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HOLONOMY THEOREM FOR FINITE SEMIGROUPS

JOHN RHODES, ANNE SCHILLING, AND PEDRO V. SILVA

Abstract. We provide a simple proof of the Holonomy Theorem using a new Lyndon–Chiswell length function on the Karnofsky–Rhodes expansion of a semigroup. Unexpectedly, we have both a left and a right action on the Chiswell tree by elliptic maps.

1. Introduction

Given a finite semigroup S generated by the alphabet A, the Holonomy Theorem states that a certain expansion of (S, A) is faithfully represented as elliptic maps on a finite tree (see [Rho91, Theorems 2.2 and 2.12]). Here we provide a new treatment of the Holonomy Theorem. Our point of departure from [Rho91, RS12] is the use of the Karnofsky–Rhodes expansion [MRS11]. We give a new explicit Lyndon–Chiswell length function on the Karnofsky–Rhodes expansion of S. The Holonomy Theorem then follows from the Chiswell construction. Unexpectedly, we have both a left and a right action on the Chiswell tree by elliptic maps.

The mixing time measures how quickly a Markov chain tends to the stationary distribution. In [ASST15a, ASST15b], a technique was developed for an upper bound on the mixing time using a decreasing statistics on the semigroup underlying the Markov chain. We expect that our new Lyndon–Chiswell length function can be used to provide bounds on the mixing time of Markov chains. This avenue of research will be pursued in a subsequent paper.

This paper is organized as follows. In Section 2, we review the Karnofsky–Rhodes expansion of the Cayley graph of a semigroup with a finite set of generators. In Section 3, we introduce the Dedekind height function and our new Lyndon–Chiswell length function. The Lyndon–Chiswell length function is used in the Chiswell construction, which provides a rooted tree associated to the Karnofsky–Rhodes expansion of the semigroup. The Chiswell construction in turn establishes the Holonomy Theorem (see Theorem 3.10). We conclude in Section 4 with several examples.

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2. The Karnofsky–Rhodes expansion

Let A be a finite alphabet and let A^+ (respectively A^*) denote the free semigroup (respectively the free monoid) on A. If $\theta\colon A^+ \to S$ is a semigroup morphism onto a semigroup S, we say that S is *generated by* A. We usually view A as a subset of S. The reference to the morphism is omitted whenever possible and we use the notation (S, A) to describe this situation.

We denote by $S^{\mathbb{1}}$ the monoid obtained by adjoining to S a (new) identity $\mathbb{1}$ (even if S is already a monoid). The *Green's quasi-orders* on S are defined by

- $a \leqslant_{\mathcal{R}} b$ if $a \in bS^{\mathbb{1}}$,
- $a \leqslant_{\mathcal{L}} b$ if $a \in S^{\mathbb{1}}b$,
- $a \leqslant_{\mathcal{J}} b$ if $a \in S^{\mathbb{1}}bS^{\mathbb{1}}$.

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Then $\mathcal{X} = \leqslant_{\mathcal{X}} \cap \geqslant_{\mathcal{X}}$ for $\mathcal{X} = \mathcal{R}, \mathcal{L}, \mathcal{J}$.

We denote by $S^{\rm op}$ the *opposite* semigroup of S, where the binary operation $\cdot$ on S is replaced by the binary operation $x \circ y = y \cdot x$. Note that the $\mathcal{L}$ relation of $S^{\rm op}$ is the $\mathcal{R}$ relation of S, and the $\mathcal{R}$ relation of $S^{\rm op}$ is the $\mathcal{L}$ relation of S.

We define the *left and right Cayley graphs* of (S, A), denoted respectively by $\mathsf{LCay}(S, A)$ and $\mathsf{RCay}(S, A)$, as follows:

- $S^{\mathbb{1}}$ is the vertex set in both graphs,
- the edge set of $\mathsf{LCay}(S, A)$ is $\{(s, a, as) \mid s \in S^{\mathbb{1}},\ a \in A\}$,
- the edge set of $\mathsf{RCay}(S, A)$ is $\{(s, a, sa) \mid s \in S^{\mathbb{1}},\ a \in A\}$.

Note that these graphs are *complete* and *deterministic*: given a vertex s and $u \in A^+$, there exists a unique path with label u starting at s. The following remark, which follows from the definitions, will allow us to use left-right symmetries:

Remark 2.1. $\mathsf{LCay}(S, A) = \mathsf{RCay}(S^{\rm op}, A)$.

An edge (p, q) of a directed graph is called a *transition edge* if there exists no path from q to p. This applies also to A-labeled graphs (in particular to left and right Cayley graphs), where (s, a, s') is a transition edge if there is no path from s' to s. Note that in a Cayley graph, edges of the form $(\mathbb{1}, a, a)$ are always transition edges.

If $s \xrightarrow{a} s'$ is an edge of $\mathsf{RCay}(S, A)$, then $s' = sa$ and so $s' \leqslant_{\mathcal{R}} s$. Hence this edge is a transition edge if and only if $s' <_{\mathcal{R}} s$. Note also that if two transitions edges occur in two different paths, they must occur *in the same order*.

The *right Karnofsky–Rhodes expansion* $\mathsf{KR}_{\rm right}(S, A)$ of (S, A) is defined as the quotient A^+/τ_r, where τ_r is the congruence on A^+ defined as follows: $u\,\tau_r\,v$ if $u = v$ holds in S and the paths $\mathbb{1} \xrightarrow{u} u$ and $\mathbb{1} \xrightarrow{v} v$ in $\mathsf{RCay}(S, A)$ have the same transition edges. Then S is a homomorphic image of $\mathsf{KR}_{\rm right}(S, A)$ in the obvious way.

The *left Karnofsky–Rhodes expansion* of (S, A) can be defined by

$$\mathsf{KR}_{\rm left}(S, A) = \mathsf{KR}_{\rm right}(S^{\rm op}, A).$$

We will be paying particular attention to $\mathsf{KR}^{\mathbb{1}}_{\rm right}(S, A)$, which is obtained by adjoining the (new) identity $\mathbb{1}$ to $\mathsf{KR}_{\rm right}(S, A)$. We can view $\mathsf{KR}^{\mathbb{1}}_{\rm right}(S, A)$ as the quotient $A^*/(\tau_r \cup \{(1, 1)\})$. Similarly, we define $\mathsf{KR}^{\mathbb{1}}_{\rm left}(S, A)$.

