

# Lawrence Berkeley National Laboratory

## Recent Work

### Title

UNSUBTRACTED DISPERSION RELATIONS IN WEAK INTERACTIONS AND A PARTIALLY  
CONSERVED AXIAL-VECTOR CURRENT

### Permalink

<https://escholarship.org/uc/item/0rv6598d>

### Author

Saito, Takesi.

### Publication Date

1965-06-23

University of California  
Ernest O. Lawrence  
Radiation Laboratory

UNSUBTRACTED DISPERSION RELATIONS IN WEAK INTERACTIONS  
AND A PARTIALLY CONSERVED AXIAL-VECTOR CURRENT

TWO-WEEK LOAN COPY

*This is a Library Circulating Copy  
which may be borrowed for two weeks.  
For a personal retention copy, call  
Tech. Info. Division, Ext. 5545*

Berkeley, California

## **DISCLAIMER**

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

Submitted for publication to Phys. Rev. Letters

UCRL-16224

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory  
Berkeley, California

AEC Contract No. W-7405-eng-48

UNSUBTRACTED DISPERSION RELATIONS IN WEAK INTERACTIONS  
AND A PARTIALLY CONSERVED AXIAL-VECTOR CURRENT

Takesi Saito

June 23, 1965

UNSUBTRACTED DISPERSION RELATIONS IN WEAK INTERACTIONS  
AND A PARTIALLY CONSERVED AXIAL-VECTOR CURRENT\*

Takesi Saito†

Lawrence Radiation Laboratory  
University of California  
Berkeley, California

June 23, 1965

Following a partially conserved axial-vector current (PCAC) hypothesis, we assume that any matrix elements of divergence of the axial-vector current  $(\partial_{\mu} a_{\mu})$  satisfy unsubtracted dispersion relations (UDR). We shall show that a combined use of UDR with PCAC, due to Nambu,<sup>1</sup> leads to a very small wave-function renormalization constant of the pion field ( $Z_3 \ll 1$ ), where an additional condition for PCAC is necessary to obtain the case  $Z_3 = 0$ . The polology version of PCAC due to Gell-Mann and others<sup>2</sup> does not generally lead<sup>3</sup> to such a conclusion, although both versions give the same Goldberger-Treiman (G-T) relation.<sup>4</sup> (One can see, however, that in the one-channel approximation or in the unitary symmetric limit both versions are equivalent.) A model of PCAC  $(\partial_{\mu} a_{\mu} = \text{const } \phi_{\pi})$  proposed by Gell-Mann and Lévy<sup>5</sup> contradicts UDR. But their model can always be modified so as to be equivalent to the polology or Nambu's version of PCAC. We treat weak interactions in lowest order and neglect all electromagnetic corrections. We do not assume time-reversal invariance in weak interactions.

Let us define  $F$ , the invariant amplitude for the  $\pi$ - $\mu$  decay, by

$$(2q_0)^{1/2} \langle \pi^- | \partial_\mu a_\mu | 0 \rangle = q^2 F = \mu^2 F, \quad (1)$$

where  $a_\mu$  denotes the axial-vector current,  $q$  is the four-momentum of the pion, and  $\mu$  is the pion rest mass. The off-shell amplitude  $F(s = q^2)$  is assumed to satisfy UDR,

$$F(s) = \frac{1}{\pi} \int_{9\mu^2}^{\infty} ds' \frac{\text{Abs } F(s')}{s' - s}. \quad (2)$$

The absorptive part of  $F(s)$  is given by

$$s \text{ Abs } F(s) = \pi \sum_n \langle 0 | J_\pi | s, n \rangle \langle s, n | \partial_\mu a_\mu | 0 \rangle \delta^4(q_n - q), \quad (3)$$

where  $J_\pi$  is the source of the pion field, and  $n$  denotes all the variables other than  $s$ . By summing up over spins and separating out kinematical factors, Eq. (3) can be written as<sup>6</sup>

$$\begin{aligned} s \text{ Abs } F(s) &= \pi \sum_m g_m^*(s) \rho_m(s) f_m(s) \\ &= \pi g^\dagger(s) \rho(s) f(s) \end{aligned} \quad (4)$$

in the sense of matrix notation. Here the two invariant amplitudes,  $g(s)$  and  $f(s)$ , represent virtual dissociation of the pion into

intermediate states and their annihilation into a lepton pair, respectively;  $\rho(s)$  is a kinematical factor.

