Title
Nonlinear wave equations

Journal
Proceedings of the ICM, 3(2)

Author
Tataru, Daniel

Publication Date
2003-04-24

Peer reviewed

ICM 2002 · Vol. III · 1–3

Nonlinear Wave Equations

Daniel Tataru*

Abstract

The analysis of nonlinear wave equations has experienced a dramatic growth in the last ten years or so. The key factor in this has been the transition from linear analysis, first to the study of bilinear and multilinear wave interactions, useful in the analysis of semilinear equations, and next to the study of nonlinear wave interactions, arising in fully nonlinear equations. The dispersion phenomena plays a crucial role in these problems. The purpose of this article is to highlight a few recent ideas and results, as well as to present some open problems and possible future directions in this field.

2000 Mathematics Subject Classification: 35L15, 35L70.
Keywords and Phrases: Wave equations, Phase space, Dispersive estimates.

1. Introduction

Consider the constant and variable coefficient wave operators in $\mathbb{R} \times \mathbb{R}^n$,

$$\Box = \partial_t^2 - \Delta_x, \qquad \Box_g = g^{ij}(t,x)\partial_i\partial_j.$$

In the variable coefficient case the summation occurs from 0 to n where the index 0 stands for the time variable. To insure that the equation is hyperbolic in time we assume that the matrix g^{ij} has signature $(1,n)$ and that the time level sets $t = const$ are space-like, i.e. $g^{00} > 0$. We consider semilinear wave equations,

$$\Box u = N(u) \qquad (SLW), \qquad \Box u = N(u, \nabla u) \qquad (GSLW)$$

and quasilinear wave equations,

$$\Box_{g(u)} u = N(u)(\nabla u)^2 \qquad (NLW), \qquad \Box_{g(u,\nabla u)} u = N(u, \nabla u) \qquad (GNLW).$$

To each of these equations we associate initial data in Sobolev spaces

$$u(0) = u_0 \in H^s(\mathbb{R}^n), \qquad \partial_t u(0) = u_1 \in H^{s-1}(\mathbb{R}^n).$$

There are two natural questions to ask: (i) Are the equations locally well-posed in $H^s \times H^{s-1}$? (ii) Are the solutions global, or is there blow-up in finite time?

*Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA. E-mail: tataru@math.berkeley.edu

Local well-posedness. In a first approximation we define it as follows:

Definition 1. *A nonlinear wave equation is well-posed in* $H^s \times H^{s-1}$ *if for each* $(v_0, v_1) \in H^s \times H^{s-1}$ *there is* $T > 0$ *and a neighborhood* V *of* (v_0, v_1) *in* $H^s \times H^{s-1}$ *so that for each initial data* $(u_0, u_1) \in V$ *there is an unique solution* $u \in C(-T, T; H^s)$, $\partial_t u \in C(-T, T; H^{s-1})$ *which depends continuously on the initial data.*

In practice in order to prove uniqueness one often has to further restrict the class of admissible solutions. In most problems, the bound T from below for the life-span of the solutions can be chosen to depend only on the size of the data.

It is not very difficult to prove that all of the above problems are locally well-posed in $H^s \times H^{s-1}$ for large s. The interesting question is what happens when s is small. One indication in this regard is given by scaling. At least in the case when the nonlinear term has some homogeneity, for instance $N(u) = u^p$ or $N(u) = u^p(\nabla u)^q$, one looks for an index α so that all transformations of the form $u(x,t) \to \lambda^\alpha u(\lambda x, \lambda t)$, $\lambda > 0$ leave the equation unchanged. Correspondingly one finds an index $s_0 = \frac{n}{2} - \alpha$ so that the norm of the initial data (u_0, u_1) in the homogeneous Sobolev spaces $\dot{H}^s \times \dot{H}^{s-1}$ is preserved by the above transformations.

Below scaling ($s < s_0$) a small data small time result rescales into a large data large time result. Heuristically one concludes that local well-posedness should not hold. Still, to the author's knowledge there is no proof of this yet.

Conjecture 2. *Semilinear wave equations are ill-posed below scaling.*

This becomes much easier to prove if one strengthens the definition of well-posedness, e.g. by asking for uniformly continuous or C^1 dependence of the solution on the initial data.