3. The Chiswell construction

3.1. **The Dedekind height function.** We shall write $T = \mathsf{KR}_{\rm right}(S, A)$ throughout this section, and let $\varphi \colon T^{\mathbb{1}} \to S^{\mathbb{1}}$ denote the canonical surmorphism.

The *Dedekind height function* $h \colon S^{\mathbb{1}} \to \mathbb{N}$ is defined as

$$h(s) = \max\{k \in \mathbb{N} \mid \text{there exists a chain } s_0 >_{\mathcal{J}} \cdots >_{\mathcal{J}} s_k = s \text{ in } S^{\mathbb{1}}\}.$$

This should be denoted h_S, but the semigroup S is usually understood, as in Proposition 3.4 below.

Finite semigroups are known to be *stable*: they satisfy the equalities

$$\leqslant_{\mathcal{R}} \cap\, \mathcal{J} = \mathcal{R}, \quad \leqslant_{\mathcal{L}} \cap\, \mathcal{J} = \mathcal{L}.$$

The following result will prove useful later.

Lemma 3.1. *If $s <_{\mathcal{K}} s'$ holds in S for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}\}$, then $h(s) > h(s')$.*

Proof. The result is immediate for $\mathcal{J}$. By symmetry, we may assume that $s <_{\mathcal{R}} s'$. It follows that $s \leqslant_{\mathcal{J}} s'$. Now since S is stable we cannot have $s \mathcal{J} s'$, thus $s <_{\mathcal{J}} s'$ and so $h(s) > h(s')$. □

A semigroup S is *regular* if every $s \in S$ is regular. That is, for each $s \in S$ there exists an element $s' \in S$ such that $ss's = s$.

Lemma 3.2. *If $t, t' \in T^{\mathbb{1}}$ satisfy $\varphi(tt't) = \varphi(t)$, then $tt't = t$.*

Proof. Let $u, v \in A^*$ represent t and t', respectively. We have paths $\mathbb{1} \xrightarrow{u} \varphi(t)$ and $\mathbb{1} \xrightarrow{uvu} \varphi(tt't)$ in $\mathsf{RCay}(S, A)$. Since $\varphi(tt't) = \varphi(t)$ and $\mathsf{RCay}(S, A)$ is deterministic, we actually have a loop labeled by vu at $\varphi(t)$. Since a loop cannot contain transition edges, it follows that $uvu \, \tau_r \, u$ and so $tt't = t$. □

Lemma 3.3. *Assume that S is a regular semigroup and let $t, t' \in T^{\mathbb{1}}$.*

(i) *If $\varphi(t) \leqslant_{\mathcal{J}} \varphi(t')$, then $t \leqslant_{\mathcal{J}} t'$.*
(ii) *$\varphi(t) <_{\mathcal{J}} \varphi(t')$ if and only if $t <_{\mathcal{J}} t'$.*

Proof. (i) Since $\varphi(t) \leqslant_{\mathcal{J}} \varphi(t')$, there exist $p, q \in T^{\mathbb{1}}$ such that $\varphi(t) = \varphi(pt'q)$. On the other hand, since S is regular, we have $\varphi(t) = \varphi(tzt)$ for some $z \in T$. Hence

$$\varphi(t) = \varphi(tzt) = \varphi(tztzt) = \varphi(tzpt'qzt)$$

and it follows from Lemma 3.2 that $t = tzpt'qzt$. Therefore $t \leqslant_{\mathcal{J}} t'$.

(ii) Assume that $\varphi(t) <_{\mathcal{J}} \varphi(t')$. By (i), we get $t \leqslant_{\mathcal{J}} t'$. Since $\leqslant_{\mathcal{J}}$ is preserved by homomorphisms, $t \, \mathcal{J} \, t'$ implies $\varphi(t) \, \mathcal{J} \, \varphi(t')$, a contradiction. Thus $t <_{\mathcal{J}} t'$.

Conversely, assume that $t <_{\mathcal{J}} t'$. Hence $\varphi(t) \leqslant_{\mathcal{J}} \varphi(t')$. Since $\varphi(t) \, \mathcal{J} \, \varphi(t')$ implies $t \, \mathcal{J} \, t'$ by (i), we get $\varphi(t) <_{\mathcal{J}} \varphi(t')$. □

Proposition 3.4. *Assume that S is a regular semigroup and let $t \in T^{\mathbb{1}}$. Then $h(t) = h(\varphi(t))$.*

Proof. By Lemma 3.3(ii), we have a chain

$$t_1 >_{\mathcal{J}} \cdots >_{\mathcal{J}} t_k = t$$

in T if and only if we have a chain

$$\varphi(t_1) >_{\mathcal{J}} \cdots >_{\mathcal{J}} \varphi(t_k) = \varphi(t)$$

in S. Thus $h(t) = h(\varphi(t))$. □

Remark 3.5. If (S, A) is not regular, computing $h(t)$ for $t \in T$ can be more challenging sometimes.

3.2. **The Lyndon–Chiswell length function.** Write

$$\ell = 2 \max\{h(s) \mid s \in S^{\mathbb{1}}\}.$$

Denote by $\mathrm{end}(E)$ the endpoint of an edge E of a directed graph.

Let $\alpha, \beta \in T^{\mathbb{1}}$. Let $(E_1, \ldots, E_m)$ and $(E'_1, \ldots, E'_n)$ be the corresponding sequences of transition edges. Since any edge starting at $\mathbb{1}$ is a transition edge, we have $m = 0$ if and only if $\alpha = \mathbb{1}$. Let

$$\xi(\alpha, \beta) = \max\{i \in \{0, \ldots, m\} \mid E_1 = E'_1, \ldots, E_i = E'_i\}.$$

Hence $\xi(\alpha, \beta)$ counts the maximum number of transition edges consecutively shared by α and β, when we start with the first and proceed in order.

