the Mathias property, and for a related result see [22, Theorem 1.20].

(ii) $\vec{y} = (y_0, y_1, \dots)$ is an $\subseteq$-decreasing sequence of sets, each in U.

Say $\langle t, \vec{z}\rangle \leq \langle s, \vec{y}\rangle$ iff

(iii) s is an initial segment of t,

(iv) $z_n \subseteq y_n$ for every $n \geq \mathrm{last}(t)$, and

(v) $t(n+1) \in y_{t(n)}$ for every $n \in [\mathrm{lh}(s), \mathrm{lh}(t) - 1)$.

If $p = \langle s, \vec{y}\rangle$, then we will often write $\mathrm{stem}(p) = s$. Following convention, we write $q \leq_0 p$ to mean that $q \leq p$ and $\mathrm{stem}(q) = \mathrm{stem}(p)$. It will also be convenient to write $\vec{z} \leq \vec{y}$ to mean that $z_n \subseteq y_n$ for every $n < \omega$.

A generic filter $G \subseteq \mathbb{P}_U$ gives rise in a natural way to a generic real g, the range of the union of all the stems of conditions in G. Clause (v) of the definition ensures that a condition $\langle s, \vec{y}\rangle$ forces the generic real to almost-diagonalize $\vec{y}$.

There is a natural Ellentuck-type neighborhood associated to a condition $\langle s, \vec{y}\rangle$. Let's first define a version for finite sequences:

$$\begin{aligned} [\![s, \vec{y}]\!] := \{ t \in \omega^{<\omega} &: t \subseteq s, \text{ or } s \subseteq t \text{ and} \\ &t(n+1) \in y_{t(n)} \text{ for every } n \geq \mathrm{lh}(s)\}. \end{aligned}$$

Informally, $[\![s, \vec{y}]\!]$ comprises all end-extensions of s that diagonalize $\vec{y}$ beyond s. Now define $[s, \vec{y}]$ to be the set of $x \in [\omega]^\omega$ such that s enumerates an initial segment of x and every initial segment of x belongs to $[\![s, \vec{y}]\!]$.

Remarks.

- $[\![s, \vec{y}]\!]$ is a subtree of $\omega^{<\omega}$, and $[s, \vec{y}]$ is the set of its branches.
- $[\emptyset, \vec{y}]$ is exactly the set of diagonalizations of $\vec{y}$.
- If $p = \langle s, \vec{y}\rangle$ is a condition and $t \in [\![s, \vec{y}]\!]$, then $[\![t, \vec{y}]\!]$ is a condition that extends p.

We have described a natural identification of conditions $\langle s, \vec{y}\rangle$ with subtrees $[\![s, \vec{y}]\!]$ of $\omega^{<\omega}$. A fusion argument (see the proof of Lemma 2.2) shows that these trees form a dense subposet of

Laver forcing guided by U, for which many of the results in this section were proved by Judah and Shelah [22]. Nonetheless, our presentation of the poset allows us to characterize $\mathbb{P}_U$-genericity by a proof very similar to Mathias's original characterization of genericity for Mathias forcing. We will therefore reprove some of the results of [22] and translate them into our language.

Lemma 2.2. Let $O \subseteq [\omega]^\omega$ be an open set. There is a condition $\langle \emptyset, \vec{y} \rangle \in \mathbb{P}_U$ such that either $[\emptyset, \vec{y}] \subseteq O$ or $[\emptyset, \vec{y}] \subseteq O^c$.

Proof. Consider the following game. Players I & II play to create a sequence $\langle y_0, k_0, y_1, k_1, \ldots \rangle$

I	y_0	$\supseteq$	y_1	$\supseteq$	y_2	$\supseteq$	$\cdots$
II		$k_0 \in y_0$		$k_1 \in y_1$		$k_2 \in y_2$	$\cdots$

such that, for every $n < \omega$,

- $y_n \in U$,
- $y_{n+1} \subseteq y_n$,
- $k_n \in y_n$, and
- $k_n < k_{n+1}$.

At the end of the game, Player I wins iff $\{k_0, k_1, \ldots\} \in O$. By the determinacy of open games, one of the two players has a winning strategy σ.

Suppose first that σ is a winning strategy for Player I. Define

$$y_n = \bigcap \{\sigma(t) : t \text{ a sequence of II's moves consistent with } \sigma, \operatorname{last}(t) \leq n\}.$$

Any diagonalization x of the sequence $\vec{y}$ is a sequence of II's moves in a complete run of the game consistent with σ, and thus $x \in O$. That is, $[\emptyset, \vec{y}] \subseteq O$.

Suppose that, on the other hand, σ is a winning strategy for Player II.

Claim. The set $z_\emptyset$ of $k < \omega$ for which there is $w_k \in U$ satisfying $k = \sigma(\langle w_k \rangle)$ belongs to U.

Proof of Claim. Suppose not, so that $v := \{k : (\forall w \in U)\, k \neq \sigma(\langle w \rangle)\} \in U$. But this means that v is a valid first move for Player I, and then $\sigma(v) \in v$. This is impossible. ∎

We can repeat the argument of the Claim to obtain a tree $T \subseteq \omega^{<\omega}$ and, for each $t \in T$, sets $w_t, z_t \in U$ satisfying

(i) every $t \in T$ has $\mathrm{succ}_T(t) \in U$;

(ii) $w_t \subseteq w_s$ whenever $s \subseteq t$;

(iii) $z_t = \left\{k < \omega : k = \sigma(\langle w_\emptyset, \ldots, w_t, w_{t^\frown\langle k\rangle}\rangle)\right\}$.

Put $y_0 = z_\emptyset$ and inductively define $y_n = y_{n-1} \cap \bigcap \{z_t : \mathrm{last}(t) = n,\ t \in T\}$ Clause (iii) and the choice of y_n imply that every $x \in [\emptyset, \vec{y}]$ is a sequence of II's plays consistent with the strategy σ. We conclude that $[\emptyset, \vec{y}] \subseteq O^c$, as desired. ∎

Definition 2.3. Say that a condition $\langle s, \vec{y}\rangle$ *captures* a dense set $D \subseteq \mathbb{P}_U$ if every $x \in [s, \vec{y}]$ has an initial segment whose enumeration $t \in [\![s, \vec{y}]\!]$ satisfies $\langle t, \vec{y}\rangle \in D$.

Compare with [30, Definition 2.2]. Informally, a condition captures D if to get into D one need only extend the *stem* of the condition, and the set of extensions that work is very rich, containing an initial segment of every diagonalization of $\vec{y}$.

If $\langle s, \vec{y}\rangle$ captures D, then $\langle s, \vec{y}\rangle$ still captures D in any outer model. To see this, observe that

$$\{x \in [\omega]^\omega : (\exists t,\ \text{an initial segment of } x)\ \langle t, \vec{y}\rangle \in D\}$$

is open in the product topology (since D is open in the forcing topology) and apply Π^1_1-absoluteness. We will use this fact in the proof of Lemma 2.6.

Lemma 2.4. Let $p \in \mathbb{P}_U$ be a condition.

(a) For every dense open set $D \subseteq \mathbb{P}_U$, there is a condition $q \leq_0 p$ that captures D.

(b) For every countable family $\{D_n : n < \omega\}$ of dense open subsets of $\mathbb{P}_U$, there is a condition $q \leq_0 p$ that captures all of the D_n.

Proof. (a) Assume for convenience that p is the empty condition; the proof of the more general result is similar. For every increasing sequence $t \in \omega^{<\omega}$ choose (if possible) a sequence $\vec{y}^t \in U^\omega$ such that $\langle t, \vec{y}^t \rangle \in D$. (If no such sequence exists, just put $\vec{y}^t = \langle \omega, \omega, \ldots \rangle$.) Construct a sequence $\vec{y}^\infty$ by defining

$$y_n^\infty = \bigcap_{\mathrm{last}(t) \le n} y_n^t.$$

By the choice of $\vec{y}^t$, if there is any $\vec{z} \in U^\omega$ such that $\langle t, \vec{z} \rangle \in D$, then $\langle t, \vec{y}^\infty \rangle \in D$. (This uses that D is open.) Consider

$$O = \{x \in [\omega]^\omega : (\exists t,\ \text{an initial segment of } x)\ \langle t, \vec{y}^\infty \rangle \in D\},$$

an open subset of $[\omega]^\omega$. By Lemma 2.2, there is a $\vec{y}^*$ (which can be taken to satisfy $\vec{y}^* \le \vec{y}^\infty$) in U^ω such that either $[\emptyset, \vec{y}^*] \subseteq O$ or $[\emptyset, \vec{y}^*] \subseteq O^c$.

If $[\emptyset, \vec{y}^*] \subseteq O$, then $\langle \emptyset, \vec{y}^* \rangle$ captures D, and we're done.

Suppose that $[\emptyset, \vec{y}^*] \subseteq O^c$. Since D is dense, there is a condition $\langle s, \vec{z} \rangle \in D$ satisfying $\langle s, \vec{z} \rangle \le \langle \emptyset, \vec{y}^* \rangle$. By choice of $\vec{y}^\infty$, $\langle s, \vec{y}^\infty \rangle \in D$, so $\langle s, \vec{y}^* \rangle \in D$ since D is open. But now $[s, \vec{y}^*] \subseteq O$, and $[s, \vec{y}^*]$ is nonempty, a contradiction.

Part (b) can be proved similarly to part (a), with an added fusion argument. Again assume for convenience that p is the empty condition. Define $\vec{y}^t$ so that for any $i \le \mathrm{last}(t)$, if there is $\vec{z}$ with $\langle t, \vec{z} \rangle \in D_i$, then $\langle t, \vec{y}^t \rangle \in D_i$. Set $y_n^\infty = \bigcap_{\mathrm{last}(t) \le n} y_n^t$.

Part (a) shows that there exists $\langle \emptyset, \vec{y}^{*n} \rangle \le \langle \emptyset, \vec{y}^\infty \rangle$ that captures D_n. Indeed for any fixed $z_0, \ldots, z_{n-1} \in U$ one can find such $\vec{y}^{*n}$ with $y_i^{*n} = z_i$ for $i < n$. To see this, for any r of length $\le n$ with $r(i+1) \in z_{r(i)}$ run the construction of $\vec{y}^*$ in part (a) below $\langle r, \vec{y}^\infty \rangle$. The option $[r, \vec{y}^*] \subseteq O^c$ is still impossible, because of the current definition of $\vec{y}^\infty$. So the construction produces $\vec{y}^{*r}$ with $[r, \vec{y}^{*r}] \subseteq O$. Set y_i^{*n} for $i \ge n$ to be $\bigcap_{\mathrm{last}(r) \le i} y_i^{*r}$, and $y_i^{*n} = z_i$ for $i < n$. Then for any $x \in [\emptyset, \vec{y}^{*n}]$, setting r to be the first initial segment of x with $\mathrm{last}(r) \ge n$ we have $x \in [r, \vec{y}^{*r}] \subseteq O$.

In particular we can obtain $\vec{y}^{*n}$ so that for all $i < n$, $y_i^{*n} = y_i^{*(n-1)}$. Set $y_n^* = \bigcap_{m \le n} y_n^{*m}$. Then $\langle \emptyset, \vec{y}^* \rangle$ captures all D_n. ∎

Lemma 2.5 (Prikry property)**.** If $p = \langle s, \vec{y} \rangle \in \mathbb{P}_U$ is a condition and σ is a sentence in the forcing language, then there is a condition $p^* \le_0 p$ that either forces σ or forces $\neg\sigma$.

Proof. Assume for convenience that $s = \emptyset$; the proof of the general case is similar. We can again assume, by shrinking the y_ns if necessary, that every sequence in $[\![\emptyset, \vec{y}]\!]$ is strictly increasing. Let D

be the (dense, open) set of conditions that decide σ:

$$D = \{q \in \mathbb{P}_U : q \Vdash \sigma \text{ or } q \Vdash \neg\sigma\}.$$

By Lemma 2.4 there is a condition $\langle \emptyset, \vec{y}^* \rangle \leq \langle \emptyset, \vec{y} \rangle$ that captures D. Now define

$$B = \{x \in [\omega]^\omega : \langle t, \vec{y}^* \rangle \Vdash \sigma \text{ for some initial segment } t \text{ of } x\}$$

and

$$C = \{x \in [\omega]^\omega : \langle t, \vec{y}^* \rangle \Vdash \neg\sigma \text{ for some initial segment } t \text{ of } x\}.$$

Notice that both B and C are open subsets of $[\omega]^\omega$ and that $[\emptyset, \vec{y}^*] \subseteq B \cup C$. Apply Lemma 2.2 to get a condition $\langle \emptyset, \vec{y}^{**} \rangle \leq \langle \emptyset, \vec{y}^* \rangle$ such that $[\emptyset, \vec{y}^{**}] \subseteq B$ or $[\emptyset, \vec{y}^{**}] \subseteq C$.

We will show that $\langle \emptyset, \vec{y}^{**} \rangle \Vdash \sigma$ in the case when $[\emptyset, \vec{y}^{**}] \subseteq B$. (The other case is similar.) Suppose for a contradiction that we can find a condition $\langle s, \vec{z} \rangle \leq \langle \emptyset, \vec{y}^{**} \rangle$ that forces $\neg\sigma$. Choose any $x \in [s, \vec{z}] \subseteq [\emptyset, \vec{y}^{**}] \subseteq B$ and let t be an initial segment of x such that $\langle t, \vec{y} \rangle \Vdash \sigma$. This fact contradicts our assumption that $\langle s, \vec{z} \rangle \Vdash \neg\sigma$; indeed, $\langle t, \vec{y} \rangle$ and $\langle s, \vec{z} \rangle$ are compatible conditions, since one of s and t is an initial segment of the other. ∎

Lemma 2.6 (Characterization of $\mathbb{P}_U$-genericity, Judah–Shelah)**.** Let $V \subseteq W$ be models of set theory, let $U \in V$ be an ultrafilter in V, and let $\mathbb{P}_U \in V$ be the associated diagonalization forcing. For a real $g \in [\omega]^\omega \cap W$, the following are equivalent.

(a) g is a $\mathbb{P}_U$-generic real over V;

(b) for every decreasing sequence $\vec{y} \in V$ of sets $y_n \in U$, g almost-diagonalizes $\vec{y}$.

Proof. First, we assume that g is a $\mathbb{P}_U$-generic real over V, by which we mean that the set

$$G = \{\langle s, \vec{y} \rangle \in \mathbb{P}_U : g \in [s, \vec{y}]\}$$

is a generic filter. Fix a sequence $\vec{y}$ of sets $y_n \in U$ that belongs to the ground model V. The set D of conditions $\langle t, \vec{z} \rangle$ such that $\vec{z} \leq \vec{y}$ is a dense set in V, so genericity provides a condition $\langle t, \vec{z} \rangle \in G \cap D$. There is $n < \omega$ such that t is the enumeration of $g \cap n$. It follows that $g \setminus n$ diagonalizes $\vec{z}$ and therefore also $\vec{y}$.

For the converse, suppose that (b) holds of g and let $D \in V$ be a dense subset of $\mathbb{P}_U$. We'll need a way to relate a condition p to a similar condition with a different stem. For any condition $p = \langle s, \vec{y} \rangle$

and any increasing sequence $a \in \omega^{<\omega}$, define $\mathrm{copy}_a(p)$ to be the condition $\langle a^\frown s, \vec{y}\rangle$. Likewise, define $\mathrm{copy}_a(D) = \{\langle t, \vec{z}\rangle : \mathrm{copy}_a(\langle t, \vec{z}\rangle) \in D\}$. Observe that $\mathrm{copy}_a(D)$ is a dense open subset of $\mathbb{P}_U$.

Apply Lemma 2.4 to the empty condition to get $p = \langle \emptyset, \vec{y}\rangle$ that captures the dense open set $\mathrm{copy}_a(D)$ for every increasing $a \in \omega^{<\omega}$. By assumption, there is $n < \omega$ such that $g \setminus n$ diagonalizes $\vec{y}$. The fact that p captures $\mathrm{copy}_{g\cap n}(D)$ means that there is $k \geq n$ such that $\langle g \cap [n,k), \vec{y}\rangle \in \mathrm{copy}_{g\cap n}(D)$. (Here we are using the absoluteness of capturing from V to W; $g \setminus n$ does not belong to V.) In other words, $\langle g \cap k, \vec{y}\rangle \in D$. And $g \in [g \cap k, \vec{y}]$, so we have shown the filter determined by g to be generic, as desired. ∎

Corollary 2.7 (Mathias property)**.** If g is $\mathbb{P}_U$-generic over a model M and $g' \subseteq g$, then g' is $\mathbb{P}_U$-generic over M too.

Happy families in the Solovay model

Let κ be an inaccessible cardinal, and let G be generic over V for the Levy collapse $\mathrm{Coll}(\omega, {<}\kappa)$. For the definition of $\mathrm{Coll}(\omega, {<}\kappa)$, its basic properties, and a proof of Solovay's theorem, see [25, §§10–11].

For our purposes, the Solovay model is $L(\mathbb{R})$ of the generic extension $V[G]$. In this section we will prove the first main theorem of this part.

Theorem 2.8. If $X \in L(\mathbb{R})^{V[G]}$ is a subset of $[\omega]^\omega$ and $H \in L(\mathbb{R})^{V[G]}$ is a happy family, then X is H-Ramsey.

Remarks.

(1) When we say that "H is a happy family" and "X is H-Ramsey," we mean that these statements are true in $V[G]$, though they are certainly absolute between $L(\mathbb{R})^{V[G]}$ and $V[G]$.

(2) Contrast this theorem with Mathias's theorem (Theorem 1.9 above): Using a Mahlo cardinal, Mathias constructs a model in which all sets in $L(\mathbb{R})$ are H-Ramsey for *every* happy family H, whereas using only an inaccessible cardinal we get this strong Ramsey property for only the happy families that belong to $L(\mathbb{R})$.

We'll need the following lemma, an essential ingredient in Solovay's proof that all sets in the Solovay model are Lebesgue-measurable.

Lemma 2.9 (Factor Lemma, Solovay [42])**.** Suppose that κ is an inaccessible cardinal and that $\mathbb{P}$ is a poset of size $< \kappa$. Let G be generic over V for the Levy collapse $\mathrm{Coll}(\omega, {<}\kappa)$. If in $V[G]$ there is a filter $h \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over V, then there is $G^* \in V[G]$ that is $\mathrm{Coll}(\omega, {<}\kappa)$-generic over $V[h]$ and such that $V[h][G^*] = V[G]$.

$$\begin{tikzcd}
V[G] & \\
& V[h] \arrow[ul, "\mathrm{Coll}(\omega,{<}\kappa)"'] \\
V \arrow[uu, "\mathrm{Coll}(\omega,{<}\kappa)"] \arrow[ur, "\mathbb{P}"'] &
\end{tikzcd}$$

Figure 2.1: Factor Lemma diagram.

Proof of Theorem 2.8.

Let $H \in L(\mathbb{R})^{V[G]}$ be a happy family, and let $X \subseteq [\omega]^\omega$ be a set of reals that belongs to $L(\mathbb{R})^{V[G]}$. Every set in $L(\mathbb{R})$ is ordinal-definable from a real, so there are a formula $\varphi = \varphi(x, y, w)$, a real $r \in V[G]$, and a sequence α of ordinals such that

$$x \in X \text{ iff } V[G] \vDash \varphi[x, r, \alpha].$$

We may assume that H is also ordinal-definable from r, by (if necessary) replacing r with a real coding both r and a real from which H is ordinal-definable.

Recall that κ is inaccessible and G is generic over V for the Levy collapse $\mathrm{Coll}(\omega, {<}\kappa)$. It follows from the chain-condition of the Levy collapse that there is an ordinal $\beta < \kappa$ such that $G \upharpoonright \beta$ is $\mathrm{Coll}(\omega, {<}\beta)$-generic over V and the real parameter r belongs to $V[G \upharpoonright \beta]$. Let $\varphi^* = \varphi^*(x, y, w)$ be the natural formula satisfying the equivalence

$$\varphi^*(x, y, w) \iff \emptyset \Vdash_{\mathrm{Coll}(\omega,{<}\kappa)} \varphi(x, \check{y}, \check{w}).$$

And let

$$\overline{X} = \{x \in [\omega]^\omega \cap V[G \upharpoonright \beta] : V[G \upharpoonright \beta] \models \varphi^*[x, r, \alpha]\}.$$

Notice that $\overline{X}$ belongs to $V[G \upharpoonright \beta]$; in fact, by the homogeneity of the collapse, it is equal to $X \cap V[G \upharpoonright \beta]$. Indeed, for $x \in [\omega]^\omega \cap V[G \upharpoonright \beta]$,

$$\begin{aligned} x \in X &\iff V[G] \models \varphi[x, r, \alpha] \\ &\iff \emptyset \Vdash_{\mathrm{Coll}(\omega, <\kappa)} \varphi[x, r, \alpha] \\ &\iff x \in \overline{X}. \end{aligned}$$

By similar arguments, $\overline{H} := H \cap V[G \upharpoonright \beta]$ belongs to $V[G \upharpoonright \beta]$. (Here we use our assumption that $H \in L(\mathbb{R})$. We only needed to find an initial segment of the collapse that absorbed the real parameter r.)

