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Essays on Macroeconomics and Finance

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy

in Economics

by

Andrés Mariano Schneider

2018

ABSTRACT OF THE DISSERTATION

Essays on Macroeconomics and Finance

by

Andrés Mariano Schneider

Doctor of Philosophy in Economics

University of California, Los Angeles, 2018

Professor Andrew Granger Atkeson, Co-Chair

Professor Mikhail Chernov, Co-Chair

The following essays contribute towards our understanding of the macroeconomic fundamentals of financial assets. The dissertation is composed of four chapters.

Chapter one—Risk Sharing and the Term Structure of Interest Rates

I propose a general equilibrium model with heterogeneous investors to explain the key properties of the U.S. real and nominal term structure of interest rates. I find that differences in investors' willingness to substitute consumption across time are critical to account for nominal and real yields dynamics. When the endogenous amount of credit supplied by risk-tolerant investors is low, the aggregate price of risk and the real interest rate are high. Thus, real bonds are risky. I study nominal bonds under both exogenous and endogenous (Taylor rule) inflation. I find that when the Taylor loading on inflation is greater than one, the nominal term structure is upward sloping regardless of the correlation between nominal and

real shocks. I use the model to shed light on two salient interest rate puzzles: (1) the secular decline of long-term real and nominal rates since the 1980s, and (2) the sudden spike in real yields at the height of the Great Recession.

Chapter 2—Endogenous and Exogenous Risk Premia

In this second chapter, I investigate how levered balance sheets amplify the effects of exogenous aggregate volatility shocks on asset prices. Risk premia are determined by the interaction of exogenous time-varying fundamentals with the endogenously determined levered balance sheets. When macro-volatility shocks hit the economy, asset prices decline, levered agent loses relatively more net worth and aggregate risk aversion rises endogenously. I find that this feedback between balance sheets and macro-volatility produces six times more volatile premiums than an economy with only cash flow shocks, thus improving the model's ability to match the data. However, the effects on investment and growth are mild

Chapter 3—Liquidity Shocks, Business Cycles and Asset Prices

The third chapter is joint work with Saki Bigio. In the aftermath of the Great Recession, macro models that feature financing constraints have attracted increasing attention. Among these, Kiyotaki and Moore (2012) is a prominent example. In this paper, we investigate whether the liquidity shocks and financial frictions proposed by Kiyotaki and Moore (2012) can improve the asset pricing predictions of the frictionless RBC model. We study the quantitative business cycle and asset pricing properties in an economy in which agents feature recursive preferences, are subject to a liquidity constraint, and suffer liquidity shocks. We find that the model predicts highly nonlinear time variation and levels of risk premia, which are driven by endogenous fluctuations in equity prices. However, the model fails to account for a basic fact: Periods of scarce liquidity are associated with high asset prices and low expected returns.

Chapter 4—A Macrofinance View of U.S. Sovereign CDS Premiums

The forth and last chapter of the dissertation is joint work with Mikhail Chernov and Lukas Schmid. Premiums on U.S. sovereign CDS have risen to persistently elevated levels since the financial crisis. We ask whether these premiums reflect the probability of a fiscal default – a state in which budget balance can no longer be restored by raising taxes or eroding the real value of debt by raising inflation. We develop an equilibrium macrofinance model in which the fiscal and monetary policy stance jointly endogenously determine nominal debt, taxes, inflation and growth. We show how CDS premiums reflect endogenous risk-adjusted fiscal default probabilities. A calibrated version of the model is quantitatively consistent with the observed CDS premiums.

The dissertation of Andrés Mariano Schneider is approved.

Stavros Panageas

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University of California, Los Angeles

2018

To all those who inspired me and I learned from.

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Chapter 1

Risk Sharing and the Term Structure of Interest Rates

1.1 Introduction

Long-term nominal yields on U.S. government bonds display a higher mean and lower volatility than short-term yields. Data from inflation-protected bonds (TIPS) show that the real yield curve exhibits similar patterns, which suggests that real risks are important for understanding the nominal term structure. Characterizing the macroeconomic fundamentals that drive these common features in the real and nominal yield curve, in a unified framework, has been a long-standing challenge for macroeconomists and financial economists (Gürkaynak and Wright, 2012). The main contribution of this paper is to propose a model in which the credit market plays a key role in understanding these salient properties of U.S. real and nominal yield curves.

Indeed, in theory and practice, interest rates are determined in the credit market, which renders it a natural starting point for the study of term structure dynamics. The basic feature of the credit market is that heterogeneous investors lend to and borrow from each other with the purpose of sharing risks; heterogeneity creates gains from trade. I incorporate this idea in a general equilibrium term structure model and find that the difference in investors' willingness to substitute consumption across time is critical in capturing the properties of the nominal and real yield curves we observe in the data.

Long-term nominal yields on U.S. government bonds display a higher mean and lower volatility than short-term yields. Data from inflation-protected bonds (TIPS) show that the real yield curve exhibits similar patterns, which suggests that real risks are important for understanding the nominal term structure. Characterizing the macroeconomic fundamentals

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In this economy, the quantity of credit generates endogenous fluctuations in asset prices. In particular, term premia and yields are endogenously time-varying due to fluctuations in a single state variable that summarizes the credit conditions in the economy: the market value of leveraged investors' net worth over the total market value of net worth. This state variable has been underscored by many macro models that feature a credit market (with and without frictions), but this paper is the first to explicitly examine its influence on the term structure of interest rates.

The economic mechanism hinges on two assumptions. First, I motivate a credit market by assuming that investors have different attitudes toward risks (Dumas, 1989; Wang, 1996; Chan and Kogan, 2002; Bhamra and Uppal, 2009; Longstaff and Wang, 2012; Gârleanu and Panageas, 2015; Barro et al., 2017; Hall, 2017a). Thus, in equilibrium, risk-tolerant investors issue short-term debt to finance leveraged positions in risky assets, which implies that their net worth is relatively more exposed to aggregate shocks. As a consequence, the effect of exogenous i.i.d. shocks on asset prices is persistently amplified by risk-tolerant investors' net worth, which generates endogenous fluctuations in the term structure. Second, I assume that investors with a high risk aversion (RA) coefficient exhibit a smaller elasticity of intertemporal substitution (EIS) than that implied by time-additive constant relative risk-aversion preferences. This assumption is key in capturing the quantitative properties of the term structure, because agents with relatively low EIS must be compensated with higher interest rates in equilibrium. Recursive preferences are essential to accommodate this feature.

The main mechanism is as follows. A negative aggregate shock generates a contraction in leveraged risk-tolerant investors' net worth, reducing their aggregate ability to supply credit. The contraction in aggregate credit produces an increase in the price of credit—i.e., the spot real rate. This is because a more risk-averse investor, who has a low EIS, must be incentivized to reallocate his portfolio and smooth consumption over time. In addition, the price of risk rises endogenously, because a more risk-averse investor is at the margin in the market for risky assets. The increase in the real rate implies that real bond prices become low in bad states: The marginal investor requires a positive premium to hold real bonds. .

This relationship between interest rates and the credit market delivers an average upward-sloping real term structure. On average, fixing investors' expectations about future short-term rates, long-term yields are higher than short-term yields. This is because long-term bonds command a higher term premium; the larger the horizon of a bond, the more likely it will drop in value during bad states, because they have higher exposure to variations in interest rates. Put differently, long-term bonds have a higher elasticity with respect to endogenous changes in the share of net worth held by risk-tolerant investors (the model's endogenous state variable, which summarizes the amount of credit in the economy). I evaluate this elasticity in the empirical section of the paper.

A further implication of the mechanism is that it takes time for the credit market to recompose after a negative shock. Simply put, a contraction in aggregate credit implies lower asset prices, which further implies that risk-tolerant investors can supply less credit. This persistence shows up in equilibrium asset prices, and in particular in long-term bonds: The longer the horizon of the bond, the larger the effect of the credit market's persistence. This translates into a higher volatility of long-term bond prices relative to short-term bonds. However, since long-term bonds are stationary, this volatility grows at a slower pace than the horizon of bonds.[1] As a result, since yields are (log) bond prices divided by the horizon of the bond, long-term yields are always less volatile than short-term yields.

After reviewing the main theoretical underpinnings of the mechanism described above, I study the nominal term structure of interest rates. For this, I consider two alternative inflation processes: exogenous and endogenous (derived via a Taylor rule).

The purpose of introducing exogenous inflation is to study a decomposition between the real and nominal components of the nominal term premium. In this analysis, the nominal component is driven by the exogenous negative correlation between cash flow and inflation

[1] I obtain an invariant distribution in the economy by using a simple OLG framework based on Blanchard (1985) and Gârleanu and Panageas (2015). I review this in detail in Section 3.

shocks (e.g., Cox et al., 1985; Wachter, 2006; Piazzesi and Schneider, 2006; Bansal and Shaliastovich, 2013). That is, if inflation occurs in bad states, the marginal investor requires a premium to hold nominal bonds. The real component is driven by the endogenous risk generated in the credit market. In this decomposition, I find that even with a large negative correlation between inflation and real shocks, the real component explains 80% of the average nominal term premium observed in the data. This result is in line with recent studies showing the importance of the real component in the nominal term structure (e.g., Abrahams et al., 2016).

Motivated by this result, I derive a nominal term structure that is purely driven by the real component. Using a Taylor rule, I derive an endogenous inflation process that is consistent with both the policy rule and the marginal investor's nominal pricing kernel (e.g., Gallmeyer et al., 2007b). As a result, inflation does not introduce new shocks, as in the exogenous case—i.e., the nominal term premium is not driven by nominal risk. I obtain an average slope of the nominal term structure that is in line with the data, driven by the fact that the Taylor loading on the policy rule is greater than one. Sensitivity analysis shows that the larger (smaller) the Taylor loading, the smaller (larger) the mean and volatility of inflation, and the flatter (steeper) the nominal yield curve.

I next evaluate the central theoretical predictions of the model. For this, I first extract aggregate shocks from macroeconomic data. I exploit the fact that I consider an aggregate endowment with i.i.d. growth rates, and therefore aggregate shocks are straightforward to identify (under the null of my model). Second, I feed the shocks into the model to study the predictions for the endogenous state variables. In this step, I compare fluctuations in the amount of credit in the model against fluctuations in the data (total credit to private sector over GDP), and I find the model captures these fluctuations relatively well. After checking the predictions for credit, I compute the implied series for the endogenous state variable in the model, and use those series to check whether the model's key predictions are verified in the term structure data.

In particular, I regress yields from the data onto the model's endogenous state variable (derived after feeding the macro shocks). The purpose of this is to test the main model's predictions: the sensitivity of both yields and slope (difference in yields) with respect to the endogenous state variable. The model predicts that long-term yields are less sensitive to the endogenous state variable than short-term yields (i.e., they are less volatile), and that the slope is positive and nonlinearly related to the endogenous state variable. Regressions using actual data for yields and the model's implied series for the state variable confirm these two central predictions.

I then study whether the endogenous state variable can capture the fluctuations in the short-term nominal interest rate. This is the key prediction of the endogenous inflation case. I find that the endogenous state variable can account for a significant portion of short-term nominal interest rate variability, even after controlling for other well-studied macro factors since Ang and Piazzesi (2003).

After validating the theoretical predictions, I provide an application of the model's mechanism to shed light on two puzzles regarding yields (Campbell et al., 2009). These are: (1) the sudden spike in the level and the reversion of the slope of the real term structure at the height of the Great Recession; and (2) the secular decline of nominal and real rates over the last 30 years. The objective is not only to provide further evidence on the mechanism I propose, but also to show that the connection between the credit market and the term structure provides a coherent perspective for important macroeconomic phenomena.

Specifically, in both applications I stress the role of the aggregate EIS. The sudden spike in real rates during the Great Recession can be rationalized as a sudden collapse in credit that produced a drastic reduction in aggregate willingness to substitute consumption into the future. The secular decline in nominal and real rates can be rationalized by the observed contemporaneous increase in the amount of credit in the economy, which in the model translates into a decrease in the price of credit (i.e., the spot risk-free rate). Due to the single factor structure of the model, the decrease in short-term rates is also reflected in long-term rates. In addition, the model implies a secular decrease in inflation expectations pinned down by the Taylor policy rule—which is consistent with survey data, as shown by Chernov and Mueller (2012), among others.

I conclude by comparing the model's prediction for the state variable with an alternative interpretation. Prior literature has interpreted risk-tolerant investors as the owners of financial institutions, or "credit suppliers" (e.g., Longstaff and Wang, 2012; Silva, 2016; Santos and Veronesi, 2016; Drechsler et al., 2017). In this view, the net worth of financial firms should be useful in capturing the credit conditions in the economy, and therefore yield dynamics. Following this alternative view, I construct the ratio of the market value of financial firms over the total market value of firms. I report the time series of this measure and compare it with those of the endogenous state variable in the model. In find the correlation of these two variables is significantly positive.

Literature. My paper fits into three strands of literature: heterogeneous agents and the credit market, macro-finance models of the term structure, and empirical literature studies that show the importance of credit measures in capturing yields dynamics.

First, my paper is related to recent papers in macroeconomics and finance that stress the role of the credit market in determining the behavior of equilibrium asset prices. A common theme in these papers is that agents exhibit heterogeneous exposure to aggregate risks (i.e., a group of agents operates with leverage in equilibrium), driven by differences in a technological feature (preferences, productivity, menu of assets, beliefs, information, etc.). Within this strand, my work is in line with studies that focus on preference heterogeneity and analyze the positive implications for asset prices and macroeconomic quantities with a frictionless credit market (e.g., Dumas, 1989; Wang, 1996; Chan and Kogan, 2002; Bhamra and Uppal, 2009; Longstaff and Wang, 2012; Gârleanu and Panageas, 2015; Barro et al., 2017; Hall, 2017a; Schneider, 2017).[2]

Specifically, Longstaff and Wang (2012) study an endowment economy in which agents feature heterogeneous constant relative risk-aversion preferences and analyze the role of the credit market on asset prices. In particular, they find that real yields on perpetual bonds are smaller than the short-term real yield (i.e., a downward-sloping real yield curve). Following this line, Hall (2017a) studies an economy with differences in risk aversion and argues that the secular decline in the average real rates can be explained by an increase in the wealth share of risk-averse agents. An implicit result in this analysis is that real bonds are hedges, and therefore the yields on long-term real bonds have a lower mean than short-term yields.

[2]Recent papers in the heterogeneous beliefs literature have analyzed the role of s is an important body of literature that analyzes the implication of heterogeneous beliefs in many

Gârleanu and Panageas (2015), extend the analysis to heterogeneous agents with recursive preferences, in which the economy has a simple OLG structure to obtain a stationary wealth distribution, and underscore the importance of heterogeneous preferences in determining the equity premium; Barro et al. (2017) studies an economy in which heterogeneous agents share aggregate risk in an economy subject to disasters and focus on the implications for the supply of safe assets; Wang (1996) considers an economy with heterogeneous agents with constant relative risk aversion and studies the theoretical properties of real yields; Schneider (2017) studies an economy in which fluctuations in premiums are driven by the interaction between endogenous changes in balance sheets and exogenous changes in macro volatility.

Relative to this first strand of literature, in this paper I show that the credit market is a key macroeconomic fundamental to understand the real and nominal term structure in a unified framework. In my results, I highlight the role of differences in investors' EIS. In fact, when investors exhibit the same EIS—but different RA—the economy exhibits a downward-sloping real term structure with only 10% of the yields' volatility we observe in the data.

A second strand this paper is related to is the macro-finance models of the term structure with a representative agent. This literature is extensive, but leading examples are Piazzesi and Schneider (2006), who study a Long Run Risk economy, where the representative agent exhibits very high risk aversion, and dislikes exogenous inflation such that more than compensates the downward sloping real yield curve; Bansal and Shaliastovich (2013), who study a Long Run Risk economy with stochastic volatility and analyze the implications for interest rates and currencies; Relative to this second strand of literature, in this paper I focus on the role of the credit market in determining the properties of the real and nominal term structure. This connection cannot be made in a representative agent setup.

Within the representative agent literature, Wachter (2006) introduces an exogenous time variation in the habits framework of Campbell and Cochrane (1999) and finds an upward-sloping nominal and real term structure of yields but also of their corresponding volatilities (the 5-year yield is more volatile than the 1-year). In the data, however, the longer the maturity of the bond (either nominal or real), the smaller the volatility of the yield. Also, Wachter's model predicts a slope of the nominal and real yield curves that are very similar. In the data I report below, also documented by Backus et al. (2017), the slope of the nominal term structure is at least twice the slope of the real term structure—the nominal term structure is steeper than the real. In my paper, in addition to providing an economic mechanism that links the term structure to credit market activity, I show that my model can also capture the fact that long-term yields are less volatile than short-term, and also that the nominal term structure is steeper than the real. Indeed, I show that the slope of the nominal term structure vis-à-vis the real can be rationalized by the reaction of monetary policy to the endogenous risks generated in the credit market.

Several papers have introduced further structure to the representative agent framework, and they study the term structure in a large scale dynamic stochastic general equilibrium model (DSGE) with production. Prominent examples are Rudebusch and Swanson (2008, 2012). The mechanism I propose in this paper, which generates endogenous time variation in the aggregate RA and EIS, can be introduced in a reduced form in such large scale DSGE models.

Lastly, my paper is related to empirical papers that stress the role of macro variables associated with the credit market in driving term premia over the business cycle. Haddad and Sraer (2015) use a measure of banks' exposure to interest rates ("income gap") to capture the key properties of term premia, using a partial equilibrium model to illustrate the mechanism. Greenwood and Vayanos (2014) show empirically how the supply and the maturity structure of government bonds affect bond yields and expected returns. Relative to these papers, the state variable in my paper can be interpreted as a macro factor that is helpful in capturing the yield's dynamics.

1.2 Model

In this section I study an endowment economy populated by heterogeneous investors, and I assume that the sole source of heterogeneity among investors is in their preferences. In particular, investors differ in their RA and EIS. I provide a sensitivity analysis regarding this assumption, and highlight the importance of heterogeneous EISs for capturing the term structure dynamics.

Setup. I consider an exchange economy in which time is continuous, denoted by $t > 0$. Uncertainty in the economy is characterized in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a standard filtration. There is a single perishable good, the numeraire. Aggregate endowment of this good follows a Geometric Brownian Motion (GBM)

$$\frac{dy_t}{y_t} = \mu dt + \sigma dW_{1,t}, \; y_0 > 0 \; , \tag{1.1}$$

where $W_1 = \{W_{1,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ is a Brownian motion on $(\Omega, \mathbb{P}, \mathcal{F})$ representing aggregate uncertainty, and parameters $\mu > 0$, $\sigma > 0$ are real numbers.

The economy is populated by two classes of investors, A and B. The aggregate population remains constant and normalized to one. To obtain a stationary solution in the model, I follow Gârleanu and Panageas (2015) and I consider a simple OLG framework in line with Blanchard (1985). Investors face an exogenous death risk $\varphi > 0$, and a fraction φ of new investors are born. The probability φ is the same for all investors regardless of age, preference, or wealth. Of the newly born investors, a constant fraction $\overline{x} \in (0, 1)$ is of type A, while $1 - \overline{x}$ is of B-type. Newly born investors receive a "start-up" endowment, perfectly tradable, in order to begin their operations in financial and goods markets.

Intuitively, since risk-tolerant investors operate with leverage in equilibrium, their net worth grow faster when there is a sequence of positive returns. This implies that they can end up dominating the economy (or disappearing, if the sequence of shocks is sufficiently negative). The OLG setup prevents this outcome without changing the fundamental risk-sharing properties driven by preference heterogeneity.[3]

To insure against exogenous death risk, investors can write contracts with perfectly competitive insurance companies. The possibility of insuring against death risk, together with the financial instruments specified below, implies that this economy has complete markets. The contract specifies that the investor receives a flow of resources φ, proportional to his

[3]There are several ways to obtain stationarity, and some papers have already used a similar OLG device I use in this paper (Drechsler et al. (2017), Dou (2017), Silva (2016), and Barro et al. (2017)). Also, Di Tella (2017) assumes that leveraged agents ("experts") face a probability of becoming unleveraged agents ("households"). Di Tella and Kurlat (2017) introduce an exogenous tax that redistributes wealth from leveraged agents to unleveraged.

net worth, per unit of time. For this, he agrees to pay his entire net worth to the insurance company upon his death. Investors find it optimal to sign this contract, provided they have no bequest motives (Blanchard, 1985). As I show below, this device is useful to introduce stationarity in the model. I next introduce investors' preferences and balance sheets.

Preferences and balance sheets. Investors feature recursive preferences, as in Duffie and Epstein (1992b). For each investor i, his utility function $\mathcal{U}_{i,t}$ is given by

$$\mathcal{U}_{i,t} = E_t^{\mathbb{P}} \left[\int_t^{\infty} f\left(c_{i,u}, \mathcal{U}_{i,u}\right) du \right],$$

where

$$f\left(c_i, \mathcal{U}_i\right) = \frac{1}{1 - 1/\psi_i} \left(1 - \gamma_i\right) \mathcal{U}_i \left\{ c_i^{1-1/\psi_i} \left(\left(1 - \gamma_i\right) \mathcal{U}_i\right)^{\frac{1/\psi_i - 1}{1-\gamma_i}} - \left(\rho + \varphi\right) \right\}. \tag{1.2}$$

In this notation, ψ_i represents the EIS and γ_i the RA, for this investor. Also, c_i represents the flow of consumption and ρ the time preference, which is adjusted by φ (Gârleanu and Panageas, 2015). These preferences are useful because they disentangle the RA coefficient from the EIS—a crucial aspect of the model that allows me to focus on the following assumption.

Assumption 1. *In the remainder of the paper, I assume*

$$i) \quad \gamma_A < \gamma_B ,$$

$$ii) \quad \psi_A > \psi_B .$$

This assumption means that A-type investors are relatively more risk tolerant and are relatively more willing to substitute consumption across time. Qualitatively, this feature is implicitly assumed under time-additive constant relative risk aversion (CRRA) preferences.

Each investor continuously trades two classes of financial assets: shares on a risky claim and positions in risk-free money market account. I denote by q_t the price of the risky asset. This asset pays, each period, a unit of the endowment minus the amount of resources allocated to the "start-up" wealth of the newly born. I denote $s_{i,t}$ the number of shares a given investor holds in this asset. The price of the risky asset follows an Itô process

$$\frac{dq_t}{q_t} + \left(\frac{y_t - \varphi e_t}{q_t}\right) dt = \mu_{q,t} dt + \sigma_{q1,t} dW_{1,t} \ , \tag{1.3}$$

where the drift $\mu_{q,t}$ and the diffusion $\sigma_{q1,t}$ are determined in equilibrium, and e_t is represents the resources for the newly born investors—which I describe below. Thus, q_t accounts for the total wealth in the economy.[4]

Let $\widetilde{t_i}$ be the investor's i birth time. Since investor's optimal decisions will not depend on their age, I simplify the notation and remove explicit dependence of variables to $\widetilde{t_i}$. The total net worth $n_{i,t}$ of an operating investor in period $t > \widetilde{t_i}$ is given by the following accounting identity

$$n_{i,t} = q_t s_{i,t} - b_{i,t} \ , \tag{1.4}$$

where $b_{i,t}$ is the value of the short-term money market account held by investor i. Positions in this account receive a return of $r_t dt$—i.e., the spot real risk-free rate.

[4] I show this in the appendix.

Using (1.3) and (3.15), I can write the law of motion for the net worth of an operating investor

$$\frac{dn_{i,t}}{n_{i,t}} = \left[r_t - \frac{c_{i,t}}{n_{i,t}} + \frac{s_{i,t}q_t}{n_{i,t}}\left(\mu_{q,t} - r_t\right) + \varphi\right]dt + \frac{s_{i,t}q_t}{n_{i,t}}\sigma_{q,t}dW_{1,t}, \ t > \widetilde{t}_i, \tag{1.5}$$

and I define $\alpha_{i,t} = \frac{s_{i,t}q_t}{n_{i,t}}$ as investor's i portfolio share. Notice that investors receive φ from the insurance company that collects his wealth upon his death.

Newly born investors receive an initial level of wealth and can immediately start operating in financial and goods markets . These resources are perfectly tradable. I follow Gârleanu and Panageas (2015) and assume that any investor, of any type, born in $\widetilde{t} < t$ receives an endowment process given by $y_{t,\widetilde{t}} = \omega y_t G\left(t - \widetilde{t}\right)$, with $\omega \in (0,1)$ and G a deterministic function that controls the investor's life-earning profile, specified below. Thus, in period t the present value of initial earnings (i.e., initial endowment) for an investor born today in t is

$$e_t = y_t E_t^{\mathbb{Q}}\left[\int_t^{+\infty} \exp\left(-\int_t^h r_u du\right)\omega\frac{y_h}{y_t}G\left(h-t\right)dh\right]. \tag{1.6}$$

The expectation is computed under the equivalent martingale measure on $(\Omega, \mathcal{F}, \mathbb{Q})$, which is guaranteed to exist since markets are complete and there are no arbitrage opportunities. At an aggregate level, in a given period t, the resources associated with initial earnings account for a total of $\widehat{e}_t$ (as a share of y_t), denoted by

$$\begin{aligned} \widehat{e}_t &= \frac{1}{y_t}\int_{-\infty}^t \varphi \exp\left(-\varphi\left(t-u\right)\right)e_u du \\ &= E_t^{\mathbb{Q}}\left[\int_t^{+\infty}\exp\left(-\int_t^h r_u du\right)\omega\frac{y_h}{y_t}dh\right]. \end{aligned} \tag{1.7}$$

The last step follows by normalizing the function $\int_{-\infty}^{t} \varphi \exp(\varphi (u-t)) G(t-u)\, du = 1$, and by a simple application of Fubini's theorem. Notice that I can write $\widehat{e}_t = \omega(q_t/y_t)$. Thus, the endowment claim (total wealth) is basically the replication of two assets: aggregate earnings $\widehat{e}_t$ and an asset $\widehat{q}_t$ that pays a dividend equal to $(1-\omega)\, y_t$ per unit of time. That is

$$\widehat{q}_t = E_t^{\mathbb{Q}} \left[\int_t^{\infty} \exp\left(- \int_t^h r_u du \right) (1-\omega) \frac{y_h}{y_t} dh \right]. \tag{1.8}$$

I can now write the dynamic problem of investor i, whose birth was in $\widetilde{t}_i$, as

$$\max_{\{s_i, c_i\}} \mathcal{U}_{i,t}$$

subject to

(1.5), (1.6),

where the control variables are the number of shares on the endowment claim, s_i, and the consumption flow, c_i. I next define a competitive equilibrium.

Definition 1 (Competitive equilibrium)**.** *A competitive equilibrium is a set of adapted stochastic processes for the investor's problem $c_A, c_B, \alpha_A, \alpha_B$, and a set of prices r, q such that: (1) Given prices, policy functions solve investors' problem; (2) and the goods and asset market clears (money market clears by Walras' Law)*

$$\begin{aligned} \int_{\mathbb{A}_t} c_{i,t} di + \int_{\mathbb{B}_t} c_{i,t} di &= y_t \ , \\ \int_{\mathbb{A}_t} s_{i,t} di + \int_{\mathbb{B}_t} s_{i,t} di &= 1 \ , \end{aligned}$$

where $\mathbb{A}_t$ and $\mathbb{B}_t$ are the sets of investors A and B in period t, respectively.

1.3 Solving for the Equilibrium

The purpose of this section is to represent the model in a recursive fashion. The equilibrium is characterized by the endogenous distribution of net worth across investors. However, the state space can be simplified by using the fact that investor's optimal choices are linear in their net worth and that investor's death risk is independent of their age. This implies investor's within a preference type undertake the same actions. Thus, I can derive the equilibrium conditions as a function of the following endogenous state variable

$$x_t = \frac{n_A}{n_A + n_B}, \tag{1.9}$$

where $n_{A,t} = \int_{\mathbb{A}_t} n_{i,t} di$ and $n_{B,t} = \int_{\mathbb{B}_t} n_{i,t} di$. The variable $x_t \in (0, 1)$ is the relative market value investor A's net worth, and it captures aggregate conditions in the credit market. Intuitively, when x_t is low, the aggregate ability of risk-tolerant investors to supply credit decreases. As shown below in Proposition 3, this type of investor choose an equilibrium portfolio share that is greater than one (i.e., they are leveraged).

The law of motion of x follows from applying Itô's lemma to ratio (1.9). This is important for pinning down the dynamics of the endogenous variables in a Markov equilibrium. In what follows, I express all aggregate endogenous state variables as a function of x. That is, I seek to solve investors' control variables (their consumption-wealth ratios and portfolio shares), the price of the endowment claim (q/y) and the interest rate (r), as a function of x. The system of ordinary differential equations that characterize the equilibrium consists of investors' value function (Hamilton-Jacobi-Bellman equations), the no-arbitrage conditions for total wealth and initial wealth, together with the market clearing conditions for consumption and shares (the money market account clears by Walras' Law).

Proposition 1 (Law of motion for x). *The endogenous state variable x follows an Itô process*

$$dx_t = \mu_{x,t} dt + \sigma_{x,t} dW_{1,t} \ , \tag{1.10}$$

where

$$\begin{aligned} \mu_{x,t} &= x_t (1 - x_t) \left(\frac{c_{B,t}}{n_{B,t}} - \frac{c_{A,t}}{n_{A,t}} + (\alpha_{A,t} - \alpha_{B,t}) \left(\mu_{q,t} - r_t - \sigma_{q1,t}^2 \right) \right) + \frac{\varphi \widehat{e}_t}{pd_t} (\overline{x} - x_t) \, , \\ \sigma_{x,t} &= x_t (1 - x_t) (\alpha_{A,t} - \alpha_{B,t}) \sigma_{q1,t}, \\ x_0 &\in (0, 1) \, , \end{aligned}$$

with functions $\alpha_{A,t} = \alpha_A (x_t) ; \alpha_{B,t} = \alpha_B (x_t) ; \frac{c_{A,t}}{n_{A,t}} = \frac{c_A}{n_A} (x_t) ; \frac{c_{B,t}}{n_{B,t}} = \frac{c_B}{n_B} (x_t) ; r_t = r (x_t) ; \mu_{q,t} = \mu_q (x_t) ; \sigma_{q1,t} = \sigma_{q1} (x_t) ; q/y = pd (x_t)$. The initial x_0 is a number in $(0, 1)$. Provided $\mu_{x,t}$ and $\sigma_{x,t}$ satisfy the usual uniform Lipschitz and linear growth condition in x, then the stochastic differential equation (1.10) is strong Markov and has a unique solution.

Proof. See appendix. □

Notice that the second term in the drift function $\mu_{x,t}$ is due to the demographic structure assumed above. This term is key for obtaining an invariant distribution of x. Informally, notice that for very small values of x, the diffusion tends to zero and the drift becomes larger and positive. Thus, the process never reaches zero. Similar logic implies an upper boundary at one.[5]

The diffusion term, $\sigma_{x,t}$, depends on the differences in investors' portfolio shares. If $\alpha_{i,t}$'s were the same for both investors, then $dW_{1,t}$ shocks would not affect x. As a result, when the economy reaches the stochastic steady state (i.e., when $\mu_{x,t} = 0$), it remains there. This implies that differences in investors' exposure to aggregate risk are critical for obtaining fluctuations in the wealth distribution in this setup.

Hamilton-Jacobi-Bellman Equation and investors' first order conditions. The investor's problem can be written recursively

$$0 = \max_{c_i, s_i} f\left(c_{i,t}, \mathcal{U}_{i,t}\right) + E^{\mathbb{P}}\left[d\mathcal{U}_{i,t},\right] \tag{1.11}$$

subject to his budget constraint (1.5) and his initial wealth (1.6). To solve the recursive problem, I appeal to the homotheticity properties of the value function and the constraints. This implies that the value function can be written in the following power form:

$$\mathcal{U}_{i,t}\left(x_t, n_{i,t}\right) = \frac{\left(\xi_{i,t}^{\frac{1}{1-\psi_i}} n_{i,t}\right)^{1-\gamma_i}}{1-\gamma_i}\ , \tag{1.12}$$

where the known function $\xi_i\left(x_t\right)$ captures the investor's valuation of the future investment

[5]Technically, the second term changes the speed of the process at the boundary. See Karlin and Taylor (1981), chapter 15, for a discussion of the boundary behavior of Itô processes.

opportunities. This function can be expressed as an Itô process,

$$\frac{d\xi_{i,t}}{\xi_{i,t}} = \mu_{\xi i,t}dt + \sigma_{\xi i,t}dW_{1,t} \ , \tag{1.13}$$

with adapted processes $\mu_{\xi i,t} = \mu_{\xi i}(x_t)$ and $\sigma_{\xi i,t} = \sigma_{\xi i}(x_t)$ determined in equilibrium. Using (1.12) and (1.13) in (1.11), the problem can be written with $\frac{c_i}{n_i}$ (i.e., the consumption-wealth ratio) and $\alpha_i = \frac{s_i q_t}{n_i}$ (the portfolio share) as control variables

$$\begin{aligned} 0 \ = \ & \max_{\left\{\frac{c_i}{n_i},\alpha_i\right\}} \frac{\psi_i}{\psi_i - 1}\left(\left(\frac{c_i}{n_i}\right)^{1-\frac{1}{\psi_i}}(\xi_i)^{\frac{1}{\psi_i}} - (\rho+\varphi)\right) + E^{\mathbb{P}}\left[\frac{dn_i}{n_i}\right] - \frac{\gamma_i}{2}E^{\mathbb{P}}\left[\left(\frac{dn_i}{n_i}\right)^2\right] \\ & + \frac{1}{1-\psi_i}\left[\mu_{\xi i} + \frac{1}{2}\left(\frac{1-\gamma_i}{1-\psi_i} - 1\right)\sigma^2_{\xi i,t}\right] + \left(\frac{1-\gamma_i}{1-\psi_i}\right)E^{\mathbb{P}}\left[\frac{d\xi_i}{\xi_i}\frac{dn_i}{n_i}\right] \ , \\ & \textit{subject to} \\ & (1.5), (1.6). \end{aligned} \tag{1.14}$$

The first-order conditions (FOC) of this problem, for investor i, are given by

$$\frac{c_i}{n_i} = \xi_i \ , \tag{1.15}$$

$$\alpha_i = \frac{\mu_q - r}{\gamma_i \sigma_q^2} + \left(\frac{1-\gamma_i}{1-\psi_i}\right)\frac{\sigma_{\xi i}}{\gamma_i \sigma_q}. \tag{1.16}$$

Investors' demand for the risky asset consists of a "myopic" term, $\frac{\mu_q - r}{\gamma_i \sigma_q^2}$, and a "hedging" term, $\left(\frac{1-\gamma_i}{1-\psi_i}\right)\frac{\sigma_{\xi i}}{\gamma_i \sigma_q}$. In the representative agent economy, $\alpha = 1$ by market clearing. However, this is not the case in heterogeneous-investor economies in which different classes of investors can participate in the market for the risky asset. In the next proposition, I characterize the

A-type investor's demand for the risky asset, and show that A-type investors operate with leverage in equilibrium if and only if $\gamma_A < \gamma_B$.

Proposition 2 (Leverage and Risk Sharing)**.** *(1) A-type investor's demand for risky assets is given by*

$$\alpha_A(x) = \frac{1-(1-x)\,x\frac{R(x)}{\gamma_B}}{x+(1-x)\left[\frac{\gamma_A}{\gamma_B}-\frac{xR(x)}{\gamma_B}\right]}\ ,$$

with

$$R(x) = \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} - \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}.$$

(2) Aggregate risk is concentrated in A-type investors (i.e., $\alpha_A > 1$), and thus positive aggregate endowment shocks increase x if and only if $\gamma_A < \gamma_B$.

Proof. See appendix. □

The variable $R(x)$ above captures the risk-sharing mechanism. Mechanically, $R(x)$ can be written as the difference in the sensitivity of the value functions with respect to x. That is,

$$R(x) = \frac{d\log\mathcal{U}_A}{dx} - \frac{d\log\mathcal{U}_B}{dx}. \tag{1.17}$$

A negative (positive) R implies that a marginal increase in x improves the utility of B (A) relatively more. Notice that R would be zero if there were no motive to share aggregate risk (and $\alpha = 1$).

Discussion of assumption 1. There is a large literature documenting heterogeneity in EIS among individuals (see Guvenen (2006) for a summary). In general, the evidence in the literature shows that people who choose to be more exposed to aggregate risk (for example by holding stocks) exhibit a larger EIS. My assumption follows this line: in the model presented above, low-RA investors choose to be more exposed to aggregate risk and I assume they have a larger EIS. In my setup, heterogeneity in risk aversion is important because aggregate risk is concentrated in risk-tolerant agents, and therefore x increases after a positive endowment shock (i.e., $\sigma_x > 0$). Put differently, x would not react to macro shocks if $\gamma_A = \gamma_B$. In contrasts, Guvenen (2009), who also studies an economy with heterogeneous EISs, finds that differences in RA are not relevant in his results. This is because he studies an economy in which there is limited market participation. This last assumption immediately implies that stockholders (i.e., those who are allowed to trade the risky asset) concentrate aggregate risk.

The assumption is qualitatively in line with time-additive preferences featuring CRRA preferences. Under CRRA preferences, the EIS is set to be the inverse of the RA coefficient. Thus, the assumption of $\gamma_A < \gamma_B$ would immediately lead to $\psi_B < \psi_A$, as stated in assumption 1. This is consistent with Longstaff and Wang (2012), Wang (1996), and Hall (2017a), among others. In the context of time-additive preferences, heterogeneous EISs can be rationalized as differences in an agent's willingness to substitute across goods (see, for example, Atkeson and Ogaki (1996)). One interpretation of $\psi_B < \psi_A$ is that type-A investors' expected consumption is more sensitive to fluctuations in spot interest rates, but less than one-to-one.[6] As I show below, the distinction between RA and EIS is crucial in capturing

[6]Suppose consumption follows an Itô process with constant drift μ_{ci} and diffusion σ_{ci} for investor i, then

$$\mu_{ci} = \psi_i (r - \rho) + (1 + \psi_i) \gamma_i \sigma_{ci}^2,$$

so the greater ψ_i, the more sensitive is expected consumption to movements in r. If $\psi_i < 1$, movements are less than one-to-one.

the quantitative properties of the yield curve.

1.4 Term Structure of Interest Rates

Equipped with the equilibrium definition and the model's solution, I can now characterize the term structure of interest rates in the economy. Since the economy features complete markets and there are no arbitrage opportunities, I can obtain a stochastic discount factor "as if" there were a representative agent (Constantinides and Duffie (1996)). The properties of the discount factor, characterized below in proposition 4, depend on the risk-sharing dynamics of the economy.

After deriving the discount factor, I value zero-coupon bonds. I start by analyzing the properties of real bonds, (i.e., assets that pay a unit of consumption in the future), and then extend to value nominal bonds (i.e., assets whose cash flow is in monetary units). In this analysis, money is solely a unit of account, and I assume the marginal investor can transform money into goods (and vice versa) without any friction whatsoever.

I next derive the real stochastic discount factor.

Proposition 3. *The state-price process* $m(x_t) > 0$ *satisfies*

$$\frac{dm_t}{m_t} = -r(x_t)\,dt - \kappa(x_t)\,dW_{1,t} \ ,$$

with

$$\kappa\left(x_t\right) = \frac{\sigma_{q1}\left(x_t\right) - x_t\left(\frac{1-\gamma_A}{(1-\psi_A)\gamma_A}\right)\sigma_{\xi A}\left(x_t\right) - \left(1-x_t\right)\left(\frac{1-\gamma_B}{(1-\psi_B)\gamma_B}\right)\sigma_{\xi B}\left(x_t\right)}{\frac{x}{\gamma_A}+\frac{1-x}{\gamma_B}}, \tag{1.18}$$

$$r\left(x_t\right) = \mu_q + \frac{1}{pd\left(x_t\right)} - \kappa\left(x_t\right)\sigma_{q1} - \frac{\widehat{e}\left(x_t\right)}{pd\left(x_t\right)}\varphi\ , \tag{1.19}$$

where $r\left(x_t\right)$ and $\kappa\left(x_t\right)$ are adapted and bounded processes. The process for x is given by (1.10).

Proof. See appendix. □

Then, I can define the process ζ_t as

$$\zeta_t = \exp\left(\int_0^t \kappa\left(x_u\right) dW_{1,u} - \frac{1}{2}\int_0^t \kappa\left(x_u\right)^2 du\right), \tag{1.20}$$

which is a martingale in $\mathbb{P}$ and represents the Radon-Nikodym derivative $d\mathbb{Q} = \xi_T d\mathbb{P}$, provided regular conditions are verified. [7] With a standard application of Girsanov's theorem, I can define a Brownian motion in the equivalent martingale measure $\mathbb{Q}$.

To derive the real yield curve, I calculate the price of real zero-coupon bonds. Let $P_t^{(T)}$ represent the price of an asset that pays a unit of consumption in T periods from now (t) (i.e., a zero-coupon bond). So $P_t^{(0)} = 1$. Then

$$P_t^{(T)} = E_t^{\mathbb{P}}\left[\frac{m_{t+T}}{m_t}\right] \equiv E_t^{\mathbb{Q}}\left[\exp\left(\int_t^{t+T} r\left(x_u\right) du\right)\right] \equiv P\left(x_t, T\right). \tag{1.21}$$

[7] In particular, Novikov's condition, $E^{\mathbb{P}}\left[\exp\left(\int_0^T \kappa_t\left(x_s\right)^2 ds\right)\right] < \infty$, which holds since $\kappa(x)$ is a bounded function and x is Markov.

The real yield can be computed from prices as $y_t^{(T)} = -\frac{\log P_t^{(T)}}{T}$, while forward rates from T to $T+j$, $y_{f,t}^{(T\to T+j)}$, follow immediately by no-arbitrage. I next characterize the value of the real bond (1.21).

Problem 1 (Valuing real bonds). *The price of the real bond $P(x,T)$ (a T-real bond) solves the following Cauchy problem:*

$$\begin{aligned} -P_T'(x,T) + \mathcal{L}P(x,T) - r(x)P(x,T) - \kappa(x)P_x'(x,T)\sigma_x(x) &= 0 \,, \qquad (1.22) \\ P(x,0) &= 1,\ \forall x, \end{aligned}$$

where $\mathcal{L}$ is the differential operator in x.

From (1.22), the term premium of a T-quarter real zero-coupon bond is given by

$$\begin{aligned} E^{\mathbb{P}}\left[\frac{dP_t^{(T)}}{P_t^{(T)}}\right] - r_t dt &= \underbrace{-cov_t^{\mathbb{P}}\left(\frac{dm_t}{m_t}, \frac{dP_t^{(T)}}{P_t^{(T)}}\right)}_{\text{T-real term premium}}, \qquad (1.23) \\ &= \underbrace{\frac{P_x'^{(T)}}{P^{(T)}}}_{>0}\underbrace{\kappa(x_t)\sigma_x(x_t)}_{>0}. \end{aligned}$$

The sign of the T-real term premium is characterized by the derivative $P_x'^{(T)}$, since $\kappa(x_t)\sigma_x(x_t) > 0\ \forall t$ by definition. Mechanically, bond prices are higher in states of nature in which the real rate is lower (i.e., there is an inverse relationship between zero-coupon bond prices and rates). In the model, r is high in states in which x is low. This implies that $P_x'^{(T)} > 0$. If those states correspond to high prices of risk, the market will compensate the marginal investor with a positive premium to hold a T-real bond. Below, I elaborate this intuition further, when I present the model solution and the term structure of interest rates. In particular, when I show numerical results for the covariance term (1.23).