Lemma 3.6. *For all $\alpha, \beta, \gamma \in T^{\mathbb{1}}$, we have:*

(i) *$\xi(\alpha, \beta) = \xi(\beta, \alpha)$;*
(ii) *$\xi(\alpha\gamma, \beta\gamma) \geqslant \xi(\alpha, \beta)$;*
(iii) *$\xi(\alpha, \gamma) \geqslant \min(\xi(\alpha, \beta), \xi(\beta, \gamma))$.*

Proof. (i) and (iii) follow from the symmetry and the transitivity of equality.

(ii) follows from the following fact: the sequence of transition edges of $\alpha\gamma$ starts with the sequence of transition edges of α. □

With the notation above, and writing $k = \xi(\alpha, \beta)$, define the *Lyndon–Chiswell length function* $D \colon T^{\mathbb{1}} \times T^{\mathbb{1}} \to \mathbb{N}$ by

$$D(\alpha, \beta) = \begin{cases} \ell & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta \text{ and } k = 0, \\ 2h(\mathrm{end}(E_k)) & \text{if } k > 0 \text{ and } E_{k+1}, E'_{k+1} \text{ both exist and } \mathrm{end}(E_{k+1}) = \mathrm{end}(E'_{k+1}), \\ 2h(\mathrm{end}(E_k)) - 1 & \text{in all remaining cases.} \end{cases}$$

Note that $\mathrm{im}(h) = \{0, 1, \ldots, \frac{\ell}{2}\}$ implies $\mathrm{im}(D) \subseteq \{0, 1, \ldots, \ell\}$. We show now that

$$(\alpha \neq \beta \ \wedge\ \xi(\alpha, \beta) < \xi(\alpha, \gamma)) \ \Rightarrow\ D(\alpha, \beta) < D(\alpha, \gamma) \tag{3.1}$$

holds for all $\alpha, \beta, \gamma \in T^{\mathbb{1}}$.

Assume that $\xi(\alpha, \beta) < \xi(\alpha, \gamma)$. We may assume that $\xi(\alpha, \beta) > 0$, otherwise $D(\alpha, \beta) = 0$. Then $\mathrm{end}(E_{\xi(\alpha,\beta)}) >_{\mathcal{R}} \mathrm{end}(E_{\xi(\alpha,\gamma)})$ because there exists in $\mathsf{RCay}(S, A)$ a path from $\mathrm{end}(E_{\xi(\alpha,\beta)})$ to $\mathrm{end}(E_{\xi(\alpha,\gamma)})$ containing transition edges. By Lemma 3.1, we get $h(\mathrm{end}(E_{\xi(\alpha,\beta)})) < h(\mathrm{end}(E_{\xi(\alpha,\gamma)}))$, yielding $D(\alpha, \beta) < D(\alpha, \gamma)$. Thus (3.1) holds.

The following properties go a little beyond those of [Rho91, Fact 1.9]. We provide a full proof.

Lemma 3.7. *The Lyndon–Chiswell length function satisfies the following properties for all $\alpha, \beta, \gamma \in T^{\mathbb{1}}$:*

(i) $D(\alpha, \beta) = D(\beta, \alpha)$;
(ii) $D(\alpha\gamma, \beta\gamma) \geqslant D(\alpha, \beta)$;
(iii) $D(\gamma\alpha, \gamma\beta) \geqslant D(\alpha, \beta)$;
(iv) ***(isoperimetric inequality)*** $D(\alpha, \gamma) \geqslant \min(D(\alpha, \beta), D(\beta, \gamma))$.

Proof. (i) It follows easily from Lemma 3.6(i).

(ii) We may assume that $\alpha\gamma \neq \beta\gamma$ and $\xi(\alpha, \beta) > 0$. In view of Lemma 3.6(ii), we may also assume that $\xi(\alpha\gamma, \beta\gamma) = \xi(\alpha, \beta) = k$. Hence we may also assume that $\mathrm{end}(E_{k+1}) = \mathrm{end}(E'_{k+1})$ (the case where we do not subtract 1). Since $E_1, \ldots, E_{k+1}$ and $E'_1, \ldots, E'_{k+1}$ are the first $k + 1$ transition edges corresponding to α and β, respectively, we immediately get $D(\alpha\gamma, \beta\gamma) = D(\alpha, \beta)$.

(iii) Similarly to the proof of (ii), we may assume that $\gamma\alpha \neq \gamma\beta$ and $\xi(\alpha, \beta) > 0$. In view of Lemma 3.6(ii), we may also assume that $\xi(\gamma\alpha, \gamma\beta) = \xi(\alpha, \beta) = k$. Hence we may also assume that $\mathrm{end}(E_{k+1}) = \mathrm{end}(E'_{k+1})$ (the case where we do not subtract 1).

Consider words $e_1u_1 \ldots e_mu_m$ and $e'_1u'_1 \ldots e'_nu'_n$ representing α and β respectively, where e_i and e'_j denote the labels of E_i and E'_j. Write $\alpha_i = e_1u_1 \ldots e_iu_i$ and $\beta_j = e'_1u'_1 \ldots e'_ju'_j$ for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. We claim that $\gamma\alpha_k \mathrel{\mathscr{R}} \gamma\beta_k$.

Clearly, the transition edges arising from γ (say $E''_1, \ldots, E''_c$) are the same. So we have to check that the subpaths labeled by α_k and β_k yield the same transition edges, and we refer to the paths labeled by α_k and β_k.

First, we note that only the original transition edges can survive as transition edges when we replace α_k by $\gamma\alpha_k$. Indeed, consider the path

$$\mathbb{1} \xrightarrow{e_1} p_1 \xrightarrow{u_1} q_1 \ldots \xrightarrow{e_k} p_k \xrightarrow{u_k} q_k$$

in $\mathsf{RCay}(S, A)$. For each $1 \leqslant i \leqslant k$, there exists a path $q_i \xrightarrow{v_i} p_i$ (for the absence of transition edges). Hence $\varphi(\alpha_{i-1}e_i) = \varphi(\alpha_{i-1}e_iu_iv_i)$ and so $\varphi(\gamma\alpha_{i-1}e_i) = \varphi(\gamma\alpha_{i-1}e_iu_iv_i)$, preventing any transition edges arising from the u_i.

