Title
Students Developing Voices in New Learning Ecologies: Voice, Identity, Position and Function as a Framework to Support Multimodal Investigations of Learning Mathematics over Multiple Timescales

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By

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Abstract

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This dissertation lives at the intersection of two essential and under-researched domains. The first concerns the impact of new pedagogies at the university level. Although lecture is still the dominant mode of teaching in university mathematics, some mathematics faculty have been exploring the use of small-group work as a primary instructional mode. Little has been documented, at a fine-grained level, regarding the impact of small-group learning at university. The second concerns the growth and change of university students' mathematical identities, especially at key points in their academic careers. Far more students begin their university careers intending to be mathematics majors than actually graduate as such, with much of the blame for attrition being placed on the teaching methods used. Moreover, many students who major in mathematics avoid pursuing advanced mathematical studies at the graduate level because of their struggle with proof-intensive courses at undergraduate level. A key question is how students grapple with the demands of these kinds of courses. This dissertation is situated in such a course, a one-semester course in number theory. It provides a detailed examination of the experiences of two focal students as they negotiated the challenges of group work in a mathematically demanding context. A new theoretical framework and methodological tools were created to unpack what took place, at a fine grained level of detail, over three timescales: in classroom groupwork, over the course, and in rapport to the major program.

Calls for evidence-based innovative teaching at the college level have been growing since the late 1980s, specifically in STEM (Science, Technology, Engineering, and Mathematics) fields, in large part to mitigate attrition. Inspired by the educational reform in K-12, some faculty have turned to the so-called active learning or student-centered pedagogies. As indicated by this dissertation and other research, such pedagogies have the potential to rekindle some new interests in collegiate students and to foster positive relationships with the disciplines. Nevertheless, there is a history of some student resistance to active learning pedagogies, either because students are not prepared for such pedagogies or have had negative past experiences with the same. To understand the failure and success of innovative pedagogies, research must

closely attend to the *voices* that students develop as they interact with the new learning ecologies.

Building on psychoanalytical, socio-linguistic, and socio-cultural theories, this dissertation (Chapter 1) proposes and leverages a specific conceptualization of *voice* that addresses students' past, present, and future. Such a conceptualization is needed to account for the realities of undergraduate students, who draw upon their resources, individualized in past experiences, as they engage in learning activities to prepare themselves for the future. The construct of *voice* is broken into three constituents: *identity*, *position*, and *function*; hence, the name of the VIP+function framework. The framework looks at the individualized *identities* that students actuate by animating *positions* in their ongoing activities to accomplish *functions* that forge their near and more distal futures. Additionally, this project (Chapter 2) enhances the investigation of identity formation and development by re-purposing an existing data collection technique, renamed as *stimulated construction of narratives about interactions* (SCNI). The SCNI technique attempts to generate data that can be jointly studied by two robust theoretical approaches of identity, narrative and situated approaches, which have been largely independent to this point.

The impact of pedagogies should be assessed not only by examining students' engagement or performance in a program or course, but also by their power of transforming students' identities. The case studies reported in this dissertation unpack the processes by which two students, Ted and Bettie, boosted their mathematical identities as they adapted to a proof-intensive course and small-group learning. Ted developed the confidence to pursue advanced mathematical studies (Chapter 3). Bettie faced events that challenged and diminished her mathematical identity, which she restored and strengthened over the course of the semester (Chapter 4). Overcoming several impediments, Bettie developed a new form of active engagement with the content, in contrast to her previous reliance on memorization and practice. (Chapter 5). The case studies of this work document some of the power of the pedagogies of small-group learning, despite their limitations. They reveal processes by which the two focal students were able to support each other's learning development through groupwork, in ways that teaching based on lecture would not afford. They also highlight some advantages of having students work in the same group over multiple sessions: In cases similar to the ones discussed here, students needed time to build accurate understandings of each other's behaviors and together establish a group culture that optimally supported each member's learning processes.

In sum, this dissertation explores the impact of new pedagogies on students' identity development by providing theoretical/methodological tools and analytic examples, which are applied to University level with the potential for application at K-12 education as well.

Keywords: voice, identity, positioning, function, undergraduate mathematics, proofs, narrative identity, situated theory, small-group learning.

Contents

Acknowledgment

As our voices travel in the air, they imprint us on the flying papers they encounter. As I complete this milestone, I recognize the imprint of many voices that shaped me throughout the journey that led to and made this dissertation.

My parents, who did not go beyond their 8th grade in school, voiced their appreciation of academic careers and the sense of integrity since my childhood. The Jesuit order imprinted in me the care for individuals as unique persons. The many voices I heard in Lebanon, Morocco, Egypt, Soudan, Syria, Jordan, France, England, and California in various contexts and activities carved a cultural sensitivity that prepared me to appreciate the socio-cultural theories that shaped this dissertation.

My advisor, Alan Schoenfeld, instilled in me the love of the field of mathematics education, a type of love ready to invest countless hours in work that would move the field forward. The voices of many professors who worked at, still work at, or visited the university of California at Berkeley rekindled my belief in academic research. I mention only few: Randi Engle, Geoffrey Saxe, Frederick Erickson, Claire Kramsch and Anna Sfard. My colleagues and friends, particularly Kimberley Seashore, Jennifer Langer-Osuna, Einat Heyd-Metzuyanim, Edward Schaefer, Matthias Beck, Anna Zarkh, and Samer Banna, supported, challenged, and boosted my identities that are voiced in this dissertation.

Transcription symbols

.␣	dot following a text and followed by a space indicates a full-stop of speech
␣.␣	dot in between two spaces indicates a pause
␣...␣	three dots in between two spaces indicate a long pause
,	indicates a vocal low rise
?	indicates a vocal high rise
::	indicates vocal elongation of precedent phoneme. The more colons, the more elongation.
[text]	indicates information accompanying or explaining speech, such as gestures or specifying object of indexical.
[…]	indicates missing, not reported speech.
"text"	indicates quoted or as-if quoted speech, such as inner thoughts, within speech.
=	indicates overlapping speech
Te-	indicates an interrupted utterance by speaker or other participants

References

Int1-mmdd-Name	Early interview on day dd of month mm with participant's Name
Int2-mmdd-Name	Exit interview on day dd of month mm with participant's Name
SCNI-mmdd-Name	SCNI interview on day dd of month mm with participant's Name

Chapter 1: A Framework

to Investigate Students' Adaptations to New Learning Ecologies

How do new pedagogies in mathematics classrooms succeed or fail? What learning processes influence students' adaptations to new learning ecologies? How can students of feeble self-efficacy develop positive learning identities in classes with new pedagogies? This dissertation attempts to investigate these questions through two multimodal case studies of two collegiate students, Ted and Bettie (all identifiers are pseudonyms), at different timescales: during small-group sessions, over a semestrial course and in relation to major programs. Ted's and Bettie's adaptations to a new subdiscipline (number theory) and a new pedagogy (small-group work) positively transformed them. While Ted, one of the focal participants, smoothly adapted to and benefited from the new learning ecology, Bettie, the other focal participant, endured a productive struggle.

The need for innovative teaching in STEM fields in the U.S. can be traced in part to concern about attrition, which influential studies (Astin, 1993; Daempfle, 2003; Seymour & Hewitt, 1994) attributed in large measure to classroom pedagogies.[1] Efforts that move away from traditional teaching towards active learning pedagogies in STEM classrooms have increased among both faculty and researchers. Indeed, a large body of research supports the learning benefits, including retention, of various student-centered pedagogies in STEM undergraduate fields (Chang, Sharkness, Hurtado, & Newman, 2014; Ferrare & Lee, 2014; Freeman et al., 2014; Gasiewski, Eagan, Garcia, Hurtado, & Chang, 2012; Laursen, Hassi, Kogan, Hunter, & Weston, 2011; Laursen, Hassi, Kogan, & Weston, 2014; Prince, 2004; Springer, Stanne, & Donovan, 1999).

Nonetheless, some faculties in STEM and other fields have been facing students' resistance and revolt against innovative teaching (Allen, Wedman, & Folk, 2001; Dembo & Seli, 2004; Ellis, 2015; Felder & Brent, 1996; Fielding, 2004; Hockings, 2005; Pepper, 2010; Seidel & Tanner, 2013; Thorn, 2003). Students' resistance can manifest through their disengagement in class, groupwork and homework assignments, and be voiced in their negative reports to the administration (Kearney & Plax, 1992). Faculty may refrain or retract from changing the established pedagogies for fear of stirring students' dissatisfaction. Seidel and Tanner (2013), comprising biology faculty, depict the current reality of college teaching as one that is caught in transition.

> It seems we are in a time of transition, in which estimates are that less than half of all college faculty members are using student-centered teaching approaches (Hurtado et al., 2012). As such, if you introduce an innovative teaching strategy into your classroom, it is

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[1] Prior to studies of the phenomenon of attrition in STEM fields, faculty in mathematics departments had been calling for a reform of teaching at college. Indeed, "calculus reform," dating back to the mid-1980s - see MAA Notes #6, *Toward a Lean and Lively Calculus* (Douglas, 1986) - was stimulated by concern about high attrition rates and ineffective pedagogy.

likely that there are students in your course who have not previously experienced anything like this before. Even if they have, their experiences may not have been positive. (p.594)

As research on classroom pedagogies continues to develop, the need for research to support educators and students in their initial implementation of novel classroom pedagogies is as crucial as the task of grounding these pedagogies in evidence.[2] While empirical evidence can enhance educators' confidence in a supported pedagogy, it does not provide them with guidance with regard to implementing it in a new context. What do educators and students need to attend to as they engage with novel learning designs? To answer this question, we need to enhance our understanding of students' behaviors concerning new learning ecologies constructed at college.

This dissertation attempts to contribute to the sparse research available on students' engagement with new learning ecologies. It will draw upon socio-cultural, socio-linguistic, and psychoanalytical theories to illuminate the processes responsible for shaping students' adaptations to new learning ecologies. Data are gathered from a number theory class in which the instructor used small-group work for the first time in his teaching career, asking students to prove theorems by themselves. Most of the students were not familiar with intensive groupwork the way it was used in the studied class. In this dissertation, two students, Ted and Bettie, were selected for an in-depth analysis of the development of their voices. A third case study, Melissa's, previously published (El Chidiac, Carlson & Ponnuswamy, 2018), will be in the backdrop while comparing Ted's and Bettie's cases.

The dissertation will focus on the two types of students noted by Seidel and Tanner (2013)—having a negative experience and not having any experience with the new pedagogy. Ted had negative experiences with small-group learning, whereas his experience of the number theory class was positive and productive. Bettie was not used to studying with peers but learned how to render groupwork beneficial during the number theory class. She was at a risk of either dropping or failing the course. Like many students, Bettie felt pride in her ability to engage with the aspects of mathematics she had mostly engaged with until this point – mathematics that primarily called (from her point of view) for memorizing procedures and implementing them. Whether in high school algebra or calculus, she did well on this "Pre-Proof Mathematics" (PPM).[3] Like many students, she found herself challenged by the new demands of producing

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[2] National policymakers (Olson & Riordan, 2012) urge STEM college institutions to adopt evidence-based pedagogies in attempt to address attrition. While evidence may convince educators to adopt one pedagogy rather than another, it may not impel students to engage in new pedagogies. In attempt to persuade students about working in small-groups in her calculus class, she showed them evidence supporting this pedagogy coming from a research conducted in their campus. She was surprised by students' indifference. Some students reacted, "yeah but this doesn't work for me."

[3] I shall use the term "Pre-Proof Mathematics" to describe instruction that is primarily focused on understanding and using mathematics, rather than deriving mathematical results (a.k.a. "proving.") Some topics, e.g., calculus, can be taught either way – that is, they can be experienced as mostly applied or as deeply theoretical. Typically, the vast majority of introductory collegiate mathematics courses, including

proofs. For her, as for many other students, this challenge resulted in a threat to her pre-proof mathematical identity (PPMI).[4]

In the remaining sections of this chapter, I first explain the theoretical framework adopted in this research and then justify its importance in investigating students' engagement with new learning ecologies. Next, I describe the content of this dissertation's chapters and their interconnectedness, to support this investigation. The selection of focal students will be discussed in the chapter that delves into the methods and methodology.

A framework for the analysis of students' voice

In this section, I unfold the theoretical framework in three subsections. In the first subsection, I delineate the phenomenon under study—students' engagement with novel learning ecologies at college—and set two desiderata for its investigation. In the second subsection, I construct a framework, to be called *VIP+function*, by building on socio-cultural, socio-linguistic, and psychoanalytical theories, and delineate how it meets the two desiderata. In the third subsection, I explain how the VIP+function framework can support the investigation of students' adaptation to new learning ecologies.

Two desiderata to investigate students' adaptation to new learning ecologies

As previously mentioned, this dissertation attempts to shed light on collegiate students' adaptation to new pedagogies and sub-disciplines. However, students' participation in a major program influences their engagement with the requirements of that course. For this reason, I investigate their engagement over three timescales[5]:

- *the timescale of a major program* (e.g. mechanical engineering or mathematics for arts),
- *the timescale of a course* (e.g. number theory or Calculus I), and
- *the timescale of class assigned activity* (e.g. classroom groupwork or doing an assigned homework).

A course, which commonly runs over a quarter or a semester, is a series of learning assigned activities that have to be fulfilled over a few minutes/hours inside or outside classroom. A program involves many, or at least two, years of successful participation in a designed collection of courses, some of which are elective. The three delineated timescales will be fleshed out through the desiderata.

Desideratum 1: A unified framework capable of operating at three timescales.

Students' engagement at each timescale is mutually interconnected with the two other timescales. The interconnectedness of timescales depends on (i) how the institution determines

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calculus and some courses beyond (e.g., differential equations, and introductory linear algebra) tread lightly on proof – hence the label PPM.

[4] This is a widely known phenomenon in the mathematics community, to the point where "transition to mathematical proof" is a major concern. Googling that phrase results in 15,400,000 hits.

[5] The theoretical construct of timescale is inspired by Lemke (2002).

the required courses for major programs and (ii) how instructors design class activities to meet the learning goals of a course, as approved by the institution.

Students' engagement at each timescale mutually influence one another. I will illustrate them two by two: class activity and course, course and program, class activity and program. First, students' engagement with assigned activities and courses mutually influence one another. For instance, a student may drop a course because of her harrowing experience in assigned activities and take the course again when another faculty teaches it. A student may also work hard to overcome barriers in assigned activities because s/he is determined to succeed in the course. Second, engagement in a course and a major program mutually influences one another. For instance, a student may change his/her major because he/she wants to drop a required course or boost his/her investment in another course because of his/her interest in that particular major. Even if a student enters college with an undeclared major, the choice of the same looms at her/his engagement with courses. Third, we can also envision a mutual influence of assigned activities and major programs, when a type of activity is shared across several courses comprising a particular program. For instance, students may change their engineering major because they dislike computations or may invest in learning computations because they want to graduate as engineers.

Due to the mutual interconnectedness of engagement at three timescales— assigned learning activities, course, and major programs—a consistent investigation of students' engagement at college requires a unified framework that is capable of operating at the three timescales listed above.

The construction of such a framework is challenging because of the diverse cultural activities involved at each timescale. Consider the following examples. Engagement in a class activity may consist of interacting with peers in order to solve a problem within classroom, working with a friend to do a given homework, and studying alone for a test. The degree of engagement in a course may range from being as simple as enrolling electronically in the course to becoming as invested as using office hours to discuss progress with the instructor. Engagement in a major program may be realized through personal reflections, discussions with advisors about elective courses, conversations with friends about prospective jobs, and attending a job fair. The desired framework must be as general as possible to encompass a wide variety of cultural activities, including individual and collective ones, while simultaneously remaining pertinent to each one of them.

Desideratum 2: Sensitivity to changes at various scales

If we assume that students can change their participative ways using innovative pedagogies for engagement, we will need an analytical framework that will be sensitive to such modifications. It would be wiser to construct a framework that is sensitive, rather than unresponsive, to changes in students' behaviors within class activities and courses. If no changes are observed through the lens of a framework which is sensitive to changes, the basic assumption of change will be weakened.

Students' perceptions of innovative pedagogies can change from the beginning to the end of a given term (Ellis, 2015). Their ways of engaging in class assigned activities, courses, and programs, e.g. level of investment and perception of peers, may also shift over time. Changes of engagement may occur from moment to moment in an assigned activity within a class, from week to week regarding a course, and from semester to semester regarding a program. The time segments are merely indicative: they give a sense of the timescale with respect to changes at the three levels of engagement.

While changes in the degree of engagement over long timescales are evident, changes over shorter timescales, e.g. within an hour of class work, may seem unlikely. Changing a major and dropping a class are ostensible changes, which may be bolstered by minute changes at any of the three relevant timescales. For instance, changes regarding peer-to-peer and student-to-instructor interactions as well as individual performance achievements through the class activities may either foster or dilute the initial engagement in a course. Indeed, I enroll in a course to indicate at least a minimal level of engagement with it. Additionally, students who receive low grades in their first STEM courses at college are likely to shift to non-STEM majors (Sage, Cervato, Genschel & Ogilvie, 2018). Grades also can be affected by the level of students' engagement with class activities. Thus, the improvement of the same, that results in improvement of grades, can prevent potential drop-outs.

Tracking both ostensive and minute changes in students' participations in class assigned activities, courses, and major programs is central to investigating their adaptation to novel pedagogies. Students' successful adaptations to novel learning ecologies may result from the cumulation of small changes, such as their perceptions of peers, adjustment of their personal schedules to make time for homework, and finding a supportive study group. On the other hand, significant changes, once detected, can be investigated further to distill the processes that spur them. For example, a student's engagement in small-group work may improve after he/she changes his/her perception of his/her groupmates. In such cases, the researcher should investigate the processes by which he/she changes his/her perception of groupmates.

The desired framework must be able to detect changes at both small (within class activities) and large (throughout a course or program) scales. Thereby, it will be able to indicate potential cumulative effects while linking small changes to ostensive ones. It will also identify further investigations that are worth undertaking.

The VIP+function framework

To investigate students' adaptations to new pedagogies, I propose a framework of four constituents: *voice*, *identity*, *position*, and *function*. It will be called VIP+function framework. In this subsection, I will define the constituents of this framework then explain them through a conversation with the established relevant literature, the phenomenon under study, and the desiderata set in the previous subsection. I will also describe how these constructs are operationalized in this dissertation.

Identity

Definition. In the VIP+function framework, *identity means an individualized set of a semiotic repertoire and entrenched habits*. The constituent of identity in the VIP+function framework attends to students' histories that they bring to bear on their engagement in a class activity, course, and major program. It covers the frames of mind and embodied habits that a particular student individualizes in relation with his/her life experiences.

Semiotic repertoires are clusters of an individual's understanding of the world, activities, other people, and him/her self (Blommaert, 2005). For example, Isabel's repertoire of mathematics may comprise her understanding of her teachers' mathematical acts, the mathematical activities in which she participated either recently or a long time ago, which acts are valued as mathematically worthy, her mathematical knowledge, and her beliefs about herself in rapport with mathematics. Among the semiotic repertoires, metaphors (Lakoff & Johnson, 1980, 1999) are worth noting. People draw on metaphors to make sense of lived situations. For instance, small-group work within classroom may be conceived as a market place where participants exchange knowledge as a commodity. As such, participants who do not contribute their knowledge will be seen as thieves who take without giving, as it were. Figured worlds (Holland, Lachicotte, Skinner & Cain, 1998) are other types of semiotic repertoires. Indeed, students in high school were observed as behaving according to a figured world of romance in groupwork within a mathematics classroom (Esmonde & Langer-Osuna, 2013).

An individual's history also includes *embodied habits* (Bourdieu, 1990; Kramsch, 2010). Our bodies have memories that associate behaviors with situations. Relevant to the phenomenon under study are learning methods. Throughout their school experiences, students build habits that let them cope with new situations and concepts. For instance, while facing a new mathematical problem at college, some students may immediately search for the answer in the internet, because this is how they previously used to complete their homework in high school. Learning habits are built concomitant to an individual's history and, thus, constitute intrinsic parts of one's identity. Also relevant to the phenomenon under study are students' social habits. Students may find in the new learning ecologies social situations towards which they have already built an entrenched dealing habit. For example, a student's habit of remaining reserved while interacting with strangers may also be enacted in his/her participation in the classroom groupwork with peers whom the student meets for the first time.

Two types of individualizations. I have unpacked the constructs of semiotic repertoires and entrenched habits, and what remains is to unfold the process of individualization as stated in the aforementioned definition of identity. Individualization is the process by which *what we see, hear, know, and do* becomes *who we are*. Some individualizations can be momentary while others can be carried over a long period of time. These two types of individualizations are found in Goffman's and Lacan's works, as explained next.

Lacan, the French psychoanalytical theorist, established that an individual defines his/her *self* in relation to an idiosyncratic symbolic world, called *L'imaginaire*, that they construct through social images (Copjec, 2015; Lacan & Fink, 2006). By this process,[6] semiotic repertoires and entrenched habits which originate in the social realm become personal or, as I prefer to call it, *individualized*. Lacan continues to state that a mismatch between *L'imaginaire* and a lived situation causes emotional distress. In Lacan's theory, individualized habits and semiotic repertoires are no longer seen as objects that people can take up and off without vital repercussion—they are part of who people are. A person who individualizes semiotic repertoires and habits may become defensive within ecologies that require a change of those elements. For this person, such change is not merely behavioral but existential.

[6] Lacan calls this process *identification*, which I try to avoid here, because in educational research it is used to indicate surface attachment to signified objects. Educators and educational researchers liken the process of identification to dressing up. Identities become like clothes that one can change at ease without vital repercussions. For Lacan, identities are like our bodies, which can change but not as fast or easy as changing clothes.

Educators and educational researchers have been influenced by social theories, such as Goffman's (Goffman, 1967, 1981), that conceive of human activities as a process of role playing. In such a paradigm, students and teachers are expected to abandon old roles and take up new ones set by novel pedagogies, as seamlessly as an actor changes his/her role within the same play. Within traditional teaching models, teachers are expected to be the deliverers of knowledge; in reformed models, they become facilitators of students' autonomous learning process. If students and teachers can change their pedagogical roles and ways of thinking about the discipline seamlessly, then their old roles will have not been used to shape their selves. As such, their old behaviors and semiotic repertoires will have had no bearing on the *imaginaire* by which they live. Some students may navigate through their education by *playing* roles, others by *living* these roles as intrinsic parts of their being. Hereby, two types of individualizations are recognized, one as playing roles and the other as shaping selves by augmenting the *imaginaire*.

The most striking difference between Lacanian psychoanalysis and Goffman's sociology is found in their respective interpretations of achievement. In the role-playing paradigm, actors perform for an audience which evaluates their performance. Researchers and educators who are influenced by the role-playing paradigm emphasize the importance of social recognition. However, this theoretical frame fails to explain why a student may experience a sense of fulfillment after solving a mathematical problem sitting alone on his/her desk, even if he/she does not get any social recognition of his/her work. For Lacan, a person experiences a sense of satisfaction when s/he realizes her/his *imaginaire*, that is, when her/his idiosyncratic symbolic world matches a lived reality. If the *imaginaire* does not involve social recognition, its realization generates an internal satisfaction regardless of social perception.

It is worth noting that for Lacan, *L'imaginaire*, i.e. identities as defined here, holds identifications with repertoires and habits that have not yet been realized in individuals' histories. Such identifications generate a desire for fulfillment. In addition to the previous definition of identity, it can also be defined as *who and what one was, is and looks forward to being.* Sfard & Prusak (2005) advance a similar definition of identity that involves aspirations and define learning as "bridging the gap between actual and desired identities."

Operationalization. Identities as defined hereabove underpin individual behaviors and can be readily identifiable in individuals' narratives about themselves and their interactions with other people and objects in the world, in a manner similar to psychoanalytical methods. For this purpose, I conducted regular and SCNI interviews which will be discussed in Chapter 2.

As for operationalization of the construct of identity, we should be looking for identifications and thematic speeches. What signals identities are absolute speeches by which speakers identify as particular types of people (e.g. "I am a boss"), performers of an act (e.g. "I have been tutoring kids since I was in high-school" and "I never talk to classmates") and subjects of desires and emotions (e.g. "I look up to engineers who invent new products" and "I hate proofs"). Additionally, we should attend to speeches that compare pedagogies and disciplines with other repertoires. Such comparisons may indicate robust metaphors and figured worlds that may be at play in shaping students' identities. More information on the methods of investigating metaphors can be found in Lakoff and Johnson's work (1980), and that on figured worlds in educational settings can be located in Langer-Osuna's dissertation (Langer-Osuna, 2009).

I assume all detected identities to be short-lived roles and put the burden of verification on claims about long-lived identities. I use the following three aspects to verify entrenched identities, ordered from weak to strong:

(1) Identities narrated in one instance but with absolute statements, such as “I always,” “I belong to,” and “I did this since long time.”
(2) Identities used to talk about a topic on two or more instances, such as when a student would say “I am a shy person I don’t go to office hours” in a regular interview and two weeks later she would comment on her groupwork interactions saying that “I didn’t defend my idea because I’m shy.”
(3) Identities used to narrate many aspects of the speaker’s life, such as a student talking about his/her computational skills at work while gambling with his/her friends and in classroom groupwork.
(4) Identities couched with emotions in participants’ narratives, such as “I feel alive when I solve mathematical problems” and “I was annoyed when she invaded my personal space.”

Desiderata. Identities, as previously defined, shape students’ engagement in class activities, course, and programs (desideratum 1). Desired identities shape and sustain students’ engagement in major programs. Students draw on their semiotic repertoires and habits to interact with authorities, such as instructors, and peers, such as groupmates. Again, their entrenched learning methods either facilitate or impede their learning experiences with new pedagogies.

Identities, as *roles* and *imaginaire*, can change (desideratum 2). Social environments and new discourses may interpellate different semiotic repertoires at different moments. A student may draw on his/her repertoire of colleagueship with an instructor while drawing upon hierarchy with another. In class activities, students may engage with groupmates at moments, actuating their social identity, and work individually at other moments, actuating a sense of autonomy.

According to Lacan, the *imaginaire* also changes, although not seamlessly, in response to new favorable social environments. He emphasizes on the role of language in shaping the self. On one hand, semiotic repertoires are linguistic products. On the other hand, individuals identify with the hidden speaker of the language by which they constructed their own *imaginaire.* Language, per se, presents itself as speech which hides the speaker, who is none of the people who speak the language. Indeed, none of the speakers of common languages have created these languages. By identifying with the hidden speaker beyond the language, individuals recognize their individualized semiotic repertoires and habits as traps. Think of students trapped in their beliefs about their mathematical abilities. The identification with the hidden speaker of the language becomes an identification with the beyond—beyond the language and its semiotic products. Experiencing identities as traps and identifying with the hidden speaker beyond the language, individuals start desiring what lies beyond their individualized semiotic repertoire. Thus, they become ready to individualize new semiotic repertoires. Due to lack of space and time, Lacan’s theory cannot be fully explained in this work. However, it is worth noting that the individualization of new semiotic repertories and entrenched habits must be mediated by the freeing speech of another person. Individuals cannot free themselves from their own identities, but appropriate speeches by other people can free them. While in Goffman’s theory the role of the social other, the audience, is to reinforce and encourage identities that are played well, in Lacan’s theory it resides in freeing others from current identities to pave the path for new ones.

Position

Definition. In the VIP+function framework, a *position is defined as a behavior framed by a cultural activity*. The construct of position is concerned with students’ behaviors in the *hic et nunc*, as opposed to identity, that covers students’ histories. Behaviors that are relevant to this inquiry can take place in a classroom, office, at the library, or anywhere outside campus. The *now* of the behavior can be oriented toward a short timescale, such as participation in a class activity, or long timescale, such as discussion with instructors about a course, and extended timescale, such as withdrawal from a major.

Positions are value-laden. Their meanings are determined by norms that are established in the activities in which they emerge. *A student looking at groupmates' work* during a test, for instance, is judged differently than the occurrence of the same during a collaborative small-group work.

Positions are more than the illocutionary forces of speech acts (Davis & Harre, 1990). They must be interpreted in relation to the cultures or micro-cultures that uphold them (Erickson & Mohatt, 1977). To illustrate how the construct of position is used in this work, consider the two questions a student may ask her groupmate during small-group work in a number theory classroom.

Question1: "what is the Euclidian algorithm?"

Question2: "are 261 and 84 co-prime?"

From a linguistic viewpoint, the two questions have one illocutionary force: to request information from the interlocutor. However, within the mathematical culture of groupwork, the type of information that is requested is set to be mathematical. An interlocutor is not expected to explain how Euclid brought about his algorithm. Additionally, the two questions involve different positions. Question1 animates the position of *soliciting an explanation from an interlocutor*, while Question2 refers to *soliciting an assessment of the speaker's idea*. The two positions are socio-mathematically different. Question1 involves minimal to no mathematical content and places the burden of stuffing the title Euclidian algorithm with mathematical content on the interlocutor's shoulders. On the contrary, Question2 involves mathematical thinking by attempting to contextualize the concept of relative primality with the numbers 261 and 84. The interlocutor for Question2 can merely reply with a "yes" or a "no," relying on his established mathematical authority within the group.

Operationalization. For the purpose of the VIP+function framework, I name the positions in the way that they would be described by any person who is familiar with the cultural activities in which they emerge. The construct of position in the VIP+function framework builds on the operationalization of Marcy Wood's construct of "micro-identity" (Wood, 2013).

The narratives of the focal participants, Ted and Bettie, did not involve sophisticated positions at extensive and long timescales. They were confined to *taking this course, dropping a course, dropping a major, leaving school, going back to school* and *joining a major program*. The positions pertaining to class assigned activities, mainly small-group work, required refinement and discernment. Small-group work aimed at enhancing mathematical learning afforded a variety of social and socio-mathematical positions. For the coding exercise of Bettie's, Melissa's and Ted's behaviors, I selected ten social and socio-mathematical positions (see P1 through P10 in Table 1-1). The coding exercise will be described in Chapter 2. Other important positions, which are not included in the coding process but were part of qualitative analysis, include the following: *calling instructor or TA to the group*, *writing on the shared dry-erase poster board*, *listening to a groupmate presenting an idea*, *looking at a groupmate's notes*, and *looking at textbooks or online resources*.

Desiderata. Evidently, positions change within activities at any given timescale. They "can be transgressed, reframed and reconstructed from moment to moment" (Wood, 2013, p.

781). Additionally, as stated in the definition of positions, they pertain to any engagement at any timescale level.

Table 1-1: The social and socio-mathematical functions pertaining to groupwork, covered by the coding in this project.

P1	Monitoring social and socio-mathematical interactions of groupwork
P2	Asking groupmate(s) to explain a mathematical idea
P3	Asking groupmate(s) to assess the speaker's own mathematical idea(s)
P4	Being approached for mathematical explanation by a groupmate
P5	Being approached for an idea assessment by a groupmate
P6	Contributing a mathematical idea to the activity
P7	Offering a mathematical explanation to the group or a groupmate
P8	Assessing a mathematical contribution to the groupwork
P9	Attending to a groupmate who is not specifically soliciting X's attention
P10	Being attended to by a groupmate whose attention X is not specifically soliciting

Function

Definition. In the VIP+function framework, a *function is a goal set to be achieved by enacting a position in a cultural activity*. Functions are concerned about the effect that a position can have beyond the duration of its animation. A behavior that animates a position for milliseconds can have intended or unintended effect(s) on a storyline that encompasses it over seconds and minutes (Davis & Harre, 1990) and/or the perpetrator's engagement at various timescales.

I illustrate the construct of function first with previous examples and then in rapport with storylines. By checking off the title of a course on the school registration software, a student animates the position of dropping the class. He may have decided to drop the class because he was annoyed by the instructor. As such, dropping the class comprises the function of avoiding interaction with the instructor during class activities. Moreover, a student may change his/her major to avoid doing activities, such as proving theorems, common to the classes that have to be attended for the given major. The same examples can be given for positions in class activities that are intended to boost students' engagement at the course and major levels.

Storylines have not been discussed in this dissertation so far. They are the stories that interlocutors attempt to live through their conversations. There are socio-mathematical storylines. The following ones are frequently observed in small-group work: (i) a student helping another to understand a mathematical concept, (ii) two students debating ideas, and (iii) groupmates building on each other's ideas to solve a problem. Recall Question2, "are 261 and 84 co-prime?" The position of *soliciting an assessment of the speaker's idea* may take on different functions in each storyline that has been mentioned previously. In the tutoring storyline, by uttering Question2, the explainer may intend to check the understanding of the interlocutor. In the debate storyline, the speaker may be leading the interlocutor to agree with his/her original position. In the building solutions storyline, the speaker may be suspecting a mistake present in the shared work. I will refer to the functions that pertain to storylines as ecological functions.

Functions can be intentional, ecological, and unintentional. For unintentional functions, think of a student who would join a study group to enhance his/her mathematical understanding. In the study group, a friendship develops between her and a groupmate. Joining a study group is, then, a position which carries the intentional function of learning and the inadvertent function of building friendship.

In mathematical groupwork, positions may be associated with various functions depending on the affordances of cultural activities and the participants' agencies. During groupwork, students may attempt to fulfill the following functions: *to build their own understanding, repair a groupmate's reasoning, enhance a groupmate's understanding, alleviate a groupmate's feeling, hurt a groupmates' feeling, support one groupmate against another,* and *reconcile opposing stances*, among many others.

Operationalization. Intentional functions of positions can be distilled by soliciting participants' perspectives on their interactions, through regular or SCNI interviews (see Chapter 2 and El Chidiac, 2017). The ecological functions of positions are studied through the conversations within which these positions are animated.

Desiderata. By definition, the construct of function meets desideratum 1. As for desideratum 2, the functions can change in connection with the same position or in case the position changes. Changes of functions may occur over seconds, minutes, hours, days, weeks, and even years (Schoenfeld, 2011; Saxe, 2012; Saxe et al., 2015).

Voice

Definition. People develop voices through *what they say and do* (Bakhtin, 1981). The power of a voice resides in its three operations. Kramsch (2003) identifies two these operations: every voice actuates an identity and animates a position. The VIP part of our framework builds on Kramsch's work. Referring to the work of Saxe (2012), Schoenfeld (2011), and Lemke (2002), I include a third operation: every voice attempts to achieve a function that may be oriented with different timescales. Hence, the notation VIP+function. An alternative notation would be V-IPF, since the framework analyses voices as actuated identities, animated positions, and set functions. However, the VIP+function is a memory-friendly notation.

To illustrate the constituents of voice, I provide two examples of the same utterance that deploys two different voices in two different situations. First, imagine a tutoring storyline. Chadi explains to a groupmate, Sara, the concept of relatively prime numbers. He goes through the decomposition of numbers into their prime factors then states that two numbers are relatively prime when they have no common prime factors other than 1. When he is done, Chadi asks Sara Question2: "are 261 and 84 co-prime?" Obviously, Chadi is actuating a tutoring identity, which involves the habit of checking on learners' understanding. Chadi's voice while uttering Question2 actuates a mathematics tutoring identity, animates the position of soliciting the assessment of a mathematical idea, and attempts to check Sara's understanding (see diagram in Figure 1-1).

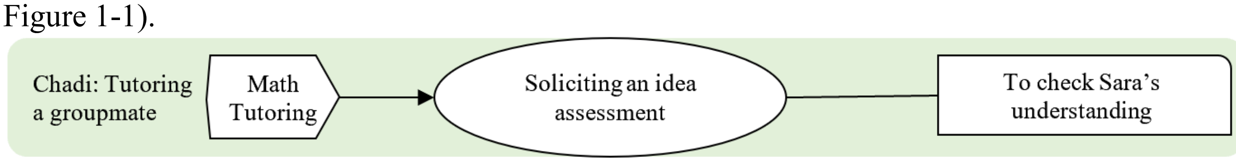

Figure 1-1: VIP+function representation of Chadi's voice in the tutoring storyline. In the VIP+function diagrams, pointed rectangles represent identities, ovals positions and rounded rectangles functions.

Now, imagine a storyline about cooperative groupwork. Fred, Gaby, and John work in the same group. Each one of them is solving an equation. They check one another's work now and then. Gaby writes,

$$84x \equiv 252\ (mod\ 261) \Rightarrow x \equiv 3\ (mod\ 261)$$

He simplifies both parts of the congruence by 84. Such an operation is accurate only if 84 and 261 are relatively prime. Fred notices Gaby's work and ask Question2 to the group. He can ask such a question because he is a member of the group where groupmates can check one another's work, and because he notices that the co-primality condition is not satisfied in Gaby's work. Through uttering Question2, Fred develops a voice that (i) actuates a dual identity, a group membership and a number theory identity, (ii) animates the position of soliciting an idea assessment, and (iii) attempts to question Gaby's work (see diagram in Figure 1-2).

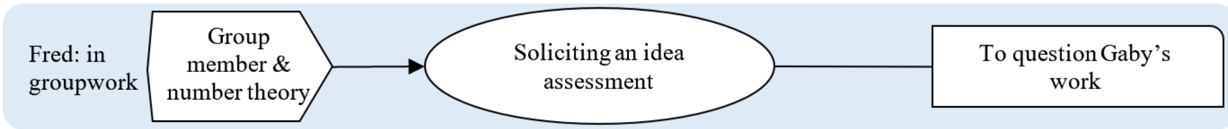

Figure 1-2: VIP+function representation of Fred's voice in the groupwork storyline.

Although Chadi and Fred utter the same question in two contexts, each one develops a different voice compared to the other. Both voices animate the same position but actuate different identities and set different functions.

Desiderata. As presented previously, since identities, positions, and functions—constituents of voices—are relevant to the three desired timescales and shift within each of them, voices shift likewise. Voices shift by changing only one, only two, or all the three constituents. This dissertation will study the development of students' voices within class activities, a number theory course, and major programs. The developmental trajectories of voices will be traced through the changes of actuated identities, animated positions, and set functions over time, as students engage in class activities, number theory class, and major programs.

The ways in which the VIP+function framework supports investigations of students' adaptation to new learning ecologies.

The overarching goal of this work is to illuminate processes by which students successfully adapt to new learning ecologies. The construct of a personal voice, which originates in Bakhtin's work (1981), is developed in Kramsch's work (2003) as identity and position, and is expanded in this work to include functionality, supports the investigation of students' adaptation to new learning ecologies.

In this work, the adaptation of students to a new learning ecology is conceived in terms of students developing personal voices. *The reconciling labor between individualized histories and present cultural and discursive powers exudes personal voices that attempt to shape the near or distant future*. A learning ecology bears cultural and discursive powers, by its design and its participants' collective behaviors, which materialize in positions and functions (Erickson & Mohatt, 1977). When a learning ecology is new to participants, the positions and functions that it promotes may conflict with, foster or change the participants' individual histories and aspirations.

The theoretical foundations of the VIP+function framework envision that the adaptation to new learning ecologies will take place in the actuation of identities, the animation of positions, and the setting of functions. The analytical aspect of the VIP+function framework allows one to track the shift of identities, positions, and functions through students' learning experiences over time. Students adapt to new learning ecologies when they do the following:

(1) *they actuate productive rather than conflicting identities with respect to the new ecology*; e.g. students start to learn proofs by going beyond their computational and

algorithmic understanding of mathematics and strive towards learning heuristics and strategies.

(2) *they animate positions that cohere with the new pedagogy*; e.g. when, within an active learning pedagogy, they start solving problems on their own instead of finding the answers in published resources.

(3) *they set functions that are conducive to learning*; e.g. participating in groupwork to enhance understanding rather than finish a project.

New learning ecologies promote positions and functions, which a student has to negotiate (Erickson, 2004; Esmonde, 2009; Langer-Osuna, 2016; McDermott & Raley, 2011; Nasir, 2002). Students may fully reject or adhere to the positions and functions of a given learning environment. They may also take up one and subvert the other. Students may take up the ecological positions but assign them personal functions. For example, students may participate in groupwork not to foster their learning but to copy their groupmates' work. Similarly, students may achieve the functions set by the pedagogical design while animating their personal positions, which may be discouraged by the design. For example, students may complete their homework by copying answers from the internet, a method that is banned in active learning environments.

Students may actuate different identities in different learning contexts, which is endorsed by situated learning theory (Hand & Gresalfi, 2015). Nonetheless, the actuated identities are shaped through an individual's history rather than the current context. Although cultural practices influence the actuation of individual identities, they do not govern the way in which these identities shape individual participations. Once actuated, the identities, as *imaginaire*, pursue their course, which is historically and narratively constructed. The debate between situated versus historicist theories will be discussed in the conclusion of this dissertation, in the light of the analyses conducted in this endeavor.

It should be noted that the analyses in this dissertation assume an ecological approach, although they focus on individual voices (as in Figure 1-1 and Figure 1-2). This approach overcomes the dichotomy of agency and socio-cultural determination (El Chidiac et al., 2018). In some cases, the evidence may indicate the predominance of personal history over socio-cultural resources, or the other way around. Nonetheless, it constantly assumes at least a minimal influence of personal and socio-cultural forces.

This dissertation focuses on what voices develop through students' engagement with new learning ecologies in classroom, how they are connected with each other, and how they shape and are shaped by students' engagement at all levels of collegiate life. When data affords, I will investigate how voices develop from the moments of students' engagement. The purpose of such investigations is to reveal the ecological features that bolster developmental steps.

Overview of chapters

The dissertation consists of one long manuscript. The current chapter sets the theoretical framework constructed for the entire project. The methods and methodology, described in Chapter 2, also remain common throughout the project. The three remaining chapters, from Chapter 3 through 5, contain reports of two case studies. In this section, I explain the ways by which the methodological choices of this dissertation project, namely the data collection techniques and the focal students, serve to illuminate the phenomenon under study. Then, I describe the particularities and the functions of the reporting chapters.

Chapter 2: Methods and methodology

Participants and site. The class selected for this study is a number theory course at a North-American university taught by professor Hoffmann (all proper names are pseudonyms), the winner of two national teaching awards. In this class, he decided to introduce small-group work for the first time in his teaching career. He believed that students would be less distracted by their cellphones, if they worked with peers. His pedagogy was inspired by the Moore method (Jones, 1977). Most of the students in his class were not used to small-group work, and none of them was used to proving important mathematical theorems in either group or solitary work, as required in Hoffmann's number theory class.

Selection of focal students. A study of how the learning process developed through groupwork in this class has been undertaken and prepared for publication in a separate project. For this dissertation, the analyses will focus on students who accomplished *significant developmental changes* in course of the studied class. Two students who attended this class reported significant learning developments:

- *Ted* decided to apply for a master's degree in advanced mathematics because of his positive experience in the number theory class.
- *Bettie* changed her learning methods from memorization to understanding through her experience in this class.

Methodology and data collection. The methodology and data collection techniques of this dissertation are strongly influenced by ethnographic (Erickson, 1992) and psychoanalytical (Copjec, 2015) methods. The study of voices, identities, positions, and functions is grounded in observed behaviors and participants' perspectives. During social interactions, students may develop voices in undertones that may not be recognizable through their behaviors in groupwork (Erickson, 2004). Moreover, students' goals, aka functions, of animating positions may not transpire through group interactions. For this reason, eliciting students' perspectives on their respective behaviors in groupwork is necessary for conducting a VIP+function analysis.

Group sessions in the studied number theory class were videorecorded throughout the semester, with the exception of the first two weeks. These videos afforded the investigation of identities, positions, and functions that were voiced through groupwork. To elicit the intended functions of students' behaviors, I conducted two types of individual interviews at different moments in the course. The SCNI (stimulated construction of narratives about interactions) interviews consisted of students commenting on a video of their recent group session. Semi-structured interviews were conducted in the first month and the last two weeks of the semester.

Interviews were also needed to elicit identities that *students* connected to their learning experiences. In this regard, I followed the basic psychoanalytical methodology: actual histories of individuals matter less than how individuals narrate their past experiences in the present (Copjec, 2015). Psychoanalytical methods study *spontaneous* narratives on selves and social interactions. I allowed room for the students' spontaneous narratives during the interviews. First, the protocols of interviews involved open-ended questions which tapped into students' psychologies and histories. Second, as I conducted the interviews, I allowed a large space for participants to guide the conversations.

Chapter 2 will describe further the data collection techniques. Full transcripts of all interviews with Bettie and Ted are provided in appendices A and B, respectively.

Strategy for data selection. In analyzing of data, I faced the problem of data management. The transcripts of Ted's and Bettie's interviews are substantial (42 and 26 single-line pages, respectively) and the 16-classroom observations of Ted's and Bettie's group span over 42 hours of videos. I followed a *retrospective systematic method* in the selection of relevant pieces of data sources for the analyses.

In selecting data sources for analyses, I let participants' perspectives determine the leads. I start first with the exit interviews to identify the self-reported significant changes in students' adaptation to a new pedagogy and subdiscipline. Then, I determine which data source—videos of group sessions, or students' narratives—can afford and is most pertinent to study the reported change. For the video analyses, I select all episodes from the 16 groupwork observations and conduct a micro analysis using the VIP+function framework. For the analyses of interviews, I detect all topics that students connect to their reported significant changes. Then, I distill the identities, positions, and functions from participants' narratives that contain the mention of the detected relevant topics.

Ted and Bettie reported three significant changes resulting from their experiences of the number theory class. A chapter will be dedicated to the analysis of each reported significant change.

- Ted reported that he decided to postpone the pursuit of his teaching credentials to pursue a master's degree in advanced mathematics. (see Chapter 3)
- Bettie reported she succeeded, by the end of the course, in affirming her pre-proof mathematical ability to her groupmates, who used to doubt her mathematical abilities. (see Chapter 4)
- Bettie reported she changed her learning methods from memorization to understanding. (see Chapter 5)

Chapters 3, 4, and 5 contain detailed reports of the data selection strategy. Their primary function is to carefully report the methodological steps and the genesis of findings from the analytical work at each step. For this reason, this write-up choice may hinder a smooth reading of the chapters for readers who are interested in the findings. Nevertheless, I have grabbed this opportunity that only a dissertation may offer, as opposed to journal papers. Journals tend to reject long reports which describe the application of methodologies to cases. Dissertations at U.C. Berkeley do not limit page numbers and become accessible to public. Readers interested in the findings will have to look for journal publications of this work. However, those who are interested in the methodology will find in these chapters all data and methodological descriptions I have used to reconstruct my arguments and findings or discover alternative ones.

Chapter 3: Ted Tao: Creativity, games, and tutoring young people

The third chapter serves two functions: while it reports the case study of Ted Tao, it illustrates the retrospective systematic data selection strategy, discussed in the overview of Chapter 2. In this chapter, the VIP+function framework is applied to narratives.

Ted changed his prospective career due to the confidence he gained from his experience in the number theory class that he attended. He had been preparing himself to become a high-school mathematics teacher and took up the number theory class because it was required for his mathematics major for teaching. After the number theory class, he dropped the notion of a high-school teaching career and applied to graduate school for advanced mathematical studies. By the end of his master's program, he won a pre-doctoral fellowship in mathematics. Ted's experience in the number theory class was instrumental in boosting his confidence to apply to graduate school. He recognized that the specific *pedagogy* of the number theory class, mainly requiring him to prove theorems within groupwork, bolstered his confidence. He reported that prior to number theory class, he used to read but had never proved theorems by himself. He also acknowledged that two of his groupmates helped him refine his mathematical thinking. Ted was used to small-group learning before he joined the number theory class. However, unlike in the number theory class, his past experiences with groupwork had been mostly discouraging.

The VIP+function analysis of Ted's narratives, reported in Chapter 3, highlights three processes that significantly contributed to his successful adaptation to the relatively new pedagogy of the number theory class:

- Ted used his *teacher identity*—tutoring young people at a community center—as a resource to animate favorable *positions* during his groupwork in the number theory class.
- Because the group design did not request consequential collective tasks, Ted was enabled to pursue *learning functions* rather than work for grades.
- Ted drew on his experience with puzzle games (*playing games identity*) to shape his mathematical thinking and sustain his engagement during hours of struggle with challenging proof problems.

Chapter 4: Bettie reaffirming her mathematical ability

The fourth chapter also serves two functions: while it studies Bettie's reaffirmation of her arithmetic (PPM) ability, it also illustrates the application of the VIP+function framework on students' voices in the moment-by-moment of groupwork over multiple group sessions.

At the start of the number theory class, Bettie was confident about her pre-proof mathematical abilities. However, her confidence in her pre-proof mathematical skills suffered because her groupmates, who constantly doubted her mathematical contributions during groupwork. However, by the end of the class, Bettie succeeded in restoring her belief in her own pre-proof mathematical ability and somehow repairing her groupmates' perceptions of the same.

The analysis reported in the fourth chapter studies the trajectory on which Bettie restored her confidence, through the VIP+function framework. It focuses on her voices during computational activities in groupwork throughout the number theory class. The observed positions Bettie animated in the first two computational activities exhibited her lack of confidence and her struggle with mathematical identities. However, in the third and fourth computational activities Bettie started to exhibit confident positions and a nascent number theory identity. The analysis highlights two following factors that boosted this change of her positions and identities:

- Bettie identified with a confident groupmate, Ted, in the third computational activity.
- The competence required for the fourth computational activity fell within Bettie's prior knowledge of arithmetic (PPM) and number theory.

Chapter 5: Bettie's individualization of new learning methods

The fifth chapter investigates Bettie's formation of a new learning method. This formation took place over time, within and across the following three learning activities: solitary study, groupwork in classroom, and study group sessions outside classroom. The report in the fifth chapter illustrates the application of the VIP+function framework across different cultural activities.

Historically, Bettie's learning method consisted of solitary study that entailed memorization of materials for tests. This learning method failed to be productive in the number theory and the proof classes. At one point, Bettie was at a risk of almost dropping the class because of her low scores on the weekly homework. She did not drop out because she believed that she was good only at "math and sucks at everything else." Bettie's confidence in her career was at stake. By the end of the semester, Bettie instilled new learning methods to enhance her mathematical understanding. She also changed her perspective on small-group learning process: she thought it was the "stupidest thing ever" early in the semester, but realized that it "was definitely more beneficial than sitting in a lecture" by the end of the course.

The study of Bettie's voices that developed over the semester revealed a complex net of shifts of positions and functions, each one of which crucially contributed to the formation of new and productive learning methods. What further supported this desired outcome was the transfer of productive positions and functions that developed within one type of learning activity to another. Her personal efforts and the affordances allowed by the learning environment supported Bettie's development of productive learning methods. Notably, Bettie's active learning habits stemmed from her passive learning habits which interacted with a favorable learning ecology. Her development of new learning methods followed the emerging rather than the substituting model. New habits became rooted in old habits.

Contributions

From a research standpoint, this dissertation attempts to contribute to the field of mathematical education in three following ways:

- It introduces *Lacan's theory* to mathematical education research on identity formation and development.
- It introduces and leverages a data collection technique, *Stimulated Construction of Narratives about Interactions (SCNI)*, which recycles the stimulated recall technique to aid socio-cultural, socio-linguistic, and psychoanalytical investigations.
- It introduces and leverages the *VIP+function framework*, which has been devised to investigate the development of students' voices within (new) learning ecologies. The framework is applied to *videos* of students' groupwork over a semester-long course and to student's *narratives* about their selves and their behaviors in groupwork, extracted at multiple moments through the semester.

As for the findings, this dissertation sheds new lights on the phenomenon of students' adaptation to new learning ecologies in various ways. I highlight three major findings here:

- Mathematical identities can be fortified by entrenching mathematical practices within social and academic individualized identities (Chapter 3).
- An active learning identity can be formed within a semestrial course. It stems from current learning habits and develops through shifts of positions and functions (Chapter 5).
- Peers can aid one another, not only in building mathematical knowledge but also in freeing each other from the grip of detrimental individualized identities (Chapter 4 and Chapter 5).

Chapter 2: Methods and Methodology

In this chapter, I describe the methods and methodologies of this project. This chapter constitutes a second pillar of the dissertation along with the first chapter. The methodological options explained in this chapter shape the remaining chapters in terms of analysis and write-up. This research project incorporates a repurposed data collection technique, a so-called stimulated construction of narratives about interactions (SCNI). A separate section is dedicated to ground the technique theoretically and position it such that it becomes contributive to the endeavor of a research on identity within mathematics education. Furthermore, I delineate the retrospective systematic methodology that I employed to select the relevant data for analysis.

Participants and pedagogical design

Participants

Information about the composition of the class is provided by age, gender and major in Table 2-1. Information about individual students is provided in Table 2-2.

The class under study was on the subject of elementary number theory offered at the mathematics department of a university that was known for admitting students generously. The majority of enrolled students in the studied class (13 out of 16) were, at that time, majoring in mathematics with a concentration on teaching. The course was a requirement for students who had enrolled in the "mathematics for teaching" program. Most students (16 out of 23) were in their senior year of college. A student was enrolled in a Mathematics graduate program and another one was enrolled in an open program. The age of students varied significantly (most of the students were between 19 and 30 years old). In the interviews, some students mentioned that they had spent few years working after they graduated from high-school, or before going to college. The class had slightly more men than women (10 women and 13 men). The students were ethnically and linguistically diverse. They self-identified as Hispanic, Caucasian, Asian, Pacific-Islander, and also had mixed ethnicities. No participant self-identified as African-American.

Table 2-1: Distribution of students over age, gender and major (N=23 students).

Age range	# students
19-22	7
23-25	4
26-29	6
33	1
64	1
Not reported	4

Gender	# students
Female	10
Male	13
Transgender	0
Other	0

Major	# students
Mathematics for Teaching	13
Mathematics for Advanced Studies	3
Mathematics for Liberal Arts	3
Applied Mathematics	1
Computer Engineering	1
Master's in Mathematics	1
Open University	1

Table 2-2: Information on students by their small-groups. Ted and Bettie, focal students, are part of G3. Majors: T (mathematics for teaching); AS (mathematics for advanced studies); LA (mathematics for liberal arts); CE (computer engineering); AM (applied mathematics); MA (master's in mathematics); OU (open university).

Group	Student's Name	Major	Student's Major	Age	Gender	Ethnicity
G1	Gaia	T	Senior	25	Female	Pacific-Islander
	Izabelle	T	Senior	26	Female	Mexican/Hispanic
	Leila	T	Senior	28	Female	Mixed: Spanish, Native American, Irish, Yugoslavian
	Nawal	T	Senior	24	Female	Hispanic
G2	Emil	LA	Senior	26	Male	-
	Melissa	T	Senior	22	Female	Caucasian
	Randi	T	Junior	29	Male	Caucasian
	Tito	T	Senior	22	Male	Vietnamese & Filipino
	Tom	AS	Senior	27	Male	White as the driven snow
G3	**Bettie**	**LA**	**Junior**	**22**	**Female**	**-**
	Boutros	T	Senior	-	Male	Mexican-American
	Jeremy	AS	Junior	20	Male	Half Chinese, a quarter Mexican, and a quarter white
	John	AM	Senior	-	Male	-
	Ted	**T**	**Senior**	**27**	**Male**	**Born in Hong Kong, I come from a Chinese ethnic background, but culturally my city has been heavily influenced by the British. Moved to the U.S. when I was 11 years old.**
G4	Alan	AS	Junior	19	Male	Mexican-American
	Howard	MA	Graduate	64	Male	Anglo-Saxon
	Jack	T	Senior	22	Male	Asian-Chinese
	Karl	OU	Freshman	-	Male	Asian
G5	Charbel	CE	Senior	33	Male	White
	Judy	T	Senior	23	Female	White, but also part Hispanic
	Laura	T	Senior	22	Female	Mexican
	Mona	T	Senior	23	Female	Full Hispanic
	Sara	LA	Junior	-	Female	-

Instructor. The instructor of the studied course, professor Martin Hoffmann (all identifiers are pseudonyms), won two national awards for his distinguished teaching. He had been using an interactional lecturing pedagogy prior to the studied class. At the university, the number theory course was assigned to be taught by different faculty every year. Hoffmann had taught the number theory class two years and seven years prior to the studied one. The previous time he taught number theory, he had a negative experience. As per his testimony, his students were disengaged, test-driven and used to text with their cellphones most of the time they spent in class. He was determined to make a radical pedagogical change in the current class. Compelled that small-group work would reduce cellphone-related distractions, he implemented small-group work throughout the entire course (the studied one). Although he had never used active learning methods in his college classes prior to the studied one, he had used some groupwork during his volunteering work at an educational community program.

Additionally, Hoffmann had worked with a graduate student of Robert Lee Moore, who introduced the so-called Moore method (Jones, 1977). The design of the worksheets for the studied class were inspired by the Moore method only regarding one aspect: asking students to reproduce proofs of major theorems while working either together in class or alone outside class. Hoffmann did not instill a competitive atmosphere, which was a feature of the original Moore method.

Nathaly, a graduate student in the Mathematics education program at the same university, assisted the instructor. She graded the homework and attended to the small groups during class sessions. The instructor graded the tests.

Focal students. Ted and Bettie were the only students who reported significant changes in their learning methods and their prospective careers over the course of the studied class. Thus, they were selected for focal studies (see Chapter 1 for further information on the relevance of Ted's and Bettie's cases to the current investigation).

Ted, a 27-year-old man, was a senior student majoring in mathematics for teaching at the time of this study. About his ethnic background, he wrote the following: "born in Hong Kong, I come from a Chinese ethnic background, but culturally my city has been heavily influenced by the British, moved to the U.S. when I was eleven years old." Ted was considered a "gifted student" when was admitted to Texas Academy of Mathematics and Sciences at the age of sixteen. He self-identified in the early survey as a "math nerd in love with teaching through proofs instead of arithmetic." Ted decided to participate in the research five weeks into the semester. His first interview took place on 10/01.

Bettie, a twenty-two-year-old woman, was a junior student majoring in mathematics for liberal arts at the time of the study. She had attended a community college before transferring to the current university. The number theory course was not required for her major; she chose it as elective because she was told that the instructor "speaks English." Bettie complained about the instructors' accents and attitudes at the current university. She confessed that she "hated all her teachers at this school."

Ted and Bettie were part of the same group, G3, along with John, Jeremy, and Boutros. A visiting student joined G3 for the first two weeks of the class, when Boutros was with another group. Boutros joined G3 after the visiting student left. Since then, the composition of G3 remained the same throughout the semester. The group members did not choose each other. They just happened to sit at the same table and stuck together, like most of the students in the class. With the exception of John, G3's members started to meet in the library to study number theory about a month into the semester. They used to complete the homework that they could not finish during classroom groupwork and supported each other's learning progress.

As for other participants, in the exit interviews they reported minimal or no changes in their learning methods. Melissa and Izabel reported that they had fostered relative confidence on learning that other students struggled as much as they did. Tito, Jack, and Boutros reported that they were becoming more comfortable asking questions to groupmates, sharing their ideas with the group, and writing on the shared dry-erase poster board. Tito also noted that he had learned to come prepared to class. Tom reported that he learned to loosen up while monitoring groupwork. Jeremy reported that he was realizing the worthiness of learning mathematics with other people, for previously he used to work alone. Alex reported that he had participated less in groupwork because his groupmate, Jack, needed the group attention to improve his grade. Howard, Nawal, Leila, and Randi reported no change in their learning methods. Ted reported that the study group helped him reinforce his understanding of mathematical concepts and procedures as he explained them to other students. Future work will include an analysis of all learning developments, minimal or otherwise, in this class.

The design of the course

Curriculum.

The instructor broke the curriculum into eleven topics, creating a worksheet for each number theory topic (full worksheets can be found in appendix E). The topics and worksheets were given to students in the following chronological order.

Worksheet 1: Euclidean Algorithm
Worksheet 2: Prime Numbers (Part I)
Worksheet 3: Modular Arithmetic
Worksheet 4: The Chinese Remainder Theorem
Worksheet 5: Cryptography
Worksheet 6: Arithmetic Functions
Worksheet 7: The Möbius Function
Worksheet 8: Primitive Roots
Worksheet 9: Primitive Numbers (Part II)
Worksheet 10: Quadratic Residues
Worksheet 11: Continued Fractions

The class spent between one and two weeks on each topic, working on selected problems from the respective worksheets. Some problems in the worksheets were designed to guide students in proving established theorems. The other problems comprised the application of these theorems to numerical problems.

The instructor encouraged students to work with two textbooks: Stein (2008) and Andrews (1971). The textbooks are available in digital versions on the internet, free of cost.

Requirements.

To complete the course, students had to participate in the class sessions (attendance was tracked and counted for 10% of overall grade), submit a weekly homework (50% of the grade), take a mid-term exam (20% of the grade) and final exams (20% of the grade). The mid-term exam took place on 10/22, and the final exam on 12/15.

The **homework** consisted of selected problems from the worksheets, which students tackled in classroom. Students could help one another in constructing the proofs and answers to their homework. However, students were asked to submit individual answers, which they were expected to write "using their own words." Over the semester, they were asked to submit a total of 13 homework.

Grading system for number theory class:

Participation	10%
Homework	50%
Midterm Exam	20%
Final Exam	20%

Classroom sessions.

The class met twice a week for 1h 15m per session over the Fall semester. Classes took place in the late afternoons. Over the semester, the class met 31 times. Two sessions were dedicated for the midterm (10/22) and the final (12/15) exams. The sessions preceding the tests, on 10/20 and 12/10, were reserved for spontaneous questions which students could ask the instructor regarding the assigned materials for the exam. The remaining 27 sessions were regular class sessions.

The regular class sessions consisted mainly of small-group work. The instructor used interactional lectures only to introduce new definitions, which ran for between 10 to 20 minutes each time. Otherwise, the major part of the learning process took place in small-groups, which the instructor and his assistant visited to check on the students' progress. At the beginning of the semester, the instructor felt the need to spend a long time with each group (sometimes up to 20 minutes). Over time, the groups learned how to rely on their internal resources.

At the beginning of each session, the instructor determined the problems that the students were expected to tackle in that day. The problems to be tackled in classroom were selected from the worksheets and were also often assigned as homework. Basically, students spent their classroom groupwork preparing their homework together. They rarely finished their assigned homework in class, having to eventually complete it on their own. Some students met outside classroom to support each other in completing their homework.

Small-group design.

In the first class of the semester, the instructor explained the functioning and the norms of the course. He asked students to compose small-groups of four to five students. He explained that they were supposed to solve problems that were assigned in worksheets while working in groups throughout the class sessions. They were encouraged to stay with the same group as long as the class worked on the same worksheet. However, after the third week, the group compositions remained the same for the rest of the semester. The class contained five small-groups: two groups of four members and three groups of five members (see group compositions in

The class under study was on the subject of elementary number theory offered at the mathematics department of a university that was known for admitting students generously. The majority of enrolled students in the studied class (13 out of 16) were, at that time, majoring in mathematics with a concentration on teaching. The course was a requirement for students who had enrolled in the "mathematics for teaching" program. Most students (16 out of 23) were in their senior year of college. A student was enrolled in a Mathematics graduate program and another one was enrolled in an open program. The age of students varied significantly (most of the students were between 19 and 30 years old). In the interviews, some students mentioned that they had spent few years working after they graduated from high-school, or before going to college. The class had slightly more men than women (10 women and 13 men). The students were ethnically and linguistically diverse. They self-identified as Hispanic, Caucasian, Asian, Pacific-Islander, and also had mixed ethnicities. No participant self-identified as African-American.

Table 2-1: Distribution of students over age, gender and major (N=23 students).

Age range	# students
19-22	7
23-25	4
26-29	6
33	1
64	1
Not reported	4

Gender	# students
Female	10
Male	13
Transgender	0
Other	0

Major	# students
Mathematics for Teaching	13
Mathematics for Advanced Studies	3
Mathematics for Liberal Arts	3
Applied Mathematics	1
Computer Engineering	1
Master's in Mathematics	1
Open University	1

Table 2-2).

The instructor encouraged students to solve the problems by counting on their own and their groupmates' knowledge. He discouraged them from looking for answers in published resources. Students reminded each other of Hoffmann's motto: "**use your brain, not the book.**" Some students downloaded the suggested textbooks on their tablets, which they brought to classroom groupwork. Nonetheless, most students respected Hoffmann's norm about the use of published resources. They occasionally took recourse to published resources, mainly textbooks, during classroom groupwork. Mostly, such moments arose when the students became stuck. *Bettie was the only student in the class who used to keep the textbooks open in front of her and refer to them during classroom groupwork.* Hoffmann noticed this but, fortunately, did not reinforce the norm.

Groups were encouraged to establish the respective group cultures that they wanted. Hoffmann did not intervene in monitoring the social dynamics of groups. He let students negotiate their group ecologies as they saw fit. The groups established their own pace of working on the problems. The faster groups covered more materials in classroom than the slower ones. Hoffman support the pace of each group. His goal was **to foster students' learning** rather than cover curriculum materials.

Interventions and classroom setup.

The instructor, Hoffmann, invited the researcher to share thoughts about improving the pedagogical design. I shared two suggestions throughout the course: one about the classroom setup, and the other about introducing dry-erase poster boards. The class was assigned a classroom that contained individual tables with folding tops. The groups had to pull the tables together and could not create a seamless shared surface to work on it. The instructor succeeded in shifting the assigned classroom to one that contained hexagonal tables, and the class moved to the new classroom by the second week of the semester.

After the midterm, the researcher suggested to provide each group with a dry-erase poster board, and some dry-erase markers and erasers. The intention was to foster shareability of individual work and enhance communication. Students could support their mathematical thoughts using visualizations, which they could draw and erase at will. Hoffmann introduced dry-erase poster boards on the session that was held after the midterm (on 10/27).

Data Collection

For the complexity of the phenomena under study, I conducted an intensive and multi-layered set of data collection, which comprised the following: surveys, videos of group sessions in classroom, unmediated individual interviews, mediated individual interviews (SCNI, to be described next), brief reports from students on their experience of the group sessions (memos), fieldnotes from classroom observations and informal encounters with participants, students' submitted homework and exams, and students' grades. The author conducted all interviews and collected the described data. Table 2-3 lists and describes the data sources of this project and mentions the dates of their administration. The surveys and interview protocols can be found in appendix D. The next section will elaborate on the SCNI interviews because it repurposes an existing one, a stimulated recall.

The early and the exit interviews (semi-structured types) were devised to engage students in narrating their histories with regard to mathematical learning in schools, mathematics outside school settings, and small-group learning. I also set a brainstorming question at the outset of each interview: I asked participants to speak aloud whatever associations with words, images and stories, their brains made when they heard a word I uttered. In the early interview, I used the word *mathematics,* while in the exit interview the words *number theory* and *Math 310* (the code of the number theory course in the school). I asked participants to elaborate on the associations they voiced. As they did, many of them shared experiences that were emotionally charged. While conducting these interviews, I paid attention to moments when participants expressed emotional stances and probed for further elaborations. I did not stop them when they wandered along with their speeches. I attempted to create a space where they could feel comfortable expressing their thoughts and themselves. Early interviews were audio-recorded, while exit interviews were video-recorded. Participants knew that the records of interviews would not be shared with the instructor, other students, faculty, or school administration.

Table 2-3: List of data sources collected throughout the number theory course, including a brief description of collecting methods and the date of administration.

Data Source	Description	Dates of administration
Surveys		
Early Survey	Proposed to all students enrolled in the class	09/05
Exit Survey	Proposed to all students enrolled in the class	12/18
Videos		
Videos of classroom	Camera following professor (only during the 2nd and 3rd weeks of the course)	09/01-03-08-09
Videos of 2 groups	Videotaping two groups, selected randomly, with stable cameras oriented over the entire group and microphones at the center of the group tables (only during the 2nd and 3rd weeks of the course)	09/01-03-08-09
Videos of group sessions	Videotaping every group session with stable cameras oriented over the entire group and microphones at the center of the group tables (administered to the four focus groups only)	09/15–12/10
Interviews		
Early Interviews	Individual interviews on students' social worlds, their prior experiences with math classrooms and small-group work, and their perceptions of their group mates (administered on focus groups only)	09/15-22*
SCNIs	Individual interviews with focus group members commenting on recently videotaped group sessions**	09/24–12/1
Exit Interviews	Individual interviews on students' overall experience with their groupwork and their perceptions of their own and their groupmates' mathematical identities (administered on focus groups only)	12/03-08-10
1st Interview with Instructor	A video-recorded interview with the instructor, Prof Hoffmann, on what led him to use groupwork, his design of the course, and his perception of students.	10/22
2d Interview with Instructor	Instructor watching and commenting on selected videos of his interaction with groups	11/13
Notes		
Memos	Prompted brief report on groupwork to be written and submitted immediately after each group session (administered on focus groups only)	09/15–12/10
Field notes	From observed classes and informal encounters between instructor and students	08/25–12/10
Students' work and grades		
Homework	Copies of students' responses and grades of the weekly homework assigned to the studied class	weekly
Mid-term Exam	Copies of students' responses and grades of the mid-term exam taken for the studied class	10/22
Final Exam	Copies of students' responses and grades of the final exam taken for the studied class	12/15

* Ted and Kim decided to participate in the research at a later date. Ted's early interview took place on 10/01, and Kim's on 10/06.

** The first SCNI interviews for G3 were cancelled, because of events that took place on campus on the scheduled date of the interviews.

Participation in the research

Students participated differentially in the research (see Table 2-4). They formed a total of five groups of four or five students each. Two students of group G5 did not consent to being videotaped. Hence, G5 was not selected for the focus group study. Only members of focal groups submitted memos after their group sessions. As for interviews (administered only to focal groups), there was one student of each focal group who did not participate in them. Ted (in G3) and Karl (in G4) decided to participate in the research later in the semester.

The SCNI sessions were administered per group but taken individually. The group members participated in SCNI sessions on the same day as the group sessions, one after the other; every other week, group members took the SCNI sessions within 24 hours of the group session. The SCNI sessions lasted between 45 to 90 minutes each, depending on students' elaboration of their comments and their willingness (or lack thereof) to watch the entire video of groupwork. Some students, who consented to participating in the interviews, skipped some SCNI sessions (as reported in Table 2-4) because they forgot to come back for the same after class or had other commitments.

Students who participated in interviews and/or wrote memos were given the choice of receiving one of two types of compensation for the same: monetary compensation or mathematics tutoring. Only Bettie and Boutros picked the tutoring compensation, although Bettie never asked for it. The remaining students chose the monetary compensation except Howard, who did not take any compensation. I was determined to compensate the participants, not only because the interviews required many hours of their time, but also to draw students who had not developed a sense of freely giving their time and effort. A research on identity and collaborative work must draw participants having diverse motivations. Research that calls upon students' generosity may, by design, exclude those who are only motivated by self-interest.

Table 2-4: Actual participation of students and groups in the multiple activities involved in the research.

Group	Student	Group (video)	Early survey	Int1	SCNI 1 9/22-10/2	SCNI 2 10/6-10/16	SCNI 3 10/20-10/30	SCNI 4 11/3-11/20	Int2	Exit survey
G1	Gaia	X	x							x
	Izabelle		x	x	x	x	x	x	x	x
	Leila		x	x	x	x	x	x	x	x
	Nawal		x	x	x	x	x	x	x	x
G2	Emil	X	x							x
	Melissa		x	x	x		x		x	x
	Randi		x	x	x	x	x	x	x	x
	Tito		x	x	x		x	x	x	x
	Tom		x	x	x	x	x	x	x	x
G3	**Bettie**	**X**		**x**		**x**			**x**	**x**
	Boutros		x	x		x		x	x	x
	Jeremy		x	x		x	x	x	x	x
	Ted		**x**	**x**			**x**	**x**	**x**	**x**
	John									
G4	Alan	X	x	x	x	x	x	x	x	x
	Howard		x	x	x	x	x	x	x	x
	Jack		x	x	x	x	x	x	x	x
	Karl		x	x						x
G5	Charbel		x							x
	Judy		x							x

Laura		x	x
Mona		x	
Sara			

The SCNI Technique

The field of mathematics education is divided between two methodological approaches taken for investigating identity development in classroom settings. On one camp stand the endorsers of “situated identities,” who study positionings that play out within ongoing interactions (Hand & Gresalfi, 2015). On the other camp stand the upholders of “narrative identities,” who study stories that people tell about themselves and others (Sfard & Prusak, 2005). The situated and narrative theories are theoretically and empirically well grounded. However, their methodologies operate autonomously: situated theory focuses on cultural practices (Wenger, 1998), while narrative theory focuses on discourse (Sfard, 2008). I believe the field will be enhanced if the two methodologies can be reconciled without subsuming one approach under the other. I propose a data collection technique to bridge the two theoretical pillars of identity research with respect to mathematics education.

In this section, I explain how a data collection technique can bridge between the situated and narrative approaches. I will first illustrate the methodological problem with the well-known case of Mrs. Oublier. Then I will reframe the problem through situated and narrative theories leading to a solution: a data collection technique based on *stimulated constructions of narratives about interactions* (SCNI). Third, I describe the conduction of an SCNI session as used for the data collection of this dissertation. I conclude by highlighting the differences between the SCNI sessions and the commonly known SR (stimulated recall) interviews.

Defined concisely, *an SCNI session consists of probing participants to construct narratives about (or comment on) their social interactions by watching a video of the activity in which they had recently participated (within 24 hours).* The medium can be video-records of participants’ behaviors, audio-records of participants’ speeches, or artifacts produced during an activity.

The case of Mrs. Oublier

Mrs. Oublier (Cohen, 1990) was a mathematics teacher who epitomized the case of teachers undergoing a transition from traditional to reformed teaching. In her interviews with a researcher, Mrs. Oublier couched her teaching style on a reformed narrative. However, her observed interactions with her students in classroom were entrenched in traditional teaching practices.

The case of Mrs. Oublier epitomizes the autonomy of two realms: pragmatics and narratives. She seems to be actuating different identities, one during her teaching practice (at a time period Δt_1) and another while narrating about herself and others in her conversations (at time period Δt_2). Cases such as Mrs. Oublier’s emerge not from schizophrenia but the divorce between her pragmatic and narrative resources (see

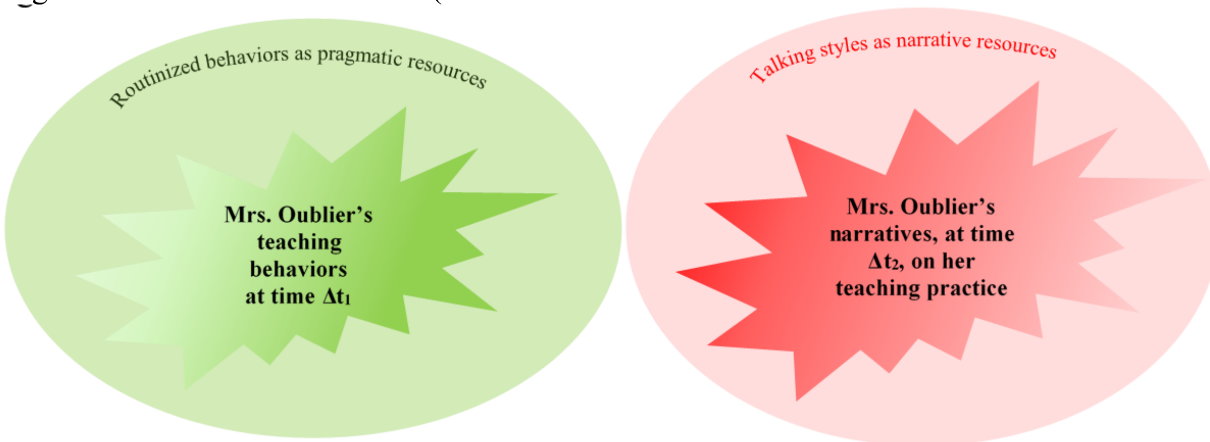

Figure 2-1).

Regarding their narrative identities, participants can negotiate subject positions of *past and future* experiences by couching them in discourses and styles of speaking that are pre-established in certain communities (Baynham, 2014). On the other hand, positions that participants animate in the *here and now* of a practice draw upon routinized behaviors (Lave & Wenger, 1991; Wenger, 1998). When individuals actuate different semiotic repertoires and entrenched habits, in a manner similar to Mrs. Oublier during the self-transition of her teaching style, they animate different voices. Mrs. Oublier internalized the semiotic repertoire of reformed teaching, whereas her teaching habits were still entrenched in the pragmatics of traditional teaching.

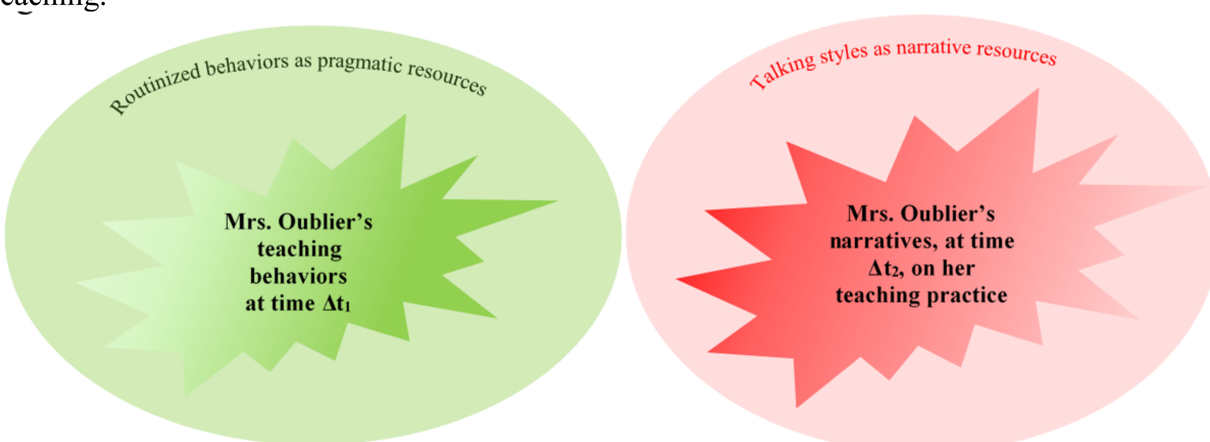

Figure 2-1: Representation of Mrs. Oubliers' voices with regard to teaching practice and conversations about self.

Theoretically reframing the problem

The divorce between Mrs. Oublier's practice and narratives can be induced by the different contexts within which they emerged: classroom activities and interviews. Both situated and narrative theories conceive of social interactions in teaching and interviews as two radically different activities, and rightly so. From a situated theory perspective, as per Wenger's community of practice theory (1998), both activities are part of different communities and practices because they involve different tasks and their respective participants negotiate different norms. From a narrative perspective, in accordance with Sfard's commognitive theory (2008), conversational activities vary depending on interlocutors and the presence or absence of people who are talked about.

SCNI sessions create a hybrid space that bridges the realm of practice and the narratives about a practice. I will show how this bridging is possible within situated and narrative perspectives, respectively. Situated theory acknowledges that communities of practice can be connected through the reification of a practice. By reification, Wenger refers to the solidification of a perspective on the practice, such as a constitution and a charter of conduct. Videos of participants engaged in their learning activities are reifications of their learning practices. They eternalize one perspective on the practice, namely the camera's perspective. When participants, in SCNI sessions, comment on the videos, they are actually negotiating (making sense of) a reification of their learning practice. As such, they prolong their engagement with the learning practice by negotiating a reification thereof. For Wenger, there are two processes by which people engage with a practice: participation and reification. This is how SCNI sessions are seen as a hybrid activity connecting two practices: interviewees participate in an interview practice by negotiating a reification of the learning practice.[7] By negotiating reifications in SCNI sessions, participants have the opportunity to elicit a personal voice, which may remain implicit or hidden in observed ongoing interactions.

The investigation of identities through narrative approaches (Heyd-Metzuyanim & Sfard, 2012; Sfard & Prusak, 2005; Wood, 2013) presents facing a dichotomy. On one hand, interview settings allow participants to elicit and elaborate on enduring identities, which they narratively connect with multiple activities over long periods of time (Sfard & Prusak, 2005). On the other hand, narratives about self and others in ongoing learning-oriented activities can elicit fleeting identities that participants can change and transgress even within the same conversational activity (Heyd-Metzuyanim & Sfard, 2012; Wood, 2013). Researchers may establish connections between enduring and fleeting identities, which highlight how narrative identities produced in interviews are connected to contextual identities constructed through participation in ongoing learning activities. However, these connections are regarded as unsupported by evidence with respect to the narrative methodology. For a connection between enduring and fleeting identities to count as being supported by evidence within the narrative methodology, participants must utter a speech that conveys the connection. However, such speeches tend to be rare in unmediated interviews and ongoing learning activities. The SCNI data collection technique offers participants an opportunity to provide the type of evidence needed for narrative theories to develop their investigations of identity. In SCNI sessions, participants narrate their behaviors in the videotaped activity; hence, the name of *stimulated construction of narratives about interactions*. By doing so, they provide narratives about fleeting identities, which may cohere with enduring identities narrated in unmediated interviews or provide new enduring identities.

Whether researching taking a situated or a narrative approach, SCNI sessions open up a new hybrid space (Figure 2-2) that can provide expedients to the development of identity

[7] One may argue that SCNI sessions are interventionist. Such a claim holds true only if the negotiations of reifications influence processes of participation. Unfortunately, I could not detect such moments; otherwise, I would have proposed that the SCNI sessions are part of educational activities that support learning. The effect of SCNI sessions on Bettie's and Ted's participations in classroom groupwork is doubtful, since they participated in only one and two SCNI sessions, respectively.

research, as discussed previously. More to the point, they provide a hybrid space where situated and narrative identities overlap. In SCNI sessions, participants narrate the identities they negotiate through their participations in an activity. Narrative identities produced in the SCNI sessions are reifications of identities negotiated through participation in a given practice.

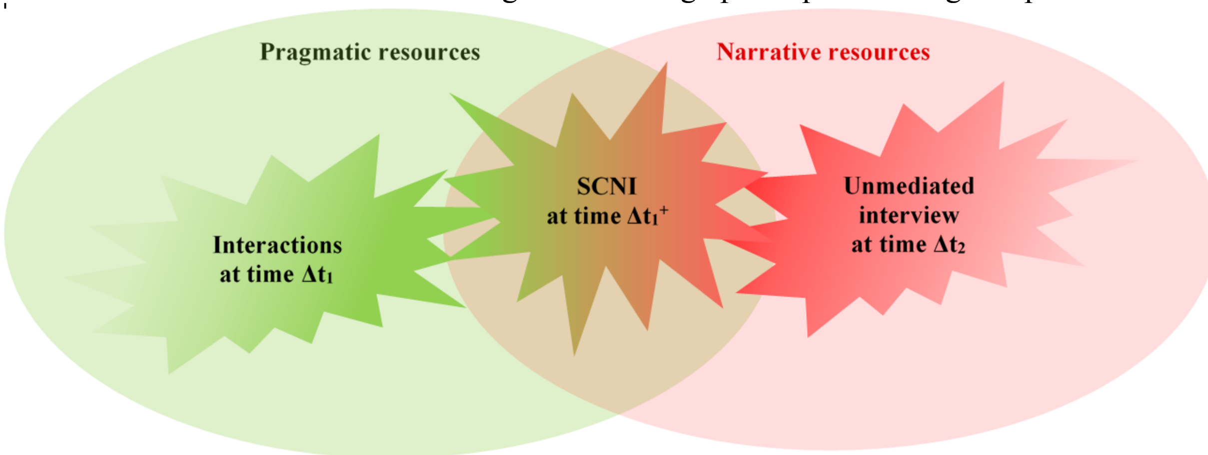

Figure 2-2: The diagram represents the third space within which the SCNI sessions is situated. Researchers can study participants' pragmatic resources (green field) by observing a cultural activity at time Δt_1. They can also study participants' narrative resources through interviews at time Δt_2 (which may take place before or after Δt_1). The SCNI sessions at time Δt_1^+ can be used to elicit how participants connect their narrative and pragmatic resources.

Conduct of SCNI

The set-up for the SCNI technique requires two phases: first, the preparation of the *media* (preferably a video of an activity under study) then an interview setting with a screen to play the recently collected media (I played the video on a laptop). I will describe the SCNI sessions just as I have conducted them for this project.

I videotaped group sessions of the participating groups. I stabilized and pointed an unmonitored video camera towards each group. The camera was oriented to capture the entire interactional space of the group, capturing the faces of group members as much as possible. A microphone connected wirelessly to the camera was placed either at the center or a corner of the table.

The SCNI sessions took place in a private office, which I borrowed from a faculty at the school. After the class ended, I would go to the office to prepare the set-up for the SCNI sessions. I would hook the camera that captured the group session of the group assigned for the interviews to a laptop. I also set up a camera to record the SCNI session.

The sitting posture was pre-meditated. The facilitator and the participant sat near each other, both of them facing the laptop which played the video. This sitting posture was intended to emphasize the difference between SCNI sessions and regular interviews, both of which were conducted by the researcher. The sitting posture would emphasize participants' interactions with the videos more than the facilitator. Unlike regular interviews, in the SCNI interviews, the facilitator was present mainly to listen and take notes, and occasionally probe.

The SCNI sessions were conducted individually in this study. Members of the selected group would come to the office one after the other. Most SCNI sessions took place after the class ended. Some of them had to be rescheduled the next day due to scheduling difficulties. All SCNI sessions took place within 24 hours after the end of the videotaped group activities.

The facilitator launched each SCNI session with this probe or a similar version of the same:

> In my study, I try to understand the interactions between people. Today, I would like you to help me see through your eyes to understand what happened in your recent group session. You will watch a video of it to help you recall what happened. You can pause the video at any point of time in which you recall your significant mathematical reasoning and your feelings about yourself or your groupmates at the moment of the interactions. Try your best not to confuse your current thoughts and emotions with those you experienced when you were working within the group. Do you have any questions, before we start?

After responding to the clarifying questions, I played the video from the start of the group session. I sat silently, waiting for participants to comment on parts of the video. Occasionally, I would pause the video and ask gentle probes, such as "what were you thinking (or doing) at this moment?" Or "what did you feel at this moment?" Or "what did you mean by saying/doing that?" Sometimes, the participants enjoyed watching the video and forgot about commenting on it. In such a case, I would pause the video and ask, "what were you doing at this moment?"

At the end of the session, the facilitator followed up with questions on selected comments. I would indicate or rewind to the section of the video and ask questions such as the following: "What did you mean when you commented on this moment saying you were frustrated?" "Is it common that you feel/think like this in such circumstances?" "At time t of the video, you said Fred is smart. Why is that?" "At time t of the video, you said you are not getting it. Did you understand it later in the group session? (If yes) when and how?"

Most SCNI sessions ended when the time was up and the next participant was supposed to come. The SCNI sessions were scheduled for over 45 minutes, most of them lasting this long. Few of them lasted as short a time as 30 minutes. Few others, when not limited by a next session that was to be held, went over as long as 90 minutes.

SCNI versus stimulated recall techniques

Stimulated recall is a well-known and used data collection technique in educational research and professional development (Anthony, 1994; Calderhead, 1981; Gass & Mackey, 2000; Keith, 1988; Lyle, 2003; Meade & McMeniman, 1992; Stough, 2001; Wear & Harris, 1994). The SCNI technique is a mediated type of interview similar to stimulated recall. However, the functionality of SCNI sessions differ from stimulated recall. The SCNI sessions are not intended to elicit what exactly took place during groupwork. The function of recall does not befit the use of that technique in this project. Operationally, stimulated recall interviews involve questions of *what* and avoid questions of *why*. Participants are readdressed to focus on describing, rather than explaining, their behaviors during an activity (Gass & Mackey, 2000). On the contrary, the main purpose of SCNI sessions is to elicit explanations of participants' behaviors during an activity.

As noted previously, the SCNI sessions are intended to create a hybrid space where participants narrate their behaviors in hope of eliciting identities and functions that they connect to the positions that they animate in their groupwork. For a narrative approach, the VIP+function compositions produced in SCNI interviews take place in participants' narratives and, thus, constitute a piece of evidence. For a situated approach, such compositions have the status of being reifications of students' participations in a practice.

Data Analysis

Multiple analyses are employed in this dissertation. A retrospective and systematic method is used to select relevant data to be analyzed within the VIP+function framework. This project produced multimodal data: videos of groupwork in classroom, videos of two types of interviews, students' written work, and students' responses to surveys and memos (see Table 2-3). Depending on the affordances of the data sources, the VIP+function analysis may investigate one, two, or three constituents of the framework. A subsection is dedicated to explaining each analytical method.

As noted in Chapter 1, the VIP+function analysis may indicate investigations that are worth undertaking for the purpose of this dissertation. Three such investigations were identified and conducted with additional established analytical methods, indicated in Table 2-5.

Table 2-5: Additional established analytical methods used for special investigations. The table names the methods, the purpose of using them, and the location, in the dissertation, where they are reported.

Additional analytical method	Purpose of use	Reported in	Page in dissertation
Competitive argumentation methodology (Schoenfeld, Smith & Arcavi, 1993).	To test a claim on why Ted leaves majors to join other majors on multiple occasions.	Chapter 3 (C1.2) Creativity (C1.3) Proofs	Analysis starts at page 43
	To analyze the psychological resources Ted utilized to sustain his engagement with constructing proofs despite struggle.	Chapter 3 Why Ted spent hours attempting to find proofs	Analysis starts at page 58
Participation structure (Erickson & Mohatt, 1977) and pronoun analysis (Fairclough, 2003)	To analyze an activity in groupwork, so-called Ted-Bettie activity, where Ted explains to Bettie and Bettie identifies with Ted.	Chapter 4 Quadratic reciprocity law (on 11/19)	Analysis starts at page 115

Retrospective systematic data selection

The analysis in this project faced a common challenge of longitudinal research: data management. Since the current project selects two students for a focal study because of their significant development through their engagement with a new learning method, the investigation is set for a retrospective methodology. It attempts to investigate how the focal students developed their voices over time and what were the psychological and environmental elements which fostered the developmental steps.

The next challenge was to select relevant data in a methodological manner. I addressed this challenge by following the recommendation of Cobb and his colleagues (2003): "a central challenge in conducting retrospective analyses is *to work systematically* through the extensive, longitudinal data sets generated in the course of a design experiment so that the resulting claims are trustworthy" (p.12; my emphasis). Next, I present the systematic method that I employed to select relevant data for analysis.

In selecting the data, I follows **an ethnographic principle** that allows participants perspectives to influence the research (Erickson, 1992, 2012; McDermott, 1996; McDermott & Raley, 2011). An ethnographic approach coheres with the theoretical foundations of this project: personal voice and psychoanalysis. In attending to personal voices, a researcher must follow what participants voice and determine as valuable. We must attend not only to their voices during their participation in the learning activities, but also to their voices as they make sense of their participation in the interviews. While the cultural dimension is investigated through the constituent of *position*, the investigation of *identities* requires the researcher to listen to participants by following how they narrate and organize their respective *imaginaire*.

The retrospective systematic method for data selection used here consists of two following moments: I first select the focal data source, and then identify the relevant pieces of data.

Selection of focal data sources.

The starting point is the students' self-reports in the exit interviews. The **first step** consists of detecting the narratives pertaining to the phenomenon under study. In the case of this dissertation, the phenomenon under study is the students' development of personal voices. The relevant narratives are ones that indicate changes in course of the number theory class. The dissertation limits itself to three significant changes reported by participants, mainly the two focal students (Table 2-6).

The focal data source for the investigation of each change is determined based on the self-reported changes and the way in which participants narrate them. The project offers three options of data sources: videos of group sessions, students' self-reports and students' mathematical work. For the investigation of a change pertaining to an extended timescale, such as Ted's change of his prospective career, the participants' narratives afford a closer look at their decision-making process than that provided by observing their participation in an activity within a short timescale. Thus, I use Ted's narratives as a primary source of data to analyze his change of prospective career. On the contrary, videos of group sessions afford a closer look at participants' behaviors in groupwork to investigate how a student restores confidence vis-à-vis his/her groupmates, such as Bettie's reaffirmation of her arithmetic (PPM) ability. As for the changing learning methods, it occurs at the confluence of classroom and outside-classroom learning activities. In Bettie's narratives about the change in her learning methods, reading the book, studying with peers, and doing homework by herself are connected to the shift. Participants' narratives are likely to connect their participations across activities. For this reason, I select Bettie's narratives as a primary data source to investigate the shift of her learning methods, while attending to other data sources while investigating specific topics.

Table 2-6: References in participants' narratives about significant changes. The table describes the changes and selected data sources to support the analysis of the same.

Narratives indicating changes	Description of change	Selected as primary data source	Other relevant data sources	Study reported in
Int2-1203-Ted lines 425–435	Ted's change of prospective career	Ted's narratives in interviews, including narratives on his participation in groupwork		Chapter 3
Int2-1202-Bettie lines 306–314	Bettie's restoration of her arithmetic (PPM) ability vis-à-vis her groupmates	Videos of group sessions	Interviews for triangulation of psychological claims	Chapter 4
Int2-1202-Bettie lines 250–279	Bettie's change of learning methods	Bettie's narratives in interviews	Videos of group sessions and students' submitted homework.	Chapter 5

Identifying relevant pieces of data

After selecting the primary data source, data need to be organized for analysis. The data management of videos defers from narratives. For **videos**, it is the researcher's responsibility to determine what parts of the video-records are pertinent to the investigation of respective change.

For the investigation into Bettie's reaffirmation of her arithmetic (PPM) ability in her group, we need to attend to computational activities throughout the semester. Such episodes in G3 are identified and prepared for a VIP+function analysis (to be discussed next). Importantly, these episodes must be analyzed chronologically to track the development of Bettie's voice over time.

As for the **narratives**, they require a distinct treatment. A large part of narrative power relies on how participants narrate and connect different topics together. Nevertheless, topics in interviews appear and disappear, being contingent on the interviewee's probes (in regular interviews) or the interactional moments within recent groupwork in the videos (in SCNI sessions). It is the analyst's task to identify, assemble, and arrange the narratives pertaining to a phenomenon under study. I conducted this task systematically, as described in the methodical five steps given below (Figure 2-3). The body of Chapter 3 unfolds the methodical steps described below, in the form of an illustration. The analysis in Chapter 5 follows the same steps with regard to Bettie's narratives, even though the write-up of the report focuses on the types of activities in which Bettie participated.

The methodical steps utilize two constructs, namely component and conversational moment, in ways specific to the retrospective systematic methodology. By **component**, I understand an entity, such as a name, an object, an activity, or a state of affairs, that is narrated in either opposition or support to the entity under study. Consider the snippet from Bettie's narrative about her new learning method (Table 2-7). It presents a **conversational moment** of a component in an instance of a participant's narrative that mentions the component. The length of a conversational moment carries over all utterances that couch the specific component. A conversational moment may consist of one or many speech turns.

The methodical steps of identifying, assembling, and arranging narrative data (Figure 2-3) comprises over multiple layers, attending to primary, secondary, and tertiary components. The purpose of this endeavor is to evaluate the narrative power of a component within the narrative net that couches the phenomenon under study. The narrative power of a component is measured by the number of components it influences either directly or indirectly. The net of components and their narrative power is illustrated in Chapter 3 (Figure 3-3, Figure 3-6, and Figure 3-11).

Table 2-7: The analysis of components pertaining to Bettie's new learning method, within a parcel of her narrative.

Int2-1202-Bettie lines 255–258	**Component**
now I realize I need to read, obviously . I have to read through the book.	Textbook
I have to . like do the homework . like slo::owly at my own pace and like do it myself.	Homework
and . um . that's like the only way I'm gonna retain anything or like know what I'm doing	Fostering understanding

Methodical steps to identify, assemble, and classify narrative data for the analysis of a phenomenon.

Step #1:	Analyze the narratives on the phenomenon under study to highlight the primary components that are narrated as constituting, generating, shaping, and directly interfering in the phenomenon. Then, study each primary component separately, as described in steps #2 and #3.
Step #2:	Identify and collect the conversational moments involving the primary component systematically throughout the narratives.
Step #3:	Analyze the collected conversational moments to detect further components that are narrated as constituting, generating, shaping, and interfering directly in the primary component. The newly identified components will be called secondary components, which are narratively connected to the phenomenon under study through the primary components.
Step #4:	Repeat steps #2 and #3 for each secondary component to find the respective tertiary components.
Step #5:	The investigation of tertiary components is left to the discretion of the researcher, since the effect of tertiary components on the phenomenon under study may be diluted due to the narrative's remoteness.

Figure 2-3: List of methodical steps used in this dissertation to select relevant data for the analysis of changes in students' learning identities.

Analysis with the VIP+function framework

The application of the VIP+function framework depends on the type of data, whether videotaped social interactions or participants' narratives.

In the investigation of participants' narratives, I focus on explicit indications of identities, positions and functions while analyzing the multiple layers of components (as explained in Figure 2-3). Implicit indications of identities and functions are analyzed through the methodology of competitive claims (see Table 2-5).

In analyzing video records of groupwork, I parse the selected parts of videos into eco-units, i.e., conversational units that start with an initiation and develop on the same (Clark & Schaefer, 1989; Engle, Langer-Osuna & McKinney de Royston, 2014). Within each eco-unit, I focus on the turns of a focal student. The speech in each turn is considered a *voice,* as defined in the first chapter. Then, I analyze the four constituents involved in each voice, as described in the first chapter and illustrated in Figure 1-1 and Figure 1-2. The focal students' speech and gestures as well as the groupmates' responses in the eco-unit provide indications of the positions animated in the voices and the functions set by the same. Narratives on interactions, when they are available in SCNI sessions, are consulted to check the function and the identity determined by the analysis of social interactions in groupwork.

The determination of the mathematical identity actuated in the voices required one to attend to the content of speech. **Arithmetic identity** (PPMI) involved the repertoires of numbers, basic operations on numbers, and the task of solving equations. **Number theory identity** involves the constructs and ideas presented in the curriculum (above), such as modular arithmetic, prime numbers, and primitive roots.

For the important current trends in mathematics placed on certain positions (Langer-Osuna, 2016; O'Connor & Michaels, 1993; Webb & Mastergeorge, 2003), I constructed the following coding scheme to track the animation of 10 social and socio-mathematical positions through groupwork done over the semester (Table 2-8). The results of this coding exercise report the number of instances in which the respective position is animated by a focal student.

Table 2-8: A scheme by which to code positions which a student animates during groupwork. The scheme is confined to 10 social and socio-mathematical positions. These positions are selected for their pertinence to the development of mathematical voices.

#	Position	Description
P1	Monitoring social and socio-mathematical interactions of groupwork	Student distributes tasks to groupmates, calls on a groupmate to take on an action, such as writing on the white board, or proposes that the group does certain actions, such as discussing a topic or moving on to solving another problem.
P2	Asking groupmate(s) to explain a mathematical idea	Student asks the group or specific groupmates to explain issues pertaining to a mathematical activity (math ideas, procedures, or group interactions).
P3	Asking groupmate(s) to assess the speaker's own mathematical idea(s)	Student asks groupmate(s) whether his/her mathematical thoughts are correct or worthy
P4	Being approached for mathematical explanation by a groupmate	Student is approached by a groupmate to explain to the latter an activity-related issue. (To identify whether a student is being approached or not, the coder must attend to participants' gazes, gestures, bodily positionings, and speeches)
P5	Being approached for an idea assessment by a groupmate	Student is approached by a groupmate to assess or evaluate a mathematical idea of the latter. (To identify if a student is being approached or not, the coder must attend to participants' gazes, gestures, bodily positionings, and speeches)
P6	Contributing a mathematical idea to the activity	Student shares with the group or some groupmates a mathematical thought either from his/her own knowledge or gathered from textbooks
P7	Offering a mathematical explanation to the group or a groupmate	Student explains an activity-related issue to the group or groupmates, usually the role explains an idea that has been worked out either individually or collectively prior to this instance
P8	Assessing a mathematical contribution to the groupwork	Student evaluates or assesses the worthiness or correctness of a groupmate's contribution.
P9	Attending to a groupmate who is not specifically soliciting X's attention	Student attends to a groupmate's task-related need; student's help is not specifically solicited. (e.g. a groupmate publicly expresses confusion without turning to student, who volunteers to support her/him)
P10	Being attended to by a groupmate whose attention X is not specifically soliciting	Student is attended to by a groupmate, who animates P9.

Lastly, I provide immense amount of data in the appendices and carefully document the analytical actions and decisions taken throughout the dissertation to allow the reader to conduct reliability tests.

Chapter 3: Ted Tao:

Creativity, Games, and Tutoring Young People

This chapter analyzes the case of a student, Ted, whose academic studies underwent multiple changes of STEM and non-STEM major programs. Ted was also a prodigal son dropping out of college and working for few years before resuming his studies. This case study provides an opportunity to simultaneously investigate two phenomena: dropping-out of and going back to school. Ted's decade-long journey was a winding path that culminated in him winning a pre-doctoral fellowship in mathematics.

The number theory class under study played a significant role in Ted's academic journey. In this class, Ted realized for the first time that he could prove theorems by himself. The resulting confidence impelled Ted to change his career orientation from pursuing teaching credentials to applying for a master's degree in advanced mathematical studies.

This chapter attempts to investigate the resources that contributed to the enhancement of Ted's mathematical confidence and the change in his prospective career. Ted's case will be investigated through the *retrospective systematic methodology* (Chapter 2) and *VIP+function framework* (Chapter 1). A section of the analysis in this chapter builds on Engle's (2012) Productive Disciplinary Engagement (PDE) framework, which is useful for its relevance to Ted's engagement with proof problems. The adaptation of the PDE framework is explained in this introduction.

This investigation, as it unfolds through the retrospective systematic methodology, requires only one data source, namely Ted's narratives produced through four interviews (see appendix B). By having recourse to Ted's narratives as primary data source, I assume that participants' *narratives* afford a closer look at their engagement with a major program over extended timescale than *videos* of classroom group sessions confined to a short timescale. Nevertheless, part of informational data from classroom group sessions are supplied by the two SCNI sessions (see Chapter 2) in which Ted participated. In other terms, through the SCNI sessions Ted interpreted his participation in two classroom groupworks by translating social interactions into narratives. By doing so, data from the two classroom observations and regular interviews can now receive one analytical treatment, i.e., a narrative analysis.

As noted in the first chapter, the report of Ted's case study is an illustration of the retrospective systematic methodology. The linearity in the write-up of this chapter follows closely the methodical five steps explained in the second chapter and reproduced below (Figure 3-1). Recall that a component is an entity, such as a name, object, activity, and state of affairs, that is narrated in opposition to or support of the entity under study.

Methodical steps to identify, assemble, and classify narrative data for the analysis of a phenomenon.

Step #1: Analyze the narratives on the phenomenon under study to highlight the primary components that are narrated as constituting, generating, shaping, and directly interfering in the phenomenon. Then, study each primary component separately, as described in steps #2 and #3.

Step #2: Identify and collect the conversational moments involving the primary component systematically throughout the narratives.

Step #3: Analyze the collected conversational moments to detect further components that are narrated as constituting, generating, shaping, and interfering directly in the primary component. The newly identified components will be called secondary components, which are narratively connected to the phenomenon under study through the primary components.

Step #4: Repeat steps #2 and #3 for each secondary component to find the respective tertiary components.

Step #5: The investigation of tertiary components is left to the discretion of the researcher, since the effect of tertiary components on the phenomenon under study may be diluted due to the narrative's remoteness.

Figure 3-1: List of methodical steps used in this dissertation to select relevant data for the analysis of changes in students' learning identities.

Table 3-1: The primary, secondary, and tertiary components in Ted's narratives that pertain to his change of prospective career, as analyzed by following the retrospective systematic methodology (see snippet above). The template of C# indicates a primary component, C#.# a secondary component, and C#.#.# a tertiary component.

Narrative hierarchy	Description of component
C1	Ted's identification with people who produced new proofs.
C1.1	Smartness.
C1.2	Creativity.
C1.2.1	Creative writing (Rebellion phase).
C1.3	Proofs.
C1.3.1	Applied mathematics.
C1.3.2	The Tao family.
C2	The challenges of proof production.
C2.1	Problematizing.
C2.2	Resources.
C2.3	Accountability.
C2.4	Authority.
C3	Ted's interest in graduate school.
C4	Ted's teaching career.
C4.5	Thinking like teaching.
C4.6	Ted's positionings in groupwork.
C4.6.1	Learning by explaining.
C4.6.2	Ted became mindful of group dynamics.
C4.6.3	The effect of grades on Ted.
C4.3	Tutoring young people.

The organization of this chapter (see Table 3-1) reports the investigations of the multilayered components found in Ted's narratives: the primary, secondary, and tertiary components pertaining to Ted's change of his prospective career. The sections of the chapter are dedicated to the primary components found in the narrative instances in which Ted reports on his change of prospective career. The subsections investigate the secondary and tertiary components pertaining to the primary component of the section.

I devised a symbol—C followed by a number—to keep track of the narrative hierarchy of the component, as found through the methodical five steps. The template of C# indicates a primary component, C#.# a secondary component, and C#.#.# a tertiary component. For example, the symbol C4 indicates that the component on *Ted's teaching career* is a primary component found in Ted's narratives on the change of his prospective career. The symbol C4.6 indicates that the component on *Ted's positionings in groupwork* is a secondary component found in the narratives on the primary component C4 (*Ted's teaching career*). The symbol C4.6.3 indicates that the component on *the effect of grades on Ted* is a tertiary component found in the narratives on the secondary component C4.6 (*Ted's positionings in groupwork*).

As noted in Table 3-1, some secondary components of C4 are missing. The missing secondary components are ones that have been analyzed under previous primary components. Furthermore, the component C4.3 is discussed last in the section dedicated to C4, because it provides a synthesis of the primary component. The numbering within each level is randomly assigned prior to the analysis. Once numbered, the components are analyzed by order. The *Tutoring of young people* is analyzed prior but reported after the analysis of C4.5 and C4.6. I opt to maintain the number C4.3 for this component as indicator of the order by which it has been analyzed, even though it is reported last in this chapter.

The hierarchical position of a component is not to be confused with its narrative importance or relevance. The hierarchy of components is merely devised to ensure a systematic treatment of data. As for the narrative power of a component, it is analyzed through its direct and indirect influence on other components as reported in the narratives. I illustrate the narrative power of components with diagrams—Figure 3-3, Figure 3-4, Figure 3-6, and Figure 3-11—which represent the interconnectedness of components that pertain to Ted's change of prospective career, according to his narratives.