If κ were a Mahlo cardinal, then we could choose β to ensure that $\overline{H}$ is closed under diagonalizations of sequences in $V[G \upharpoonright \beta]$ and then find in $V[G \upharpoonright \beta]$ a Ramsey ultrafilter $U \subseteq \overline{H}$. That approach is not open to us, and it is for this reason that we must use the diagonalization forcing, since — unlike Mathias forcing — it has the Mathias and Prikry properties even when guided by a non-Ramsey ultrafilter.

Because H is a coideal in $V[G]$, its intersection $\overline{H}$ with $V[G \upharpoonright \beta]$ must be a coideal in $V[G \upharpoonright \beta]$. Working in $V[G \upharpoonright \beta]$, we can extend the filter dual to the ideal $\overline{H}^c$ to obtain an ultrafilter $U \subseteq \overline{H}$.

Claim 2.10. Let $g \in V[G]$ be a real that is generic over $V[G \upharpoonright \beta]$ for a poset $\mathbb{P} \in V[G \upharpoonright \beta]$. Then $g \in X$ iff $V[G \upharpoonright \beta][g] \models \varphi^*[g, r, \alpha]$.

Proof of Claim 2.10. This is just an application of the Factor Lemma 2.9. Work in $V[G]$. Because $\mathrm{Coll}(\omega, <\beta) * \dot{\mathbb{P}}$ is a poset in V of size $< \kappa$, we can apply the Factor Lemma to find a filter G^* that is $\mathrm{Coll}(\omega, <\kappa)$-generic over $V[G \upharpoonright \beta][g]$ and such that

$$V[G] = V[G \upharpoonright \beta][g][G^*].$$

Suppose first that $g \in X$, so $V[G] \models \varphi[g, r, \alpha]$. The homogeneity of the Levy collapse guarantees that in $V[G \upharpoonright \beta][g]$ the empty condition forces $\varphi(\dot{g}, \check{r}, \check{\alpha})$. That is,

$$V[G \upharpoonright \beta][g] \models \varphi^*[g, r, \alpha].$$

For the converse, suppose that $V[G \restriction \beta][g] \models \varphi^*[g, r, \alpha]$. Now $V[G] = V[G \restriction \beta][g][G^*]$ must satisfy $\varphi[g, r, \alpha]$, since the empty condition in $\mathrm{Coll}(\omega, {<}\kappa)$ forces it and G^* is $\mathrm{Coll}(\omega, {<}\kappa)$-generic over $V[G \restriction \beta]$. That is, $g \in X$. ∎

Claim 2.11. In $V[G]$, for every $\langle \emptyset, \vec{y} \rangle \in \mathbb{P}_U$ there is a real $g \in H$ that is $\mathbb{P}_U$-generic below $\langle \emptyset, \vec{y} \rangle$ over $V[G \restriction \beta]$.

Proof of Claim 2.11. Work in $V[G]$. By the characterization of $\mathbb{P}_U$-genericity (Lemma 2.6), we need only ensure that g almost-diagonalizes every $\subseteq$-decreasing sequence of sets in U that belongs to $V[G \restriction \beta]$. But $[\omega]^\omega \cap V[G \restriction \beta]$ is countable, so we can find a single sequence $\vec{z} = \langle z_n : n < \omega \rangle$ of subsets of U such that any real that almost-diagonalizes $\vec{z}$ also almost-diagonalizes every sequence in $U^\omega \cap V[G \restriction \beta]$. Moreover, we can arrange that $\vec{z} \leq \vec{y}$.

It remains to show that there is a real $g \in H$ that diagonalizes the sequence $\vec{z}$, but this follows immediately from the definition of *happy family*, since each z_n belongs to H. ∎

Using the Prikry property of $\mathbb{P}_U$, find a condition $\langle \emptyset, \vec{y} \rangle \in V[G \restriction \beta]$ that forces either $\varphi^*(\dot{g}, \check{r}, \check{\alpha})$ or its negation. (Here $\dot{g}$ is the natural name for the generic real g.) Now the real $g \in [\emptyset, \vec{y}] \cap H$ provided by Claim 2.11 will witness that X is H-Ramsey. The argument is symmetric in X and X^c, so we may assume without loss of generality that $g \in X$. It follows by Claim 2.10 that $\langle \emptyset, \vec{y} \rangle$ forces $\varphi^*(\dot{g}, \check{r}, \check{\alpha})$, not its negation. Working in $V[G]$, consider any real $g' \in [g]^\omega$. Since $\mathbb{P}_U$ has the Mathias property, g' is also $\mathbb{P}_U$-generic over $V[G \restriction \beta]$. And g diagonalizes $\vec{y}$, so g' does also. That is, $\langle \emptyset, \vec{y} \rangle$ belongs to the $\mathbb{P}_U$-generic associated to g'. It follows that

$$V[G \restriction \beta][g'] \models \varphi^*[g', r, \alpha],$$

since $\langle \emptyset, \vec{y} \rangle$ forces it. Now apply Claim 2.10 to conclude that $g' \in X$. This completes the proof that X is H-Ramsey. ∎

By taking $X = H$ in Theorem 2.8 and applying Lemma 1.5, we obtain the following corollary.

Corollary 2.12. If $A \in V[G]$ is an infinite mad family, then A does not belong to the Solovay model $L(\mathbb{R})^{V[G]}$, and neither does the ideal that it generates.

CHAPTER 3

Meager filters and determinacy

To answer Törnquist's question, whether AD implies that there are no infinite mad families, one might hope to show that the ideal generated by an infinite mad family fails to have a regularity property that all sets enjoy under AD. Unfortunately, the Baire property is inadequate for this approach. It can be seen directly from the definition that the ideal generated by an infinite mad family has the Baire property; in this section we give a game-theoretic proof that all ideals are meager under AD.

Notice first that an ideal on ω has the Baire property if and only if it is meager; this is a well known application of 0–1 laws. See Oxtoby [37, Thm. 21.4]. (Likewise, an ideal is measurable if and only if it has measure zero.)

For a real $x \in [\omega]^\omega$, write e_x for the unique increasing function $\omega \to \omega$ that enumerates x. We say a filter F on ω is *bounded* if the family $\{e_x : x \in F\}$ is bounded in the eventual-domination ordering on ω^ω.

Recall Talagrand's characterization of filters on ω with the Baire property.

Theorem (Talagrand [46])**.** A filter on ω is meager iff it is bounded.

The proof of Talagrand's theorem is very nearly contained in Blass [7]; for the reader's convenience, we give the rest of the details here.

Claim. Let F be a filter on ω containing every cofinite set. The following are equivalent.

(i) F is meager.

(ii) There is an interval partition $(I_0, I_1, \dots)$ of ω such that every set in F meets all but finitely many of the pieces I_n.

(iii) There is an increasing function $f \in \omega^\omega$ such that every set in F meets the interval $[n, f(n))$ for all but finitely many n.

(iv) F is bounded.

Proof of claim. The equivalence of (i) and (ii) is established in Proposition 9.4 of [7], so it suffices to prove the implications (ii) $\leftrightarrow$ (iii), (ii) $\to$ (iv), and (iv) $\to$ (i).

(ii) $\to$ (iii). Define $f(n)$ to be $\max(I_{k+1}) + 1$ for the unique k such that $n \in I_k$. Now let $x \in F$ and let n be large enough that $x \cap I_{k+1} \neq \emptyset$, where $n \in I_k$. Notice that $I_{k+1} \subseteq [n, f(n))$, so x meets $[n, f(n))$.

(iii) $\to$ (ii). Put $I_n = [f^n(0), f^{n+1}(0))$. Then every $x \in F$ meets I_n for sufficiently large n.

(ii) $\to$ (iv). Define $f(n) = \max I_{2n}$. Let $x \in F$ and choose k large enough that x meets I_m for all $m \geq k$. In particular, x meets each of the intervals $I_k, I_{k+1}, \ldots, I_{2k}$, so $|x \cap \max I_{2k}| \geq k$. This implies that $e_x(k) \leq \max I_{2k} = f(k)$, from which we conclude that $e_x \leq^* f$.

(iv) $\to$ (i). Suppose that $f \in \omega^\omega$ dominates the enumerating function of every member of F. This means that F is included in the set

$$\{x \in 2^\omega : \forall^\infty k \, |x \cap f(k)| \geq k\} = \bigcup_m \{x \in 2^\omega : \forall k \geq m \, |x \cap f(k)| \geq k\}.$$

It's easy to see that for each m the set $\{x \in 2^\omega : \forall k \geq m \, |x \cap f(k)| \geq k\}$ is a nowhere-dense subset of 2^ω, so we conclude that F is meager. ∎

The game we study here is already well known. It provides a convenient proof, apparently part of the folklore, that there are no nonprincipal ultrafilters under AD.

Players I & II alternate playing nonempty intervals $I_k \in [\omega]^{<\omega}$

I	I_0		I_2		I_4		$\cdots$
II		I_1		I_3		I_5	$\cdots$

subject to the following rules.

(i) Each play I_k is a nonempty finite interval, and

(ii) $\min(I_{k+1}) = \max(I_k) + 1$.

So the game board looks like this:

$$\begin{array}{c|ccccc}
\mathrm{I} & [0, n_0] & & [n_1+1, n_2] & & \cdots \\
\hline
\mathrm{II} & & [n_0+1, n_1] & & [n_2+1, n_3] & \cdots
\end{array}$$

We call an interval partition $(I_0, I_1, \ldots)$ a *run* of the game, and an interval partition of some initial segment $[0, n]$ of ω is a *partial run* of the game. If, in a run of the game, $n \in I_{2k}$, then we say Player I *claims* n, and similarly Player II *claims* every member of each interval I_{2k+1}.

To determine the winner of a run of the game, we require a filter F on ω. Player I wins the game iff $\bigcup_{n\in\omega} I_{2n} \in F$. That is, Player I wins iff he claims a measure-one set of integers.

It's clear that Player I has a winning strategy if F is a principal ultrafilter. Conversely, if Player I has a winning strategy, then a strategy-stealing argument shows that the filter is principal. A strategy-stealing argument also shows that Player II cannot have a winning strategy if F is an ultrafilter and $\{0\} \notin F$. (If $\{0\} \in F$, then Player II still does not have a winning strategy since Player I does.) This gives an easy proof that there are no nonprincipal ultrafilters on ω under AD.

In fact, the game can be used to show that every filter is meager under AD. We provide that argument here, since it does not seem to appear in the literature. Bartoszyński and Scheepers [6], however, give a similar characterization of meager filters using an integer game, with some applications.

Proposition 3.1. A filter F on ω is meager if and only if Player II has a winning strategy for the partition game for F.

Proof. This can be proved directly, but it will be easier to go through Talagrand's characterization of meager filters. Suppose first that F is meager, hence bounded. Let $g \in \omega^\omega$ be a function that dominates e_x for every $x \in F$; we can assume without loss that g is increasing. We describe a strategy σ for Player II. Suppose it is Player II's turn and that Player I has claimed exactly k integers so far. Player II simply plays the shortest nonempty interval I such that $\max(I) \geq g(k+1)$. (Notice that $g(k+1)$ may already have been claimed.) Explicitly,

$$\sigma(I_0, \ldots, I_{2n}) = \begin{cases} [\max(I_{2n})+1, g(k+1)] & \text{if } g(k+1) \geq \max(I_{2n})+1 \\ \{\max(I_{2n})+1\} & \text{if } g(k+1) \leq \max(I_{2n}) \end{cases}.$$

Now consider a run $(I_0, I_1, I_2, \dots)$ of the game according to σ, and put $x = I_0 \cup I_2 \cup \cdots$, the set of integers claimed by Player I. If Player I has claimed exactly k integers before at the beginning of II's turn $2n+1$, then we have

$$g(k+1) \leq \max I_{2n+1} < \min I_{2n+2} = e_x(k+1).$$

We have found infinitely many k for which $g(k+1) < e_x(k+1)$; this means that g does not dominate e_x. (Our infinitely many k are those of the form $k = |I_0| + |I_2| + \cdots + |I_{2n}|$.) Because g dominates the enumerating function of every member of F, x cannot belong to F. So Player I loses, and σ is a winning strategy for Player II.

For the converse, suppose that F is unbounded, and we'll show that Player I can defeat any strategy σ for Player II. For a fixed n, there are only finitely many partial runs $(I_0, \dots, I_k)$ of the game such that both

- $I_0 \cup I_1 \cup \cdots \cup I_k = [0, n]$, and
- k is even, so it is Player II's turn to play after the partial run.

Therefore, there are only finitely many intervals Player II might play immediately after such a partial run, if Player II plays according to σ. With this in mind, let $f(n)$ be the largest m contained in any such interval. This definition guarantees that in any run $(I_n)_{n\in\omega}$ of the game according to σ, $\max(I_{2n+1}) \leq f(\max(I_{2n}))$ for every n.

Since F is unbounded, there is some set $x \in F$ whose enumerating function e_x is not dominated by the function $n \mapsto f^{2n}(0)$. (The exponent here indicates how many times f should be iterated.) That is, there are infinitely many $n \in \omega$ such that $|f^{2n}(0) \cap x| < n$. For such n, the set x meets fewer than n of the $2n$ intervals $(f^k(0), f^{k+1}(0)]$, $0 \leq k < 2n$. It follows that there are infinitely many k for which

$$x \cap (f^k(0), f^{k+1}(0)] = \emptyset. \tag{$*$}$$

On his turn, Player I should claim all integers up to $f^k(0)$ for some k satisfying $(*)$. Player II, if using her strategy σ, must respond by playing an interval $I \subseteq (f^k(0), f(f^k(0))]$. Therefore Player II never claims any members of x, so Player I claims all members of x. Player I claims a set in F and wins, so σ is not a winning strategy for Player II. ∎

CHAPTER 4

Determinacy and large cardinals

Background

This chapter includes a proof of our partial answer to Törnquist's question. We begin by clarifying some relevant parts of the literature.

In [14] Farah isolates the weakest analogue of *happy family* for which the key facts of Mathias's original paper [30] can still be proved.

Definition 4.1 (Farah)**.** Let H be a coideal on ω. If $\langle D_n \rangle_{n<\omega}$ is a sequence of dense subsets of the poset $\langle H, \subseteq \rangle$, then a set $x \in [\omega]^\omega$ is a *diagonalization* of $\langle D_n \rangle_{n<\omega}$ if there exists a sequence of sets $d_n \in D_n$ so that x is a diagonalization of $\langle d_n : n < \omega \rangle$, meaning that $f(n+1) \in d_{f(n)}$ for every $n < \omega$, where $f \colon \omega \to \omega$ is the increasing enumeration of x. We say that H is *semiselective* if for every sequence $\langle D_n \rangle_{n<\omega}$ of dense subsets and every $y \in H$, there is $x \in H \cap [y]^\omega$ that diagonalizes the sequence $\langle D_n \rangle_{n<\omega}$.

When guided by a happy family, Mathias forcing is σ-closed $*$ ccc, whereas when guided by a semiselective coideal it is σ-distributive $*$ ccc.

In [14, §4] Farah presents a theorem of Todorcevic, that, in the presence of a supercompact cardinal, every set in $L(\mathbb{R})$ is H-Ramsey for every semiselective coideal H. Lemma 4.3 in that paper is not true as stated, but it is also not used in Todorcevic's argument; rather Todorcevic used a connection between the Ramsey property and the Ellentuck topology (see also Todorcevic [47, Ch. 7]) and the conclusion of Lemma 4.4 of the paper, which is true assuming the existence of a supercompact cardinal, by Theorem 4.1 of Feng–Magidor–Woodin [15].

Large cardinals

We prove Todorcevic's theorem by different methods, from weaker large cardinals, in the region of Woodin cardinals. Our large cardinal assumptions are weak enough to follow from determinacy assumptions. This allows us to then prove a version of the theorem under determinacy. Our methods also give a correct version of Lemma 4.3 of [14]. The restriction of the lemma to proper posets, or more generally to the class of reasonable posets of Foreman and Magidor [16], is true and follows from Theorem 4.3 below.

Theorem 4.3 is a triangular version of the following Embedding Theorem of Neeman–Zapletal [35] and [36].

Theorem 4.2 (Neeman–Zapletal [36])**.** (Large cardinals) Suppose that $\mathbb{P}$ is a proper (or reasonable) poset and that G is $\mathbb{P}$-generic over V. There is an elementary embedding $j\colon L(\mathbb{R})^V \to L(\mathbb{R})^{V[G]}$ that fixes every ordinal.

Theorem 4.3. (Large cardinals, see remarks following Corollary 4.5 for the exact assumptions) Suppose M is a countable transitive model that embeds into a sufficiently large rank-initial segment of V, say by $\pi\colon M \overset{\sim}{\longrightarrow} V_\theta$. If G is $\mathbb{P}$-generic over M for some proper (or reasonable) poset $\mathbb{P}$ in M, then there is an embedding $\widehat{\pi}\colon L(\mathbb{R})^{M[G]} \to L(\mathbb{R})^{V_\theta}$ that agrees with π on ordinals and reals, and completes a commuting system of elementary embeddings:

$$\begin{tikzcd} & L(\mathbb{R})^{V_\theta} & \\ L(\mathbb{R})^M \arrow[ur, "\pi"] \arrow[rr] & & L(\mathbb{R})^{M[G]} \arrow[ul, "\widehat{\pi}"'] \end{tikzcd}$$

Using Theorem 4.3, we prove Todorcevic's theorem by imitating Mathias's proof (Theorem 1.7) that there are no analytic infinite mad families.

Theorem 4.4. In the presence of large cardinals (specifically $|\mathbb{R}|^+ + 1$-iterable model with the sharp of ω Woodin cardinals), every set $X \subseteq [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every happy family H.

Proof. Let $X \in L(\mathbb{R})$ be a subset of $[\omega]^\omega$ and let H be any happy family. Find a formula φ, a real parameter z, and a sequence α of ordinal parameters such that

$$x \in X \iff L(\mathbb{R}) \vDash \varphi[x, \alpha, z].$$

Let Z be a countable elementary submodel of a sufficiently large rank-initial segment of V containing z, α, and H, and let M be the transitive collapse of Z. Let $\pi\colon M \to Z$ be the anti-collapse map. Put $\overline{H} = \pi^{-1}(H)$, and $\overline{\alpha} = \pi^{-1}(\alpha)$. Notice that $\overline{H} = H \cap Z$, and $\overline{H}$ is a happy family in M. It follows that Mathias forcing $\mathbb{M}^M_{\overline{H}}$ has the Prikry and Mathias properties in M. So there is a condition $\langle 0, x\rangle \in M$ that either forces (over M) $\varphi^{L(\mathbb{R})}(\dot{g}, \overline{\alpha}, z)$ or forces $\neg\varphi^{L(\mathbb{R})}(\dot{g}, \overline{\alpha}, z)$. (Here $\dot{g}$ is the usual name for the generic real.) For any real g that is $\mathbb{M}^M_{\overline{H}}$-generic over M below $\langle 0, x\rangle$, we have the following equivalence.

$$\begin{aligned} g \in X &\iff L(\mathbb{R}) \vDash \varphi[g, \alpha, z] \\ &\iff M[g] \vDash \varphi^{L(\mathbb{R})}[g, \overline{\alpha}, z] \\ &\iff \langle 0, x\rangle \Vdash_{\mathbb{M}^M_{\overline{H}}} \varphi^{L(\mathbb{R})}[\dot{g}, \overline{\alpha}, z]. \end{aligned} \tag{$*$}$$

The second equivalence is a consequence of Theorem 4.3.

Since $\overline{H}$ is countable, there is a real $g \in H$ that is $\mathbb{M}^M_{\overline{H}}$-generic over M below $\langle 0, x\rangle$. (To see this, argue exactly as we did in the proof of Claim 2.11 while proving Theorem 2.8.) Since $\mathbb{M}^M_{\overline{H}}$ has the Mathias property in M, every subset $g' \subseteq g$ is generic too. Suppose without loss of generality that $g \in X$. Then the $\Rightarrow$ direction of $(*)$ implies that $\langle 0, x\rangle \Vdash \varphi^{L(\mathbb{R})}[\dot{g}, \overline{\alpha}, z]$ over M, and the $\Leftarrow$ direction of $(*)$ implies that $g' \in X$ for every $g' \subseteq g$. ∎

We apply Lemma 1.5 to obtain

Corollary 4.5. In the presence of large cardinals (specifically $|\mathbb{R}|^+ + 1$-iterable model with the sharp of ω Woodin cardinals), there are no infinite mad families in $L(\mathbb{R})$.

Remarks.

(1) The large cardinal assumption we use to prove Theorem 4.3 is the existence of $M^\sharp_\omega$, the minimal iterable model for the sharp of ω Woodin cardinals, and its $|\pi(\mathbb{P})|^+ + 1$-iterability.