On the other hand, consider the paths

$$\mathbb{1} \xrightarrow{\gamma} r_0 \xrightarrow{e_1} s_1 \xrightarrow{u_1} r_1 \ldots \xrightarrow{e_k} s_k \xrightarrow{u_k} r_k \tag{3.2}$$

and

$$\mathbb{1} \xrightarrow{\gamma} r'_0 \xrightarrow{e_1} s'_1 \xrightarrow{u'_1} r'_1 \ldots \xrightarrow{e_k} s'_k \xrightarrow{u'_k} r'_k \tag{3.3}$$

in $\mathsf{RCay}(S, A)$. Denote by F_i the edge $r_{i-1} \xrightarrow{e_i} s_i$. The equality $E_i = E'_i$ yields $\varphi(\alpha_{i-1}) = \varphi(\beta_{i-1})$, hence $\varphi(\gamma\alpha_{i-1}) = \varphi(\gamma\beta_{i-1})$ and so F_i must coincide with the edge $r'_{i-1} \xrightarrow{e_i} s'_i$.

It follows from the above remarks that there exists some subsequence $t_1, \ldots, t_w$ of $1, \ldots, k$ such that $\{E''_1, \ldots, E''_c, F_{t_1}, \ldots, F_{t_w}\}$ is the set of transition edges of both paths (3.2) and (3.3). Hence $\gamma\alpha_k \mathrel{\mathscr{R}} \gamma\beta_k$. Note that $\varphi(\alpha_k) \geqslant_{\mathscr{J}} \varphi(\gamma\alpha_k) \mathrel{\mathscr{R}} \varphi(\gamma\alpha_{t_w})$, hence $h(\varphi(\gamma\alpha_{t_w})) \geqslant h(\varphi(\alpha_k))$. Therefore, to prove (iii), we may assume that $\varphi(\alpha_k) \mathrel{\mathscr{J}} \varphi(\gamma\alpha_k)$.

Now

$$\varphi(\alpha_k e_{k+1}) = \mathrm{end}(E_{k+1}) = \mathrm{end}(E'_{k+1}) = \varphi(\beta_k e'_{k+1}). \tag{3.4}$$

We claim that e_{k+1} labels a transition edge in any path in $\mathsf{RCay}(S,A)$ representing $\gamma\alpha$.

Suppose it does not. Then

$$\varphi(\alpha_k) \geqslant_{\mathscr{J}} \varphi(\alpha_k e_{k+1}) \geqslant_{\mathscr{J}} \varphi(\gamma\alpha_k e_{k+1}) \mathrel{\mathscr{R}} \varphi(\gamma\alpha_k) \mathrel{\mathscr{J}} \varphi(\alpha_k),$$

hence $\varphi(\alpha_k) \mathrel{\mathscr{J}} \varphi(\alpha_k e_{k+1})$. Since $\varphi(\alpha_k) \geqslant_{\mathscr{R}} \varphi(\alpha_k e_{k+1})$ and finite semigroups are stable, it follows that $\varphi(\alpha_k) \mathrel{\mathscr{R}} \varphi(\alpha_k e_{k+1})$, contradicting E_{k+1} being a transition edge.

Thus e_{k+1} labels a transition edge in any path in $\mathsf{RCay}(S,A)$ representing $\gamma\alpha$. Similarly, e'_{k+1} labels a transition edge in any path in $\mathsf{RCay}(S,A)$ representing $\gamma\beta$.

If these edges labeled by e_{k+1} and e'_{k+1} coincide, we immediately get $D(\gamma\alpha,\gamma\beta) > D(\alpha,\beta)$. Therefore we may assume they differ, and in that case they have the same endpoint in view of (3.4). Hence $D(\gamma\alpha,\gamma\beta) = D(\alpha,\beta)$ and we are done.

(iv) We may assume that α,β,γ are all different. Let $(E_1,\ldots,E_m)$, $(E'_1,\ldots,E'_n)$ and $(E''_1,\ldots,E''_p)$ be the sequences of transition edges corresponding to α,β,γ, respectively.

In view of Lemma 3.6(i) and (iii), and (3.1), we may assume that

$$0 < k = \xi(\alpha,\gamma) = \xi(\alpha,\beta) \leqslant \xi(\beta,\gamma).$$

It follows that $E_i = E'_i = E''_i$ for $i = 1,\ldots,k$.

Unless $\mathrm{end}(E_{k+1}) = \mathrm{end}(E'_{k+1})$, we obtain $D(\alpha,\beta) = 2h(\mathrm{end}(E_k)) - 1$ and we are done. Hence we may assume that $\mathrm{end}(E_{k+1}) = \mathrm{end}(E'_{k+1})$. Now, unless $\mathrm{end}(E'_{k+1}) = \mathrm{end}(E''_{k+1})$, we get $D(\beta,\gamma) = 2h(\mathrm{end}(E_k)) - 1$ and we are done as well. But then $\mathrm{end}(E_{k+1}) = \mathrm{end}(E''_{k+1})$ and so $D(\alpha,\gamma) = 2h(\mathrm{end}(E_k)) = D(\alpha,\beta)$. Therefore (iv) holds. □

Remark 3.8. In view of these properties, D can indeed be called a length function for (unexpectedly) both a left and right action because of Lemma 3.7 (ii) and (iii).

3.3. **Representations as elliptic maps on a rooted tree.** Let $\Gamma = (V,E)$ be a simple undirected graph. Then Γ is a tree if it is connected and admits no cycles. If we distinguish a vertex $v_0 \in V$, we have the *rooted tree* (Γ, v_0).

Given a rooted tree (Γ,v_0), we obtain a *depth function* $\delta\colon V \to \mathbb{N}$ as follows: $\delta(v)$ is the edge length of the shortest path connecting v to v_0. An *endomorphism* of the rooted tree (Γ,v_0) is a function $\varphi\colon V \to V$ such that:

- $\delta(\varphi(v)) = \delta(v)$ for every $v \in V$;
- if v —— w is an edge of Γ, so is $\varphi(v)$ —— $\varphi(w)$.