The chapter will unfold as follows. The adaptation of the PDE framework to Ted's narratives is explained and followed by information on Ted's background and participation in the reported research. (Further information on the methods and methodologies, such as the site, the classroom design, the conduct of interviews, and methodologies for data analysis, are explained in the second chapter). The following sections report the steps of the retrospective systematic methodology as applied to Ted's narratives. The first section analyzes and extracts the primary components in Ted's narratives on his change of prospective career. The following four sections are dedicated to the analysis of each primary component, where secondary and tertiary components are identified. They also analyze the *voices*—i.e., *identities*, *positions,* and *functions*—as they appear in the conversational moments about the identified components. The conclusion summarizes the self-reported shifts of identities, positions, and functions at the extend timescale that led Ted to change his prospective career and at the long timescale that resulted in development of Ted's participation in the number theory course.

Adapting the principles of productive disciplinary engagement

Why use an additional framework? The narrative instances that mention a primary component may reach a large number, in which case the identification of the secondary components becomes unmanageable. This situation happens to the study of the primary component C2 (*The challenges of proof production*), where the number of instances mentioning the challenge of proofs exceeds forty in Ted's narratives. I use the principles of productive disciplinary engagement (PDE) (Engle, 2012; Engle & Conant, 2002) to classify the overwhelming references in clusters that are manageable for analysis.

The PDE framework consists of four principles, i.e., problematizing, resources, authority and accountability, that sustain students' engagement with disciplinary tasks and make their social interactions in groupwork conducive to learning. The PDE principles, as established by Engle and Conant, are oriented to address either pedagogical design or group dynamic aspects. Here, I reformulate the principles to make them relevant for narrative analysis. I will investigate whether and in which ways Ted mentions any of the PDE principles when he narrates his engagement with the proof problems.

Applied to proof activities, the four principles of PDE can be reformulated as follow. *Problematizing* is the process by which a mathematical activity is rendered more challenging to participants. The principle of *resources* comprises all accessible sources of knowledge that are relevant to the mathematical task, whether they are published resources or human beings. The principle of *authority* is concerned with the three ways by which participants generate mathematical ideas: they can author unprecedented proofs, reproduce existing proofs, and read published proofs. The merit of authoring new proofs consists of contributing to the advancement of the field, while reading published proofs in understanding them. As for the act of reproducing existing proofs by counting on one's own knowledge, it provides a sense of authorship even though the product does not advance the field. The principle of *accountability* is concerned with the ways in which participants relate to the socio-mathematical norms established in the discipline and classrooms.

The PDE framework endorses the fact that in order to sustain an esteemed productive engagement with a disciplinary task, a dual balance must be maintained (Figure 3-2). On one hand, the generation of challenges—problematization—must be counter-balanced with the available resources that participants can employ to address these challenges. Learners may disengage from an activity that extends a challenge beyond the resources available to them, while an overflow of resources may impede the engagement with the challenge from generating new knowledge. On the other hand, authority and accountability are two principles caught in a tension with each other. They need to counter-balance one another. A strong enforcement of disciplinary norms may suffocate authorship, which needs a safe space to err and get refined. An authorship that does not abide by disciplinary norms will suffer rejection by the disciplinary community.

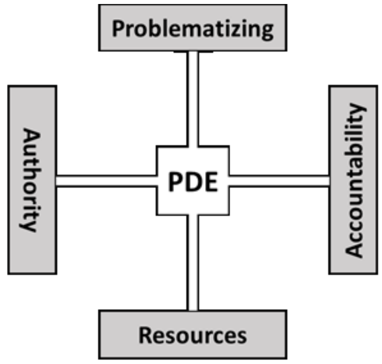

Figure 3-2: A representation of the balance between the four principles of productive disciplinary engagement as stated by Engle (2012).

The focal participant: Ted

Ted, a 27-year old male, was a senior student majoring in mathematics with a concentration on teaching. In the early survey, he elaborated on his ethnic background in a nuanced way:

> *18. If you wish, describe your ethnic background.*
> [Ted's response:] born in Hong Kong, I come from a Chinese ethnic background, but culturally my city has been heavily influenced by the British, moved to the U.S. when I was eleven years old.

In describing himself in rapport to mathematics in the early survey, Ted wrote a statement that involved multiple overlapping identifications and positionings.

> *8. How do you consider yourself in relation to mathematics (e.g., accomplished mathematician, math student, math nerd, in love with numbers, like graphs, master geometry, logic-lover, dislike proofs, curious about math, ...)? Elaborate why you see yourself as such.*
> [Ted's response:] math nerd in love with teaching through proofs instead of arithmetic.

Ted's statement involved multiple overlapping identifications and positionings. In this statement, Ted's love for teaching mediated his identification with mathematics. He discriminated between two activities within mathematics, proofs and arithmetic (PPM), and self-identified with a preference for teaching one of them, the "proofs" activity. Furthermore, in this statement Ted positioned himself with regard to the prompt (question #8). He subverted two expressions offered as potential answers to the question: "in love with numbers" and "dislike proofs." Ted was "in love with *teaching* through proofs." The insertion of the teaching identity was a personal addition, not inspired by the prompt. In the beginning of the semester, Ted's teaching identity seemed to mediate his identification with the activity of constructing proofs, as opposed to arithmetic (PPM).

Although selected from a suggestion in the prompt, the "math nerd" identification expressed Ted's appreciation of mathematics. Indeed, he was admitted to a prestigious pre-college school, Texas Academy of Mathematics and Sciences, at the age of sixteen. Moreover, he had a cat called Euclid.

In his number theory class, Ted joined a group, G3, composed of four other students: Bettie, Boutros, Jeremy, and John. He played a central role in initiating a study group outside his classroom, which most of his groupmates, except John, joined. He also shared with his groupmates a link to his google drive, where he stored his homework and his notes of the number theory class. Ted used to upload his homework to the google drive prior to the submission deadline. Ted took that class because he was “generally interested in math and proofs, also needed the class for [his major]” (reported in the early survey). By the time he took the number theory class, Ted had shifted majors thrice, from applied mathematics to creative writing to mechanical engineering to mathematics for the purpose of teaching (see Int1-1001-Ted lines 58–77).

Further information on the site, the pedagogical design, and the data collection can be found in the second chapter. Suffice it here to note two types of interviews: regular and SCNI. Ted decided to participate in the research a month after the semester started. His first interview (int1) took place on 10/01. He then participated in two SCNI interviews on 10/15 and 11/12. His exit interview (int2) took place on 12/03. All interviews are transcribed with lines that are numbered in appendix B. The lines reboot at the start of a new interview session. References to the interviews are informative, as follows: datasource-date-studentname lines #–#. For example, Int2-1203-Ted lines 105–120 refers to the text between lines 105 and 120 of Ted’s exit interview on 12/03.

The primary components pertaining to Ted’s change of prospective career

The first step of the methodical selection of data, as described earlier (Figure 3-1), consists of identifying the conversational moments that mention the phenomenon under study: in this case, Ted’s change of prospective career. Once these narratives are identified, the primary components narratively connected to the phenomenon under study must be extracted from the selected narratives.

In the exit interview, Ted reported only one change through his number theory class: by the end of the course he was considering pursuing a master’s in mathematics and postponing the pursuit of his teaching credentials. This shift was mentioned in only one instance, the so-called triggering instance, that took place in the exit interview on 12/03 (see the following excerpt).

<u>Int2-1203-Ted lines 407–421</u> -- *The triggering narrative instance*

Fady: […] Did this experience make you rethink other choices of your career? Or give you something to do . to look differently for your future?

Ted: Well. yeah. instead of going straight for the credential program for teaching . I am considering going into master's program.

Fady: Master's program? what are you? … can you say more why it is?

Ted: I . I think I . I have haven’t really been tested like this in terms of proofs . and I was not conf . even though I really liked it . I wasn't confident that I could do it. you know? Like uh . I . I look at people who prove things like this the first time . and I think wow. you know? [laughs] I'm like how could I ever be that smart. and I'm like . now . now I realize that you know proofs is something doable . it's it just takes a lot of time. and patience.

Fady: Thank you . I am going to ask you about other changes that you experienced in this class . and tell me what were they . if any? Did you experience any change in the ways of learning?

Ted: um … before . I would . I would read the proofs and just understand it . and then just move on. and now . now I try to actually . prove things.

Table 3-2: The extraction of primary components from Ted's narrative instance on his change of prospective career.

Int2-1203-Ted lines 407–421	Description of Component	Component #
Well. yeah. instead of going straight for the credential program for teaching .	Ted's teaching career	C4
I am considering going into master's program […]	Ted's interest in graduate school	C3
I . I think I . I have haven't really been tested like this in terms of proofs . and I was not conf . even though I really liked it . I wasn't confident that I could do it. you know? Like uh . […] and I'm like . now . now I realize that you know proofs is something doable . it's it just takes a lot of time. and patience. […] um … before . I would . I would read the proofs and just understand it . and then just move on. and now . now I try to actually . prove things.	The challenges of proof production	C2
I . I look at people who prove things like this the first time . and I think wow. you know? [laughs] I'm like how could I ever be that smart.	Ted's identification with people who produced new proofs	C1

As evidenced in the above excerpt (see also Table 3-2), Ted narrated his emerging consideration of career change with regard to four primary components:

(C1) Ted's identification with people who produced new proofs.

(C2) The challenges of proof production.

(C3) Ted's interest in graduate school.

(C4) Ted's teaching career.

Ted used to admire the intellectual ability of mathematicians who proved theorems (C1). He found in proving theorems a type of "smartness" that he desired for himself and yet had doubts about whether it was within his reach. The pedagogical design of the number theory class, which required students to reproduce proofs of theorems as part of small-group work, exposed Ted to the challenges associated with proving theorems (C2). Heretofore, he used to "read" and "understand" published proofs but had never been given the opportunity to construct proofs by himself. In the number theory class, he "realized" that he was capable of "doing" rather than merely understanding mathematicians' works, if he devoted "time" and applied "patience." By gaining this confidence, Ted felt impelled to consider pursuing a master's degree (C3) and abandon pursuing his teaching credentials (C4). The following sections report the systematic analyses of each primary component present throughout Ted's narratives.

(C1) Ted's identification with people who produced new proofs.

Ted's identification with mathematicians who produced new proofs was noted in only two instances throughout Ted's narratives (excerpts given below): at the beginning of the early interview, when Ted reflected on the thoughts that the word *mathematics* evoked in his mind, and in the middle of the exit interview, when he talked about considering a career shift.

<u>Int1-1001-Ted lines 9–14</u> -- *Ted looking up to mathematicians who produced new proofs*

Fady: and um you mentioned it's poetic and driven by creativity . how so?

Ted: I often times like look at like the people who like first came up with calculus for [unintelligible]. I look at that and I think to myself . could I have come up with that if it didn't exist first? And I think to myself how creative must somebody be to come up with something like this. and how it defines the world around you. How mathematics is that. That's like creativity.

<u>Int2-1203-Ted lines 412–416</u>

Ted: I have haven’t really been tested like this in terms of proofs . and I was not conf . even though I really liked it . I wasn't confident that I could do it. you know? Like uh . I . I look at people who prove things like this the first time . and I think wow. you know? [laughs] I'm like how could I ever be that smart. and I'm like . now . now I realize that you know proofs is something doable . it's it just takes a lot of time. and patience.

Although the two instances were recorded two months apart, Ted used the same sentence structure and five similar words to express his admiration for people who produced original mathematical work (see the comparison below). Although this admiration was mentioned only twice in Ted’s narratives, it could be an identification that “often times” sustained and gave meaning to Ted’s experience in the number theory class, where he was first offered the opportunity to become like the people whom he admired (“how could I ever be that smart!”).

(Int1) **I** often times like **look at** like the **people who** like **first** <u>came up with calculus.</u>

(Int2) **I look at people who** <u>prove things</u> like this the **first** time.

The two aforementioned instances specify the objects of Ted’s admiration in different ways. In the early interview, Ted highlighted the *creativity* involved in producing new rules and theorems in calculus. In the exit interview, Ted praised the *smartness* involved in producing new proofs.

Creativity, smartness and proofs/calculus are, henceforth, considered three secondary components which are connected to the primary component C1. The next subsections are dedicated to investigating each one of them in the order of increasing analytical complexity: (C1.1) smartness, (C1.2) creativity, and then (C1.3) proofs, which include calculus.

(C1.1) Smartness.

Ted used the word “smart” only twice throughout his interviews. The second instance was the one mentioned above (Int2-1203-Ted lines 412–416), which led to this analysis. The first instance was in the early interview when he talked how people around him perceived mathematics and mathematicians (see excerpt below).

Ted: In my family circle . is all immigrants from China. The culture around mathematics is more like if you can do math then you are good student. right? like and mathematics is kind of like a benchmark for all the other subjects like if you can do math you should be able to do science and everything else just fine. you know? so . that's uh whenever math comes up with my family it's it's kind of like talked about . like benchmark for “are you smart enough? alright . are you worthy of being a Tao [i.e. Ted’s family]?” [laugh]. (Int1-1001-Ted lines 23–30)

In the abovementioned instance (Int1-1001-Ted lines 23–30), the word “smart” was used as a narrative voice that imbibed the opinions of Ted’s family, who were immigrants from China and perceived mathematics as an indicator of smartness, the proficiency of which was required to own the Tao family name. It was not Ted’s voice that used the word “smart.” Additionally, soon after that instance, Ted shared the story of his rebellion against his parents when he was eighteen years old. After he was admitted to Texas Academy of Mathematics and Science (TAMS), a prestigious early college admission from school, Ted revolted against his parents, dropped his studies at TAMS, and went to study creative writing in Georgia (Int1-1001-Ted lines 58–68). Contrary to his family’s view of mathematics, Ted perceived it as a democratic discipline accessible to anybody who was motivated enough to invest in it.

Ted: I feel like if we can change that culture, the way that parents look at math, the way that kids look at math; instead of looking at something that they hate, it could be something fun. (Int1-1001-Ted lines 329–330)

The few occurrences of the word "smart" in Ted's narratives (only two instances) are striking, especially in comparison with the narratives of Bettie (26 occurrences) and Melissa (10 occurrences). This fact indicates that the secondary component C1.1, i.e., smartness as human faculty, provides a weak justification of the primary component C1, Ted's admiration for people who produced original proofs. The other secondary component, i.e. (C1.2) creativity, is more frequently present in Ted's discourse than smartness (C1.1), as reported in the next subsection.

The tertiary component of the Tao family (C1.1.1) is present in five instances throughout Ted's narratives (Int1-1001-Ted lines 23–30, 60–62, 70–71, 91–95, and Int2-1203-Ted 647–650). Two of these instances (Int1-1001-Ted lines 23–30 and 60–62) have been studied earlier in relation to the value of mathematics perceived by the Tao family and Ted's rebellious phase. The three other instances (Int1-1001-Ted lines 70–71, 91–95, and Int2-1203-Ted lines 647–650) indicate that Ted's family had a fluctuating influence on Ted, along with meagre resources to support his decision-making and his application to graduate schools. The component C1.1.1 played an implicit role in connection to C1.2 (creativity), as studied in the next section.

(C1.2) Creativity.

Ted's narratives involve five instances on creativity (see Table 3-3), divided between creativity in mathematics (Int1-1001-Ted lines 2, 9–14 and 39–50) and creative writing (60–68 and 114–116). As explained in this subsection, the secondary component C1.2 (creativity) is connected to the secondary component C1.3 (proofs) and a tertiary component C1.2.1 (creative writing).

Table 3-3: Ted's narratives on the secondary component C1.2, creativity.

Data source	Reference	*Selective quote*
Int1-1001-Ted	2	*[Mathematics is] poetic, driven by creativity.*
Int1-1001-Ted	9–14	*I think to myself how creative must somebody be to come up with something like this.*
Int1-1001-Ted	39–50	*I feel like a champion for the creative side of math. [...] you have to be creative to generate these proofs and to be in the front line of math. You are being a creative person. [...] the people that I know, besides those who are already in the math department, [exhalation] tend to look at math as if it is just memorization and arithmetic.*
Int1-1001-Ted	60–68	*I went through a [...] hyper rebellion where I just completely threw everything my parents said out the window and I said, I wanna do creative writing. [...] I was so focused on the creative writing that I stopped succeeding in math.*
Int1-1001-Ted	114–116	*I love [mathematics]. It is a core part of my personality. I look at everything through math. That is just how I am. So I've never deviated far from the math part path except for my super rebellious state of creative writing.*

For Ted, the creative part of mathematics resided in the production of proofs (see the third row in Table 3-3). Ted's appreciation of proofs in mathematics (C1.3) stemmed from the creativity (C1.2) involved in the production of new proofs ("you have to be creative to generate these proofs and to be in the front line of math" 39–50). Creativity in mathematics was an entrenched identity for Ted. He defined himself as "the champion for the creative side of math" (further investigation of Ted's identity regarding proofs can be found in the subsection dedicated to C1.3).

As for creative writing (C1.2.1), Ted mentioned it in only two instances (reported in the last two rows of Table 3-3). Ted mentioned creative writing for the first time when he was narrating about the "winding path" that led him to choose mathematics with a focus on teaching for his major (Int1-1001-Ted lines 58–77). At the age of eighteen, he "went through a hyper rebellion," when he dimmed the voices of his parents ("I completely threw everything my parents said out the window"), which presumably included their voice on smartness in mathematics ("are you smart enough? alright . are you worthy of being a Tao?" 29–30). After silencing his parents' voices, Ted went on to develop a new voice that was rooted in "creative writing." The rebellion against his parents affected Ted's identity regarding mathematics, since the Tao family identified as being smart in mathematics. In effect, Ted "focused on the creative writing" and "stopped succeeding in math."

In the second instance (Int1-1001-Ted lines 111–116), Ted was justifying his choice of math as a major ("because I love it"), when he realized that he had "never deviated from the math path except for [his] rebellious state of creative writing." Thus, creative writing became the name of Ted's rebellious phase, which lasted for many years and targeted his parents as well as his successful mathematical identity.

> Ted: Then eventually I went back to the school . and got into [name of current college] initially as a mechanical engineering major . because my dad said . "You should go make money if you go to school." And . I said . "okay. Engineering sounds . sounds . like something I can do." right . hum . and I discovered that I don't like the real world very much. [laugh] I like the math of engineering . but I didn't like applying it to the science. I didn't like translating . and back to the real world. (Int1-1001-Ted lines 69–74)

After the rebellious phase faded away, Ted reconciled with his parents and his mathematical identity (see excerpt above). He heeded his dad's voice ("you should go make money if you go to school") and chose mechanical engineering as a major when he went back to school. He also reconciled himself with mathematics by changing his major from engineering to mathematics, specifically concentrating on teaching. By choosing mathematics with a focus on teaching, Ted reconciled himself with his mathematical identity but went against his dad's voice, since teaching at a high school (C4) was not as remunerating a job as one that could be obtained in the field of engineering (C1.3.1).

There was a pattern emergent in Ted's shift of his majors. He joined TAMS to major in applied mathematics, which was aligned with his parents' voice. During his rebellious phase, he dropped the applied mathematics major to undertake studies in creative writing. When he came back to school, he chose mechanical engineering as a major—being a section of applied mathematics—which was aligned with his father's voice. Two years into this program, he again changed his major to mathematics. From this observed pattern, I put forth the following claim for consideration.

> **Claim 1: The voice that pushed Ted away from mechanical engineering towards mathematics in his college years was the same one that he developed by dropping out of applied mathematics at TAMS in order to pursue studies in creative writing.**

The alternative to claim 1 is its negation. Claim 1 can be tested as follows. It can be confirmed if the same decision-making elements are central to the earlier and the later shifts; otherwise, it can be refuted. For instance, if Ted's decision in both instances was made on the basis of his ability, the claim should be refuted since the abilities relevant to mathematics would significantly differ from those required for creative writing.

The first instance where "creativity" was mentioned in Ted's narratives (Int1-1001-Ted line 2) supported this claim. During the brainstorming activity of the early interview (line 2), Ted included "poetic" and "driven by creativity" within the notion of what the word "mathematics" evoked for him. Then, he elaborated that he had found creativity in the production of new theorems and patterns (lines 9–14). If Ted conceived of mathematics as poetic and driven by creativity, he had possibly been associating mathematics with creative writing (C1.2.1). The next subsection titled (C1.3) "Proofs" will serve to test this claim further.

Before moving on to the next subsection, it is worth noting that all instances of creativity are witnessed only in the early interview (see Table 3-3). As it will become clear in the next subsection, Ted's understanding of creativity involved challenges that stimulated new ways of thinking, rather than the application of memorized formulas and procedures. Thus, the primary component C2 (the challenges of proof production) should be seen as bolstering the secondary component C1.2 (creativity), since challenging situations call for the application of creativity.

In fact, in the early interview Ted associated the following words with *mathematics*: fun, visual, spatial, patterns, poetic, driven by creativity and beautiful (Int1-1001-Ted lines 2–18). In the exit interview, Ted associated the following words with *number theory*: fun, challenging, new, interesting, complicated but simple, and beautiful (Int2-1203-Ted lines 12–23). While Ted emphasized "creativity" in the early interview, he conjoined the metaphor for creativity ("new") with another metaphor that involved "challenge."

(C1.3) Proofs.

Ted's narratives include 52 occurrences of the word "proof." The investigation in this subsection is limited to the eight instances (see Table 3-4) where Ted described his perception of proof as a subdiscipline within mathematics. Most of the remaining instances of proofs in Ted's narratives pertain to the investigation of proof production (C2).

Table 3-4: Ted's narratives on the secondary component C1.3: how Ted perceived the subdiscipline of proofs and integrated it as a part of his identity.

Data source	Reference	*Selective quote*
Int1-1001-Ted	44–50	*I feel like a champion for the creative side of math [...] You have to be creative to generate these proofs and to be in the front line of math. [...] [People] tend to look at math as if it is just memorization and arithmetic.*
Int1-1001-Ted	138–146	*I would only pay attention to the teacher when they're doing the proofs, cause that's the part that I cared about. The arithmetic, I felt, if I know why it works, then I can figure out how it works.*
Int1-1001-Ted	152–156	*I appreciate the proofs and the generalized forms much more than applying the conclusions of the proofs to problems. So that's I think a big reason why I had very little interest in engineering classes, [which is all about] just applying that same rule over and over.*
Int1-1001-Ted	158–161	*Math is a string of proofs and the arithmetic is just like some extra bonus to what we have.*
Int1-1001-Ted	168–177	*Doing the arithmetic is like I have to keep doing the same thing over and over using the same rule, and I think that gets boring to me. That's why I have that preference for the proofs and part of it is like laziness. [...] laziness for not doing something that almost somebody else can do. A calculator can do that.*
Int2-1203-Ted	20–23	*[Number theory class is] fun because it's requiring me to prove things instead of just do computations. I have a strong bias towards like proofs and that part of math that I am part of it.*
Int2-1203-Ted	81–87	*[The number theory class] challenged a different part of my brain, because it wasn't just applications [...] it's like we are proving what we need to know.*
Int2-1203-Ted	412–421	*I realize that proofs is something doable.*

Mathematics constituted an entrenched identity for Ted. He attested, “I look at everything through math, that’s just who I am” (Int1-1001-Ted lines 114–115). His friends who knew him for two or three years confirmed his mathematical identity. They used to tell him, “why wouldn’t you be a math major;” for his friends “knew who [he was]” (106). Through his experiences in mathematics-related classes, Ted carved a specific mathematical identity by differentiating the disciplines within mathematics.

In his narratives, Ted delineated two mathematics-related disciplines: mathematics and applied mathematics. He self-identified with the former and dissociated himself from the latter. He defined *mathematics* as “a string of proofs” (Int1-1001-Ted line 160, 194–201, and Int2-1203-Ted 368–371) and a construction of “generalized forms” (152–153). For Ted, proofs were explanations of why rules and formulas worked (138–146). As for applied mathematics, Ted defined it as being opposed to mathematics. For Ted, *applied mathematics*, which included arithmetic (PPM) and engineering, consisted of the task of “applying the conclusions of the proofs to problems” (152–156)—“memorizing formulas and applying them” repetitively (Int2-1203-Ted line 484). Ted noted that engineering was all about “applying that same rule over and over” (Int1-1001-Ted lines 155–156), and arithmetic (largely algorithmic mathematics) was about “doing the same thing over and over using the same rule” (172–173).

The voice underpinning the shifts of majors.

Since middle school, Ted used to be bored with repetitive applications of formulas (Int1-1001-Ted lines 131–145). This situation continued to bother him all the way through college. He explicitly remarked that his lack of interest in engineering classes stemmed from this feature: “I like the math of engineering, but I didn’t like applying it to the science” (72–74) and “applying the conclusions of the proofs to problems, that's I think a big reason why I had very little interest in engineering classes” (154–156). He illustrated his idea as follows: “Ohm's Law works the same way with a simple circuit to a super complex circuit; to me, that is not very interesting” (158–159). Ted lost interest in mechanical engineering as soon as he realized that it was a branch of applied mathematics, the main practice of which involved *doing the same thing* by repetitively applying the same “one or two rules” (155). Thus, like applied mathematics, engineering did not expose Ted to new rules and challenges enough, that would sustain his active engagement with this discipline.

Ted declared his profound identification with proofs on several occasions:

- I feel like a champion for the creative side of math [aka proofs]. (Int1-1001-Ted 44)
- [In eighth grade] I would only pay attention to the teachers when they were doing the proofs. (139–140)
- [In college] I definitely still had the attitude where I appreciate the proofs and like the generalized forms much more than applying […] the conclusions of the proofs […] to problems. (152–154)
- I have a strong bias towards like proofs and that part of math that I am part of it. (Int1-1203-Ted 22–23)

Ted shared two features of proofs that explained his strong attachment to this sub-discipline:

(i) the primal status of proofs in mathematics.
(ii) the continual challenge for the brain.

As early as in the eighth grade (Int1-1001-Ted 138–146), Ted appreciated proofs because he thought that if he knew the reason why the formulas work, he could figure out how they work.

He used to irritate his teachers with his "why it works" questions, when all they cared about was the "how it works" instructions. Ted sustained this understanding of proofs as a primal activity in mathematics even at college. For Ted, "to be in the front line of math," one must "generate proofs" (46). He defined mathematics as "a string of proofs," while arithmetic entailed an application of the conclusions of proofs—"some extra bonus to what we have" (160–161).

In addition to the primal status of proofs in mathematics, Ted found the production of proofs stimulating to his brain: "doing a proof is really hard sometimes" (Int1-1001-Ted 168–177) and "it challenged a different part of my brain" (Int2-1203-Ted 81–87). He often contrasted the cognitive processes of producing proofs with the notions of "memorizing" (Int1-1001-Ted 45–46) and repeating familiar procedures (Int1-1001-Ted 168–177 and Int2-1203-Ted 81–87) mechanistically, like a "calculator" (Int1-1001-Ted 177 and 196–198). Although Ted did not unfold his understanding of "the creative part of mathematics," he seemed to have meant that the cognitive demand for new ways of thinking required the construction of proofs. Indeed, each proof problem would bring a new challenge, even though one could use prior knowledge to address it.

For Ted, while engineering and applied mathematics required memorization and application of familiar procedures, proof problems presented new challenges that required the application of new ways of thinking. Contrary to engineering classes, as perceived by Ted, mathematics classes—specifically those related to proof activities—provided him with opportunities to *author mathematics* by developing a personal voice, rather than re-voicing previously established formulas and procedures.

To become an author, Ted needed opportunities to construct ideas he could claim as his own: "doing a proof is really hard sometimes, but once you do it […] that conclusion is yours forever" (see full excerpt below). If the proof was not copied from (and only checked by using) another resource, Ted could still claim it as his own, even when "other people" had found the solution as well. This would not have been the case consequent to the application of preestablished formulas and procedures. Ted rightly resorted to the metaphor of a calculator: a calculator could not appropriate an answer that it produced, because it only executed orders which were authored by somebody else.

<u>Int1-1001-Ted lines 162–177</u> -- *Laziness*

Fady: […] why . you had this resistance or this feeling towards applying mathematics to real world?

Ted: Laziness.

Fady: Laziness?

Ted: Laziness.

Fady: How so?

Ted: like . hum .I feel like yes . doing a proof is really hard sometimes . but like once you do it once . and maybe you look at other people solution that . that . that takes different path to the same conclusion. Once you have that conclusion. That conclusion is yours forever.

Fady: Right.

Ted: Right . whereas like with you doing the arithmetic . it's like . I have to keep doing the same thing over and over using the same rule . and I think that gets boring to me. That's . that's why I have that preference for the proofs . and it . it's part of it is like laziness.

Fady: yeah . laziness mean not doing something boring?

Ted: Right . and . laziness for not doing something that almost feels . somebody else can do that. A calculator can do that. You know?

By delineating two idiosyncratic understandings of mathematics and applied mathematics, Ted was not attempting to carve a mathematical identity to fit his abilities. Rather, he was searching for ecologies that would allow him to author mathematics and, by doing so, author his self *a la* "people who prove things for the first time."

Reaching this conclusion, claim 1 is thus confirmed. Out of an act of rebellion, Ted undertook two shifts, when he dropped his applied mathematics major to pursue his studies in creative writing and when he dropped engineering major for mathematics. In the former shift, Ted revolted against his parents to develop a personal voice that took its roots in studies related to creative writing. In the latter shift, he revolted against the repetitive duplication of existing formulas and procedures to develop a sense of authorship, which seemed more feasible to him within mathematics than within engineering classes.

Figure 3-3 summarizes and illustrates the findings of the analysis pertaining to Ted's identification with "the people who prove things the first time" (C1). I note that this section does not include tables showing the instances of tertiary components C1.2.1 and C1.3.1, because the relevant instances were already present in the tables of their corresponding secondary components C2 (Table 3-3) and C3 (Table 3-4), respectively.

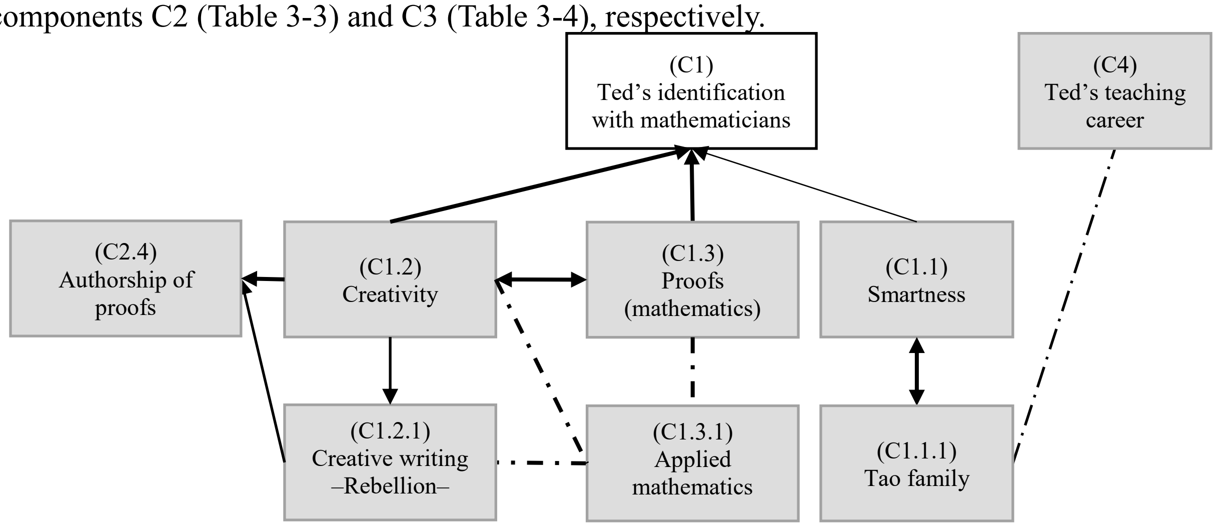

Figure 3-3: Diagram illustrating the interconnectedness of secondary and tertiary components pertaining to the primary component C1 (*Ted's identification with people who produced new proofs*). In this narrative net, the component C1.2 (*creativity*) is the most to influence other components. Note also the negative connections C1.3.1 (applied mathematics) holds with three other components, namely C1.3, C1.2, and C1.2.1.

Legend of arrows: A→B signifies that component A bolsters component B, A↔B means the two components A and B mutually bolster each other, and A - - B (interrupted line) means that the components A and B are positioned in opposition to each other. For example, Ted positioned (C1.3.1) applied mathematics as comprising a void in creativity (C1.2). Proofs (C1.3) and Creativity (C1.2) mutually bolster each other, since for Ted proof problems offered opportunities to create new, non-memorized ways of thinking; moreover, he was interested in proofs because he valued creativity. Ted admired "the people who prove things for the first time" (C1) because of their creativity (C1.2); the converse (C1→C1.2) was found to be untrue (Ted did not appreciate creativity because of genius mathematicians). Likewise, Ted's appreciation of proofs led him to admire mathematicians who produced new proofs (C1.3→C1), and not the other way around. The three thicknesses of lines and arrows indicate the strength of connections between the components: 0.5 pt for weak connections, 1 pt for regular connections, and 1.5 pt for strong connections. For example, Ted admired mathematicians more for their creativity (C1.2) than their smartness (C1.1). Thus, the arrow for C1.2→C2 is 1.5 pt, while the arrow for C1.1→C2 is 0.5 pt.

(C2) The challenges of proof production.

Ted mentioned the challenges related to production of proofs in forty-one references throughout the four interviews (see Table 3-5). He repeatedly noted that he used to spend several hours on certain proof problems. In the early interview, he reported that he used to spend only about an hour trying to construct proofs on his own before consulting published resources (Int1-1001-Ted lines 181–192). In the later interviews, he reported that he had devoted many hours to solving problems on his own without the aid of published resources. The period of time most commonly used in relation to Ted's unaided solo work on proof production was "three hours" (in five following instances: SCNI-1029-Ted 97–104, Int2-1203-Ted 20–30, 313–318, 385–389, and 656–659). This section will investigate the factors that bolstered Ted's stamina in proof production.

The PDE framework is employed to manage the large amount of references related to proof challenges. The PDE principles will be used to group the references into smaller components, more manageable for analysis, that are relevant to productive engagement in constructing proofs.

Each reference, i.e., each row in Table 3-5, consists of the optimal conversational moment necessary to accurately understand the instance pertaining to the challenges related to proof production. References are conjoined if they overlap. Table 3-6 indicates which of the PDE principles are mentioned in each reference of Table 3-5. Figure 3-4 represents the results of the classification. As observed in Table 3-6 and noted in Figure 3-4, some references mention multiple principles and the relationships between them. Thus, the resulting classes are not orthogonal.

Table 3-5: Ted's narratives on the challenges associated with proof production.

Data source	Reference	*Selective quote*
Int1-1001-Ted	168–170	*Doing a proof is really hard sometimes. […] But once you have that conclusion. That conclusion is yours forever.*
Int1-1001-Ted	181–192	*I spend a big chunk of my time not looking at the textbook and just trying to figure [the proof] out using like what I already know. After about an hour looking for a proof and I can't figure it out then I might consult the books. But usually I give it a good attempt to actually try to prove it myself first before I look at any resources. […] Whenever I'm stuck –happens once in a while–, I tend to complain to my cat. [chuckles] That's a little bit of it. I'm like "can you tell me the answer?" And another way that I deal with it, besides looking it up, would be just to take a long walk and just come back to it.*
Int1-1001-Ted	268–281	*The only thing that I found sometimes challenging [in my group] is when we have different approaches to the same problem. […] The intention of the conversation really is to get to the elegant proof, one that's not only correct but nothing extra, nothing less. When we have conversations around things like that, I feel like there can sometimes be some push back simply because […] people want to defend their answers.*
Int1-1001-Ted	290–311	*Jeremy is very high standish for himself in term of what he wants the proof to look like when he's working on it. He likes that first draft to be sufficient as a proof. […] John is very confident in his math […]and good about sharing his math, he is the one that gets defensive the easiest when we do have conversations about whether or not it is sufficient and so on.*
SCNI-1029-Ted	19–20	*[On the midterm] I had a 110 in my proofs class, the prerequisite for [number theory] class, essentially the first class that I'm taking concurrently with it.*
SCNI-1029-Ted	81–84	*This was a breakthrough moment, when John really talked about showing they can't be equal. [I got the idea of proving it by] contradiction. That's the basis of solving this problem.*
SCNI-1029-Ted	97–104	*I felt like a boss. We all broke through with this problem. […] I guess I just been staring at this problem for like 3 or 4 hours before class. I would just look at it for like 10 or 15 minutes at random intervals when I'm working or when I'm waiting for the bus or anything like that, just like look at the problem. Think for a little bit and put it away.*

Data source	Reference	***Selective quote***
SCNI-1029-Ted	111–113	*And like at that moment, it was like all those little moments of looking at it coming together. You know just clicked, all of it falling into place. It's like all the big pieces are there now and just like thinking about how to order, so that it sounds logical when you are writing the proof. That is what's going on in my mind.*
SCNI-1029-Ted	225–229	*I try to do the work ahead of time, whenever I have time on the weekends. I'm done with the homework by Tuesday usually. And then I'll show up and be like here it is.*
SCNI-1029-Ted	277–293	*We were struggling with that [i.e. manipulating powers in modular equivalences] quite a bit at this point.*
SCNI-1029-Ted	347–349	*Professor Hoffmann came by to talk about the divisibility, to use number two that like drove us to get to a breakthrough.*
SCNI-1029-Ted	371–375	*When Bettie is here, she's fairly positive and keeps it humorous. Plus she tends to be the one to bring out the references […] and we can sometimes like get around problems without Professor Hoffmann's push, by looking at the references that she provides.*
SCNI-1029-Ted	385–396	*There are times where we are presented with really new definitions and we struggle with the definition of things for quite a while before we get fluid into the work.*
SCNI-1112-Ted	121–122	*I was spending some time thinking about how to justify it to John and Jeremy cause they're good skeptics.*
SCNI-1112-Ted	151–164	*I try really hard not to take pictures of other people's work. So that was inefficient use of time. But it helps me understand what [Boutros] was doing cause I was doing it too.*
SCNI-1112-Ted	225–230	*When I have no idea how to approach a proof I do that. I just write down everything that I know and then find out some stuff that we can deduce from what we know before starting.*
SCNI-1112-Ted	265–283	*I get into my best teacher voice and I try to explain it to myself.*
SCNI-1112-Ted	286–290	*This was I think important for like finishing up the proof: to see that it is an even number.*
SCNI-1112-Ted	292–296	*We stay stuck like this for quite a while. And then eventually […] [Bettie] showed us, she actually found the solution to this part in the book and I glanced at it. […] and then we basically solved it.*
SCNI-1112-Ted	317–331	*I just enjoy the struggle, when I'm dealing with a math problem that's just the right level of out of reach. […] I feel a mild sense of shame every time I have to look into the book to get the answer.*
Int2-1203-Ted	20–30	*I need to be really patient to myself. And that was a big deal. Sometimes it's like you know when you stare at a problem for two, three hours, you got frustrated sometimes. But like to control that and use that as fuel for finishing it up.*
Int2-1203-Ted	81–87	*[The number theory class] challenged a different part of my brain […] we are proving what we need to know.*
Int2-1203-Ted	95–113	*That's kind of a new step for me on top of a new step. so that's why that one was challenging.*
Int2-1203-Ted	114–129	*[With the Chinese remainder theorem] I was very stubborn. […] I try to not look at any references and just look at the problem and bang my head against it for a few hours before I finally look at the book. […] [I solved this by] eventually looking it up in the book and other references online […] Wikipedia and books on scholar.google.com*
Int2-1203-Ted	130–134	*I was refreshing the idea of what contrapositive is for example, little things that you need to know to prove things. I was just like refreshing that. Numerically I understood divisibility. But how to write it properly, that was the beginning challenge in the class.*
Int2-1203-Ted	135–166	*I did take [a proof] class up to the half way point. but then I dropped from school for that semester […] [the proof class taken concurrently] the timing was off [with number theory] for some topics. But right at the beginning, it was really matched up. […] This class [i.e. number theory] is making it so that class [i.e. proof] is just so easy. Because everything I do here is application of what I have to learn in proofs.*
Int2-1203-Ted	167–180	*[Helpful resources are] referencing the books for sure and talking to other people about it […] Other people would challenge your proofs. And that challenge makes your proof more refined.*
Int2-1203-Ted	182–191	*I would often tackle a problem and then go to sleep. Then come back and look at the solution [the next day] to review it.*
Int2-1203-Ted	219–225	*[In groupwork outside classroom, we would be] checking with one another. […] if we were all at the same problem and we feel stuck, then we both write down what we have known and then we would look at each other's –cross reference.*

Data source	Reference	*Selective quote*
Int2-1203-Ted	261–267	*The homework we didn't get to work on the class, we work on it in group outside. […] sometimes they'll be proofs.*
Int2-1203-Ted	313–318	*[When I'm stuck, I think overnight] I will try my best for like three, four hours and then look at the books. Sometimes I would ask Jeremy or John.*
Int2-1203-Ted	339–351	*[I submitted an incomplete homework] once, […] the one about the mu function. […] I wanted to dedicate time to study for the final […] I read the proof […] online. But I have no idea what it meant. And I wasn't about to write down something that I didn't understand.*
Int2-1203-Ted	352–366	*My only frustration is to never get a perfect grade […] The things that are marked off are like that I cannot dispute it. I did that wrong. […] That feel would make me a better mathematician.*
Int2-1203-Ted	385–389	*I feel I could be more efficient because I am too stubborn. Cause I spend so much time like on problems. I know that I spent three or four hours on problems that I should have spent half an hour on them and then look it up.*
Int2-1203-Ted	391–406	*[This class is] five out of five [because of] the challenge and the chance to actually prove things. […] There's no more hand holding. but there's just the right amount of guidance when the professor comes around to check-up on you.*
Int2-1203-Ted	411–421	*Now I realize that proofs is something doable. It just takes a lot of time. and patience. […] I would read the proof and just understand it, and then just move on. And now I try to actually prove things.*
Int2-1203-Ted	465–473	*We have been able to bring forth […] what they're good at to the table to solve things together. […] I can be kind of negative and stressed out a lot. Bettie's energy will balance that out. Boutros' calm demeanor cools down the group, cause like John and I can get really intense whenever we have these talks. Jeremy is just like a great addition too, cause he's just like so able to appreciate the beauty within what we do. […] We all just take a moment to pause and appreciate the beauty of what we were doing.*
Int2-1203-Ted	477–480	*John is extremely rigorous with his proofs. He likes to be the skeptic and the questioner. and I really like what he does with that. Whenever he reads anybody's proofs, he kinda goes in with the mind set of "let me critique this" and he's a very critical thinker.*
Int2-1203-Ted	482–486	*[Bettie] is like "I just gotta go back and memorize it" and I think it's making proofs really hard for her.*
Int2-1203-Ted	523–527	*Me, John and Jeremy are being very stubborn and trying to do it without any references. [Bettie] would be the one to be like "hey wait but it's in the book. look." And then we would read the book and then go back and forth to explain the proof in the book.*
Int2-1203-Ted	656–659	*The way I approach personal life is similar to how I approach proof problems. When I find something unsolvable, I would think about it for three or four hours before I go ask for help.*

Table 3-6: The classification of Ted's 40 conversational moments with regard to the secondary components and the four principles of PDE.

Data source	Reference	Problematizing	Resources	Authority	Accountability
Int1-1001-Ted	168–170	x		x	
Int1-1001-Ted	181–192	x	x	x	
Int1-1001-Ted	268–281	x		x	
Int1-1001-Ted	290–311	x		x	x
SCNI-1029-Ted	19–20		x		
SCNI-1029-Ted	81–84		x	x	
SCNI-1029-Ted	97–104		x	x	
SCNI-1029-Ted	111–113			x	x
SCNI-1029-Ted	225–229			x	
SCNI-1029-Ted	277–293	x			x
SCNI-1029-Ted	347–349		x		
SCNI-1029-Ted	371–375		x		
SCNI-1029-Ted	385–396	x			x
SCNI-1112-Ted	121–122	x			x
SCNI-1112-Ted	151–164		x	x	
SCNI-1112-Ted	225–230		x	x	

SCNI-1112-Ted	265–283		x	x	
SCNI-1112-Ted	286–290			x	
SCNI-1112-Ted	292–296		x		x
SCNI-1112-Ted	317–331	x	x	x	
Int2-1203-Ted	20–30		x	x	
Int2-1203-Ted	81–87	x		x	x
Int2-1203-Ted	95–113	x			
Int2-1203-Ted	114–129	x	x	x	
Int2-1203-Ted	130–134	x			x
Int2-1203-Ted	135–166	x	x		
Int2-1203-Ted	167–180	x	x		x
Int2-1203-Ted	182–191			x	x
Int2-1203-Ted	219–225		x	x	
Int2-1203-Ted	261–267		x	x	
Int2-1203-Ted	313–318		x	x	
Int2-1203-Ted	352–366	x	x	x	x
Int2-1203-Ted	339–351			x	x
Int2-1203-Ted	385–389	x	x	x	
Int2-1203-Ted	391–406	x	x	x	
Int2-1203-Ted	411–421			x	
Int2-1203-Ted	465–473		x		
Int2-1203-Ted	477–480	x			x
Int2-1203-Ted	482–486			x	x
Int2-1203-Ted	523–527		x	x	x
Int2-1203-Ted	656–659		x	x	
Total by principles (N=41)		**18**	**24**	**28**	**15**

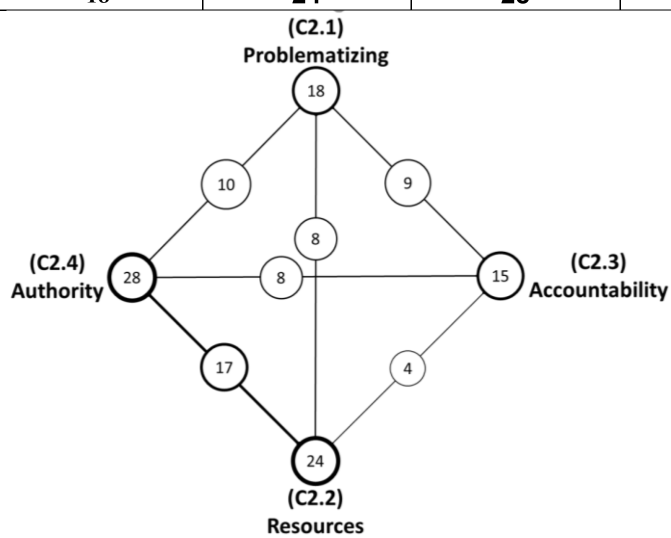

Figure 3-4: The classification of Ted's references to the challenges of proof production (N = 41) across the PDE principles and the second components. The circled numbers near the components' names represent the numbers of conversational moments in Ted's narratives where the component is mentioned. The circled numbers on the line segments represent the number of conversational moments in Ted's narratives which mention the relationship between the components, joined by the lines. For instance, 14 conversational moments mention the rapport between authoring mathematics by counting on self as opposed to looking for solutions in published resources. The size of contours is proportional to the number of conversational moments.

The results of this classification (Figure 3-4) show that Ted's narratives on proof production involved elements pertaining to all the four principles of PDE. Subsequently, every PDE principle can be validly considered to be a secondary component of C2. The distribution of principles over the references is skewed towards the principles of authority (68%) and the resources (58.5%). The four principles overlap with each other in about 22% of the references on average. The overlap of the principles of authority and the resources is strikingly referenced most often (41%), while the overlap of accountability and resources is referenced least often (10%). The references of other combinations of overlapping principles are ranged between 19% and 25%.

A subsection is dedicated to analysis of and report on each of the secondary components, C2.1 through C2.4, as well as their connections to the principles of PDE.

C2.1: Elements that increased the degree of the challenging aspect of proofs for Ted (Problematizing).

C2.2: Ted's use of available resources to address the challenges (Resources).

C2.3: Ted's and his groupmates' accountability to the socio-mathematical norms established in their group, in their classroom, and in the subdiscipline of proofs (Accountability).

C2.4: Ted's authorship of proofs (Authority).

Note that an element gains narrative power not only by the frequency of its occurrence but also as a result of its wording and its connections to other central components of the narrative. Thus, the classification made here is indicative and cannot preclude the necessity of consulting Ted's speeches, which is shown in the following subsections.

(C2.1) Problematizing.

Four processes supported the problematization of mathematical tasks for Ted: the lack of prerequisite knowledge, the cumulation of new definitions, the groupwork design, and Ted's groupmates, namely John and Jeremy. The first three processes were reported as remaining the same throughout the semester, while the fourth endured a change within the same period. Ted indicated that a shift had occurred in his understanding of John's positions in groupwork. This subsection describes the four problematizing processes reported in Ted's narratives and then presents an analysis of the shift of Ted's perception, through the VIP+function framework.

First, Ted was simultaneously taking the number theory class and its prerequisite, the proof class (SCNI-1929-Ted lines 19–20, and Int2-1203-Ted 130–134 and 135–166). At the beginning of the semester, the materials learned in the proof class supported Ted's learning in the number theory class. But soon, the opposite became the only case: materials covered in the number theory class supported Ted's learning in the proof class. He had to find alternative resources to build his prerequisite knowledge.

Second, the curriculum of the number theory class introduced new definitions within a short amount of time, which did not allow Ted to reinforce new concepts (SCNI-1029-Ted 385–396 and Int2-1203-Ted 95–113). Ted and his groupmates struggled whenever a new definition was introduced, especially when it built on previous definitions.

Third, Ted noted that the group design regarding the number theory class provided a good balance between challenging mathematical tasks and the right amount of help from the teacher. Ted rated the number theory class as "five out of five," because of "the challenge and the chance to actually prove things […] there's no more hand holding, but there's just the right amount of guidance, when the professor [came] around to check-up on you" (Int2-1203-Ted 391–406).

Fourth, Ted identified two groupmates, John and Jeremy, who were "skeptical" and mathematically demanding (Int1-1001-Ted 268–281, 290–311, SCNI-1112-Ted 121–122, and

Int2-1203-Ted 167–180 and 477–480). In groupwork, Jeremy pushed his peers to produce high-quality, neat proofs while John often critiqued his groupmates' ideas. Hence, Ted realized that he needed to refine his thoughts to meet his groupmates' mathematical exigencies.

(C2.2) Resources.

The resources Ted reported in his narratives can be grouped by their functions in two categories: to provide mathematical knowledge and to sustain engagement when one became stuck. When the mathematical resources failed to support a *productive* engagement with a particular task, Ted took aid of some surprising resources in order to sustain at least an *engagement* with the task and refrain from giving up.

The resources that supported Ted with mathematical knowledge are mentioned as follows:

- a learned set of *heuristics* (SCNI-1112-Ted 225–230 and Int2-1203-Ted 219–225),
- *books and other online resources*, such as Wikipedia (Int1-1001-Ted 181–192, Int2-1203-Ted 114–129, 167–180, 313–318, 339–351, and 385–389),
- *Proof class* taken simultaneously with the number theory class (Int2-1203-Ted 135–166),
- *John*, for having a robust mathematical knowledge and knowing how to share his ideas (Int1-1001-Ted 290–311, SCNI-1029-Ted 81–84, and Int2-1203-Ted 313–318),
- *other students* who helped him refine his proofs (Int2-1203-Ted 167–180), and
- professor *Hoffmann* (SCNI-1029-Ted 347–349 and Int2-1203-Ted 391–406).

The resources that sustained Ted's engagement during resourceless situations are mentioned as follows:

- *talk to his cat* in solitary work (Int1-1001-Ted 181–192),
- *take a break* in solitary work (Int1-1001-Ted 181–192 and Int2-1203-Ted 313–318),
- *Bettie's humor* and *Boutros' calm demeanor*, that balanced Ted's and John's intense conversations during groupwork (SCNI-1029-Ted 371–375 and Int2-1203-Ted 465–473), and
- *Jeremy's appreciation of mathematical beauty* that impelled his groupmates to "pause and appreciate the beauty of what [they] were doing" (Int2-1203-Ted 465–473).

Ted enjoyed the struggle with proof production when the "math problem [was] just the right level of out of reach" (SCNI-1112-Ted 317–331). Nonetheless, he maintained an engagement with the tasks for a long period, sometimes up to four hours, even when the mathematical problems far outweighed his resources (Int1-1001-Ted 181–192, Int2-1203-Ted 114–129, 313–318, and 656–659). He attributed this behavior to his stubborn character (studied in the subsequent subsection).

(C2.3) Accountability.

Strikingly, all the instances that mentioned the principle of accountability occurred in references which also mentioned at least one other principle (see Table 3-6). More importantly, the talk regarding accountability co-occurred with the reference to three other principles (see Figure 3-4). Although the number of references that mentioned the principle of accountability (15 references) was the smallest compared to the other principles of PDE (18, 23, and 26 references), the principle of accountability meaningfully tainted the other principles. This subsection reports how Ted's accounts of the principles of authority, the resources and the notion of problematizing were connected to the principle of accountability towards the classroom norms, the group members and the disciplinary norms.

Additionally, the references of C2.3 reports one identity-related shift: the shift of Ted's perception regarding John's critiques of his groupmates' ideas. This shift is closely analyzed and reported in this subsection.

Ted expected of his mathematical *authorship* to manifest an accurate understanding of the discipline. He abhorred the methods of writing which required memorization without understanding (Int2-1203-Ted 482–486). On the contrary, he refused to complete a homework problem because he could not make sense of the proof that he found online (339–351). Ted strived to make his proofs as logical as possible (SCNI-1029-Ted 111–113). Indeed, after writing his proofs, he often took time to revise them to ensure that they were logically sound (Int2-1203-Ted 182–191). As per the design of the number theory classroom, the graded weekly homework reinforced the principle of accountability. Ted appreciated the rigor of the grading, for it made him "a better mathematician" (352–366).

As for accountability and *resources*, Ted's group was flexible with the classroom norm that Professor Hoffmann attempted to establish. Since the proofs and answers to most of the problems in the worksheets were published either within the textbooks or online, Professor Hoffmann encouraged his students to count on their knowledge in solving problems without looking at published answers. While Ted, John, and Jeremy resisted looking into the books, they tolerated Bettie's recourse to textbooks for solutions (SCNI-1112-Ted 292–296). At certain moments, they appreciated the fact that she pointed to solutions present in a book because they could consequently avoid calling the professor for support (SCNI-1029-Ted 371–375). Even when the group ended up using the solution given in a book, they would spend sufficient time discussing and explaining the same so that it made sense to them (Int2-1203-Ted 523–527).

In his narration, Ted held the *problematization* processes as accountable to the group members and the discipline. He felt impelled to construct robust mathematical arguments that would meet John's and Jeremy's mathematical exigencies (SCNI-1112-Ted 121–122). He recognized Jeremy and John as students who were accountable to the mathematical discipline; they were "rigorous with proofs" and "very high standish" regarding how proofs should look like (Int1-1001-Ted 290–311 and Int2-1203-Ted 477–480). In fact, they helped Ted refine his proofs (167–180). Ted described the culture of his groupwork in the classroom as one that was carefully attentive to the proof norms. They cared about translating their intuitions into mathematical statements (130–134). Furthermore, the group spent sufficient time to accurately grasp new definitions (SCNI-1029-Ted 385–396). Also, they attended to justifications of their mathematical actions and the steps within the proofs (277–293 and Int2-1203-Ted 81–87).

At the beginning of the semester, Ted perceived John as someone who problematized the task in order to defend his authorship instead of the disciplinary norms. As per Ted's testimony, John used to render the group dynamics unproductive (Int1-1001-Ted 268–281). However, Ted seemed to change his perception of John's behavior by the end of the semester. The following subsection presents the analysis of Ted's perception of John's behavior.

Shift in Ted's perception of John's position

Ted shifted his understanding of John's critiques in course of the semester. Early in the semester, Ted perceived John as someone who brought his "ego into play" within groupwork and easily became defensive while discussing his groupmates' multiple solutions (Int1-1001-Ted 268–283 and 300–302). He reconstructed John's inner voice when he defended his own solution as follows: "this is my answer I want to defend it. I am only doing the way this . is and give some some explanation or excuse" (278–279).

However, as the semester went by, Ted ended up appreciating John's position in groupwork (see excerpt below).

> Ted: [John] likes to be the skeptic and the questioner. and I I really like what he does with that. umm whenever he reads anybody's proofs . he kinda goes in with the mind set on . "let me critique this." and he's a very critical thinker. (Int2-1203-Ted 477–480).

Ted's shift of perception regarding John's critiques can be analyzed through the VIP+function framework as follows. Early in the semester, Ted ascribed to John's *position* (p1: critiquing his groupmates' ideas) an ego-centric authorship *identity* (i1) and an egotistic *function* (f1: to defend his answers). As Ted worked with John in the same group for a long period of time, he got to adjust his perception of John's position (p1). He realized that John opposed his groupmates' proofs whether or not they critiqued his ideas. Thus, he understood that critiquing others' ideas (i2) was an entrenched identity for John and, thus, substituted it with his previous misperceived interpretation (i1). His new understanding of John's identity allowed Ted to reevaluate his perception of John's function (f1) and replace it with a new one, i.e., to problematize contributions (f2). Evidently, the new composition of the VIP+function (i2-p1-f2) prevented unduly social interactions, which were bolstered by its earlier composition (i1-p1-f1), and instilled productivity within group dynamics. This shift provided an instance where the task of building an accurate understanding of groupmates' identities fostered productive disciplinary engagement.

Table 3-7: A VIP+function analysis of Ted's perceptions regarding John's position in the early (first row) and late (second row) interviews.

Reference	Voice	Interpreted identity	Behavioral position	Inferred function
Early: Int1-1001-Ted 268–283 and 300–302.	**John's voice in Ted's speech:** "this is my answer I want to defend it. I am only doing the way this. is and give some explanation or excuse"	**i1:** Ego-centric authorship	**p1:** Critiquing groupmates' solutions	**f1:** To defend his answers.
Late: Int2-1203-Ted 477–480	**John's voice in Ted's speech:** "let me critique this"	**i2:** Skeptic and questioner	**p1:** Critiquing groupmates' solutions	**f2:** To problematize the task for accountability to the discipline.

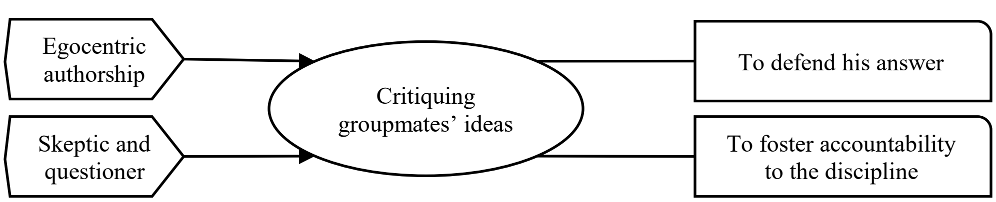

Figure 3-5: VIP+function of Ted's interpretation of John's behavior within groupwork. The diagram accompanies the table above it. When Ted repaired his interpretation of John's identity, from egocentric authorship to a skeptical worldview, he adjusted his understanding of John's intentions from his critiques on others' ideas.

(C2.4) Authority.

The principle of authority regarding the challenges of proof production prevailed throughout Ted's references (in 27 references within Ted's narratives). Most striking was the number of references that addressed the tension between authority and resources: in 10 references, Ted mentioned mathematical authorship and the temptation of looking into references for the solutions. He claimed to have spent up to four hours on solving some problems and traced his avoidance of looking at references for solutions to his stubborn nature. The investigation of Ted's stamina in proof production comprises the subject of a subsection, after providing a report of the remaining aspects of the principle of authority in his narratives.

For Ted, authorship yielded ownership ("doing a proof is really hard sometimes […] but once you have that conclusion, that conclusion is yours forever" Int1-1001-Ted 170) and must have been built on a robust understanding of concepts, as set by the discipline (discussed under the Accountability section). Ted noted two moments in his method of authoring proofs: deducing, then organizing. In the first moment, Ted would generate "stuff" from the given problem and his own mathematical knowledge (SCNI-1112-Ted 225–230). In the second moment, he would organize all the pieces of information he had possessed and produced in way that would adhere to sound logic, in order to make a proof (SCNI-1029-Ted 111–113).

Ted appreciated the fact that the number theory classroom offered him an opportunity to author mathematical proofs (Int2-1203-Ted 81–87 and 391–406). Despite the classroom design that set students to work in groups, Ted's references on authorship were exclusively individualistic. Although he acknowledged moments where group members built proofs off each other's ideas (SCNI-1029-Ted 81–84, Int2-1203-Ted 219–225 and 261–267), he claimed the write-up of proofs and answers as his own. He even went as far as interacting with the study group having already done homework problems on his own (SCNI-1029-Ted 225–229).

Why Ted spent hours attempting to find proofs.

Ted emphasized in the interviews that he strongly resisted consulting references and used to endure failed attempts at finding proofs for many hours (Int1-1001-Ted 181–192, Int2-1203-Ted 114–129, 313–318, and 656–659). By the end of the semester, Ted regretted this resistance: "I feel I could be more efficient because I am too stubborn […] I know that I spent three or four hours on problems that I should have spent half an hour on them and then look it up" (Int2-1203-Ted 385–389). Why would Ted torture himself for many hours by attempting to prove theorems that had been already proven? Was his behavior an instantiation of a personality trait, namely being stubborn, or an attempt to foster a mathematical identity related to authorship? I put forth the two claims to be tested.

Claim A: Ted's endurance with the challenges of proofs stemmed from his stubborn personality trait.

Claim B: Ted's endurance with the challenges of proofs was an attempt to foster his mathematical authorship.

Ted traced this behavior back to his stubborn personality: "I'm fairly stubborn so I will try really really hard to not look at anybody else's solution" (SCNI-1112-Ted 91–97), "I was very stubborn […] I try to not look at any references" (Int2-1203-Ted 114–129), "I am too stubborn cause I spend so much time like on problems" (385–389), and "me, John and Jeremy are being very stubborn and trying to do it without references" (523–527).

Ted’s so-called stubborn character was in fact reflective of his entrenched habit of relying on himself. In fact, Ted notified this to the interviewer: “the way I approach personal life is similar to how I approach proof problems; when I find something unsolvable, I would think about it for three or four hours before I go ask for help” (Int2-1203-Ted 656–659). In a subsequent conversational moment, he brought up the story of his time in Utah, where he first moved to live on his own. Instead of seeking help from his parents, against whom he was revolting, or friends, he endured a whole month of snow without heat nor food. The reliance on self, in this case, seemed to achieve a sort of autonomy or independence from parental authority.

Int2-1203-Ted lines 674–681 -- *Ted’s reliance on self despite resourcelessness*

Ted: I'm not as broke as I was four years ago. uh I think.

Fady: now cause you live on your own . right?

Ted: Yeah . when I first started living on my own . this is . [sighs] I remember one month where I was in the winter in Utah. and it was like snowing and all that . I didn't have money for heat . and I didn't have money for food besides water. That was all I had. I had water for a whole month in the snow with no heat. That was . that was . brutal.

Fady: Very survival man. Nice.

Ted: That's why I’m like “oh this math class is tough but . it is not winter with [unintelligible]”

The reliance on self was also evident in Ted’s behavior during groupwork. Take, for instance, the group session on 11/12. Students were asked to compute the quadratic residues of 2, 3, 5, 7, and 11 (see Wk10#1). Ted had been using a time-consuming computational strategy. Then, he noticed that Boutros had already finished most of the computations on the shared dry erase poster board. Such computations consisted of the application of a simple procedure “over and over again,” an activity of negligible interest for Ted (see C1.3.1). However, Ted started making the computations on his side of the dry erase poster board, instead of merely taking a picture of Boutros’ work, which was what his groupmates did. He acknowledged that relying on himself to make the computations gave him an autonomous understanding thereafter.

SCNI-1112-Ted lines 91–97 -- *Ted’s comment on when he looked at Boutros’ computations.*

Ted: [pauses video] I'm fairly stubborn. So I will try really really hard to not look at anybody else's solution . and try to come up with my own before . I reference everybody else's work. Once I get to the point I feel kind of stumped . like 90 percent sure I can’t do it alone . then I look at other people's.

SCNI-1112-Ted lines 151–164 -- *Ted’s comments on when he turned to do the computations by himself using Boutros’s strategy.*

Ted: I try really hard not to take pictures of other people's work. So that was inefficient use of time. But it helps me understand what [Boutros] was doing cause I was doing it too.

The evidence consulted thus far supports the qualified claim A: the reliance on self was an entrenched habit for Ted, which was actuated in different contexts. In Utah, he relied on himself and tried to construct his autonomy vis-à-vis his parents. In the number theory classroom, his self-reliance led to self-attempts made to enhance his understanding of mathematics.

To complete the investigation of Ted’s endurance regarding proof production, I investigated Ted’s feelings when he succeeded in constructing a proof after many hours of struggle. Such an event happened in the group session on 10/29, which was followed by an SCNI interview (see SCNI-1029-Ted lines 50–104). On 10/29, students worked on reproducing the proof of Euler's theorem (Wk3#4). As per his testimony, Ted spent up to four hours thinking about the proof with no avail, before going to class. He arrived half an hour late, when his groupmates had already made a few attempts. At some point, John remarked that two elements of the residue system could not be equal. John's remark triggered Ted to construct the proof by contradiction strategy; he tried the same, and found that it worked. Ted commented on his celebration after finding the solution as follows.

SCNI-1029-Ted lines 96–104 -- *Ted's comment on when he found a proof*

Fady: How did you feel here?

Ted: What was I feeling? ummm . at that point . I . I felt like a boss. [laughs] I like . felt . like . like we all broke through with this problem . and I don't know. so . I guess I just been staring at this problem for . like 3 or 4 hours . you know before class . I think I think like I would just look at it for like like 10 or 15 minutes at random intervals when I'm working or when I'm waiting for the bus or anything like that. Just like look at the problem . Think for a little bit . and put it away and like at that moment . it was like all those those little moments of looking at it coming together. You know just clicked . all of it falling into place.

The cognitive experience of reorganizing pieces of knowledge into a coherent whole generated a feeling of authority for Ted ("I felt like a boss"). His use of the "boss" metaphor seemed to support claim B: as mentioned, Ted endured many hours of failed attempts at proving theorems because he wanted to foster a mathematical authorship, which would become undermined if he took aid of the existing proofs. Authorship can be seen as an integral moment in the path of gaining authority (Engle, 2012). Ted's ultimate goal of testing his ability in constructing proofs might have been aimed at supporting his discernment with regard to whether he could undertake studies in advanced mathematics (see the section on C3: Ted's interest in master's programs).

Thus far, the investigation found evidence in support of both claims. In search of further relevant evidence, I pursued the investigation of metaphors, starting with the ones that transpired in Ted's jubilation on finding the solution (see excerpt above). Ted described his cognitive processes, by which he found the proof, as one requiring work that was similar to solving puzzle games: "think for a little bit and put it away," "all those little moments," and "all of it falling into place." Ted happened to have used the puzzle metaphor in the beginning of the early interview. He was asked to share any words and images which the word "mathematics" evoked in his mind.

Int1-1001-Ted lines 2–8 -- *Ted's comments on how mathematics is fun*

Ted: Fun . visual . spatial . patterns . poetic . driven by creativity.

Fady: Why do you see it fun?

Ted: well ummm . If you want to win a strategy game you'd better be good at math. So ever since I was little I played a lot of games. You know board games. And to me . Math was the way that I could win games. Like when I was little [unintelligible] then eventually math became like puzzles to me. I was always better at these games because of math.

In the brainstorming activity of the early interview, the first word that came to Ted's mind on hearing the word "mathematics" was "fun." Ted elaborated how mathematics was fun for him: "to win a strategy game you'd better be good at math." The first image in his mind that connected with mathematics was his memory of winning games since his childhood because of his mathematical knowledge ("I was always better at these games because of math"). In another conversational moment, Ted noted that during high school, he used to impose his solutions on his groupmates in class to finish soon and "go back to playing cards with [his] friends" (Int1-1001-Ted 214–218).

Strikingly, in the brainstorming activity of the exit interview, "fun" was the first image that Ted evoked on hearing the word "number theory" (Int2-1203-Ted 6–23). This time, however, the image of "fun" involved the challenge of proving theorems.

<u>Int2-1203-Ted lines 20–23</u> -- *Ted elaborating on number theory as fun*

Fady: You said that is fun . why?

Ted: mmm . because it is challenging. Um and because . because it's requiring me to prove things instead of um just do computations. and I have a strong bias towards like proofs and that part of math that I am part of it.

How could proving theorems be fun? While elaborating on mathematics as fun in the early interview, Ted asserted that "math became like puzzles to me" (Int1-1001-Ted 7–8). In a different conversational moment, Ted defined mathematics as "a string of proofs" (160). When he studied for the midterm, he "stringed" all definitions, axioms, and proofs covered in the number theory class "in a logical progression" (see excerpt below and also read Int1-1001-Ted 193–201). During an SCNI interview, he described his thought process with regard to the construction of proofs as a puzzle-solving task: "all the big pieces are there now and just like thinking about how to order, so that it sounds logical when you are writing the proof; that is what's going on in my mind" (SCNI-1029-Ted 111–113). As rendered evident, Ted did perceive mathematics as a puzzle, the pieces of which were axioms, definitions, and theorems that were connected to each other (or "stringed") through proofs. Ted had fun proving theorems despite enduring many hours of struggle, because (refer to claim C) he was seeking to relive a specific historicity, namely "being always better at" board and puzzle games.

<u>Int1-1001-Ted lines 193–201</u> -- *Ted shares how he studies for the midterm*

Fady: How did you prepare for the midterm?

Ted: hum . I made a list of every single theorem that we covered. um . and then I made a list of all the . definitions and the axioms that are related to the theorems. and then stringed them all together in a logical prof . progression. um . and then I looked at the proofs for everything that was a theorem. Like to everything.

With this new understanding of Ted's embodied meaning of mathematics, which was strongly associated with winning games as per his own admission (claim C), we can reevaluate claims A and B. Ted's acclamation "I felt like a boss," when he found the proof of the first step of Euler's theorem after hours of struggle (SCNI-1029-Ted 98), can be interpreted as an acclamation of winning, being the first one to have pieced the puzzle together. In this case, "boss" would have meant "winner." Nonetheless, winning a game may involve assuming some sense of authority and power over other people. Recall that Ted voiced his feelings on winning by using a comparative expression, "I was always *better* at these games" (Int1-1001-Ted 8). Thus, Ted was authoring his own self as a winner by putting together mathematical proofs, just like he did he used to solve a puzzle game.

As for claim A, the habit of relying on self could be interpreted within claim C, i.e., Ted was seeking to relive his history of winning games by constructing proofs. Note that Ted claimed individual superiority: "*I* was always better at these games" (Int1-1001-Ted 8) and "*I* felt like a boss" (SCNI-1029-Ted 98). To take full credits for a win, one must rely on oneself solely. Claim C could explain why Ted felt a "sense of shame every time [he had] to look into the book to get the answer" (SCNI-1112-Ted lines 329). Ted used to go to the group study outside the classroom having already solved the homework, in order to have a chance to reinforce his knowledge by explaining *his* answer to his groupmates (see C4.6.1). Additionally, Ted used to type his answers to the homework in a Pdf document and share it with his groupmates as his own work. Had he looked for answers in references, Ted would not have been able to claim a genuine win and take full credits of his work. According to claim C, reliance on external resources would have doctored the win.

Overall, claim C—Ted was seeking to relive his historicity of winning games by constructing proofs—was strongly supported by the evidence. Additionally, it encompassed the other two alternative claims, A and B. Thus, Ted's identity as "games player and winner" was the most powerful resource that sustained his engagement with the production of proofs for many hours, without looking at external resources.

Ted's motivated identity that was entrenched in the notion of winning games would not suffice to render his engagement with proof challenges meaningful and productive. Ted's groupmates, his instructor, the group design, and the course curriculum significantly contributed to his successful experience with the proof challenges, as noted through the narrative analysis of component C2. Ted described his engagement with the proof challenges in ways that accounted for the four principles of productive disciplinary engagement, with power mostly balanced across them.

(C3) Ted's interest in graduate school.

Ted's interest in pursuing advanced studies in mathematics appeared only in the exit interview, when the interviewer asked whether the experience in the number theory class led Ted to change his career orientation. Ted answered the question by noting that he did consider postponing the pursuit of his teaching credentials and applying to master programs that would let him undertake advanced studies of mathematics (Int2-1203-Ted lines 407–421). Then, the topic of the master programs reappeared when the end of the interview organically evolved into a conversation on career counseling (632–742).

Ted specified that he started considering the option of pursuing advanced studies in mathematics after he realized that he was able to prove theorems on his own, an awareness that took place through his experience in the number theory class. However, his narratives did not specify the exact moment when he started considering master programs in mathematics as a career choice. Although he seemed to have been thinking about it prior to the exit interview, he did not act on it thereafter (Int2-1203-Ted 639–641). He was still hesitant about whether he should undertake that step for several reasons. First, he was not confident about being "good enough" to succeed in graduate courses, which might have turned out to be more demanding than the number theory class (Int2-1203-Ted 716–718). Second, he was concerned about the research element involved in graduate school, with regard to which he had no experience (727–732). Third, he was worried whether graduate programs would still allow him to continue his work with the youth (740–741) and whether the expertise he gained through teaching the youth could translate into a career of teaching at college (688–696). As discussed in C4.3, in Ted's teaching career, his work with the youth touched many aspects of his life, including his cognitive processes and his positioning within groupwork, and showed him the path of a promising career (see Figure 3-11).

Notwithstanding his concerns, Ted was adamant to honor what he had discovered through his experience in the number theory class—the import of mathematical challenges to his life. His successful teaching career was only partially fulfilling for him. With an emphatic voice, he declared, "I would need to be continually challenged . mathematically . you know? […] I need the challenge of mathematics to keep me alive" (709–718). Teaching at high school could not fulfill Ted's existential need, since the mathematics taught at high school were too basic for Ted. His interest in going to graduate school was unequivocally entwined with the search for mathematical challenges. The interviewer pointed out to Ted the master's programs related to mathematics-oriented education in order to fulfill his teaching and mathematics identities. Ted replied by asking whether he would still "learn a lot of math" in mathematics-oriented education programs. Ted's attachment to mathematical challenges, which led him to consider a career change, indicated that his childhood experiences with mathematics and games (claim C) were not merely fleeting moments of the past, but rather determinant of his being.

The question of whether Ted would drop the idea of going to graduate school remained, given the precedents of him changing career paths four times in the past (from applied mathematics to creative writing, then to work, then back to school for engineering, then to mathematics for teaching). In fact, Ted ended up joining the master's program at his current school. Soon after his graduation, he received a prestigious pre-doctoral fellowship, which set him up for a Ph.D. program.

(C4) Ted's teaching career

Table 3-8 lists the instances in Ted's narratives that involve his orientation towards a career of teaching. The third column in Table 3-8 provides only a selective quote from the referenced text (first and second columns). Subsequently, I summarize and analyze information related to a career of teaching in Ted's narratives. If the reader desires to consult the conversational flows of the narratives, he/she may find them in appendix B.

Table 3-8: Instances in Ted's narratives mentioning his teaching career. Full references are found in appendix B.

Data source	Reference	***Selective quote***
Int1-1001-Ted	53–77	*It was a winding path […] teaching could be a path that I can take.*
Int1-1001-Ted	87–90	*With my girlfriend when I talk about the future, I always talk about classrooms.*
Int1-1001-Ted	338–357	*Bring social justice into math curriculum […] especially [in] high school.*
SCNI-1112-Ted	266–283	*[When I'm stuck] I get into the best teacher voice and I try to explain it to myself.*
Int2-1203-Ted	299–305	*[In groupwork] I ask the teacher questions.*
Int2-1203-Ted	688–696	*I want to keep teaching […] but I don't know if teaching high school or […] higher level.*
Int2-1203-Ted	690–699	*[In high school] teaching is a challenge, but I need the challenge of mathematics to keep me alive.*

Ted attempted several career paths before landing on a career of teaching (Int1-1001-Ted lines 53–77). He was admitted to college at an early age, when he was sixteen years old, and chose applied mathematics as his major. After the early entrance college program, he went through a rebellious phase against his parents, during which he abandoned mathematics and "focused on" creative writing. Then, he led a life of work and travel for some time, which ended by going back to school and joining his current university. He first picked engineering as a major for his undergraduate program. Two years later, he shifted his major to mathematics with a focus on teaching. Thus, Ted chose to teach mathematics as his career after a series of unsatisfying career attempts, including the study of applied mathematics, creative writing, and engineering.

When he started the engineering program at his current university, Ted decided to teach at a community center because he found that tutoring was a job which he was "able to do well" (Int1-1001-Ted lines 74–76). His teaching experience at the community center impelled him to pursue teaching as a career (Int1-1001-Ted lines 76–77). He realized that teaching mathematics was more befitting his identity than engineering, because he "always talk[ed] about classrooms" and "never talk[ed] about creating or designing products" with his girlfriend, while discussing his future (Int1-1001-Ted lines 87–90).

For Ted, teaching mathematics became a strongly entrenched identity. First, he modeled his thinking processes and behavior in groupwork according to teaching practices. When stuck on a problem, he would talk to himself as he would speak to one of his young students. During the SCNI interview on 11/12, he commented on how he self-reflected during groupwork: "I get into that voice in my head. and I'm like teaching myself like as a- talking to myself as if I'm a little kid. I'm like 'well . you know this and this is true and you know this and this is true so:o what do you think is the next step?'" (SCNI-1112-Ted lines 279–281). Additionally, he described his role in groupwork as a "humble" teacher who would monitor his groupmates' mathematical works and group conversations by asking "teacher questions."

> Ted: I ask questions a lot of others. and it's like uhm . kind of . kind of like very subtly guiding the conversation. like "hey what are you working on?" little questions like that. the teacher questions. [laughs] "what are you working on?" "what did you get so far?" "oh . I think . I am not sure what is going on" . sometimes I am like "can you explain me what is happening?" just to see where everybody is at. So that's kind of my . my role. (Int2-1203-Ted lines 300–305)

Although Ted started considering master's programs on mathematics after he gained confidence in constructing mathematical proofs in the number theory class, he continued to frame his future career in terms of teaching. After gaining confidence, he was hesitant about whether he should teach at high school or college (Int2-1203-Ted lines 688–696).

Table 3-9: The secondary components related to the analysis of Ted's gain of confidence through the primary component of the career of teaching. The full references can be found in appendix B.

#	Secondary component	Reference	*Selective quote*
C4.1	Applied mathematics and engineering.	Int1-1001-Ted lines 59–60 & 70–74.	*I like the math of engineering, but I didn't like applying it to the science.*
C4.2	The challenge of mathematics.	Int2-1203-Ted lines 711–718.	*[In high school] teaching is a challenge, but I need the challenge of mathematics to keep me alive.*
C4.3	Tutoring young students.	Int1-1001-Ted line 74–77 & 692–696.	*I figured out that [tutoring]'s something I am able to do well. [...] Working with [...] youth [...] is when I discovered that [...] teaching could be a path that I can take. [...] A lot of my skillsets besides math is like working with youth with [...] emotional and [...] disabilities.*
C4.4	Ted's girlfriend.	Int1-1001-Ted line 87–90.	*Conversations with [my girlfriend] help me reflect and drive me towards wanting to teach.*
C4.5	Thinking like teaching	SCNI-1112-Ted line 268.	*I get into my best teacher voice and I try to explain it to myself.*
C4.6	Ted's positionings in groupwork.	Int2-1203-Ted lines 300–305.	*I ask a lot of [...] teacher questions: "what are you working on?" "what did you get so far?" [...].*

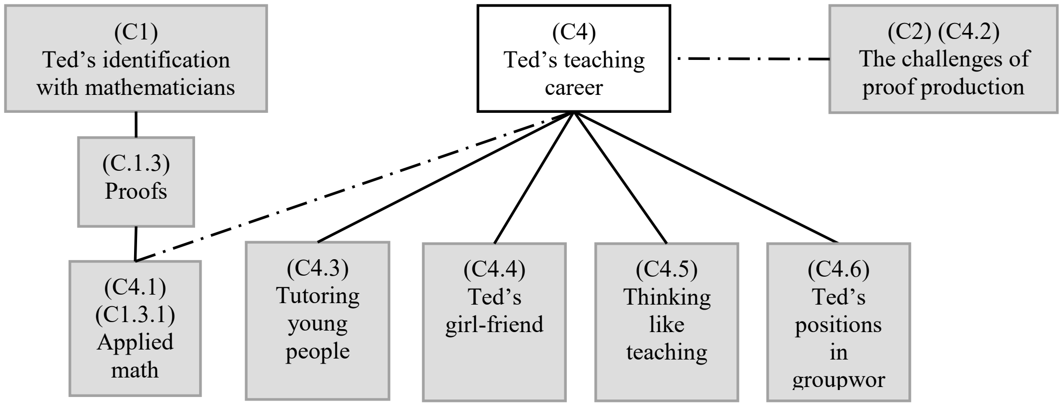

Figure 3-6: Diagram of secondary components pertaining to the primary component C4, Ted's teaching career. Two secondary components of C4 are confounded by other components: C4.2 with the primary component C2 and C4.1 with the tertiary component C1.3.1. For the meaning of arrows, revisit the legend in Figure 3-3.

Ted narrated his career choice of teaching in connection with six other components, which are to be treated as secondary components with respect to the analysis of Ted's confidence in constructing proofs (see Table 3-9 and Figure 3-6). The secondary component C4.1, engineering and applied mathematics, held a negative connection with the primary component C4, as Ted dropped his engineering major to pursue a teaching career. The nexus of engineering and applied mathematics has been studied as a tertiary component related to Ted's identification with mathematicians, namely C1.3.1. The secondary component C4.2 was positioned in opposition to the primary component C4, since Ted felt that teaching was insufficiently stimulating for his mind. He expressed his need of mathematical challenges in an existential manner ("I need the challenge of mathematics to keep me alive"). The challenge of mathematics (C4.2) was studied under the primary component C2 ("The challenge of proof production").

The five remaining secondary components, from C4.3 to C4.6, will be studied and reported under this section. The fourth component (C4.4), i.e., "Ted's girlfriend," was not mentioned in any other instance within Ted's narratives. Thus, its significance was limited to only one instance—supporting Ted's shift of his own major from engineering to mathematics for teaching. Ted's narratives provided rich information on the remaining three secondary components. The following subsections are dedicated to reporting and analyzing them in the following order: C4.5, C4.6, then C4.3.

(C4.5) Thinking like teaching

The instance that mentions Ted's thinking like teaching (SCNI-1112-Ted lines 268–282) is noted under the discussion of the resources (C2.2) that Ted employed to tackle the challenges of proof production. In this subsection, I conduct a close analysis of this reference because it sheds light on the analysis of Ted's teaching career as well. The following excerpt reproduces the transcript of the conversational flow, yielding the instance under study. Table 3-10 provides the VIP+function analysis of Ted's thought processes, as described in this instance being studied.

The instance of interest took place in the second SCNI interview on 11/12. During the group session on that same day, Ted's groupworked on proving Euler's second theorem. The problem in the worksheet was broken into two parts and included hints (see Wk10#2). Students managed to prove the first part together (2.a) but struggled with the second part (2.b). During the SCNI interview, Ted commented on his behavior in the videotaped group session, when he was sitting still, staring, and immersed in his thoughts.

SCNI-1112-Ted lines 260–282 -- *Ted describing how he thinks*

Ted: I think I stare like this meda- [Fady pauses video] like I sit like this meditating about the problem for quite a while. cause I-I wasn't sure what to do next for part b.

Fady: oh okay.

Ted: Yeah so I'll just kinda be . look like I'm [resumes video] lost in thought.

Fady: [pauses video] When you're meditating is it- how do you think? What's . what do you do?

Ted: I get into my best teacher voice and I try to explain it to myself . like say things that are obvious and then st- st- try to make connection.

Fady: okay

Ted: But . yeah it's very very . amorphous . you know what I mean? Like my thought process is very much like . this is [gestures with his hands swirling in front of him . then shrugs] I . I . can't really put it into words. It's not like a straight path. I just like have all these like little points of memory in my head and I like try to draw connections between those points. it's not a straight line . you know?

Fady: And who is your best teacher's voice? Who- who is your best teacher?

Ted: Ummm well I mean when I teach little kids.

Fady: Oh.

Ted: I get into that voice in my head. and I'm like teaching myself like as a- talking to myself as if I'm a little kid. I'm like "well . you know this and this is true and you know this and this is true so:o what do you think is the next step?" Then I'll ask myself that question and just like silence in my mind and try to see if I can find the next step. [laughs] It's a weird thought process. [resumes video]

Ted started commenting on his look ("I stare like this" and "I sit like this") when he would be "lost in thought." Although it was the interviewer's move to probe for a further narrative on Ted's thought process ("how do you think?"), he immediately responded by using a metaphor ("I get into my best teacher voice and I try to explain it to myself") and describing his mechanism of thinking ("say things that are obvious and then st- st- try to make connection"). Note that both parts of Ted's speech, the metaphor and the mechanism, were uttered in a one-breath unit, with an ending intonation only coming at the word "connection." The mechanistic part of Ted's response could have sufficed to answer the interviewer's question. Nonetheless, he responded by first stating the teaching metaphor that encapsulated his thinking mechanism, i.e., his own voice while teaching "little kids."

Put in Lacanian terminology, Ted identified with the social image of teaching to operate on his self as a thinker. What operation? Ted might have likely been using this thought process, i.e., stating the obvious and then making connections, prior to his teaching experience. However, his teaching identity gave this thinking mechanism a new label and reality. By identifying this thinking process with his teaching practice, Ted brought the reality of his teaching experience, a dialogical reality, into his solitary notion of mathematical problem-solving or proof production.

Note that the two-step thinking process, stating the obvious and then making connections, cohered with the puzzle-solving process—setting the relevant pieces and then bringing them together. The teaching metaphor, a dialogic reality, was thus competing with the existing puzzle metaphor, a solitary reality. Two identities of Ted, the games identity and the tutoring identity, animated his solitary thoughts as he constructed proofs and solved mathematical problems.

A VIP+function analysis sheds further light on Ted's processes of thinking, such as teaching by attending to the subversion of positionings. Recall the following context: when the teaching metaphor was actuated, it constituted the notion of Ted being resourceless—an unpleasant situation—during mathematical proof production. By bringing in the teaching reality to bear on this situation, Ted split his unitary self into a self as teacher (his own voice) and a self as a student who is stuck. By positioning himself as a teacher, Ted subverted his resourceless predicament: the teacher, who was none other than Ted himself as teacher, was now a fictitious, but nonetheless empowering, resource. By positioning himself as one of the little kids whom he used to teach, Ted reframed his unpalatable situation as a pleasant one, because he appreciated, devoted himself to, and had pleasant times with his young students. The dual self-positionings, hence, provided "fuel" (Int2-1203-Ted 30) for Ted to sustain his engagement with mathematical challenges through the resourceless moments.

Table 3-10: A VIP+function analysis of Ted's description of his thought processes: thinking like teaching the self.

Reference	Voice	Identity	Position	Function
SCNI-1112-Ted lines 268–269.	Ted talks to self: "well . you know this and this is true and you know this and this is true so:o what do you think is the next step?"	Self as teacher of "little kids."	Asking probing questions to self as a "little kid".	To sustain engagement with task despite being resourceless.

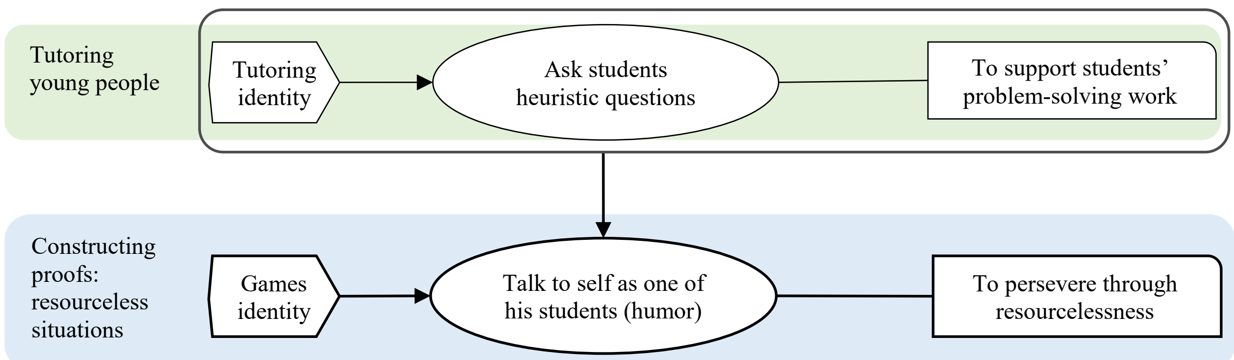

Figure 3-7: VIP+function onfiguration of the two realities, puzzle and teaching metaphors, which Ted animated during his solitary thinking process as he solved and proved mathematical problems.

Noteworthy is Ted's explicit use of the word *voice*: "I get into that voice in my head. and I'm like teaching myself like as a- talking to myself as if I'm a little kid." As set by the theoretical framework of this dissertation, a voice actuates an identity, animates a position, and attempts to achieve a function. The power of a voice resides in these three components—identity, position and function—that travel with the voice and impact new contexts other than the one within which the voice originates. The VIP+function analysis of Ted's teaching voice (Table 3-10 and Figure 3-7) instantiates the power of a voice. As analyzed in this section, Ted's teaching was animated to bear on his engagement with the proof challenges. The positionings and functions of Ted's teaching voice, that originated in his workplace, reframed his unpalatable position and turned it into an emotional resource that emanated encouragement.

Ted expressed his frustration with challenging proof problems: "when you stare at a problem for two three hours . you got frustrated sometimes. but like to . to control that. and use that as like . fuel for finishing it up" (Int2-1203-Ted lines 29–30). Drawing on his teaching voice to alleviate the dire situation of his mathematical challenges comprised one of Ted's tactics of digging up “fuel” to sustain his engagement.

(C4.6) Ted’s positionings in groupwork

Ted reported that he used to enact a teaching role during the study group, when his group in the number theory class met outside the classroom (Int1-1203-Ted lines 300–305). This section investigates this claim systematically through Ted’s narratives, by looking at the positions and functions he enacted in groupwork according to his self-report. The instances in Ted’s narratives that involve his positionings in groupwork are reported in Table 3-11.

Table 3-11: Instances involving Ted’s positionings in groupwork throughout his narratives (interviews).

Data source	Reference	*Selective quote*
Int1-1001-Ted	206–208	*In elementary school in Hong Kong, there wasn't anything like groupwork outside of P.E..*
Int1-1001-Ted	208-213	*In middle school, I was very disruptive [in groupwork].*
Int1-1001-Ted	214–220	*In high school, if it’s a math and science related [...] groupwork, I tend to be the leader of the group [...] That’s my attitude back then, I was so arrogant.*
Int1-1001-Ted	219–220	*I got to college and I started to care about other people's feelings [...] I knew when to shut up and let people work things out.*
Int1-1001-Ted	222–223	*I tend to like facilitate the conversations cause I see [...] the path to the solution.*
Int1-1001-Ted	224–242	*I only go study with people if they are wanting to learn from me [...] I'm like one of those type of people that learn by explaining [...]by looking at what mistakes they make, I would remember what mistakes I shouldn't make.*
Int1-1001-Ted	290–294	*I have to really prompt [Boutros] to be like “Hey so where you at? Can you can you tell me what you are doing?” that he'll share with the group.*
SCNI-1029-Ted	165–182	*The pictures [of the shared work] that I take are shared on [...] [my] google drive. and every colleague in my group got access to that.*
SCNI-1029-Ted	194–196	*When we work in a group [outside classroom] I would have [my homework] as pdf [...] and refer to that [...] they’ll pull it up, we will talk about the problem using that as a background.*
SCNI-1029-Ted	198–214	*I personally learn by explaining. [...] before almost all my classes I pick couple people anyway. and [...] study with them. [...] I do the homework ahead of time and explain it to them. [...] If taught, I remember it. That’s how I study.*
SCNI-1029-Ted	225–229	*That is how I manage [my busy schedule]. For newer concepts [...] I will try [...] to explain it to myself that is how I learn it initially. And then I have to explain it to others.*
SCNI-1112-Ted	1–8	*Today [...] I went over in detail with Bettie on the solutions for number two and three.*
SCNI-1112-Ted	20–25	*[Boutros] looked over my solution for number 4 for the last worksheet. We just like talked about it to see if my solution made sense. Cause I was doing it when I was sick. I don't know if like what I wrote was just rambling or if it made sense.*
SCNI-1112-Ted	70–77	*A lot of times, I like to explain to help me reinforce a new idea.*
Int2-1203-Ted	192–194	*Fady: you created this group [to study outside classroom]? Ted: yeah.*
Int2-1203-Ted	205–207	*At that point [when we started to meet outside classroom to study] I decided to share my Google Drive with everyone.*
Int2-1203-Ted	270–272	*Sometimes, it's like me who asked and, sometimes, it's other people who are asking to meet [outside the class to study together].*
Int2-1203-Ted	300–305	*I try to remain humble [...] I ask the teacher questions.*
Int2-1203-Ted	308–311	*I find more clarification through the process of explanation.*
Int2-1203-Ted	324–338	*If I do bad [in groupwork] it's I'm about to learn something.*
Int2-1203-Ted	372–378	*I was in [my groupmates’] presence [in the study group]. but I feel like I was in my own zone when I was doing the midterm review.*
Int2-1203-Ted	422–435	*The groupwork in this class is not pressured [...] [by imposing] a group project.*
Int2-1203-Ted	448–451	*Because the group [...] wasn’t [...] striving for a final project, I was able to be more chill than usual. [...] I tend to be the one who pushes a lot when it is a group project that has a grade.*
Int2-1203-Ted	465–473	*I can be kind of negative and stressed out a lot. and like Bettie’s energy will balance that out. [...] Boutros' calm demeanor really cools down the group. cause like John and I can get really intense whenever we have these talks.*

Data source	Reference	*Selective quote*
Int2-1203-Ted	552–572	*Often times I find myself being the one […] who would work on the math the entire time […] I'm the slightly anti-social one who is just like "I'm gonna keep writing and listen to these jokes and be amused but not really show it."*
Int2-1203-Ted	603–628	*I tried to […] not critique anyone's work until they ask me to do it […] At the beginning […] I was doing that a little bit. And then I realized "wait wait let me wait until they ask me to look at their work."*
Int2-1203-Ted	699–701	*I felt more conscious about [group dynamics] than before I started working with the youth group certainly. Cause before that I would just do it. I just want my grade. […] I didn't care about the group dynamics before.*

The narratives on Ted's positionings in groupwork (Table 3-11) point to three significant factors that shaped his behavior in the groupwork related to the number theory class:

- Ted's way of learning by explaining (Int1-1001-Ted lines 224–242, SCNI-1029-Ted lines 194–196, 198–203, 225–229, SCNI-1112-Ted lines 1–8, 20–25, 70–77, and Int2-1203-Ted lines 308–311),
- Ted became mindful of group dynamics (Int1-1001-Ted lines 219–220, Int2-1203-Ted 603–615 and 699–701),
- The effect of graded group projects on Ted's behavior in groupwork (Int2-1203-Ted lines 422–435 and 448–451).

(C4.6.1) Learning by explaining.

Repetitively, Ted noted that his method of learning new mathematical concepts was to explain them first to himself and then to others. By explaining his reasoning process to others, Ted hoped to reinforce his memory of the new concepts: "I remember the concept much better when I am explaining it to a peer" (Int1-1001-Ted lines 228–229), "it's more dynamic way to study and refresh your knowledge" (242), and "If taught, I remember it; that's how I study" (SCNI-1029-Ted line 203). Additionally, as he would explain how his peers could fix their mistakes, he would learn to avoid those mistakes himself: "by looking at what mistakes they make, I would remember what mistakes I shouldn't make […] They got their mistake and then I can pinpoint it and fix it" (Int1-1001-Ted lines 237–241).

By the end of the semester, Ted noted a new function of such explanations to others: it helped him refine his processes of thinking by virtue of receiving feedback on his answers to the mathematical problems.

<u>SCNI1112-Ted lines 177–80</u> -- *John reacts to Ted's explanation*

Ted: [pauses video] Whenever John says "I'm not thoroughly convinced" that's like a signal for me to like reinforce what I know. So he said that "I'm not fully convinced about something" and then . [resumes video] we go on to talk about it.

<u>Int2-1203-Ted</u> -- *Groupmates' contributions to Ted's understanding*

Ted: I think talking to other people about it it's it's important . […] other people would challenge your proofs . and that challenge makes your proof more refined. you know. like when they're like "I don't understand why you jump from this to this" . that is just means you need to explain it better. you just need to insert a few lines to justify that part. and I found that like just talking with other people or showing them my proof . "does this make sense?" Then . that . that peer review really helps. (lines 175–180)

Ted: [my groupmates] helped me understand a lot when they asked . what I wrote. You know like . when when they . when they asked for more explanations. because I . I personally find more clarification through the process of explanation. Like I feel more clear about it after I talked about it. You know. (lines 309–311)

The investigation of Ted's narratives regarding groupwork outside the classroom shed light on the function that he attempted to achieve through the study group. By inviting his groupmates to study outside the classroom, Ted had been creating an environment where he could practice his study method, i.e., learning by explaining. He needed peers to whom he could explain, so that he could foster this process of learning. Ted noted this motivation on several instances in his narratives (see excerpts below).

<u>Int1-1001-Ted lines 226–235</u> -- *Ted's motivation for creating study groups*

Ted: I only go study with people if they are wanting to learn from me . in the sense that I am not very good at listening to other people's instructions when they're my peers.
[…]
I started doing this when I was 17. I would just take a few classmates who I know are really really bad in the class are doing really poorly and I say . "Let's study together" and I'm already at like a B- or an A and so they're they're willing to listen to me.

<u>SCNI-1029-Ted lines 222–229</u> -- *Ted's busy schedule*

Fady: […] when you mention about your busy schedule and you just mention that you are very very busy this semester like four classes . and your jobs[8]. So . uh . h . how are you managing the [time for a study group]. Is it helpful?

[00:25:15.24] Ted: Hum . lots of coffee. [laughs] . That is how I manage it. uhh. I mean to me . it's . I have to study by explaining. Like I mean . especially for newer concepts . once I . I will try to gra- to explain it to myself that is how I learn it initially. And then I have to explain it to others . to . reinforce it. so I try to do the work ahead of time . whenever I have time on the weekends . I'm done with the homework by Tuesday usually. And then I'll show up . and be like "here it is."

<u>SCNI-1112-Ted lines 70–77</u> -- *Ted's function from explaining to others*

Ted: [pauses video at the moment when he responds to Bettie's request for explanation] A lot of times . I like to . uh explain to help me reinforce a new idea. so I try to gather everything that I know that I know for certain is true then I relay it and try to see if I gain understanding through the explanation process . so that's what I was doing with Bettie.

Fady: so what you are doing with Bettie is more for yourself?

Ted: Yeah. It's kind of selfish. but it helps her out too . you know . ends up being good for both of us. but the intention was more for myself. [resumes video]

Earlier in the analysis, Ted noted that he took up the position of a teacher in the study group, by asking his groupmates "teacher questions" (the reason behind C4.6). A broader investigation of Ted's interest in the study group revealed a learning function, rather than teaching function, of his questions. Although Ted's peers might have inadvertently benefited from his explanations in the study group, his primary function was to foster his knowledge (see excerpt above for SCNI-1112-Ted lines 70–77).

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[8] Ted was taking four classes and working two jobs during the semester when this research was undertaken (see SCNI-1029-Ted lines 1–21).

Recall the "teacher questions" Ted would ask in the study group: "what are you working on?" "what did you get so far?" "oh . I think . I am not sure what is going on . can you explain me what is happening?" (Int1-1203-Ted lines 300–305). Ted noted that the purpose of these questions was "just to see where everybody [was] at." In the study group, students spent a good amount of their time working individually, which would undercut the purpose that Ted wished to fulfill from the study group. By probing students to share their work (Int1-1001-Ted lines 290–294), Ted would create opportunities for explanations. He used to go to the study group having already solved the worksheet problems and would look for opportunities to explain his solutions to others. One tactic Ted used to spur explanations was to point out the mistakes in others' works and repair them (Int1-1001-Ted lines 236–242).

Thus, during groupwork in the number theory class at school, Ted animated a particular position—checking on students' work—that originated within teaching-oriented settings, where Ted tutored or taught young people. However, he repurposed this position to fit his learning identity. Thus, the predominant function of checking on his groupmates' work was the enhancement of his understanding. The original function of checking on others, i.e., attending to their needs, persisted only as a contingent function. The VIP+function configuration in Figure 3-8 represents the repurposing of the "checking on others" position, when Ted animated it during his work with his peers.

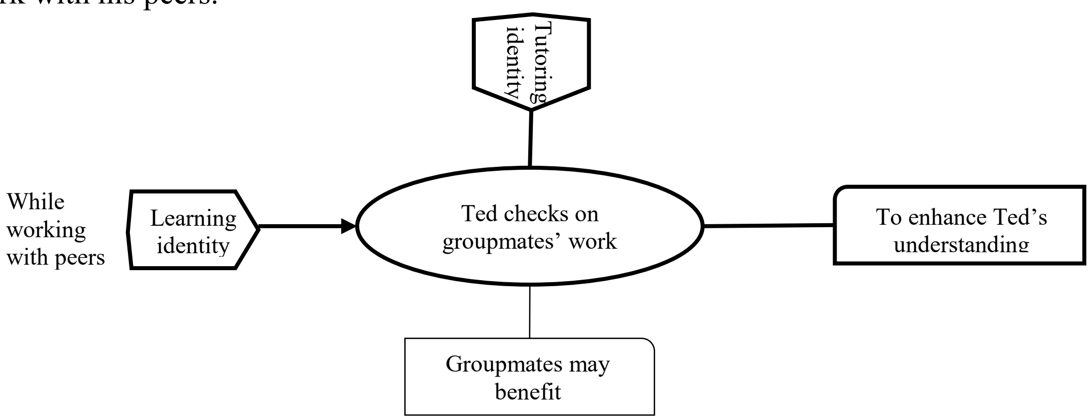


Figure 3-8: Ted's predominant participation position and function while working with his peers, as per his self-report. His behavior in groupwork repurposed a position that originated within tutoring and teaching-oriented settings.

(C4.6.2) Ted became mindful of group dynamics.

Table 3-12 contains all references on Ted's mindfulness about group dynamics within his narratives.

Table 3-12: Ted's references in his narratives on his mindfulness about group dynamics.

Data source	Reference	*Selective quote*
Int1-1001-Ted	208–220	*I didn't know to care about other people's feelings [...] That's my attitude back then, I was just so arrogant and [laughs] I was complete asshole a lot of times. [when] I got to college and I started to care about other people's feelings [chuckles] and I knew when to shut up and let people work things out.*
Int1-1001-Ted	268–283	*When we talk about how some members' solutions might not be sufficient, people get defensive.*
Int1-1001-Ted	290–307	*Boutros is really quiet [...] Jeremy wants the first draft to be sufficient as proof [...] John gets defensive the easiest [...] Bettie is the one to consult the book first.*
SCNI-1029-Ted	385–396	*making sure everybody's on the same page with what the definitions are. [...] All the group members needed to arrive at that point before we could move on.*
SCNI-1112-Ted	101–108	*[Bettie] hasn't been very happy about the grades she's gotten [...] [In groupwork] I pretend I don't know the solution and be like why don't you explain it to me, when I see somebody that is struggling with their confidence a little bit.*

Data source	Reference	*Selective quote*
SCNI-1112-Ted	182	*Needed to make sure everybody was on the same page.*
Int2-1203-Ted	284–298	*Bettie brings out a very positive energy [...] Boutros tends to be a solo worker [...] Jeremy likes to explain.*
Int2-1203-Ted	448–460	*[The fear of making mistakes] dissipated over time. [...] We were kind to one another even though we were making mistakes.*
Int2-1203-Ted	500–536	*Jeremy seems to be a little less confident in the beginning of the semester [...] Bettie is more under pressure now.*
Int2-1203-Ted	545–572	*Jeremy like to joke [...] John is like "no nonsense" [...] Boutros likes to draw [...] Bettie likes to chime in and joke.*
Int2-1203-Ted	573–583	*I was really nervous to show anyone my grade. [...] a couple of my groupmates did not do so well and I didn't wanna put it out there.*
Int2-1203-Ted	603–625	*I tried to make it a point to not critique anyone's work until they ask me to do it [...] I've been having those moments a lot recently, just realizing my behavior in a way. Maybe I'm finally growing up.*
Int2-1203-Ted	699–701	*I felt more conscious about [group dynamics] than before I started working with the youth group certainly. Before that, I would just do it. I just want my grade. I don't care [laugh] I didn't care about the group dynamics.*

Ted highlighted that his positioning in groupwork shifted significantly from his school experiences to his college experiences. He described himself as "arrogant," an "asshole", and careless about "other people's feelings", during his years at school (Int1-1001-Ted lines 208–220). For instance, during groupwork in high school, he would say the following to his groupmates who disagreed with him: "No I don't care what you have to say. this is the answer. if you disagree with me . you can get a bad grade . go ahead" (217–218). Ted claimed that he changed his arrogant attitude and started to care for "other people's feelings" in college (219), though not quite at the very beginning (148).

Indeed, in the number theory class, Ted was aware of his groupmates' socio-emotional and socio-mathematical profiles (Int1-1001-Ted lines 268–283, 290–307, Int2-1203-Ted 284–298, 500–536, and 545–572). In his narratives, he also emphasized four considerate positions that he animated frequently in groupwork. First, he avoided showing his grades to his groupmates out of the fear of making them feel inferior, although the grades were significantly meaningful for him (SCNI-1112-Ted lines 101–108 and Int2-1203-Ted lines 573–583). Second, he also avoided showing off his knowledge, especially to students who were "struggling with their confidence" (SCNI-1112-Ted lines 101–108). Third, he avoided critiquing his groupmates' works unless they would request his opinion (Int2-1203-Ted lines 603–625). Fourth, he attempted to ensure that everyone in the group were "on the same page" while tackling new definitions or critical steps in the proofs (SCNI-1029-Ted lines 385–396 and SCNI-1112-Ted line 182).

In the exit interview, Ted acknowledged that recently he became reflective about the impact of his behavior on other people, not only in classroom groupwork but also in his personal life (Int2-1203-Ted lines 621–628). He claimed that his work at the community center with young people made him become aware of the intricacies of group dynamics (699–701). The community center, where Ted worked, welcomed young people with social, emotional, and cognitive challenges (692–696). Educating them seemed to have enfeebled his arrogant attitude and had taught him to be sensitive.