Nominal Term Structure: Exogenous Inflation. I first consider the case in which inflation is exogenous, as in other papers in the macro-finance term structure literature (e.g., Piazzesi and Schneider (2006), Bansal and Shaliastovich (2013), among others). That is, I compute the nominal stochastic discount factor—which is used to discount future cash flows denominated in dollars—by introducing exogenous fluctuations in the purchasing power of a dollar (i.e., exogenous fluctuations in the price level).

The objective is to study the role of inflation risk, since in this case the nominal term premium is driven by the assumption that inflation and real shocks are negatively correlated: inflation occurs in high marginal utility states. In other words, this assumption implies that the purchasing power of nominal payments decreases precisely when the marginal investor require those resources the most. Therefore, the market has to compensate the marginal investor with a premium to hold such an asset. In the quantitative analysis below, I provide a decomposition of the nominal term premium in order to quantify the role of this negative correlation vis-à-vis the real component.

I introduce an exogenous price process p_t (CPI), as in Cox et al. (1985). That is,

$$\begin{aligned} \frac{dp_t}{p_t} &= \pi_t dt + \sigma_p \sigma\left(\pi_t\right) dW_{2,t},\ p_0 > 0 \ , \\ d\pi_t &= \lambda_\pi \left(\overline{\pi} - \pi_t\right) dt + \sigma\left(\pi_t\right) dW_{3,t},\ \pi_0 > \pi_L, \qquad (1.24) \\ & \textit{with} \\ \sigma\left(\pi_t\right) &= \sigma_\pi \sqrt{\pi_t - \pi_L} \ , \end{aligned}$$

where $W_2 = \{W_{2,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ and $W_3 = \{W_{3,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ are aggregate Brownian motions in the probability space $(\Omega, \mathbb{P}, \mathcal{F})$ representing shocks to inflation and shocks to expected inflation, respectively. The parameters $(\lambda_\pi, \overline{\pi}, \pi_L)$ are real numbers, and are associated with the persistence, mean, and the lower bound on inflation. Importantly, the exogenous process π_t is stationary (see appendix).

I assume that processes W_1 and W_3 are correlated; that is, $\langle dW_1 dW_3 \rangle_t = \phi_{13} dt$. In particular, I assume that $\phi_{13} < 0$, so shocks to *pi* and shocks to aggregate endowment are negatively correlated (Piazzesi and Schneider, 2006). This implies that a nominal asset is expected to produce lower real payments (i.e., inflation erodes the purchasing power of nominal payments) in periods of low growth, which creates persistent inflation risk. I assume that contemporaneous shocks to the CPI process are uncorrelated with W_1 and W_3. Similarly, I assume that W_2 and W_3 are uncorrelated. It is worth emphasizing that $\langle dW_2 dW_3 \rangle_t = \langle dW_1 dW_2 \rangle_t = 0$ is without loss of any generality, either from a quantitative or a qualitative perspective. This is because these shocks are i.i.d., so they have a minor role (whereas dW_3 are persistent). I assume this to focus on the role of persistent inflation risk.

Then, I can define a nominal pricing kernel, $m_t^\$ = m_t / p_t$. Using Itô's lemma

$$
\begin{aligned}
\frac{dm_t^\$}{m_t^\$} &= \frac{dm_t}{m_t} - \frac{dp_t}{p_t} + \left(\frac{dp_t}{p_t}\right)^2 - \frac{dp_t}{p_t}\frac{dm_t}{m_t}, \\
&= -i_t dt - \kappa_t dW_{1,t} - \sigma_p dW_{2,t} \ ,
\end{aligned}
$$

where i_t represents the nominal interest rate

$$i(x_t, \pi_t) = r(x_t) + \pi_t - \sigma_p^2 \sigma(\pi_t)^2 . \tag{1.25}$$

Notice that (1.25) is the Fisher equation, plus an "Itô adjustment", $\sigma_p^2 \sigma(\pi_t)^2$, that is quantitatively small. With these elements, I next value zero-coupon nominal bonds. Let $P_t^{\$,(T)}$ be the price of a nominal zero-coupon bond paying one dollar T periods from now. Thus

$$P_t^{\$,(T)} = E_t^{\mathbb{P}}\left[\frac{m_{t+T}^{\$}}{m_t^{\$}}\right] \equiv E_t^{\mathbb{Q}}\left[\exp\left(\int_t^{t+T} i(x_u, \pi_u)\, du\right)\right] \equiv P^{\$}(x, \pi, T) \ .$$

Problem 2 (Valuing nominal bonds: Exogenous inflation). *The price of the nominal bond $P^{\$}(x, \pi, T)$, a T-nominal bond when inflation is exogenous, solves the following Cauchy problem:*

$$\begin{aligned} -P_T'^{\$}(x,\pi,T) + \mathcal{L}P^{\$}(x,\pi,T) - i(x,\pi)\, P^{\$}(x,\pi,T) &= \kappa(x)\, P_x'^{\$}(x,\pi,T)\, \sigma_x \\ &\quad + \kappa(x)\, P_\pi'^{\$}(x,\pi,T)\, \sigma(\pi)\, \phi_{13} \ , \\ P^{\$}(x,\pi,0) &= 1, \ \forall (x,\pi) \ , \end{aligned} \tag{1.26}$$

where $\mathcal{L}$ is the differential operator in x and π.

Equation (1.26) shows that the nominal term premium can be decomposed into a real component and a nominal component. That is,

$$
\begin{aligned}
T - \textit{nominal term premium} &= -cov_t^{\mathbb{P}}\left(\frac{dm_t^{\$}}{m_t^{\$}}, \frac{dP_t^{\$,(T)}}{P_t^{\$,(T)}}\right) \\
&= \overbrace{\underbrace{\frac{P_x^{\$,(T)}}{P^{\$}}}_{>0}\underbrace{\sigma_x\left(x_t\right)\kappa\left(x_t\right)}_{>0}}^{\text{real}} + \overbrace{\underbrace{\frac{P_\pi^{\$,(T)}}{P^{\$}}}_{<0}\underbrace{\phi_{13}}_{<0}\underbrace{\sigma\left(\pi_t\right)\kappa\left(x_t\right)}_{>0}}^{\text{nominal}}
\end{aligned} \tag{1.27}
$$

Both terms in (1.27), the real and the nominal, are positive. The real component is positive primarily because $P_x^{\prime\$,(T)} > 0\ \forall\left(x, \pi, T\right)$, and the intuition is the same as the one described above for the real bond. The sign of the nominal component, however, depends on the sign of the correlation between endowment shocks and inflation expectation shocks, ϕ_{13}. This is because $P_\pi^{\prime\$,(T)} < 0\ \forall\left(x, \pi, T\right)$: An increase in inflation expectation increases the spot nominal rate (via the Fisher identity established in (1.25)). Thus, the price of the nominal bond price, for any finite maturity, decreases when inflation expectations increases—i.e., the derivative with respect to π is negative across the state space. But since $\phi_{13} < 0$, then positive endowment, or "supply," shocks are associated with negative shocks to inflation expectations. Economically, this means that nominal payments are expected to be eroded by inflation during periods in which investors value those resources the most. So the sign of the $\phi_{13}P_\pi^{\prime\$,(T)}$ determines the sign of nominal component of the nominal term premium.

The Nominal Term Structure: Endogenous Inflation. Instead of extending the state space by adding an exogenous inflation process, another alternative is to derive a process for π_t via a simple monetary policy rule, conducted by a monetary authority. Thus, I consider a monetary authority that determines the inflation rate $\frac{dp_t}{p_t}$ in a way that is consistent with the marginal investor's stochastic discount factor (e.g., Gallmeyer et al. (2007b)). For this, I consider a standard specification of such a rule in the form of a so-called Taylor rule

$$i_t^{MP} dt = \delta_0 dt + \delta_\pi \left(\frac{dp_t}{p_t} - \overline{\pi} dt \right) , \tag{1.28}$$

where i_t^{MP} represents the monetary policy rate, δ_0 is a constant ("intercept"), and δ_π is the "Taylor loading," $\overline{\pi}$ the inflation target, and $\frac{dp_t}{p_t} = \pi_t dt$ is the instantaneous change in the CPI.[8] Since I consider a fully flexible-prices endowment economy, there is no output gap in this rule.

The nominal interest rate i_t^{MP} has to clear the nominal bond market, and for this it must be consistent with the nominal pricing kernel. This implies

$$\begin{aligned} i_t^{MP} dt &= -E_t^{\mathbb{P}} \left[\frac{dm_t^{\$}}{m_t^{\$}} \right] , \\ \delta_0 + \delta_\pi \left(\pi_t - \overline{\pi} \right) &= r\left(x_t\right) + \pi_t , \end{aligned} \tag{1.29}$$

which is the standard Fisher equation. Thus, I can solve for the endogenous $\pi\left(x_t\right)$ by solving (1.29). That is,

$$\pi\left(x_t\right) = \frac{\delta_0 - \delta_\pi \overline{\pi}}{\left(1 - \delta_\pi\right)} + \frac{r\left(x_t\right)}{\delta_\pi - 1}. \tag{1.30}$$

Equation (1.30) shows that under $\delta_\pi = 1$, inflation expectations are not well defined (i.e., a version of the Taylor principle is violated). Then, using (1.30), the nominal interest rate

[8]The monetary authority implements the rule such that, in equilibrium, the stochastic process for the price level p_t is locally "smooth," i.e., $\sigma_{p,t} = 0$. That is, the monetary policy is consistent with the conditional expectation of the stochastic discount factor.

takes the form of

$$i_t = i_t^{MP} \equiv \delta_0 + \delta_\pi \frac{\delta_0 - \delta_\pi \overline{\pi}}{(1-\delta_\pi)} + \frac{\delta_\pi}{\delta_\pi - 1} r(x_t).$$

This means that when $\delta_\pi > 1$ (which is commonly used in the literature) the loading on the real component, $\frac{\delta_\pi}{\delta_\pi - 1}$, is greater than one. In other words, the nominal interest rate magnifies fluctuations in the real risk-free rate.

With the derived $\pi_t = \pi(x_t)$, I can value nominal bonds. It is worth emphasizing that inflation is not a state variable to value nominal bonds, as opposed to (1.26). Instead, the sole state variable (other than time to maturity) is x_t. That is, $P_t^{\$,(T)} = P^{\$}(x,T)$. This implies that the problem of valuing nominal bonds is similar to (1.22).

Problem 3 (Valuing nominal bonds: Endogenous inflation) *The price of the nominal bond $P^{\$}(x,T)$, a T-nominal bond when inflation is endogenous, solves the following Cauchy problem:*

$$\begin{aligned} P_T'^{\$}(x,t) + \mathcal{L}P^{\$}(x,T) - i^{MP}(x) P^{\$}(x,T) &= \kappa(x) P_x'^{\$}(x,T)\sigma_x \qquad (1.31) \\ P^{\$}(x,0) &= 1, \ \forall x \ , \end{aligned}$$

where $\mathcal{L}$ is the differential operator in x.

Before concluding this section and proceeding to the quantitative analysis, I provide a proposition for the representative agent benchmark. In that case, all prices and quantities can be solved in closed form. Under this benchmark, real yields are constant and exhibit zero volatility.

Proposition 4 (Infinitely lived investor). *If preferences are the same (i.e., $\gamma_A = \gamma_B$ and*

$\psi_A = \psi_B$) and there is no mortality risk (i.e., $\varphi \to 0$), then

(i) the real risk-free rate is constant $r_t = \overline{r}$, with

$$\overline{r} = \rho + \frac{\mu}{\psi} - \left(1 + \frac{1}{\psi}\right) \frac{\gamma \sigma^2}{2};$$

(ii) the real term structure is flat and the volatility of yields is zero at all maturities

$$\begin{aligned} y_t^{(T)} &= r_t = \overline{r}, \quad \forall (t, T) \ , \\ var\left(y_t^{(T)}\right) &= 0, \quad \forall (t, T) \ ; \end{aligned}$$

(iii) the price-dividend ratio is constant, $pd_t = \overline{pd}$;

(iv) under exogenous inflation, the nominal term structure depends on inflation expectations only. Nominal bond prices can be solved in closed form and equal to

$$P^{\$,(T)} (\pi, t) = A(t) \exp \left(B(t) \pi + C(t) \sqrt{\pi - \pi_L} \right) \ ,$$

where coefficients $A(t)$, $B(t)$, and $C(t)$ solve the system reported in the appendix; and [9]

(v) under endogenous inflation, the nominal term structure is flat and the volatility of nominal yields is zero at all maturities.

[9] The resulting partial differential equation for the nominal bond price is similar to the one resulting from the "double squared root process" for the interest rate. I thank Francis Longstaff for bringing this point to me.

Proof. See appendix. □

1.5 Quantitative analysis

In this section I explore the quantitative properties of the model. To that end, I solve the model—and the corresponding partial differential equations for bond prices—numerically. I use a global solution technique based on spectral methods (Trefethen, 2000; Boyd (2001)). I start by describing the calibration procedure and then discuss the model's solution. I continue with an analysis of the real term structure, and conclude by studying the nominal term structure (with both exogenous and endogenous inflation).

Calibration. I report the calibration in Table 1.2, in which I divide parameters into groups: preferences, endowment and demography, and inflation. I calibrate parameters at a quarterly frequency.

Regarding preferences, there are mainly four parameters: $\gamma_A, \gamma_B, \psi_A$, and ψ_B. I set $\gamma_B = 10 > \gamma_A = 1.5$, which implies, on average, an aggregate γ of 5.1.[10] These values for risk aversion are within the range that have been used in the asset-pricing literature. Regarding the EIS, I set values for the ψ_A and ψ_B as free parameters, and explore different alternative specifications below. Intuitively, the larger the difference between ψ_A and ψ_B (ceteris paribus), the larger the increase of the spot real rate after an endogenous reduction in aggregate credit. In the baseline calibration, I use a ψ_A, ψ_B very similar to those in

[10] Aggregate γ is displayed in the denominator (1.18), which is the inverse of $\frac{x}{\gamma_A} + \frac{1-x}{\gamma_B}$.

Gârleanu and Panageas (2015).

I calibrate the endowment parameters, the drift μ and diffusion σ, to match the mean and volatility of time-integrated U.S. consumption data. Regarding demographic parameters, I set a value of φ such that investors have an expected time operating in the financial market of 30 years, and lastly I set $\overline{x} = 0.11$ which stabilizes the share of risk-tolerant agents net worth around 0.15 (I report the invariant distribution below). Lastly, I specify a function $G(t) = G_1 e^{-g_1 t} + G_2 e^{-g_2 t}$ (i.e., a double exponential) to be consistent with the hump-shaped income pattern over the life cycle of the investor. I set $(G_1; G_2) = (30.72/4; -30.29/4)$, which implies a similar pattern to that of Gârleanu and Panageas (2015), but at a quarterly frequency.

To derive the endogenous inflation expectations, I calibrate the Taylor coefficient δ_π=1.5 as a baseline (see Taylor (1993), and many others). I set the inflation target $\overline{\pi}$=0.005, which implies a 2% annual target. For the exogenous inflation expectation process, I set parameters $\sigma_\pi, \theta_\pi, \pi_L, \overline{\pi}$ such that the mean of inflation expectations is 0.9% per quarter to match the level of the nominal yield curve, and inflation spends 99% of time in the range [-0.5%, 2%] in quarterly terms. This captures the dynamics of observed inflation. I plot the invariant distribution for π in the appendix. Lastly, I set the correlations between inflation, endowment, and inflation expectation shocks as free parameters to illustrate their role in the decomposition between nominal and real term premium. In particular, I set $\phi_{12} = \phi_{23} = 0$, and I focus on the correlation between shocks to inflation expectations and shocks to the real economy. I set $\phi_{13} = -0.5$ as a plausible lower bound on this correlation, as many previous studies have found a greater number in the data (for example, Piazzesi and Schneider (2006) find -0.2).[11]

The Economic Mechanism in the Model. Figure 1.1 shows the solution of the relevant objects in the model that summarize the economic mechanism. First, notice that both the real risk-free rate and the price of risk move in tandem across the state space: Real interest rates are high (bond prices are low) when the aggregate price of risk is high. This occurs when the relative market value of risk-tolerant equity is low, which implies that the total amount of credit in the economy is low.

As shown in Proposition 3, negative aggregate shocks affect risk-tolerant investor's net worth relatively more. When risk-tolerant investors lose net worth, their ability to supply credit at an aggregate level is reduced. Thus, total credit as a fraction of total equity in the economy goes down. This produces an increase in the price of credit—the real rate —because the market has to compensate agents with a lower EIS to smooth consumption over time. On the other hand, since relatively more risk-averse investors are clearing the market, the price of the risky asset goes down and the aggregate price of risk increases, as shown in the Figure.

From a risk-sharing perspective, $R(x)$ represents how changes in x affect investors' utility,

[11] The closer ϕ_{13} is to zero, the smaller the nominal component of the nominal term premium. Even with $\phi_{13} = -0.5$, as discussed below, the nominal component is already relatively small in my model.

as shown in (1.17). In Figure 1.1, $R(x) < 0$ across the state space, which means

$$\begin{aligned} R(x) &= \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} - \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B} < 0 \\ &\Rightarrow \\ \frac{d\log\mathcal{U}_A}{dx} &< \frac{d\log\mathcal{U}_B}{dx} \end{aligned}$$

Thus, changes in x improve B-type investor's utility relatively more: An increase in x implies that they have to bear less aggregate risk, and since they dislike risk relatively more, their utility increases relatively more than that of an A-type. If there were no gains from sharing risk, R would be zero.

The Real Yield Curve. Figure 1.2 shows the results for the term structure up to 80 quarters. On average, the slope of the real yield curve is upward sloping: Real bonds are risky assets, since bond prices go down in states in which the price of risk increases. I report the term premium (i.e., the covariance in expression (1.22)) below. Interestingly, the yield curve features endogenous fluctuations across the business cycle. In particular, when there is a contraction in risk-tolerant investors' net worth (i.e., low x), the level of the real yield curve increases and its slope becomes negative. That is, x is negatively correlated with the level factor, and positively correlated with the slope of the curve.

A useful way to further understand the implications of the model is to study the interest rate dynamics. That is, $r(x_t)$, where the state variable x_t follows the law of motion in (1.10).

Then, using Itô's lemma, the interest rate dynamics are given by

$$\begin{aligned} dr_t &= \mu_{r,t} + \sigma_{r,t} dW_{1,t}, \\ & \quad with \\ \mu_{r,t} &= r'_x \mu_{x,t} dt + \frac{1}{2} r''_{xx} \sigma^2_{x,t}, \\ \sigma_{r,t} &= r'_x \sigma_{x,t} \; . \end{aligned} \tag{1.32}$$

Panel (b) of Figure 1.2 shows the drift and diffusion associated with r. In particular, notice that the expected change of r, $\mu_{r,t}$, becomes more negative when x decreases; because real rates are mean reverting, they expected to fall. The expectation that the short-term rate will decrease in the futures is strong enough to imply the reversion of the slope in panel (a).

I next study the term premium. Long-term rates consists of two components: the expectations of short rate dynamics and the term premium. More precisely, the premium a long-term bond commands is represented in equation (1.23),

$$E^{\mathbb{P}}\left[\frac{dP_t^{(T)}}{P_t^{(T)}}\right] - r_t dt = -cov_t^{\mathbb{P}}\left(\frac{dm_t}{m_t}, \frac{dP_t^{(T)}}{P_t^{(T)}}\right).$$

In Figure 1.3 I show the model's prediction for this covariance. The left-hand panel shows the covariance across the state space, for three different maturities (4, 20, and 80 quarters). The larger the horizon, the larger the premium the bond carries. Intuitively, the longer the horizon of the bond, the more likely it will lose value in a bad state at some point of its lifetime. The right-hand panel of the figure shows the mean term premium across horizons. This panel conveys the idea that long-term bonds are riskier than that of short-term, and therefore should pay a higher return on average.

I next study the very long end of the yield curve, which may have several practical purposes (from social security to government budget projections). To that end, I first solve the real term structure that matches the short part (up to 40 quarters), but up to a horizon where the yield curve becomes almost flat.[12] Then, I compute the volatility of 10-year forward contracts, which can be easily derived from bond prices. Figure 1.4 shows the real term structure up to 800 quarters (i.e., 200 years), and the volatility of 10-year forward rates. The figure shows that the average real term structure becomes flat at nearly 700 quarters, and forward rates have substantial volatility up to 10-year contracts between 280 quarters to 320 quarters (i.e., 70 years to 80 years).

Lastly, Table 2.3 displays the theoretical moments and illustrates the role of EIS heterogeneity in the model. As shown in Figure 1.2, the model matches the slope and volatility of the real term structure. Qualitatively, the model captures the fact that the volatility of real yields is downward sloping, although the volatility of the 40-quarters yield is higher than in the data (42 basis points versus 30 basis points at a quarterly frequency, respectively). The table also illustrates the risk-sharing mechanism that drives this result. Indeed, under the baseline calibration, A-type investors consume 0.0108 of their net worth (represented by ξ_A), whereas B-type consume a higher fraction, 0.0156. This is basically due to the fact that A-type investors are operating with leverage, so they consume a smaller fraction of

[12] I show the properties of a perpetual consol bond in the appendix.

their net worth on average. Also, as expected, the volatility of the consumption-wealth ratio of B-type investors is larger (they are less willing to smooth consumption intertemporally). This implies that investors are sharing aggregate risk, a measure denoted by R in the Table.

When B-type investors feature CRRA preferences, i.e., $\psi_B = 1/\gamma_B$, the gains from sharing risk are lower: Fluctuations in x have a relatively similar impact on investors' utility (R is close to zero). Table 2.3 shows that the consumption-wealth of both agents is relatively similar and less volatile than in the baseline calibration. This implies that leveraged investors are borrowing against investors who have a similar willingness to smooth consumption intertemporally, and therefore the equilibrium spot rate does not fluctuate much. As a consequence, the volatility of yields is roughly 3 times lower than in the data, and the slope of the yield curve is significantly smaller: Long-term bonds are less risky, since interest rates (and real bond prices) are not expected to fluctuate much.

In the case of $\psi_B = \psi_A$, investors are heterogeneous along the RA dimension only. This implies that the volatility of real yields is virtually zero, and the slope of the real yield curve is almost flat (indeed, slightly downward sloping).

The Nominal Yield Curve with Exogenous Inflation. Figure 1.5 shows the results for the nominal yield curve when inflation follows an exogenous stochastic process. The left panel fixes x at the steady-state value and displays the nominal yield curve for different values of inflation expectations in the bivariate stationary density (shown in Figure 1.7). On average, the yield curve is upward sloping, because both the real component and the nominal components render nominal bonds risky assets, as shown in equation (1.27). I discuss the decomposition between these two sources below. The real source of risk is explained above.

The nominal source of risk comes from $\phi_{13} < 0$: An exogenous sequence of positive inflation expectation shocks is associated with negative shocks to the real economy. This means that inflation is expected to erode the purchasing power of nominal payments precisely when the marginal investor values those resources the most.[13] Therefore, nominal bonds are risky.

Over the business cycle, the left panel of Figure 1.5 shows that when is π high, the nominal term structure is downward sloping. This is denoted by the gray line. Intuitively, when current inflation is high, nominal rates are expected to go down in the future (i.e., π is mean reverting). A similar logic applies when π is low, since in such states of nature the nominal interest rate is expected to increase. Thus, as shown by the blue line, the nominal curve is even more upward sloping than on average.

On the right-hand side of Figure 1.5, I show the nominal term structure when π is a steady state. The red line fixes the steady state of both x and π, which means it is the same as on the left-hand side. In this case, when π is fixed to the steady state, the properties of the nominal term structure are driven by x, so the intuition is very similar to that one developed for the real term structure.

To understand the role of inflation risk, driven by ϕ_{13}, I next study the decomposition of the nominal and real components of (1.27). This is a useful analysis, because in models in which the real term structure is flat, 100% of the nominal term premium is driven by inflation risk. Even more, in models in which the real term structure is downward sloping (such as the long-run risk models, e.g., Bansal and Yaron (2004a)), inflation risk has to more than compensate for the negative real term premium to obtain an upward-sloping nominal

[13]This is clear from Figure 1.7, where low x states (blue line on panel (b)) are associated with higher inflation states.

curve consistent with the data.

Figure 1.6 illustrates this decomposition for an 80-quarter nominal bond.[14] On the left-hand side, the figure depicts the real and nominal components across the x dimension (i.e., fixing π at different values); on the right-hand side I shows the real and nominal component across the π dimension (i.e., fixing x at different values).

On average, the real component explains about 80% of the nominal term premia. As shown in the upper-left panel, an increase in x reduces the real component. This is because effective risk aversion decreases as x increases and risk-tolerant investors rebuild their balance sheets. This is scaled by the level of π: The greater π is (gray line), the smaller the real component. The upper-right panel shows this from a different perspective: It fixes x and shows the real component for different levels of π. The intuition for the dynamics over the state space is similar to the one above: An increase in π means a reduction in the real components, and this effect is scaled by the level of x.

The Nominal Yield Curve with Endogenous Inflation. Motivated by the previous decomposition, in which the real component drives the nominal term premium, I next study the nominal term structure with endogenous inflation expectations. As shown in equation (1.30), endogenous inflation expectations depend on policy parameters, δ_0 and δ_π, and also on the real interest rate $r(x)$. In particular, when $\delta_\pi > 1$, the nominal interest rate moves in the same direction as the real interest rate, by a factor $\frac{\delta_\pi}{\delta_\pi - 1} > 1$.

The difference in the magnitudes implies that the monetary authority anchors inflation expectations by adjusting the policy instrument more than one-to-one to fluctuations in the

[14]Results in this decomposition are very similar for different maturities other than 80 quarters.

real economy (represented by $r(x)$); and this will be captured by fluctuations in nominal bond prices. In other words, the sensitivity of nominal bond prices to fluctuations in x will be higher than that of real bond prices. That is the derivative with respect to x is higher in a nominal bond than in a real bond

$$P_x^{\prime\$}(x,T) > P_x^{\prime}(x,T) \ \ \forall (x,T) . \tag{1.33}$$

Expression (1.33) implies that the Taylor coefficient magnifies the positive real term premium. In Figure 1.8, panel (a), I show the results for the nominal term structure under endogenous inflation. The average nominal yield curve, with $\delta_\pi = 1.5$, is upward sloping and in line with the evidence. Indeed, the slope of the nominal term structure in the figure is higher than that of Figure 1.2 (almost twice, as in the data), precisely because of the effect of $\frac{\delta_\pi}{\delta_\pi - 1}$ and its impact in the derivative of bond prices. Interestingly, the nominal term structure inherits the properties of the real economy, and thus exhibits endogenous fluctuations across the business cycle.

In panel (b) of Figure 1.8, I show the properties of endogenous inflation expectations and the corresponding nominal yield curves. The lower the Taylor coefficient δ_π (which can be interpreted as a relatively "loose" monetary rule), the greater the unconditional mean and standard deviation of inflation expectations. A lower coefficient then translates into a steeper nominal yield curve, because monetary policy reacts relatively more to changes in the real economy, which implies a more volatile nominal rate and thus a greater derivative $P_x^{\prime\$,(T)}(x,t)$. This can be seen on the right-hand side of panel (b) in Figure 1.8, where I show the normalized yield curves. (I normalize yields to 0 at maturity 0.)

1.6 Empirical Analysis

In this section I evaluate the empirical predictions of the model. I begin by extracting macro shocks from the data, using the fact that the aggregate endowment is i.i.d. in the model. I then introduce the realized sequence of shocks into the model and compute the time series of the endogenous state variable x.

The first exercise consists of regressing yields from the data onto the implied series of x. In particular, I consider two regressions that intend to capture the main theoretical prediction of the model: Yields are persistently negatively exposed to x (i.e., $P'_x(x, T)$ is positive, and thus yields are negatively exposed to x). I regress yields onto x precisely to capture this sensitivity at different maturities. I then regress the slope of the term structure onto x. I compare the regressions results with the model's prediction for both the sensitivity of yields and slope.

The second exercise is to regress the short-term (1 quarter) nominal interest rates against the model's implied x, but controlling for several macroeconomic variables. In this analysis, I follow (Ang and Piazzesi (2003)), and investigate whether x contains information to explain fluctuations in the short-term nominal interest rate beyond other well-studied macroeconomic factors (GDP growth, inflation, and unemployment). In this exercise I evaluate the predictions in the endogenous inflation case, where the short-term nominal rate depends on x.

The third exercise is an application of the model to shed light on two salient interest-rates puzzles (Campbell et al., 2009): (1) the sudden spike of real rates in the Great Recession; (2) the secular decline of real and nominal long-term rates since the 80s. The purpose is to provide further evidence on the mechanism I propose, which relates the credit market with the term structure of interest rates. In these exercises, I use the time series implied by the model.

I conclude this section by comparing x with an alternative interpretation of the model. Previous literature (cited below) interprets risk-tolerant investors as financiers. According to this view, the relative net worth of financial firms should be indicative of credit conditions in the economy, and should be related to yields. I construct a "credit factor" that intends to capture this view and compare it with x.

Business Cycle: Preliminaries. I begin by feeding the model with macro shocks. To that end, I take advantage of the assumption that aggregate endowment is a geometric Brownian motion, which means that log growth rates are i.i.d. at an aggregate level. Then, shocks can be easily identified (under the null of the model):

$$\begin{aligned} d \log y_t - \left(\mu - \frac{1}{2}\sigma^2\right) dt &= \sigma dW_{1,t}, \\ \Delta \log y_t - \left(\mu - \frac{1}{2}\sigma^2\right) \Delta &= \sigma \left[W_{1,t+\Delta} - W_{1,t}\right]. \end{aligned} \tag{1.34}$$

I discretize Δ to make it equal to one quarter and use NIPA data for real personal consumption expenditures at quarterly frequency. Figure 1.11 displays the series of the index and the shocks. I then feed these shocks into the model, starting from the stochastic steady state in

1971:Q2, to obtain predictions for the endogenous state variables. I start in 1971:Q2 to be consistent the sample periods for which yields data are available (reported in Table 3.1).

Business Cycle, Credit, and Yields. I start by analyzing the model prediction for credit. Figure 1.12 shows results for credit over total equity in the model and credit over GDP in the data. The figure indicates the model captures the fluctuations in credit well. Motivated by this, I compute the time series for the endogenous state variable x to compare it with yields data. As described in the previous section, the model predicts that low x implies a high level of rates and lower slope. In the next figure I show that the data show a similar pattern.

Figure 1.13 compares fluctuations in the endogenous state variable in the model with yields data. The implied series for x shows a negative correlation of -0.35 with the first principal component of the real term structure, and -0.54 with the first principal component of the nominal term structure. As has been shown by many previous studies (e.g., Litterman and Sheinkman (1991)), the first principal component —the level of the curve—explains the vast majority of yield curve fluctuations (more than 90% in any sample). The figure also shows a positive correlation of 0.25 between the slope of the real term structure and the implied series for x. This correlation is weaker in the case of the nominal term structure (0.14).

Elasticities. I next study the sensitivity of yields, at different maturities, with respect to x. This is a useful first step to verify the key theoretical prediction of the model, which is that the derivative of bond prices with respect to x is positive. That is,

$$\frac{P'_x(x,T)}{P(x,T)} > 0.$$

More precisely, the idea is to use yields from data to capture the sensitivity of bonds to x

$$E^{\mathbb{P}}\left[\frac{\partial \log P(x,T)}{\partial x}\right] = E^{\mathbb{P}}\left[\frac{P'_x(x,T)}{P(x,T)}\right] \equiv E^{\mathbb{P}}\left[T.y'^{(T)}_x\right],$$

where the last equivalence follows from the relationship between yields and prices of zero-coupon bonds. To capture this relationship, I use data on real yields described in section 2, with maturities $N = (4, 8, 12, 20, 28, 40)$ quarters. I specify the following linear regression:

$$y_t^{(N)} = \alpha_1^{(N)} + \beta_1^{(N)} x_t + \varepsilon_{1,t}^{(N)}. \tag{1.35}$$

I use the model's implied series for x and yields from the data to estimate (1.35). Panel (A) of Table 1.4 shows estimates for real yields. Results indicate that coefficients are all negative and statistically significant, and they display the following pattern:

$$-\beta_1^{(10)} > -\beta_1^{(7)} > ... > -\beta_1^{(1)}.$$

In other words, long-term real yields are less sensitive to x (in absolute value). A similar pattern holds, but is mechanically opposite, for bond prices: longer-term bond prices are more sensitive to x. Intuitively, this indicates that bond prices are persistent (but stationary) processes. Yields inherit this property, but since they are proportional to maturity (we divide by N), the persistence of bonds is offset by N. The longer the maturity, the stronger this effect.

The model can capture this very well. Figure 1.9 shows the model's prediction for the sensitivity of yields with respect to x. The figure shows the unconditional derivative for yields $E^{\mathbb{P}}\left[y_x^{\prime(T)}\right]$. The left-hand panel shows the derivative of yields with respect to x, over the state space, for three different maturities. On the right-hand side, I show the unconditional mean across maturities. Both are in line with the estimations reported in Table 1.4. In other words, short-term yields are unconditionally more volatile (i.e., more sensitive to x) than long-term yields, both in the data and in the model.

Panel (B) contrasts the results for nominal yields. The coefficients are larger than those for real yields, which is consistent with the prediction in the endogenous inflation case (nominal bonds are more sensitive to x, because Taylor loading is greater than 1). Although the R^2 are higher, the coefficients are not statistically different from each other as, they were for the real yields.

Slope. I now evaluate whether the model's predictions for the slope of the term structure are consistent with the data. Figure 1.10 shows that the model predicts an average positive slope, but with a nonlinear relationship against x. The intuition comes from the mechanism elaborated on above: When x is low and real rates are high, rates are expected to fall in the future; they are mean reverting. This effect is strong enough to imply that during low-x states, long-term rates are lower than short-term short. When x is at its mean, the effect of x on the slope is close to zero (i.e., the derivative of the slope against x, at the steady state, is close to zero).

To evaluate this prediction, I compute the slope of real yields at different horizons in the data—that is, the difference in yields (i.e., the slope) as

$$\mathit{slope}\,(N) = y_t^{(N)} - y_t^{(4)} \text{ , for } N = (8, 12, 20, 28, 40)\,.$$

To capture the nonlinear aspect of the relationship predicted by the model, I specify the following regression

$$y_t^{(N)} - y_t^{(4)} = \alpha_3^{(N)} + \beta_3^{(N)} x_t + \beta_4^{(N)} x_t^2 + \varepsilon_t^{(N)}, \tag{1.36}$$

where the left-hand side represents the slope at different horizons and the quadratic term intends to capture the nonlinearity predicted by the model, and is reported in Figure 1.10, panel (a). In Figure 1.10, panels (b) and (c), I fit a kernel regression that indicates that the quadratic specification in (1.36) is enough to capture the nonlinearities in the data.

In Table 1.5, I report the estimates of (1.36). The coefficient associated with x is positive, but the coefficient associated with x^2 is negative and larger (in absolute value). This implies that changes in the model's endogenous state variable produce significant nonlinear changes in the slope of the real term structure. A marginal deviation of x from its mean, however, does not create a significant change in the slope. This is what the row "Net effect" reports: It evaluates whether the derivative of (1.36) is different from zero on average. This is consistent with the model prediction, indicating that a marginal change in x, starting from the steady state, is very small. But when x is small, an increase in x produces an increase in the slope.

When x is large, a decrease in x produces an increase in the slope.

Model's x as a Macro Factor. In this subsection I evaluate the key theoretical prediction of the endogenous inflation case: I study how the short-term nominal rate changes with the endogenous state variable x. For this, I follow Ang and Piazzesi (2003) and regress the short-term nominal rate against several macro factors, in which I include x (the endogenous state variable implied by the model). The macro factors I include have been widely documented in the macro-finance term structure literature (proxies for inflation, GDP growth, and unemployment). Since Ang and Piazzesi (2003), many papers have incorporated macroeconomic variables into affine term structure models to provide an interpretation of the previous latent factor models (e.g., Litterman and Sheinkman (1991)).

Table 1.6 shows the correlations between short-horizon nominal yields, x, x^2, and x^3. The purpose of incorporating x^2 and x^3 in the analysis is to capture the nonlinear dynamics implied by the model. As can be seen in the table, the yields' dynamics are negatively correlated with x. This negative correlation was implicitly described in Figure 1.13, where I showed only the first principal component of nominal yields. Notice higher order terms are also relevant.

I then regress the one-quarter nominal rate $y_t^{\$,(1)}$ onto different macro factors f_t. The regression is specified as in Ang and Piazzesi (2003):

$$y_t^{\$,(1)} = \alpha_4 + \beta_4' f_t + v_t, \tag{1.37}$$

where f_t is a vector of macroeconomic factors and v_t is a shock that captures orthogonal information to macro variables (e.g., policy shocks). The factors I consider, in addition to x, are: an inflation factor, a real activity factor, the CPI core, and the unemployment gap.[15] I construct the inflation and real activity macro factors in the same way as Ang and Piazzesi (2003). This consists of computing the first principal component of various inflation and real activity indexes. The CPI core and unemployment gap are representative of the "policy factors" typically used by the Monetary Authority when considering adjusting the short-term rate (Bauer and Rudebusch, 2017), so they are also useful controls for x.

Table 1.7 report regressions' results. The first two columns are the specifications in which x is not included. This is a useful benchmark to compare with. Notice that in column (1) and (2), the only significant component is the one associated with inflation. This is consistent with Ang and Piazzesi (2003), who report that real activity is sensitive to the sample period considered. Also, column (1) indicates that the CPI core delivers a higher goodness of fit than that of column (2); R^2 is 0.54 with the CPI core and 0.18 with the inflation factor.

Column (3) shows the result of regressing the one-quarter nominal yield $y_t^{\$,(1)}$ onto x. As expected, based on the correlation structure reported in Table 1.6, the coefficient is negative and significant. The R^2 is almost the same as the regression including inflation and the real activity factor (0.17 versus 0.18, respectively). Indeed, as shown in column (5), when x is included in the regression of $y_t^{\$,(1)}$ against the inflation and real activity factors, the goodness of fit is more than twice (0.40 versus 0.18, respectively). Importantly, x remains negative and statistically significant. Also, notice that in column (5), the coefficients for inflation and real activity are 1.50 and 0.55, very close to those typically used in calibrations of the Taylor

[15] The unemployment gap is the difference between actual unemployment and the natural rate of unemployment reported by the Congressional Budget Office.

rule since Taylor (1993).[16] Column (4) shows similar results but with unemployment gap and CPI core: x remains negative and statistically significant and improves the goodness of fit (although not as much as column (4) against (2)).

In column (6) I evaluate the effect of x, x^2 and x^3. The result indicates that x and x^2 are significant, although the goodness of fit of does not increase much (it increases only 0.01). Then, in columns (7) and (8), I report the same specification as (4) and (5), but include the higher-order terms x^2 and x^3. In both (7) and (8), the introduction of x, x^2, and x^3 increasea the goodness of fit vis-à-vis (1) and (2). Importantly, x remains negative and statistically significant.

These results are in line with the theory predicted above: They imply that when short-term nominal interest rate is high in the data, the market value of leveraged risk-tolerant investors is low in the model. Even more, these results indicates that x contains information that is beyond the standard macroeconomic factors commonly studied in the literature.

Puzzle I: Sudden spike in real rates in the Great Recession. Early in Fall 2008, real rates (measured by TIPS) showed a sudden spike, and the real term structure was reversed (i.e., the short-term rate was above the long-term rates). As noted by Campbell et al. (2009), there were several institutional and liquidity influences on TIPS yields during this episode. These may have distorted, at least partially, their prices.

However, from a macroeconomic perspective, using the standard Fisher equation logic, it was evident that real rates, on impact, increased. More precisely, on December 15, 2008, the Federal Reserve set the short-term interest rate at 0%-0.25%. Also, according to the

[16] This does not indicate the coefficients are identified (Backus et al., 2016).

Survey of Professional Forecasters (SPF), during the first and second quarters of 2009 the one-quarter-ahead median inflation expectation was -9.5% and -2.4%, respectively (in annual terms). Through the lens of the Fisher equation, this implies a very large spot real rate. For example, in 2009:Q1,

$$r_t = \underbrace{i_t}_{=0} - \underbrace{E^{\mathbb{P}}\left[\frac{dp}{p}\right]}_{-9.5\%} \tag{1.38}$$

Thus, even though certain distortions may have contributed to the sudden spike in TIPS, the Fisher equation's logic also indicates that real rates actually increased.

In Figure 1.14, I show the model's time series predictions using the real macro shocks reported in Figure 1.11. The left-hand side shows the business cycle fluctuations of the 10-year real rate and the 1-year real rate. As it is evident from the plot, on average the real term structure is upward sloping (black line is above red line). The model predicts that during the Great Recession, the level of real rates increased pari passu with the drastic decrease in credit—a reduction in x which implies the aggregate willingness to substitute consumption intertemporally. Even more, it predicts an inversion of the real yield curve (red line crosses the black line). Qualitatively, this is consistent with the evidence.

During 2009, the monetary authority started to intervene in a variety of markets, and its balance sheet was multiplied by five. These interventions are not captured in the model, but several studies have argued they have affected the behavior of yields (e.g., Krishnamurthy and Vissing-Jorgensen (2011)). In the right-hand panel, I show the result of subtracting the one-year real rate produced by the model from the nominal real rate in the data. This is a proxy of inflation expectations. As shown in the figure, during the crisis the model predicts an expected deflation in line with the SPF. However, the model predict a more persistent dynamics: it takes longer for the credit market to be rebuilt.

An intuitive interpretation of the Fed's interventions during the Great Recession is that they introduced willingness to substitute consumption intertemporally into the markets. At the height of the financial crash (2008:Q4-2009:Q2), the marginal investor required a large compensation to postpone his current consumption into the future. Thus, the market would have to compensate him by providing a higher incentive (i.e., a high real rate) to perform such delay in his consumption. Since the nominal rate was set to zero, the adjusting economic force was deflation expectation (as shown in the SPF and predicted by the model). Thus, when the Fed started to intervene, those policies prevented the scenario predicted by the model, by "introducing" willingness to smooth consumption, thus reducing real interest rates—even though the credit market remained impaired.

Puzzle II: Secular decline in real and nominal long-term rates. Several papers have documented the fact that long-term nominal and real rates have been declining in the last 30 years (Caballero et al. (2008), Bernanke et al. (2011), Hall (2017a), among others). This period also witnessed a significant increase in the size of the credit market. For example, Philippon (2015) shows that the amount of assets intermediated in the financial sector rose from approximately 2.5% of GDP in 1980 to 4% of GDP in 2008.

The theoretical mechanism in the model predicts that an increase in the amount of credit is associated with a reduction in the price of credit—i.e., the spot real rate. Put differently, a credit expansion produces an increase in aggregate EIS, and this implies that the market has to compensate the marginal investor with a lower interest rate to incentivize him to smooth consumption over time. Due to the single-factor structure of the model and the endogenous persistence in the credit market fluctuations, this reduction in the level of rates translates into a decrease in long-term real rates.

To capture these dynamics, I study a transitional dynamics exercise (e.g., King and Rebelo (1993)) by starting the economy 2 standard deviations below the stochastic steady state of the endogenous state variable x. Then, introduce the same macro shocks reported in the previous subsection and shown in Figure 1.11 and I compute long-term real rates in the model at each period of time. I pin down nominal rates with the same Taylor rule reported in the endogenous inflation term structure. That is, I keep the same calibration already shown above.