Endomorphisms of rooted trees are also known as *elliptic maps*. We denote by $\mathrm{EM}(\Gamma,v_0)$ the monoid of all elliptic maps of (Γ,v_0).

A *representation* of a monoid M as elliptic maps on a rooted tree (Γ,v_0) is a monoid homomorphism $\theta\colon M \to \mathrm{EM}(\Gamma,v_0)$. The representation is *faithful* if φ is one-to-one.

3.4. **The Chiswell construction and the holonomy theorem.** We adapt next the Chiswell construction described in [Rho91, Proof of Theorem 1.12] and [RS12, Proof of Theorem 4.7] (see also [Chi76]).

Let $T = \mathsf{KR}_{\mathrm{right}}(S,A)$ and let $D\colon T^{\mathbb{1}} \times T^{\mathbb{1}} \to \mathbb{N}$ be the Lyndon–Chiswell length function defined in Section 3.2 (with maximum value ℓ). Write

$$C = \{(k,\alpha) \mid 0 \leqslant k \leqslant \ell, \alpha \in T^{\mathbb{1}}\}.$$

We define a relation $\sim$ on C by $(k,\alpha) \sim (k',\beta)$ if:

- $k = k'$;
- $D(\alpha,\beta) \geqslant k$.

It follows from Lemma 3.7(i) and (iv) that $\sim$ is indeed an equivalence relation on C. Note that $(0,\alpha) \sim (0,\beta)$ for all $\alpha, \beta \in T^{\mathbb{1}}$.

Denote by $[k,\alpha]$ the equivalence class of $(k,\alpha) \in C$. Define an undirected graph $\mathcal{C}$ with vertices $[k,\alpha]$ and edges $[k,\alpha]$ — $[k+1,\alpha]$ when $0 \leqslant k < \ell$ and $\alpha \in T^{\mathbb{1}}$. Note that

$$\text{if } [k,\beta] \text{ — } [k+1,\alpha] \text{ is an edge of } \mathcal{C} \text{ then } [k,\beta] = [k,\alpha]. \tag{3.5}$$

Indeed, if there exists such an edge then there exists some $\gamma \in T^{\mathbb{1}}$ such that $[k,\beta] = [k,\gamma]$ and $[k+1,\alpha] = [k+1,\gamma]$. If $\alpha = \gamma$ we are done, otherwise we must have $D(\alpha,\gamma) \geqslant k+1 > k$. Hence $[k,\alpha] = [k,\gamma]$ and (3.5) holds.

With minimal adaptations from [Rho91] and [RS12], we prove the following lemma for the sake of completeness.

Lemma 3.9. $(\mathcal{C}, [0,\mathbb{1}])$ *is a rooted tree.*

Proof. We have a path

$$[0,\mathbb{1}] = [0,\alpha] \text{ — } [1,\alpha] \text{ — } \cdots \text{ — } [k,\alpha]$$

for every vertex $[k,\alpha]$, hence $\mathcal{C}$ is connected.

Suppose that

$$[k_0,\alpha_0] \text{ — } [k_1,\alpha_1] \text{ — } \cdots \text{ — } [k_n,\alpha_n] = [k_0,\alpha_0]$$

is a cycle in $\mathcal{C}$. We may assume that $k_0 \geqslant k_i$ for every $0 \leqslant i \leqslant n$. Then $k_1 = k_{n-1} = k_0 - 1$ and it follows from (3.5) that $[k_1,\alpha_1] = [k_0-1,\alpha_0] = [k_{n-1},\alpha_{n-1}]$, a contradiction. Therefore $\mathcal{C}$ is a tree as required. ☐

This rooted tree is the *Chiswell construction* induced by the Lyndon–Chiswell length function $D\colon T^{\mathbb{1}} \times T^{\mathbb{1}} \to \mathbb{N}$. Note that the depth function is given by $\delta([k,\alpha]) = k$.

Theorem 3.10 (Holonomy Theorem)**.** *Let* (S,A) *be a finite semigroup* S *with generators* A*. Then* $\mathsf{KR}^{\mathbb{1}}_{\text{right}}(S,A)$ *and* $\mathsf{KR}^{\mathbb{1}}_{\text{left}}(S,A)$ *are faithfully represented as elliptic maps on a finite rooted tree.*

Proof. Once again, we adapt the proof from [Rho91, RS12].

Let

$$\begin{array}{rcl} \epsilon\colon \mathsf{KR}^{\mathbb{1}}_{\text{right}}(S,A) & \to & \mathrm{EM}((\mathcal{C},[0,\mathbb{1}])) \\ \alpha & \mapsto & \epsilon_\alpha \end{array}$$

be defined by

$$\epsilon_\alpha([k,\beta]) = [k,\alpha\beta].$$

First, we show that ϵ_α is well-defined. Suppose that $[k,\beta] = [k',\beta']$. Then $k = k'$ and $D(\beta,\beta') \geqslant k$. This implies $D(\alpha\beta,\alpha\beta') \geqslant k$ by Lemma 3.7(iii) and so $[k,\alpha\beta] = [k',\alpha\beta']$. Thus ϵ_α is well-defined.

It is obvious that $\delta(\epsilon_\alpha([k,\beta])) = k = \delta([k,\beta])$. On the other hand, if $[k,\beta]$ — $[k+1,\beta]$ is an edge of $\mathcal{C}$, so is $\epsilon_\alpha([k,\beta]) = [k,\alpha\beta]$ — $[k+1,\alpha\beta] = \epsilon_\alpha([k+1,\beta])$. Therefore ϵ_α is an elliptic map on the finite rooted tree $(\mathcal{C},[0,\mathbb{1}])$ and so ϵ is well-defined.

Given $\alpha,\alpha' \in \mathsf{KR}^{\mathbb{1}}_{\text{right}}(S,A)$, we have

$$\epsilon_{\alpha\alpha'}([k,\beta]) = [k,\alpha\alpha'\beta] = \epsilon_\alpha(\epsilon_{\alpha'}([k,\beta])),$$

hence $\epsilon_{\alpha\alpha'} = \epsilon_\alpha\epsilon_{\alpha'}$. On the other hand, $\epsilon_{\mathbb{1}}([k,\beta]) = [k,\beta]$ and so $\epsilon_{\mathbb{1}}$ is the identity map. Thus ϵ is a monoid homomorphism.