If $s = s_0$ then for small initial data local well-posedness is equivalent to global well-posedness. The same would happen for large data if we were to strengthen the definition of well-posedness and ask for a lifespan bound which depends only on the size of the data. This is the only case where this distinction makes a difference.

If $s > s_0$ then a local well-posedness result gives bounds for life-span T_{max} of the solutions in terms of the size of the data,

$$\|(u_0, u_1)\|_{H^s \times H^{s-1}} \leq M \implies T_{max} \gtrsim M^{s_0 - s}.$$

The better localization in time makes the problems somewhat easier to study. However, besides scaling there are also other obstructions to well-posedness. These are related to various concentration phenomena which can occur depending on the precise structure of the equation.

Global well-posedness. We briefly mention that there is a special case in which the global well-posedness is well understood, namely when the initial data is small, smooth, and decays at infinity. This is not discussed at all in what follows.

Consider first the case when s is above scaling, $s > s_0$, and local well-posedness holds in $H^s \times H^{s-1}$. Then any solution can be continued as long as its size does not blow-up. Hence the goal of any global argument should be to establish a-priori bounds on the $H^s \times H^{s-1}$ norm of the solution. All known results of this type are for problems for which there are either conserved or quasi-conserved positive definite quantities. Such conserved quantities can often be found for equations which are physically motivated or which have some variational structure. For simplicity suppose that there is some index s_c and an energy functional E in $H^{s_c} \times H^{s_c-1}$ which is preserved along the flow. The index s_c needs not be equal to the scaling index s_0. There are three cases to consider:

(i) The subcritical case $s_c > s_0$. Then a local well-posedness result at $s = s_c$ implies the global result for $s \geq s_c$. Furthermore, in recent years there has been considerable interest in establishing global well-posedness also for $s_0 < s < s_c$. This is based on an idea first introduced by Bourgain [5] in a related problem for the Schröedinger equation, and followed up by a number of authors.

(ii) The critical case $s_c = s_0$. Here the energy is not needed for small data, when local and global well-posedness are equivalent. For large data, however, the energy conservation is not sufficient in order to establish the existence of global solutions. In addition, one needs a non-concentration argument, which should say that the energy cannot concentrate inside a characteristic cone.

(iii) The supercritical case, $s_c < s_0$. No global results are known:

Open Problem 3. Are supercritical problems globally well-posed for $s \geq s_0$?

A simple example is the equation (NLW) with $N(u) = |u|^{p-1}u$. The energy is

$$E(u) = \int |u_t|^2 + |\nabla_x u|^2 + \frac{1}{p+1}|u|^{p+1}.$$

Then $s_c = 1$, while $s_0 = \frac{n}{2} - \frac{2}{p-1}$. In $3+1$ dimensions, for instance, $p = 3$ is subcritical, therefore one has global well-posedness in $H^1 \times L^2$. The exponent $p = 5$ is critical and in this case the problem is known to be globally well-posed in $H^1 \times L^2$; the non-concentration argument is due to Grillakis [7]. The exponent $p = 7$ is supercritical.

Blow-up. Not all nonlinear wave equations are expected to have global solutions. Quite the contrary, generic equations are expected to blow up in finite time; only for problems with some special structure it seems plausible that global well-posedness may hold. A simple way to produce blow-up is to look for self-similar solutions, $u(x,t) = t^\gamma u(\frac{x}{t})$. If they exist, self-similar solutions disprove global well-posedness. Because they must respect the scaling of the problem, they are not so useful when trying to disprove local well-posedness.

Another way to produce blow-up solutions is the so-called ode blow-up. In the simplest setting this means looking at one dimensional solutions (say $u(x,t) = u(t)$) which solve an ode and blow up in finite time. Then one can truncate the initial data spatially and still retain the blow-up because of the finite speed of propagation. This is still not very useful for the local problem.

A better idea is to constructs blow-up solutions which are concentrated essentially along a light ray, see Lindblad [10],[11] and Alihnac [1]. In this setup the actual blow-up occurs either because of the increase in the amplitude, in the semilinear case, or because of the focusing of the light rays, in the quasilinear case. As it turns out, the counterexamples of this type are often sharp for the local well-posedness problem.