(2) Theorem 4.2 applies not only to proper posets, but to the larger class of so-called *reasonable* posets (introduced by Foreman and Magidor [16]), which includes all posets of the form

σ-distributive $*$ ccc. (A poset $\mathbb{P}$ is *reasonable* if for all uncountable cardinals κ, $([\kappa]^\omega)^V$ is stationary in $([\kappa]^\omega)^{V^{\mathbb{P}}}$. See [16].) Our triangular version, Theorem 4.3, also applies to this broader class. We therefore obtain Todorcevic's theorem for semiselective coideals, not just happy families.

(3) Schindler [41] has computed the consistency strength of Theorem 4.2 for proper posets to be exactly the existence of a *remarkable* cardinal (See Definition 8.2), which is stronger than an ineffable cardinal but weaker than the existence of $0^\sharp$. The consistency strength of the version for reasonable posets remains open. (See Part II, especially Chapter 9.)

Schindler and William Chan have observed (unpublished) that our triangular version of the Embedding Theorem, Theorem 4.3, is also consistent relative to a remarkable cardinal.

AD^+

We assume ZF throughout; theorems listed below under AD^+ are theorems of $\mathsf{ZF} + \mathsf{AD}^+$. For the precise definition of AD^+ and a thorough discussion of it, the reader is encouraged to consult [8, §§1–3]. AD^+ implies AD, and it is open whether the two are equivalent.

A version of Theorem 4.4 holds under AD^+, and implies a version of Corollary 4.5 under AD^+. These are proved using an AD^+ version of the triangular embedding theorem. The theorem applies to posets in the following class:

Definition 4.6. Call poset $\mathbb{P} \subseteq \mathbb{R}$ *absolutely proper* (respectively *absolutely reasonable*) if there exists a club $C \subseteq [\mathbb{R}]^\omega$ and $A \subseteq \mathbb{R}$ so that for every $U \in C$ and every transitive countable model N of a large enough fragment of ZFC with $\mathbb{R}^N = U$ and $\mathbb{P} \cap U, A \cap U \in N$, $\mathbb{P} \cap U$ is proper (respectively reasonable) in N.

Note that absolute properness (reasonableness) is a Σ_1 property in $\{\mathbb{R}\}$ over $\mathbb{N} \cup \mathbb{R} \cup \mathcal{P}(\mathbb{R})$, with predicates for membership and coding finite and countable sequences of reals by reals, in other words it is a Σ^2_1 statement. In particular it reflects up from transitive models containing $\{\mathbb{R}\}$. It is phrased without reference to countable elementary substructures of initial segments of the universe, which need not exist in contexts where DC fails. Posets which are provably proper (reasonable) under AC by sufficiently absolute arguments are often absolutely proper (reasonable). For example:

Lemma 4.7. Let H be a happy family. Then Mathias's forcing $\mathbb{M}_H$ (with conditions coded as reals in a natural way) is absolutely proper.

Proof. Let C be the club of countable $U \subseteq \mathbb{R}$ which are elementary under the coding predicates for countable sequences and for conditions in $\mathbb{M}_H$, and so that $U \cap H$ satisfies the requirements in the definition of a happy family with upward closure restricted to $y \in U$ and selectivity restricted to descending sequences that are coded by a real in U. Then for any N as in Definition 4.6, $U \cap H$ is a happy family in N. Hence in particular $\mathbb{M}^N_{U\cap H}$ is proper in N. Using elementarity under coding, $\mathbb{M}^N_{U\cap H} = \mathbb{M}_H \cap U$. ∎

Theorem 4.8. (AD$^+$) For every $\alpha < \Theta$, for every $A \subseteq \mathbb{R}$, and for stationarily many countable $Z \preceq L_\alpha(\mathbb{R}, A)$, if M is the transitive collapse of Z, $\pi\colon M \xrightarrow{\sim} L_\alpha(\mathbb{R}, A)$ is the anti-collapse embedding, $\overline{\alpha} = M \cap \mathrm{On}$, $\overline{A} = \pi^{-1}(A)$, and $\mathbb{P}$ is absolutely proper (or absolutely reasonable) in M, then there exists a countable transitive model N of ZFC with $\mathbb{R}^N = \mathbb{R} \cap M$ and $\overline{\alpha}, \overline{A} \in N$, and there exists a $\mathbb{P}$-name $\dot{A}^* \in N$, so that for every G which is $\mathbb{P}$-generic over N, there is an embedding $\widehat{\pi}\colon L_{\overline{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G]) \to L_\alpha(\mathbb{R}, A)$ that agrees with π on ordinals and reals, maps $\dot{A}^*[G]$ to A, and completes a commuting system of elementary embeddings:

$$\begin{tikzcd} & L_\alpha(\mathbb{R}, A) & \\ M \arrow[ur, "\pi"] \arrow[rr] & & L_{\overline{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G]) \arrow[ul, "\widehat{\pi}"'] \end{tikzcd}$$

Figure 4.1: The triangular embedding theorem.

Theorem 4.9. (AD$^+$) Every set $X \subseteq [\omega]^\omega$ is H-Ramsey for every happy family H. Consequently, there are no infinite mad families.

Proof. Let $A \subseteq \mathbb{R}$ code X and H. Fix a countable Z elementary in $L_\omega(\mathbb{R}, A)$ for which Theorem 4.8 holds, with $X, H, A \in Z$. Let M and π be as in the theorem. By Lemma 4.7, $\mathbb{M}^M_H$ is absolutely proper in M. Now proceed as in the proof of Theorem 4.4, but using Theorem 4.8 over N instead of Theorem 4.3, and forcing a truth value to $\varphi^{L_\omega(\mathbb{R},\dot{A}^*)}(\dot{g},\dot{A}^*)$, where φ is a formula so that $\varphi(x,A)$ holds in $L_\omega(\mathbb{R},A)$ iff $x \in X$. ∎

Remarks.

(1) Since Theorem 4.8 applies to absolutely reasonable posets, the last theorem can be strengthened to cover absolutely semiselective coideals, where absoluteness is meant in the manner of Definition 4.6, replacing proper there with semiselective throughout. The *absoluteness* of the semiselecitivity is necessary in several places beyond the use of Theorem 4.8, including primarily for the arguments that use the Prikry and Mathias properties (as in the proof of Theorem 4.4) over N.

(2) Some contexts where semiselectivity is absolute were obtained by Larson and Raghavan [27]. For example the notion asserting that player II wins the strategic selectivity game there is clearly absolute, and is equivalent under $\mathsf{AD}_{\mathbb{R}}$ to semiselectivity that persists to some outer model of choice with the same reals, by [27, Proposition 1.4].

(3) Versions of Theorem 4.8 that provide less elementarity can be proved with less than full determinacy. For example, the restriction of the theorem to projective sets A, where we weaken stationarity to existence, weaken elementarity throughout to just Σ_1 elementarity in the structure $(\mathbb{R},A)$, and remove the club in the definition of absolute properness requiring the conditions of the definition for all Σ_1 elementary countable substructures in $(\mathbb{R},A)$, is provable under projective determinacy. The argument is less involved than the proof of the full theorem, simply using the facts that projective truth is absolute to iterable models with a large enough finite number of Woodin cardinals, the ability to iterate to make any real generic for the collapse of the first Woodin cardinal, and the existence of these models under projective determinacy.

(4) This projective version of Theorem 4.8 is enough to run the proof of Theorem 4.9 for projective sets H and X. In particular projective determinacy implies that there are no projective infinite mad families. This was proved independently by Karen Haga.

An application of the triangular embedding theorem

The AD^+ triangular embedding theorem has additional applications. For example it can be used to obtain results similar to these obtained under $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ for proper ideals in Chan–Magidor [9], working instead under the more general AD^+, but restricting to absolutely proper ideals:

Theorem 4.10. (AD^+) Let I be a σ-ideal on ω^ω, so that $\mathbb{P}_I$ is absolutely proper. Let $\mathbf{\Gamma}$ be a pointclass closed under Borel substitutions, with a universal set. Let E be an equivalence relation whose equivalence classes are all in $\mathbf{\Gamma}$ (respectively in both $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$). Then there exist densely many Borel sets C in I^+ so that $E \restriction C$ is in $\mathbf{\Gamma}$ (respectively in both $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$).

Recall that $\mathbb{P}_I$, studied in Zapletal [50], is the poset of Borel sets in I^+, ordered by reverse inclusion mod I. Zapletal presented many of the standard cardinal invariants forcing notions in this form. For many of the ideals he considered, the posets $\mathbb{P}_I$ are provably proper, and by methods similar to the proof of Lemma 4.7 one can show these specific posets are also absolutely proper. In the context of absolute properness we assume some coding of Borel sets to view conditions in $\mathbb{P}_I$ as reals.

Proof of Theorem 4.10. We prove that there exists C as in the theorem. Working throughout below an arbitrary $B \in I^+$ the same argument would show there exist densely many such C. We prove only the case of membership in $\mathbf{\Gamma}$. The proof handling both $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$ simultaneously is similar.

Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be universal for $\mathbf{\Gamma}$. Let $A \subseteq \mathbb{R}$ code I, U, E, and a club witnessing that $\mathbb{P}_I$ is absolutely proper, so that the poset is absolutely proper in $L_\omega(\mathbb{R}, A)$. Note that $\mathbb{P}_I$ is also proper in $L_\omega(\mathbb{R}, A)$. Otherwise there is a stationary set of substructures without master conditions. But this can be reflected into a model N as in Definition 4.6, contradicting properness in N.

Fix a countable Z elementary in $L_\omega(\mathbb{R}, A)$ for which Theorem 4.8 holds, with $I, U, E, A \in Z$. Since $\mathbb{P}_I$ is proper we can pick Z for which there are master conditions in $\mathbb{P}_I$. Let M and π be as in the theorem. Let $\overline{A} = \pi^{-1}(A)$, $\overline{I} = \pi^{-1}(I)$, $\overline{U} = \pi^{-1}(U)$, $\overline{E} = \pi^{-1}(E)$. Let N and $\dot{A}^*$ be as in Theorem 4.8 for the forcing $\mathbb{P}^M_{\overline{I}}$. Let $\dot{U}^*$ name the set coded by $\dot{A}^*$ in the way U is coded by A, and similarly with $\dot{E}^*$.

Recall from [50, Proposition 2.1.2] that if G is $\mathbb{P}^M_{\overline{I}}$ generic over N then $\bigcap G$ is a singleton real. Let $\dot{x}$ name this real. The E-equivalence class of $\dot{x}[G]$ is in $\mathbf{\Gamma}$. By universality it is equal to the section U_y

for some real y. Using the elementarity given by Theorem 4.8 it follows that there is a name $\dot{y} \in N$ so that over N the $\dot{E}^*$-equivalence class of $\dot{x}$ is forced to be equal to the section $\dot{U}^*_{\dot{y}}$. Again by the theorem, for any generic G, the E-equivalence class of $\dot{x}[G]$ is then equal to $U_{\dot{y}[G]}$.

Let $C = \{\dot{x}[G] : G \text{ is } \mathbb{P}^M_{\bar{I}}\text{-generic over } N\}$. Since N is countable, C is non-empty. In fact since there is a master condition for Z in $\mathbb{P}_I$, $C \in I^+$, see the proof of [50, Proposition 2.2.2]. Each $x \in C$ uniquely determines the generic which induces it, denoted G_x, through the condition that $B \in G_x$ iff $B \in \mathbb{P}^M_{\bar{I}} \wedge x \in B$.

We claim that $E \restriction C$ is in $\mathbf{\Gamma}$. To see this, note simply that for any $x_1, x_2 \in C$, $x_1 \; E \; x_2$ iff $x_2 \in U_{\dot{y}[G_{x_1}]}$ iff $\langle \dot{y}[G_{x_1}], x_2 \rangle \in U$. The condition on the right is in $\mathbf{\Gamma}$ using closure under Borel substitutions since N is countable. ∎

Proof of the triangular embedding theorem

We derive both versions from the same lemma.

Call $\langle Q, \Sigma, \delta \rangle$ a *germ* which *captures* $F \subseteq \mathbb{R}$ just in case that:

1. Q is a model of a large enough fragment of ZFC with infinitely many Woodin cardinals, whose supremum is δ.
2. Σ is an $\omega_1 + 1$ iteration strategy for Q.
3. (Condensation) If $\mathcal{T}$ is an iteration tree by Σ, and $\pi : N \to N^*$ is an elementary embedding, with N, N^* transitive models of enough of ZFC and $\mathcal{T} \in \mathrm{range}(\pi)$, then $\pi^{-1}(\mathcal{T})$ is according to Σ.
4. (Capturing) For every iteration map $j : Q \to Q^*$ by Σ, there is a $\mathrm{Coll}(\omega, j(\delta))$-name $\dot{F}^* \in Q^*$, so that for every generic H for $\mathrm{Coll}(\omega, j(\delta))$ over Q^*, every $\eta < j(\delta)$, and every $x \in Q^*[H \restriction \eta]$, $x \in \dot{F}^*[H]$ iff $x \in F$.

Condition (3) typically holds for uniquely iterable fine structural models. In particular it holds for $Q = M^\sharp_\omega$, the minimal model for the sharp of ω Woodin cardinals, and its unique iteration strategy.

Let $j\colon Q \to Q^*$ be according to Σ. Let H be generic over Q^* for $\mathrm{Coll}(\omega, < j(\delta))$. The *derived model* of Q^* using H is $L_\alpha(\mathbb{R}^*, F^*)$ where $\alpha = Q^* \cap \mathrm{On}$, $\mathbb{R}^*$ consists of the reals in the extensions of Q^* by strict initial segments of H, and $F^* = F \cap \mathbb{R}^*$. We refer to $\mathbb{R}^*$ as the *derived reals*. By the capturing condition, F^* belongs to $Q^*[H]$, and in fact there is a name in Q^* which produces the right F^* independently of H.

By a *composed Σ-iteration of Q with direct limit $j\colon Q \to Q^*$* we mean a sequence of embeddings $j_n\colon Q_n \to Q_{n+1}$ so that $Q_0 = Q$, each j_n is the embedding of a countable iteration tree $\mathcal{T}_n$ on Q_n with final model Q_{n+1}, for each n the composition of the trees $\mathcal{T}_0, \dots, \mathcal{T}_n$ is according to Σ, Q^* is the direct limit of the models Q_n, and j is the direct limit embedding. We write $j_{n,m}\colon Q_n \to Q_m$ for the composed embeddings, and $j_{n,\infty}\colon Q_n \to Q^*$ for the direct limit embedding from Q_n.

If the sequence of trees belongs to V then the composition of all ω trees is by Σ, and hence $j\colon Q \to Q^*$ is itself an iteration embedding by Σ. But we will use the definition in models which do not know enough of Σ to recognize the entire sequence as an iteration by Σ.

Lemma 4.11 (by Neeman–Zapletal [36])**.** Let N be a countable transitive model of a large enough fragment of ZFC. Let $\mathbb{P}$ be a proper (or reasonable) poset in N. Let $\langle Q, \Sigma, \delta\rangle$ be a germ. Suppose that $Q \in N$ and that the restriction of Σ to trees of length $\le |\mathbb{P}|^+$ in N belongs to N. Let G be $\mathbb{P}$-generic over N. Then there exists a composed Σ-iteration $\langle j_n \colon n < \omega\rangle$ of Q with direct limit $j\colon Q \to Q^*$ so that both $\mathbb{R}^N$ and $\mathbb{R}^{N[G]}$ can be realized as the derived reals of derived models of Q^*. Moreover such $\langle j_n \colon n < \omega\rangle$ can be found inside any forcing extension of $N[G]$ by $\mathrm{Coll}(\omega, \mathbb{R}^N) \times \mathrm{Coll}(\omega, \mathbb{R}^{N[G]})$, the finite restrictions of the sequence belong to N, and the sequence of critical points of the maps $j_{n,\infty}$ is increasing and cofinal in $j(\delta)$.

Proof. This follows by the construction in Section 2 of Neeman–Zapletal [36], building iterates of the model Q of the current lemma using the iteration strategy Σ, and working over the model N of the current lemma rather than V. We only note that the use of unique iterability in Lemma 3 of [36] is replaced here by a use of the condensation assumption on Σ, which implies condensation in N for the restriction of Σ to trees in N. ∎

Proof of Theorem 4.3. Fix the relevant objects. It is enough to prove for every $\beta \in M \cap \mathrm{On}$, every $\vec{\gamma} \in \beta^{<\omega}$, every $\vec{y} \in (M[G] \cap \mathbb{R})^{<\omega}$, and every formula φ, that $L_\beta(\mathbb{R}^{M[G]}) \models \varphi(\vec{\gamma}, \vec{y})$ iff $L_{\pi(\beta)}(\mathbb{R}) \models$

$\varphi(\pi(\vec{\gamma}), \vec{y})$. Since every element of $L(\mathbb{R})$ is definable from ordinals and reals, one can then set $\widehat{\pi}$ to map elements definable from $\vec{\gamma}, \vec{y}$ in $L_\beta(\mathbb{R})^{M[G]}$ to elements definable in the same way from $\pi(\vec{\gamma}), \vec{y}$ in $L_{\pi(\beta)}(\mathbb{R})$, define the horizontal embedding $k: L(\mathbb{R})^M \to L(\mathbb{R})^{M[G]}$ to map elements definable from $\vec{\gamma}, \vec{y}$ in $L_\beta(\mathbb{R})^M$ to elements definable in the same way from $\vec{\gamma}, \vec{y}$ in $L_\beta(\mathbb{R})^{M[G]}$, and verify that the two embeddings, which obviously commute with π, are elementary.

Fix β, $\vec{\gamma}$, $\vec{y}$, and φ. Suppose that $L_\beta(\mathbb{R}^{M[G]}) \models \varphi(\vec{\gamma}, \vec{y})$. We prove that $L_{\pi(\beta)}(\mathbb{R}) \models \varphi(\pi(\vec{\gamma}), \vec{y})$.

The large cardinal assumption we use is the existence of $M_\omega^\sharp$, and its $|\pi(\mathbb{P})|^+ + 1$-iterability. Let $Q = M_\omega^\sharp$, let δ be the supremum of the Woodin cardinals of Q, and let Σ be the unique iteration strategy for Q. Then $\langle Q, \Sigma, \delta\rangle$ is a germ (capturing, for example, the empty set). By the elementarity of π, Q belongs to M, and so does the restriction of Σ to trees of length $\leq |\mathbb{P}|^+$ in M.

Let $j_n: Q_n \to Q_{n+1}$ and $j: Q \to Q^*$ be given by Lemma 4.11 applied with $N = M$. Let Q^{**} be obtained by iterating the sharp of Q^* past β. Q^* does not belong to M, but it belongs to $M[G][h]$ for some generic h for $\mathrm{Coll}(\omega, \mathbb{R}^M) \times \mathrm{Coll}(\omega, \mathbb{R}^M[G])$ over $M[G]$. Q^{**} then also belongs to $M[G][h]$. Let $\dot{j}_{n,m}$, $\dot{Q}_n$, $\dot{Q}^*$, $\dot{Q}^{**}$, and $\dot{j}_{n,\infty}$ in M name the corresponding objects in $M[G][h]$.

Fix H witnessing that $\mathbb{R}^{M[G]}$ can be realized as the derived reals of Q^{**}. Let $\eta < j_{0,\infty}(\delta)$ be large enough that $\vec{y}$ belongs to $Q^{**}[H \restriction \eta]$. Fix a name $\dot{u} \in Q^{**}$ so that $\dot{u}[H \restriction \eta] = \vec{y}$. Using the symmetry of the collapse we can assume that it is outright forced in $\mathrm{Coll}(\omega, j_{0,\infty}(\delta))$ over Q^{**} that $L_\beta(\dot{\mathbb{R}}) \models \varphi(\vec{\gamma}, \dot{u})$, where $\dot{\mathbb{R}}$ names the derived reals.

Let m be large enough that the critical point of $j_{m,\infty}$ is above $(2^\eta)^{Q^{**}}$. The statements above all hold in $M[G][h]$, and Q_m, $j_{0,m}$ belong to M. Let $q \in G * h$ force these statements, and force the values of $\dot{Q}_m$ and $j_{0,m}$.

We now apply π to shift the statements from M to V_θ. Let $G' * h'$ be generic for $\pi(\mathbb{P}) * (\mathrm{Coll}(\omega, \mathbb{R}) \times \mathrm{Coll}(\omega, \mathbb{R}^{V[\dot{G}]}))$ over V, below $\pi(q)$. Let $Q' = \pi(\dot{Q}^{**})[G'][h']$, $Q'_n = \pi(\dot{Q}_n)[G'][h']$, $j'_{0,n} = \pi(\dot{j}_{0,n})[G'][h']$, $j'_{n,\infty} = \pi(\dot{j}_{n,\infty})[G'][h']$. Note that $Q'_m = Q_m$, $j'_{0,m} = j_{0,m}$, and the critical point of $j'_{m,\infty}$ is above $(2^\eta)^{Q'}$, because $G' * h'$ is below q. In particular $H \restriction \eta$ is generic for $\mathrm{Coll}(\omega, \eta)$ over Q'.