Finally, assume that $\epsilon_\alpha = \epsilon_{\alpha'}$. Then in particular

$$[\ell,\alpha] = \epsilon_\alpha([\ell,\mathbb{1}]) = \epsilon_{\alpha'}([\ell,\mathbb{1}]) = [\ell,\alpha'],$$

hence $D(\alpha,\alpha') \geqslant \ell = \max(\mathrm{im}(D))$.

Suppose that $\alpha \neq \alpha'$. Since

$$D(\alpha,\alpha') = \ell = 2\max\{h(s) \mid s \in S^{\mathbb{1}}\},$$

	$\cdot a$	$\cdot b$
aaa	aaa	aab
aab	aba	$aabb$
aba	baa	bab
baa	aaa	aab
bab	aba	$babb$
$aabb$	$abba$	$abbb$
$abba$	baa	bab
$abbb$	$bbba$	$bbbb$
$babb$	$abba$	$abbb$
$bbba$	baa	bab
$bbbb$	$bbba$	$bbbb$

TABLE 1. The right action semaphore code in $\{a,b\}^4$ associated to the ideal generated by aaa, aab, aba, baa, bab.

we are not subtracting 1, which implies that α possesses transition edges beyond $E_{\xi(\alpha,\beta)}$. This contradicts the fact that h should reach its maximum value at $\text{end}(E_{\xi(\alpha,\beta)})$). Thus $\alpha = \alpha'$ and so ϵ is one-to-one. Therefore the representation is faithful.

Recall now that $\mathsf{KR}_{\text{left}}(S,A) = \mathsf{KR}_{\text{right}}(S^{\text{op}},A)$. It follows from the first part that $\mathsf{KR}^{\mathbb{1}}_{\text{left}}(S,A)$ is also faithfully represented as elliptic maps on a finite rooted tree. ☐

Remark 3.11. When we consider the right action of $\mathsf{KR}_{\text{right}}(S,A)$ on the Chiswell tree $(\mathcal{C},[0,\mathbb{1}])$ given by

$$[k,\beta]\alpha = [k,\beta\alpha],$$

we obtain an injective monoid homomorphism

$$\epsilon' \colon \mathsf{KR}^{\mathbb{1}}_{\text{right}}(S,A) \to (\text{EM}((\mathcal{C},[0,\mathbb{1}])))^{\text{op}},$$

since here the elliptic mappings must compose from left to right.

Remark 3.12. Note that the Lyndon-Chiswell length function on $\mathsf{KR}^{\mathbb{1}}_{\text{right}}(S^{\text{op}},A) \times \mathsf{KR}^{\mathbb{1}}_{\text{right}}(S^{\text{op}},A)$ would be the version of the Lyndon-Chiswell length function built from S when we replace its right Cayley graph by its left Cayley graph. And Lemma 3.7(ii) ensures that left-right symmetry is preserved at all levels of the proofs, so we could replicate all the preceding proofs using $\mathsf{LCay}(S,A)$ and $\mathsf{KR}^{\mathbb{1}}_{\text{left}}(S,A)$.

In the next paper we will expand this theory and apply it to mixing times.

4. EXAMPLES

4.1. **Right action on semaphore codes.** Let A be a finite alphabet and denote by $\text{RC}(A^k)$ the set of right congruences on A^k. As shown in [RSS16a, RSS16b], every right congruence can be approximated by a special right congruence and special right congruences are in bijection with semaphore codes. A *semaphore code* [BPR10] is a suffix code $\mathcal{S}$ over A (i.e., all elements in the code are incomparable in suffix order) for which there is a right action in the following sense: If $u \in \mathcal{S}$ and $a \in A$, then ua has a suffix in $\mathcal{S}$. The right action $u.a$ is the suffix of ua that is in $\mathcal{S}$.

If (S,A) is the finite semigroup associated to a right congruence, then the special right congruence corresponds to $\mathsf{KR}_{\text{left}}(S,A)$ as shown in [RS19].

Example 4.1. Let $A = \{a,b\}$ be a two letter alphabet and I the ideal in $A^\star$ generated by aaa, aab, aba, baa, bab. Then the right action suffix semaphore code in A^4 is given in Table 1. Hence S has 11 elements. To compute $\mathsf{KR}_{\text{left}}(S,A)$, we compute the action of the various subwords of the elements in S on S and record the images, see Figure 1. The elliptic right action of a on the Chiswell tree is given in Figure 2, whereas the elliptic right action of b is given in Figure 3.

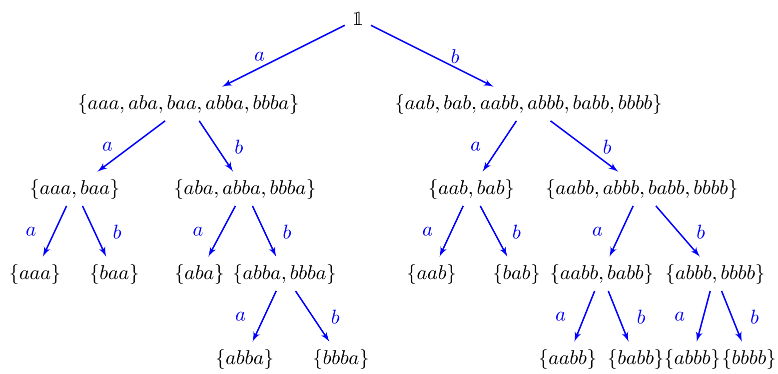

Figure 1. The Chiswell tree $\mathsf{KR}_{\text{left}}(S, A)$ of Example 4.1.