Ted carried this self-maturity to the number theory class. Bettie, Ted's groupmate who had been struggling with proofs, appreciated his "approachable" personality and his patience while he explained mathematics to her. Indeed, Ted played a significant role in Bettie's identity development, mainly due to his considerate identity (see subsequent chapters). Ted's patience, which Bettie praised, was likely forged through his experience of educating young people with diverse capabilities.

> Bettie: Ted is more like ca . like not calm but he's more . patient . in teaching . or like not even teaching just like . going through what he did. and I can ask him and he like doesn't . I can tell he would never . like he doesn't get annoyed. he loves to help in people. cause obviously he wants to be a teacher. so he's like he'll go through it . and he'll like take his time to come . to school . just to help me . to like . understand things. I think that's really cool. and I feel like he's probably like . he's just really smart. He's so smart. (Int2-1202-Bettie lines 324–329).

This analysis provides another instance where Ted's positioning in the number theory group built on the development of his tutoring identity that he underwent in his workplace. As Ted clarified, the considerate and patient positions were not spuriously enacted in his workplace and in the number theory class, but were genuinely internalized into his identity as a teacher, which further expanded to constitute his entire personal life (628). For this reason, the VIP+function configuration in Figure 3-9 shows the transfer of positions as mediated by Ted's tutoring identity.

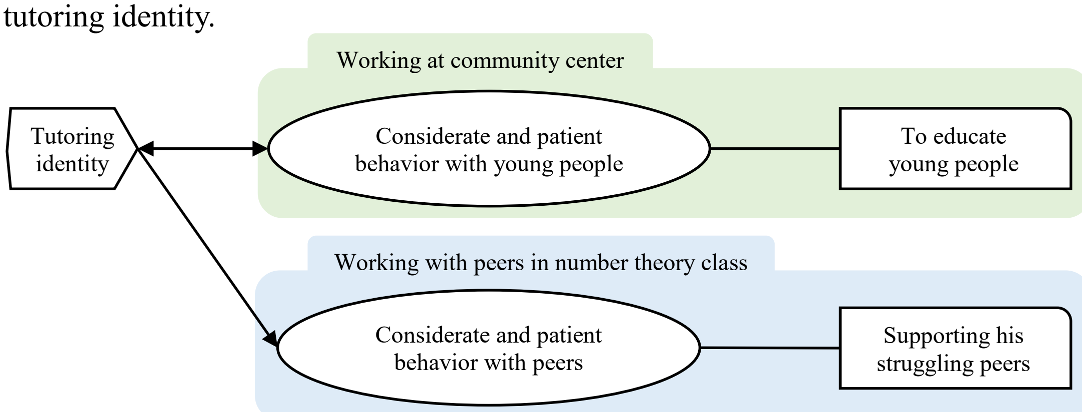


Figure 3-9: VIP+function configuration of Ted's transfer of his considerate and patient positions from his workplace to the number theory classroom, as mediated by Ted's tutoring identity.

(C4.6.3) The effect of grades on Ted.

Although Ted's narratives on group design and grades (see Table 3-13) are indirectly connected to his teaching career (primary component C4) and his gain of confidence (the developmental shift under study), they are relevant by virtue of comprising a testing task for claims A, B and C. Table 3-14 provides Ted's scores on his weekly homework.

Table 3-13: Ted's narratives on grades and scores.

Data source	Reference	***Selective quote***
Int1-1001-Ted	214–219	*"No I don't care what you have to say. this is the answer. if you disagree with me, you can get a bad grade. go ahead." That's my attitude back then [in high school].*
SCNI-1029-Ted	13–21	*All my midterms are good. I had a 110 in my proofs class [...] got a 94 for my stats class and got a 97 or something like that for this one.*
SCNI-1112-Ted	101–108	*[Bettie] hasn't been very happy about the grades she's gotten [...] I was just lying and saying that I've never gotten a perfect score.*
Int2-1203-Ted	352–366	*My only frustration is to never get a perfect grade [on the homework]. I keep getting 6.8 out of 7, 5.9 out of 6. [...] I feel satisfied but [...] I want more.*
Int2-1203-Ted	424–435	*There's a part of your grade that's dependent on your interaction with your group.*
Int2-1203-Ted	448–451	*I like to get good grades [...] I tend to be the one who pushes a lot when it is a group project that has a grade.*
Int2-1203-Ted	573–583	*I was really nervous to show anyone my grade.*
Int2-1203-Ted	699–701	*I just want my grade [...] I didn't care about the group dynamics before.*

Table 3-14: Ted's scores on weekly homework.

HW1	HW2	HW3	HW4	HW5	HW6	HW7	HW8	HW9	HW10	HW11	HW12	HW13
58%	80%	90%	97%	88%	100%	85%	99%	97%	98%	98%	100%	97%

Ted strove towards getting "good grades" (Int2-1203-Ted line 448). Although he was satisfied with the high scores he obtained on his homework, he still wanted full scores (Int2-1203-Ted lines 352–366). He reported that his behavior in groupwork was influenced by his entrenched desire for good grades.

<u>Int2-1203-Ted lines 422–435</u> -- *The effect of group design on Ted's behavior*

Fady: Uh . did you have the chance to work in groups before? Is this something new in this class . or?

Ted: yeah . I felt like the groupwork in this class is not pressured. Whereas like . like . you know . before this a lot of group projects was like . "okay you have a set group project for this amount of time and you have to have it finished by the end." There's a part of your grade that's dependent on your interaction with your group. You know what I mean? Like um . where . whereas this one . you have individual accountability the whole time. You're [next word is uttered with high pitch] allowed to work in groups. So more . more natural that way. um . cause a lot of group projects . if . if a finished project is required then . there are people who are gonna contribute at different rates. And there's gonna be friction among the group . because the design [unidentified] for an outcome. You know?

Fady: okay.

Ted: whereas . whereas this type of group dynamic I feel like . is . is much better. you know? Like if there's friction at all . well then just don't meet. [laughs]

If the groupwork required a collective project that would be graded, Ted would become irritated if some groupmates would contribute poorly to the project (Int2-1203-Ted lines 424–435). He would also feel impelled to push his groupmates to give the best results for the sake of getting a high grade on the collective project (Int2-1203-Ted lines 448–451).

Ted's competitive behavior supports claim C, which highlights an identity entrenched in his history of frequently playing board and puzzle games in the past (claim C). Ted's narrative on consequential tasks can be classified according to the VIP+function frame, as represented in Figure 3-10. The initial function is produced by the environment and the group design, with respect to the production of a consequential collective task. Setting up a prize (grades to gain or lose) would trigger Ted's gaming identity, with its internalized position, competitive behavior, and function, to win the best prize "perfect score."

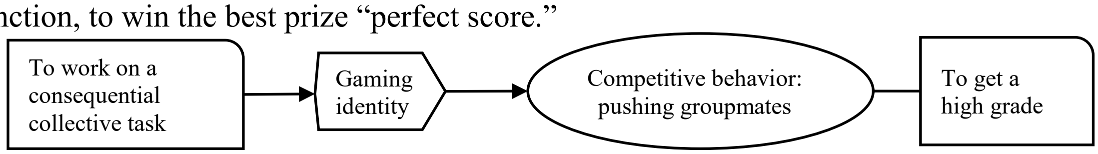


Figure 3-10: The VIP+function configuration of Ted's behavior when group design requires a consequential collective task, which was not the case in number theory class.

Fortunately, the number theory class imposed an individual grading system instead of giving a collective grade. Students were supposed to support one another in solving the homework problems during groupwork, but would have to submit individual homework for grading. Thus, the main function of groupwork in the number theory class was learning as much as possible, so that individuals could complete their homework at home. This function of the group design of the number theory class attenuated Ted's gaming identity and its subsequent negative effect on the group dynamics.

Ted reported that, unlike in other classes, he did not experience frictions in his group within the number theory class: “we were kind to one another even though we were making mistakes” (Int2-1203-Ted 460). Ted was not “pushy” with his groupmates. On the contrary, he “was able to be more chill than usual” (450), allowing his groupmates time to process their knowledge. He was considerate of his groupmates’ feelings, about which he would otherwise “not care” in competitive settings (699–701). He was even sensitive enough not to show his grades to his groupmates, because some of them would have been getting unsatisfactory scores on their homework (SCNI-1112-Ted lines 101–108 and Int2-1203-Ted lines 573–583).

Thus, Ted’s approachable personality, which he transferred from his workplace consisting of young people to the number theory class (Figure 3-9) and which Bettie praised and benefited from, could have been compromised, had the group design involved a consequential collective task. Moreover, the animation of his “explaining to learn” position could have suffered the same fate (Figure 3-8). Ted would not have spent time in groupwork on reinforcing his mathematical understanding, had the groupwork required him to participate in the completion of a consequential collective task within a given period of time.

In summary, the component C4.6.3 (the effect of grades on Ted) interferes with a smooth productive operation of the components C4.6.1 (Ted’s learning by explaining) and C4.6.2 (Ted’s mindfulness). The interference is represented as interrupted lines which connect the corresponding components in the diagram of Figure 3-11, which represents the connections between all components pertaining to C4 (Ted’s teaching career, to be discussed next).

(C4.3) Tutoring young people

Ted’s narratives mentioned his work as a tutor or teacher of young people in eleven references, all of which connect the component C4.3 to other significant components in his narratives (see Table 3-15). Analysis of the primary component C4 (Ted’s teaching career) is summarized and represented by the diagram of Figure 3-11, which reveals the influence of the component C4.3 (Ted’s tutoring young people) within his narratives.

The secondary component C4.3 strongly bolsters the primary component C4 in Ted’s narratives. Tutoring and teaching young people showed Ted a promising career path, following a series of failed attempts at constructing a career (Int1-1001-Ted lines 58–77 and Int2-1203-Ted lines 688–696). Due to his work with the youth, Ted subsequently changed his major from engineering to mathematics for the purpose of teaching. He was deeply engaged with this work as he attempted to understand the factors that hindered young people from liking mathematics (Int1-1001-Ted lines 321–335) and tried to explore ways to address them (340–358).

Table 3-15: Ted’s narratives on his work with the youth and the connections of C4.3 with other components (fourth column).

Data source	Reference	***Selective quote***	Component
Int1-1001-Ted	31–33	*I have friends from work who genuinely just don't like math. They would tell me like “hey this kid really needs help with all these math problems . why don’t you go help him.”*	C4.5
Int1-1001-Ted	75–77	*I've been tutoring. So I figured out that that’s something I am able to do well. and then I went into working with a community center called JTCC. And they worked with a youth administration. That is when I discovered that “hey teaching could be a path that I can take.”*	C4 & C4.5
Int1-1001-Ted	321–335	*There's this sense that math is […] a necessary evil at best, for a lot of kids as well as parents.*	C4
Int1-1001-Ted	340–358	*I’ve been reading about bring social justice into math curriculum […] especially with high school age . youth.*	C4

Data source	Reference	*Selective quote*	Component
SCNI-1029-Ted	335–341	*That was a little thing that the middle school that I work at does consensus. You do consensus by like you [thumb is up] love it . you can live with it [a thumb pointing parallel to the ground] . absolutely no [thumb is down]*	C4
SCNI-1112-Ted	268–283	*I get into my best teacher voice [...] I mean when I teach little kids*	C4.5
Int2-1203-Ted	31–37	*I would just sit on the white board doing my homework during that time [i.e. when students are doing homework lab] I love those little kids' reactions. [mimicking little kids] "That's my homework?" [laughs]. Like . "no no no . that's my homework."*	C2
Int2-1203-Ted	636–637	*So much of my time has been spent on like class and school and class and work. [...] I just haven't had the time to look at the [master's] programs.*	C3
Int2-1203-Ted	688–696	*A lot of my skillsets besides math is like working with youth.*	C4
Int2-1203-Ted	699–701	*I felt more conscious about [group dynamics] than before I started working with the youth group certainly.*	C4.6.2
Int2-1203-Ted	706–708	*Working with little kids that would just scream profanity at you [laughs].*	C4

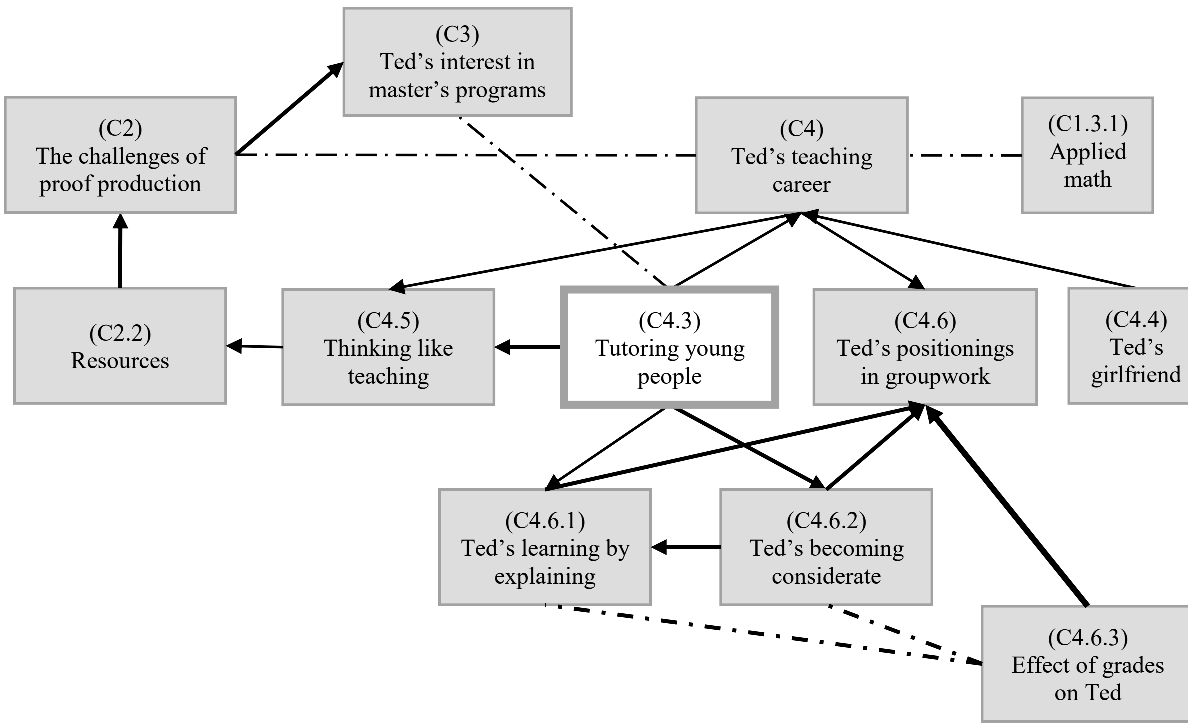


Figure 3-11: Diagram of the connections summarizing the analysis of the primary component C4. The secondary component C4.3—Ted's tutoring young people—is revealed as influential, within Ted's narrative, on his gain of confidence in proof production. For the meaning of the directions of arrows, revisit the legend in Figure 3-3.

Although the interviews took place in Ted's school, not his workplace, he brought up the time he spent tutoring young people in connection with many other narrative components (see the connections stemming out of C4.3 in Figure 3-11). As per Ted's narratives, it seems to have a direct influence on five components (C4, C3, C4.5, C4.6.1, and C4.62), through which it further influences other components. The component C4.3 has eight paths of influence:

(i) C4.3 → C4 - - C1.3.1
(ii) C4.3 → C4 - - C2 → C3
(iii) C4.3 → C4 → C4.6
(iv) C4.3 → C4.5 → C2.2 → C2
(v) C4.3 → C4.6.1 → C4.6

(vi) C4.3 → C4.6.2 → C4.6
(vii) C4.3 → C4.6.2 → C4.6.1 → C4.6
(viii) C4.3 - - C3

Ted reported three ways in which his tutoring identity and his voice supported and shaped his learning experience in the number theory class:

- Ted's internalized voice as a teacher was a resource through which he could sustain his engagement with "really challenging proof problems"; see Figure 3-7 and path (iv).
- Ted's position of checking on his students was repurposed while he worked with his peers on the homework given in the number theory class; see Figure 3-8 and path (v).
- As a result of working with young people, Ted fostered a sensitive and considerate identity, which he carried to the number theory class, thereby impelled his groupmates to approach him for mathematical explanations, and, thus, opened up further opportunities to learn by explaining to others; see Figure 3-9, and path (vi) and (vii).

Path (i) represents the effect of Ted's tutoring experience on the change of his major from engineering to mathematics for the purpose of teaching. Path (ii) represents the value that the tutoring experience added to Ted's teaching career, although he was seeking further fulfilment through mathematical challenges that eventually impelled him to pursue master's programs in mathematics. Path (iii) represents the extension of Ted's teaching identity to groupwork in the number theory class. Path (viii) represents Ted's delay in applying to master's programs, because of his busy schedule of vacillating between work with the youth and studies at college.

The absence of consequential collective tasks from the group design of the number theory class averted the animation of Ted's revolting positions (see Figure 3-10). Such positions could have significantly reduced his learning opportunities in two ways: by distracting him from learning and enforcing productivity functions (represented in connection with C4.6.3 - - C4.6.1) and by pushing his groupmates away, in which case they would not have responded to his call for making a study group outside classroom (C4.6.3 - - C4.6).

Conclusion

The narrative instance where Ted reports his consideration of a career shift (Int2-1203-Ted lines 407–421) comprises merely the tip of an iceberg. It can be misleading if it is taken without the excavation that has been reported in this chapter. On its own, it tells the story of Ted, who became aware of his ability in constructing proofs through the number theory class and decided to undertake advanced studies in mathematics. However, the two-level—in some cases, three-level—investigation of primary and secondary components revealed two underpinned stories: one being existential (about identities), over an extensive timescale, and the other being situated (about positions and functions in context) over a semester-long course.

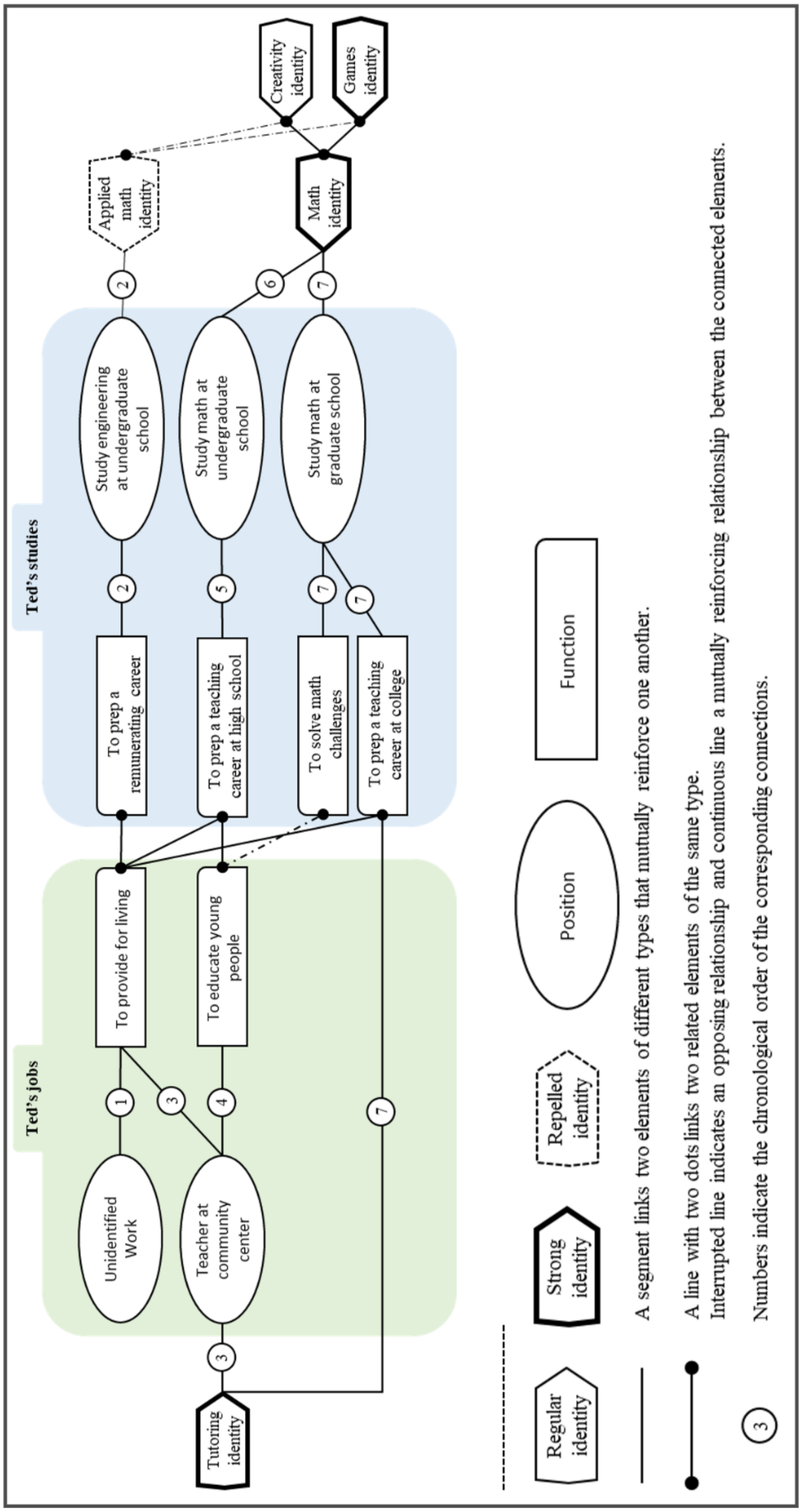


Figure 3-12: Diagram tracing the shifts of identities, positions and functions over a 4-year winding path that led Ted to select the field of advanced mathematics as a career choice.

The story unfolding over an extensive timescale (4 years)

Figure 3-12 summarizes Ted's 4-year winding path that led him to select the field of mathematics as his career. Due to the scarcity of information regarding his early shift from applied mathematics to creative writing, the diagram in Figure 3-12 starts with Ted's decision of going back to school to study engineering. This section summarizes the findings of this chapter that pertain to development over a long timescale (that consists of several years). It reports the shifts of identities, positions, and functions that took place over four years of Ted's life in undergraduate school. The numbers in the text refer to the connections in the diagram of Figure 3-12.

Ted went through a phase where he worked to provide for his living and his travels **(1)**, until he decided to go back to school. Encouraged by his father, he selected engineering as a major because it would supposedly prepare him for a remunerating career **(2)**. This shift hinged on the same function (To provide for a living) and the sense that he could succeed at engineering classes just like he did at Texas Academy of Mathematics and Science. However, his engineering studies triggered a repelled identity (Applied mathematics), which Ted experienced as "boring" because the discipline required application of the "same rule over and over again."

While undertaking engineering studies, Ted needed to contribute to the income of the household that he shared with his girlfriend in an expensive city. Therefore, he decided to work as a tutor and teacher because he had been successfully tutoring kids since he was 17 years old **(3)**. The decision of teaching at a community center was bolstered by a function (To provide for a living)—having commonality with his previous state **(2)**—and an entrenched identity (Tutoring identity). Ted's identity as a tutor thrived at the community center. He enjoyed working with the youth and was a good influence on many of them.[9] Ted's work at the community center was not only meant to provide for a living but also to support the youth, for whom Ted deeply cared **(4)**. Thus, his position as teacher at the community center had a dual function (To provide for a living + To support the youth at the center).

At school, Ted was wrestling with his applied-mathematics-oriented identity. On the contrary, at his work he used to thrive: this boosted his identity as a teacher. Teaching at high school became a potential career option for Ted, which further responded to a calling (To support the youth) and a need (To provide for his living). To prepare for obtaining teaching credentials, he shifted majors **(5)**—he stopped studying for engineering and started studying for the purpose of teaching mathematics. This shift hinged on Ted's newly-enhanced identity as a teacher as well as on the function of building his career for a living.

The mathematics classes boosted Ted's mathematical identity **(6)**, which was strongly rooted in his desire for creativity and winning games. Ted's entrenched identities of creativity and playing puzzle games, which the engineering classes had suffocated, could find a place to be animated within mathematics classes. Specifically, by authoring proofs in the number theory class, Ted authored himself as a winner of challenges much like the creative mathematicians who

[9] After the research ended, Ted added me to his Facebook account. I read many positive comments on his work at the community center. Besides, a faculty at the school told me that it will be a big loss to the youth, if Ted leaves the community center to pursue graduate studies.

proved theorems. It is worth noting that this shift hinged on his position of studying mathematics, which originally was animated by his teaching identity and then rekindled as his mathematical identity.

Once rekindled, Ted's mathematical identity thirsted for mathematical challenges. At this junction (7), the *teacher* and *mathematical* identities were positioned in opposition because of a conflict between the functions that they animated: *To educate young people* and *To solve mathematical challenges*. As per Ted's worldview, the conflict resided in the fact that advanced mathematics programs, which would fulfill the latter function, might have demanded long hours of study, in which case he would have to reduce his hours at the community center. Additionally, Ted might have needed to relocate to a place far away from the center.

Ted's mathematical identity was bolstered by two other entrenched identities, namely games and creativity. The performativity of mathematics was entrenched in Ted's mind through many years of winning games and admiring the creativity of mathematicians who constructed new proofs. Additionally, the components of proofs, creativity, and games influenced many other components of Ted's narrative world (see Figure 3-3 for proofs—C1.3—and creativity—C1.2—and claim C for games). Thus, Ted's mathematical identity was powerful enough to sway his decision towards going to graduate school, notwithstanding the risk of eventually dropping his work with young people.

Nonetheless, Ted's teaching identity was also influential within his narrative components (see C4.3 in Figure 3-11) and its performativity was entrenched in many years of successful tutoring—since Ted was 17 years old. He could not silence it. Ted set a twofold goal for his graduate studies: to be challenged mathematically and trained for a teaching career at college itself.

The story over long timescale (a semester)

Ted ascribed his emerging confidence in pursuing advanced mathematical studies to the awareness of his ability of constructing mathematical proofs, that took place through the number theory course. The pedagogy of the class was inspired by the Moore method insofar as it encouraged students to produce proofs of established theorems without consulting references. Ted throve in this pedagogy. However, a close analysis of his narratives on his engagement in the class over the semester revealed a complex system of factors—regrouped under the four principles of PDE—which bolstered his successful experience.

The PDE principles were devised to investigate pedagogical designs. In this chapter, they were applied as a usable framework containing Ted's narratives on his engagement with proof challenges. Ted reported elements ranging from classroom design, teacher's practice, groupmates' behaviors, to his individualized identities, that bolstered each principle of PDE (see section C2). Worth noting was Ted's emphasis on the importance of socio-emotional resources, such as humor in groupwork and individual work. Ted's success in the number theory class should be ascribed to his personal resources as well as the learning ecologies that he was presented with.

The PDE framework endorses a balance between, on one hand, the principles of resources and problematization and, on the other hand, the principles of authority and accountability (see Figure 3-2). Ted's learning development depended on mechanisms by which both the principles of problematization and resources affected the balance of authority and accountability (see Figure 3-13).

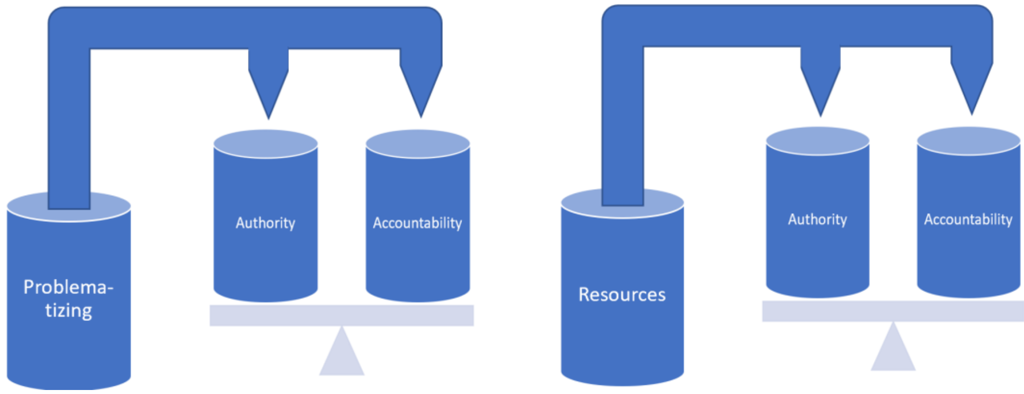


Figure 3-13: The principles of resources and problematization affecting the balance between authority and accountability.

First, Ted reported two functionalities of problematizing processes, depending on whether they attempted to boost authority or accountability. The former function of problematizing processes attempted to problematize authority by defending somebody's stance in the group against somebody else's, with the focus being on the people rather than the merit of their ideas (Int1-1001-Ted 268–283). The latter function of problematizing processes attempts to increase the challenging aspect of an activity by opposing or expanding ideas and contributions, with focus being on the norms of the discipline (e.g. SCNI-1112-Ted 121–122). If social interactions problematized authorship despite accountability to the discipline, learning ecologies would turn unproductive as participants would be using excuses rather than mathematical justifications to defend their stances. Engle and her colleagues (2014) analyzed another case of undue influence. As the analysis of Ted's perception of John's critiques showed (Figure 3-5), groupmates understand each other's problematizing functions based on their perceptions of each other's identities. In such cases, it would be recommended to have students spend enough time working together in order to adjust their perceptions of each other's identities, which is what happened with Ted and John. Recall that Ted acknowledged the fact that John played a significant role in challenging him mathematically and refining his mathematical justifications as well as the write-up of his proofs.

Second, Ted behaved differently with respect to taking aid of available resources depending on whether or not their function would boost authorship. Ted avoided taking help of resources that provided fully accountable answers, such as textbooks or his groupmates' finished work. However, he fully participated in group interactions when his groupmates were in the process of generating answers, i.e., an ecology that proved resourceful in boosting authorship. A VIP+function analysis of Ted's narratives showed that he actuated his entrenched gaming identity when he tackled the proof challenges (claim C). By authoring mathematical proofs, Ted was attempting to author himself as a winner.

Ted's gaming identity prevented him from consulting resources of acclaimed accountability to the discipline, for it would betray the rules of the game, so to speak. While the gaming identity productively bolstered Ted's authorship-oriented positions during his individual work, it could have hindered his productive experience in groupwork had it been actuated in this context (Figure 3-10). Fortunately, Ted drew on his tutoring identity rather than his gaming identity while working as part of the group (Figure 3-7, Figure 3-8, and Figure 3-9). His tutoring identity informed the positions that he inhabited during groupwork, which he repurposed to successfully enhance his mathematical understanding. Thus, Ted built upon the repertoires of two identities, playing games and tutoring young people, while dealing with mathematical resources.

The PDE framework turned out to be generative while being used to investigate individual accounts on disciplinary engagement. It highlighted the influence of the principles of resources and problematization on the balance between authority and accountability. The additional analysis of the VIP+function framework identified the influence of entrenched identities on the principles of resources and problematization, both of which determine the productivity of a disciplinary engagement depending on how they affect the balance between authority and accountability.

Summary of identity development mechanisms

To conclude, the VIP+function analysis of Ted's narratives, in line with the research regarding his gain of confidence in proof production, leveraged seven mechanisms of identity development:

- Identities can comprise who somebody was and what he/she can become; they can encapsulate the past in its performative aspects (habits) and project a future purposed to fulfill the desire of becoming like a person from the social realm who is other than one's self.
    - Ted's identities of tutoring, games, and creativity gained influence through their repetitive performance over many years, and his admiration of creative mathematicians fueled his performance in authoring proofs in the number theory class, which resulted in the emergence of a new prospective career.
- Repelled identities can be as powerful as the ones that are embraced.
    - Ted's identity related to applied mathematics was rejected because of the other two opposed and embraced identities, namely creativity and gaming.
- Identities can travel across contexts through common functions or goals.
    - The empowerment of Ted's teaching identity in his workplace affected his studies at school through the translation of the following functions: from e*ducating young people*, animated in his workplace, to *training for a career of teaching at high school*.
- Although identities are commonly known to animate positions, they can be dimmed and then reanimated by inhabiting relevant positions.
    - Ted's mathematical identity was dimmed during his study of creative writing and then reanimated when he took up mathematics classes between changing his major from engineering to mathematics for the purpose of teaching. Recall that number theory was a required class for a mathematics major related to teaching.
- Identities can increase their power by clustering with each other and can influence various aspects of one's life.
    - Ted's mathematical identity was bolstered by identities that traditionally did not count as part of the mathematical discipline. Because of this cluster, the "fun" and "challenge" features found in playing games also tinged the actuations of his mathematical identity.

- Identities can be foregrounded in one context and backgrounded in another. The functions pursued in an activity can either trigger individualized identities or hinder the same.
    - Notwithstanding the dominance of Ted's entrenched identity of playing games throughout the number theory class, it was deflated during the groupwork with his peers because the group design did not require a consequential collective task to be completed.
- Participants can develop voices with hybrid identities: one identity may animate the position and the other can animate the function.
    - During individual work and groupwork, Ted developed voices wherein his tutoring identity animated the positions and his gaming identity set the functions.

Chapter 4: Bettie Reaffirming Her Mathematical Ability

This chapter introduces the case of Bettie then investigates one of the two self-reported changes in Bettie's experience of number theory class, namely Bettie's restoration of her arithmetic (Pre-Proof Mathematics) ability. The other self-reported change is investigated in the fifth chapter. In the exit interview, Bettie noted that she succeeded in repairing her confidence in her computational skills, which some groupmates treated as unreliable. Through the lens of the VIP+function framework, I investigate how Bettie's confidence in her arithmetic (PPM) ability suffered from her groupmates' behaviors and how it was restored throughout the number theory class. Furthermore, this chapter illustrates how the VIP+function framework can be applied to video-records of social interactions.

Bettie started the number theory class being confident of her arithmetic (PPM) skills but lacking confidence in her proof skills. She was aware that the number theory class might create a challenge to her, because it required proof skills. Furthermore, Bettie was used to learn in traditional teaching settings, mainly by listening to instructors' lectures. The analysis of Bettie's individualized identities at the beginning of the semester—to be reported in this chapter—shows that Bettie was an epitome of traditional learning methods. The active learning involved in the number theory class presented an additional challenge to Bettie. In the number theory class, Bettie had to adapt to a challenging subdiscipline, "proof," and a new pedagogy, learning through small-group work.

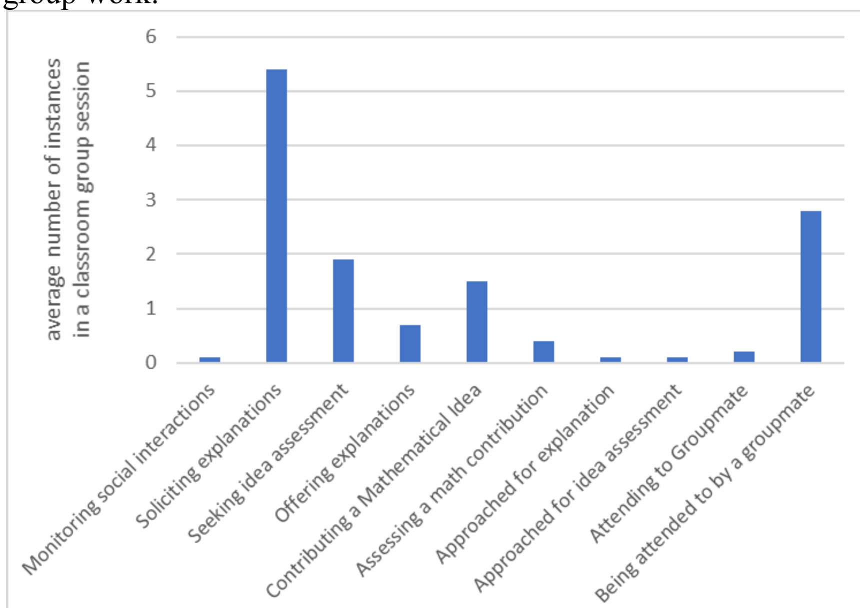


Figure 4-1: This histogram represents the average number of instances (y-axis) of listed positions (x-axis), as animated by Bettie in a group session (N = 11 group sessions). The histogram represents the results of a coding exercise of all video-recorded group sessions in which Bettie participated (N=11). Details about the coding scheme and rubrics can be found in the third chapter (see Table 2-8).

Recall that Bettie is selected for a focal study because of her significant self-reported changes in the number theory class. Nevertheless, the analysis of Bettie's participation in classroom groupwork (Figure 4-1) does not indicate a successful adaptation to active learning. Bettie's participation in classroom group sessions was coded using the coding scheme presented in Chapter 2 (revisits Table 2-8). The results indicate that on average in a 75-minute groupwork, Bettie predominantly animated passive positions by soliciting information and receiving attention from her groupmates. Her active participation, such as offering explanations and contributing mathematical ideas, was sparse on average. Indeed, during classroom groupwork, Bettie was observed working alone most of the time, looking at her tablet and textbook, writing on her notebook, and occasionally interacting with her groupmates.

These aforementioned observations of Bettie's behaviors in classroom groupwork shed some doubts on the authenticity of her self-reports. To closely investigate Bettie's learning development throughout the number theory class, I conduct two analyses, one analysis in this chapter and the other in the next chapter. In this chapter, I look closely at Bettie's behaviors in specific classroom activities, namely the ones that involve computations. In the next chapter, I look at learning activities in which Bettie participated outside the classroom.

A closer look at Bettie's participation in the computational activities of groupwork—as reported in this chapter—displays results that cohere with her self-reports on her significant learning changes. For a closer analysis, I use the VIP+function framework to document not only positions but also identities and functions animated through Bettie's participation in classroom groupwork over the semester. Since Bettie's self-reports indicate an improvement of her confidence in computational skills, the VIP+function framework is applied to the computational activities in Bettie's groupwork (G3) throughout the semester. A total of four computational activities are detected tackling the following mathematical concepts:

- Primitive roots (on 10/27),
- Quadratic residues (on 11/12),
- Quadratic reciprocity law (on 11/19), and
- Continued fractions (on 12/01).

The VIP+function analysis, to be reported in this chapter, reveals shifts of Bettie's mathematical identities, positions and functions as animated over the four computational activities. These shifts in identities, positions and functions indicate a change in Bettie's voice, which appears lacking confidence in the first two computational activities but gaining confidence in the last two computational activities. Further investigation is conducted to study the factors that boost the confidence in Bettie's voices during the third and fourth computational activities. It highlights an unexpected process that boosted Bettie's confidence in the third computational activity, while working on a complex mathematical construct (the quadratic reciprocity law). To the best of my knowledge, this process is undocumented in the research on learning mathematics. I call it **the mirroring process, by which an individual can animate confident voices reflecting the interlocutor's confidence**. As discussed in the conclusion of the chapter, the mirroring process provides access to a new learning ecology for learners who are not yet familiar with the same. By the mirroring process, learners can voice confidence prior to building the required knowledge that could support such confidence.

The chapter builds on two sections. The first section documents and studies Bettie's individualized identities and positions as reported in her early interview (see appendix A). The second section investigates how the documented identities and positions in Bettie's early interview play out through her participation in computational activities in the number theory class. In the second section, a subsection is dedicated to each one of the four abovementioned computational activities. Further information on the methods and methodologies is provided in Chapter 2. I make relevant methodological reminders when appropriate in the chapter.

Bettie's individualized identities at the beginning of the semester

In this section, I document Bettie's individualized identities found in her early interview (Int1-0922-Bettie, see appendix A). Some identities are readily identifiable in Bettie's narratives as semiotic repertoires and entrenched habits (reported in Table 4-1). Bettie's narratives of her early interview involve entrenched positionings about herself and groupmates without manifesting the semiotic repertoires that generate them. Such positionings are reported in a separate table (Table 4-2). The documented individualized identities and positionings are abundant, 11 identities and 14 positionings. They will be made relevant when future analyses encounter them actuated and animated in Bettie's participation in classroom groupwork or her other interviews. Each one of the individualized identities and positions is assigned a number for future references.

Two major identities, mathematical and learning identities, must hold our attention. Although only the mathematical identity is relevant to the investigation of computational activities (reported in this chapter), the next subsections will study the two identities. The learning identity (memorization) is relevant to the investigation of the next chapter (Chapter 5). In the following two subsections, I scrutinize the documented mathematical and learning identities by looking at data from other interviews and sources.

Table 4-1: Bettie's individualized identities at the start of the class, as analyzed in the narratives of her early interview on 9/22.

#	Bettie's individualized identity	References in her narratives	***Key phrase, quoting Bettie***
Id1	Mathematics is hard to understand but rewarding when it is understood.	Int1-0922-Bettie lines 1-23, 42-43.	*You really have to work hard to understand mathematics*
Id2	Smartness is to be fast.	Int1-0922-Bettie lines 31-34, 284-289.	*He's hella smart [...] he finishes the problem before any of us get to it.*
Id3	Learning is fostered by seeing more than listening.	Int1-0922-Bettie lines 33-37.	*I can't hear words and understand what they say; I have to see.*
Id4	There is an urge to understand mathematics	Int1-0922-Bettie lines 41-43, 131-135, 186-191, 195-200, 205-207.	*I like to fully understand the homework.*
Id5	A way to memorize something is to repeat, copy, and write it.	Int1-0922-Bettie lines 96-101.	*Writing down, repeating it, just memorizing.*
Id6	Grades are reliable indicators of ability.	Int1-0922-Bettie lines 118-120.	*I was getting As obviously I was good at mathematics.*
Id7	Arithmetic (PPM) is about manipulating formulas that are to be memorized.	Int1-0922-Bettie lines 149-158.	*Algebra that's it, all you have to do is memorize the stinkin formulas, plug in these things, try to solve.*
Id8	Asking questions is a sign of weakness (not smartness).	Int1-0922-Bettie lines 168-169, 214-219.	*I feel stupid asking [...] I'd rather not talk to them, they'd think I'm stupid.*
Id9	Solitary study, by reading textbooks and online resources, is most efficient for learning.	Int1-0922-Bettie lines 181-191, 198-200, 205-207, 216, 247-249.	*I like to work alone because it takes me a pretty long time to figure out [the mathematics of ongoing conversations].*
Id10	Go over homework to prepare for tests.	Int1-0922-Bettie lines 194-200, 235-237.	*When there's an exam, I just like go over the homework multiple times.*

#	Bettie's individualized identity	References in her narratives	*Key phrase, quoting Bettie*
Id11	Memorize what cannot be understood.	Int1-0922-Bettie lines 205-207.	*If I don't understand the work, I memorize it.*

Table 4-2: Bettie's individualized positionings of self and others documented in her early interview on 9/22.

#	Bettie's positioning of self or others	References in her narratives	*Key phrase, quoting Bettie*
Po1	Bettie is not smart. (influenced by Id2)	Int1-0922-Bettie lines 31-37, 214-216.	*I'm not super smart, it takes me a while to understand things.*
Po2	Bettie is good only at mathematics.	Int1-0922-Bettie lines 49-53.	*I seem to be good at math and suck at everything else.*
Po3	Instructors, especially of the proof classes, at her current college are bad explainers.	Int1-0922-Bettie lines 76-78, 81-87, 162-173.	*Every single proof teacher I've had at this school has been [mean and impatient]*
Po4	Bettie is shy and gets intimidated.	Int1-0922-Bettie lines 84-85, 171-172, 214-216, 278-280.	*I'm a quiet person, don't speak up or go to office hours.*
Po5	Bettie cannot understand the number theory textbooks.	Int1-0922-Bettie lines 130-137.	*I can't really read a paragraph and understand what it's talking about […] It's hard in this class.*
Po6	Betties avoids asking questions so that she would not look stupid. (influenced by Id8)	Int1-0922-Bettie lines 167-169, 214-219.	*I feel stupid asking […] I'd rather not talk to them, they'd think I'm stupid.*
Po7	Bettie tends to ask basic questions in groupwork.	Int1-0922-Bettie lines 214-216, 253-255.	*I ask a lot of stupid questions.*
Po8	Bettie finds online resources helpful, sometimes.	Int1-0922-Bettie lines 247-249.	*Sometimes they [online videos or webpages] show us some work where I can understand what they're saying.*
Po9	Bettie does not understand group talk on mathematics.	Int1-0922-Bettie lines 253-255.	*I don't understand half of the stuff that we talk about.*
Po10	Jeremy explains the best; the other groupmates have complex minds.	Int1-0922-Bettie lines 274-280.	*It makes sense when Jeremy explains it to me […] when they explain it to me, I don't get it.*
Po11	Boutros is quiet during groupwork.	Int1-0922-Bettie lines 281-282.	*Boutros is really quiet.*
Po12	John and Ted are smart.	Int1-0922-Bettie lines 284-293.	*John's hella smart.* *Ted is smart too.*
Po13	Ted tends to explain to others.	Int1-0922-Bettie lines 296-298, 306.	*If I really don't understand it, Ted will be like "oh okay this is how you do this."*
Po14	Ted is the group leader.	Int1-0922-Bettie line 297-298, 305-306.	*He's kind of like the group leader among us, asks questions, explain, [and urges groupmates to work together after class]*

Bettie's individualized mathematical identity

Bettie grounded her mathematical identity in her computational ability. In the early interview, she reported the following.

Bettie: [in] arithmetic like calculus and algebra, I was A, an A student and I feel like that's what wanted me to be a math major cause I was getting As obviously I was good at it. (Int1-0922-Bettie lines 118–122).

However, her experience with proof classes demonstrated the opposite, as reported in the following excerpt of her early interview (Int1). Bettie's experience with proofs changed in the studied class, as she reported in her exit interview (Int2).

<u>Int1-0922-Bettie lines 81-87</u>

Bettie: Um yeah. I guess like uh my first um proof class [unidentified] I don't even remember my teacher's name but he was kind of like mean, and when we asked questions he would get like pis, like pissed that we were asking questions when we didn't understand what he was trying to say . and I don't know I kind of just felt like intimidated a lot. I'm like a really quiet person so I don't speak up or go to office hours or anything so it kinda just like . and like every single proof teacher I've had at this school has been like that.

<u>Int2-1202-Bettie lines 26-40</u>

Bettie: Um. Well I mean just for me I feel like theory in general is . just like learning proofs and . I don't know . it's just been really difficult for me. But . uhh out of all the proof classes I've taken this is probably the most that I've . like . learned . I guess you can say. Cause a lot of the time I kinda just got by. and I feel like this one I'm actually understanding like . why. just cause . yeah.

Fady: okay uh. did you take a class on proofs here [at this school]?

Bettie: yeah

Fady: And how was it?

Bettie: I didn't learn anything.

Fady: In that class?

Bettie: Yeah. so when I went into um. Modern Algebra I got a tutor . which kinda helped me but I still didn't really . fully get it. I just kinda memorized and wrote down what I remembered. uh yeah like I feel like this [number theory] is the one class that's actually helping me understand. and like come up with things on my own instead of like finding the answer and writing it down.

Since the start of the semester, Bettie distinguished between the two following sub-disciplines in mathematics: on one side "arithmetic" (PPM including algebra, calculus and computations), and on the other side "proofs" (involves theoretical constructs). Her grades in "arithmetic" positioned her as being good at math, while her struggle with proof classes as weak. She voiced this dilemma in her soliloquy (present below), reaffirming her arithmetic (PPM) ability and implicitly attempting to salvage her position within mathematics (note the repair in "m. arithmetic" by which she possibly replaced mathematics).

Bettie: Dude don't doubt me when it comes to m . arithmetic because. I'm . there's a reason why I'm a math major like I'm smart . I just don't understand proofs. (Int2-1202-Bettie lines 309-312).

Proofs, a missing identification.

Did Bettie identify, *a la* Lacan, with the "proof" sub-discipline at the start of the semester? Doubtful. Bettie claimed, "me and my brother and sister all seem to be really good at math and suck at everything else" (Int1-0922-Bettie lines 49–50). For her, the "proof" sub-discipline was merely a mathematics-related "thing" that disturbed her mathematical positioning and, because of which, she could no longer affirm "don't doubt me when it comes to mathematics." Her domain of self-efficacy was consequently reduced to "arithmetic" (see the preceding excerpt).

Per se, the word "proof" was emptied of significance, since Bettie "didn't learn anything" throughout all previous proof classes. She succeeded in the quizzes of proof classes by "winging" answers that she had memorized from her homework (Int2-1202-Bettie lines 267–277). Additionally, Bettie did not take the number theory class seriously at the beginning (see the next section). When asked why she chose number theory as an elective, she responded as follows.

> Bettie: […] it just seemed interesting I guess. I wanna know kinda like why we do things like I see like I know formulas and I can just do them like plug numbers in but I want it's kind of interesting to see like why. (Int1-0922-Bettie line 72-73; underlines for analysis purposes)

Notice the wavering motivation, in the above excerpt, between "it's interesting" (single underline) and "I want" (double underline). The phrase "I want" interpellates an engaged subject of desire, whereas "it's interesting" objectifies the content into a facultative curiosity. In the short excerpt above, Bettie starts with an unsure, ("I guess") objectified, and optional curiosity, moves after a pause into an engaged subject ("I wanna know kinda like why we do things"), elaborates on her individualization of mathematical computations ("like I see like I know formulas and I can just do them like plug numbers in"), then states a missing individualization ("I see … but [I don't] see …"), that is, she does not know why the formulas work. Towards this missing element, she immediately repairs her current identification "I want" with an objectified curiosity "it's kind of interesting to see."

Bettie therefore started number theory class having realized a missing element within her individualized mathematics but was irresolutely open to embracing the new identification that was required. Throughout the class, this irresolution shifted to a determination and even actual identification. This shift has been analyzed and reported here.

Bettie's individualized "arithmetic."

How did Bettie individualize the "arithmetic" (PPM) sub-discipline? She mentioned, in the early interview (Int1-0922-Bettie lines 114–173), her high-school algebra teacher who helped her become good at mathematics. Bettie learned to memorize formulas since that class.

> Bettie: she [Bettie's favorite high-school algebra teacher] kinda just made it we couldn't walk into the classroom until we could like repeat the formulas to her so I would be standing outside standing out there trying to memorize the stinkin formulas [laugh] and they kinda just stuck with me and I feel like algebra like that's it all you have to do is memorize. (lines 149-152)

Bettie also mentioned a trigonometry class she enjoyed at her community college, because it was all about solving problems with formulas and equations (lines 153–158). Although she "hated" all her teachers heretofore at the current university, she "probably" appreciated her statistics class the most (lines 159–164). In her early interview, Bettie asserted that she knew how to work with formulas (“I know formulas and I can just do them like plug numbers in”, line 72). Thus, for Bettie, her so-called “arithmetic” (PPM) sub-discipline consisted of **memorizing and manipulating formulas**, which she had assertively individualized.

Bettie exhibited her individualized “arithmetic” identity (PPMI) in her homework, which she used to complete by copying answers from textbooks, as per her testimony (Int2-1202-Bettie lines 260–266). Indeed, an analysis of her homework (see appendix C) revealed that Bettie heavily relied on the textbooks for her homework. More importantly, Bettie consistently transformed the answers of the textbooks into formulas, especially for the first five homework (Hw1 through Hw5). She would either disregard the explanatory English sentences or transform the English phrases into logico-mathematical symbols. Consider, for instance, the vignette of Figure 4-2

As Bettie copied from the textbook, she adjusted the variables to the ones given in the prompt (WK3#4). Consider, for example, the proof of the corollary 5-2 in the textbook and part c of Bettie’s homework. The variable used in the textbook was called *n* (i.e. $n^p \equiv n$). But the prompt in 4(c) named the variable a (i.e. $a^p \equiv a$). Bettie substituted a to all *n* in the textbook, except in one instance, namely $\gcd(p, n) = 1$. This oversight, along with the same exact steps in both documents, textbook and Bettie’s homework, provided evidence that Bettie copied her homework from the textbook.

Bettie had a recognizable and predictable style of copying from the textbooks. In her early homework, she mostly attended to the formulas in the proofs and only occasionally to explanatory and justifying English sentences. Notice the scripts undelined in blue in the textbook (Figure 4-2), representing all inscriptions that did not have a counter-part in Bettie’s homework. All of them turned out to be English sentences. On the contrary, Bettie did not miss copying any formula in the proof of the textbook. Bettie seemed to value formulas more than English sentences in mathematical writings, a behavior that can be traced back to her individualized “arithmetic” identity (PPMI).

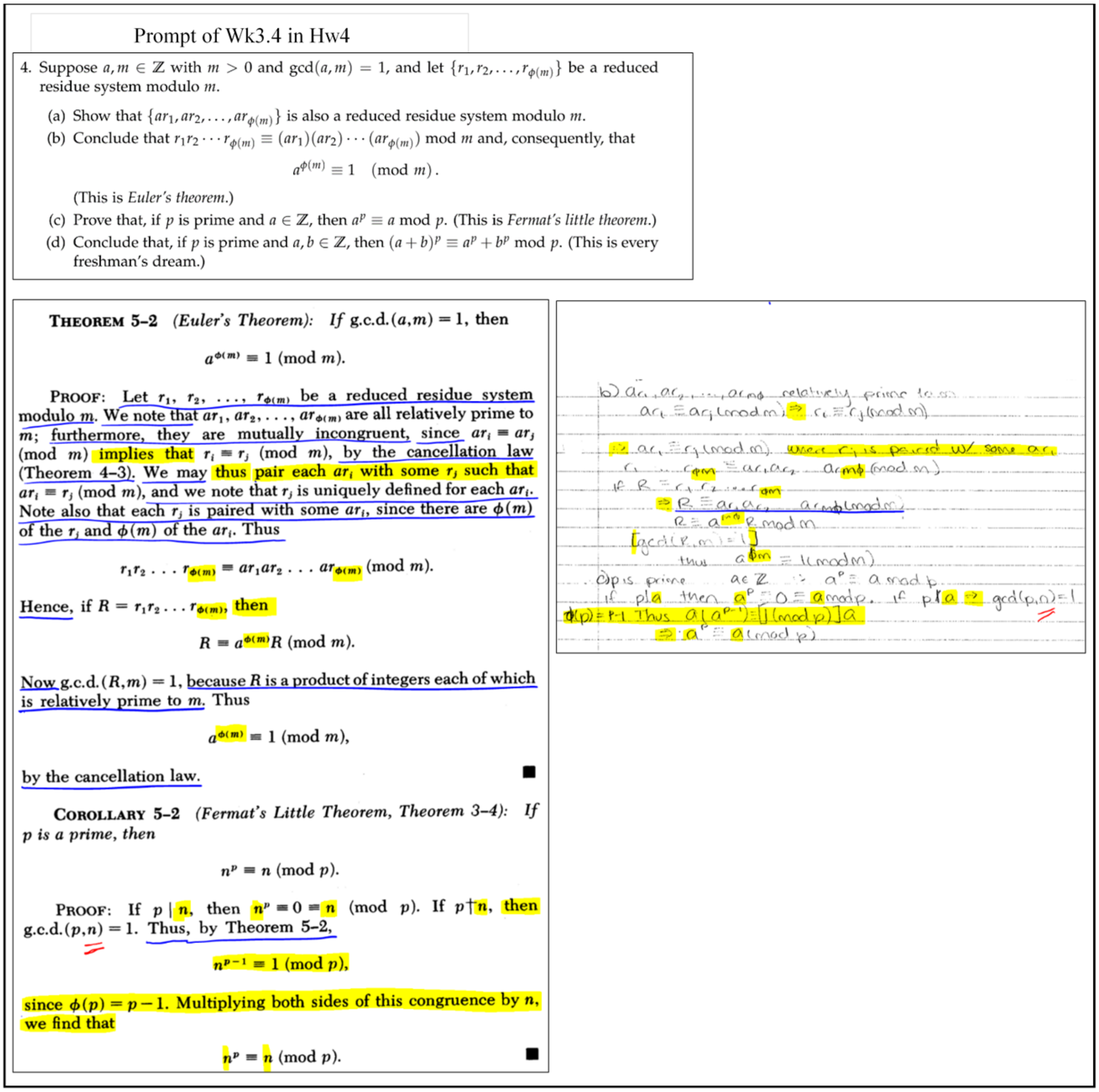
The lack of attention to the explanatory and justifying sentences in proofs interfered in Bettie’s learning development with reagrd to proof skills. Consider, for instance, the second sentence in the proof of the textbook and its counter-part in Bettie’s homework (reproduced in Table 4-3).

Table 4-3: Comparison of a sentence in the textbook and its counter-part in Bettie’s homework.

Textbook	Bettie’s homework
We note that $ar_1, ar_2, \ldots, ar_{\varphi(m)}$ are all relatively prime to m; furthermore, they are mutually incongruent, since $ar_i \equiv ar_j \ (mod\ m)$ implies that $r_i \equiv r_j \ (mod\ m)$, by the cancellation law (Theorem 4-3).	$ar_1, ar_2, \ldots, ar_{\varphi(m)}$ relatively prime to m $ar_i \equiv ar_j \ (mod\ m) \Rightarrow r_i \equiv r_j \ (mod\ m)$

By disregarding the statement “they are mutually incongruent,” Bettie created a counter sense. In the book, the implication $[ar_i \equiv ar_j \ (mod\ m) \ \Rightarrow r_i \equiv r_j \ (mod\ m)]$ was a proof by contradiction to the claim $ar_i \equiv ar_j$. As per Bettie’s writing in her homework, she approved that $ar_i \equiv ar_j$. Thus, had Bettie read and understood the second sentence in the proof, she would have translated it into mathematical symbols as follows:

$ar_i \not\equiv ar_j \ (mod\ m)$ because $r_i \not\equiv r_j \ (mod\ m)$ and $\gcd(a, m) = 1$.



Prompt of Wk3.4 in Hw4

4. Suppose $a, m \in \mathbb{Z}$ with $m > 0$ and $\gcd(a, m) = 1$, and let $\{r_1, r_2, \dots, r_{\phi(m)}\}$ be a reduced residue system modulo m.
   (a) Show that $\{ar_1, ar_2, \dots, ar_{\phi(m)}\}$ is also a reduced residue system modulo m.
   (b) Conclude that $r_1 r_2 \cdots r_{\phi(m)} \equiv (ar_1)(ar_2) \cdots (ar_{\phi(m)}) \bmod m$ and, consequently, that
   $$a^{\phi(m)} \equiv 1 \pmod{m}.$$
   (This is *Euler's theorem*.)
   (c) Prove that, if p is prime and $a \in \mathbb{Z}$, then $a^p \equiv a \bmod p$. (This is *Fermat's little theorem*.)
   (d) Conclude that, if p is prime and $a, b \in \mathbb{Z}$, then $(a+b)^p \equiv a^p + b^p \bmod p$. (This is every freshman's dream.)

THEOREM 5–2 *(Euler's Theorem): If* g.c.d.$(a, m) = 1$, *then*

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

PROOF: Let $r_1, r_2, \dots, r_{\phi(m)}$ be a reduced residue system modulo m. We note that $ar_1, ar_2, \dots, ar_{\phi(m)}$ are all relatively prime to m; furthermore, they are mutually incongruent, since $ar_i \equiv ar_j \pmod{m}$ implies that $r_i \equiv r_j \pmod{m}$, by the cancellation law (Theorem 4–3). We may thus pair each ar_i with some r_j such that $ar_i \equiv r_j \pmod{m}$, and we note that r_j is uniquely defined for each ar_i. Note also that each r_j is paired with some ar_i, since there are $\phi(m)$ of the r_j and $\phi(m)$ of the ar_i. Thus

$$r_1 r_2 \dots r_{\phi(m)} \equiv ar_1 ar_2 \dots ar_{\phi(m)} \pmod{m}.$$

Hence, if $R = r_1 r_2 \dots r_{\phi(m)}$, then

$$R \equiv a^{\phi(m)} R \pmod{m}.$$

Now g.c.d.$(R, m) = 1$, because R is a product of integers each of which is relatively prime to m. Thus

$$a^{\phi(m)} \equiv 1 \pmod{m},$$

by the cancellation law. ■

COROLLARY 5–2 *(Fermat's Little Theorem, Theorem 3–4): If p is a prime, then*

$$n^p \equiv n \pmod{p}.$$

PROOF: If $p \mid n$, then $n^p \equiv 0 \equiv n \pmod{p}$. If $p \nmid n$, then g.c.d.$(p, n) = 1$. Thus, by Theorem 5–2,

$$n^{p-1} \equiv 1 \pmod{p},$$

since $\phi(p) = p - 1$. Multiplying both sides of this congruence by n, we find that

$$n^p \equiv n \pmod{p}. \quad ■$$

b) $ar_1, ar_2, \dots, ar_{m\phi}$ relatively prime to m
$ar_i \equiv ar_j \pmod{m} \Rightarrow r_i \equiv r_j \pmod{m}$
$\Rightarrow ar_i \equiv r_j \pmod{m}$ where r_j is paired w/ some ar_i
$r_1 \dots r_{\phi m} \equiv ar_1 ar_2 \dots ar_{m\phi} \pmod{m}$
if $R = r_1 r_2 \dots r_{\phi m}$
$\Rightarrow R \equiv ar_1 ar_2 \dots ar_{m\phi} \pmod{m}$
$R \equiv a^{m\phi} R \bmod m$
[$\gcd(R, m) = 1$]
thus $a^{\phi m} \equiv 1 \pmod{m}$
c) p is prime $a \in \mathbb{Z}$ $\therefore$ $a^p \equiv a \bmod p$
if $p \mid a$ then $a^p \equiv 0 \equiv a \bmod p$. if $p \nmid a \Rightarrow \gcd(p, n) = 1$
$\phi(p) = p-1$ Thus $a(a^{p-1}) \equiv [1 \pmod{p}]a$
$\Rightarrow a^p \equiv a \pmod{p}$

Figure 4-2: Comparing proofs in Andrews textbook (on the left) and Bettie's homework on Wk3#4b-c (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source. The underlined letter in red (two lines) is an evidence of an oversight in copying from the textbook; the variable n is otherwise consistently substituted with a.

Bettie's individualized learning identity

Within the identity of "arithmetic" (PPMI), heretofore considered as mathematics, Bettie individualized a memorization-learning method—mathematics was about memorizing formulas (Int1-0922-Bettie lines 95–101, 149–152, Int2-1202-Bettie lines 36–38, 269–279, 285–286, 584–588). Ted, Bettie's tutor-like groupmate during study group, also mentioned Bettie's learning method of memorization (see the next excerpt of Ted's Int2).

Int2-1203-Ted lines 482–486

Ted: Bettie's approach is . to . to number theory . is making her life a little bit more difficult. because uh she is approaching it with . like I think what we do with applied math . which is just memorize formulas and apply them. umm and she mentioned that multiple times with the group she's like . “I just gotta go back and memorize it” and I think it's making proofs really hard for her.

In the several instances where Bettie talked about her learning by memorization, its method and reason were highlighted. First, Bettie clearly stated that her way of memorizing was through writing (see the next excerpts of Bettie’s Int1 and Int2). Thus, copying her homework from published resources has dual functions for Bettie—completing the homework and training her memory.

Int1-0922-Bettie lines 96-101

Bettie: […] [Tutor of modern algebra] would repeat like definition after definition like just to make me memorize it and it kind of just stuck like the best thing that ever happened, yeah so that was really good.

Fady: Okay and uh repetition is what makes this experience very helpful

Bettie: Yeah and writing it down. Yeah like keep doing it, repeating it, just memorizing

Int2-1202-Bettie lines 275-276

Bettie: […] my way of memorizing is like writing. so if I write down for homework I kinda remember how I wrote it out and I'll write it down . for like the quiz.

Second, Bettie had an urge for understanding mathematics; memorizing was a fallback option short of understanding the materials (see the next two excerpts of Int1). Bettie expressed a twofold disatisfaction with memorizing. First, the content to be memorized was “stinkin formulas” (Int1-0922-Bettie line 151). Second, copying mathematical answers without understanding them caused her frustrations (line 189). However, Bettie’s identification with the alternative learning method, seeking understanding, depended on her stamina. “If I don't understand I just like try to memorize” (line 206). However, how far was Bettie willing to go before giving up? Bettie’s engagement also shifted throughout the semester; she became committed only by the second month of the class (see third subsection).

Int1-0922-Bettie lines 188-191

Bettie: […] I just like to read over because I like to understand things cause like it's really frustrating when I'm just like copying work I have to really just like understand what I'm doing and why I’m doing it so I kind of just like to work alone because it takes me a pretty long time to figure out

Int1-0922-Bettie lines 204-208

Fady: Do you find it helpful, the resources that you find online?

Bettie: Sometimes. There's just like different notation so it kind of confuses me but I like try to make sense of it and then if I don't understand I just like try to memorize the work that they put and then just write it down for whenever I see the problem again.

Bettie’s voices in computational activities: “Dude, don’t doubt me when it comes to arithmetic”

In this section, I report the analysis of Bettie’s reaffirmation of her arithmetic identity (PPMI) by following the retrospective systematic methodology of data selection (see Chapter 2).

Recall the first step starts with Bettie's narratives on the reported change, followed by the selection of the relevant data source—in this case, video-records of groupwork. Then, I identity pertinent parts of the data—in this case, computational activities. The selected episodes of video-records are parsed into eco-units. An eco-unit is a conversational unit that starts with an initiation and develop on the same (Clark & Schaefer, 1989; Engle, Langer-Osuna & McKinney de Royston, 2014). A conversational initiation can be a question or a contribution of an idea generating a sort of discontinuity with previous speech. The eco-unit covers the entire utterances and speech turns that follow up on the initiation. Eco-units commonly end with a moment of silence or an interjection of a new initiation. The eco-units are the optimal ecological context in which Bettie's voices, i.e. her talk at every speech turn, are to be interpreted and understood. A pertinent analysis of the identities, positions and functions that constitute a voice must look at the eco-unit within which the voice is animated.

Bettie reported in the exit interview (Int2-1202-Bettie lines 304–312) that her groupmates, who could not recognize her mathematical ability early in the semester due to her struggle with proofs, eventually realized her ability in arithmetic (PPM). In her narrative, Bettie blamed her groupmates for not trusting her arithmetic (PPM) ability while working with computations.

<u>Int1-0922-Bettie lines 306-314</u>

Fady: Did you experience any change in your ways of participating in the group?

Bettie: Uh Yeah. I feel like I can like uh I like . if it has to do with arithmetic I feel like I'm just . I feel like . I can do it. like I. Maybe they don't take me as like serious so when I have the answer they're like "whatever like it's probably wrong." but I usually do get the right answer and I feel like "hah" like "told you". And they're like "wait" "but what?" Jeremy always questions me and I'm like "Dude don't doubt me when it comes to m . arithmetic because." I'm . there's a reason why I'm a math major like I'm smart . I just don't understand proofs.

After the midterm, the instructor started to include computational activities at the beginning of some worksheets in order to facilitate the understanding of new definitions (to consult the worksheets go to appendix E). He included computational activities for the following definitions: primitive roots (Wk8#1 group session on 10/27), quadratic residues (Wk10#1 group session on 11/12), the Jacobi symbol (Wk10#4 group session on 11/19), and continued fractions (Wk11#1&2 group session on 12/01).

This subsection analyzes Bettie's socio-mathematical interactions within her group through each of the four computational activities. The analysis shows that her groupmates challenged Bettie's "arithmetic" (PPM) confidence, which she exhibited in the first computational activity. Thereafter, Bettie exhibited a lack of confidence in the subsequent computational activity. It was only in the third and fourth computational activities that Bettie started to animate confident mathematical voices.

Primitive roots (on 10/27).

The worksheet on primitive roots (Figure 4-3) was the first worksheet to include a computational activity (Wk8#1). Hoffmann started the class on 10/27 with a four-minute talk on the dry-erase poster boards, introducing them to the class for the first time—the midterm tests were returned on this day. Then he engaged the class in a twenty-five-minute interactive instruction, introducing and connecting the following two definitions: the order of an integer and its primitive roots.

a and m are coprime integers.

The **order** of a modulo m is the smallest integer k such that $a^k \equiv 1\ (mod\ m)$.

If the order of a is $\varphi(m)$, then a is a **primitive root**.

Hoffmann then moved to compute the primitive roots modulo m for $m = 2, 3, 4\ and\ 5$. For each case of m, he started by computing $\varphi(m)$ and then checking whether the order of each integer coprime with m was $\varphi(m)$. For instance, for $m = 5$, $\varphi(5) = 4$, he computed $2^2, 2^3$ and found that none was congruent to 1 modulo 5. Thus, 2 was the primitive root modulo 5. Similarly, 3 was a primitive root because $3^2\ and\ 3^3$ were not congruent to 1 modulo 5. However, 4 was not primitive root because $4^2 \equiv 1\ (mod\ 5)$.

He ended his instruction by pointing out a proposition that would be proven in WK8#4b, which was useful in guessing the number of primitive roots modulo m. To be precise, if m had any primitive root, the total number of primitive roots was $\varphi(\varphi(m))$. He illustrated with $m = 5, where\ \varphi(\varphi(5)) = 2$, and they found that only 2 and 3 were primitive roots modulo 5.

Worksheet 8: Primitive Roots

1. Compute all primitive roots mod 6, 7, and 8.
2. Suppose a has order n mod m, and $a^k \equiv 1$ mod m. Show that $n|k$.
3. Show that, if a is a primitive root mod m, then $\{a, a^2, \ldots, a^{\phi(m)}\}$ is a reduced residue system mod m.
4. Suppose a has order n mod m, and $\gcd(k, n) = g$. Show that a^k has order $\frac{n}{g}$ mod m. Conclude that this implies the following two corollaries:
   (a) If a is a primitive root mod m, then a^k is also a primitive root mod m if and only if $\gcd(k, \phi(m)) = 1$.
   (b) If there exists a primitive root mod m, then there are precisely $\phi(\phi(m))$ primitive roots.
5. Andrews 7.1.6, 7.2.15, Stein 2.8, 2.30.
6. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Figure 4-3: Worksheet number 8 on primitive roots, introduced on 10/27.

During the groupwork on computational activities, Bettie animated a voice her by active participation in ten eco-units, analyzed herein. Overall, Bettie exhibited confidence early in the groupwork but struggled afterwards. She attempted to develop two types of voices: contributive, and in search of explanations.

Bettie's contributive voice.

Throughout the collective computational activity, Bettie made mathematical contributions in three eco-units (starting at 0:43:36, 0:46:34, and 0:55:42) along with two quasi-contributions in two eco-units (at 0:43:36 and 1:01:42).

<u>Eco-unit of 1027-g3 at 0:43:31-0:43:47</u> – *Computing the primitive roots modulo six*

Jeremy: you already got all the primitive roots?

Boutros: there is only one primitive root.

Bettie: [looks at Boutros's notes] when you get five?

Boutros: five

Jeremy: oh yeah. there is only one primitive root. [hand gesture towards Boutros and looks to his left -Bettie sitting to his right]

Bettie: [Looks at her notes, as if talking to self] oh yeah. phi of phi of six [$\varphi(\varphi(6))$]
[writes on her notebook]

In her first vocal act ("when you get five?"), Bettie volunteered to complete Boutros' statement ("there is only one primitive root") by animating a position of seeking an idea assessment. What idea was Bettie seeking to assess? She might have been trying to either check the answer of her computation (five is a primitive root modulo six) or justify Boutros' statement. Bettie's behavior in the remainder of the eco-unit supports the latter claim. After Boutros confirmed the answer as "five," Bettie continued to search in her notebook as if something was still missing. She proclaimed a "eureka", voicing "oh yeah. phi of phi of six," which was the instructor's formula for finding the number of primitive roots. Thus, Bettie was attempting to provide a justification to Boutros' statement but failed the first time. Bettie's first vocal act was an enthusiastic but misplaced and, otherwise, correct mathematical contribution.

In her first attempt to justify Boutros' statement, Bettie appealed to the computation of primitive roots to justify the total number thereof. In her second attempt, she realized that she needed the right formula, $\varphi\big(\varphi(m)\big)$. In both situations, Bettie was actuating her individualized "arithmetic" identity (PPMI), a discipline associated with formulas and plugging in numbers (see Id7 in Table 4-1).

<u>Eco-unit of 1027-g3 at 0:46:34-0:46:48</u> – *Computing primitive roots modulo seven*

Jeremy: so what's the second one?

Boutros: it has to be:::: =six

Bettie: =three

Boutros: oh three. Are you sure? [turning his face to Bettie, who nods]

Ted: three for seven right?

Boutros: alright. okay. I probably did something wrong then.

<u>Eco-unit of 1027-g3 at 0:46:48-0:47:01</u> – *Use of calculator*

Jeremy: [unclear; probably asking if they're allowed to use calculators]

Bettie [looks at her calculator]: yeah. I think so.

[Jeremy takes out his calculator from his backpack]

Bettie: here is my one that I checked out. [giggles]

<u>Eco-unit of 1027-g3 at 0:47:01-0:47:40</u> – *Bettie's contribution discussed*

Boutros: alright you're right. wait. (18 sec) I don't think three is.

Jeremy: [types on his calculator] no it is. Wait. yeah it is.

Bettie's third vocal act ("= three"), which also completed Boutros' speech, was a genuine mathematical contribution in its position and function. However, her answer conflicted with Boutros' ("six"). Boutros challenged Bettie by asking her whether she was confident about her answer. She only nodded. Boutros accepted the need to revise his computation after Ted indicated that he had also got "three" as the answer. Boutros remained uncertain about "three" (see eco-unit above, at 0:47:01–0:47:40) but Jeremy asserted that "three" was a primitive root modulo seven. While Bettie's answer was correct, she did not defend it through mathematical argumentation and her feeble social influence could not support it. Her groupmates, namely Ted and Jeremy, had a stronger influence on Boutros. Notice that neither Ted nor Jeremy offered arguments about their answers. Yet they, unlike Bettie, swayed Boutros.

More precisely, in the eco-unit of 0:46:48–0:47:01, Bettie assertively responded to Jeremy, who had been questioning the use of calculators. All groupmates were using calculators. Jeremy had been calling upon the classroom authority to check whether the current practice was allowed. By responding, Bettie positioned herself as a monitor of group interactions, speaking on behalf of the classroom authority. Nonetheless, when the instructor visited the group (at 0:49:04), he remarked that "ay you don't need calculators." He looked at Jeremy, who at that moment was the only group member using a calculator. Bettie and Jeremy smiled at Hoffmann's remark. Bettie was found unreliable with regard to relaying the classroom authority's expectations.

Eco-unit of 1027-g3 at 0:55:42-0:56:15 – *Computing primitive roots modulo eight*

Jeremy: [unclear] them all I didn't get a primitive root

Bettie: yup five is one

Boutros [to Jeremy]: I did it this way. this way you doing it?

Bettie: yeah

Boutros: does it go like that? [Jeremy stares at Boutros's notebook]

Bettie: I got five.

Jeremy: five squared is congruent to one mod eight. so it isn't the five. [Jeremy looks at Bettie]

[Bettie makes facial gestures -like puzzled- and turns away to look at Boutros' notes and then engages in off-topic conversation]

Bettie challenged Jeremy's finding, i.e., there were no primitive roots modulo eight, by asserting that five was a primitive root. Boutros attempted to investigate Jeremy's computational method along with his own. While Jeremy had been looking at Boutros' notebook, Bettie interjected a reaffirmation of her result—"I got five." Looking at Bettie, Jeremy explained why five could not be a primitive root modulo eight, i.e., $5^2 \equiv 1 \ (mod\ 8)$. At that point, Bettie evaded the conversation.

Bettie stood up for the answer of her computation against a socio-mathematically influential groupmate, Jeremy. Nevertheless, she did not defend her stance. She just reaffirmed her result, which resounded her learning method that was based on repetition and devoid of understanding (Id4 in Table 4-1), and evaded the conversation when she was pressured by Jeremy's argument regarding why her answer did not hold. Thus, in this eco-unit, Bettie actuated her arithmetic identity (PPMI), which consisted of formulas and plugging in numbers and did not involve mathematical argumentation. She employed social tactics, through proclamation and puzzled silence, in order to evade an engagement with mathematical argumentation.

Table 4-4: VIP+function analysis of Bettie's contributive voice in the computational group activities on 10/27.

Eco-unit	Voice	Identity	Position	Function
1027-g3 at 0:43:31-0:43:47	[Looks at Boutros's notes] when you get five?	Group member's right and "arithmetic" sub-discipline (PPM) [computations justify a result]	Seeking an idea assessment	Primarily, to complete a groupmate's statement and, possibly, to check her answer.
1027-g3 at 0:43:31-0:43:47	[looks at her notes, as if talking to self] oh yeah. phi of phi of six [$\varphi(\varphi(6))$] [writes on her notebook]	Group members' expectations and "arithmetic" sub-discipline (PPM) [find the right formula]	Offering an explanation	To indicate she understands now, agreeing with Boutros and Jeremy.
1027-g3 at 0:46:34-0:46:48	=three	Group members' expectations and "arithmetic" sub-discipline (PPM) [find the results of computations]	Contributing mathematically	To answer a groupmate's question.

1027-g3 at 0:46:34-0:46:48	Boutros: oh three. Are you sure? [turning his face to Bettie, who nods]	Social aspect of groupwork	Minimizing debate [nodding]	To defend herself.
1027-g3 at 0:46:48-0:47:01	Bettie [looks at her calculator]: yeah. I think so.	Classroom and group membership	Relaying classroom authority	To justify her and her groupmates' use of calculators.
1027-g3 at 0:46:48-0:47:01	Here is my one that I checked out. [giggles]	[unidentified]	Off-topic talk	[unidentified]
1027-g3 at 0:55:42-0:56:15	Yep five is one	Group members' expectations and "arithmetic" sub-discipline (PPM) [checking the answer]	Sharing her result	To contradict Jeremy's finding.
1027-g3 at 0:55:42-0:56:15	Yeah […] I got five	Group members' expectations and "arithmetic" sub-discipline (PPM) [checking the answer]	Proclaiming her result	To foster social influence.
1027-g3 at 0:55:42-0:56:15	[Bettie makes facial gestures – as if puzzled – and turns away to look at Boutros' notes]	Social aspect of groupwork	Puzzled [facial gesture]	To evade mathematical argumentation.

Throughout the first computational activity, Bettie volunteered a significant amount of contributions which, no matter how small, surpassed (twice as often) as her average enactment of a contributing role—1.5 instances—per group session (see Figure 4-1). These unprecedented contributive instances reflected Bettie's self-efficacy (see Po2 in

) and her individualized "arithmetic" (PPM) sub-discipline (Id7 in Table 4-1). Yet, they suffered. Bettie contributed twice with a shy confidence, as she hinged her contributions to Boutros' initiatives. Once, she misplaced her contribution and the other times, she did not defend them either with mathematical argumentation or through social influence. She did attempt to exercise certain tactics of social influence in the eco-unit 0:55:42–0:56:15, against Jeremy. Yet, Jeremy reinforced the discourse of mathematical argumentation which Boutros had initiated and, thus, frightened Bettie.

Bettie's voice soliciting explanation.

When Bettie realized her lack with respect to understanding the arithmetic procedures of computing the primitive roots, she started to ask questions to her groupmates as well as her instructor, Hoffmann. She explicitly solicited explanations in six instances within five eco-units (see the transcripts given below) in order to enhance her understanding. All six instances were either without or with a referred mathematical identity (see Table 4-5). Bettie merely exercised her right, as a group member, to ask questions, which she voiced as general, without mathematical content and/or by repeating the mathematical speeches of her interlocutors (analytical evidence follows; rushed readers may skip to the discussion section).

Table 4-5: VIP+function analysis of Bettie's knowledge-soliciting voice during the computational group activities on 10/27.

Eco-unit	Voice	Identity	Position	Function
1027-g3 at 0:52:26-0:54:39	when is there is none again? I didn't understand.	Group member's right and a referred mathematical identity	Soliciting explanations from participants	To enhance her understanding of the procedures.
1027-g3 at 0:52:26-0:54:39	it's just you just look until you find?	Group member's right without mathematical identity	Soliciting an idea assessment	To enhance her understanding of the procedures.

1027-g3 at 0:59:30-0:59:56	[pointing to Boutros's notes on the dry-erase poster board] is this how you need to write the integers?	Group member's right and a referred mathematical identity	Soliciting an idea assessment	To enhance her understanding of the procedures.
1027-g3 at 1:00:33-1:00:39	wait. so eight has none?	Group member's right and referred mathematical identity	Soliciting an idea assessment	To enhance her understanding of the procedures.
1027-g3 at 1:01:18-1:01:42	why isn't three a primitive root?	Group member's right and a referred mathematical identity	Soliciting an explanation from Boutros	To enhance her understanding of the procedures.
1027-g3 at 1:01:42-1:02:51	it has to be . wait. it has to be under four?	Group member's right and a referred mathematical identity	Soliciting an explanation from Boutros	To enhance her understanding of the procedures.
1027-g3 at 1:01:42-1:02:51	so if this is a prime . if this is uh . oh I get it . yeah yeah yeah. because it's suppose . it's a prime root if this is four […] and it's more than four.	Group member's right and a personalized mathematical identity	Sharing her understating with Boutros	To allow Boutros to evaluate her current understanding.

In the eco-unit at 0:52:26–0:54:39, Hoffmann explained the nuance in the significance of $\varphi(\varphi(m))$ by pointing out that in cases where there were no primitive roots, the expression could no longer (obviously) designate the number of primitive roots. Bettie felt lost and asked him, "when is there is none again? I didn't understand." The specification of her lack of understanding was merely a re-voicing of Hoffmann's speech—"possibility number two is there is none" (underlined in transcript). Hoffmann replied, and Bettie followed up by soliciting an idea assessment ("you just look until you find?"); the idea was emptied of mathematical content.

<u>Eco-unit of 1027-g3 at 0:52:26-0:54:39</u> – *Discussing the proposition about* $\varphi(\varphi(m))$

Hoffmann: it is a subtle statement. the statement is if there is one . if you can find one then there is phi of phi of m

Jeremy: I see::::

Hoffmann: Okay? so for example for eight . there is two possibilities here. number one you find one and then you may know there has to be another one. <u>possibility number two is there is none</u>. that could happen okay? Are you really with me? so check . this is going ahead . little bit ahead of yourselves. but that's what I meant. [points to Bettie's worksheet] can you see 4b? they say there is phi of phi of m primitive roots. but it's a subtle statement. check out the whole sentence. the sentence says . if there is a primitive root . then I'm guaranteeing there is phi of phi of m ones. But that's really an if there is one. okay? You will see there is some ms for which there is no primitive roots period. sorry. in which case this sentence is vacuous. you know what I mean? I'm only telling you . if you can find me one . I guarantee you there is total of phi of phi of m primitive roots. yeah? If you can't find anyone . I'm not saying anything. Actually, if you can't find one, there is none. [laugh] right?

[0:53:49.13] Bettie: when is there is none again? I didn't understand.

Hoffmann: No body knows.

Bettie: it's just you just look until you find?

Hoffmann: that's right.

John: Wait. when you say there is no

Hoffmann [to Bettie]: not quite sure what I have just said. you can think about that.

Bettie: yeah

Hoffmann: so so that's comp. so again the easy answer to your question is "too complicated".

Bettie: mhm

Hoffmann: so a priori we just don't know whether or not there is any primitive roots. okay? but what what you gonna prove probably on thursday in 4b is if you can find one . okay? then there is gonna be a total of phi of phi of m ones. yeah? so if there is one . then we can count them actually. and I promise there will always be phi of phi of m one.

Bettie: okay.

Bettie initiated a new eco-unit at 0:59:30–0:59:56 in order to ask about the write-up of the results ("is this how you need to write the integers?"). To improve her understanding, she animated a question, soliciting an idea assessment. The idea was transferred from Boutros' notes to the dry-erase poster board, referred by the indicative "this." Where Bettie had to include her own mathematical wording, she used the general term "integers" to denote the key word of the lesson, "primitive roots." Here again, she voiced a referred or a generalist mathematical identity.

<u>Eco-unit of 1027-g3 at 0:59:30-0:59:56</u> – *Asking about the write-up of results*

Bettie: [pointing to Boutros's notes on the dry-erase poster board] is this how you need to write the integers?

Hoffmann: it's up to you. I mean I like this kind of stuff I like to just write a sentence at the end. to just say the primitive roots mod eight are . and here's the list. Just because . with primitive roots you're gonna do a lot of computational . and it gets like . confusing.

Hoffmann guided Boutros through the steps of checking the primitive roots modulo eight. They tried number three. Hoffmann concluded that it was "not (a) primitive root." Then they moved to number five. Hoffmann concluded that "this is one of those cases where you don't have a primitive root" (1027-g3 at 1:00:20). Jeremey interjected with a question and then Bettie followed-up, "wait. So eight has none?" She solicited an assessment of an idea while re-voicing Hoffmann, without (again) the use of "primitive roots." Herein, Bettie solicited the knowledge that she requested to understand through a referred mathematical identity.

<u>Eco-unit of 1027-g3 at 1:00:33-1:00:39</u> – *Computing the primitive roots modulo eight*

Bettie [to Hoffmann]: wait. so eight has none?

Hoffmann: has none.

[Bettie erases writings on her notebook]

As soon as the instructor had left the group, Bettie turned to Boutros and inquired about his earlier computation with Hoffmann—"why isn't three a primitive root?" (eco-unit at 1:01:18–1:01:42). She did so by re-voicing the instructor, who earlier concluded after checking number three, "so. not primitive root."

Boutros attempted to explain when a number was primitive root. He ended by stating that "so it [exponent] has to be under four." Bettie could not make sense of this statement. She re-voiced it with an intonation of confusion, "it has to be . wait. it has to be under four?" (eco-unit 1:01:42–1:02:51). Both Bettie's and Boutros' understandings were riddled with misconceptions at this stage. In this eco-unit, each one of them constructed a new, partially accurate appreciation of the definition of primitive roots. Bettie shared her current understanding aloud, "so if this is a prime . if this is uh . oh I get it . yeah yeah yeah. because it's suppose . it's a prime root if this is four. […] and it's more than four." Given the context, sharing her thoughts provided a way to allow her interlocutor to evaluate them. Per se, it took the position of offering an explanation but its function was to enhance her understanding. In fact, Boutros assessed her understanding, "exactly." Herein, Bettie built on Boutros' explanation, "okay this is a primitive root, right?" (underlined in the transcript) and constructed her own connections of the ideas, relative to Boutros', about the exponent being less or greater than $\varphi(m)$. This was the only instance, during the computational activity, where Bettie actuated a personal mathematical identity.

Eco-unit of 1027-g3 at 1:01:18-1:01:42 – *Question on primitive root modulo eight*

Bettie [to Boutros]: why isn't three a primitive root?

Boutros: umm. because when you take this out [writing on his section of dry-erase poster board see pict20151027_163519-g3] you take three two . that's equal to one mod eight. because it's nine.

Bettie: uh

Boutros: so if this number is less than the phi . then it's not . it can't be a primitive root when this is three

Bettie: aha

Boutros: so it has to be under four.

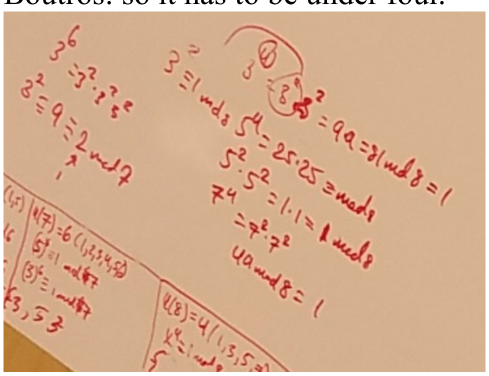


Figure 4-4: Picture of the dry-erase poster board on Boutros's side with his notes, to which the conversation at 1:01:18–1:02:51 refers.

Eco-unit of 1027-g3 at 1:01:42-1:02:51 – *Talk on how to compute primitive roots*

Bettie: it has to be . wait. it has to be under four?

Boutros: yeah. so let's say . five . three to the six . this is a primitive root because it's not . because this is . wait. I donno how to explain it. Okay this is a primitive root, right? we found that for phi of seven.

Bettie: mhm

Boutros: so and that's because if we take one out . This is the same as three two three two three two right?

Bettie: yeah

Boutros: so we take one of these and get nine. That's two mod seven. So if this isn't one, that means this isn't prime roots. this is the next prime root. [unclear]

Bettie: I know

Boutros: I'm trying to figure it out myself . when I'm explaining it to you.

Bettie: yeah.

Boutros: it's probaly not helpful

Bettie: so if this is a prime . if this is uh . oh I get it . yeah yeah yeah. because it's suppose . it's a prime root if this is four.

Boutros: exactly

Bettie: and it's more than four.
Boutros: it doesn't count
Bettie: yeah. it has to be four [circling the power of three in the inscription]
Boutros: so that was the case for every single one of these set.
Bettie: okay

Discussion

Despite the social damage to her voice, through the thwarted contributions and the solicitation of explanation through a referred mathematical identity, Bettie left the class satisfied. In her memo on 10/27, she reported that it was a "fun class [today] because I wasn't completely confused." She also noted that her particular learning moment came during the computational activity ("problem 1 in ws 8 I understood and felt cool"). As these two statements reflected, Bettie's feeling of an enhanced understanding engulfed the demotion of her unprecedented contributive voice. Indeed, Bettie improved her understanding of the two new definitions throughout the computational activity. Notwithstanding the fact that her voice reflected either weak or no mathematical identity, she constructed her knowledge by identifying with the participants' mathematical talks. Through her voice, that solicited an explanation, she collected pieces of knowledge from the mathematical group talks irrespective of whether she actively participated in them.

A close analysis of Bettie's notes shows that she had started the groupwork with a completely inaccurate understanding of primitive roots. Prior to her solicitation of explanations, she took the primitive roots to be the solutions of the equation $a^{\varphi(m)} \equiv 1 \; mod \; (m)$ (notice the ∴ under the equation in the highlighted part of Bettie's notes, in Figure 4-5). This early understanding was probably due to the influence of Bettie's entrenched "arithmetic" identity (PPMI) ("seeing an equation with a variable" meant, to Bettie, "solving for the variable"). Her computation of the primitive roots must have involved plugging integers in the equation for every case of m ($m = 6, 7$ & 8). She stopped when she reached a number of solutions that were equal to the value of $\varphi(\varphi(m))$.[10]

When Hoffmann sat with the group, he remarked that they needed to check only the integers which were co-prime with m. Bettie noted this remark (see the highlighted frame #1 in Figure 4-5). Yet, such remark would not repair her understanding. She probably continued to check only the co-prime integers for the equation $a^{\varphi(m)} \equiv 1 \; mod \; (m)$. A few minutes later, Hoffmann guided John through the computation of the primitive roots modulo eight. Bettie paid attention and copied the steps on her notebook (see the highlighted frame #2 in Figure 4-5). This identification brought a new piece of knowledge which was different and disconnected from her prior understanding. While it is doubtful that Bettie understood the new checking procedure, she must have noticed the discrepancy in her prior procedure of checking. In her method, she would check different integers for the same exponent. On the other hand, John and Hoffmann were checking the same integer for different exponents.

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[10] Evidently, Bettie had forgotten the Euler theorem, which Hoffmann mentioned during his instruction. Had she continued to search for solutions beyond the limited number of $\varphi(\varphi(m))$, she would have found that all integers that were co-prime with m were solutions to the equation.

Possibly, Bettie connected the different pieces of knowledge which she had collected from the participants when she worked with Boutros. She exclaimed, "yeah yeah yeah. because it's suppose . it's a prime root if this is four […] and it's more than four […] yeah. it has to be four [circling the power of three in the inscription]" (1027-g3 at 1:02:32). The group interactions at least support this improvement in her understanding—Bettie learned that primitive roots were not solutions to Euler's equation. She learned from Jeremy that five was not the primitive root modulo eight (eco-unit of 1027-g3 at 0:55:42–0:56:15). She learned, from Hoffmann, that there were no primitive roots modulo eight (eco-unit of 1027-g3 at 1:00:33–1:00:39) while she knew that three and five were solutions to $a^4 \equiv 1\ (mod\ 8)$.

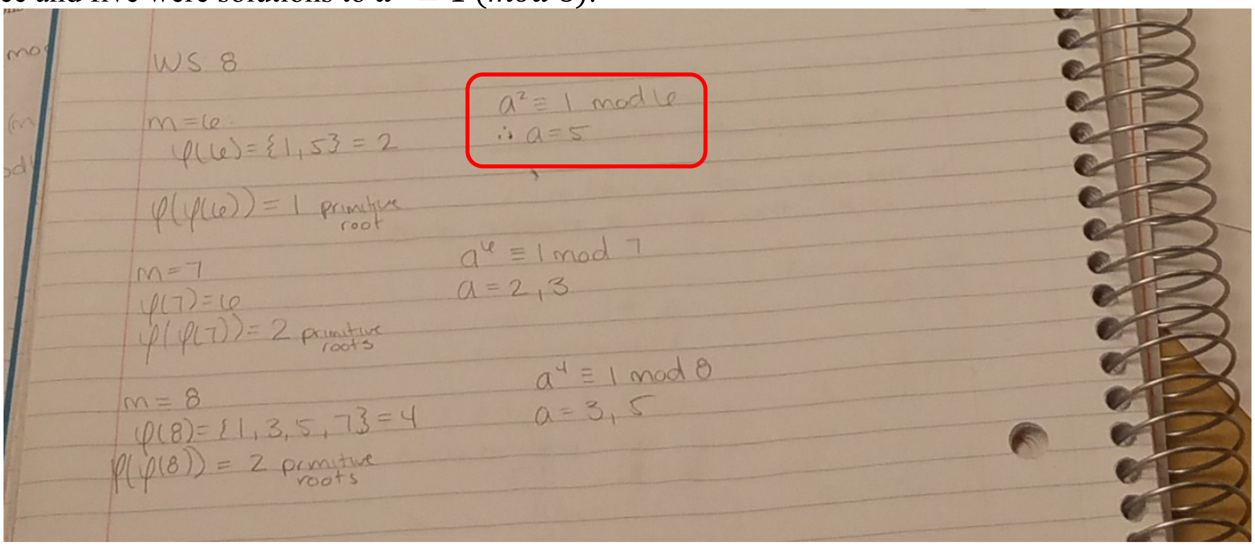


Figure 4-5: Picture of Bettie's notebook at 0:52:49 in the 1027-g3 video. Instructor was with the group, explaining the nuance of the significance of $\varphi(\varphi(m)$. When Bettie, for the first time in the computational activity, solicited an explanation. The red frame has been added to highlight Bettie's understanding at that moment.

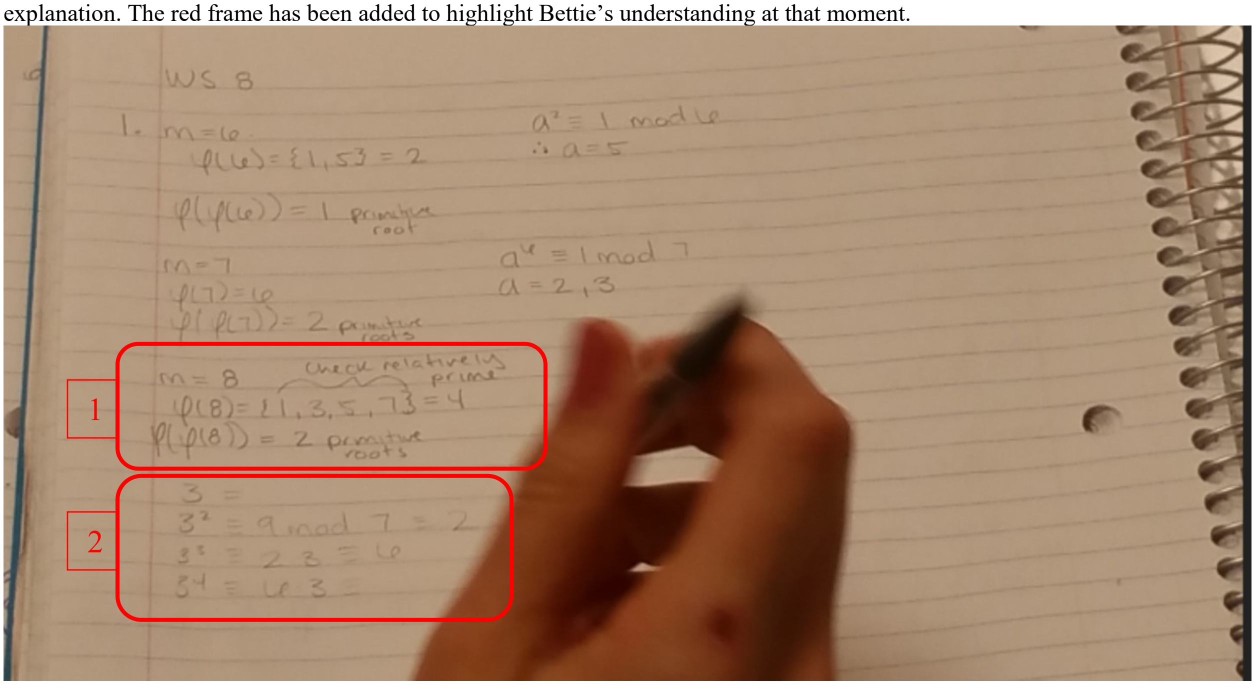


Figure 4-6: Picture of Bettie's notebook at 0:57:49 in the 1027-g3 video. The instructor was still with the group, guiding John in checking whether three was a primitive root modulo seven. Bettie was copying their computations.

Quadratic residues (on 11/12).

The class, on 11/12, started with a twelve-minute instruction introducing the definition of quadratic residues and then computing the quadratic residues of three (see Figure 4-7). Hoffmann defined the quadratic residues as follows:

a is a quadratic residue $mod\ m$ iff there exists x such that $x^2 \equiv a\ (mod\ m)$.

There are two ways to compute the lists of quadratic residues modulo an integer. The first method, strategy A, literally follows the definition. Take $m = 7$, for example. The residue system of seven is $\{0, 1, 2, 3, 4, 5, 6\}$.

For $a = 0$, $\quad 0^2 \equiv 0\ (mod\ 7)$. Thus, $x = 0$.
For $a = 1$, $\quad 1^2 \equiv 1\ (mod\ 7)$. Thus, $x = 1$.
For $a = 2$, $\quad 3^2 \equiv 2\ (mod\ 7)$. Thus, $x = 3$.
For $a = 3$, there exists no such x.
For $a = 4$, $\quad 2^2 \equiv 4\ (mod\ 7)$. Thus, $x = 2$.
For $a = 5$, $\quad$ there exists no such x.
For $a = 6$, $\quad$ there exists no such x.

Therefore, the quadratic residues of seven are 0, 1, 2, and 4.

The second method, strategy B, computes the values of all the squared residues modulo m. In the case of $m = 7$, the squares of the residues modulo seven are as follows:

$$(0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2) = (0, 1, 4, 2, 2, 4, 1)\ (mod\ 7)$$

Therefore, the quadratic residues of seven are 0, 1, 2, and 4.

Most students in Hoffmann's class were confused about the notion of quadratic residue. In most groups, different students unknowingly adopted different methods and confused each other. Additionally, some students took the values of x to be the quadratic residue rather than a. Bettie herself suffered from these confusions during her computational work.

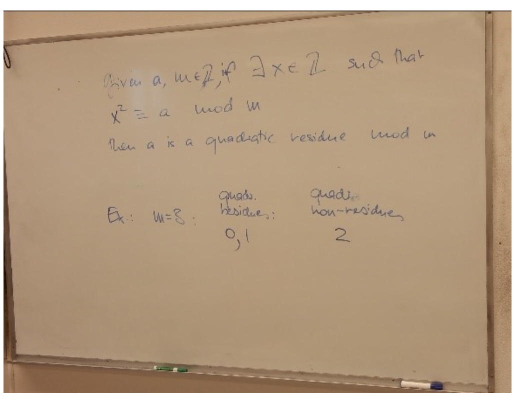


Figure 4-7: The definition of the quadratic residues modulo m, as stated by Hoffmann, on the class board.

Worksheet 10: Quadratic Residues

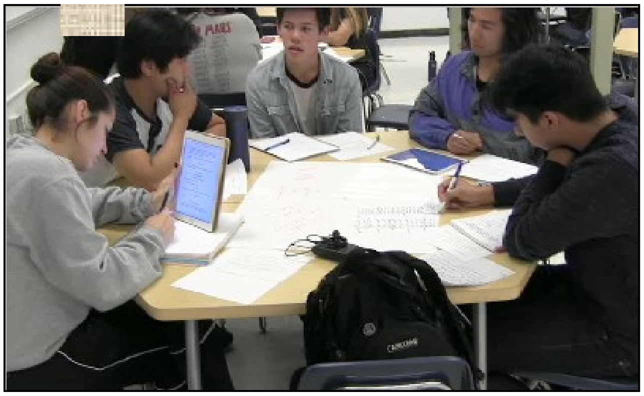
1. Make a list of all quadratic residues mod 2, 3, 5, 7, and 11.
2. In this exercise, we'll prove another one of Euler's theorems: If p is an odd prime, then a is a quadratic residue mod p if and only if $a^{\frac{p-1}{2}} \equiv 1 \bmod p$.
   (a) Prove the "$\Longrightarrow$" direction, e.g., by recalling another theorem by Euler.
   (b) For the "$\Longleftarrow$" direction, you may assume that there exits a primitive root r mod p (which is true, although we haven't prove it). Assuming $a^{\frac{p-1}{2}} \equiv 1 \bmod p$, use the fact that $a \equiv r^n$ for some n, and show that n is even.
3. Use Euler's theorem to prove, given a primitive root r mod p (as above, an odd prime), that g^n is a quadratic residue mod p if and only if n is even. Conclude that, for an odd prime p, exactly half the integers between 1 and $p-1$ are quadratic residues mod p.
4. Let p be and odd prime not dividing a and b. Show that:
   (a) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$
   (b) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \bmod p$
5. Andrews 9.2.2.
6. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

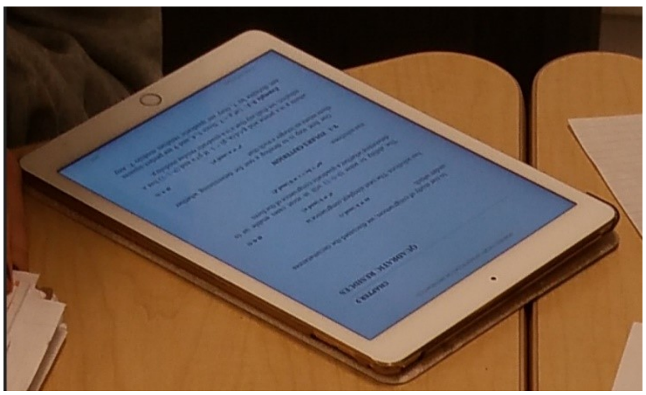
Figure 4-8: Worksheet 10 on quadratic residues.

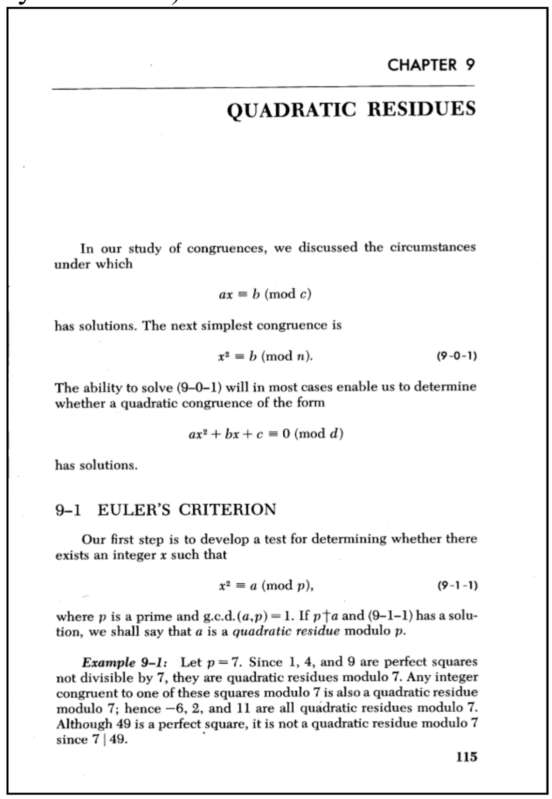
The groupwork was launched at 0:12:15 with Hoffmann probing, "okay start talking." Bettie immediately launched the first eco-unit of groupwork by seeking clarification. The group members discussed the computations of quadratic residues (see WK10#1 in Figure 4-8) for about twenty-three minutes. Salient in this group session was Boutros's work on the shared board, where he systematically tested every residue of cases where $m = 3, 5, 7, 11 \; and \; 13$ (see Figure 4-10). Jeremy doubted Boutros' work but Ted pointed to it and lauded it twice ("I like the way [pointing to Boutros' work on the board] he's doing it" in 1112-g3 at 0:19:24, and "yeah this way [pointing to Boutros' work on the board] of doing it is like the most brutal way [gesture of hands moving apart] of like generating all of them," at 0:21:39).

As for Bettie, she fully attended to and engaged in group discussions for about ten minutes of the early groupwork (0:12:15–0:23:00). Then (0:23:00–0:35:00), she picked out her tablet from her bag and started browsing, reading, copying, and then working on her notebook while her tablet remained in front of her. Most of the time, her tablet was displaying chapter nine, on Quadratic Residues, in page 115–116 of Andrews textbook (see Figure 4-9). During this individual work, she joined a collective conversation on the Modern Algebra course (0:33:36–0:34:16). At 0:35:00, Bettie looked at Boutros' work on the shared board and approached him to verify her interpretation of his work. Meanwhile, Hoffmann joined the group (at 0:35:11). Bettie pursued her solitary work after receiving Boutros' answer. Hoffmann and Ted talked about Wk10#2. Bettie attended to them when Hoffmann used Boutros' work on the board to test the theorem in Wk10#2 (at 0:36:18). She returned to her individual work (at 0:37:25) once the testing got over. She seemed to be doing computations (Bettie's notes on her individual work were captured by a picture only after she flipped the page; see Figure 4-11). She looked at Boutros' work on the board for a little bit (0:38:31–0:38:49), returned to continually writing on her notebook, looked again at Boutros' work (0:39:27–0:39:43), and then she flipped the page and focused on her individual work till she finished the problem (at 0:42:11).

Throughout the computational activity on 11/12, Bettie animated several voices in seven eco-units (see the transcripts in Table 4-6). All nine of Bettie's voices solicited explanations meant to enhance her understanding (see the VIP+function analysis in Table 4-7). The two variables, $a \; and \; x$, in the definition of quadratic residues confused Bettie. Most of her questions sought clarification on this particular issue. By the end of her individual work (0:23:00–0:42:11), Bettie seemed to have adopted strategy B (discussed above). She was influenced by Boutros' work, to which Ted oriented her attention (more analysis follows).