Figure 1.15 reports the results and compares them with the evidence for long-term rates. Panel (a) shows the model's prediction for credit/total equity. In particular, notice that the figure shows that the amount of credit as a fraction of total wealth in the model is approximately multiplied by two. In the same period, the data on domestic credit to private sector over GDP went from 92.4% to 188.0%[17], which indicates that the increase in credit predicted by the model is on the order of magnitude of that in the data. The red bars in panels (b) and (c) show the average dynamics of the 10-year real rate in the model and in the data. Nominal rates display a similar pattern, because they are pinned down by the same Taylor rule (with $\delta_\pi > 1$) as shown in (1.30). That is, the monetary authority anchors inflation expectations by moving the nominal rate in tandem with the real rate. Thus, inflation expectations are also trending downwards—which is consistent with the evidence

[17]Source: World Development Indicators http://databank.worldbank.org/wdi

reported in Chernov and Mueller (2012).

An Alternative Interpretation for x. Prior studies in the heterogeneous-agents literature that interpret risk-tolerant investors as financiers (see Silva (2016), Drechsler et al. (2017), Longstaff and Wang (2012), Santos and Veronesi (2016), among others). According to this interpretation, the equity of financial firms should be important to capture credit conditions in the economy; in theory, therefore, it should be useful to understand the behavior of yields. In this line, I compute the market value of the financial sector equity [18] over the total market value of equity in CRSP, and I define this as cf (credit factor):

$$cf_t = \frac{\text{market value of financial sector equity}}{\text{market value of total equity}}.$$

Under this alternative interpretation, a higher cf_t implies that a larger quantity of credit is being supplied, which translates to lower real and nominal rates. In Figure 1.16, I compare cf against other related measures. Panels (a) and (b) compare against the proposed measure by He et al. (2016), in levels and in shocks. Panel (c) constructs cf_t using the market value of equity in financial firms over the total market value of equity reported by the Flow of Funds. Panel (d) compares, at an annual frequency, with the flow of intermediated assets in the financial sector in Philippon (2015).

To understand how sensible this proposed factor is, I compare cf with the endogenous state variable in the model. For this, I proceed as before and feed the model with the macro shocks reported in Figure 1.11. Figure 1.17 compares the fluctuations in cf with the implied

[18] I consider SIC codes 60-64, which include a broad range of financial institutions.

series for the endogenous state variable in the model, x. As shown in the figure, x and cf exhibit a high correlation (0.8). Put differently, cf in the data is high in periods in which risk-tolerant investors' balance sheets are relatively well capitalized in the model.

Thus, in this interpretation, cf could be used in term-structure empirical analysis to further understand yields' properties (with some guidance from the theory elaborated on this paper).

1.7 Conclusion

In this paper, I propose a model where the credit market is a key macroeconomic fundamental for understanding the salient properties of the U.S. real and nominal term structure. In this, I depart from the representative agent framework and propose a general equilibrium term structure model with heterogeneous investors in which the amount of credit in the economy is key in characterizing the equilibrium.

I find that differences in investors' willingness to substitute consumption across time is critical to match the salient properties of both the nominal and real term structure. Endogenous contractions in the amount of credit lead to increases in the real interest rate and the aggregate price of risk, to incentivize investors with high risk aversion and low willingness to substitute consumption to clear the markets. Thus, real bonds are risky and they are negatively exposed to the endogenous risk created by the credit market. This implies that the marginal investor must be compensated with a premium to hold real bonds. At an aggregate level, this mechanism generates dynamics for the real rate and aggregate price of risk that can be interpreted as a representative agent with time-varying, and negatively correlated, risk aversion and elasticity of intertemporal substitution.

I provide a decomposition of the nominal term premium, between the endogenous source of risk created by the credit market and exogenous inflation shocks. I find that, consistent with recent studies, the model's real term premium explains a significant portion of the nominal term premium. Motivated by this, I derive a nominal term structure by introducing a Taylor rule. I show that when the monetary authority adjusts the nominal rate more than one-to-one to deviations of inflation from its target, this makes nominal bonds more sensitive to real risks. Thus, the nominal term structure is steeper than the real term structure for any correlation between inflation and real shocks. Put differently, the economy exhibits a significant nominal term premium, even when inflation shocks play no role.

To validate the model's key theoretical prediction, I introduce macro shocks to the model and obtain the series of the endogenous state variable. I find that fluctuations in credit in the model capture well the fluctuations in credit in the data. I use the implied series for the endogenous state variable and data for yields to evaluate the model's main theoretical predictions: the relationship of yields and slope of the term structure with respect to the endogenous state variable. I find that the data validate the model's predictions. In addition, I find that the implied series of the model's endogenous state variable contain information to explain short-term nominal interest rate variability that extends beyond well-studied macro variables (GDP, inflation, and unemployment).

I then use the model to study two interest rate puzzles: the secular decline in long term real and nominal bonds since the 1980s; and the sudden spike in real rates during the Great

Recession. I show that these puzzles can be rationalized by the connection between the credit market and yields. In particular, the sudden spike in real rates during the Great Recession can be attributed to an endogenous collapse in the aggregate of credit (i.e., a drastic reduction in the aggregate elasticity of intertemporal substitution). The secular decline in real rates can be attributed to the contemporaneous increase in the amount of credit during since the 1980s. Using the Taylor rule, nominal yields inherit the properties of real yields, as discussed in Section 5. Thus, the model implies—also consistent with the evidence—a decline in inflation expectations.

This work provides several avenues for future research. For example, it provides a framework to study how unconventional monetary policies generated a reduction in real rates together with an increase in inflation expectations after Spring 2009. In the model's prediction, in which policy interventions are not incorporated, the spike in real yields would have been more persistent (the credit market takes time to rebuild). Also, incorporating the credit factor into the empirical macro-finance term structure model can improve our understanding of how monetary policy affect long-term rates through the credit channel. Lastly, the mechanism that generates time variation in the aggregate risk aversion and elasticity of intertemporal substitution can be introduced, in a reduced form, into larger scale models.

1.8 Tables and Figures

Table 1.1: Evidence

		Maturity (quarters)							
Panel A. Full Sample		4	8	12	20	28	40	80	diff(40-4)
Mean	Nominal	0.0133	0.0139	0.0144	0.0152	0.0158	0.0165	0.0174	0.0033
	TIPS	0.0043	0.0044	0.0046	0.0051	0.0055	0.0058		0.0015
St. Dev	Nominal	0.0090	0.0088	0.0085	0.0080	0.0076	0.0072	0.0068	-0.0018
	TIPS	0.0050	0.0046	0.0043	0.0038	0.0034	0.0030		-0.0020
Panel B. Short Sample I									
Mean	Nominal	0.0037	0.0042	0.0048	0.0061	0.0072	0.0083	0.0099	0.0047
	TIPS	0.0002	0.0003	0.0005	0.0013	0.0021	0.0029		0.0027
St. Dev	Nominal	0.0043	0.0040	0.0037	0.0032	0.0030	0.0028	0.0025	-0.0018
	TIPS	0.0041	0.0036	0.0033	0.0029	0.0027	0.0024		-0.0017
Panel C. Short Sample II									
Mean	Nominal	0.0161	0.0167	0.0171	0.0177	0.0182	0.0187	0.0194	0.0026
	TIPS	0.0053	0.0055	0.0057	0.0062	0.0065	0.0068		0.0015
St. Dev	Nominal	0.0075	0.0072	0.0069	0.0066	0.0064	0.0061	0.0058	-0.0013
	TIPS	0.0045	0.0039	0.0035	0.0030	0.0027	0.0024		-0.0021

NOTES: Full sample is 1971:Q3-2016:Q4. Short sample I is 2003:Q1-2016:Q4. Short sample II is 1971:Q3-2008:Q2. Numbers are in decimals, at quarterly frequency. Source: Chernov and Mueller (2012), Gürkaynak et al. (2007), and Gürkaynak et al. (2010).

Table 1.2: Baseline Calibration

PARAMETERS (QUARTERLY)			
1. *Preferences*		**Value**	**Description**
	γ_A	1.5	risk aversion investor A
	γ_B	10	risk aversion investor B
	ψ_A	0.7	EIS investor A
	ψ_B	0.02	EIS investor B
	ρ	0.001/4	time preference
2. *Endowment and demography*			
	μ	0.0055	drift growth
	σ	0.019	diffusion growth
	φ	0.008	birth/death rate
	$\overline{x}$	0.11	fraction of new investors A
3. *Inflation*			
	δ_π	1.5	Taylor coefficient
	λ_π	0.08	persistence inflation expec.
	σ_π	0.012	diffusion inflation expec.
	π_L	-0.01	inflation expec. lower bound
	$\overline{\pi}$	0.009	mean inflation expec.
	ϕ_{12}	0	$\text{cov}(dW_1, dW_2)$
	ϕ_{13}	-0.5	$\text{cov}(dW_1, dW_3)$
	ϕ_{23}	0	$\text{cov}(dW_2, dW_3)$

NOTES: I describe the calibration in the main text.

Table 1.3: Theoretical Moments in the Model

		$\psi_B = baseline$		$\psi_B = 1/\gamma_B$		$\psi_B = \psi_A$	
Variable	Description	Mean	St.dev	Mean	St.dev	Mean	St.dev
<u>*Model*</u>							
ξ_A	consumption/wealth investor A	0.0108	0.0006	0.0103	0.0002	0.0101	0.0001
ξ_B	consumption/wealth investor B	0.0156	0.0022	0.0131	0.0005	0.0091	0.0001
R	risk sharing	-2.4514	1.2677	-0.0932	0.0277	-1.6006	0.1924
$\mu_q - r$	expected excess return	0.0146	0.009	0.0058	0.0019	0.001	0.002
σ_q	vol. returns	0.1311	0.0503	0.0572	0.0101	0.007	0.002
r	real risk free rate	0.0043	0.0046	0.0047	0.0016	0.0044	0.0004
$y^{(4)}$	real yield 4 quarter	0.0044	0.0046	0.0047	0.0016	0.0044	0.0004
$y^{(40)}$	real yield 40 quarter	0.0058	0.0042	0.0050	0.0015	0.0043	0.0004
$y_t^{(80)}$	real yield 80 quarter	0.0066	0.0037	0.0052	0.0014	0.0042	0.0003
<u>*Data*</u>							
$y^{(4)}$		0.0043	0.0050				
$y^{(40)}$		0.0058	0.0030				

NOTES: This table reports theoretical moments from the model and in yield's data. Numbers are in decimal, at a quarterly frequency. The first column, $\psi_B = baseline$, corresponds to the parametrization in Table 1.2. The second column, $\psi_B = 1/\gamma_B$, corresponds to the case in which B-type investors have CRRA preferences (i.e., $\psi_B = 1/\gamma_B = 0.1$). The third column, $\psi_B = \psi_A$, corresponds to the case in which both types of investors have the same EIS. Data for real yields are as in Table 3.1. Risk sharing R is as in equation (1.17), $R(x) = \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} - \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}$.

Table 1.4: Regression: Elasticities

A. Estimates of the price elasticity,real : $y_t^{(N)} = \alpha_1^{(N)} + \beta_1^{(N)} x_t + \varepsilon_{1,t}^{(N)}$						
	$\beta_1^{(4)}$	$\beta_1^{(8)}$	$\beta_1^{(12)}$	$\beta_1^{(20)}$	$\beta_1^{(28)}$	$\beta_1^{(40)}$
OLS estimates	-0.042***	-0.041***	-0.039***	-0.037***	-0.035***	-0.033***
Conf. Int.	[-0.076;-0.007]	[-0.070;-0.011]	[-0.066;-0.014]	[-0.059; -0.015]	[-0.054; -0.016]	[-0.049; -0.016]
R^2	0.054	0.068	0.079	0.096	0.107	0.119
B. Estimates of the price elasticity, nominal: $y_t^{\$,(N)} = \alpha_2^{(N)} + \beta_2^{(N)} x_t + \varepsilon_{2,t}^{(N)}$						
	$\beta_2^{(4)}$	$\beta_2^{(8)}$	$\beta_2^{(12)}$	$\beta_2^{(20)}$	$\beta_2^{(28)}$	$\beta_2^{(40)}$
OLS estimates	-0.125**	-0133***	-0.136***	-0.138***	-0.136***	-0.134***
Conf. Int.	[-0.174;-0.075]	[-0.177;-0.086]	[-0.178;-0.093]	[-0.176;-0.099]	[-0.172;-0.101]	[-0.166;-0.101]
R^2	0.174	0.212	0.239	0.272	0.292	0.310

NOTES: Significance at 1%, 5%, and 10% is indicated with ***, ** and *. Hubert-White standard errors. Sample period is 1971:Q3-2008:Q2 (i.e., Short sample II in Table 3.1), and the source of the data for yields is Chernov and Mueller (2012), Gürkaynak et al. (2007), and Gürkaynak et al. (2010). x is the implied endogenous state variable after feeding the model with the shocks described in Section 7.

Table 1.5: Regression: Term Structure Slope

$y_t^{(N)} - y_t^{(4)} = \alpha_3^{(N)} + \beta_3^{(N)} x_t + \beta_4^{(N)} x_t^2 + \varepsilon_t^{(N)}$					
	$y^{(8)} - y^{(4)}$	$y^{(12)} - y^{(4)}$	$y^{(20)} - y^{(4)}$	$y^{(28)} - y^{(4)}$	$y^{(40)} - y^{(4)}$
Constant	-0.001***	-0.010***	-0.016***	-0.020***	-0.024***
$\beta_3^{(N)}$	0.071***	0.123***	0.19***	0.239***	0.290***
$\beta_4^{(N)}$	-0.197***	-0.336***	-0.52***	-0.654***	-0.788***
Net effect	[-0.005, 0.003]	[-0.009; .0.006]	[-0.013;0.01]	[-0.016;0.014]	[-0.016;0.017]
R^2	0.050	0.051	0.054	0.059	0.066

NOTES: Significance at 1%, 5%, and 10% is indicated by ***, ** and *. Hubert-White robust standard errors. Net effect is the confidence interval for the marginal effect $\frac{d\left(y_t^{(N)} - y_t^{(4)}\right)}{dx} = \beta_3^{(N)} + 2\beta_4^{(N)} x_t$. Sample period is 1971:Q3-2008:Q2 (i.e., Short sample II in Table 3.1). Source of data is Chernov and Mueller (2012), Gürkaynak et al. (2007), and Gürkaynak et al. (2010). x is the implied endogenous state variable after feeding the model with the shocks described in Section 7.

Table 1.6: Correlation x and Nominal Yields

	$y^{\$,(1)}$	$y^{\$,(4)}$	$y^{\$,(20)}$	x	x^2
$y^{\$,(4)}$	0.985***				
$y^{\$,(20)}$	0.925***	0.954***			
x	-0.413***	-0.441***	-0.564***		
x^2	-0.105***	-0.116	-0.199**	0.571***	
x^3	-0.333***	-0.369***	-0.474***	0.842***	0.721***

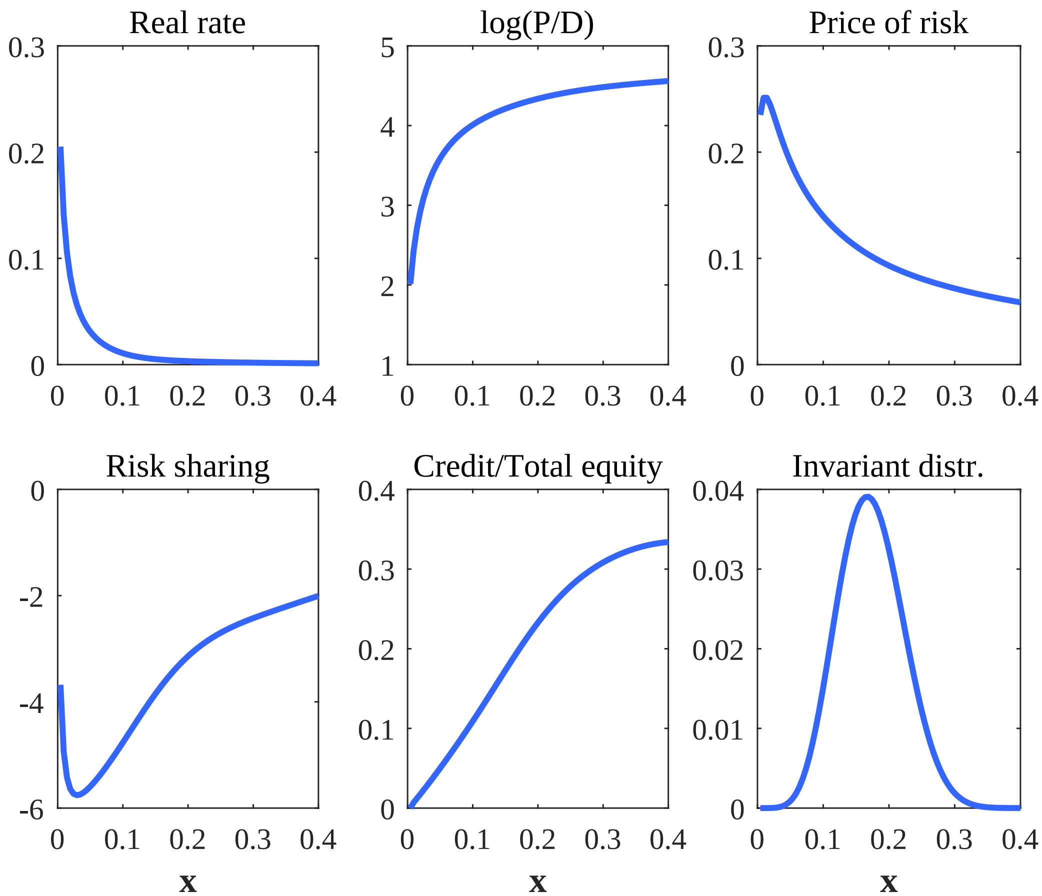
NOTES: Significance at 1%, 5%, and 10% is indicated by ***, ** and *. Sample period is 1971:Q3-2008:Q2 (i.e., Short sample II in Table 3.1). One quarter (3 months) nominal yield $y_t^{\$,(1)}$ is from Fama CRSP Treasury Bill files. Four-quarter and ($y_t^{\$,(4)}$) and 20-quarter ($y_t^{\$,(20)}$) Fama CRSP zero-coupon files . x is the endogenous state variable in the model, after feeding the shocks reported in Figure 1.11.

Table 1.7: Short-Term Nominal Rate Regressions, $y_t^{\$,(1)} = \alpha_0 + \alpha_1' f_t + v_t$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Constant	5.58***	5.55***	6.81***	5.52***	6.71***	5.78**	5.45***	5.94***
CPI core	2.22***			1.98***			1.94***	
Unemp.gap	-0.18			-1.02***			-1.42***	
Inflation		1.43***			1.50***			1.49***
Real activity		0.02			0.55***			0.86***
x			–2.04***	-1.67***	-2.50***	-2.11***	-2.31***	-2.53***
x^2						1.49**	1.90***	2.31***
x^3						-0.65	-0.62	-1.23***
adj-R^2	0.54	0.18	0.17	0.60	0.40	0.18	0.63	0.46

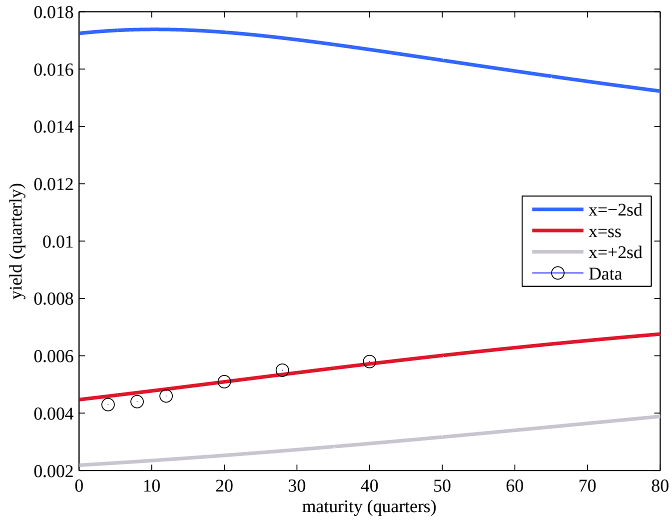
NOTES: Sample period is 1971:Q3-2008:Q2 (i.e., Short Sample II in table 3.1). Significance at 1%, 5%, and 10% is indicated by ***, ** and *. $y_t^{\$,(1)}$ is the 1-quarter nominal interest rate from Fama CRSP Treasury Bill files. f_t is a vector of different macroeconomic variables considered in each of the table's columns. Inflation and real activity are constructed as in Ang and Piazzesi (2003). Inflation is the first principal component of the CPI, PPI, and spot commodity prices. Real activity is the first principal component of the growth rate of employment and growth rate of industrial production. The unemployment gap is the difference between actual unemployment and the natural rate of unemployment from the Congressional Budget Office, as considered by Bauer and Rudebusch (2017). cf is described at the end of Section 7.

Figure 1.1: Model Solution

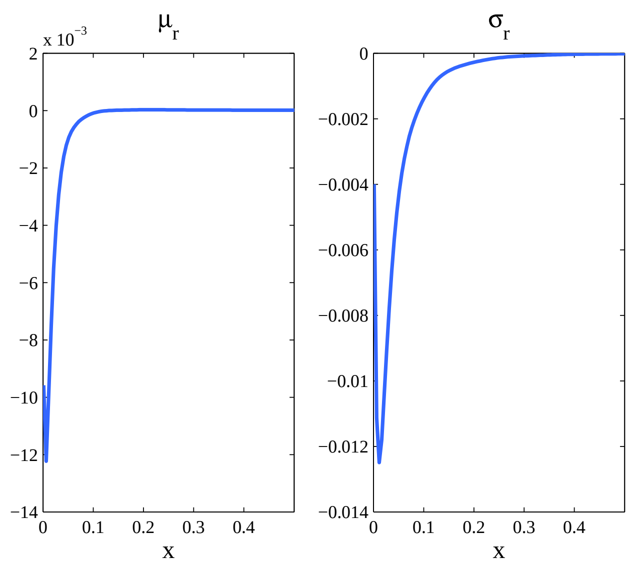


NOTES: This figure shows the model solution, with the calibrated parameters from Table 1.2 (quarterly frequency).

Figure 1.2: Real Yield Curve



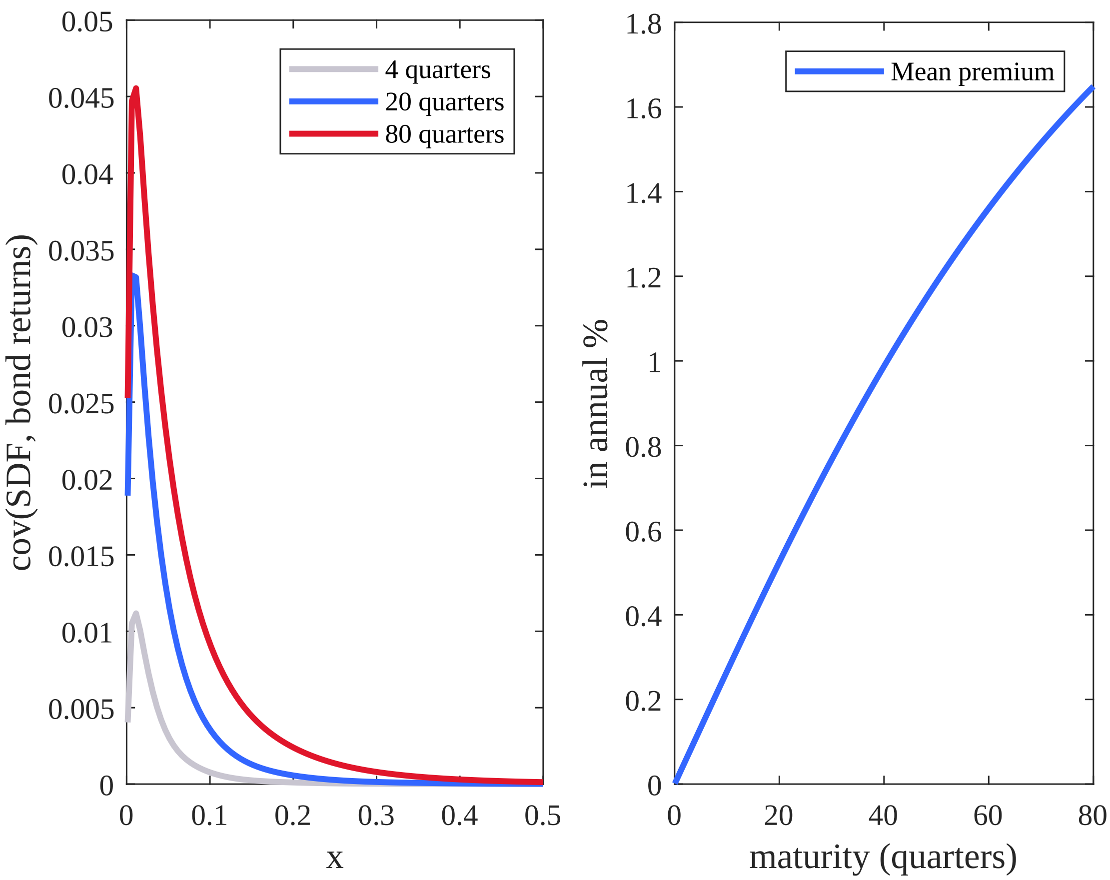
(a) Real yield curve



(b) $\mu_{r,t}$ and $\sigma_{r,t}$

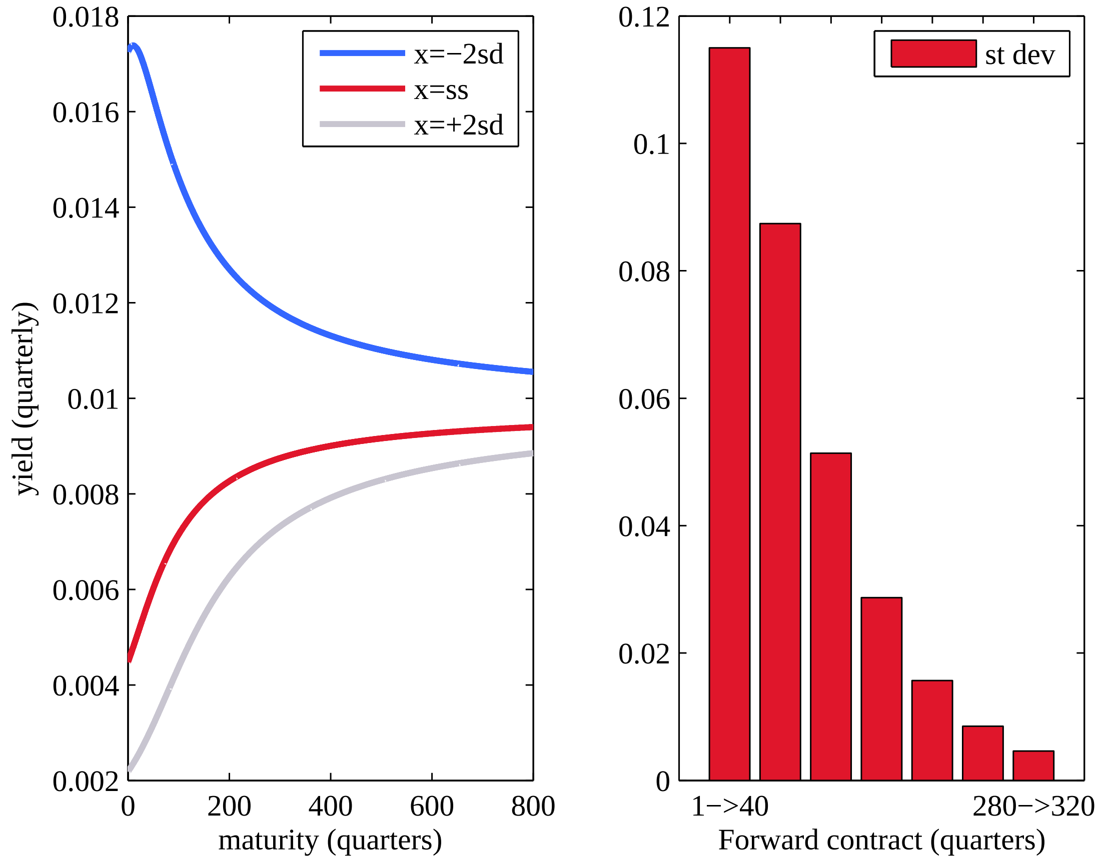
NOTES: Panel (a) displays the real yield curve in the model for three different levels of x. The blue (gray) line represents the real yield curve when x is 2 standard deviations below (above) its mean. The red line represents the real yield curve when x is at its unconditional mean. Panel (b) displays the expected change in the short-term rate, μ_r, and the diffusion for the short-term rate, σ_r, reported in (1.32). Data is from Table 3.1, full sample.

Figure 1.3: Term Premia in the Model: $cov_t^{\mathbb{P}}\left(\frac{dm_t}{m_t}, \frac{dP_t^{(T)}}{P_t^{(T)}}\right)$



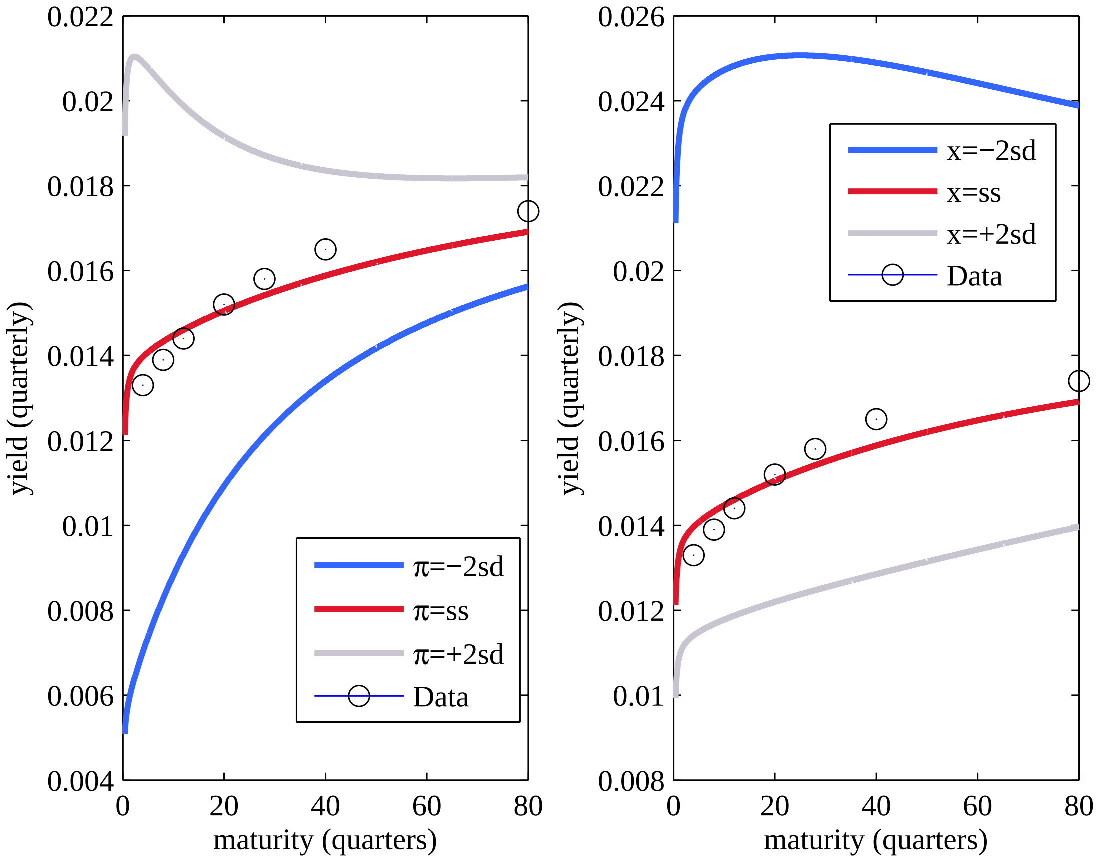
NOTES: The left-hand panel shows the conditional covariance between the stochastic discount factor and real bond returns, across the state space (red, blue, and gray lines correspond to 80, 20, and 4 quarters, respectively). The right-hand panel shows the unconditional covariance across maturities.

Figure 1.4: The Long-Term Real Yield Curve and Volatility of Forward Contracts



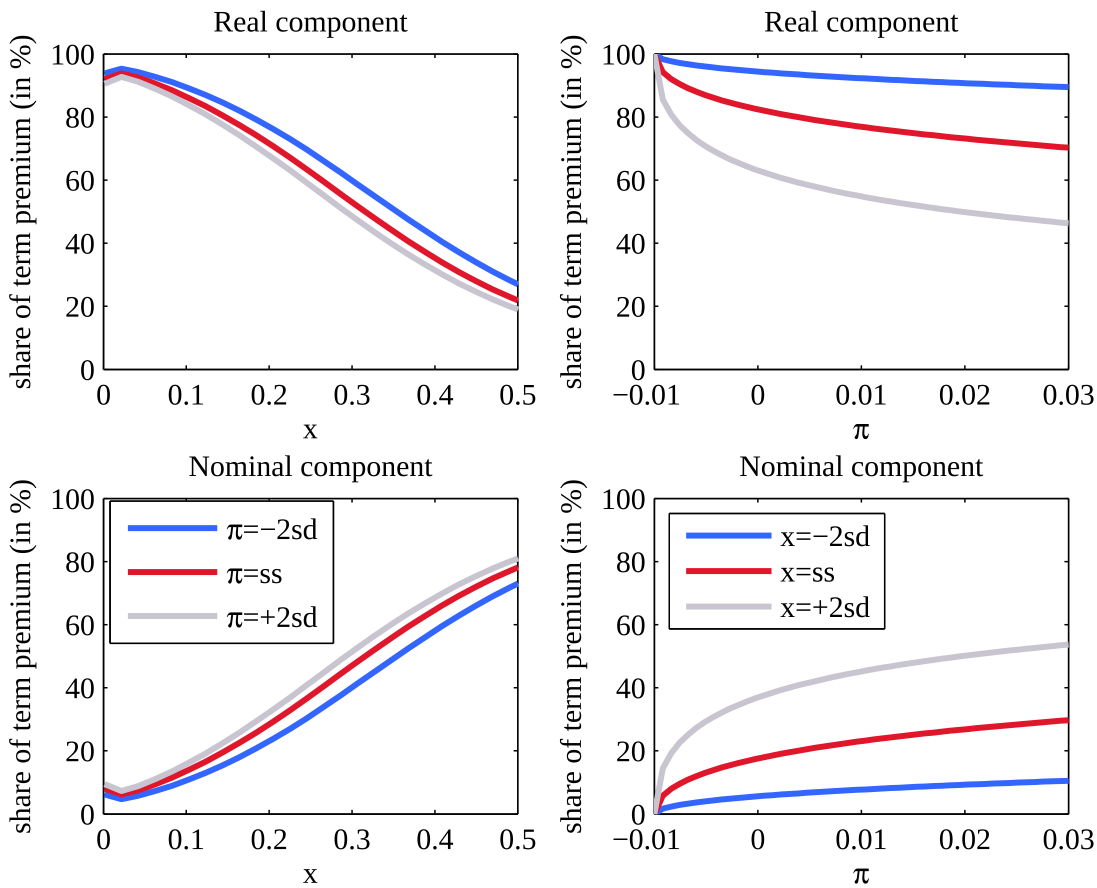
NOTES: The left panel shows the real yield curve conditional on different values of the endogenous state variable x. The blue (gray) line represents the real yield curve when x is 2 standard deviations below (above) its mean. The red line represents the real yield curve when x is at its stochastic steady state. The right panel is the standard deviation of 10 forward contracts, starting with the contract for $1q \rightarrow 40q$, continuing with $40q \rightarrow 80q$, $80 \rightarrow 120$, $120 \rightarrow 160$, and so on, until $280 \rightarrow 320$ in the last bar.

Figure 1.5: The Nominal Term Structure: Exogenous Inflation Case



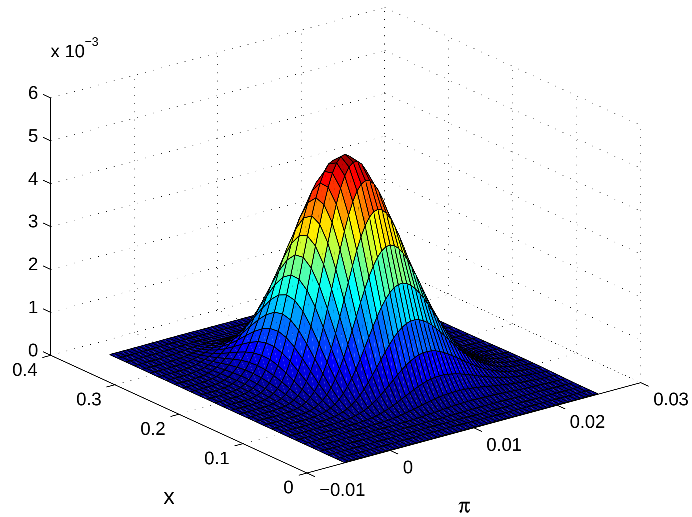
NOTES: This figure shows the nominal term structure of interest rates in the exogenous inflation case. In the left-hand panel, I set x to its unconditional mean, and I show the yield curve for three different values of π . The red line is the nominal term structure when π is at its unconditional mean level; the gray (blue) line is when π is two standard deviations above (below) its unconditional mean level. In the right-hand panel, I set π to its unconditional mean, and I show the yield curve for three different values of x . The red line is the mean x; the gray (blue) line is for x two standard deviations above (below) $x's$ mean. By definition, the red line is the same in both panels. Data from nominal yields are from Table 3.1, full sample.

Figure 1.6: Decomposition of 80-Quarters' Nominal Yield

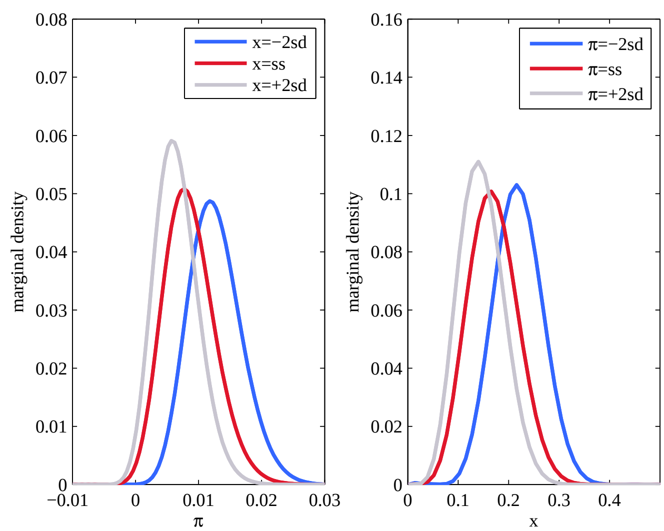


NOTES: This figure shows a decomposition of a 20-year nominal bond term premia. The left-hand panel (both upper and lower) display the real and nominal components over the x state space. The three lines represent difference levels for the other state variable, π. The red line is when π is at its unconditional mean; the blue (gray) line is when π is two standard deviations below (above) the steady state. The right-hand panel (both upper and lower) display the real and nominal components over the π state pace. The three lines represent difference levels for the other state variable, x. The red line is when x is at its unconditional mean; the blue (gray) line is when x is two standard deviations below (above) the steady-state level.

Figure 1.7: Invariant Distribution (x, π): Exogenous Inflation



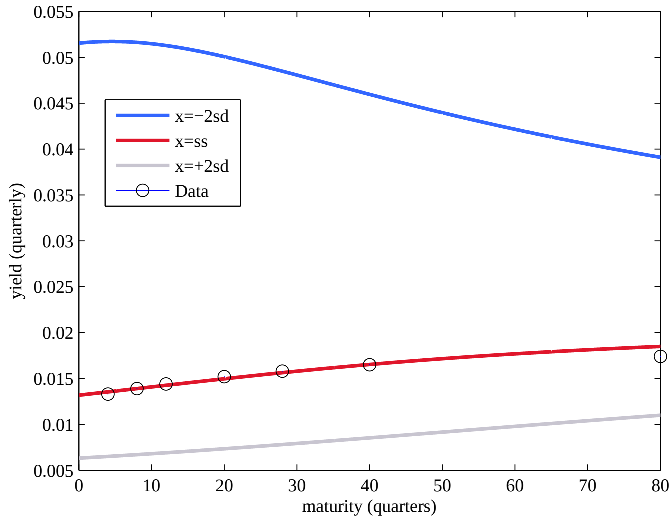
(a) Invariant distribution of (x, π)



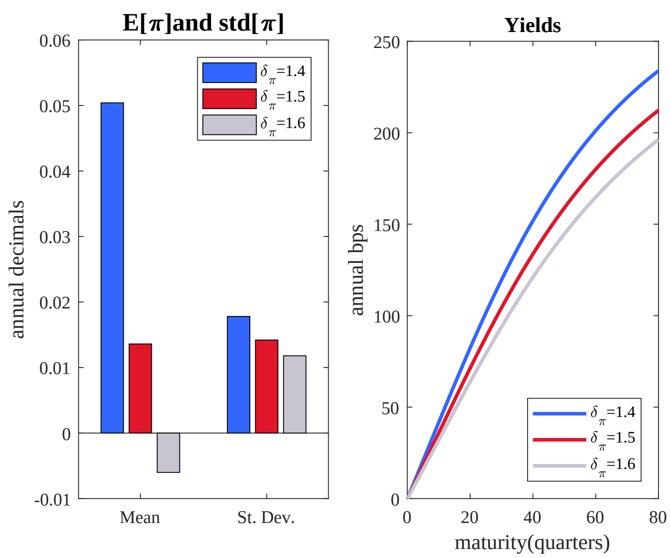
(b) Marginal distributions

NOTES: (a) shows the invariant bi-distribution in (x, π). (b) depicts the marginal distributions. In (b), the left-hand panel shows the marginal invariant distribution for π, for different levels of x: when x is 2 standard deviations below the mean (blue), when it is at its mean (red), and when it is 2 standard deviations above the mean (gray). Similarly, the right-hand panel in (b) illustrates the marginal invariant distributions for x, for different values of π. Marginal distributions are computed by integrating the bivariate mass accordingly.