2. Semilinear wave equations

Usually, a fixed point argument is used to obtain local results for semilinear equations. We first explain this for the case when $s = s_0$. We define the homogeneous and inhomogeneous solution operators, S and $\Box^{-1}$ by

$$S(u_0, u_1) = u \Longleftrightarrow \{\Box u = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1\},$$

$$\Box^{-1} f = u \Longleftrightarrow \{\Box u = f, \quad u(0) = 0, \quad \partial_t u(0) = 0\}.$$

Then the equation (NLW) for instance can be recast as

$$u = S(u_0, u_1) + \Box^{-1} N(u).$$

To solve this using a fixed point argument one needs two Banach spaces X and Y with the correct scaling and the following mapping properties:

$$S : H^s \times H^{s-1} \to X, \quad \Box^{-1} : Y \to X, \quad N : X \to Y.$$

The first two are linear, but the last one is nonlinear. The small Lipschitz constant is always easy to obtain provided the initial data is small and that N decays faster than linear at 0. The solutions given by the fixed point argument are global.

In the case $s > s_0$ the scaling is lost, and with this method one can only hope to get results which are local in time. To localize in time one chooses a smooth compactly supported cutoff function χ which equals 1 near the origin. The fixed point argument is now used for the equation

$$u = \chi S(u_0, u_1) + \chi \Box^{-1} N(u).$$

A solution to this solves the original equation only in an interval near the origin where $\chi = 1$. The modified mapping properties are

$$\chi S : H^s \times H^{s-1} \to X, \quad \chi \Box^{-1} : Y \to X, \quad N : X \to Y.$$

How does one choose the spaces X, Y? One approach is to use the energy estimates for the wave equation and set

$$X = \{u \in L^\infty(H^s), \nabla u \in L^\infty(H^{s-1})\}, \quad Y = L^1(H^{s-1}).$$

The first two mapping properties are trivial. However, if the third holds then we must also have $N : X \to L^\infty(H^{s-1})$. The one unit difference in scaling between L^1 and L^∞ implies that this can only work for $s \geq s_0 + 1$.

What is neglected in the above setup is the dispersive properties of the wave equation. Solutions to the linear wave equation cannot stay concentrated for long time intervals. Instead, they will disperse and decay in time (even though the energy is preserved). In harmonic analysis terms, this is related to the restriction theorem (see [17]) and is a consequence of the nonvanishing curvature of the characteristic set for the wave operator, namely the cone $\xi_0^2 = \xi_1^2 + \cdots + \xi_n^2$. Here ξ stands for the Fourier variable. One way of quantifying the dispersive effects is through the Strichartz estimates. They apply both to the homogeneous and the inhomogeneous equation (see [8] and references therein):

$$S : H^\rho \times H^{\rho-1} \to L^p L^q, \qquad |D|^{1-\rho_1-\rho}\Box^{-1} : L^{p_1'} L^{q_1'} \to L^p L^q$$

where (ρ, p, q) and (ρ_1, p_1, q_1) are subject to

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \rho, \quad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}, \quad 2 \leq p, q \leq \infty, \quad (\rho, p, q) \neq (1, 2, \infty).$$

The worst case in these estimates occurs for certain highly localized approximate solutions to the wave equation, which are called wave packets. A frequency λ wave packet on the unit time scale is essentially a bump function in a parallelepiped of size $1 \times \lambda^{-1} \times (\lambda^{-\frac{1}{2}})^{n-1}$ which is obtained from a $\lambda^{-1} \times (\lambda^{-\frac{1}{2}})^{n-1}$ parallelepiped at time zero which travels with speed 1 in the normal direction. Because of the uncertainty principle, this is the best possible spatial localization which remains coherent up to time 1. Of course one can rescale and produce wave packets on all time scales.