Using the elementarity of π, and since $\mathbb{R}^M$ can be realized as the derived reals of Q^{**}, $\mathbb{R} = \mathbb{R}^{V_\theta}$ can be realized as the derived reals of Q'. Let E be generic witnessing this. Since $H \restriction \eta$ can be coded by an element of $\mathbb{R}^V$, it belongs to the derived model of Q' using E. By standard arguments E can then be rearranged to get $E \restriction \eta = H \restriction \eta$ without changing the derived reals. Then $\dot{u}[E] = \vec{y}$. By the elementarity of π, it is forced in $\mathrm{Coll}(\omega, j'_{0,\infty}(\delta))$ over Q' that $L_{\pi(\beta)}(\dot{\mathbb{R}}) \models \varphi(\pi(\vec{y}), \dot{u})$. Interpreting this using E it follows that $L_{\pi(\beta)}(\mathbb{R}) \models \varphi(\pi(\vec{y}), \vec{y})$. ∎

To prove Theorem 4.8 we need the following fact from inner model theory.

Fact. (AD^+) Every true Σ^2_1 statement has a witness F that is captured by a germ.

Proof sketch. Let φ have only bounded quantifiers, and suppose that $(\exists F \subseteq \mathbb{R})\varphi(F, \mathbb{R}, \mathbb{N})$ is true. By Woodin's basis theorem for Σ^2_1, see Steel [44, Theorem 9.10], there is then a witness F which belongs to $\boldsymbol{\Delta}_0 = \boldsymbol{\Gamma}_0 \cap \check{\boldsymbol{\Gamma}}_0$, for a pointclass $\boldsymbol{\Gamma}_0$ which is good in the sense of Steel [45, Definition 3.1]. Among other things this means that there is a universal set U for $\boldsymbol{\Gamma}_0$, and a $\boldsymbol{\Gamma}_0$ scale on this set. Since U is universal, there is a real z so that $F = U_z$.

These facts themselves can be witnessed using $\boldsymbol{\Gamma}_0$ and U which belong to $\boldsymbol{\Delta}_1 = \boldsymbol{\Gamma}_1 \cap \check{\boldsymbol{\Gamma}}_1$ for a good $\boldsymbol{\Gamma}_1$, so that $\boldsymbol{\Gamma}_0 \subseteq \boldsymbol{\Delta}_1$, and similarly we can further arrange to have a good $\boldsymbol{\Gamma}_2$ so that $\boldsymbol{\Gamma}_1 \subseteq \boldsymbol{\Delta}_2 = \boldsymbol{\Gamma}_2 \cap \check{\boldsymbol{\Gamma}}_2$. By [45, Lemma 3.13] there is then a coarse $\boldsymbol{\Gamma}_0$-Woodin mouse P, with $z \in P$, and an iteration strategy Σ_P for P so that all iterates P^* of P by Σ_P are themselves $\boldsymbol{\Gamma}_0$-Woodin structures. This means that the ordinal height of P^* is a Woodin cardinal in $L(P^* \cup \{T, P^*\})$, where T is the tree of a $\boldsymbol{\Gamma}_0$ scale on U, see [45, Definition 3.11, Theorem 3.4]. Moreover the proof is such that P is not $\boldsymbol{\Gamma}_1$-Woodin, and Σ_P is the unique strategy that correctly moves witnesses for this. This in particular gives branch condensation for Σ_P, a property similar to our condensation condition above but holding for more general hulls.

Now by an argument similar to the proof of Sargsyan-Steel [38, Lemma 2.5], there exists a (P, Σ_P) mouse Q satisfying an arbitrarily large finite fragment of ZFC, with ω Woodin cardinals. In particular this means that a real coding P belongs to Q, Q is closed under Σ_P and has a predicate for Σ_P, and Q is iterable by a strategy Σ which moves Σ_P correctly. This last property determines Σ uniquely, and implies that Σ has our condensation property. The argument for obtaining Q and Σ is similar to the construction of fine structure uniquely iterable models with infinitely many Woodin cardinals under AD, but replacing closure under ordinary constructibility with closure under constructibility *and* Σ_P.

The branch condensation properties of Σ_P are used to see this hierarchy has condensation properties similar to the hierarchy of constructibility.

Let δ be the supremum of the Woodin cardinals of Q. It remains to prove that $\langle Q, \Sigma, \delta\rangle$ captures F. Fix an iteration map $j\colon Q \to Q^*$ by Σ. Recall that $P \in Q^*$ is $\mathbf{\Gamma}_0$-Woodin, U is a universal set for $\mathbf{\Gamma}_0$, z is a real so that $F = U_z$, and T is a $\mathbf{\Gamma}_0$ scale on U. The iteration strategy Σ_P is moved correctly by j, and in particular Q^* is closed under Σ_P and has a predicate for Σ_P. Working in Q^*, let $\sigma\colon P \to P^*$ be an iteration of P by Σ_P to reach a model $L(P^* \cup \{P^*, T\})$ so that every real generic for $\mathrm{Coll}(\omega, \eta)$ over Q^* for $\eta < j(\delta)$ is generic for Woodin's extender algebra at $P^* \cap \mathrm{On}$ over $L(P^* \cup \{P^*, T\})$. This is a variant of Woodin's second genericity iteration used in [36, §2], but working with many names for reals simultaneously. The named reals can be made generic for Woodin's extender algebra because (for any iterate P^*), the ordinal height of P^* is Woodin in $L(P^* \cup \{P^*, T\})$.

Let A be a maximal antichain of conditions in Woodin's extender algebra in $L(P^* \cup \{P^*, T\})$ which force the generic real to belong to $p[T]_z$. Note $A \in L(P^* \cup \{P^*, T\})$. Using the $P^* \cap \mathrm{On}$-chain condition for the extender algebra, A is bounded in $P^* \cap \mathrm{On}$. In particular $A \in P^*$, and hence $A \in Q^*$. Recall that conditions in A are identities in an infinitary language, and if x is generic for the extender algebra then the generic induced by x, call it W_x, consists of the identities satisfied by x. Now for any real x in a generic extension of Q^* by $\mathrm{Coll}(\omega, \eta)$ for $\eta < j(\delta)$ we have $x \in F$ iff $x \in p[T]_z$ iff $(\exists a \in A) a \in W_x$ iff x satisfies an identity $a \in A$. Letting $\dot{F}^* \in Q^*$ be a $\mathrm{Coll}(\omega, j(\delta))$-name for the set of reals in small extensions of Q^* which satisfy an identity in A, it follows that $\langle Q, \Sigma, \delta\rangle$ captures F. ∎

Proof of Theorem 4.8. Given $A \subseteq \mathbb{R}$ and a prewellorder $\leq$ on $\mathbb{R}$ of ordertype α, let $\psi_{A,\leq}\colon \mathbb{R} \to L_\alpha(\mathbb{R}, A)$ be a standard surjection defined uniformly in A and $\leq$. This can be done since every element of $L_\alpha(\mathbb{R}, A)$ is definable in the structure from reals and ordinals below α. Let $T_{A,\leq}$ be the truth predicate for $L_\alpha(\mathbb{R}, A)$ in the codes, meaning that $\langle n, x_0, \ldots, x_i\rangle \in T_{A,\leq}$ if and only if the nth formula in a standard enumeration holds of $\psi_{A,\leq}(x_0), \ldots, \psi_{A,\leq}(x_i)$ in $L_\alpha(\mathbb{R}, A)$. Given a function $e\colon L_\alpha(\mathbb{R}, A)^{<\omega} \to L_\alpha(\mathbb{R}, A)$ let $e_{A,\leq} = \{\langle x_0, \ldots, x_i, y\rangle \colon \psi_{A,\leq}(y) = e(\psi_{A,\leq}(x_0), \ldots, \psi_{A,\leq}(x_i))\}$.

If Theorem 4.8 fails then there exists $\alpha < \Theta$, $A \subseteq \mathbb{R}$, and a function $e\colon L_\alpha(\mathbb{R}, A)^{<\omega} \to L_\alpha(\mathbb{R}, A)$, so that the conclusion of the theorem fails for every Z which is closed under e, and every such Z

is elementary in $L_\alpha(\mathbb{R}, A)$. Using the coding in the previous paragraph this can be expressed as a Σ^2_1 statement, quantifying over $\leq$ and $e_{A,\leq}$ instead of α and e. By Woodin's Σ^2_1 basis theorem, if the theorem fails then there are witnesses A, $\leq$, $e_{A,\leq}$ as above in a good pointclass, and in particular in a pointclass satisfying uniformization. So there exists $\widehat{e}_{A,\leq}\colon \mathbb{R}^{<\omega} \to \mathbb{R}$ uniformizing $e_{A,\leq}$ in its last coordinate. For $x_0, \ldots, x_i \in \mathbb{R}$ then $\widehat{e}_{A,\leq}(x_0, \ldots, x_i)$ is an element of $\mathbb{R}$, and we view it as a function from ω to ω. Let $\widehat{e}^r_{A,\leq}$ be the relation $\{\langle x_0, \ldots, x_i, n, k\rangle\colon \widehat{e}_{A,\leq}(x_0, \ldots, x_i)(n) = k\}$.

Suppose for contradiction that Theorem 4.8 fails. By the previous fact from inner model theory, we can find a germ $\langle Q, \Sigma, \delta\rangle$ capturing a set F that codes A, $\leq$, and $\widehat{e}^r_{A,\leq}$ witnessing the failure in the sense of the previous paragraph, and codes the truth predicate $T_{A,\leq}$.

Let $N = L_\theta(u)[\Sigma]$ where u is a real coding Q and θ is least so that $N \models \mathsf{ZFC}$. By $L_\theta(u)[\Sigma]$ we mean the model obtained by constructing over u relative to the predicate $\{\langle \mathcal{T}, \xi\rangle\colon \mathcal{T}$ is by Σ and ξ belongs to cofinal branch through $\mathcal{T}$ given by $\Sigma\}$. Note $\theta < \omega_1$ since ω_1 is measurable under AD. N is a model of choice, closed under Σ, and the restrictions of Σ to strict initial segments of N are in N.

Let α be the ordertype of $\leq$, and let $e\colon L_\alpha(\mathbb{R}, A)^{<\omega} \to L_\alpha(\mathbb{R}, A)$ be the function coded by $\widehat{e}^r_{A,\leq}$. Let $Z = \psi''_{A,\leq}\mathbb{R} \cap N$. We will prove that Z is closed under e and that the conclusion of Theorem 4.8 holds for Z, contradicting the fact that A, $\leq$, and $\widehat{e}^r_{A,\leq}$ form a counterexample to the theorem.

We will make several uses of Lemma 4.11. First, we use it to see that $F \cap N$ belongs to N. By the lemma, with the trivial poset $\mathbb{P}$, inside every extension $N[h]$ of N by $\mathrm{Coll}(\omega, \mathbb{R})^N$ there is an iteration map $j\colon Q \to Q^*$ by Σ so that $\mathbb{R}^N$ can be realized as the derived reals of Q^*. Since $\langle Q, \Sigma, \delta\rangle$ captures F it follows using the capturing condition that $F \cap \mathbb{R}^N$ belongs to $N[h]$. This is true for all h, so $F \cap \mathbb{R}^N \in N$.

It follows that $\leq \cap N$, $A \cap N$, $\widehat{e}^r_{A,\leq} \cap N$, and $T_{A,\leq} \cap N$ all belong to N. From the membership of $\widehat{e}^r_{A,\leq} \cap N$ in N it follows that $\mathbb{R}^N$ is closed under $\widehat{e}_{A,\leq}$. This in turn implies that Z is closed under e. Let M be the transitive collapse of Z. M can be determined from $T_{A,\leq} \cap N$, and therefore belongs to N. In particular $\mathbb{R} \cap Z = \mathbb{R} \cap M \subseteq \mathbb{R}^N$. Without loss of generality we can assume the coding $\psi_{A,\leq}$ is such that for every real x, $\psi_{A,\leq}(0^\frown x)$ is a code for x. Then since $\mathbb{R}^N$ is closed under prepending 0, it follows that $\mathbb{R}^N \subseteq \mathbb{R} \cap M$. So $\mathbb{R}^N = \mathbb{R} \cap M$.

Let $\pi\colon M \to L_\alpha(\mathbb{R}, A)$ be the anticollapse embedding, $\overline{A} = \pi^{-1}(A)$, and $\overline{\alpha} = M \cap \mathrm{On}$. π is elementary since $Z = \mathrm{range}(\pi)$ is closed under e. Let $\mathbb{P}$ be absolutely proper in M. Then $\pi(\mathbb{P})$ is absolutely proper in $L_\alpha(\mathbb{R}, A)$, and moreover a club witnessing this belongs to Z, hence in particular $M \cap \mathbb{R} = N \cap \mathbb{R}$ belongs to this club. By definition it follows that $\mathbb{P} = \pi(\mathbb{P}) \cap (\mathbb{R} \cap M)$ is proper in N. Similarly if $\mathbb{P}$ is absolutely reasonable in M then it is reasonable in N.

Let G be $\mathbb{P}$-generic over N. By Lemma 4.11, inside every extension of $N[G]$ by $\mathrm{Coll}(\omega, \mathbb{R}^N) \times \mathrm{Coll}(\omega, \mathbb{R}^{N[G]})$, there is an iteration map $j \colon Q \to Q^*$ by Σ so that both $\mathbb{R}^N$ and $\mathbb{R}^{N[G]}$ can be realized as the derived reals of Q^*. Arguing as above but over $N[G]$ instead of N it follows that $F \cap \mathbb{R}^{N[G]}$ belongs to $N[G]$. This in turn implies, again arguing as above, that $\mathbb{R}^{N[G]}$ is closed under $\widehat{e}_{A,\leq}$, that $Z^* := \psi''_{A,\leq}\mathbb{R} \cap N[G]$ is elementary in $L_\alpha(\mathbb{R}, A)$, and that $Z^* \cap \mathbb{R} = \mathbb{R}^{N[G]}$. Let M^* be the transitive collapse of Z^*, and let $\widehat{\pi}$ be the anticollapse embedding. By elementarity M^* has the form $L_\nu(\mathbb{R}^{N[G]}, A \cap \mathbb{R}^{N[G]})$.

Note that $Z^* \supseteq Z$ so we can define an elementary embedding $k = \widehat{\pi}^{-1} \circ \pi \colon M \to L_\nu(\mathbb{R}^{N[G]}, A \cap \mathbb{R}^{N[G]})$. We claim that $\nu = \overline{\alpha}$ and k is the identity on ordinals. For both statements it is enough to show that $Z^* \cap \mathrm{On} \subseteq Z \cap \mathrm{On}$. Without loss of generality, through an appropriate choice of $\psi_{A,\leq}$, we can assume that $Z \cap \mathrm{On} = \varphi''\mathbb{R}^N$ and $Z^* \cap \mathrm{On} = \varphi''\mathbb{R}^{N[G]}$ where φ is the norm associated to $\leq$. It is then enough to show that every real in $N[G]$ has a $\leq$-equivalent real in N.

Fix H and H^* witnessing that $\mathbb{R}^N$ and $\mathbb{R}^{N[G]}$ respectively can be realized as the derived reals of Q^*. Using the capturing condition, and since F codes $\leq$, we have a name $\dot{r} \in Q^*$ so that $\dot{r}[E]$ is equal to the intersection of $\leq$ with the derived reals of Q^* using E, for every generic E for $\mathrm{Coll}(\omega, \delta)$ over Q^*. Let $\dot{\varphi}$ name the norm associated to $\dot{r}$. Fix $x \in N[G]$ and let $\mu = \dot{\varphi}[H^*](x)$. Fix $\eta < j(\delta)$ so that $x \in Q^*[H^* \restriction \eta]$, and a $\mathrm{Coll}(\omega, \eta)$-name $\dot{u} \in Q^*$ so that $\dot{u}[H^* \restriction \eta] = x$. Using the symmetry of the collapse we can assume it is outright forced in $\mathrm{Coll}(\omega, j(\delta))$ that $\dot{\varphi}(\dot{u}) = \mu$. Let $y = \dot{u}[H] \in N$. Since $H \restriction \eta$ can be coded by a real in $N \subseteq N[G]$, it belongs to the derived model of Q^* by H^*. Using standard arguments one can then rearrange H^* to a generic E^* which produces the same derived reals as H^*, but with $E^* \restriction \eta = H \restriction \eta$. The former fact implies that $\dot{r}[E^*] = \dot{r}[H^*]$ and hence $\dot{\varphi}[E^*](x) = \dot{\varphi}[H^*](x) = \mu$, while the latter implies that $\dot{\varphi}[E^*](y) = \mu$. Putting the two together it follows that x and y belong to the same $\dot{r}[E^*]$-equivalence class, and hence to the same $\leq$-equivalence class.

We now have the elementarity and agreement requirements of Theorem 4.8. It remains to find $\dot{A}^* \in N$ so that for every $\mathbb{P}$-generic G, $\dot{A}^*[G] = A \cap \mathbb{R}^{N[G]}$.

Let δ_0 be the first Woodin cardinal of Q. Working in N find an iteration $\sigma\colon Q \to Q'$ of Q so that every real in any generic extension of N by $\mathbb{P}$ is generic over Q' for Woodin's extender algebra at $\sigma(\delta_0)$. This is a variant of Woodin's second genericity iteration used in [36, §2], but working with all names for reals simultaneously. In N, the iteration terminates at some length $< |\mathbb{P} \times U|^+$ where U is the set of canonical $\mathbb{P}$-names for reals.

Using the capturing condition, and since F codes A, there is a name $\dot{A}' \in Q'$ so that for every generic E for $\mathrm{Coll}(\omega, \sigma(\delta))$, and any x added by a strict initial segment of E, $x \in A$ iff $x \in \dot{A}'[E]$. Woodin's extender algebra at $\sigma(\delta_0)$ is subsumed by $\mathrm{Coll}(\omega, \sigma(\delta))$. From this using the symmetry of the collapse it follows that there is an extender algebra name $\dot{A}'' \in Q'$ so that for every generic g for the extender algebra, $\dot{A}''[g] = A \cap Q'[g]$. From every real generic for the extender algebra one can definably construct a generic filter g_x for the algebra so that $x \in Q'[g_x]$. Now, working in N, let $\dot{A}^*$ be a $\mathbb{P}$-name for the set of reals x (in the extension by $\mathbb{P}$) so that $x \in \dot{A}''[g_x]$. Then for every x in an extension $N[G]$ of N by $\mathbb{P}$, $x \in \dot{A}^*[G]$ iff $x \in \dot{A}''[g_x]$ iff $x \in A$, with the last equivalence using the fact that by construction x is generic for the extender algebra over Q'. ∎

Remark. One can prove versions of Theorems 4.3 and 4.8 dropping the properness or reasonableness assumptions, with the conclusions weakened to allow $\widehat{\pi}$ and π to differ on ordinals, and (in Theorem 4.8) to allow $\widehat{\pi}$ to have domain $L_\beta(\mathbb{R}^{N[G]}, \dot{A}^*[G])$ for some β possibly different from $\overline{\alpha}$. Equivalently, this allows the horizontal embeddings in the theorems' diagrams to move ordinals.

To see this, for example in the case of Theorem 4.8, instead of constructing a single iterate Q^* so that both $\mathbb{R}^N$ and $\mathbb{R}^{N[G]}$ are realized as derived reals of Q^*, construct an iteration $j\colon Q \to Q_1^*$ so that $\mathbb{R}^N$ can be realized as the derived reals of Q_1^*, and a *further* iteration $i\colon Q_1^* \to Q_2^*$ so that $\mathbb{R}^{N[G]}$ can be realized as the derived reals of Q_2^*. This can be done with a direct use of Woodin's genericity iterations, working over N (after collapsing its reals) to construct j and then over $N[G]$ (after collapsing its reals) to construct i, noting that the initial segments of Q_1^* to any of its Woodin cardinals belong to N and therefore to $N[G]$. The proof of Theorem 4.8 can then be continued with the natural modifications, except that, since $\mathbb{R}^N$ and $\mathbb{R}^{N[G]}$ are realized over different models, one can no longer prove that $Z^* \cap \mathrm{On} \subseteq Z \cap \mathrm{On}$, and consequently k need not be the identity on ordinals. One can prove the modified Theorem 4.3 similarly.

CHAPTER 5

Questions

We conclude this part of the thesis with a list of open questions.

Much remains open about how the existence of infinite mad families relates to the existence of other pathological sets of reals. Mathias's original conjecture is the most well-known example.

Question 1 (Mathias [30])**.** (In ZF + DC) If every set has the Ramsey property, does it follow that there are no infinite mad families?

Question 1 seems to be open even under additional hypotheses like AD.

There has been some further study of semiselective coideals since Farah's original paper.

Theorem (Di Prisco–Mijares–Uzcátegui [11])**.** Suppose that G is generic over V for the Levy collapse of a weakly compact cardinal. Then, in $V[G]$, every set $X \subseteq [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every semiselective coideal H.

Question 2 (Di Prisco–Mijares–Uzcátegui)**.** Is the weakly compact in the theorem necessary?

It is natural to wonder whether the inaccessible in Theorem 2.8 is necessary.

Question 3. Is it consistent relative to ZFC alone that every set of reals in $L(\mathbb{R})$ is H-Ramsey for every $H \in L(\mathbb{R})$?

A positive answer to Question 3 would also answer the longstanding open question of whether "every set has the Ramsey property" is consistent relative to ZFC alone. A negative answer might indicate something special about $[\omega]^\omega$, even among definable happy families.