Figure 1.8: The Nominal Term Structure: Endogenous Inflation Case



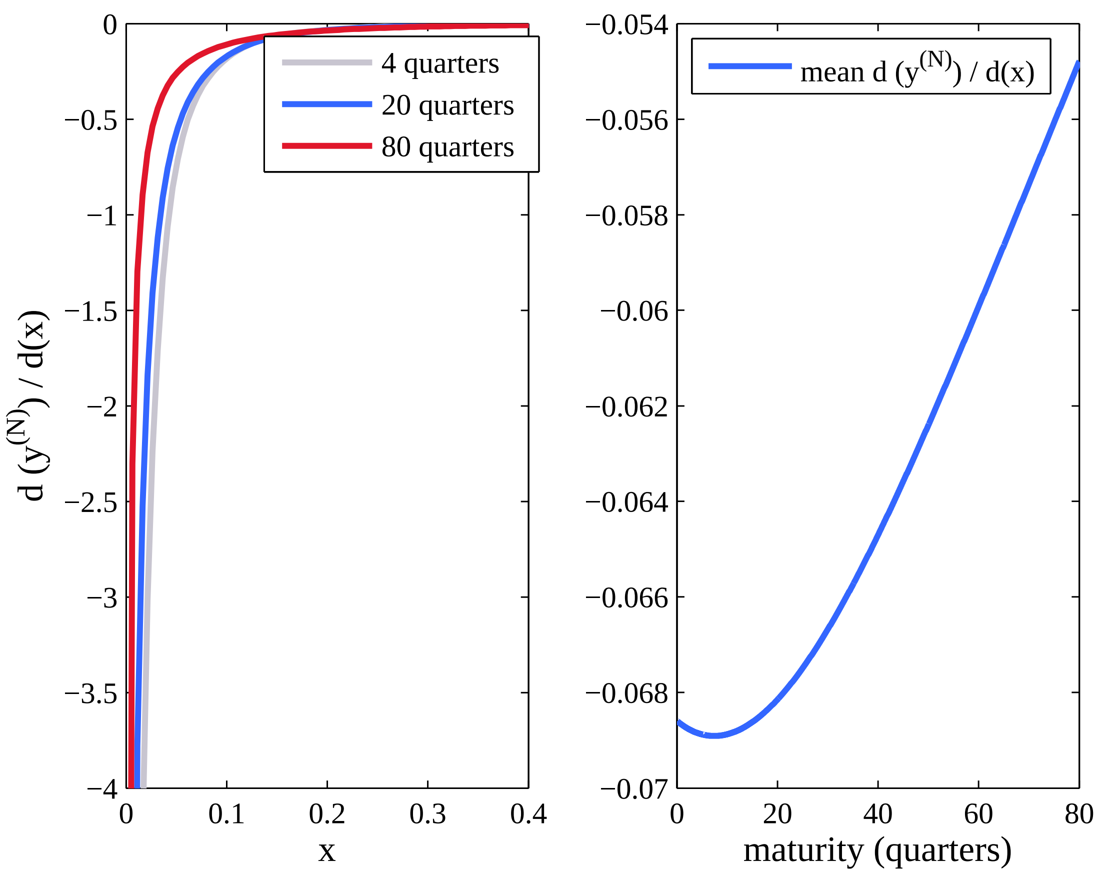
(a) Nominal yield curve: endogenous π with $\delta_\pi = 1.5$



(b) Mean π, std π, and normalized yields for different δ_π

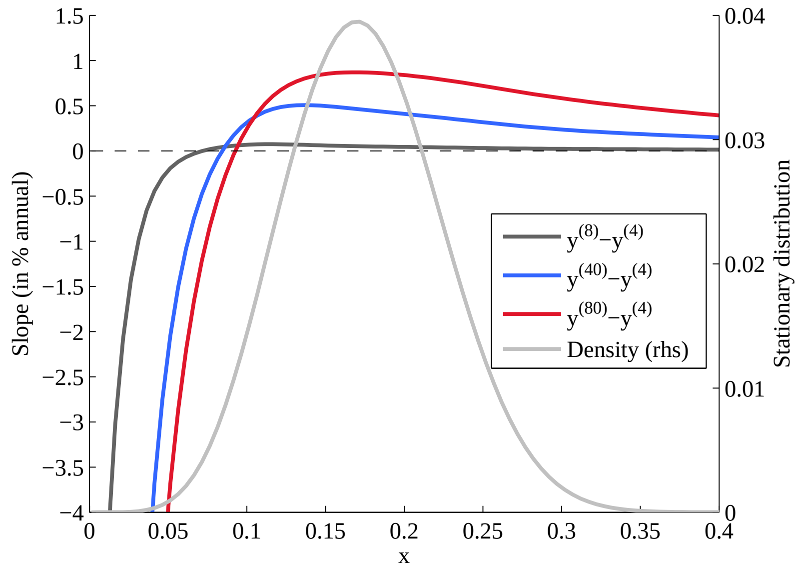
NOTES: (a) shows the nominal term structure in the endogenous inflation case. The red line is when x is at the steady state; the blue (gray) line is when x is two standard deviations below (above) the mean. In (b), the left panel shows how the mean and volatility of inflation for different Taylor coefficients δ_π, and right panel shows the slope of the nominal term structure for different Taylor coefficients δ_π.

Figure 1.9: Average Sensitivity Real Bond Yields in the Model: $E\left[\frac{dy_t^{(T)}}{dx}\right]$

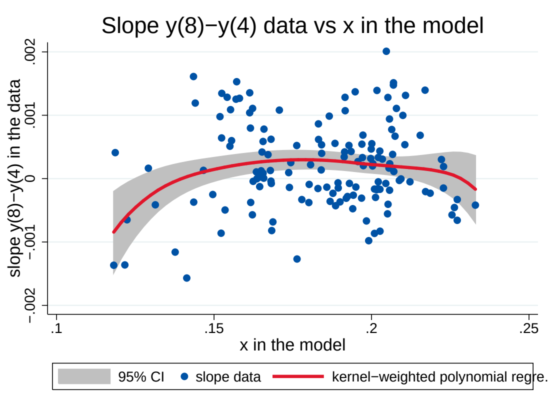


NOTES: The left panel shows the derivative $y_x^{\prime(T)}$ across the state space (red, blue, and gray lines correspond to 80, 20, and 4 quarters). The right panel shows the unconditional mean across maturities.

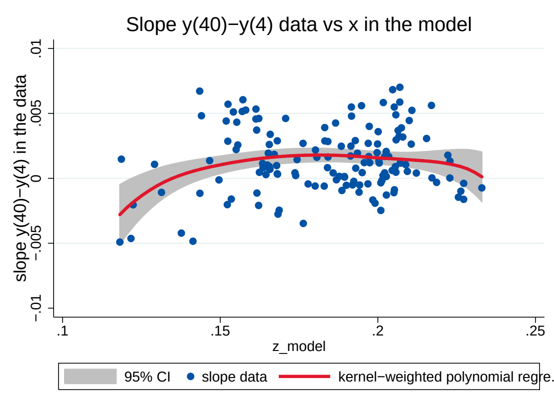
Figure 1.10: Term Structure Slope



(a) Slope in the model



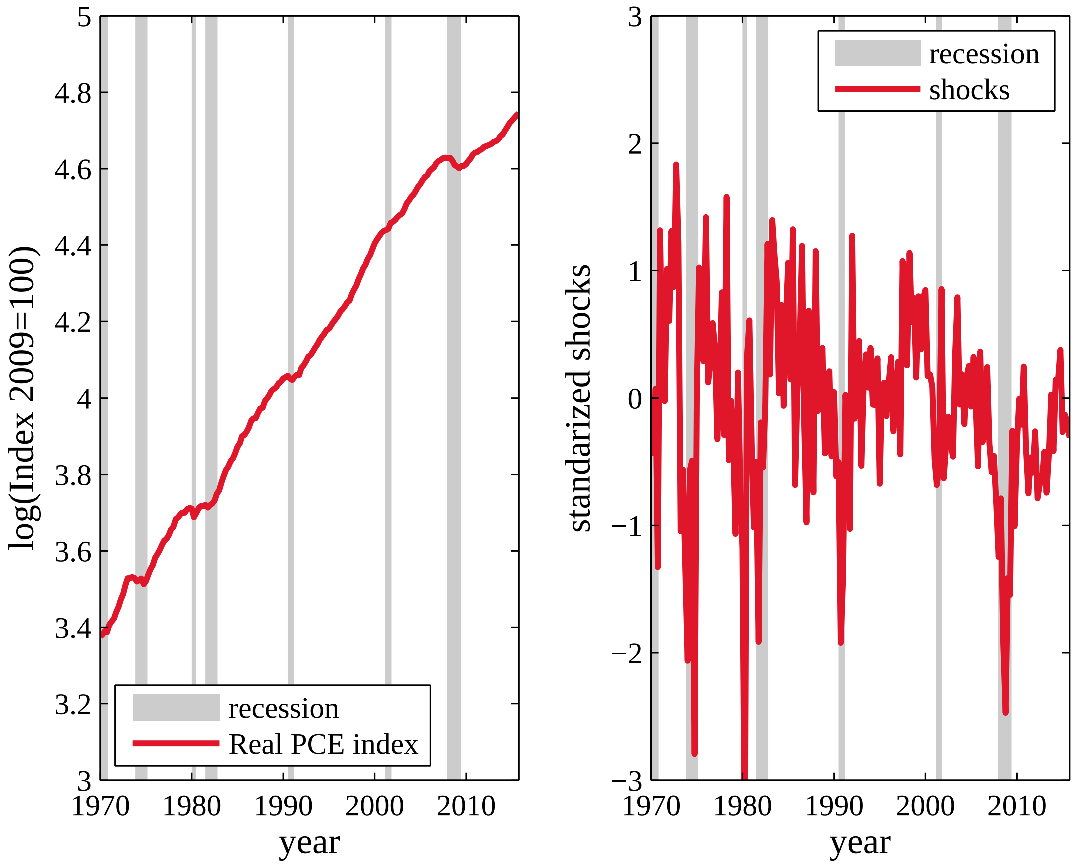
(b) Slope $y^{(8)} - y^{(4)}$ in the data and x in the model



(c) Slope $y^{(40)} - y^{(4)}$ in the data and x in the model

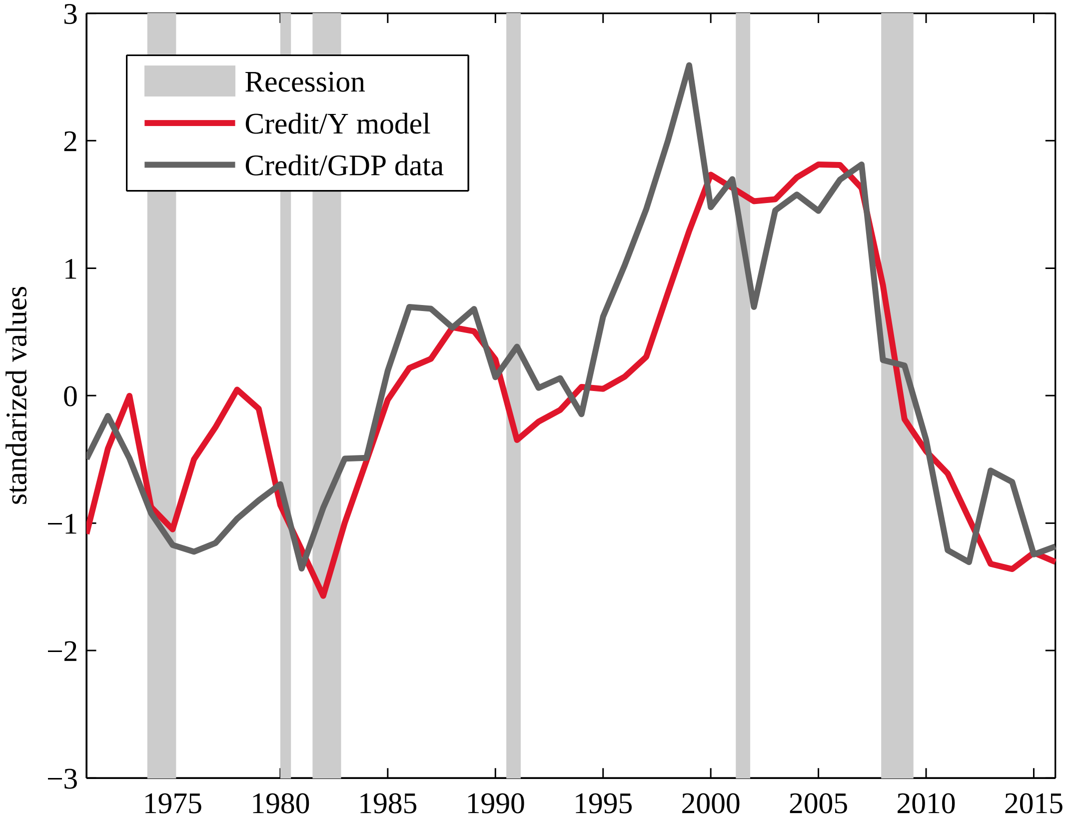
NOTES: Panel (a) shows the slope of the real term structure in the model. The red line is the spread of an 80-quarter yield minus a 4-quarter yield. The blue line is the spread of a 40-quarter yield minus a 4-quarter yield. The black line is the spread of an 8-quarter yield minus a 4-quarter yield. The gray line is the invariant distribution. Panels (b) and (c) show $y^{(8)} - y^{(4)}$ and $y^{(40)} - y^{(4)}$ from the data and x predicted by the model. The red line is the kernel-weighted local polynomial regression. I use an Epanechnikov kernel function.

Figure 1.11: Personal Consumption Expenditure Shocks and Series



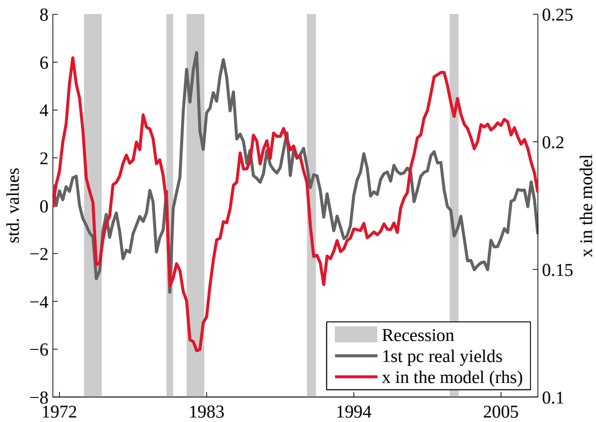
NOTES: The left panel shows the log index of personal consumption expenditures from NIPA Table 2.3.3. The right panel shows the shocks; following equation (1.34), $\Delta \log y_t - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta = \sigma\left[W_{1,t+\Delta} - W_{1,t}\right]$.

Figure 1.12: Business Cycle Analysis: Credit/GDP Data vs Credit/Y model

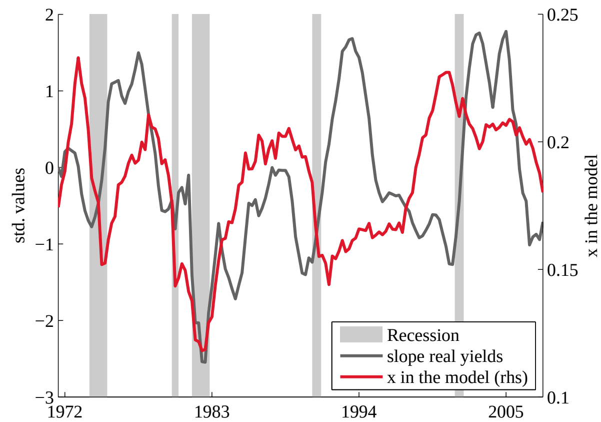


NOTES: The red line shows the implications for credit over total equity in the model after introducing the macro shocks in Figure 1.11. I start the economy from the stochastic steady state in 1971:Q3 . The black line is the fluctuations in total credit to the private sector over GDP in the U.S. (source: The World Bank).

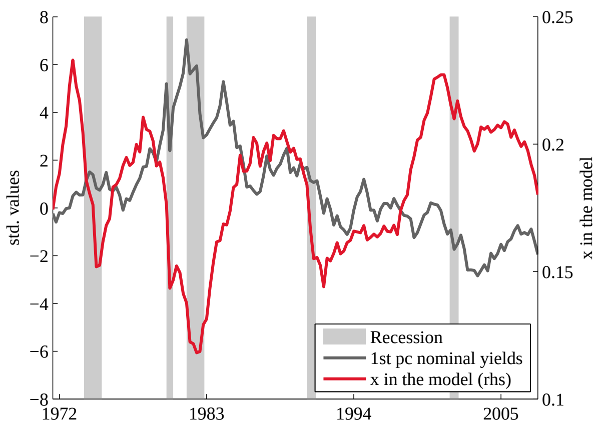
Figure 1.13: Model Implied x and Yield Curve Data



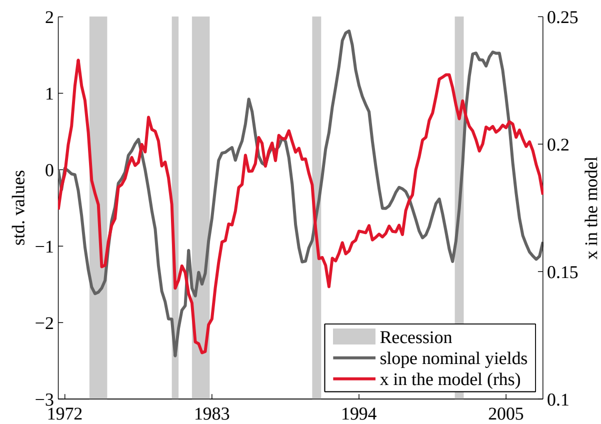
(a) x and the 1pc of real yields



(b) x and the slope of real yields



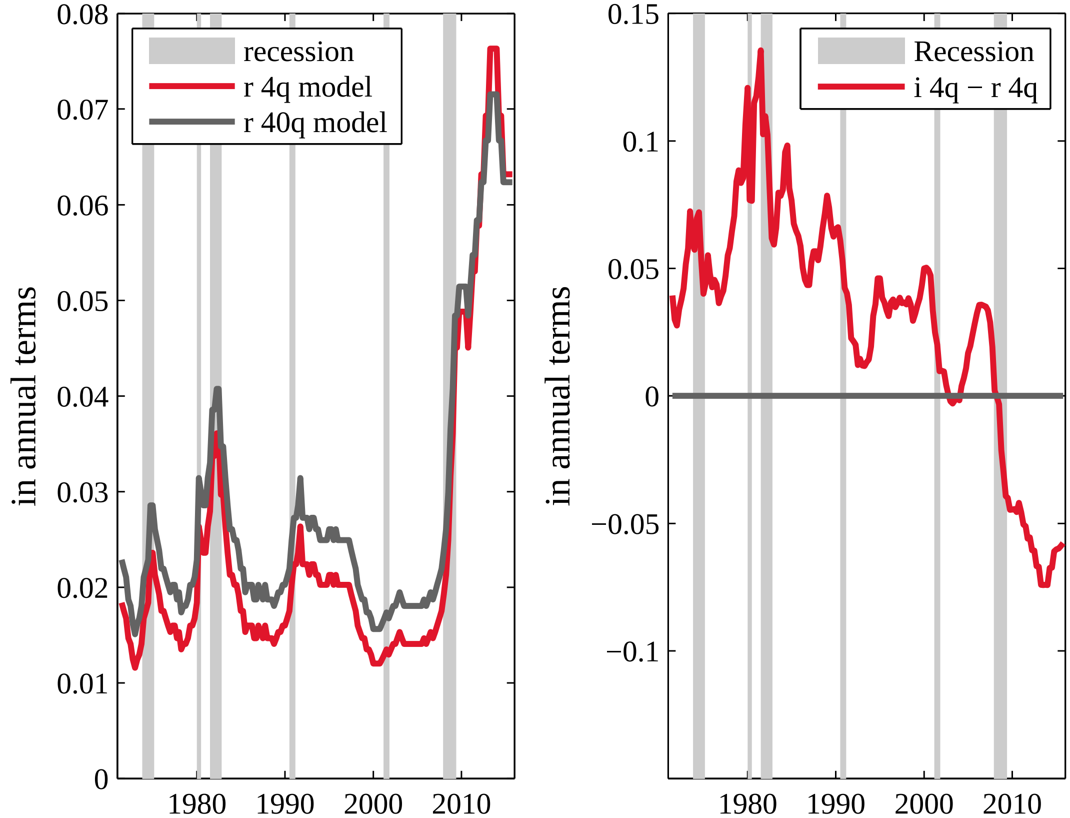
(c) x and the 1pc of nominal yields



(d) x and the slope of nominal yields

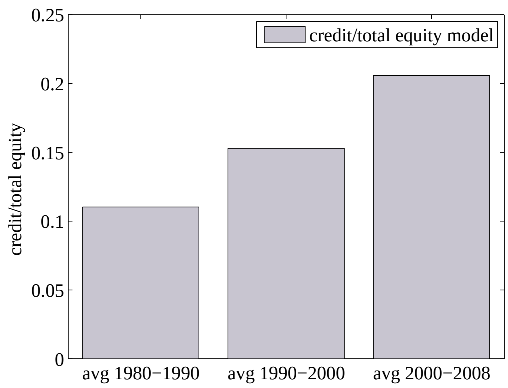
NOTES: Variable x is the endogenous state variable in the model, after feeding the sequence of macro shocks reported in Figure 1.11, and explained in Section 7. The first principal component of nominal and real yields is computed over the yields considered in Table 3.1. The slope is the 10-year minus 1-year yield (i.e. ,40 quarters minus 4 quarters), and I compute the annual average of this spread each quarter.

Figure 1.14: Puzzle I: Spike in Real Rates in the Great Recession

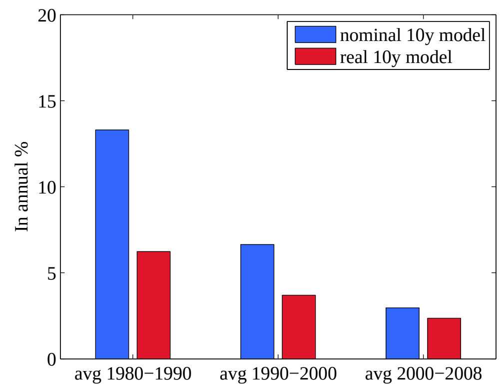


NOTES: The left panel shows the 40-quarter (10-year) yield in black, and the 4-quarter (1-year) in red, predicted by the model when I feed the series of macroeconomic shocks reported in Figure 1.11. The right panel shows the result of subtracting the model-implied 4-quarter real rate from the - quarter nominal rate in the data. That is, this implies a proxy for implied inflation expectations.

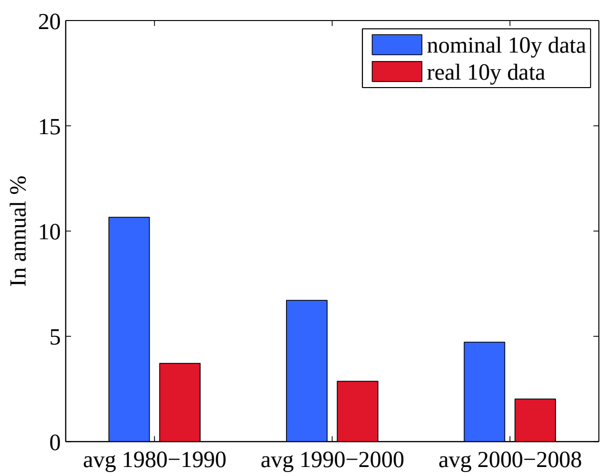
Figure 1.15: Puzzle II: Secular Decline in Long-Term Rates



(a) Credit/Total Equity: Model



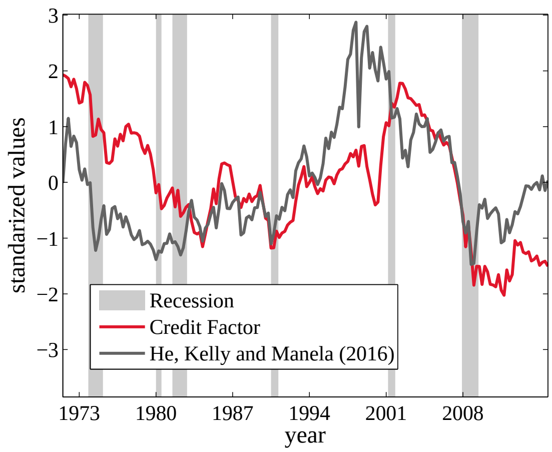
(b) Real and Nominal Long-Term Rates: Model



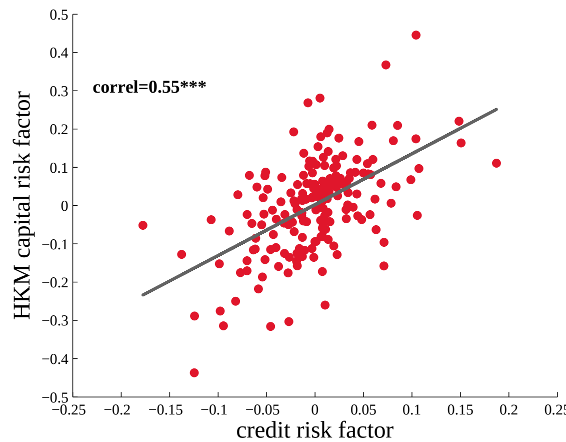
(c) Real and Nominal Long-Term Rates: Data

NOTES: Panels (a) and (b) show the model's predictions after feeding the macro shocks reported in Figure 1.11, when analyzing the transitional dynamics from 2 standard deviations below the mean of x. Panel (a) shows total credit/total equity in the model. Panel (b) shows the implications for 10-year nominal and real rate in the model. Panel (c) shows 10-year nominal and real rates in the data.

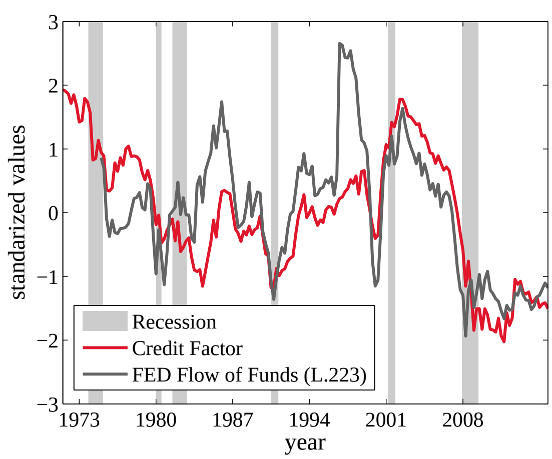
Figure 1.16: Comparing the Credit Factor



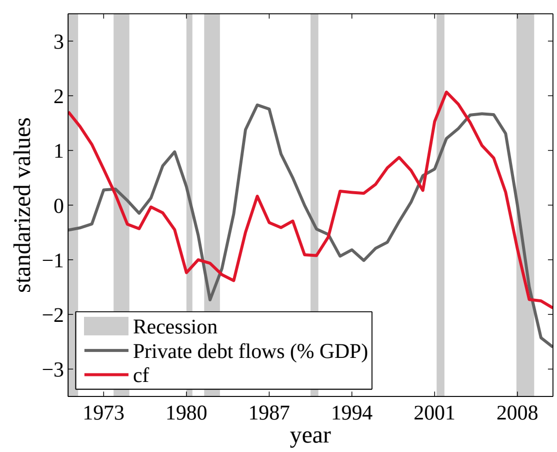
(a) cf and He, Kelly and Manela (2016) factor: Levels



(b) cf and He, Kelly and Manela (2016) factor: Shocks



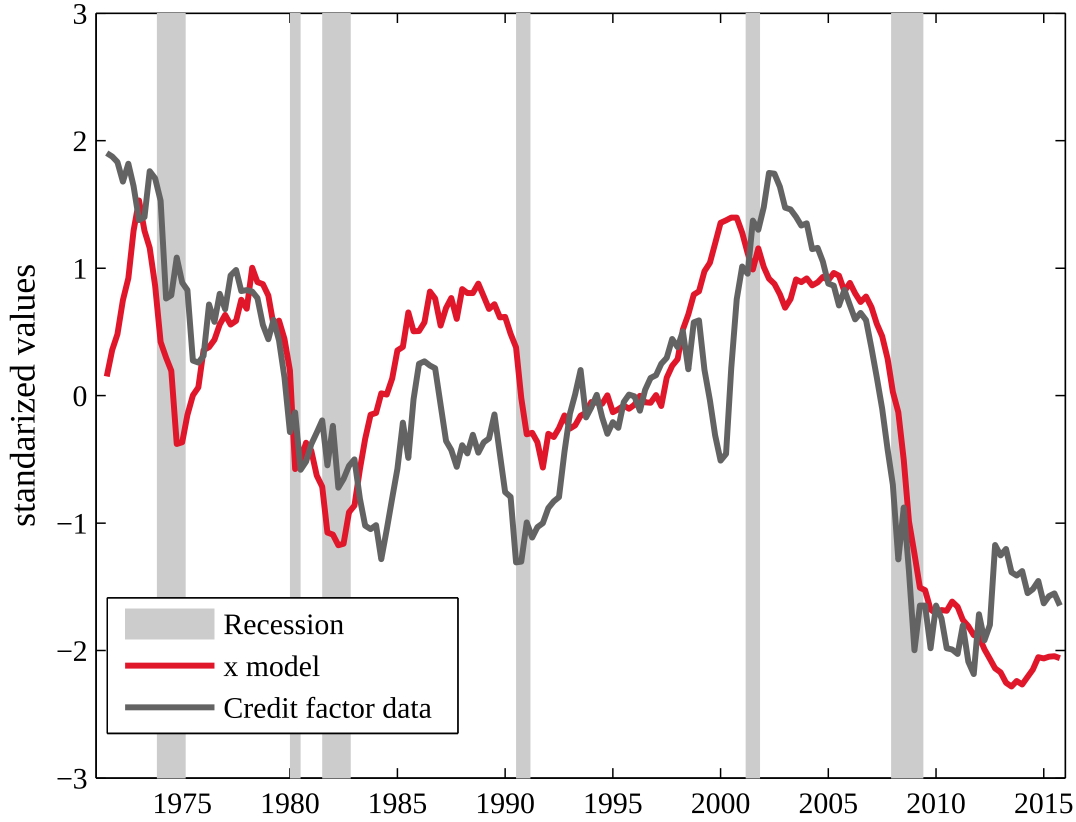
(c) cf and cf from Flow of Funds



(d) cf and debt flows/GDP from Philippon (2015) (annual)

NOTES: Variable Credit Factor (cf), displayed in all panels, is the market value of net worth of SIC codes 60-64 over total net worth (source: CRSP). I do a rolling linear detrending each quarter to remove the persistent component. The availability of cf in CRSP is from 1926, although the figure shows since 1971:Q3 to be consistent with the yields. Panels (a) and (b) compare with the factor used in He et al. (2016). Panel (c) compares with the data in the Flow of Funds, Table L.223. Panel (d) compares, at annual frequency, with the flow of intermediated assets in the financial sector (source: Philippon (2015)).

Figure 1.17: Credit Factor vs x in Model



NOTES: This figure shows the implications for x in the model, after introducing the macro shocks in Figure 1.11 . The black line is cf defined in Section 7, and displayed in Figure 1.16.

1.9 Appendix

Proof proposition (law of motion for x). The law of motion follows by applying Ito's lemma in (1.9).

$$\frac{dx_t}{x_t} = \frac{dn_{A,t}}{n_{A,t}} - \frac{dq_t}{q_t} + \left(\frac{dq_t}{q_t}\right)^2 - \left(\frac{dq_t}{q_t}\right)\left(\frac{dn_{A,t}}{n_{A,t}}\right) , \tag{1.39}$$

where the aggregate wealth for A-type investors can be computed as $n_{A,t} = \overline{x} \int_{-\infty}^{t} \varphi e^{-\varphi(t-\widetilde{u})} n_{A,\widetilde{u}} d\widetilde{u}$, and $q_t = n_{A,t} + n_{B,t}$. Then

$$\frac{dq_t}{q_t} = x_t \frac{dn_{A,t}}{n_{A,t}} + (1 - x_t) \frac{dn_{B,t}}{n_{B,t}} ,$$

so, the terms in (1.39) are

$$\begin{aligned}
\frac{dn_{A,t}}{n_{A,t}} - \frac{dq_t}{q_t} &= (1 - x_t)\left(\frac{dn_{A,t}}{n_{A,t}} - \frac{dn_{B,t}}{n_{B,t}}\right) , \\
\left(\frac{dq_t}{q_t}\right)\left(\frac{dq_t}{q_t} - \left(\frac{dn_{A,t}}{n_{A,t}}\right)\right) &= \left(\frac{dq_t}{q_t}\right)\left((1 - x)\left(\frac{dn_{B,t}}{n_{B,t}} - \frac{dn_{A,t}}{n_{A,t}}\right)\right) , \\
&= \sigma_{q1}^2 \left(x_t \alpha_{A,t} + (1 - x_t)\, \alpha_{B,t}\right)(1 - x)\left(\alpha_{B,t} - \alpha_{A,t}\right) \\
&= \sigma_{q1}^2 (1 - x)\left(\alpha_{B,t} - \alpha_{A,t}\right)
\end{aligned}$$

where the last step follows for market clearing for shares. Using Itô's lemma in $n_{A,t}$ and $n_{B,t}$

$$\begin{aligned}
\frac{dn_{A,t}}{n_{A,t}} &= \left[r_t + \varphi - \frac{c_{A,t}}{n_{A,t}} + \alpha_{A,t}\left(\mu_{q,t} - r_t\right)\right] dt + \alpha_{A,t}\sigma_{q,t} dW_{1,t} + \varphi\left(\frac{\overline{x}}{x_t}\frac{\widehat{e}_t}{pd_t} - 1\right) dt\ , \\
\frac{dn_{B,t}}{n_{B,t}} &= \left[r_t + \varphi - \frac{c_{B,t}}{n_{B,t}} + \alpha_{B,t}\left(\mu_{q,t} - r_t\right)\right] dt + \alpha_{B,t}\sigma_{q,t} dW_{1,t} + \varphi\left(\left(\frac{1-\overline{x}}{1-x_t}\right)\frac{\widehat{e}_t}{pd_t} - 1\right) dt,
\end{aligned}$$

Then

$$\begin{aligned}
\sigma_x &= x\left(1-x_t\right)\left(\frac{dn_{A,t}}{n_{A,t}} - \frac{dn_{B,t}}{n_{B,t}}\right) = x\left(1-x_t\right)\left(\alpha_{A,t} - \alpha_{B,t}\right)\ , \\
\mu_{x,t} &= x_t\left(1-x_t\right)\left(\frac{c_{B,t}}{n_{B,t}} - \frac{c_{A,t}}{n_{A,t}} + \left(\alpha_{A,t} - \alpha_{B,t}\right)\left(\mu_{q,t} - r_t - \sigma^2_{q1,t}\right)\right) + x_t\left(1-x_t\right)\frac{\varphi\widehat{e}_t}{pd_t}\left(\frac{\overline{x}}{x_t} - \frac{1-\overline{x}}{1-x_t}\right),
\end{aligned}$$

and then $x_t\left(1-x_t\right)\left(\frac{\overline{x}}{x_t} - \frac{1-\overline{x}}{1-x_t}\right) = \left(\overline{x}\left(1-x_t\right) - \left(1-\overline{x}\right)x_t\right) = \left(\overline{x} - \overline{x}x_t - x_t + \overline{x}x_t\right) = \left(\overline{x} - x_t\right)$.

■

Proof proposition (leverage and risk sharing). As stated in FOC, the portfolio share of A-type agents is

$$\alpha_A = \frac{\mu_q - r}{\gamma_A\sigma_q^2} + \left(\frac{1-\gamma_A}{1-\psi_i}\right)\frac{\sigma_{\xi A}}{\gamma_A\sigma_q}\ ,$$

where, notice,

$$\sigma_{\xi A} = \frac{\xi_{x,A}}{\xi_A} x\left(1-x\right)\left(\alpha_B - \alpha_A\right)\sigma_q$$

which means

$$\begin{aligned}
\frac{\mu_{q,t} - r_t}{\sigma_q^2} &= \gamma_A\alpha_A - \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\sigma_{\xi A}}{\sigma_q} \\
&= \gamma_A\alpha_A - \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} x\left(1-x\right)\left(\alpha_A - \alpha_B\right).
\end{aligned}$$

Use market clearing for shares

$$\begin{aligned} x\alpha_A + (1-x)\,\alpha_B &= 1 \\ (1-x)\,(\alpha_A - \alpha_B) &= (\alpha_A - 1) \end{aligned}$$

means

$$\begin{aligned} \frac{\mu_{q,t} - r_t}{\sigma_q^2} &= \gamma_A \alpha_A - \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} x\,(\alpha_A - 1) \\ &= \alpha_A \left[\gamma_A - \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} x\right] + \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} x. \end{aligned}$$

So, we can use this in the conditions for B

$$\begin{aligned} \alpha_A \left[\gamma_A - \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} x\right] + \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} x + \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B} x\,(\alpha_A - 1) &= \gamma_B \alpha_B \\ \alpha_A \left[\gamma_A - \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} x + \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B} x\right] + \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} x - \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B} x &= \gamma_B \alpha_B. \end{aligned}$$

Define

$$R_t^x = \left[\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} - \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}\right] x$$

then

$$\alpha_A \left[\frac{\gamma_A}{\gamma_B} - \frac{R_t^x}{\gamma_B}\right] + \frac{R_t^x}{\gamma_B} = \alpha_B,$$

so

$$\begin{aligned}
x\alpha_A + (1-x)\,\alpha_B &= 1 \\
x\alpha_A + (1-x)\left[\alpha_A\left[\frac{\gamma_A}{\gamma_B} - \frac{R_t^x}{\gamma_B}\right] + \frac{R_t^x}{\gamma_B}\right] &= 1 \\
x\alpha_A + (1-x)\,\alpha_A\left[\frac{\gamma_A}{\gamma_B} - \frac{R_t^x}{\gamma_B}\right] + (1-x)\,\frac{R_t^x}{\gamma_B} &= 1 \\
\alpha_A &= \frac{1-(1-x)\,\frac{R_t^x}{\gamma_B}}{x+(1-x)\left[\frac{\gamma_A}{\gamma_B} - \frac{R_t^x}{\gamma_B}\right]},
\end{aligned}$$

so

$$\begin{aligned}
\alpha_A - 1 &= \frac{1-(1-x)\,\frac{R_t^x}{\gamma_B}}{x+(1-x)\left[\frac{\gamma_A}{\gamma_B} - \frac{R_t^x}{\gamma_B}\right]} - 1 \\
&= \frac{(1-x)\,(\gamma_B-\gamma_A)}{\gamma_B x + (1-x)\left[\gamma_A - R_t^x\right]}.
\end{aligned}$$

Notice $\alpha_A > 1 \Leftrightarrow$

$$\begin{aligned}
1-(1-x)\,\frac{R_t^x}{\gamma_B} &> x+(1-x)\,\frac{\gamma_A}{\gamma_B} - (1-x)\,\frac{R_t^x}{\gamma_B} \\
1-x &> (1-x)\,\frac{\gamma_A}{\gamma_B} \\
\gamma_B &> \gamma_A
\end{aligned}$$

■

Proof proposition (stochastic discount factor). Suppose there is a unique stochastic discount factor m, with a drift given by a process r and diffusion (price of risk) given by κ.

Then, the absence of arbitrage implies that

$$\mu_q - r_t = E_t^{\mathbb{P}}\left[\frac{dm_t}{m_t}\frac{dq_t}{q_t}\right] \tag{1.40}$$
$$= \sigma_{q1,t}\kappa_t.$$

We can solve for κ_t using the agent's FOCs and market clearing-conditions. In particular, I define $\varepsilon_{i,t} = \alpha_{i,t}\sigma_{q1,t}$ as the exposure chosen by agent i to W_1 shocks. The price of this exposure is κ_t. Then,

$$\alpha_{i,t}\left(\mu_q - r_t\right) = \varepsilon_{i,t}\kappa_t.$$

Then the FOC for $\varepsilon_{i,t}$ are

$$\varepsilon_{i,t} = \frac{\kappa_t}{\gamma_i} + \left(\frac{1-\gamma_i}{\left(1-\psi_i\right)\gamma_i}\right)\sigma_{\xi i},$$

and using market clearing for shares, I get $x\varepsilon_A + (1-x)\,\varepsilon_B = \sigma_{q1}$, so

$$\kappa\left(x\right) = \frac{\sigma_{q1} - x\left(\frac{1-\gamma_A}{(1-\psi_A)\gamma_A}\right)\sigma_{\xi A} - (1-x)\left(\frac{1-\gamma_B}{(1-\psi_B)\gamma_B}\right)\sigma_{\xi B}}{\frac{x}{\gamma_A} + \frac{1-x}{\gamma_B}}$$

where

$$\sigma_{\xi i} = \frac{\xi_{ix}'}{\xi_i}\sigma_x.$$

Then, the risk free rate follows by the no-arbitrage condition (1.40). Following expression

(1.9), and incorporating the laws of motion $\frac{dn_{A,t}}{n_{A,t}}$, $\frac{dn_{B,t}}{n_{B,t}}$, I obtain

$$\begin{aligned}\frac{dq_t}{q_t} &= \left[r_t + \delta + \left(x_t\alpha_{A,t} + (1-x_t)\,\alpha_{B,t}\right)\left(\mu_{q,t} - r_t\right) - x\frac{c_{A,t}}{n_{A,t}} - (1-x_t)\,\frac{c_{B,t}}{n_{B,t}}\right] dt \\ &+ \left(x_t\alpha_{A,t} + (1-x_t)\,\alpha_{B,t}\right)\sigma_{q1,t}dW_{1,t} \\ &+ \left[\varphi\frac{e_t}{q_t} - \varphi\right] dt.\end{aligned}$$

Thus, using market clearing for goods and shares, and then canceling out

$$\begin{aligned}\frac{dq_t}{q_t} &= \left[\mu_{q,t} + \varphi\frac{e_t}{q_t} - \frac{y_t}{q_t}\right] dt + \sigma_{q1,t}dW_{1,t}, \\ \frac{dq_t}{q_t} + \frac{y_t - \varphi e_t}{q_t}dt &= \mu_{q,t}dt + \sigma_{q1,t}dW_{1,t}.\end{aligned}$$

By no-arbitrage, I obtain the expression in equation (1.19)

$$E^{\mathbb{P}}\left[\frac{dq_t}{q_t}\right] + \frac{y_t - \varphi e_t}{q_t}dt - r_t - \sigma_{q1,t}\kappa_t = 0.$$

Using $pd_t = q_t/y_t$, this can be written as an ordinary differential equation in $pd\,(x_t)$. That is, using Itô's lemma in the function $pd_t y_t = q_t$, I have $\mu_q = \mu_{pd}\,(x) + \mu + \sigma_{pd}\,(x)\,\sigma$ and $\sigma_{q1}\,(x) = \sigma_{pd}\,(x) + \sigma$. The functions are simply $\mu_{pd}\,(x) = \frac{pd'_x}{pd}E\,[dx] + \frac{1}{2}\frac{pd''_{xx}}{pd}E\,[dx^2]$ and $\sigma_{pd} = \frac{pd'_x}{pd}\sigma_x$. ■

Proof proposition (infinitely lived investor). I first solve for the value function of the representative investment investor in the economy with aggregate endowment (2.3). To that end, I use the same power form as in (1.12), together with the first-order condition for consumption. Then, I can substitute to get

$$\mathcal{U} = \frac{c^{1-\gamma}}{1-\gamma} \xi^{\frac{(1-\gamma)\psi}{1-\psi}}, \tag{1.41}$$

with ξ being a constant (i.e., there are no endogenous fluctuations in the investment opportunity set). I can then use (1.41) in $0 = f(\mathcal{U}, c) + E^{\mathbb{P}}[d\mathcal{U}]$ to solve for ξ

$$\begin{aligned} 0 &= \frac{\rho}{1-\frac{1}{\psi}}(\xi - 1) + \mu - \frac{\gamma}{2}\sigma^2 \\ \xi &= \left[\frac{\gamma}{2}\sigma^2 - \mu\right]\left(\frac{1-\frac{1}{\psi}}{\rho}\right) + 1 \end{aligned}$$

The stochastic discount factor, following the martingale approach developed in Schroder and Skiadas (1999), is given by

$$m_t = \exp\left(\int_0^t f'_{\mathcal{U},u} du\right) f'_{c,t}, \tag{1.42}$$

where the derivatives with respect to the value function $\mathcal{U}$ and c are given by

$$\begin{aligned} f'_c &= \rho c^{-\frac{1}{\psi}} \left((1-\gamma)\mathcal{U}\right)^{\frac{\left(\frac{1}{\psi}-\gamma\right)}{1-\gamma}}, \\ f'_{\mathcal{U}} &= \frac{\rho}{1-\frac{1}{\psi}}\left(\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\right) c^{1-\frac{1}{\psi}} (1-\gamma)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} \mathcal{U}^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} - \frac{\rho(1-\gamma)}{1-\frac{1}{\psi}}, \end{aligned}$$

so the risk-free rate is

$$r_t = -E^{\mathbb{P}}\left[\frac{dm_t}{m_t}\right]$$

Using Itô's lemma in (1.42) and computing the expectation yields

$$r_t = \overline{r} \equiv \frac{\rho\left(1-\gamma\right)\psi}{1-\psi}\left[\xi - 1\right] + \frac{1}{\psi}\mu - \frac{1}{\psi}\left(\frac{1}{\psi}+1\right)\sigma^2.$$

To compute real bond prices, I use a guess-and-verify procedure. That is, I guess that real bond prices are exponentially affine in the time dimension

$$\begin{aligned} P^{(T)}\left(t\right) &= \exp\left(A\left(t\right)\right), \\ &\quad \mathit{with} \\ P^{(0)}\left(t\right) &= 1,\ \forall t \end{aligned}$$

where A_t is an unknown function of time. Real bond prices are characterized by the same Cauchy problem as in the main text,

$$\begin{aligned} P_t^{\prime(T)} &= \overline{r}P \\ A_t^{\prime} &= r \\ &\quad \mathit{with} \\ A\left(0\right) &= 0 \end{aligned} \tag{1.43}$$

so the ODE (1.43) is very simple: $A\left(t\right) = \overline{r}t$. Then yields $y_t^{(T)} = -\frac{1}{T}\log P_t^{(T)} = \overline{r}\ \forall\left(t,T\right).$

Then the price-dividend ratio is characterized by

$$pd_t = \frac{q_t}{y_t} = E_t^{\mathbb{P}}\left[\int_t^{\infty} \frac{m_{t+u}}{m_t}\frac{y_u}{y_t}du\right].$$

The $pd_t = \overline{pd}$ is constant, and it has the standard Gordon growth expression. The partial differential equation characterizing the nominal bond is

$$\begin{aligned}
\frac{P_t'(\pi,t)}{P(\pi,t)} &= -\overline{r} - \pi + \frac{P_\pi'(\pi,t)}{P(\pi,t)}\lambda_\pi(\overline{\pi} - \pi_t) + \frac{1}{2}\frac{P_{\pi\pi}''(\pi,t)}{P(\pi,t)}\sigma_\pi^2(\pi - \pi_L) \\
&\quad - \frac{P_\pi'(\pi,t)}{P(\pi,t)}\gamma\varphi_{13}\sigma\sigma_\pi\sqrt{\pi - \pi_L}, \qquad (1.44) \\
P(\pi,0) &= 1\ \forall\pi.
\end{aligned}$$

I next change variables and assume that $4(\overline{\pi} + \pi_L)\lambda_\pi = \sigma_\pi^2$ to ease calculations[19]

$$z = \pi - \pi_L\ ,$$

and notice

$$\begin{aligned}
P(\pi) &= P(z + \pi_L), \\
\frac{P_\pi'}{P} &= \frac{P_z'}{P}.
\end{aligned}$$

Use the solution

$$P = \widetilde{A}(t)\exp\left(B(t)z + C(t)\sqrt{z}\right)\ , \qquad (1.45)$$

[19]Notice that when $\pi_L = 0$, this assumption would led to a violation of the so-called Feller condition. Because $\pi_L < 0$, the process is reflected at a point that is below 0.

where $\widetilde{A}(t)$ is adjusted for the change in variables. Substituting (1.45) in (1.44), functions $\widetilde{A}, B$, and C solve a system of ordinary differential equations. In particular, the solution for B follows a particular case of the Riccati equation:

$$\begin{aligned} B_t' &= \frac{\sigma_\pi^2}{2} B(t)^2 - \lambda_\pi B(t) - 1 \ , \\ B(0) &= 0 \ , \end{aligned}$$

the function $C(t)$, associated with the $\sqrt{x}$ term solves

$$\begin{aligned} 0 &= -C_t' - \lambda_\pi C(t) - B(t)\gamma\varphi_{13}\sigma\sigma_\pi + \frac{\sigma_\pi^2}{2} B(t) C(t) \ , \\ C(0) &= 0 \ , \end{aligned}$$

and the constant

$$\begin{aligned} 0 &= -\frac{\widetilde{A}_t'}{\widetilde{A}} - \overline{r} + \pi_L + \lambda_\pi(\overline{\pi} + \pi_L) B(t) - \frac{1}{2} C(t)\gamma\varphi_{13}\sigma\sigma_\pi + \frac{\sigma_\pi^2}{2} C(t)^2 \\ \widetilde{A}(0) &= 1. \end{aligned}$$

Although the solution for $A(t), B(t)$, and $C(t)$ can be solved in closed form, I omit it in the interest of space(see Longstaff (1989)). However, this system of ODEs can be solved numerically with any standard routine.