In low dimension $n = 2, 3$ the Strichartz estimates provide a complete set of results for generic equations of both (NLW) and (GNLW) type. Consider the following two examples, of which the second is wrong but almost right:

$$\Box u = u^3, \quad n = 3, \quad s = s_0 = \frac{1}{2}, \quad X = L^4, \quad Y = L^{\frac{4}{3}},$$

$$\Box u = u\nabla u, \quad n = 3, \quad s_0 = \frac{1}{2}, \quad s = 1 \quad X = |D|^{-1} L^\infty L^2 \cap L^2 L^\infty \quad Y = L^2.$$

For $n \geq 4$, however, the Strichartz estimates no longer provide all the results. The reason is as follows. The worst nonlinear interaction in both (NLW) and (GNLW) occurs for wave packets which travel in the same direction. One can use the Strichartz estimates to accurately describe the interaction of same frequency wave packets. But in the interaction of two wave packets at different frequencies, the low frequency packet is more spread, and only a small portion of it will interact with the high frequency packet. However, unlike in low dimension, the Strichartz estimates do not provide sharp bounds for this smaller part of a wave packet.

A more robust idea due to Bourgain [4] and Klainerman-Machedon [12] is to use the $X^{s,b}$ spaces associated to the wave equation very much in the same way the Sobolev spaces are associated to the Laplacian:

$$\|u\|_{X^{s,b}} = \|(1 + |\xi|)^s (1 + ||\xi_0| - |\xi'||)^b \hat{u}\|_{L^2}.$$

Then one chooses $X = X^{s,\frac{1}{2}}$ and $Y = X^{s-1,-\frac{1}{2}}$. The Strichartz information is not lost since for ρ, p, q as above we have the dual embeddings

$$X^{\rho,\frac{1}{2}+} \subset L^p L^q, \qquad L^{p'} L^{q'} \subset X^{-\rho,-\frac{1}{2}-}.$$

Within the framework of the $X^{s,b}$ spaces one can prove bilinear estimates which provide a better description of the interaction of high and low frequencies, see [6] and references therein. The bilinear estimates are obtained as weighted convolution estimates in the Fourier space, by using the above embeddings, or by combining the two methods. Sometimes even this setup does not suffice and has to be modified further, see [22].

Conjecture 4. *The equation $\Box u = u^p$ is locally well-posed in $H^s \times H^{s-1}$ for $n \geq 4$, $0 \leq s \leq \frac{1}{2}$, $p(\frac{n+1}{4} - s) \leq (\frac{n+5}{4} - s)$.* (see [19] for more details)

The null condition. A natural question to ask is whether there are equations which behave better than generic ones. This may happen if the worst interaction (between parallel wave packets) does not occur in the nonlinearity. A good example is (GNLW) with a quadratic nonlinearity $Q(\nabla u, \nabla u) = q^{ij}\partial_i u\, \partial_j u$. The cancellation condition, called null condition, asserts that

$$q^{ij}\xi_i\xi_j = 0 \qquad \text{in the characteristic set } g^{ij}\xi_i\xi_j = 0.$$

All such null forms are linear combinations of

$$Q_{ij}(\nabla u, \nabla v) = \partial_i u \partial_j v - \partial_i v \partial_j u, \qquad Q_0(u,v) = g^{ij}\partial_i u \partial_j v.$$

Open Problem 5. Study semilinear wave equations corresponding to variable coefficient wave operators for $n \geq 4$ (generic case) or $n \geq 2$ (with null condition).

In the constant coefficient case one can easily use the null condition in the context of the $X^{s,b}$ spaces. This is done using inequalities of the following form:

$$|q_0(\xi,\eta)| \leq c(|p(\xi)| + |p(\eta)| + |p(\xi+\eta)|)$$

respectively

$$|q_{ij}(\xi,\eta)| \leq c|\xi|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}|\xi+\eta|^{\frac{1}{2}}(|p(\xi)|^{\frac{1}{2}} + |p(\eta)|^{\frac{1}{2}} + |p(\xi+\eta)|^{\frac{1}{2}})$$

where by $p(\xi)$ we denote the symbol of the constant coefficient wave operator, given by $p(\xi) = \xi_0^2 - \xi_1^2 - \cdots - \xi_n^2$. Combining this with the embeddings above one can lower the s in the local theory whenever the null condition is satisfied. Unfortunately, this does not always give optimal results. The problem of obtaining improved L^pL^q estimates for null forms has also been explored, see [28][20] [26], but without immediate applications to semilinear wave equations. We limit the following discussion to two of the more interesting models.