CHAPTER 9

QUADRATIC RESIDUES

In our study of congruences, we discussed the circumstances under which

$$ax \equiv b \pmod{c}$$

has solutions. The next simplest congruence is

$$x^2 \equiv b \pmod{n}. \qquad (9\text{-}0\text{-}1)$$

The ability to solve (9–0–1) will in most cases enable us to determine whether a quadratic congruence of the form

$$ax^2 + bx + c \equiv 0 \pmod{d}$$

has solutions.

9–1 EULER'S CRITERION

Our first step is to develop a test for determining whether there exists an integer x such that

$$x^2 \equiv a \pmod{p}, \qquad (9\text{-}1\text{-}1)$$

where p is a prime and g.c.d.$(a,p) = 1$. If $p \nmid a$ and (9–1–1) has a solution, we shall say that a is a *quadratic residue* modulo p.

Example 9–1: Let $p = 7$. Since 1, 4, and 9 are perfect squares not divisible by 7, they are quadratic residues modulo 7. Any integer congruent to one of these squares modulo 7 is also a quadratic residue modulo 7; hence -6, 2, and 11 are all quadratic residues modulo 7. Although 49 is a perfect square, it is not a quadratic residue modulo 7 since $7 \mid 49$.

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Figure 4-9: Picture at the top-left: about Bettie copying from her tablet in 1112-g3 at 0:28:22. Picture at the bottom-left: Bettie's tablet in 1112-g3 at 0:40:15, which contained the first page of chapter 9 of Andrews' textbook (picture at the bottom-right). Bettie had been looking at her tablet since 0:23:00.

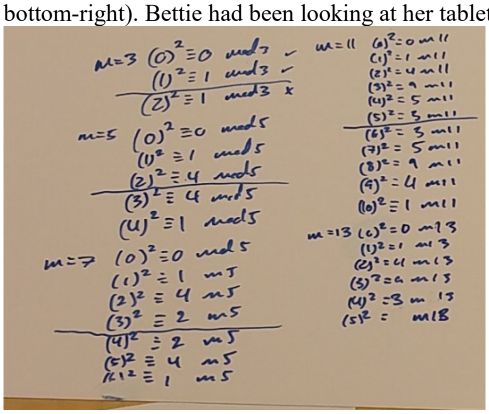


Figure 4-10: Boutros' notes on the dry-erase poster board in 1112-g3 at 0:23:12, to which Bettie referred at 0:35:08.

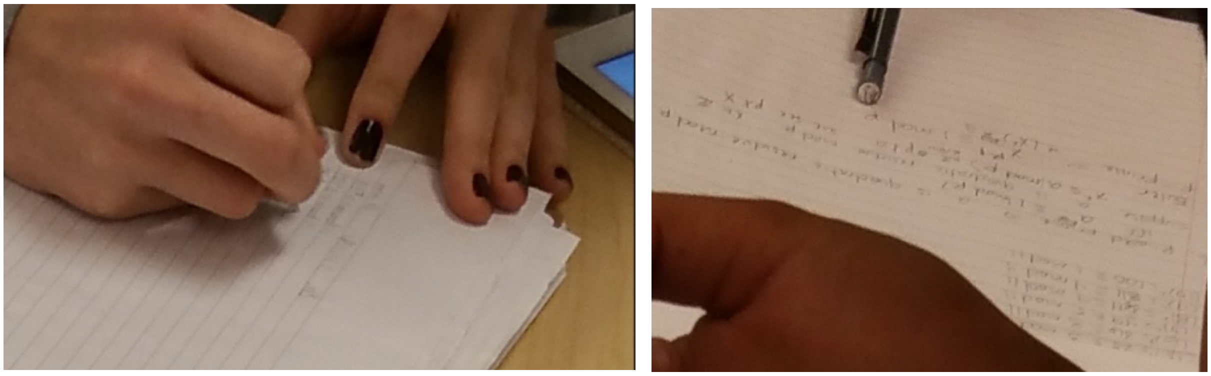


Figure 4-11: Bettie's notebook in 1112-g3 at 0:40:14 (left) and at 0:52:44 (right). She was computing the quadratic residues of eleven, starting with six, copying from Boutros's notes (see figure above in Figure 4-10), and adding the middle-part of the computation, e.g. Boutros writes $(9^2) \equiv 4\ m11$, Bettie writes $(9^2) \equiv 81 \equiv 4\ mod\ 11$.

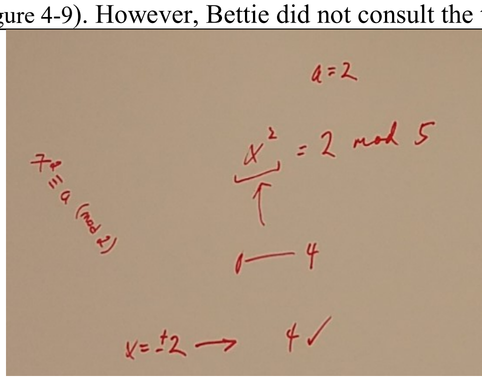
Table 4-6: Transcripts of the seven eco-units when Bettie exercised a voice, during the computational activity, regarding the quadratic residues during the groupwork on 11/12 (video of 1112-g3).

Time stamp of eco-unit	Excerpt of transcript
1112-g3 at 0:12:15 – 0:12:53	Bettie: so he [Hoffmann] is saying zero squared is congruent to a. Is a just a given? I don't get it. John: No a is not given. You have to plug numbers into x and a. Ted: [unclear, as they move the dryearse board on table] John: you use all xs up to eight. Right? Ted: not sure.
1112-g3 at 0:12:53 – 0:14:03	Bettie: [looks toward the classroom board] I don't know what he's [Hoffmann] doing. He's just saying number xs. Ted: where are the markers. So what happens here is zero is quadratic residue mod three because there is exists some x you can say . some any x that will make this true. because zero times zero is equivalent to zero mod m. So zero is a quadratic residue. Same with one. plus or minus one are both for x. but no number you would square will give you two mod three. Bettie: will give you two for x or two for a? Ted: so any integer that you choose and you square it you will not be able to when modulo three you would not see it equivalent to two. This is another way of looking at it.
1112-g3 at 0:17:14 – 0:17:41	Bettie: [looks at Ted's notes on dry-erase board, see Figure 4-12] so you're saying four mod five [unidentifiable two syllables with t in the second]? Ted: no I'm saying um four . like is a quadratic residue in here. I know that [erases the inscription x=±2] this can be rewritten as [writes (2)^2] two squared. [looks at Bettie] so there is exists some number [pointing to 2] squared so that when moded by five [points to 5] it is equal to four.
1112-g3 at 0:18:53 – 0:19:08	Bettie: So you're just making up an a . that makes it . true? Ted: a's range from 0 up to . m minus one Bettie: [writes notes on her notebook]
1112-g3 at 0:19:33 – 0:19:44	Bettie: And x's range from zero to m minus one? Jeremy: No. a is. Bettie: Then when do you solve for x? Jeremy: we don't know [laughs] that's all we're trying to figure out Bettie: aw [looks at the classroom board then starts writing on her notebook]
1112-g3 at 0:20:03 – 0:20:12	Ted: I think that x's just need to range from zero up to . this was m then this m minus one =Bettie: [looks at the classroom board] and m has to be prime? =Jeremy: x's to the m minus one. that's all? Ted [turns his face to Jeremy]: mhm
1112-g3 at 0:35:08 – 0:35:21	Bettie: so for this [pointing at Boutros's notes (see Figure 4-10)] one? m=7. Boutros: mhm Bettie: you stop here [dividing line] because you start getting the same? Boutros: yeah Bettie: Oh. Okay.

Table 4-7: VIP+function analysis of Bettie's voice during computational group activities on 11/12.

Eco-unit	Voice	Identity	Position	Function
1112-g3 at 0:12:15 – 0:12:53	1st so he [Hoffmann] is saying zero squared is congruent to a. Is a just a given? I don't get it.	Group member's right and interpreting and "arithmetic" (number theory) identity (PPMI)	Soliciting an idea assessment	To enhance her understanding of the procedures.
1112-g3 at 0:12:53 – 0:14:03	2nd [looks towards the classroom board] I don't know what he's [Hoffmann] doing. He's just saying number xs	Group member's right and interpreting a mathematical identity	Soliciting explanations from groupmates	To enhance her understanding of the procedures.
1112-g3 at 0:12:53 – 0:14:03	3rd will give you two for x or two for a?	Group member's right and interpreting an "arithmetic" identity	Soliciting an idea assessment	To enhance her understanding of the procedures.
1112-g3 at 0:17:14 – 0:17:41	4th [looks at Ted's notes on dry-erase board, see Figure 4-12] so you're saying four mod five [unidentifiable two syllables with t in the second]?	Group member's right and interpreting an "arithmetic" (number theory) identity	Soliciting an idea assessment	To enhance her understanding of the procedures.
1112-g3 at 0:18:53 – 0:19:08	5th So you're just making up an a . that makes it . true?	Group member's right and interpreting "arithmetic" identity	Soliciting an idea assessment	To enhance her understanding of the procedures.
1112-g3 at 0:19:33 – 0:19:44	6th And x's range from zero to m minus one?	Group member's right and interpreting "arithmetic" (~number theory) identity	Soliciting idea assessment	To enhance her understanding of the procedures.
1112-g3 at 0:19:33 – 0:19:44	7th Then when do you solve for x?	Group member's right and "arithmetic" identity*	Soliciting explanation from groupmates	To enhance her understanding of the procedures.
1112-g3 at 0:20:03 – 0:20:12	8th [looks at the classroom board] and m has to be prime?	Group member's right and interpreting "arithmetic" (number theory) identity	Soliciting idea assessment	To enhance her understanding of the procedures.
1112-g3 at 0:35:08 – 0:35:21	9th so for this [pointing at Boutros's notes (see Figure 4-10)] one? m=7. […] you stop here [dividing line] because you start getting the same?	Group member's right and interpreting "arithmetic" (~number theory) identity	Soliciting idea assessment	To enhance her understanding of the procedures.

* Andrew's textbook presented the quadratic definition in the context of solving modular quadratic equations (see Figure 4-9). However, Bettie did not consult the textbook until 0:23:00, after this eco-unit.



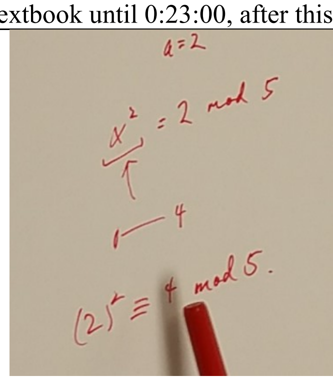


Figure 4-12: Ted's notes on the dry-erase poster board in 1112-g3 at 0:17:14 (left) and at 0:17:36 (left).

Discussion

Compared to the previous computational activity (Primitive roots on 10/27), Bettie made two significant shifts pertaining to the development of her voice. First, she silenced her contributive voice and animated only solicitations of knowledge for her own benefit. Second, contrary to her soliciting voices during the computational activity on 10/27 (which actuated referred mathematical identities), her soliciting voices on 11/12 actuated personalized mathematical identities, which were mostly "arithmetic" and occasionally "number theory" identities (evidence supporting this observation follows). As noted in the discussion of the primitive roots activity, Bettie did enhance her understanding of the computation even though her soliciting voices predominantly symbolized either referred or no mathematical identities. During the computational activity on 11/12, Bettie voiced her personal mathematical thoughts as she solicited knowledge from her groupmates.

In most (seven out of nine, see Table 4-7) of her solicitations during the computational activity on 11/12, Bettie positioned herself as seeking an assessment of ideas. Her ideas were not the product of her computations but, rather, her interpretations of what she struggled to understand in the participants' mathematical talks and notes. In her first voice during the activity (see the first row of Table 4-7), Bettie interpreted Hoffmann's instruction through her individualized "arithmetic" identity (PPMI). She interpreted $x^2 \equiv a\ (mod\ m)$ through the lens of the formal quadratic equation commonly written as $ax^2 + bx + c = 0$, where $a, b, and\ c$ are given numbers and x is the unknown variable that has to be found. In computing the quadratic residues, Hoffmann was in search of a and x, which got Bettie confused about her interpretation. The ideas in the first, third, fifth, and seventh solicitations of the assessment (see the corresponding rows in Table 4-7) resulted from Bettie's interpretative act wherein she used the common meanings of mathematical symbols in a situation that required flexibility.

Bettie explicitly actuated an "arithmetic" identity (PPMI) of number theory (in the 4th and 8th voices) and implicitly (in the 6th and 9th voices) (see Table 4-7). In the ninth voice, the "stop" referred to computing the congruence of the squared residues, which Bettie had been doing on her own (see Figure 4-11). In the sixth voice, the "range from zero to m minus one" implicitly indicated the residue system. In the fourth voice, Bettie checked her interpretation of Ted's notes on the board (see Figure 4-12).

Most significantly, a "number theory" identity was actuated in Bettie's eighth voice. She was looking at Boutros' work (see Figure 4-10) when she suddenly looked at the classroom board and then interjected, "and m has to be prime?" when Ted was also mentioning m in his speech. Bettie must have realized the primality condition by reflecting upon the values of which they were asked to compute the quadratic residues, namely 2, 3, 5, 7, and 11 (see Figure 4-8). Notably, the definition of quadratic residues, stated on the classroom board (see Figure 4-7), did not mention the primality condition. In this instance, Bettie could have asked, "why are we using an interrupted sequence of integers." Yet, Bettie reflected upon the sequence and voiced an "arithmetic" "number theory" identity.

Through the computational activity on 11/12, Bettie accomplished a significant twofold development. First, she actuated an individualized mathematical identity which occasionally took up the concepts of number theory. In fact, the mathematical ideas that she sought to assess were interpretations of either her instructor's or her groupmates' mathematical work (see Table 4-7; 1st voice, "he is saying …"; 3rd voice, "will give you …", following-up on Ted's speech, "no number […] will give you two mod three"; 4th voice, "so you're saying …"; 5th voice, "so you're just making up …"; 9th voice, "so you stop here because …"). Contrary to merely re-voicing somebody's mathematical ideas, which prevailed in her computational activity on 10/27 (see Table 4-5), Bettie exercised a minimal mathematical identity during the computational activity on 11/12.

Second, she predominantly animated positions whereby she solicited idea assessments, which was a smart tactic through which to voice her confidence in "arithmetic" (PPMI) while acknowledging the threats from her more-knowledgeable groupmates. During the computational activity on 10/27, Bettie started, with a burst of confidence, animating contributive attempts by voicing her mathematical identity. Then, because those contributions failed, she shifted to soliciting knowledge without an individualized mathematical voice. The disappointment from her failed contributions led her to make a major shift of positioning on 10/27.

Bettie animated a more stable position during the computational activity on 11/12, as compared to 10/27. The position from where she solicited the idea assessment allowed Bettie to voice her self-constructed "arithmetic" ideas and safeguard them from social threats. By couching the ideas as questions to be confirmed by her groupmates, Bettie acknowledged their superior positioning but self-positioned, socially, as not yet committed to these ideas. When she voiced her contributions as affirmative statements (see Table 4-4), she was identified with these ideas (Boutros to Bettie: "are you sure" in eco-unit of 1027-g3 at 0:46:34–0:46:48) and the attacks against them abated her mathematical identity (Bettie's puzzled face after Jeremy's critique in eco-unit of 1027-g3 0:55:42–0:56:15). By seeking the assessment of ideas, Bettie positioned herself as an inquirer of these ideas and, thus, prevented subjective evaluations. Furthermore, seeking the idea assessment set the stage for an objective evaluation of the ideas with minimal subjective repercussions. Responses such as "you're wrong" or "this is right" would not befit questions such as "must m be prime?"[11] Given the socio-cognitive forces involved in Bettie's case, i.e., lacking grasp over mathematical concepts and positioning her groupmates as more knowledgeable than herself, her act of soliciting the idea assessment to her enhance own understanding was the optimal position-function that could be animated.

Quadratic reciprocity law (on 11/19).

Hoffmann proceeded differently for the 11/19 class. Aware that half of his students were distracted by a take-home exam for another major course, he decided to alternate his classroom activities between lecturing and groupwork. He had introduced the Legendre and the Jacobi symbols by the end of his previous class, on 11/15. He started the 11/19 class (1119-g3 at

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[11] Tito, belonging to group G2, endured a social context similar to Bettie's, except that he avoided being self-positioned as weak or unreliable. He said, "I hate to be criticized in public." He realized that he was under a high risk, given the mathematical knowledge his groupmates were exhibiting. He ended up checking his ideas with Emil, a groupmate and a bus-mate. Tito started testing his ideas by sharing them with Emil, during their bus ride to school, before venturing them in the group. During the groupwork, he would toss his ideas to Emil in private conversations before sharing them with the group. Such a tactic would work for somebody who wanted to contribute to the group. Bettie, however, was interested in enhancing her understanding.

0:08:16) by setting a specific goal, i.e., to prove the quadratic reciprocity law, and then introduced Gauss's lemma (see Figure 4-13). Afterwards (at 0:18:53), he proved the corollary for the case $m = -1$, where $\mu = \frac{p-1}{2}$ (see Figure 4-14). Then (at 0:22:42), he used the corollary to find whether integers were quadratic residues modular primes ($12 \ mod \ 13 \ and \ 18 \ mod \ 13$). The students asked a couple of questions.

Bettie came to class at 0:25:44 (of the 1119-g3 video), when Hoffmann had just started the corollary for $m = 2$. He set the strategy for finding the value of μ by breaking the values of p into four cases (see Figure 4-15). Then he assigned groups to compute the value of μ for each case. Bettie's group, G3, was in charge of computing μ for $p = 8k - 1$. Computations in the groups lasted for about nine minutes (1119-g3 at 0:35:00–0:44:02). Then, Hoffmann engaged the students in a whole-class activity (1119-g3 at 0:44:02–1:12:35), during which he completed the proof of the case $m = 2$, proved the quadratic reciprocity law (see Figure 4-13 bottom), and applied it to compute $\left(\frac{101}{127}\right) \ and \ \left(\frac{5}{33}\right)$. Then, he set the students to compute $(\frac{10}{33})$ by working in groups for about six minutes (1119-g3 at 1:12:35–1:17:42). During the last six minutes of the class, he verified the answers and talked about the meaning of the Jacobi symbol, in contrast to the Legendre symbol.

Bettie spent most of her early groupwork catching up with what she missed from the class (see the first three eco-units in Table 4-8). When the groups were asked to compute $(\frac{10}{33})$, Ted solved it quickly on the shared board (see the top-left side of the dry-erase poster board under the turkey-hand drawing, in Figure 4-17). Hoffmann happened to be near the group, observing Ted's computation. When Ted claimed that he was done, getting negative one as the answer, Hoffmann asked him to repeat the computation slowly and Bettie concurred with a personal request, "can we do that for me? I don't understand it" (see transcript of eco-unit 1:14:13–1:14:36).

As Ted was erasing the side of the board (opposite to where he did the earlier computation), Bettie launched a question but John and Jeremy interrupted Ted's response to her (at 1:14:36). Once responding to Jeremy and John, Ted started over his computation (at 1:15:02) by writing on the erased side (see the bottom-left side of the poster board, in Figure 4-17). Ted was mainly interacting with Bettie, who interjected with small contributions (see Table 4-8 for transcript). Boutros was only listening and occasionally nodding. John and Jeremy were doing their own computations and interrupted the Ted-Bettie micro-activity to check their answers (1:15:43–1:15:47). Henceforth, they followed Ted's explanation. John asked Ted about his notes (1:17:00–1:17:41). Ted had to stop the computation at the stage where $(\frac{5}{11})$ had been left untreated (see Figure 4-17), because Hoffmann called the class to pay attention (at 1:17:42). During the whole class discussion, Bettie attempted to compute $(\frac{5}{11})$ and solicited Ted's attention in order to share and check her ideas (see eco-units of 1:18:06 and 1:20:11, in Table 4-8).

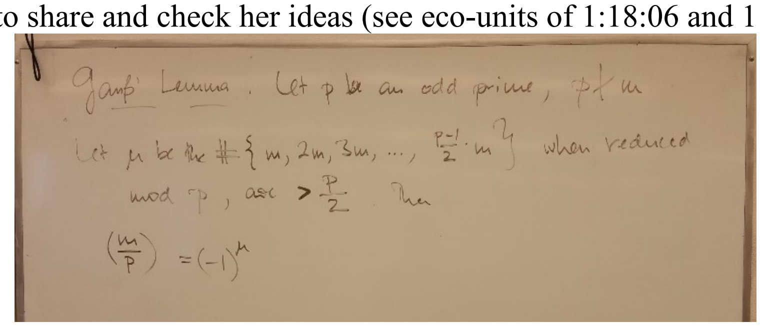

Figure 4-13: Gauss Lemma, as stated by professor Hoffmann on the classroom board, on 11/19.

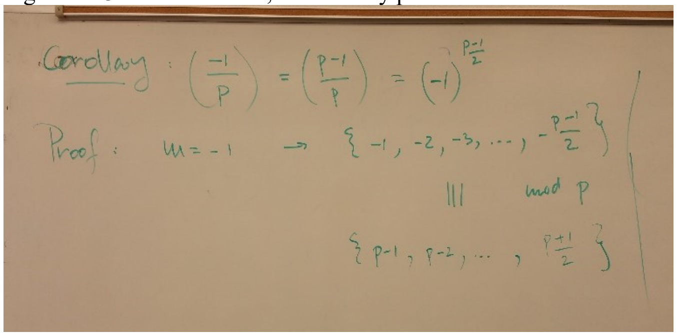

Figure 4-14: Picture of the classroom board depicting the proof of the Gauss corollary for m = -1, as explained by Hoffmann on 11/19.

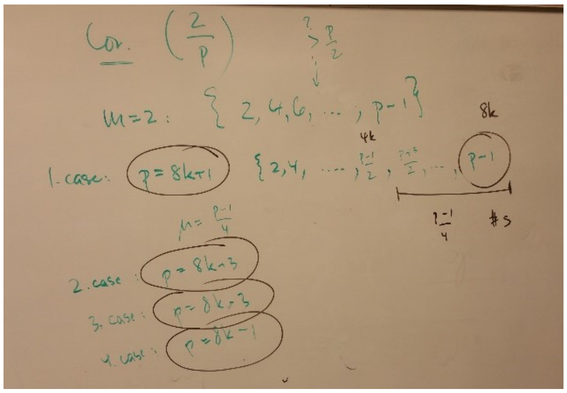

Figure 4-15: Picture of the classroom board stating the organization of the proof of Gauss corollary for m = 2, as laid out by Hoffmann on 11/19.

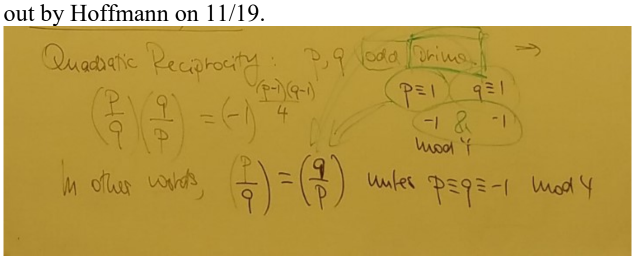

Figure 4-16: The quadratic reciprocity law, as stated on the classroom board by professor Hoffmann on 11/19.

Table 4-8: Bettie's animated voices during the groupwork on 11/19 – lesson on quadratic reciprocity for the Jacobi symbol. The highlighted pronouns will be used in the discussion of this section.

Time stamp of eco-unit	**Excerpt of transcript**
1119-g3 at 0:35:00-0:44:02 – G3 is working on the case of $p = 8k - 1$	
1119-g3 at 0:35:55 – 0:36:17	Bettie [copies from classroom board]: what does it say . number what? minus one in the bottom? Ted: four Bettie: it says number x. Ted: yeah. number x. like the count of elements Bettie: got it.
1119-g3 at 0:37:01 – 0:37:16	Bettie [speaks as Jeremy is thinking aloud]: So wait. What now we're working on [unidentified] [Jeremy and Ted are already in conversation, no one attended to Bettie; she takes out her tablet and opens the e-textbook]
1119-g3 at 0:44:02-1:12:35 – Whole class activity: Hoffmann explains the quadratic reciprocity	
1119-g3 at 1:02:00 – 1:02:27	[Bettie is still working with the proof of corollary for m=2] Bettie [to Ted]: so he [Hoffmann] uses the k just to have any prime? Jeremy: I don't know Ted: if you have two over some prime then you check that prime mod eight. This specifically when you have the case . [starts writing on group board] two over p and you check for this [circles the p and writes mod 8]
1119-g3 at 1:12:35-1:17:42 – G3 is computing the Jacobi of 10/33	

1119-g3 at 1:14:13 – 1:14:36	Hoffmann: negative one. … can you do it slowly again? Bettie: Yeah can we do that for me? I don't understand it Ted: an era::ser [looks for eraser then takes his pencil box and erases the notes on his right side of the shared board]
1119-g3 at 1:14:36 – 1:15:02	Bettie [to Ted]: so why did you split the ten into two and five [interrupted] John: Is three a quadratic residue of five? Ted: uhu? John: Is three [interrupted] Jeremy [to Ted]: Is two a quadratic residue of eleven? John [to Jeremy]: you can reduce that one though. Ted [to Jeremy]: you mod eleven by four and see if it's one . positive or minus one if it is a three than it's not so you put negative one. once you get down to two then you can replace it that way. Jeremy: okay. so I get five over eleven [1:14:55-1:15:11 Bettie writes on her notebook]
1119-g3 at 1:15:02 – 1:15:43	[at Hoffmann's request, Ted is redoing the Jacobi 10/33 slowly so everyone in the group can follow. In fact, only Bettie and Boutros follows Ted's explanation. Jeremy and John are doing the computation on their own] Ted: [writes on the shared board, see Figure 4-17] If we start out with the ten over thirty three . we have to first split up anything that's not prime into its primes. [Bettie looks up and Ted looks at Bettie] so when we do that Bettie: wait. sorry say it again. Ted: when we do this we have to first split any number that is not a prime into its prime numbers. Bettie: okay Ted: so first I'm doing the ten. Bettie: got it [she goes back to working on her notebook] Ted[continues]: so that's two over thirty three and five over thirty three. [Bettie looks at board for 3 sec] then I have to split the bottom for each one of these. giving me . [Bettie writes on her notebook] two over three . two over eleven. five over eleven. and five over three.
1119-g3 at 1:15:43 – 1:15:47	John [to Ted]: did you get negative one? Jeremy: I get negative one also. Ted: yeah I did.
1119-g3 at 1:15:47 – 1:16:04	[Ted looks at Bettie] Bettie: [looks upward] got it. Ted: Ted: okay. then umm. I'm gonna reduce this one . five mod three [glances at Bettie] is just two mod three. [looks at Bettie] Bettie: [writes on her notebook] so p is five mod three Ted: uhum Bettie: which is [glances at Ted's notes] =two Ted: =two mod three Bettie: yup
1119-g3 at 1:16:04 – 1:16:39	Ted: [Bettie looks at poster board] and then from there . um . [Bettie looks at her notebook] I know that this one [pointing (2/3) the one on the right] and this one [pointing to (2/3) the one on the left] is negative one. [looks at Bettie] [1:16:14] Bettie: two over three is negative [looks at her notes] Ted: because you look at Bettie: yeah Ted: the three [circles the 3 in (2/3) right side and draws an arrow and continues writing about p, mod 4 and μ, see Figure 4-17] when three mod. whenever you have plus or minus three mod four when p is equal to two . mu is equal to negative one . in this case, right? [looks at Bettie] [01:16:26.29] Bettie: [looks at the shared board reflecting for 3 seconds] Yeah. because it's prime. Ted: right. so when you mod the bottom by four . if you get plus or minus three . this whole thing [circles (2/3) on the right side] becomes just negative one. Same thing with this one [circles (2/3) on the left side]. Bettie: [nods]
1119-g3 at 1:16:39 – 1:17:00	Ted: and then this one [pointing to (2/11)] mod ... what. whatever it is um [Bettie looks at her tablet] Ted: mod four is this one [points to 11 in (2/11); Bettie looks at his work] is three mod four so this is also negative one [draws an arrow downward and write -1]. these two [-1 under (2/3) and -1 under the other (2/3) in the inscriptions, see Figure 5-17] canceled leaving you with this [circling (5/11)] times this [circling (2/11) and scratching (2/3) on both sides, then looking at Bettie]. Bettie: [as she writes on her notebook] so this two over three is negative one. Ted: uhum Bettie: and tw= [interrupted by John]

1119-g3 at 1:17:00 – 1:17:41	John: =what do you referring to when you do this way? [circling Ted's side notes about p, mod 4 and mu] [Bettie looks at Ted's work] Ted: [Bettie looks at her notebook] and if you . if that p is congruent to plus or minus one mod four or congruent to plus or minus three mod four then you get your dividend plus or minus . you know? so if it's congruent to plus or minus one [Bettie glances at Ted's work] mod four is . one. congruent to plus or minus three mod four will give you . negative one. [Bettie glances at Ted's work then writes on her notebook] John: yeah
1119-g3 at 1:17:42-1:20:42 – Whole class activity Hoffmann discussing the difference between Jacobi and Legendre symbols	
1119-g3 at 1:17:42 – 1:18:06	[These time stamps report the shifts of Bettie's postural positions] [1:17:45.13] Bettie looks at classroom board. [1:17:49.05] Bettie turns to look at Ted's recent work on shared board. [1:17:50.12] Bettie looks back at classroom board. [1:17:57.09] Bettie leans forward and stares at Ted's recent work. [1:18:00.25] Bettie writes on her notebook.
1119-g3 at 1:18:06 – 1:18:42	Bettie [to Ted]: [extends her hand to point to Ted's work on board; while Hoffmann is addressing the whole class] then you inverse. maybe you inverse eleven over five. Ted: uh? Bettie: and you do negative eleven over. this equals negative eleven over five and you do it mod five. Ted: oh what you do from there is whenever we have [writes (5/11)] the top and bottom where one of them is moded by [turns to look at class board] moded by four is equal plus minus one? [stares at the class board, then writes on board -unidentifiable notes]
1119-g3 at 1:20:11 – 1:20:26	Bettie: [extends her hand to Ted's most recent notes on computing (5/11)] it's mod five . right here. Ted: [shakes his head in agreement and writes mod 5. He looks again at Bettie and shakes his head in approval] Bettie: [smiles] okay [writes on her notebook]

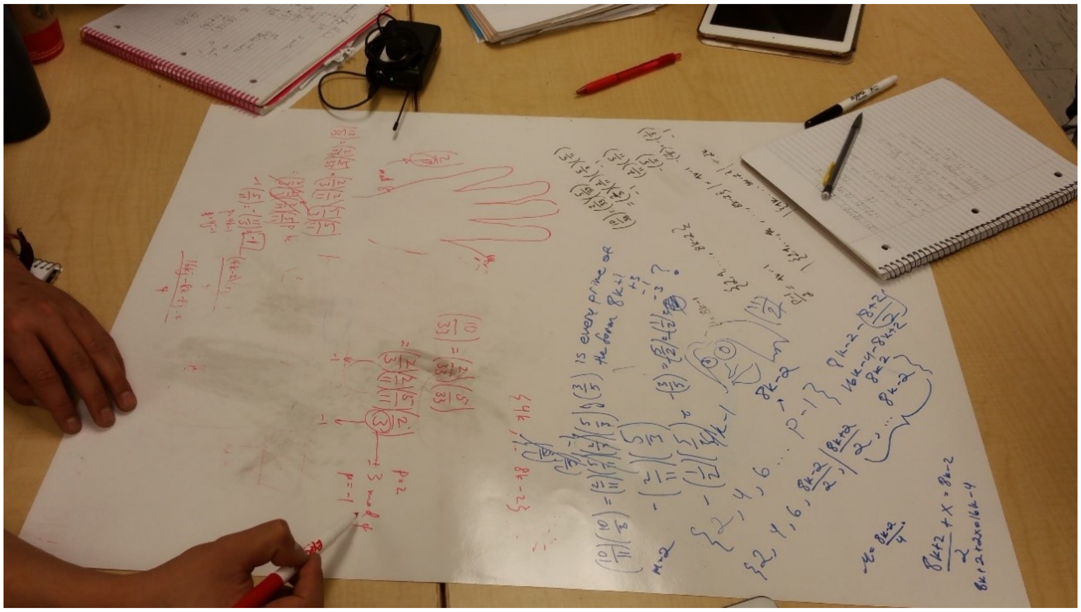

Figure 4-17: The group postural positioning (picture on the top) and the dry-erase poster board (picture on the bottom) in 1119-g3 at 1:16:27. The inscriptions under the turkey-hand drawing is Ted's earlier computation of $(\frac{10}{33})$. The computation where the hand [Ted's] points with the pen, is the computation of the Ted-Bettie micro-activity. Bettie is sitting.

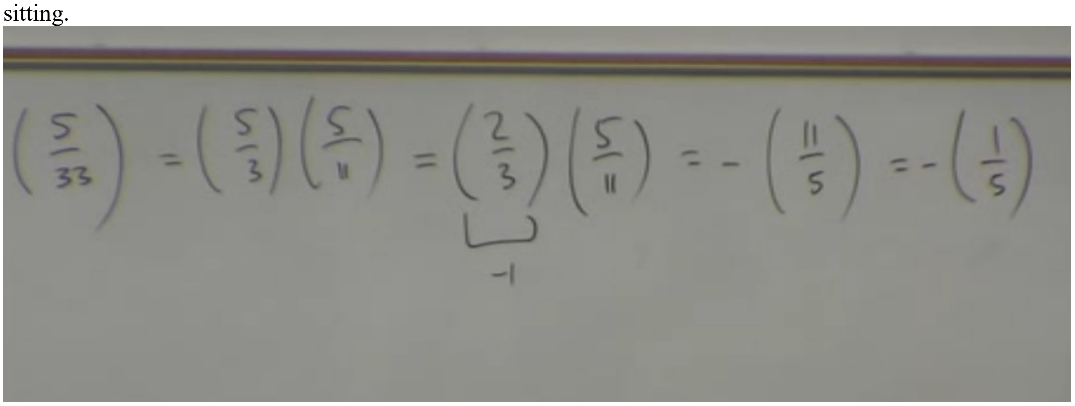

Figure 4-18: Inscriptions on the classroom board when the groups where computing $(\frac{10}{33})$.

Table 4-9: VIP+function analysis of Bettie's voice during the computational group activity on 11/19.

Eco-unit	Voice	Identity	Position	Function
1119-g3 at 1:14:13 – 1:14:36	1st Yeah can we do that for me? I don't understand it	Group member's right without mathematical identity	Monitoring group interactions (in the form of a request)	To enhance her understanding of the computation.
1119-g3 at 1:14:36 – 1:15:02	2nd so why did you split the ten into two and five [interrupted]	Group member's right and interpreting an "arithmetic" number theory identity*	Soliciting an explanation	To enhance her understanding of the procedures.
1119-g3 at 1:15:02 – 1:15:43	3rd wait. sorry say it again? [Bettie was looking at her notebook when Ted started explaining]	Group member's right	Monitoring interactions (command and request intonation)	To catch up with what she missed when she wasn't attending the group.
1119-g3 at 1:15:47 – 1:16:04	4th [writes on her notebook] so p is five mod three	Group member's right and a number theory identity	Contributing a justification	To allow Ted to evaluate her understanding.

	[…] which is two [Ted: mod three]			
1119-g3 at 1:16:04 – 1:16:39	5th two over three is negative [looks at her notes]	Group member's right and a referred number theory identity	Re-voicing Ted	To check if she is aligned with Ted
1119-g3 at 1:16:04 – 1:16:39	6th [looks at the shared board reflecting for 3 seconds] Yeah. because it's prime.	Group member's right and number theory identity**	Contributing a justification	To allow Ted to evaluate her understanding.
1119-g3 at 1:16:39 – 1:17:00	7th [as she writes on her notebook] so this two over three is negative one.	Group member's right and referred mathematical identity	Re-voicing Ted	To indicate where she was situated in her understanding.
1119-g3 at 1:18:06 – 1:18:42	8th [extends her hand to point to Ted's work on shared board] then you inverse. maybe you inverse eleven over five?***	Group member's right and a (minimal) number theory identity****	Soliciting an idea assessment	To complete the computation and confirm Bettie's added steps.
1119-g3 at 1:18:06 – 1:18:42	9th and you do negative eleven over. this equals negative eleven over five and you do it mod five.	Group member's right and a number theory identity*****	Contributing a mathematical idea	To complete the computation and confirm Bettie's added steps.
1119-g3 at 1:20:11 – 1:20:26	10th [extends her hand towards poster board] it's mod five . right here.	Group member's right and a number theory identity	Contributing a justification	To confirm Bettie's understanding.

* Although Bettie asked about numbers, she interpreted the splitting of the Jacobi 10/33 into two, 2/33 and 5/33, as the splitting of a numerator. For Bettie to ask about the splitting, she must have been thinking about the Jacobi as irregular fractions, since splitting two regular fractions should be evident for a mathematics undergraduate student. Additionally, the Jacobi definition, as introduced in class on 11/15, required splitting the denominator, and not the numerator as in Ted's case, into its prime factorization. Bettie's question might have been expressing this confusion.

** In fact, primality is merely the condition for the justification of $\left(\frac{2}{3}\right) = -1$, which Ted explained in his following-up for Bettie ("right. so when you […]"). At this moment, Ted was confused—the modular four is used for the condition of the quadratic reciprocity law and the computation of $(\frac{-1}{m})$, which he interpreted as the computation of $(\frac{2}{m})$. The latter requires modular eight.

*** I separate the 8th and 9th voices, although in the eco-unit they were separated only by an "uh?" Ted's uncertainty allowed Bettie to reboot and produce a different voice. The 8th voice was guessing ("maybe"), asking ("over five?") and stating the generic rule ("inverse"). The 9th voice asserted the answer ("negative eleven over five") and the next step ("do it mod five").

**** The idea of inversing $\left(\frac{5}{11}\right)$ was most likely inspired by the inscriptions on the classroom board (see Figure 4-18 and recall Bettie that looked twice at classroom board between 1:17:42 and 1:18:06. The mathematical identity in the 8th voice qualifies as personal, because Bettie thought of attending to another resource (she had not looked, heretofore, at the classroom board during the Ted-Bettie micro-activity), interpreting its inscriptions and connecting the computations in both the resources, i.e., Ted's work on the group board and Hoffmann's work on the classroom board.

***** In fact, the inverse of $\left(\frac{5}{11}\right)$ is positive $(\frac{11}{5})$. But Bettie might have meant the negative one of $\left(\frac{2}{11}\right)$ or might have been simply reproducing the result of the computation on the classroom board (see Figure 4-18). The idea of mod five comes from Bettie's understanding, as it was not noted on the classroom board (Hoffmann only said it verbally when computing $\left(\frac{5}{11}\right)$ as part of $\left(\frac{5}{33}\right)$.

Discussion

In this discussion, I attempt a close analysis of Bettie's emergent number theory identity and her contributive positioning during the Ted-Bettie micro-activity on 11/19 (1:14:13–1:20:26). The close analysis has been in pursuit of the factors that bolstered the significant actuation of her number theory identity as well as her animation of her contributive positioning, which, heretofore, was unprecedented (see the analysis of earlier computational activities in Table 4-4, Table 4-5, and Table 4-7).

During the Ted-Bettie micro-activity, Bettie predominantly actuated a number theory identity in six out of ten instances (2nd, 4th, 6th, 8th, 9th, and 10th voices, in Table 4-9). Compared to the previous computational activity, Bettie showed a significant improvement in her mathematical voice within this micro-activity. On 11/12, Bettie mainly voiced an interpreted mathematical identity (see Table 4-7). On 11/19, she voiced a number theory identity through her contributions, that were meant to justify Ted's statements (4th, 6th, and 10th voices) and complete the computation (8th and 9th voices).

Bettie's mathematical voices during this micro-activity manifested a partial understanding. On one hand, she learned how to reduce a numerator in a Legendre symbol (4th, 9th and 10th voices), which was the topic of the previous lesson (on 11/15). On the other hand, she had not yet fully grasped the two following topics of the current lesson: how to justify $\left(\frac{2}{3}\right) = -1$ (see note for the 6th voice) and, possibly, how to apply the quadratic reciprocity law (see note on the 9th voice). Despite her hesitation regarding the reciprocity law ("maybe you inverse" in the 8th voice), she ventured to voice it to Ted through assertive linguistic forms ("you do negative eleven over. this equals negative eleven over five" in the 9th voice).

As far as the positions were concerned, the Ted-Bettie micro-activity was not a regular explanatory activity, where the explainer would be commonly positioned as resourceful and addressee(s) would act as seekers of knowledge. This seemed to be peculiar to Bettie and Ted, starting with Bettie's uncommon request, "can we do this for me?" (notice the pronouns). In this request, the activity was depicted as a collective "doing" ("we do") for the benefit of Bettie ("for me"). Thereafter, instead of remaining a passive listener to Ted's explanation, Bettie animated an active "doer" positioning multiple times throughout the micro-activity (in the 4th, 6th, 9th, and 10th voices, see Table 4-8).

To investigate the learning nature of the relationship between Ted and Bettie, I consulted both of their interviews and found two illuminating narratives in Bettie's exit interview.

<u>Int2-1202-Bettie lines 315-332.</u>

Fady: Did you experience any change in the group dynamics as a whole from the beginning to now?

Bettie: Oh yeah.

Fady: What did it change?

Bettie: mm. Probably?

Fady: What. How was it in the beginning and how was it now?

Bettie: So in the beginning I felt like I looked more to Jeremy to like help me . kinda like explain it to me. in like he . I've noticed like he's more like . uh . I don't know how to explain it like . I don't know. Anyways I don't even like ask him anymore, I ask Ted. cause Ted is more like ca . like not calm but he's more . patient . in teaching . or like not even teaching just like . going through what he did. and I can ask him and he like doesn't . I can tell he would never . like he doesn't get annoyed. he loves to help in people. cause obviously he wants to be a teacher. so he's like he'll go through it . and he'll like take his time to come . to school . just to help me . to like . understand things. I think that's really cool. and I feel like he's probably like . he's just really smart. He's so smart.

Fady: Uh okay.

Bettie: And I'd always think I always thought like Jeremy had a better way of explaining things, but no. Now I look to Ted.

Int2-1202-Bettie lines 546-550, 557-559

Fady: Let's suppose that you if next time you go into a classroom and they require uh small groupwork. What do you keep, what do you change in your behaviors, in your choice as a group, and whatever happens in your experience? For your next experience?

Bettie: Umm. I'll probably . take the group with the most approachable people . […] I kinda like the people that are low key like really smart . but they're not trying to be like "I'm the smartest" [right hand moved up and back at head level]. I don't know . the most approachable [both hands extended at chest level] . like **people that I can like see myself** (emphasis for analytical purpose).

Early in the semester, Bettie identified Jeremy as reliable and a good explainer. Over the semester, she shifted her identifications and realized that, for her, Ted was more suitable in explaining mathematical ideas as compared to Jeremy. The reason was socio-emotional rather than intellectual. She came to understand Ted as being "more patient," "doesn't get annoyed," and "loves to help in people," based on her experience with him ("he'll take time to come to school just to help me understand things"). She specified that Ted's way of supporting his peers did not involve a superior positioning, wherein an explainer would be positioned as more knowledgeable than her/his addressees. Ted, as she said, would support his peers by reconstructing his answers anew ("not even teaching just going through what he did"). Bettie's description of Ted's way of supporting his peers fitted his behavior during the Ted-Bettie micro-activity on 11/19—he started his computation anew and explained the steps he had undertaken.

When asked about what she had learned from her experience of the groupwork in the number theory class (Int2-1202-Bettie lines 546–559), Bettie highlighted the significance of working with "approachable" students. She would not feel comfortable working with students who would turn the groupwork into an affirmation of their superiority ("trying to be like 'I'm the smartest'"). According to her report, she never approached the instructors of her current school for support, because they were "sorta superior" (Int1-0922-Bettie lines 166–169). Bettie felt comfortable working with people who were "really smart" but also "low key." The approachable aspect of "smart" students was significant because, for Bettie, it bolstered her identification with them ("people that I can see myself"). In Bettie's eyes, Ted was an approachable person, low-key for not playing the role of a superior teacher, and "really smart. so smart" (Int2-1202-Bettie lines 324–329).

Was Bettie identifying herself with Ted during the Ted-Bettie micro-activity on 11/19? Note that the act of copying somebody else's work constitutes a referred identity and not Lacanian identification with the author of the original work. A Lacanian identification requires an operation on self. Thus, to count as a mathematical identification an imitation of others must involve mathematical cognitive operations on the imitator's part. Identification with others entails a back-and-forth reflection of the identifier, i.e., the self as reflected in the image, and the identified, i.e., the (social) image as reflecting the self. Two pieces of evidence in the Ted-Bettie micro-activity on 11/19, gestural and linguistic, support such identification between Bettie, the identifier self, and Ted, the identified social image.

Throughout the micro-activity, Ted alternated between looking at the board while writing and making eye contact with Bettie, while Bettie alternated between looking at Ted's notes and writing on her notebook (see Table 4-8). She did not merely copy Ted's notes but also re-did the computation. In effect, Bettie wrote on her notebook only when she understood and agreed with Ted's mathematical reasoning. The first time she wrote on her notebook during the micro-activity was after she understood why Ted split the ten in the numerator of the Jacobi. She said "got it" and immediately started writing on her notebook (see eco-unit of 1119-g3 at 1:15:02–1:15:43). When she finished copying, she looked at Ted's notes when he said that "I have to split the bottom for each one of these." Bettie would understand and agree with him based on the same reason as previously mentioned (Ted had said, "we have to first split *any number that is not a prime* into its prime numbers"). As soon as she heard that he was about to split the denominators, she started writing on her notebook (same eco-unit). At the end of this eco-unit, we could expect the following, or similar inscriptions, on Bettie's notebook:

$$\left(\frac{10}{33}\right) = \left(\frac{2}{33}\right)\left(\frac{5}{33}\right) = \left(\frac{2}{3}\right)\left(\frac{2}{11}\right)\left(\frac{5}{3}\right)\left(\frac{5}{11}\right)$$

Ted was distracted by John's and Jeremy's interruptions (eco-unit 1:15:43–1:15:47). After cursorily responding to their request (taking four seconds), Ted checked on Bettie by looking at her. She looked at him and replied, "got it" (eco-unit 1:15:47–1:16:04). Not only was Bettie attentive to being aligned with Ted, but Ted, her social image (the-identified-with), reflected the same attention. Ted moved to reduce $(\frac{5}{3})$ into $(\frac{2}{3})$ and then looked at Bettie. She wrote on her notebook as she re-voiced Ted, by adding a hint of justification ("so p is five mod three"). The added p in Bettie's speech might be referring to the generic form of the Legendre symbol $(\frac{p}{q})$, where p must be reduced to its smallest residue, a rule which Hoffmann repeatedly used in the classes held on 11/15 and 11/19. To make sure that she got the same answer as Ted, she glanced at his notes to check the answer of five modular three. Ted noticed and said the response aloud. After Bettie approved, he moved on to the next step. Bettie's current notes would look like the following:

$$\left(\frac{10}{33}\right) = \left(\frac{2}{33}\right)\left(\frac{5}{33}\right) = \left(\frac{2}{3}\right)\left(\frac{2}{11}\right)\left(\frac{5}{3}\right)\left(\frac{5}{11}\right) = \left(\frac{2}{3}\right)\left(\frac{2}{11}\right)\left(\frac{2}{3}\right)\left(\frac{5}{11}\right)$$

Ted moved on to compute $\left(\frac{2}{3}\right)$ (1119-g3 at 1:16:04–1:16:39). He pointed to the two Legendre of two over three and stated they were equal to negative one, then looked at Bettie (again, checking the reflection). She hesitated, took what Ted had said and started reflecting upon it by looking at her notebook. Ted noticed a misalignment and explained further ("because you look at…") and then checked back with Bettie (he said "right?" while looking at her). Bettie stared at Ted's notes for three seconds and then accepted his idea by finding her own reason behind it ("yeah because it's prime). Ted followed up, because the primality was just a surface justification of the case. He re-explained the condition and, as soon as Bettie nodded, he moved to computing $\left(\frac{2}{11}\right)$. Bettie went back to her notebook when Ted stumbled a bit ("mod … what. whatever it is um"). But then, she quickly looked back at Ted's work when he said, "this one is three mod four." During this speech, Ted seemed to be thinking aloud, focused on his thoughts for a longer time than usual. As expected, he looked at Bettie when he finished his speech-unit. Bettie took the turn and started from the computation that two over three was negative one, which she had not had the chance to write it in her notebook yet. Then John interrupted her utterance when she seemingly moved to two over eleven. Thereafter, Hoffmann called for a whole-class activity, attracting Ted's attention. Most likely, Bettie was re-doing the Legendre of two over eleven during the whole-class activity, until she called back Ted's attention (at 1:18:06).

While Ted attended to the activity, Bettie attempted to complete the last step on her own and started computing $(\frac{5}{11})$. She attended to a new resource, the classroom board, and connected the mathematical computations in Ted's work on the group board to Hoffmann's work on the classroom board (see 1119-g3 at 1:17:42–1:18:42 in Table 4-8). Bettie's 8th and 9th voices (Table 4-9) identified Ted as the perpetrator of the mathematical moves which Bettie had figured out on her own (Bettie to Ted: "**you** inverse […] **you** inverse […] **you** do […] **you** do […]"). Thus, she shifted to position herself no longer on the side of the image in the mirror but become the agent, as she attempted to predict what Ted would do in the final computational step.

Notice the difference between the 8th and 9th voices (Table 4-9). The 8th voice was guessing ("maybe") and soliciting an idea assessment (question form, "five?"), whereas the 9th voice was assertively contributive ("you do […] this equals […] you do"). The only change that occurred between the 8th and 9th voices was that Ted reconnected with the activity, that is, reestablished his identification with the activity. In the 8th voice, when Ted had been attending to the whole-class activity, Bettie had been guessing Ted's moves. In the 9th voice, when the identification resumed, Bettie animated Ted, who spoke assertively. Going back to the voices that Bettie animated throughout the Ted-Bettie micro-activity (Table 4-9), those which were voiced when the identification was active, i.e., the 4th, 5th , 6th, 7th, 9th, and 10th voices in Table 4-9, were all assertive. In contrast, her voices when the identification was inactive, i.e., the 1st, 2nd and 8th voices, were animated as seeking a solicitation of knowledge and exhibiting a lack of confidence.

Bettie's use of the pronoun "you" in her 8th and 9th voices indicated an identification, as discussed above. Ted's use of pronouns also evidenced an identification activity. Ted shifted between the pronouns, started speaking in the first plural pronoun, then the first, then the second and, in responding to Bettie's 9th voice, the second and the first plural pronouns (read through the pronouns highlighted in red, in Table 4-8). The use of diverse pronouns invited interlocutors to identify with either one or the other. Ted used the first plural pronoun when he launched the micro-activity ("we start out"; "we have to first split"; "we do that"; "we do this"; "we have to first split" in 1119-g3 at 1:15:02–1:15:43). Ted's "we" could refer to the current interlocutors or the fictive members of a mathematical community. Doubtfully, Ted's "we" involved a hierarchical positioning such as the group of experts versus the novices. It was most likely a welcoming "we," open for interlocutors to identify with at their will. More significant and univocal was Ted's shift from the first singular pronoun ("First I'm doing," "I have to split the bottom," "giving me," "I'm gonna reduce this one," and "I know that this one and this one is negative one," in 1119-g3 at 1:15:20–1:16:15) to the second pronoun ("you look at," "you have plus or minus," "you mod," "you get plus or minus," "leaving you with," in 1119-g3 at 1:16:15–1:17:00). Earlier, Ted was describing his actions. Then (at 1:16:15), he started identifying Bettie as the perpetrator of the actions he would take. The shift took place when Bettie expressed hesitation about accepting Ted's computational move, $(\frac{2}{3}) = -1$. The shift to the second pronoun can be seen as a tactic to repair an emerging breach in the identification. Ted wanted Bettie to remain connected to his mathematical actions despite the experienced discrepancy in her mathematical understanding. He positioned Bettie as the perpetrator of his actions and wanted to guide her to agree with him, first in the narrative realm and, subsequently as well as hopefully, in the actual realm. Notice his narrative structure, resembling the navigating instructions: "when you have [situation1], you get [situation 2]" (repeated twice in the eco-unit of 1119-g3 at 1:16:04–1:16:39).

To appreciate Ted's tactic of shifting from the first to the second pronoun, imagine another potential scenario where he could have started defending his position and arguing why his computation was right. Such a positioning was possible for Ted, who used it on the occasions when John criticized his work. But with Bettie, he must have had another goal primal to defending his intellectual stance. Taking up a defensive position would presume a broken identification and create another genre of activity than the one that Bettie requested, "**we do** that **for me**."

The Ted-Bettie micro-activity was, thus, composed of two alternating participation structures (as defined by Frederick Erickson (1992)): (i) Ted writing on the group board and sharing his mathematical moves in pieces, then (ii) Bettie processing Ted's talk and his notes by reflecting upon them and writing on her notebook. Bettie's notebook mirrored Ted's notes on the shared board. The mirroring of these two minds was checked at every eye-contact between them. The use of the second pronoun, found in Ted's and Bettie's speeches, animated the interlocutor as an agent in a speech principled by the speaker. Therefore, the enacted postural positionings and the use of pronouns throughout the micro-activity evidenced an identification activity *a la* Lacan.

During the computational activity on 11/12, Bettie voiced her solicitations of idea assessments as questions (e.g. "and m has to be prime?" see Table 4-7). Such objectifying questions, as I argue, prevented face-threats from her groupmates, such as when, on 10/27, Boutros questioned Bettie's contribution and Jeremy criticized another one (see Table 4-4). In the Ted-Bettie micro-activity on 11/19, Bettie animated herself according to the image and the likeness of Ted,[12] when she voiced assertive contributions (from the 4th to the 7th, the 9th, and the 10th voice in Table 4-9; most striking was the shift of position from the 8th to the 9th voice). Note that Bettie's voices through the micro-activity could have well been animated as questions, since they attempted to enhance her lack of knowledge (1st voice). By animating herself as Ted's image, Bettie organized her mathematical understanding in his likeness, conditioned by the affordances of her mathematical knowledge. From her 4th to her 7th voice, Bettie hinged onto Ted's work as he re-constructed his computation anew. As such, during the first part of the micro-activity (prior to whole-class discussion), Ted was the principal of the work and, occasionally, Bettie re-voiced him (in her 5th and 7th voices) and authored justifications (in her 4th and 6th voices). In the second part of the micro-activity (during the whole-class discussion), Bettie was the principal of her ideas (from her 8th through her 10th voice), which she animated with her usual threatened confidence in the 8th voice and, in the likeness of Ted's usual confident position, in the 9th and the 10th voices.

Through the Ted-Bettie micro-activity on 11/19, Bettie went beyond reaffirming her "arithmetic" identity (PPMI). Through this activity, a nascent number theory identity materialized, which was not yet individualized but identified with another person. Bettie's confidence in this computational activity was executed through a mirroring process, by which one constructs "self" at the image and likeness of other(s).

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[12] The phrase "being an image" is commonly devalued. Nonetheless, images in mirrors have their own dignifying world and are never fully controlled by the principal agent. You cannot make your image in the mirror move its right hand when you move your right hand. The principal agent must accommodate the world of the image in the mirror.

Continued fractions (on 12/01).

On 12/01, Hoffmann introduced the truly "arithmetic" notation of continued fractions by computing the continued fractions of π, $\frac{723}{327}$ and e (1201-g3 0:12:31–0:26:00). Thereby, he showed an example with a fraction form and two examples with decimal forms. To transform the numbers into continued fractions, Hoffmann repeated the following procedure: take the integer part of the number, then inverse the decimal or the fraction of the unit part and start over with the denominator (see Figure 4-19).

In group G3, the students computed the continued fractions of Wk11#1 (see Figure 4-20) individually. They talked to check their answers and, occasionally, their understandings. Boutros, Ted, and Jeremy worked on the shared poster board; each one of them wrote on his side of the poster board. John worked on his tablet while Bettie worked in her notebook. Hoffmann sat with the group for about three minutes (1202-g3 0:33:38–0:36:41), during which he helped Ted in solving Wk11#2a (see Figure 4-20), while the other groupmates had been still working on Wk11#1. Then, Ted joined the group, which was computing all the numbers of Wk11#1. When Jeremy moved to Wk#2, Ted shared what he learned from Hoffmann at 0:1:00:14–1:01:01, during which Bettie was focused on computing Wk11#1f. Later (at 1:01:28–1:02:19), John and Ted discussed the strategy of using a variable x for Wk11#2a, but Bettie was still focused on Wk11#1f. For a third time (at 1:04:15–1:05:33), John and Jeremy discussed how to solve the quadratic equation for Wk11#2a, while Bettie had been still focusing on Wk11#1f. Hoffmann visited the group a second time (at 1:05:33–1:07:10), commented on Bettie's computation of Wk11#1f, and then talked to Ted. When he left the group, he engaged the whole class in a talk about the golden ratio. During the class discussion, Bettie approached Boutros and asked him about the variable x (at 1:09:40–1:10:15). By the time she asked Boutros about Wk11#2a (at 1:13:05–1:13:19), the group went into talking about registrations for the next semester and stopped working on the worksheet (at 1:13:43).

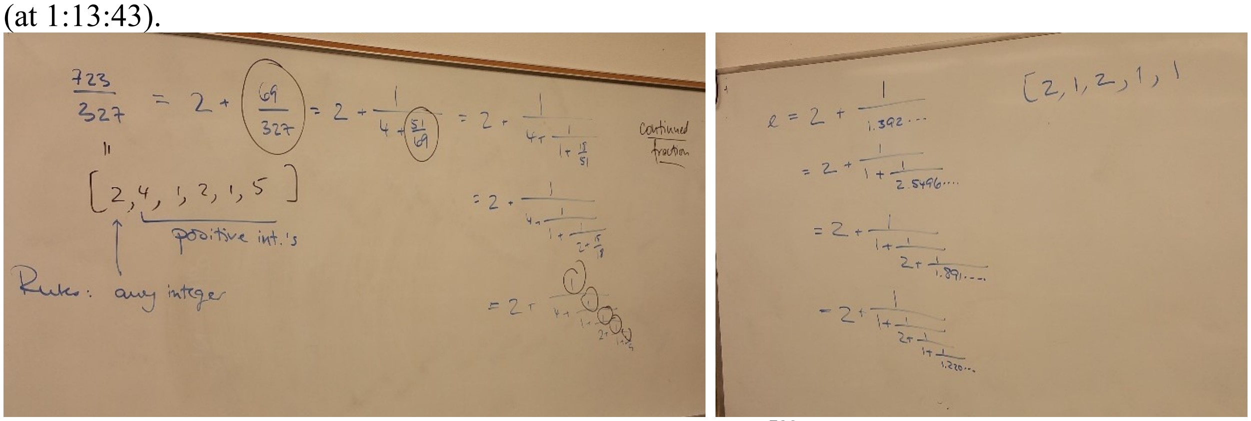

Figure 4-19: Hoffmann's computations of the continued fractions for $\frac{723}{327}$ (picture on the left) and e (picture on the right).

Worksheet 11: Continued Fractions

1. Find continued fraction expansions for

   (a) $\frac{100}{37}$ (c) $\frac{21}{13}$ (e) $\frac{13}{35}$

   (b) $\frac{1001}{45}$ (d) $\frac{1000}{301}$ (f) $\frac{\sqrt{5}-1}{2}$

2. Compute

   (a) $[2,3,2,3,\ldots]$ (b) $[1,2,1,2,\ldots]$ (c) $[1,2,2,2,\ldots]$

Figure 4-20: The computational problems, Wk11#1 and Wk11#2, in class the lesson on 12/01.

Table 4-10: Bettie's animated voices in the groupwork on 12/01 – lesson on continued fractions.

Time stamp of eco-unit	Excerpt of transcript
Working on Wk11#1	
1201-g3 at 0:28:53 – 0:29:06	Bettie: [unidentified] guys. just something one over what? [no one responds to her, as they seem very engaged with their individual computations] Bettie: [leans to her left side to sneak a peek at Boutros's work]
1201-g3 at 0:30:31 – 0:31:55	[Bettie is attending to this conversation; see Figure 4-21 to follow with Ted's inscriptions] Jeremy: [looks at Ted's work on shared board] did I do this wrong? [looks at other groupmates' notes] Ted [to Jeremy]: [looks at Jeremy's work] yes. but . that way [points to Jeremy's notes] is like a sure fire way to do it for even like irrational numbers. the way you're doing it? Jeremy: how are you doing it? Ted: the rational numbers there is an easier way like you can do it in Euclidean algorithm [looks at Jeremy] another way. um [writes on shared board 100=2.37+26] one hundred is equal to two times thirty-even plus what was it twenty something [computes on calculator] twenty-six. and then you write um you take these two [circles 37 and 26] and you're dividing them. [looks at Jeremy] cause this is really just saying [writes 100/37=2+26/37] one hundred over thirty-seven is equal to two plus twenty-six over thirty-seven. [0:31:21] [looks straight and notices Bettie attending to his work] and then we flip this one [circles 26/37] that's is actually saying [writes 37÷26] thirty-seven divided by twenty-six and =we Bettie: =oh [and she erases her work on her notebook and starts writing; she is no longer looking at Ted's work] Ted: move this over [draws two oblique arrows] Jeremy: okay Ted: [writes 31=1 26+] plus something [holds the calculator] and then each of these numbers [drawing squares around the divisors 2 and 1] in the boxes . will be . these here [points to the continued fraction] [0:31:44.09] Williams was teaching history . and that's one of the things he went over I think [looks at Boutros] Boutros: [looks at Ted's work and shakes his head] Jeremy: yes [unidentified] Ted: yeah
1201-g3 at 0:38:49 – 0:39:32	[Boutros, Ted and Jeremy were talking about Wk11#1b for about a minute. Bettie interjects once she finished her computation] Bettie: [finishes writing on her notebook and looks at Jeremy] wait. isn't that supposed to be easy? Jeremy: what? Bettie: b? Ted: one thousand one over forty-five? Bettie: yeah Ted: I'm checking it right now, It should be . it should go okay I think Bettie: Cause I'm like done after two steps. Ted: did you take the twenty -wo and divide that? Bettie: yeah Ted: by forty-five? Bettie: twenty-two. I mean I did [points to her notebook and Ted looks at it] twenty-two plus eleven over forty-five. Ted: yeah [nods] Bettie: yeah. it's right. Right? Ted: that's correct. Bettie: yeah [Ted leans forward toward Bettie's notebook] and then I did the forty five over eleven . it's four plus one over eleven. right? [looks at Ted] Ted: [nods] yeah. Bettie: and done. twenty-two and four.
1201-g3 at 0:39:45 - 0:39:54	Boutros: [points to Ted's work on poster board] this is four . right? Bettie: yeah. Boutros: [points to 22 in his work on poster board] so it is this. so you can have a double digits. Bettie: I think so. Boutros: o::oh. Okay.
1201-g3 at 0:44:12 – 0:44:58	Bettie: [counts on her notebook] one two three four Ted [to Boutros]: I think for and five. so it's four ones? and one five? for c? [flips page and resumes working on her notebook] Boutros: no Bettie: I mean two I mean. one one one one and two?

	Ted: hanging on there yet Boutros: over here? [points to his notebook] Bettie: [leans towards Boutros's work] the c. Is the last one two? I mean. what did I get? [looks at her notebook]. yeah two. Boutros: oh. [completes computation on dry-erase poster board] three wherever two one last two. yeah. you're right. [writes on his notebook] the last one is two.
1201-g3 at 0:49:10 – 0:49:17	Bettie to Boutros: [unintelligible, logically asking about #1d. Boutros moves his hand away to allow Bettie to look at his notebook] Bettie: [points to her notebook] three three nine one two . Boutros: [looks and nods]
1201-g3 at 0:49:17 – 0:49:24	Ted: I'm doing d right now. started with a leading three? [looks at Bettie] Bettie: yeah Ted: okay.
1201-g3 at 0:49:31 – 0:49:49	[Bettie asks about #1e where numerator is smaller than denominator] Bettie: wait, how did you guys do thirteen over thirty-five? Jeremy [thinking aloud]: I'm just gonna do this with the Euclidean algorithm Boutros [to Bettie]: we start with zero so . thirteen over thirty-five equals zero plus thirteen. and you just ran the same. Bettie: alright. [turns to write on her notebook]
1201-g3 at 1:05:42 – 1:05:51	Hoffmann [to Bettie]: what are you doing here [points to her notebook, her computation of #2f]? Bettie: I'm just doing . nothing Hoffmann: do you see a pattern? Bettie: yeah Hoffmann: you should stop doing it.
1201-g3 at 1:05:53 – 1:07:25	[parallel conversation between Jeremy, Boutros and Bettie around the formula of solving quadratic equations then talk about Modern Algebra course]
Bettie is working on Wk11#2a	
1201-g3 at 1:09:40 – 1:10:15	[Hoffmann is engaging the class in an interactive lecturing] Bettie [turns to Boutros and points to his work of #2a on group board]: [uncaptured sound, voice of Hoffmann overwhelms in the video] Boutros: nods Bettie: [points to top and down on Boutros's work where he transformed a continued fraction into a quadratic equation and wrote the two solutions, uncaptured sound] Boutros: I think so Bettie: why? Boutros: um. I don't know. So if you write this out [points to the continued fraction form on his notebook] and that's this [points to the development of the continued fraction in his work on group board, see Figure 4-21]. You know it's like something you [unidentifiable]
1201-g3 at 1:13:05 – 1:13:30	Bettie [to Boutros]: Did you [unintelligible]? Bettie: starting with this [points to Boutros's work on board, the first formula where he puts x in the continued fraction; see Figure 4-21]. why x? [Boutros seems puzzled by the question and attends to Ted explaining to John; Bettie then moves to attend to Ted's work]

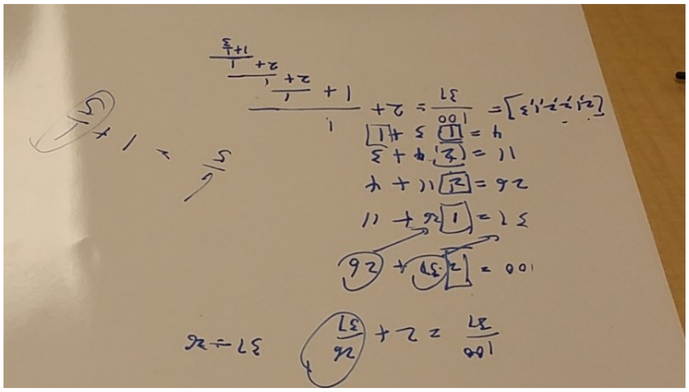

Figure 4-21: Ted’s work on the dry-erase poster board, explaining, at Jeremy’s initiative, the procedure of transforming rational numbers into continued fractions for the example 100/37. See the eco-unit 1201-g3 at 0:30:31–0:31:55.

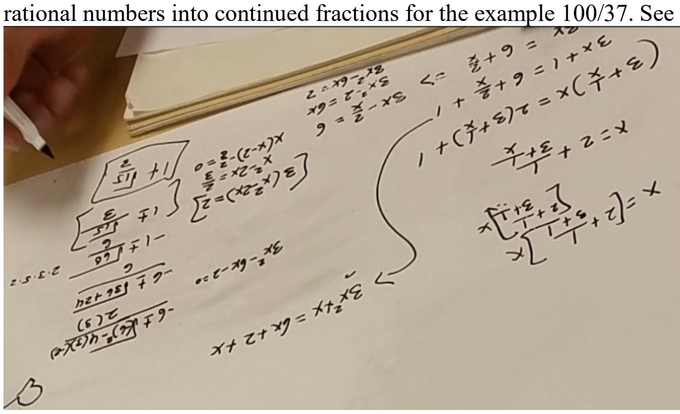

Figure 4-22: Boutros’s work on the poster board, regarding Wk11#2a.

Table 4-11: Bettie’s voices, on 12/01, during the computation of continued fractions.

Eco-unit	Voice	Identity	Position	Function
1201-g3 at 0:28:53 – 0:29:06	1st [unidentified] guys. just something one over what?	Group member’s right	Soliciting an explanation	To enhance the speaker’s understanding.
1201-g3 at 0:38:49 – 0:39:32	2nd [finishes writing on her notebook and looks at Jeremy] wait. isn't that supposed to be easy? […] b?	Group member’s right and duty	Sharing the perplexity	To draw attention or critique groupmates.
1201-g3 at 0:38:49 – 0:39:32	3rd Cause I'm like done after two steps.	Group member’s right and an “arithmetic” identity (PPMI)	Justifying her perplexity	To contribute a mathematical idea.
1201-g3 at 0:38:49 – 0:39:32	4th twenty-two. I mean I did [points to her notebook and Ted looks at it] twenty-two plus eleven over forty-five.	Group member duty and an “arithmetic” identity (PPMI)	Contributing a mathematical idea	To explain her computation to her groupmate.
1201-g3 at 0:38:49 – 0:39:32	5th it's right. Right?	Group member’s right	Soliciting an idea assessment	To seek her groupmate’s acknowledgment.
1201-g3 at 0:38:49 – 0:39:32	6th yeah [Ted leans forward toward Bettie's notebook] and then I did the forty five over eleven . it's four	Group member’s right and “arithmetic” identity (PPMI)	Contributing a mathematical idea	To seek groupmate’s acknowledgment.

	plus one over eleven. right?			
1201-g3 at 0:38:49 – 0:39:32	7th and done. twenty-two and four. [looks at Ted]	Group member's right and an "arithmetic" number theory identity	Contributing a mathematical idea	To proclaim her result.
1201-g3 at 0:39:45 - 0:39:54	8th [Boutros: is this four?] yeah.	Group member duty and an "arithmetic" number theory identity	Assessing a groupmates' idea	To enhance a groupmate's understanding.
1201-g3 at 0:39:45 - 0:39:54	9th [Boutros: you can have a double digits.] I think so.	Group member duty and an "arithmetic" number theory identity	Sharing her stance	To support a groupmates' stance.
1201-g3 at 0:44:12 – 0:44:58	10th one two three four […] and five. so it's four ones? and one five? for c? [flips the page and resumes working on her notebook]	Group member's right and an "arithmetic" number theory identity	Soliciting an answer assessment	To announce her answer with group and confirm it.*
1201-g3 at 0:44:12 – 0:44:58	11th I mean two I mean. one one one one and two? […] [leans towards Boutros's work] the c. Is the last one two? I mean. what did I get? [looks at her notebook]. yeah two. [Boutros: you're right]	Group member's right and an "arithmetic" number theory identity	Sharing her answer and soliciting assessment for its last part (two?)	To confirm her answer.
1201-g3 at 0:49:10 – 0:49:17	12th [points to her notebook] three three nine one two .	Group member's right and an "arithmetic" number theory identity	Contributing a mathematical answer	To confirm her answer.
1201-g3 at 0:49:17 – 0:49:24	13th [Ted to Bettie: started with a leading three?] Yeah	Group member duty and an "arithmetic" number theory identity	Assessing a groupmate's answer	To support her groupmate.
1201-g3 at 0:49:31 – 0:49:49	14th wait, how did you guys do thirteen over thirty-five?	Group member's right	Soliciting an explanation	To enhance her understanding.
1201-g3 at 1:05:42 – 1:05:51	15th I'm just doing . nothing	Group member duty	Offering an explanation	To divert the instructor's attention.
1201-g3 at 1:05:53 – 1:07:25	16th Uncaptured sound	Group member's right		
1201-g3 at 1:09:40 – 1:10:15	17th Uncaptured sound	Group member's right		
1201-g3 at 1:09:40 – 1:10:15	18th Uncaptured sound [points to top and down on Boutros's work where he transformed a continued fraction into a quadratic equation and wrote the two solutions]	Group member's right and (~ an "arithmetic" identity)		
1201-g3 at 1:09:40 – 1:10:15	19th Why?	Group member's right	Soliciting an explanation	To enhance her own understanding.

1201-g3 at 1:13:05 – 1:13:30	20th starting with this [pointing to Boutros's work on board]. why x?	Group member's right and an "arithmetic" identity (PPMI)	Soliciting an explanation	To enhance her own understanding.
* The function of Bettie's 10th voice is equivocal. She could be looking to confirm her answer and announce her results. Her gestural behaviors, mainly flipping the page and resuming her work on her notebook as soon as she announced her results, support the latter function. The former function is only implicit in her position whereby she solicited an assessment. It was reanimated in the 11th voice, after her answer had been challenged.				

Discussion

The most striking fact in this group session was that Bettie never consulted her notebook. She was confused, at the start of the groupwork, about the inversion procedure (1st voice in the eco-unit 1201-g3 at 0:28:53, which no one attended). She seemed lost until Jeremy checked his answer with Ted (at 0:30:31). Ted went on to explain how to use Euclidean algorithm for rational fractions (eco-unit at 0:30:31–0:31:55, see the transcript in Table 4-10). Bettie attended to Ted's explanation while Ted noticed her only halfway through the same (at 0:31:21). Notice Ted's shift of pronouns, from "you" (directed towards Jeremy) to "we," when he noticed that another groupmate was also listening. She expressed that she had understood Ted when he explained the inversion of the remaining fraction; she straightaway went to write the same on her notebook. Indeed, her computations following this instance yielded the correct answers. She worked systematically for every number in Wk11#1, using either the Euclidean algorithm or the decimal method with a calculator.

Bettie made all the computations on her own, without any support from a published or a group resource. Since the first computational activity (on 10/27), Bettie had not voiced an individualized "arithmetic" identity (PPMI) as intensely as in her computation of the continued fractions (on 12/01). She voiced an individualized "arithmetic" identity in four instances (3rd, 4th, 6th, and 20th voices) and an "arithmetic" number theory identity in ten instances (from the 7th to the 13th voices). Note that the definition of continued fractions was the closest notion, in number theory, to arithmetic (PPM). Bettie must have felt confident once she had learned the inversion procedure. However, we should not underestimate the computational aspect, with can get confusing (notice Jeremy's and Boutros' struggles in eco-units at 0:30:31–0:31:55, 0:39:45–0:39:54 and 0:44:12–0:44:58). Although the computations of continued fractions might look trivial for experts, they were not so for the students. Bettie's work was, indeed, praiseworthy.

Bettie's confidence emanated in various ways throughout the computational activity of the continued fractions. In an unprecedented manner, she fought for her answers in two eco-units (0:38:49–0:39:32 and 0:44:12–0:44:58) by producing multiple voices in an eco-unit. In the previous computational activities, Bettie was observed animating one or, at most, two voices per eco-unit. On 12/01, she animated six voices in the eco-unit, at 0:38:49–0:39:32, and two voices in each of the subsequent three eco-units. When working on #1b, Boutros was confused about the number of digits that the integers in a continued fraction should contain. However, he did not know how to voice his confusion to Ted productively, and Jeremy diverted the conversation (see the transcript below 1201-g3 at 0:37:54–0:38:49). As soon as Bettie finished her computation of #1b, she looked at Jeremy, the main speaker in the previous eco-unit (see the transcript below), and voiced an opposing perplexity, "isn't that supposed to be easy?" [why are you debating about it?] (see the eco-unit of 1201-g3 at 0:38:49–0:39:32 in Table 4-10).

<u>Eco-unit of 1201-g3 at 0:37:54-0:38:49</u> – *Boutros is confused about the number of digits*

Boutros: this [pointing to Ted's method on board] doesn't work for b.

Ted: for b?

Boutros: =uhum
Jeremey: =it does. It's the same thing.
Ted: wait let's try it.
Jeremy: this algorithm should work if it is still a rational number. it's the same thing.
Ted: one hundred over forty five. well we have to start doing decimals with calculator starting with f I think.
Boutros: ah. o::kay
Jeremy: yeah you can't. this one'd be harder to doing by hand I think
Boutros: uhm [nods]
Jeremy: because it's bigger numbers. but you could do it like this [pointing to his work on board, after Ted's method] are you doing it just like the Euclidean algorithm right now?
Boutros: work through this [pointing to Ted's work on board]. but see here [pointing to his work on board] I'm getting double digits
Jeremy: and so you write down all like the quotients?
Boutros: [nods]
Jeremy: actually yeah. I don't think [unintelligible]

In the eco-unit of 1027-g3 at 0:55:42–0:56:15, Bettie opposed Jeremy by insisting that five was a primitive root. She lost the battle as she could not defend her answer. However, in the eco-unit of 1201-g3 at 0:38:49–0:39:32, Bettie opposed Jeremy (as she looked at him) and went on supporting her stance throughout (the transcripts of both eco-units are reproduced for convenience). Ted's role in this eco-unit implicitly boosted Bettie's assertive stance. Notice that while addressing Jeremy (her 2nd voice split in Table 4-12 with two turns), Bettie animated an uncertain position in asking the questions. Surprisingly, she responded to Jeremy's question (what?) with an inquisitive intonation (b?). Recall that Bettie reported in her exit interview that she was no longer feeling comfortable discussing mathematics with Jeremy. However, with Ted (from her 3rd to her 7th voice), she explained her computational steps using assertive statements, to which she added a solicitation of acknowledgment. The 4th voice was an assertive statement, followed by a request of acknowledgment from Ted in the 5th voice. Notice that the fifth voice has two parts: one was assertive ("it's right") and the following was requesting approval ("right?"). The 6th voice animated both aspects together, starting with an assertive description of the steps ("I did the forty five over eleven. it's four plus one over eleven.") followed by the approval request ("right?"). The answer was announced assertively, "and done. Twenty-two and four."

Table 4-12: Comparing two eco-units when Bettie opposed Jeremy on 10/27 (left column) and 12/01 (right column).

Eco-unit of 1027-g3 at 0:55:42–0:56:15 ***Computing the primitive roots modulo eight***	**Eco-unit of 1201-g3 at 0:38:49–0:39:32** ***Computing the continued fraction of 1001/45***
Jeremy: [unclear] them all I didn't get a primitive root Bettie: yup five is one Boutros [to Jeremy]: I did it this way. this way you doing it? Bettie: yeah Boutros: does it go like that? [Jeremy stares at Boutros's notebook] Bettie: I got five.	Bettie: [finishes writing on her notebook and looks at Jeremy] wait. isn't that supposed to be easy? Jeremy: what? Bettie: b? Ted: one thousand one over forty-five? Bettie: yeah Ted: I'm checking it right now, It should be . it should go okay I think Bettie: Cause I'm like done after two steps. Ted: did you take the twenty-two and divide that? Bettie: yeah Ted: by forty-five?

Eco-unit of 1027-g3 at 0:55:42–0:56:15 *Computing the primitive roots modulo eight*	Eco-unit of 1201-g3 at 0:38:49–0:39:32 *Computing the continued fraction of 1001/45*
Jeremy: five squared is congruent to one mod eight. so it isn't the five. [Jeremy looks at Bettie] Bettie: [makes facial gestures -like puzzled- and turns away to look at Boutros' notes and then engages in off-topic conversation]	Bettie: twenty-two. I mean I did [points to her notebook and Ted looks at it] twenty-two plus eleven over forty-five. Ted: yeah [nods] Bettie: yeah. it's right. Right? Ted: that's correct. Bettie: yeah [Ted leans forward toward Bettie's notebook] and then I did the forty five over eleven . it's four plus one over eleven. right? [looks at Ted] Ted: [nods] yeah. Bettie: and done. twenty-two and four.

On 10/27, Boutros doubted and questioned Bettie's answer that three was a primitive root modulo seven. In response, Bettie only nodded, asserting her stance, and Ted approved her answer, which temporarily swayed Boutros (in the eco-unit of 1027-g3 at 0:46:34–0:46:48; transcript is reproduced in Table 4-13). On 12/01, Boutros opposed Bettie's answer for the continued fraction of $\frac{21}{13}$. However, in this eco-unit (1201-g3 at 0:44:12–0:44:58; transcript is reproduced in Table 4-13 for convenience) she pursued the investigation on her own in order to discover who was right. She looked at Boutros's work and checked her own. Boutros ended up noticing his incomplete computation. Contrary to the situation on 10/27, Bettie was ready to follow-up and defend her answer on 12/01.

Table 4-13: Comparing the two situations, when Boutros opposed Bettie on 10/27 (left column) and 12/01 (right column).

Eco-unit of 1027-g3 at 0:46:34–0:46:48 *Computing the primitive roots modulo seven*	Eco-unit of 1201-g3 at 0:44:12–0:44:58 *Computing the continued fraction of 21/13*
Jeremy: so what's the second one? Boutros: it has to be:::: =six Bettie: =three Boutros: oh three. Are you sure? [turning his face to Bettie, who nods] Ted: three for seven right? Boutros: alright. okay. I probably did something wrong then.	Bettie: [counts on her notebook] one two three four Ted [to Boutros]: I think for and five. so it's four ones? and one five? for c? [flips page and resumes working on her notebook] Boutros: no Bettie: I mean two I mean. one one one one and two? Ted: hanging on there yet Boutros: over here? [points to his notebook] Bettie: [leans towards Boutros's work] the c. Is the last one two? I mean. what did I get? [looks at her notebook]. yeah two. Boutros: oh. [completes computation on dry-erase poster board] three wherever two one last two. yeah. you're right. [writes on his notebook] the last one is two.

More importantly, Bettie aminated two voices (8^{th} and 13^{th}) to assess her groupmates' ideas, which denoted a boost in her mathematical authority. In total, she animated three voices (8^{th}, 9^{th}, and 13^{th}) oriented to the benefit of her groupmates, a function that was not observed in the previous computational activities except the first one, where Bettie was enthusiastic. In the last computational activity on 12/01, Bettie indeed affirmed an "arithmetic" (PPM) ability which was socially recognized.

Analysis of Bettie's confidence over four computational activities

The VIP+function analysis produced in this chapter is summarized in this section by focusing on Bettie's confidence. The findings (Table 4-4, Table 4-5, Table 4-7, Table 4-9, and Table 4-11) will be condensed to a 3-level coding of confidence applied to the identities, positions, and functions of Bettie's voices which she animated during the four computational activities in the number theory class. In this section, I explain the coding exercise and the composition of Table 4-14 which reports the coding of Bettie's voices.

The rows represent Bettie's voices in a chronological order during the computational activities in the number theory class. The first column provides the date and topic of the computational activity, while the second column indicates which of Bettie's voice is being coded in the respective activity.

The third column in Table 4-14 represents the mathematical dimension of the identities that the voices actuate: the lighter grey cells represent a non-mathematical identity, such as questions that do not involve any mathematical idea ("Can you say it again?"), the regular grey cells represent arithmetic identities (PPMI) ("how to solve for x?"), and the darker grey cells number theory identities ("eight has no primitive roots").

The fourth column represents the confidence level conveyed at the positions that Bettie's voices animate. Positional confidence is broken into three levels. The first level of positional confidence, i.e., nascent confidence (lighter green cells), is considered to be the positions of asking others, such as groupmates or instructor, to explain mathematical ideas that the speaker does not understand. For example, Bettie asks the group, "Can you explain to me what Hoffmann has just said?" This confidence merely supports participation in learning settings. The second level of positional confidence, i.e., hesitant confidence (regular green cells), is considered to be the positions when the speaker seeks the assessment of a mathematical idea that she/he generates by herself/himself, such as computing a continued fraction then checking the answer with a groupmate. The third level of positional confidence, i.e., settled confidence (darker green cells), is considered to be the affirming positions, such as when Bettie shares her mathematical idea or holds onto her epistemic stance.

Functions can also exhibit confidence. This is evaluated in the fifth column. The evaluation of functional confidence in this work builds on Engle's (2012) levels of authorship, as presented in the third chapter (Page 39). The functions of evading an engagement in mathematical conversations, such as not answering a groupmates' question or shrugging shoulders and remaining silent when a groupmate criticizes the speaker's ideas, are considered to exhibit a rudimentary level of confidence (lighter blue cells in Table 4-14). The functions oriented to enhancing one's own understanding are regarded as the voices participating in group activities to foster confidence in one's own knowledge (regular blue cells in Table 4-14). The functions such as contributing to the advancement of groupwork or enhancement of a groupmates' understanding exhibit a confidence in one's own ideas as valuable for others (darker blue cells in Table 4-14).

Conclusively, Table 4-14 represents a three-level analysis of Bettie's confidence along three dimensions: identity, position, and function. The confidence levels go from weaker to stronger and are represented using lighter to darker colors, respectively. The confidence exhibited in identity is evaluated through the type of mathematical identity actuated in Bettie's voices: social without mathematical identity (lighter grey), arithmetic identity (PPMI) (regular grey), and number theory identity (darker grey). Bettie's confidence is also analyzed through the forms of her participation in the activities as positions expressing a nascent confidence (lighter green), a hesitant confidence (regular green), or a settled confidence (darker green). The confidence involved in the functionality of Bettie's voices is evaluated through the authorship degree of her engagement: evading engagement (lighter blue), enhancing own understanding (regular blue), and contributing to group or groupmates' understanding (darker blue).

Table 4-14: Chronological representation of the confidence in Bettie's voices along the constituents of identities, positions, and functions, throughout the four computational activities in the number theory class, as analyzed in this chapter. The legend for the significance of colors is provided at the end of the table. Note that the 16th, 17th, and 18th voices in the fourth computational activity (on 12/01) are not included in this coding exercise because of the noise causing disturbances in the sound quality captured in the video-recording at those moments.

Date & Topic	Order in activity	Identity	Position	Function
10/27 Primitive roots	1st voice			
	2nd voice			
	3rd voice			
	4th voice			
	5th voice			
	6th voice			
	7th voice			
	8th voice			
	9th voice			
	10th voice			
	11th voice			
	12th voice			
	13th voice			
	14th voice			
	15th voice			
11/12 Quadratic residues	1st voice			
	2nd voice			
	3rd voice			
	4th voice			
	5th voice			
	6th voice			
	7th voice			
	8th voice			
	9th voice			
11/19 Quadratic reciprocity law	1st voice			
	2nd voice			
	3rd voice			
	4th voice			
	5th voice			
	6th voice			
	7th voice			
	8th voice			
	9th voice			
	10th voice			
12/01 Continued fractions	1st voice			
	2nd voice			
	3rd voice			
	4th voice			
	5th voice			
	6th voice			
	7th voice			
	8th voice			
	9th voice			
	10th voice			
	11th voice			
	12th voice			
	13th voice			
	14th voice			
	15th voice			
	19th voice*			
	20th voice			

Legend of colors for the 3-level coding of confidence along identity, position, and function:		
Identity	**Lighter grey**	Social identity without mathematical content
	Regular grey	Arithmetic identity (PPMI)
	Darker grey	Number theory identity
Position	**Lighter green**	Position expressing a nascent confidence (mainly by seeking knowledge from groupmates)
	Regular Green	Position expressing a hesitant confidence (mainly by asking groupmates to confirm a personal idea)
	Darker green	Position expressing a settled confidence (mainly by contributing to group or groupmates' works).
Function	**Lighter blue**	To evade a situation
	Regular Blue	To enhance own understanding
	Darker blue	To advance groupwork or enhance groupmates' understanding

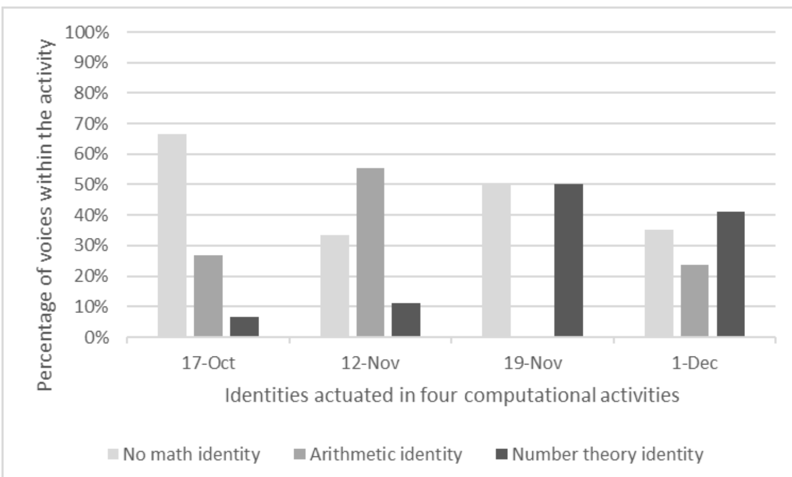

Figure 4-23: The relative frequencies of the 3-level confidence about identities as actuated in Bettie's voices within each of the four computational activities. N=15 for 17-Oct, N=9 for 12-Nov, N=10 for 19-Nov, and N=17 for 1-Dec.

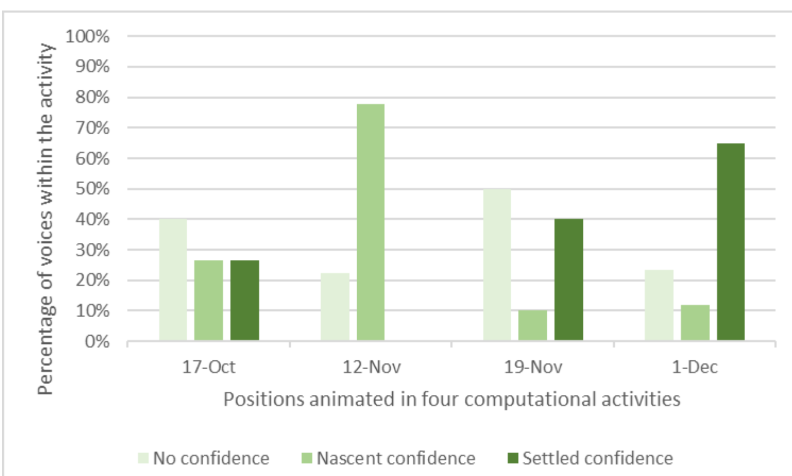

Figure 4-24: The relative frequencies of the 3-level confidence about positions as animated in Bettie's voices within each of the four computational activities. N=15 for 17-Oct, N=9 for 12-Nov, N=10 for 19-Nov, and N=17 for 1-Dec.

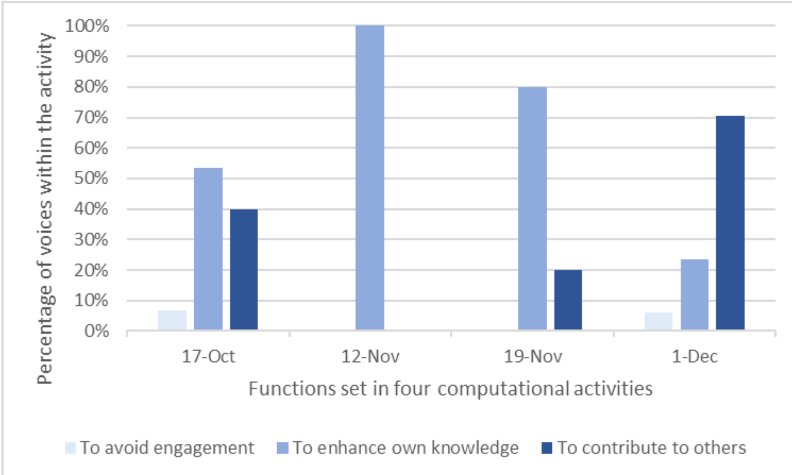

Figure 4-25: The relative frequencies of the 3-level confidence about the functionalities set by Bettie's voices within each of the four computational activities. N=15 for 17-Oct, N=9 for 12-Nov, N=10 for 19-Nov, and N=17 for 1-Dec.

While The VIP+function analysis produced in this chapter is summarized in this section by focusing on Bettie's confidence. The findings (Table 4-4, Table 4-5, Table 4-7, Table 4-9, and Table 4-11) will be condensed to a 3-level coding of confidence applied to the identities, positions, and functions of Bettie's voices which she animated during the four computational activities in the number theory class. In this section, I explain the coding exercise and the composition of Table 4-14 which reports the coding of Bettie's voices.

The rows represent Bettie's voices in a chronological order during the computational activities in the number theory class. The first column provides the date and topic of the computational activity, while the second column indicates which of Bettie's voice is being coded in the respective activity.

The third column in Table 4-14 represents the mathematical dimension of the identities that the voices actuate: the lighter grey cells represent a non-mathematical identity, such as questions that do not involve any mathematical idea ("Can you say it again?"), the regular grey cells represent arithmetic identities (PPMI) ("how to solve for x?"), and the darker grey cells number theory identities ("eight has no primitive roots").

The fourth column represents the confidence level conveyed at the positions that Bettie's voices animate. Positional confidence is broken into three levels. The first level of positional confidence, i.e., nascent confidence (lighter green cells), is considered to be the positions of asking others, such as groupmates or instructor, to explain mathematical ideas that the speaker does not understand. For example, Bettie asks the group, "Can you explain to me what Hoffmann has just said?" This confidence merely supports participation in learning settings. The second level of positional confidence, i.e., hesitant confidence (regular green cells), is considered to be the positions when the speaker seeks the assessment of a mathematical idea that she/he generates by herself/himself, such as computing a continued fraction then checking the answer with a groupmate. The third level of positional confidence, i.e., settled confidence (darker green cells), is considered to be the affirming positions, such as when Bettie shares her mathematical idea or holds onto her epistemic stance.

Functions can also exhibit confidence. This is evaluated in the fifth column. The evaluation of functional confidence in this work builds on Engle's (2012) levels of authorship, as presented in the third chapter (Page 39). The functions of evading an engagement in mathematical conversations, such as not answering a groupmates' question or shrugging shoulders and remaining silent when a groupmate criticizes the speaker's ideas, are considered to exhibit a rudimentary level of confidence (lighter blue cells in Table 4-14). The functions oriented to enhancing one's own understanding are regarded as the voices participating in group activities to foster confidence in one's own knowledge (regular blue cells in Table 4-14). The functions such as contributing to the advancement of groupwork or enhancement of a groupmates' understanding exhibit a confidence in one's own ideas as valuable for others (darker blue cells in Table 4-14).

Conclusively, Table 4-14 represents a three-level analysis of Bettie's confidence along three dimensions: identity, position, and function. The confidence levels go from weaker to stronger and are represented using lighter to darker colors, respectively. The confidence exhibited in identity is evaluated through the type of mathematical identity actuated in Bettie's voices: social without mathematical identity (lighter grey), arithmetic identity (PPMI) (regular grey), and number theory identity (darker grey). Bettie's confidence is also analyzed through the forms of her participation in the activities as positions expressing a nascent confidence (lighter green), a hesitant confidence (regular green), or a settled confidence (darker green). The confidence involved in the functionality of Bettie's voices is evaluated through the authorship degree of her engagement: evading engagement (lighter blue), enhancing own understanding (regular blue), and contributing to group or groupmates' understanding (darker blue).

Table 4-14 provides a chronological coding of the confidence level in Bettie's voices, the precedent figures provide the relative frequencies of the same. Figure 4-23 represents the relative frequencies of the three levels of confidence in the identities that Bettie actuated in each computational activity. Figure 4-24 and Figure 4-25 pertain to the positions and functions of the same, respectively.

The results of the 3-level coding of Bettie's confidence along the constituents of voice, identity, position, and function (

Table 4-14) portray the changes in her confidence during the computational activities. Recall (Figure 4-1) that on average, Bettie animated two or three instances of contributive positions during a group session. Unlike the average trend of her participation in group work, during the computational activity on 10/27, Bettie exhibited confidence on two dimensions: she was observed animating four voices with contributive positions and three other voices with contributive functions. This confidence faded away after the 9th voice through the 14th voice of the same activity and during the next computational activity (on 11/12). Bettie's confidence reappeared in the third and fourth computational activities: highest levels of confidence are observed in identities, positions, and functions on 11/19 and 12/01. The most striking observation was the actuation of number theory identities in 50% and 40% of the voices animated during the computational activities on 11/19 and 12/01, respectively, as compared to 5% and 10% on 10/27 and 11/12, respectively (see Figure 4-23). The ecologies that bolstered Bettie's confidence on 11/19 and 12/01 were discussed under the respective subsections. The following section will summarize the processes by which Bettie's confidence spiked during the last two computational activities.

Conclusion: Two processes to enhance confidence

The ecologies that bolstered Bettie's confidence during the computational activity on 11/19 differed from the ones that bolstered her confidence on 12/01.

Bettie's reaffirmation of her "arithmetic" (PPM) ability on 12/01 was made possible due to the arithmetic nature of the task. The computation of continued fractions fell within Bettie's zone of productive struggle. Indeed, by 12/01, Bettie had mastered basic arithmetic skills as well as the Euclidian algorithm which facilitated her grasp over the computations related to continued fractions. Furthermore, on 12/01, Bettie could exhibit confident voices during her interactions with Ted as opposed to her interactions with Jeremy on both 10/27 and 12/01.

Ted's effect on Bettie's confidence was obvious on 11/19. During the Ted-Bettie micro-activity, Bettie did not voice her own confidence but instead she exhibited Ted's confidence through the so-called mirroring process. Early in the computational activity, Bettie was lost. She did not understand the quadratic reciprocity law, partly because she came to class 20 minutes after the class started. During the Ted-Bettie micro-activity, Bettie mirrored and built on Ted's knowledge and positions. She did not have the sufficient knowledge that could warrant her confident positions. She imitated Ted's thinking process and enmeshed it with her nascent understanding of the Jacobi symbol. The mirroring process enhanced first, Bettie's position then, her mathematical identity. The confidence that Bettie gained by imitating Ted's positions paved the path for her to voice her nascent number theory identity. Moreover, during the Ted-Bettie micro-activity, Bettie did not actuate any arithmetic identity (PPMI) in which she was confident. She had the courage to maintain a number theory voice throughout the computational activity.

Through the analysis of Bettie's restoration of her arithmetic (PPM) ability, we encounter two processes by which she actuated a number theory identity. The first process is documented in the scholarship (Engle & Conant, 2002; Engle, 2012). It involves keeping the problematization of the mathematical task within the reach of students' resources (Bettie's case on 12/01). The second process, i.e., mirroring, is not documented in the literature of mathematical education. Through the mirroring process, peers can support each other in learning new concepts. Because of the lack of confidence, such as in Bettie's case, students may silence their voices during small-group work. In this case, students would miss learning opportunities, such as building on peer's ideas and having their thoughts refined. By animating the confident positions of a peer, a student who is learning new concepts can start feeling comfortable in sharing and thus receive feedback on his/her nascent understanding of new mathematical concepts.

In summary, confidence can be boosted by providing students with opportunities to (i) exercise their prior knowledge with peers or (ii) identify with peers who have already built confidence. One path of boosting confidence is based on personal knowledge while the other on positive social interactions. The next chapter will continue to investigate Bettie's learning development through the number theory class. It will show how the social process of identification can lead to the generation of new learning habits.

Chapter 5: Bettie's Individualization of New Learning Methods

The previous chapter investigates how Bettie restores her mathematical (PPM) identity and initiates a number theory identity. The current chapter investigates the instillment of new learning methods in Bettie's identities. Although the 10% improvement of Bettie's grades from the midterm (52%) to the final tests (62%) indicates a learning gain, her most esteemed achievement in the number theory class is her change in learning how to learn. The analysis reported in this chapter traces Bettie's learning development back to a multifaceted organization of individual and ecological factors. It reveals an entangled chain of multiple shifts of positions and functions within multiple learning activities leading to the emergence of an *active learning* identity.

The investigation in this chapter follows the retrospective systematic methodology for data selection and the VIP+function analysis, as described in the second chapter. The primary data source used for this endeavor is Bettie's narratives (see appendix A), which were generated by two regular interviews—early in the semester (Int1-0922-Bettie) and toward the end of the semester (Int2-1203-Bettie)—and a mediated interview (SCNI-1015-Bettie). Other data sources, such as videos of Bettie's group (G3) sessions in classroom, her notes during groupwork, her memos after the group sessions, and her groupmates' submitted homework, are consulted for specific investigations. Information on the methods and methodologies can be found in the second chapter.

The introduction continues by instantiating the VIP+function theoretical framework and making it relevant to the investigation of learning methods. Since Bettie's development of her new learning identity takes intricate paths, I provide an overview of her development to support the reading of the analytical branches that occur through the report.

The reporting part of the chapter is divided into three sections. The first section is dedicated to studying Bettie's coping mechanism with face-saving dynamics, which will reveal central in the subsequent studies. The second section investigates the development of Bettie's voice within the learning activities in which she participated over the semester. This investigation studies how Bettie adapted to small-group learning. The third section focuses on the development of Bettie's voice across the learning activities. It studies how the active learning identity was instilled.

The investigation of voices in rapport to learning methods

How can the VIP+function framework be instantiated for investigating the development of *learning methods*? In this chapter, learning is defined in an individualistic way, since the investigation is concerned with Bettie. For the purpose of this chapter, learning is conceived of as the process by which individuals construct their knowledge by interacting with resources, such as textbooks, websites, classmates' notes, family members, friends, groupmates, instructors, tutors, and one's own prior knowledge. The *identities* pertaining to learning methods are semiotic repertoires about knowledge-building and entrenched habits by which people construct their knowledge. A traditionalist identity of learning conceives knowledge as produced by experts and transmitted to students who are not fit to question the produced knowledge. Within such a worldview of knowledge-building, memorization becomes the most befitting learning method for students. Bettie had internalized a traditional identity of learning when she joined the number theory class (Id5, Id7, and Id11 in Table 4-1). Active learning identity holds a democratic perception of knowledge-building resulting in demanding expectations from students: learners can generate knowledge when they understand the workings of concepts. Perfunctory explorations of knowledge resources do not befit active learning identity, which requires learners to understand the materials, check their understanding with peers and experts, and generate knowledge by themselves and in collaboration with peers, among other habits.

In investigating the learning methods, *positions* become ways by which participants resort to a learning resource, such as listening versus talking to groupmates, reading versus copying from a textbook, and going to office hours or asking the instructor questions in the classroom. As for *functions*, they are concerned with the purposes that learners set for their interactions with resources of knowledge. In groupwork, for example, functions pertaining to learning methods can consist of enhancing one's understanding, supporting groupmates to find the solution, rehearsing one's thinking process, and attempting to look smart, among many other idiosyncratic functions that learners may set for themselves. Solitary study can also take several functions. Students may spend time studying alone for a course to get high grades, to look smart when working in group, or to determine the topic of their master's thesis. A *voice* pertaining to learning methods is any actuation of a learning identity that animates a learning position in pursuit of a learning function.

Overview of Bettie's learning development

The development of Bettie's learning voice underwent several shifts, from passive to active learning voices, within and across multiple learning activities. She overcame multiple impediments along her developmental path. In this overview, I list the shifts in learning methods, impediments, and learning activities pertaining to the instillment of an active learning identity.

Bettie reported **three shifts** pertaining to her learning methods throughout this class:

- from "hating" groupwork to finding it extremely beneficial
- from memorizing formulas to understanding the mathematical concepts
- from copying answers present in textbooks and online resources to doing the homework by herself

Bettie's shift in her perception on small-group learning encompassed the other shifts in learning methods. At the end of the exit interview, Bettie emphatically highlighted the shift in her views on collaborative learning when asked if she had anything else to say that was not covered in the interview.

> Bettie: […] my view on groupwork from beginning to now. I really was against it and I thought it was the stupidest thing ever in the beginning. And I thought that. I really hated it. I didn't even like it. I didn't even want to go to class. Because I thought it was just stupid. because . I wasn't um . I wasn't getting taught anything. And I was just like . working with . people I could just work by myself at home. I kinda just thought it was stupid and I hated it. but . um towards the end I felt it was definitely more beneficial than sitting in a lecture. Like . by fa:ar. (Int2-1202-Bettie lines 594–600).

The number theory class was the first to offer Bettie an opportunity to learn through small-group work.

> Bettie: This is the one class that's actually helping me understand. and like come up with things on my own instead of like finding the answer and writing it down (Int2-1202-Bettie lines 38–40).

To achieve the three aforementioned shifts, Bettie had to overcome **three impediments**:

- face-saving dynamics
- difficulty in comprehending the ongoing mathematical conversations in groupwork
- excessive reliance on published resources (e.g. textbooks and webpages)

First, Bettie had to alleviate her concern about her groupmates' perceptions of her mathematical abilities. At the beginning of the semester, Bettie "was too embarrassed to ask for help" from her groupmates because she was "nervous to look stupid."

Second, Bettie reported a difficulty in understanding her teachers' and groupmates' mathematical talks not only during instructional moments but also during groupwork. About teachers, she said, "I can't just like hear words and understand what they're trying to say" (Int1-0922-Bettie lines 34–36). She coped with her difficulty in comprehending the ongoing mathematical conversations, which lessened her benefit from groupwork early in the semester.

Third, Bettie started this class having established an excessive reliance on textbooks and online resources to fulfill her mathematical tasks. Thereby, her habit clashed with the classroom norm, by which students were encouraged to rely on their own knowledge as they solved the worksheet problems. Hoffmann explicitly told the students, "use your brain—not the textbook."

Three learning activities. Throughout her exit interview, Bettie repeatedly reported that the pedagogical nature of the number theory class helped her move away from the method of learning by memorizing to seeking mathematical understanding by reading textbooks, working with classmates inside and outside the classroom, and doing the homework on her own (e.g. Int2-1202-Bettie lines 27–30, 36–40, 96, 250–266). As per Bettie's report, the development of an active learning voice took place within and across three learning activities: solitary study, groupwork in classroom, and studying with a group outside the class time. The following diagram (Figure 5-1) summarizes the shifts of identities, positions, and functions that took place within and across the three learning activities. The shifts indicated in the diagram are discussed at various places in this report. The diagram may serve as a map to help the reader maintain a bird's eye view of Bettie's development while reading through the analytical branches.