■

Evidence for the U.K. In the table below, I report evidence for real and nominal yields in the U.K.. The source of the data is the Bank of England.[20] I use the same criteria adopted for U.S. data and consider the Full Sample the largest available sample for real yields. This includes 1985:Q1-2016:Q4 for the U.K. Short sample I and Short sample II are as in the U.S. data, 2003:Q1-2016:Q4 and 1985:Q1-2008:Q2. I report real rates starting from 3 years, because the shorter maturity available is 2.5 years.

		Maturity (quarters)							
Panel A. Full Sample		4	8	12	20	28	40	80	diff(40-12)
Mean	Nominal	0.0134	0.0136	0.0138	0.0143	0.0147	0.0150	0.0139	0.0012
	Real			0.0044	0.0048	0.0050	0.0053	0.0046	0.0006
St. dev	Nominal	0.0094	0.0089	0.0085	0.0080	0.0077	0.0072	0.0062	-0.0012
	Real			0.0051	0.0047	0.0044	0.0043	0.0040	-0.0011
Panel B. Short Sample I									
Mean	Nominal	0.0053	0.0057	0.0061	0.0071	0.0078	0.0085	0.0096	0.0028
	Real			0.0000	0.0001	0.0001	0.0013	0.0015	0.0013
St. Dev	Nominal	0.0051	0.0048	0.0045	0.0040	0.0036	0.0031	0.0022	-0.0017
	Real			0.0044	0.0038	0.0034	0.0030	0.0024	-0.0014
Panel C. Short Sample II									
Mean	Nominal	0.0177	0.0177	0.0178	0.0179	0.0180	0.0179	0.0160	0.002
	Real			0.0071	0.0072	0.0073	0.0074	0.0067	0.003
St. Dev	Nominal	0.0070	0.0065	0.0062	0.0061	0.0060	0.0059	0.0061	-0.006
	Real			0.0022	0.0021	0.0022	0.0022	0.0024	0.000

As can be seen in the table, the nominal and real term structures share similar properties:

[20] http://www.bankofengland.co.uk/statistics/pages/yieldcurve/default.aspx

They are upward sloping on average up to 40 quarters (10 years) but then the average yield of an 80-quarter bond is smaller than the 40-quarter. Small sample II is the exception, in which on average both real and nominal yield curve are upward sloping. The volatility of long-term rates is smaller than the volatility of short-term yields. Indeed, the volatility of real and nominal yields is very similar in Short Sample I (as in the U.S.). **Extension:**

Consol bond. In this extension, I consider the pricing of a perpetual (real) bond with an exponentially decaying coupon, denoted δ. The purpose is to illustrate the main results in the paper using these alternative financial instruments. To avoid redundancy with the analysis presented above, I include this subsection in the appendix. The price of the consol bond, denoted C_t, is

$$C_t = E_t^{\mathbb{Q}}\left[\int_t^{\infty} e^{-\int_t^s (r_u+\delta)du} ds\right] \equiv C(x_t)$$

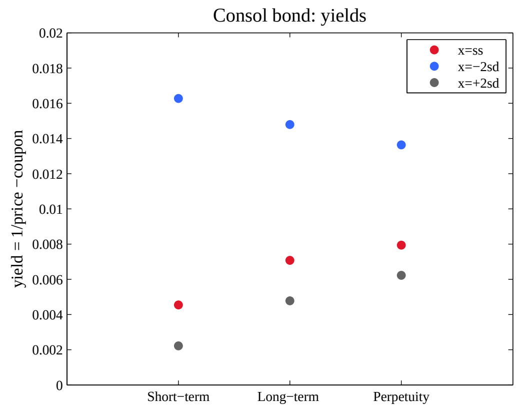
Thus, the yield of the bond is $y_{C,t} = 1/C_t - \delta$. The maturity of the bond is determined by $1/\delta$, and when $\delta = 0$, the bond is a perpetuity. The consol bond is characterized by the following ordinary differential equation:

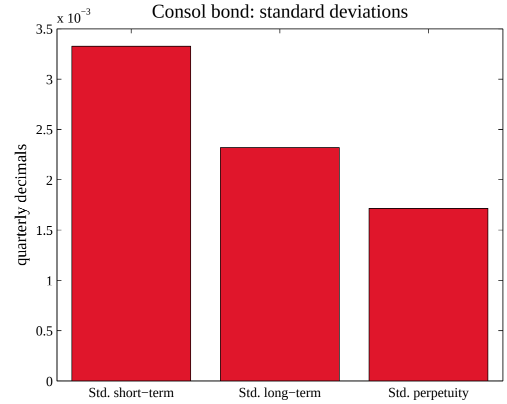
$$-(r(x) + \delta) + \frac{1}{C} + \frac{C_x'}{C}\mu_x(x) + \frac{1}{2}\frac{C_{xx}''}{C}\sigma_x(x)^2 - \frac{C_x'}{C}\sigma_x(x)\kappa(x) = 0.$$

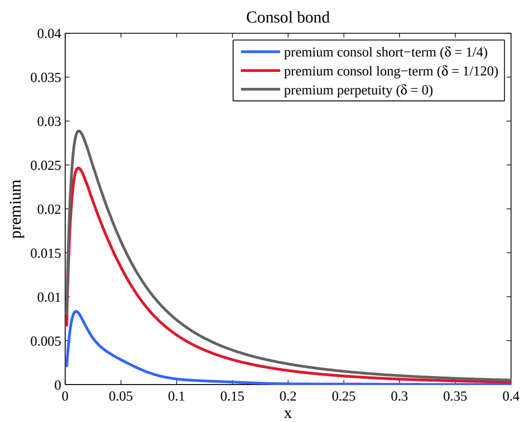
In the next figure, I show the yield of C_t for $\delta = 1/4 (= \delta_{short-term})$, $\delta = 1/120 (= \delta_{long-term})$, and $\delta = 0 (= \delta_{perpetuity})$. These are proxies for a 4-quarter, 120-quarter, and perpetual zero-coupon bonds. The left panel shows the yield y_C for different levels of x. The right panel shows the standard deviation of the three yields. As in the main text, the term structure is upward sloping and the volatility of yields decreases with maturity.

Lastly, the next figure shows the premium associated with each consol bond (i.e., $cov_t^{\mathbb{P}}\left(\frac{dm}{m}, \frac{dC}{C}\right) = \frac{C_x'}{C}\sigma_x(x)\kappa(x)$). The figure shows that long-term bonds pay an average higher compensation for risk.

Numerical procedure. As mentioned in the text, I use a spectral collocation method based on Chebyshev polynomials of the first kind to solve the problems numerically (model,







real yield curve, nominal yield curve, consol bonds, invariant distribution for x, invariant bivariate distribution for x and π). This technique yields a highly accurate global solution.

The solution of the model consists of 4 functions that depend on x_t: two value functions, ξ_A, ξ_B, the valuation of the aggregate earnings and the valuation of the endowment claim. Equilibrium is characterized by a system of nonlinear ordinary differential equations with the 2 HJB equations, the no-arbitrage condition for the endowment claim and the initial earnings, the market clearing conditions for goods ($x\xi_A+(1-x)\xi_B = y/q$), the market clearing condition for shares, and the first order conditions. Real bond prices are characterized by the partial differential equation in (1.22), where prices depend on two state variables (x and t). Nominal bond prices with exogenous inflation are characterized by the partial differential in (1.26), where prices depend on three state variables (x, π and t).

The procedure is as follows. Consider a generic function $h(x) : (0,1) \rightarrow R$. Then, the function can be written in a polynomial form as

$$h(x) = \sum_{i=0}^{K} a_i \Psi_i(\omega_i(x)) + O(K), \tag{1.46}$$

where K is the order of the polynomial, Ψ is the basis function (which in this case is the Chebyshev polynomials), $\{a_i\}_{i=0}^{K}$ are unknown coefficients, ω_i are the Chebyshev nodes, and $O(K)$ is an approximation error (which is of order 10^{-15} in the solutions I provide). The Chebyshev nodes are

$$\omega_i = \cos\left(\frac{2i+1}{2(K+1)}\pi\right), \quad i = 0, ..., K.$$

Therefore, $\omega_i \in [-1, 1]$. Since in the model $x \in (0,1)$, I express the domain as [21] $x_i = \frac{1}{2}(1+\omega_i)$, and therefore x never reaches 0 or 1 for finite K. The Chebyshev polynomials of order $j > 2$ can be represented in the following recursive form:

$$\begin{aligned} \Psi_0 &= 1 \\ \Psi_1 &= \omega \\ \Psi_{j+1} &= 2\omega\Psi_j - \Psi_{j-1}. \end{aligned} \tag{1.47}$$

Based on (1.47), it is straightforward to compute the derivatives of $h(x)$ using (1.46). The rest of the procedure is to solve for the associated set of unknown coefficients as $\{a_i\}_{i=0}^{K}$

[21] For π, which is between π_L and π^{max} —where I π^{max} is set to 5 standard deviations above the mean of π —the nodes are $\pi_i = \pi_L + \frac{\pi^{max} - \pi_L}{2}(1+\omega_i)$.

in each function, such that equilibrium conditions are verified. Since the state variable x is strong Markov, based on Duffie and Lions (1992), the founded solution for the value functions is unique. Solving PDEs for a function $h(x, \pi)$ is a direct extension of this logic, by extending the argument to a tensor grid to represent the two-dimensional state space.

Invariant distribution of π. Let $g(\pi, t, ; \pi_0)$ be the density process associated with (1.24). Informally, the invariant distribution is when $\frac{\partial[g(\pi,t;\pi_0)]}{\partial \pi} = 0$ (i.e., g does not depend on time). The Kolmogorov Forward Equation for the process π with initial condition $\pi_0 > \pi_L$ is

$$0 = -\frac{\partial\left[g(\pi)\mu(\pi)\right]}{\partial \pi} + \frac{1}{2}\frac{\partial^2\left[g(\pi)\sigma(\pi)^2\right]}{\partial \pi^2},$$

where $\mu(\pi) = \lambda_\pi(\overline{\pi} - \pi_t)$ and $\sigma(\pi) = \sigma_\pi\sqrt{\pi_t - \pi_L}$, as in (1.24). So, $0 = \frac{\partial}{\partial \pi}\left\{-g(t,\pi)\mu(\pi) + \frac{1}{2}\frac{\partial\left[g(t,\pi)\sigma(\pi)^2\right]}{\partial \pi}\right\}$. I can omit the constant for now, it will be used to integrate the density to one. Changing variables $\widetilde{g}(\pi) = g(\pi)\sigma(\pi)^2$. Then,

$$\frac{\frac{\partial}{\partial \pi}\widetilde{g}(\pi)}{\widetilde{g}(\pi)} = 2\frac{\mu(\pi)}{\sigma(\pi)^2},$$

which means

$$g(\pi) = \frac{1}{\sigma(\pi)^2}\exp\left(2\int^u \frac{\mu(u)}{\sigma(u)^2}du\right).$$

The integral boils down to

$$\int^u \frac{\mu(u)}{\sigma(u)^2}du = \frac{\lambda_\pi\overline{\pi}}{\sigma_\pi^2}\int^u \frac{du}{(u-\pi_L)} - \frac{\lambda_\pi}{\sigma_\pi^2}\int^u \frac{u}{u-\pi_L}du$$

so

$$g(\pi) = \frac{\overline{g}}{\sigma_\pi^2}(\pi - \pi_L)^{2\left(\frac{\lambda_\pi(\overline{\pi}-\pi_L)}{\sigma_\pi^2}\right)-1}\exp\left(-\frac{2\lambda_\pi}{\sigma_\pi^2}\pi\right)\ ,\ \pi \in (\pi_L, \infty] \tag{1.48}$$

where $\overline{g}$ is a constant to integrate to one. The following is a picture of g under the calibration in the main text.

Chapter 2

Endogenous and Exogenous Risk Premia

2.1 Introduction

Understanding why risk premia fluctuate is a key question in macroeconomics and finance (Cochrane, 2011b). The Great Recession has underscored the importance of the credit market and the role of levered balance sheets in amplifying aggregate uncertainty in the economy. This body of research argues that risk premia fluctuate endogenously because of the presence of levered agents. Yet, this view offers an alternative perspective to leading theories where fluctuations in risk premia are driven by exogenous aggregate shocks to time-varying fundamental volatility (for example, the Long Run Risk paradigm). How do these two apparently disconnected views about risk premia interact in general equilibrium? What is the quantitative importance of each? What are the effects on growth and investment?

In this paper I study an economy with a frictionless financial market where balance sheet dynamics (i.e., the credit market) amplify exogenous macro-volatility shocks. When the economy is hit by macro-volatility shocks, asset prices go down, which affect risk-tolerant agents' balance sheet relatively more (they are levered). As a consequence, they reduce their positions in risky assets and asset prices go down even further, since relatively more risk-averse agents must clear the market—and this affects risk-tolerant agents' net worth again.

In my calibrated model, I find that: i) The feedback loop triggered by macro-volatility shocks creates risk premia fluctuations that are 6 times higher (and closer to the data) than a model with only standard cash flow shocks (such as Longstaff and Wang (2012)); ii) Balance sheets are responsible for a 20% of risk premia fluctuations, macro-volatility for 50% and the interaction of both account for the remainder 30%; iii) In the production economy setup commonly used in the literature[1], I find the effect of risk premia fluctuations on growth and investment is mild.

The model has two central ingredients. First, the economy is populated by heterogeneous agents, and the sole difference between them is their attitude toward risk. This is a parsimonious assumption to motivate the credit market, and it allows me to focus on the mechanism, since one class of agents operates with leverage in equilibrium. Importantly, the economy features perfect risk-sharing (e.g, Longstaff and Wang (2012), Santos and Veronesi (2016), among others). That is, I abstract from any sort of financial friction that interrupts the flows in the credit market. Therefore, my model provides a frictionless benchmark for other classes of models[2], and could potentially help to dissect the precise quantitative impact of those frictions.

Second, agents in the economy exhibit recursive preferences and share two sources of aggregate risks: the standard cash flow (or TFP) and, crucially, aggregate uncertainty shocks—i.e., exogenous fluctuations in aggregate volatility. Recursive preferences are essential because they allow me to disentangle risk aversion from the elasticity of intertemporal substitution (EIS). This feature is key in the amplification of macro-volatility shocks via the credit market. For example, EIS is less than one, high macro-volatility increases asset prices, strengthens levered agents' balance sheets and increases the amount of credit in the economy. This ultimately mitigates the effect of negative TFP shocks. However, an EIS

[1]He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014), Di Tella (2017).

[2]Such as models that feature different types of financing constraints and/or idiosyncratic shocks.

greater than one implies a negative price of macro-volatility risk across the state space and therefore macro-volatility shocks get amplified.

The main contribution of this paper is to show (theoretically and empirically) how exogenous fluctuations in aggregate volatility are amplified in a meaningful and nonlinear way throughout the credit market. The presence of levered balance sheets increase both the level and the volatility of the equity risk premium, but the amplification of only cash flow shocks produce quantitatively small fluctuations in premiums. In this sense, the introduction of macro-volatility improves drastically the ability of the model to match asset prices data. Put differently, a model with balance sheet dynamics where only cash flow shocks are amplified have difficulties in reproducing realistic fluctuations in premiums.

I start my analysis by first studying an endowment economy. I show that a risk aversion greater than one and an EIS greater than one are key to obtaining that both TFP and macro-uncertainty shocks affect levered agents more than proportionately. Then, I investigate the business cycle properties of the model by feeding an estimated series of shocks from macroeconomic data. I estimate these shocks using Markov Chain Monte Carlo. In this exercise I compare the series implied by the model vis-à-vis the ones produced by the representative agent benchmark (i.e., no balance sheet dynamics) and by the economy without macro-volatility shocks (i.e., only cash flow shocks). I find a highly non linear interaction between macro-volatility and balance sheets: The model's implied series of the equity premium have almost twice the level and volatility than the representative agent benchmark, and 6 times higher volatility than the model with only cash flow shock.

I next estimate the two central predictions of the model. As stated above, the mechanism

I propose relies on the fact that macro-uncertainty shocks reduce aggregate asset prices, thus affecting levered agents' balance sheets more than proportionally. A deterioration of levered agents balance sheets' lead to further changes in prices. The key objects capturing this mechanism are the price-elasticities with respect to balance sheets and macro-volatility. In other words, for a given price $p(v, z)$, where v is macro-volatility and z is the levered agents' relative net worth, the model predicts $p'_z/p > 0$ and $p'_v/p < 0$. The magnitude of these derivatives pin down the volatility of risk premia in the model.

To accomplish this, I estimate the price-elasticities of the U.S. aggregate stock market. The price-elasticity with respect to z is key in the heterogeneous agents and the credit market literature (with or without frictions). The unambiguous prediction of these models is that the price of the risky asset increases with levered agents' relative net worth (i.e., $p'_z/p > 0$). Recent literature on intermediary asset pricing (e.g., Adrian et al. (2014) and He et al. (2016)) has tested the impact of shocks to intermediaries on asset returns. The argument is that leverage is useful to capture financial intermediaries' marginal utility of wealth. However, the implications for leverage are model dependent: high leverage could be representing high or low marginal utility of wealth, depending on the modeling assumptions —leverage is an endogenous variable. To circumvent this, I test for p'_z/p, which is ultimately what governs the amplification mechanism.

I find that both the signs and the magnitudes of the estimates are consistent with the predictions of the model. More precisely, I find statistically significant elasticities coefficients of 0.304 and -0.146 for z (balance sheets) and v (macro-volatility), respectively. I show the model is able to capture these two elasticities. Indeed, the model predicts a price-elasticity of 0.174 for z and -0.096 for v. This suggest the model can capture relatively well the role of the credit market in amplifying aggregate uncertainty.

I conclude the paper by studying how the dynamics of risk premia affect growth and investment. For this, I extend the analysis to consider an AK growth model with adjustment costs to investment. I find that results for risk premia are qualitatively and quantitatively similar than in the endowment economy, while growth and investment rates are mildly affected. This result is in line with Backus et al. (2015), although in a different setup, suggesting that fluctuations in discount rates may affect the macroeconomy through another channel (for example, through the labor market, as suggested in Hall (2017b)).

Related literature. This paper contributes mainly to two strands of literature. First, it relates to the literature on exogenous fluctuations in aggregate uncertainty (i.e., stochastic macro-volatility) and their impact on asset prices and macroeconomic quantities. It is worth emphasizing the distinction between idiosyncratic (or micro) uncertainty and aggregate (or macro) uncertainty, since the objective of this paper is the latter. After the Great Recession, there has been a fast-growing literature studying aggregate effects of micro-uncertainty, particularly using models featuring financing frictions.

For example, Di Tella (2017) studies a production economy where an increase in idiosyncratic uncertainty generates an endogenous increase of aggregate risk. The key ingredient is a moral hazard problem that creates incentives for some agents to take levered positions in aggregate TFP shocks. Christiano et al. (2014) introduce agency problems into a standard general equilibrium economy with production, and they allow the volatility of cross-sectional idiosyncratic uncertainty to fluctuate over time. They argue that this is a key risk driving the business cycle. In a similar line, Dou (2017) shows that the impact of idiosyncratic shocks on asset prices depends mainly on the source of that shock and the degree of risk sharing in the economy.

In general, those papers suggest a negative correlation between the volatility of financial intermediaries equity and aggregate asset prices/returns. As Figure 2.1 illustrates, the evidence is not conclusive about this channel, with the exception of the Great Recession. In fact, recent papers such as Herskovic et al. (2016) predict a positive relationship between idiosyncratic volatility of the cross-sectional returns and price-dividend ratios at a firm level. Figure 2.1 illustrate a similar relationship, but with respect to the aggregate price-dividend ratio: the idiosyncratic volatility of financial sector equity and the aggregate price-dividend ratio.[3] Thus, based on this evidence, and in contrast to those papers, I take a different route: I study the role of aggregate volatility (or macro-uncertainty shocks, in the sense of Bloom (2014)) and its amplification in the credit system.

Aggregate volatility shocks have been widely documented by financial economists (e.g., Bansal and Yaron (2004a), Bansal et al. (2014), Campbell et al. (2016), Segal et al. (2015), among others). In fact, Figure 2.1 shows that macro-volatility has a negative impact on aggregate asset prices. Relative to this strand of literature, this paper contributes by studying, theoretically and empirically, how these shocks are amplified in the credit market. For this, heterogeneous agents are key.

The second strand of literature studies the role of heterogeneous agents and the credit market in the amplification of aggregate shocks. This literature can be traced back to

[3] Several papers have documented that idiosyncratic volatility is a key factor to price the cross section of returns. See Herskovic et al. (2016). My paper is silent about cross-sectional implications.

Dumas (1989) and Wang (1996). This class of models study the implications of investors' heterogeneity in a Merton (1971)/Lucas (1978) setup. Longstaff and Wang (2012) study an economy with heterogeneous agents with different attitudes toward risks. Bhamra and Uppal (2009) study an economy similar economy to show that the introduction of a risk-sharing instrument generates endogenous volatility. A key distinction of my paper is that they analyze time-additive preferences and do not incorporate macro-uncertainty shocks. Also, Gârleanu and Panageas (2015) extends the analysis to recursive preferences, but they also focus on cash flow shocks. Barro and Mollerus (2014) study a discrete time economy with disaster risks, where agents exhibit recursive preferences, different risk aversion coefficients, and a unitary EIS. Drechsler et al. (2017) and Silva (2016) apply this framework to study the risk channel of conventional and unconventional monetary policy, respectively. Borovicka (2016) studies an economy with recursive preferences where agents have different beliefs about the fundamentals. In this sense, my paper contributes to this strand of literature by taking the step of incorporating aggregate uncertainty shocks and quantifying the role of balance sheets. This comes with the cost (and the benefit) of introducing an additional state variable into the analysis but enriching our view of this class of models.

Structure of the paper. The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 characterizes the equilibrium and the main analytical properties. Section 4 elaborates the quantitative exercises. Section 5 extends the analysis into a production economy. Section 6 concludes.

2.2 Model

I consider a continuous-time, infinite-horizon economy in which time is indexed by $t \geq 0$. Uncertainty is represented in a probability space $(\Omega, P, \mathcal{F})$ satisfying the usual conditions.[4] The economy is populated by two types of agents: households and experts, denoted by h and e, respectively. Total size of the population is normalized to one, and the sole difference between these two groups of agents is their attitude toward risk. In the rest of the paper, I assume experts (e) are the relatively more risk-tolerant agent, whereas households (h) are the relatively more risk-averse agent.

Preferences and demography. All agents in the economy have recursive preferences as in Duffie and Epstein (1992a). That is

$$U_{i,t} = \mathbb{E}_t \left[\int_t^\infty f\left(c_{i,u}, U_{i,u}\right) du \right] , \tag{2.1}$$

with

$$f\left(c_i, U_i\right) = \frac{\rho}{(1-1/\psi)} \left(1-\gamma_i\right) U_i \left\{ c_i^{1-1/\psi} \left(\left(1-\gamma_i\right) U_i\right)^{\frac{1/\psi - 1}{1-\gamma_i}} - 1 \right\} , \tag{2.2}$$

where γ_i is the risk aversion (RA) of agent $i \in \{h, e\}$ and c_i represents his level of consumption. Both the discount rate ρ and EIS ψ are the same for h and e. As is well-known, recursive preferences allow for separation between the EIS and the risk aversion. The special case where $1/\psi = \gamma_i > 1$ is the time separable CRRA additive preferences. Despite the fact that the setup can be extended to incorporate differences in other parameters, in this paper

[4]Information flows in the filtration $\{\mathcal{F}_t, t \geq 0\}$, also with regular properties. See Øksendal (2014) or Protter (1990).

I consider only the case of differences in $\gamma : \gamma_h > \gamma_e$. This is a parsimonious assumption to introduce a credit market and leverage in equilibrium and therefore a tractable alternative to focus on the mechanism.

I introduce turnover risk to ensure a non-degenerate wealth distribution.[5] There are several alternative assumptions to obtain this.[6] I follow Gârleanu and Panageas (2015), Drechsler et al. (2017), Dou (2017), and Silva (2016). In particular, I assume agents die and are born at each instant, with exponential probability λ. Agents do not have a bequest motive, and their death risk is not measurable under the filtration generated by aggregate Brownian processes (defined below), denoted by $\mathcal{F}_t$. New agents are born in constant proportions $\overline{z}$ and $1 - \overline{z}$ for e and h, respectively, and they are equally endowed in a per-capita basis. The idea is that net worth of perished agents is pooled and then redistributed pro-rata. The consequence of introducing mortality risk is that agents' effective time preference parameter is higher[7]—namely $\lambda + \rho$ for each agent.

Endowment. The aggregate endowment in the economy is given by

$$d \log y_t = \mu dt + \sqrt{v_t} dW_{1,t}, \; y_0 > 0 \;, \tag{2.3}$$

$$dv_t = \kappa_v \left(\overline{v} - v_t \right) dt + \sigma_v \sqrt{v_t} dW_{2,t}, \; v_0 > 0 \;, \tag{2.4}$$

[5] Intuitively, there is a non-zero probability that one group will dominate in the long term because shocks affect agents differently

[6] For instance, Brunnermeier and Sannikov (2014) specify different time-preference parameters; Di Tella (2017) introduce a probability of switching roles;. Di Tella and Kurlat (2017) introduce proportional taxes and subsidies.

[7] In the solution, I explore low values for λ. Intuitively, if the death probability is high, agents will behave as myopic investors.

where $W_1 = \{W_{1,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ and $W_2 = \{W_{2,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ are standard Brownian motions representing aggregate uncertainty in $(\Omega, P, \mathcal{F})$. All parameters in the process (2.3)-(2.4) $\mu, \overline{v}$, κ_v, and σ_v have standard properties. The instantaneous expected percentage change of log-output ($\log y$) is represented by μ, while parameters $\overline{v}$, κ_v, and σ_v related to the mean, persistance, and volatility of the process v_t, respectively. I explain them in more detail below. Lastly, I assume $d\left\langle W_1 W_2 \right\rangle_t = \varphi dt$. That is, macro-uncertainty shocks (represented by W_2) and cash flow shocks (represented by W_1) covary with coefficient $\varphi < 0$. A version of this parameter has been intensively used in the option pricing literature since Heston (1993). In (2.3)-(2.4), this assumption intends to capture the stylized fact that aggregate volatility is higher when output growth is lower (and vice-versa). Below, I also investigate a production economy where capital accumulation is subject to macro-uncertainty shocks.

Markets and balance sheets. I assume frictionless financial markets. All agents can continuously trade shares of the aggregate endowment and locally riskless debt. Specifically, each share pays y_t per unit of time, and the total amount of shares is normalized to one. I define q_t as the price of such an asset and let $s_{i,t}$ be the number of shares agent i holds. Then, the cumulative return of this asset consists of the capital gains dq_t/q_t and the dividend yield y_t/q_t. That is,

$$dR_t = \frac{dq_t + y_t dt}{q_t} = \mu_{R,t} dt + \sigma_{q1,t} dW_{1,t} + \sigma_{q2,t} dW_{2,t} \ , \tag{2.5}$$

where μ_R, σ_{q1}, and σ_{q2} are unknown functions to be solved in equilibrium. I assume $n_{i,0} > 0$ for $i \in \{e, h\}$. Agents can also trade risk-free debt. In that market, they can borrow and lend from each other at a risk-free rate r_t. I next define the agent's i net worth n_i as

$$n_{i,t} = s_{i,t}q_t - b_{i,t} \ , \tag{2.6}$$

where $b_{i,t}$ is the value of short-term debt issued at time t. Based on (2.5) and (2.6), the evolution of agent's i net worth is given by

$$\frac{dn_{i,t}}{n_{i,t}} = \left[r_t - \frac{c_{i,t}}{n_{i,t}} + \frac{q_t s_{i,t}}{n_{i,t}} \left(\mu_{R,t} - r_t \right) \right] dt + \frac{q_t s_{i,t}}{n_{i,t}} \left(\sigma_{q1,t} dW_{1,t} + \sigma_{q2,t} dW_{2,t} \right) \ ,$$

and as defined in (2.5), σ_{q1} is the diffusion associated to "cash flow" shocks and σ_{q2} to macro-uncertainty shocks. With these elements in place, I next describe agents' problem.

Agents' problem. In this decentralized formulation of the economy, each agent solves a standard dynamic problem as in Merton (1973). That is, each agent decides how much to consume and how to wealth allocate in his portfolio of savings. More precisely, in this economy both investors solve the following dynamic problem:

$$\begin{aligned} & \max_{c_i, s_i} U_{i,t} \\ & s.t. \\ \frac{dn_{i,t}}{n_{i,t}} &= \left[r_t - \frac{c_{i,t}}{n_{i,t}} + \frac{q_t s_{i,t}}{n_{i,t}} \left(\mu_{R,t} - r_t \right) \right] dt + \frac{q_t s_{i,t}}{n_{i,t}} \left(\sigma_{q1,t} dW_{1,t} + \sigma_{q2,t} dW_{2,t} \right) \ , \\ n_{i,0} &> 0, \end{aligned} \tag{2.7}$$

where $U_{i,t}$ is defined in (2.1), and the control variables $(s_{i,t}, c_{i,t})$ are the number of shares he buys and the level of consumption per instant, respectively. I define agent's i portfolio share, denoted by $\alpha_{i,t}$as the market value of his asset holdings over the market value of his equity: $\alpha_{i,t} = q_t s_{i,t}/n_{i,t}$.

I next define a competitive market equilibrium in this economy. Despite the fact the economy exhibits a continuum of heterogeneous agents, they are identical within each group/type. And because their preferences are homogeneous, each agent within each group features the same policy decisions. That is, agents within a group $i \in \{i, e\}$ have identical consumption rates $c_{i,t}/n_{i,t}$ and choose the same portfolio holdings. Notice that demographic turnover does not change the composition of the population and hence it is possible to aggregate variables within each class of agents. I turn to the model solution after the following definition.

Definition 4 (Competitive equilibrium in the endowment economy)**.** *A competitive equilibrium is a set of aggregate stochastic processes adapted to the filtration generated by* W_1 *and* W_2*. These processes are prices* (q, r) *and policy functions for each household and each expert* (c_i, α_i) *and net worth* n_i *such that:*

i) Given prices and initial net worth, each group of agents $i \in \{e, h\}$ *solves his problem.*

ii) Market clearing, defined in terms of policies for each group of agents $i \in \{e, h\}$*, is as follows*

$$
\begin{aligned}
c_{e,t} + c_{h,t} &= y_t \ \forall t, \\
s_{e,t} + s_{h,t} &= 1 \ \forall t, \\
n_{e,t} + n_{h,t} &= q_t \ \forall t,
\end{aligned}
$$

where the last condition $n_{e,t} + n_{h,t} = q_t$ *holds by market clearing in the market for risk-free* *debt.*[8]

2.3 Solving the Model

All agents face a dynamic problem and operate in frictionless markets. I solve for the decentralized version of the economy, since closed form solutions are unfeasible and therefore there are no analytical gains from the Pareto formulation. A key property of the decentralized specification is that the state-space can be simplified. Instead of keeping track of the aggregate wealth of each agent, the system can be summarized with the wealth share of agents e as a fraction of total net worth. I define that variable as

$$z_t = \frac{n_{e,t}}{n_{e,t} + n_{h,t}}. \tag{2.8}$$

The solution method consists of looking for a Markov equilibrium in the state space $(v, z) \in (0, \infty) \times (0, 1)$. I next define the key elements of the model solution.

Exogenous state variable. The exogenous state variable is v and its law of motion is given by (2.4), which is a standard square root process (e.g.,Cox et al. (1985)). This process is stationary, and reflected at zero, provided $2\kappa_v \sigma^2 / \sigma_v^2 > 1$.[9] The stationary distribution is Gamma with shape parameter $2\kappa_v \sigma^2 / \sigma_v^2$ and scale $\sigma_v^2 / 2\kappa_v$.

Endogenous state variable. As stated above, the homogeneity properties of preferences and policies allow for a simplification of the state space. Instead of keeping track of the

[8] By Walras law, $b_e + b_h = 0$. Introducing this into agents' balance sheets yields $n_{e,t} + n_{h,t} = q_t$. I introduce this in the specification of the equilibrium because it is going to be relevant below.

[9] See Feller (1951) and Karlin and Taylor (1981) for a general treatment.

aggregate net worth for each type of agent, I defined the wealth share of experts as a fraction of total wealth in (2.9). Proposition 2 establishes the law of motion for z. The proof (in the appendix) consists of the application of Ito's lemma to equation (2.9).

Proposition 5 (Law of motion for z)**.** *The law of motion for z is given by*

$$dz_t = \mu_{z,t}dt + \sigma_{z1,t}dW_{1,t} + \sigma_{z2,t}dW_{2,t} \tag{2.9}$$

with

$$\begin{aligned}
z\sigma_{z1} &= z\left(1-z\right)\left(\alpha_e - \alpha_h\right)\sigma_{q1}\ , \\
z\sigma_{z2} &= z\left(1-z\right)\left(\alpha_e - \alpha_h\right)\sigma_{q2}\ , \\
z\mu_z &= z\left(1-z\right)\left(\frac{c_h}{n_h} - \frac{c_e}{n_e} + \left(\alpha_h - \alpha_e\right)\left(\mu_R - r - \sigma_{q2}^2 - 2\sigma_{q1}\sigma_{q_2}\varphi - \sigma_{q1}^2\right)\right) + \lambda\left(\overline{z} - z\right).
\end{aligned}$$

where the functions are $\alpha_e = \alpha_e\left(v,z\right);\ \alpha_h = \alpha_h\left(v,z\right);\ \sigma_{z1} = \sigma_{z1}\left(v,z\right)\ ;\ \sigma_{z2} = \sigma_{z2}\left(v,z\right);\ \frac{c_h}{n_h} = \frac{c_h}{n_h}\left(v,z\right);\ \frac{c_e}{n_e} = \frac{c_e}{n_e}\left(v,z\right);\ \mu_R = \mu_R\left(v,z\right);\ r = r\left(v,z\right);\ \sigma_{q1} = \sigma_{q1}\left(v,z\right)$ *and* $\sigma_{q2} = \sigma_{q2}\left(v,z\right).$

Proof. See appendix. □

Hamilton-Jacobi-Bellman (HJB) and first-order conditions. The value function of each agent is given by $U_{i,t}\left(n_{i,t}, z_t, v_t\right)$, and the recursive formulation of problem (2.7) is represented by the following HJB for each agent:

$$0 = \max_{c_i, s_i} f\left(c_i, U_i\right) + \mathbb{E}\left[dU\right]. \tag{2.10}$$

Since preferences are homothetic, the value function $U_{i,t}(n_{i,t}, z_t, v_t)$ has the following power form:

$$U_{i,t}(n_{i,t}, z_t, v_t) = \frac{n_{i,t}^{1-\gamma}}{1-\gamma} J_i(z_t, v_t),$$

where J_i is an auxiliary function capturing the marginal utility of wealth for each agent[10], which in fact fluctuates with the stochastic investment opportunity set. Notice that the higher is J, the higher the marginal utility of wealth n_i. To ease algebra, I will define a monotone transformation of J and instead solve for $\xi_i(z,v)^{\frac{1-\gamma}{1-\psi}} = J_i(z,v)$. Notice that $\frac{1-\gamma}{1-\psi} > 0$ provided $\gamma > 1$, $\psi > 1$, and therefore this transformation is increasing. The unknown function $\xi_i(z,v)$ follows an Ito process—that is:

$$\frac{d\xi_i}{\xi_i} = \mu_{\xi i,t} dt + \sigma_{\xi i1,t} dZ_{1,t} + \sigma_{\xi i2,t} dZ_{2,t},$$

where $\mu_{\xi i,t}$, $\sigma_{\xi i1,t}$, and $\sigma_{\xi i2,t}$ are functions that must be solved in equilibrium. Then, using $(1-\gamma) U_i = \left(n_i \xi_i^{\frac{1}{1-\psi}} \right)^{1-\gamma}$ together with (2.2), the recursive formulation given in (2.10) can be written as (omitting subscript t)

$$\begin{aligned} 0 &= \max_{c_i,\alpha_i} \frac{\psi}{1-\psi} \left[\left(\frac{c_i}{n_i} \right)^{\frac{\psi-1}{\psi}} \xi_i^{\frac{1}{\psi}} - (\rho+\kappa) \right] + r - \frac{c_i}{n_i} + \alpha_i (\mu_R - r) - \frac{\gamma}{2} \mathbb{E} \left[\left(\frac{dn_i}{n_i} \right)^2 \right] \\ &\quad + \frac{1}{1-\psi} \left(\mathbb{E} \left[\frac{d\xi_i}{\xi_i} \right] + \frac{1}{2} \left(\frac{\psi-\gamma_i}{1-\psi} \right) \mathbb{E} \left[\left(\frac{d\xi_i}{\xi_i} \right)^2 \right] \right) + \frac{1-\gamma_i}{1-\psi} \mathbb{E} \left[\frac{dn_i}{n_i} \frac{d\xi_i}{\xi_i} \right], \end{aligned} \tag{2.11}$$

[10] The derivative of $U_{i,t}$ with respect to $n_{i,t}$ depends on J_i

with $\alpha_i = s_i q/n_i$ —i.e., the portfolio share. First-order conditions of this problem are

$$\begin{aligned} \frac{c_i}{n_i} &= \xi_i \ , \qquad (2.12) \\ \gamma_i \alpha_i &= \frac{\mu_R - r}{\sigma_{q1}^2 + \sigma_{q2}^2 + 2\sigma_{q1}\sigma_{q2}\varphi} - \underbrace{\left(\frac{1-\gamma_i}{1-\psi}\right)\left(\frac{\sigma_{\xi i1}\sigma_{q1} + \sigma_{\xi i2}\sigma_{q2} + \left(\sigma_{q1}\sigma_{\xi i2} + \sigma_{q2}\sigma_{\xi i1}\right)\varphi}{\sigma_{q1}^2 + \sigma_{q2}^2 + 2\sigma_{q1}\sigma_{q2}\varphi}\right)}_{\textit{hedging term}}. \end{aligned}$$

As is usual in portfolio problems à-la Merton (1973), the demand for risky assets has a *myopic* component and a *hedging* component. While the former is a standard mean-variance term, the latter contains several terms that illustrate how agent's demand for risky assets changes with the endogenously time-varying investment opportunity set. Before proceeding with further analytical properties of the solution, I define a Markov equilibrium in the (v, z) space. This definition will guide the solution for the following functions:

$$\begin{aligned} p\left(v_t, z_t\right) &= q_t/y_t = p_t \ , \\ \xi_i\left(v_t, z_t\right) &= \xi_{t,i} \ \text{ for } i \in \{e, h\} \ , \\ r\left(v_t, z_t\right) &= r_t. \end{aligned}$$

Definition 6 (Markov equilibrium in the endowment economy)**.** *A Markov equilibrium in* $(v, z) \in (0, +\infty) \times (0, 1)$ *is a set of adapted stochastic processes* $p(v, z), r(v, z), \xi_e(v, z), \xi_h(v, z)$ *and policy functions* $\alpha_e(v, z), \alpha_h(v, z), \frac{c_e}{n_e}(v, z), \frac{c_h}{n_h}(v, z)$*, and a law of motion for* z *such that:*

i) ξ_e *and* ξ_h *solve for experts' and households' HJB, and* α_e*,* α_h*,* c_e*, and* c_h *are the corresponding policy functions.*

ii) Markets clear

$$z\widehat{c}_e + (1-z)\,\widehat{c}_h = 1/p\ ,$$
$$z\alpha_e + (1-z)\,\alpha_h = 1.$$

iii) The law of motion of the endogenous state variable z satisfies and (2.9).