Wave maps. These are functions from $\mathbb{R}^n \times \mathbb{R}$ into a complete Riemannian manifold (M, g) which are critical points for

$$I(\phi) = \int_{\mathbb{R}^n \times \mathbb{R}} |\partial_t \phi|_g^2 - |\nabla_x \phi|_g^2 dx\ dt.$$

In local coordinates the equation for wave maps has the form

$$\Box\phi^k = \Gamma^k_{ij}(\phi)Q_0(\phi^i\phi^j)$$

where Γ^k_{ij} are the Riemann-Christoffel symbols. The energy functional is

$$E(u) = \int_{\mathbb{R}^n} |\partial_t\phi|^2_g + |\nabla_x\phi|^2_g dx.$$

The scaling index is $s_0 = \frac{n}{2}$ and $s_c = 1$. Local well-posedness for $s > s_c$ can be obtained using the $X^{s,b}$ spaces. For $s = s_c$, using some modified $X^{s,b}$ spaces, local (and therefore small data global) well-posedness was established first in homogeneous Besov spaces $B^{\frac{n}{2}}_{2,1} \times B^{\frac{n}{2}-1}_{2,1}$ in Tataru [25] and then in Sobolev spaces by Tao [21] (for the sphere, $n \geq 2$) and other authors (general target manifold, $n \geq 3$). Large data global well-posedness is false in the supercritical case $n \geq 3$, where self-similar blowup can occur. This leaves open problems in the critical case $n = 2$:

Conjecture 6. *(i) The two dimensional wave maps equation is globally well-posed for small data in $H^1 \times L^2$ for any complete target manifold.*

(ii) The two dimensional wave maps equation is globally well-posed for large data in $H^1 \times L^2$ for "good" target manifolds.

The Yang Mills equations. Given a compact Lie group $\mathcal{G}$ whose Lie algebra $\mathbf{g}$ admits an invariant inner product $\langle\cdot,\cdot\rangle$ one considers $\mathbf{g}$ valued connection 1-forms $A_j dx^j$ in $\mathbb{R}^n \times \mathbb{R}$. The covariant derivatives of $\mathbf{g}$ valued functions are defined by

$$D_j B = \partial_j B + [A_j, B].$$

The ($\mathbf{g}$ valued) curvature of the connection A is

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

This is invariant with respect to gauge transformations

$$A_j \to O A_j O^{-1} - \partial_j O O^{-1}, \qquad O \in \mathcal{G}.$$

A Yang-Mills connection is a critical point for the Yang-Mills functional

$$I(A) = \int_{\mathbb{R}^n\times\mathbb{R}} \langle F_{ij}, F^{ij}\rangle dx\ dt$$

where indices are lifted with respect to the Minkovski metric. Then the Yang-Mills equations have the form

$$D^j F_{ij} = 0$$

and the energy functional is

$$E(A) = \int_{\mathbb{R}^n\times\mathbb{R}} \langle F_{ij}, F_{ij}\rangle dx\ dt.$$

A Yang-Mills connection is not a single connection, but instead it is a class of equivalence with respect to the above gauge transformation. In order to view the Yang-Mills equations as semilinear wave equations and solve them one has to fix the gauge, i.e. select a single representative out of each equivalence class. Common gauge choices include: (i) the temporal gauge $A_0 = 0$, (ii) the wave gauge $\partial_j A^j = 0$ and (iii) the Coulomb gauge $\sum_{j=1}^n \partial_j A_j = 0$. To understand the equation better it may help to look first at an oversimplified version, namely

$$\Box u = (u \cdot \nabla_x)u + \nabla_x p, \qquad \nabla_x \cdot u = 0.$$

This exhibits a Q_{ij} type null condition. The scaling index is $s_0 = \frac{n-2}{2}$ and $s_c = 1$. Using the $X^{s,b}$ spaces one can improve the local theory somewhat, but certain more subtle modifications of this are needed in order to handle high-low frequency interactions. In [9] such an approach is used to prove that local well-posedness holds for $s > s_0$, $n \geq 4$.