The amplitudes,  $G(s)$  and  $f(s)$ , are assumed to satisfy the following dispersion relations, respectively:

$$g(s) = g + \frac{s-\mu^2}{\pi} \int_{9\mu^2}^{\infty} ds' \frac{T^\dagger(s')\rho(s')g(s')}{(s'-\mu^2)(s'-s)}, \quad (5)$$

$$f(s) = \frac{\mu^2}{\mu^2-s} Fg + \frac{1}{\pi} \int_{9\mu^2}^{\infty} ds' \frac{T^\dagger(s')\rho(s')f(s')}{s'-s}, \quad (6)$$

where  $T(s)$  is the scattering amplitude in the pseudoscalar sector, and  $f(s)$  has been written in the unsubtracted form in accordance with the PCAC hypothesis. The solutions of Eqs. (5) and (6) are given by

$$g(s) = D^{-1}(s) g(0), \quad (7)$$

$$f(s) = \epsilon(s) - \frac{\mu^2}{s-\mu^2} F g(s), \quad (8)$$

where

$$\begin{aligned} \epsilon(s) &= D^{-1}(s) [f(0) - F g(0)] \\ &= G(s) - F g(s), \end{aligned} \quad (9)$$

$$G(s) = D^{-1}(s) f(0) \quad (10)$$

Here  $D(s)$ , normalized to one at  $s = 0$ , is the denominator function of  $T(s)$  in the N/D method, but it has no poles and its determinant has no zeros. The second term in (8) is a special solution of Eq. (6), while  $\epsilon(s)$  is a solution of the homogeneous equation of Eq. (6), normalized at  $s = 0$ .

If the pole term in (8) dominates for small  $s$ , we get the G-T relation  $f(o) \approx F g(o)$ . This is the poleology version<sup>2</sup> of PCAC. The G-T relation follows also from Nambu's version<sup>1</sup> of PCAC defined by

$$G(s) \approx F G(s), \quad (11)$$

which becomes rigorous at the high-energy limit  $\left[ \lim_{s \rightarrow \infty} \epsilon(s) = 0 \right]$ . More precisely, Eq. (11) means

$$|F g(s)| \gg |\epsilon(s)| \text{ for all } s. \quad (12)$$

If  $G(s) = F g(s)$  for all  $s$ , we have

$$f(s) = - \frac{\mu^2}{s-\mu^2} F g(s), \quad (13)$$

a result known to Gell-Mann and Levy,<sup>5</sup> who conjectured the relation

$$\partial_\mu a_\mu = \mu^2 F \phi_\pi, \quad (14)$$

from which Eq. (13) immediately follows.



But Eq. (13) contradicts Eq. (2), because, after inserting Eq. (13) into Eq. (2) and putting  $s = \mu^2$ , what we get is  $F = 0$ . So, we must abandon this case. Equation (14) should be modified as

$$\partial_{\mu} a_{\mu} = \mu^2 F \phi_{\pi} + R, \quad (15)$$

from which the poleology or Nambu's version of PCAC always follows under an appropriate condition for  $R$ . Both versions of PCAC give the same G-T relation, but they are generally not equivalent to each other.<sup>7</sup>

Now we show, using Eqs. (2) and (12), that  $Z_3$  must be much smaller than one. For this purpose we recall the formulae

$$\begin{aligned} (s-\mu^2)^2 \sigma(s) &= \sum_n |\langle 0 | J_{\pi} | s, n \rangle|^2 \delta^4(q_n - q) \\ &= g^{\dagger}(s) \rho(s) g(s), \end{aligned} \quad (16)$$

$$Z_3^{-1} = 1 + \int ds \sigma(s), \quad (17)$$

where  $\sigma(s)$  is the Lehmann weight function for the pion propagator.

A similar function  $\gamma(s)$  will be introduced by

$$(s-\mu^2)^2 \gamma(s) = g^{\dagger}(s) \rho(s) G(s). \quad (18)$$

Then  $\text{Abs } F(s)$  can be written in terms of these functions,

$$s \text{ Abs } F(s) = \pi(s-\mu^2)^2 \left[ \gamma(s) - F\sigma(s) - \frac{\mu^2}{s-\mu^2} F\sigma(s) \right]. \quad (19)$$

Substituting Eq. (19) into Eq. (2) and putting  $s = \mu^2$ , we have<sup>8,9</sup>

$$F = \int ds \frac{s-\mu^2}{s} [\gamma(s) - F\sigma(s)] / \left[ 1 + \int ds \frac{\mu^2}{s} \sigma(s) \right]. \quad (20)$$