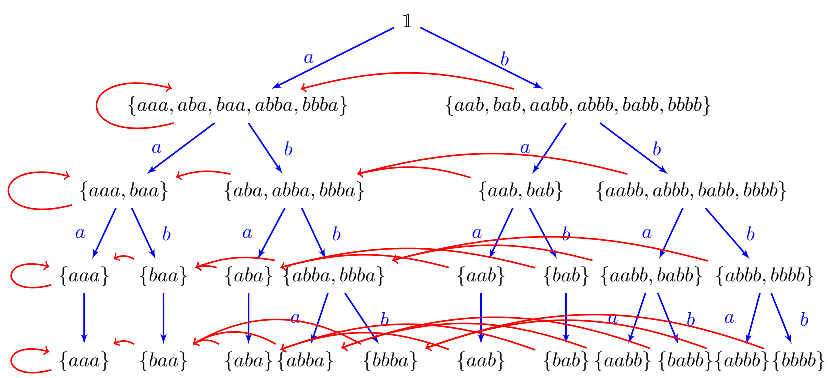

Figure 2. The action of $\cdot a$ on the semaphore code induces the action level-by-level on the the Chiswell tree of Example 4.1 (in red).

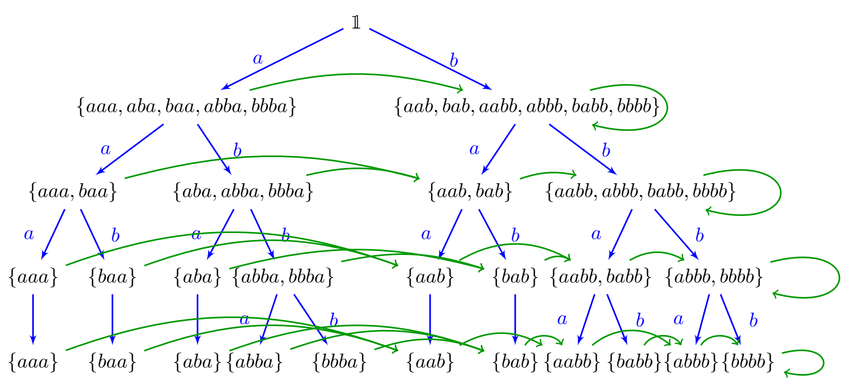

Figure 3. The action of $\cdot b$ on the semaphore code induces the action level-by-level on the the Chiswell tree of Example 4.1 (in green).

4.2. **Left zero semigroup with two generators.** Let (S, A) be the left zero semigroup $\mathsf{LZ}(2)$ (that is $xy = x$ for all $x, y \in \mathsf{LZ}(2)$) with two generators $A = \{a, b\}$. The Karnofsky–Rhodes expansion of the left Cayley graph of (S, A) is depicted in Figure 4. Then the Chiswell construction is given in Figure 5. The left and right actions of a and b on the Chiswell construction are given in Figures 6, 7, 8 and 9, respectively.

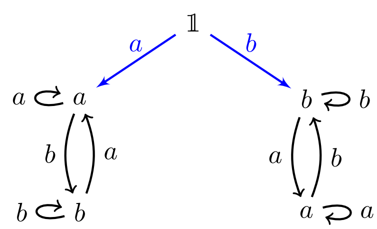

FIGURE 4. $\mathsf{KR}_{\text{left}}(\mathsf{LZ}(2), \{a, b\})$.

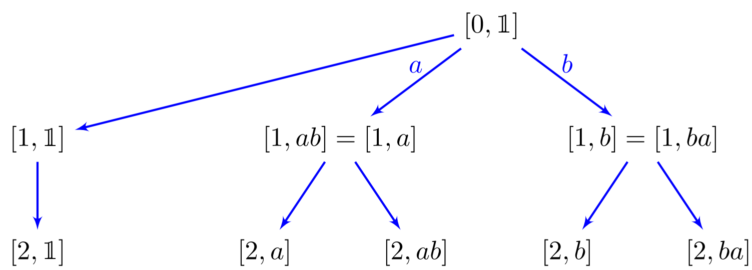

FIGURE 5. The Chiswell construction for $\mathsf{KR}_{\text{left}}(\mathsf{LZ}(2), \{a, b\})$.

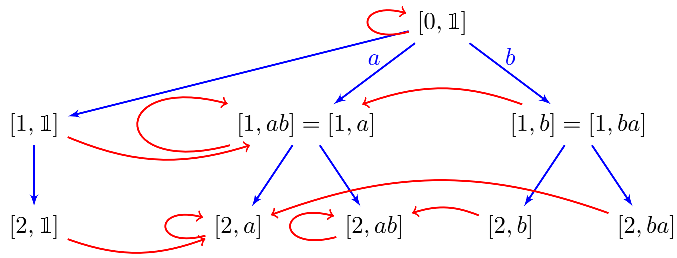

FIGURE 6. The (left) action of $a\cdot$ on the Chiswell tree for $\mathsf{KR}_{\text{left}}(\mathsf{LZ}(2), \{a, b\})$.

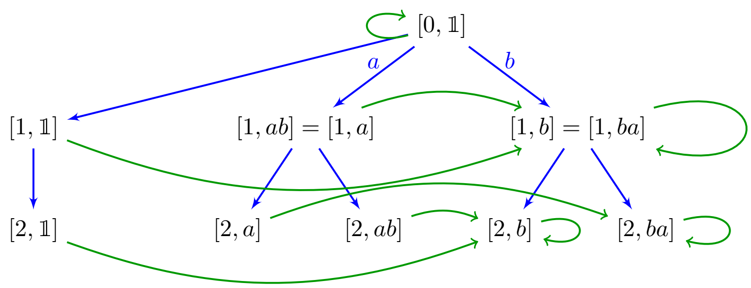

FIGURE 7. The (left) action of $b\cdot$ on the Chiswell tree for $\mathsf{KR}_{\text{left}}(\mathsf{LZ}(2), \{a, b\})$.

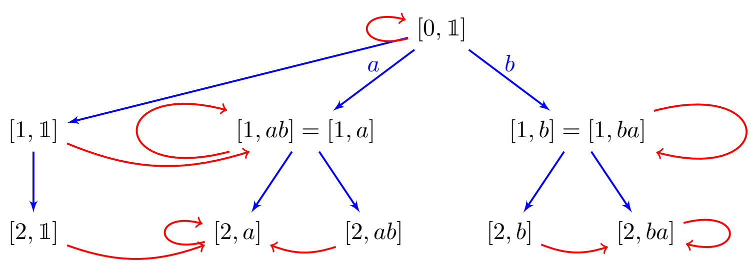

FIGURE 8. The (right) action of $\cdot a$ on the Chiswell tree for $\mathsf{KR}_{\text{left}}(\mathsf{LZ}(2), \{a, b\})$.