Open Problem 7. Is the Yang-Mills equation well-posed for $s > s_0$, $n = 2, 3$? (Likely not for $n = 2$. For $n = 3$ one can obtain $s > \frac{3}{4}$ using the $X^{s,b}$ spaces.)

Conjecture 8. *(i) The Yang-Mills equation is globally well-posed for small data in* $H^{s_0} \times H^{s_0-1}$ *for* $n \geq 4$.

(ii) The Yang-Mills equation is globally well-posed for large data in $H^{s_0} \times H^{s_0-1}$ *for* $n = 4$.

3. Nonlinear wave equations

We consider (NLW), since (GNLW) reduces to it by differentiation. The fixed point argument in the semilinear case cannot be applied in the nonlinear case, because the wave equation parametrix is not strongly stable with respect to small changes in the coefficients. Instead, one must adopt a different strategy: (i) show that local solutions exist for smooth data, (ii) obtain a-priori bounds for smooth solutions uniformly with respect to initial data in a bounded set in $H^s \times H^{s-1}$ and (iii) prove continuous dependence on the data in a weaker topology, and obtain solutions for $H^s \times H^{s-1}$ data as weak limits of smooth solutions. Steps (i) and (iii) are more or less routine, it is (ii) which causes most difficulties. A good starting point is Klainerman's energy estimate

$$\|\nabla u(t)\|_{H^{s-1}} \lesssim \|\nabla u(0)\|_{H^{s-1}} \exp\Big(\int_0^t \|\nabla u(s)\|_{L^\infty} ds\Big).$$

This shows that all Sobolev norms of a solution remain bounded for as long as $\|\nabla u\|_{L^1 L^\infty}$ stays bounded. It remains to see how to obtain bounds on $\|\nabla u\|_{L^1 L^\infty}$. The classical approach uses energy estimates and Sobolev embeddings, but, as in the semilinear case, it only yields results one unit above scaling, namely for $s > \frac{n}{2} + 1$.

Better results could be obtained using the Strichartz estimates instead. However, this is very nontrivial as one would have to establish the Strichartz estimates for the operator $\Box_{g(u)}$, which has very rough coefficients. Compounding the difficulty, the argument is necessarily circular,

$$\begin{matrix}\text{coefficients} \\ \text{regularity}\end{matrix} \Longrightarrow \begin{matrix}\text{Strichartz} \\ \text{estimates}\end{matrix} \Longrightarrow \begin{matrix}\text{solution} \\ \text{regularity}\end{matrix} \Longrightarrow \begin{matrix}\text{coefficients} \\ \text{regularity}\end{matrix}$$

One can get around this with a bootstrap argument of the form

$$\left. \begin{array}{c} \|(u_0, u_1)\|_{H^s \times H^{s-1}} \leq \epsilon \\ \|\!|g(u)|\!\| \leq 2 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{Strichartz estimates for } \Box_{g(u)} \text{ in } [-1, 1] \\ \|\!|g(u)|\!\| \leq 1 \end{array} \right.$$

where the (possibly nonlinear) triple norm contains the needed information about the metric. Still, apriori there is no clear way to determine exactly how it should be defined. A starting point is to set $\|\!|g(u)|\!\| = \|\nabla g\|_{L^1 L^\infty}$, but this only leads to partial results. Following partial results independently obtained by Bahouri-Chemin [3],[2] and Tataru [23], [18] and further work of Klainerman-Rodnianski [13], the next result represents the current state of the problem:

Theorem 9. (Smith-Tataru [16]) *The equation (NLW) is locally well-posed in* $H^s \times H^{s-1}$ *for* $s > \frac{n}{2} + \frac{3}{4}$ $(n = 2)$ *and* $s > \frac{n}{2} + \frac{1}{2}$ $(n = 3, 4, 5)$*. In addition, the Strichartz estimates with* $q = \infty$ *hold for the corresponding wave operator* $\Box_{g(u)}$.

Lindblad's counterexamples correspond to $s = \frac{n+3}{4}$ and show that this result is sharp for $n = 2, 3$. The restriction to $n \leq 5$ is not central to the problem, it can likely be removed with some extra work.