I next study the key analytical properties of the model that shed light on the mechanism behind the results. I first discuss the role of balance sheets and the amplification of macro-uncertainty shocks. The idea is to understand how z and v affect prices. I next solve for the portfolio share in the first-order conditions (2.12). That is, I obtain an expression for α_e as a function of the endogenous functions p, ξ_e and ξ_h. And finally I derive the prices of risk in the economy.

Amplification and balance sheet effects. In economies where there is a credit market, the slope of the price function, $\frac{\partial p}{\partial z} > 0$, is what generates the amplification of shocks (see (Brunnermeier and Sannikov, 2016). The greater this slope, the higher the volatility of returns. The intuition is as follows: A negative shock to dividends affects experts' balance sheets more than proportionately, provided they operate with leverage. Therefore, z decreases, *ceteris paribus*. Lower z implies experts are reallocating their portfolios by selling risky assets. And this implies the price of the risky asset goes down again. The higher the sensitivity of prices to z, the more pronounced this feedback loop between shocks, prices, and balance sheets.

To understand the role of z and v, I next characterize the equilibrium diffusion processes associated to cash flow and macro-uncertainty shocks in (2.5). These diffusion processes are functions $\sigma_{q1}(v,z)$ and $\sigma_{q2}(v,z)$ that show whether returns increases or decreases after a W_1 or W_2 shock hit the economy. The diffusions, derived in the appendix as a step to prove proposition 3, are given by

$$\sigma_{q1}(v,z) = \frac{\sqrt{v}}{1-\frac{p_z(v,z)}{p(v,z)}z\left(\alpha_e(v,z)-1\right)}, \tag{2.13}$$

$$\sigma_{q2}(v,z) = \frac{\frac{p_v(v,z)}{p(v,z)}\sigma_v\sqrt{v}}{1-\frac{p_z(v,z)}{p(v,z)}z\left(\alpha_e(v,z)-1\right)}. \tag{2.14}$$

The terms $\sqrt{v}$ and $\sigma_v\sqrt{v}$ in the numerators of (2.13) and (2.14), respectively, represent fundamental volatility. On the other hand, the term in the denominator captures the role of balance sheets in the amplification of both types of aggregate shocks. Mathematically, this term captures the geometric series of the feedback loop explained above. A key term is $p_v(v,z)$ in (2.14). Its sign determines whether macro-uncertainty shocks increase or decrease prices. Provided the intertemporal substitution effect dominates for both agents ($\psi > 1$), then $p_v(v,z) < 0$ across the state space. Intuitively, irrespective of which agent is holding the asset, an increase in v (a positive shock in W_2) reduces prices. And this reduction in asset prices affect experts more than proportionately because they operate with leverage. Thus, macro-volatility shocks reduce z and therefore prices again. In other words, macro-uncertainty shocks are amplified by triggering the feedback loop described above.

With this mechanism in mind, it is useful to use the first-order conditions and other equilibrium conditions to characterize the portfolio share of experts (α_e) and analyze the risk-sharing properties of the mechanism. I do that in the following proposition.

Proposition 7 (solving for α_e)**.** *Experts' demand for risky assets, $\alpha_e(v,z)$, is given by*

$$\alpha_e(v,z) = \frac{1-(1-z)\Lambda_0(v,z)}{z+(1-z)\Lambda_1(v,z)}, \tag{2.15}$$

where Λ_0 and Λ_1 are reported in the appendix, and

$$\begin{aligned} R^z(v,z) &= (1-\gamma_e)\frac{\xi_{e,z}(v,z)}{\xi_e(v,z)} - (1-\gamma_h)\frac{\xi_{h,z}(v,z)}{\xi_h(v,z)}, \\ R^v(v,z) &= (1-\gamma_e)\frac{\xi_{e,v}(v,z)}{\xi_e(v,z)} - (1-\gamma_h)\frac{\xi_{h,v}(v,z)}{\xi_h(v,z)}, \\ T(v,z) &= \frac{\sigma_v\left(\varphi + \frac{p_v(v,z)}{p(v,z)}\sigma_v\right)}{1+2\sigma_v\varphi\left(\frac{p_v(v,z)}{p(v,z)}\right)+\left(\frac{p_v(v,z)}{p(v,z)}\right)^2\sigma_v^2}. \end{aligned} \tag{2.16}$$

Proof. See appendix. □

Proposition 4 solves for α_e as a function of relevant objects in the model. In particular, there are three functions: R^z, R^v, and T. I define expressions R^z and R^v as relative risk-sharing elasticities with respect to z and v, respectively. Those functions are the difference in the semi-elasticities $\xi_{x,i}/\xi_i$ (adjusted by agents' i risk aversion). This shows the percentage change in utility after a unit change in state variable $x \in \{z,v\}$.[11] I next analyze the signs of $T(v,z)$ and $R^v(v,z)$ because they are important to understanding the risk-sharing dynamics. Since I solve the model numerically, I explore the intuition using the signs of the semi-elasticities.

Provided $\psi > 1$, the intertemporal substitution effect dominates for both types of agents. Thus, an increase in volatility will induce agents to consume a higher fraction of their wealth,

[11] Taking log of U_i, then $\frac{\partial \log U_i}{\partial x} = \frac{1-\gamma_i}{1-\psi}\frac{\xi_{i,x}}{\xi_i}$.

save more in risk-free debt, and reduce their holdings of the risky asset. But this effect is particularly strong for households, since $\gamma_h > \gamma_e$, and therefore they exhibit a stronger "precautionary motive." This implies their consumption-wealth ratio is more sensitive to v, $\xi_{v,h}(v,z) > \xi_{v,e}(v,z) > 0$, which also means $\frac{\xi_{h,v}}{\xi_h} - \frac{\xi_{e,v}}{\xi_e} > 0$: after an increase in volatility, households increase the fraction of net worth they consume by more than experts. Therefore, $R^v(v,z) < 0$. On the other hand, $p_v(v,z) < 0$ implies $T(v,z) < 0$ provided $\varphi \leq 0$.

Understanding the signs of $R^v(v,z)$ and $T(v,z)$ is important because they characterize $\sigma_{z1}(v,z)$ and $\sigma_{z2}(v,z)$, as I show in the next proposition. These functions $\sigma_{z1}(v,z)$ and $\sigma_{z2}(v,z)$ represent how z is changes with W_1 and W_2.

Proposition 8 (Risk sharing and concentration of risk)**.** *Experts are relatively more exposed to both cash flow and macro-uncertainty shocks than households (i.e., $\sigma_{z1} > 0$ and $\sigma_{z2} < 0$) iff $\psi > 1$ and $\gamma_h > \gamma_e > 1$.*

Proof. See appendix. □

This means that a positive cash flow shock improves experts' net worth relatively more than that of households ($\sigma_{z1} > 0$), and macro-uncertainty shocks deteriorate experts' net worth relatively more ($\sigma_{z2} < 0$). This is at the core of the risk-sharing mechanism.

Figure 2.2 provides an illustrative characterization of the relationship between the EIS, the risk aversion coefficients, and the impact of balance sheets and macro-volatility. As I discuss in the calibration section, in the numerical solution I set $\psi > 1$ and $\gamma_h > \gamma_e > 1$. This parameter configuration is central in the analysis because macro-uncertainty reduces prices ($p_v < 0$) and therefore drives z down ($\sigma_{z2} < 0$). The reduction in the relative wealth of experts produces another round of price reduction ($p_z > 0$).

A useful feature of this setup is that agents trade in frictionless financial markets where there no arbitrage opportunities. This allows me to derive the implied prices of risk in the economy "as if" the economy were populated by a representative agent. In other words, there exists a state-price deflator $\pi(z, v)$ in the economy following an Ito process

$$\frac{d\pi_t}{\pi_t} = -rdt - \sigma_{\pi 1} dZ_{1,t} - \sigma_{\pi 2} dZ_{2,t}. \tag{2.17}$$

The next proposition characterizes the prices of risk $\sigma_{\pi 1}$ and $\sigma_{\pi 2.}$

Proposition 9 (Prices of risk)**.** *Prices of risk $\sigma_{\pi 1}$ and $\sigma_{\pi 2}$ are given by*

$$\begin{aligned} \sigma_{\pi 1} &= -\varphi \sigma_{\pi 2} \\ &\quad + \gamma(z) \left(\sigma_{q1} + \varphi \sigma_{q2} - \frac{1}{1-\psi} \left[z \frac{1-\gamma_e}{\gamma_e} (\sigma_{\xi e1} + \sigma_{\xi e2} \varphi) + (1-z) \frac{1-\gamma_h}{\gamma_h} (\sigma_{\xi h1} + \sigma_{\xi h2} \varphi) \right] \right) \\ \sigma_{\pi 2} &= -\varphi \sigma_{\pi 1} \\ &\quad + \gamma(z) \left(\sigma_{q2} + \varphi \sigma_{q1} - \frac{1}{1-\psi} \left[z \frac{1-\gamma_e}{\gamma_e} (\sigma_{\xi e2} + \sigma_{\xi e1} \varphi) + (1-z) \frac{1-\gamma_e}{\gamma_e} (\sigma_{\xi h2} + \sigma_{\xi h1} \varphi) \right] \right) \ , \end{aligned} \tag{2.18}$$

with

$$\gamma(z) = \frac{\gamma_e \gamma_h}{z \gamma_h + (1-z) \gamma_e} \ .$$

Equity premium. With the state-price deflator defined in (2.17) and (2.18), I can price any security. Following Campbell (2003) and Abel (1999), I model the dividend processes of the aggregate stock market as $div_t = y_t^\theta$. Applying Ito's lemma to div_t implies

$$\frac{d\left(div_t\right)}{div_t} = \left[\theta\mu + \frac{1}{2}\theta\left(\theta-1\right)v_t\right]dt + \theta\sqrt{v}dW_{1,t}\ .$$

Let $\widehat{q}_t$ denote the price of the claim to future dividends div_t and let $\widehat{p}_t = \widehat{q}_t/div_t$ be the price-dividend ratio. By absence of arbitrage,

$$\widehat{q}_t = E_t\left[\int_t^\infty \frac{\pi_s}{\pi_t} div_s ds\right] \tag{2.19}$$

Proposition 10. $\widehat{p}_t$ *solves the following PDE:*

$$-r + \frac{1}{\widehat{p}} + E\left[\frac{d\widehat{p}}{\widehat{p}} + \frac{d\left(div_t\right)}{div_t} + \frac{d\left(div_t\right)}{div_t}\frac{d\widehat{p}}{\widehat{p}} + \frac{d\widehat{p}}{\widehat{p}}\frac{d\pi}{\pi} + \frac{d\left(div_t\right)}{div_t}\frac{d\pi}{\pi}\right] = 0. \tag{2.20}$$

Proof. See appendix. □

2.4 Results

In this section I describe the quantitative results, discuss the role of macro-uncertainty shocks in the model, and perform an empirical evaluation of the theoretical predictions of the model. I organize the content as follows. I first discuss the calibration of the parameters used in the numerical solution and I study its properties. Next, I feed the model with an estimated sequence of macro-uncertainty and cash flow shocks. This exercise allow me to study the quantitative properties of the model using the discipline imposed by the shocks.[12] I estimate the sequence of shocks using the calibrated parameters and Random-Walk Markov Chain Monte Carlo (RW-MCMC), a widely used estimation technique to extract latent shocks from data. Next, I validate the model empirically by focusing on the central prediction about the role of balance sheets (z) and macro-uncertainty (v) in determining the price of risky assets. That is, I estimate the elasticity of the price of the risky asset with respect to z and v, which are at the core of the mechanism described in the model.

Calibration. Table 3.1 summarizes the calibration of the model at a quarterly frequency. First, I calibrate the expected growth rate of the economy μ and the mean variance $\overline{v}$ to match the mean and the variance of real per-capita growth of U.S. consumption expenditures in the postwar period (1947:Q2-2015:Q4). The source of this data is the NIPA tables from the Bureau of Economic Analysis. The persistence (κ_v) and volatility (σ_v) of the variance factor are from Zviadadze (2016).[13] The value for $\kappa_v = 0.0649$ implies a half-life of macro-uncertainty shocks of 10.5 quarters, very similar to the one considered by the long-run risk literature. Volatility of v, σ_v=0.0019, implies that the standard deviation of d$log(y)$ is in the range [0.6% to 2.96%] (annualized) with 0.99 probability. I choose a correlation parameter between $W1$ and $W2$ (i.e. φ) of -0.5 to capture the stylized fact that macro-uncertainty shocks are negatively correlated with cash-flow shocks.[14] Quantitative results are not significantly affected by this choice, and they are robust to values in the interval

[12]Se, for instance, Hansen and Prescott (1993) and more recently Hansen and Ohanian (2016).

[13]I adapt the ARG(1) variance factor from Zviadadze (2016) into the square-root process for v by matching the moments. The ARG(1) process estimated is the discrete time counterpart of the square-root process.

[14]See, for instance, Bansal et al. (2014), Berger et al. (2016), and Bloom (2014).

$\varphi \in (-1, 0]$. The parameter θ is chosen to match the fact that aggregate dividends are 5.5 times more volatile than aggregate consumption in the data.

Second, I consider the calibration of agents' preferences. As established in the main text, the sole difference between the two classes of agents is their relative risk aversion parameter, and I assume $\gamma_h > \gamma_e$. I choose $\gamma_h = 20$ and $\gamma_e = 3$ to match an average level of leverage of 3.65 in the stochastic steady state, following He and Krishnamurthy (2017)—they use 3.77.

It is worth emphasizing that there are several alternatives regarding the level of leverage in the data. The main distinction has to do with the precise identification of the marginal investor operating with leverage in the market for risky securities. For example, Adrian et al. (2014) argue that those are security broker dealers, He et al. (2016) propose the official list of primary dealers that are counterparts of the New York Fed, and Gertler et al. (2016) argue a distinction between institutions within the FEeds regulatory framework ("whosale banks") and commercial banks ("retail"). Instead, He and Krishnamurthy (2017) take a comprehensive view of the financial sector and consider a variety of heterogeneous sophisticated intermediaries as marginal investors. I follow this line and consider a rather conservative overall level of leverage of 3.5.

Another important preference parameter is the EIS. I follow the asset pricing literature and set EIS=1.5. As mentioned above, this is important because a key consequence is that the price of risky assets goes down when macro-volatility is high. I then calibrate the time-preference ρ to match a low level of the risk-free rate. The last two parameters, $\overline{z}$ and λ, are used to ensure the stationarity of the endogenous state variable. I choose an average experts' population size of $\overline{z}$ =8% to match the relative market capitalization of the financial sector over the total market capitalization in the CRSP/Compustat database. Lastly, I use a low death (and birth) rate of 0.004, which means that investors' expected life is 250 quarters. So agents' discount factor is $\rho + \lambda = 0.0041$.

Macro-uncertainty shocks and risk premiums. Figure 2.3 reports the behavior of the equity premium across the state space. It is clear from both panels that this relationship is nonlinear, even in the frictionless model environment considered in this paper.[15] The left panel shows that a deterioration in levered agents' balance sheet is associated with higher premiums. However, notice this dynamics depend on the level of aggregate uncertainty: a drop in experts' net worth affects premiums disproportionately when aggregate uncertainty is high. The right panel shows it from a different perspective: an increase in macro-volatility (i.e., v_t) affects the risk premium disproportionately more when levered agents' net worth is weak.

To gain intuition about these dynamics, it is useful to observe Figure 2.4, which illustrates $\sigma_{z,2}$ (the diffusion associated with macro-uncertainty shocks in the Ito process for dz) and the risk-free rate. The bottom panels show that positive macro-uncertainty shocks have negative balance sheet effects—that is, $\sigma_{z,2} < 0$ across the state space. This is useful to understand the intuition: positive macro-volatility shock will induce experts to reallocate their portfolio by reducing their exposure to risky assets. When they sell, they share the risk with relatively more risk-averse agents (households), that would be willing to take such a risk at a lower price and higher expected return. Given that experts operate with leverage, such a reduction in prices affects their net worth more than proportionately, and therefore

[15]For instance, by introducing an occasionally binding constraint, one could, in principle, obtain even more nonlinear results.

z declines. The decline in z affects the endogenous investment opportunity set, triggering another round of sells, and the dynamics explained in Figure 2.3 follow. This is what means $\sigma_{z,2} < 0$.

On the other hand, the top panels show that the model has tight predictions for the risk-free interest rate. Its dynamics are qualitatively similar to a "flight-to-quality"—i.e., episodes of high macro-volatility, coupled with a deterioration of levered agents' net worth. In those episodes, there is a higher demand for the risk-free asset, driving its price up and reducing its instantaneous return. However, those movements are quantitatively smaller than the ones predicted by other models featuring financial frictions, such as He and Krishnamurthy (2017) or Di Tella (2017). In those models, interest rate dynamics are qualitatively similar but reach far more negative values in episodes when experts have low net worth.[16] This is important because that could explain why in those models risk premia spikes. In He and Krishnamurthy (2017) this could be a result of log-preferences (coupled with the particular financial constraint), while in Di Tella (2017) it could be a result of the calibration of the parameters controlling the strength of the moral hazard problem. Here, instead, the real risk-free rates do not exhibit a drastic swing in those episodes.

Invariant distribution and theoretical moments. I compute the invariant bivariate distribution by solving the corresponding Kolmodorov Forward Equation (KFE) associated

[16]For instance, in Di Tella (2017), the real rate reaches -60% and in He and Krishnamurthy (2017) -5%. Although they use a production economy model, my results for the interest rate are similar both in the endowment and production setup.

with v and z. That is, I define the density function $g(v,z)$, and the KFE is given by

$$\begin{aligned} 0 &= -\frac{\partial\left[g\left(v,z\right)\mu_v\left(v\right)\right]}{\partial v}-\frac{\partial\left[g\left(v,z\right)\mu_z\left(z,v\right)\right]}{\partial z} \\ &+\frac{1}{2}\frac{\partial^2\left[g\left(v,z\right)\left(\sigma_{z,1}^2\left(z,v\right)+\sigma_{z,2}^2\left(z,v\right)+2\sigma_{z,1}\left(z,v\right)\sigma_{z,2}\left(z,v\right)\varphi\right)\right]}{\partial z^2} \\ &+\frac{1}{2}\frac{\partial^2\left[g\left(v,z\right)\sigma_v^2 v\right]}{\partial z^2}+\frac{\partial^2\left[g\left(v,z\right)\left(\varphi\sigma_{z1}\left(v,z\right)+\sigma_{z2}\left(v,z\right)\right)\sigma_v\sqrt{v}\right]}{\partial z\partial v}. \end{aligned}$$

Figure 1.7 shows the bivariate distribution, together with the corresponding marginal distributions. This plot shows a similar logic to the one developed for the equity premium and the real rate above. As shown in the left-hand side of panel (b), the marginal distribution of z has a lower mean, the higher is v—since $\sigma_{z2} < 0$ as shown above. The right-hand side of panel (b) shows this from another perspective: in states of high z, macro-volatility has a lower mean.

Equipped with the invariant distribution, I next analyze the theoretical moments of the relevant endogenous variables in the model. Table 2.2 show the results. Conditional moments are computed with the marginal distribution of z, when macro-volatility is at its unconditional mean. This helps to illustrate the role of each factor in explaining the results.

In particular, the table highlights two key results. First, the model predicts almost 6 times more volatile premiums than when v is set to its mean. Thus, macro-uncertainty shocks introduce significantly more variation in premiums relative to an economy with only balance sheet dynamics and amplification of cash flow shocks. Second, risk premia dynamics in the model have a higher level of premiums (approximately 70% higher) and almost twice the volatility than the equivalent representative agent economy. In other words, the meaningful interaction between balance sheets and macro-volatility described above not only generates time variation in premiums but also in levels. Indeed, the table suggest approximately 50% of the total volatility of risk premia (1.2%) can be attributed to macro volatility (i.e., the representative agent benchmark), 20% to balance sheets, and the remainder 30% to the interaction of both.

In the second column of Table 2.2 I report the theoretical moments for an alternative parametrization of θ and γ_h. I set an alternative θ to 7.5, which is indeed in line with the relative volatility of real dividend growth and real consumption growth in the postwar period when dividends are computed as in Cochrane (2008) instead of 12-month rolling sum of dividends. I also set γ_h=30, which increases the level of leverage and the mean size of levered investors. Notice now the (harmonic) mean risk aversion is 14.84 instead of 12.68 in the baseline. In this alternative calibration I find higher (and more volatile) expected excess returns. Leverage, however, is lower: experts have a higher unconditional mean (i.e., their net worth has a higher relative market capitalization) and operate with lower leverage.

Endogenous and exogenous risk: The equity premium. I next study the quantitative predictions of the model using macroeconomic data. To that end, I estimate both cashflow and macro-uncertainty shocks. As in the calibration, I use real per-capita U.S. consumption expenditures to filter the model (2.3)-(2.4) and obtain series for W_1 and W_2 using RW-MCMC. In Figure 2.6 and Table 3.2 I show both the mean estimated series and the statistical properties of the estimation. I then introduce the sequence of shocks into the model, starting from the stochastic steady state, and I evaluate the predictions for the equity premium.

Figure 2.7 illustrates the results. The solid line displays the equity premium implied by the model, while the dashed line is the one implied by the model when I shut down macro-uncertainty shocks—i.e., $dW_{2,t} = 0, \forall t$. These results suggest that introducing macro-uncertainty generates substantial time-variation in expected returns relative to a model with heterogeneous agents driven solely by cash flow shocks. For example, Figure 2.7 shows that during the last recession, the expected return on equity predicted by the model is about 8% while the one predicted by the model without macro-uncertainty shocks is about 4%.

In Figure 2.7, the blue dashed line shows the fluctuations in the equity premium in a representative agent economy with the average risk aversion.[17] The series clearly shows the notion of amplification due to balance sheets: when solid black and dashed blue line are compared, we see the former having a higher level and higher fluctuations.

Because the equity premium is not observable in the data, I next illustrate the predictions of the model for the dividend-price ratio. It is important to highlight that the model does not exhibit fluctuations in dividend news, and movements in the dividend-price ratio are driven by changes in the discount rates. Figure 2.8 illustrates the behavior of the dividend-price ratio. The level is in line with historical numbers for the aggregate stock market (i.e., between 4% and 5% per year), and I also show both the black and red lines for the actual model and without macro-uncertainty shocks, respectively. Both display countercyclical behavior—dividend-price ratios are increasing during crisis.

The figure also shows that fluctuations in dividend yields in the data are higher than in the model. One plausible alternative to explain this is that the model does not feature

[17] The average risk aversion is the harmonic mean of the individual risk aversions $\gamma = \left[\frac{z}{\gamma_A} + \frac{1-z}{\gamma_B}\right]^{-1}$.

fluctuations in dividend growth rates—the drift of dividends is constant. But this can help to address the extent to which the equity premium predicted by the model is sensible or not, since changes in dividend yields can be decomposed into changes in discount rates or changes in dividend growth rates. In other words, if changes in dividend growth do not account for a substantial fraction of changes in dividend yields, then the premiums predicted by the model are rather conservative predictions.

Through the lens of the results described for the equity premium, a natural question to ask is, what is the role of balance sheets in these results? In other words, how large is the amplification mechanism generated by balance sheets, coupled with macro-uncertainty shock? I describe this in the next subsection.

Endogenous and Exogenous risk: Quantifying the amplification mechanism. Recall in the main text, the price-dividend on the dividend claim $\widehat{p} = \widehat{q}/div$ is a function on the state space $(v, z) \in (0, \infty) \times (0, 1)$. The variance of returns is given by

$$var\left(dR\left(v,z\right)\right) = \sigma_{\widehat{q}1}^{2}\left(v,z\right) + \sigma_{\widehat{q}2}^{2}\left(v,z\right) + 2\sigma_{\widehat{q}1}\left(v,z\right)\sigma_{\widehat{q}1}\left(v,z\right)\varphi, \tag{2.21}$$

where, using Ito's lemma, the expression for $\sigma_{\widehat{q}1}$ and $\sigma_{\widehat{q}2}$ are given by

$$\begin{aligned} \sigma_{\widehat{q}1}\left(v,z\right) &= \left(\theta + \mathcal{A}\left(v,z\right)\right)\sqrt{v}, &(2.22)\\ \sigma_{\widehat{q}2}\left(v,z\right) &= \left(\frac{\widehat{p}_v\left(v,z\right)}{\widehat{p}\left(v,z\right)} + \mathcal{A}\left(v,z\right)\frac{p_v\left(v,z\right)}{p\left(v,z\right)}\right)\sigma_v\sqrt{v}, &(2.23)\\ &\quad with \\ \mathcal{A}\left(v,z\right) &= \frac{\frac{\widehat{p}_z(v,z)}{\widehat{p}(v,z)}z\left(\alpha_e\left(v,z\right)-1\right)}{1-\frac{p_z(v,z)}{p(v,z)}z\left(\alpha_e\left(v,z\right)-1\right)}, &(2.24) \end{aligned}$$

where notice p is the price of the endowment claim and $\widehat{p}$ denotes the price of the dividend claim. I refer to the term $\mathcal{A}(v,z)$ as the amplification factor. A central element of $\mathcal{A}(v,z)$ is the elasticity $\left(\widehat{p}_z(v,z)/\widehat{p}(v,z)\right)z$, which I discuss in detail in the next subsection. Also, notice that when there are no balance sheet effects (i.e. $p_z = \widehat{p}_z = 0$), then $\mathcal{A}(v,z) = 0$.

Next, I define the fundamental variance of returns, which is given by (2.21) when $\mathcal{A}(v,z) = 0$. This is when $\overline{\sigma_{\widehat{q}1}} = \theta\sqrt{v}$ and $\overline{\sigma_{\widehat{q}2}} = \left(\widehat{p}_v/\widehat{p}\right)\sigma_v\sqrt{v}$. Finally, I define the ratio of the variance of returns when there are balance sheet effects and where there are not

$$\mathcal{R}(v,z) = var\left(dR(v,z)\right)/var\left(\overline{dR(v)}\right). \tag{2.25}$$

Figure 2.9 shows the predicted series for (2.25), together with the time series for z. The average amplification level 20.1%, and it is negatively correlated with z. This means that, in the model, the returns to the risky asset would have a 20.1% higher variance than the fundamental variance, due to the balance sheet effects summarized by (2.25). It is also clear that the amplification mechanism is higher during recessions (periods of negative cashflow shocks and positive macro-uncertainty shocks). This is because z is countercyclical, as shown by the red dashed line. Notice that prior to the Great Recession, the relative size of experts was 9%, and the fundamental shocks that hit the economy drove the size to roughly 5%.

Empirical validation of the model. The key elements to evaluate the theory are the pro-cyclical behavior of p with respect to z and the countercyclical behavior with respect to v. That is, $\widehat{p}'_z(\cdot) > 0$ and $\widehat{p}'_v(\cdot) < 0$, which are the model's predictions. For this, I consider the following two elasticities: i) $\beta_z = \left(\widehat{p}'_z/\widehat{p}\right)z$ capturing the magnitude of the amplification effect, and ii) $\beta_v = \left(\widehat{p}'_v/\widehat{p}\right) v$ capturing the role of macro-volatility on the risky price.

To accomplish this, I run several exercises based on the following regression:

$$\log \widehat{p}_t\left(z_t, v_t\right) = \alpha + \beta_z \log z_t + \beta_v \log v_t + \varepsilon_t \ . \tag{2.26}$$

This specification allows me to capture the relevant elasticities because β_z and β_v are indeed $\beta_z = \left(\widehat{p}'_z/\widehat{p}\right) z$ and $\beta_v = \left(\widehat{p}'_v/\widehat{p}\right) v$. In order to run (2.26), I need data on $\widehat{p}$, z, and v. For these three variables, which I describe next, I use monthly data since January 1926 to December 2015. Notice that this period is different than the postwar period considered in the calibration. This is because I use the calibration to estimate the quarterly shocks for real consumption expenditures and this data are only available in the postwar period. The results presented in this section are robust in the postwar sample.

For $\widehat{p}$, I use CRSP data on the value-weighted portfolio, and I follow a standard practice to construct the price-dividend ratio following Campbell and Shiller (1989). For v, I use the realized variance of industrial production over the previous 12 months as in Bansal et al. (2014).[18]

Lastly, to construct z I use the market equity of financial firms over the total market equity of the firms listed in CRSP/Compustat. This measure is a proxy for the conditions in the credit market. Even more, a central aspect of this sector is that they operate with leverage: they issue short-term debt while they hold risky investments in their asset side, which is

[18] I run several robustness checks with different alternatives for v. I don't report the results here, because they are similar to the ones I report. These alternatives are, I consider i) expected variance (i.e. $E_t\left[v_{t+1}\right]$ from an AR(1)), ii) a GARCH(1,1) over industrial production growth rates, iii) realized variance over the previous 6 months, iv) realized variance over the previous 24 months.

a business model reminiscent to what risk-tolerant agents do in the model. Indeed, in the data, financial firms hold five time higher leverage than any other sector in the economy.[19] For this reason, I use the Industry Classification (SIC) codes 60 to 64. That is, I exclude sector 65 and 67 ("Real State" and "Holding And Other Investment Offices", respectively) from the main Finance Division (i.e. Division "H": Finance, Insurance, And Real Estate). I compute market equity as in He et al. (2016), which is the number of shares outstanding times the closing price at the end of each month.

By considering SIC codes 60 to 64, I include a variety of intermediaries that have been documented in the literature (i.e., depositary institutions, non-depositary institutions, and-security dealers, among others). Importantly, using z in the empirical analysis allows me to circumvent the discussion of whether leverage is procyclical or countercyclical. In the data, however, there is consensus for market leverage to be countercyclical (as in the model presented above). But another family of models predicts leverage is procyclical (as book value of leverage in the data).[20] However, the object I estimate has a tighter relationship with theory: virtually any model in the literature of heterogeneous agents with a credit market predicts $\widehat{p}'_z(z, \cdot) > 0$.

Table 2.3 shows the statistical properties of these variables and Table 2.5 shows the main regression for equation (2.26). I estimated all the regressions in this section using GMM.[21]

I first run the regression without v, in column (1), and I obtain a significantly positive

[19]Using the marked-to-market measure of leverage in He et al. (2016)

[20]See Adrian et al. (2014) and He et al. (2016), among others, for a discussion on the cyclical properties of leverage.

[21]I have also ran 2SLS with several lags as instruments to avoid potential endogeneity issues.

β_z. Notice the order of magnitude is around 0.5, meaning that a 1% increase in the relative capitalization of the financial sector is associated with a 0.5% increase in the price-dividend ratio. I next run the regression without z, but including only v. I obtain a statistically significant negative β_v. In column (3) I run precisely equation (2.26)—the main specification. Results confirm the qualitative prediction of the model: the estimated coefficients show $\widehat{\beta}_z > 0$ and $\widehat{\beta}_v < 0$. Two observations are worthwhile. First, the prediction for $\beta_v < 0$ is a central element in the theory I presented in section 2, because that is the mechanism through which macro-uncertainty shocks are amplified: positive macro-uncertainty shocks reduce prices, affecting levered agents more than proportionately, triggering the amplification mechanism discussed in section 2. Secondly, notice that the estimation show $\left|\widehat{\beta}_z\right| > \left|\widehat{\beta}_v\right|$, suggesting the relative importance of balance sheet vis-à-vis macro-volatility. Columns (4)-(6) show the same regressions but exclude the Great Depression.[22]

In Table 2.6 I show a key robustness exercise.[23] I control the regression by including a term that captures the idiosyncratic volatility of the financial sector. This control is important, since many researchers have been underscoring the role of idiosyncratic volatility, as I have reviewed in the literature section. To compute that, I follow Herskovic et al. (2016), equation 1, and I run a regression of the changes in the market value of equity onto the Fama-French factors. That is, I run $dn_{i,t} = a_i + \mathbf{b}_i'\mathbf{F}_t + \widetilde{\varepsilon}_{i,t}$, where $dn_{i,t}$ is the log-change in the daily market equity of firm i in the financial sector[24], and $\mathbf{F}_t$ are the three Fama-French factors.

[22]Results excluding the Great Recession and/or the Great Depression are similar.

[23]I performed several other robustness checks, such as computing macro-volatility with a GARCH(1,1), with an AR(1), exclude SIC code 62 from the definition of z, regress only high-frequency components after filtering data with Baxter-King filter, among others. In the interest of space, those are available upon request.

[24]The definition of the financial sector here is the same as above: SIC 60-64.

Idiosyncratic volatility $\widetilde{v}_t$ is then calculated as the standard deviation of the residuals over each calendar month. I include $\widetilde{v}_t$ as a regressor in regression (2.26).

Column (1) in Table (2.6) shows the following: When running a regression of the price-dividend ratio of the aggregate value-weighted portfolio with respect to $\widetilde{v}_t$ (a sensible proxy for the idiosyncratic volatility of levered agents), the results show that higher levels of idiosyncratic volatility are associated with higher prices. In column (2), I show that when including z, idiosyncratic volatility does not have a significant effect on prices. In column (3) I run the regression with macro-volatility and idiosyncratic volatility, without introducing the role of balance sheets. Results still suggest the opposite sign for idiosyncratic volatility (with a t-statistic of 1.9) and a strongly significant effect for v. And in column (4) I show that when I include both z and v, and control with idiosyncratic volatility ($\widetilde{v}_t$), the macro-volatility elasticity β_v is still negative and significant, and the elasticity β_z is still positive and significant.

The main lesson from these exercises is that the estimates validate the theoretical predictions, namely $\beta_z < 0$ and $\beta_v > 0$, with the salient feature that $|\beta_z| > |\beta_v|$.

I conclude by comparing the estimated β_z and β_v with the theoretical counterparts in the model, given by $\mathbb{E}\left[(\widehat{p}'_v/\widehat{p})\, v\right]$ and $\mathbb{E}\left[(\widehat{p}'_z/\widehat{p})\, z\right]$. Figure 2.10 shows the model not only predicts the qualitative feature that $\beta_z < 0$ and $\beta_v > 0$ but also has the quantitative prediction for the relative magnitude $\left|\widehat{\beta}_z\right| > \left|\widehat{\beta}_v\right|$, showing that the balance sheet factor affects the price-dividend relatively more than macro-volatility. The red bars illustrate the model's predictions, whereas the black bars show the results from column (3) in Table 2.5 (i.e., the main specification). The model predicts quantitatively smaller balance sheet effects than those estimated in the data: while in the model a 1% increase in z is associated with an increase of 0.174% in the price-dividend ratio, in the data is associated with a 0.280% increase. And for v, the model predicts quantitatively higher effects than in the data: while in the model a 1% increase in macro-volatility is associated with a reduction of -0.091% in the price-dividend ratio, in the data it is associated with a -0.024% decrease.

2.5 Extension to Production Economy

The objective of extending to a production economy is twofold. First, to study the implications of risk premia fluctuations over growth and investment. Second, to study the extent to which asset pricing dynamics are qualitatively robust to endowment economy assumption. I purposefully follow the literature and specify an AK production technology with adjustment costs to investment (see Brunnermeier and Sannikov (2014), He and Krishnamurthy (2017), and Di Tella (2017), among others).

Technology. Both agents operate an AK technology and I assume they have the same productivity to transform capital into final good, in order to keep differences in γ_i as the sole source of heterogeneity. The production function for each agent is given by

$$y_{i,t} = ak_{i,t} \ ,$$

where the change in "effective units" of capital is given by

$$\frac{dk_{i,t}}{k_{i,t}} = g_{i,t}dt + \sqrt{v_t}dW_{1,t} \ .$$

The function $g_{i,t}$ represents the expected growth rate of capital. As in Brunnermeier and Sannikov (2014), I assume there are adjustment costs in the accumulation of capital. That is, agent i has to invest $\iota(g_{i,t})\, k_{i,t}$ to achieve a growth rate $g_{i,t}$, where $\iota'(g) > 0$, $\iota'(g) > 0$. As before, aggregate uncertainty is given as

$$dv_t = \kappa_v (\overline{v} - v_t)\, dt + \sigma_v \sqrt{v_t} dW_{2,t}.$$

Capital accumulation is subject to TFP shocks[25], represented by W_1, and aggregate uncertainty shocks, represented by W_2. I define W_1 and W_2 as in the endowment economy. Next, the price of capital is denoted by q_t

$$\frac{dq_t}{q_t} = \mu_{q,t} dt + \sigma_{q1,t} dW_{1,t} + \sigma_{q2,t} dW_{2,t} \quad ,$$

while total return on capital is represented in a standard way (capital gains plus dividend yield)

$$dR_{i,t} = \left[\frac{a - \iota_i (g_{i,t})}{q_t} + \mu_{q,t} + g_{i,t} + \sigma_{q1} \sqrt{v_t} \right] dt + \sigma_{R1,t} dZ_{1.t} + \sigma_{R2,t} dW_{2,t} \quad ,$$

where I have defined $\mu_{Ri,t} = \mathbb{E}_t [dR_{i,t}]$, $\sigma_{R1,t} = \sqrt{v_t} + \sigma_{q1,t}$, and $\sigma_{R2,t} = \sigma_{q2,t}$.

Balance sheets are as in Section 2, and since the static investment decision is the same for all agents, the dynamic problem faced by each agent is analogous to that of section 2 (i.e., the HJB). I next define a Markov equilibrium in the production economy.

[25] W_1 can be interpreted as TFP shocks if we let k be the "effective" units of capital.

Definition 11 (Markov Equilibrium in the Production Economy)**.** *A Markov equilibrium in* (v, z) *is a set of adapted stochastic processes* $\xi_e, \xi_h, q, r,$ *and policy functions* $\alpha_e, \alpha_h, \frac{c_e}{n_e}, \frac{c_h}{n_h}$*, and a law of motion for* z *such that:*

i) ξ_e *and* ξ_h *solves for experts' and households HJB, and* α_e*,* α_h*,* c_e*, and* c_h *are the corresponding policy functions.*

ii) Market clears

$$\begin{aligned} z\widehat{c}_e + (1-z)\,\widehat{c}_h &= \frac{a - \iota\left(g\left(q\right)\right)}{q} \\ z\frac{q_t k_e}{n_e} + (1-z)\,\frac{q_t k_h}{n_h} &= 1. \end{aligned}$$

iii) The law of motion of the endogenous state variable z *satisfies (2.9).*

The convex cost function $\iota(g)$ have standard properties are $\iota' > 0, \iota'' > 0$. A commonly used specification for ι is[26]

$$i\left(g\right) = \kappa_1 \left(g+\delta\right)^2 + \kappa_2 \left(g+\delta\right), \tag{2.27}$$

where κ_1 and κ_2 are constants calibrated to match levels of investment and growth that are consistent with the data. Therefore, the static optimization problem is to maximize the

[26] See Brunnermeier and Sannikov (2014), Brunnermeier and Sannikov (2015), Di Tella (2017), and He and Krishnamurthy (2017), among others.

expected return on capital by choosing g. That is:

$$\begin{aligned} \iota' &= q, \\ g(q) &= \frac{q-\kappa_2}{2\kappa_1} - \delta, \end{aligned} \tag{2.28}$$

and for investment

$$\iota(q) = \kappa_1 \left(\frac{q-\kappa_2}{2\kappa_1}\right)^2 + \kappa_2 \left(\frac{q-\kappa_2}{2\kappa_1}\right).$$

From (2.28), it is clear that an x% decrease in q will produce an approximately $\frac{x}{2\kappa_1}$% decrease in the *expected growth* rate of the economy.

Figure 2.11 illustrates the behavior of the risk premium on productive capital and the real risk-free interest rate. Overall, results are qualitatively similar to those in the endowment economy: macro-uncertainty shocks affect premiums and rates more when experts' balance sheets are impaired. In Figure 2.12, I show the results for the investment rate (ι) and expected growth rate of the economy (g). From a qualitative point of view, results are in line with the analysis elaborated so far. However, both the expected growth rate of the economy and the investment rate do not fluctuate as much as we observed generally in the data, and in particular during in the Great Recession (a period characterized by a deterioration of levered agents' net worth).[27]

These results are consistent, for instance, with He and Krishnamurthy (2017), where the investment rate moves from around 10.5% (in the first best) to 9% in a financial crisis.

[27]The QoQ annualized growth rate of GDP was -8.2% in 2008:Q3. The QoQ annualized % change of private investment was -10% in that period with a minimum of -38.7% in 2009:Q1.

It is important to emphasize that the results are robust to a specification where agents feature different productivity levels as in Brunnermeier and Sannikov (2014), for instance. As a matter of fact, if agents exhibit different productivity levels, when the low-productivity type holds a larger fraction of capital, the expected growth rate of the economy and the investment rate will go further below what is shown in Figure 2.12. However, the differences in productivity needed to reconcile the empirical evidence may be extremely large. In other words, the price of risky capital must show a drastic swing to put the *expected* growth rate of the economy at a close level to the one *realized* in the Great Recession.

Overall, the results suggest that changes in volatility and discount rates may affect the real economy though another channel (as in Backus et al. (2015)). Hall (2017b) shows that the labor market is a relevant channel to understand the connection between discount rates and the real economy.

2.6 Conclusion

There is a growing body of research in macroeconomics and finance investigating the role of the credit market and the presence of leverage in generating endogenous fluctuations in risk premia. In this paper I provide a quantitative and theoretical investigation of how the credit market interacts in general equilibrium with exogenous fluctuations in aggregate volatility.

I develop a model with a frictionless credit market where macro-volatility shocks are amplified in a highly nonlinear fashion. In periods of high macro-volatility, asset prices are low, and therefore levered agents lose relatively more net worth, which generates further declines in asset prices. I quantify this mechanism over the business cycle and find that: i) The feedback loop triggered by macro-volatility shocks creates risk premia fluctuations that are 6 times higher (and closer to the data) than a model with only standard cash flow shocks (such as Longstaff and Wang (2012)); ii) Balance sheets are responsible for 20% of risk premia fluctuations, macro-volatility for 50% and the interaction of both the remainder 30%; iii) Using the commonly analyzed AK production framework with adjustment costs of investment, I find that results are similar for risk premia, but fluctuations in growth and investment are mild.

The setup I use can be extended to introduce frictions in the credit market and evaluate the quantitative implications of those vis-à-vis the ones implied by the frictionless benchmark used in this paper. A comprehensive understanding about the quantitative implications of endogenous changes in risk aversion, exogenous volatility and financial frictions is a task for future research.

2.7 Tables and Figures

Table 2.1: Calibration

DESCRIPTION		SYMBOL	VALUE
1. Technology			
	Expected growth rate	μ	0.0052
	Mean variance	$\overline{v}$	0.0079^2
	Variance persistence	κ_v	0.0649
	Variance volatility	σ_v	0.0019
	Corr(W_1, W_2)	φ	-0.5
	Dividend variance	θ	5.5
	Capital/output	a	3
2. Preferences			
	Time preference	ρ	0.0001
	Risk aversion e	γ_e	3
	Risk aversion h	γ_h	20
	EIS	ψ	1.5
3. Levered agents			
	Mean proportion e	$\overline{z}$	0.08
	Turnover	λ	0.004

NOTES: This table shows the model's calibration at a quarterly frequency. I describe the procedure in the main text.