We conclude with Törnquist's question that motivated our work in Chapter 4, which remains open.

Question 4 (Törnquist [48])**.** Does AD imply that there are no infinite mad families?

Part II

Trees and remarkable cardinals

CHAPTER 6

Introduction and motivation

The work in the second part of this thesis was motivated by a desire to better understand the Embedding Theorem of Neeman and Zapletal [36]. Its consistency strength was computed exactly by Schindler [41]; in this part of the thesis we show that the restriction of the Embedding Theorem to σ-closed $*$ ccc posets has the same consistency strength. This requires a new proof of the lower bound in Schindler's theorem, using a poset in the smaller class.

There is a rich tradition in set theory of investigating the extent to which forcing can change the truth value of assertions about ordinals and reals, especially in the presence of large cardinals. It is a now-classical theorem of Woodin that, in the presence of suitable large cardinals, the theory of $L(\mathbb{R})$ cannot be changed by set forcing, and, conversely, if the theory of $L(\mathbb{R})$ cannot be changed by set forcing, then AD holds in $L(\mathbb{R})$. (See Steel [43].) Adding parameters to the theory changes the situation quite drastically, though: for example, any forcing collapsing ω_1 changes the theory of $L(\mathbb{R})$ with ordinal parameters. Neeman and Zapletal [36] showed that, assuming roughly a proper class of Woodin cardinals, the theory of $L(\mathbb{R})$ with ordinal and real parameters cannot be changed by any proper forcing. We call the conclusion of their theorem $L(\mathbb{R})$-*absoluteness for proper posets*. Likewise, if $\mathcal{C}$ is a class of posets, then $L(\mathbb{R})$-*absoluteness for all posets in* $\mathcal{C}$ is the assertion that the theory of $L(\mathbb{R})$ with ordinal and real parameters cannot be changed by any forcing in $\mathcal{C}$.

In [41] Schindler identified the consistency strength of $L(\mathbb{R})$-absoluteness for proper posets to be exactly what he called a *remarkable* cardinal (see Definition 8.2). The definition is a natural weakening of Magidor's characterization of supercompactness [29], in which the embedding is only required to exist in some generic extension of V. This large-cardinal assumption sits far below the level of Woodin cardinals and $\mathsf{AD}^{L(\mathbb{R})}$: while a remarkable cardinal must be weakly compact, if $0^\sharp$ exists then every Silver indiscernible is remarkable in L.

Theorem 6.1 (Schindler)**.**

(a) Assume $V = L$. If κ is a remarkable cardinal, then $L(\mathbb{R})$-absoluteness for proper posets holds in the extension by the Levy collapse to make $\kappa = \aleph_1$.

(b) Conversely, if $L(\mathbb{R})$-absoluteness for proper posets holds, then $\aleph_1$ is remarkable in L.

Schindler's lower-bound argument uses the reshaping and almost-disjoint coding methods of Jensen (see Jensen & Solovay [24]). The reshaping poset to which he applies $L(\mathbb{R})$-absoluteness is not σ-closed $*$ ccc or indeed proper in any strong sense. (For more on the properness, distributivity, and stationary-preservation of the reshaping poset, see [40].) Lower bounds for the Proper Forcing Axiom are typically obtained using anti-square posets, which take the form σ-closed $*$ ccc. (For a particularly relevant example, see [4].) One might expect by analogy with the forcing-axiom case that *proper* in Theorem 6.1(b) can be replaced by *σ-closed $*$ ccc*. Our main theorem confirms that expectation.

Theorem 6.2. If $L(\mathbb{R})$-absoluteness holds for σ-closed $*$ ccc posets, then $\aleph_1$ is remarkable in L.

Whereas Schindler's proof uses almost-disjoint coding, our proof expands on the coding methods introduced by Kunen, who showed that $L(\mathbb{R})$-absoluteness for ccc posets is equiconsistent with the existence of a weakly compact cardinal. (A version of this theorem appears in a paper of Harrington–Shelah as [20, Theorem C]. See [19] for the upper-bound portion of the argument.) Assuming that $\aleph_1^V$ is not weakly compact in L, Kunen codes an uncountable sequence of reals into a single real by specializing an Aronszajn tree. Since $\aleph_1$ can be weakly compact in L without being remarkable in L, we must expand these coding techniques to trees that may have large levels. The ccc part of our poset will be a modified specializing poset, following Kunen, but first we must add the tree to be specialized. The trees we use bear some resemblance to those of [31], and in Chapter 10 we adapt the methods of that article to obtain a finer lower bound on the consistency strength of $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets.

CHAPTER 7

The coding argument

In this chapter we show how to use our generic-absoluteness assumption to establish a principle about trees on ω_1. In the next chapter, we will use that principle in a σ-closed extension to deduce that $\aleph_1^V$ is remarkable in L.

The first difficulty in adapting the coding methods of [20] is that our trees will not belong to L, so they too will have to be coded. A more substantive difficulty is that, although our trees will have size $\aleph_1$, they will not typically have countable levels. With a view toward defining a suitable countable analogue of the α^{th} level of a tree, we present our trees as increasing sequences of countable subtrees.

Tree presentations

Definition 7.1. A *tree presentation* is a sequence $\vec{T} = \langle T_\alpha : \alpha \leq \omega_1 \rangle$ of countable trees satisfying the following conditions.

(i) $\vec{T}$ is *concrete*: Each T_α is a tree on a subset of ω_1. Moreover, the height of T_α is a (countable) limit ordinal.

(ii) $\vec{T}$ is *increasing*: if $\alpha < \beta$ then $T_\alpha \subseteq T_\beta$. Moreover, if $\alpha < \beta$, then T_β is an end-extension of T_α: that is, if $s \leq_{T_\beta} t$ and $t \in T_\alpha$, then $s \in T_\alpha$ also.

(iii) $\vec{T}$ is *continuous*: if $\alpha \leq \omega_1$ is a limit ordinal, then $T_\alpha = \bigcup_{\beta < \alpha} T_\beta$.

Remarks.

(1) Any two presentations of the same tree agree on a club. That is, if $\langle T_\alpha : \alpha \leq \omega_1 \rangle$ and $\langle U_\alpha : \alpha \leq \omega_1 \rangle$ are two tree presentations with $T_{\omega_1} = U_{\omega_1}$, then $T_\alpha = U_\alpha$ for a club of $\alpha < \omega_1$.

(2) If T_{ω_1} is an Aronszajn tree, then one naturally obtains a tree presentation of T_{ω_1} by taking T_α to be the set of nodes on level $< \alpha$.

We will typically use $\leq$ or $\leq_{\vec{T}}$ to refer to the order on any tree in a tree presentation.

Definition 7.2. Let $\vec{T} = \langle T_\alpha : \alpha \leq \omega_1 \rangle$ be a tree presentation. The α^{th} *boundary of* $\vec{T}$ is the set of suprema in $T_{\omega_1} \setminus T_\alpha$ of branches through T_α. That is, a node $t \in T_{\omega_1}$ lies on the α^{th} boundary of $\vec{T}$ iff $t \notin T_\alpha$ but

$$\{s \in T_{\omega_1} : s < t\} \subseteq T_\alpha.$$

Definition 7.3. Let X be a subset of ω_1 and $\vec{T}$ be a tree presentation. We say that X is *codable along* $\vec{T}$ if T_{ω_1} has no uncountable branches, but the set

$$\{\alpha < \omega_1 : T_\alpha \text{ has a cofinal branch in } L[X \cap \alpha]\}$$

is club in ω_1.

The *Tree Reflection Principle at* $\aleph_1$, abbreviated $\mathrm{TRP}(\aleph_1)$, is the assertion

$$(\forall X \subseteq \omega_1)(\forall \text{tree presentations } \vec{T})\ X \text{ is not codable along } \vec{T}.$$

Here we are primarily interested in presentations of trees that have uncountable levels, but Definition 7.3 is relevant even for $\aleph_1$-Aronszajn trees. Even if T_α has many cofinal branches (as will be the case if T_{ω_1} has countable levels, for example), it may not have any cofinal branches in $L[X \cap \alpha]$. In fact, $\mathrm{TRP}(\aleph_1)$ is consistent (see Theorem 7.4), so it does not contradict the existence of $\aleph_1$-Aronszajn trees.

Our main theorem, Theorem 6.2, factors conveniently through the Tree Reflection Principle.

Theorem 7.4. $L(\mathbb{R})$-absoluteness for ccc posets implies $\mathrm{TRP}(\aleph_1)$.

Since σ-closed posets don't add reals, $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets holds if and only if $L(\mathbb{R})$-absoluteness for ccc posets holds in every extension by a σ-closed poset. With this observation in hand, we obtain the following corollary.

Corollary 7.5. $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets implies that $\mathrm{TRP}(\aleph_1)$ holds in every σ-closed forcing extension.

Combining Schindler's Theorem 6.1 with Corollary 7.5 gives the following consistency result.

Theorem 7.6. Assume $V = L$. If κ is a remarkable cardinal, then in $V^{\mathrm{Coll}(\omega,<\kappa)}$ the following holds: $\mathrm{TRP}(\aleph_1)$ holds in every extension by a σ-closed forcing.

Theorem 7.6 can be proved directly, without going through Theorem 7.4.

in Chapter 8 the following theorem will be proved as Theorem 8.1, completing the proof of Theorem 6.2.

Theorem. If $\mathrm{TRP}(\aleph_1)$ holds in every σ-closed forcing extension, then $\kappa = \aleph_1^V$ is remarkable in L.

The rest of this chapter is devoted to a proof of Theorem 7.4.

Coding along trees

The following lemma gets to the heart of our coding argument.

Lemma 7.7. Suppose that X is codable along $\vec{T}$. Then, in a ccc generic extension, there is a real g such that $L[g]$ has uncountably many reals.

We may assume that $\aleph_1^{L[X\cap\alpha]}$ is countable for every $\alpha < \omega_1$; otherwise we could take the trivial poset and let g be any real coding $X \cap \alpha$.

Let $C \subseteq \omega_1$ be a club witnessing that X is codable along $\vec{T}$. In a series of claims we modify $\vec{T}$ until it is suitable for coding.

Claim 7.8. There is a tree presentation $\vec{T}' = \langle T'_\alpha : \alpha \leq \omega_1 \rangle$ such that T'_{ω_1} has no cofinal branches and for every $\alpha \in C$, the subtree T'_α has infinitely many cofinal branches in $L[X \cap \alpha]$. Moreover, for any finite subset F of T_{ω_1}, there are infinitely many cofinal branches of T'_α in $L[X \cap \alpha]$ each disjoint from F.

Proof of Claim 7.8. Let T'_α be the image under a definable bijection $\omega_1 \times \omega \to \omega_1$ of $T_\alpha \times \omega$ ordered as the disjoint union of infinitely many copies of T_α. ∎

By replacing $\vec{T}$ with $\vec{T}'$ if necessary, we may assume that $\vec{T}$ satisfies the conclusion of Claim 7.8.

Claim 7.9. There is a tree presentation $\vec{T}' = \langle T'_\alpha : \alpha \leq \omega_1 \rangle$ satisfying the conclusion of Claim 7.8 such that T'_{ω_1} has no cofinal branches and if $b \in L[X \cap \alpha]$ is a cofinal branch through T_α, then b has a supremum in $T_{\alpha+1}$.

Proof of Claim 7.9. Notice first that T_α has countably many cofinal branches in $L[X \cap \alpha]$, by our assumption that $\aleph_1^{L[X\cap\alpha]}$ is countable. Recursively define

- $T'_0 = T_0$,
- $T'_\gamma = \bigcup_{\beta<\gamma} T'_\gamma$ for γ limit, and
- $T'_{\alpha+1} = T'_\alpha \cup S_\alpha$,

where the order on T'_α is extended to make S_α exactly the set of suprema of cofinal branches through T_α that belong to $L[X \cap \alpha]$. Notice that T_α and T'_α have the same branches, so in fact $T'_{\alpha+1}$ has suprema of every branch in $L[X \cap \alpha]$ through T'_α.

To satisfy concreteness, the S_α should be chosen to be sets of countable ordinals, and in the end a definable bijection $\omega_1 \times 2 \to \omega_1$ should be used to ensure that the T'_α are trees on ω_1. We leave it to the reader to fill in the details. ∎

The point of Claim 7.9 is to guarantee that the specializing function will not diverge along a countable branch of the tree. Again, by replacing $\vec{T}$ with $\vec{T}'$ if necessary, we may assume that $\vec{T}$ satisfies the conclusion of Claim 7.9.

Now we can mimic the coding argument in [20], using the presentation $\langle T_\alpha : \alpha \leq \omega_1 \rangle$ in place of the sequence of initial segments of the tree.

We review the definition of the modified specializing poset defined in [20], tweaked for our purposes.

Definition 7.10. Let $\langle d_\alpha : \alpha < \omega_1 \rangle$ be a sequence of reals, construed here as subsets of ω. Conditions in the poset $\mathbb{P}(d_\alpha : \alpha < \omega_1)$ are finite partial specializing functions $T_{\omega_1} \rightharpoonup \mathbb{Q}$ that code $\vec{d}$ along $\vec{T}$. Precisely, a condition is a finite partial function $p \colon T_{\omega_1} \rightharpoonup \mathbb{Q}$ with the following properties:

(a) $s < t$ in T_{ω_1} implies $p(s) < p(t)$ in $\mathbb{Q}$, and

(b) if t is on the α^{th} boundary of $\vec{T}$ and belongs to the domain of p, then either $p(t) \in d_\alpha$ or $p(t)$ is not an integer.

Baumgartner's original argument for the ccc of the specializing poset (see [23, Lemma 16.18], for example) can be repeated to show:

Claim 7.11. If T_{ω_1} has no uncountable branches, then $\mathbb{P}(d_\alpha : \alpha < \omega_1)$ has the ccc.

Suppose that G is $\mathbb{P}(d_\alpha : \alpha < \omega_1)$-generic and consider the generic specializing function $f = \bigcup G \colon T_{\omega_1} \to \mathbb{Q}$.

Remarks.

(1) As in [20], a crucial observation for us is that f is continuous: if t is a node on a limit level of T_{ω_1} then $f(t) = \sup\{f(s) : s < t\}$.

(2) The other crucial observation is that $f''B_\alpha \cap \mathbb{Z} = d_\alpha$ for $\alpha \in C$, where B_α is the intersection of the α^{th} boundary of $\vec{T}$ with $L[X \cap \alpha]$. The $\subseteq$ inclusion follows directly from condition (b) in the definition of the poset; the other inclusion follows from a genericity argument, using the properties of the presentation obtained in Claims 7.8 and 7.9.

Proof of Lemma 7.7. It will be convenient and harmless to assume that T_0 has infinite height. For $\alpha < \omega_1$ we will write α^* for $\min(C \setminus (\alpha + 1))$.

The ccc poset will be a length-ω finite-support iteration of posets of the form $\mathbb{P}(d_\alpha : \alpha < \omega_1)$. Since each iterate is ccc, the full iteration is ccc.[1]

First, let $\langle d^0_\alpha : \alpha < \omega_1\rangle$ be any sequence of ω_1 distinct reals such that, for all $\alpha \in C$, d^0_α codes α^*, T_{α^*}, and $X \cap \alpha^*$. Suppose inductively that f_n is the generic specializing function added by $\mathbb{P}(d^n_\alpha : \alpha < \omega_1)$. Let d^{n+1}_α be a real coding $f_n \restriction T_{\alpha^*}$.

In the extension by the full iteration, let g be any real coding (the countable sequence of reals) $\langle d^n_\beta : n < \omega, \beta < \min(C)\rangle$.

[1] Note that the tree T_{ω_1} is special after the first forcing, so none of the later posets can add uncountable branches to it.

We will verify that X, C, $\langle d^n_\alpha : \alpha < \omega_1 \rangle$, and $\vec{T}$ all belong to $L[g]$. This suffices, since it implies in particular that $\langle d^0_\alpha : \alpha < \omega_1 \rangle \in L[g]$. We prove by induction on $\alpha \in C$ that d^n_α belongs to $L[g]$. Moreover, the proof gives a uniform definition of d^n_α in $L[g]$ from n and α; this uniformity in turn is used for the limit case of the proof.

The base case of $\alpha = \min(C)$ is immediate from the choice of g.

Suppose first that the induction hypothesis holds for $\alpha \in C$; we will show that it holds for α^*. From d^0_α we can decode α^*, T_{α^*}, and $X \cap \alpha^*$. From d^{n+1}_α we can decode $f_n \restriction T_{\alpha^*}$, which in turn gives us $d^n_{\alpha^*}$ by Remark (2).

The interesting case is when $\alpha \in \lim(C)$. As in Remark (2), let B_α be the intersection of the α^{th} boundary of $\vec{T}$ with $L[X \cap \alpha]$. We claim that B_α belongs to $L[g]$. By the inductive assumption and the *uniformity* of the proof, the sequence $\langle d^n_\beta : n < \omega,\ \beta \in C \cap \alpha \rangle$ belongs to $L[g]$. From this sequence one can decode $X \cap \alpha$, T_α, and $f \restriction T_\alpha$. From T_α and $X \cap \alpha$ one can decode B_α.

Moreover, if $s \in B_\alpha$, then we can (in $L[g]$) compute $f_n(s)$: the continuity of f implies that $f_n(s) = \sup\{f_n(t) : t \leq s,\ t \in T_\alpha\}$. Now $d^n_\alpha = f_n'' B_\alpha \cap \mathbb{Z}$ belongs to $L[g]$, and the induction is complete. ∎

We need one more fact, which is well known.

Theorem 7.12 (Woodin)**.** Suppose that $L(\mathbb{R})$-absoluteness holds for ccc posets. Then $\aleph_1$ is inaccessible to the reals, meaning that $\aleph_1^{L[x]} < \aleph_1$ for every real x.

Theorem 7.12 follows from a lemma of Woodin [49]: If X is an uncountable sequence of reals in V and c is Cohen-generic over V, then in $V[c]$ there is no random real over $L(X, c)$. In fact, full $L(\mathbb{R})$-absoluteness for ccc posets is not needed for Theorem 7.12; the theorem can be proved from the absoluteness of Σ^1_4 sentences to Cohen and Random extensions. (See [1, Theorem 2.1.1.3].) Likewise, Theorem 7.4 requires only the absoluteness of Σ^1_4 sentences to ccc extensions, since the existence of a real x for which $\aleph_1^{L[x]}$ is uncountable can be expressed as a Σ^1_4 sentence.

Proof of Theorem 7.4. The theorem follows immediately from Lemma 7.7, the definition of "codability along $\vec{T}$," and Theorem 7.12. ∎

CHAPTER 8

The reflection argument

In this chapter we complete our proof of Theorem 6.2 by proving the following theorem.

Theorem 8.1. Suppose that $\mathrm{TRP}(\aleph_1)$ holds in every extension by a σ-closed poset. Then $\kappa = \aleph_1^V$ is a remarkable cardinal in L.

Remarkable cardinals

Definition 8.2 (Schindler)**.** Let κ be a cardinal and $\lambda \geq \kappa$ another cardinal. We say that κ is λ-*remarkable* if there is a cardinal $\overline{\lambda} < \kappa$ and in the extension $V^{\mathrm{Coll}(\omega,<\kappa)}$ an elementary embedding $j\colon H(\overline{\lambda}) \to H(\lambda)$ such that $j(\overline{\kappa}) = \kappa$, where $\overline{\kappa}$ is the critical point of j.

If κ is λ-remarkable for every $\lambda > \kappa$, then we just say that κ is *remarkable*.

Remarks.

(1) If we replaced $V^{\mathrm{Coll}(\omega,<\kappa)}$ with V in the definition, we would get supercompactness, by a theorem of Magidor [28]. (See [25, Theorem 22.10].)

(2) The embedding j in the definition is (in $V^{\mathrm{Coll}(\omega,<\kappa)}$) a countable object; its existence is therefore absolute between any two extensions of V where $\kappa = \aleph_1$.

(3) By standard arguments, if κ is λ^+-remarkable, then in $V^{\mathrm{Coll}(\omega,<\kappa)}$ the set of $X \in [H(\lambda)]^\omega$ for which the anticollapse of X witnesses the λ-remarkability of κ is stationary in $[H(\lambda)]^\omega$. (See [39].)

See [41, Lemma 1.6] for a characterization of remarkability that does not refer to forcing.

In consistency strength, remarkable cardinals sit strictly between ineffable and ω-Erdős cardinals. (This is proved in [41].) For a more detailed analysis of the consistency strength of remarkable cardinals, see [17, §4], but for our purposes the following will suffice.

Fact (Schindler [41])**.**

(a) Every remarkable cardinal is weakly compact.

(b) If $0^\sharp$ exists, then every Silver indiscernible is remarkable in L.

(c) Remarkability is downward-absolute to L.

Since their discovery by Schindler, remarkable cardinals have been analyzed from many perspectives. For instance, Cheng and Gitman [10] define a version of the Laver Preparation for remarkable cardinals and use it to obtain an indestructibly remarkable cardinal, and their ideas are extended in [4] to obtain a weak version of the Proper Forcing Axiom from a remarkable cardinal. It is also shown there that this weak version of PFA implies that $\aleph_2^V$ is remarkable in L.