Open Problem 10. Improve the above result in dimension $n \geq 4$.

Wave equation parametrices. In most approaches, the key element in the proof of the Strichartz estimates is the construction of a parametrix for the wave equation. There are many ways to do this for smooth coefficients, however, as the regularity of the coefficients decreases, they start to break down. Let us begin with the classical Fourier integral operator parametrix, used in the work of Bahouri-Chemin:

$$K(x, y) = \int a(x, y, \xi) e^{i\phi(x,y,\xi)} d\xi.$$

The phase ϕ is initialized by $\phi(x, y, \xi) = \xi(x - y)$ when $x_0 = y_0$ and must solve an eikonal equation, while for the amplitude a one obtains a transport equation along the Hamilton flow. The disadvantage is that all spatial localization comes from stationary phase, which seems to require too much regularity for the coefficients.

One way to address the issue of spatial localization is to begin with wave packets, which have the best possible spatial localization on the unit time scale. In the variable coefficient case the frequency λ wave packets are bump functions on curved parallelepipeds of size $1 \times \lambda^{-1} \times (\lambda^{-\frac{1}{2}})^{n-1}$. These parallelepipeds are images of $\lambda^{-1} \times (\lambda^{-\frac{1}{2}})^{n-1}$ parallelepipeds at the initial time, transported along the Hamilton flow for $\Box_g$ corresponding to their conormal direction. Then one can seek approximate solutions for $\Box_g$ as discrete superpositions of wave packets, $u = \sum_T u_T$. It is not too difficult to construct individual wave packets, the more delicate point is to show that the wave packets are almost orthogonal. This approach, which is used in [16], was originally introduced by Smith [14] and used to prove the Strichartz estimates in 2 and 3 dimensions for operators with C^2 coefficients.

Another parametrix with a better built in spatial localization can be obtained by doing a smooth phase space analysis:

$$K(\tilde{y}, y) = \int_C a(x,\xi) e^{i(\phi(y,x,\xi) - \overline{\phi(\tilde{y},x_t,\xi_t)}} dx\, d\xi\, dt \quad \phi(y,x,\xi) = \xi(x-y) + i|\xi|(x-y)^2.$$

Here $(x,\xi) \to (x_t,\xi_t)$ is the Hamilton flow for $\Box_g$ on the characteristic cone $C = \{g^{ij}(x)\xi_i\xi_j = 0\}$. One can factor this into a product of three operators, namely an FBI transform, a phase space transport along the Hamilton flow and then an inverse FBI transform. Neglecting the first one, i.e. setting $x = y$ above, produces an operator which is similar to the Fourier integral operators with complex phase. However, it seems to be more useful to keep the Gaussian localizations at both ends. Parametrices of this type were introduced in Tataru [24] and used to prove Strichartz estimates for operators with C^2 coefficients in all dimensions. The C^2 condition was later relaxed in [18] to $\nabla^2 g \in L^1 L^\infty$. Localization and scaling arguments lead also to weaker estimates for operators whose coefficients have less regularity. Such estimates are known to be sharp, see the counterexamples in Smith-Tataru [15].

The null condition. As in the semilinear case, one may ask whether better results can be obtained for equations with special structure. However, unlike the semilinear case, little is known so far. We propose the following

Definition 11. *We say that the equation (GNLW) satisfies the null condition if*

$$\frac{\partial g^{ij}(u,p)}{\partial p_k}\xi_i\xi_j\xi_k = 0 \qquad \text{in } g^{ij}(u,p)\xi_i\xi_j = 0.$$

Conjecture 12. *If the null condition holds then the equation (GNLW) is well-posed in $H^s \times H^{s-1}$ for some $s < \frac{n}{2} + \frac{3}{4}$ ($n = 2$) respectively for some $s < \frac{n}{2} + \frac{1}{2}$ ($n = 3$).*

In $3+1$ dimensions a problem which does not quite fit into the above setup but still satisfies some sort of null condition is the Einstein's equations in general relativity. It is similar to the Yang Mills equations in that it has a gauge invariance, and the null condition is only apparent after fixing the gauge. Klainerman-Rodnianski have obtained a different proof of Theorem 9 for this special case of (NLW).