Table 2.2: Theoretical moments

	BASELINE				$\gamma_h = 30$, $\theta = 7.5$				REP. AGENT			
	Unconditional		*Conditional*		*Unconditional*		*Conditional*		*Unconditional*		*Conditional*	
Model	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev
x	9.86	4.42	9.54	4.45	11.35	4.64	10.91	4.69	0	0	0	0
$\alpha_e(v,z)$	3.64	0.49	3.67	0.51	3.12	0.43	3.15	0.45	0	0	0	0
$\mu_q - r$	3.67	1.17	3.48	0.21	9.87	3.40	9.41	0.90	2.21	0.64	0	0
dp	4.41	0.12	4.42	0.09	7.79	0.44	7.87	0.33	5.59	0.07	0	0
Data												
x	8.27	6.03										
α_e	3.77											
$\mu_q - r$	8.07	4.63										
dp	3.54	1.49										

NOTES: Numbers are in percentage annual terms. Baseline refers to the calibration in Table 3.1, where Rep. Agent is the solution of the model with a single agent with the (harmonic) mean risk aversion. Unconditional moments are computed with the bivariate distribution for v and z. Conditional moments are computed with the marginal distribution for z when $v = \overline{v}$ (i.e., $f(z|v = \overline{v})$). In the data, x is the market value of sectors SIC 60-64 over total market cap. in 1960 to 2016. α_e is from He and Krishnamurthy (2017). Expected excess return, $\mu_q - r$ is from Cochrane (2011b), table 1. And the dividend-price ratio is computed as in Cochrane (2011b).

Table 2.3: Sample moments: Regressions

	Mean	StDev	Skew.	Kurt	Min	Max	Obs	P-Perron	AR(1)
$\log p_t$	3.3953	0.4411	0 .27623	2.7839	1.9995	4.5456	1068	-3.45**	0.99
$\log z_t$	0.0778	0.34699	0.03337	3.9798	-1.6199	1.740	1068	-7.54***	0.88
$\log v_t$	-9.5435	1.4524	0.75929	3.1411	-12.6009	-5.0496	1068	-5.90***	0.98

NOTES: P-Perron is the test statistic from Phillip and Perron (1988). The null hypothesis is that the variable contains a unit root, and the alternative is that the variable was generated by a stationary process. I use fourth-order Newey-West (1987) error correction. I compare the critical values for the statistic from the interpolated Dickey-Fuller table, in Fuller (1976): 1% -3.96 (*** in the table), 5% -3.41 (** in the table), 10% -3.12 (* in the table)

Table 2.4: Correlations

	Full sample (1926-2015)			Postwar (1947-2015)		
	log(p)	*log(z)*	*log(v)*	*log(p)*	*log(z)*	*log(v)*
log(p)	1			1		
log(z)	0.4044	1		-0.5401	1	
	(3.4521)			(-2.4282)		
log(v)	-0.5630	-0.3677	1	-0.5401	0.3474	1
	(-2.1379)	(-2.8453)		(-2.4282)	(1.1419)	
Observations	1069	1069	1069	828	828	828

NOTES: This table show the correlation of the variables for both samples. In parentheses, I show the t-statistic adjusted as in Agiakloglou and Tsimpanos (2012). That is,

$$t = \frac{\widehat{corr}}{\sqrt{\frac{1}{T}\left(\frac{1+\widehat{\rho}_i\widehat{\rho}_j}{1-\widehat{\rho}_i\widehat{\rho}_j}\right)}},$$

where $\widehat{\rho}_i$ and $\widehat{\rho}_j$ are the estimated AR(1) coefficient associated to the series considered in the correlation.

Table 2.5: Main regression

	(1)	(2)	(3)	(4)	(5)	(6)
	Full sample (1926-2015)			Ex-Great Dep. (1935-2015)		
log(z)	0.512		0.304	0.568		0.314
	(0.0965)		(0.276)	(0.204)		(0.167)
log(v)		-0.171	-0.146		-0.182	-0.154
		(0.0365)	(0.0363)		(0.0360)	(0.0337)
Constant	3.356	1.763	1.977	3.373	1.664	1.900
	(0.119)	(0.276)	(0.279)	(0.0688)	(0.337)	(0.316)
Observations	1069	1069	1069	972	972	972

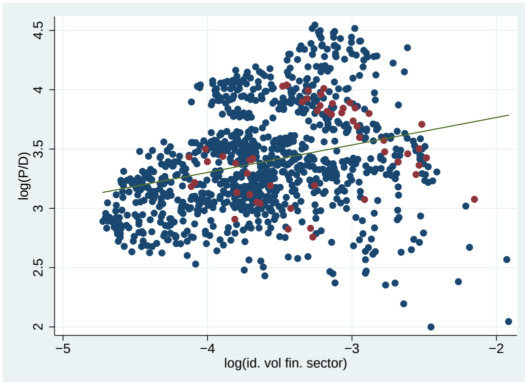
NOTES: I estimate regressions using GMM, with heteroskedastic standard errors (in parentheses) based on Bartlett kernel with optimal lag selection (Newey and West, 1994). Columns (1)-(3) are the results for the full sample. Columns (4)-(6) exclude the Great Depression. Results are similar for the period 1935 to 2006 (i.e., excluding both the Great Depression and Great Recession), and I do not report them to save space. The main specification of equation (2.26) is column (3) and (6). The rest of the columns should the results. The data are monthly and the sample is 1926-2015. z_t is the market equity of the financial sector over total market equity. I linearly detrend the series of log(z_t) in real time. Financial sector is defined as SIC 60-64 (i.e., excluding Real State and Other Investment Offices). v_t is the realized variance of Industrial Production growth rates over the last 12 months (see Bansal et al. (2014)). p_t is the price-dividend ratio of the value-weighted portfolio (see Campbell and Shiller (1989)).

Table 2.6: Main regression controlling for idiosyncratic volatility

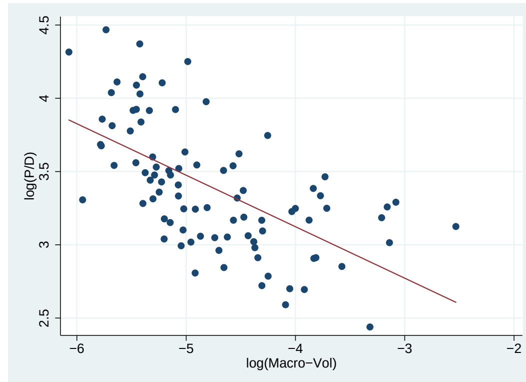
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	Full sample (1926-2015)				Ex-Great Dep. (1935-2015)			
$\log \widetilde{v}_t$	0.232	0.159	0.162	0.1291	0.342	0.252	0.206	0.135
	(0.116)	(0.106)	(0.085)	(0.081)	(0.098)	(0.094)	(0.840)	(0.081)
$log(z)$		0.454		0.2639		0.418		0.296
		(0.171)		(0.133)		(0.198)		(0.158)
$log(v)$			-0.161	-0.1422			-0.155	-0.135
			(0.033)	(0.0289)			(.033)	(0.028)
Constant	4.229	3.935	2.434	2.482	4.676	4.303	2.671	2.574
	(0.453)	(0.413)	(0.441)	(0.3719)	(0.384)	(0.373)	(0.442)	(0.350)
Observations	1069	1069	1069	1069	972	972	972	972

NOTES: I estimate regressions using GMM, with heteroskedastic standard errors (in parentheses) based on Bartlett kernel with optimal lag selection (Newey and West, 1994). Columns (1)-(3) are the results for the full sample. Column (4)-(6) exclude the Great Depression. Results are similar for the period 1935 to 2006 (i.e. excluding both the Great Depression and Great Recession), and I do not report them to save space. The main specification of equation (2.26) is column (4) and (6). The data is monthly and the sample is 1926-2015. z_t is the market equity of the financial sector over total market equity. I linearly detrend the series of $\log(z_t)$ in real time (results are very similar.). Financial sector is defined as SIC 60-64 (i.e. excluding Real State and Other Investment Offices). v_t is the realized variance of Industrial Production growth rates over the last 12 months (see Bansal et al. (2014)). p_t is the price-dividend ratio of the value-weighted portfolio (see Campbell and Shiller (1989)). $\widetilde{v}_t$ is the idiosyncratic volatility of the financial sector, computed as in Herskovic et al. (2016), equation (1). That is, I run a regression $\log dn_{i,t} = \alpha_i + \mathbf{b}_i'\mathbf{F}_t + \varepsilon_{i,t}$, where $\log dn_{i,t}$ is the change in the market equity of the financial sector and $\mathbf{F}_t$ are the 3 Fama-French factors. Idiosyncratic volatility is then calculated as the standard deviation of residuals within each calendar month.

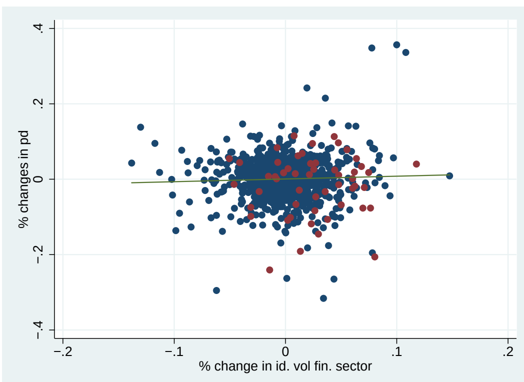
Figure 2.1: Macro-volatility, idiosyncratic volatility, and asset prices



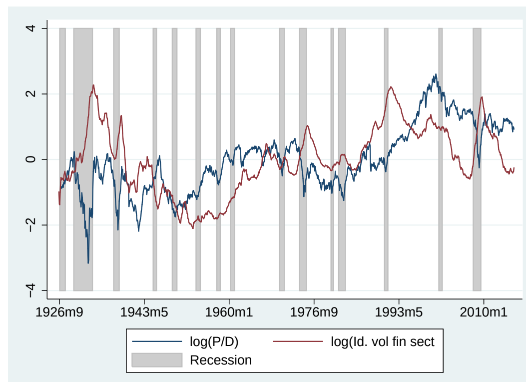
(a) Id. vol. and log(p/d)



(b) Macro-volatility and log(p/d)



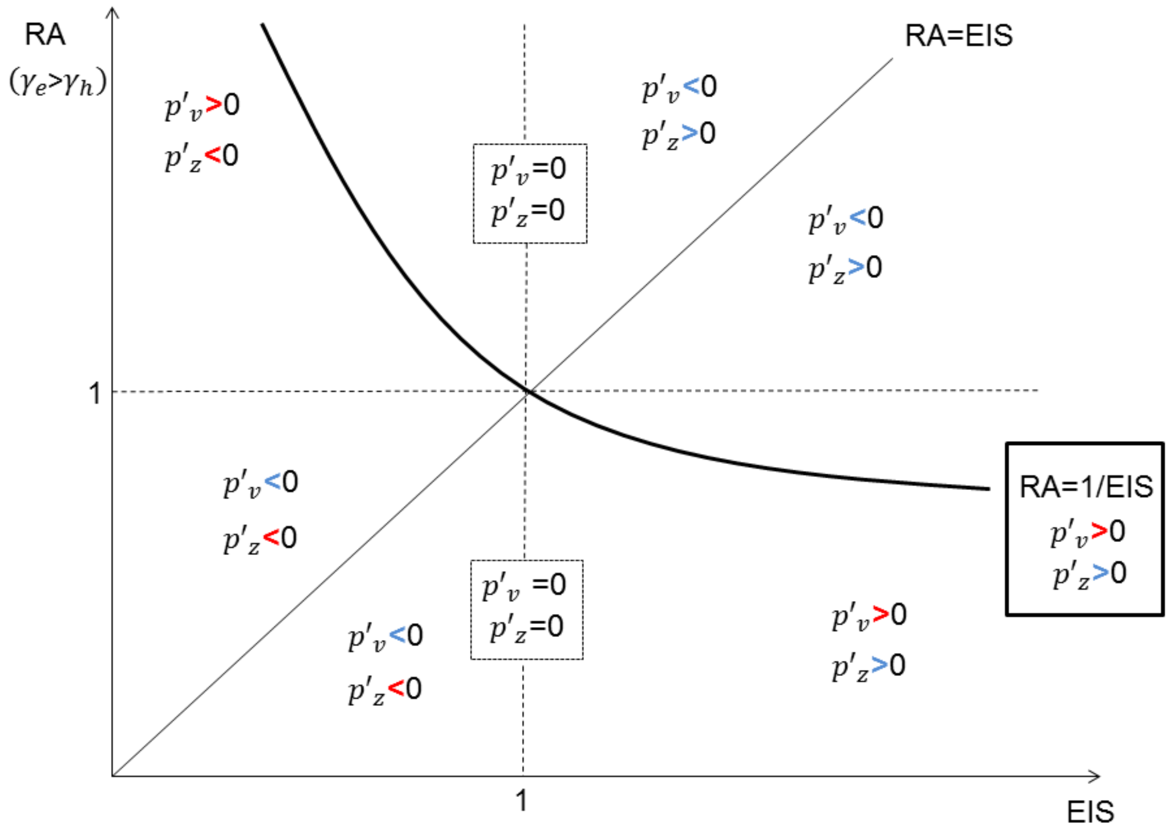
(c) % change id. vol. and CRSP returns



(d) Id. vol. and log(p/d)

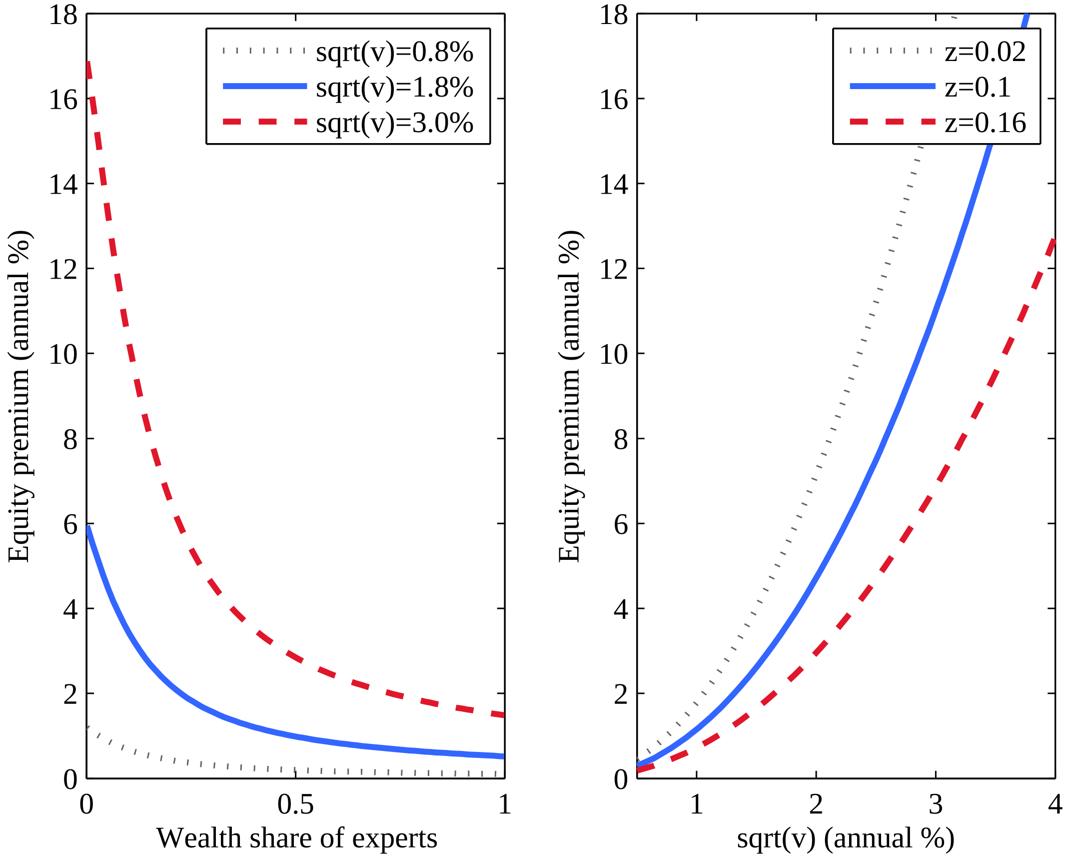
NOTES: Panel (a) shows the idiosyncratic volatility of financial firms equity and the aggregate p/d ratio. Id. vol. is constructed as in Herskovic et al. (2016), equation (1). That is, I run a regression $\log dn_{i,t} = \alpha_i + \mathbf{b}_i'\mathbf{F}_t + \varepsilon_{i,t}$, where $\log dn_{i,t}$ is the change in the market equity of a firm in the financial sector (SIC 60-64) and $\mathbf{F}_t$ are the 3 Fama-French factors. Id. vol. is calculated as the standard deviation of residuals within each calendar month. The data are monthly and the sample is 1926 to 2015. The aggregate price-dividend ratio is based on CRSP data, following Campbell and Shiller (1989). Panel (b) shows macro-volatility and aggregate p/d ratio. Macro-volatility is constructed as in Bansal et al. (2014). That is, I compute the variance of the 12-month variance of industrial production growth rates. Panel (c) shows changes in id. vol. and changes in the p/d ratio. Panel (d) displays the time series of log id. vol. and the aggregate log p/d (standarized values).

Figure 2.2: The role of γ_e, γ_h, and ψ



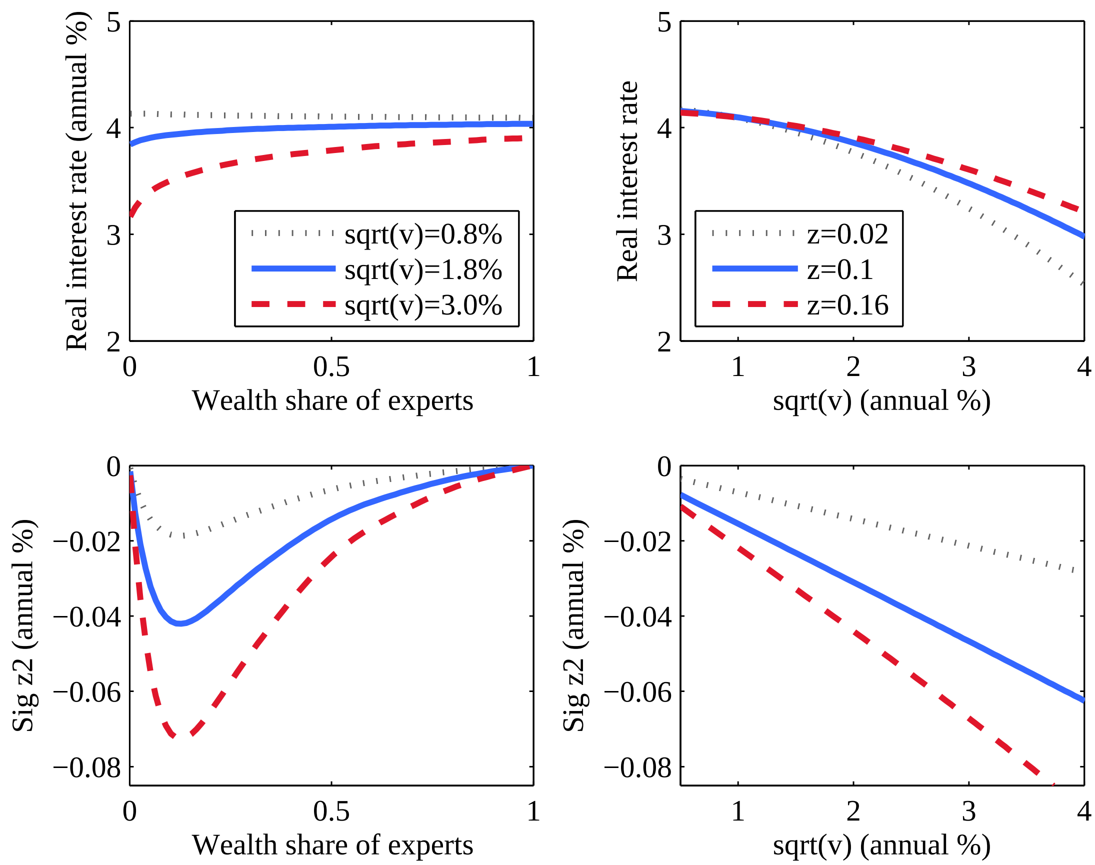
NOTES: This figure illustrates the relationship between the risk aversion coefficients and the elasticity of intertemporal substitution. The case of RA=1/EIS is when both agents have different risk aversions and therefore different EIS.

Figure 2.3: Equity premium in the endowment economy



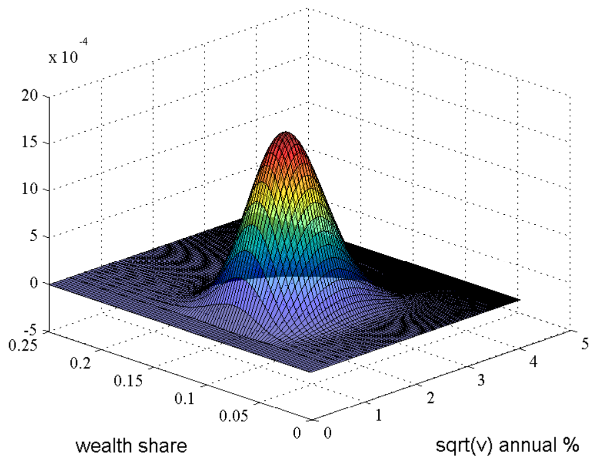
NOTES: This figure illustrates the equity premium implied by the model. The left panel considers the endogenous state variable z in the x-axis and shows the equity premium for three different levels of the exogenous state variable v. The right panel considers the exogenous state variable v in the x-axis, and shows the equity premium for three different levels of the endogenous state variable z. The levels are indicated in the legend, where "sss" means stochastic-steady state (i.e., when the drift of the corresponding state variable is zero).

Figure 2.4: Real Interest rate and $\sigma_{z,2}$

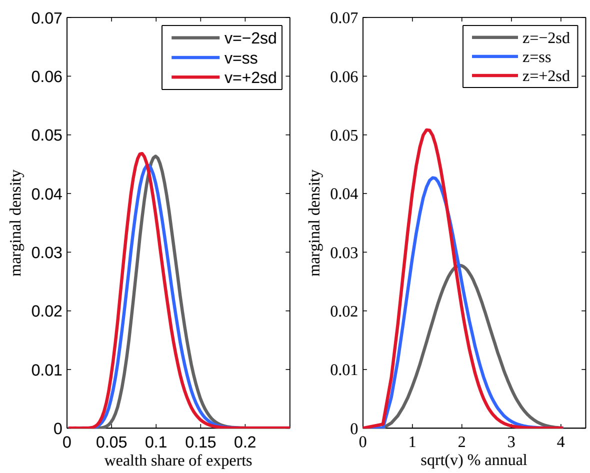


NOTES: This figure illustrates the real interest rate and $\sigma_{z,2}$. The top panels show the real interest rate. The upper-left panel considers the endogenous state variable z in the x-axis and shows the real risk-free interest rate for three different levels of the exogenous state variable v. The upper-right panel considers the exogenous state variable v in the x-axis, and shows the real risk-free rate for three different levels of the endogenous state variable z. Similarly, the lower panels show for $\sigma_{z,2}$. The levels are indicated in the legend, where "sss" means stochastic-steady state (i.e., when the drift of the corresponding state variable is zero).

Figure 2.5: Invariant distribution (v, z) in the endowment economy



(a) Invariant distribution of (x, v)



(b) Marginal distributions

NOTES: Panel (a) shows the invariant distribution in two dimensions. Panel (b) shows the invariant distribution at different point of the state space . The left-hand side shows the marginal invariant distribution for v for different levels of x: when x is at 2 standard deviations below the mean (blue), at its mean (red), and 2 standard deviations above the mean (gray). Similarly, the right-hand side illustrates the marginal invariant distributions for x for different values of v. The marginal distributions are computing by integrating the bi-variate mass accordingly.

Figure 2.6: Estimated shocks (series)

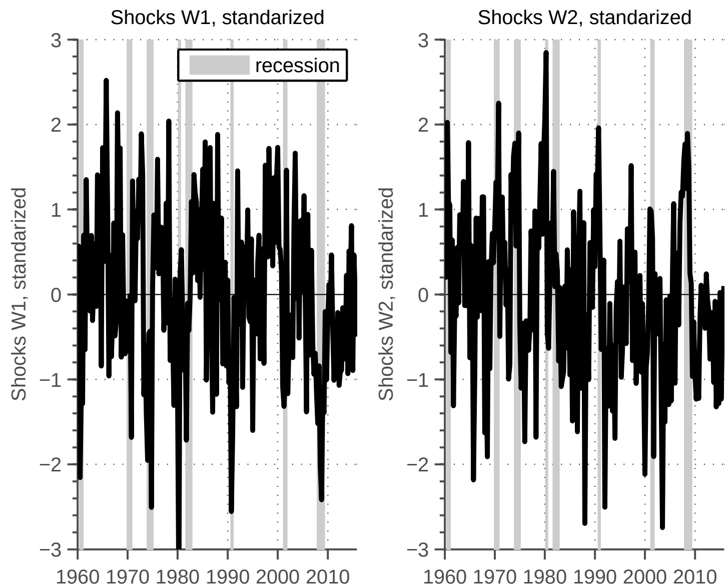


Table 2.7: Estimated shocks (summary statistics)

A. STATISTICS

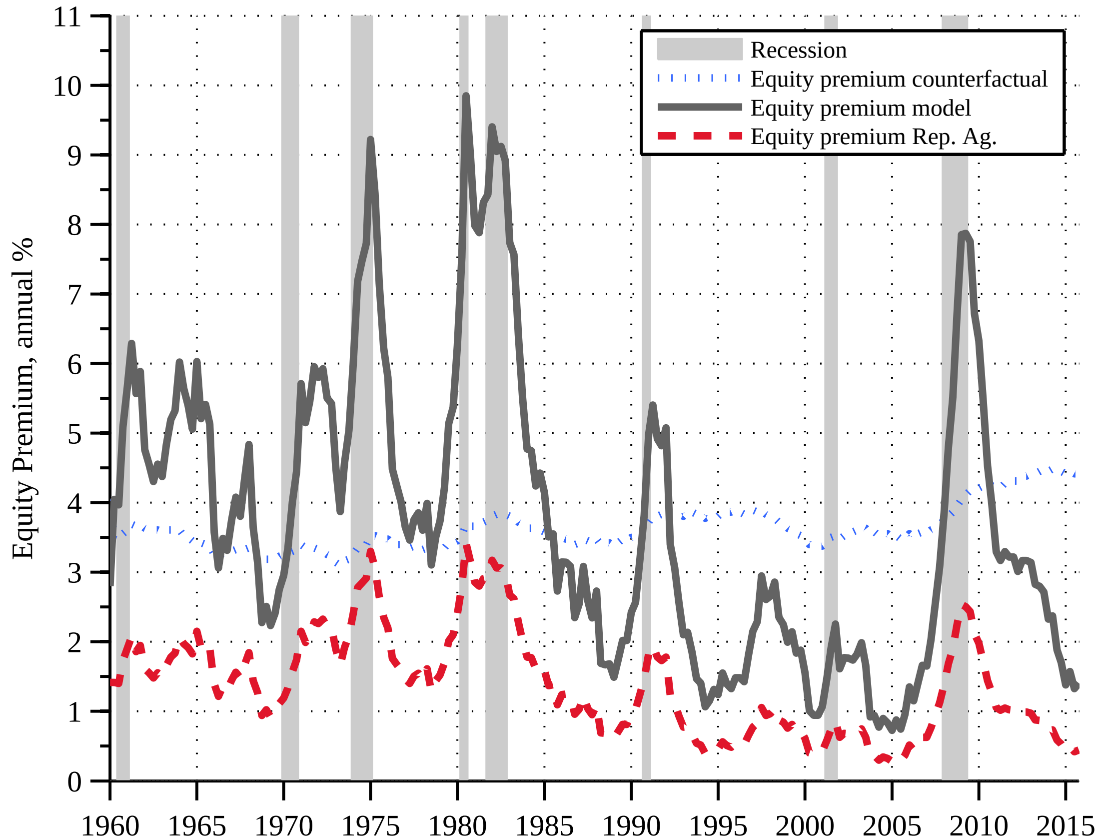
	MEAN	STD	SKEW	KURT
W_1	0.00624	0.9938	-0.16339	3.2342
	[-0.02729 0.04102]	[0.91248 1.07269]	[-0.37222 0.03761]	[2.81364 3.87996]
W_2				
	-0.02460	0.9899	0.06004	2.9393
	[-0.07399 0.029246]	[0.90508 1.07576]	[-0.2415 0.372240034]	[2.4457 3.6730]

B. CORRELATIONS

corr$(\varepsilon_1, \varepsilon_2)$	TRUE	MEAN
	$\rho = -0.5$	-0.4332
		[-0.523728727 -0.3383]

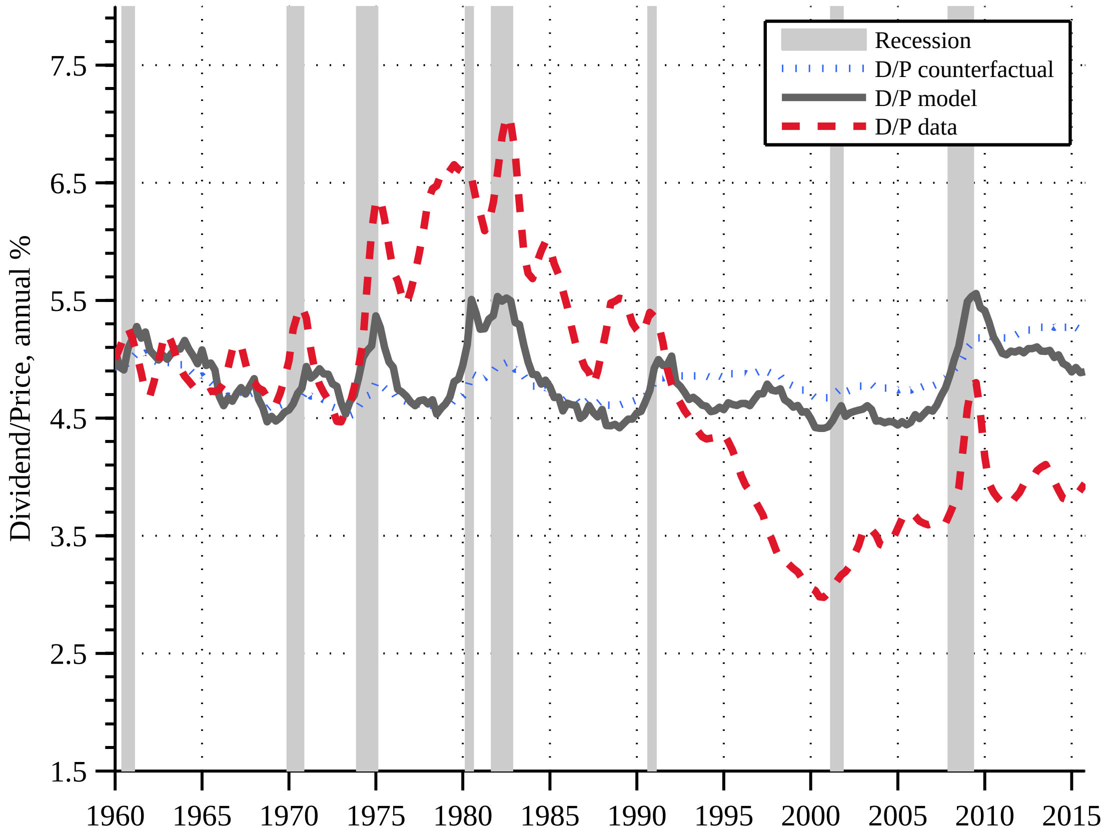
NOTES: The top panels illustrate the estimated shocks using per-capita total real consumption expenditures in the United States for the postwar period. Series are standardized so they have zero mean and unit variance. I report the mean estimation of the shock series. The table reports the estimated moments, where 2.5% and 97.5% are in parentheses. I use 500,000 draws and I set a tuning parameter to get around 50% acceptance rate in the RW-MCMC.

Figure 2.7: Equity premium in the endowment economy, implied series



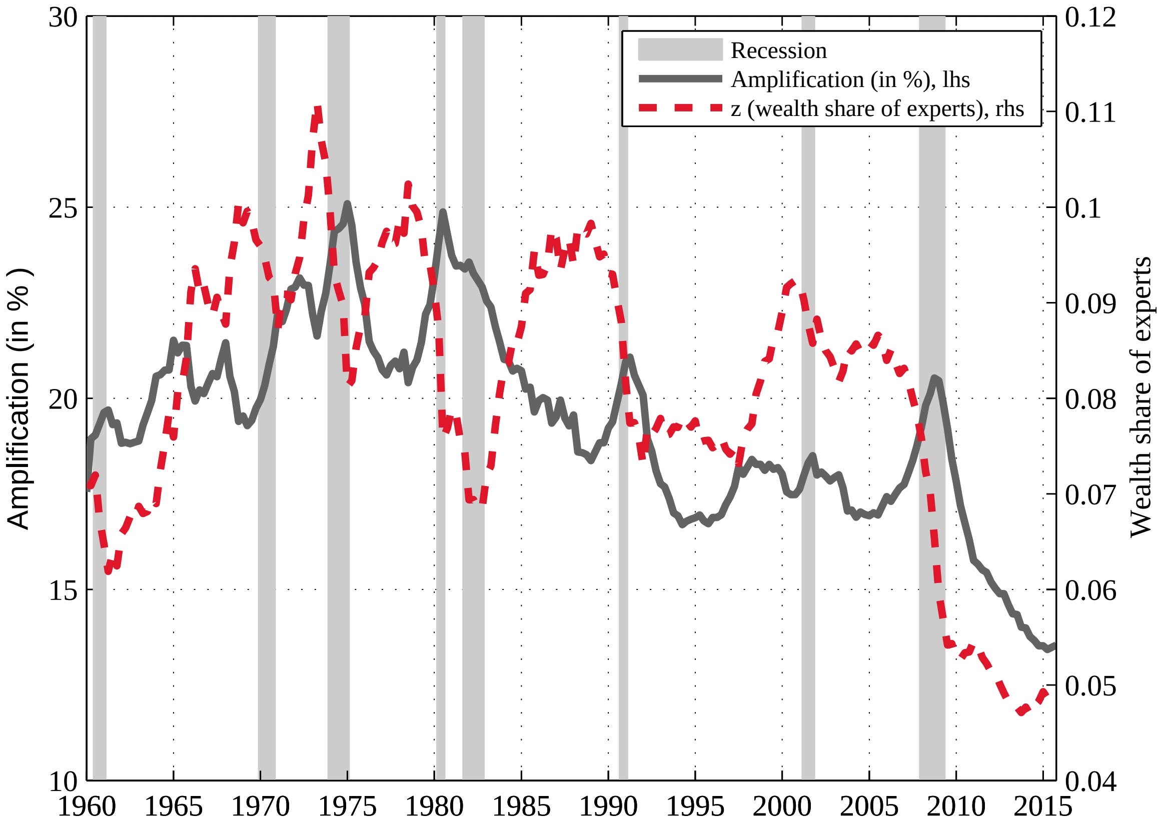
NOTES: This figure illustrates the model's prediction of the equity premium. The solid black line shows the equity premium when both cash flow and macro-uncertainty shocks are considered. The dashed red line is the equity premium when there I shut down macro-uncertainty shocks (i.e., $v_t = \overline{v}\ \forall \dot{t})$. Both series are initiated at the stochastic steady state in 1960:Q1. I consider the mean estimation of the shock series of $W_1 and W_2$.

Figure 2.8: Dividend/price in the endowment economy, implied series



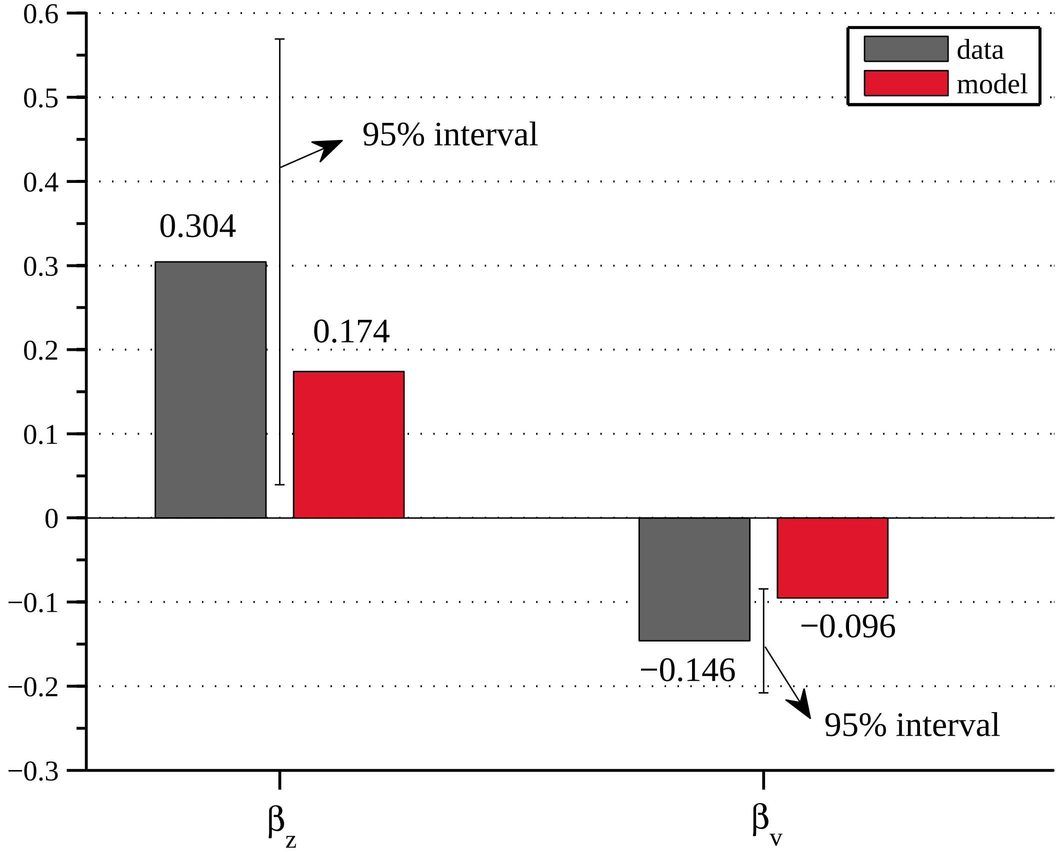
NOTES: This figure illustrates the models' implication of the divided-price ratio. The solid black line shows the dividend-price ratio when both cash-flow and macro-uncertainty shocks are considered. The dashed red line is the equity premium when I shut down macro-uncertainty shocks (i.e. . $v_t = \overline{v}$ $\forall \dot{t}$).). Both series are initiated at the stochastic steady state in 1960:Q1, and I consider the mean estimation of the shock series. The dashed blue line is the dividend-price ratio of the aggregate stock market (NYSE/NASDAQ/AMEX) from CRSP. I rescale the series to coincide with the level of the steady state implied by the model in 1960:Q1.

Figure 2.9: Amplification of shocks, the ratio $\mathcal{R}(v, z)$ in the endowment economy



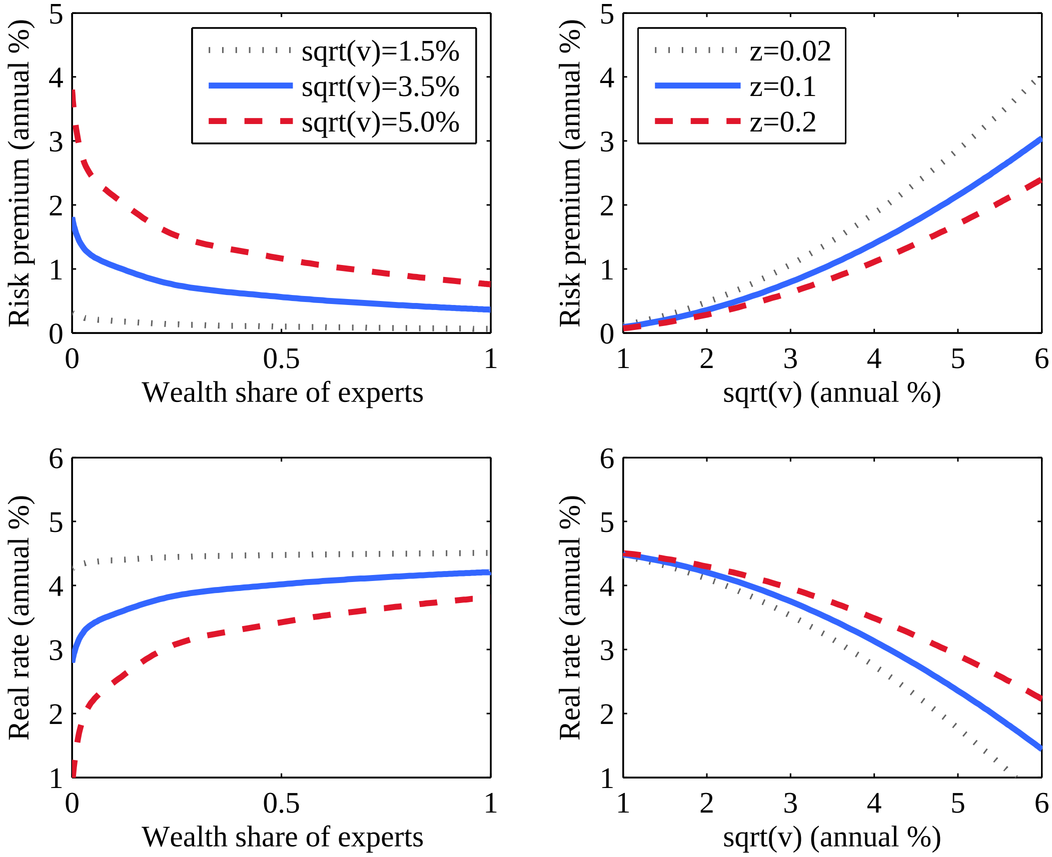
NOTES: This figure illustrates the amplification of shocks as defined in (2.25) and I express it in %. The left panel shows the time series for $\mathcal{R}$, the right panel shows the time series for the endogenous state variable z. I consider the mean estimation of the shock series.