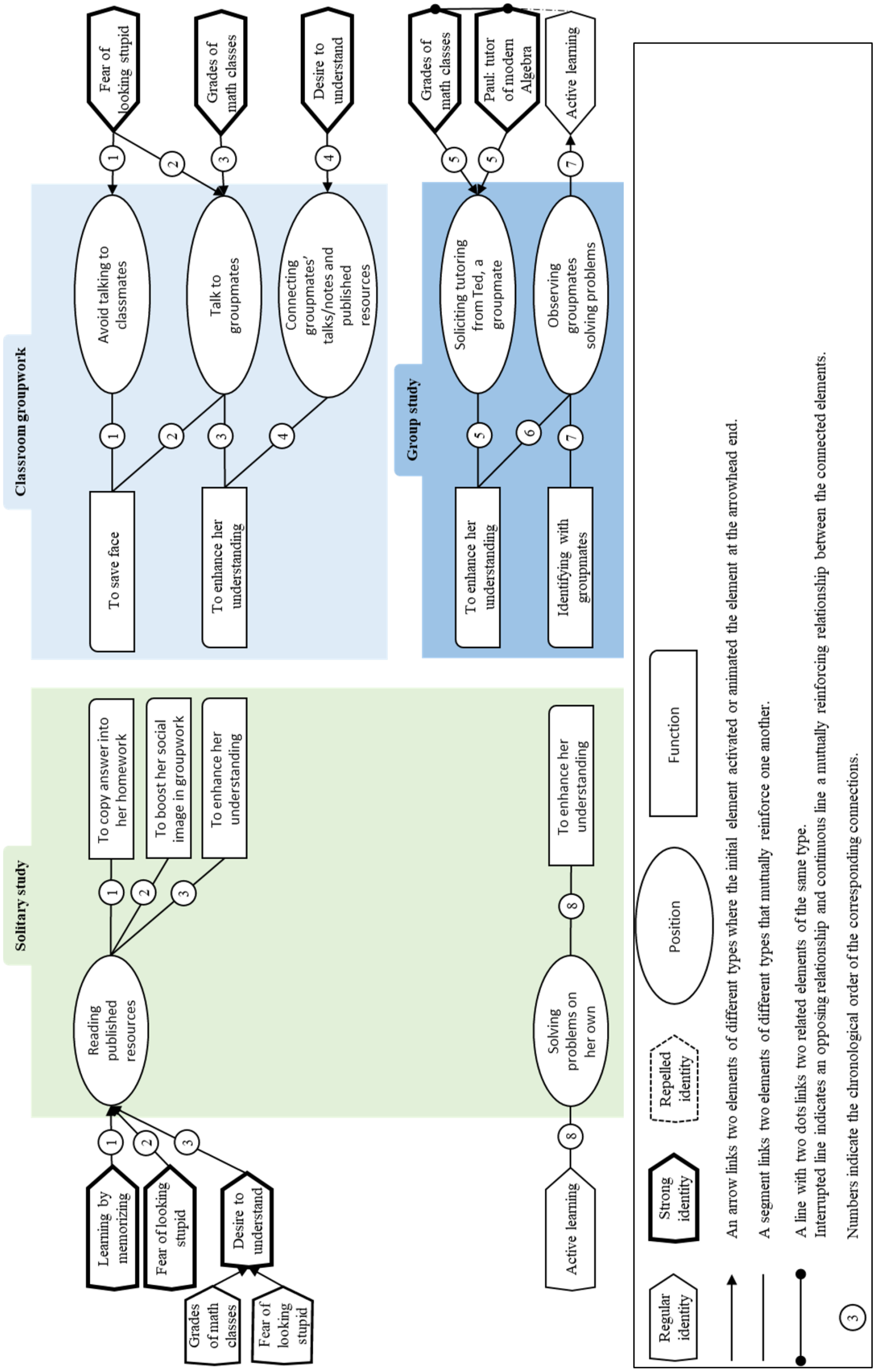

Figure 5-1: Betties' shifts of learning identities, positions and functions throughout her experience in number theory class across three learning activities: solitary study, classroom groupwork, and study group outside of the classroom.

Attenuating face-saving

The first hurdle Bettie faced in small-group learning was her low self-esteem in mathematics (see Po1, Po5, and Po9 in Table 4-2) and her concern about her social image. She overcame this hurdle in two steps, as studied in this section. Bettie first "acted" as a knowledgeable group member to save face. Then, when her position of interacting with her groupmates generated comfort, she shifted the function of her participative position to enhancing her mathematical knowledge.

"In the beginning, I was too embarrassed to ask for help"

Bettie was not used to studying with classmates and avoided talking to them. When asked about her learning experience in the past, Bettie reported that she did not "really talk to people in [her] classes" because "they'd think [she was] stupid or something" (Int1-0922-Bettie lines 214–219). For Bettie, the act of refraining from interacting with instructors and classmates was a face-saving tactic. She avoided situations where she would need to expose her knowledge and, thus, risk "looking stupid" (see Int1-0922-Bettie lines 168, 216, & 219 and Int2-1202-Bettie lines 235).

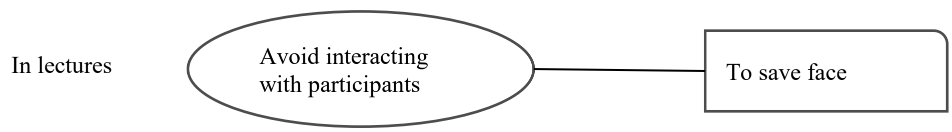

Figure 5-2: Bettie's entrenched position and function association at the start of the number theory class.

Figure 5-2 illustrates the position and function that Bettie used to animate during the classroom lectures. As per her testimony, she used to avoid any interaction with instructors and classmates to hide her self-perceived lack of "smartness."

Bettie defined smartness by the ability to understand and quickly solve mathematical problems (see Id2 in Table 4-1), a quality that she ascribed to her groupmates in group G3 (see the excerpt below). She self defined as having difficulty in comprehending the ongoing mathematical conversations as well as the number theory textbooks. Thus, as per her definition of smartness, she did not qualify as an owner of this quality ("I'm not super smart, it takes me a while to understand things," see Po1 in Table 4-2). Bettie cared about animating a positive image in her mathematical social world. Yet, she ended up not animating voices with people perceived as smart, in order to hide her perceived self-image as being not-smart.

> Bettie: I was like a bit the stupidest person in the world like in my group . everyone's so smart and like everyone knows . everything like they read math for fun […] I've always felt like I'm around like geniuses and I'm like the stupid one. (Int2-1202-Bettie lines 366–376).

"Groupwork forces me to talk to people"

As early in the semester as the first interview (on 9/22), Bettie indicated that groupwork "forced [her] to like ask questions and like talk to people" (int1-0922-Bettie line 236). In her exit interview, she brought back and elaborated on this point:

> Bettie: Because like [groupwork] forces you to talk to people and you don't want to be the one person that doesn't get it, I guess. Because that was me in the beginning

and I felt so dumb and I hated it . I hated groups. But it forced me to talk and I had to try to act like I knew what was going on. I kinda like it kinda pushed me to read and learn. (Int2-1202-Bettie lines 46–49).

During the groupwork, Bettie could no longer hide as she did in the lectures. Participating in the groupwork became her new face-saving tactic. Hiding and remaining silent, while all the other groupmates were chiming in to solve the problem at hand, became a sign of resourcelessness, as per Bettie's worldview. She resolved to play the role of active participation, even though her acting did not reflect a genuine knowledge ("I had to try to act like I knew what was going on"). The purpose of her acting was to hide her lack of knowledge. Based on Bettie's self-report, groupwork forced her to shift her participative position in learning activities, but she maintained the function of face-saving ("you don't want to be the one person that doesn't get it"). In her testimony, Bettie insinuated that what started as fake developed into a learning experience mediated by reading ("it kinda pushed me to read and learn").

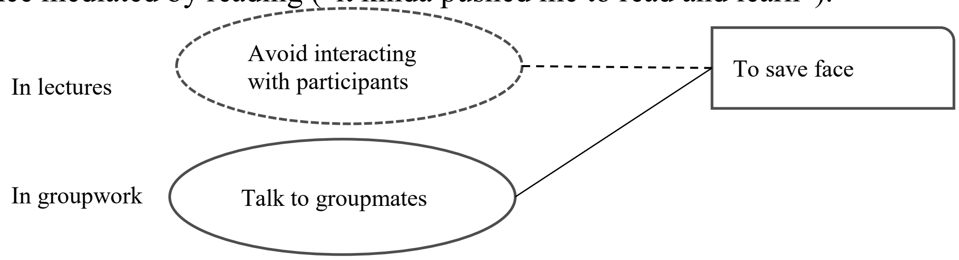

Figure 5-3: Bettie's first position-function shift, related to group participation and face-saving mechanism.

Bettie's position in groupwork, playing a fake role of active learning, might have been fostered by her learning identities that operated on knowledge without understanding (Id5, Id7, and Id11 in Table 4-1):

Id5: A way to memorize something is to repeat, copy, and write it

Id7: Arithmetic is about manipulating formulas that are to be memorized

Id11: Memorize what cannot be understood

Bettie had a special way of drawing upon prior knowledge, which was mediated by memorization with surface-level understanding (read Int2-1202-Bettie lines 269–278). She used to memorize answers of her homework in association with keywords in the questions and, during the test, she would look for keywords and "wing" the answer that she memorized in association with the keywords. Bettie acknowledged that her answers had "probably nothing to do with the questions" on the test but hoped they had "partially the right idea."

<u>Int2-1202-Bettie lines 269–278</u>

Bettie: uh I do the same thing. but I would just find the answer and memorize how I like I would . I would see the type of question and then I would like . I guess I would just remember key words like . like "gcd" I remember. "oh I remember that one problem with gcd now just kinda like wing it and hope" . probably memorize what I wrote down for that answer and then like put it on this answer kinda thing.

Fady: uhu uhu

Bettie: Because like my way of memorizing is like writing. so if I write down for homework I kinda remember how I wrote it out and I'll write it down . for like the quiz. Like the quiz question or anything, and like . it probably has nothing to do with the question, but I'm just hoping like partially I have the right idea.

In groupwork of the number theory class, Bettie might have animated the same learning position that she used to animate in taking tests, that was, memorizing associations without mathematical understanding of the connections. Indeed, in the first videotaped group session (09/03), Bettie was seen proposing a memorized procedure without understanding the current question. As the group was trying to solve the following problem (Wk2#1), Bettie suggested (excerpt below) dividing the initial expression by g on both sides of the equal sign, without realizing that the initial expression was merely a definition of the variable g rather than an equation.

Wk2#1: $Let\ a, b\ \in \mathbb{Z}_{>0}. Show\ that, if\ g = \gcd(a, b)\ then \gcd\left(\frac{a}{g}, \frac{b}{g}\right) = 1.$

Bettie's suggestion: "it sounds like. by looking at it. like "oh" g equals gcd(a,b) so divide by g and then you get 1 [giggles]" (in video of 0903-g3 at 0:26:03).

A minute later (in video of 0903-g3 at 0:27:10), Bettie brought to the group the idea of linear combination, by memorizing an equation. She said, "what was that thing? it was like a x plus b y equal g [$ax + by = g$] [unidentified speech]." Soon after (in video of 0903-g3 at 0:27:36–0:28:10), Bettie acknowledged, during an off-topic conversation with groupmates, that she had copied her answer from the textbook. It seemed that early in the semester, Bettie used to read the textbook to show off knowledge in groupwork. However, her knowledge was based on surface associations with lack of accurate mathematical understanding. In the case of Wk2#1, the principle of dividing the two sides of an equation by g was applied to the linear combination $ax + by = g$ rather than the definition of the variable g, as Bettie suggested (at 0:26:03).

"Now I don't care"

In the exit interview (Int2-1202-Bettie), Bettie acknowledged having overcome her early embarrassment about looking "stupid" to her groupmates in three instances: when she (i) spoke about submitting incomplete homework (lines 136-144), (ii) evaluated her learning in group (lines 232-242) and (iii) recalled her experience in modern algebra class with a "smarter" friend (lines 436-446).

<u>Int2-1202-Bettie lines 136–144</u> -- *Submitting incomplete homework*

Fady: Did it happen that you submitted incomplete homeworks or you kept some exercises blank?

Bettie: Yeah in the beginning I did all the time.

Fady: In the beginning of the? Why?

Bettie: Um just cause I was too embarrassed to ask for help. And uh I was like trying. I was just being lazy. I wasn't taking the homework serious. I was just doing it the day before or the day of and try to get it finished. And the ones that were super hard and I was stuck on I just skipped and go to the next one. And then like once I started seeing how low my homework scores were, uhhh like "oh shoot, I really need to step it up."

<u>Int2-1202-Bettie lines 232–242</u> -- *Assessing learning in small-group*

Fady: […] how would you assess your learning in this group? About this uh. this uh class.

Bettie: um I like it. I . I'm . In the beginning I really hated it, I thought it was like pretty stupid just because . I was uh I was nervous to like . look stupid . and I didn't want to ask questions like to my group. […] But after . um . I got over that uh . self-conscious.

Fady: uh what did help you to do that?

Bettie: um just talking to them more. talking to the group more. So I started to feel more comfortable and I was just "now I don't care."

Int2-1202-Bettie lines 436–446 -- *A classmate friend in modern algebra class*

Bettie: […] I had a friend in Modern . and she was just so smart too and that was my only . like that was the first friend I've ever met in any math class. and she like understood everything. and she like spoke . like . I don't know how to explain it. I just felt like intimidated . kinda . Like she's smarter than me. or better than me. so I would like . I would pretend to understand things and like . pretend like I was like at that level, but I wasn't. I was just like go with it. Like "oh I don't know what you're doing, but okay" "yeahhh. It's right." [laughs] kind of thing. That's how I felt like in the beginning with the guys [my groupmates]. because I was just like oh these guys are just like so smart. but then now I'm like I don't care . like "how do you do that" . "what are you doing?" [giggles] "Teach me" . "Go slow" [giggles] and they're like "okay" . like "whatever".

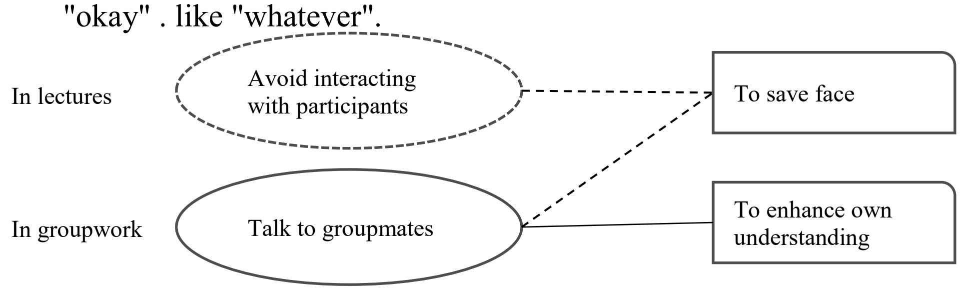

Figure 5-4: Bettie's second position-function shift related to group participation and face-saving mechanism.

In each one of the three aforementioned excerpts, Bettie reported a shift of dispositions ("shoot, I really need to step it up," "but after um I got over that uh. self-conscious," and "but then now I'm like I don't care"). The disposition she renounced was filled with embarrassment ("I was too embarrassed to ask for help"), laziness ("I was just being lazy. I wasn't taking the homework serious"), nervousness ("I was nervous to like. look stupid. and I didn't want to ask questions like to my group"), intimidation ("I just felt like intimidated. kinda. Like she's smarter than me […] That's how I felt like in the beginning with the guys [my groupmates]"), and fake participation ("I would pretend to understand things and like. pretend like I was like at that level, but I wasn't"). Bettie was trapped within the face-saving mechanisms and laziness. In the new disposition she undertook, the face-saving mechanisms were backgrounded: "now I don't care" (repeated in two excerpts).

Bettie renounced the fake participation and adopted a genuine one, by which she had the courage to ask her groupmates for help ("how do you do that?" . "what are you doing?" [giggles] "Teach me" . "Go slow" [giggles]). Indeed, by the time a group session was held on 9/17, Bettie started to frequently animate positions whereby she could solicit knowledge from her groupmates (see Figure 5-5). In the first videotaped group session (9/03), Bettie enacted contributive rather than soliciting positions slightly more often. However, by the second videotaped group session (9/17), this trend reversed. For almost all the times in the rest of the semester, the solicitation of explanations predominated, sometimes reaching twice and thrice as many instances as compared to the contributive voices. Note that the relative frequency of contributive voices after 10/06 fluctuated between 20 and 30% of Bettie's total voices in the group session, except on 12/01, where her contributive voices reached about 42%. Recall that Bettie reaffirmed her arithmetic (PPM) skills in this session (see Chapter 4, page 121).

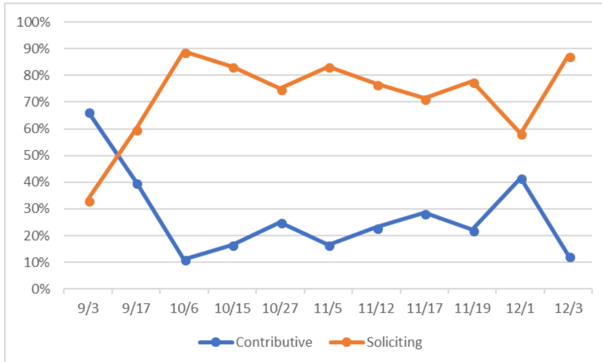

Figure 5-5: The graph represents the relative frequencies of Bettie's contributive versus soliciting positions, as animated during eleven group sessions in the number theory class. The contributive positions included offering an explanation, contributing a mathematical idea, and assessing a groupmate's idea. The soliciting positions included the act of asking groupmates to explain to her and assess her mathematical ideas.

What bolstered Bettie's shift in functions from *saving face* to *enhancing her understanding*? In the narratives where she noted this shift of functions (Int2-1202-Bettie lines 136-144, 232-242, and 436-446; excerpts reproduced above), Bettie presented two factors that produced this shift: the gained familiarity with her groupmates (Int2-1202-Bettie lines 232–242) and low scores in homework (lines 136–144).

First, Bettie became familiar with her groupmates by "talking to the group more." Through her fake participation during the early group sessions, Bettie built a familiarity with the group's culture that helped her renounce the face-saving demand. She expressed her comfort in the possibility of asking questions to her groupmates as soon as when she submitted her first homework (on 09/03). Hoffmann had asked students to write a short text describing their feelings about their groups in their first homework (see Bettie's response in Figure 5-6). In the exit interview, Bettie highlighted that her groupmates were "approachable," unlike two other classmates (Int2-1202-Bettie lines 550–562). She noted that she would feel uncomfortable working with the two other classmates who, according to her perspective, turned their groupwork into a combat about who was the smartest. The group culture, which G3 constructed through the early sessions, seemed to have induced comfort in Bettie to legitimize her lack of knowledge and start soliciting knowledge from her groupmates.

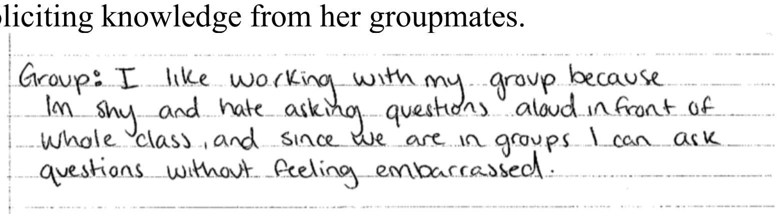
Group: I like working with my group because Im shy and hate asking questions aloud in front of whole class, and since we are in groups I can ask questions without feeling embarrassed.

Figure 5-6: Copy of Bettie's comment on her groupwork in her first homework (Hw1).

Table 5-1: Bettie's scores on the submitted homework, sorted by the due dates.

Hw	Hw1	Hw2	Hw3	Hw4	Hw5	Hw6	Hw7	Hw8	Hw9	Hw10	Hw11	Hw12	Hw13
due date	09/03	09/10	09/17	09/24	10/01	10/08	10/15	10/22	11/05	11/12	11/19	12/03	12/10
Grade	87%	52%	-	84%	36%	68%	70%	79%	80%	93%	94%	71%	94%

Second, the low scores in Bettie's early homework (see Table 5-1) also urged her to abandon the lackadaisical attitude and take up a serious learning disposition. The low scores could have affected Bettie's behavior only because she valued the high grades (see Id6 in Table 4-1). Melissa, a classmate who struggled like Bettie with face-saving dynamics and low scores in homework, never shifted to productive learning dispositions in this class. A previous work (El

Chidiac et al., in press) compared Bettie's and Melissa's learning trajectories in this class. The study showed that grades had a concerning effect on Bettie but not Melissa because of their different individualized identities. Bettie thought of herself as "really good at math and sucks at everything else" (Int1-0922-Bettie line 49–50), a status that she measured by her grades ("I was getting As obviously I was good at it" Int1-0922-Bettie line 120). Low grades would threaten the unique identity that Bettie performed well. On the contrary, Melissa used to tolerate low grades throughout her college experience. In fact, Melissa's predominant identity was attached to dancing; she chose to major in mathematics with focus on teaching only to be able to afford a long-term living. She planned to teach at elementary schools, a career that did not require the advanced mathematics offered at college.

Although both factors, familiarity with groupmates and the low scores on homework, attenuated an unproductive coping mechanism of face-saving, the former started to take root earlier than the latter. Evidence, mainly in Bettie's homework on 09/03, traced Bettie's comfort with the group G3 to the second week of class (on 09/03). However, the low scores started to appear on 9/15, when Hoffmann returned the second homework, the first one on which Bettie received a low score (52%). This observation was important to explain how Bettie could approach a groupmate, Ted, asking him to help her with her homework outside the classroom, after she realized her low scores on two homework assignments. Had Bettie remained within the face-saving mechanism, she would have not approached a groupmate to tutor her. She would have found support in the tutoring service provided by the department freely. While Bettie felt impelled to take a serious learning disposition because of her low scores on the weekly homework, she approached a groupmate for help because of the non-competitive ecology that G3 constructed and that made Bettie feel comfortable showing her lack of relevant mathematical knowledge.

Learning to learn from groupmates: Bettie's voice within learning activities

Bettie reported having trouble comprehending her instructors' or groupmates' mathematical discussions, a problem which she endured throughout her college experience. As such, she could benefit neither from classroom lectures nor from groupwork. Not being able to learn from others, she fostered the method of learning through textbooks and other resources on the internet (either YouTube videos or mathematics-related websites), which she could re-read and re-watch at her own pace and will. Consider the following typical excerpts, where Bettie talked about her solitary work.

Int1-0922-Bettie

Bettie: […] I feel like I'm not super um I don't know I guess smart so it takes me a while to understand things I have to see it done a couple times and like I have to do it a couple times to like completely fully understand it. I can't just like hear words and understand what they're trying to say, like I have to see and a lot of times you don't get teachers to do that so you just have to so it's hard like you have to find the way of learning yourself. (lines 32–37)

[…]

Bettie: […] I feel like read. comprehension for me is really hard. so having to read something and like fully understand it . is just like . puhhh. so like these classes have been really hard for me. but . like I'm barely sliding by. like just trying to get them over with. just cause like I can't really read a paragraph and understand what it's talking about I need like work I need like to see why. like what it means. I don't I don't. it's kind of hard in this class . just cause I feel like he's just like do it. so like we'll do it . like as a group . but still like I don't know if I'm doing it right so then I'm just. (lines 131–137)

[…]

Fady: Do you work with others? What do you do when you're stuck?

Bettie: Uh I go online

Fady: Okay

Bettie: But yeah I work alone, unless like uh . I'll like ask in class or something "oh how did you do this one" and then I'll see like how they did it. but most of the time I don't really even take cause I just like . to read over because I like to understand things cause like it's really frustrating when I'm just like copying work I have to . really just like . understand what I'm doing and why I'm doing it so . I kind of just like to work alone because it takes me . a pretty long time to figure out. (lines 183–191)

Bettie mended her failing comprehension of the fleeting mathematical conversations by participating in three types of activities: solitary study, groupwork in the classroom, and the study group outside classroom. I will investigate the shifts in Bettie's positions and functions animated in each one of these activities and across them. Regrading her positions, I will consider Bettie's ways of using the knowledge resources. Bettie mentioned three types of knowledge resources, namely published resources (hard copies and electronic texts and videos, available on her tablet as well as the internet), her groupmates' talks, and their written products (notes and homework).

Solitary work

In her exit interview, Bettie admitted to copying answers for the homework problems, from the textbook and resources on the internet.

<u>Int2-1202-Bettie lines 260–266</u>

Bettie: before [early in the semester] I would just like do the homework. but I wouldn't really . like know what I was just . because I didn't understand what the question was asking and I just didn't . know learn anything in my proofs class so it was just kinda bullshitted through that . and I was just like copying and pasting . finding answer online . and writing it out and like hoping it was the right answer.

An investigation of Bettie's homework corroborated the fact that she was copying her homework mainly from the textbooks (see appendix C and recall discussion in Chapter 4 around Figure 4-2). Indeed, she was attempting to couch the textbooks' answers in her own words to hide the copying and abide by the instructor's norm, "write the homework in your own words." However, Bettie's reformulations of the textbooks' answers exhibited lack of accurate understanding of the mathematical concepts in those answers. Based on Bettie's self-report (Int2-1202-Bettie 260–266) and the analysis of her homework (appendix C), we could claim that, early in the semester, Bettie was copying her homework from answers in published resources without accurate understanding of the mathematics involved therein. This position of copying answers without understanding was a habit that Bettie used to apply in tests: she used to memorize answers and "wing" them in hopes of hitting the right answer (Int2-1201-Bettie lines 269–279).

At the beginning of the number theory class, Bettie seemed to have actuated her memorizing identity and the habit of putting random answers as she completed her homework.

Later, Bettie changed the function of her position in doing the homework, as per her testimony (Int2-1202-Bettie lines 291–305). While she continued to use the textbook, she started to strive to understand what she was reading. In fact, she started reading the textbook from the beginning to build an accurate mathematical understanding of the concepts covered in the previous lessons.

<u>Int2-1202-Bettie lines 291–305</u>

Fady: did you experience any change in your ease of understanding the materials throughout the semester?

Bettie: oh yeah.

Fady: Did it become more complex or did it become easier? Or.

Bettie: It's becoming easier. I feel like I kinda . cause before [early in the semester]. I wasn't . I wasn't even really doing homework before, so when we move on to the next like worksheet, I would just like oh like fuck maybe I should've done the last worksheet. and then I'm just like I don't know what I'm doing ever. so then I kinda just like . I like started from the beginning of the book and literally read through the whole entire first like half of the book. like this is in the beginning. Then I'm like "wow" why didn't I get that? "you good stupid." And then now . like cause it takes a lot . now I don't know if I feel like it's based off of what we've already learned just like kinda doing things a little different.

Fady: uh

Bettie: And then I'm like "Oh. Well it makes sense." So now I'm understanding it faster not that it's like easier, it's just that . um . I'm able to . figure it out faster.

Bettie's self-report hence highlights the emergence of a new functionality of reading the textbooks, namely to enhance her mathematical understanding. The initial functionality, i.e. to copy for the homework, persisted but assumedly with the new functionality, i.e. to build mathematical understanding. Figure 5-7 represents the functionality shift of the reading-published-work position.

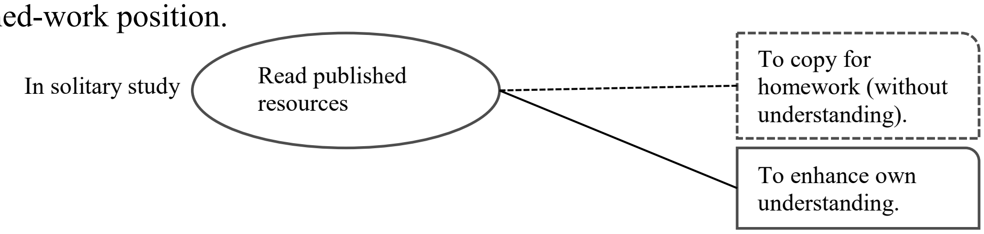

Figure 5-7: Shift in the function relating to the use of published resources during Bettie's solitary study. This shift occurred at an unidentified date.

In her narrative about the new functionality of reading the textbooks, Bettie noted that the textbook was her recourse to be able to do the homework and understand the mathematical concepts, so that she could participate meaningfully in groupwork ("I wasn't even really doing homework before, so *when we move on to the next like worksheet,* I would just like oh like fuck maybe I should've done the last worksheet. and then I'm just like I don't know what I'm doing ever"). Recall that Bettie maintained a desire to understand mathematical concepts, despite the overwhelming memorization habit (see Id4 and Id11 in Table 4-1 and analysis of Bettie's individualized learning identity in chapter 4, page 91). While Bettie's narrative foregrounds her desire to understand the mathematics discussed in groupwork as the main reason for her resorting to reading the textbook, it does not explain why this desire was rekindled only later in the semester. She acknowledged that her desire to understand the mathematical concepts was not actuated when she was doing her homework early in the semester.

Based on the previous investigation of face-saving (Figure 5-4), we could identify two identities that might have spurred Bettie's desire to understand mathematical concepts. Bettie's concern about understanding was not actuated during solitary study, but was felt as an urge in groupwork ("when **we** move on to the next worksheet"). Bettie's search for mathematical understanding might be underpinned by her fear of "looking stupid" when working with her groupmates ("you don't want to be the one person that doesn't get it" Int2-1202-Bettie line 46). The other situation that might have rekindled Bettie's search for mathematical understanding could be the low scores on her homework ("I was just being lazy. I wasn't taking the homework serious […] once I started seeing how low my homework scores were, uhhh like "oh shoot, I really need to step it up"" Int2-1202-Bettie lines 136–144).

Groupwork in the classroom

The availability of several data sources, videos of group sessions, interviews and copies of Bettie's notebook and homework, makes the investigation of Bettie's behavior in classroom's groupwork rich. I will first report findings from Bettie's interviews then test them with data sources from the group session on 10/15 of G3.

Bettie's reports in the interviews indicate a heavy reliance on notetaking by drawing from textbooks and group conversations. I select the group session on 10/15 because this was the only group session that was followed by an SCNI interview, during which pictures of Bettie's notebook were taken. The three data sources, i.e., the video of group session, the SCNI interview, and the pictures of Bettie's notebook, afforded the two relevant studies reported hereafter. Study A closely analyzed two moments of Bettie's copying process from groupmates' notes and her textbook, during the 10/15 group session. Study B analyzed the moment-by-moment development of her notes through the session, which allowed the evaluation of her mathematical learning. Before reporting the findings of Study A and Study B, I report the findings based on Bettie's interviews then provide an overview of the group session on 10/15.

Bettie's narratives on groupwork

Once Bettie renounced her concern about face-saving (see Figure 5-4), she seriously started investing in learning methods during groupwork, as noted in previous analyses. Yet, she had to cope with her entrenched visual learning identity (Id3 in Table 4-1) and resolve her difficulty in understanding the ongoing mathematical conversations (Po9 in Table 4-2).

During groupwork, Bettie almost always put a hard copy of Andrews textbook and her tablet, which contained electronic versions of Stein and Andrew's textbooks, in front of her. She consulted and perused them frequently. Along with her groupmates' notes, these published resources offered Bettie mathematical visualizations to aid her understanding of the ongoing conversations. As such, the textbooks served also as resources to aid in making sense of the mathematical concepts. Bettie concisely and accurately described her group participation in the following excerpts.

<u>SCNI-1015-Bettie lines 112–115</u>

Bettie: That's basically all I do. like when I'm in class. I just listen to what they're saying and look at the book . cause if I don't understand it then . when they're like talking .

I don't know. so I just zone people out . until I look at it myself because . otherwise it just confuses me more.

Int2-1202-Bettie lines 109–115

Bettie: […] when . [my groupmates] usually seem to get it. they just like go through it. and I feel I feel like annoying or stupid trying to ask them for help all the time, so I just look on my own and then I figure it out kinda like a little bit by myself and then . Then I look to them and they like help me and . or I'll just look at their work and see how they're doing it and then put two and two together. but I like to learn it on my own because I don't like having to ask questions and then I don't like people getting irritated with me.

In the aforementioned excerpts, Bettie highlighted three positions, i.e., listening, soliciting explanations, and looking at written texts. She had recourse to four types of knowledge resources: group talk, groupmates, groupmates' notes, and published resources. As per her reports, she highlighted four participative positions that she animated in classroom's groupwork in order to enhance her understanding of the mathematical materials (see Figure 5-8). Bettie alternated between the four participative positions depending on the ecological affordances as well as the constraints. She used published resources when her groupmates were not available to explain to her or when their conversations did not make sense to her.

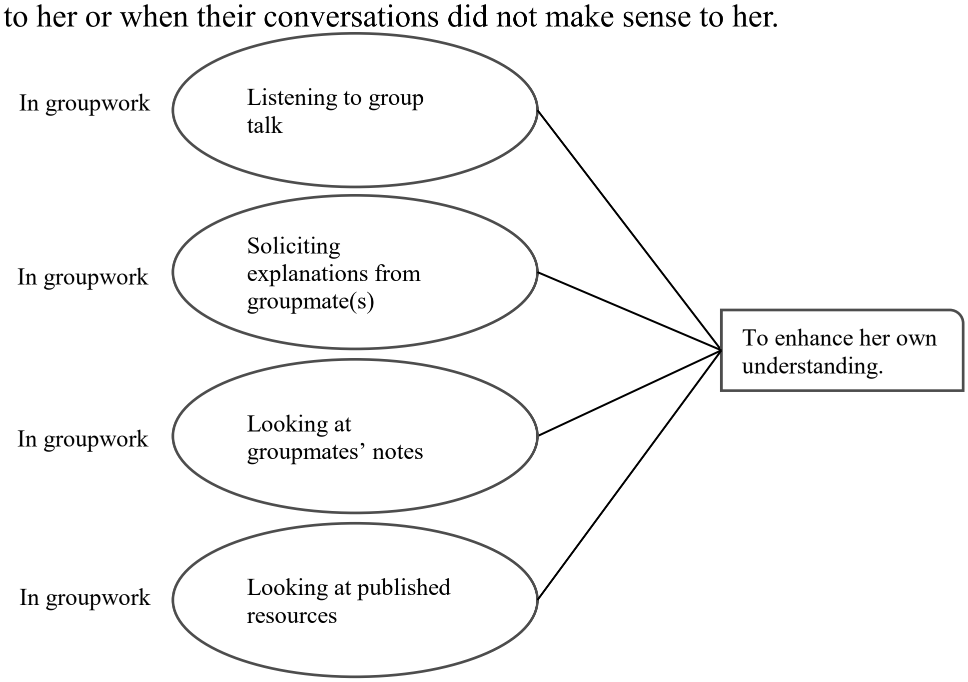

Figure 5-8: Bettie's participation forms, as highlighted in her reports, in the classroom's groupwork after she attenuated her concern with face-saving.

Bettie built her mathematical understanding within the groupwork by connecting the knowledge she encountered in the resources that she tapped upon. She emphasized the connections that she made during the group sessions in her interviews ("put two and two together" Int2-1202-Bettie line 109–113). The connecting work also transpired through her interactions in the group sessions, which were the subject of a close study (see Study B given below). Because Bettie was simultaneously attentive to group talk and knowledge in published resources during groupwork, she could occasionally support her groupmates' works by pointing to relevant resources within the textbooks (see Study B for a detailed analysis of Bettie's report, in the following excerpt). Thus, Bettie occasionally shifted the function of her connecting work, during the group sessions, from enhancing her knowledge to supporting her groupmates' endeavors (depicted in Figure 5-9). She reported this aspect in the exit interview (see the excerpt below). In fact, in the group session on 10/15, Bettie contributed three times by drawing information available in textbooks (see Table 5-2, below, at [0:24:30–0:33:07], [1:01:07–1:02:50] and [1:13:40–1:18:04]).

> Bettie: […] my contribution [to groupwork] would be finding the:e answers in the book. [laughs] Like the definitions in the book and like . try to showing them [i.e. groupmates] "Look I found it right here you can read it" and . that's like the most I did for the group. (Int2-1202-Bettie lines 461–464)

In addition to the four participative positions highlighted in her narratives (see Figure 5-8), Bettie continually took down notes during the groupwork. Although she inadvertently mentioned her notetaking process in her interviews, it actually epitomized her learning processes in groupwork. First, her notes materialized the connections between knowledge that was recruited from various resources and further supported such connections (see Study B below, depicted in Figure 5-20). Second, the way Bettie copied from her groupmates' notes and textbooks into her notebook comprised an identification process (see Study A below).

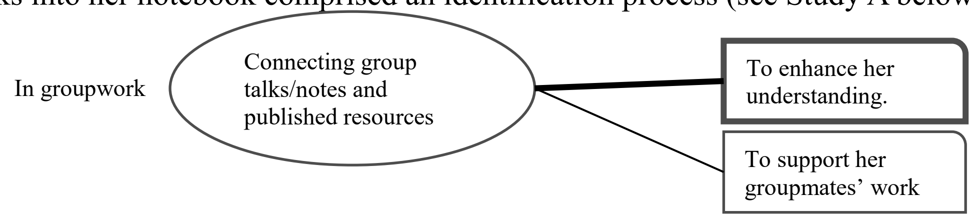

Figure 5-9: Functional shifts describing Bettie's participation in groupwork. The occasional shift from building her own understanding to supporting her groupmates took place due to the connection between various sources of knowledge, resulting from the shifts of participative positions (see Figure 5-8).

Group session of G3 on 10/15: An overview.

Students started working on Wk7#1 at the start of class on 10/15 and were also tasked with working on #2 and #3 (see the worksheet in Figure 5-10). Hoffmann had introduced the Möbius function in the previous class (on 10/13), which Bettie had not attended. Thus, she spent a long time, at the beginning of the group session on 10/15, trying to understand the definition of the Möbius function. At some moments, she interacted with her groupmates (see her comments in SCNI-1015-Bettie lines 23, 47–54, 56, 109, 148–152, 156–157) and at other moments, she worked on her own, with the aid of textbooks (SCNI-1015-Bettie lines 37–44, 60–62, 103–108, 132–134).

The group interactions are thoroughly described in Table 5-2, followed by pictures of the textbooks' content as well as the group materials which the students significantly used.

Worksheet 7: The Möbius Function

1. Show that $\mu(n)$ is multiplicative.
2. Prove that
$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$
Hint: for $n > 1$, try induction on the number of prime factors of n.
3. Prove the *Möbius Inversion Formula*:
$$f(n) = \sum_{d|n} g(d) \qquad \text{if and only if} \qquad g(n) = \sum_{d|n} \mu(d) f(\tfrac{n}{d}).$$
Hint: write sums like the one on the right-hand side as
$$\sum_{d|n} \mu(d) f(\tfrac{n}{d}) = \sum_{de=n} \mu(d) f(e).$$
4. Andrews 6.4.1, 6.4.3, 6.4.7, 6.4.8, 6.4.11.
5. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Figure 5-10: Worksheet 7 of the number theory class.

Table 5-2: Descriptions of the group interactions in G3, on 10/15.

Time stamp in 1015-g3 video	Description of episodes
0:08:31 – 0:09:45	Bettie copies the definition of the Möbius functions from John's notes while John, Ted, and Jeremy settle down.
0:09:45 – 0:14:49	John, Ted, and Jeremy are working individually, with occasional group conversations on how to prove the multiplicativity of the Möbius function (Wk7#1). Meanwhile, Bettie is alternating between flipping her textbook, scrolling on her tablet, attending to the group talk, and writing on her notebook.
0:14:49 – 0:18:00	Hoffmann visits the group. John presents his thoughts about #1. Immediately, Hoffmann reacts by clarifying about how the Möbius function works. Then (at 0:17:36), Ted presents his strategy, by cases, about proving #1. Hoffmann approves the same. Bettie alternates between the previously mentioned actions. Boutros joins the group (at 0:17:45).
0:18:00 – 0:21:40	John explains to Bettie how the Möbius function works, at her request, while Ted is writing the full proof of #1. Afterwards (at 0:19:46), Ted asks John about the presumed condition of relative primality for the multiplicative arithmetic functions. Then (at 0:20:59), Ted presents the complete proof of #1 to the group by pointing to his work, on a single sheet, at the center of the table (not captured by the camera).
0:21:40 – 0:24:30	While Bettie reads Ted's proof of #1 and then writes it down on her notebook, John, Ted, and Jeremy move to tackle #2 by considering the case where n is a prime ($n = p$). They work individually, with intermittent collective discussions. They discuss the value of the Möbius of a prime number (first at 0:24:05–0:24:34), when Bettie is attending to the group talk and takes note of the same (at 0:24:05). Jeremy corrects the agreement about the Möbius of a prime and both Ted and John approve $\mu(p) = -1$ (at 0:25:16–0:25:47). However, Bettie seems distracted and does not take note of the new collective understanding.
0:24:30 – 0:28:40	John and Ted share their strategies for #2. Ted works on the case of the powers of primes ($n = p^{\alpha}$) (see Figure 5-14). Their discussion leads to a debate on whether they need to use the induction, hinted in #2 (see Figure 5-10). Ted's strategy does not use the induction. TA stands near the group and participates in the discussion (at 0:26:40–0:28:40).
0:28:40 – 0:33:07	Bettie supports Ted's method by showing the textbook's proof on her tablet (at 0:28:42). She hands him the tablet and starts writing on her notebook. Ted, Jeremy, and John take the tablet, read, and discuss the proof. Then, they work individually. John holds the tablet for a while and then returns it to Bettie (at 0:31:27). Ted starts testing an idea by thinking aloud and taking both John's and Jeremy's inputs (at 0:31:40).
0:33:07 – 0:43:10	Hoffmann sits with the group (at 0:33:07–0:35:50), first listens to the conversation and then objects to Ted's "base case." Ted learns that the case $n = p$ is a part of $n = p^{\alpha}$. Bettie takes note (at 0:35:40–0:36:36) and then turns the page of her notebook. Ted and Jeremy discuss how the induction method applies to #2 (0:35:50–0:36:54). Bettie interjects with a clarifying question about Ted's notes and Ted explains them to her (at 0:36:56–0:38:03). However, this leads to Ted

	become confused. Bettie consults her tablet and then starts writing on her notebook (see Figure 5-19) by looking at the tablet (at 0:38:52–0:42:43). Then, she looks at the definition of the Möbius function in John's worksheet and writes again on her notebook, until she is interrupted by the instructor. Meanwhile, John asks a question about the symbol conventions, which leads to a discussion about arithmetic functions (at 0:38:10–0:40:30). Then, Ted, John, and Jeremy engage in writing and discussing a formal proof of #2 using the induction method (at 0:40:30–0:42:02). Individual work follows.
0:43:10 – 0:47:50	The instructor calls the attention of the whole class to talk about the midterm and invites the group to talk about the groupmates' behaviors that hinder the learning process. John requests to use Bettie's tablet (at 0:47:23).
0:46:27 – 0:48:48	Bettie and John talk about midterms in other classes. The other group members are working individually, in silence.
0:48:48 – 0:54:21	John reads in Bettie's tablet. Bettie takes Ted's sheet, reads his proof by induction (see Figure 5-15), and then returns it to the center of the table (at 0:49:11–0:49:40). John and Ted discuss their works (at 0:50:07–0:54:57). Meanwhile, Bettie takes her tablet, scrolls, and reads through it (at 0:50:22), attends to John's and Ted's conversation (at 0:51:05), turns back to read in her tablet (at 0:51:42), and then starts writing on her notebook by looking at her tablet and occasionally looking at Ted's notes (at 0:52:51–0:57:05).
0:54:21 – 1:01:07	Hoffmann sits with the group. Ted presents his proof by induction and discusses it with the instructor, who approves it (at 0:54:21–0:56:50). Jeremy, John, and Boutros are attending to the discussion, while Bettie is still focused on reading in her tablet and writing on her notebook. Then they move to talk about #3 (at 0:57:07–1:01:07). Bettie attends to this conversation and writes in her notebook, in three moments (at 0:58:22, 0:59:28, and 1:00:58).
1:01:07 – 1:02:50	Bettie asks John to check her computation of $\mu(3)$, who corrects it (at 1:01:10–1:02:06). Then (at 1:02:06–1:02:50), she shares a formula (see box in Figure 5-19) that she found in the ebook and Ted connects it to his proof.
1:02:50 – 1:13:40	Members work on #3 individually. Occasionally, Ted, John, and Jeremy engage in conversations. Bettie focuses on reading in her tablet and writing on her notebook. She takes a minute to write the homework assignment by looking at the classroom board (at 1:08:25–1:09:14).
1:13:40 – 1:18:04	The group session gradually fades into off-task conversations and ends. Meanwhile, Bettie has the chance to share her understanding of the proof of #3, in the ebook, with Jeremy (at 1:16:16–1:17:39).

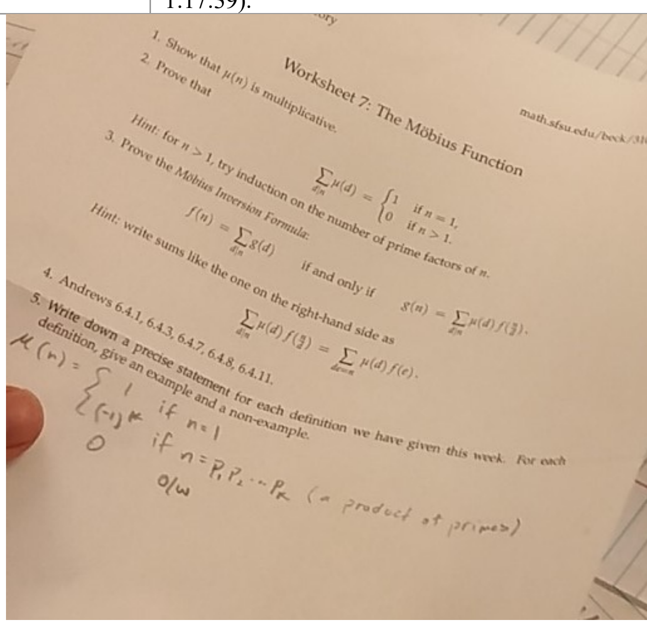

Figure 5-11: Definition of the Möbius function on John's worksheet, which Bettie copied in her notebook. Picture taken at 1015-g3 at 0:37:44.

We now define the Möbius function $\mu(n)$, which will be useful in the next theorem.

DEFINITION 6–1:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^2 \mid n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1 p_2 \ldots p_r \text{ where the } p_i \text{ are distinct primes.} \end{cases}$$

Example 6–3: $\mu(2) = -1$, $\mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$, and $\mu(6) = 1$.

Figure 5-12: The definition of the Möbius function in Andrew's textbook, Definition 6-1 on page 77.

6–4 THE MÖBIUS INVERSION FORMULA

In Section 6–2, we saw two similar formulae related to $\phi(n)$, namely

$$n = \sum_{d \mid n} \phi(d) \quad \text{and} \quad \phi(n) = \sum_{d \mid n} \mu(d) \frac{n}{d}.$$

These two formulae represent a special case of a general theorem on the Möbius function. However, before we can prove this general theorem, we must prove a special result about the Möbius function.

THEOREM 6–5: $\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$

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PROOF: The assertion is clearly true if $n = 1$. We proceed by mathematical induction on the number of different prime factors of n when $n > 1$.

First, if $n = p^\alpha$, then

$$\begin{aligned} \sum_{d \mid n} \mu(d) &= \mu(1) + \mu(p) + \mu(p^2) + \ldots + \mu(p^\alpha) \\ &= 1 - 1 + 0 + \ldots + 0 \\ &= 0. \end{aligned}$$

Suppose the theorem is true for integers with at most k prime factors. Assuming that $n = n'p^\alpha$, where n' has k distinct prime factors and p is a prime that does not divide n', we have the equation

$$\begin{aligned} \sum_{d \mid n} \mu(d) &= \sum_{d \mid n'} \mu(d) + \sum_{d \mid n'} \mu(pd) + \sum_{d \mid n'} \mu(p^2 d) + \ldots + \sum_{d \mid n'} \mu(p^\alpha d) \\ &= \sum_{d \mid n'} \mu(d) - \sum_{d \mid n'} \mu(d) + 0 + \ldots + 0 \\ &= 0. \quad \blacksquare \end{aligned}$$

Figure 5-13: The proof of Wk7#2 in Andrew's textbook, Theorem 6-5 on pages 86–87.

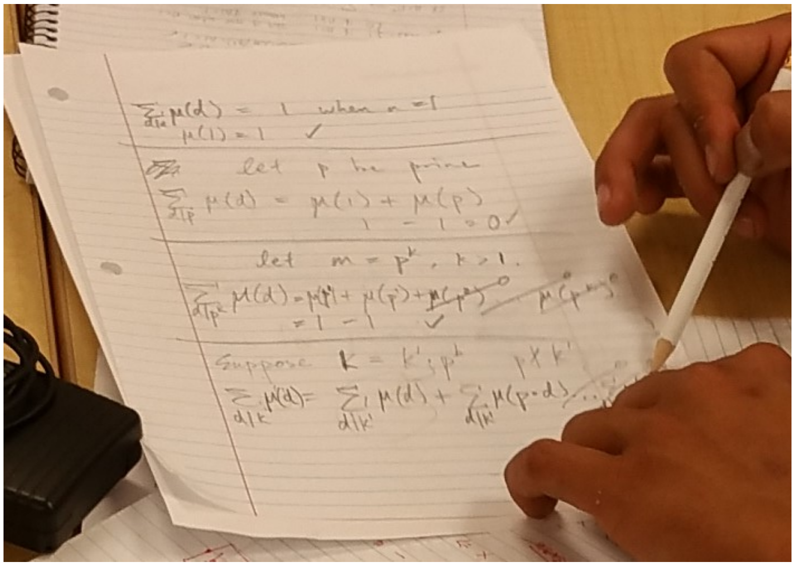


Figure 5-14: Ted's work on Wk7#2. Picture taken in 1015-g3, at 0:37:39.

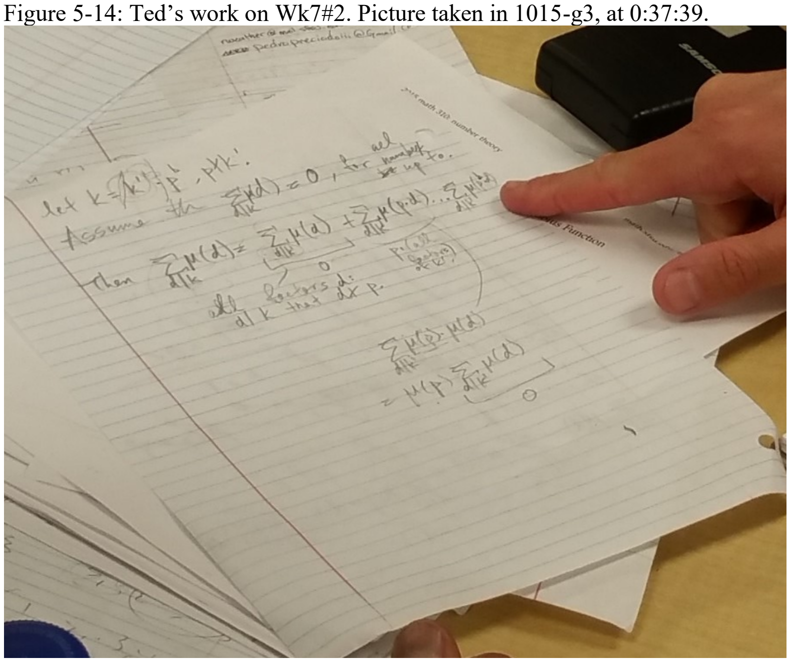


Figure 5-15: Ted's work on Wk7#2 using the induction method. Picture taken in 1015-g3 at 0:56:01, when Ted was presenting his work to Hoffmann.

6–1 COMBINATORIAL STUDY OF $\phi(n)$ 79

multiples of p. Hence,

$$\phi(n) = p^{\alpha}\phi(n') - p^{\alpha-1}\phi(n'). \tag{6-1-1}$$

For example, let $n = 30 = 2 \cdot 3 \cdot 5$ and $n' = 6 = 2 \cdot 3$; below, we have underlined the $5 \cdot \phi(6)$ numbers in $\{1,2,\ldots,30\}$ that are relatively prime to 6.

$\underline{1}$ 2 3 4 $\underline{5}$ 6 $\underline{7}$ 8 9 10 $\underline{11}$ 12 $\underline{13}$ 14 15 16 $\underline{17}$ 18

$\underline{19}$ 20 21 22 $\underline{23}$ 24 $\underline{25}$ 26 27 28 $\underline{29}$ 30

To obtain the $5^{1-1}\phi(6) = \phi(6)$ numbers in $\{1,2,\ldots,30\}$ that are relatively prime to 6 but not to 5, we take all of the numbers less than or equal to $30/5 = 6$ that are relatively prime to 6, and multiply each by 5. Thus we obtain $5 \cdot 1 = 5$ and $5 \cdot 5 = 25$. The exclusion of these numbers from those underlined above leaves 1, 7, 11, 13, 17, 19, 23, and 29; that is, all numbers in $\{1,2,\ldots,30\}$ that are relatively prime to 30.

Hence, by (6–1–1),

$$\begin{aligned}
\phi(n) &= p^{\alpha}\phi(n') - p^{\alpha-1}\phi(n') \\
&= p^{\alpha}\sum_{d\mid n'}\mu(d)\frac{n'}{d} - p^{\alpha-1}\sum_{d\mid n'}\mu(d)\frac{n'}{d} \\
&= \sum_{d\mid n'}\mu(d)\frac{n}{d} - \frac{1}{p}\sum_{d\mid n'}\mu(d)\frac{n}{d} \\
&= \sum_{\substack{d\mid n \\ p\nmid d}}\frac{\mu(d)n}{d} + \sum_{d\mid n'}\mu(pd)\frac{n}{pd} \quad \text{[here we have used the obvious fact that if } p\nmid d\text{, then } \mu(pd) = -\mu(d)\text{]} \\
&= \sum_{\substack{d\mid n \\ p\nmid d}}\frac{\mu(d)n}{d} + \sum_{\substack{pd\mid n \\ p\nmid d}}\mu(pd)\frac{n}{pd} \\
&= \sum_{\substack{d\mid n \\ p\nmid d}}\frac{\mu(d)n}{d} + \sum_{\substack{pd\mid n \\ p\nmid d}}\mu(pd)\frac{n}{pd} \\
&\qquad + \sum_{\substack{p^2d\mid n \\ p\nmid d}}\mu(p^2d)\frac{n}{p^2d} + \ldots + \sum_{\substack{p^{\alpha}d\mid n \\ p\nmid d}}\mu(p^{\alpha}d)\frac{n}{p^{\alpha}d} \\
&= \sum_{d\mid n}\mu(d)\frac{n}{d}.
\end{aligned}$$

Figure 5-16: Andrew's textbook, page 79. Bettie used the highlighted equation, the justification of a step in a different theorem, for Wk7#2 in 1015-g3 at 0:56:53.

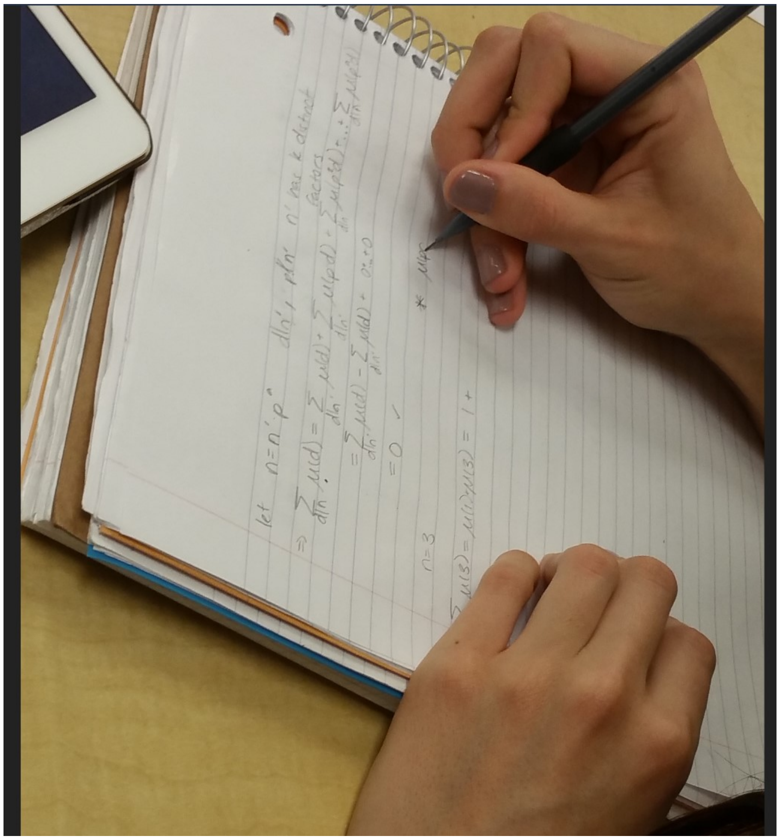
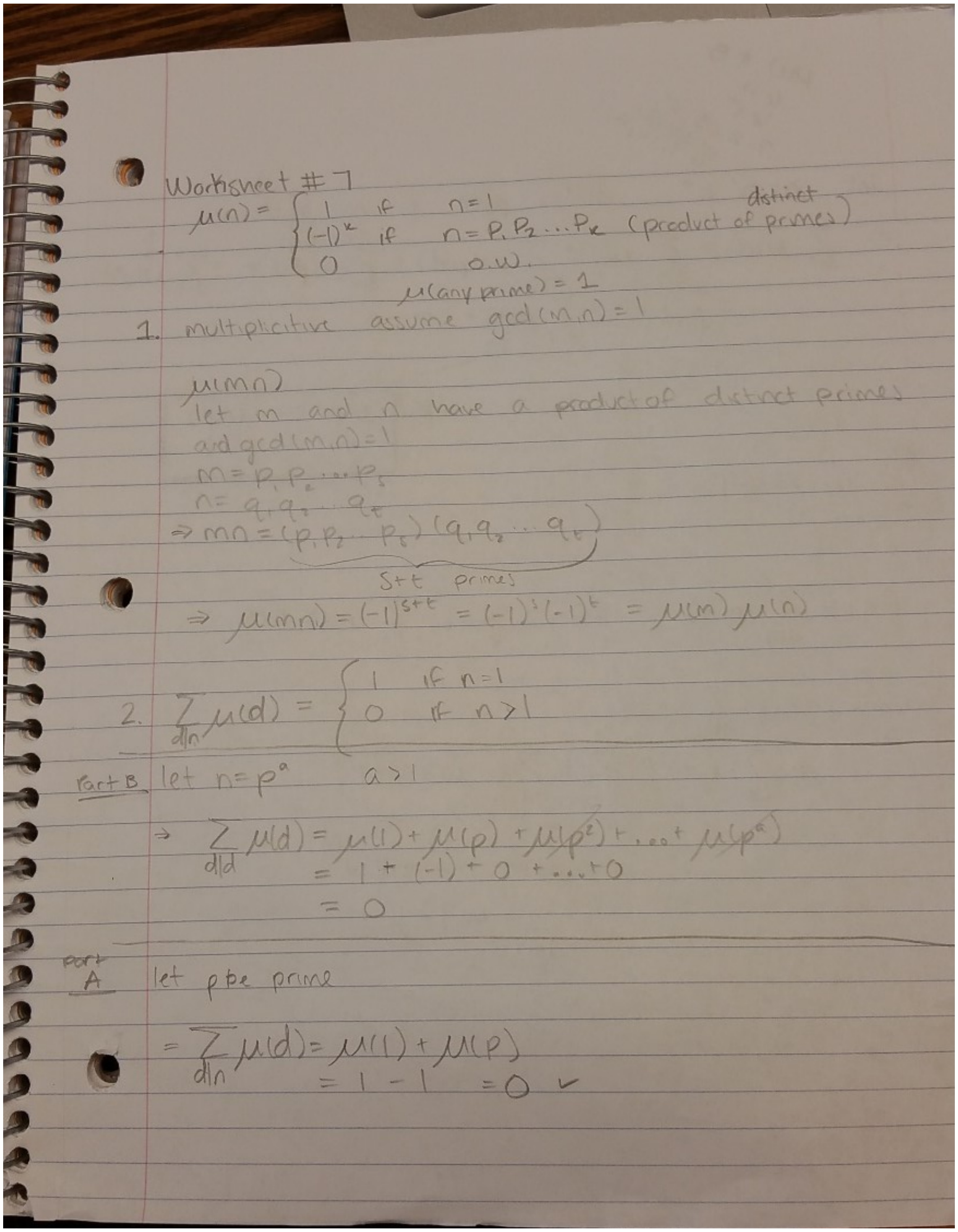


Figure 5-17: Picture of Bettie copying (bottom) the formula $\mu(pd) = -\mu(d)\ if\ p \nmid d$ from the tablet (left). The pictures were taken at the time stamps 0:56:47 and 0:56:53 in the video 1015-g3. She was looking at Andrew's, page 79 (see Figure 5-16), on her tablet.



Worksheet # 7

$\mu(n) = \begin{cases} 1 & \text{if} \quad n=1 \\ (-1)^k & \text{if} \quad n = p_1 p_2 \dots p_k \text{ (product of distinct primes)} \\ 0 & \quad \text{o.w.} \end{cases}$

$\mu(\text{any prime}) = 1$

1. multiplicitive assume $\gcd(m,n)=1$

$\mu(mn)$

let m and n have a product of distinct primes

and $\gcd(m,n)=1$

$m = p_1 p_2 \dots p_s$

$n = q_1 q_2 \dots q_t$

$\Rightarrow mn = \underbrace{(p_1 p_2 \dots p_s)(q_1 q_2 \dots q_t)}_{s+t \text{ primes}}$

$\Rightarrow \mu(mn) = (-1)^{s+t} = (-1)^s(-1)^t = \mu(m)\mu(n)$

2. $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$

Part B let $n = p^a$ $\quad a>1$

$\Rightarrow \sum_{d|d} \mu(d) = \mu(1) + \mu(p) + \cancel{\mu(p^2)} + \dots + \cancel{\mu(p^a)}$

$= 1 + (-1) + 0 + \dots + 0$

$= 0$

Part A let p be prime

$= \sum_{d|n} \mu(d) = \mu(1) + \mu(p)$

$= 1 - 1 = 0$ ✓

Figure 5-18: Bettie's notebook from the 10/15 group session. This page, referred to as page 1, includes her notes for the definition of the Möbius function, Wk7#1, and the first part of Wk7#2. Picture was taken during the SCNI interview taken after the group session.

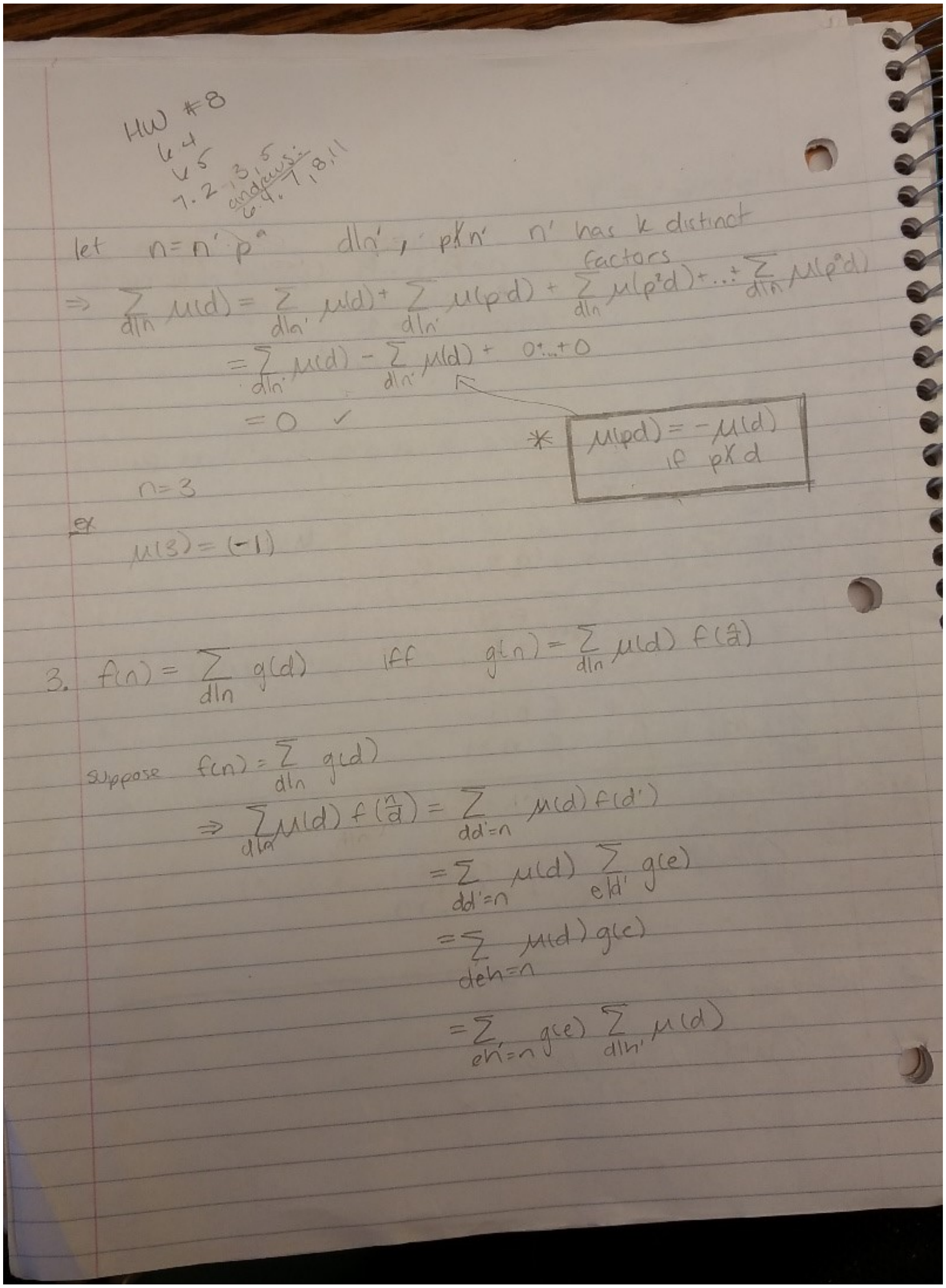
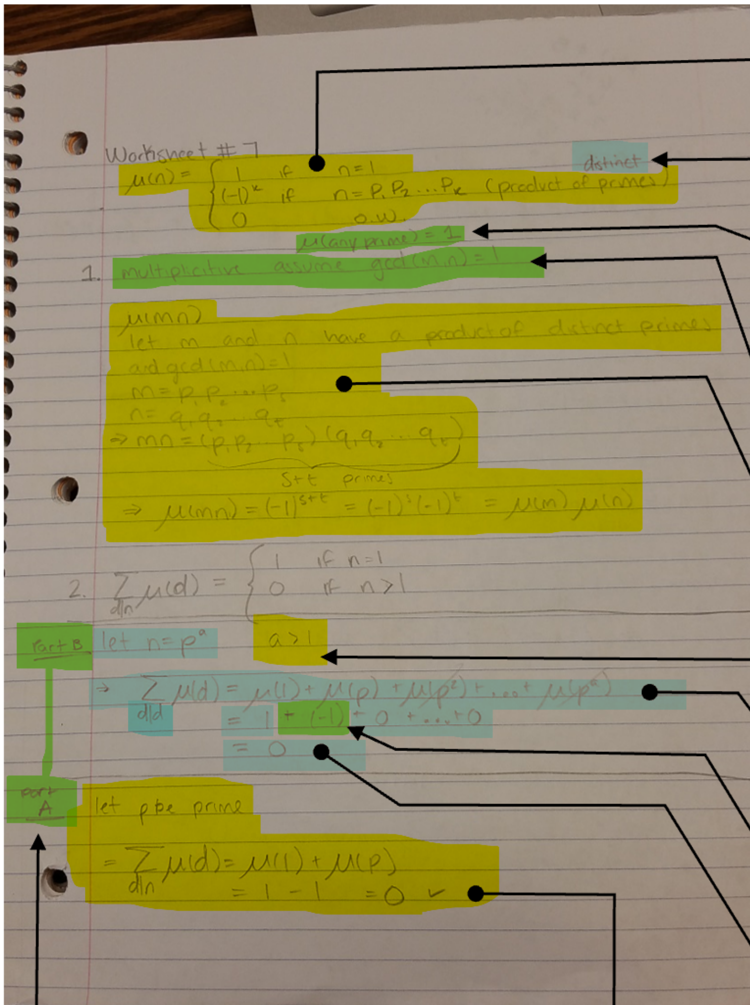


Figure 5-19: Bettie's notebook from the 10/15 group session. This page, referred to as page 2, includes the second part of Wk7#2 and the first part of Wk7#3. Picture taken during the SCNI interview taken after the group session.

Table 5-3: Moment-by-moment analysis of Bettie's notes during the group session on 10/15. I call this analytical drawing chart a table because it involves the logic of rows and columns. Although rows and columns are not spatially aligned, every connection between a highlighted text in Bettie's notebook (to be regarded as first column) and an explanatory box (to be regarded as second column) can be considered a row, which is numbered in the box.



[1] Yellow highlight: definition of the Mobius function from John's notes (see Figure 5-11) at 0:08:27–0:09:45.

[2] distinct: note added while reading from ebook (see Figure 5-12) at 0:17:37.

[3] $\boldsymbol{\mu(any\ prime) = 1}$: note added at 0:24:20–0:24:29, following the group conversation at 0:24:05.

[4] Green highlight: note added after a conversation between Ted and John at 0:20:00–0:20:16.

[5] Yellow highlight: proof of #1 mediated copy from Ted's notes at 0:21:52–0:24:08, after Ted explained his work.

[6] Blue highlight: base case of #2 mediated copy from the ebook (see Figure 5-13) at 0:29:16–0:30:48. Slashes are in Ted's proof (see Figure 5-14).

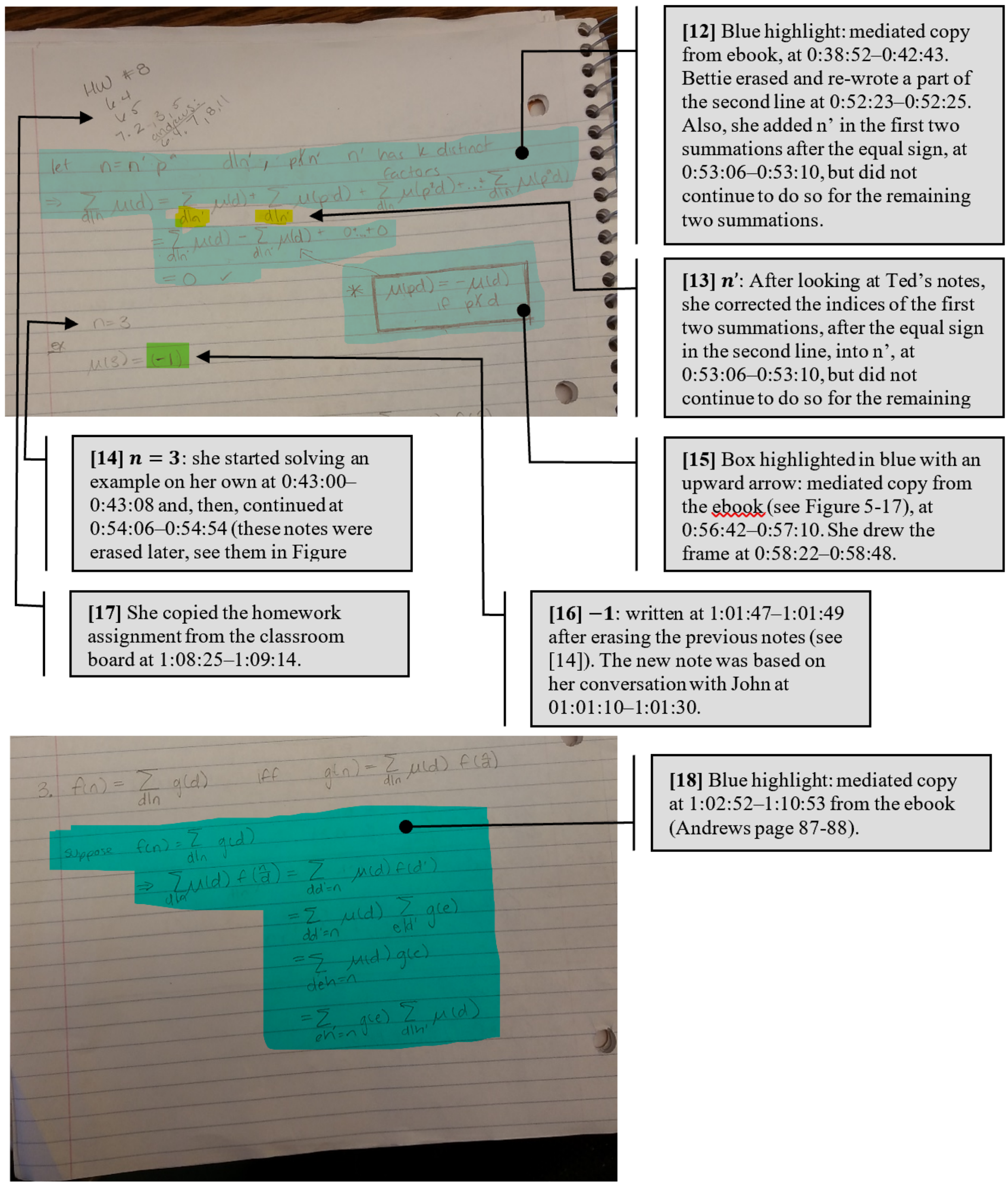
[7] + (- 1): written at 0:31:11–0:31:19 after Ted had explained to Bettie why $\mu(p) = -1$.

[8] Blue after green highlight: written without looking at the tablet, at 0:31:11–0:31:18.

[9] a > 1: added at 0:32:09–0:32:10 when Bettie was looking at Ted's notes (see

[10] Yellow highlight: written at 0:35:40–0:36:22, mediated copy from Ted's work (see Figure 5-14).

[11] part A - Part B: written at 0:36:22–0:36:36, following Hoffmann's comment on Ted's choice of the base case.



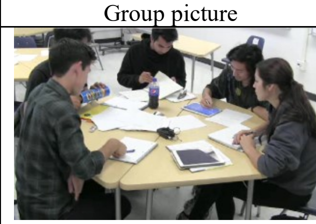
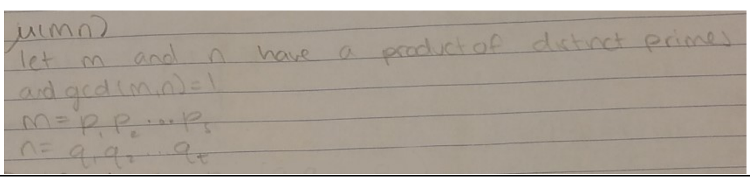
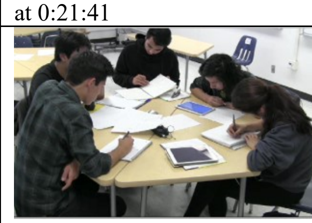
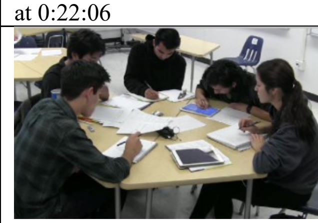
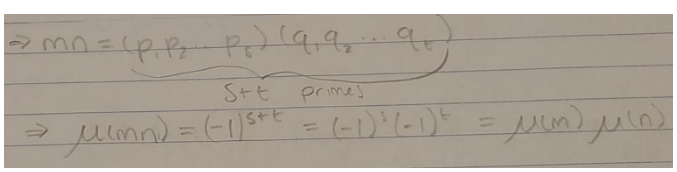
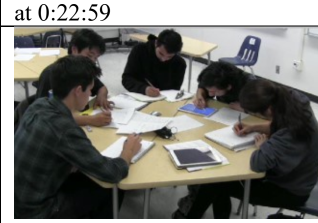
Study A: Bettie's behaviors while copying from written resources.

Bettie was continuously taking notes during groupwork. She wanted to keep a track of the knowledge produced during groupwork for future use. She noted, "I like to write on my paper just cause . I like to keep it. so I can look at it" (Int2-1202-Bettie lines 350–351). She associated an additional function to writing, i.e., to memorize, which was an entrenched habit for her. In the exit interview, she noted, “my way of memorizing is like writing. so if I write down for homework I kinda remember how I wrote it out” (Int2-1201-Bettie lines 275–276).

This close study of Bettie’s notetaking routine during groupwork revealed the involvement of two cognitive faculties: short-term memorizing and mathematical sense-making. Recall Bettie’s urge for understanding, that she narrated as an individualized identity (Id4 in Table 4-1). Bettie’s routine of notetaking comprised cognitive mediation, which was particularly evident in her act of copying her groupmates’ work or the textbook proofs. Her copying routine involved two moments: (i) reading original notes and, then, (ii) writing what she could keep in mind with little or no resort to the original notes.

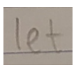
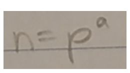
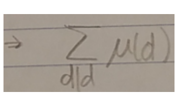
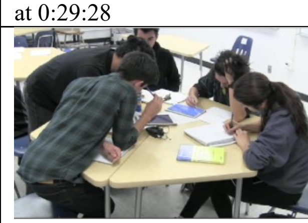
On 10/15, Bettie engaged in six copying activities, either from her groupmates notes or proofs in the textbook (see Table 5-3 rows [1], [5], [6-8], [10], [12, 15], and [18]). This study focuses on the second and the third copying activities ([5] and [6-8]), since the subsequent copying activities involved the same mechanisms. The first copying activity consisted of copying the definition of the Möbius function from John’s notes, because Bettie had missed the previous lesson wherein the definition was introduced (an atypical activity).

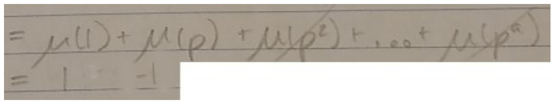
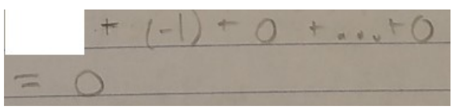
Table 5-4: Micro-analysis of Bettie’s act of copying Ted’s proof of Wk7#1 in 1015-g3, at 0:21:29–0:24:08. The first column provides the duration of the segment. The transcript of Bettie’s actions is present in the second column. The third column reproduces her notes and the fourth column contains a picture during the corresponding time period.

Duration	Bettie’s actions	Group picture
23 sec.	**[1]** Bettie looks at Ted’s notes after he finishes presenting his proof of Wk7#1.	 at 0:21:41
4 sec.	**[2]** Bettie writes on her notebook by looking at it. 	 at 0:22:06
7 sec.	**[3]** Bettie looks at Ted’s notes from the other side of the table.	 at 0:22:59
5 sec.	**[4]** She writes on her notebook by looking at it. 	 at 0:23:42

The second copying activity consisted of copying Ted’s proof of Wk7#1 (at 0:21:52–0:24:08). Bettie listened to Ted while he presented his proof to the group, by leaning closer to heed his notes (1015-g3 at 0:20:59–0:21:29). When Ted finished his proof, Bettie kept looking at his notes for about twenty-three seconds then started writing (see Table 5-4). She alternated between carefully reading Ted's proof and writing it on her notebook without looking at his notes. She did this dual move twice. While writing Ted's proof on her notebook, she counted on her memory as well as her understanding, as she did not look at the original notes. The chunks of information she processed in each reading-writing unit (see pictures of Bettie's notes in Table 5-4 rows [2] and [4]) were substantial enough to suspect at least a minimal reflective faculty in action with her short-term memory. The proof involved six variables, two of them, s and t, were used as indices and exponents. Additionally, Bettie paused her action of writing at a strategic place in the proof, which indicated a reflective process in the action of copying. First, she wrote the given and the variables (row [2]) and, then, looked back at Ted's notes to focus on the details of the proof (rows [3] and [4]).

Table 5-5: Micro-analysis of Bettie's act of copying of the base case of Wk7#2 from Andrews' theorem 6-5 (see Figure 5-13 and Table 5-3[6]). The analyzed episode is in 1015-g3 at 0:29:16–0:31:18. The first column provides the duration of the segment. The transcripts of Bettie's actions are present in the second column. The third column reproduces Bettie's notes and the fourth column contains a picture during the corresponding time period.

Duration	Bettie's actions and notes	Group picture
5 sec.	**[1]** Bettie writes on her notebook. Her tablet is present at the center of table. 	 at 0:29:19
3 sec.	**[2]** She looks at her tablet by leaning forward.	 at 0:29:22
4 sec.	**[3]** She writes on her notebook by looking down at it. 	 at 0:29:25
5 sec.	**[4]** She looks up at the tablet.	 at 0:29:28
11 sec.	**[5]** She writes on her notebook by looking down at it. 	 at 0:29:36

22 sec.	**[6]** She leans forward and looks at the tablet closely.	 at 0:29:46
42 sec.	**[7]** She writes on her notebook by looking down at it. 	 at 0:30:09
17 sec.	**[8]** [Bettie asks why $\mu(p) = -1$.] Bettie: [looks at Ted] why is *mu* of *p* negative one? Ted: it's uh because it's negative one to the power of that *p* of =that [gesturing with thumb] Bettie: =Oh yeah [looks at Ted, smiles, and nods] Ted: no. [hand gesture] negative one to the number of prime factors. Bettie: okay. [looks at her notebook] Ted: so Jeremy: I think it's one prime so it's one factor. Bettie: =[looks at Jeremy and nods] Ted: =yeah.	 at 0:30:56
13 sec.	**[9]** She writes, erases, and then writes by looking down at her notebook. 	 at 0:31:07

The third copying activity drew the first step in the proof of Wk7#2 from the textbook (at 0:29:16–0:31:18) (see Table 5-5). Prior to copying from the textbook, Ted and John were debating whether to use the induction method to solve Wk7#2 (at 0:24:30–0:28:40). Bettie had found the proof of the theorem in her ebook (see Figure 5-13) and, mistakenly, thought it was by cases, like Ted (since the proof starts with "first" and proceeds, in a similar way, to Ted's case of $n = p^\alpha$, see Ted's work in Figure 5-14). Bettie handed her tablet to the group to be sure that Ted's method was correct. She looked at the tablet present in the center of the table for eight seconds (at 0:28:59–0:29:08), got distracted by a comment from a TA who was passing by, (at 0:29:08–0:29:16) and then she started writing on her notebook (continue details in Table 5-5).

Bettie enacted the aforementioned reading-writing moves in this copying activity as well (four times). In the beginning, she had to look more frequently and process a small amount of information (see [1-5]). However, the sum of the Möbius of primes (in [7] and [9]), the large and the main body of the base case, was written on the notebook after a close look, lasting twenty-two-seconds, at the tablet. She processed the entire computation in one unit.

Clearly, Bettie was engaged in sense-making as she copied the proof. First, after she wrote the sum of the Möbius of the powers of p (see Table 5-5 [7]), she crossed off the Möbius of powers greater than one, a move that did not imitate the textbook (see Figure 5-13) but did resemble Ted's notes (see Figure 5-14). Thus, as Bettie was writing the proof, she had in mind both resources—the textbook's and Ted's proofs—through which she undertook a cognitive act of making connections. Second, when Bettie moved to the second line, she put one for $\mu(1)$ and already knew the $\mu(p^{\alpha>1})$ was zero, but still asked her groupmates why $\mu(p) = -1$. Note that she had memorized all these values. She interrupted her writing to solicit an explanation in order to improve her understanding.

Bettie's acts in writing the proofs from other resources, her groupmates, and her textbooks, involved cognitive processes mainly comprising the act of memorizing with sense-making. She appeared to animate her entrenched habits of writing-to-memorize, which she utilized while doing her homework in the previous classes (Int2-1201-Bettie lines 275–276). In the groupwork, she enacted the same habit not only with her textbook but also with her groupmates' notes. Since she endeavored at least a minimal reflection on the content, she ended up connecting the content from both the resources.

Figure 5-20 depicts the old learning practice and its associated function (writing-to-memorize) and the shift to a new learning practice—copying from her groupmates' notes—and the new actuated functions.

The activation of two resources organically led Bettie to make connections across their contents, which actuated a new function, maybe inadvertently, defined as cognitively mediated copying. The new learning practice stood in discontinuity as well as in continuity with the old one. Bettie enacted the same reading-writing practice with both the resources. However, the students' notes were not as refined as those present in the textbooks. They were more like drafts of thoughts that supported the students' mathematical talks in the group. Study B investigated the connections that Bettie made between her notes and her groupmates' mathematical conversations.

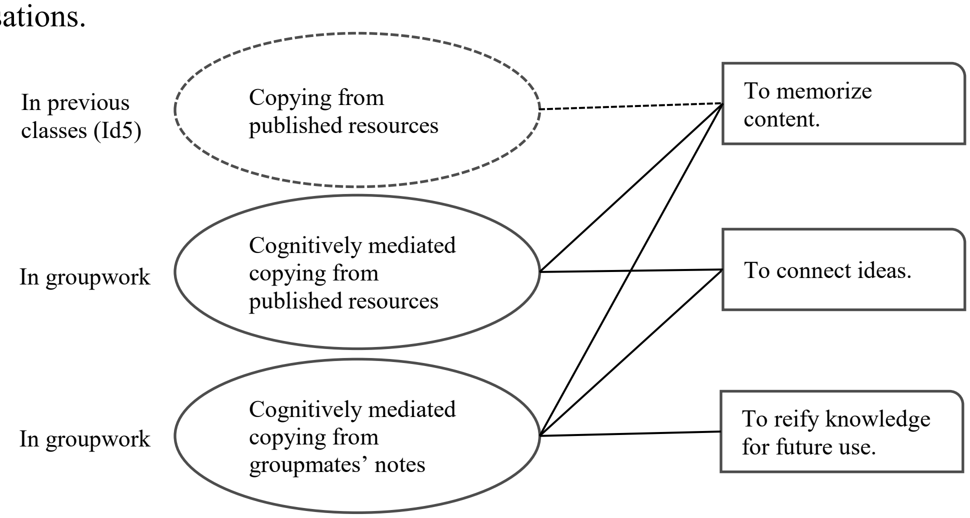


Figure 5-20: Resources used for mediated copying and the associated functions that Bettie animated in the classroom's groupwork.

Study B: A moment-by-moment analysis of Bettie's notes during the group session on 10/15.

In this group session, Bettie learned the definition of the Möbius function and worked on Wk7#1, 2, & 3 (Bettie's original notes can be found in Figure 5-18 and Figure 5-19). The current analysis, reported in Table 5-3, investigated not only the temporal development but also the learning mechanisms of Bettie's note-production. Her notes were broken down into pieces by the resources from which she drew them. The texts highlighted in yellow were copied from her groupmates' notes and the texts highlighted in blue were taken from textbooks. The texts highlighted in green were notes which Bettie added after listening to a group conversation. The pieces of her notes in Table 5-3 were assigned a description and the time period of their productions were included in boxes, to which I shall refer by their numbers (e.g. Table 5-3[4]). The time stamps in the boxes refer to the video of the group session of G3 on 10/15 (entitled 1015-g3).

Three results from the current analysis (Table 5-3) will be highlighted:

(i) Bettie's notes, as connected information, across multiple resources,

(ii) The role of group conversations in Bettie's moment-by-moment learning process,

(iii) The productive learning process about the Möbius of a prime number.

In Bettie's notes, the definition of the Möbius function, along with the proofs of Wk7#1 and #2, were constructed by drawing information from the textbook, her groupmates' notes, as well as their talks (notice the presence of different highlight colors in Table 5-3 rows [1–16]). Take, for example, Bettie's notes on the definition of the Möbius function (Table 5-3 rows [1–3]). Bettie copied this from John's worksheet in the beginning of the group session (see Table 5-3 row [1]). But, as per her comments in the SCNI interview, she was attempting to make sense of how the definition works, until John explained it to her (at 0:18:00).

<u>SCNI-1015-Bettie lines 32–44</u> – *1015-g3 video paused at 0:12:31*

Fady: What were you looking for here?

Bettie: Uh I was looking for the answer.

Fady: Uh oh the answer of finding the solution in the book?

Bettie: Yeah. cause I didn't know I was trying to find a better understanding of the definition . because he [John] gave me like the written out definition that . he [Hoffmann] gave us but I didn't get it so.

Fady: Oh the definition that John gave you at the beginning of the class was different from?

Bettie: Wasn't different, but I was just tryna find like . a writing out of what we were do . what we were looking for, what we're doing, what it does. I didn't get it. but he [John] helped me like soon.

Thus, when Hoffmann sat with the group and Ted was presenting his strategy for Wk7#1 to him (at 0:14:49–0:18:00), Bettie was still focused on understanding the definition of the Möbius function. In fact, as Ted was presenting his strategy, she was looking at her tablet, noticed the book's definition, and added the condition that was "distinct" to her notes (see Table 5-3 row [2]). Almost seven minutes later, after she had finished copying Ted's proof of Wk7#1, she heard her groupmates agreeing that the Möbius of any prime was one and added a note under her definition (see Table 5-3 row [3]).

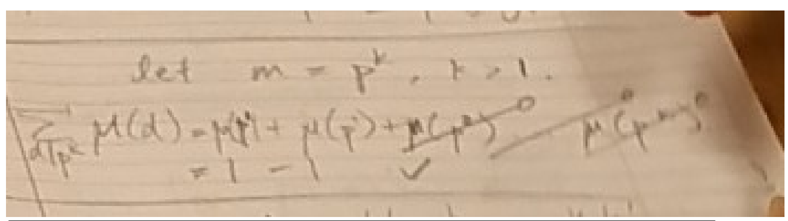
The analysis of Bettie's notes from the group sessions on 10/15 qualified the description she gave, in the SCNI interview, about her participation forms: alternating between listening to groupmates' mathematical conversations and reading in the textbooks (SCNI-1015-Bettie lines 112–115). She also attended to her groupmates' notes. Her notebook was the space where she materialized the connections that she had constructed across the resources. In Lacanian terminology, her notebook mirrored her cognitive processes, which consisted of building mathematical understanding by drawing pieces of knowledge from the available resources.

Study A investigated Bettie's interactions with her groupmates' notes. Now, I shall investigate her learning from the group's mathematical conversations, with which Bettie struggled, given her individualized learning identity (Id3 in Table 4-1).

Bettie's attention to the group conversations was unstable. At some moments (e.g. in Table 5-3 rows [3], [4] and [10–11] and Table 5-2 at 0:24:30–0:33:07), Bettie provided evidence of taking up elements from these conversations, such as when she noted the group agreement on $\mu(p) = 1$ (see Table 5-3 row [3]). At other moments, she seemed disconnected from them, such as when her groupmates corrected their prior agreement on the Möbius of a prime number but Bettie did not update her understanding (see Table 5-2 at 0:21:40–0:24:30). She explained this phenomenon in her SCNI interview—she "zoned out" her groupmates when their mathematical conversations stopped making sense to her (SCNI-1015-Bettie lines 61–62 & 114–115). Thus, Bettie made her individual learning a central function of her participation in the groupwork, which, sometimes, worked to her benefit and, in other instances, made her miss out on important conversations. For example, when Ted delivered a confusing explanation to Bettie (Table 5-2 at 0:33:07–0:43:10), she went on to scroll on her tablet and look at her groupmates' notes, thereby missing a significant conversation about how the induction method worked.

In two instances during the class on 10/15, the group conversations fulfilled and oriented Bettie's attention to written mathematical work. First, when the debate about whether they should use the induction method in proving Wk7#2 heated up between Ted and John (Table 5-2 at 0:24:30–0:28:40), Bettie listened to Ted explaining that he could prove the first case without using the induction method. She looked at his notes (see Figure 5-14), remembered its similarities with the textbook's proof (see Figure 5-13), which she had read earlier, took her tablet to retrieve the proof given in the book, and concluded that Ted was right (see her comments on this episode in SCNI-1015-Bettie lines 145–152). She told Ted, "you're right," and handed him the tablet which displayed the proof. Ted's work and the proof in the textbook (reproduced below for convenience) shared similar computations. However, the textbook used the induction method, as noted at the start of the proof ("we proceed by mathematical induction"). Bettie seemed to primarily attend to the computations present in the textbooks, as per her entrenched "arithmetic" identity (PPMI) (Id7 in Table 4-1Table 4-1). She had a weak background in proof methods, which did not help her gain an accurate understanding of Ted-John debate as well.

Ted's work on WK7#2.



Proof of Wk7#2 in Andrew's textbook.

PROOF: The assertion is clearly true if $n = 1$. We proceed by mathematical induction on the number of different prime factors of n when $n > 1$.

First, if $n = p^\alpha$, then

$$\sum_{d|n} \mu(d) = \mu(1) + \mu(p) + \mu(p^2) + \ldots + \mu(p^\alpha)$$
$$= 1 - 1 + 0 + \ldots + 0$$
$$= 0.$$

Second, when Hoffmann tried to convey to Ted that he chose the wrong base case and that his case $n = p$ was redundant for Wk7#2 (see Table 5-3 at 0:33:07–0:43:10), Bettie started copying Ted's work on the case $n = p$ (see Table 5-3 rows [10–11]). Obviously, she did not understand the conversation but was oriented enough with Ted's work to realize that he had invented a case, which she dubbed as "part A," which was not included in the proof of the book, which she dubbed as "part B." She commented on this discovery in the SCNI interview.

SCNI-1015-Bettie lines 160–163 – *1015-g3 video paused at 0:34:07*

Bettie: that's when I realized that I didn't have uh. that I skipped this part [pointing to part A on her notes, see Table 5-3 row [11]] and then I went straight from here [part B on her notes] and went straight to here [correcting herself] instead of going here [part A on notes] first.

In both instances, Bettie constructed partial understandings of the mathematical group conversations in the classroom, a fact that was aligned with her acknowledged struggle with ongoing mathematical conversations. Bettie's entrenched visual learning identity (Id3 in Table 4-1) could be the factor that animated her disposition towards written mathematical work during the group sessions.

Bettie learned from the group session on 10/15 that the Möbius of a prime number was negative one. Her SCNI interview offered evidence of this gain in her learning (see SCNI-1015-Bettie lines 90–99). Yet, notice that Bettie attended to the Möbius of prime numbers in three moments during the group session (see Table 5-3 rows [3], [7], and [16]). In the first exposure (at 0:24:50), her groupmates spoke about the "Möbius of any prime." In the second exposure (at 0:29:16), Bettie and Ted used "*mu* of p" and agreed that it was negative one. Bettie doubtfully understood p was a prime number in the second exposure, because later (at 0:58:22) she could not find the value of $\mu(3)$ and, instead, turned to John. In the third exposure (at 1:01:10), Bettie and John spoke about "*mu* of three" and "*mu* of any prime number."

It is important to notice that multiple exposures to a concept, in different modalities, was productive for Bettie. The sessions of the study group outside the classroom offered Bettie an increased exposure to concepts that she encountered in the classroom (next subsection).

Before moving to the next subsection, we can retain, from the analysis of Bettie's learning in the classroom's groupwork, that she shifted to a serious learning disposition by predominantly actuating the function of enhancing her own understanding. Prior to this class, Bettie used to rely on published resources to enhance her mathematical understanding. In this class, Bettie had positive experience of learning from her groupmates during classroom's group sessions. Recall, on 10/15, for instance, Bettie understood how the Möbius function works due to John's explanation, after failing to make sense of the textbook's definition and the teacher's explanation. Thus, groupwork in classroom expanded Bettie's access to productive resources, mainly by adding her groupmates' knowledge materialized in speech and notes to textbooks. However, her individualized learning identities, namely "arithmetic" (PPMI in Id7) and visual learning (Id3), interfered in her learning experience in the group session on 10/15. Observations of other group sessions spread over the semester approved the typicality of Bettie's behavior in groupwork as observed on 10/15. Bettie's "arithmetic" identity (PPMI) prevented her from accurately comprehending the definitions and proofs in the textbooks. Additionally, her visual learning identity distracted her from keeping up with the group conversations by probing her to consult the textbooks. During groupwork in classroom, Bettie's attention was divided over group conversations, groupmates' notes and textbooks.

Study group outside the classroom

Since the fourth week of the semester, Ted had been encouraging his groupmates to meet and study outside class hours (see Int1-0922-Bettie lines 296–304). The idea was first actualized on either 9/29 or 10/01 (see Int2-1202-Bettie line 219 and the group session in 0929-g3 at 1:25:58), when Bettie first reserved a study room in the library (Int2-1203-Ted lines 203–204). Since then, the study group became a regular activity for Ted, Bettie, Jeremy, and Boutros. John could not join them because he lived far away from campus. Other classmates occasionally joined the group. The study group sessions took place in the study rooms at the library, which were equipped with large white-board walls, and lasted for about two hours each. Group members would text each other whenever one of them was free to study at the library, so that whoever was available could join. They primarily worked on finishing the homework of the number theory class, since sometimes they could not go through all problems in the classroom's groupwork. Sometimes they worked on the homework they got in other classes.

The group members reported that they had created a relaxed, flexible, and productive learning ecology during the study sessions. Each one was free to do his or her own work and seek each other's support whenever they needed. Those who wanted to solve problems together could do so. Moreover, the group members also acknowledged the central role that Ted played in the study sessions. He commonly explained his methods to other students and was approached by other students who asked him to assess their ideas.

<u>SCNI-1112-Ted lines 5–8</u>

Ted: […] we actually communicate with one another . like before the homework is due . and we meet up at around 1 o'clock. 1:30ish. at the library uhh. Today . there was me . Boutros . and C . Bettie. and we worked together on the last homework . and we . I went over on detail with Bettie on the solutions for number two and three.

<u>Int2-1203-Jeremy at 0:12:41-0:13:42</u>

Jeremy: Yeah we would uh there'd be a big white board wall . so everyone would have a marker . and then we could all do the problem together. on the wall. and like if you thought of something . you could like write it on the wall . and like kind of play teacher. and uh. show your idea to everybody. and then they would tell you what they think about it. […] I did do some of the writing and like . explaining what I was thinking to people. but I also did some of the just watching. usually though I would watch like Ted cause he would do a lot of the writing. He wrote a lot. Ted's really good at this class. so he would … explain stuff. yeah.

<u>Int2-1202-Bettie lines 185–186</u>

Bettie: Well we get like a room. Me and Ted mostly and then Ted like teaches all of us. how to do it. so that's cool.

Why did Bettie join a study group, knowing she faced difficulties in learning from group conversations?

Bettie took the first initiative to actualize Ted's suggestion about studying outside class. As per her account, she approached Ted and asked him to start helping her in doing her homework, after a group session wherein she realized that he understood the homework.

<u>Int2-1202-Bettie lines 197–205</u>

Fady: And you mentioned this [study group] did not happen in the beginning [of the semester]- this happened later on. What do you know when this happened? and how?

Bettie: Uhh just cause I asked umm Ted . I forgot what we were doing. we were just doing homework when Ted said like "I'll be in the library if anyone needs help on their homework just stop when I'm here." and then we started doing the homework and he understood it completely so I asked him if he can just start helping me on my homework. and he was like "Yeah I'm always here." and we started a group message. and then now everyone texts everyone when we're in the library and we just like meet up if we're in school or not.

Her initial intention was to have Ted as her tutor for the number theory class rather than to join a study group. Bettie had signed up for free private tutoring sessions, which the researcher had proposed as compensation for the time participants put in the interviews. However, she never asked for any tutoring from the researcher. In the exit interview, the researcher asked her whether she took a tutor for number theory class.

Int2-1202-Bettie lines 533–534

Fady: I see. Uhhh. Did you work with any tutor this year?

Bettie: Just Ted.

Fady: Just Ted. Okay.

In the early interview, Bettie had emphatically narrated a successful tutoring experience for the modern algebra class in the previous semester (Int1-0922-Bettie lines 90–101). She described it as “awesome” and a “really good” experience in a school where she had very few positive learning experiences. Paul, her tutor for modern algebra class, taught her to memorize by writing down and repeating definitions as well as formulas. He also helped her to get a high grade in that class (Int2-1202-Bettie line 288). Hence, Bettie’s request to Ted could be seen as being rooted in her previous positive learning experience with Paul. Yet, unlike Paul, who was a graduate student, Ted was a groupmate. Given Bettie’s initial face-saving tactic, i.e. to avoid talking to classmates because of fear of “looking stupid” (see Figure 5-4), the choice of a groupmate as tutor was significant. It also corroborated her genuine shift away from worrying about face-saving.

In sum, Bettie initially intended to repeat with Ted the productive tutoring experience, which she had with Paul. It was Ted who invited her to join the study group sessions, where he could help her. As such, the participation in a study group, which was a foreign practice to Bettie, occur in continuity and discontinuity with her regular learning practices. In this case, the discontinuity stemmed from an external agent rather than Bettie’s internal motivation.

What development did Bettie achieve in the study sessions?

The intended tutoring session turned into study group. Ted invited all the group members to the study group. Bettie was doubly privileged in study sessions. First, she received privileged moments with Ted and, second, found herself learning from her groupmates’ work. Unlike the classroom’s groupwork, where Bettie’s attention was divided between the groupwork and the textbooks, the study sessions made Bettie focus on listening to her groupmates work. In comparison to the classroom’s groupwork, she said the following about the study sessions:

Bettie: […] outside of uh . the classroom we even go get beers we'll just go drink and then like . get off topic . and then finally we'll like go back and start studying again. And like I don't know we're just kind of . I feel we're more friends . like outside I guess. and we just chill like . we do math and . we're not super like . trying to rush through the problems . and just like slow it do:own. They go on the board like do all the problems their own ways and work together and then I'm kinda like just sitting there . and taking it all in. [giggles] or just going over other problems but . I feel like m . I feel like it's . cooler outside class. just cause we . have more time just . doing whatever we want and not having to stay so on topic .

our brains can rest. And then come back . then go off and come back. (Int2-1202-Bettie lines 511–520)

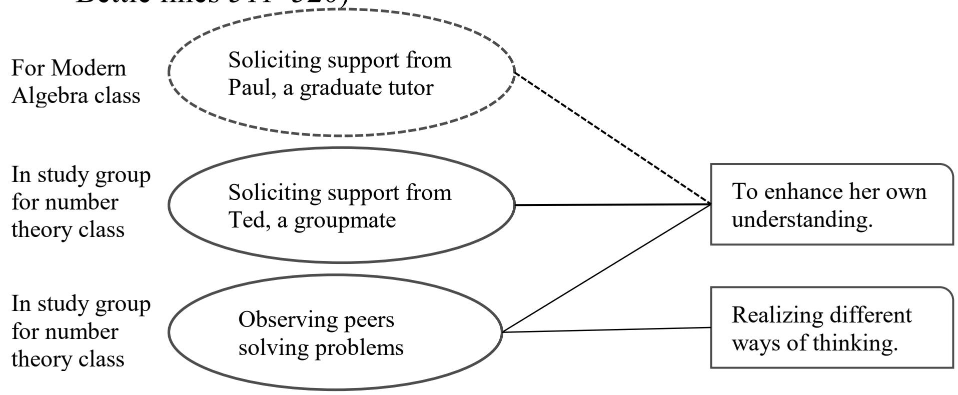


Figure 5-21: Position-function shifts that took place during the study group outside the classroom.

In the study sessions, collective productivity was no longer a part of group norms. There was no assigned goal in the study sessions, except to study and support each other. The group members slowed down their work and allowed themselves to go off-task. Hence, Bettie felt comfortable interrupting her groupmates' work for soliciting explanations.

During groupwork in classroom, Bettie used to disengage from groupwork, when she did not understand the mathematical group conversations, and focus on reading the textbooks (SCNI-1015-Bettie lines 61–62). In the study group sessions, the relaxed learning ecology kept Bettie connected with groupwork by permitting interruptions for soliciting clarifications at any moment. While Bettie's attention was divided over multiple knowledge resources in classroom, she was focused on her groupmates' work in the study sessions outside classroom.

Although there were no videos of the study sessions to investigate Bettie's behavior therein, her narratives therein reflected a focus on her groupmates' work (see excerpts below). On one side, textbooks and published resources were never mentioned when Bettie talked about the study sessions. On the other side, Bettie elaborated on her groupmates' mathematical profiles and how she drew useful knowledge from the diverse richness of their works.

Int2-1202-Bettie – *narratives on study sessions*

Bettie: But there [in study sessions]. Everyone is always different. Like they do different ways . and then . they always come to the same conclusion . But it's easier . cause like I can look at all theirs . and like kinda like . come up with my own . way . of understanding it. cause it's just like. Ted is more like detailed . like he . he like goes rea:ally into depth with everything. And then I'll look at Jeremy's and he's like quick and short and like done. But it's like the same . and same conclusion. which is like cool . then I can like read into details on this side [waving right hand] and look at what it looks like . just like math . on this side [waving left hand] and then . and then it makes stuff of it that makes sense. (lines 390–397)

[…]

Boutros's smart too. but he's just. he's like me . like we sit there and do it ourselves . before we like talk . about it. And then like . like when we met . when we meet in like the lab that's when I hear him talking like "Oh wow. You're so smart." (lines 406–409).

In addition to the relaxed learning ecologies, the study sessions offered Bettie another opportunity at attempting to understand the ongoing mathematical conversations. Between the classroom's groupwork and the next study session, Bettie used to review her notes and enhance her understanding of the materials so that she could hone her questions to her groupmates and have the basic requisite knowledge to understand their mathematical conversations. In her SCNI interview (excerpt below), Bettie explained the factors that made the study sessions helpful.

> Bettie: […] I don't know maybe because I've already looked through it myself so I know what to ask. I don't know I like to look over things by myself before I start talking to other people cause I like to know what I'm talking about. So in class when we're going over a problem for the first time I don't really know what we're doing so like when they're talking I don't really understanding what they're even saying. so it takes me a while to get something so at home I do it by myself and then I come to group and I have a better understanding what we're talking about so it makes it a lot easier to understand what they're saying. (SCNI-1015-Bettie lines 6–13).

Throughout the study sessions, Ted shared his notes and his homework with his group members. They were typed in LaTeX and saved in his google drive. Bettie consulted Ted's resources when she was completing her homework (see Int2-1202-Bettie lines 126–135). Ted's written homework, which consisted of rich and accurate mathematics (see Table 3-14 and Appendix C), were connected to his work during the classroom and the study sessions. Thus, Bettie had an additional opportunity to engage with the reified knowledge of groupwork and familiarize herself further with Ted's ways of reasoning, this time by reading his work.

Instilling an active learning identity across learning activities: A synthesis

This section investigates the third reported shift in Bettie's learning identity: solving problems on her own rather than copying them from published resources. The previous section highlights Bettie's productive use of published resources during learning activities. However, an excessive reliance on published resources does not boost confidence in one's own knowledge. While active learning pedagogies encourage students to know how to seek and use external resources of knowledge, they also encourage students to build and rely on their own knowledge.

This section analyzes how Bettie started to rely on her own mathematical knowledge in doing her homework. I first present evidence of the change in her learning habits, then track the shifts of identities, positions and functions from the beginning to the end of the semester. Most of these shifts are studied in the previous two sections and included in Figure 5-22. While the previous sections focus on the shifts within each learning activity, solitary study, classroom groupwork, and study group sessions, the current section highlights the connections across the activities.

