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Following a partially conserved axial-vector current (PCAC) hypothesis, we assume that any matrix elements of divergence of the axial-vector current $(\partial_\mu a_\mu)$ satisfy unsubtracted dispersion relations (UDR). We shall show that a combined use of UDR with PCAC, due to Nambu,[1] leads to a very small wave-function renormalization constant of the pion field $(Z_3 \ll 1)$, where an additional condition for PCAC is necessary to obtain the case $Z_3 = 0$. The polology version of PCAC due to Gell-Mann and others[2] does not generally lead[3] to such a conclusion, although both versions give the same Goldberger-Treiman (G-T) relation.[4] (One can see, however, that in the one-channel approximation or in the unitary symmetric limit both versions are equivalent.) A model of PCAC $(\partial_\mu a_\mu = \text{const}\ \phi_\pi)$ proposed by Gell-Mann and Lévy[5] contradicts UDR. But their model can always be modified so as to be equivalent to the polology or Nambu's version of PCAC. We treat weak interactions in lowest order and neglect all electromagnetic corrections. We do not assume time-reversal invariance in weak interactions.

Let us define F, the invariant amplitude for the π-μ decay, by

$$(2q_o)^{1/2} \langle \pi^- | \partial_\mu a_\mu | 0 \rangle = q^2 F = \mu^2 F , \tag{1}$$

where a_μ denotes the axial-vector current, q is the four-momentum of the pion, and μ is the pion rest mass. The off-shell amplitude $F(s = q^2)$ is assumed to satisfy UDR,

$$F(s) = \frac{1}{\pi} \int_{9\mu^2}^{\infty} ds' \frac{\text{Abs } F(s')}{s' - s} . \tag{2}$$

The absorptive part of $F(s)$ is given by

$$s \text{ Abs } F(s) = \pi \sum_n \langle 0 | J_\pi | s,n \rangle \langle s,n | \partial_\mu a_\mu | 0 \rangle \, \delta^4(q_n - q) , \tag{3}$$

where J_π is the source of the pion field, and n denotes all the variables other than s. By summing up over spins and separating out kinematical factors, Eq. (3) can be written as[6]

$$s \text{ Abs } F(s) = \pi \sum_m g_m^*(s) \, \rho_m(s) \, f_m(s)$$

$$= \pi \, g^\dagger(s) \, \rho(s) \, f(s) \tag{4}$$

in the sense of matrix notation. Here the two invariant amplitudes, $g(s)$ and $f(s)$, represent virtual dissociation of the pion into intermediate states and their annihilation into a lepton pair, respectively; $\rho(s)$ is a kinematical factor.

The amplitudes, $G(s)$ and $f(s)$, are assumed to satisfy the following dispersion relations, respectively:

$$g(s) = g + \frac{s-\mu^2}{\pi} \int_{9\mu^2}^{\infty} ds' \frac{T^{\dagger}(s')\rho(s')g(s')}{(s'-\mu^2)(s'-s)} , \tag{5}$$

$$f(s) = \frac{\mu^2}{\mu^2-s} Fg + \frac{1}{\pi}\int_{9\mu^2}^{\infty} ds' \frac{T^{\dagger}(s')\rho(s')f(s')}{s'-s} , \tag{6}$$

where $T(s)$ is the scattering amplitude in the pseudoscalar sector, and $f(s)$ has been written in the unsubtracted form in accordance with the PCAC hypothesis. The solutions of Eqs. (5) and (6) are given by

$$g(s) = D^{-1}(s)\, g(o) , \tag{7}$$

$$f(s) = \epsilon(s) - \frac{\mu^2}{s-\mu^2} F\, g(s), \tag{8}$$

where

$$\epsilon(s) = D^{-1}(s) \left[f(o) - F\, g(o)\right]$$

$$= G(s) - F\, g(s) , \tag{9}$$

$$G(s) = D^{-1}(s)\, f(o) \tag{10}$$

Here $D(s)$, normalized to one at $s = 0$, is the denominator function of $T(s)$ in the N/D method, but it has no poles and its determinant has no zeros. The second term in (8) is a special solution of Eq. (6), while $\epsilon(s)$ is a solution of the homogeneous equation of Eq. (6), normalized at $s = 0$.

If the pole term in (8) dominates for small s, we get the G-T relation $f(o) \approx F\, g(o)$. This is the polology version[2] of PCAC. The G-T relation follows also from Nambu's version[1] of PCAC defined by

$$G(s) \approx F\, G(s), \tag{11}$$

which becomes rigorous at the high-energy limit $\left[\lim_{s\to\infty} \epsilon(s) = 0 \right]$. More precisely, Eq. (11) means

$$|F\, g(s)| \gg |\epsilon(s)| \quad \text{for all} \quad s \;. \tag{12}$$

If $G(s) = F\, g(s)$ for all s, we have

$$f(s) = -\frac{\mu^2}{s-\mu^2}\, F\, g(s) \;, \tag{13}$$

a result known to Gell-Mann and Levy,[5] who conjectured the relation

$$\partial_\mu a_\mu = \mu^2 F\, \phi_\pi \;, \tag{14}$$

from which Eq. (13) immediately follows.