References

[1] Serge Alinhac. *Blowup for nonlinear hyperbolic equations,* Boston: Birkhäuser, 1995. Progress in Nonlinear Differential Equations and their Applications, 17.

[2] Hajer Bahouri and Jean-Yves Chemin, Equations d'ondes quasilineaires et effet dispersif, *Int. Math. Res. Not.*, 1999(21):1141–1178, 1999.

[3] Hajer Bahouri and Jean-Yves Chemin, Equations d'ondes quasilineaires et estimations de Strichartz, *Am. J. Math.*, 121(6):1337–1377, 1999.

[4] Jean Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations II. The KdV-equation, *Geom. Funct. Anal.*, 3(3):107–156, 209–262, 1993.

[5] Jean Bourgain, Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity, *Internat. Math. Res. Notices*, no. 5:253–283, 1998.

[6] Damiano Foschi and Sergiu Klainerman, Bilinear space-time estimates for homogeneous wave equations, *Ann. Sci. École Norm. Sup. (4)*, 33(2):211–274, 2000.

[7] Manoussos Grillakis, Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity, *Ann. of Math.*, 132(3):485–509, 1990.

[8] Markus Keel and Terence Tao, Endpoint Strichartz estimates, *Amer. J. Math.*, 120(5):955-980, 1998.

[9] Sergiu Klainerman and Daniel Tataru, On the optimal local regularity for Yang-Mills equations in R^{4+1}, *J. Amer. Math. Soc.*, 12(1):93–116, 1999.

[10] Hans Lindblad, Counterexamples to local existence for semi-linear wave equations, *Amer. J. Math.*, 118(1):1–16, 1996.

[11] Hans Lindblad, Counterexamples to local existence for quasilinear wave equations, *Math. Res. Lett.*, 5(5):605–622, 1998.

[12] Sergiu Klainerman, Matei Machedon, Space-time estimates for null forms and the local existence theorem, *Comm. Pure Appl. Math.*, 46(9): 1221–1268, 1993.

[13] Sergiu Klainerman and Igor Rodnianski, Improved local well posedness for quasilinear wave equations in dimension three, preprint.

[14] Hart Smith, A parametrix construction for wave equations with $C^{1,1}$ coefficients, *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[15] Hart Smith and Daniel Tataru, Counterexamples to Strichartz estimates for the wave equation with nonsmooth coefficients, *Math. Res. Lett.*, to appear.

[16] Hart Smith and Daniel Tataru, Sharp local well-posedness results for the nonlinear wave equation. `http://www.math.berkeley.edu/~tataru/nlw.html`

[17] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993.

[18] Daniel Tataru, Strichartz estimates for operators with nonsmooth coefficients III, to appear, J. Amer. Math. Soc.

[19] Terence Tao, Low regularity semi-linear wave equations *Comm. Partial Differential Equations*, 24(3-4):599–629, 1999.

[20] Terence Tao, Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates, *Math. Z.*, 238(2):215-268, 2001.

[21] Terence Tao, Global regularity of wave maps, II, Small energy in two dimensions, *Comm. Math. Phys.*, 224(2):443–544, 2001.

[22] Daniel Tataru, On the equation $\Box u = |\nabla u|^2$ in $5+1$ dimensions, *Math. Res. Lett.*, 6(5-6):469–485, 1999.

[23] Daniel Tataru, Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation, *Am. J. Math.*, 122(2):349–376, 2000.

[24] Daniel Tataru, Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients, II, *Amer. J. Math.*, 123(3):385–423, 2001.

[25] Daniel Tataru, On global existence and scattering for the wave maps equation, *Amer. J. Math.*, 123(1):37–77, 2001.

[26] Daniel Tataru, Null form estimates for second order hyperbolic operators with

rough coefficients. http://www.math.berkeley.edu∼tataru/nlw.html

[27] Michael E. Taylor, *Pseudodifferential operators and nonlinear PDE.* Birkhäuser, Boston, 1991.

[28] Thomas Wolff, A sharp bilinear cone restriction estimate, *Ann. of Math. (2)*, 153(3):661–698, 2001.