Remarkability from branchless trees

Let $\kappa = \aleph_1^V$. Let $\lambda \geq \kappa$ be an ordinal and $f\colon \kappa \to \lambda$ a bijection. (We will later assume that λ is a cardinal of L and force to add f.) The rest of the definitions in this section are made relative to f and λ.

In order to reflect properties of $\langle \kappa, \lambda \rangle$ to a pair $\langle \overline{\kappa}, \overline{\lambda} \rangle$ of countable ordinals, we need to arrange that $\overline{\kappa}$ is $\aleph_1$ in a suitable inner model. For this we use the following tree, a version of the tree of attempts to express κ as an ordinal of countable cofinality, modified to have height ω_1.

Definition 8.3. Let S, a tree of height κ, be defined as follows. Nodes are pairs $\langle \alpha, s \rangle$, where α is an ordinal $< \kappa$ and s is a strictly increasing finite sequence of ordinals, each $< \kappa$. We also require that $\alpha \in [s_{n-1}, s_n)$, where $s = \langle s_0, s_1, \ldots, s_{n-1}, s_n \rangle$. Nodes are ordered by the ordinal order in the first coordinate and by extension of sequences in the second.

If we define S_β as in Definition 8.3 but after replacing "$< \kappa$" with "$< \beta$," it is easy to see that we get a tree presentation $\langle S_\beta : \beta \leq \omega_1 \rangle$ in the sense of Definition 7.1.

Let $\beta \leq \omega_1$. If b were an uncountable branch through S_β, then the first coordinates of nodes on b would have to be unbounded in β. Conversely, any cofinal ω-sequence in β defines a cofinal branch through S_β. That is, we have:

Lemma 8.4. S_β has a cofinal branch iff β has countable cofinality.

Our second tree will capture the fact that λ is a cardinal of L.

Definition 8.5. If δ is an ordinal, then we write $\gamma(\delta)$ for the least ordinal γ (if any exist) such that δ is not a cardinal in $L_{\gamma+1}$.

Remarks. Notice that if $\gamma(\delta)$ is defined, then

(1) $\gamma(\delta) < (\delta^+)^L$, and

(2) every element of $L_{\gamma(\delta)+1}$ is definable in $L_{\gamma(\delta)+1}$ from parameters in δ.

The following definition will apply to the rest of this chapter. We will define *point* differently in Chapter 10.

Definition 8.6. Let $\alpha < \kappa$. Define ε_α to be the ordertype of $f''\alpha$.

A *point* is a countable ordinal α such that $\gamma(\varepsilon_\alpha)$ is defined. Points are ordered as follows. If α and α^* are points, then we say $\alpha <_T \alpha^*$ iff:

(i) $\alpha < \alpha^*$ (as ordinals).

(ii) Let $j\colon \varepsilon_\alpha \to f''\alpha$ and $j^*\colon \varepsilon_{\alpha^*} \to f''\alpha^*$ be the anticollapse embeddings. Let $\pi\colon \varepsilon_\alpha \to \varepsilon_{\alpha^*}$ be the composite $(j^*)^{-1} \circ j$. Then π extends to an elementary embedding $\widehat{\pi}\colon L_{\gamma(\varepsilon_\alpha)+1} \to L_{\gamma(\varepsilon_{\alpha^*})+1}$ with $\widehat{\pi}(\varepsilon_\alpha) = \varepsilon_{\alpha^*}$.

Notice that the extended embedding in (ii) is determined uniquely from π, by Remark (2) above.

The definition of *point* depends on both the ordinal λ and the function f. When we wish to emphasize this dependence, we will call a point a $\langle\lambda, f\rangle$*-point*.

Let $T = T(\lambda, f)$ be the tree of increasing sequences of $\langle\lambda, f\rangle$-points.

Lemma 8.7. If there is an uncountable branch through $T(\lambda, f)$, then λ is not a cardinal of L.

Proof. Let $\langle \alpha_\xi : \xi < \omega_1 \rangle$ be a branch through T of length ω_1 and for convenience put $\varepsilon(\xi) = \varepsilon_{\alpha_\xi}$. For $\xi < \zeta < \omega_1$ let $\pi_{\xi,\zeta} \colon \varepsilon(\xi) \to \varepsilon(\zeta)$ and $\widehat{\pi}_{\xi,\zeta} \colon L_{\gamma(\varepsilon(\xi))+1} \to L_{\gamma(\varepsilon(\zeta))+1}$ be the maps determined by (ii) above. It is easy to see that the $\pi_{\xi,\zeta}$ commute, so the maps $\widehat{\pi}_{\xi,\zeta}$ also commute. Let M_∞ be the direct limit of the system $\langle L_{\gamma(\varepsilon(\xi))+1}, \widehat{\pi}_{\xi,\zeta} : \xi < \zeta < \omega_1 \rangle$, which is wellfounded since it is taken along a sequence of length ω_1. Let $\widehat{\pi}_{\xi,\infty} \colon L_{\gamma(\varepsilon(\xi))+1} \to M_\infty$ be the direct-limit maps. Because the maps $\widehat{\pi}_{\xi,\zeta}$ extend the maps $\pi_{\xi,\zeta}$ determined by the anticollapses of $f''\alpha$, because $\sup_\xi \alpha_\xi = \kappa = \omega_1$, because $\widehat{\pi}_{\xi,\zeta}(\epsilon(\xi)) = \epsilon(\zeta)$, and because $f''\kappa = \lambda$, the model M_∞ is an end-extension of $L_{\lambda+1}$, and $\widehat{\pi}_{\xi,\infty}(\varepsilon(\xi)) = \lambda$ for every ξ. But then, by the elementarity of these maps, λ is not a cardinal in M_∞, so λ is not a cardinal in L. ∎

Lemma 8.8. If λ is not a cardinal of L, then there is in $L[f]$ a branch of length ω_1 through $T(\lambda, f)$.

Proof. Let $\gamma^* \geq \lambda$ be least so that λ is not a cardinal in L_{γ^*+1}. For each $\alpha < \omega_1$, let H_α be the hull in L_{γ^*+1} of $f''\alpha$. There is a club C of α for which $H_\alpha \cap \lambda = f''\alpha$.

For every $\alpha \in C$, let M_α be the transitive collapse of H_α, let $j_\alpha \colon M_\alpha \to H_\alpha$ be the anticollapse embedding, so that $j_\alpha(\varepsilon_\alpha) = \lambda$ and $M_\alpha = L_{\gamma(\varepsilon_\alpha)+1}$ by elementarity. Every $\alpha \in C$ is a point, and, for $\alpha < \alpha'$ each in C, the map $(j_{\alpha'})^{-1} \circ j_\alpha$ witnesses that $\alpha <_T \alpha'$. So C is a branch through T.

It is clear that this construction can be carried out in any inner model with f as an element, in particular in $L[f]$. ∎

The tree $T = T(\lambda, f)$ has a natural tree presentation, namely $\langle T \cap \alpha : \alpha \leq \omega_1 \rangle$.

Proof of Theorem 8.1. Assume the Tree Reflection Principle at $\aleph_1$ and let $\lambda \geq \kappa$ be a cardinal of L. We must show that κ is λ-remarkable in L. That is, we must find $\bar{\kappa} < \bar{\lambda} < \kappa$ and (possibly in a generic extension of V) an embedding $j \colon L_{\bar{\lambda}} \to L_\lambda$ such that $\bar{\kappa}$ is the critical point of j and $j(\bar{\kappa}) = \kappa$. The definition requires the domain of j to be $H(\bar{\lambda})^L$, so we must also arrange that $\bar{\lambda}$ is a cardinal of L; this is the main difficulty.

By first forcing with the (countably closed) poset $\mathrm{Coll}(\aleph_1, \lambda)$ if necessary, we can assume that $|\lambda| = \aleph_1$ in V. (By assumption, $\mathrm{TRP}(\aleph_1)$ holds in this extension.) Fix a bijection $f \colon \kappa \to \lambda$. The embedding we find to witness remarkability will extend the anticollapse of a set $f''\bar{\kappa}$.

The following is routine.

Claim 8.9. There are a set $X \subseteq \kappa$ and a club of $\overline{\kappa} < \kappa$ satisfying the following. Let H be the hull in L_λ of $f''\overline{\kappa}$, and let $\rho \colon H \to L_{\overline{\lambda}}$ be the collapsing map.

(a) $f''\overline{\kappa} \cap \kappa = \overline{\kappa}$.

(b) $H \cap \mathrm{On} = f''\overline{\kappa}$.

(c) The map $\rho \circ f \colon \overline{\kappa} \to \overline{\lambda}$ belongs to $L[X \cap \overline{\kappa}]$.

(d) For every $\alpha < \overline{\kappa}$, $L[X \cap \overline{\kappa}] \vDash |\alpha| \leq \aleph_0$. ∎

Fix $X \subseteq \kappa$ as given by Claim 8.9. We can use Lemma 8.4 to see that S_κ has no uncountable branches in V. Since λ is a cardinal of L, we can use Lemma 8.7 to see that $T(\lambda, f)$ has no uncountable branches in V. We can therefore apply $\mathrm{TRP}(\aleph_1)$ to the natural tree presentation of $S \cup T(\lambda, f)$ to obtain a stationary set of $\overline{\kappa} < \kappa$ such that $S_{\overline{\kappa}}$ and $T(\lambda, f) \cap \overline{\kappa}$ each have no cofinal branches in $L[X \cap \overline{\kappa}]$. Fix such a $\overline{\kappa}$ that also satisfies the conclusions of Claim 8.9. Lemma 8.4 applied in $L[X \cap \overline{\kappa}]$ to $S_{\overline{\kappa}}$ implies that $\overline{\kappa}$ has uncountable cofinality in $L[X \cap \overline{\kappa}]$; combining this fact with item (d) of Claim 8.9, we see that $\overline{\kappa} = \aleph_1^{L[X \cap \overline{\kappa}]}$.

Let H be the hull in L_λ of $f''\overline{\kappa}$, and notice that H is countable. By Condensation, H collapses to an initial segment $L_{\overline{\lambda}}$ of L; let $\rho \colon H \to L_{\overline{\lambda}}$ be the collapsing map. We aim to show that ρ^{-1} witnesses the λ-remarkability of κ. By clauses (a) and (b) of Claim 8.9, ρ^{-1} has critical point $\overline{\kappa}$, and the image of its critical point is κ.

Let $\overline{f} \colon \overline{\kappa} \to \overline{\lambda}$ be the map $\rho \circ f$. Clause (c) of Claim 8.9 ensures that $\overline{f}$ belongs to $L[X \cap \overline{\kappa}]$.

Claim 8.10. The tree $T(\lambda, f) \cap \overline{\kappa}$ is exactly the tree $T(\overline{\lambda}, \overline{f})$ as computed in the model $L[X \cap \overline{\kappa}]$.

Proof of Claim 8.10. The key observation is that ρ restricts to an order isomorphism $f''\alpha \to \overline{f}''\alpha$ for every $\alpha < \overline{\kappa}$. This and the fact that $\overline{\kappa} = \aleph_1^{L[X \cap \overline{\kappa}]}$ imply that $\alpha < \overline{\kappa}$ is a $\langle \lambda, f \rangle$-point in V iff it is a $\langle \overline{\lambda}, \overline{f} \rangle$-point in $L[X \cap \overline{\kappa}]$.

To see that the tree orderings agree for $\langle \lambda, f \rangle$-points and for $\langle \overline{\lambda}, \overline{f} \rangle$-points, notice that the maps π in item (ii) of Definition 8.6 are the same for $\langle \lambda, f \rangle$-points and $\langle \overline{\lambda}, \overline{f} \rangle$-points, since the order-isomorphisms given by ρ commute with the maps j and j^* and their inverses. ∎

The point of Claim 8.10 is that

$$L[X \cap \overline{\kappa}] \vDash T(\overline{\lambda}, \overline{f}) \text{ has no cofinal branches,}$$

so we can apply Lemma 8.8 in $L[X \cap \overline{\kappa}]$ to $T(\overline{\lambda}, \overline{f})$, concluding that $\overline{\lambda}$ is a cardinal of L. (Here we have used the fact that $L[X \cap \overline{\kappa}] \supseteq L[\overline{f}]$, which follows from item (c) of Claim 8.9.) This concludes the proof that $\rho^{-1}\colon L_{\overline{\lambda}} \to L_{\lambda}$ witnesses the λ-remarkability of κ in L. ∎

CHAPTER 9

Generic absoluteness from remarkability

In this chapter we review the upper-bound portion of Schindler's theorem (Theorem 6.1(a)), that $L(\mathbb{R})$-absoluteness for proper posets holds in the extension by the Levy collapse to turn a remarkable cardinal into $\aleph_1$. The point of this diversion is to emphasize the level-by-level upper bound that his argument gives. We will need to strengthen our definition of *λ-remarkability* so that it better resembles the definition of *λ-subcompactness*.

Definition 9.1. Let $\kappa \le \lambda$ be cardinals. We say that κ is *strongly λ-remarkable* if for every $X \subseteq \lambda$ there is in $V^{\mathrm{Coll}(\omega,<\kappa)}$ an embedding $j\colon \langle H(\overline{\lambda}), \overline{X}, \overline{\kappa}\rangle \to \langle H(\lambda), X, \kappa\rangle$ such that $\overline{\kappa} < \kappa$ and $j \restriction \overline{\kappa}$ is the identity.

A standard argument shows that if κ is strongly λ-remarkable, then in $V^{\mathrm{Coll}(\omega,<\kappa)}$ the set of $M \in [H(\lambda)]^\omega$ whose anticollapses witness the strong λ-remarkability of κ is stationary in $[H(\lambda)]^\omega$.

A cardinal is remarkable if and only if it is strongly λ-remarkable for all λ, since

$$\lambda^+\text{-remarkable} \quad \Rightarrow \quad \text{strongly } \lambda\text{-remarkable} \quad \Rightarrow \quad \lambda\text{-remarkable.}$$

Recall that a poset $\mathbb{P}$ is *μ-linked* if there is a function $f\colon \mathbb{P} \to \mu$ such that $f(p) = f(q)$ only if p and q are compatible conditions.

We show that Schindler's argument gives the following refined version of Theorem 6.1(a).

Theorem 9.2. Assume $V = L$. If κ is strongly μ^+-remarkable, then in the extension $V^{\mathrm{Coll}(\omega,<\kappa)}$, $L(\mathbb{R})$-absoluteness holds for posets that are proper and μ-linked.

By well-known arguments (see [41, Lemma 2.2] and also [2, Lemma 1.2]), it suffices to show the following:

Theorem. Assume $V = L$. Suppose that κ is strongly μ^*-remarkable, G is $\mathrm{Coll}(\omega, {<}\kappa)$-generic over V, and H is generic over $V[G]$ for some proper, μ-linked poset $\mathbb{P} \in V[G]$. Suppose further that x is a real in $V[G][H]$. Then in V there are a poset Q_x of size $< \kappa$ and a Q_x-generic F over V such that $x \in V[F]$.

Assume $V = L$. We first reduce the problem to posets of size μ^+.

Claim 9.3. If $\mathbb{P}$ is a μ-linked poset and τ is a $\mathbb{P}$-name for a real, then $\mathbb{P}$ has a μ-linked complete subposet $\overline{\mathbb{P}}$ of size μ^+ such that τ is a $\overline{\mathbb{P}}$-name.

Consequently, the empty condition forces $\tau[\dot{G} \cap \overline{\mathbb{P}}] = \tau[\dot{G}]$, where $\dot{G}$ is the canonical name for the generic.

Proof. Since μ-linked posets have the μ^+-cc, we can assume that the name τ has size μ. Let $f\colon \mathbb{P} \to \mu$ witness the linkedness of $\mathbb{P}$. Choose θ sufficiently large, and let $X \prec H(\theta)$ be an elementary submodel of size μ^+ such that $X^\mu \subseteq X$ and $\tau \in X$. Then $\mathbb{P} \cap X$ is as desired. Notice that $f \restriction \mathbb{P} \cap X$ witnesses that $\mathbb{P} \cap X$ is μ-linked, and every maximal antichain of $\mathbb{P} \cap X$ is maximal in $\mathbb{P}$, by the elementarity of X and the fact that X is closed under μ-sequences. ∎

In $V[G]$, let $\mathbb{P}$ be a μ-linked, proper poset, and let τ be a $\mathbb{P}$-name for the real x. In light of Claim 9.3, we can assume that $\mathbb{P}$ has size μ^+. By replacing $\mathbb{P}$ with an isomorphic copy, we can further assume that $\mathbb{P} \subseteq \mu^+$. Let $f\colon \mu^+ \to \mu$ witness that $\mathbb{P}$ is μ-linked. Both $\mathbb{P}$ and f are coded by subsets of μ^+, so we can apply the strong μ^+-remarkability of κ to find (in $V[G]$) an embedding $j\colon L_{\overline{\mu}^+} \to L_{\mu^+}$ such that

- j has critical point $\overline{\kappa}$,
- $j(\overline{\kappa}) = \kappa$,
- there are $\overline{\mathbb{P}}$, $\overline{\tau}$, and $\overline{f}$ such that j is an elementary map $\langle L_{\overline{\mu}^+}, \overline{\mathbb{P}}, \overline{\tau}, \overline{f}\rangle \to \langle L_{\mu^+}, \mathbb{P}, \tau, f\rangle$,
- and $\overline{\mu}$ is a cardinal of L.

As noted earlier, images of embeddings of this type form a stationary subset of $[L_{\mu^+}]^\omega$. We can therefore extend j to an embedding $L_{\overline{\theta}} \to M \prec L_\theta$, which we also call j, for which M is countable and $\mathbb{P}, \tau, f \in M$. (We do not assume that $\overline{\theta}$ is a cardinal of L.) Take θ large enough to witness the properness of $\mathbb{P}$ in $V[G]$.

Claim 9.4. $M[G] \cap \mathrm{On} = M \cap \mathrm{On}$.

Proof of Claim. This just uses the κ-cc of the collapse. Suppose that $\sigma \in M$ is a $\mathrm{Coll}(\omega, {<}\kappa)$-name for an ordinal. Since $\mathrm{Coll}(\omega, {<}\kappa)$ has the κ-cc, we can assume that σ has size $< \kappa$. By elementarity of M the size of σ belongs to $M \cap \kappa = \overline{\kappa}$. Since $\overline{\kappa} \subseteq M$ it follows that $\sigma \subseteq M$. It follows by standard forcing arguments that $\sigma[G] \in M$. ∎

Claim 9.5. $M[G] \prec L_\theta[G]$.

Proof. This is a standard forcing argument, using no special properties of the poset. ∎

So we have an extended anticollapse embedding, which we also call j, from a model of ordinal height $\overline{\theta}$ onto its image $M[G]$. By a condensation argument, the domain of j must be exactly $L_{\overline{\theta}}[\overline{G}]$, where $\overline{G} = j^{-1}{}''G = G \cap \overline{\kappa}$. (This follows from the fact that $M \cap \kappa = \overline{\kappa}$.)

Claim 9.6. The set of countable $N \prec L_\theta[G]$ such that $N = M[G]$ for some $M = N \cap V \prec L_\theta$ is club in $[L_\theta[G]]^\omega$.

Proof. This is another straightforward application of the collapse's chain condition. The set is clearly closed. Consider some countable set $X \subseteq L_\theta[G]$. There is a countable set X^* of $\mathrm{Coll}(\omega, {<}\kappa)$-names, each of size $< \kappa$ (i.e., countable in $V[G]$), for the members of X. Choose a countable model $M \prec L_\theta$ such that $X^* \subseteq M$. Since each name $\sigma \in X^*$ is countable, it is a subset of M, so $M[G] \supseteq X$. ∎

In light of Claim 9.6 and the properness of $\mathbb{P}$, we can choose our model M so that $\mathbb{P}$ has master conditions for $M[G]$; that is, we can choose $M \prec L_\theta$ so that H is $\mathbb{P}$-generic over $M[G]$.

We have an embedding $j\colon L_{\overline{\theta}}[\overline{G}] \to M[G] \prec L_\theta[G]$ with critical point $\overline{\kappa}$. Recall that $\overline{\mathbb{P}} = j^{-1}(\mathbb{P})$ and $\overline{\tau}$ is the $\overline{\mathbb{P}}$-name $j^{-1}(\tau)$. Let $\overline{H} = j^{-1}{}''H$. The elementarity of j implies that $\overline{H}$ is $\overline{\mathbb{P}}$-generic over $L_{\overline{\theta}}[\overline{G}]$.

Claim 9.7. $\overline{H}$ is $\overline{\mathbb{P}}$-generic over $L[\overline{G}]$.