As represented in Figure 5-22 and discussed below, Bettie's change of behavior in doing her homework was mediated by her participation in classroom groupwork and study group. The most influential process that led Bettie to rely on her own knowledge in doing her homework was the process of identification with her groupmates, whom she was delighted to observe as they solved problems on their own during the study group sessions. A close look at her development showed that the accumulation of small changes throughout her solitary study and classroom groupwork were necessary to allow the identification process to take place.

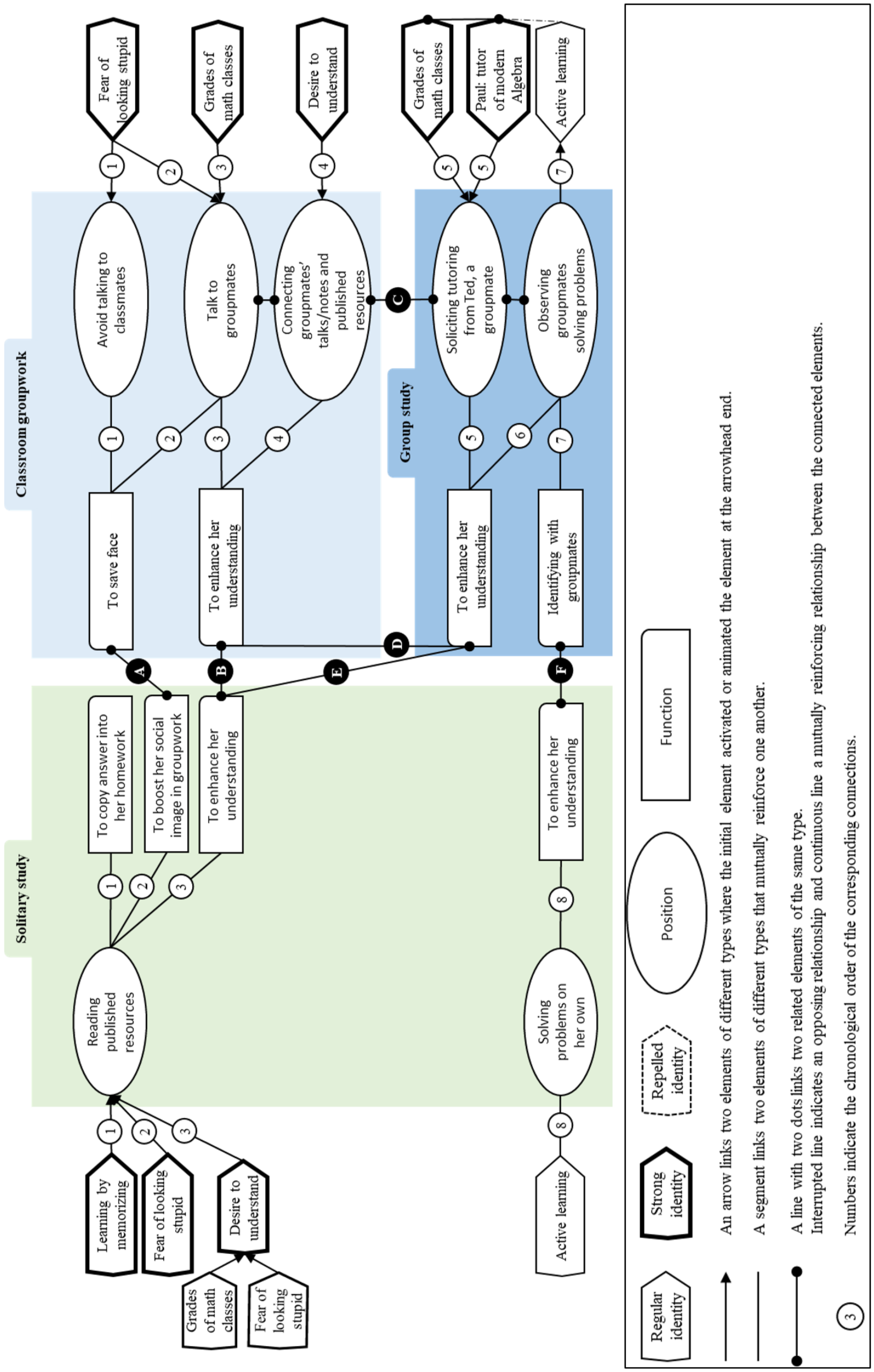


Figure 5-22: Betties' shifts of identities, positions and functions throughout her experience in number theory class across three learning activities: solitary study, classroom groupwork, and study group outside of the classroom.

Evidence of the change in Bettie's way of doing the homework

The reliance on published resources, for doing homework, dominated Bettie's narratives in the early interview (see Id9 with Id5, Id7, & Id11 in Table 4-1), all of which lacked any mention of doing homework by relying on self. The only reference in the early interview to "doing" homework, rather than "copying" homework, was a complaint.

Bettie: [Hoffmann]'s just like "do it." so like we'll do it . like as a group . but still like I don't know if I'm doing it right. (Int1-0922-Bettie lines 136–137)

On the contrary, throughout her exit interview, Bettie repeatedly and emphatically noted the reliance on self in doing the homework. The excerpts below reproduce three such conversational moments.

Int2-1202-Bettie

Bettie: This is the one class that's actually helping me understand. and like come up with things on my own instead of like finding the answer and writing it down (lines 38–40)

[…]

I have to read through the book. I have to . like do the homework . like slo::owly at my own pace and like do it myself. and . um . that's like the only way I'm gonna retain anything or like know what I'm doing" (lines 256–257)

[…]

now that . I'm actually like reading the book, working with friends, like doing the homework, actually doing the homework myself" (lines 264–265).

The most striking reference about Bettie's reliance on herself for the homework, was when she reevaluated her experience of being tutored by Paul in the previous semester. She provided two strikingly opposite evaluations of the tutoring experience with Paul (read Table 5-6). What was "awesome," the "best thing that ever happened", and "really good" early in the semester (first column in Table 5-6), became, by the end of the number theory class, a flawed learning method ("I didn't really learn"), which was to be repaired by relying on the self while doing the homework ("I should've done it myself"). After the number theory class, Bettie became critical of the methods of learning that involved copying and memorizing, which she praised highly in her early interview (recall Id5, Id7, & Id11). The alternative learning method became "doing the homework by herself."

Table 5-6: Comparing Bettie's perspective on her experience of being tutored by Paul, in the early and the exit interviews.

Int1-0922-Bettie lines 88–101	Int2-1202-Bettie lines 280–290
Fady: Okay, uh are there other experiences that are . maybe positive or something . is there? Bettie: Yeah well last semester when I was taking modern I um found a tutor his name was . you know Paul. he worked in the math tutoring room. and he ended up just being my private tutor for modern and he like completely made sense with the whole entire class for me. which was awesome Fady: tutoring Bettie: Yeah so I would just go to his every like once a week and he would help me figure out the homework. go over the notes. and like . he would re like um he would repeat like definition after definition like just to make me	Fady: You mentioned that you uh you used to have a tutoring and you mentioned at the beginning of the semester you had a tutor that helped you a lot by repeating the same exercise. Do you use this method? How do you think about this? Bettie: oh . I feel like Modern really has nothing to do with number theory, like some . somewhat but not really. and um . I feel like I kinda learned and I probably wouldn't remember it now. I learned at the time . Modern. and his way was like . cool like it made me kinda memorize but I wasn't really motivated to learn. I just wanted to pass the class. so I just kinda going like "Oh yeah" just copying the answer. I didn't really learn. He helped me . in which I got an A like in the class. because he gave me

memorize it. and it kind of just stuck. like the best thing that ever happened. yeah so that was really good. Fady: Okay and uh repetition is what makes this experience very helpful Bettie: Yeah and writing it down. Yeah like keep doing it. repeating it. just memorizing.	basically all the answers, but I didn't learn. and now I see that. now I see like "Oh" maybe I should've done it myself.

A new position in classroom groupwork and new function in

This subsection describes the two connectors A and B between functions set within solitary study and classroom groupwork as represented in Figure 5-22.

As analyzed above, Bettie animated fake participative positions during groupwork for a short phase early in the semester (Bettie: "I had to try to act like I knew what was going on" Int2-1202-Bettie lines 46–49). Fearing her groupmates would think she was "stupid," Bettie started reading the answers of the problems in the textbook to pick ideas that she could use during classroom groupwork to show some knowledge (Bettie: "[groupwork] kinda pushed me to read and learn" ibid). However, her ways of connecting mathematical knowledge was surface, as they reflected her entrenched habit of memorizing without understanding (recall discussion on page 139–141).

Notwithstanding the weak ownership of learning reflected by Bettie's fake learning positions, they accomplished a significant two-fold change (represented in Figure 5-22 by shifts from #1 to #2 under classroom groupwork and solitary study activities). Bettie's individualized fear of "looking stupid" coupled with small-group pedagogy led to a shift of a learning position in classroom groupwork and a change of function in solitary study. In classroom groupwork, Bettie went against her initial habit of not talking to classmates in her classes and started interacting with her groupmates. Moreover, to convey during groupwork a social image of herself as knowledgeable in mathematics, she started hunting knowledge in the textbook, which she initially used to copy answers into her homework.

When Bettie realized the insufficiency of her investment in her homework (scoring low), she repurposed her participative position in classroom groupwork and her reading of the textbook toward building an accurate knowledge (shifts from #2 to #3 in Figure 5-22; recall analyses in the subsections entitled *"Now I don't care"* and *Solitary work*). At this point, Bettie's individualized desire for understanding mathematical concepts (see Id1, Id4, and Id11 in Table 4-1) resurged and predominated almost all her *voices* throughout the semester. The most significant change was the increase of her positions that solicited knowledge from her groupmates during groupwork (recall Figure 5-5). Bettie started to feel more comfortable soliciting knowledge from her groupmates than showing off her knowledge by the group session on 9/17. At this second stage, Bettie mainly attempted to connect the knowledge she gathered from reading published resources, listening to her groupmates' mathematical conversations and copying their work in classroom.

The following subsections study the development of the new function–to enhance her mathematical understanding–and new position–interacting with groupmates–respectively.

The function of enhancing understanding across three learning activities

This subsection describes the connectors D and E in Figure 5-22, which represent Bettie's construction of her knowledge across groupwork, solitary study and study group sessions.

Bettie joined the study group to enhance her understanding of number theory, since the homework continued to return low grades (see scores of Hw2 and Hw5 in Table 5-1). Her solitary study and participation in classroom groupwork were not sufficient. Through the mediation of reified knowledge–**textbooks** and **Ted's documents**, she productively connected between her participation in her three learning activities: solitary study, classroom groupwork, and study group sessions.

As per her testimony, classroom groupwork was the place where she encountered new mathematical concepts for the first time. Between the class and next study group session, Bettie spent time in consulting the **textbooks** to foster her understanding of the new concepts. Her purpose was to hone down her questions and be able to understand her groupmates' explanations during the study group sessions.

Ted used to go to the study group sessions having solved the homework problems to be able to reinforce her understanding by explaining them to peers (recall the component C4.6.1 discussed on pages 69–71). As per Bettie's testimony, Ted used to explain to others by guiding his listeners through his thinking processes that led him to find the proof ("Ted is […] patient in teaching, not even teaching just like going through what he did and I can ask him and he would never get annoyed" Int2-1202-Bettie lines 324–327). He also shared with Bettie and other groupmates his number theory folder on Google drive, where he placed his homework and notes from classes. When working on her homework at home, Bettie used to consult **Ted's documents** when she struggled with problems.

The study group created rich learning opportunities for Bettie: she could have a second exposure to the mathematical concepts discussed in class after fostering her understanding by consulting published resources on her own and she could have a second chance to understand Ted's explanations during the study sessions by consulting his reified thinking in the shared documents.

The position of studying with peers across three learning activities

This subsection describes the connectors B and F in Figure 5-22, which represent the development of Bettie's position of studying with peers across groupwork, solitary study and study group sessions.

Bettie was not used to animate the position of studying with classmates prior to the number theory class. Being asked to work in group with peers pushed her to talk about mathematics with her groupmates. She animated fake contributive positions. Then, as she became comfortable with her groupmates by interacting with them, she started to animate genuine learning positions, mainly soliciting knowledge from others, conducive to her learning. The comfort she gained by working with her groupmates in classroom encouraged her to ask one of them, Ted, to help her with her homework outside classroom. This initiative led to the creation of a study group.

During the study group sessions, Bettie transformed her learning position from merely soliciting knowledge to watching her peers solving problems on their own (#6 in Figure 5-22). She described the study group sessions as follows.

> Bettie: They [groupmates in study sessions] go on the board like do all the problems their own ways and work together and then I'm kinda like just sitting there . and taking it all in. (Int2-1202-Bettie lines 516–517).

During the study group sessions, Bettie's attention was focused on observing how her groupmates were solving mathematical problems by relying on their own knowledge and modes of thinking. Recall that during classroom groupwork, Bettie's attention was divided between looking at the textbooks and listening to group conversations (Figure 5-8). Nevertheless, Bettie's desire of becoming like her groupmates was actuated during classroom groupwork as well. Bettie narrated this desire in her memos on two group sessions (11/12 and 12/01).

<u>Bettie's memo on 11/12</u>: "I love my group when I understand what we are doing. They are definitely ahead of me but I was happy to catch up and feel involved."

<u>Bettie's memo on 12/01</u>: "we all understood #1 including me" and "I love when I can keep up with the smart kids."

Recall that Bettie's understood the mathematical materials and felt involved during the group session on 11/12, because of the Ted-Bettie activity by which Bettie identified with Ted, through the so-called mirroring process (see Chapter 4 subsection entitled *Quadratic residues*). Moreover, on 12/01, Bettie could "keep up with the smart kids" because she computed the continued fractions by herself and reaffirmed her arithmetic identity (see Chapter 4 subsection entitled *Continued fractions*).

Bettie appreciated her groupmates' modes of smartness ("I was happy to catch up" and "I love when I can keep up with the smart kids"), which she recognized as lacking in her individualized identities (Po1 in Table 4-2). To put this in Lacanian terminology, Bettie identified with her groupmates' mode of smartness, that was, desiring to acquire what they manifested. The groupmates' smartness manifested as solving problems by relying on their own knowledge. By observing her groupmates solving mathematical problems during the study group sessions, Bettie was not only trying to understand the mathematical content but also attempting to acquire the mode of smartness she did not have.

The identification with her peers, as problem solvers relying on themselves, was a first step towards the individualization of the same. Bettie reported a significant moment of relying on herself in solving mathematical problems that took place while she was doing her homework during her solitary work.

<u>Int2-1202-Bettie lines 496–504</u> – *Bettie's most positive experience in groupwork*

Fady: Do you remember recall a very positive or your most positive experience in the group.

Bettie: hmmm. I don't know which homework it was, but I just remember understanding the whole thing and like . I just went through the whole entire homework by myself. Didn't even look at the google doc [i.e., Ted's documents] or anything. I just did it. I remember feeling like super. I felt really smart. But then I got to class and I was like "wow I did this homework by myself" and they're like . and I think John was like "Yeah the homework was pretty easy". I'm like "ope. [laughs] well." [laughs] I was [moves right hand and thumb down]. It was great. Whatever. I did it.

Note that the question was about a positive experience in the group, while Bettie's response was about her homework. In this narrative, the solitary work and group work were merged regarding the reliance on self in solving mathematical problems. When Bettie solved an entire homework by herself without consulting external resources, she felt "super" and "really smart." Despite John's remark that qualified her enthusiasm, she celebrated her achievement ("It was great, whatever, I did it"). The sense of accomplishment conveyed by the proclamation "I did it" indicated an endeavor, by which Bettie aspired to rely on her own knowledge when solving mathematical problems.

In summary, the pedagogical demand of relying on one's own knowledge for doing the homework was a nuisance for Bettie at the beginning of the number theory class (Int1-0922-Bettie lines 136–137). Nevertheless, she observed her groupmates continually animating this position and celebrating their achievements when they found solutions, during groupwork inside and outside the classroom throughout the semester. By the end of the semester, she celebrated her own achievement by animating the same position. Bettie first encountered this learning position, intrinsic to active learning pedagogies, in the social realm, among her peers. Then, she learned to appreciate and desire it before she individualized it.

Conclusion

The reliance on one's own knowledge in tackling mathematical problems is one defining feature, among others, of active learning (Jones, 1977; Mahavier, 1999; Yoshinobu et al., 2011). It is most commonly translated into a classroom norm that forbids the use of published resources during classroom group or individual work. Students, like Bettie, who are not prepared for active learning pedagogies and lack the needed disciplinary knowledge may suffer from such norm. Such students need to develop two distinct competences: learning positions coherent with the new pedagogy and disciplinary competences. The development or formation of new pedagogical positions will have to develop out of individualized identities (semiotic repertoires and entrenched habits). As Bettie's case illustrates, such students need time to repurpose their learning positions and functions to make them productive within the new learning environment. During this period of adjustment, students need to be able to access their habitual resources, so they can keep up with the knowledge that their more advanced peers are constructing. If students are provided with appropriate learning ecologies, such as in Bettie's case, the initial learning positions and functions eventually change to allow for the desired new identities to take roots.

Bettie's case is good news to innovative educators. For new learning habits to take roots, students do not need to silence their habitual voices or do away with their entrenched habits, if at all they can do such things. On the contrary, in Bettie's case, individualized identities belonging to an opposing pedagogy, such as the reliance on published resources or experts/tutors for knowledge building and test/grade-oriented dispositions, became resources that spurred productive shifts of learning positions and functions within the new learning ecologies. We should note that the development of Bettie's voice in the number theory class–the trajectory from where she started to where she ended–is remarkable and rarely observed in classrooms using new pedagogies. As the analysis in this chapter shows, three learning activities, namely solitary work, classroom groupwork, and study group sessions outside classroom, conjoined to afford Bettie's development. Each learning activity created different and complementary learning opportunities for Bettie. Rather than confining themselves into the designed activity, innovative educators of active learning may want to encourage students to participate in learning activities outside the designed one. By doing so, students may not only enhance their knowledge of the discipline but also learn to animate the needed pedagogical positions.

Discussion, Conclusion, and Outlook

This dissertation was motivated by both practical and theoretical/methodological issues. On the practical side, the problem of attrition and the challenges of small-group learning in undergraduate mathematics-based programs were central. On the theoretical/methodological side, the dissertation was concerned about the challenges in unpacking the interpersonal and intrapersonal dynamics to understand the development of identities within various contexts over various time spans (group work, a semester-long course, a student's mathematical career to date, and envisioned). The development of new theoretical/methodological tools in this dissertation afforded the shedding of some light on the two practical issues. The analytical chapters focused on two case studies (Ted and Bettie), with little discussion of the practical issues. Here, I discuss the practical issues then conclude with the theoretical/methodological contributions of this dissertation.

Practical issues: A discussion on attrition and small-group learning

Educational research in STEM is split with respect to the problem of attrition. While some research trace the problem of attrition back to a deficiency in traditional teaching methods at college (e.g., Daempfle, 2003; Ferrare & Lee, 2014; Seymour & Hewitt, 1994), other research trace it back to students' abilities as they graduate from school (e.g. Crisp, Nora, & Taggart, 2009; Green & Sanderson, 2018; Heilbronner, 2011; Thompson & Bolin, 2011; Whalen & Shelley, 2010). Each set of research suggests a different treatment approach. The former proposes to use evidence-based student-centered pedagogies at college (Olson & Riordan, 2012). The latter, part of which dismiss the effect of college teaching styles on attrition and attainment (Green & Sanderson, 2018), suggests to foster robust mathematical understanding at school level.

Since most student-centered pedagogies rely on groupwork, faculty who want to reform their teaching may wonder whether they open Pandora's box by using small-group learning. Indeed, several research projects have scrutinized small-group learning, although more so at school than college levels. On one hand, a large number of studies provide evidence of the multifaceted impact of small-group work at school and college levels (e.g. Freeman et al., 2014; Gresalfi, 2009; Hassi & Laursen, 2015; Laursen, Hassi, Kogan, Hunter, & Weston, 2011). On the other hand, faculty have faced some resistance to innovative teaching styles, mainly the pedagogies that require students to take ownership of their learning (Ellis, 2015; Seidel & Tanner, 2013). In addition there are various detrimental processes, such as undue power and emotionally unsupportive dynamics, observed during peer interactions in learning settings (e.g. Callahan, 2008; Engle et al., 2014; Langer-Osuna, 2016; Webb, 2013).

Faculty can indeed find in contemporary research reasons to dismiss small-group learning in their classrooms and doubt whether a change of their teaching practices will ever mitigate attrition or foster attainment. Notwithstanding, the cases studied in this dissertation demonstrate the possibilities inherent in those pedagogies.

This dissertation was situated in what might be called an existence proof. Professor Hoffmann's number theory course was based on small-group work, which contributed in significant ways to the changes in identities of the two focal students, Ted and Bettie. The delineation of their changes—Ted pursuing a career as a mathematician and Bettie becoming (and seeing herself as) a genuine learner/doer of mathematics—demonstrates not only the positive impact reformed pedagogies can have at the college level, but the details of what supported student growth and the mechanisms by which it took place.

The analyses in this dissertation highlighted the importance of *the identifications with others* as a central process for both Ted's and Bettie's identity development. The process of identification with others was understood as the desire of what is lacking in one's own individualized identities but found in others. Ted's identification with mathematicians who proved theorems set him on a productive path in the number theory class, which permitted him to realize his mathematical ability and desire. Ted did not need to see an expert reproducing the proofs of theorems on the board, but he needed to produce them by himself. Groupwork provided him with opportunities to refine his mathematical reasoning. As for Bettie, she needed to see peers who solve mathematical problems in ways other than reproducing memorized procedures. Her participation in groupwork stimulated her identification with such peers, which led her to start doing her homework by relying on her own knowledge. Lectures would not provide such learning opportunities for both Ted and Bettie.

Both students noted that their learning experiences in small-group could have turned unprofitable, had the learning ecologies been different. Ted would have become oddly competitive in groupwork requiring a consequential collective task. Bettie needed to use the textbooks as she transitioned to an active learning disposition. Her transition would have been impeded by a rigid enforcement of the classroom norm that discouraged students from using textbooks. Moreover, the amiable interactions in her group encouraged her to be a genuine learner and work with her groupmates not only in classroom but also outside classroom.

This dissertation proposes a promising orientation that educators and researchers can take as they continue to learn how to implement small-group learning at college and school levels. We need to focus our attention on creating learning ecologies that generate and foster students' productive identifications with others. Instead of regarding small-group learning as Pandora's box, we should be encouraged to conceive of it as Alibaba's den, which holds an astounding treasure despite inadvertent troubles caused by thieves.

Theoretical and methodological contributions

The theoretical and methodological tools developed in this dissertation enabled the identification of significant moments in the development of both students' identities and the factors that contributed to them. The VIP+function framework and the supporting data collection techniques, mainly SCNI, were developed as a means of

(i) considering students' past, present, and future
(ii) in investigating their engagement with activities at different timescales (i.e., groupwork sessions, course, and major program)
(iii) by analyzing multiple modalities of data (i.e., observations of ongoing practice and narratives).

At the crux of its contribution, this dissertation constructed a new conceptualization of '*voice*' by building on socio-cultural, socio-linguistic, and psychoanalytical theories. It suggested understanding '*voices*' as

(i) the actuation of narratively mediated and historically individualized *identities*
(ii) by animating *positions* in moments of an ongoing cultural activity
(iii) to target *functions* beyond the immediate time frame of the animated position.

Exemplified by the cases of Ted and Bettie, the VIP+function framework proposes a renewed appreciation of identity development. Rather than looking at identities as behaviors appearing in some contexts and disappearing in others, the VIP+function framework enables the observation of the continuities within the discontinuities in the voices that actuate different identities across contexts. As shown in Ted's and Bettie's identity development (Figure 3-12 and Figure 5-22 respectively), voices can maintain continuity by sustaining the same positions and/or functions across learning activities. Through continuities across contexts, students can build on learned experiences in activities other than—and prior to—the current one. However, due to the possibility of discontinuities between learning activities a development into something different can occur. This dual mechanism of discontinuity and continuity, as illustrated in the VIP+function analysis of Ted's and Bettie's voice development, can explain the commonly observed positive effect of participating in different-but-related learning activities on attainment in STEM fields (Callahan, 2008).

My hope is that over time the theoretical and methodological tools developed in this dissertation will prove useful, not simply at the undergraduate level, but in K-12 STEM education more broadly. As the field comes to develop tools for inquiry into identity that refine our understanding of the processes by which personal identities grow and change within and across learning activities, it may become increasingly possible to develop richer learning environments, attracting and keeping students in STEM.

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Appendix A:

Transcript of Bettie's Interviews

Int1-0922-Bettie

[00:00:08.09] F: When you will be done, you'll just . it's good to . if you have time to do this right now, this is part of the survey, this is part of the interview as well, but you can do this when you are done or uh . umm alright so whenever,

This is an interview that where, I will ask you questions. Uh don't worry about this right now, uh I want to from these interviews to help me see through your eyes like help me introduce me into your world to learn who you are that can help understand uhh what what you are and uh in the classroom and with the group. Uh for now for this interview I'd like to ask you about your history concerning mathematics and your history with small group work and your learning habits. We're gonna go from one to the other uh through the activity. Uh is that okay you can skip any questions that you feel you don't want to answer or just say I skip okay? It's o kay. Uh let's start with the area I want to mean as part of this to be as informative as possible so I can really understand uh who you are and how you think uh and but also very much as genuine as possible. Let's start with brainstorming, so I'm going to tell you a word, whatever pops up in your mind, say it out loud, okay? Whether emotions, events, words, sentences, memories, anything, okay? And the word is, uh, mathematics.

[00:02:15.21] B: Hard. [sigh]

[00:02:24.00] F: What else?

[00:02:26.23] B: More? Uh rewarding I guess

[00:02:31.19] F: Hard, rewarding, okay. Uhh why is it hard?

[00:02:40.11] B: Uh just cause you have to really work hard to understand it. Not so much that it's just like uh this is too hard but just like you really have to be committed . and rewarding is because when you actually do understand it it's kind of fun.

[00:03:00.05] F: uh huh uh huh uhh okay. There anything else you want to say about mathematics that pops up in your mind?

[00:03:10.23] B: uh probably like teaching styles, uh a lot of time you won't get a teacher that.

Interviewer: Teach cells?

B: like teaching styles.

Interviewer: ah ok.

[00:03:25.16] B: ah. cause a lot of teachers just teach what they know they don't really dumb it down kinda. I feel like I'm not super um I don't know I guess smart so it takes me a while to understand things I have to see it done a couple times and like I have to do it a couple times to like completely fully understand it. I can't just like hear words and understand what they're trying to say, like I have to see and a lot of times you don't get teachers to do that so you just have to so it's hard like you have to find the way of learning yourself.

[00:03:55.17] F: okay thank you. Uh when you socialize with other people, uh with friends or family or uh relatives and they happen to speak about mathematics or mathematicians, what do you they usually say?

[00:04:16.17] B: mmm well I guess when they ask me like what my major is I'll say math and then they're just like oh wowww like you're smart or like I guess, I don't know. Like it's hard like it is for you but I just put in more work to understand it.

[00:04:32.21] F: okay uh huh. And you're majoring in mathematics for?

[00:04:38.27] B: uh liberal arts. Mathematics for liberal arts.

[00:04:43.02] F: okay. What do your parents think about your major?

[00:04:51.07] B: uhh . They think I'm crazy.

[00:04:54.12] F: why?

[00:04:56.07] B: uh I don't know. Me and my brother and sister all seem to be really good at math and suck at everything else so I mean I guess for them it kind of makes sense why that's my major but uh I don't know, they kind of just see like I'm still in school so they're just proud that I'm trying to get a degree. They're not really too involved in like what I do.

[00:05:16.19] F: oh. Do you live uh with them do you live on your own?

[00:05:22.07] B: yeah, I live on my own here

[00:05:24.22] F: okay and what do you plan to do with the degree in the future as is a thinking comes to your mind

[00:05:31.28] B: mm I kind of just want to get into business, like business field finance kinda, I don't know like I really don't even know, I just want to get a degree and try to start from there.

[00:05:42.12] F: okay. Uh is there any specific way where number theory specifically can help you uh with your future career or

[00:05:53.27] B: I guess just help me in like my future classes that I plan on taking just cause a lot of the proofs that we do now I've seen from other classes and like a lot I'm sure proofs I'll see in this class will come up in other classes.

[00:06:10.19] F: okay and uh is um uh is it required for your major

[00:06:18.02] B: uh it's an I'm taking it as an elective.

[00:06:20.21] F: oh okay why did you choose that

[00:06:23.11] B: umm kind of just word of mouth people would tell me that it was a good class and it was um . I had like three more electives that I need to take so I'm just taking them all. And it just seemed interesting I guess. I wanna know kinda like why we do things like I see like I know formulas and I can just do them like plug numbers in but I want it's kind of interesting to see like why

[00:06:50.10] F: okay and when people told you that it's a good class, uh what did they mean by that?

[00:06:56.03] B: Uhh well I mean that the teacher they told me that the teacher was really good and I I haven't had a really good experience with teachers at this school so I was kind of excited just to have somebody that . spoke English I guess

[00:07:14.26] F: okay. okay. uh . how is you mentioned tell me more about your experience with other mathematics courses that you've taken here

[00:07:30.23] B: Um yeah. I guess like uh my first um proof class [unidentified] I don't even remember my teacher's name but he was kind of like mean, and when we asked questions he would get like pis, like pissed that we were asking questions when we didn't understand what he was trying to say . and I don't know I kind of just felt like intimidated a lot. I'm like a really quiet person so I don't speak up or go to office hours or anything so it kinda just like and like every single proof teacher I've had at this school has been like that.
[00:08:03.23] F: Okay, uh are there other experiences that are . maybe positive or something . is there?
[00:08:12.26] B: Yeah well last semester when I was taking modern I um found a tutor his name was . you know Paul. he worked in the math tutoring room. and he ended up just being my private tutor for modern and he like completely made sense with the whole entire class for me. which was awesome
[00:08:31.22] F: tutoring
[00:08:33.01] B: Yeah so I would just go to his every like once a week and he would help me figure out the homework. go over the notes. and like . he would re like um he would repeat like definition after definition like just to make me memorize it. and it kind of just stuck. like the best thing that ever happened. yeah so that was really good.
[00:08:53.14] F: Okay and uh repetition is what makes this experience very helpful
[00:08:57.04] B: Yeah and writing it down. Yeah like keep doing it. repeating it. just memorizing.
[00:09:06.14] F: Uh huh uhh okay. Uh I’m gonna ask you right now about school about your history with mathematics in general we can go back as far as you wish. We can go back to your childhood or to recent math, high school, and recently. You’re a senior, right?
[00:09:28.07] B: Uh yeah
[00:09:29.23] F: So you've been here for, this is your fourth year? Or fifth year?
[00:09:32.04] B: Well my yeah this is my second year at state but I'm going on my fifth year.
[00:09:38.24] F: Where did you where did you go before?
[00:09:40.26] B: I went to Santa Barbara City College
[00:09:42.25] F: Okay and then two years there?
[00:09:46.10] B: Three. I did three years there and now
[00:09:49.25] F: Wonderful. Uhh so about your history with mathematics and as much as you can tell concerning your joys and uh struggles with it your likeness your dislikeness uh how what is your history with these things. And you mentioned already that you were good at math, your parent said this?
[00:10:14.12] B: Yeah so arithmetic like calculus and like algebra, I was A, an A student and then I feel like that's what like kind of wanted me to be a math major just cause I was getting As obviously I was good at it. But like I don't know I'm not really um . I didn't start getting into math until I was probably a sophomore it took me three years to get out of like algebra and then I feel like after that I was like fine.
[00:10:49.09] F: Okay

[00:10:51.14] B: But yeah, so . I think it just took like one good teacher and then it just clicked for me and then it was just easier since. Cause like yeah.
[00:11:03.15] F: Can you tell me more about this teacher? What
[00:11:05.19] B: Yeah. Well I guess she kind of just pushed me cause I didn't really take high school serious at all
[00:11:11.08] F: Uh huh
[00:11:12.29] B: And then uh she kind of like forced us to learn so it was like oh okay that's why I do that so . and then . like once I I feel like read. comprehension for me is really hard. so having to read something and like fully understand it . is just like . puhhh. so like these classes have been really hard for me. but . like I'm barely sliding by. like just trying to get them over with. just cause like I can't really read a paragraph and understand what it's talking about I need like work I need like to see why. like what it means. I don't I don't. it's kind of hard in this class . just cause I feel like he's just like “do it.” so like we'll do it . like as a group . but still like I don't know if I'm doing it right so then I'm just.
[00:12:02.23] F: Yeah . uh we're going to go back to the uh to the current things can you can you go back to your history and tell me like more about the majorships that you did in with mathematics. So you mentioned this teacher that was an important shift and this was which class this here?
[00:12:28.20] B: It was algebra. Cause I did algebra A and then algebra B and then I took just full algebra
[00:12:34.16] F: Okay
[00:12:35.16] B: And then um
[00:12:36.25] F: And this was which uh year of high school
[00:12:39.25] B: My freshman year
[00:12:40.21] F: Freshman okay.
[00:12:41.23] B: And uh she kinda just made it we couldn't walk into the classroom until we could like repeat the formulas to her so I would be standing outside standing out there trying to memorize the stinkin formulas [laugh] and they kinda just stuck with me and I feel like algebra like that's it all you have to do is memorize
[00:13:02.08] F: Okay and uh other places and let's say at Santa Barbara is there any specific class that you really enjoyed or did well
[00:13:18.16] B: Um yeah probably like my trig class I thought trig was really fun
[00:13:24.06] F: Trig okay why?
[00:13:25.07] B: Just cause like uh you have like a game kinda like plug in these things try to solve when you finally get it right it's just kind of just like a good feeling I guess
[00:13:35.25] F: Okay and when you came here is there uh a specific class that really uh
[00:13:46.27] B: Mm
[00:13:50.00] F: Impressed you?
[00:13:51.05] B: Probably my stats class that I'm taking right now it's like honestly I haven't had a good experience with math here like I I probably hated every one of my teachers so far so
[00:14:04.12] F: Okay alright
[00:14:07.25] B: Not so much them as a person but just like their style of teaching like I just don't understand like I'm just like okay like whatever I like asked friends out of class

to help me because I feel like stupid asking cause I feel like they just like sorta like sup superior

[00:14:24.12] F: Huh?

[00:14:25.15] B: like they [teachers] think like oh like you're dumb I just get like intimidated a little bit so I’ll just go ask my friends after. So yeah I haven't had a good experience . at this school

[00:14:38.17] F: Okay. You couldn't find students or other students whom you could approach for help or?

[00:14:45.06] B: Yeah

[00:14:48.18] F: Uhh how do you prepare for, how do you work for your homeworks. Usually what do you do?

[00:14:57.15] B: How do I prepare for homework? Uh I just do it I don't really like prepare

[00:15:03.05] F: I mean uh like do you work alone?

[00:15:06.21] B: Oh yeah

[00:15:07.22] F: Do you work with others? What do you do when you're stuck?

[00:15:12.11] B: Uh I go online

[00:15:13.24] F: Okay

[00:15:14.24] B: But yeah I work alone, unless like uh . I'll like ask in class or something "oh how did you do this one" and then I'll see like how they did it. but most of the time I don't really even take cause I just like . to read over because I like to understand things cause like it's really frustrating when I'm just like copying work I have to . really just like . understand what I'm doing and why I’m doing it so . I kind of just like to work alone because it takes me . a pretty long time to figure out.

[00:15:43.01] F: Mhm

[00:15:44.13] B: A problem

[00:15:46.06] F: Okay. Uh and for tests? How do you prepare for exams and?

[00:15:52.22] B: Well. I usually just try to really understand the homework so like when I'm doing the homework I like to fully understand it and if I can't fully understand it then I just skip the whole problem like I just don’t waste my time but I like to fully understand the homework so when there's an exam I just like go over the homework multiple times and then I kind of like read through the book and skim through the book and notes but I really just focus on the homework

[00:16:16.25] F: Okay. Do you go online as well if you don't understand something uhh as you prepare for the test?

[00:16:24.27] B: Uh yeah. I'm like always online

[00:16:25.17] F: Do you find it helpful, the resources that you find online?

[00:16:29.13] B: Sometimes. There's just like different notation so it kind of confuses me but I like try to make sense of it and then if I don't understand I just like try to memorize the work that they put and then just write it down for whenever I see the problem again

[00:16:44.14] F: Is there a specific uh website uh that you check

[00:16:50.11] B: No I just type in the question and stuff comes up, [laugh] or I'll just go on like, no not for these classes

[00:17:02.14] F: Okay what what's the effect of uh other students in your classroom has been for you when you are in a math classes what other students usually how do they affect your learning your ways
[00:17:18.23] B: Umm . They help me I guess in like uh asking the questions that I would ask I guess. uh I feel like I have a lot of questions like but I just don't ask because I feel like it's a stupid question but I don't really talk to people in my classes so uh
[00:17:39.29] F: Why is that?
[00:17:40.29] B: I don't know . I'd just rather just not talk to them. They’d think I'm stupid or something.
[00:17:49.24] F: Ohh. Okay. uh can you talk a little bit about your experience with the group work did you did you have the chance to work in groups uh in high school
[00:18:02.24] B: No
[00:18:03.10] F: Not at all, okay.
[00:18:04.25] B: Well I guess like oh go work on this problem in your group or but not like what we do here
[00:18:11.24] F: Uh so is it frequent now like uh how often
[00:18:16.08] B: No. I don't. I don't even remember some years but yeah
[00:18:23.07] F: And at Santa Barbara?
[00:18:25.11] B: No group work at all
[00:18:26.29] F: No group work okay. So this is pretty much your first class where you take groups
[00:18:32.16] B: Yeah
[00:18:33.04] F: So let's talk about your experience here so you mentioned a little bit about it earlier can you develop?
[00:18:37.29] B: Yeah I like it because we're just basically working on our homework and it kinda forces me to like ask questions and like talk to people, I guess that's why I like it
[00:18:52.19] F: Okay. You mentioned that sometimes you have difficultly like you're not sure if you understood things or not
[00:19:00.16] B: Uh huh
[00:19:01.01] F: Uhh how how does this work for you like uh what do you do about this
[00:19:08.04] B: Um nothing
[00:19:10.17] F: Okay
[00:19:11.15] B: I just take what I have
[00:19:13.15] F: When you go home and you start writing the homework does it click or not
[00:19:18.15] B: Sometimes sometimes I just like look online or look at my book and like reread the definitions over and over and then sometimes they kinda like they show us some work where I can kinda understand what they're saying
[00:19:30.19] F: Okay uh . Can you tell me so far what is what in your personality skills that you have or uh your personality is helping the group or is uh contributing to the group?

[00:19:47.17] B: Um I guess just like my questions I guess it's just like making them understand it more because I ask a lot of stupid questions just cause I don't understand half of the stuff that we talk about
[00:19:59.03] F: Okay
[00:19:59.24] B: And then it kind of makes them like when they have to explain it to me it kind of like forces them to . I don't know know how to answer the questions
[00:20:08.14] F: Okay can you let's go a little bit one after one in your group. Uhh I think you've been in this group for some time right?
[00:20:18.22] B: Mhmm
[00:20:19.09] F: Uh did you start with another group and then shifted to this one? I don't remember
[00:20:23.23] B: I don't even remember. I feel like I've been with them the whole time
[00:20:29.07] F: Okay can you tell me describe to me the profile of every other group member like how they think their habits of dealing with the group and thinking, their ways
[00:20:40.24] B: Umm . Honestly I forget all their names
[00:20:48.03] F: Okay I can help you
[00:20:49.23] B: Okay
[00:20:50.12] F: Which one do you want to start with?
[00:20:52.16] B: Umm .
[00:20:53.27] F: Uhh so today you have at your we'll start at your left, Jeremy.
[00:21:05.22] B: Okay yeah Jeremy. Umm . I feel like he he knows how he::e knows how to explain it best. like when I look at his work cause he just like it's kind of just like short and to the point it makes sense usually when I look at his work or like when he explains it to me um . I feel like he I don't know it's just kind of an easier way of understanding it like a lot of their like mind is so like smart when they try to explain it to me it’s like uhh why don't you get it I'm just like uh I don't know like sorry I mean you could like keep trying and trying so I just don't bother asking but I feel like I would ask Jeremy probably.
[00:21:53.12] F: Okay. And then Boutros? He’s the one next to Jeremy
[00:22:01.12] B: Umm I feel like he's really quiet . he's really quiet.
[00:22:07.26] F: Okay and then there is um is it John?
[00:22:13.18] B: Oh yeah John. He’s hella smart.
[00:22:16.10] F: In which way
[00:22:19.04] B: Huh
[00:22:19.29] F: In which way smart? Like how does it reveal?
[00:22:23.09] B: Like he like he like just does his work and gets it he like writes the problem he finishes the problem before any of us like get to it
[00:22:30.27] F: Okay
[00:22:34.01] B: I don't know I feel like he he'll ask sometimes but . yeah usually we look at his and try to like make sense of it and then we can like dumb it down a little bit uh Tom?
[00:22:49.26] F: yeah Ted maybe
[00:22:52.15] B: Oh Ted . yeah uh he's really smart too. He’s kind of like . uh really nice about it. Like if I really don't understand it he'll be like oh okay like this is how you do

this this is how you do this like we should work after class like all of us and try to get our homework done and he's kinda like
[00:23:16.24] F: And did it happen, do you go, do you meet outside the classroom?
[00:23:22.09] B: Uh I haven't yet, no, but probably this week because
[00:23:27.04] F: Do they meet after class?
[00:23:29.27] B: No we like just exchanged numbers but yeah he like he kinda initiated it.
[00:23:35.15] F: Okay
[00:23:37.11] B: So I would say he's kind of like the group leader among us. Cause he like makes people like he'll ask questions. Like explain how you did that or I don't know
[00:23:52.26] F: Okay
[00:23:54.15] B: Yeah
[00:23:55.26] F: I think that's it. Thank you very much.
[00:24:00.24] B: Thank you, I need to go home and sleep
[00:24:04.13] F: Yeah you look tired
[00:24:05.14] B: I'm so sick
[00:24:06.29] F: Alright. Are you taking medicine?
[00:24:10.04] B: No I like I woke up this morning feeling like crap I've been at school all day so I haven't had time to do anything
[00:24:17.10] F: This is like a cold right?
[00:24:20.25] B: Yeah uh my roommate just finished...

SCNI-1015-Bettie

[00:00:06.15] F: Okay uhhh how often do you meet with the group outside classroom?
[00:00:15.10] B: probably once a week at least. like next week we're probably going to meet up more because our midterms on Thursday.
[00:00:27.12] F: And uh can you tell what exactly things in that midterm action outside. the same group outside the classroom you find helpful?
[00:00:40.17] B: ummm I don't know maybe because I've already looked through it myself so I know what to ask. I don't know I like to look over things by myself before I start talking to other people cause I like to know what I'm talking about. So in class when we're going over a problem for the first time I don't really know what we're doing so like when they're talking I don't really understanding what they're even saying, so it takes me a while to get something so at home I do it by myself and then I come to group and I have a better understanding what we're talking about so it makes it a lot easier to understand what they're saying.
[00:01:26.25] F: People's behavior differ from the group in the class and the group outside the class.
[00:01:32.13] B: Uh no.
[00:01:33.24] F: It's the same. And who usually attends this?
[00:01:38.00] B: Well it's just been me, Ted, and Jeremy.
[00:01:52.26] F: I think the video is....
Video at 0:09:46
[00:02:45.14] F: So you can pause whenever you have something to say and uh. What were you doing at this time do you remember?
[00:02:58.01] B: Uh John gave me the definition of mu n.
[00:03:05.28] F: [unidentified] *Video at 0:10:20*
[00:03:42.24] [00:03:42.24] B: so I don't . I don't know what we're doing until you'll see I'm gonna ask him what it . I get a better understanding of the definition because I didn't know.
Video at 0:11:20
[00:04:48.08] F: What did Jeremy say?
[00:04:50.00] B: Oh he just pointed at a number. It was a really long number.
[00:04:53.11] F: Oh
[00:04:54.29] B: It was a sage calculation.
Video at 0:12:31
[00:06:11.14] F: What were you looking for here?
[00:06:18.29] B: Uh I was looking for the answer.
[00:06:15.26] F: Uh oh the answer of finding the solution in the book?
[00:06:15.26] B: Yeah. cause I didn't know I was trying to find a better understanding of the definition . because he [John] gave me like the written out definition that . he [Hoffmann] gave us but I didn't get it so.
[00:06:34.11] F: Oh the definition that John gave you at the beginning of the class was different from?

[00:06:38.29] B: Wasn't different, but I was just tryna find like . a writing out of what we were do . what we were looking for, what we're doing, what it does. I didn't get it. but he [John] helped me like soon.
Video at 0:14:22
[00:08:46.02] F: What's here?
[00:08:46.08] B: He's just showing me his work and he's like this is just showing a whole lot of nothing and I just thought it was funny.
[00:08:53.16] F: There was nothing on the paper that he was showing?
[00:08:56.10] B: No like he writes a bunch of words and work and he's just like ignore it it's a bunch of nothing. That's why I was laughing.
[00:09:06.01] F: Why do you think he showed you?
[00:09:08.13] B: um Because he could tell that I didn't know what I was doing. I kinda looked at his paper like to see and he just turned it around to show me.
Video at 0:15:00 [Prof visits group at 0:14:48 till 0:18:00]
[00:09:37.24] B: I don't really know what I'm doing until after he [professor Hoffmann] leaves. then they talk to me.
Video running [prof is still with group]
[00:11:34.21] F: So you could not follow what he was doing at all?
[00:11:39.10] B: Well my mind was still kind of stuck on . what . what we're even doing. So when I hear . when I hear people talking and I don't understand I just zone them out because it confuses me more so I just like keep do . I just keep looking on my own.
[00:11:55.02] F: well let's move forward. do you want to move forward to where we are?
Video stopped at 0:19:45 [Bettie asks Ted to explain to her "what the hell prof was talking about" -- John takes it up and explain to Bettie starting with the definition of the mobius function]
[00:14:21.13] F: were you satisfied with explanation?
[00:14:25.13] B: Yeah. it made sense.
//// Bettie asserts here that she works on #2 and no longer #1 ; but this is confusing because the book has examples for #1 but not #2. For this sentence to make sense, she might be taking cases of #2 for examples ///
Video at 0:22:00
[00:14:57.10] B: And then basically for the rest of the time I was just looking through the um the examples in the book and I was getting the answers, but I didn't start understanding it until the end so I had to relook over it what they were trying to say and it started making sense.
[00:15:13.02] F: Okay you did not understand what exactly?
[00:15:16.10] B: Uh for number 2 I forget what it was, it was like uh oh the sum of mu d and then I ended up understanding it at the end so yeah.
[00:15:32.19] F: But here I think you were working on number one, right?
[00:15:35.01] B: Two
[00:15:36.29] F: Two already?
[00:15:37.21] B: yeah we moved on to number 2.
Video at 0:23:11
[00:16:14.11] F: What are you solving here?

[00:16:18.09] B: We're doing um we're doing number 2. And I was like looking at the one in the book and I was looking at what Ted was doing and then basically I was just trying to make sense of it all.
[00:16:33.07] F: Okay
Video at 0:24:08 [when John claims that mu of any prime is 1]
[00:17:10.25] B: Isn't it negative one or is it 1?
[00:17:12.21] F: Sorry?
[00:17:14.01] B: The mu of any prime number is -1 or 1?
[00:17:19.01] F: What do you think?
[00:17:20.00] B: Negative one?
[00:17:24.08] F: uhuh. Why? Because it has?
[00:17:25.18] B: Because it's 1 prime so it is - 1 to the 1
[00:17:29.13] F: yeah its prime composition is 1 so if 1 is an odd number so
B: uhum
Video at 0:25:38
[00:18:50.14] F: Do you mind, do you mind pulling up your notebook so you can remember maybe further what you were writing at the moment and what was going on?
[00:19:04.11] B: I was really just writing out of the book. [Bettie pulls her notebook and points] So this is where we're at?
[00:19:12.29] F: Like here what were you writing before just before that?
[00:19:18.04] B: Umm. I was just copying off the worksheet and then the question I was copying the question and then I found it in the book so I was going through it and trying to figure it out. I was just writing it, I didn't understand what I was doing. and then afterwards when like I was listening to them talk until I start realizing "like oh" that's why.
Interviewer: Okay. okay.
B: That's basically all I do. like when I'm in class. I just listen to what they're saying and look at the book . cause if I don't understand it then . when they're like talking . I don't know. so I just zone people out . until I look at it myself because . otherwise it just confuses me more.
*Video at *
[00:20:50.08] F: Let's move a little bit. [F fast forward the video] *Video at 0:28:22 [TA visits the group]*
[00:21:05.05] F: Okay. You recall exactly here at this stage what you started to understand.
[00:21:17.01] B: Well I found part - I went straight to this part [pointing to her notes part B] when . I had p to the power . and then . [Bettie's cellphone sends notification] I didn't really understand what they were saying but then I missed the whole like the whole beginning part [pointing to notes below part A] . which let p be prime without a power. then that started making sense. and then I realized that anything above 1 was zero. so .
F: okay.
B: I just didn't understand why mu of p was negative 1 until . I found this [pointing to page 2 of her notes] where mu uh pd automatically goes to negative mu(d). but . yeah it soon started making sense. And this one. number 3 . I just copied out of the book. but

then I started realizing that they're just plugging in. so then it started making sense. and then I wasn't even working with the group at this point.
[00:22:31.08] F: okay. Uh when when did you split from the group at this point do you remember?
[00:22:35.01] B: Three
[00:22:36.17] F: At three? Completely went on your own?
[00:22:40.25] B: Uh
F: because?
B: I was . I found it in the book and they were just like talking amongst themselves. so I was just trying to understand it for myself cause . I didn't get where they were getting all um all the summations and everything. so I just figured it out.
F: okay.
B: And then at the end I got it and then I started telling Jeremy but then we got really aside. so I just packed myself up I was tryna go.
[00:23:06.02] F: Okay. But now there was the
Video at 0:28:45 [Jeremy and Ted are arguing about #2; Ted admits he could be totally wrong; Bettie claims Ted is right and shows the book on tablet]
[00:23:20.26] F: Do you remember this part?
[00:23:23.02] B: Oh because we are on number 2 still and he was like going over this part for the p to the a and I was like you're right because I found it in the book so I showed him the answer like what the book said and he was just like yeah!!! because he got it right. Because he and John were arguing over the answer . John didn't believe what Ted was saying was true so I was like no here it is in the book
Video at 0:32:11
[00:25:32.19] F: So here do you recall that there was a conversation between them that you handed in -- did you follow the conversation?
[00:25:42.01] B: Um no I I I was kinda I guess I was following what they were saying but . it didn't make sense until I found out . until I figured out that mu of prime is -1. so until then I don't know . I don't know what I was thinking. I was just wasn't understanding completely
Video at 0:34:07
[00:28:06.16] B: that's when I realized that I didn't have uh. that I skipped this part [pointing to part A on her notes] and then I went straight from here [part B on her notes] and went straight to here [correcting herself] instead of going here [part A on notes] first.
F: okay.
*Video at
[00:28:44.28] B: I really have to leave
F: Okay no problem.
/// Bettie continues to watch the video and forward///
F: Okay um do you want to watch the time while you understand the mu the mu of prime. Can you just show me that time? Where was it?
This is the first video probably it's in the second.
[00:29:24.23] B: So I think it's right here.
//// two stops noted 0:18:50 and 0:24:09 ////

*Video at
*Video skips to
[00:29:54.10] F: Here?
*Video at
[00:32:21.28] B: I was talking about how I have 2 midterms next week.
*Video at
[00:32:28.00] B: we're talking about stats.
*Video at
[00:36:52.02] B: And then I'm done with them?
[00:36:54.21] F: From here you..? You're done? Can I take a picture of your notes because we talked about them? Okay so this is. one minute ok and and this one. perfect! Thank you very much B I appreciate it! You can take these cookies on the road.
[00:37:45.22] B: No I shouldn't - Thank you!

Int2-1202-Bettie

[00:00:00.00] B: And it's like for graduation here like upper division that doesn't have to do with
[00:00:05.24] F: Oh it's not related?
[00:00:06.23] B: Yeah so I'm just taking it. mhm.
[00:00:11.13] F: You enjoy it?
[00:00:12.13] B: Yeah. It's easy. Just try to get out of it. It's actually pretty interesting. Just cause I do workout a lot and it's like you learn new stuff.
[00:00:25.01] F: okay. so just a sec. ok. so today is 12.
[00:00:35.15] B: 02.
[00:00:36.18] F: Yeah. uh okay don't crash [talking to the laptop]. okay. uh. okay. so today B we're not going to do the regular videos it's going to be more thinking, reflecting, uh about the entire experience with you. okay so yeah. everything is working and and okay. uh. starting. okay so I'm going to ask you questions about general issues concerning your experience in this class and you can skip any question if you don't want to answer it just say I skip that.
[00:01:32.19] B: okay.
[00:01:34.19] F: uh I would like to start with the brainstorming like uh I'll say a word and whatever pops up in your mind say it out loud. and.. ok. whether word, verbs pop in your mind or experiences, feelings let them emerge and express them out loud. uh and so what do you what does pop up in your mind when I say number theory.
[00:02:00.14] B: uh hard. do I have to explain why?
[00:02:05.10] F: okay. no just let it other words also come out we can go with it. So it's hard. Why?
[00:02:13.06] B: Why?
[00:02:13.27] F: yeah. You can say more about this?
[00:02:15.07] B: Um. Well I mean just for me I feel like theory in general is just like learning proofs and . I don't know . it's just been really difficult for me. But . uhh out of all the proof classes I've taken this is probably the most that I've . like . learned . I guess you can say. Cause a lot of the time I kinda just got by. and I feel like this one I'm actually understanding like . why. just cause . yeah.
[00:02:40.13] F: okay uh. did you take a class on proofs here?
[00:02:44.25] B: yeh
[00:02:45.05] F: And how was it?
[00:02:47.09] B: I didn't learn anything.
[00:02:48.29] F: In that class?
[00:02:49.23] B: Yeah. so when I went into um. Modern Algebra I got a tutor, which kinda helped me, but I still didn't really . fully get it. I just kinda memorized and wrote down what I remembered. uh yeah like I feel like this is the one class that's actually helping me understand. and like come up with things on my own instead of like finding the answer and writing it down.
[00:03:15.00] F: Is there any particular thing that you find that helped you?
[00:03:18.21] B: uh
[00:03:19.21] F: in learning better than other classes?

[00:03:23.00] B: Just . maybe . not trying . trying not to look stupid. [smiles]
F: aha
B: Because like it forces you to talk to people and you don't want to be the one person that doesn't get it, I guess. Because that was me in the beginning and I felt so dumb and I hated it . I hated groups. But it forced me to talk and I had to try to act like I knew what was going on. I kinda like it kinda pushed me to read and learn.
[00:03:48.17] F: Did you usually come prepared to group work?
[00:03:51.18] B: Uh sometimes, but or just from homework . doing homework. uh I'll have to read to even like do the problem. and Ted would like have his own little google doc and then like he would write he goes in full detail for every homework assignment. and it's easy to read him. and see what he's doing and read the book and then figure out your own way of doing it.
[00:04:16.18] F: Okay so you used what Ted used to post a lot?
[00:04:23.12] B: mhm. Definitely.
[00:04:25.08] F: Okay I think we're going to go back to some of these topics in the next questions. Uh I would like to first ask you if you got the chance to talk about topic related to number theory with people outside the class like your family, friends, colleagues at work?
[00:04:42.11] B: No (chuckles)
F: it didn't happen.
B: Yeah they have no idea what number theory would even mean.
F: okay.
[00:04:50.07] F: Uh and which theorems, definitions, notions, methods whatever you learned in this class that you liked most?
[00:05:02.03] B: ummm.
F: if any [laughs]
B: if that I like the most. Probably just like greatest common divisors. I feel like that was more like . arithmetic What was that? I think it's Euc . Eucledian Algorithm. It feels like using math uh arithmetic, I kinda just like doing that. I'm more a fan of arithmetic rather than writing and theory.
[00:05:29.09] F: ok ok. That's ok. Good. Uh would you feel ready for example to give an example or explain this Eucledian Algorithm?
[00:05:37.28] B: no. Heck no. I just do it. I don't even know how I do it. I do it though. I have to write it down full step for step. Don't know what I'm doing, but I'm doing steps.
[00:05:50.06] F: Is there any topics that you struggled with? Do you recall which ones?
[00:05:54.26] B: oh yeah. Probably uh. Chinese. I remembered Chinese Remainder Theorem was the hardest thing I did and I didn't even learn it until the midterm. The midterm came up. cause even for the homework I just like [waving right hand in the air] woof . that was it . I just did it.
F: feel free to say more.
B: But um when I uh when I was watching youtube videos on how to do Chinese Remainder Theorem I finally understood how to do it. But it was different from the way uhh the professor did it . a little bit. so I like . remember on the midterm I was trying to use my calculator and he was like no calculators and I was like "oh my shit" because I

needed . the calculator. Either way I figured it out. That was hard. It was really hard for me to comprehend.

[00:06:43.10] F: And other notions like primitive roots. Uh quadratic residues. How did you feel about those notions?

[00:06:51.03] B: Uh. I thought those were uh nice because I was able to understand them. I could do the problems by myself rather than ask how.

[00:07:03.15] F: aha ok. Good. Did you find your best ways of learning number theory? What were your best learning methods.

[00:07:15.14] B: Probably just doing it by myself and working with a group after.

[00:07:20.27] F: okay. After you do it?

[00:07:22.16] B: Myself. Like trying to go through the problems myself and learning it my . like slow pace . at my own time. And then when I come to the group and then we talk about it, it all made sense.

F: ok

B: But I had I had to like step back and do it . and do the learning on my own. because like . other people will explain things the way they understand it and . the way I understand it is completely different.

[00:07:48.06] F: And uh the group work I noticed and you mentioned in the memos that sometimes you worked on your own in the group and you were using mostly the textbook. Uh what is that so uh why you prefer to work on your own rather than working with the group? Is there a reason for that?

[00:08:09.10] B: Uhh just at first cause um when . they usually seem to get it. they just like go through it. and I feel I feel like annoying or stupid trying to ask them for help all the time, so I just look on my own and then I figure it out kinda like a little bit by myself and then . Then I look to them and they like help me and . or I'll just look at their work and see how they're doing it and then put two and two together. but I like to learn it on my own because I don't like having to ask questions and then I don't like people getting irritated with me.

[00:08:41.24] F: Oh. if you [unidentified] them

[00:08:43.13] B: oh yeah It's because everyone else is getting it and I'm like the one "What" "Why" "How?" And they're like moved on to the next problem and I'm just like "Oh, I'm still on number one."

[00:08:52.26] F: Did you try to ask them sometimes? Are you saying this because sometimes they got irritated or?

[00:08:59.18] B: Oh no no. Just cause I felt.

F: how you think they would do.

B: uhu. I felt like I would get annoyed if someone kept asking me about number one when I'm trying . like . work on number four or . whatever.

[00:09:14.28] F: Umm. Okay. And um When you were stuck with your homework, what did you do?

[00:09:23.00] B: Uh just copied and pasted it. and looked online. kinda. Or like I would just type in the subject and try to just watch videos and try to understand from step 1 because . uh . I really like to . know what like what I'm doing and why I'm doing it. I hate just copying down work, so I'll look at um I'll watch videos and try to figure out my own

way of doing it or . if I can't . I literally just can't do it, I'll look at Ted's powerpoint and see how he did it and like use two of them and make my own. and I'll re-read it and then I'll rewrite it in my own way. just because I feel like . then that way I'll remember it . if I can like put it in my own words.
[00:10:03.22] F: Did it happen that you submitted incomplete homeworks or you kept some exercises blank?
[00:10:08.28] B: Yeah in the beginning I did all the time.
[00:10:11.18] F: In the beginning of the? Why?
[00:10:11.18] B: Um just cause I was too embarrassed to ask for help. And uh I was like trying. I was just being lazy. I wasn't taking the homework serious. I was just doing it the day before or the day of and try to get it finished. And the ones that were super hard and I was stuck on I just skipped and go to the next one. And then like once I started seeing how low my homework scores were, uhhh like "oh shoot, I really need to step it up."
[00:10:40.28] F: How did you prepare for the midterm?
[00:10:43.07] B: Uh homework.
[00:10:45.11] F: Going through the homeworks?
B: uhu
F: Like doing it again? On your own?
B: uhu
F: And then checking with the solution?
[00:10:51.02] B: The group.
[00:10:52.21] F: Oh the group?
[00:10:53.03] B: Yeah I met with the group too and a bunch others from the class that I never even knew existed, which is cool.
[00:11:03.06] F: uh. So with whom did you work? Can you just tell me?
[00:11:07.22] B: Well my whole entire group. and then Melissa. and then I forget her name, she's in the group behind me... She doesn't do interviews.
[00:11:20.27] F: Laura?
B: Not Laura, the other one.
F: Jennifer?
B: No.
F: Judy not Jennifer. uhu Sara?
[00:11:24.21] B: Maybe it is Sara. Does she have a nose?
[00:11:28.01] F: Mona? Everyone has a nose.
[00:11:31.09] B: Yeah I know. I don't know her name. But one of those girls and then I worked with um who else? I think that was it. mostly Boutros, Ted, and um Boutros, Ted, and uh ... um What's his name?
[00:12:02.22] F: In your group? Jeremy?
[00:12:02.22] B: Yeah. Them three in my group. I didn't really work with John.
[00:12:09.07] F: How did this happen? Like you said you did not know each other. How did this work?
[00:12:16.00] B: Just cause others knew other people.
[00:12:17.27] F: Who? Like who? I'm very interested in knowing details about that.

[00:12:22.14] B: I think Boutros knew Melissa and the other girl cause they had classes together. but then I found out I had them in other classes but I didn't even . like . I didn't even know Boutros was in my proofs class and I think Melissa was in my proofs class too. and then
[00:12:35.15] F: This was years ago?
[00:12:36.18] [00:12:36.18] B: This was last year. I had like never ever talked to anyone in my classes. just because I'm like shy and embarrassed. But yeah so I thought that was cool.
[00:12:48.00] F: Okay Boutros. You met with Boutros because he was in your group and you used to meet with him outside to study?
[00:12:56.04] B: Well we get like a room. Me and Ted mostly and then Ted like teaches all of us. how to do it, so that's cool.
[00:13:05.10] F: And then he brought Melissa to the group?
[00:13:06.04] B: Uh Boutros did.
[00:13:09.00] F: Boutros brought Melissa to the group? And Sara as well?
[00:13:10.09] B: Yeh when we were studying.
[00:13:14.10] F: So Sara was with Boutros or someone else?
[00:13:15.27] B: I think she knew Melissa. So it was like people knew people and told them we were like studying and so then they just showed up.
[00:13:24.17] F: So how did all this started?
[00:13:26.06] B: Because we just got a room to study for a group. we just got a room down in the library.
[00:13:33.13] F: And you mentioned this did not happen in the beginning - this happened later on. What do you know when this happened? and how?
[00:13:40.23] B: Uhh just cause I asked umm Ted . I forgot what we were doing. we were just doing homework when Ted said like "I'll be in the library if anyone needs help on their homework just stop when I'm here." and then we started doing the homework and he understood it completely so I asked him if he can just start helping me on my homework. and he was like "Yeah I'm always here." and we started a group message. and then now everyone texts everyone when we're in the library and we just like meet up if we're in school or not.
[00:14:06.08] F: Okay. Is uh . like do you remember approximately give an estimate of when this happened?
[00:14:13.28] B: when? oh.
F: Which week?
B: Probablyyy like.
[00:14:20.20] F: Is it before the midterm?
B: yeh. Probably:y
F: Way ahead before the midterm?
[00:14:22.05] B: yeah probably like - mid - yeah right before the midterm. Like mid midterm if that makes sense. Like a fourth into.
[00:14:33.21] F: Like a fourth into the semester? So you probably started in September. End of September?
[00:14:39.07] B: right at the end of September.

[00:14:42.07] B: End of September - early October.

[00:14:45.13] F: okay. thank you. How are you planning on preparing for the finals?

[00:14:50.03] B: Uh homework. and just talking with the group. That's it.

[00:14:54.22] F: Did you already set time for that?

[00:14:56.22] B: Uh yeah.

[00:14:59.11] F: You already meet regularly?

[00:15:00.18] B: We already meet regularly. We're probably going to meet today.

[00:15:05.16] F: aha. So you like how does it happen? So every time you meet to schedule another time? Or.

[00:15:11.21] B: yeah . or we sometimes . we don't even just plan it we just say "Oh I'm in the library"

[00:15:16.14] F: And you know each other’s phone numbers and stuff?

[00:15:17.27] B: Yeah.

[00:15:22.28] F: and now you already mentioned this - how would you assess your learning in this group? About this uh. this uh class.

[00:15:33.14] B: um I like it. I . I'm . In the beginning I really hated it, I thought it was like pretty stupid just because . I was uh I was nervous to like . look stupid . and I didn't want to ask questions like to my group. and . uh I feel like Hoffmann was kinda leaving it to us to learn . the material. uh I didn't know how like exactly he wanted it and I was just kinda like guessing . the format and I was just . not . I don't know . I don't know how to explain it. But after . um . I got over that uh . self conscious.

[00:16:15.06] F: uh what did help you to do that?

[00:16:16.13] B: um just talking to them more. talking to the group more. So I started to feel more comfortable and I was just "now I don't care."

[00:16:25.04] F: Did the meetings uh outside the class help you with this?

[00:16:29.29] B: Uh yeah probably? Probably just a little bit. because we can talk more outside of math and number theory. Like not just talk about number theory we can talk about personal things and just like uh . I even met Ted's girlfriend. which was cool.

F: Ah. ok. this makes you feel closer.

B: uhu. Like just because we didn't. I don't know . we didn't have to like just focus on one thing.

[00:16:55.27] F: Ok. Thank you. Um . I'm going to ask you to recall the entirety of your experience and tell me if you experience any change in the followings: did you experience any change in your ways of learning?

[00:17:13.12] B: Uh yeah definitely. I kinda just . not really change it but it made me realize what . my style of learning is . kinda thing. I would just study and I didn't really know what was beneficial and what wasn't. and . um . now I . now I realize I need to read, obviously . I have to read through the book. I have to . like do the homework . like slo::owly at my own pace and like do it myself. and . um . that's like the only way I'm gonna retain anything or like know what I'm doing.

F: Ok

B: because before I would just like do the homework. but I wouldn't really . like know what I was just . because I didn't understand what the question was asking and I just didn't . know learn anything in my proofs class so it was just kinda bullshitted through that . and I was just like copying and pasting . finding answer online . and writing it out and like hoping it was the right answer. but now that . I'm actually like reading the book, working with friends, like doing the homework, actually doing the homework myself, I just feel like this is just what I need to start doing.

[00:18:22.07] F: okay. Uh you told me that . you said that this changed, so what was before that? What was your methods, learning methods before that?

[00:18:33.08] B: uh I do the same thing. but I would just find the answer and memorize how I like I would . I would see the type of question and then I would like . I guess I would just remember key words like . like "gcd" I remember. "oh I remember that one problem with gcd now just kinda like wing it and hope" . probably memorize what I wrote down for that answer and then like put it on this answer kinda thing.

F: uhu uhu

B: Because like my way of memorizing is like writing. so if I write down for homework I kinda remember how I wrote it out and I'll write it down . for like the quiz. Like the quiz question or anything, and like . it probably has nothing to do with the question, but I'm just hoping like partially I have the right idea. [laughs] so that's what I was doing. I didn't know . I didn't know what I was doing . I just memorized things.

[00:19:22.17] F: You mentioned that you uh you used to have a tutoring and you mentioned at the beginning of the semester you had a tutor that helped you a lot by repeating the same exercise. Do you use this method? How do you think about this?

[00:19:39.01] B: oh . I feel like Modern really has nothing to do with number theory, like some . somewhat but not really. and um . I feel like I kinda learned and I probably wouldn't remember it now. I learned at the time . Modern. and his way was like . cool like it made me kinda memorize but I wasn't really motivated to learn. I just wanted to pass the class. so I just kinda going like "Oh yeah" just copying the answer. I didn't really learn. He helped me . in which I got an A like in the class. because he gave me basically all the answers, but I didn't learn. and now I see that. now I see like "Oh" maybe I should've done it myself.

[00:20:23.09] F: Okay. Um. Now to another question - did you experience any change in your ease of understanding the materials throughout the semester?

B: oh yeah.

F: Did it become more complex or did it become easier? Or.

[00:20:40.06] B: It's becoming easier. I feel like I kinda . cause before . I wasn't . I wasn't even really doing homework before, so when we move on to the next like worksheet, I would just like oh like fuck maybe I should've done the last worksheet. and then I'm just like I don't know what I'm doing ever. so then I kinda just like . I like started from the beginning of the book and literally read through the whole entire first like half of the book. like this is in the beginning. Then I'm like "wow" why didn't I get that? "you good stupid. " And then now . like cause it takes a lot . now I don't know if I feel like it's based off of what we've already learned just like kinda doing things a little different.

F: uh

B: And then I'm like "Oh. Well it makes sense." So now I'm understanding it faster not that it's like easier, it's just that . um . I'm able to . figure it out faster.

[00:21:34.10] F: Did you experience any change in your ways of participating in the group?
[00:21:38.11] B: Uh Yeah. I feel like I can like uh I like . if it has to do with arithmetic I feel like I'm just . I feel like . I can do it. like I. Maybe they don't take me as like serious so when I have the answer they're like "whatever like it's probably wrong." but I usually do get the right answer and I feel like "hah" like "told you". And they're like "wait" "but what?" Jeremy always questions me and I'm like "Dude don't doubt me when it comes to m . arithmetic because." I'm . there's a reason why I'm a math major like I'm smart . I just don't understand proofs.
[00:22:14.27] F: uhu. umm So in terms of participating? ok. Did you experience any change in the group dynamics as a whole from the beginning to now?
[00:22:26.23] B: Oh yeah.
[00:22:28.17] F: What did it change?
[00:22:29.20] B: mm. Probably?
[00:22:32.23] F: What. How was it in the beginning and how was it now?
[00:22:35.04] B: So in the beginning I felt like I looked more to Jeremy to like help me . kinda like explain it to me. in like he . I've noticed like he's more like . uh . I don't know how to explain it like . I don't know. Anyways I don't even like ask him anymore, I ask Ted. cause Ted is more like ca . like not calm but he's more . patient . in teaching . or like not even teaching just like . going through what he did. and I can ask him and he like doesn't . I can tell he would never . like he doesn't get annoyed. he loves to help in people. cause obviously he wants to be a teacher. so he's like he'll go through it . and he'll like take his time to come . to school . just to help me . to like . understand things. I think that's really cool. and I feel like he's probably like . he's just really smart. He's so smart.
[00:23:28.26] F: Uh okay.
[00:23:30.27] B: And I'd always think I always thought like Jeremy had a better way of explaining things, but no. Now I look to Ted.
[00:23:40.02] F: okay . And what about the group as a whole like is there anything that changed between other group mates between how you used to work in the beginning and uh?
[00:23:52.01] B: No
[00:23:53.10] F: You previously said that you participated more at the end because at the beginning you were shy you said.
[00:23:59.21] B: Yeah. I didn't really understand what I was doing.
[00:24:01.21] F: But you do now?
[00:24:02.23] B: Yeah it's umm everyone's basically the sa:ame. like they never really changed, I feel like it was mostly me. and like . I like the white boards because I can see their work because rather than having to . look at their paper and the pencil just wasn't . so now we have the boards and I like . how the professor is like giving us . some examples and like doing . more like showing us how he wants it . and it's kinda like a guideline . he's not giving us the answer, but he's giving us like a guideline to go off of. which is easier for me . because . I'm more of a visual learner . rather than him just speaking . and . trying to like comprehend what he's saying. it's just really hard for me.
[00:24:44.11] F: ok. ok. uhh did you use the whiteboard yourself?

[00:24:49.25] B: Sometimes. I like to write on my paper just cause . I like to keep it. so I can look at it . rather than writing. If I was more confident . in what I'm doing . I feel like white board would be pretty . legit. but I just need . my papers so I can like look at it like look back at it, and I'll have it for like future.
[00:25:08.03] F: And the few times when you used it . what was purpose of using it? Or what do you use it for?
[00:25:16.28] B: Umm.
[00:25:17.09] F: Do you remember that you used it?
[00:25:18.22] B: I used it like a few times. but I only did it . because uhh I think it's cause just like erasing a lot or just messing around like not really being serious and I was just trying to figure something out.
[00:25:30.09] F: ok okay. Thank you. Uhhh so let's take a . like . if you were to give um an overall experience if your group what would you say . like briefly . like overall experience?
[00:25:49.04] B: Overall experience. I feel like I met a bunch of smart-ass people. [laugh] Like I've met Melissa, I feel like me and her were like . on the same level, so I was like . wow . this is crazy because I thought that I was like a bit the stupidest person in the world like in my group . everyone's so smart and like everyone knows . everything like they read math for fun. And I'm over here like "uhhh" . like "what are we doing right now?" They're like "Oh I read this. And I haven't even taken number theory and I'm like I already know how to prove this." And I'm like "what!" . I just thought that was crazy. These guys are fricking ridiculous. like so smart. All of them.
[00:26:25.19] F: How did Melissa help you?
[00:26:27.22] B: No. it just made me feel better about myself. cause I was like feeling rea:ally . like . I don't know. I just it made me she's like she's an equal. I feel like we're at the same level. and I've never met. I've never felt like that before. I've always felt like I'm around like geniuses and I'm like the stupid one. like this is not where I should be.
[00:26:46.13] F: okay. Uhhh. So this will bring us to the question how can you go about uh people that you met in your group and outside and talk about their number theory ability? How do you find it um like one by one.
[00:27:04.08] B: Like what? Sorry
[00:27:05.00] F: Like describe their abilities like mathematical abilities with number theory. Not general mathematics. So how do you see each one of them skills in mathematics and number..
[00:27:18.27] B: Uhh. I feel like, I just feel like they're all pretty advanced um. Ummmm.
[00:27:28.01] F: So Ted you said he is..
[00:27:29.25] B: He [Ted] knows everything. Like I think he just kn . I don't know. I think he's already read the book before or something . I don't know. He just knows it all. so I always usually look to him he always knows the answer. Or like he knows how to go about . he knows how to start it off. And then . like Jeremy pret . Jeremy's pretty smart too. He. But there. Everyone is always different. Like they do different ways . and then . they always come to the same conclusion . But it's easier . cause like I can look at all theirs . and like kinda like . come up with my own . way . of understanding it. cause it's just like. Ted is more like detailed . like he . he like goes rea:ally into depth with

everything. And then I'll look at Jeremy's and he's like quick and short and like done. But it's like the same . and same conclusion. which is like cool . then I can like read into details on this side [waving right hand] and look at what it looks like . just like math . on this side [waving left hand] and then . and then it makes stuff of it that makes sense.

[00:28:26.16] F: Yeah. Okay. Uh John?

[00:28:29.14] B: I don't really ask him for help.

[00:28:32.00] F: But how do you see him?

[00:28:34.00] B: He's smart. He's smart. But he's . like . He's mo:ore of his own. I don't know I feel like he's more to himself. He like talks to them, meaning like Jeremy and Ted . but I don't know. I don't really ask him for help. I feel like he's learning at the same time I'm learning . but he's like . he gets it faster. and he just like learns. I don't know.

[00:28:58.11] F: Boutros?

[00:28:59.20] B: Umm. Boutros's smart too. but he's just. he's like me . like we sit there and do it ourselves . before we like talk . about it. And then like . like when we met . when we meet in like the lab that's when I hear him talking like "Oh wow. You're so smart."

[00:29:17.13] F: Oh he's different from um.

[00:29:19.07] B: From them? Yeah. Cause they're more like verbal like "oh like. this is. What did you do from there?" They like talk and figure it out. Then me and Boutros are like looking at our books . like or like doing it on our own . rather than . like talking about it because we're still trying to figure it out.

[00:29:34.24] F: And when you work with uh in this small group outside class is Boutros different?

[00:29:41.15] B: well uh . he talks more and he like knows what he's doing. like he'll like go up on the board . and he'll like do the whole problem . and I'm like "wow" [amazed face gesture].

[00:29:49.29] F: Okay.

[00:29:52.22] B: I think that's really good. Like that makes sense . and he's like he just gets it . like he can just do it. but he's more to him. like he's more . like personal . when it comes to doing the problems, but when we work together then that's when I see like oh okay . he gets this.

[00:30:07.02] F: Uhh. And . so you talked about Melissa and you see that her mathematical abilities are like yours.

[00:30:16.23] B: Well I feel like we're both like . uh . we learn . at our own pace. We like learn it. but we like learn it in our own way, if that makes sense. Like we're not . like so reading and like genius . like . these guys . like math is their life.

[00:30:35.19] F: aha. How do you see about. What about the other girl that joins you outside the class.

[00:30:40.21] B: Um that was only one time. That was only once. Yeah and she seemed to understand it too.

F: aha. Okay.

B: But like I'm not saying she's dumb or anything we're . like . She's smart too . but . obviously. she's in math . Number Theory. but we're just like . we were confused. I had a friend in Modern . and she was just so smart too and that was my only . like that was the first friend I've ever met in any math class. and she like understood everything. and she like spoke . like . I don't know how to explain it. I just felt like intimidated . kinda . Like she's smarter than me. or better than me. so I would like . I would pretend to understand things and like . pretend like I was like at that level, but I wasn't. I was just like go with it. Like "oh I don't know what you're doing, but okay" "yeahhh. It's right." [laughs] kind of thing. That's how I felt like in the beginning with the guys . because I was just like oh these guys are just like so smart. but then now I'm like I don't care . like "how do you do that" . "what are you doing?" [giggles] "Teach me" . "Go slow" [giggles] and they're like "okay" . like "whatever".

[00:31:50.25] F: Uhh. Now can you describe the roles that each group member tended to take in the group . group work? Go one by one. Like what do you see that they tended to do.

[00:32:06.29] B: I just feel like.

[00:32:08.00] F: If this changed from the beginning till' the end that would be interesting to see.

[00:32:12.00] B: No. I just feel like Ted is the leader and everybody else kind does their own thing . or like sometimes . um . I don't know. Ted is the::e like . he's the leader of the group for sure.

[00:32:27.18] F: And how do you describe your contributions to the . How did you uh contribute to it?

[00:32:34.27] B: Mmm. I don't know. I feel like no one really took me that serious so when I get an answer they're like . "oh" . like . "alright" . But when I get the right answer they're like "How" "What" "Let me see" . Then they'll be like "oh okay kinda makes sense" and I'm like "yeah". Or I'll find like . my contribution would be finding the:e answers in the book. [laughs] Like the definitions in the book and like . try to showing them "Look I found it right here you can read it" and . that's like the most I did for the group.

F: uhu

B: Ted . Ted made a freaking whole google doc . he put all the homeworks up there from like the beginning. He like . puts the work up from class. He does. like he does everything for us. and just gives us the link. So he like he takes initiative definitely to help all of us.

[00:33:25.14] F: uhu. Interesting. Can you recall a negative experience or experiences during classwork?

[00:33:32.25] [00:33:32.25] B: Probably one time I remember almost wanting to cry . cause I didn't understand what was going on . and like all of them were talking and understanding and like . I was kinda like asking for he:elp but not really I was just being like shy about it. And . um . people like they seemed like they didn't want to help . not like they were being rude or anything . but they're just like in . the zone and you know like not understanding everything and it was just like . I felt really stupid and like I didn't even want to be in the class. cause like I was already just frustrated because I couldn't. I remember I think I had just came from trying to do the homework and I was like pissed because I couldn't do it . and then I got to class and I was already like discouraged and everyone was like . doing super like going fast through the problem and I got stuck on

one . Then I remembered wanting to cry . I hated it and like . I didn't want to go class. I don't think I even went to class the next day.
F: ah
B: because I was so irritated and everything.
[00:34:23.01] F: Do you remember when - which day?
[00:34:26.17] B: No. Probably the Chinese Remainder Theorem because I remember that was just like . wah [waving from head outward-up with right hand] I didn't know . what the hell was going on. I think it was that. Probably like that worksheet.
[00:34:35.03] F: Before the midterm?
[00:34:36.08] B: Yeah. Before definitely. Before I was like comfortable around them . I guess.
[00:34:42.22] F: You've experienced something similar after that or something other things negative?
[00:34:46.15] B: No. That was it.
[00:34:49.01] F: Do you remember recall a very positive or your most positive experience in the group.
[00:34:54.00] B: hmmm. I don't know which homework it was, but I just remember understanding the whole thing and like . I just went through the whole entire homework by myself. Didn't even look at the google doc or anything. I just did it. I remember feeling like super. I felt really smart. But then I got to class and I was like "wow I did this homework by myself" and they're like . and I think John was like "Yeah the homework was pretty easy". I'm like "ope. [laughs] well." [laughs] I was [moving right hand and thumbs down] It was great. Whatever I did it.
[00:35:31.12] F: ok. Great. So you uh. I wanna go back little bit between. If you can compare how the um the ambience or environment of the group in classroom and your study group outside classroom. So if you're asked to compare between these two what do you find similar, what do you find different? What is .
[00:36:01.15] B: Um. I feel like overall we get like off topic pretty easily. But um.
[00:36:08.02] F: Where? in both?
[00:36:08.02] B: Yeah. We're just pretty like we just go off topic randomly. But outside of uh . the classroom we even go get beers we'll just go drink and then like . get off topic . and then finally we'll like go back and start studying again. And like I don't know we're just kind of . I feel we're more friends . like outside I guess. and we just chill like . we do math and . we're not super like . trying to rush through the problems . and just like slow it do:own. They go on the board like do all the problems their own ways and work together and then I'm kinda like just sitting there . and taking it all in. [giggles] or just going over other problems but . I feel like m . I feel like it's . cooler outside class. just cause we . have more time just . doing whatever we want and not having to stay so on topic . our brains can rest. And then come back . then go off and come back.
[00:37:04.19] F: Okay. How long do you usually spend outside class when you study together?
[00:37:10.15] B: Probably two hours.
[00:37:11.26] F: In one chunk?

[00:37:13.19] B: Yeah . that's just because that's how long we get the room for. um and sometimes we don't just do number theory like we'll do other classes . I had . a kinesiology . there was another guy I forgot his name. He's in Melissa's group. He's short.
[00:37:29.04] F: Tito?
[00:37:30.15] B: Tito. yes.
[00:37:32.06] F: Short slim?
[00:37:32.06] B: Yeah. So it's me like Boutros and Tito just like in the room. and we're just doing other classes like working on other finals so that it's cool.
[00:37:40.29] F: I see. Uhhh. Did you work with any tutor this year?
[00:37:49.15] B: Just Ted.
[00:37:51.08] F: Just Ted. Okay.
[00:37:53.26] F: Uh at the beginning you mentioned that you uh wanted to pursue tutoring.
[00:37:59.01] B: I know.
[00:37:59.01] F: Uhh. Is this you didn't feel the need for that?
[00:38:04.08] B: Yeah. well I mean in the beginning I did. that's why I said it. But um. no no. I mean I feel like I'm . I'm taking in as much as I possibly can. I'm actually kinda learning.
[00:38:16.09] F: You found what you're getting from the group and to study and from Ted was enough.
[00:38:23.15] B: Yeah.
[00:38:26.08] F: Uhh. Let's suppose that you if next time you go into a classroom and they require uh small group work. What do you keep, what do you change in your behaviors, in your choice as a group, and whatever happens in your experience? For your next experience?
[00:38:46.12] B: Umm. I'll probably . take the group with the most approachable people . kinda. because I feel like some of the guys in our class are kinda just like too smart and they're kinda just like older. And you know who I'm talking about? Like the two older guys? Like the one guy behind me . he's always talking . and I feel like that's kind of annoying. But . then the other guy the older guy he's just like really smart and he like . kinda like shows off kinda. I think that's annoying. Like I don't know.
F: uhu
B: I kinda like the people that are low key like really smart . but they're not trying to be like "I'm the smartest" [right hand moved all up]. I don't know . the most approachable [both hands extended at chest level] . like people that I can like see myself. I feel like my group is perfect. Like I would've uh I didn't even pick them. But I got really lucky. I didn't even know what we were doing when we were sitting down at the table. I thought we were just picking seats. I just sat on an open chair.
F: okay
B: I didn't even like . look who I was sitting next to. I didn't even think I was going to have to like [chuckles] talk to them ever. But . I yeah . it worked out definitely in my favor.
[00:39:51.21] F: And what would you change if any?

[00:39:53.26] B: Nothing. Maybe . if I . maybe if I like tried harder in the beginning. cause I feel like that's kinda gonna get like catch up to me when my final grade comes out. It's like really gonna like bite me in the ass. but . I'm like doing really good now.
[00:40:12.14] F: Okay. Uhh. How confident are you for the final?
[00:40:17.02] N: no no. Not confident at all.
F: you're not confident
B: Um yeah. I'm gonna get there. Hopefully.
F: okay
B: but we'll see.
[00:40:25.00] F: All the best.
[00:40:25.29] B: Yeah it was hard because he . he didn't really tell us what was gonna be on the midterm. So I'm guessing that's what he's gonna do for the final too. He just kinda said "just learn everything".
F: [laughs]
B: like "uh okay" . like "cool" "great". Cause I know nothing so. [laughs] [00:40:41.18]
F: Did you feel that there was a lot of materials to learn during this class?
[00:40:44.13] B: uhh . kinda. I feel like it's kinda just based off like certain things that we learned. just like memorizing definitions I guess. You kinda just go off of the same thing like . gcd or just like . Euclidian or whatever Euler . like you just have to memorize those things and it kinda just like . everything else falls into place. You just remember the definitions of what you're doing.
F: okay
B: but . even then I struggle. So I'm making it sound like it's easy but . yeah.
[00:41:18.13] F: Uh is there anything else you would like to say. These are my questions. If you'd like to say anything about the class that these questions did not cover. And you think it's relevant that I know about it. If you want to say anything about the research.
[00:41:33.09] B: Umm. Probably just like . that my view on groupwork from beginning to now. I really was against it and I thought it was the stupidest thing ever in the beginning. And I thought that. I really hated it. I didn't even like it. I didn't even want to go to class. because I thought it was just stupid. because . I wasn't um . I wasn't getting taught anything. and I was just like . working with . people I could just work by myself at home. I kinda just thought it was stupid and I hated it. But . um towards the end I felt it was definitely more beneficial than sitting in a lecture. Like . by fa:ar.
F: Okay. Oh. uhu
B: So I really enjoyed it. And . yeah . I'm glad . I took this class.
[00:42:16.17] F: Great. Thank you very much.
B: Thank You.

Appendix B: Transcript of Ted's Interviews

Int1-1001-Ted

// Transcript from the back-up audio//

Ted: Fun . visual . spatial . patterns . poetic . driven by creativity.

[0:00:28]

Fady: Why do you see it fun?

Ted: well ummm . If you want to win a strategy game you'd better be good at math. So ever since I was little I played a lot of games. You know board games. And to me . Math was the way that I could win games. Like when I was little [unintelligible] then eventually math became like puzzles to me. I was always better at these games because of math.

Fady: and um you mentioned it's poetic and driven by creativity . how so?

Ted: I often times like look at like the people who like first came up with calculus for [unintelligible]. I look at that and I think to myself . could I have come up with that if it didn't exist first? And I think to myself how creative must somebody be to come up with something like this. and how it defines the world around you. How mathematics is that. That's like creativity.

Fady: poetic in which sense?

Ted: poetic in the sense that … it's so pure . and the patterns within within things that you discover in math . when we draw it out it's really beautiful sometimes you know?

Fady: okay. Thank you.

[00:02:00]

Fady: Now I'm gonna go a little bit talk about your … your social network. neighborhood . friends . family all the people when you sit together and if the mathematics topic comes up . What do they usually talk about. How do they talk about mathematics or mathematicians?

Ted: I have different circles. Like in my family circle . is all immigrants from China. The culture around mathematics is more like if you can do math then you are good student.

// Transcript from regular audio starts here//

[00:00:01.02] Ted: right? like and mathematics is kind of like a benchmark for all the other subjects like if you can do math you should be able to do science and everything else just fine. you know? so . that's uh whenever math comes up with my family it's it's kind of like talked about . like benchmark for "are you smart enough? alright . are you worthy of being a Tao [i.e. Ted's family]?" [laugh] and with my friends . it kind of depends on what groups of friends . friends that you're talking about um. Cause I have friends from work who genuinely just don't like math. uh . they would tell me like "hey this kid really needs help with all these math problems . why don't you go help him." so that's . that's that relationship. And . yeah . I mean . I also have friends who are really interested in math . but uh quite a few them in . in my life recently . mainly look at math as if it is something hard . and that it is something you just have to know . instead of something you want to know. Um . When I was younger . uhm . when I went to high school called Texas Academy of Math and Science . everybody that I knew loved math . and loved science. so was a . kind of . it's kind of shift from that time.

[00:01:22.12] Fady: Okay . and what do you do when they talked about these differences whether in family? or in these others circles . when they mention that mathematics is hard . or in your family when they try to test you are you uh . do you qualify to be a Tao? Ahh . what is your response to that . or what do you feel . how do you feel when you are in such a position or situations?

[00:01:45.26] Ted: Mmm . I feel like a champion for the creative side of math often . they talk about like how it is something much . It's not you . not memorize anything . you have to be creative to like generate these proofs . and to be like in the front line of math . you are being a creative person . like so . no . no matter what. I'm . still like . they . the base assumption of math . like for the people that I know . besides those who are already in the math department. [exhalation] tend to look at math as if it is a uhm just memorization and arithmetic. And . I often times I am the person who would argue that math is not just that. you know.

[00:02:29.19] Fady: Yeah. okay. Umm so do you ah . what is the sort of choice of becoming or taking mathematics as your major? And you are doing which program?

[00:02:42.14] Ted: Oh . with the concentration on teaching.

[00:02:44.23] Fady: Math for teaching?

[00:02:45.24] Ted: Uh . huh.

[00:02:46.04] Fady: Okay. So why does . why this choice . so what is the what did lead you to . to make this choice of majoring? Did you have other choices?

[00:02:55.22] Ted: I was a . it was a winding path. Um . I first started college when I was sixteen um at Texas Academy of Math and Science. And at that point . I was umm . I was applied math major . when I first started . and then . I went through a super rebellious phase . like hyper rebellion where I just completely threw everything my parents said out the window . and I said . I wanna do creative writing.

[00:03:22.16] Fady: Which age is that?

[00:03:24.29] Ted: That was when I was eighteen.

[00:03:26.29] Fady: Eighteen

[00:03:27.02] Ted: So I switch over to creative writing . um . and like . you know . hum . got into . got into a university over in Georgia . but I was so focused on the creative writing that I stopped being succeeding in math . uh . then I went through a phase just working . and traveling. Then eventually I went back to the school . and got into [name of current college] initially as a mechanical engineering major . because my dad said . “You should go make money if you go to school.” And . I said . “okay. Engineering sounds . sounds . like something I can do.” right . hum . and I discovered that I don't like the real world very much. [laugh] I like the math of engineering . but I didn’t like applying it to the science. I didn't like translating . and back to the real world. hum . and then . all throughout this time starting from when I was seventeen or so . I've been tutoring. So . I figured out that that’s something I am able to do well. and then I went into working with a community center called JTCC . and they worked with a youth administration. That is when I discovered that . “hey teaching could be a path that I can take.”

[00:04:43.21] Fady: Okay.

[00:04:46.01] Ted: Yeah.

[00:04:46.12] Fady: And when did you make this shift? from uh engineering to uh through engineering to math for teaching? So you entered [name of current college] as an engineering? For engineering? and when the shift you shifted major?

[00:05:03.29] Ted: About two years ago. No . a year and a half ago?

[00:05:08.29] Fady: A half . uh. So after two years? of being in on campus . you did this shift . and uh so it was just motivated because you like this teaching and you did not uh . you did not uh . feel that engineering is made for you?

[00:05:25.15] Ted: right . and I . whenever I have conversations um with my girlfriend when I talk about the future . I always talk about classrooms . I never talk about like creating products . or designing products. so conversations with her . kind of also help me reflect and drive me towards tea- wanting to teach.

[00:05:44.12] Fady: uh . and what do your parents think about this shift?

[00:05:49.29] Ted: My parents. um . they said to me at this point . just get the . get the hell out of college . graduate . that is all we want. [Laugh] and that they realized at this point that they can't make me do anything.
[00:06:02.19] Fady: Okay.
[00:06:02.28] Ted: Yeah.
[00:05:58.24] Fady: Do you live with them or you live with on your own?
[00:06:07.02] Ted: I don't live with them anymore. So . that was another reason why I worked so much . was to get that initial financial stability.
[00:06:15.20] Fady: Okay. I see . thank you. umm . and when you tell your friends that you are majoring in mathematics . what do they tell you?
[00:06:28.01] Ted: Ummm . they usually ask just why? [laughs] . why? why do you do that like. Ummm . besides the ones who are also math majors . you know. Hum . that's . that's the common response I get from most from my friends. Ummm . and there are those who know me for more than two . three years . then . they're just like: why wouldn't you be a math major . cause they know who I am.
[Interruption by a lady saying that there is a class coming in 4 minutes]
[00:07:12.06] Fady: Uh . finish this sentence then maybe we can go to [inaudible]
[00:07:18.20] Ted: I kind of lost my train of thought
[00:07:21.06] Fady: Yes. Okay. Then you were thinking about what your friends say when you tell them that uh you are majoring in mathematics . and they tell you why . and then your response is?
[00:07:31.28] Ted: Because I love it . and that it is a core part of my personality . I look at everything through math . that is just how I am. so I've never deviated far from the math part path except for my super rebellious state of creative writing.
[00:07:52.05] Fady: Let's continue that
[nosy sound] change classrooms
[00:07:58.22] Fady: now we are gonna talk about a little bit about more about uh you mentioned already your history with mathematics. uh . the joys and um ummm joys and things that you did not like . disliked . and . and your all the painful things and with your history of mathematics. Can you mention a little bit about the teaching . that you receive what kind of teaching you . uh . you felt relatable . what kind of teaching you disliked through your high school and here in college as well?
[00:08:38.09] Ted: Ummm .In Hong Kong . the teaching like . oh . I am an immigrant from Hong Kong . and I was . I had my elementary schooling over there about 1^{st} through 5^{th} grade I was in Hong Kong. Hum . I was just very very intensive . hum I-I remember there was a shock coming over to the US . because by 5^{th} grade . I was working on hum . you know volume formulas . like basic geometry . and basic algebra already in 5^{th} grade. and I come over here we're working on fractions. I was so confused as to why that the skill level was different . and then . at that point . um I was very very bored with the math . in about 6^{th} grade or so . so I just started ummm . trying to generalize everything that the teacher was saying. That was my way of stop . stopping being bored. Right. Hum . so that if um . like if uh if a teacher would give me a formula for like the area of a regular polygon that's like 3-sided . then I would try to generalize it to n-sided. You know? Like . that's . that's what I was doing . just like sitting in the back of the class .
[00:09:56.03] Fady: And this is already in your 6^{th} grade?

[00:10:00.26] Ted: 6th grade or so . yeah. And then at about 8th grade . that's when I felt confident that I could teach myself whenever . whenever I have a book . and often times . hum . I would only pay attention to the teachers when they're doing the proofs. Cause that's the part that I cared about . the arithmetic I felt . if I know why it works . then I can figure out how it works . why do I need to memorize the arithmetic . so I ignored all the teacher's notes . that like memorize how to do this . and I just kept asking them why. And some teachers hated that. you know. They are just like “this is how. I don't care about why. focus on how to get the answer.” I am like “obviously . I already know how. can you tell me why?” [Laugh] that is where I was . in the high school age.

[00:10:47.18] Fady: this is high school . and how did it start to look like at college?

[00:10:52.06] Ted: At college it turned into this really weird brand of arrogance.

[00:10:56.14] Fady: Yeah . I'd imagine . changing from one subject to another. hum . but when you came back to engineering and mathematics. How did you see the . how did you relate to the teaching and the styles of classrooms?

[00:11:12.08] Ted: I definitely still had the attitude where I appreciate the proofs . and like the generalized forms much more than applying what the conclusions of the proofs . you know . to problems. So that's I think a big reason why I had . very little interest in engineering classes. Cause once you have like one . one or two rules . then it's really just applying that same rule over and over to come more and more complex systems . but the rule remains the same . you know.

[00:11:46.17] Fady: Yeah.

[00:11:46.25] Ted: so like . for example. Ohm's Law . it works the same way with . you know a simple circuit to a super complex circuit. To me . that is not very interesting. so . I . I think that I like I have . I've never really stopped thinking about math as if like it's a string of proofs and the arithmetic is just like some extra bonus to what we have.

[00:12:10.17] Fady: Hum. what do you think why . you had this resistance or this feeling towards applying mathematics to real world?

[00:12:21.17] Ted: Laziness.

[00:12:23.04] Fady: Laziness?

[00:12:23.15] Ted: Laziness.

[00:12:24.06] Fady: How so?

[00:12:25.03] Ted: like . hum .I feel like yes . doing a proof is really hard sometimes . but like once you do it once . and maybe you look at other people solution that . that . that takes different path to the same conclusion. Once you have that conclusion. That conclusion is yours forever.

[00:12:42.04] Fady: Right.

[00:12:43.06] Ted: Right . whereas like with you doing the arithmetic . it's like . I have to keep doing the same thing over and over using the same rule . and I think that gets boring to me. That's . that's why I have that preference for the proofs . and it . it's part of it is like laziness.

[00:13:00.22] Fady: yeah . laziness mean not doing something boring?

[00:13:05.05] Ted: Right . and . laziness for not doing something that almost feels . somebody else can do that. A calculator can do that. You know?

[00:13:13.16] Fady: Okay . uhhh .now I [clears throat] can we . try to see how you study for your work for your homeworks like do you have a certain strategies . technique you do when you are given a homework? What do you do?

[00:13:34.06] Ted: When I am given a homework first I . I definitely look through the all the problems I that I need to do and I just target the ones that I can do right away um and I finish those first and then I spend a big chunk of my time not looking at the textbook and just trying to figure it out using like what I already know and . after about an hour looking for a proof and I can't figure it out then I might consult the books . but usually I give it a good attempt to actually try to prove it myself first before I look at any resources.

[00:14:13.07] Fady: Okay . and what do you do when you are stuck?

[00:14:16.04] Ted: [inhales deeply then clears throat] . mmm whenever I'm stuck . happens once in a while and . I tend to complain to my cat. [chuckles] That's a little bit of it. I'm like "can you tell me the answer?" Ummm and another way that I-I deal with it um besides looking it up . would be just to take a long walk and just come back to it. Like just take a break from the problem and come back to it.

[00:14:52.16] Fady: mm . uh thank you and how do you prepare for tests?

[00:14:56.01] Ted: Uh prepare for tests I . go through . I read through ev-every single proof that that is involved in all the umm all the things that's being tested uh and make sure that I understand th. the meaning behind the proofs and what they're used for and then I look at example problems . but when I look at the example problems umm I don't really do the number part because I can use a calculator. It's I've I focus on like what proof or what theorem is being used for any particular arithmetic problem and then from there . if I understand the proof and which proof applies to which . then I can generate it during the test. So that's that's how I study for it.

[00:15:44.15] Fady: Okay thank you. Uhhh let's talk right now about your history with small group work if any so can you recall like moments throughout your schooling: middle . elementary-middle . or high school and even here at college . like uh what is your experience when you worked with groups whether inside the classroom or outside the classroom?

[00:16:06.27] Ted: Well . elementary school in Hong Kong . there wasn't anything like group work outside of P.E. outside of team sports you are competing against one another it's very competitive there. it's [whispers] it's crazy [laughs]. ummm once I got to middle school I was too busy learning English to . and . to be like really focused on the group work. I was very disruptive cause I felt . since I . I didn't . I had that language barrier uh and I was young enough that I didn't know to care about other people's feelings. I would just disrupt and like "you guys aren't listening to me" but not . without realizing . that you know it's because they can't really understand what I'm saying. So I was very disruptive from about 6th through 7th or 8th grade-ish. Umm in high school . I if it's a math and science related group type of group work . I tend to be the leader of the group back then. I would just like "I want to get this done so that I can go back to playing cards with my friends." [laughs] so in the middle of the class work I would just be like "No I don't care what you have to say. this is the answer. if you disagree with me . you can get a bad grade . go ahead." That's my attitude back then I was just so arrogant and [laughs] I was complete asshole a lot of times. Um. then . I got to college and I started to care about other people's feelings [chuckles] umm and I knew when to shut up and let people work things out.

[00:17:42.05] Fady: Okay

[00:17:43.18] Ted: but in the end I tend to like facilitate the conversations cause I see the a- the path to the solution to the of the . of the of what we're working on usually.

[00:17:56.25] Fady: Okay. Umm . Can you uh . so outside the classroom . do you work . study with groups?

[00:18:12.22] Ted: [deep inhale] usually it's . people ask . I only go study with people if they are wanting to learn from me . in the sense that I am not very good at listening to other people's instructions when they're my peers um but I remember the concept much better when I am

explaining it to a peer as I am studying it so like I'm like one of those type of people that learn by explaining . you know what I mean?
[00:18:39.28] Fady: aha
[00:18:40.14] Ted: Yeah so is in outside of outside of sc . sc. school and outside of the classroom setting . I do have small groups and I started doing this when I was 17. I would just take a few classmates who I know are really really bad in the class are doing really poorly and I say . “Let's study together” and I'm already at like a B- or an A and so they're they're willing to listen to me. it's just practicing the same thing and I get to explain it. And I love doing that because [deep inhale] by looking at what mistakes they make . I would remember what mistakes I shouldn't make
[00:19:15.00] Fady: mhhm
[00:19:15.26] Ted: in a way . and I would have to really like . understand what they're doing to see where they got their mistake and then I can pinpoint it and fix it. So to . I . I feel like it's more dynamic way to study and to refresh your knowledge.
[00:19:33.21] Fady: Can you talk a little bit about your experience with the current group with whom you are working? uuuhh in Number theory . did you shift groups from the beginning or . uh
[00:19:46.00] Ted: still the same one . still the same one.
[00:19:46.20] Fady: since the first lesson right?
[00:19:48.16] Ted: mmhmm
[00:19:50.11] Fady: uhhh .
[00:19:51.01] Ted: wait . the first lesson . and then . the very first one we had like some groupwork and then we split up again and everybody got shuffled once . right?
[00:19:59.21] Fady: yeah
[00:20:00.22] Ted: and then after that it
[00:20:01.07] Fady: so you did not .
[00:20:01.17] Ted: was the same one.
[00:20:02.09] Fady: Okay . you you you changed these shufflings or is it you who shifted from a group to another or .?
[00:20:12.16] Ted: hmmm .
Fady: How
Ted: I remember I arrived to class a little bit late for the first session so I might have like and I just joined a group that only had two or three people so I think the professor assigned me to one to this group or something I don't . I don't remember exactly how it happened . but for as long as I've been in this class I mainly remember this group
Fady: okay.
Ted: yeah
[00:20:34.07] Fady: And can you describe to me how the interactions are in the group and how do you feel about the group?
[00:20:42.21] Ted: mmm I think we have a g-good flow of like um a little independent work flowing back together take a lean and talk about it and then going back to some independent work . umm the only thing that I . I found sometimes challenging is uh when we have different approaches to the same problem an . or like when we umm talk about how . some . some members’ solutions might not be sufficient umm . people get defensive.
Fady: [slight chuckle]

Ted: You know what I mean? Like umm the ego comes into play and they get they start defending their answer . but in real . reality the intention of the conversation really is to get to the . to a . elegant proof . you know? One that's not only correct . but nothing extra . nothing less and . and when we have conversations around things like that I feel like there can sometimes be some push back simply because . “This is my answer I want to defend it. I am only doing the way this . is and give some some explanation or excuse” and it's . that's common I think with like all math and science groups . you know? People want to defend their answers. Umm and I think that it would be more efficient if people can drop their egos.

Fady: Okay.

Ted: But that's hard for every group. [Fady and Ted laugh]

[00:22:08.14] Fady: ehhh can we go through ehh talk a little bit about every individual or group member in terms of you can tell me like describe me the profile of each one of them uhh. in term of how they tend to think mathematically their mathematical habits . and at the same time . their social habits like in terms of behaving in the group or doing things.

[00:22:32.01] Ted: Let's see . who do I start with?

[00:22:34.08] Fady: uh just one second . let me .okay

[00:22:42.05] Ted: Okay so . Let's see. Boutros is . super chill. like he he's like uh I I can't ever picture him being angry kind of guy [chuckles] personality wise. Ummm. mathematically he's he's really quiet like when we when we have conversations in the group . I have to really prompt him to be like . “Hey so where you at? Can you can you tell me what you are doing?” that he'll share with the group . but umm he doesn't share what . what he does very . often umm. Let's see. for Jeremy he . he is very sh- very umm high standish for himself in term of what he wants the proof to look like when he's working on it and he likes his . that first draft to be sufficient as a proof and then uh he could just write it neatly kind of thing. umm and in that there are times when like he wants to get it to that . to that level standard uh when conceptually the rest of the group wants to move on umm so that's . that's something that we work with sometimes [deeply inhales] for John he's very confident in his math and he's he's he's very good about about like sharing his math umm he he is the one that gets defensive the easiest umm when we do have conversations about whether or not it is sufficient and so on. Ummm . for Bettie she is very humble. I think I think she is better than . better at math than what she says. umm and yeah I like to see her share what she does a little more uhh. she likes to do the research in the book. I've noticed that a lot. almost every time we're working together . she is the one to consult the book first. Diff . different style of learning. ummm who else is in the group? I think that that covers all of them? Right?

[00:24:51.15] Fady: Yeah I think we went through all of them.

Ted: and Ted is awesome.

Fady: Ted is awesome

Ted: [laughs]

Fady: uhhh . okay . wonderful. That's about it. I don't know if you want to say something else about that is very significant for you as a mathematician . or a someone studying mathematics . that is not said.