In Eqs. (2) and (20) the convergence condition is necessary :

$$\lim_{s \rightarrow \infty} s\sigma(s) [\gamma(s)/\sigma(s) - F] = 0. \quad (21)$$

Therefore we must consider two cases:

Case A:  $\lim_{s \rightarrow \infty} s\sigma(s) = 0.$

This means  $Z_3$  is finite. Let us rewrite (12) in terms of  $\sigma(s)$  and  $\gamma(s)$ :

$$\begin{aligned} (s-\mu^2)^2 |\gamma(s) - F\sigma(s)| &= |g^\dagger(s)\rho(s) \epsilon(s)| \\ &\leq \sum_m |g_m(s)| |\epsilon_m(s)| \rho_m(s) \ll |F| \sum_m |g_m(s)|^2 \rho_m(s) \\ &= (s-\mu^2)^2 |F| \sigma(s). \end{aligned} \quad (22)$$

Then we have the following inequality:

$$\begin{aligned}
 |F| &\leq \int ds \frac{s-\mu^2}{s} |\gamma(s) - F\sigma(s)| / \left[ 1 + \int ds \frac{\mu^2}{s} \sigma(s) \right] \\
 &\ll |F| \int ds \frac{s-\mu^2}{s} \sigma(s) / \left[ 1 + \int ds \frac{\mu^2}{s} \sigma(s) \right], \quad (23)
 \end{aligned}$$

which reduces to

$$1 < 1 + \int ds \frac{2\mu^2}{s} \sigma(s) \ll \int ds \sigma(s). \quad (24)$$

This means that  $Z_3$  must be much smaller than one, but does not vanish.

Case B:  $\lim_{s \rightarrow \infty} s\sigma(s) \neq 0.$

This means  $Z_3$  vanishes. From the convergence condition (22) we must have a relation<sup>8</sup>

$$F = \lim_{s \rightarrow \infty} \gamma(s)/\sigma(s), \quad (25)$$

which can be regarded as an additional condition for Nambu's version of PCAC. In order to get the case  $Z_3 = 0$ , this additional condition is necessary. It is worthwhile to note that in the one-channel approximation Eq.(25) reduces to  $F = f(o)/g(o)$ . This means  $\epsilon(s) \equiv 0$ , and contradicts Eq.(2). This difficulty has already

been pointed out by many authors,<sup>10</sup> and it has been shown recently by Ida<sup>8</sup> that this difficulty is due to the inadequate one-channel approximation.

If  $f(s)$  does not vanish at high-energy limit,  $f(s)$  must satisfy the once subtracted dispersion relation instead of Eq. (6). But even in this case we have the same solution as (8), and conclusions obtained above never change, as long as we adopt UDR for the  $\pi$ - $\mu$  decay amplitude.

I would like to express my sincere thanks to Dr. Y. Fujii, Dr. M. Ida, and Dr. W. Rarita for useful discussions. Thanks are also due to Dr. David L. Judd for his hospitality at the Lawrence Radiation Laboratory.

FOOTNOTES AND REFERENCES

- \* Work done under auspices of the United States Atomic Energy Commission.
- † Formerly Takesi Ogimoto. On leave of absence from Department of Physics, Osaka University, Osaka, Japan.
1. Y. Nambu, Phys. Rev. Letters 4, 380 (1960).
  2. J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento 17, 757 (1960).
  3. Other consistency conditions on strong interactions implied by PCAC have been derived by S.L. Adler, Phys. Rev. 137, B1022 (1965).
  4. M.L. Goldberger and S.B. Treiman, Phys. Rev. 110, 1178 (1958).
  5. M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).
  6. The functions,  $g_m(s)$ ,  $\rho_m(s)$ , and  $f_m(s)$ , depend on several variables other than  $s$ . The suffix  $m$  includes such variables;  $\rho_m(s)$  is defined to be positive for  $s > 9\mu^2$  and  $m$  physical.
  7. In the one-channel approximation it follows from the G-T relation  $f(o) \approx F g(o)$  that  $G(s) \approx F g(s)$ , but generally not.
  8. M. Ida, Phys. Rev. 132, 401 (1963).
  9. K. Nishijima, Phys. Rev. 133, B1092 (1964).
  10. E. R. McCliment and K. Nishijima, Phys. Rev. 128, 1970 (1962);  
B. Barrett and G. Barton, Nuovo Cimento 29, 703 (1963).

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