Proof. The strength of the remarkability assumption in L comes from the ability to choose the height of the embedding's domain to be a *cardinal* of L, and it is here that we make use of that. By the $\overline{\kappa}$-cc of $\mathrm{Coll}(\omega, <\overline{\kappa})$, the L-cardinal $\overline{\mu}^+$ remains a cardinal in $L[\overline{G}]$, the extension by $\mathrm{Coll}(\omega, <\overline{\kappa})$. Every antichain of $\overline{\mathbb{P}}$ has size $\leq \overline{\mu}$, since $\overline{f}$ witnesses the $\overline{\mu}$-linkedness of $\overline{\mathbb{P}}$. It follows that every antichain of $\overline{\mathbb{P}}$ in $L[\overline{G}]$ belongs to $L_{\overline{\mu}^+}[\overline{G}]$. And now, since $\overline{H}$ is $\overline{\mathbb{P}}$-generic over $L_{\overline{\theta}}[\overline{G}] \supseteq L_{\overline{\mu}^+}[\overline{G}]$, it must also be generic over $L[\overline{G}]$. ∎

Now we're ready to argue that the real x belongs to $L[\overline{G}][\overline{H}]$, a generic extension of L by the small poset $Q_x := \mathrm{Coll}(\omega, <\overline{\kappa}) * \dot{\overline{\mathbb{P}}}$. This follows, in $L[\overline{G}]$, from the fact that $\overline{\tau}$ is a $\overline{\mathbb{P}}$-name for x:

$$\begin{aligned} n \in x &\text{ iff } (\exists p \in H)\, p \Vdash_{\mathbb{P}} \check{n} \in \tau \\ &\text{ iff } (\exists p \in H \cap M[G])\, p \Vdash_{\mathbb{P}} \check{n} \in \tau \\ &\text{ iff } (\exists \overline{p} \in \overline{H} \cap L_{\overline{\theta}}[\overline{G}])\, \overline{p} \Vdash_{\overline{\mathbb{P}}} \check{n} \in \overline{\tau} \\ &\text{ iff } \check{n} \in \overline{\tau}[\overline{H}]. \end{aligned}$$

The second "iff" uses the fact that H is generic over $M[G]$, the third "iff" uses the elementarity of j, and the fourth uses the Forcing Theorem, for which we needed to know that $\overline{H}$ was $\overline{\mathbb{P}}$-generic over $L[\overline{G}]$. ∎

Remark. Unlike the proofs of generic absoluteness in Part I, which also work for reasonable posets (see Remark (2) on Page 26), Schindler's proof here seems to use properness in an essential way. Indeed, Schindler has shown [40] that $L(\mathbb{R})$-absoluteness for reasonable posets is strong enough to give an inner model with a strong cardinal, much stronger in consistency strength than the existence of a remarkable cardinal.

CHAPTER 10

A better lower bound

In the previous chapter, we showed that the consistency of a strongly λ^+-remarkable cardinal is enough to imply the consistency of $L(\mathbb{R})$-absoluteness for λ-linked proper posets. The arguments of Chapters 7 and 8 give a naive level-by-level lower bound, too: $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets that are λ-linked implies the λ-remarkability of $\aleph_1^V$ in L. While we do have an equiconsistency between full remarkability and $L(\mathbb{R})$-absoluteness for all σ-closed $*$ ccc posets, we do not have a level-by-level equiconsistency. In this chapter, we improve the naive lower bound to get closer to a level-by-level equiconsistency. To do this, we adapt the methods of [31].

Definition 10.1 (See [31])**.** Let λ be a cardinal. A Σ^2_1 *truth for* λ is a pair $\langle Q, \psi\rangle$ such that $Q \subseteq \lambda$, ψ is a first-order formula with one free variable, and there is a set $B \subseteq H(\lambda^+)$ (called the *witness* for $\langle Q, \psi\rangle$) such that $\langle H(\lambda^+), \in, B\rangle \models \psi[Q]$.

An interval $[\kappa, \lambda]$ of cardinals is called Σ^2_1*-indescribable* if for every Σ^2_1 truth $\langle Q, \psi\rangle$ for λ, there are cardinals $\overline{\kappa} \leq \overline{\lambda} < \kappa$, $\overline{Q} \subseteq \overline{\lambda}$, and an embedding $j\colon H(\overline{\lambda}) \to H(\lambda)$ such that $\langle \overline{Q}, \psi\rangle$ is a Σ^2_1 truth for $\overline{\lambda}$, and j is elementary from $\langle H(\overline{\lambda}), \in, \overline{\kappa}, \overline{Q}\rangle$ to $\langle H(\lambda), \in, \kappa, Q\rangle$ with $j \restriction \overline{\kappa} = \mathrm{id}$.

We define what it means for the interval $[\kappa, \lambda]$ to be Σ^2_1*-remarkable* by weakening the definition of "Σ^2_1-indescribable" to require only that the embedding j exist in $V^{\mathrm{Coll}(\omega,<\kappa)}$.

Notice that the interval $[\kappa, \lambda]$ is Σ^2_1-indescribable for every $\lambda \geq \kappa$ if and only if κ is supercompact. (See [34].) A similar argument shows that, likewise, the interval $[\kappa, \lambda]$ is Σ^2_1-remarkable for every $\lambda \geq \kappa$ if and only if κ is remarkable.

Theorem 10.2. Let $\kappa = \aleph_1^V$, and let $\lambda \geq \kappa$ be a cardinal of L. Assume $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets that are also $\mathfrak{c} \cdot |\lambda|$-linked. Then, in L, the interval $[\kappa, \lambda]$ is Σ^2_1-remarkable.

The reader may, of course, replace $\mathfrak{c} \cdot |\lambda|$ with $|\lambda|$ at the cost of assuming CH, but there are models where $L(\mathbb{R})$-absoluteness for proper posets holds but $2^{\aleph_0} \geq \aleph_2$. For example, force over L with $\mathrm{Coll}(\omega, {<}\kappa)$, where κ is remarkable, and then force over the extension to add $\aleph_2$ Cohen reals.

To prove Theorem 10.2 we must repeat the argument of Chapter 8, finding an embedding $j\colon L_{\bar{\lambda}} \to L_\lambda$ that witnesses the λ-remarkability of κ and also reflects a prescribed Σ^2_1 truth $\langle Q, \psi\rangle$ for λ.

We adapt the methods of [31], especially those of the first section of that paper, though we will reflect a gap $[\kappa, \lambda]$ of cardinals instead of a single cardinal κ (which in that paper is ω_2^V). Whereas the embeddings of [31] extend inclusion maps $\bar{\varepsilon} \to \varepsilon$, ours will extend reflections of a bijection $\kappa \to \lambda$.

Our strategy will initially resemble our strategy in Chapter 8, though we will have to work harder to reflect a Σ^2_1 truth. Our strategy is first to collapse λ to have size κ. Fix a bijection $f\colon \kappa \to \lambda$ in the extension. It will be convenient to assume that $f(0) = \kappa$, so that in what follows we needn't worry about taking α large enough that $\kappa \in f''\alpha$. We will ultimately capture the Σ^2_1 truth by applying the Tree Reflection Principle to many more trees than we considered in Chapter 8.

If we could collapse λ^{+L}, then we could establish the Σ^2_1-remarkability of $[\kappa, \lambda^{+L}]$, which implies our desired conclusion. In particular, we needn't consider the case when λ^{+L} has countable cofinality in V.

Basic definitions

Definitions in this section are made relative to a function f whose domain is $\kappa = \aleph_1$ and whose range is an ordinal λ. We caution the reader that our reflection argument will later require us to relativize these definitions to a different function $\bar{f}$ whose domain, while countable, is $\aleph_1$ in an inner model of V. The dependence on f will not always be made explicit, both to maintain readability and because later we will show that many definitions do not change after relativizing.

Definition 10.3. For $\alpha \leq \kappa$, let π_α denote the collapsing map of $f''\alpha$, and let ε_α denote $\mathrm{range}(\pi_\alpha)$. Finally, we write $f_{\alpha,\alpha'}\colon \varepsilon_\alpha \to \varepsilon_{\alpha'}$ for the function $\pi_{\alpha'} \circ \pi_\alpha^{-1}$.

Fix a club $C \subseteq \kappa$ on which $\alpha \mapsto \varepsilon_\alpha$ is one-to-one.

An *f-point* is a limit ordinal β such that there is $\alpha \in C \cup \{\kappa\}$ satisfying the following conditions.

- $L_\beta \models \varepsilon_\alpha$ is the largest cardinal, and
- β is not a cardinal of L.

The α in the definition is unique (by choice of C), is called the *level* of β, and is denoted $\alpha(\beta)$. We will write $\varepsilon(\beta)$ for $\varepsilon_{\alpha(\beta)}$. (The reader is encouraged to think of α as reflecting κ and of ε_α as reflecting λ.)

Notice that $\alpha(\beta)$ depends on the club C, but only on $C \cap \beta$.

NB. Unlike in [31], the level of an f-point β is not the largest cardinal in L_β.

We will continue to use Definition 8.5: write $\gamma(\beta)$ for the least γ (if any exist) such that β is not a cardinal in $L_{\gamma+1}$.

Let β be a point on level α. Let $\overline{\alpha} < \alpha$ and let H be the hull in $L_{\gamma(\beta)+1}$ of $\mathrm{range}(f_{\overline{\alpha},\alpha})$ $(= \pi_\alpha{}''(f''\overline{\alpha}))$. We say that $\overline{\alpha}$ is *stable* in the f-point β if $\overline{\alpha} \subseteq f''\overline{\alpha}$ and $H \cap \varepsilon_\alpha = \mathrm{range}(f_{\overline{\alpha},\alpha})$. In that case, the anticollapse of H is an embedding $M \to L_{\gamma(\beta)+1}$ that extends $f_{\overline{\alpha},\alpha}$ and has critical point $\overline{\alpha}$. Its domain M must be a level of L; in fact, it must be $L_{\gamma(\overline{\beta})+1}$ for some $\overline{\beta}$, which we denote $\mathrm{proj}_{\overline{\alpha}}(\beta)$, and $\overline{\beta}$ must be an f-point on level $\overline{\alpha}$. The antiprojection embedding, which we call $j_{\overline{\beta},\beta}\colon L_{\gamma(\overline{\beta})+1} \to L_{\gamma(\beta)+1}$, is unique because $\overline{\beta}$ projects below $\varepsilon_{\overline{\alpha}}$.

Claim 10.4. Suppose that β is an f-point and that $\overline{\alpha} < \alpha$ is an ordinal. There is at most one f-point $\overline{\beta}$ on level $\overline{\alpha}$ such that there is an elementary embedding $j\colon L_{\gamma(\overline{\beta})+1} \to L_{\gamma(\beta)+1}$ with critical point $\overline{\alpha}$.

The proofs of the following two claims are straightforward and analogous to the proofs of Claims 1.3–1.4 of [31].

Claim 10.5 (Commutativity of projections)**.** Suppose that $\overline{\alpha} < \alpha < \alpha^*$ and that $\overline{\beta}$, β, and β^* are all f-points, respectively on levels $\overline{\alpha}$, α, and α^*. Suppose further that $\beta = \mathrm{proj}_\alpha(\beta^*)$ and $\overline{\beta} = \mathrm{proj}_{\overline{\alpha}}(\beta)$. (We mean this to imply that $\overline{\alpha}$ is stable in β and α is stable in β^*.) Then $\overline{\alpha}$ is stable in β^*, $\mathrm{proj}_{\overline{\alpha}}(\beta^*) = \overline{\beta}$, and $j_{\overline{\beta},\beta^*} = j_{\beta,\beta^*} \circ j_{\overline{\beta},\beta}$.

Claim 10.6 (The projection ordering is "treelike")**.** Suppose that $\overline{\alpha} < \alpha < \alpha^*$ and that $\overline{\beta}$, β, and β^* are all f-points, respectively on levels $\overline{\alpha}$, α, and α^*. Suppose that $\beta = \mathrm{proj}_\alpha(\beta^*)$ and $\overline{\beta} = \mathrm{proj}_{\overline{\alpha}}(\beta^*)$. Then $\overline{\beta} = \mathrm{proj}_{\overline{\alpha}}(\beta)$.

The following claim is analogous to Claim 1.5 of [31] and is proved similarly. Notice that item (2) follows from Claim 10.4.

Claim 10.7 (Points on the same level I)**.** Suppose that β and β^* are points on the same level α, with $\beta < \beta^*$. Let $\overline{\alpha} < \alpha$ be stable in β^*, let $\overline{\beta}^* = \mathrm{proj}_{\overline{\alpha}}(\beta^*)$, and let j^* denote $j_{\overline{\beta}^*,\beta^*}$. Suppose that β is definable in $L_{\gamma(\beta^*)+1}$ from parameters in $\mathrm{range}(f_{\overline{\alpha},\alpha})$ (that is, β belongs to the range of j^*). Then:

(1) $\overline{\alpha}$ is stable in β.

Let $\overline{\beta} = \mathrm{proj}_{\overline{\alpha}}(\beta)$ and let $j = j_{\overline{\beta},\beta}$.

(2) $\overline{\beta} = (j^*)^{-1}(\beta)$. (In particular, $\mathrm{proj}_{\overline{\alpha}}(\beta) < \mathrm{proj}_{\overline{\alpha}}(\beta^*)$.)

(3) $j = j^* \restriction L_{\gamma(\overline{\beta})+1}$.

A *thread* of f-points is a sequence $T = \langle \beta_\alpha : \alpha \in D\rangle$ such that:

(i) D is club in ω_1, and, for each $\alpha \in D$, β_α is an f-point.

(ii) Let $\alpha \in D$ and let $\overline{\alpha} < \alpha$. Then $\overline{\alpha} \in D$ iff $\overline{\alpha}$ is stable in β_α.

(iii) Let $\overline{\alpha} < \alpha$ both belong to D. Then $\beta_{\overline{\alpha}} = \mathrm{proj}_{\overline{\alpha}}(\beta_\alpha)$.

By Claim 10.5, the system

$$\langle L_{\gamma(\beta_\alpha)+1}, j_{\overline{\beta},\beta} : \overline{\alpha}, \alpha \in C, \overline{\alpha} < \alpha\rangle$$

commutes. We write $\mathrm{dlm}(T)$ to denote the direct limit of this system. Since this direct limit is taken along a sequence of uncountable cofinality, the direct limit is well-founded and therefore a level of L; in fact, it must equal $L_{\gamma(\beta^*)+1}$ for some β^*, and the direct-limit embeddings must be the antiprojection embeddings j_{β_α,β^*}. We call β^* the *limit* of T and write $\beta^* = \lim(T)$.

Claim 10.8. Let β be an f-point on level ω_1. Then there is a thread $T \in L[f, C]$ with $\lim(T) = \beta$.

Contrast with Claim 1.7 of [31], which draws the stronger conclusion $T \in L$.

Proof. The hulls in $L_{\gamma(\beta)+1}$ of $f''\alpha$ for α stable in β form a thread whose limit is β. ∎

Capturing the Σ^2_1 statement

In [31], a Σ^2_1 statement $\langle Q, \psi\rangle$ is captured by taking direct limits of levels of L that capture $\langle Q \cap \alpha, \psi\rangle$ for various α; here, we will need to approximate Q using f, rather than by simply using initial segments of Q.

The definitions in this section are made relative to a function f as in the previous section and also to a subset Q of the range of f. Let ψ be a formula with one free variable. For $\alpha \leq \kappa$, put $Q_\alpha := \pi_\alpha''(Q \cap f''\alpha)$.

We say that an f-point β *captures* $\langle Q, \psi\rangle$ if the following hold:

(i) $Q_\alpha \in L_\beta$.

(ii) There is $\eta < \gamma(\beta)$ and there is $B \subseteq L_\beta$ in $L_{\eta+1}$ such that $\langle L_\beta; \in, B\rangle \models \psi[Q_\alpha]$.

The *witness* of β, denoted $\eta(\beta)$, is the least η witnessing condition (ii). There is a subset of L_β in $L_{\eta(\beta)+1} \setminus L_{\eta(\beta)}$, and so the following holds.

Claim 10.9. Every element of $L_{\eta(\beta)+1}$ is definable in $L_{\eta(\beta)+1}$ from parameters in L_β.

If $\beta < \beta^*$ are f-points on the same level α that each capture $\langle Q, \psi\rangle$, then we say β and β^* are *compatible* if there is an elementary embedding $L_{\eta(\beta)+1} \to L_{\eta(\beta^*)+1}$ with critical point β. If β and β^* are compatible f-points on level α, then the embedding witnessing this is unique by Claim 10.9 and it is denoted φ_{β,β^*}.

Claim 10.10. $\varphi_{\beta^*,\beta,^{**}} \circ \varphi_{\beta,\beta^*} = \varphi_{\beta,\beta^{**}}$.

Let Y be a set of compatible f-points on the same level α. The direct limit of the system $\langle L_{\eta(\beta)+1}, \varphi_{\beta,\beta'} : \beta, \beta' \in Y \wedge \beta < \beta'\rangle$ is denoted $\operatorname{hlim}(Y)$ and is called a *horizontal direct limit* to emphasize that f-points in Y are on the same level. If $\operatorname{hlim}(Y)$ is wellfounded, then it is a level of L and by elementarity of the direct-limit embeddings it must be the first level of L satisfying

$$(\exists B \subseteq L_{\beta^*})\, \langle L_{\beta^*}, \in, B\rangle \models \psi[Q_\alpha].$$

Notice that $\beta^* = \sup(Y)$, because $\operatorname{crit}(\varphi_{\beta,\beta'}) = \beta$ and $\varphi_{\beta,\beta'}(\beta) = \beta'$.

There is no reason to expect $Q \cap f''\alpha$ to belong to L, since f need not. Luckily, we can reflect membership of Q in L to membership of Q_α in L_β for many points β, as the next claim shows.

Claim 10.11. Suppose that $\overline{\beta}$ and β are f-points and that $\overline{\beta} = \mathrm{proj}_{\overline{\alpha}}(\beta)$. Let $j = j_{\overline{\beta},\beta}$. Then $(j^{-1})''Q_\alpha = Q_{\overline{\alpha}}$. In particular, if $Q_\alpha \in L_\beta$, then $j^{-1}(Q_\alpha) = Q_{\overline{\alpha}}$.

Proof. This follows immediately from the definition of Q_α and the fact that j extends $f_{\overline{\alpha},\alpha} = \pi_\alpha \circ \pi_{\overline{\alpha}}^{-1}$. ∎

Claim 10.12 (Points on the same level II)**.** Suppose that β and β^* are points on the same level α, with $\beta < \beta^*$. Let $\overline{\alpha} < \alpha$ be stable in β^*, let $\overline{\beta}^* = \mathrm{proj}_{\overline{\alpha}}(\beta^*)$, and let j^* denote $j_{\overline{\beta}^*,\beta^*}$. Suppose that β is definable in $L_{\gamma(\beta^*)+1}$ from parameters in $\mathrm{range}(f_{\overline{\alpha},\alpha})$ (that is, β belongs to the range of j^*).

Recall from Claim 10.7 that $\overline{\alpha}$ is stable in β. Let $\overline{\beta} = \mathrm{proj}_{\overline{\alpha}}(\beta)$ and let $j = j_{\overline{\beta},\beta}$.

Suppose also that β and β^* capture $\langle Q, \psi \rangle$ and that $\overline{\alpha}$ is large enough that Q_α is definable in $L_{\gamma(\beta^*)+1}$ from parameters in $\mathrm{range}(f_{\overline{\alpha},\alpha})$.

Then:

(1) $\overline{\beta}$ and $\overline{\beta}^*$ capture $\langle Q, \psi \rangle$.

(2) $\overline{\beta}$ and $\overline{\beta}^*$ are compatible iff β and β^* are compatible.

Assume that β and β^* are compatible. Let $\varphi = \varphi_{\beta,\beta^*}$ and $\overline{\varphi} = \varphi_{\overline{\beta},\overline{\beta}^*}$.

(3) $\varphi = j^*(\overline{\varphi})$.

(4) $j^* \circ \overline{\varphi} = \varphi \circ j$.

Proof. The proof resembles the proof of Claim 1.9 in [31], except that our Claim 10.11 is needed for item (1). ∎

The forcing and the trees

Now we suppose that $f\colon \kappa \to \lambda$ is a bijection and that $\langle Q, \psi \rangle$ is a Σ^2_1 truth for λ in L.

Following [31], we express the Σ^2_1 truth as a statement about a club E of points on level κ and then force to add a set K so that limits of threads through K are exactly the points in E, effectively turning the Σ^2_1 statement into a Π^1_1 statement.

Claim 10.13. There is a club $E \subseteq \lambda^{+L}$ such that:

(a) every $\beta \in E$ is an f-point on level κ,

(b) β captures $\langle Q, \psi \rangle$,

(c) for any two $\beta, \beta^* \in E$, the points β and β^* are compatible, and

(d) $\mathrm{hlim}(E)$ is wellfounded.

Proof. Follow the proof of Claim 1.11 of [31], replacing κ with λ throughout. ∎

Definition 10.14. As in [31], we add a system $K \subseteq \kappa$ by countable conditions as follows. (The only clause that differs materially from its analogue in [31] is clause (g).) A condition in $\mathbb{A}$ is a countable set $p \subseteq \kappa \cup (\lambda, \lambda^{+L})$ of f-points satisfying the following conditions.

(a) Every point in p captures $\langle Q, \psi \rangle$, and if $\beta \in p \setminus \kappa$ then $\beta \in E$.