[00:25:19.15] Ted: Mmm . I wish one day that we can change the culture around how the U.S. like general population . views math. That's that's

Fady: what do you wish

Ted: my wish like if we if I if I could give a genie a wish and just make it just come true . that would be my wish.

[00:25:49.08] Fady: which way your sense . can you say more about this?
[00:25:52.00] Ted: There's there's this sense that math is some a necessary evil at best for a lot of . a lot of kids as well as parents . cause I work in the after school program so . I talk to them all the time . "Well if they finish the problem and did it right or they got the right answer . okay. like move on." why do we have to go back and talk about fractions? When when they're getting a C in percentages? "Well maybe your kid doesn't . can get an A right? right?" umm but yeah just . just the fact that all the . [deep inhale] not all . but like most of the students that I've encountered at that age has been . very resistant to the idea that math can be something useful outside of the classroom. And . it's not until like . later on in the grade school maybe 11th 12th grade when they really make the connection and I feel like if we can change that culture like the way that parents look at math . the way that kids look at math . in . instead of looking at like something that they hate . it could be something fun . you know? It's almost like chen . culturally accepted as a norm to say . "Math is hard. I'm not good at that." and there is no shame in that . right? Whereas you can't go out and say . "Oh English is hard. I'm really bad at . like really terrible at English." Well . people will judge you for that . whereas with with math nobody judges you for that and I wonder why.
Fady: mhm . okay.
Ted: you know? yeah.
[00:27:27.07] Fady: How do you think . what can help them to understand that mathematics is useful?
[00:27:34.14] Ted: Umm . I think uh . I've been reading a little bit about like bring social justice into like math curriculum. uh . like . would . especially like with high school age . youth.
[00:27:46.00] Fady: What are you reading? which resources?
[00:27:48.04] Ted: Ummm . Professor Carol has been giving us some resources to read. Ummm . I don't remember the author's name . but I can give that to you if you want. Ummm . just like thinking about . like how statistics is used in law . for example. And how people justify passing laws or fighting against it using numbers . and how that can really genuinely directly affect your life? Ummm . For example . like in the [name removed] district . how you define low income housing? Right? What is that number really significantly changes this whether or not. [city removed] looks like it has sufficient low-income housing . right? And for the kids in the [area removed] . they feel this first hand . some of their friends . and some of their friends' parents . or even themselves are getting evicted. So . to bring that type of urgency. And for them to realize that knowing math . could give them power over subjects . like that. I think . I think it's gotta be key. it's the younger ages . that's like even more challenging . you know . cause to talk about social justice . you were . it requires some like good amount of literacy . skills . or as you know. for the . for the umm . also you need to be able to make it inferences . for certain age groups . it's just not quite reasonable to expect them to have those skills. So how can you bring it to the younger groups? Anyway . I'm rambling . I am totally rambling . you know that . right? hehehhe .
[00:29:28.26] Fady: I don't think so. uh . yeah. Thank you very much. give me or for making this time . and for participating . actually . it's gotta be very helpful . to get those your perspective. Today . we didn't have the group to do the normal things . but you will enjoy this study I think .

SCNI-1029-Ted

[00:00:02.19] Fady: Uhhm I was trying just to .How are you? You look very busy.
[00:00:13.27] Ted: Feeling good. Humm. I am always busy.
Fady: Yeah
Ted: I have. cause after school . I have two jobs.
[00:00:21.25] Fady: Two jobs. How much per? How many hours per week?
[00:00:25.25] Ted: uh . I am not supposed to . to say. because I did promise my advisor .
[00:00:34.09] Fady: 23 hours?
[00:00:36.03] Ted: oh. like 15 plus . 36 hours or so?
[00:00:42.20] Fady: A week?
[00:00:43.07] Ted: en hum .
[00:00:44.07] Fady: Wow . Okay . so .
Ted: sneaky sneaky
Fady: Yeah. So you need that? You doing because you need money. And how many course are you taking?
[00:00:56.10] Ted: Four.
[00:00:56.24] Fady: Four courses? Handling it well?
[00:01:00.26] Ted: Feels like it right now after I got the midterms back.
[00:01:04.12] Fady: All your midterms were good?
[00:01:05.22] Ted: All my midterms are good. I had uhhh a 110 in my proofs class . like the prereqs here for this class essentially the first class that I'm taking concurrently with it. umm got a 94 for my stats class and got a 97 or something like that . for this one . so .
[00:01:25.13] Fady: Yeah . uhhh. yeah . It's actually . I did read your auhhh . . yeah . I did read your midterm you handed. It's really good . impressive. So . 10 . 29 [typing date on note sheet] . uh . so here we go. why is it? okay. So what we're going to do today . is umm . a you are going to watch a video . a video of the group session that you just did. And you're going to help me see through your eyes. Like . bring me into yourself in the video. I'm not interested in what you are feeling or thinking right now after the group session happened . but I am interested in how your feelings. What happen about your thinking and feelings in the moment of the time of the interaction. So I am interested in knowing what were you thinking . mathematically . but also what are you feeling about yourself . about other groupmates . you know? If you got distracted as well is interesting . so all of this. We are going to watch this . and you can stop it . click the space bar . and say whatever you want to say and bringing in me in there . okay? Let me just test the sound. I think you are right here.
[00:03:12.15] Ted: Yeah . I showed up . Half an hour late.
[00:03:16.25] Fady: was half an hour?
[00:03:17.29] Ted: Yeah.
[00:03:19.08] Fady: Okay.
[00:03:20.23] Ted: Just about.
[00:03:22.08] Fady: Coming from?
[00:03:24.10] Ted: From work.
[00:03:26.18] Fady: Okay.
[00:03:27.22] Ted: But I do call out from work like on the midterm weeks.
Fady: Okay.
Ted: so to really put time into that.
[00:03:35.08] Fady: And to do more studies and .
[00:03:36.28] Ted: Yeah.

[00:03:37.16] Fady: Okay. Please can you get closer to the camera so the camera can capture you and feel at ease . uh [00:04:08.23] [audio inaudible/cuts out]
* Video paused * *//The timestamps in this section refer to the audio recording//*
[2:19] Ted: I am getting organized. [pauses video] This is like . where I am just like . catching up seeing what problem they are working on. I did think about this problem after the last cl- last time we've met.
Fady: you haven't or-
Ted: I thought about it .
Fady: you thought about it.
Ted: outside the class. [resumes video]
Fady: Okay.
* Video paused *
[3:05] Ted: [pauses video] Here I'm um reading um what they have to see if it . matches with what I have written or if there is something significantly different. [plays video]
* Video paused *
[3:29] Ted: [pauses video] here I'm being impressed about Jeremy's handwriting. [plays video]
Fady: in which sense?
Ted: [pauses video] he's super neat. he is like so neat when he writes with his hands it feels like he's typing it up.
Fady: uhu.
* Video paused *
[4:04] Ted: [pauses video] here I agreed with everything that he was writing but I added a constraint on n that is definitely true. [resumes video]
* Video paused *
[5:35] Ted: [pauses video] here we're talking about just generating the reduced residue system with the a to some power. If a is a primitive root . then a and a squared and so on all the way up to a to the phi of m or something like that . is a reduced residue system. And I think we were clarifying the definition of what um it means to not repeat in the set. Meaning that they wanted to show that a to some i is not equivalent to a to some j mod m. [resumes video]
Fady: uhu.
* Video paused *
[7:03] Ted: [pauses video] I like to stare in the space when I am thinking [giggles]
Fady: uhu.
//Henceforth . the timestamps refer to the video recording//
[00:09:41.04] [audio returns in video]
[00:10:00.08] Ted: Breakthrough. This was a breakthrough moment like whe- when he [John] really talked about showing they can't be equal . "why don't we say they're equal and show that there is a contradiction?" and that's the basis of solving this problem.
* Video paused *
[00:10:51.16] Fady: Do you recall . uh how did you . mean? why did you think about this material . How did this come about? Is it .?
[00:11:02.11] Ted : I just kind of feel it .
Fady: okay.
Ted: I I I feel the solution come to me . and then I am like . okay. Let me write it out. and see if this feeling is right.
[00:11:11.21] Fady: Hummm .

[00:11:12.04] Ted: So I let myself like have the freedom to make a mistake. and then just test out. test it out. I don’t know if what I am writing is right . Write it out . and then. verify that's true. And then . knock it out.

* Video paused *

[00:11:28.24] Fady: How did you feel here?

[00:11:32.15] Ted: What was I feeling? ummm . at that point . I . I felt like a boss. [laughs] I like . felt . like . like we all broke through with this problem . and I don't know. so . I guess I just been staring at this problem for . like 3 or 4 hours . you know before class . I think I think like I would just look at it for like like 10 or 15 minutes at random intervals when I'm working or when I’m waiting for the bus or anything like that. Just like look at the problem . Think for a little bit . and put it away and like at that moment . it was like all those those little moments of looking at it coming together. You know just clicked . all of it falling into place.

* Video paused *

[00:12:47.22] Ted: We are checking . after . after everybody seems satisfied with the solution . I am kind of stubborn like maybe I made a mistake . let me make sure . that I didn't mess up somewhere.

[00:13:01.13] Fady: aha

* Video paused *

[00:13:54.05] Ted: It's like hum all the big pieces are there now . and just like thinking about how to order . so that it sounds logical when you are writing the proof. That is what's going on in my mind.

[00:14:06.24] Fady: aha.

* Video paused *

[00:14:20.06] Ted: coffee addict. yeah.

[00:14:26.11] Fady: [laughs]

[00:14:39.27] Ted: like I think I was trying to decide whether or not I wanted to write all that on paper again or if I wanted to take a picture of it. At this point.

* Video paused *

[00:16:04.17] Ted: When I write it “what direction?” [laughs]

* Video paused *

[00:17:46.01] Ted: There are so many things in it . [laughs]

* Video paused *

[00:17:49.25] Fady: Well . It's actually [inaudible] uhhh . so here you went to . to check the theorems . and you have them offer that's . that's okay. here is . let me . one second. [rewinding the video]

* Video paused *

[00:18:53.27] Fady: Do you remember this conversation here about if k+c is less than phi of m?

Ted: uhu.

Fady: okay . what do you think about this conversation?

[00:19:04.18] Ted: I was uncertain about about whether or not uhm . c less than phi of m has to be the constraint or if k+c less than phi of m has to be the constraint. cause we were considering all the numbers a to the one all the way up to the phi of m . and then anything bigger than that would just be congruent back to something.

Fady: exactly.

Ted: so before . I was thinking k+c . we just need to consider all k+c up to phi of m . we don't need to go further up. But above it . this still works . so it was just like . one of those like does the boundary need to be there? [laughs]
[00:19:45.09] Fady: Actually now . I am watching the video . it . it must be there. Because I followed with you another style of proof that happened in different styles . so here it's not a condition . it's the given. You start up with 2 powers of a . where the powers are less than phi of m. cause you want to prove that if 2 powers of a less than phi of m . they are not equal.
[00:20:18.18] Ted: Yeah.
[00:20:18.27] Fady: So you . you suppose that hey are equal . and then you prove then they cannot to be equal.
[00:20:25.02] Ted: Yeah. Cause ya .
[00:20:26.08] Fady: So it's a given. In your proof . so I was talking about it as a condition . but it's not a condition. So . this is where the mistake happened. cause I didn't understand why you were . why you use this proof for . you see . it came before after that .
Ted: Yeah.
Fady: so . just. I just want to point this out. But I think you . yeah. uh . yeah . I wonder why you could not explain this to me . like challenge me in this point . like . this is what we want. I have the feeling that you lost the what were you doing here. Right?
Ted: Yeah.
Fady: or not?
[00:21:10.11] Ted: It was . it was like I was a tiny bit lost . and now I started to . to question whether or not it was a given or was it a condition or just was some constrain that needs to be there or not. Yeah . I forgot that was like part of the first sentence.
[00:21:26.09] Ted: Okay. This is where I left off.
* Video paused *
[00:21:40.25] Fady: Okay. I am going to just for the sake of time. [skips forward in video] I'm not going to keep you here all night. uh . what was it again? picture? yourself? are you taking picture?
[00:21:50.10] Ted: We were all taking pictures. All of us. Yeah . the pictures that I take are shared on the google doc . uh google drive. and every colleague in my group got access to that.
[00:22:00.01] Fady: Oh . Wonderful. Can you . uh . can you .
Ted: Do you want to see that drive?
Fady: add me to the drive. [providing his email address]
Ted: yeah
[00:22:08.27] Ted: hum . actually . I maybe I can just . or .
[00:22:11.24] Fady: Or you can do here. The question I don’t have Internet from here.
[00:22:16.09] Ted: Let me . let me see.
[00:22:17.29] Fady: But . let me . let me just send you . oh . you have my email actually.
[00:22:20.29] Ted: I can uhhm. send it to you by phone or it's just uh . it's share by link anybody with the link can see it . they can edit it . and they will see it.
[00:22:29.08] Fady: okay . so if I have the link . so I can access.
[00:22:34.08] Ted: oh . you are just like take a look right now . before I send it even. hum . so . it's organized like this. All the homework it's like here. and this is the hum . pdf version of the paper thing that I showed you with all the theorems. and this is just the homework’s . oh yeah. and within each of the homework folders I'll have like pictures . or pdfs of the homework . things like that.

[00:22:59.22] Fady: Okay
T[00:23:02.25] Ted: that . you can look.
[00:23:06.19] Fady: So this is your answer to the homework right?
Ted: uhu
Fady: so you post your answers.
[00:23:11.27] Ted: Yes. and then like wait for the critiques. [laughs]
[00:23:16.12] Fady: Okay. So you post it before you submit it?
[00:23:18.18] Ted: Yeah. and then see if there . see if anybody can
Fady: you shared with others
Ted: Yup.
[00:23:24.06] Fady: Yeah.
[00:23:24.28] Ted: uhhh . times when we work in a group I would have that as pdf . and just being like . refer to that . and then like they'll pull it up . we will talk about the problem using that as a background.
[00:23:35.26] Fady: Okay . yeah. Okay. How did this idea come about?
[00:23:38.22] Ted: I don't know. I just enjoy it. like [laughs] . doing that . so . because I . I personally learn by explaining. I'm kind of weird. I have to explain it for it to go into my brain. so I always like . before almost all my classes I pick couple people anyway. and just try to study with them. and I do it by . do the homework ahead of time . and explain it to them.
[00:24:04.21] Fady: Okay.
[00:24:04.28] Ted: if taught . I remember it. that is how I study.
[00:24:07.23] Fady: Yeah . and you share this with everyone . are they consulting it?
[00:24:11.28] Ted: hmmm . some of them are taking a look and then they will be like . wait this isn't clear. and it's kind of nice. you know.
[00:24:18.29] Fady: do . do you receive feedback . like telling you this?
[00:24:21.27] Ted: uh . usually it would .
Fady: They don't understand?
Ted: be face to face . yeah.
[00:24:24.19] Fady: Okay.
Ted: yeah.
Fady: so they . they usually read it? Read what you post?
[00:24:29.07] Ted: Yeah.
[00:24:29.21] Fady: Okay. and I also heard that you initiated some group . group work outside of classroom?
[00:24:38.23] Ted: uh . that. Hmmm . we were sharing group work outside of the classroom. yeah . we do that.
[00:24:44.00] Fady: Like . with some of the some of the group members . so how's it going?
[00:24:50.03] Ted: Hum . whenever we can get our busy schedules together and actually meet up. It's it's nice . you know to go through and do all the homework together.
[00:25:00.04] Fady: Okay . so exactly when you mention about your busy schedule and you just mention that you are very very busy this semester like four classes . and your jobs. So . uh . h . how are you managing the . Is it helpful?
[00:25:15.24] Ted: Hum . lots of coffee. [laughs] . That is how I manage it. uhh. I mean to me . it's . I have to study by explaining. Like I mean . especially for newer concepts . once I . I will try to gra- to explain it to myself that is how I learn it initially. And then I have to explain it to others

. to . reinforce it. so I try to do the work ahead of time . whenever I have time on the weekends . I'm done with the homework by Tuesday usually. And then I'll show up . and be like “here it is.
[00: 25:44.21] Fady: So this is part of you studying? or something .
Ted: mmhm
Fady: How . how many hours did you study for the midterms?
[00:25:52.05] Ted: It was kind of crazy. Because I was . I had no idea . what the midterm was gonna look like . so I studied everything for like 20 hours.
Fady: 20 hours?
Ted: about 20 hours . yeah.
[00:26:03.09] Fady: Wonderful. okay
[00:26:05.03] Ted: Because I was just so nervous . [laughs] .
[00:26:06.10] Fady: [laughs] This was outside the group work? or 20 hours including group work?
[00:26:11.19] Ted: Including group work.
[00:26:14.11] Fady: Okay . and how many hours group work? Do you remember? Can you? How many hours did you put in working with others in group?
[00:26:23.02] Ted: Hmm. Probably good like 15 . No probably like 10 to 15 hours of it was included . and then like . everything else was me . studying by myself late . late night.
[00:26:34.18] Fady: Okay okay. uhum [forwarding video]
* Video paused *
[00:26:34.16] Fady: Okay . okay. uh . just now. know how to take pictures. Okay. so here . you started number 4 . I think.
[00:26:51.11] Ted: Yeah.
[00:26:51.19] Fady: Do you recall what was your first thinking about the problem? like here at this moment? when you were writing this one?
[00:27:01.00] Ted: Number 4 . I was a . translating . what the problem was into the mathematical definitions . so a has a order of n mod m . then I translate that to a to the power of n equals 1 mod m . just like writing it out equations like . just like this starting block . what can we do with this. You know?
[00:27:23.13] Fady: Okay. [playing the video]
[00:27:27.26] Ted: Thank God. Jeremy is also doing the same. [chuckles]
* Video paused *
[00:28:51.00] Ted: [pauses video] At this point . I was thinking umm “Well first I gotta show that it works before I gotta show anything else so:o that should be that first step. So I was kind of constructing uh what I need to show in order to show that a to the power of k has the order of n over g. Well first we gotta show that a to the power of k to the power n over g will be equivalent to one mod m and then we gotta show somehow that n over g has to be the smallest one. and then uh and then I was like . “let me just worry about the first part” [plays video].
[00:29:25.28] Fady: uhu.
[00:29:51.05] Ted: Oh that's . uh?
* Video paused *
[00:30:35.18] Ted: I remember playing around with these equations and being really bothered by the order in which I wrote them [chuckles]
Fady: mmhmm
Ted: cause it was completely out of order. in terms of the logic that I need to use to show that it is true. so I had to write down the step number on the side . to remember? [plays video]

[00:32:23.04] Ted: [laughs]

[00:32:26.20] Fady: Now I can see. [laughs together] [in video . Ted drinks from the coffee mug]

[00:32:30.17] Ted: really coffee . I'm just like I wish you .

* Video paused *

[00:32:48.25] Ted: I think this is where we were starting to struggle.

Fady: [pauses video]

Ted: This is where we were starting to struggle.

Fady: okay.

Ted: because there are different ways to look at like how . how you can test um numbers that are smaller. and Jeremy was uh talking about . “let some some p be equal or less than n over g or something like that.” And then showing it by manipulation of the exponents. but we couldn't do that with the same logic that was . that we used. because you can certainly take the powers like a . let's say a is equivalent to one mod b. you can say a to the c is equivalent to one mod b because you can take the power of c on both sides . but you can't remove the powers. so we weren't. we were struggling with that quite a bit. at this point.

Fady: uhu

Ted: cause we're both tackling it that way. He [Jeremy] was dealing with it like . “oh some . some smaller divisor” and I was dealing with subtract some integer . but logically it was very similar. what we were trying to do.

[00:33:55.24] Fady: uhu.

Ted: [plays video]

* Video paused *

[00:34:33.09] Ted: [pauses video] I really wished I was taller and I kept thinking that for a while . when I was writing like that [in video . Ted leans all over the table to write on the poster board]. [chuckles] [plays video]

* Video paused *

[00:35:19.02] Ted: [pauses video] I like to look on the . worksheet to see if there is anything on previous problems that I could use . but I totally didn't see uh that we could use number two.

* Video paused *

[00:36:07.00] Ted: [pauses video] I immediately thought about how you can generate the entire residue syst- reduced residue system using a single number. but I thought that was just too trivial. [laughs] that wasn't his . that wasn't what he was looking for. so I was like . “let somebody else say it” . as of this point. [plays video]

[00:36:42.21] Ted: Two exclamation points next to three because he said it was most important. [chuckles]

[00:36:47.02] Fady: Oh okay. I see.

* Video paused *

[00:37:04.00] Ted: [pauses video] [laughs] um sorry how do I go back like a couple of frames?

Fady: [rewinds video back and plays it again]

* Video paused *

[00:37:37.25] Ted: My emotions were kinda really clearly written on my face thinking about it [pauses video] very doubtful [chuckles] shaking my head . I'm not a very subtle person [chuckles]

* Video paused *

[00:38:21.22] Ted: [pauses video] At this point . I was thinking . “Do I use this one to prove number 4?” I'm still like thinking about number 4. [resumes video]

[00:39:44.06] Ted: [laughs]
* Video paused *
[00:39:54.21] Ted: [pauses video] I got so distracted by doing math that I completely forgot about the congratulation card [laughs]
* Video paused *
[00:41:32.50] Fady: What did you write here? [pauses video] Do you remember?
[00:41:36.00] Ted: um we . we discussed it actually. like uh I mean uh we ended up going with John's question because it was the most reasonable. I think Jeremy and I were not taking it very seriously. [laugh] umm I think we made. made suggestions verbally. [resumes video]
[00:42:18.27] Ted: We were just like . "Let's talk about math" . John's like . "Let's ask a useful question." [chuckles]
* Video paused *
[00:42:39.00] Ted: [pauses video] The one thing you knew as you're undergrad. at this point . I've stopped thinking about that question and went back to thinking about the math.
Fady: uhu.
* Video paused *
[00:42:57.00] Ted: [pauses video] That was a little thing that uh the middle school that I work at . does consensus. You do consensus by like you [thumb is up] love it . you can live with it [a thumb pointing parallel to the ground] . absolutely no [thumb is down]
Fady: okay.
Ted: and you get to point where everybody is here [thumb is horizontal] or here [thumb is up].
Fady: okay
[00:43:13.07] Ted: [resumes video] Making more things tougher than I should have.
* Video paused *
[00:44:31.28] Fady: [pauses video] uh I don't want to take too much of your time . and I know that your time is very valuable. uh Is there any part from here to the end you'd like to watch?
Ted: um
Fady: or you think it's important . something significant in the group interaction?
[00:44:47.00] Ted: Towards the end . when um when Professor Hoffmann came by to talk about the divisibility. to use number two that like drove us to get to a breakthrough towards the end.
[00:44:58.28] Fady: Okay what happened there . do you recall?
[00:45:01.20] Ted: Um well between then and there we . discover like the things that we wrote down part of it was uh we can't take away the exponents and assume that's just true . like for a to the k to the power c is equal to one . we can't say a to the power k is equal to one.
Fady: uhu
Ted: So like we discover that at that point where we were just like . "wait but what do we do then?"
[00:45:26.08] Fady: Okay
Ted: hum.
Fady: Oh here is where . [plays video]
[00:45:30.13] Ted: Yeah that's where . the teacher comes up
* Video paused *
[00:46:15.02] Ted: Yeah that that one little push that he gave us finished- uh helped us finish that first part of a.
Fady: uhu okay

[00:46:21.23] Fady: uh I have just a quick question like how do . there are two people absent here. so how do you .. how do you compare this interaction . or this group work . this session to previous ones?
[00:46:40.00] Ted: hum Boutros is like usually really quite . when we work together.
Fady: uhu.
Ted: so it's . he would he'd we would usually have to prompt him for him to say to . say where he's at . umm so I'd say not . that huge of a difference. uh but when Bettie is here is it's it's . she's fairly positive and keeps it humorous. um plus she tends to be the one to . like bring out the references like she would tell us . "hey we looked it- I looked it up in the book and this is what it is." and we can sometimes like get around problems without Professor Hoffmann's push . by looking at the references that she provides. So she contributes in that way.
[00:47:25.07] Fady: Okay . but uh in in this session . you did uh number 3?
Ted: uhu
Fady: and number?
[00:47:30.23] Ted: Beginning of four.
[00:47:31.10] Fady: and beginning of number 4.
[00:47:33.19] Ted: Which I finished . when I was downstairs waiting for this
[00:47:35.23] Fady: Oh waiting for the interview. Okay wonderful. So this is quite productive right?
[00:47:41.13] Ted: mhm I noticed that.
[00:47:42.18] Fady: In these groups . are these other previous sessions as productive as this one?
[00:47:47.22] Ted: I think it fluctuates umm there are times where we are presented with really new definitions and we we struggle with the definition of things for quite a while before we get fluid into the work. you know? making sure everybody's on the same page with what the definitions are. I think that was .
[00:48:04.06] Fady: Like previous session was . less productive you think? When you started talking uh about order and proof . it was about definition.
[00:48:14.02] Ted: hum . right when we begin to talk about order like . "find the order or like find all the primitive roots of six . seven . and eight." and we were . I think we were struggling with like whether or not we need to test numbers where the gcd is not equal to one well there's no order for it . for numbers with gcd not equal to one . but we needed to I think all the group members needed to arrive at that point before we could move on.
[00:48:41.23] Fady: okay okay. That's great. Thank you. So uh our meeting is in two weeks.
Ted: okay.

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[00:02:42.20] Fady: This is our last time . for the views. we will have only one . last one next time. So um first before we start watching the video . is there anything you'd like to say about the group . how the group is working outside the classroom? When I don't have access to videotape.
[00:03:25.26] Ted: I think . hum . It's . it's good . cause like we actually communicate with one another . like before the homework is due . and we meet up at around 1 o'clock. 1:30ish. at the library uhh. Today . there was me . Boutros . and C . Bettie. and we worked together on the last homework . and we . I went over in detail with Bettie on the solutions for number two and three.
[00:03:53.04] Fady: The homework submitted today. Did you . did you do any of the today's exercises?
[00:03:58.08] Ted: Yes yes yes . that was
[00:03:59.15] Fady: Did you also the worksheet did you work today's worksheet? Before the class?
[00:04:06.02] Ted: No . cause um he handed this one out today during the section. [coughing] I'm still like mildly
[00:04:13.12] Fady: Bless you. I can see that the death is going away.
[00:04:19.26] Ted: Yeah. Not . not yet. Not today. Not today
Fady: It's like still hinging
Ted: It's always . I have no control over when I will die. I really don't . [laughs]
[00:04:36.03] Fady: So . uh . wonderful. uh . so this is and the uh what did you do with Boutros today?
[00:04:44.28] Ted: uh . we just all met up umm . and he looked over my solution for number 4 for the last worksheet. I think hum . when they met in class last time . they didn't finish it. so we just like talked about it . to see if my solution made sense. cause I was doing it when I was sick. I don't know if like what I wrote was just rambling or if it made sense.
[00:05:08.08] Fady: And . Okay.
[00:05:11.11] Ted: But I was sitting there . exploring different ways to choose a for the compositeness test.
Fady: Okay
Ted: That's why I brought that up towards the end of the class with professor Hoffmann.
[00:05:27.02] Fady: uh . okay. okay. Do you want to say more about this?
[00:05:36.04] Ted: hum . I think I am good. yeah. [plays video]
[00:05:59.00] Ted: [laughs] sitting there stuffing [inaudible]
* Video paused *
[00:06:29.28] Fady: [pauses video] Okay . just a reminder that what I want you to do is to help me be in your shoes . let see through your eyes what's going on from your perspective. What you felt about yourself . about other groupmates . and also what you thought about the mathematics that were going on in the moment?
Ted: Okay.
Fady: and how this developed? Okay? Feelings and thoughts. [resumes video]
* Video paused *
[00:07:09.12] Fady: [pauses video] Is this the first time you get the quadratic . the definition of quadratic numbers?
Ted: Yeah.
Fady: Okay. [resumes video]
* Video paused *

[00:07:33.23] Ted: [pauses video] I was looking at the worksheet . and I was thinking it looked kind of scary cause I had no idea what it was . it was . I didn't even know what the definition of quadratic residue yet was yet . So I was kind of intimidated by the worksheet at that moment. [resumes video]

* Video paused *

[00:08:32.11] Ted: [pauses video] Texting my student who I was going to meet after the class right there. [resumes video]

[00:08:35.07] Fady: Okay.

* Video paused *

[00:10:39.11] Fady: [pauses video]

Ted: Oh that . I was going to pause it. yeah . that was there . I was like I wasn't entirely sure but I'm pretty sure zero is a quadratic root.

Fady: Okay

Ted: Yeah . so . I didn’t really say it out loud . but I think he-he saw me. [resumes video]

[00:11:17.25] Ted: Like maybe::e .

* Video paused *

[00:13:51.02] Ted: [pauses video] here . I think we were like starting to . like getting more . specific about like what numbers we actually need to test and then I think um . figuring out like that we needed to test only for for a's . we only need to test um from 0 up to x minus 1. Oh . sorry. yeah . it's um . x squared equivalent to a mod m . so we only test the a from 0 up to m minus 1. Instead of testing any more . because those are just congruent back to some a from the before . right? So we just really need to test the . reduced residue system of the number. um . and I think like clarifying that took a while. [resumes video]

* Video paused *

[00:15:17.15] Ted: [pauses video] A lot of times . I like to . uh explain to help me reinforce a new idea. so I try to gather everything that I know that I know for certain is true then I relay it and try to see if I gain understanding through the explanation process . so that's what I was doing with Bettie.

[00:15:35.28] Fady: so what you are doing with Bettie is more for yourself?

[00:15:39.03] Ted: Yeah. It's kind of selfish. but it helps her out too . you know . ends up being good for both of us. but the intention was more for myself. [resumes video]

* Video paused *

[00:16:31.20] Ted: [pauses video] uh . this is me being kind of shaky. trying to reword the definition in another way to explain it. [resumes video]

* Video paused *

[00:18:23.08] Ted: So this was um . [pauses video] this was us still trying to figure out an efficient way to generate all the um . quadratic residues like by brute force. And I think Boutros was like doing it already . and it took us as a while to recognize “hey . his work was like awesome” [resumes video].

* Video paused *

[00:19:08.24] Ted: [pauses video] so once we have to find the:e what numbers we need to test for a . I think we also needed to settle on like what numbers like we need to test for x. and I think at this point . Boutros was using like the different possible values for x and just generating all the . quadratic residues. [resumes video]

* Video paused *

[00:20:20.00] Ted: [pauses video] I'm fairly stubborn. So I will try really really hard to not look at anybody else's solution . and try to come up with my own before . I reference everybody else's work. Once I get to the point I feel kind of stumped . like 90 percent sure I can’t do it alone . then I look at other people's.
[00:20:38.29] Fady: This is now you are looking at Boutros’?
[00:20:40.23] Ted: Yeah. [resumes video]
* Video paused *
[00:21:39.00] Ted: [pauses video] This was a lie. [resumes video]
[00:21:41.04] Fady: Why did you lie? [pauses video]
[00:21:43.14] Ted: Um . cause I saw like the look on Bettie's face. She ha- she hasn't been very happy about the grades she's gotten . and I've had many math classes where I've lost friends or like people stop talking to me because they're like “oh . this guy already knows everything.” [laughs] Like I've been in that situation a lot . so I always lie basically and say “Oh . I have no idea what's going on. Why don't you tell me?” or like I was I pretend like I don’t know the solution and be like why don’t you explain it to me when I see somebody that is struggling with their confidence a little bit. So here . I was just you know lying . and saying that I've never gotten a perfect score you know [laughs] [resumes video] it wasn’t necessary . but .
* Video paused *
[00:23:53.12] Ted: So we're still debating that at this point. cause I'm pretty I . I was pretty sure at this point that we only need to test x up from 0 up to m minus 1.
[00:24:04.20] Fady: Um . you say you mention that you looked at Boutros’s work at the beginning. Did you . what did you see when you looked . what did you get for that?
[00:24:16.19] Ted: It was very brief glance I saw that . I I thought he was just like trying different numbers . trial and error kind of thing . but here I didn't realize he was systematically doing it.
Fady: Okay.
Ted: I thought they were both just playing with the numbers for now. [resumes video]
[00:24:31.24] Fady: Thank you.
* Video paused *
[00:24:53.17] Ted: [pauses video] I was spending some time thinking about how to justify it to John and Jeremy cause they're they're good skeptics [resumes video]
* Video paused *
[00:25:19.10] Ted: [laughs] [looks at Fady and pauses video] 6.8 out of 7 [laughs]. point two off because I didn't . I didn't say why 8 doesn’t have primitive roots. I just wrote none. cause I got lazy. [laughs] [resumes video]
* Video paused *
[00:27:03.22] Ted: This is where John . John get's really insightful. [pauses video] John get's really insightful with what he is about to say. [resumes video]
[00:27:23.16] Ted: Yeah. He's observing the pattern [inaudible]
* Video paused *
[00:28:06.19] Ted: [laughs] Jeremy is really funny.
[00:28:09.15] Fady: Yeah. [pauses video] What do you think about this?
[00:28:11.04] Ted: I think that . he . he knows how to appreciate the little things in math. that just like makes you smile a little bit . you know. “It sounds like a cool word.” yeah. he's right it is a cool word. [laughs] [resumes video]
[00:28:24.15] Fady: Okay

* Video paused *

[00:30:02.28] Fady: [pauses video] What were you saying here?

[00:30:04.28] Ted: Um this was me- me trying to see like what's . what's the point of finding quadratic residues. just to see like is- if we're only looking at like numbers from zero to n minus one . what if we have some numbers that are bigger? Well we just mod them back down.

Fady: uhu

Ted: and then find that there is- it's congruent to a quadratic root . then there will be numbers that when multiplied together would be the same- will be equivalent to that. so . it is just like extending it to see the pattern of like “oh first we have a bunch of primitive or no not prim- . we have a bunch of quadratic roots between zero and n minus one and what happens when we repeat this chain later on.” I was thinking about that. [resumes video]

Fady: uhu.

* Video paused *

[00:31:50.21] Fady: [pauses video] What were you doing at this time? What you think?

[00:31:53.13] Ted: hmm oh I was uh being stubborn again cause I don't ever take pictures of other people's work. Yeah . so I saw what he [Boutros] was doing and I did it neatly on my paper instead of taking a picture of what he did.

Fady: oh okay.

Ted: I try really hard not to take pictures of other people's work. So that was kinda inefficient use of time. [resumes video] But it helps me understand what he was doing . cause I was doing it too.

[00:32:20.07] Fady: So at this moment . you understood it?

Ted: uhu [pauses the video]

Fady: I mean at this moment you understood fully what he was doing?

Ted: Yeah.

Fady: before that . no?

Ted: that's right.

Fady: okay. [resumes video]

* Video paused *

[00:32:47.13] Ted: [pauses video] At this point . John and um Boutros were . put all of the roots in like a row . and then he was able to see that there's like . mirror symmetry but then I was trying to see if it’s like behaving in some pattern like Pascal’s triangle or something. so that's what they . were playing around. [resumes video]

* Video paused *

[00:33:41.12] Ted: I have no idea what they're talking about [laughs]

[00:33:45.12] Fady: Were you listening to them? [pauses video] Do you remember?

[00:33:47.00] Ted: [shakes hand negatively] hu’u. During this time I was basically zoned out. I wanted to finish number one all the way and then move on to number two.

[00:33:52.25] Fady: uhu. Okay.

Ted: [resumes video]

* Video paused *

[00:34:30.14] Ted: [pauses video] Whenever John says “I'm not thoroughly convinced” that's like a signal for me to like reinforce what I know. So he said that “I'm not fully convinced about something” and then . [resumes video] we go on to talk about it.

[00:35:50.13] Ted: [comments on video] Four mod three is equivalent to one mod three.

[00:36:30.00] Ted: [comments on video] Needed to make sure everybody was on the same page. [giggles]

* Video paused *
[00:37:13.00] Fady: And you were? [pauses video] What's your work here? Is it your
[00:37:18.17] Ted: I was I think on . like . number eleven already at this point . like almost done.
[00:37:25.01] Fady: oh trying number eleven?
[00:37:27.01] Ted: Or to find the quadratic residues of eleven.
[00:37:31.06] Fady: uhu. okay.
Ted: [resumes video]
[00:38:27.12] Ted: I think . we went on- on like this for a while.
[00:38:29.27] Fady: For trying numbers?
[00:38:33.07] Ted: uhu.
[00:38:45.26] Ted: Can probably fast-forward a little bit here.
[00:38:48.02] Fady: What you can do is press and hold the forward.
* Video paused *
[00:39:46.05] Ted: This is where I start
[00:39:46.24] Fady: What is this talk? [pauses video]
[00:39:48.05] Ted: Um they're just talking about um next semester what classes they're taking. if it's like- like modern or real . and they're making comments about what class is hard . and so on.
Fady: okay
Ted: You know I was like "well if it's hard I'll study more." [laughs]
[00:40:04.23] Fady: And did you take any of them?
[00:40:06.01] Ted: Not yet.
Fady: Not yet.
Ted: I have to take modern . real analysis . geometry all in one semester along with the capstone. It will be fun. [resumes video]
[00:40:14.14] Fady: Yes
[00:40:17.00] Fady: [Inaudible]
[00:40:19.27] Ted: Yeah I think I'm going to actually quit some of my jobs.
* Video paused *
[00:41:12.20] Ted: Oh [pauses video] I was stumped with like a really really dumb question at this point. I was like "what the heck is an odd prime?" [laughs]
Fady: oh.
Ted: [resumes video] I was searching special types of prime that are odd. Like odd function . even function . some other definition?
* Video paused *
[00:41:53.26] Fady: [pauses video] Yeah how did you feel about this?
[00:41:55.20] Ted: I thought- I felt a little silly . but I- I- I thought it was a humorous moment. yeah. [resumes video]
[00:42:01.10] Fady: okay . yeah
[00:43:17.22] Ted: I thought he [Hoffmann] was kinda . holding our hand . guiding us through this calculation here.
[00:43:59.10] Ted: So as John was saying. there's a way to do it that's more clever.
* Video paused *
[00:45:30.20] Ted: [pauses video] I had no idea where we were going with what I was writing. I was just writing things up I like . they knew was kinda true and then just like kind of mapping out different things. lot of times when I have no idea how to approach a proof I do that. just write

down everything that I know. and then find out some stuff that we can deduce from what we know . before starting. [resumes video]
[00:45:49.20] Fady: okay
* Video paused *
[00:47:15.25] Ted: [pauses video] I think what I wrote actually didn't quite fully make sense. but umm yeah. I think all of us were . lost enough that we were like “maybe there's some hope in what we wrote down or something” [laughs] [resumes video]
* Video paused *
[00:49:23.18] Fady: [pauses video] Do you recall what he was asking because I cannot hear it.
[00:49:26.19] Ted: uh he said um “how did you jump from here to here?” quote unquote. Um and it was um just the equation x squared is equivalent to a mod m and we took the . both sides of the equation . I mean equivalent relation . to the power of phi of . p. That was it.
[00:49:50.21] Fady: oh was that the question?
[00:49:51.24] Ted: yeah so it was like [hand gesture] [resumes video]
[00:50:18.08] Ted: [comments on him drinking coffee in video] I'm such an addict.
[00:50:20.25] Fady: what? coffee?
[00:50:22.00] Ted: yeah [laughs]
[00:50:28.08] Ted: It can't be healthy [chuckles]
* Video paused *
[00:51:46.08] Ted: [pauses video] That caught our attention the uler . Euler. cause I always did that in my head. [chuckles]
[00:51:51.05] Fady: Oh you when you [incomprehensible and overlapping talk] but they're not talking about you . right?
[00:51:55.18] Ted: No . but in my- in my head when I was reading . when I was younger I would always say “uler” because I didn't know it was Euler. [resumes video]
[00:52:01.04] Fady: uhu.
* Video paused *
[00:52:17.18] Ted: [laughs] John drew a portrait of- of Euler [Fady pauses the video] John drew a portrait of Euler the class before or something like that .
Fady: oh
Ted: and Jeremy took a picture . yeah it was kinda cute. [resumes video]
* Video paused *
[00:53:20.22] Ted: I think I stare like this meda- [Fady pauses video] like I sit like this meditating about the problem for quite a while. cause I . I wasn't sure what to do next for part b.
Fady: oh okay.
Ted: Yeah so I'll just kinda be . look like I'm [resumes video] lost in thought.
* Video paused *
[00:53:50.13] Fady: [pauses video] When you're meditating is it- how do you think? What's . what do you do?
[00:53:59.02] Ted: I get into my best teacher voice and I try to explain it to myself . like say things that are obvious and then st- st- try to make connection.
Fady: okay
Ted: But . yeah it's very very . amorphous . you know what I mean? Like my thought process is very much like . this is [gestures with his hands swirling in front of him . then shrugs] I . I . can't really put it into words. It's not like a straight path. I just like have all these like little points of

memory in my head and I like try to draw connections between those points. it's not a straight line . you know?
[00:54:34.27] Fady: And who is your best teacher's voice? Who- who is your best teacher?
[00:54:39.04] Ted: Ummm well I mean when I teach little kids.
Fady: Oh.
Ted: I get into that voice in my head. and I'm like teaching myself like as a- talking to myself as if I'm a little kid. I'm like "well . you know this and this is true and you know this and this is true so:o what do you think is the next step?" Then I'll ask myself that question and just like silence in my mind and try to see if I can find the next step. [laughs] It's a weird thought process.
[resumes video]
[00:55:42.28] Ted: [commenting on video] Review [laughs]
* Video paused *
[00:56:46.21] Ted: [pauses video] This was I think important for like finishing up the proof to- to see that it is an even number. cause umm we'll have like some x to the like n over two to the phi of m or something like that. so we needed to see that it was even. [resumes video]
[00:57:30.50] Ted: I'm trying to gather as much as we move forward. like going through all the theorems that we went through . to see if there was anything useful in my head.
* Video paused *
[01:00:05.23] Ted: We stay stuck like this for quite a while . and then um [pauses video] eventually umm . right before Bettie's iPad ran out of battery. She show . she showed us like uh . she actually found the solution to this part in the book. and I glanced at it. [resumes video]
Fady: uhu . and then you-
Ted: and then we basically solved it.
* Video paused *
[01:02:04.23] Fady: What were you trying here? Do you remember? [resumes video]
[01:02:07.22] Ted: I s::started out with a to the p minus one that is equivalent to one mod p and then wrote down like . a to the . oh wait no what we were given which is a to the p minus one over two is equivalent to one mod p.
Fady: uhu
Ted: and then I wrote that it implies that a to the p minus one is equivalent to one mod p. and then I said "well that's also obvious because it's Euler's theorem" . [laughs] and then I . yeah I was- I was still feeling pretty lost. right about then.
[01:02:41.17] Fady: uh you were still- okay
[01:02:44.00] Ted: uhum.
[01:02:43.03] Fady: Let me just check because I just checked the time uh . probably Boutros is here. We'll just start back up and continue right there where we left off.
[01:02:52.24] Ted: mmkay
[01:02:53.17] Fady: Oh he is here.
[01:02:57.25] Ted: sup?
[01:02:58.11] Boutros: what's up man?
[01:02:58.27] Ted: Okay
[01:02:59.16] Fady: So uh actually we can stop here.
[01:03:01.21] Ted: Alright .
[01:03:04.09] Fady: Do you still have something to say . in general?
[01:03:07.14] Ted: mmm
[01:03:08.25] Fady: About the rest of the .

[01:03:10.12] Ted: I enjoy the struggle [hits the table with his hand].
[01:03:13.08] Fady: uh you enjoy this struggle [pointing at the laptop] or is .
[01:03:15.08] Ted: like I just enjoy the struggle. when I'm dealing with a math problem that's just the right level of out of reach.
[01:03:23.15] Fady: Okay. uh here you got the solution from the book.
[01:03:31.03] Ted: yeah.
[01:03:33.00] Fady: and what do you feel about this?
[01:03:35.20] Ted: I feel . a mild sense of shame every time I have to look into the book to get the answer. [laughs] that's where I'm at like ev- but it's a very mild sense of shame cause I'm not like . a professor or anything you know?
Fady: okay.
Ted: So looking stuff up isn't bad. [handshake]
[01:04:01.02] Fady: okay. Get well soon
[01:04:02.04] Ted: Yeah
[01:04:03.00] Fady: Take good care of yourself.

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[00:02:17.09] Fady: Today . we're gonna . I'm gonna ask you a little bit about the overall experience of the class . and I have questions . and if you have things to say that the questions did not cover . and I will be very interested to see . and if there is any questions that you don't like to answer . just tell me skip . and I will skip it.
[00:02:42.11] Ted: Okay.
[00:02:43.07] Fady: Okay. so we are going to start with brainstorming . kind of let . uh . telling me whatever pops up in your mind when I say a word.
[00:02:49.13] Ted: Word association?
[00:02:51.20] Fady: Verbs . associations . images . feelings . events . memories .
Ted: okay.
Fady: when I say number theory.
[00:02:59.19] Ted: Fun.
[00:03:01.21] Fady: what else?
[00:03:04.06] Ted: Um . challenging . uh new . interesting. It's um . it's complicated . but simple. [laughs] . um . beautiful.
[00:03:26.27] Fady: Okay . and what does pop up in your mind when I say math 310?
[00:03:32.04] Ted: Math three ten? um in what school? first question. uh . and then that's [inaudible]. and .then I think . I think about the school much more than the class when I . when you mention the . when you mention that.
[00:03:52.28] Fady: You said that is fun . why?
[00:03:56.27] Ted: mmm . because it is challenging. Um and because . because it's requiring me to prove things instead of um just do computations. and I have a strong bias towards like proofs and that part of math that I am part of it.
[00:04:18.26] Fady: uh . and how did you handle the challenge?
[00:04:21.28] Ted: uh . not sleeping. [laughs] uh . lots and lots and lots of coffee? umm I guess uh . definitely a lot of uh time spent researching and a lot of patience with myself . cause it was . it was more challenging than a lot of the classes that I've taken. So to me . I need to be really patient to myself . and that was . that was a big deal. Sometimes it's like you know when you when you stare at a problem for two three hours . you got frustrated sometimes. but like to . to control that. and use that as like . fuel for finishing it up.
[00:05:08.16] Fady: Okay . thank you. Did you get the chance to talk about number theory outside class . with family . friends . partners . colleagues?
[00:05:21.10] Ted: umm . I did talk about it with some colleagues uh cause I was doing it on the white board during homework lab. For the . for the little kids . they have homework lab . so they have 20 minutes of reading. So I would just sit on the white board doing my homework during that time [inaudible] I love those little kids' reactions. [mimicking little kids] "That's my homework?" [laughs]. Like . "no no no . that's my homework." And then um . some of my colleagues like uh . are also attend [name of current university]? so they they asked me about it.
[00:05:54.21] Fady: they attend what?
[00:05:55.26] Ted: [name of current university]
[00:05:56.22] Fady: oh . [name of current university]
[00:05:57.15] Ted: so there um . one of them is an economics major . so he has some math background. and we talked about what this is all about . so . it's it's fun to talk him about it . like how.
[00:06:07.13] Fady: Do you remember which topics was raised during this conversation with your colleagues?

[00:06:12.12] Ted: It's um . it's pretty fundamental stuff like the definition of mod. um what does it mean to be equivalent relation. things like that. It's like not [unintelligible] part.
[00:06:21.08] Fady: Equivalence . you mean congruence?
[00:06:25.01] Ted: yeah. equivalent . uh the what the triple equal side means instead of the double equal side.
[00:06:33.21] Fady: Okay. uh . and did this happen a lot? Or just a couple of .
[00:06:37.27] Ted: Twice . and they got . they were like okay. like don't want to talk about it anymore. [laughs]
[00:06:44.22] Fady: Okay. uh . which topics in number theory that you feel comfortable discussing with others?
[00:06:54.08] Ted: Um . well. the congruence and like how you can add and subtract . um both sides. like the general arithmetic that you can do with modular arithmetic. um . and I am starting to get a feel for like things like reduce residue system. and what it means to apply that to proofs. um . and I guess like the homeworks that we have turned in those are the topics that I feel comfortable talking about. um . as long as I have something to reference when I am talking .
Fady: okay
Ted: it wouldn't be like free verbal.
[00:07:34.14] Fady: yeah . sure. But you feel comfortable if a conversation is engaged for example if you are . if you are . hired in a company . and they needed number theory topics . and you colleagues at this company raised topic and you feel?
[00:07:51.00] Ted: I will be able to chime in. and if it's like the brainstorming phase . safe space to talk . yes certainly. and then if it's like related to performance in the profession then I would say I need a part-time.
Fady: Okay
Ted: So it's like yeah.
Fady: some details.
Ted: If I was have to give a presentation give me a week before I do it.
[00:08:14.08] Fady: uh . is this also how you felt every time you finished the coursework?
[00:08:19.13] Ted: um . math related course works.
[00:08:23.24] Fady: Math courses?
[00:08:25.12] Ted: yeah . not . not always science . yeah.
Fady: oh
Ted: I get by in science classes using the math. and that's it. if I have to explain general science concepts . I can't do that.
[00:08:36.20] Fady: uh . all of then you felt very confident? is there any difference from other math classes that you felt in this class?
[00:08:47.17] Ted: mmm. It . it challenged a different part of my brain though. because it wasn't just applications of like . arithmetic. right. cause cause . in up to about Calc III it was about just like . "oh they've already proven this" . we're gonna skip the proofs . and work on just applying these proofs . these things people have proven. But this time it's like we are proving what we need to know. so I feel like that was . that's the main difference.
[00:09:20.08] Fady: and you took modern algebra? and [inaudible]?
Ted: no . not yet.
Fady: Not yet? Both of them?
[00:09:26.07] Ted: Not modern . and [inaudible] next semester.
[00:09:27.12] Fady: oh . next semester. Okay . so this is kind of the first uh . upper .class? did

you take after . this upper . first upper uh?
[00:09:38.15] Ted: just about. yeah. I took game theory . that was more . that was more elective.
[00:09:45.04] Fady: Yeah. okay . is there any topics that you struggled with?
[00:09:55.23] Ted: uh . the quadratic residues? um . dealing with that . it's . and and the beginning of the Chinese Remainder theorem.
Fady: Okay
Ted: Those two . oh and I think at the very very beginning . just like working out how to prove divisibility rules. and just like refreshing myself what it means to prove things. that was . that was also challenging.
[00:10:19.19] Fady: oh . divisibility. So can you talk a little bit about each one of those? The quadratic . what was really challenging?
[00:10:27.13] Ted: It . it's just a new concept built upon a new concept. It's like to me . mod is new.
Fady: okay.
Ted: and then were building on top of like . uh there is a modular equation and we have a variable. And now . instead of just a x a linear type of relationship now it's quadratic. So like that . that's kind of a new step for me. On top of a new step . so that's why that one was challenging. But um . yeah.
[00:10:56.28] Fady: uh . so the definition . making sense of the definition? is that what was challenging?
Ted: yeah.
Fady: The Chinese Remainder Theorem . so what was the . the challenging part? how did you learn it first? did you learn it first from the book? or from the? um
[00:11:16.00] Ted: I was very stubborn after that point . and try to not look at any references and just look at the problem and bang my head against it for a few hours before . I . finally look at the book. [laughs]
[00:11:29.10] Fady: I think the worksheet that did not explain it clearly if I remember very well. so you try to make sense of it out of the worksheet?
[00:11:39.06] Ted: Right . and just like to . given what . what . what was proven before this . and what I can prove with.
[00:11:45.07] Fady: Oh . okay . yeah. So and how did you solve this challenge?
[00:11:49.22] Ted: eventually looking it up in the book . for . and other references online.
[00:11:55.00] Fady: Like what? the references?
[00:11:56.18] Ted: mmm . Wikipedia actually is quite helpful sometimes. it . Wikipedia sometimes is very . high-level for the material. so .um . I sometimes look at um . just different . books that I find on uh scholar.google.com or yeah. It's like um . and searching the topic itself. and looking at cite.edu . I look at that.
[00:12:25.18] Fady: And uh you mentioned also the proof rules. rules of divisibility?
[00:12:32.05] Ted: yeah . that's just like refreshing the idea . of um . like . what is to . what contrapositive is for example. like little . little things that you need to know to prove things. I was just like refreshing that. so like numerically I understood divisibility . but how to write it properly. that was the beginning challenge in the class.
[00:12:54.17] Fady: uh . did you take the proof class before this one?
[00:12:57.15] Ted: concurrently with this one.
[00:12:58.25] Fady: oh . concurrently?
[00:12:59.27] Ted: Yeah.

Fady: okay . so:o

Ted: I-I-I I mean . I for one of the semesters before I did take the class up to the half way point. but then I dropped from school for that semester.

[00:13:09.07] Fady: okay . and did you find the proof class helpful? for this one? Did? Or .

[00:13:16.01] Ted: Yeah. I mean . um . the proofs class covers things like injection . bijection right after we did it in number theory.

[00:13:24.20] Fady: oh . after that

[00:13:25.09] Ted: yeah . so so the timing was off for some topics. but . right at the beginning . it was really matched up. and .

Fady: okay . okay

Ted: at the beginning was super helpful. and then later on it became . just I just had to look for references.

[00:13:42.15] Fady: in proofs?

[00:13:44.01] Ted: um.

[00:13:45.16] Fady: Okay. do you remember like with regards to which topics?

[00:13:48.25] Ted: Oh . like how . how is it to like . write in symbols . like specifically what it means is to be injection or like . surjection? and how . how you get the bijection from. I remember like one of the . one of the worksheet question 4 or 5 . and had a proof that required showing it bijection for two sets or something.

Fady: yeah.

Ted: show . show that . there is a function and doing bijection . and then therefore . the set . the set size is the same. Alright . and then it was making that connection and I didn't realize that that oh . bijection . oh that means the domain and the codomain are the same cardinality. That connection was . it's like .[laughs]

[00:14:34.01] Fady: and how do you find the proof class that you're taking? do you learn from it a lot?

[00:14:39.02] Ted: I feel like . this class is making it so that class is just . so easy. Because everything I do here is application of what I have to learn in proofs so .

[00:14:58.09] Fady: Okay . alright. uh . okay. so now let's talk about the best way that you found helpful for learning number theory.

[00:15:08.01] Ted: uhu.

[00:15:09.13] Fady: yeah . it's the time

[00:15:13.06] Ted: I meant this is part of the answer . I'm pointing at the coffee cup . so . yeah . coffee.

Fady: check the video.

Ted: yeah . check the video. let's see. what other things? uhm . referencing the books for sure . hum. But I think . I think talking to other people about it it's it's important . like . it's really important to be able to like cause other people would challenge your proofs . and that challenge makes your proof more refined. you know. like when they're like "I don't understand why you jump from this to this" . that is just means you need to explain it better. you just need to insert a few lines to justify that part. and I found that like just talking with other people or showing them my proof . "does this make sense?" Then . that . that peer review really helps.

[00:16:13.22] Fady: uh . okay. So these things . what else? if any?

[00:16:25.20] Ted: um . a lot of time and patience. And this dedication to the subject itself . right?

[00:16:34.28] Fady: so what helped you in your personal studies. so when you used to study

alone . how . is there any specific method that you use to learn better?
[00:16:45.23] Ted: I would often tackle a problem and then go to sleep. and then come back . and look at the solution. like I won't try to review it at the same day. Like I will do all the homework and then sleep . come back . and look at it . and see if then it makes sense. because that time gap to sleep really.
Fady: Okay.
Ted: like helps you with review. it captures mistakes.
[00:17:10.22] Fady: uh . would you like to . to talk about your experience with this . group study outside of the uh . outside of the . the class . you created this. this. group?
Ted: Yeah.
Fady: and people started to come and can you record like the development . the history issue that I am asking you to trace the history like how the idea started . how you launched it . and what happened . why ? so if you can go back to the . this time . this moment probably the beginning of the course? and then tell me as much as you can in details about this group study?
[00:17:55.07] Ted: So I think we started talking about like studying outside of class just like . like you know in our conversation in class. For about the first three to four weeks. and then we stopped talking about it for a while. and then the midterm started . to come around. about a week before the midterm that's when people started to be serious about actually meeting. and then uh . we arranged to meet up and study for the midterm. I think Bettie was the one who scheduled the study room. and that really helped out. we all like met up in that room and we were able to talk and use the white board. So that was the beginning of it. uh . and then at that point I decided to share my Google Drive with everyone as well. so . but I don't think everybody looked at it too much. it's just like when I reference specific things I don't remember.
[00:18:48.16] Fady: Okay. did the Google Drive idea started after the midterm or before the midterm?
[00:18:55.14] Ted: the google drive idea . I mean that one is just happened that I have for all my classes. I keep everything on my Google Drive . and I share with my classmates. so . That idea was like pretty long ago.
[00:19:07.13] Fady: so you started for yourself. but when did you share it was what . what uh . what moment of the class?
[00:19:16.14] Ted: um . It was when I started do . like two weeks after when I started turning in typed homework. So that was homework 4 or 5?
[00:19:31.15] Fady: so it was before . uh midterm?
[00:19:35.05] Ted: Yeah . so that's just something that I generally do.
[00:19:40.05] Fady: okay. and how . how was the dynamic . so you meet what happens? uh?
[00:19:50.19] Ted: it's a lot more casual. there's usually like parts of it like we have like [unclear] . and like solo work . we'll just work our own thing . and then come back and meet. Or like we . we would . a couple of us would get up to the white board and work. because we are not working at the same pace. and we were checking with one another. It will be like “hey . what are we working on?” if we were all at the same problem . and we feel stuck . then we both write down what we have known.
Fady: Okay.
Ted: and then we would look at each other's cross reference. and then yeah.
[00:20:23.07] Fady: uh . uh. okay. some people mention that you used to go to the bar before or just hang out before you start studying?
[00:20:34.22] Ted: I . no . that would be . that would be the other members of the group. I don’t

have time to go to the bar. [laughs] yeah . I haven't I haven't stepped a foot in the bar. No.
Fady: To the bar.
Ted: no I have seen . I have seen them. and I like they've also invited me at some point . or maybe or maybe not. I don't remember these things. but like . yeah.
Fady: Okay.
Ted: I don't go out to . drink. [laughs]
[00:21:01.29] Fady: I mean I think they . they didn't go only to drink sometimes they went to a coffee shop or something.
[00:21:09.06] Ted: oh . yeah. Sometimes it's like we get food or something. yeah . or go . go to the city center to get a burrito right before class.
[00:21:15.26] Fady: Yeah how long will this time last? and what happens?
[00:21:20.07] Ted: uh? personally . I intend such things for fifteen to thirty minutes.
[00:21:23.26] Fady: Okay.
[00:21:26.00] Ted: like a very little time. [laughs]
[00:21:28.05] Fady: how many times you were going with them to the ?
[00:21:29.17] Ted: two times? Three times?
Fady: two times?
Ted: Not very often.
[00:21:32.27] Fady: for fifteen or thirty minutes? and then came back? okay. and how many times in total did you meet?
[00:21:38.16] Ted: oh . that one's hard. cause like not everybody met up each time we meet. so sometimes only it's like me and Bettie . and Boutros or Jeremy comes a lot. John is here sometimes cause he lives over in [city name]. so it's harder for him. um . and let's see. uh. Melissa . from another group has joined us. and it's like this.
[00:22:06.11] Fady: did other people from other groups join you?
[00:22:08.07] Ted: um . Tito showed up couple of times . yeah.
[00:22:11.25] Fady: okay. uh . do you remember the topics that you tackled? in general?
[00:22:18.22] Ted: um. generally we just work on the homework. generally.
[00:22:21.15] Fady: the homework?
Ted: yeah.
Fady: is there a specific topics about the homework?
[00:22:26.15] Ted: Oh . yeah. A lot of the Andrews problems cause we don't work on that in class.
Fady: Okay
Ted: so the homework we didn't get to work on the class . we work on it in group outside.
[00:22:36.10] Fady: it's applications. right? the Andrews?
[00:22:40.10] Ted: sometimes it is . sometimes they'll be proofs.
[00:22:45.12] Fady: um . okay. What is your role in this group . so you are the one who initiated it . like or launched it ? is that right? am I getting it right?
[00:22:57.23] Ted: um . I . I . I mean I asked people if they're here. I feel like it happens kind of naturally . organically. there's no like really one person who pushes. like sometimes . it's like me who asked . and sometimes it's other people who are asking to meet.
[00:23:15.25] Fady: I . I meant the original idea? started? is it you who asked them to meet? or someone else?
Ted: I think.
Fady: at the beginning.

Ted: At the very beginning.
Fady: and then everyone took initial or freedom to ask if other people are around? they would gather . I mean if people are around . and they respond to the call . they will meet . is that how it works?
[00:23:38.12] Ted: Yeah . and like one or two of us usually takes the role of picking a study room. and whoever picks the study room usually is just like set to be there. and everybody comes in and out.
[00:23:51.09] Fady: okay. wonderful. and in this is there people who tend to be more explainers than to be more people who ask for help? like can you describe the personalities or the . the behaviors of people in those moments?
[00:24:10.07] Ted: Are they going to see these?
[00:24:13.05] Fady: No. they are not.
[00:24:17.04] Ted: um. Let's see. Think. uh starting with Bettie . she's very positive. But . you know. She brings out a very positive energy . and. but she's also like the one who seeks the help the most. um . and then Boutros tends to be more like a solo worker. um . he does contribute with . especially with like the application part of the proofs. and I think that he um . yeah . he . he is . he’s fairly quiet when . when we were
Fady: in the study group?
Ted: yeah. and then Jeremy likes to explain for sure. Jeremy likes to explain. likes to explain.
[00:25:04.02] Fady. Ah . likes to explain. Ah. And to whom does he usually explain?
[00:25:12.14] Ted: hum . the re . the whole group. Like when we progress usually like the whole group is listening to one person.
[00:25:19.01] Fady: oh . okay. okay. uh . what's your role in those places?
[00:25:25.05] Ted: um . I try to remain humble. and always like ask questions . I . I . I ask questions a lot of others. and it's like uhm . kind of . kind of like very subtly guiding the conversation. like “hey what are you working on?” little questions like that. the teacher questions. [laughs] “what are you working on?” “what did you get so far?” “oh . I think . I am not sure what is going on” . sometimes I am like “can you explain me what is happening?” just to see where everybody is at. So that's kind of my . my role.
[00:26:04.10] Fady: okay. uh . so you got to explain also to people. Did they help you understand things? or it's usually uh you helping them? Or ?
[00:26:15.09] Ted: They helped me understand a lot when they asked . what I wrote. You know like . when when they . when they asked for more explanations. because I . I personally find more clarification through the process of explanation. Like I feel more clear about it after I talked about it. You know.
[00:26:34.20] Fady: Okay. Okay . wonderful. uh . and probably you answered this question . but if you have additional information about it . that would be nice. What did you do when you get stuck on homework?
[00:26:49.17] Ted: hm. I think I have said the most . most of my strategies . yeah.
[00:26:57.06] Fady: Yeah . which is think overnight
[00:27:01.29] Ted: Yeah . I will try . try my . try my best for like three . four hours . and then . and then look at the books. [inaudible] Sometimes I would ask Jeremy or John.
[00:27:13.05] Fady: Okay.
[00:27:13.17] Ted: John is like have a good mastery of the . of how to explain it.
Fady: John?
Ted: uhum.

[00:27:20.09] Fady: Do you ask him in class . outside the class?
[00:27:23.00] Ted: Yeah . I feel . I feel like in class. That's like . there's some form of pressure. and like John and I feel most free to have conversations. During the class . so he and I . here . he and I are tend to have a lot of back and forth . and Jeremy chimes in too. And then I feel like during class time that's when.
[00:27:43.28] Fady: Yeah. what do you mean by pressure? What's that pressure in class you feel that will make you ?
[00:27:50.02] Ted: like you're in a group . and you feel like you want to perform. You want to perform well. Like cause there's class the professor's also there. You know?
Fady: I see.
Ted: So I feel like that might be a pressure . but that is my assumption. my perspective.
[00:28:08.27] Fady: Okay . uh . Okay How do you think uh . John with regard to this?
[00:28:14.09] Ted: He and I . I feel like he and I are similar and that's like that we don't really care what anybody thinks of our math abilities. We are . we're doing this four ourselves. kind of thing. you know? So I feel like those who have that attitude tend to be immune to that pressure which is like well if I do bad . it's I'm about to learn something. [laughs].
[00:28:37.17] Fady: uhm. uh and I think we didn't talk about the . I am sorry did you submit an incomplete homework? or did you leave some exercises blank or something?
[00:28:53.16] Ted: Once.
[00:28:54.08] Fady: When was that?
[00:28:56.04] Ted: The homework right before the final. There was one about the . the mu function. and it was like the product of the mobius . like that involved the mobius function . to proof like the product of f of d.
Fady: Okay.
Ted: Like . yeah. I . I
Fady: Why? Because
Ted: I wanted to dedicate time to study for the final. and that one took so long. and I did . I read the proof like I found it somewhere . like online. but I have no idea what it meant. and I wasn't about to write down something that I didn’t understand. [laughs]
[00:29:30.07] Fady: Okay. okay. uh. How did the . How did the grades on the homework and the midterm? um . effected you? positively . negatively or what? in which sense? which ways?
[00:29:45.13] Ted: My only frustration is to never get a perfect grade. [laughs].
Fady: Yeah . right
Ted: Like I keep getting 6.8 out of 7. 5.9 out of 6. and . and I look at it . then there is. The things that are marked off are like that I cannot dispute it. I did get that wrong. and then it if . it feels like that there is a lot rigor in grading. And that feel would make me a better mathematician. That's how I feel when I look at the grading you know?
[00:30:17.03] Fady: because the grader is rigorous?
[00:30:19.19] Ted: yes.
[00:30:21.15] Fady: umm. okay. but did you feel good about these grades? or not good . like satisfied?
[00:30:28.09] Ted: I feel satisfied. but that kind of
Fady: You want more
Ted: I want more.
[00:30:34.29] Fady: Okay. How did you prepare for the midterm?
[00:30:38.25] Ted: hum . I made a list of every single theorem that we covered. um . and then I

made a list of all the . definitions and the axioms that are related to the theorems. and then stringed them all together in a logical prof . progression. um . and then I looked at the proofs for everything that was a theorem. Like to everything.

[00:31:06.20] Fady: um. did you work with others?

[00:31:10.16] Ted: I was in their presence. but I feel like I was in my own zone when I was doing the midterm review.

[00:31:15.23] Fady: oh . the group.

Ted: um.

Fady: the group study or something.

Ted: yeah.

Fady: and how are you planning to prepare for the finals?

[00:31:24.01] Ted: um . probably a very similar process . to what I did before. Just go through all the definitions and theorems.

Fady: okay.

Ted: try to make sense them all.

Fady: um.

[00:31:35.20] Fady: How would you assess your learning in this class?

[00:31:40.07] Ted: I am having fun . but I feel like I could be more efficient because I am too stubborn. cause I spend so much time like on problems that like I know that I spent three or four hours on problems that I should have spent half an hour on them and then look it up.

Fady: Okay

Ted: [laughs].

[00:32:01.21] Fady: but . okay. but whatever you learned after that . is it satisfying? I mean the overall uh learning outcome of this entire class. if .

[00:32:10.18] Ted: Five out of five. Five out of five.

[00:32:15.05] Fady: five out of five . uh . okay. Is this how you also felt with other classes? Math classes?

[00:32:21.19] Ted: Not necessarily.

[00:32:23.01] Fady: Not always?

[00:32:24.00] Ted: No.

[00:32:24.17] Fady: What is specific in this class that makes you feel like this? five out of five? that makes if perfect almost?

[00:32:32.23] Ted: um … the challenge and the chance to actually prove things. those two things and also um . after we try to prove things as a group . I feel like there is good really good feedback . on the content. and yeah like I feel like there's there's . not . there's no more hand holding. but there's just the right amount of guidance when like . the professor comes around to just like . check-up on you . and be like . "hey . what are you doing?" and gives us a hint here or there when we were stuck. But not the solution. just the right amount . you know?

[00:33:21.02] Fady: Okay. did this experience make you rethink other choices of your career? Or give you something to do . to look differently for your future?

[00:33:33.03] Ted: Well. yeah. instead of going straight for the credential program for teaching . I am considering going into master's program.

[00:33:44.01] Fady: Master's program? what are you? … can you say more why it is?

[00:33:52.08] Ted: I . I think I . I have haven't really been tested like this in terms of proofs . and I was not conf . even though I really liked it . I wasn't (Schoenfeld, Smith, & Arcavi, 1993)t that I could do it. you know? Like uh . I . I look at people who prove things like this the first time .

and I think wow. you know? [laughs] I'm like how could I ever be that smart. and I'm like . now . now I realize that you know proofs is something doable . it's it just takes a lot of time. and patience.
[00:34:28.14] Fady: Thank you . I am going to ask you about other changes that you experienced in this class . and tell me what were they . if any? Did you experience any change in the ways of learning?
[00:34:39.09] Ted: um … before . I would . I would read the proof and just understand it . and then just move on. and now . now I try to actually . prove things.
[00:34:58.12] Fady: Uh . did you have the chance to work in groups before? Is this something new in this class . or?
[00:35:06.13] Ted: yeah . I felt like the groupwork in this class is not pressured. Whereas like . like . you know . before this a lot of group projects was like . "okay you have a set group project for this amount of time and you have to have it finished by the end." There's a part of your grade that's dependent on your interaction with your group. You know what I mean? Like um . where . whereas this one . you have individual accountability the whole time. You're [next word is uttered with high pitch] allowed to work in groups. So more . more natural that way. um . cause a lot of group projects . if . if a finished project is required then . there are people who are gonna contribute at different rates. And there's gonna be friction among the group . because the design [unidentified] for an outcome. You know?
Fady: okay.
Ted: whereas . whereas this type of group dynamic I feel like . is . is much better. you know? Like if there's friction at all . well then just don't meet. [laughs]
[00:36:05.12] Fady: Did you experience any change in your ease of understanding the materials? Did it become easier . complex . fluctuated over the semester?
[00:36:14.24] Ted: Yes definitely fluctuating cause uh certain new concepts are easy to grasp. uh for example like after the introduction of the J- Jacobian symbols umm . we talked about how to apply it .when we're looking at larger numbers. right? when in classroom . when we were doing the flipped and stuff [gestures with hands] that was . that was quick. you know? that was just a short application to a concept. but when we introduced like- things like the . least residues and using that to do Gauss' lemma. that . that was a new progression. but that- each one was like a step . but the step seemed a lot bigger than the other one. So I feel like when you have those big jumps . that's when you have the fluctuation.
[00:37:06.21] Fady: Do you experience any changes in your ways of participating in the group? In class.
[00:37:17.10] Ted: I feel like because the group work was so central . umm and it was it was not like . it wasn't the potential for friction from like . striving for a final project . umm I was able to be more chill than usual. Cause I get pretty intense . like I . I . I like to get good grades . you know? So I tend to be the one who pushes a lot when it is a group project that has a grade.
[00:37:48.09] Fady: uh okay did you experience any change in the group dynamics? Did you see that the group changed in some ways from the beginning till now?
[00:38:04.19] Ted: I feel like people got more familiar with one another. there wasn't like that . there wasn't as much of the fear of like you know "If I'm going to say something stupid" kind of fear . you know? umm but . tha:at kinda dissipated over time.
[00:38:19.05] Fady: Umm okay . do you know why? Can you tell why?
[00:38:26.20] Ted: I think umm all of us have been fairly conscious about being positive . even when people make mistakes. Like it . that was like a caution that I feel like from everyone's part.

So . we were . we were kind to one another even though we were making mistakes.

[00:38:51.17] Fady: Okay. So in general . like briefly . how would you describe your experience in the group . overall. in class.

[00:39:05.16] Ted: It's been good.

[00:39:10.18] Fady: Okay mm . can you say more? Why . it's good?

[00:39:23.23] Ted: hm I feel like we were . because of the frictionless type of . type of interaction between all the members that we have umm been really able to bring forth everybody's like . what they're are good at to the table to . to solve things together. you know? Like . like some . I I can be kind of negative and stressed out a lot. and like Bettie's energy will balance that out. um [laughs] like Boutros' calm demeanor really like . cools down the group. cause like John and I can get really intense whenever we have these talks. umm and Jeremy is just like a great addition too . cause he's just like so able to appreciate the beauty . within what we do. just like . "wow this is beautiful." We all just take a moment to pause and appreciate the beauty of what we were doing . you know?

[00:40:28.20] Fady: okay. uh . okay starting my next question. I wanted umm can you go around the groupmates a::and tell me how you think about their number theory ability . like skills . personalities how . how would you describe them as number theory persons?

[00:40:59.10] Ted: John is extremely rigorous with his proofs. He . he likes to be the skeptic and the questioner. and I I really like what he does with that. umm whenever he reads anybody's proofs . he kinda goes in with the mind set on . "let me critique this." and he's a very critical thinker. um … Jeremey has the best handwriting for sure [snickers] umm but he . let's see . he is very confident with his umm with his number theory. and he has skills to back it up. I I think he uh he has a very agile mind. Let's see. mm … I think uh Bettie's approach is . to . to number theory . is making her life a little bit more difficult. because uh she is approaching it with . like I think what we do with applied math . which is just memorize formulas and apply them. umm and she mentioned that multiple times with the group she's like . "I just gotta go back and memorize it" and I think it's making proofs really hard for her.

[00:42:29.21] Fady: Did you try to address this with her directly? or talk with her about this?

[00:42:36.00] Ted: hmm I mean not directly . but I would mention . in conversation with the whole group . I would say like "I don't memorize."

Fady: but you don't say that directly?

Ted: I don't push it upon anybody else. cause how can I say what's best for her to learn? She has to discover that herself. Even if we give her advice . she has to try it out. umm let's see .[pulls out a bottle of water] water.

[00:43:07.15] Fady: We are now at uh Boutros'.

[00:43:17.06] Ted: Boutros' uh I feel like he is a pretty good mathematician too. and . but he . he kind of lacks this confidence to share his work. he would . he would kinda work for a long time and then when he's really really sure that's when he would share something. and I think that umm it would be to his benefit if he . you know . share his work and be okay with making . mistakes more.

[00:43:45.08] Fady: Do you see any these members changing behaviors throughout the semester? Like John . did he become like this or he was like this at the whole time . Boutros . Bettie?

[00:44:05.17] Ted: umm . I feel like John was . was pretty constant his . his he was solid the whole time. umm I think . mmm … not a lo . not a lot of changes for Boutros. and Jeremy seems to be a little less confident in the beginning of the semester . just a little bit. and .

Fady: okay . why so?
Ted: I'm not sure. But . it it's just like a feeling I get umm it's . you know . it's not like I have a device to measure confidence [laughs]
Fady: It's just an impression?
Ted: yeah an impression. umm Bettie . she . she's she fe . it feels like she is more under pressure now. At the . in the beginning . she was like little more care-free . but now I feel like she is under pressure.
[00:45:13.29] Fady: Under pressure . but how is this pressure making her- how do you see this pressure making her more serious or more careless? or uh how is it affecting her learning or behaviors in group?
[00:45:27.14] Ted: I just hear her saying like . "I I don't get it." And then stop more. often in . during class. Like outside of class like we're . we have more time like we can work through it. but umm sometimes in classes it's like . she's like . "I don't get it" and then just doesn't . just kind of drops out from the groupwork. you know?
Fady: oh okay
Ted: So that's happening a little bit.
[00:45:54.19] Fady: uh was she more insistent in the beginning?
[00:45:57.21] Ted: yeah . umm she tends to be the one who would pull out the references from the book. and help . help us break through. umm when we . when me John . Jeremy are being very stubborn and trying to do it without any references . she would be the one to be like . "hey wait but . but it's in the book. look." and then we would read the book and then go back and forth to just to explain the proof in the book. you know?
[00:46:24.06] Fady: yeah . okay. So you . you find that Boutros did not change in terms of uh venturing out and talking in the group?
[00:46:30.29] Ted: That's right. he's he's always been fairly quiet.
[00:46:33.01] Fady: Is he different in the study group than in the class group?
[00:46:37.24] Ted: I think all of us uh feel more casual. cause it's not a class environment so we . we joke around. and we don't . we don't feel like there's any boundaries to . to like the class boundary.
[00:46:52.06] Fady: Does he express himself more in study groups?
Ted: Yeah.
Fady: and share ideas and discuss and talk?
[00:47:01.21] Ted: and I feel . I feel like that's . that's generally the same for all.
Fady: For all? They talk more in study group than-?
Ted: and that might be . might . part of of the factor it might be they had a chance to look at the homework before. uh cause uh you often when we're working in class is the very first time we see it. and then during the group work they have time to prep a little bit before we meet up.
Fady: Okay . I see.
Ted: That might affect it.
[00:47:26.13] Fady: So you mentioned a little bit also . the role that each one is kind of trying to play like when you said to Bettie is the person who would bring . the reference from the books. uh let's go also make a round about the roles in terms . the roles that they tended to play or the behaviors when the soci . social dynamics of the . of the group. Not uh only about mathematics . but also about other things. I mean . yeah lets go larger and not only mathematics and see what . what would you say about each one? How the behaviors of each one tended to be in the group and in class?

[00:48:12.04] Ted: Often times I find myself being the one who can . who . who would work on the math the entire time. like I I I don't stop when I'm there. I mean . I pause and like “haha” at the joke . that's great. And often times I would be the one who keeps writing while everybody is joking. um.
Fady: yeah. Okay.
Ted: then . let's see. mmm as before . Jeremy is also the one who likes to joke and bring laughter to the group.
Fady: Jeremy.
Ted: Jeremy likes to joke. umm John . John sometimes is like . “no nonsense.” like that's . that's kinda what I get from him like . “no nonsense.” and then a switch would flip and then he would be joking with Jeremy and like it's like . he's open. right? umm I think once we start getting distracted then we're not really thinking about math anymore. [laughs] like Boutros likes to draw.
Fady: draw?
Ted: yeah draw on the whiteboard. Like have you ever found the art on the whiteboard?
Fady: This is last time right?
Ted: Yeah last time . and like I think we had a portrait of Euler at some point.
Fady: and he . okay.
Ted: and that's just like the doodler. umm let's see ... Bettie's really humble. she is the humble one in the group for sure. and . but she also likes to chime in and joke. It's like . I'm the . I'm the slightly anti-social one who is just like “I’m gonna keep writing and listen to these jokes and be amused but not really show it.”
[00:50:13.15] Fady: Is there any negative experience that happened during . this class in group?
[00:50:30.22] Ted: umm … Not really from my perspective.
Fady: Kind of frustration . disappointment uh
Ted: I think . I think right after we got the test back . I was really nervous . to show anyone my grade.
Fady: Oh okay . because it was good grade?
Ted: [laughs] yeah I mean I think a couple of my groupmates did . did not do so well and I didn't wanna . put it out there.
[00:51:01.03] Fady: okay. and you succeeded? Did they know about your grade?
[00:51:05.22] Ted: Eventually they found out. like they just looked at it. [laughs] cause I had it in my binder and they saw it.
[00:51:14.03] Fady: And what are the . like significant positive experiences? Was there any moment where . you recall a very positive experience?
[00:51:32.21] Ted: mmm … oh . when Jeremy and I started working on number 4 for the last homework where we were not required to do . and we were both finished . and like just hanging out with like his roommate or something at this study room. and we were like . “okay let's try number 4 anyway. just for the fun of it.” Like when we were doing that . just for the fun of it . that was amazing to me.
Fady: okay uh . nice.
[00:51:59.26] Fady: And did you experience joy in group work in class?
[00:52:05.01] Ted: Yeah. I mean it’s it’s fun to be chatting with other people about math.
Fady: okay. good.
[00:52:15.03] Fady: Let’s say that next time . uh like next semester . if next semester you are in a class that requires also group work . what would you keep and what would you change from the ways you behaved in the group work?

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[00:52:43.21] Ted: … I definitely will try to keep using Latex . so that I can upload stuff on google drive again. uh my hand writing is atrocious so doing that would be important. and sharing that work . I think . would be great. Definitely would try to ask people to critique the work.
Fady: That you are doing? you mean these pages . these . uh your homework?
Ted: yeah and I haven't . I have kind of like tried to make it a point to not critique anyone's work until they ask me to do it . and I think that is important.
[00:53:22.07] Fady: Did you do that in this class?
[00:53:25.13] Ted: Not at the beginning.
Fady: At the beginning?
Ted: not . not like not at the beginning. Like at the beginning I was just like . "no wait wait wait. no wait wait wait." [laughs]
Fady: At the beginning you did that?
Ted: yeah I was doing that a little bit. and then I realized "wait wait let me . wait until they ask me to look at their work."
Fady: Okay.
Ted: cause I would want people to do that to me.
Fady: uhu.
[00:53:45.10] Fady: Did anyone tell you about this?
Ted: No.
Fady: or you just realize it for yourself?
Ted: I just realized it.
Fady: What made you realize that?
[00:53:51.15] Ted: I don't know. Like it was just like one of those . "Oh hey maybe I shouldn't do that." I've been having those moments a lot recently. just realizing my behavior in a way. Maybe I'm finally growing up.
Fady: Okay wow. Congratulations.
Ted: [laughs]
[00:54:12.28] Fady: This is happening also in your personal life as well?
[00:54:15.16] Ted: Right. Yeah. just looking back and having a new perspective on certain memories and be like . "oh wait I was a jerk. I shouldn't do that again."
[00:54:31.06] Fady: Okay. Is there other things that you'd like to say that these questions did not cover?
[00:54:38.00] Ted: Pretty thorough questions. I think . think they're . good.
[00:54:45.11] Fady: Good. alright . so you have uhh one more. Is this your last umm still have one more year to go?
Ted: One more semester.
Fady: One more semester . and you will graduate. So any plans for next year?
[00:54:58.17] Ted: I have been so- like so much of my time has been spent on like class and school and . uh class and work. that-
Fady: you didn't plan for the future?
Ted: I kno:ow. I think that . that's not good right? Um I keep saying that I would consider programs but I just haven't had the time to look at the programs themselves and I feel like I need to start.
Fady: uhu.
[00:55:30.15] Fady: Okay and you have good people around you actually to . to support you and

. and did you decide about the master . about if you are going to do to a master or not?
[00:55:49.10] Ted: Um . I . I feel like I'm procrastinating. because I'm like dreading it
Fady: Dreading it?
Ted: Dreading it like . like I . I don't know what the process is like . like at this step like nobody in my family . at this like . at this step nobody in my family can really help me because nobody had gone to grad school in my family. Well I mean only two people graduated from college in my generation so far.
Fady: Okay.
[00:56:21.10] Fady: I think you can find support here from the school right?
Ted: I .
Fady: In terms of . I mean in terms of the process for applying to grad school . is that what you are talking about?
[00:56:34.18] Ted: Yeah . I . I . I am very much like uhh . the way I approach personal life is similar to how I approach proof problems. and then when I . when I find something unsolvable . I would think about it for three four hours before . I go ask for help. I find it difficult to ask for help.
[00:56:58.18] Fady: Okay I see. uh so you may ask for help maybe two days before the deadline for submissions?
[00:57:07.14] Ted: [sighs] That is a good question [laughs]
[00:57:12.25] Fady: yeah . good luck. I think it's a . I don't remember when was the deadline in [name of a university] . but it could be . I think it is in January for submissions. Yeah. At least . I think what you need to think about is if you want to do a master or not . that's important. I mean no one else will help you with that . but once you decide and if you want to do a master and apply for a master program . this is where your professors will be glad . actually . to help that. You- are you gonna ask them anyway . to write recommendation letters for you? And this can be a moment where you can discuss with them a- what do you suggest . which program is best fit for you? Is it here at [current university] or go other places? uh cause you are telling me that you are looking for programs that can provide also some uh financial support right?
Ted: Yeah. I am so broke. [laughs]
Fady: That's fine. We've all been there.
[00:58:21.20] Ted: Yeah . I'm not as broke as I was four years ago. uh I think.
Fady: now cause you live on your own . right?
Ted: Yeah . when I first started living on my own . this is . [sighs] I remember one month where I was in the winter in Utah. and it was like snowing and all that . I didn't have money for heat . and I didn't have money for food besides water. That was all I had. I had water for a whole month in the snow with no heat. That was . that was . brutal.
[00:58:57.04] Fady: Very survival man. Nice.
Ted: That's why I'm like "oh this math class is tough but . it is not winter with [unintelligible]"
Fady: yeah nice. it's uh yeah let me know I think I . yeah let me know if there is anything that I can help you with in terms of . I think you need to decide if you want to think aloud with someone.
Ted: I do want to think about it, like cause-
Fady: I don't know how you think or how you make your decisions. but if thinking aloud with someone can help you . I can help.
[00:59:32.17] Ted: I do . I . I want to . want to keep teaching math . I feel like I want to go into teaching. but I don't know umm if teaching high school is better for me . or if I should go into

teaching at a higher level.
Fady: like college or-
Ted: … and a lot of my skillsets besides math is like working with youth . with like umm emotional and [unintelligible] and like umm disabilities and things like that you know? so I uh . I kind of cultivated that skill set throughout these three years working with the kids. and uh I wonder if that can be still applied at higher levels. cause like . like the higher you go it's less and less students will be there.
[01:00:36.25] Fady: yeah. You know . I think you can use what did you feel about . with your peer relationship like . with the study group . with this one?
[01:00:47.03] Ted: I felt more conscious about it than like . than ac . before I started working with the youth group. certainly. cause before that I would just do it. I just want my grade I don't care [laugh] I didn't care about the group dynamics before.
[01:01:02.24] Fady: Yeah. I mean . these are not like the kids? so the kids help you . how to be more conscious with them? but when you worked with them did you feel certain joy like sharing your knowledge with these people or working . because this is going to be the quality of the . of the type of uh population that you are going to work with . or work for . if you teach in college.
[01:01:27.10] Ted: Yeah. There was never any moment of frustration . cause you know? working with little kids that would just scream profanity at you [laughs] so that makes this seem like any group interaction with adults seem so much easier.
[01:01:48.20] Fady: Yeah so so . But which one would you enjoy if you were to compare . working with kids or working with your peers . where would you see yourself?
[01:02:00.29] Ted: … I think I would be comfortable doing either . but at the same time I feel like I would need to be continually challenged . mathematically . you know? Like there . there teaching is a challenge . but I need the challenge of mathematics to keep me alive.
[01:02:16.12] Fady: and this is . you find it more with uh in a college level? Is th
at what you are saying?
[01:02:21.29] Ted: Yeah. and because of all of that I've been more and more leaning towards teaching at a college level. But I am so intimidated by the process. Like how do I know I am good enough . you know?
[01:02:41.28] Fady: uhh I mean you still have this semester. I think grades will help you ask professor's opinions how they see . and your peers . how they see in you. uh it's not only something you discover on your own through your own personal reflection it is good to also get feedback from . you know . how professors see you.
[01:03:08.05] Ted: What do you think?
// Fady went into presenting different masters programs and answering some of Ted's questions about them. I include only Ted's questions here. The omitted parts include information about the interviewer and are not relevant to the conducted investigations. //
[01:05:31.06] Ted: When you become a professor . do you have to do research in mathematics? If you are a math professor?
[…]
[01:07:03.19] Ted: I'm nervous, but I really would like to do the higher level math research. But I don't feel that confident that I can get there.
[…]
[01:09:07.28] Ted: Is this a decision that I- like what I would research later on like if I go into like college umm teaching programs like do I have to decide what I want to research now? or is this . is this something I have to talk about?

[…]

[01:11:16.06] Ted: So going into the math education path there will be . I will learn a lot . a lot of math still?

[…]

[01:13:30.01] Ted: is it like that for master's? like are you allowed to work while you're getting your masters?

[…]

Appendix C:

Analysis of Bettie's Homework

I- The first set compares Bettie's homework to answers in textbooks and, occasionally groupmates' work.

THEOREM 2–1 *(Euclid's Division Lemma): For any integers k $(k > 0)$ and j, there exist unique integers q and r such that $0 \leq r < k$ and*

$$j = qk + r. \quad (2\text{-}1\text{-}1)$$

PROOF: Note that we have simply rewritten a division problem in terms of multiplication and addition. In the notation used above, j is the dividend; k, the divisor; q, the quotient; and r, the remainder.

If $k = 1$, r must be zero, so that $q = j$.

If $k > 1$, suppose first that $j > 0$. (We shall consider the cases in which $j = 0$ and $j < 0$ later.) By the basis representation theorem (Theorem 1–3), j has a unique representation to the base k, say

$$\begin{aligned} j &= a_s k^s + a_{s-1}k^{s-1} + \ldots + a_1 k + a_0 \\ &= k(a_s k^{s-1} + a_{s-1}k^{s-2} + \ldots + a_1) + a_0 \\ &= kq + r, \end{aligned}$$

where $0 \leq r = a_0 < k$.

If a second pair q' and r' existed, we could find a representation for q' to the base k, say

$$q' = b_t k^t + \ldots + b_1 k + b_0,$$

so that

$$\begin{aligned} j &= kq' + r' \\ &= b_t k^{t+1} + \ldots + b_1 k^2 + b_0 k + r', \end{aligned}$$

but

$$j = a_s k^s + a_{s-1}k^{s-1} + \ldots + a_1 k + a_0.$$

By the uniqueness of the representation of j to the base k, we see that $t = s - 1$, $b_i = a_{i+1}$, $r' = a_0 = r$, and thus

$$\begin{aligned} q' &= b_t k^t + \ldots + b_1 k + b_0 \\ &= a_s k^{s-1} + \ldots + a_2 k + a_1 \\ &= q. \end{aligned}$$

Consequently, the theorem is true for positive values of j.

Prompt of WS1.3 in Hw1

3. The *division algorithm* says that every division problem has a unique quotient and remainder. Come up with a precise mathematical statement for the division algorithm and prove it.

3. want to show uniqueness for quotient and remainder
$j = qk + r$ (j=dividend, k=divisor, q=quotient, r=remainder)
if $k=1$ r must be 0 so that $j=q$
if $k>1$ consider all cases of j.
suppose $j > 0$
basis representation Thm. $j = a_m k^m + a_{m-1}k^{m-1} + a_1 k^1 + a_0$
$= k(a_m k^{m-1} + a_{m-1}k^{m-2} + \ldots + a_1) + a_0$
$= k(q) + r$
where $0 \leq r < k$
if second pair q' and r' existed,
$q' = b_n k^n + \ldots + b_1 k + b_0$
so that $j = k q' + r'$
$= b_n k^{n+1} + \ldots + b_1 k^2 + b_0 k + r'$
but, $j = a_m k^m + a_{m-1} k^{m-1} + \ldots + a_1 k + a_0 \Rightarrow$
we see $n = m-1$; $b_i = a_{i+1}$, $r' = a_0 = r$
Thus $q' = b_n k^n + \ldots + b_1 k + b_0$
$= a_m k^{m-1} + \ldots + a_2 k + a_1$
$= q$
Therefore thm. is true for $j > 0$.

[Continued]

If $j=0$, it is easy to verify that $q=r=0$ is the only possible solution of (2–1–1) with $0 \le r < k$.

14 THE FUNDAMENTAL THEOREM OF ARITHMETIC

If $j<0$, then $-j>0$, and there exist unique integers q'' and r'' such that

$$-j = kq'' + r''.$$

If $r''=0$, then $j=k(-q'')$; thus we may take $q=-q''$ and $r=0$. If $r'' \neq 0$, then

$$j = -kq'' - r'' = k(-q''-1) + (k-r''),$$

and we may take $q=-q''-1$, and $r=k-r''$.

In either case, q and r satisfy equation (2–1–1). Uniqueness for negative j follows from uniqueness for $-j$, which is then positive. ■

if $j=0$, $q=r=0$ w/ $0 \le r < k$

if $j<0$ then $-j>0$ $\exists$ unique $q'', r'' \in \mathbb{Z}$ s.t.

$-j = kq''+r''$.

if $r''=0$; $j=k(-q'')$ $\Rightarrow$ $q=-q''$, $r=0$.

if $r'' \neq 0$,

$j = -kq''-r''$

$= k(q''-1)+(k-r'')$

$q=-q''-1$, $r=k-r''$

There fore q, r satisfy $j=kq+r$

Comparison of proofs in Andrews textbook (on the left) and Bettie's homework of WS1.3 (on the right). The highlited texts are modified in Bettie's work. The underlined texts in bleu are not present in the other source.

THEOREM 5–2 (*Euler's Theorem*): *If* g.c.d.$(a,m)=1$, then

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

PROOF: Let $r_1, r_2, \ldots, r_{\phi(m)}$ be a reduced residue system modulo m. We note that $ar_1, ar_2, \ldots, ar_{\phi(m)}$ are all relatively prime to m; furthermore, they are mutually incongruent, since $ar_i \equiv ar_j \pmod{m}$ implies that $r_i \equiv r_j \pmod{m}$, by the cancellation law (Theorem 4–3). We may thus pair each ar_i with some r_j such that $ar_i \equiv r_j \pmod{m}$, and we note that r_j is uniquely defined for each ar_i. Note also that each r_j is paired with some ar_i, since there are $\phi(m)$ of the r_j and $\phi(m)$ of the ar_i. Thus

$$r_1 r_2 \ldots r_{\phi(m)} \equiv ar_1 ar_2 \ldots ar_{\phi(m)} \pmod{m}.$$

Hence, if $R = r_1 r_2 \ldots r_{\phi(m)}$, then

$$R \equiv a^{\phi(m)} R \pmod{m}.$$

Now g.c.d.$(R,m)=1$, because R is a product of integers each of which is relatively prime to m. Thus

$$a^{\phi(m)} \equiv 1 \pmod{m},$$

by the cancellation law. ■

COROLLARY 5–2 (*Fermat's Little Theorem, Theorem 3–4*): *If p is a prime, then*

$$n^p \equiv n \pmod{p}.$$

PROOF: If $p \mid n$, then $n^p \equiv 0 \equiv n \pmod{p}$. If $p \nmid n$, then g.c.d.$(p,n)=1$. Thus, by Theorem 5–2,

$$n^{p-1} \equiv 1 \pmod{p},$$

since $\phi(p) = p-1$. Multiplying both sides of this congruence by n, we find that

$$n^p \equiv n \pmod{p}. \quad ■$$

Prompt of WS3.4 in Hw4

4. Suppose $a, m \in \mathbb{Z}$ with $m>0$ and $\gcd(a,m)=1$, and let $\{r_1, r_2, \ldots, r_{\phi(m)}\}$ be a reduced residue system modulo m.

(a) Show that $\{ar_1, ar_2, \ldots, ar_{\phi(m)}\}$ is also a reduced residue system modulo m.

(b) Conclude that $r_1 r_2 \cdots r_{\phi(m)} \equiv (ar_1)(ar_2)\cdots(ar_{\phi(m)}) \bmod m$ and, consequently, that

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

(This is *Euler's theorem*.)

(c) Prove that, if p is prime and $a \in \mathbb{Z}$, then $a^p \equiv a \bmod p$. (This is *Fermat's little theorem*.)

(d) Conclude that, if p is prime and $a, b \in \mathbb{Z}$, then $(a+b)^p \equiv a^p + b^p \bmod p$. (This is every freshman's dream.)

b) $ar_1, ar_2, \ldots, ar_{m\phi}$ relatively prime to m

$ar_i \equiv ar_j \pmod{m}$ $\Rightarrow$ $r_i \equiv r_j \pmod{m}$

$\Rightarrow$ $ar_i \equiv r_j \pmod{m}$ where r_j is paired w/ some ar_i

$r_1 \ldots r_{\phi m} \equiv ar_1 ar_2 \ldots ar_{m\phi} \pmod{m}$

if $R = r_1 r_2 \ldots r_{\phi m}$

$\Rightarrow$ $R \equiv ar_1 ar_2 \ldots ar_{m\phi} \pmod{m}$

$R \equiv a^{m\phi} R \bmod m$

[gcd(R, m) = 1]

thus $a^{\phi m} \equiv 1 \pmod{m}$

c) p is prime $a \in \mathbb{Z}$ $\Rightarrow$ $a^p \equiv a \bmod p$

if $p \mid a$ then $a^p \equiv 0 \equiv a \bmod p$. if $p \nmid a$ $\Rightarrow$ gcd(p, n) = 1

$\phi(p) = p-1$ Thus $a[a^{p-1} \equiv [1 \pmod{p}]]a$

$\Rightarrow$ $a^p \equiv a \pmod{p}$

Comparing proofs in Andrews textbook (on the left) and Bettie's homework on WS3.4b-c (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source. The underlined letter in red (two lines) is an evidence of an oversight in copying from the textbook.

THEOREM 5–3: *The congruence* $(m-1)! \equiv -1 \pmod m$ *holds if and only if* m *is a prime.*

5–2 THE THEOREMS OF FERMAT AND WILSON REVISITED 63

PROOF: First suppose m is a prime, and consider the $m-1$ positive integers $1, 2, \ldots, m-1$. Corollary 5–1 confirms that if a is such an integer, then there exists an inverse $\bar{a}$ of a modulo m ($\bar{a}$ is unique if we require that $1 \le \bar{a} \le m-1$) for which

$$a\bar{a} \equiv 1 \pmod m.$$

It may happen that a is its own inverse; in other words, that

$$a^2 \equiv 1 \pmod m.$$

In that case, $m \mid (a-1)(a+1)$; since m is prime, we see by Corollary 2–3 that either $m \mid a-1$ or $m \mid a+1$; therefore, $a \equiv \pm 1 \pmod m$. In the product $(m-2)(m-3)\ldots 2$, we pair each number with its inverse modulo p. Thus, for $m=7$, we write that $5\cdot4\cdot3\cdot2=(5\cdot3)\cdot(4\cdot2)$. In general, we see that

$$(m-1)! \equiv (m-1)\cdot 1 \cdot \cdots \cdot 1 \pmod m$$
$$\equiv m-1 \equiv -1 \pmod m.$$

Conversely, suppose that m is not a prime. Then there exists an $a(1<a<m)$ such that $a \mid m$; note also that $a \mid (m-1)!$. If $(m-1)! \equiv -1 \pmod m$, then there exists an integer k such that $(m-1)!+1=km$. Since $a \mid m$ and $a \mid (m-1)!$, it follows from the last equation that $a \mid 1$; but this is an impossibility, because $a>1$. Thus the congruence $(m-1)! \equiv -1 \pmod m$ cannot hold if m is not a prime. ■

Prompt of WS3.6 in Hw5

6. Suppose $m \in \mathbb{Z}_{>0}$.

(a) Show that, if $m>4$ is not prime, then $(m-1)! \equiv 0 \bmod m$.

(b) Now suppose m is prime. Show that if $a \not\equiv 0, \pm1 \bmod m$ then there exists $b \not\equiv 0, \pm1, a \bmod m$ such that $ab \equiv 1 \bmod m$.

(c) Conclude *Wilson's theorem*: $(m-1)! \equiv -1 \bmod m$ if and only if m is prime.

b) $m=p$ some prime int. $a \not\equiv \pm1 \pmod p$

$b \not\equiv a \pmod p$

$ab \equiv 1 \pmod p$ $b \not\equiv \pm 1 \pmod p$

let $a=b$ $\Rightarrow$ $a^2 \equiv 1 \pmod p$ no solution $\Rightarrow a \not\equiv \pm 1 \bmod p$

own inverse $\Rightarrow p \mid (a^2-1)$

$p \mid (a+1)(a-1)$ $\Rightarrow$ $p \mid (a+1)$ or $p \mid (a-1)$

$\Rightarrow a \equiv 1 \pmod p$ $a \equiv -1 \pmod p$

which is contradiction because we stated that a is not its own inverse.

Comparing proofs in Andrews textbook (on the left) and Bettie's homework on WS3.6b (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source. Bettie brought knowledge from group discussion in class to bear on her homework.

[Continued]

Stein Proposition 2.1.22 (page 28)

Proof. The statement is clear when $p=2$, so henceforth we assume that $p>2$. We first assume that p is prime and prove that $(p-1)! \equiv -1 \pmod p$. If $a \in \{1, 2, \ldots, p-1\}$, then the equation

$$ax \equiv 1 \pmod p$$

has a unique solution $a' \in \{1, 2, \ldots, p-1\}$. If $a=a'$, then $a^2 \equiv 1 \pmod p$, so $p \mid a^2-1=(a-1)(a+1)$, so $p \mid (a-1)$ or $p \mid (a+1)$, so $a \in \{1, p-1\}$. We can thus pair off the elements of $\{2, 3, \ldots, p-2\}$, each with their inverse. Thus

$$2\cdot3\cdot\cdots\cdot(p-2) \equiv 1 \pmod p.$$

Multiplying both sides by $p-1$ proves that $(p-1)! \equiv -1 \pmod p$.

Andrews Theorem 5-3 (page 63)

$$(m-1)! \equiv (m-1)\cdot 1 \cdot \cdots \cdot 1 \pmod m$$
$$\equiv m-1 \equiv -1 \pmod m.$$

Conversely, suppose that m is not a prime. Then there exists an $a(1<a<m)$ such that $a \mid m$; note also that $a \mid (m-1)!$. If $(m-1)! \equiv -1 \pmod m$, then there exists an integer k such that $(m-1)!+1=km$. Since $a \mid m$ and $a \mid (m-1)!$, it follows from the last equation that $a \mid 1$; but this is an impossibility, because $a>1$. Thus the congruence $(m-1)! \equiv -1 \pmod m$ cannot hold if m is not a prime. ■

c) $(m-1)! \equiv -1 \pmod m$ iff m is prime

$a \in \{1, 2, \ldots, m-1\}$ $\Rightarrow$ $a' \in \{1, 2, \ldots, m-1\}$

$a = a'$

$\Rightarrow$ $a^2 \equiv 1 \pmod m$

$\Rightarrow$ $m \mid (a+1)(a-1)$ $\Rightarrow$ $m \mid (a+1)$ or $m \mid (a-1)$

$\Rightarrow$ $a \equiv \pm 1 \pmod m$

$(m-1)! \equiv (m-1\cdot 1 \cdots 1 \pmod m)$

$\Rightarrow$ $m-1 \equiv -1 \pmod m$

conversely suppose [illegible] not prime

$\exists a \in (1<a<m)$ s.t. $a \mid m$

$a \mid (m-1)!$ if $(m-1)! \equiv -1 \pmod m$

$\Rightarrow (m-1)!+1 = mk$, $k \in \mathbb{Z}$

Since $a \mid m$ $a \mid (m-1)!$ $a \mid 1$

but $a>1$ thus

$(m-1)! \equiv -1 \pmod m$ cannot hold

Comparing proofs in Stein and Andrews textbooks (on the left) and Bettie's homework on WS3.6c (on the right). The green line in Bettie's homework is mine; it delineates the sections copied from Stein (top) and Andrews (bottom) textbooks. The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the comparative source.

Theorem 5-4: *Suppose $m_1, m_2, \ldots, m_s$ are s integers, no two of which have a common factor other than 1. Let $M = m_1 m_2 \ldots m_s$, and suppose that $a_1, a_2, \ldots, a_s$ are integers such that* g.c.d.$(a_i, m_i) = 1$ *for each i. Then the s congruences*

$$a_1 x \equiv b_1 \pmod{m_1},$$
$$a_2 x \equiv b_2 \pmod{m_2},$$
$$\ldots,$$

and

$$a_s x \equiv b_s \pmod{m_s}$$

have a simultaneous solution that is unique modulo M.

Proof: From the solutions of each particular congruence, we shall construct one common to the entire set. First we choose integers $c_1, c_2, \ldots, c_s$ such that

$$a_i c_i \equiv b_i \pmod{m_i}$$

(such integers c_i exist by Theorem 5-1). Now let $n_i = M/m_i$. Since no two of the m_i have a common factor, we see that g.c.d.$(n_i, m_i) = 1$. Hence, by Corollary 5-1, there exists an $\bar{n}_i$ such that $n_i \bar{n}_i \equiv 1 \pmod{m_i}$.

We now show that the number x_0 defined by

$$x_0 = c_1 n_1 \bar{n}_1 + c_2 n_2 \bar{n}_2 + \ldots + c_s n_s \bar{n}_s$$

is a solution of the original system of s congruences. First note that m_i divides each n_j except for n_i. Thus

$$\begin{aligned} a_i x_0 &= a_i c_1 n_1 \bar{n}_1 + a_i c_2 n_2 \bar{n}_2 + \ldots + a_i c_s n_s \bar{n}_s \\ &\equiv a_i c_i n_i \bar{n}_i \pmod{m_i} \\ &\equiv a_i c_i \pmod{m_i} \\ &\equiv b_i \pmod{m_i}. \end{aligned}$$

5-3 THE CHINESE REMAINDER THEOREM 69

If y is a solution of the system of s congruences, then, by Theorem 5-1, $x_0 \equiv c_i \equiv y \pmod{m_i}$. Hence $m_i \mid x_0 - y$ for each m_i; since no two of the m_i have a common factor, $m_1 m_2 \ldots m_s \mid x_0 - y$, that is, $M \mid x_0 - y$. Thus $y \equiv x_0 \pmod{M}$.

Prompt of WS4.3 in Hw5

2. Suppose $m, n \in \mathbb{Z}_{>0}$ are relatively prime, and $a, b \in \mathbb{Z}$. Prove that

$$x \equiv a \bmod m \qquad \text{and} \qquad x \equiv b \bmod n$$

has a solution $x \in \mathbb{Z}$ and that x is unique modulo mn.

3. Generalize the statement (and your proof) of 2. to a system of k congruences.

3. suppose $m_1, m_2, \ldots, m_k$ are k int. no common factor but 1.
let $M = m_1 m_2 \ldots m_k$; $a_1, a_2, \ldots, a_k \in \mathbb{Z}$
st. $\gcd(a_i, m_i) = 1$ for each i

$c_1, c_2, \ldots, c_k \in \mathbb{Z}$
$\Rightarrow a_i c_i \equiv b_i \pmod{m_i}$
$n_i = M/m_i \Rightarrow \gcd(n_i, m_i) = 1$
$\exists n' \in \mathbb{Z}$ st $n_i n_i' \equiv 1 \pmod{m_i}$
$(x_0 = c_1 n_1 n_1' + c_2 n_2 n_2' + \ldots + c_k n_k n_k')$
$\Rightarrow a_i (x_0 = c_1 n_1 n_1' + \ldots + c_k n_k n_k')$
$\Rightarrow a_i x_0 = a_i c_1 n_1 n_1' + a_i c_2 n_2 n_2' + \ldots + a_i c_k n_k n_k'$
$\equiv a_i c_i n_i n_i' \pmod{m_i}$
$\equiv a_i c_i \pmod{m_i}$
$\equiv b_i \pmod{m_i}$
$\Rightarrow x_0$ is a solution of each congruence
$x_0 \equiv c_i \equiv y \bmod (m_i)$
$\Rightarrow m_i \mid x_0 - y$ for each m_i
$m_1 m_2 \ldots m_k \mid x_0 - y$
$\Rightarrow M \mid x_0 - y$
$\Rightarrow y \equiv x_0 \pmod{M}$ ✓

Comparing proofs in Andrews textbook (on the left) and Bettie's homework on WS4.3 (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source.