But Eq. (13) contradicts Eq. (2), because, after inserting Eq. (13) into Eq. (2) and putting $s = \mu^2$, what we get is $F = 0$. So, we must abandon this case. Equation (14) should be modified as

$$\partial_\mu a_\mu = \mu^2 F \phi_\pi + R , \tag{15}$$

from which the polology or Nambu's version of PCAC always follows under an appropriate condition for R. Both versions of PCAC give the same G-T relation, but they are generally not equivalent to each other.[7]

Now we show, using Eqs. (2) and (12), that Z_3 must be much smaller than one. For this purpose we recall the formulae

$$(s-\mu^2)^2 \sigma(s) = \sum_n |\langle 0|J_\pi|s,n\rangle|^2 \delta^4(q_n - q)$$

$$= g^\dagger(s)\, \rho(s)\, g(s) , \tag{16}$$

$$Z_3^{-1} = 1 + \int ds\, \sigma(s) , \tag{17}$$

where $\sigma(s)$ is the Lehmann weight function for the pion propagator. A similar function $\gamma(s)$ will be introduced by

$$(s-\mu^2)^2 \gamma(s) = g^\dagger(s)\, \rho(s)\, G(s) . \tag{18}$$

Then Abs F(s) can be written in terms of these functions,

$$s \text{ Abs } F(s) = \pi(s-\mu^2)^2 \left[\gamma(s) - F\sigma(s) - \frac{\mu^2}{s-\mu^2} F\sigma(s)\right] . \quad (19)$$

Substituting Eq. (19) into Eq.(2) and putting $s = \mu^2$, we have[8,9]

$$F = \int ds \frac{s-\mu^2}{s}\left[\gamma(s) - F\sigma(s)\right] / \left[1 + \int ds \frac{\mu^2}{s} \sigma(s)\right] . \quad (20)$$

In Eqs. (2) and (20) the convergence condition is necessary :

$$\lim_{s\to\infty} s\sigma(s)\left[\gamma(s)/\sigma(s) - F\right] = 0 . \quad (21)$$

Therefore we must consider two cases:

<u>Case A:</u> $\lim_{s\to\infty} s\sigma(s) = 0$.

This means Z_3 is finite. Let us rewrite (12) in terms of $\sigma(s)$ and $\gamma(s)$:

$$(s-\mu^2)^2 |\gamma(s)-F\sigma(s)| = |g^\dagger(s)\rho(s)\,\epsilon(s)|$$

$$\leqslant \sum_m |g_m(s)|\;|\epsilon_m(s)|\;\rho_m(s) \ll |F| \sum_m |g_m(s)|^2 \rho_m(s)$$

$$= (s-\mu^2)^2 \;|F|\;\sigma(s) . \quad (22)$$

Then we have the following inequality:

$$|F| \leq \int ds \frac{s-\mu^2}{s} \; |\gamma(s) - F\sigma(s)| / \left[1 + \int ds \frac{\mu^2}{s} \sigma(s)\right]$$

$$\ll |F| \int ds \frac{s-\mu^2}{s} \sigma(s) / \left[1 + \int ds \frac{\mu^2}{s} \sigma(s)\right], \tag{23}$$

which reduces to

$$1 < 1 + \int ds \frac{2\mu^2}{s} \sigma(s) \ll \int ds \, \sigma(s). \tag{24}$$

This means that Z_3 must be much smaller than one, but does not vanish.

<u>Case B:</u> $\lim_{s\to\infty} s\sigma(s) \neq 0$.

This means Z_3 vanishes. From the convergence condition (22) we must have a relation[8]

$$F = \lim_{s\to\infty} \gamma(s)/\sigma(s), \tag{25}$$

which can be regarded as an additional condition for Nambu's version of PCAC. In order to get the case $Z_3 = 0$, this additional condition is necessary. It is worthwhile to note that in the one-channel approximation Eq.(25) reduces to $F = f(o)/g(o)$. This means $\epsilon(s) \equiv 0$, and contradicts Eq.(2). This difficulty has already been pointed out by many authors,[10] and it has been shown recently by Ida[8] that this difficulty is due to the inadequate one-channel approximation.

If $f(s)$ does not vanish at high-energy limit, $f(s)$ must satisfy the once subtracted dispersion relation instead of Eq. (6). But even in this case we have the same solution as (8), and conclusions obtained above never change, as long as we adopt UDR for the π-μ decay amplitude.

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FOOTNOTES AND REFERENCES

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4. M.L. Goldberger and S.B. Treiman, Phys. Rev. 110, 1178 (1958).

5. M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

6. The functions $g_m(s)$, $\rho_m(s)$, and $f_m(s)$, depend on several variables other than s. The suffix m includes such variables; $\rho_m(s)$ is defined to be positive for $s > 9\mu^2$ and m physical.

7. In the one-channel approximation it follows from the G-T relation $f(o) \approx F\, g(o)$ that $G(s) \approx F\, g(s)$, but generally not.

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