(b) For every $\alpha < \kappa$ all the points in p on level α are compatible, and their horizontal direct limit is wellfounded.

(c) The set $\{\alpha < \kappa : p \text{ has points on level } \alpha\}$ is closed, with a largest element.

The set of f-points in p on levels $< \kappa$ is called the *stem* of p, denoted $\mathrm{stem}(p)$, and the set of f-points in p on level κ is called the *commitment* of p, denoted $\mathrm{cmit}(p)$. Let $\mathrm{levels}(p)$ denote the set of $\alpha < \kappa$ so that p has f-points on level α. The ordering on $\mathbb{A}$ is defined by setting $q \leq p$ iff the following conditions are satisfied.

(d) $p \subseteq q$.

(e) If $\alpha \in \mathrm{levels}(p)$ then p and q have the same f-points on level α. If $\alpha \in \mathrm{levels}(q) \setminus \mathrm{levels}(p)$ then $\alpha \geq \sup(\mathrm{levels}(p))$.

(f) If $\alpha \in \operatorname{levels}(q) \setminus \operatorname{levels}(p)$ then α is stable in β for every $\beta \in \operatorname{cmit}(p)$, and

$$\left\{\operatorname{proj}_\alpha(\beta) : \beta \in \operatorname{cmit}(p)\right\} \subseteq q.$$

(g) If $\alpha \in \operatorname{levels}(q) \setminus \operatorname{levels}(p)$ then α must be large enough that both of the following hold.

  (i) For every pair $\beta < \beta'$ such that β and β' both belong to $\operatorname{cmit}(p)$, β is definable in $L_{\gamma(\beta')+1}$ from parameters in $f''\alpha$; and

  (ii) for every f-point $\beta \in \operatorname{cmit}(p)$, Q is definable in $L_{\gamma(\beta)+1}$ from parameters in $f''\alpha$.

(h) If $\alpha \in \operatorname{levels}(q) \setminus \operatorname{levels}(p)$, $\beta, \beta' \in \operatorname{cmit}(p)$, and there are no elements of E between β and β', then there are no f-points in q between $\operatorname{proj}_\alpha(\beta)$ and $\operatorname{proj}_\alpha(\beta')$. Similarly, if there are no elements of E below β, then there are no f-points in q on level α below $\operatorname{proj}_\alpha(\beta)$.

The proofs of the following two claim follow the proofs of their analogues Claims 1.12 and 1.14 of [31], with the use and proof of clause (g) modified in an obvious way.

Claim 10.15. Let $\langle p_n : n < \omega\rangle$ be a decreasing sequence of conditions in $\mathbb{A}$. Then there is a condition q such that $q \leq p_n$ for every n.

Claim 10.16. Let p be a condition in $\mathbb{A}$ and let $\xi < \kappa$. There is $q \leq p$ such that q has f-points on levels above ξ.

For Claim 10.16, notice that the proof of Claim 1.14 in [31], even though $\kappa = \aleph_2^V$ there, needs only that κ has uncountable cofinality in V.

Notice that $\mathbb{A}$ is κ-linked, since any two conditions with the same stem are compatible.

Let G be $\mathbb{A}$-generic over V, and let $K = \bigcup_{p \in G} \operatorname{stem}(p)$. The remaining definitions in this section depend on K, f, and C, and will later be relativized to $K \cap \overline{\kappa}$, $\overline{f}$, and $C \cap \overline{\kappa}$. A thread T (of height ω_1) is a thread *through* K if unboundedly many f-points of T belong to K.

We will reflect the Σ^2_1 statement by reflecting the branchlessness of several trees, which we define now.

Definition 10.17. Let R_1 be the tree of attempts to build a thread through K with unboundedly many f-points on levels in $\operatorname{levels}(K)$ that do not belong to K. More precisely, a node in R_1 is an f-point β with $\alpha(\beta) \in \operatorname{levels}(K)$ so that

(1) for unboundedly many $\overline{\alpha} < \alpha$, $\mathrm{proj}_{\overline{\alpha}}(\beta) \in K$, and

(2) for unboundedly many $\overline{\alpha} < \alpha$ in $\mathrm{levels}(K)$, $\mathrm{proj}_{\overline{\alpha}}(\beta) \notin K$ (possibly because $\overline{\alpha}$ is not stable in β and so $\mathrm{proj}_{\overline{\alpha}}(\beta)$ is not defined).

The ordering on R_1 is defined by projection: $\beta <_{R_1} \beta'$ iff $\beta = \mathrm{proj}_{\overline{\alpha}}(\beta')$. This is a tree ordering by Claim 10.6. For $\alpha < \kappa$ write $R_1 \restriction \alpha$ for the tree R_1 restricted to nodes $< \alpha$. This gives a tree presentation $\langle R_1 \restriction \alpha : \alpha \leq \kappa \rangle$ in the sense of Definition 7.1.

The following claim is proved exactly like Claim 1.15 of [31], except that (as usual) "definable from parameters in ν" should be replaced by "definable from parameters in $f''\nu$."

Claim 10.18. Let T be a thread of height κ and let $\beta = \lim(T)$. The following are equivalent.

(1) T is a thread through K.

(2) $\beta \in E$.

(3) All sufficiently large points in T on levels in $\mathrm{levels}(K)$ belong to K.

It follows that in $V[G]$, R_1 has no cofinal branches.

Definition 10.19. Let R_2 be the tree of attempts to build a thread with only boundedly many points of K to its right. That is, a node in R_2 is a pair $\langle \xi, \delta \rangle$ so that δ is an f-point, $\alpha(\delta) \in \mathrm{levels}(K)$, $\xi < \alpha(\delta)$, and for every $\overline{\alpha}$ that is stable in δ and greater than ξ, there are no points $\overline{\beta}$ of K on level $\overline{\alpha}$ with $\overline{\beta} > \mathrm{proj}_{\overline{\alpha}}(\delta)$. The ordering on R_2 is given by equality on the first coordinate and projection on the second: $\langle \xi, \delta \rangle <_{R_2} \langle \xi', \delta' \rangle$ iff $\xi = \xi'$ and $\delta = \mathrm{proj}_{\alpha(\delta)}(\delta')$. The fact that this gives a tree ordering again follows from Claim 10.6.

For $\alpha < \kappa$ we write $R_2 \restriction \alpha$ to mean the tree R_2 restricted to nodes $\langle \xi, \delta \rangle \in \alpha \times \alpha$. This gives a tree presentation $\langle R_2 \restriction \alpha : \alpha \leq \kappa \rangle$ in the sense of Definition 7.1.

To prove the following claim, copy the proof of Claim 1.17 in [31], replacing "definable from parameters in $\overline{\alpha}$" by "definable from parameters in $f''\overline{\alpha}$."

Claim 10.20. In $V[G]$, R_2 has no cofinal branches.

Definition 10.21. For an f-point δ define $\beta(\delta)$ to be the smallest $\beta > \delta$ in K on the same level as δ if there is one, and leave $\beta(\delta)$ undefined otherwise. If T is a thread of height ω_1, then this function $\delta \mapsto \beta(\delta)$ is defined on unboundedly many points of T. (See the proof of Claim 1.17 in [31].)

A node in R_3 is an f-point δ such that $\alpha(\delta) \in \text{levels}(K)$ and for every $\nu < \alpha(\delta)$ there are $\overline{\alpha} \neq \overline{\alpha}'$ between ν and $\alpha(\delta)$ such that $\beta(\text{proj}_{\overline{\alpha}}(\delta))$ and $\beta(\text{proj}_{\overline{\alpha}'}(\delta))$ are each defined, but neither is a projection of the other. The ordering on R_3 is given by projection: $\delta <_{R_3} \delta'$ iff $\delta = \text{proj}_{\alpha(\delta)}(\delta')$.

For $\alpha < \kappa$ we use $R_3 \upharpoonright \alpha$ to denote the restriction of R_3 to nodes $\delta < \alpha$. This gives a tree presentation $\langle R_3 \upharpoonright \alpha : \alpha \leq \kappa \rangle$ in the sense of Definition 7.1.

The following claim is analogous to Claim 1.18 of [31] and is proved similarly.

Claim 10.22. In $V[G]$, R_3 has no uncountable branches.

Relativizing definitions

Anticipating our need to reflect $f : \kappa \to \lambda$ to a countable function $\overline{f} : \overline{\kappa} \to \overline{\lambda}$ and work in an inner model in which $\overline{\kappa} = \aleph_1$, we collect some claims relating the definitions of the preceding sections relative to f to those same definitions relative to $\overline{f}$. For now we need only assume that $\overline{\kappa}$ is (in V) a countable ordinal and that $\overline{f} = \pi_{\overline{\kappa}} \circ (f \upharpoonright \overline{\kappa})$. Write $\overline{\lambda}$ for $\varepsilon_{\overline{\kappa}}$, the range of $\overline{f}$.

Where necessary, we will use superscripts to distinguish between a definition relative to f and its analogue relative to $\overline{f}$, so for instance we write π^f_α for the collapsing map $f''\alpha \to \varepsilon_\alpha$ and $\pi^{\overline{f}}_\alpha$ for the collapsing map $\overline{f}''\alpha \to \varepsilon_\alpha$.

The definition of $\alpha(\beta)$ depends on the club C, but only on $C \cap \beta$, so it will be harmless to ignore this dependence.

Claim 10.23. Let $\beta < \overline{\kappa}$.

(a) β is an f-point iff it is an $\overline{f}$-point, and in that case $\alpha^f(\beta) = \alpha^{\overline{f}}(\beta)$ and $\varepsilon^f(\beta) = \varepsilon^{\overline{f}}(\beta)$.

(b) $\pi^f_\alpha = \pi^{\overline{f}}_\alpha \circ (\pi^f_{\overline{\kappa}} \upharpoonright f''\alpha)$.

Let $\alpha = \alpha(\beta)$ and let $\overline{\alpha} < \alpha$.

(c) $f_{\overline{\alpha},\alpha} = \overline{f}_{\overline{\alpha},\alpha}$.

(d) $\overline{\alpha}$ is f-stable in β iff it is $\overline{f}$-stable in β, and the definitions of the projection map and antiprojection map do not depend on whether f- or $\overline{f}$-points are considered.

Proof. (a) It is enough to check that $\varepsilon^f(\beta) = \varepsilon^{\overline{f}}(\beta)$, and this follows from the definition of $\overline{f}$ and the fact that $\pi^f_{\overline{\kappa}}$ is an order-isomorphism.

(b) is very important but follows immediately from the uniqueness of the Mostowski collapse.

(c) Follows from the definition of $f_{\overline{\alpha},\alpha}$ and item (b):

$$\begin{aligned} f_{\overline{\alpha},\alpha} &= \pi^f_\alpha \circ (\pi^f_{\overline{\alpha}})^{-1} = \pi^{\overline{f}}_\alpha \circ (\pi^f_{\overline{\kappa}} \restriction f''\alpha) \circ ((\pi^f_{\overline{\kappa}})^{-1} \restriction \overline{f}''\overline{\alpha}) \circ (\pi^{\overline{f}}_{\overline{\alpha}})^{-1} \\ &= \pi^{\overline{f}}_\alpha \circ (\pi^{\overline{f}}_{\overline{\alpha}})^{-1} = \overline{f}_{\overline{\alpha},\alpha}. \end{aligned}$$

(d) Assuming that β is either f- or $\overline{f}$-stable, we get $\overline{f}''\overline{\alpha} = f''\overline{\alpha}$, since $\pi^f_{\overline{\kappa}}$ is the identity on $\overline{\alpha}$. The rest follows from item (c). ∎

Claim 10.24. Let $\overline{Q} = Q_{\overline{\kappa}} = \pi^f_{\overline{\kappa}}{}''(Q \cap f''\overline{\kappa})$. The definition of Q_α can be reinterpreted using $\overline{Q}$ instead of Q; that is, by $\overline{Q}_\alpha$ we mean $\pi^{\overline{f}}_{\overline{\alpha}}{}''(Q_{\overline{\kappa}} \cap \overline{f}''\alpha)$.

(a) $Q_\alpha = \overline{Q}_\alpha$.

(b) An f-point β f-captures $\langle Q, \psi\rangle$ iff it $\overline{f}$-captures $\langle \overline{Q}, \psi\rangle$, and in that case the definitions of $\eta(\beta)$, compatibility, the horizontal embeddings $\varphi_{\beta,\beta'}$, and horizontal direct limit do not depend on whether f-points capturing $\langle Q, \psi\rangle$ or $\overline{f}$-points capturing $\langle \overline{Q}, \psi\rangle$ are considered.

Proof. (a) Since $\pi^f_{\overline{\kappa}}$ is a bijection, $\overline{Q} \cap \overline{f}''\alpha = \pi^f_{\overline{\kappa}}{}''(Q \cap f''\alpha)$. Apply Claim 10.23(b).

(b) All of this follows immediately from (a). ∎

The definitions of the trees R_1, R_2, and R_3 depend on C, f, and K. The dependence on C and K can be safely ignored, since we are reflecting each of C and K to one of its initial segments. (Contrast this with the situation in [31], where K is not reflected to an initial segment of itself.) So we write $R_i(f)$ to emphasize the dependence on f.

The next claim follows easily from Claim 10.23.

Claim 10.25. Suppose that M is an inner model in which $\overline{\kappa} = \aleph_1$. For each $i = 1, 2, 3$, $R_i(f) \restriction \overline{\kappa} = R_i^M(\overline{f})$.

Proof of the theorem

After forcing with $\mathrm{Coll}(\kappa, \lambda)$, a σ-closed, $\mathfrak{c}\cdot|\lambda|$-linked poset to collapse λ to have size $\aleph_1$, fix a bijection $f\colon \kappa \to \lambda$. Assume for convenience that $f(0) = \kappa$. Suppose that $\langle Q, \psi\rangle$ is a Σ^2_1 truth for λ in L. Fix E as given by Claim 10.13 and force with the σ-closed poset $\mathbb{A}$ to add the system K of f-points. Let G be the $\mathrm{Coll}(\kappa, \lambda) * \dot{\mathbb{A}}$-generic over V. $\mathrm{Coll}(\kappa, \lambda)$ has size $\mathfrak{c} \cdot |\lambda|$, and $\mathbb{A}$ is κ-linked.

$L(\mathbb{R})$-absoluteness for ccc posets of size κ will hold in $V[G]$, and that is enough to apply Theorem 7.4, since the posets used in the proof of that theorem all have size $\leq \kappa$.

We will need the trees S and T from Chapter 8 to ensure that $\overline{\kappa}$ and $\overline{\lambda}$ are cardinals in $L[\overline{X}]$.

Choose $X \subseteq \kappa$ so that, for a club of $\overline{\kappa}$, $X \cap \overline{\kappa}$ codes $C \cap \overline{\kappa}$, $\pi^f_{\overline{\kappa}} \circ f \restriction \overline{\kappa}$, and $K \cap \overline{\kappa}$. The trees R_1, R_2, R_3, S, and T are trees of size $\aleph_1$ with natural tree presentations, and they have no uncountable branches in $V[G]$. We can therefore apply Theorem 7.4 to find $\overline{\kappa} < \kappa$ satisfying the following:

(1) $\overline{\kappa}$ is a regular cardinal of L.

(2) $\overline{X} := X \cap \overline{\kappa}$ codes $\overline{C} := C \cap \overline{\kappa}$, $\overline{f} := \pi^f_{\overline{\kappa}} \circ f \restriction \overline{\kappa}$, and $\overline{K} := K \cap \overline{\kappa}$.

(3) $f''\overline{\kappa} \cap \kappa = \overline{\kappa}$, and $\mathrm{Hull}_{L_\lambda}(f''\overline{\kappa}) \cap \lambda = f''\overline{\kappa}$.

(4) $R_1 \restriction \overline{\kappa}$, $R_2 \restriction \overline{\kappa}$, $R_3 \restriction \overline{\kappa}$, $S \restriction \overline{\kappa}$, and $T \restriction \overline{\kappa}$ all have no cofinal branches in $L[\overline{X}]$.

Remark. We do not require the tree R_0 of [31] for item (1), since we can use Kunen's theorem that $L(\mathbb{R})$-absoluteness for ccc posets implies the weak compactness of κ in L.

We will show that the anticollapse $j\colon L_{\overline{\lambda}} \to L_\lambda$ of $\mathrm{Hull}_{L_\lambda}(f''\overline{\kappa})$ witnesses the Σ^2_1-remarkability of $[\kappa, \lambda]$. Argue as in the proof in Chapter 8 of Theorem 6.2 to see that $\overline{\kappa} = \aleph_1^{L[\overline{X}]}$ and that $\overline{\lambda}$ is a cardinal of L, using the fact that $S \restriction \overline{\kappa}$ and $T \restriction \overline{\kappa}$ have no cofinal branches in $L[\overline{X}]$. It remains to show that $\langle j^{-1\prime\prime}Q, \psi\rangle$ is a Σ^2_1 truth for $\overline{\lambda}$ in L.

First, notice that

$$j^{-1}{}''Q = \pi^f_{\overline{\kappa}}{}''Q = \pi^f_{\overline{\kappa}}{}''(Q \cap f''\kappa) = \overline{Q}.$$

So we need to show that $\langle \overline{Q}, \psi \rangle$ is a Σ^2_1 truth for $\overline{\lambda}$ in L.

Work in $L[\overline{X}]$. Let $\overline{E}$ be the set of limits of threads through $\overline{K}$. $\overline{K}$ is a set of f-points below $\overline{\kappa}$, so by Claim 10.24 every point in $\overline{K}$ is an $\overline{f}$-point capturing $\langle \overline{Q}, \psi \rangle$. So $\overline{E}$ is a set of points on level $\overline{\kappa}$, each capturing $\langle \overline{Q}, \psi \rangle$.

Claim 10.26. $\overline{E}$ consists of compatible points.

Proof. Repeat the proof of Claim 1.21 in [31], except that there is no need for us to appeal to condition (3) on page 10 there, since our $\overline{K}$ is just an initial segment of K. (As usual, "definable from parameters in α" should be changed to "definable from parameters in $f''\alpha$" and $Q \cap L_{\overline{\kappa}}$ should be replaced by $Q_{\overline{\kappa}}$.) ∎

Claim 10.27. $\mathrm{hlim}(\overline{E})$ is wellfounded.

Proof. The proof of Claim 1.22 of [31] can be repeated, changing definability from parameters in ν to definability from parameters in $f''\nu$ and changing $Q \cap L_{\overline{\kappa}}$ to $Q_{\overline{\kappa}}$ throughout. ∎

Claim 10.28. $\overline{E}$ is unbounded in $\overline{\lambda}^{+L}$.

Proof. The proof follows the proof of Claim 1.24 in [31], but we give it here in full, since it is the heart of the argument. Fix an $\overline{f}$-point $\delta \in (\overline{\lambda}, \overline{\lambda}^{+L})$. We must find a $\beta \in \overline{E}$ with $\beta > \delta$. Let B be the set of $\alpha < \overline{\kappa}$ that are stable in δ. Apply Claim 10.8 (in $L[\overline{X}]$) to conclude that B is club in $\overline{\kappa}$ and $\langle \mathrm{proj}_\alpha(\delta) : \alpha \in B \rangle$ is a thread with limit δ. (This differs slightly from the proof in [31]: there, the thread belongs to L, whereas here it only belongs to $L[\overline{X}]$.)

Let D be the set of $\alpha \in \mathrm{levels}(\overline{K})$ such that there is a $\overline{f}$-point β in $\overline{K}$ on level α with $\beta > \mathrm{proj}_\alpha(\delta)$. Let β_α be the least such β.

Since $R_2 \restriction \overline{\kappa}$ has no cofinal branches in $L[\overline{X}]$, D is unbounded in $\overline{\kappa}$. Since $R_3 \restriction \overline{\kappa}$ has no cofinal branches in $L[\overline{X}]$, there is $\nu < \overline{\kappa}$ such that for all $\alpha, \alpha' \in D \cap (\nu, \overline{\kappa})$, one of β_α and $\beta_{\alpha'}$ is a projection of the other. It follows that $\{\beta_\alpha : \alpha \in D \wedge \alpha > \nu\}$ generates a thread. This is a thread through $\overline{K}$, which has limit greater than δ by Claim 10.7. ∎

Let $M = \mathrm{hlim}(\overline{E})$. By Claim 10.27, M is wellfounded and is therefore a level of L. (Since $\overline{\lambda}^{+L}$ is countable in V, this is nontrivial.) Each of the points in $\overline{E}$ captures $\langle \overline{Q}, \psi \rangle$, so by the elementarity of the horizontal embeddings $\varphi_{\beta,\beta'}$ it follows that M satisfies, "There exists $B \subseteq L_{\beta^*}$ such that $\langle L_{\beta^*}, B \rangle \models \psi[\overline{Q}]$." Here $\beta^* = \sup(\overline{E})$, which by Claim 10.28 equals $\overline{\lambda}^{+L}$. Since M is an initial segment of L, we have shown that $\langle \overline{Q}, \psi \rangle$ is a Σ^2_1 truth for $\overline{\lambda}$ in L, completing the proof of the theorem. ∎

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