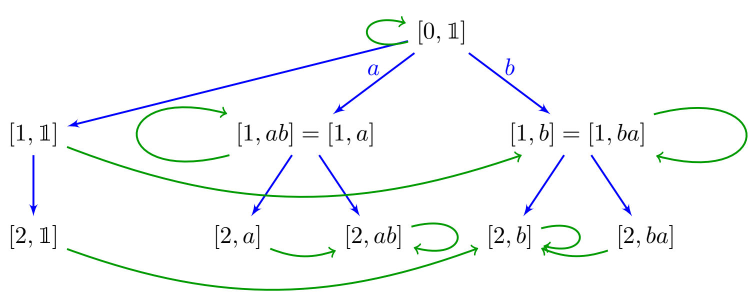

FIGURE 9. The (right) action of $\cdot b$ on the Chiswell tree for $\mathsf{KR}_{\text{left}}(\mathsf{LZ}(2), \{a, b\})$.

4.3. The monoid T_2 of total transformations. Let T_2 denote the monoid of total transformations on the set $\{1, 2\}$. We denote $\varphi \in T_2$ by $(1\varphi\ 2\varphi)$ (so in $\varphi\psi$ the map φ acts first). Let $A = \{a, b\}$ and let $\varphi\colon A^* \to S^{\mathbb{1}}$ be the monoid homomorphism defined by $\varphi(a) = (2\ 1)$ and $\varphi(b) = (1\ 1)$. It is routine to check that φ is onto and $\mathsf{LCay}(T_2, A) = \mathsf{RCay}(T_2^{op}, A)$ is depicted in Figure 10. The Karnofsky–Rhodes expansion is given in Figure 11, the Chiswell construction is drawn in Figure 12, and the left action of a on the Chiswell tree is depicted in Figure 13.

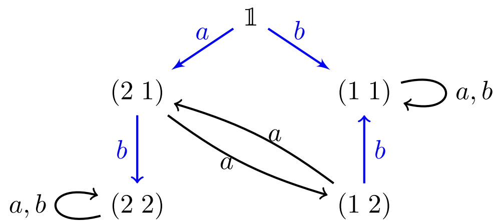

FIGURE 10. $\mathsf{LCay}(T_2, A)$ with the transition edges in blue.

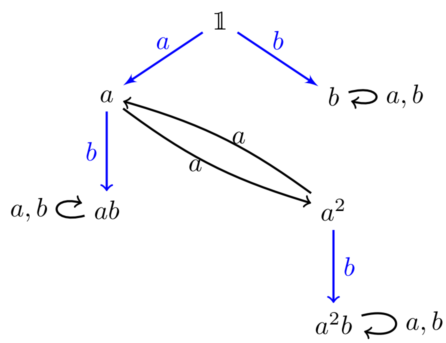

FIGURE 11. $\mathsf{KR}_{\text{left}}(T_2, A)$.

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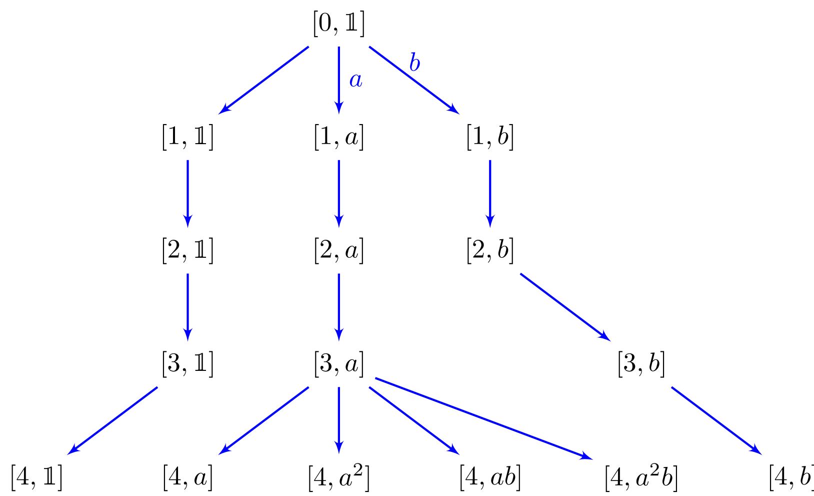

FIGURE 12. The Chiswell construction for $\mathsf{KR}_{\text{left}}(T_2, A)$.

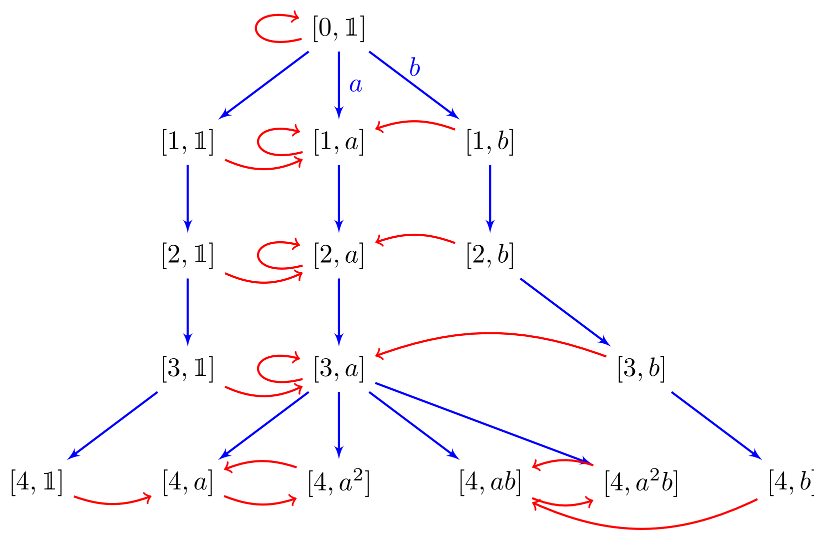

FIGURE 13. The (left) action of $a\cdot$ on the Chiswell tree for $\mathsf{KR}_{\text{left}}(T_2, A)$.

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