52 3. Public-key Cryptography

1. Together Michael and Nikita choose a 200-digit integer p that is likely to be prime (see Section 2.4), and choose a number g with $1 < g < p$.
2. Nikita *secretly* chooses an integer n.
3. Michael *secretly* chooses an integer m.
4. Nikita computes $g^n \pmod p$ on her handheld computer and tells Michael the resulting number over the phone.
5. Michael tells Nikita $g^m \pmod p$.
6. The shared secret key is then

$$s \equiv (g^n)^m \equiv (g^m)^n \equiv g^{nm} \pmod p,$$

which both Nikita and Michael can compute.

Here is a simplified example that illustrates what they did, that involves only relatively simple arithmetic.

1. $p = 97$, $g = 5$
2. $n = 31$
3. $m = 95$
4. $g^n \equiv 7 \pmod p$
5. $g^m \equiv 39 \pmod p$
6. $s \equiv (g^n)^m \equiv 14 \pmod p$

Stein 3.4

$P = 3743$

$g = 7$

$m = 1208$

$n < P$.

We know that

$$7^n \equiv 454 \pmod{3743}$$

So, our decryption S is given by

$$S = 454^{1208} \pmod{3743}$$

which gives us

$$S = 2156.$$

Prompt of Stein 3.4 in Hw6

3.4 You and Nikita wish to agree on a secret key using the Diffie-Hellman key exchange. Nikita announces that $p = 3793$ and $g = 7$. Nikita secretly chooses a number $n < p$ and tells you that $g^n \equiv 454 \pmod p$. You choose the random number $m = 1208$. What is the secret key?

3.4) 1. $p = 3793$ $g = 7$

2. $n < p$

3. $m = 1208$

4. $g^n \equiv 454 \pmod p$ $\quad e \equiv d^b \bmod m$

5. $g^m \equiv g^n \pmod p$ $\quad d \equiv e^c \bmod m$

6. $s \equiv (g^n)^m \equiv (g^m)^n \equiv g^{nm} \pmod p$

$7^n \equiv 454 \pmod{3793}$

$\Rightarrow s = 454^{1208} \pmod{3793}$

$= 2156$

Comparing proofs in Stein (top left) and Jeremy's work (bottom left) and Bettie's homework on Stein 3.4 (on the right). The green line in Bettie's homework (mine) delineates the sections copied from Stein (top) and Jeremy (bottom). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the comparative source.

PROOF: The assertion is clearly true if $n = 1$. We proceed by mathematical induction on the number of different prime factors of n when $n > 1$.

First, if $n = p^\alpha$, then

$$\sum_{d \mid n} \mu(d) = \mu(1) + \mu(p) + \mu(p^2) + \ldots + \mu(p^\alpha)$$
$$= 1 - 1 + 0 + \ldots + 0$$
$$= 0.$$

Suppose the theorem is true for integers with at most k prime factors. Assuming that $n = n'p^\alpha$, where n' has k distinct prime factors and p is a prime that does not divide n', we have the equation

$$\sum_{d \mid n} \mu(d) = \sum_{d \mid n'} \mu(d) + \sum_{d \mid n'} \mu(pd) + \sum_{d \mid n'} \mu(p^2 d) + \ldots + \sum_{d \mid n'} \mu(p^\alpha d)$$
$$= \sum_{d \mid n'} \mu(d) - \sum_{d \mid n'} \mu(d) + 0 + \ldots + 0$$
$$= 0. \qquad \blacksquare$$

Prompt of WS7.2 in Hw8

2. Prove that

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Hint: for $n > 1$, try induction on the number of prime factors of n.

7.2 prove $\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$

Prove using induction $\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$

when $n = 1$

$\sum_{d \mid 1} \mu(d) = 1$. clearly from assertion

we now use mathematical induction on different prime factors of n when $n > 1$

let $n = p^a = \{1, p, p^2, \ldots, p^a\}$

$\Rightarrow \sum_{d \mid n} \mu(d) = \mu(1) + \mu(p) + \mu(p^2) + \ldots + \mu(p^a)$

$= 1 + (-1) + 0 + \ldots + 0$

$= 0$

now let n have at most k prime factors

assume $n = n'p^a$ where n' has k prime factors

$p \nmid n'$

$\Rightarrow \sum_{d \mid n} \mu(d) = \sum_{d \mid n'} \mu(d) + \sum_{d \mid n'} \mu(pd) + \sum_{d \mid n'} \mu(p^2 d) + \ldots + \sum_{d \mid n'} \mu(p^a d)$

$= \sum_{d \mid n} \mu(d) - \sum_{d \mid n} \mu(d) + 0 + \ldots + 0$

$= 0$ ✓

Comparing proofs in Andrews textbook (on the left) and Bettie's homework on WS7.2 (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source. Bettie's additions in her work show attempts of understanding (mainly the display of the divisors of p^α which explains the unfolding of the summation). However, the change of the phrase "suppose the theorem is true for integers" into "let n have" generates a contradiction, which indicates a lack of grasp of how the induction operates in this proof.

Proof: First suppose that

$$f(n) = \sum_{d \mid n} g(d);$$

then

$$\sum_{d \mid n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{dd' = n} \mu(d) f(d')$$

$$= \sum_{dd'=n} \mu(d) \sum_{e \mid d'} g(e)$$

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$$= \sum_{deh = n} \mu(d) g(e)$$

$$= \sum_{eh' = n} g(e) \sum_{d \mid h'} \mu(d).$$

By Theorem 6–5, the sum $\sum_{d \mid h'} \mu(d)$ has the value 0 if $h' > 1$, and the value 1 if $h' = 1$. Hence

$$\sum_{d \mid n} \mu(d) f\left(\frac{n}{d}\right) = g(n).$$

Conversely, suppose $g(n) = \sum_{d \mid n} \mu(d) f\left(\frac{n}{d}\right)$. Then

$$\sum_{d \mid n} g(d) = \sum_{d \mid n} \sum_{d' \mid d} \mu(d') f\left(\frac{d}{d'}\right)$$

$$= \sum_{d'ef = n} \mu(d') f(e)$$

$$= \sum_{eh' = n} f(e) \sum_{d' \mid h'} \mu(d).$$

As before, Theorem 6–5 implies that the sum $\sum_{d' \mid h'} \mu(d')$ has the value 0 if $h' > 1$, and the value 1 if $h' = 1$. Hence

$$\sum_{d \mid n} g(d) = f(n). \quad \blacksquare$$

Prompt of WS7.3 in Hw8

3. Prove the *Möbius Inversion Formula*:

$$f(n) = \sum_{d \mid n} g(d) \quad \text{if and only if} \quad g(n) = \sum_{d \mid n} \mu(d) f(\tfrac{n}{d}).$$

Hint: write sums like the one on the right-hand side as

$$\sum_{d \mid n} \mu(d) f(\tfrac{n}{d}) = \sum_{de=n} \mu(d) f(e).$$

7.3 Mobius Inversion

$f(n) = \sum_{d \mid n} g(d)$ iff $g(n) = \sum \mu(d) f(\frac{n}{d})$

suppose, $f(n) = \sum_{d \mid n} g(d)$

$\sum_{d \mid n} \mu(d) f(\frac{n}{d})$ let $\frac{n}{d} = d'$

$\Rightarrow \sum_{dd'=n} \mu(d) f(d') = \sum_{dd'=n} \mu(d) \sum_{a \mid d'} g(a)$ $a, b \in \mathbb{Z}$ $a \cdot b = d'$ $dd' = dab = n$

$= \sum_{dab=n} \mu(d) \cdot g(a) = \sum_{ah=n} (g(a) \cdot \sum_{d \mid h} \mu(d'))$ $h = db$

$$\sum_{d \mid n} \mu(d') = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

$\therefore \sum_{ah=n} g(a) \cdot \sum_{d \mid h} \mu(d') = \sum_{ah=n} g(a) = f(n)$ ✓

conversely suppose $g(n) = \sum \mu(d) f(\frac{n}{d})$

$\sum_{d \mid n} g(d) = \sum_{d \mid n} \sum_{d' \mid n} \mu(d) f(\frac{n}{d})$

$= \sum_{d'ef=n} \mu(d') f(e)$

$= \sum_{eh'=n} f(e) \sum_{d' \mid h'} \mu(d)$

where $\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 0 \end{cases}$

$\therefore \sum_{d \mid n} g(d) = f(n)$ ✓

Comparing proofs in Andrews textbook (on the left) and Bettie's homework on WS7.3 (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source.

Theorem 7–3: *If g is a primitive root modulo m, then $g, g^2, \ldots, g^{\phi(m)}$ are mutually incongruent and form a reduced residue system* modulo m.

Proof: Suppose $1 \le s < r \le \phi(m)$ and

$$g^r \equiv g^s \pmod{m}.$$

Then $m \mid g^r - g^s$, that is, $m \mid g^s(g^{r-s} - 1)$. Hence, by Theorem 2–3, $m \mid g^{r-s} - 1$. Consequently, $g^{r-s} \equiv 1 \pmod{m}$. Thus, $r - s$ is a positive integer less than $\phi(m)$ such that

$$g^{r-s} \equiv 1 \pmod{m}.$$

This contradicts the fact that g belongs to the exponent $\phi(m)$, and the theorem is proven. $\blacksquare$

Prompt of WS8.3 in Hw9

3. Show that, if a is a primitive root mod m, then $\{a, a^2, \ldots, a^{\phi(m)}\}$ is a reduced residue system mod m.

8.3 If a is a prim root mod $m \Rightarrow \{a, a^2, \ldots, a^{\phi(m)}\}$ is reduced residue system (mod m)

$a^k \equiv 1 \pmod{m}$

if a is prim root it has order $\phi(m)$ Thus $\gcd(k, \phi(m)) = 1$

$\Rightarrow a^{\phi(m)} \equiv 1 \pmod{m}$

suppose $1 \le r < k \le \phi(m)$

$a^k \equiv a^r \pmod{m}$

$\Rightarrow m \mid a^k - a^r \Rightarrow m \mid a^r(a^{k-r} - 1)$ we know from 8.1 $a^r = 1$

$\Rightarrow m \mid (a^{k-r} - 1) \Rightarrow$

$\Rightarrow a^{k-r} \equiv 1 \pmod{m}$

which contradicts a has order $\phi(m)$, since $\gcd(k, \phi(m)) = 1$

$\therefore a^k \not\equiv a^r \pmod{m}$ [illegible] $\{a, \ldots, a^{\phi(m)}\}$

is a reduced residue system.

Comparing proofs in Andrews textbook (on the left) and Bettie's homework on WS8.3 (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source.

Theorem 1.2.1 (Euclid). *There are infinitely many primes.*

Proof. Suppose that $p_1, p_2, \ldots, p_n$ are n distinct primes. We construct a prime p_{n+1} not equal to any of $p_1, \ldots, p_n$, as follows. If

$$N = p_1 p_2 p_3 \cdots p_n + 1, \tag{1.2.1}$$

then by Proposition 1.1.20 there is a factorization

$$N = q_1 q_2 \cdots q_m$$

with each q_i prime and $m \geq 1$. If $q_1 = p_i$ for some i, then $p_i \mid N$. Because of (1.2.1), we also have $p_i \mid N - 1$, so $p_i \mid 1 = N - (N-1)$, which is a contradiction. Thus the prime $p_{n+1} = q_1$ is not in the list $p_1, \ldots, p_n$, and we have constructed our new prime. □

Prompt of WS 9.1 in Hw10

1. Prove that there are infinitely many primes. One way to do this is by means of contradiction: assuming $p_1, p_2, \ldots, p_k$ are the only primes, consider the number $p_1 p_2 \cdots p_k + 1$.[1]

Homework #10

9.1 $\exists$ k primes $\{p_1, p_2, \ldots, p_k\}$ $p_i \in \mathbb{Z}_{>0}$ $i = 1, 2, \ldots, k$

assume $N = p_1 \cdot p_2 \cdots p_k + 1$

if we factor N into primes

$\Rightarrow N = q_1 \cdot q_2 \cdots q_l$ $q_j \in \mathbb{Z}_{>0}$ $j = 1, 2, \ldots, l$

$q_1 = p_i$ for some i $\Rightarrow p_i \mid N$ we know $p_i \mid N - 1$

thus $p_i \mid N - (N-1)$ but $p_i \nmid 1$ which is contradiction

$\Rightarrow$ the prime $p_{k+1} = q_1$ is not in our list of k primes

and we have constructed a new prime

$\therefore$ there are infinitely many primes.

Comparing proofs in Stein textbook (on the left) and Bettie's homework on WS9.1 (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source.

SECTION 9–3

1. If $c = p_1 p_2 \ldots p_r$, then

$$\left(\frac{-1}{c}\right) = \left(\frac{-1}{p_1}\right)\left(\frac{-1}{p_2}\right) \cdots \left(\frac{-1}{p_r}\right)$$

$$= (-1)^{\frac{1}{2}(p_1-1)} (-1)^{\frac{1}{2}(p_2-1)} \cdots (-1)^{\frac{1}{2}(p_r-1)}$$

$$= \begin{cases} 1 & \text{if an even number of the } p_i \text{ are congruent to } 3 \pmod 4 \\ -1 & \text{if an odd number of the } p_i \text{ are congruent to } 3 \pmod 4 \end{cases}$$

$$= \begin{cases} 1 & \text{if } c \equiv 1 \pmod 4 \\ -1 & \text{if } c \equiv -1 \pmod 4 \end{cases}$$

$$= (-1)^{\frac{1}{2}(c-1)}.$$

Prompt of Andrews 9.3.1 in Hw12

EXERCISES (In Exercises 1 through 3, the symbol appearing is the Jacobi symbol, defined prior to the exercises for Section 9–2).

1. Prove that if c is odd, then $\left(\frac{-1}{c}\right) = (-1)^{\frac{1}{2}(c-1)}$.

9.3.1 prove if c is odd, then $\left(\frac{-1}{c}\right) = (-1)^{\frac{1}{2}(c-1)}$

Suppose c is odd $c \in \mathbb{Z}$

if $c = p_1 p_2 \cdots p_r$ st. p_i = odd primes

$m = -1$

$\Rightarrow \left(\frac{-1}{c}\right) = \left(\frac{-1}{p_1}\right)\left(\frac{-1}{p_2}\right) \cdots \left(\frac{-1}{p_r}\right)$

$= (-1)^{\frac{1}{2}(p_1-1)} (-1)^{\frac{1}{2}(p_2-1)} \cdots (-1)^{\frac{1}{2}(p_r-1)}$

$= \begin{cases} 1 & \text{if even \# } p_i \equiv 3 \bmod 4 \\ -1 & \text{if an odd \# } p_i \equiv 3 \bmod 4 \end{cases}$

$= \begin{cases} 1 & \text{if } c \equiv 1 \pmod 4 \\ -1 & \text{if } c \equiv -1 \pmod 4 \end{cases}$

$= (-1)^{\frac{1}{2}(c-1)}$

Comparing hints in Andrews textbook (on the left) and Bettie's homework on Andrews 9.3.1 (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source.

Prompts of Andrews 9. 4.1-5 in Hw12

Comparing hints in Andrews textbook (top left), John's work (bottom left) and Bettie's homework on Andrews 9.4.1-4 (on the right). The highlighted texts are modified in Bettie's work. The underlined texts in bleu (one line) are missing in the other source.

EXERCISES

1. Does $x^2 \equiv 17 \pmod{29}$ have a solution?
2. Does $3x^2 \equiv 12 \pmod{23}$ have a solution?
3. Does $2x^2 \equiv 27 \pmod{41}$ have a solution?
4. Does $x^2 + 5x \equiv 12 \pmod{31}$ have a solution? [Hint: Complete the square.]
5. Does $x^2 \equiv 19 \pmod{30}$ have solutions? [Hint: Use the Chinese Remainder Theorem.]

SECTION 9–4

1. $\left(\frac{17}{29}\right) = \left(\frac{29}{17}\right) = \left(\frac{12}{17}\right) = \left(\frac{4}{17}\right)\left(\frac{3}{17}\right) = \left(\frac{3}{17}\right) = \left(\frac{17}{3}\right) = \left(\frac{2}{3}\right) = -1.$
Hence, $x^2 \equiv 17 \pmod{29}$ has no solutions.

2. This problem is equivalent to $x^2 \equiv 4 \pmod{23}$, and $x = 2$ is an obvious solution.

4. Hint: $(x+a)^2 + b \equiv x^2 + 5x - 12 \pmod{31}$ holds provided that $2a \equiv 5 \pmod{31}$, and $a^2 + b \equiv -12 \pmod{31}$. Hence, we may take $a = 18$ and $b = 5$. To finish we need only solve the congruence $y^2 \equiv -5 \pmod{31}$.

9.4.4) $x^2 + 5x \equiv 12$ mod 31

$(x+a)^2 + b \equiv x^2 + 5x - 12$

$\Rightarrow 2a \equiv 5$ mod 31 $\Rightarrow a \equiv 18$ mod 31

and $a^2 + b \equiv -12$ mod 31 $\Rightarrow b \equiv 5$ mod 31

$\Rightarrow x^2 \equiv -5$ mod 31

$\left(\frac{-5}{31}\right) = \left(\frac{31}{5}\right)(-1) = \left(\frac{1}{5}\right)(-1) = (-1)^{\frac{5-1}{2}} \equiv (-1)^2 = 1$

So there are solutions

9.4.1. Does $x^2 \equiv 17 \pmod{29}$ have a solution?

$\left(\frac{17}{29}\right) = \left(\frac{29}{17}\right) = \left(\frac{12}{17}\right) = \left(\frac{3}{17}\right)\left(\frac{4}{17}\right) = \left(\frac{3}{17}\right) = \left(\frac{17}{3}\right) = \left(\frac{2}{3}\right) = -1$

Hence $x^2 \equiv 17 \pmod{29}$ has no solutions.

9.4.2. Does $3x^2 \equiv 12 \pmod{23}$ have a solution?

$3x^2 \equiv 12 \pmod{23}$

$\Rightarrow x^2 \equiv 4 \pmod{23}$

$\left(\frac{4}{23}\right) = \left(\frac{1}{23}\right) = 1$ $\therefore$ has solution

9.4.4 does $x^2 + 5x \equiv 12 \pmod{31}$ have a solution?

$(x+a)^2 + b \equiv x^2 + 5x - 12 \pmod{31}$

$2a \equiv 5$ mod 31 $a^2 + b \equiv -12$ mod 31

$a = 18$ $b = 5$ $\Rightarrow y^2 \equiv -5$ mod 31

$\left(\frac{-5}{31}\right) = -1\left(\frac{31}{5}\right) = -1\left(\frac{1}{5}\right) = \left(\frac{-1}{5}\right) = (-1)^{\frac{5-1}{2}} \equiv (-1)^2 = 1$

$\therefore$ solution

II- The second set is dedicated to the comparison of Bettie's homework to Ted's homework shared through Google drive.

Stein 2.30 in Hw6

Stein 2.30 Compute the last two digits of 3^{34}.

Note that this is similar to asking what is equivalent to $3^{45} \pmod{100}$. Since $\gcd(3,100) = 1$, we know that $3^{\phi 100} \equiv 1 \pmod{100}$. Additionally, $\phi(100) = \phi(2^2 \cdot 5^2) = \phi(2^2) \cdot \phi(5^2) = (2^2 - 2)(5^2 - 5) = 40$

$$\therefore 3^{45} \equiv 3^{40+5} \equiv 3^5 \bmod 100$$

$$\underline{43} \equiv 243 \bmod 100$$

2.30) compute last two digits of 3^{45}

$\gcd(3,100) = 1$

$3^{\varphi(100)} \equiv 1 \pmod{100}$

$\varphi(100) = \varphi(4 \cdot 25) = \varphi(2^2 \cdot 5^2) = (2^2-2)\cdot(5^2-5)$

$= (2)(20) = 40$

$\Rightarrow 3^{40} \equiv 1 \pmod{100}$

$\Rightarrow 3^{45} \equiv (3^{40})(3^5) \pmod{100} \equiv 3^5 \pmod{100}$

Comparing Ted's (left) and Bettie's (right) homework of Stein 2.30 in HW6. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.

Bettie's work follows the same steps as Ted's, which could have been placed otherwise. It includes two additional computational steps and a restate of $3^{\varphi(100)} \equiv 1(mod\ 100)$ after computing $\varphi(100)$. The additional text exhibits good understanding of the steps of the proof. Nonetheless, Bettie's work does not answer the question (find the last two digits). It misses the first and last statements of Ted's work, which describe the strategy and provides the answer to the problem. While Bettie's work shows attention to the small steps of the computation, it lacks attention to the big picture (strategy and goal).

Andrews 6.1.1 in Hw7

Andrews 6.1.1 Proof that if $\phi(m) \mid (m-1)$, then there exists no prime p such that $p^2 \mid m$.

Proof. Suppose $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ and that $\exists e_i \geqslant 2$. Then

$$\phi(m) = \phi(p^{e_i}) \cdot \phi\left(\frac{m}{p^{e_i}}\right)$$

We can guarantee that $\frac{m}{p^{e_i}} \in \mathbb{Z}$ because $p_i^e \mid m$. Then

$$\phi(m) = (p^{e_i} - p^{e_i - 1})\phi\left(\frac{m}{p^{e_i}}\right)$$

We can guarantee that $p \nmid m-1$ because $p \geqslant 2$ and if $m = kp, k \in \mathbb{Z}^+$, then $(k-1)p$ is the closest number smaller than m to be also divisible by p.
Thus, $(p \nmid (m-1)$ and $p \mid \phi(m)) \to \phi(m) \nmid (m-1)$. By contrapositive,

$$\exists p : p^2 \mid m \to \phi(m) \nmid (m-1),$$

$$\therefore \phi(m) \mid (m-1) \to \nexists \text{ prime } p : p^2 \mid m.$$

□

andrews

6.1.1 $\varphi(m) \mid m-1 \Rightarrow \nexists p \in \mathbb{Z}$ s.t. $p^2 \mid m$

let $m = p^a \cdot k$, $a \geq 2$

$\varphi(m) = \varphi(p^a \cdot k) = \varphi(p^a)\,\varphi(k)$

$= (p^a - p^{a-1})\,\varphi(k)$

$= p^{a-1}(p-1)\varphi(k)$

$\therefore p \mid m \Rightarrow p \nmid m-1$

$p^2 \mid m \Rightarrow \varphi(m) \nmid (m-1)$

$\varphi(m) \mid (m-1) \Rightarrow \nexists p \in \mathbb{Z} : p^a \mid m$, $a \geq 2$

Comparing Ted's (left) and Bettie's (right) homework on Andrews 6.1.1 in HW7. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.

Bettie's work follows the same steps of Ted's work: (1) writing m as product of a power of a prime, (2) computing $\varphi(m)$ as a function of p, (3) noting that p cannot divide m-1 and (5) lastly "stating" the proof. Notice that Ted's note, "by contrapositive," which is omitted in Bettie's work, explains the meaning of the last two statements; otherwise, they do not make sense as they stand in Bettie's work. Bettie's work exhibits an effort about understanding the steps but lacks all elements of a proof by contradiction: (i) noting it is a proof by contradiction, (ii) stating the element of contradiction and (iii) proving the statement to be contradicted otherwise.

Andrews 6.2.2 in Hw7

Andrews 6.2.2 Prove that $\prod_{d|n} d = n^{\tau(n)/2}$

Translation to plain words: The product of all of the factors of n is n to the power of half the number of factors of n. Makes sense because factors can basically be written in unique pairs (a, b) : $ab = n$, and there are about $\tau(n)/2$ many of these pairs. Following is the formal proof:

Proof.

Let $D := \{d \in D : d \mid n,\ d_i < d_j\ \forall i < j\}$.

Case 1: $n = p^j$, p is prime:

Note that $D = \{p^0, p^1, p^2, \ldots, p^j\}$ and note that the first factor times the last is n: $(p^0)(p^j) = n$, $(p^1)(p^{j-1}) = n$, so on until you get to the "middle" of the list. There can be two cases for j:

$$j \text{ is odd: } \prod_{d|p^j} d = \underbrace{(p^0)(p^j)\cdot(p^1)(p^{j-1})\cdots(p^{(j-1)/2})(p^{(j+1)/2})}_{\text{There are } (j-1)/2+1 \text{ terms here}}$$

$$= n^{(j+1)/2} = n^{(\tau(p^j)/2)}$$

$$j \text{ is even: } \prod_{d|p^j} d = \underbrace{(p^0)(p^j)\cdot(p^1)(p^{j-1})\cdots(p^{j/2-1})(p^{j/2+1})}_{\text{There are } j/2 \text{ terms here}}\cdot p^{j/2}$$

$$= n^{j/2}\cdot p^{j/2} = n^{j/2}n^{1/2} = n^{(j+1)/2} = n^{(\tau(p^j)/2)}$$

Case 2: n is composite:

Consider $n = ab, a = p^j, b = q^k$ where p and q are distinct primes. Then we know that $\prod_{d|p^j} d = a^{(\tau(p^j)/2)}$ and $\prod_{d|q^k} d = b^{(\tau(q^k)/2)}$ Since a and b share no factors, the product of all the factors of ab is simply the product

$$\prod_{d|n} d = \left(\prod_{d|p^j} d\right)\left(\prod_{d|q^k} d\right) = (a^{(\tau(p^j)/2)})(b^{(\tau(q^k)/2)})$$

$$= (ab)^{(\tau(p^j)\tau(q^k)/2)} = n^{(\tau(p^j q^k)/2)} = n^{(\tau(n))/2}$$

This logic can be extended for prime factorization, and therefore will work for all natural numbers.

$$\therefore \prod_{d|n} d = n^{\tau(n)/2}\quad \forall n \in \mathbb{Z}^+$$

6.2.2 Prove $\prod_{d|n} d = n^{\tau(n)/2}$

$D := \{d \in D : d|n,\ d_i < d_j\ \forall_i < j\}$

Case 1: $n = p^j$; p is prime $\quad D = \{p^0, p^1, p^2, \ldots, p^j\}$

$(p^0)(p^j) = n = p^1(p^{j-1}) = n$

j is odd: $\prod_{d|p^j} d = \underbrace{(p^0 p^j)(p^1 p^{j-1})\ldots(p^{(j-1)/2})(p^{(j+1)/2})}_{(j-1)/2+1 \text{ terms}}$

$= n^{j+1/2} = n^{\tau(p^j)/2}$

j is even: $\prod_{d|p^j} d = \underbrace{p^0(p^j)\,(p^1 p^{j-1})\ldots(p^{j/2-1}\, p^{j/2+1})}_{j/2 \text{ terms}}\, p^{j/2}$

$= n^{j/2}\cdot p^{j/2} = n^{j/2} n^{1/2} = n^{j+1/2} = n^{\tau(p^j)/2}$

Case 2: n is composite

Let $n = ab$; $a = p^j \quad b = q^k \quad p, q$ distinct primes.

$\Rightarrow \prod_{d|p^j} d = a^{\tau(p^j)/2}$ and $\prod_{d|q^k} d = b^{\tau(q^k)/2}$

Since $\gcd(a,b) = 1$ and share no factors

$ab = \prod_{d|n} d = \left(\prod_{d|p^j} d\right)\left(\prod_{d|q^k} d\right) = a^{\tau(p^j)/2}\, b^{\tau(q^k)/2}$

$= ab^{[\tau(p^j)\tau(q^k)/2]} = n^{\tau(p^j q^k)/2} = n^{\tau(n)/2}$

$\therefore \prod_{d|n} d = n^{\tau(n)/2} \quad \forall n \in \mathbb{Z}^+$

Comparing Ted's (left) and Bettie's (right) homework on Andrews 6.2.2 in HW7. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.

Bettie's work is a literal copy of Ted's work without the explanatory English sentences. The modifications in Bettie's work indicate an attempt of understanding the steps (grouping the factors in case 1 and noting that gcd(a,b)=1). Ted's work involves a mistake (last computation) that is found in Bettie's work. Bettie's work explicates carefully the strategy of the proof, although it literally copies Ted's.

Andrews 6.2.9 in Hw7

Andrews 6.2.9 If $\sigma(n) = 2n$, n is a perfect number.

Prove that if n is a perfect number, then $\sum_{d|n}\frac{1}{d} = 2$.

Note: Perfect number is a number whose factors sum up to be itself.

Proof. Example for Clarity: Let $k = 6$. Then

$$\sigma(k) = 1 + 2 + 3 + 6 = 12 = 2k.$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{6}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6} = \frac{\sigma(k)}{k} = 2$$

Let $d_i \in d : d_i \mid n$ If we order the d_i so that $d_i < d_j \forall i < j$, then from observation, the first factor $d_1 = 1$, and the last factor $d_{\tau(n)} = n$. Note that $d_1 \cdot d_{\tau(n)} = n$. Similarly, $d_2 \cdot d_{\tau(n)-1} = n$. In general:

$$d_i \cdot d_{\tau(n)-i+1} = n$$

So the sum becomes:

$$\sum_{d|n}\frac{1}{d} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{\tau(n-1)}} + \frac{1}{n}$$

$$= \frac{n}{n} + \frac{d_{\tau(n-1)}}{n} + \cdots + \frac{d_2}{n} + \frac{1}{n} = \frac{\sigma(n)}{n} = \frac{2n}{n} = 2$$

□

6.2.9 $\sigma(n) = 2n$, n perfect prime, prove n is perfect number

then $\sum_{d|n}\frac{1}{d} = 2$

Perfect number $:= n \in \mathbb{Z}^+ : n = \sum_{d|n} d,\ n \neq d$

Let $d_i \in d : d_i | n \quad d_i < d_j \ \forall i < j$

$n = d_1 \cdot d_{\tau(n)} \Rightarrow d_2 \cdot d_{\tau(n-1)} = n$

$\Rightarrow d_i\, d_{\tau(n-i+1)} = n$

$\sum_{d|n}\frac{1}{d} = \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} + \ldots \frac{1}{d_{\tau(n-1)}} + \frac{1}{n}$

$= \frac{n}{n} + \frac{d_{\tau(n-1)}}{n} + \ldots + \frac{d_2}{n} + \frac{d_1}{n} = \frac{\sigma(n)}{n} = \frac{2n}{n} = 2$

Comparing Ted's (left) and Bettie's (right) homework on Andrews 6.2.9 in HW7. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.

In Bettie's work, the definition of perfect numbers is more accurate than in Ted's. Notice the use of logic symbols in Bettie's work. Nonetheless, the other modifications (indices) and additions (implcation symbols) are inaccurate.

WS 9.2 in Hw10

WS 9.2 Prove that there are infinitely primes $\equiv 3 \bmod 4$. (*Hint:* assuming $p_1, p_2, \ldots, p_k$ are the only primes $\equiv 3 \bmod 4$, consider the number $4p_1p_2\cdots p_k - 1$.) Explain why this is much easier than to prove that there are infinitely primes $\equiv 1 \bmod 4$.[2]

Proof. Suppose that there are a finite number of primes $p_1, p_2, \ldots, p_k$ that are equivalent to 3 (mod 4). Then consider the integer $N = 4\left(\prod_{i=1}^{k} p_i\right) - 1$.
Since $\gcd(N, N+1) = 1$, we know that N and $N-1$ share no common prime factors. Also, $N-1$ is simply the product of 4 and is a product of "all" the primes that are equivalent to 3 (mod 4).
N is never even, since we have an even term plus one. Thus, 2 is not in the prime factorization of N, implying that all the remaining primes must be equivalent to 1 (mod 4) based on our assumption. However, examining an arbitrary $t \in \mathbb{Z} : t \equiv N \pmod 4$:

$$4\left(\prod_{i=1}^{k} p_i\right) - 1 \equiv t \bmod 4$$

$$4 \mid 4\left(\prod_{i=1}^{k} p_i\right) - 1 - t$$

$$4 \mid (-1-t) \quad \text{Justification: } [a \mid (b+c) \text{ and } a \mid b] \to a \mid c$$

$$-1 \equiv 3 \equiv t \bmod 4$$

$$N \equiv 3 \bmod 4$$

The above contradicts that the prime factorization of N only has integers equivalent to 1 (mod 4) because if the assumption was true, N is a product of primes that are the residue of 1 (mod 4), forcing N to also be a residue of 1 (mod 4).
Therefore, $\exists$ infinitely many primes that are equivalent to 3 (mod 4). □

9.2. assume there are exactly n-primes $\{p_1, p_2, \ldots, p_k\}$ $\equiv 3 \pmod 4$

suppose $N = 4\underbrace{p_1 p_2 \cdots p_k}_{\text{even}} - 1$ $\quad p_i \in \mathbb{Z}_{>0}$

we know N is never going to be even

let the prime factorization of $N = q_1 q_2 \cdots q_\ell$; $q_i \in \mathbb{Z}_{>0}$

we know $\gcd(N, N-1) = 1$ and $N-1 = 4(p_1 p_2 \cdots p_k)$ which is $\equiv 3 \bmod 4$

Since N is never going to be even since we have an even term subtracting 1 thus 2 is not in prime factorization of N $\Rightarrow$ $N \equiv 1 \bmod 4$

However examining arbitrary t ; $t \in \mathbb{Z}_{>0} \mid t \equiv N \bmod 4$

$\Rightarrow 4p_1 p_2 \cdots p_k - 1 \equiv t \pmod 4$

$4 \mid (4p_1 p_2 \cdots p_k) - 1 - t$ $\quad$ we know $4 \mid 4p_1 p_2 \cdots p_k$

$\Rightarrow 4 \mid (-1-t)$

(if $a \mid c + b$ and $a \mid b \Rightarrow a \mid c$)

$\Rightarrow -1 \equiv t \bmod 4$

$\equiv 3 \bmod 4$ $\quad$ which is a contradiction

basic arithmetic shows us that the prime factorization of N must be congruent to $3 \pmod 4$

Therefore, $\exists$ infinitely many primes $\equiv 3 \pmod 4$

Comparing Ted's (left) and Bettie's (right) homework on WS 9.2in HW10. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.
Bettie's work follows the same strategy, steps and organization of the write-up as Ted's work. Both works contains the same mistake (underlined in red). We notice that Bettie writes a number of justifications in English sentences (rather than usual logic symbols or omission) either copied or modified from or not included in Ted's work. Although Bettie's work shows attempts to understand and justify the steps of the the proof, it does not state where the contradiction resides. Her reader would guess that her proof is by contradiction as is noted and may conclude that she got on one hand $N \equiv 1\ (mod\ 4)$ and on the other hand $N \equiv 3\ (mod\ 4)$.
WS9.3 in Hw10

WS 9.3 Show that if n is composite, then so is $2^n - 1$. Thus the *Mersenne number* $M_p := 2^p - 1$ can only possibly be prime if p is prime. Find the first five Mersenne primes, and the first five composite Mersenne numbers M_p for which p is prime.

Proof. Suppose $n = pq : p, q \in Z^+$. Then

$$2^n - 1 = 2^{pq} - 1 = (2^p - 1)\left(2^{p(q-1)} + 2^{p(q-2)} \ldots + 2^p + 1\right)$$

Therefore, if n is a product of two (or more) integers, $2^n - 1$ MUST be composite with a factor $2^p - 1$, where p is one of the factors of n. □

Numerical Examination of *Mersenne Numbers*:

n	$2^n - 1$	Prime Factorization
2	3	3
3	7	7
5	31	31
7	127	127
11	2047	23 × 89
13	8191	8191
17	131071	131071
19	524287	524287
23	8388607	47 × 178481
29	536870911	233 × 1103 × 2089
31	2147483647	2147483647
37	137438953471	223 × 616318177
41	2199023255551	13367 × 164511353

The first 5 primes in the form of $2^p - 1$ occur when $p = \{2, 3, 5, 7, 13\}$
The first 5 composite numbers in the form $2^p - 1$ occur when $p = \{11, 23, 29, 37, 41\}$.

9.3. n is composite $\Rightarrow 2^n - 1$ composite

$M_p := 2^p - 1$ prime iff p-prime

let $n = pq$ where $p, q \in \mathbb{Z}_{>0}$

$\Rightarrow 2^n - 1 = 2^{pq} - 1$; we know that $2^{pq} - 1$ has a factor of $(2^p - 1) \Rightarrow 2^n - 1 = (2^p - 1)(2^{p(q-1)} + 2^{p(q-2)} + \cdots + 2^p + 1)$

$\therefore$ If n is a product of two or more integers, $2^n - 1$ is composite with a factor of $2^p - 1$, where $p \mid n$

$n > 1$	$2^n - 1$	prime factorization
2	3	3
3	7	7
5	31	31
7	127	127
11	2047	23×89
13	8191	8191
17	131071	131071
19	524287	524287
23	8388607	47×178481
29	536870911	233×1103×2089
31	2147483647	2147483647
37	137438953471	223×616318177
41	2199023255551	13367×164511353

First 5 prime numbers in form $2^p - 1$, when $p = \{2, 3, 5, 7, 13\}$

First 5 composite numbers in form $2^p - 1$, when $p = \{11, 23, 29, 37, 41\}$

Comparing Ted's (left) and Bettie's (right) homework on WS 9.3 in HW10. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source. The check marks denotes the circles added in Bettie's work. The similarity between Bettie's and Ted's work is striking. In this work of Bettie, the inclusion of English sentences to explain and state the conclusion are noticeable.

WS 9.4 in Hw10

WS 9.4 Prove that $p \in \mathbb{Z}_{>1}$ is prime if and only if $a^{p-1} \equiv 1 \bmod p$ for all $a \not\equiv 0 \bmod p$.[3] Explain how this can be used for a test for *compositeness* of an integer without actually factoring it.

Proof. Let $m \in \mathbb{Z}^+$.
Let R be the statement: "m is prime."
Let S be the statement: "$a^{m-1} \equiv 1 \bmod m$ for all $a \not\equiv 0 \bmod m$"

(i.) $R \to S$
First we examine the case when $p \in \mathbb{Z}^+$ is known to be prime. Then by Euler's Theorem,

$$a^{\phi(p)} = 1 \bmod p$$

$$a^{p-1} = 1 \bmod p$$

(ii.) $S \to R$
By contrapositive, it is logically equivalent to show $\neg S \to \neg R$
Let $p, q \in \mathbb{Z}^+$: $p < q < m$. Suppose $m = pq$: Then we consider the case when $a = p$.

$$a \equiv p \not\equiv 0 \bmod m$$

If $\gcd(p,m) = p$, then $p \mid \gcd(p^n, m)$. $p^n \equiv 1 \pmod m$ has no solution $\forall n \in \mathbb{Z}^+$ because $\gcd(p,m) \neq 1$.

□

If you can find some $a \not\equiv 0 \pmod m$: $a^{m-1} \not\equiv 1 \pmod m$, then we can say for certain that we have a composite number.

I imagine that this can be done "efficiently" by choosing a as a product of primes. Since a is fairly large, each time we do this we should consider the following:

$$a^{m-1} \bmod m \equiv (a \bmod m)^{m-1} \bmod m$$

By doing the above, we can efficiently check to see if m is composite.

However, there exists pseudo-primes that can pass this test. In particular, if a composite number that is sufficiently large is tested with an insufficiently large a, there is a chance that $\gcd(a,m) = 1$ and m would "pass" the test.

Thus this test can only tell us for certain when a number is composite, not if a number is prime.

9.4 $p \in \mathbb{Z}_{>1}$ - prime iff $a^{p-1} \equiv 1 \pmod p$, $\forall a \not\equiv 0 \pmod p$
let P define the statement "m is prime"
let Q define the statement "$a^{m-1} \equiv 1 \bmod m$ for $\forall a \not\equiv 0 \bmod m$"
first case: $P \to Q$
Supose $p \in \mathbb{Z}_+$ is known to be prime. By Eulers Thm,

$$a^{\phi(p)} \equiv 1 \bmod p$$
$$\Rightarrow a^{p-1} \equiv 1 \bmod p$$

second case: $Q \to P$
let $p, q \in \mathbb{Z}_{>0} \mid p < q < m$. Suppose $m = pq$. consider when $a = p$
$a \equiv p \not\equiv 0 \bmod m$
if $\gcd(p,m) = p$ then $p \mid \gcd(p^n, m)$ ∴
$p^n \equiv 1 \pmod m$ has no solution for $\forall n \in \mathbb{Z}_{>0}$ because $p \nmid 1$
If we can find some $a \not\equiv 0 \bmod m \mid a^{m-1} \not\equiv 1 \pmod m$) then we know we have a composite number. We can do this by choosing a as a product of primes. However, pseudo primes exist and can pass test so that is why we can only know for certain if a is composite not if it is a prime.

Comparing Ted's (left) and Bettie's (right) homework on WS 9.4 in HW10. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source. Noticeable the copying of the sentences that explain the strategy of the proof (let P … let Q … First case … Second case …), although the so-called cases are the two ways of the equivalence of two statements. Also, Ted explains the method for the so-called "second case" ("by contrapositive") but not Bettie. In the paragraph where Bettie explains how the primality test, she seems to make an intelligent choice of what to copy and what to drop. The omitted ideas in Ted, i.e. the contrapositive for the proof of the other implicature and the large number a for the test, are not mathematically solid.

WS 10.2 in Hw 11

WS 10.2 In this exercise, we'll prove another one of Euler's theorems: If p is an odd prime, then a is a quadratic residue mod p if and only if $a^{\frac{p-1}{2}} \equiv 1 \bmod p$.
Let "a is a quadratic residue mod p" be statement R.
Let "$a^{\frac{p-1}{2}} \equiv 1 \bmod p$" be statement S.

(a) Prove the "$\Longrightarrow$" direction, e.g., by recalling another theorem by Euler.

Proof. $R \to S$
Given R, we can write:

$$\exists x \in \mathbb{Z} : x^2 \equiv a \bmod p$$

$$a^{\frac{p-1}{2}} \equiv (x^2)^{\frac{p-1}{2}} \equiv x^{p-1} \equiv 1 \bmod p$$

$$\therefore a^{\frac{p-1}{2}} \equiv 1 \bmod p$$

□

(b) For the "$\Longleftarrow$" direction, you may assume that there exists a primitive root r mod p (which is true, although we haven't prove it). Assuming $a^{\frac{p-1}{2}} \equiv 1 \bmod p$, use the fact that $a \equiv r^n$ for some n, and show that n is even.

Page 1 of 5

Proof. $S \to R$
Assuming that there is a primitive root r for prime p, we can generate a reduced residue system $\{r, r^2, \ldots, r^{\phi(p)}\}$. Since all the integers in the interval $[1, p-1]$ have a GCD of 1 with p, there are $p-1$ elements in the reduced residue system, and each number will be equivalent to a member in r^n. Thus, $\exists n \in \mathbb{Z} : 1 \leqslant n \leqslant p-1$ and $r^n \equiv a \pmod p$. Then consider:

$$(r^n)^{\frac{p-1}{2}} \equiv (r^{\frac{n}{2}})^{p-1} \equiv 1 \bmod p$$

Note that $r^{\frac{n}{2}} \in \mathbb{Z}$ iff $2 \mid n$. Also note that since r has an order of $p-1$, from previous proofs we know that $(p-1) \mid \frac{n}{2}(p-1)$, implying $\frac{n}{2}$ is an integer. Thus, whenever we have $n = 2k, k \in \mathbb{Z}$,

$$(r^{2k})^{\frac{p-1}{2}} \equiv (r^k)^{p-1} \equiv 1 \bmod p$$

$$\therefore a^{p-1} \equiv 1 \bmod p$$

□

2 p = odd prime $\therefore x^2 \equiv a \bmod p$ iff $a^{p-1} \equiv 1 \bmod p$

a) Suppose p = odd prime $\Rightarrow a^{\frac{p-1}{2}} \equiv 1 \bmod p$
$x^2 \equiv a \bmod p$
$\exists x \in \mathbb{Z}$ st. $x^2 \equiv a \bmod p$ $p \nmid a \Rightarrow p \nmid x$
$a^{\frac{p-1}{2}} \equiv (x^2)^{\frac{p-1}{2}} \equiv x^{p-1} \equiv 1 \bmod p$
$\therefore a^{\frac{p-1}{2}} \equiv 1 \bmod p$

p = odd prime
b) Suppose $a^{\frac{p-1}{2}} \equiv 1 \bmod p \Rightarrow x^2 \equiv a \bmod p$
assume $a^{\frac{p-1}{2}} \equiv 1 \bmod p$ where $a \equiv g^n$ for some $n \in \mathbb{Z}$
Suppose primitive root for prime p
$\Rightarrow$ reduced residue system $\{g, g^2, \ldots, g^{\phi(p)}\}$
$p-1$ elements each equivalent to members of g^n
$\exists n \in \mathbb{Z}$ st. $1 \le n \le p-1$, $g^n \equiv a \bmod p$
$\Rightarrow (g^n)^{\frac{p-1}{2}} \equiv (g^{\frac{n}{2}})^{p-1} \equiv 1 \bmod p$ where $g^{\frac{n}{2}} \in \mathbb{Z}$ iff $2 \mid n$
Since g has order $p-1$ we know $(p-1) \mid \frac{n}{2}(p-1) \Rightarrow \frac{n}{2} \in \mathbb{Z}$
Then whenever $n = 2k$ $k \in \mathbb{Z}$
$(g^{2k})^{\frac{p-1}{2}} \equiv (g^k)^{p-1} \equiv 1 \bmod p$
$\therefore a^{\frac{p-1}{2}} \equiv 1 \bmod p$

Comparing Ted's (left) and Bettie's (right) homework on WS 10.2 in HW11. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source. The underlined text in red is worth noticing. It is worth noticing that Bettie's work adds the to-be proven statements in a different writing style than Ted's. The statement of what needs to be proven in part a is worng in Bettie's while right in Ted's work. However, she follows the proof as in Ted's answer of part a. As for part b, the to be proven statement is right in Ted's but partially right in Bettie's work ("p is an odd prime" is a given for both implications). In any case, neither Ted's not Bettie's proofs of part b exhibit clarity on what needs to be proven (i.e. there is exist a number the square of which is equivalent to a modulo p; the proof is to prove that $g^{n/2}$ is the desired number because n is even).
WS 10.3 in Hw11

WS 10.3 Use Euler's theorem to prove, given a primitive root r mod p (as above, an odd prime), that r^n is a quadratic residue mod p if and only if n is even. Conclude that, for an odd prime p, exactly half the integers between 1 and $p-1$ are quadratic residues mod p.

Let "r^n is a quadratic residue mod p" be statement R.

Let "n is even" be statement S.

Proof. $R \to S$

Given $\exists x \in \mathbb{Z} : x^2 \equiv r^n \bmod p$, consider

$$r^{p-1} \equiv 1 \bmod p$$

From the previous problem, we know that

$$(r^n)^{\frac{p-1}{2}} \equiv 1 \bmod p$$

Since r has order $\phi(p) = p-1$, or $r^{p-1} \equiv 1 \pmod p$ from WS 8.2 we know that

$$(p-1) \mid \frac{n}{2}(p-1)$$

Therefore, $\frac{n}{2} \in \mathbb{Z}$, implying that $2|n$. □

Proof. $S \to R$

Given that $\exists k \in \mathbb{Z} : n = 2k$,

$$r^n \equiv r^{2k} \bmod p$$

$$(r^n)^{\frac{p-1}{2}} \equiv (r^k)^{p-1} \equiv 1 \pmod p$$

□

3. $\exists x \in \mathbb{Z}$ where $x^2 \equiv r^n \bmod p$,

Suppose $r^{p-1} \equiv 1 \bmod p$ from before, we know

$(r^n)^{\frac{p-1}{2}} \equiv 1 \bmod p$ and r has order $p-1$

$\Rightarrow (p-1) \mid \frac{n}{2}(p-1)$

$\therefore \frac{n}{2} \in \mathbb{Z} \Rightarrow 2|n$ ✓

Suppose, $\exists k \in \mathbb{Z}$ s.t. $n = 2k$

$r^n \equiv r^{2k} \bmod p$

$(r^n)^{\frac{p-1}{2}} \equiv (r^k)^{p-1} \equiv 1 \bmod p$ ✓

Comparing Ted's (left) and Bettie's (right) homework on WS 10.3 in HW11. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.

Ted's and Bettie's answers to WS 10.3 are sound. However, Bettie's work lacks explanations of the steps (the two parts each of which proves an implication of the equivalence).

WS 10.4 in Hw11

WS 10.4 Let p be and odd prime not dividing a and b. Show that:

(a) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

Proof.

Case 1: Both a and b are quadratic residues modulo p:

First, $\left(\frac{a}{p}\right) = 1 = \left(\frac{b}{p}\right)$.

Then, suppose there exists a primitive root r modulo p. Then $\exists m, n \in \mathbb{Z}$: $r^n \equiv a \pmod p$ and $r^m \equiv b \pmod p$. Since a and b are both quadratic residues, $\exists j, k \in \mathbb{Z} : n = 2j$ and $m = 2k$.

$$ab \equiv 1 \equiv (r^n)(r^m) \equiv r^{n+m} \equiv r^{2(j+k)} \pmod p$$

From the above, we can see that ab is congruent with a $r^{2(j+k)}$ where the exponent is even, implying that ab is also a quadratic residue modulo p.

$$\therefore \left(\frac{ab}{p}\right) = 1 = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

Case 2: Only one of them is a quadratic residue modulo p:

Suppose a is a quadratic residue and b is not.

First, $\left(\frac{a}{p}\right) = 1$ and $\left(\frac{b}{p}\right) = -1$.

Then from WS 10.3, we observe that $\exists n \in \mathbb{Z} : r^n \equiv a \pmod p$ where $\exists j \in \mathbb{Z} : n = 2j$. Since b is not a quadratic residue, it must be congruent with a member of the reduced residue system that is not in the set of quadratic residues, meaning the members generated from the primitive root with odd powers. Thus, $\exists m \in \mathbb{Z} : r^m \equiv b \pmod p$, where $\exists k \in \mathbb{Z} : m = 2k+1$.

$$ab \equiv (r^m)(r^n) \equiv r^{m+n} \equiv r^{2j+2k+1} \equiv r^{2(j+k)+1} \pmod p$$

From the above, we can see that ab is congruent to $r^{2(j+k)+1}$, which has an odd power. Thus, we know that ab is not a quadratic residue modulo p.

$$\therefore \left(\frac{ab}{p}\right) = -1 = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

Case 3: Neither a nor b are quadratic residues modulo p: First, $\left(\frac{a}{p}\right) = -1 = \left(\frac{b}{p}\right)$.

Now following similar logic from above, let $n, m, j, k \in \mathbb{Z}$:

$$r^n \equiv a \pmod p \quad n = 2j+1$$
$$r^m \equiv b \pmod p \quad m = 2k+1$$

Page 3 of 5

4. a) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

case 1: a and b quadratic residues mod p

first $\left(\frac{a}{p}\right) = 1 = \left(\frac{b}{p}\right)$

$\exists$ primitive root r modulo p. $\Rightarrow \exists m, n \in \mathbb{Z}$

s.t. $r^n \equiv a \pmod p$ and $r^m \equiv b \bmod p$.

Since a, b quadratic residues $\Rightarrow \exists j, k \in \mathbb{Z}$

s.t. $ab \equiv 1 \equiv (r^n)(r^m) = r^{n+m} = r^{2(j+k)} \bmod p$

$\Rightarrow ab$ is also quadratic residue mod p

$\therefore \left(\frac{ab}{p}\right) = 1 = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

case 2: only one is quadratic residue mod p

let $\left(\frac{a}{p}\right) = 1$ and $\left(\frac{b}{p}\right) = -1$

we know $\exists n \in \mathbb{Z}$ s.t. $r^n \equiv a \pmod p$ from last problem

where $\exists j \in \mathbb{Z}$ s.t. $n = 2j$. Since b is not quadratic residue,

Thus $\exists m \in \mathbb{Z}$ st $r^m \equiv b \bmod p$ where $\exists k \in \mathbb{Z}$ st $m = 2k+1$

$ab = (r^m)(r^n) \equiv r^{m+n} \equiv r^{2j+2k+1} \equiv r^{2(j+k)+1} \bmod p$

$\Rightarrow 2(j+k)+1$ is odd power so not quadratic residue

$\therefore \left(\frac{ab}{p}\right) = -1 = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

Case 3: neither a nor b are quadratic residues mod p

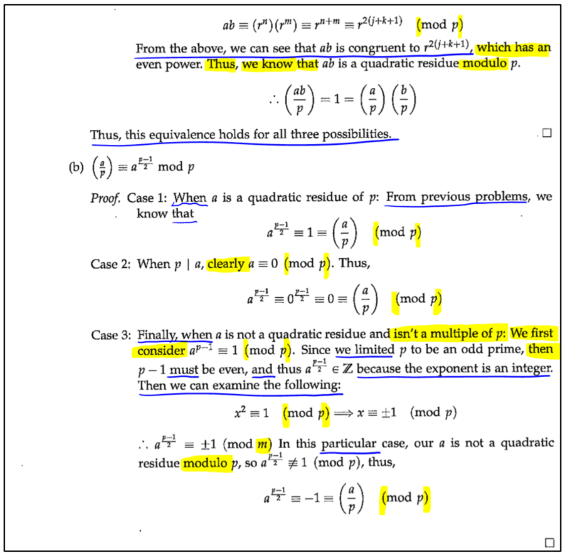
let $\left(\frac{a}{p}\right) = -1 = \left(\frac{b}{p}\right)$

let $n, m, j, k \in \mathbb{Z}$

$r^n \equiv a \pmod p$ $n = 2j+1$

$r^m \equiv b \pmod p$ $m = 2k+1$

$ab \equiv (r^n)(r^m) = r^{n+m} = r^{2(j+k+1)} \bmod p$



$$ab \equiv (r^n)(r^m) \equiv r^{n+m} \equiv r^{2(j+k+1)} \pmod p$$

From the above, we can see that ab is congruent to $r^{2(j+k+1)}$, which has an even power. Thus, we know that ab is a quadratic residue modulo p.

$$\therefore \left(\frac{ab}{p}\right) = 1 = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

Thus, this equivalence holds for all three possibilities. □

(b) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \bmod p$

Proof. Case 1: When a is a quadratic residue of p: From previous problems, we know that

$$a^{\frac{p-1}{2}} \equiv 1 = \left(\frac{a}{p}\right) \pmod p$$

Case 2: When $p \mid a$, clearly $a \equiv 0 \pmod p$. Thus,

$$a^{\frac{p-1}{2}} \equiv 0^{\frac{p-1}{2}} \equiv 0 \equiv \left(\frac{a}{p}\right) \pmod p$$

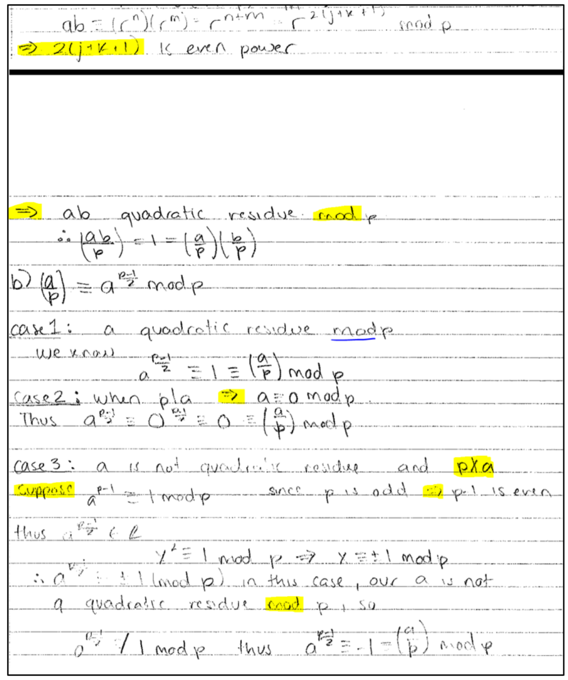
Case 3: Finally, when a is not a quadratic residue and isn't a multiple of p: We first consider $a^{p-1} \equiv 1 \pmod p$. Since we limited p to be an odd prime, then $p-1$ must be even, and thus $a^{\frac{p-1}{2}} \in \mathbb{Z}$ because the exponent is an integer. Then we can examine the following:

$$x^2 \equiv 1 \pmod p \implies x \equiv \pm 1 \pmod p$$

$\therefore a^{\frac{p-1}{2}} \equiv \pm 1 \pmod m$ In this particular case, our a is not a quadratic residue modulo p, so $a^{\frac{p-1}{2}} \not\equiv 1 \pmod p$, thus,

$$a^{\frac{p-1}{2}} \equiv -1 = \left(\frac{a}{p}\right) \pmod p$$

□



$ab \equiv (r^n)(r^m) = r^{n+m} = r^{2(j+k+1)} \bmod p$

$\Rightarrow 2(j+k+1)$ is even power

$\Rightarrow ab$ quadratic residue mod p

$\therefore \left(\frac{ab}{p}\right) = 1 = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

b) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \bmod p$

case 1: a quadratic residue mod p

we know $a^{\frac{p-1}{2}} \equiv 1 \equiv \left(\frac{a}{p}\right) \bmod p$

case 2: when $p \mid a \Rightarrow a \equiv 0 \bmod p$

Thus $a^{\frac{p-1}{2}} \equiv 0^{\frac{p-1}{2}} \equiv 0 \equiv \left(\frac{a}{p}\right) \bmod p$

case 3: a is not quadratic residue and $p \nmid a$

suppose $a^{p-1} \equiv 1 \bmod p$ since p is odd $\Rightarrow p-1$ is even

thus $a^{\frac{p-1}{2}} \in \mathbb{Z}$

$x^2 \equiv 1 \bmod p \Rightarrow x \equiv \pm 1 \bmod p$

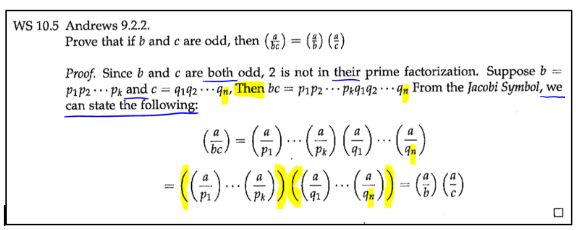
$\therefore a^{\frac{p-1}{2}} \equiv \pm 1 \pmod p$. in this case, our a is not a quadratic residue mod p, so

$a^{\frac{p-1}{2}} \not\equiv 1 \bmod p$ thus $a^{\frac{p-1}{2}} \equiv -1 \equiv \left(\frac{a}{p}\right) \bmod p$

Comparing Ted's (left) and Bettie's (right) homework on WS 10.4 in HW11. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.

Bettie's and Ted's works are very similar. The proofs are laid out clearly with multiple cases. The sentences of Ted's work that do not exist in Bettie's work are redundant.

Andrews 9.2.2 in Hw11



WS 10.5 Andrews 9.2.2.

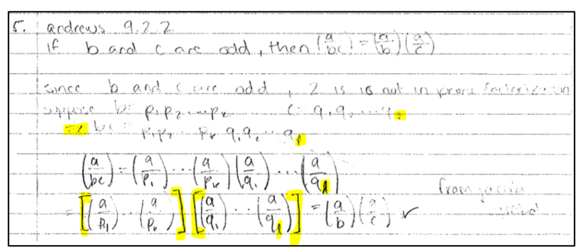
Prove that if b and c are odd, then $\left(\frac{a}{bc}\right) = \left(\frac{a}{b}\right)\left(\frac{a}{c}\right)$

Proof. Since b and c are both odd, 2 is not in their prime factorization. Suppose $b = p_1 p_2 \cdots p_k$ and $c = q_1 q_2 \cdots q_n$. Then $bc = p_1 p_2 \cdots p_k q_1 q_2 \cdots q_n$. From the Jacobi Symbol, we can state the following:

$$\left(\frac{a}{bc}\right) = \left(\frac{a}{p_1}\right)\cdots\left(\frac{a}{p_k}\right)\left(\frac{a}{q_1}\right)\cdots\left(\frac{a}{q_n}\right)$$

$$= \left(\left(\frac{a}{p_1}\right)\cdots\left(\frac{a}{p_k}\right)\right)\left(\left(\frac{a}{q_1}\right)\cdots\left(\frac{a}{q_n}\right)\right) = \left(\frac{a}{b}\right)\left(\frac{a}{c}\right)$$

□



5. andrews 9.2.2

if b and c are odd, then $\left(\frac{a}{bc}\right) = \left(\frac{a}{b}\right)\left(\frac{a}{c}\right)$

since b and c are odd, 2 is not in prime factorization

suppose $b = p_1 p_2 \cdots p_k$ $c = q_1 q_2 \cdots q_n$

$\Rightarrow bc = p_1 p_2 \cdots p_k q_1 q_2 \cdots q_n$

$\left(\frac{a}{bc}\right) = \left(\frac{a}{p_1}\right)\cdots\left(\frac{a}{p_k}\right)\left(\frac{a}{q_1}\right)\cdots\left(\frac{a}{q_n}\right)$ from jacobi symbol

$= \left[\left(\frac{a}{p_1}\right)\cdots\left(\frac{a}{p_k}\right)\right]\left[\left(\frac{a}{q_1}\right)\cdots\left(\frac{a}{q_n}\right)\right] = \left(\frac{a}{b}\right)\left(\frac{a}{c}\right)$ ✓

Comparing Ted's (left) and Bettie's (right) homework on WS 10.4 in HW11. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.

WS 11.3 in Hw13

3. $\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \dots}}}$

So, when $n=4$, our continued fraction looks like $[3, 7, 15, 1]$.

Putting this into a decimal expansion we get $[3,7,15,1] = 3.14159292$,

so comparing this with the decimal expansion of Pi 3.14159265, we see that the continued fraction of pi with $n=4$ is accurate up to 7 digits. Amazing!

Jeremy's answer to WS 11.3 in HW13

3. $\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} \Rightarrow \pi = 3 + \cfrac{1}{7 + \cfrac{1}{16}} \Rightarrow 3 + \frac{16}{113} = \frac{355}{113} = 3.141592920$

Boutros' answer to WS 11.3 in HW13

WS 11.3 Compute the start of a continued fraction expansion for π and compare the accuracy of $[a_0, a_1, \dots, a_n]$ with that of the decimal expansion up to n digits, for $n = 1, 2, 3, 4$.

Proof.

n	a_n	$\frac{p_n}{q_n}$	Decimal(π)	$\left\lvert\frac{p_n}{q_n} - \pi\right\rvert$	$\lvert\text{Decimal}(\pi) - \pi\rvert$
0	3	3	3	$1.41593\cdots \times 10^{-1}$	$1.41593\cdots \times 10^{-1}$
1	7	$\frac{22}{7}$	3.1	$1.26449\cdots \times 10^{-3}$	$4.15926\cdots \times 10^{-2}$
2	15	$\frac{333}{106}$	3.14	$8.3196\cdots \times 10^{-5}$	1.59265×10^{-3}
3	1	$\frac{355}{113}$	3.142	$2.66764\cdots \times 10^{-7}$	5.92653×10^{-4}
4	292	$\frac{103993}{33102}$	3.1416	$5.779\cdots \times 10^{-10}$	$9.26535\cdots \times 10^{-5}$

□

3.

n	a_n	$\frac{p_n}{q_n}$	X = Decimal π_n	$\frac{p_n}{q_n} - \pi$	$X - \pi$
0	3	3	3	0.145926...	0.1415926...
1	7	22/7	3.1	0.001264...	0.0415926...
2	15	333/106	3.14	$8.321962.. \times 10^{-5}$	.00159265...
3	1	355/113	3.142	$2.667764... \times 10^{-7}$	$4.073464... \times 10^{-4}$
4	292	103993/33102	3.1416	$5.779... \times 10^{-10}$	$7.346641... \times 10^{-6}$

Comparing Ted's (left) and Bettie's (right) homework on WS 11.3 in HW13. The highlighted texts are modified in the two sources. The underlined texts in bleu (one line) are missing in the other source.

WS 11.5 Prove that $[a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}$ where

$$p_0 = a_0 \qquad p_1 = a_1 p_0 + 1 \qquad p_n = a_n p_{n-1} + p_{n-2}$$
$$q_0 = 1 \qquad q_1 = a_1 \qquad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geqslant 2$$

Proof. Base case:

- $n = 0 : \frac{p_0}{q_0} = \frac{a_0}{1} = [a_0]$
- $n = 1 : \frac{p_1}{q_1} = \frac{a_1(p_0)+1}{a_1} = a_0 + \frac{1}{a_1} = [a_0, a_1]$

Induction Step:
Suppose the proposition is true for all continued fraction of length $n-1$. Then

$$\begin{aligned}
[a_0, \dots, a_n] &= \left[a_0, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n}\right] \\
&= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-2} + p_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-2} + q_{n-3}} \\
&= \frac{(a_{n-1}a_n + 1)p_{n-2} + a_n p_{n-3}}{(a_{n-1}a_n + 1)q_{n-2} + a_n q_{n-3}} \\
&= \frac{a_n(a_{n-1}p_{n-2} + p_{n-3}) + p_{n-2}}{a_n(a_{n-1}q_{n-2} + q_{n-3}) + q_{n-2}} \\
&= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} \\
&= \frac{p_n}{q_n}
\end{aligned}$$

□

5. Prove: $[a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}$ where $a_i > 0$; $\forall i > 0$

$p_0 = a_0 \quad p_1 = a_1 p_0 + 1 \quad p_n = a_n p_{n-1} + p_{n-2}$
$q_0 = 1 \quad q_1 = a_1 \quad q_n = a_n q_{n-1} + q_{n-2}$ } for $n \geq 2$

prove by induction:
let $n = 0$ and 1
when $n = 0$
$\Rightarrow [a_0] = \frac{a_0}{1} = \frac{p_0}{q_0}$ ✓

when $n=1 \Rightarrow [a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1}$ ✓

now let $x > 0$

$[a_0, a_1, \dots, a_{n-1}, x] = [a_0, a_1, \dots, a_{n-2}, [a_{n-1}, x]]$
$= \left[a_0, a_1, \dots, a_{n-2}, \frac{x a_{n-1} + 1}{x}\right] = \dfrac{p_{n-2}\left(\frac{x a_{n-1}+1}{x}\right) + p_{n-3}}{q_{n-2}\left(\frac{x a_{n-1}+1}{x}\right) + q_{n-3}}$

$= \dfrac{p_{n-2}(x a_{n-1} + 1) + x p_{n-3}}{q_{n-2}(x a_{n-1} + 1) + x q_{n-3}}$

$= \dfrac{x(a_{n-1}p_{n-2} + p_{n-3}) + p_{n-2}}{x(a_{n-1}q_{n-2} + q_{n-3}) + q_{n-2}} = \dfrac{x p_{n-1} + p_{n-2}}{x q_{n-1} + q_{n-2}}$

now substitute a_n for x
$\Rightarrow [a_0, a_1, \dots, a_n] = \dfrac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \dfrac{p_n}{q_n}$ ✓

WS 11.10 Stein 5.1 If $c_n = \frac{p_n}{q_n}$ is the n-th convergent of $[a_0, a_1, ..., a_n]$ and $a_0 > 0$, show that

$$[a_n, a_{n-1}, \ldots, a_1, a_0] = \frac{p_n}{p_{n-1}}$$

and

$$[a_n, a_{n-1}, \ldots, a_2, a_1] = \frac{q_n}{q_{n-1}}$$

Proof. For p

Base case:

$$n = 1 : [a_1, a_0] = a_1 + \frac{1}{a_0} = \frac{a_1 a_0 + 1}{a_0} = \frac{p_1}{p_0}$$

Induction Step:
Now suppose $[a_n, a_{n-1}, \ldots, a_0] = \frac{p_n}{p_{n-1}}$ is true for all integers up to n. Consider case $n + 1$:

$$\frac{p_{n+1}}{p_n} = \frac{a_{n+1}p_n + p_{n-1}}{p_n} = a_{n+1} + \frac{p_{n-1}}{p_n} = a_{n+1} + \frac{1}{\frac{p_n}{p_{n-1}}} = a_{n+1} + \frac{1}{[a_n, \ldots, a_0]}$$

Thus the statement holds for p.
Similarly, for q:
Consider base case $n = 2$, $[a_2, a_1] = \frac{a_2 a_1 + 1}{a_1} = \frac{q_2}{q_1}$. Now suppose $[a_n, a_{n-1}, \ldots a_1] = \frac{q_n}{q_{n-1}}$ for all integers up to n. Consider case $n + 1$:

$$\frac{q_{n+1}}{q_n} = \frac{a_{n+1}q_n + q_{n-1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n} = a_{n+1} + \frac{1}{\frac{q_n}{q_{n-1}}} = a_{n+1} + \frac{1}{[a_n, \ldots, a_1]}$$

Thus, the statement holds for q. □

Stein 5.1 - 4
if $C_n = \frac{P_n}{q_n}$ is the nth convergent of $[a_0, a_1, \ldots, a_n]$
and $a_0 > 0$, show that

$$[a_n, a_{n-1}, \ldots, a_1, a_0] = \frac{p_n}{p_{n-1}} \text{ and } [a_n, a_{n-1}, \ldots, a_2, a_1] = \frac{q_n}{q_{n-1}}$$

let $n=1$

$$[a_1, a_0] = \frac{P_1}{P_0} \Rightarrow [a_0, a_1] = a_1 + \frac{1}{a_0}$$

now suppose $[a_n, a_{n-1}, \ldots, a_0] = \frac{p_n}{p_{n-1}}$ holds for all n.

now let $n = n+1$

$$\frac{p_{n+1}}{p_n} = \frac{a_{n+1}P_n + P_{n-1}}{p_n} = \frac{a_{n+1}P_n}{P_n} + \frac{P_{n-1}}{P_n} = a_{n+1} + \frac{P_{n-1}}{P_n}$$

$$= a_{n+1} + \frac{1}{\frac{P_n}{P_{n-1}}}$$ since it holds for $\frac{P_n}{P_{n-1}}$

$$\Rightarrow \frac{P_{n+1}}{P_n} = a_{n+1} + \frac{1}{[a_{n+1}, a_n, \ldots, a_1]} \Rightarrow \frac{P_{n+1}}{P_n} = [a_{n+1}, a_n, a_{n-1}, \ldots, a_1]$$

as desired.

5.1 part 2
for second case let $n=1$

$$[a_1, a_0] = \frac{q_n}{q_0} \Rightarrow [a_1] = \frac{q_1}{q_0} \Rightarrow [a_1] = a_1$$

suppose $[a_n, a_{n-1}, \ldots, a_0] = \frac{q_n}{q_{n-1}}$ for all n

let $n = n+1$

$$\frac{q_{n+1}}{q_n} = \frac{a_{n+1}q_n + q_{n-1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n} = a_{n+1} + \frac{1}{\frac{q_n}{q_{n-1}}}$$

since it holds for $\frac{q_n}{q_{n-1}} \Rightarrow \frac{q_{n+1}}{q_n} = a_{n+1} + \frac{1}{[a_{n+1}, a_n, \ldots, a_1]}$

$$\Rightarrow \frac{q_{n+1}}{q_n} = [a_{n+1}, a_n, \ldots, a_0]$$ as desired

Stein 5.2 Show that every nonzero rational number can be represented in exactly two ways by a finite simple continued fraction.

Proof. Clearly, any fraction with a denominator greater than two can be rewritten thus: $\frac{a}{b} = \frac{a}{(b-1)+1/1}$. Therefore, any continued fractions can be extended by one term this way. Examples:

$$[2] = 2 + 0 = 1 + 1/1 = [1,1]$$

$$[2,3,4] = 2 + \cfrac{1}{3+\frac{1}{4}} = 2 + \cfrac{1}{3+\cfrac{1}{3+\frac{1}{1}}} = [2,3,3,1]$$

When $a_n = 1$, we can "shrink" the continued fraction using similar logic above, giving us a total of two cases.

$$\therefore [a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, (a_n - 1), 1]$$

Specifically, a continued fraction expansion of a rational number can be written with the last term as one or an integer greater than one. □

5.2 Show that every nonzero rational number can be represented in exactly 2 ways by a finite simple continued fraction.

It is obvious that any fraction can be rewritten.

$\Rightarrow \frac{a}{b} = \frac{a}{(b-1)+\frac{1}{1}}$

∴ allowing us to rewrite any continued fraction this way. . .

$[2] = 2 + 0 = 1 + \frac{1}{1} = [1,1]$ ✓

$[0,2,1] = 0 + \cfrac{1}{2+\frac{1}{1}} = 0 + \frac{1}{3} = [0,3]$ ✓

$\therefore [a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots (a_n - 1), 1]$

Stein 5.4 Determine the infinite continued fraction of $\frac{1+\sqrt{13}}{2}$. Let $x = \frac{1+\sqrt{13}}{2}$. $x \approx 2.30278$.

$$x \approx 2 + 0.30278\cdots \approx 2 + \frac{1}{3.30278\ldots} \approx 2 + \cfrac{1}{3+\frac{1}{3.30278\ldots}}$$

Guess: $x = [2,3,3,3,\ldots]$ Verification: $x = 2 + \frac{1}{1+x}$

$$(x-2)(x+1) - 1 = 0$$

$$x = \frac{\sqrt{13}+1}{2} = [2,3,3,3,\ldots]$$

5.4. $x = \frac{1+\sqrt{13}}{2} \approx 2.302778 \approx 2 + .302778 \approx 2 + \frac{303}{1000}$

$= 2 + \cfrac{1}{3+\frac{91}{303}} = 2 + \cfrac{1}{3+\cfrac{1}{3+\frac{30}{91}}} = 2 + \cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\ldots}}}$

$\frac{1000}{303} = 3 + \frac{91}{303}$

$\frac{303}{91} = 3 + \frac{30}{91}$

$\boxed{\therefore x = \frac{1+\sqrt{13}}{2} = [2,3,3,3,\ldots]}$

Check:

$x = [2,3,3,3,\ldots] = 2 + \frac{1}{x+1} = \frac{2x+2+1}{x+1}$

$x+1 = [3,3,3,\ldots]$

$x(x+1) = 2x+3$

$x^2 + x - 2x - 3 = 0$

$= 0$

$\frac{1 \pm \sqrt{1-4(1)(-3)}}{2(1)} = \frac{1 \pm \sqrt{13}}{2}$ ✓

Appendix D:

Surveys and Interview Protocols

Early survey
(1st – 2nd weeks of the semester)

Learning Mathematics with Peers at College: Survey#1

Thank you for willing to participate in this survey, the purpose of which is to inform a research on math learning in college.

Please answer the following questions to the best of your knowledge.

Please be informative when using a free writing. Remain true to your experience.

Your identity will remain strictly confidential.

1. *Mark only one oval.*

   ◯ Okay

2. **Why did you decide to take a course on number theory?**

   ____

3. **What do you know about number theory?**

   ____

4. **Are there people among your family, friends, acquaintances or peers whom you consider mathematicians or mathematician-like?**
   Mark only one oval.

   ◯ Yes
   ◯ No *Skip to question 7.*

5. **How many people among your family, friends, acquaintances or peers you consider mathematicians or mathematician-like people?**

   ____

6. **Please provide info of each one of them: give the name (you can use pseudonyms), her/his relationship to you, how often you meet with her/him and say briefly why you consider him/her a mathematician or mathematician-like.**

7. **What does a person need to do or have for you to call her/him a mathematician?**

8. **How do you consider yourself in relation to mathematics (e.g., accomplished mathematician, math student, math nerd, in love with numbers, like graphs, master geometry, logic-lover, dislike proofs, curious about math, ...)? Elaborate why you see yourself as such.**

9. **How much do you agree with the following statements about math?**

Mark only one oval per row.

	Strongly Disagree	Disagree	Agree	Strongly Agree
Some people are good at math and others will never be no matter how hard they try.	⬭	⬭	⬭	⬭
There is no creativity in mathematics.	⬭	⬭	⬭	⬭
Mathematics is about proofs and algorithms.	⬭	⬭	⬭	⬭
Anyone can become great at math if they work hard enough.	⬭	⬭	⬭	⬭
Mathematics is all about puzzles to be solved.	⬭	⬭	⬭	⬭
Mathematics is all about logic and nothing about beauty	⬭	⬭	⬭	⬭

10. **How often did you have the chance to work in small groups during high school?**
Mark only one oval.

- Never
- Few group sessions per year
- Few group sessions per month
- Couple group sessions per week
- Every day

11. **How often did you have the chance to work in small groups after high-school?**
Mark only one oval.

- I don't know yet
- Never
- Few group sessions per year
- Few group sessions per month
- Couple group sessions per week
- Every day

12. **What type of peers you found most helpful, productive or engaging to be working with?**

13. **How was your experience with group work overall?**
Please check all statements that apply to your experience.
Check all that apply.

- [ ] I like to argue with groupmates
- [ ] Groupmates with strong characters intimidate me
- [ ] My experiences depended on who is in my group
- [ ] Most of my groupmates happened to be unhelpful
- [ ] Mostly, I had very positive experiences with group work
- [ ] I usually receive helpful support from groupmates
- [ ] I usually have difficulty seeking help from groupmates
- [ ] I tend to help other groupmates
- [ ] Usually I am the one who ends up doing most of the work
- [ ] Working in group is usualy enjoyable
- [ ] It is hard to convince others
- [ ] I hated small group works
- [ ] I dislike arguments erupting through group work

14. **How often the following claims were true for you?**
Mark only only one oval per row.

	Never	Rarely	Frequently	Always
I learned a good deal of math by listening to lectures	⬭	⬭	⬭	⬭
I learned a good deal of math by studying with friends who know math as much as I do	⬭	⬭	⬭	⬭
I learned a good deal of math by discussing math topics with more knowledgeable people	⬭	⬭	⬭	⬭
I learned a good deal of math by working in small groups in classes	⬭	⬭	⬭	⬭
I learned a good deal of math by studying with friends outside class sessions	⬭	⬭	⬭	⬭
I learned a good deal of math by working alone on materials from classes or internet	⬭	⬭	⬭	⬭

Personal Info

15. **Please type your student ID #**

____

16. **Which year in college are you?**
Mark only one oval.

- ⬭ Freshmen
- ⬭ Sophmore
- ⬭ Junior
- ⬭ Senior
- ⬭ Graduate Student

17. **If you wish, provide an average of your grades in all mathematics courses at college.**
Mark only one oval.

- ⬭ A
- ⬭ B
- ⬭ C
- ⬭ D
- ⬭ E
- ⬭ F

18. **If you wish, describe your ethnic background.**

____

____

____

____

____

19. **What is your gender?**
Mark only one oval.

- ( ) Male
- ( ) Female
- ( ) Transgender

Powered by Google Forms

Protocol of Early Interview Int1
(4^{th} – 5^{th} weeks of the semester)

After welcome, interviewer introduces the session:

> Today we will do a different style of interview. If you don't mind, I would like to know about your background and aspirations. You can skip any question you don't want to answer. Is that ok with you?
> [Answer] [Reply accordingly]

The interview will be semi-structured around the lines the following question will open:

About math:

1.Let's do a brainstorming exercise… could you say outloud any words, verbs, adjectives, sentences, images, events that pop up in your mind when you think of mathematics…

2.What do people think about mathematics and mathematicians? For example, what do you hear your friends, siblings, parents and relatives say about mathematics and mathematicians when you hangout and socialize with them?

a. Can you recall moments when you mentioned math in such talk?
b. How do people react when you tell them you are majoring in math?

3.What does your parents, siblings and friends think about your choice of your major?

4.What career are you considering to pursue?

5.In your opinion, how what you are learning in number theory will help you in the future, whether career wise or personal life wise?

About self:

6.Let's go back from the future to here and now. Would you mind telling about your experiences with mathematics, the joys and struggles, what you like or dislike?

a. [probe using brainstorming]
b. Algebra
c. Numbers
d. Proofs
e. Logic
f. theorems
g. Problem solving
h. Math Classes, teachers, students,
i. Homework
j. Tests

7.How do you learn mathematics? What do you do in class and outside class to learn mathematics?

8.What do you do when you get stuck on a mathematical problem or feel like you don’t understand a mathematical concept?

9.How do you usually prepare for a test?

a. What's the difference between studying for the test and for the homework?

Experience with groupwork:

10. Now, we will shift to the last topic. Would you like to tell me about your past experiences in small group? Let's say for the past two - three years.

a. Please describe you best groupwork experience, with whom you were woring? In which class?what happened that made it befit you.
b. Please describe you worst groupwork session. With whom? In which class? What happened?

11. What part of your personality does your group usually most benefit from? How and what do you tend to contribute to the group work?

a. [if need more examples] Which one of the following roles you tend to play when working with others in small groups: listener, explainer, helper, monitor, note-taker, resolve conflicts, coordinator?

12. Do you study math or science with friends outside class?

a. [If yes,] how does your experience in the out-of-class study group compare to the Number Theory group?

13. Can we talk now about your current group work experience in Math 310? Could you please describe what and how each group member participates in the groupwork? How they tend to be? What are their social habits, to the best of your knowledge?

14. Let's imagine a tutoring company is hiring. It asks you to write a report or recommendation letter about your groupmates. What will you write in each letter, knowing that you should be true?

===

15. Does your group work well together? Why or why not?
    a. Is there anything causing you dissatisfaction, annoyance or frustration?

16. In your opinion, how does each member tend to think mathematically? What are their habits of mind?

17. If you were unsure or confused about a mathematical idea, to which member do you turn for discussing the idea? Why?

18. If you had to mimic or take on the qualities your group members have, what would they be?

19. Let's suppose you apply for a job at a company that needs math skills (such as tutoring company, an after-school program, research program or a start-up). Now, the company contacts your group members in Number Theory class asking each one to inform the company about your mathematical and social skills. What do you think each one will write about? Why?

Protocol of Late Interview Int2
(Next to the last week of instruction)

After welcome, interviewer introduces the session:

Today we will do a different style of interview. If you don't mind, I would like to know your impression now that the class is at its end. I would like to ask you about your perception of yourself and the group, with regard mathematics and social dynamics in your group.
You can skip any question you don't want to answer. Is that ok with you?
[Answer] [Reply accordingly]

The interview will be semi-structured around the lines the following question will open:

About math:

1. Let's do a brainstorming exercise… could you say outloud any words, verbs, adjectives, sentences, images, events that pop up in your mind when I say

a. number theory
b. Math 310 with Beck

2. Did you get the chance to talk about number theory outside class, with family, friends or colleagues?

3. Which theorems, definitions or problems you tackled in Math 310 that you liked?
4. Which ones you feel ready to talk about?
5. With which topics did or do you still struggle most?

About self:

6. What were your best ways of learning number theory?

7. What did you do when you get stuck on a homework or problem about number theory?
    a. Did you submit incomplete homework?

8. How did you prepare for the midterm?
9. How are you planning to prepare for your final exam?

10. How do you assess your learning in this class?
    a. How confident are you about what you learned in this class?

11. Did you experience any change in the following?
    a. your ways of learning,
    b. Your ease of understanding the materials,
    c. Your ways of participating in the groupwork
    d. The group dynamics //// what did the use of a white board on the table do to group dynamics?

Experience with groupwork:

12. How best you would describe your experience with your group overall?

13. Please describe your groupmates abilities in number theory. Did these abilities change?

14. Now please describe the roles they tended to take up in the group dynamics. Did these role change?

15. Please describe one or two group sessions that produced a POSITIVE experience for you

16. Please describe one or two group sessions that produced a NEGATIVE experience for you

17. What do you learn about participating in group work? What behaviors will you change or continue to do the next time you take a class based on small group?

Exit Survey
(Last class session 12/10)

Learning Mathematics with Peers at College: Survey#2

Thank you for willing to participate in this survey, the purpose of which is to inform a research on math learning in college.

Please answer the following questions to the best of your knowledge.

Please be informative when using a free writing. Remain true to your experience.

Your identity will remain strictly confidential.

1. *Mark only one oval.*

    - ◯ Okay

Personal Info

2. **Please type your name**

    ____

3. **Year of birth**

    ____

4. **What is your major(s)?**
    Mark only one oval.

    - ◯ Math for teaching
    - ◯ Math for liberal arts
    - ◯ Math for advanced studies
    - ◯ If other, what? ________________________

5. **How much did you learn from the Number Theory course compared to what you usually take from other lecture based courses?**
    Mark only one oval.

    - ◯ Minimal
    - ◯ Average
    - ◯ Well
    - ◯ A lot

6. **Did you use office hours?**
Mark only one oval.

- ⬭ Yes. Total time spent in office hours: ________
- ⬭ No.

7. **If you went to office hours, did you find the office hours helpful? Why so?**

8. **Did you take any tutoring for Number theory course?**
Mark only one oval.

- ⬭ Yes. How many total hours? __________
- ⬭ No.

9. **Complete the following using your rich knowledge. Number theory is about ...**

10. **What does it take to call someone an expert in number theory?**

11. **How do you consider yourself in relation to number theory (e.g., expert, curious student, numbers nerd, dislike thinking about numbers, ...)? Elaborate why you see yourself as such.**

12. **What did you learn from your experience of group work in this class?**

13. **Was the dry-erase poster board useful for you or/and the group? If yes, say in which way this tool was helpful?**

14. **How confident are you about knowing how to study for the final exam?**

15. **Did your ways of working in group change throughout the semester? If yes, please describe what changed and what caused the change.**

16. **How much do you agree with the following statements about number theory?**
Mark only one oval per row.

	Strongly Disagree	Disagree	Agree	Strongly Agree
Some people are good at number theory and others will never be no matter how hard they try.				
There is no creativity in number theory.				
Number theory is about proofs and algorithms.				
Anyone can become great at number theory if they work hard enough.				

17. **How was your overall experience with your groupmates?**
Mark only one oval per row.

	Never	Rarely	Frequently	Always
My groupmates were resourceful and knowledgeable	◯	◯	◯	◯
I had difficulty learning from my groupmates	◯	◯	◯	◯
Some of my groupmates engaged in enjoyable off-topic conversations	◯	◯	◯	◯
Some of my groupmates engaged in distracting off-topic conversations	◯	◯	◯	◯
My groupmates were ahead of me knowing the materials	◯	◯	◯	◯
I contributed to the progress of groupwork	◯	◯	◯	◯
I believed that my groupmates can help me when I'm stuck	◯	◯	◯	◯

18. **How was your experience with your group work overall?**
Please check all statements that apply to your experience.
Check all that apply.

- ☐ I learned how to make best use of groupwork as the semester went on
- ☐ Mostly, I had very positive experience with my group work
- ☐ Often I was the one who ended up doing most of the work
- ☐ I often learned how definitions and theorems work through my group sessions
- ☐ I received helpful support from groupmates
- ☐ I did not engage much in group work because I might be proven wrong
- ☐ I disliked working in my small group
- ☐ I often learned how to solve the homework in my group

(Survey continues)

19. **How often the following claims were true for you?**
Mark only one oval per row.

	Never	Rarely	Frequently	Always
In the first half of the semester, I used to come to class prepared to work on the problems with my group				
In the second half of the semester, I used to come to class prepared to work on the problems with my group				
I learned a good deal of number theory by listening to lectures				
I learned a good deal of number theory by studying with friends who know math as much as I do				
I learned a good deal of number theory by discussing math topics with more knowledgeable people				
I learned a good deal of number theory by working in small groups in classes				
I learned a good deal of number theory by studying with friends outside class sessions				
I learned a good deal of number theory by working alone (with textbooks or online resources)				

20. **How often the following claims were true about your use of the dry-erase poster board?**
Mark only one oval per row.

	Not really	Sometimes	Very often
The dry-erase feature of the poster was helpful			
I used the dry-erase board as scratch paper to try out ideas			
I used the poster to communicate my ideas to other group members			
I understood my groupmates' ideas better when they use the poster board			
I communicated my ideas more efficiently when I wrote on the poster board			
The poster board made more group members engage in the collaborative work			

21. **How confident are you about knowing the following**
Mark only one oval per row.

	I don't know this	I know a little about it	I know it well enough	I know it very well
What is Euler function	◯	◯	◯	◯
What is Euler theorem about congurence	◯	◯	◯	◯
How to use Euler theorem	◯	◯	◯	◯
What is Fermat's little theorem	◯	◯	◯	◯
How to use Fermat's little theorem	◯	◯	◯	◯
What is the chinese remainder theorem	◯	◯	◯	◯
How to use the Chinese remainder theorem	◯	◯	◯	◯
What is the mobius funtion	◯	◯	◯	◯
What is the Mobius inversion formulae	◯	◯	◯	◯
how to use the Mobius inversion formulae	◯	◯	◯	◯
what is a primitive root	◯	◯	◯	◯
how to find the primitive roots of an integer	◯	◯	◯	◯
What is a quadratic residue	◯	◯	◯	◯
How to quickly find out if a number is a quadratic residue modular an odd prime	◯	◯	◯	◯
What is Euler theorem about quadratic residues	◯	◯	◯	◯
How to use Euler's theorem about quadratic residues	◯	◯	◯	◯
What is Legendre symbol	◯	◯	◯	◯
What is Jacobi symbol	◯	◯	◯	◯
How to prove statements about Legendre and Jacobi symbols	◯	◯	◯	◯
How to find the continued fraction expansion of a number?	◯	◯	◯	◯
How to find the number that equates a continued fraction expansion?	◯	◯	◯	◯

22. **Which overall grade do you EXPECT to get on this course (Math 310)?**
Mark only one oval.

◯ A
◯ B
◯ C
◯ D
◯ E
◯ F

23. **Which overall grade you think you DESERVE for this course (Math 310)?**
Check all that apply.

- [ ] A
- [ ] B
- [ ] C
- [ ] D
- [ ] E
- [ ] F

Appendix E:

Worksheets of Number Theory Class

Worksheet 1: Euclidean Algorithm

1. In your group, remind each other about tests for divisibility by 2, 3, and 5. Prove that these tests work.
2. Let $a, b, c \in \mathbb{Z}$ with $c \neq 0$. Prove that if $c \mid a$ and $c \mid b$ then $c \mid (ax + by)$ for any $x, y \in \mathbb{Z}$.
3. The *division algorithm* says that every division problem has a unique quotient and remainder. Come up with a precise mathematical statement for the division algorithm and prove it.
4. Come up with a definition of the *greatest common divisor* of two integers. There are various ways to define the gcd; discuss advantages and disadvantages in your group.
5. Pick two 3-digit positive integers $a > b$ and run the division algorithm when b is divided into a. Run the algorithm again when the remainder is divided into b; repeat until you get remainder 0. What are you computing? Why?
6. Let $a, b \in \mathbb{Z}$, not both zero. Prove that there exist $x, y \in \mathbb{Z}$ such that
$$ax + by = \gcd(a, b)\,.$$
More generally, prove that
$$ax + by = c$$
has a solution $(x, y) \in \mathbb{Z}^2$ if and only if $\gcd(a, b) \mid c$.
7. Andrews 2.3.1.
8. Experiment with the `sage` command `divmod`. Use it with two arguments, say a 6-digit and a 3-digit number, and check that `sage` gives the correct answer.
9. Experiment with the `sage` command `xgcd`. Use it with two 5-digit arguments and check that `sage` gives the correct answer.
10. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 2: Primes

1. Let $a, b \in \mathbb{Z}_{>0}$. Show that, if $g = \gcd(a, b)$ then $\gcd(\frac{a}{g}, \frac{b}{g}) = 1$.
2. Give a careful definition of a *prime number*.
3. Let $a, b, c \in \mathbb{Z}_{>0}$.
   (a) Prove that, if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.
   (b) Conclude that if p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.
   (c) Give a counterexample that shows the previous sentence is wrong if p is not prime.
4. Prove the *Fundamental Theorem of Arithmetic*: for every integer $n \geq 2$ there exist unique primes $p_1, p_2, \ldots, p_k$ and positive integers $a_1, a_2, \ldots, a_k$ such that

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.$$

   (a) For existence, try induction on n.
   (b) For uniqueness, you may use 3(b).
5. Andrews 2.4.5 & 6.
6. Experiment with the `sage` commands `factor` and `is_prime`. Try them with a 100-digit number and a 150-digit number and compare the four running times (e.g., by using `%time` before the command). What's going on here?
7. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 3: Modular Arithmetic

1. Fix a positive integer m, and define the relation $x \sim y$ by $x \equiv y \bmod m$. Prove that $\sim$ is an equivalence relation.

2. Let $a, b, c, m \in \mathbb{Z}$ with $m > 0$.

   (a) Show that, if $\gcd(c, m) = 1$, then
   $$ac \equiv bc \pmod{m} \qquad \text{implies} \qquad a \equiv b \pmod{m}.$$

   (b) Give an example that shows that the gcd condition is necessary.

3. Suppose $a, b, m \in \mathbb{Z}$ with $m > 0$, and let $g := \gcd(a, m)$. Prove:

   (a) If $g \nmid b$ then $ax \equiv b \bmod m$ has no solution $x \in \mathbb{Z}$.
   (b) If $g \mid b$ then $ax \equiv b \bmod m$ has g distinct solutions x modulo m.
   (c) If $g = 1$ then a has a multiplicative inverse modulo m.

4. Suppose $a, m \in \mathbb{Z}$ with $m > 0$ and $\gcd(a, m) = 1$, and let $\{r_1, r_2, \dots, r_{\phi(m)}\}$ be a reduced residue system modulo m.

   (a) Show that $\{ar_1, ar_2, \dots, ar_{\phi(m)}\}$ is also a reduced residue system modulo m.
   (b) Conclude that $r_1 r_2 \cdots r_{\phi(m)} \equiv (ar_1)(ar_2) \cdots (ar_{\phi(m)}) \bmod m$ and, consequently, that
   $$a^{\phi(m)} \equiv 1 \pmod{m}.$$
   (This is *Euler's theorem*.)
   (c) Prove that, if p is prime and $a \in \mathbb{Z}$, then $a^p \equiv a \bmod p$. (This is *Fermat's little theorem*.)
   (d) Conclude that, if p is prime and $a, b \in \mathbb{Z}$, then $(a+b)^p \equiv a^p + b^p \bmod p$. (This is every freshman's dream.)

5. Suppose p is prime. Prove that $x^2 \equiv 1 \bmod p$ has precisely the two solutions $x \equiv \pm 1 \bmod p$.

6. Suppose $m \in \mathbb{Z}_{>0}$.

   (a) Show that, if $m > 4$ is not prime, then $(m-1)! \equiv 0 \bmod m$.
   (b) Now suppose m is prime. Show that if $a \not\equiv 0, \pm 1 \bmod m$ then there exists $b \not\equiv 0, \pm 1, a \bmod m$ such that $ab \equiv 1 \bmod m$.
   (c) Conclude *Wilson's theorem*: $(m-1)! \equiv -1 \bmod m$ if and only if m is prime.

7. Andrews 5.1.1–3, 3.2.3, 5.2.3, and 5.2.19.

8. Experiment with the `sage` command `mod`. Compare the running times of $2^{1000000000000} \bmod 3$ and $(2 \bmod 3)^{1000000000000}$. What do you think `sage` does?

9. Compute $7^{43} \bmod 11$ without `sage`. Check your answer with `sage`.

10. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 4: Chinese Remainder Theorem

1. Let a by the day (of the month) you were born and b the month.[1] Find $x \in \mathbb{Z}$ such that
$$x \equiv a \bmod 31 \qquad \text{and} \qquad x \equiv b \bmod 12\,.$$
2. Suppose $m, n \in \mathbb{Z}_{>0}$ are relatively prime, and $a, b \in \mathbb{Z}$. Prove that
$$x \equiv a \bmod m \qquad \text{and} \qquad x \equiv b \bmod n$$
has a solution $x \in \mathbb{Z}$ and that x is unique modulo mn.
3. Generalize the statement (and your proof) of 2. to a system of k congruences.
4. Andrews 5.3.1. (Feel free to use `sage`.)
5. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 5: Cryptography

1. Compute $2^{222} \bmod 101$.
2. Our goal is to come up with a *code* modulo 101; that is, we want to send a message consisting of 2-digit numbers, and we'd like to encode it in such a way that only our friends can decode it. The first coding scheme we'll describe is due to Diffie and Hellman. It is a *public-key code* because part of the code is known to everyone. Here's how it works: you and your friend choose a prime number p (such as 101) and an integer g between 2 and $p-1$. Both of these numbers are public (so, e.g., you two can safely discuss these numbers on the phone or over the internet—if someone wiretaps you, no problem). Now you *secretly* choose an integer m, and your friend *secretly* chooses an integer n. You compute $g^m \bmod p$ and tell your friend the result. Your friend computes $g^n \bmod p$ and tells you the result. The secret key that you both can use is
$$s \equiv g^{mn} \equiv (g^m)^n \equiv (g^n)^m \bmod p\,.$$
The last two equalities explain why both you and your friend can easily compute s. You can now use s to encode messages, e.g., using multiplication mod p, and s^{-1} to decode. Can you see why it's hard to compute s if you know p, g, g^m, and g^n? How could you make this cryptosystem safer? Do you see a way to "break" it?
3. Our second coding scheme is the *RSA cryptosystem*.[1] Here's how it works: You need two prime numbers p and q, compute their product $m = pq$, find a number b that is relatively prime to $\phi(m) = (p-1)(q-1)$, and compute an inverse c of b modulo $\phi(m)$, i.e., $bc \equiv 1 \bmod \phi(m)$. You keep all of this private except for the numbers m and b which you make public (in particular, your friends know m and b). To send you a message d, your friend encodes it as
$$e = d^b \bmod m\,.$$
You can decode your friend's message by computing
$$d = e^c \bmod m\,.$$
Explain why this decoding works. What makes this cryptosystem safe? How could you make it safer? What would one need to break it?
4. Stein 2.10, 2.30, 3.4, 3.5.
5. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 6: Arithmetic Functions

1. Let p be prime and $k \in \mathbb{Z}_{>0}$. Compute
   (a) $\phi(p^k)$
   (b) $\tau(p^k)$
   (c) $\sigma(p^k)$

2. Prove that τ and σ are *multiplicative*, that is, $\tau(mn) = \tau(m)\,\tau(n)$ and $\sigma(mn) = \sigma(m)\,\sigma(n)$ whenever $\gcd(m,n) = 1$. (*Hint:* start with the case $m = p^j$, $n = q^k$ for distinct primes p and q.)

3. Fix $m, n \in \mathbb{Z}_{>0}$ with $\gcd(m,n) = 1$. Consider the function $f : \mathbb{Z}_{mn}^* \to \mathbb{Z}_m^* \times \mathbb{Z}_n^*$ given by
$$f(k) := (k \bmod m,\ k \bmod n)\,.$$
   (a) Show that f is well defined.
   (b) Show that f is one-to-one.
   (c) Show that f is onto. (*Hint:* Chinese Remainder Theorem.)
   (d) Conclude that $\phi(mn) = \phi(m)\,\phi(n)$.

4. Derive formulas for $\phi(n)$, $\tau(n)$, and $\sigma(n)$ in terms of the prime factorization of n.

5. Fix $n \in \mathbb{Z}_{>0}$, and for $d|n$, let
$$S_d \ := \ \{m \in [n] \ : \ \gcd(m,n) = d\}\,.$$
   (a) Come up with a bijection $S_d \to \mathbb{Z}_{\frac{n}{d}}^*$.
   (b) Convince yourself that
$$[n] \ = \ \bigcup_{d|n} S_d$$
   as a disjoint union, and conclude that
$$\sum_{d|n} \phi(d) \ = \ n\,.$$

6. Andrews 6.1.1, 6.1.4, 6.2.2, 6.2.9.

7. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 7: The Möbius Function

1. Show that $\mu(n)$ is multiplicative.

2. Prove that
$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$
Hint: for $n > 1$, try induction on the number of prime factors of n.

3. Prove the *Möbius Inversion Formula*:
$$f(n) = \sum_{d|n} g(d) \qquad \text{if and only if} \qquad g(n) = \sum_{d|n} \mu(d)\, f(\tfrac{n}{d}).$$
Hint: write sums like the one on the right-hand side as
$$\sum_{d|n} \mu(d)\, f(\tfrac{n}{d}) = \sum_{de=n} \mu(d)\, f(e).$$

4. Andrews 6.4.1, 6.4.3, 6.4.7, 6.4.8, 6.4.11.

5. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 8: Primitive Roots

1. Compute all primitive roots mod 6, 7, and 8.

2. Suppose a has order n mod m, and $a^k \equiv 1$ mod m. Show that $n|k$.

3. Show that, if a is a primitive root mod m, then $\{a, a^2, \ldots, a^{\phi(m)}\}$ is a reduced residue system mod m.

4. Suppose a has order n mod m, and $\gcd(k, n) = g$. Show that a^k has order $\frac{n}{g}$ mod m. Conclude that this implies the following two corollaries:
   (a) If a is a primitive root mod m, then a^k is also a primitive root mod m if and only if $\gcd(k, \phi(m)) = 1$.
   (b) If there exists a primitive root mod m, then there are precisely $\phi(\phi(m))$ primitive roots.

5. Andrews 7.1.6, 7.2.15, Stein 2.8, 2.30.

6. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 9: Primes Again

1. Prove that there are infinitely many primes. One way to do this is by means of contradiction: assuming $p_1, p_2, \ldots, p_k$ are the only primes, consider the number $p_1 p_2 \cdots p_k + 1$.[1]
2. Prove that there are infinitely primes $\equiv 3 \bmod 4$. (*Hint:* assuming $p_1, p_2, \ldots, p_k$ are the only primes $\equiv 3 \bmod 4$, consider the number $4p_1 p_2 \cdots p_k - 1$.) Explain why this is much easier than to prove that there are infinitely primes $\equiv 1 \bmod 4$.[2]
3. Show that if n is composite, then so is $2^n - 1$. Thus the *Mersenne number* $M_p := 2^p - 1$ can only possibly be prime if p is prime. Find the first five Mersenne primes, and the first five composite Mersenne numbers M_p for which p is prime.
4. Prove that $p \in \mathbb{Z}_{>1}$ is prime if and only if $a^{p-1} \equiv 1 \bmod p$ for all $a \not\equiv 0 \bmod p$.[3] Explain how this can be used for a test for *compositeness* of an integer without actually factoring it.
5. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 10: Quadratic Residues

1. Make a list of all quadratic residues mod 2, 3, 5, 7, and 11.
2. In this exercise, we'll prove another one of Euler's theorems: If p is an odd prime, then a is a quadratic residue mod p if and only if $a^{\frac{p-1}{2}} \equiv 1 \bmod p$.

   (a) Prove the "$\Longrightarrow$" direction, e.g., by recalling another theorem by Euler.

   (b) For the "$\Longleftarrow$" direction, you may assume that there exits a primitive root r mod p (which is true, although we haven't prove it). Assuming $a^{\frac{p-1}{2}} \equiv 1 \bmod p$, use the fact that $a \equiv r^n$ for some n, and show that n is even.
3. Use Euler's theorem to prove, given a primitive root r mod p (as above, an odd prime), that g^n is a quadratic residue mod p if and only if n is even. Conclude that, for an odd prime p, exactly half the integers between 1 and $p-1$ are quadratic residues mod p.
4. Let p be and odd prime not dividing a and b. Show that:

   (a) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

   (b) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \bmod p$
5. Andrews 9.2.2.
6. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.

Worksheet 11: Continued Fractions

1. Find continued fraction expansions for

   (a) $\frac{100}{37}$ (c) $\frac{21}{13}$ (e) $\frac{13}{35}$

   (b) $\frac{1001}{45}$ (d) $\frac{1000}{301}$ (f) $\frac{\sqrt{5}-1}{2}$

2. Compute

   (a) $[2,3,2,3,\dots]$ (b) $[1,2,1,2,\dots]$ (c) $[1,2,2,2,\dots]$

3. Compute the start of a continued fraction expansion for π and compare the accuracy of $[a_0,a_1,\dots,a_n]$ with that of the decimal expansion up to n digits, for $n=1,2,3,4$.

4. Compute the continued fraction expansions of e, $\sqrt{19}$, and $\tan(1)$ with `sage`.

5. Prove that $[a_0,a_1,\dots,a_n]=\frac{p_n}{q_n}$ where

$$\begin{array}{llll} p_0=a_0 & p_1=a_1\,p_0+1 & p_n=a_n\,p_{n-1}+p_{n-2} & \text{for } n\geq 2 \\ q_0=1 & q_1=a_1 & q_n=a_n\,q_{n-1}+q_{n-2} & \end{array}$$

6. Keeping the notation from 5., show that

$$p_n\,q_{n-1}-q_n\,p_{n-1} \;=\; (-1)^{n-1}.$$

   (*Hint:* an easy way to proceed is to extend the definition of the p_j's and q_j's by setting $p_{-2}=q_{-1}=0$ and $q_{-2}=p_{-1}=1$.) Conclude that $\gcd(p_n,q_n)=1$, i.e., the fraction $[a_0,a_1,\dots,a_n]=\frac{p_n}{q_n}$ is written in lowest terms.

7. Prove that the sequence $(\frac{p_n}{q_n})_{n\geq 1}$ converges. (*Hint:* first show that $(q_n)_{n\geq 1}$ is strictly increasing, and then prove that $(\frac{p_n}{q_n})_{n\geq 1}$ is a Cauchy sequence.)

8. Show that

$$p_n\,q_{n-2}-q_n\,p_{n-2} \;=\; (-1)^n\,a_n$$

   and conclude that $(\frac{p_{2n}}{q_{2n}})_{n\geq 1}$ increases and $(\frac{p_{2n+1}}{q_{2n+1}})_{n\geq 1}$ decreases.

9. Suppose $a\in\mathbb{R}\setminus\mathbb{Q}$ has an eventually periodic continued fraction expansion, i.e.,

$$a \;=\; [a_0,a_1,\dots,a_n,a_{n+1},\dots,a_{n+k},a_{n+1},\dots,a_{n+k},\dots]$$

   for some positive integers n and k. Prove that $[a_{n+1},\dots,a_{n+k},a_{n+1},\dots,a_{n+k},\dots]$ satisfies a quadratic equation and conclude that $a=b+cx$ where $b,c\in\mathbb{Q}$ and $x\in\mathbb{R}$ satisfies a quadratic equation, i.e., a is a *quadratic irrational*.

10. Stein 5.1–4.

11. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.