Figure 2.10: Elasticities in the endowment economy: Model versus data



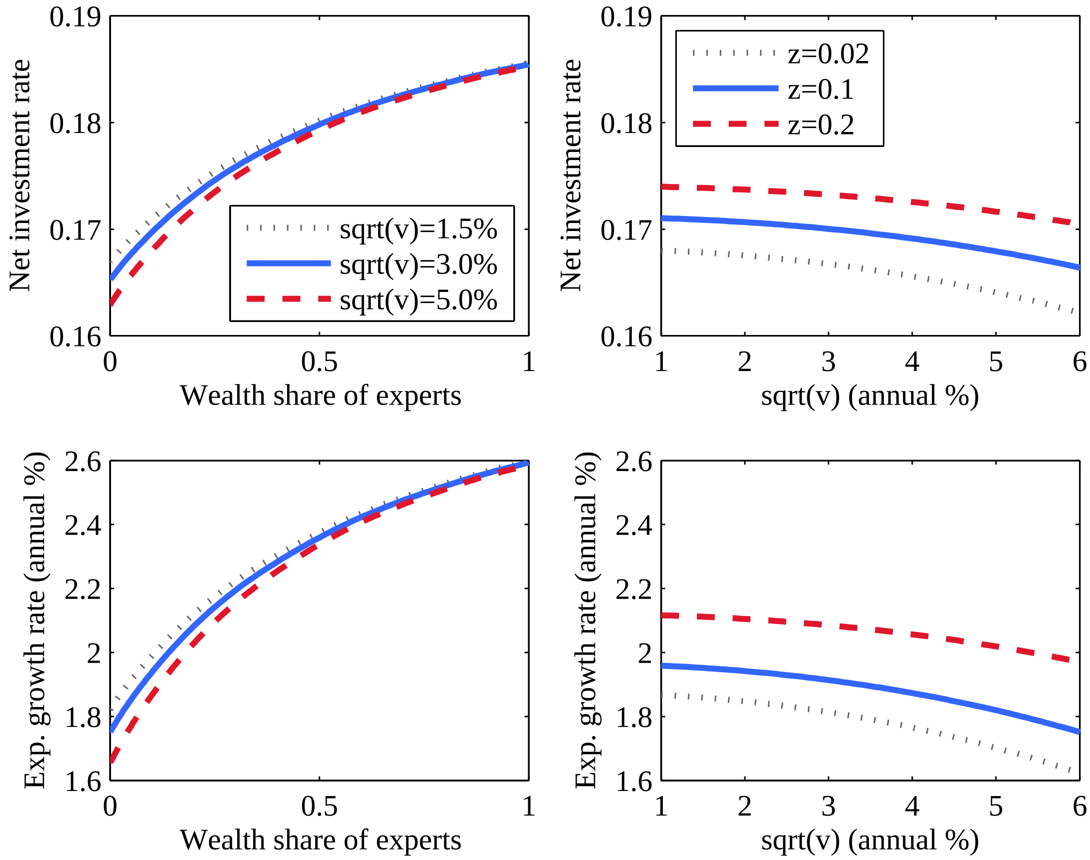
NOTES: This figure illustrates the estimated $\widehat{\beta}_z > 0$ and $\widehat{\beta}_v > 0$ in the black bars from equation (2.26). The numbers are those reported in column 3, Table 2.5. The red bars are the elasticities from the calibrated model.

Figure 2.11: Risk premium and risk-free rate in the production economy



NOTES: This figure illustrates the excess return on capita (upper panel) and the risk free rate (bottom panel) in the production economy. In both panels, the endogenous state variable z is in horizontal axis and the exogenous state variable v is on the right-hand side. The left panels (with z in the horizontal axis) indicate the behavior of the variables for three levels of v. The right panels (with v in the horizontal axis) indicates the behavior of the variables for three levels of z.

Figure 2.12: Investment and expected growth rate in the production economy



NOTES: This figure illustrates the investment rate (upper panel) and the expected growth rate of the economy (bottom panel) in the production economy. In both panels, the endogenous state variable z is in horizontal axis and the exogenous state variable v is on the right-hand side. The left panels (with z in the horizontal axis) indicates the behavior of the variables for three levels of v. The right panels (with v in the horizontal axis) indicate the behavior of the variables for three levels of z.

2.8 Appendix

Proof of proposition 1 (Law of motion for the endogenous state variable[28]). By applying Ito's lemma to (2.9), we get

$$\frac{dz}{z}=\frac{dn_e}{n_e}-\frac{dD}{D}+\left(\frac{dD}{D}\right)^2-\left(\frac{dD}{D}\right)\left(\frac{dn_e}{n_e}\right), \tag{2.29}$$

where

$$\frac{dD}{D}=z\frac{dn_e}{n_e}+(1-z)\frac{dn_h}{n_h}.$$

Therefore we can write (2.29) as

$$\frac{dz}{z}=(1-z)\left(\frac{dn_h}{n_h}-\frac{dn_e}{n_e}\right)-\left(\frac{dD}{D}\right)\left(\frac{dD}{D}-\frac{dn_e}{n_e}\right),$$

and we observe that

$$\begin{aligned} z\sigma_{z1} &= z(1-z)(\alpha_e-\alpha_h)\sigma_{q1}, \\ z\sigma_{z2} &= z(1-z)(\alpha_e-\alpha_h)\sigma_{q2}. \end{aligned}$$

The drift $z\mu_z$ requires some additional steps of algebra. In particular, notice that

$$\left(\frac{dD}{D}\right)\left(\frac{dD}{D}-\frac{dn_e}{n_e}\right)=(1-z)(\alpha_h-\alpha_e)\left(\sigma_{q2}^2+2\sigma_{q1}\sigma_{q_2}\varphi+\sigma_{q1}^2\right),$$

[28] I omit time subindex to ease notation.

where I have used market clearing conditions for the shares. On the other hand,

$$\mathbb{E}\left[(1-z)\left(\frac{dn_h}{n_h}-\frac{dn_e}{n_e}\right)\right]=(1-z)\left(\frac{c_h}{n_h}-\frac{c_e}{n_e}+(\alpha_h-\alpha_e)(\mu_q-r)\right),$$

and therefore

$$z\mu_z=z(1-z)\left(\frac{c_h}{n_h}-\frac{c_e}{n_e}+(\alpha_h-\alpha_e)(\mu_q-r)-(\alpha_h-\alpha_e)\left(\sigma_{q2}^2+2\sigma_{q1}\sigma_{q2}\varphi+\sigma_{q1}^2\right)\right)+\lambda(\overline{z}-z),$$

where the term $\lambda(\overline{z}-z)$ follows from the demographic turnover. ■

Proof of Proposition 4 (Solving for α_e). I first characterize the hedging term. For this, I need the following terms $\sigma_{q1},\sigma_{q2},\sigma_{\xi i1}$, and $\sigma_{\xi i2}$. Note that

$$\begin{aligned}\sigma_{\xi i1} &= \frac{\xi_{z,i}}{\xi_i}z\sigma_{z1},\\ \sigma_{\xi i1} &= \frac{\xi_{z,i}}{\xi_i}z\sigma_{z2}+\frac{\xi_{v,i}}{\xi_i}\sigma_v\sqrt{v},\end{aligned}$$

where $z\sigma_{z1}$ and $z\sigma_{z2}$ are given by (2.9), and they are functions of σ_{q1} and σ_{q2}. Therefore, the hedging term depends entirely on the diffusions σ_{q1} and σ_{q2}. Those functions can be solved as follows. First, notice that following the definition for p

$$\frac{dy}{y}+\frac{dp}{p}+\frac{dp}{p}\frac{dy}{y}=\frac{dq}{q}, \tag{2.30}$$

and on the other hand, we know $p(v, z)$ so using Ito's lemma, it turns out

$$\frac{dp}{p} = \mu_p dt + \frac{p_z}{p} z\sigma_{z1} dW_1 + \frac{p_z}{p} z\sigma_{z2} dW_2 + \frac{p_v}{p}\sigma_v\sqrt{v}dW_2, \tag{2.31}$$

where $\mu_p = E[\mathcal{L}p(x, \eta)]$, using $\mathcal{L}$ as the infinitesimal operator. So with (2.30), (2.31), and (2.9) I can get σ_{q1} and σ_{q2}:

$$\sigma_{q1} = \frac{\sqrt{v}}{1 - \frac{p_z}{p} z(\alpha_e - 1)}, \tag{2.32}$$

$$\sigma_{q2} = \frac{\frac{p_v}{p}\sigma_v\sqrt{v}}{1 - \frac{p_z}{p} z(\alpha_e - 1)}, \tag{2.33}$$

and I have used the market clearing condition. Then, the portfolio share can be written as

$$\gamma_i\alpha_i = \frac{\mu_q - r}{\sigma_{q1}^2 + \sigma_{q2}^2 + 2\sigma_{q1}\sigma_{q2}\varphi} - \left(\frac{1-\gamma_i}{1-\psi}\right)\left(\frac{\xi_{z,i}}{\xi_i}(\alpha_e - 1) z + \frac{\xi_{v,e}}{\xi_e} T\left(1 - \frac{p_z}{p}(\alpha_e - 1) z\right)\right), \tag{2.34}$$

with the auxiliary function $T = T(v, z)$, which is not indexed by i,

$$T = \frac{\left(\varphi + \frac{p_v}{p}\sigma_v\right)\sigma_v}{\left(1 + 2\varphi\frac{p_v}{p}\sigma_v + \left(\frac{p_v}{p}\right)^2\sigma_v^2\right)}.$$

The next step is to use (2.34) and get an expression for the Sharpe ratio

$$\frac{\mu_q - r}{\sigma_{q1}^2 + \sigma_{q2}^2 + 2\sigma_{q1}\sigma_{q2}\varphi} = \gamma_e\alpha_e - \left(\frac{1-\gamma_e}{1-\psi}\right)\left(\frac{\xi_{z,e}}{\xi_e}(\alpha_e - 1) z + \frac{\xi_{v,e}}{\xi_e} T\left(1 - \frac{p}{p_z}(\alpha_e - 1) z\right)\right)$$

and then the demand for households

$$\alpha_h = \frac{\gamma_e}{\gamma_h}\alpha_e + \left\{ z\left(\alpha_e - 1\right) R^z + \left[1 - \frac{p_z}{p} z\left(\alpha_e - 1\right)\right] T^v R^v \right\} \frac{1}{\left(1-\psi\right)\gamma_h}$$

and with a few steps

$$\alpha_h = \left[\frac{\gamma_e}{\gamma_h} + \frac{z}{\left(1-\psi\right)\gamma_h}\left(R^z - R^v T\left(\frac{p_z}{p}\right)\right)\right]\alpha_e + \frac{R^v T\left(1 + \frac{p_z}{p} z\right) - R^z z}{\left(1-\psi\right)\gamma_h,}$$

which means

$$\alpha_h = \Lambda_0 + \Lambda_1 \alpha_e$$

with

$$\begin{aligned} \Lambda_1 &= \frac{\gamma_e}{\gamma_h} + \frac{z}{\left(1-\psi\right)\gamma_h}\left(R^z - T^v R^v\left(\frac{p_z}{p}\right)\right) \\ \Lambda_0 &= \frac{\left(1 + \frac{p_z}{p} z\right) T^v R^v - z R^z}{\left(1-\psi\right)\gamma_h} \end{aligned}$$

and R^v, R^z, and T are defined in the main text. Using market clearing condition $z\alpha_e + \left(1-z\right)\alpha_h = 1$, I get

$$\alpha_e = \frac{1 - \left(1-z\right)\Lambda_0}{z + \left(1-z\right)\Lambda_1,}$$

which is expression (2.15). ■

Proof of proposition 5 (Risk sharing and concentration of risk). I show the "only if" part of the proof. That is I assume $\alpha_e > 1$ and derive the conditions under which this is true. The "if" is similar. Using market clearing conditions for the shares market,

$$\begin{aligned} z\sigma_{z1} &= z\left(\alpha_e - 1\right)\sigma_{q1} \\ z\sigma_{z1} &= z\left(\alpha_e - 1\right)\sigma_{q2} \end{aligned}$$

Since $z > 0$, I first show the $\alpha_e > 1$. Using (2.15), then

$$\begin{aligned} \alpha_e &> 1 \\ &\leftrightarrow \\ 1 - \Lambda_0 &> \Lambda_1 \end{aligned}$$

and I can write $1 - \Lambda_0 > \Lambda_1$, I find

$$\gamma_h > \frac{T^v\left(\frac{\xi_{e,v}}{\xi_e} - \frac{\xi_{h,v}}{\xi_h}\right)}{\left(1 - \psi + T^v\frac{\xi_{h,v}}{\xi_e}\right)} + \gamma_e\left(\frac{1 - \psi + T^v\frac{\xi_{e,v}}{\xi_e}}{1 - \psi + T^v\frac{\xi_{h,v}}{\xi_e}}\right).$$

Now, I know $T^v < 0$ provided the substitution effect dominates for both agents ($\psi > 1$). Also, $\frac{\xi_{e,v}}{\xi_e} - \frac{\xi_{h,v}}{\xi_h} < 0$ because the precautionary motives are stronger for households than experts. Also, the term $\left(1 - \psi + T^v\frac{\xi_{h,v}}{\xi_e}\right) < 0$ provided $\psi > 1$ together with $T^v\frac{\xi_{h,v}}{\xi_e} < 0$. This means

$$\frac{T^v\left(\frac{\xi_{e,v}}{\xi_e} - \frac{\xi_{h,v}}{\xi_h}\right)}{\left(1 - \psi + T^v\frac{\xi_{h,v}}{\xi_e}\right)} < 0.$$

And the term multiplying γ_e positive and less than one since $\frac{\xi_{h,v}}{\xi_h} > \frac{\xi_{e,v}}{\xi_e} > 0$

$$0 < \left(\frac{1 - \psi + T^v \frac{\xi_{e,v}}{\xi_e}}{1 - \psi + T^v \frac{\xi_{h,v}}{\xi_e}} \right) < 1$$

Then since $\gamma_h > \gamma_e$ and $\psi > 1$ are assumed, this means $\alpha_e > 1$ in equilibrium. It follows $z\sigma_{z1} > 0$ since $\sigma_{q1} > 0$, from (2.32). Also $z\sigma_{z2} < 0$ since $p_v < 0$ and experts are levered ($\alpha_e > 1$). ■

Proof of proposition 6 (Prices of risk) Define ε_{i1} and ε_{i2} as the desired "exposures" of agent i to W_1 and W_2, respectively:

$$\begin{aligned} \varepsilon_{i1} &= \alpha_i \sigma_{q1}, \\ \varepsilon_{i2} &= \alpha_i \sigma_{q2}. \end{aligned}$$

The idea is to restate agent's i problem with control variables c_i, ε_{i1}, and ε_{i2} instead of c_i and α_i. To that end, define excess returns as

$$\begin{aligned} \mu_q - r &= -cov\left(\frac{d\pi}{\pi}, \frac{dq}{q}\right), \\ &= \sigma_{\pi 1}\sigma_{q1} + \sigma_{\pi 2}\sigma_{q1} + \left(\sigma_{\pi 1}\sigma_{q2} + \sigma_{\pi 2}\sigma_{q1}\right)\varphi. \end{aligned}$$

So I can write

$$\alpha_i\left(\mu_q - r\right) = \varepsilon_{i1}\sigma_{\pi 1} + \varepsilon_{i2}\sigma_{\pi 2} + \left(\sigma_{\pi 1}\varepsilon_{i2} + \sigma_{\pi 2}\varepsilon_{i1}\right)\varphi.$$

First-order conditions (FOC) for ε_{i1} and ε_{i2} are

$$
\begin{aligned}
\left[\varepsilon_{i1}\right] &: \sigma_{\pi 1}+\varphi\sigma_{\pi 2}-\gamma_i\left(\varepsilon_{i1}+\varepsilon_{i2}\varphi\right)+\frac{1-\gamma_i}{1-\psi}\left(\sigma_{\xi i1}+\sigma_{\xi i2}\varphi\right)=0\ , \\
\left[\varepsilon_{i2}\right] &: \sigma_{\pi 2}+\varphi\sigma_{\pi 1}-\gamma_i\left(\varepsilon_{i2}+\varepsilon_{i1}\varphi\right)+\frac{1-\gamma_i}{1-\psi}\left(\sigma_{\xi i2}+\sigma_{\xi i1}\varphi\right)=0.
\end{aligned}
$$

Using market clearing condition for the shares, we have

$$
z\varepsilon_{e1}+(1-z)\,\varepsilon_{h1} = \sigma_{q1} \tag{2.35}
$$

$$
z\varepsilon_{e2}+(1-z)\,\varepsilon_{h2} = \sigma_{q2} \tag{2.36}
$$

so using FOC on (2.35),

$$
\begin{aligned}
& z\left(\frac{\sigma_{\pi 1}}{\gamma_e}+\varphi\frac{\sigma_{\pi 2}}{\gamma_e}-\varepsilon_{e2}\varphi+\frac{1-\gamma_e}{(1-\psi)\,\gamma_e}\left(\sigma_{\xi e1}+\sigma_{\xi e2}\varphi\right)\ \right) \\
& +(1-z)\left(\frac{\sigma_{\pi 1}}{\gamma_h}+\varphi\frac{\sigma_{\pi 2}}{\gamma_h}-\varepsilon_{h2}\varphi+\frac{1-\gamma_h}{(1-\psi)\,\gamma_h}\left(\sigma_{\xi h1}+\sigma_{\xi h2}\varphi\right)\right) \\
= \ & \sigma_{q1} \\
& z\left(\frac{\sigma_{\pi 2}}{\gamma_e}+\varphi\frac{\sigma_{\pi 1}}{\gamma_e}-\varepsilon_{e1}\varphi+\frac{1-\gamma_e}{(1-\psi)\,\gamma_e}\left(\sigma_{\xi e2}+\sigma_{\xi e1}\varphi\right)\ \right) \\
& +(1-z)\left(\frac{\sigma_{\pi 2}}{\gamma_h}+\varphi\frac{\sigma_{\pi 1}}{\gamma_h}-\varepsilon_{h1}\varphi+\frac{1-\gamma_h}{(1-\psi)\,\gamma_e}\left(\sigma_{\xi h2}+\sigma_{\xi h1}\varphi\right)\right) \\
= \ & \sigma_{q2}
\end{aligned}
$$

and inserting the following expressions in the σ_{q1} and σ_{q2} above

$$\begin{aligned} z\varepsilon_{h2}\varphi + (1-z)\,\varepsilon_{h2}\varphi &= \varphi\sigma_{q2} \;, \\ z\varphi\frac{\sigma_{\pi 2}}{\gamma_e} + (1-z)\,\varphi\frac{\sigma_{\pi 2}}{\gamma_h} &= \varphi\sigma_{\pi 2}\left[\frac{z}{\gamma_e} + \frac{1-z}{\gamma_h}\right] \;, \end{aligned}$$

we arrive to the $\sigma_{\pi 1}$ and $\sigma_{\pi 2}$ in (2.18). ■

Proof of proposition 7 (Equity premium) Since the state-price deflator π follows an Ito process with regular properties. M^{π}

$$M_t^{\pi} = E_t \int_0^{\infty} \frac{\pi_s}{\pi_0} div_s ds \;,$$

which can be written as

$$M_t^{\pi} = \int_0^{t} \frac{\pi_s}{\pi_0} div_s ds + \frac{\pi_t}{\pi_0}\widetilde{q}_t \;, \tag{2.37}$$

where $\widehat{q}_t$ is defined in (2.19). In the absence of arbitrage M^{π} is a martingale with continuous paths. Therefore, the expected infinitesimal change of M_t^{π} must be zero. That yields equation (2.20). ■

Solution procedure. The model solution is characterized by a system of Partial Differential Equations (PDEs). I solve it using projection methods. The main idea consists on constructing a tensor grid[29] for the state space $(v, z) \in (0, \infty) \times (0, 1)$, and then project the unknown functions in the state space,[30] In particular, I have to solve for three functions:

[29]See Judd (1998).

[30]For a comprehensive review, see Boyd (2001) and Trefethen (2000).

$p(v,z), \xi_h(v,z)$ and $\xi_e(v,z)$. I use the HJB for each agent and the market clearing condition for the consumption good. The remaining objects, are solved from their own definitions: $\alpha_e(v,z), \alpha_h(v,z), \mu_q(z,v), \sigma_{q1}(v,z), \sigma_{q2}(v,z)$, and $\pi(v,z)$. A key property of the model is that both state variables are stationary and therefore a solution for U_e and U_h is guaranteed to exists and to be unique, following Duffie and Lions (1992). Therefore, although I do not provide a verification theorem, the argument for existance and uniqueness of the equilibrium follows Duffie and Lions (1992).

MCMC procedure. In this part of the appendix I describe the MCMC procedure to estimate the shocks $\{W_1, W_2\}$. I follow Johannes and Polson (2010) and Gamerman and Lopes (2006) closely. I first discretize the process (2.3)-(2.4), where I consider the time step a quarter, be consistent with the calibration of the parameters. Define $g_t = \log y_{t+\Delta t} - \log y_t$, where Δ is the time-discretization interval.

$$
\begin{aligned}
g_{t+\Delta t} &= \mu\Delta t + \sqrt{v_t\Delta t}\varepsilon_{1,t+\Delta t} \\
v_{t+\Delta t} - v_t &= \kappa_v(\overline{v} - v_t)\Delta t + \sqrt{v_t\Delta t}\varepsilon_{2,t+\Delta t} \\
\varepsilon_{1,t+\Delta t} &\sim N(0, \Delta t) \\
\varepsilon_{2,t+\Delta t} &\sim N\left(0, \sigma^2\Delta t\right) \\
corr(\varepsilon_{1,t}, \varepsilon_{2,t}) &= \rho\Delta t
\end{aligned}
$$

I assume the decision interval is a quarter. That is, I discretize at the same frequency of the data. Then

$$\varepsilon_{1,t+1} = \frac{g_{t+1} - \mu}{\sqrt{v_t}} \tag{2.38}$$

$$\varepsilon_{2,t+1} = \frac{v_{t+1} - \kappa_v \overline{v} - (1 - \kappa_v) v_t}{\sqrt{v_t}} \tag{2.39}$$

where[31]

$$\begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\sigma_v \\ \rho\sigma_v & \sigma_v^2 \end{pmatrix} \right)$$

Notice that (2.38) and (2.39) are implicitly describing

$$g_{t+1} = \mu + \sqrt{v_t}\varepsilon_{1,t+1}$$

$$v_{t+1} = \alpha + \beta v_t + \sqrt{v_t}\varepsilon_{2,t+1}$$

where I have denoted $\alpha = \overline{v}\kappa_v$, $\beta = (1 - \kappa_v)$ and, I define $\varepsilon_2 = \sigma_v \rho \varepsilon_1 + \sqrt{\sigma_v^2 (1 - \rho^2)}\varepsilon_3$, with $\varepsilon_1 \perp \varepsilon_3$. Following Jacquier et al. (2004), I use the following change of variable to ease notation

$$\sigma_v \rho = \Psi$$

$$\sigma_v^2 \left(1 - \rho^2\right) = \Omega$$

[31]If

$$\varepsilon_{2,t+1} = \frac{v_{t+1} - \kappa_v \overline{v} - (1 - \kappa_v) v_t}{\sigma\sqrt{v_t}}$$

then

$$\begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix} \right)$$

The next step is to compute the joint likelihood $p\left(g, v\middle|\,\Theta\right)$, where Θ is the vector of parameters. Using the bivariate normal function

$$\begin{aligned} p\left(g, v\middle|\,\Theta\right) &\propto \prod_{t=0}^{T-1} p\left(g_{t+1}, v_{t+1}\middle|\,v_t, \Theta\right) \\ &= \prod_{t=0}^{T-1} \frac{1}{v_t} \left| \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix} \right|^{-1/2} \\ &\quad \exp\left(-\frac{1}{2} trace \left(\begin{pmatrix} 1 & \Psi \\ \Psi & \Psi^2 + \Omega \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t+1} & \varepsilon_{2,t+1} \end{pmatrix} \right) \right) \end{aligned}$$

simplifying, the joint likelihood is

$$p\left(g, v\middle|\,\Theta\right) = \Omega^{-T/2} \left(\prod_{t=0}^{T-1} \frac{1}{v_t} \right) \exp\left(-\frac{1}{2\Omega} \sum_{t=0}^{T-1} \left(\left(\Omega + \Psi^2\right) \varepsilon_{1,t+1}^2 - 2\Psi \varepsilon_{1,t+1}\varepsilon_{2,t+1} + \varepsilon_{2,t+1}^2 \right) \right)$$

The last step is to find the posterior distribution to sample the latent state. Using Bayes Theorem

$$\begin{aligned} p\left(v_t\middle|\,v_{t-1}, v_{t+1}, g, \Theta\right) &= \frac{p\left(v_t, v_{t-1}, v_{t+1}, g, \Theta\right)}{p\left(v_{t-1}, v_{t+1}, g, \Theta\right)} = \frac{p\left(g_{t+1}, v_t, v_{t+1}\middle|\,v_{t-1}, \Theta\right)}{p\left(g_{t+1}, v_t, v_{t+1}\middle|\,\Theta\right)} \frac{p\left(v_{t-1}, \Theta\right)}{p\left(\Theta\right)} \\ &\propto \frac{p\left(g_{t+1}, v_t, v_{t+1}\middle|\,v_{t-1}, \Theta\right)}{p\left(g_{t+1}, v_t, v_{t+1}\middle|\,\Theta\right)} p\left(v_{t-1}\middle|\,\Theta\right) \\ &\propto p\left(g_{t+1}, v_t, v_{t+1}\middle|\,v_{t-1}, \Theta\right) \end{aligned}$$

I omit the algebra, the density $p\left(v_t \middle| v_{t-1}, v_{t+1}, g, \Theta\right)$ is proportional to the following 5 terms

$$
\begin{aligned}
&\exp\left(-\frac{\left(\Omega+\Psi^2\right)}{\Omega}\frac{\left(g_{t+1}-\mu\right)^2}{2\exp\left(h_t\right)}\right) \\
&\exp\left(\frac{\Psi}{\Omega}\frac{\left(g_{t+1}-\mu\right)\left(v_{t+1}-\alpha\right)}{\exp\left(h_t\right)}\right)\exp\left(-\frac{1}{2\Omega}\frac{\left(v_{t+1}-\alpha-\beta\exp\left(h_t\right)\right)^2}{\exp\left(h_t\right)}\right) \\
&\exp\left(\frac{\Psi}{\Omega}\frac{\left(g_t-\mu\right)\left(\exp(h_t)-\alpha\right)}{v_{t-1}}\right)\exp\left(-\frac{1}{2\Omega}\frac{\left(\exp(h_t)-\alpha-\beta v_{t-1}\right)^2}{v_{t-1}}\right)
\end{aligned}
$$

where $h_t = \log\left(v_t\right)$, to obtain positive draws while sampling.

Chapter 3

Liquidity Shocks, Business Cycles and Asset Prices

3.1 Introduction

In the aftermath of the Great Recession, there has been increased interest by academics and policy makers in macro models that feature financing constraints. The Kiyotaki and Moore (2012) model (hereafter KM) of collateral constraints and liquidity shocks is a leading example. The main idea is that liquidity shocks constrain the fraction of assets that may be traded in a given period. These changes in liquidity can lead to fluctuations in aggregate activity and asset prices by tightening firms' ability to pledge collateral. Numerous studies have followed KM's lead (see Ajello (2016), Bigio (2015), Del Negro et al. (2016), Kurlat (2013), Venkateswaran and Wright (2014)).

In this paper, we study the quantitative properties of liquidity shocks in a real business cycle (RBC) framework, focusing on asset pricing properties and business cycle implications. To accomplish this, we use a stripped-down version of KM, characterize the equilibrium, and study the effects of liquidity shocks via global nonlinear analysis.

Our main findings are that liquidity shocks: (i) improve quantitative prediction of the levels and volatility of equity premiums relative to the frictionless RBC; (ii) predict highly nonlinear dynamics for premiums akin to models that feature balance sheet dynamics; (iii) have negligible effects on the risk-free rate; (iv) improve the relative volatility of investment to output growth rate; (v) on impact, have mild effects on the levels of investment and output; and (vi) produce counterfactual dynamics for the correlation between liquidity and the equity premium—i.e., periods of abundant liquidity are associated with higher expected returns. Lastly, after we decompose expected returns into a liquidity component and a market component, we find that the liquidity component does not account for a large share of the total premium. We demonstrate, in detail, why liquidity shocks fail to account for the basic fact that in tranquil times—which are typically associated with abundance in liquidity—expected excess returns are high.

The main mechanism in the KM framework is as follows. Investment has two characteristics that cause liquidity to become a source of business cycles. First, access to investment projects is limited to a fraction of the population, which means that resources must be reallocated from agents who do not possess these opportunities to those who do. Such reallocation requires a credible promise to deliver investment projects, which in turn requires collateral. Without collateral (or, in the model, less liquidity), that process is interrupted. Second, repayment cannot be guaranteed. This characteristic requires that investment be financed, in part, internally (investment requires a down payment). The combination of these two features creates gains from trading existing assets, and doing so enables agents to obtain internal funds to relax external financing constraints. Liquidity shocks interrupt the amount of trade, which affects the supply side of credit. In other words, periods of low liquidity are associated with a contraction in the supply schedule of claims to investment projects. When liquidity shocks are sufficiently large, they drive aggregate investment below its frictionless level. As a result, these shocks also drive a wedge between the price of capital and its replacement cost, which is a measure of inefficiently low investment.

From an asset pricing perspective, we highlight the fact that the model predicts highly nonlinear dynamics for risk premiums, together with sensible endogenous time variation. It also predicts higher levels of premiums relative to those implied by the frictionless benchmark.

In other words, the expected change in the price of equity is very sensitive to liquidity shocks. These elements of the model can significantly improve quantitative asset pricing predictions relative to the frictionless RBC setup.

The KM framework faces some challenges that we underscore throughout the paper. From a business cycle perspective, liquidity shocks produce countercyclical consumption—it increases in the recession triggered by a liquidity shock. However, the quantitative effect of this shock on macro variables is small overall: On impact, a major liquidity dry-out produces a decrease in output of -0.1% from its mean. Regarding asset prices and returns, the model predicts lower expected returns during periods of scarce liquidity. Thus, the model is counterfactual in the sense that tranquil times (those of abundant liquidity) are associated with high expected returns. Our results complement and reinforce other findings in the KM literature (i.e., Shi (2015)), but our focus on asset pricing offers further insight into the KM's implications for risk premiums dynamics.

We next provide a brief review of the literature.

Literature. Our paper is directly related to the literature that follows the KM model. Del Negro et al. (2016) is an example of this strand. A central element of that paper is the introduction of nominal rigidities and a monetary policy that is subject to a zero lower bound. Our paper complements theirs in the sense that we observe that without additional features, liquidity shocks cannot account for the dynamics of premiums and macro quantities. More precisely, we observe that liquidity shocks can improve frictionless predictions and come closer to the observed evidence, although this is not enough. In addition, our focus on risk premium dynamics emphasizes a central counterfactual element embedded in the KM constraint: In periods of abundant liquidity, expected excess returns are high. Lastly, our paper differs from Del Negro et al.'s because we study the behavior of the model globally, whereas theirs is restricted to a log-linearized version of the model.

Shi (2015) proposed an similar to ours, with a key difference in the analysis. In our paper, the labor supply schedule is fixed; liquidity shocks reduce capital accumulation, and this contracts labor demand. Thus, wages and hours decline in recessions. In contrast, Shi (2015) uses a "single family" framework with strong leisure-consumption complementarities. As in our setup, liquidity shocks lead to declines in the investment-consumption ratio. With leisure-consumption complementarities, labor supply contracts when liquidity shocks increase consumption. As a result, although liquidity shocks trigger strong reductions in hours, they also produce counterfactual movements in wages. In addition, our main focus is on asset prices and risk premium dynamics. We use recursive preferences Epstein and Zin (1989a), and we calibrate both the elasticity of intertemporal substitution and the risk-aversion coefficient, as in the leading papers in the asset pricing literature. As noted by Shi (2015), our results also suggest a relatively higher unconditional level of the equity premium, but we extend the analysis to study its behavior across the state space.

Recent literature in macroeconomics has also predicted highly non-linear risk premium dynamics, driven by fluctuations in agents' balance sheets in models with financial frictions (Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013) are prominent examples). Our results are in this vein, but with the crucial distinction that in our framework, endogenous fluctuations in expected returns are driven purely by occasionally binding constraints. This is because, as in KM, we assume that investment opportunities are i.i.d., and therefore we do not have to keep track of investor's balance sheets in the state vector.

The paper contains four sections. In Section 2, we present the model, characterize the solution, and discuss the intuition behind the effects. We explore the quantitative predictions of the model in Section 3, and Section 4 concludes.

3.2 Model

We begin by describing a version of the KM model, abstracting from fiat money. We consider an infinite-horizon economy in which time is discrete and denoted by $t = 0, 1, ...$. There are two populations, entrepreneurs and workers, each with unit measure . The former do not work, but invest in physical capital, while the latter do not invest but supply labor. In each period, entrepreneurs are randomly assigned to one of two types: investors or savers, labeled by superscripts i and s.

The economy is subject to two sources of aggregate uncertainty: productivity and liquidity shocks. We represent the productivity level in the economy by $A_t \in \mathbb{A}$ and liquidity by $\phi_t \in \Phi \subset [0, 1]$. The nature of liquidity shocks will affect the ability to sell equity and will soon become clear. Importantly, we assume that both A_t and ϕ_t follow a finite-space stationary Markov process with a stationary transition probability $\Pi : (\mathbb{A}\times\Phi) \times (\mathbb{A}\times\Phi) \to [0, 1]$.

While ϕ_t and A_t are exogenous aggregate variables, there is endogenous capital accumulation. We denote the level of capital in the economy by $K_t \in \mathbb{K}$; this completes the description of the state space. Hence, we define $s_t = \{A_t, K_t, \phi_t\} \in S_t \equiv (\mathbb{A} \times \mathbb{K}\times\Phi)$ as the vector that represents a point in the state space.

We next describe the production side of the economy and then the entrepreneur's problem.

3.2.1 Production

A representative firm produces final consumption goods. The firm hires labor from workers and capital from entrepreneurs. Production is carried out according to a Cobb-Douglas technology

$$y_t = A_t F\left(k_t, L_t\right) \ ,$$

where y_t is output and $F\left(k, L\right) = k^{\alpha} L^{1-\alpha}$. The profit maximization problem is standard, and the firm demands $\left\{L^d, k^d\right\}$ to solve

$$\pi_t = \max_{L^d, k^d} A_t (k_t^d)^{\alpha} (L_t^d)^{1-\alpha} - w_t L_t^d - r_t k_t^d \ , \tag{3.1}$$

where w_t is the real wage and r_t the net return on capital.

3.2.2 Entrepreneurs and investment

Preferences. Entrepreneurs feature recursive preferences with Kreps-Porteus aggregator, as in Epstein and Zin (1989a):

$$V_{j,t} = \left[c_{j,t}^{\rho} + \beta E_t \left[V_{j,t+1}^{1-\gamma}\right]^{\frac{\rho}{1-\gamma}}\right]^{\frac{1}{\rho}} \ . \tag{3.2}$$

In this notation, we use $\rho = 1 - 1/\psi$, where ψ is the elasticity of intertemporal substitution (EIS) and γ the risk-aversion parameter. We represent the certainty equivalent as $\mu_t =$

$E_t \left[V_{j,t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$. Notice that both types of entrepreneurs $j = (savers, investors)$ have the same preferences, and the only source of heterogeneity is their investment opportunities.

Investment opportunities. Investment opportunities arrive randomly, are distributed i.i.d across time and agents, and are independent of aggregate shocks.[1] In particular, an investment opportunity arrives with probability π. We label entrepreneurs who have an investment opportunity "investors"—whose mass is π—and those without such opportunity "savers"—whose mass is $1 - \pi$. Entrepreneurs face the following budget constraint:

$$c_t + i_t^d + q_t \Delta e_{t+1}^+ = r_t n_t + q_t \Delta e_{t+1}^-. \tag{3.3}$$

The right-hand side of (3.3) corresponds to the resources available to the entrepreneur. The first term is the return to equity holdings, where r_t is the return on equity and n_t is the amount of equity held by the entrepreneur. The second term on the right is the value of equity sales, $q_t \Delta e_{t+1}^-$, where q_t represents the price of equity. This term is the difference between the next period's stock of equity e_{t+1}^- and the nondepreciated fraction of equity owned in the current period λe_t^-. The entrepreneur uses these funds to consume c_t, to finance down payments for investment projects, i_t^d and to purchase outside equity $q_t \Delta e_{t+1}^+$. Each unit of e_t^- entitles other entrepreneurs to rights to the revenues generated by the entrepreneur's capital, and e_t^+ entitles the entrepreneur to revenues generated by other entrepreneurs. Then, the

[1] This assumption facilitates aggregation, as we explain below.

net equity for each entrepreneur and the law of motion for capital are given by

$$
\begin{aligned}
n_t &= k_t + e_t^+ - e_t^- \ , && (3.4) \\
k_{t+1} &= \lambda k_t + i_t \ , && (3.5)
\end{aligned}
$$

where i_t is the aggregate investment we define below. The sole difference between investors and savers is that the latter are not allowed to invest directly by using internal funds—they do not have an investment opportunity. That is, they are constrained to $i_t^d = 0$. Then, the law of motion for outside equity and issued equity is given by

$$
\begin{aligned}
e_{t+1}^+ &= \lambda e_t^+ + \Delta e_{t+1}^+, && (3.6) \\
e_{t+1}^- &= \lambda e_t^- + \Delta e_{t+1}^- + i_t^s,
\end{aligned}
$$

and we next describe i_t^s, which is the entitlements to others entrepreneurs (given by an investment contract).

Budget and financing constraints. When an investment opportunity is available, entrepreneurs choose a scale for an investment project, i_t. Projects increment their capital stock one for one with the size of the project. Each project is funded by a combination of internal funding, i^d, and external funds i_t^f. External funds are obtained by selling equity that entitles other entrepreneurs to the proceeds of the new project. Thus, the following accounting identity holds: $i_t = i_t^d + i_t^f$. The investment project i_t can be also be divided into final ownership i_t^i of the investor and the ownership of other entrepreneurs, i_t^s. That is,

$i_t = i_t^s + i_t^i$.

Since we assume that equity is homogeneous and equity markets are competitive, investors can raise $q_t i_t^s$ funds. Thus, external financing is equal to $i_t^f = q_t i_t^s$. Notice that at the end of the period, the investing entrepreneur increases his equity in $i_t^i = i_t - i_t^s$, while he has contributed only $i_t - q_t i_t^s$.

In addition to the accounting principles stated above, investment is subject to moral hazard: Investors can divert funds from projects. By diverting funds, they are able to increment their equity up to a fraction $1-\theta$ of the total investment. If they divert funds, the remainder of the project is lost. No enforcement or commitment technologies are available. Therefore, the incentive compatibility condition for external financing is equivalent to

$$(1-\theta)\, i_t \leq i_t^i \Rightarrow i_t^s \leq \theta i_t \ , \tag{3.7}$$

which means that outside equity holder's stake in the project may not be higher than θ. It is worth emphasizing that the distinction between inside and outside equity renders the KM model a q-theory of investment. The wedge occurs as a combination of two things: First, only a fraction of agents have access to investment opportunities, which generates a demand for outside equity. Limited enforcement causes the supply of outside equity to be limited by the incentive compatibility constraints. The value of q must adjust to equate demand with supply and this price may differ from 1.

In addition to the incentive compatibility constraint implied by moral hazard (3.7), there is a constraint on the sales of equity created in previous periods. Resellability constraints impose a limit the equity that can be sold at every period. These constraints depend on the liquidity shock ϕ_t

$$\Delta e^{-}_{t+1} - \Delta e^{+}_{t+1} \leq \lambda \phi_t n_t \ . \tag{3.8}$$

Kiyotaki and Moore (2012) motivates these constraints by adverse selection in the equity market. Bigio (2015) and Kurlat (2013) show that such a constraint can follow from adverse selection in the quality of assets. There are multiple alternative explanations for why liquidity may vary over the business cycle.What matters here is that liquidity shocks ϕ_t prevent equity markets from materializing gains from trade. Our focus is to study how shocks to liquidity can improve the predictions of business-cycle and asset-pricing theory without prying on where these shocks come from.

The investor's objective is to maximize her proceeds from total investment i^i_t by choosing a positive amount of i^s_t. We solve that program subject to the accounting identities $i_t = q_t i^s_t + i^d_t = i^i_t + i^s_t$, and, importantly, the constraint $i^s_t \leq \theta i_t$. Once we substitute the accounting identities, the investment decision —given by i^d_t— is characterized by

$$\begin{aligned} & \max_{\{i^s_t\}} i^d_t + (q_t - 1)\, i^s_t \ \ , \\ st.\ i^s_t \ \ & \leq \ \ \theta \left(i^d_t + q_t i^s_t \right) \ . \end{aligned} \tag{3.9}$$

Interpretation of this problem is clear. For every project, the investing entrepreneur increases her stock of equity $i^d_t + (q_t - 1)\, i^s_t$, which is the sum of the down payment plus the gains from selling equity corresponding to the new project, i^s_t. The constraint says that the amount of outside funding is limited by the incentive compatibility constraint. If $q_t \in \left(1, \frac{1}{\theta}\right)$, the problem is maximized at the points at which the incentive compatibility constraint binds. Therefore, within this price range, for every unit of investment i_t, the investing entrepreneur finances the amount $(1 - q_t\theta)$ units of consumption and owns the fraction $(1-\theta)$. This problem characterizes investors' resource constraint, which we state in the following proposition.

Proposition 12. *[Investors' resource constraint] When $q_t > 1$, we can re-write investors' budget constraint as follows*

$$c_{i,t} + q_t^R n_{t+1} = \left(r_t + \lambda q_t^i\right) n_t,$$

where $q_t^R = \frac{1-\theta q_t}{1-\theta}$ and $q_t^i = \phi_t q_t + (1-\phi_t)\, q_t^R$. When $q_t = 1$, the budget constraint is identical to that of savers, and is given by

$$c_t + q_t n_{t+1} = (r_t + \lambda q_t)\, n_t.$$

Proof. See appendix. □

3.2.3 Agents' problems

Workers. We assume that workers are hand to mouth, and therefore face a static problem: choosing how much to consume and work. Their preferences are

$$U_t\left(c_{w,t}, L_t\right) = c_{w,t} - \frac{\kappa}{1+\nu} L_t^{1+\nu} \ , \tag{3.10}$$

and the budget constraint is just $c_{w,t} = w_t L_t^s$, where L_t^s is the supply of labor chosen to maximize U_t.

Savers. For savers, the problem consists of maximizing the value function (3.2) by choosing the level of consumption and equity

$$\begin{aligned} V_{s,t} &= \max_{c_{s,t},n_{s,t+1}} \left[c_{s,t}^{\rho} + \beta E_t \left[V_{s,t+1}^{1-\gamma} \right]^{\frac{\rho}{1-\gamma}} \right]^{\frac{1}{\rho}} \qquad (3.11) \\ & s.t. \\ c_{s,t} + q_t n_{s,t+1} &= (r_t + q_t \lambda) n_t \equiv \omega_{s,t}. \end{aligned}$$

Investors. Similarly, investors maximize their value function, but their investment opportunity set is different from that of savers because their budget constraint encodes the optimal investment decision through the price q_t^i.

$$\begin{aligned} V_{i,t} &= \max_{i_t,c_{i,t},n_{i,t+1}} \left[c_{i,t}^{\rho} + \beta E_t \left[V_{i,t+1}^{1-\gamma} \right]^{\frac{\rho}{1-\gamma}} \right]^{\frac{1}{\rho}} \qquad (3.12) \\ & s.t. \\ c_{i,t} + q_t^R n_{i,t+1} &= \left(r_t + q_t^i \lambda \right) n_t \equiv \omega_{i,t}. \end{aligned}$$

We next define the equilibrium and elaborate on the model's solution.

3.2.4 Equilibrium

In this section, we first define equilibrium and characterize the optimality conditions for each class of agents. We then provide a decomposition of the returns on wealth.

Definition 13. *[Recursive competitive equilibrium] A recursive competitive equilibrium is a set of price functions* $\{q, w, r\}$*, real quantities* c_j, n'_j, i_j, L *for* $j = s, i$*, a sequence of distribution of individual equity holdings* Λ_t*, and a transition (law of motion) for the aggregate state of the economy* Ξ*, such that: (1) given prices, optimal policies solve agents problems (i.e., those of firms, workers and entrepreneurs); (2) goods market clears; (3) labor market clears at price* w*; (4) equity market clears at price* q*; (5) aggregate capital evolves as* $K' = \lambda K + I$*; and (6) both* Λ *and* Ξ *are consistent with the policy functions that solve agents' problems.*

Notice that in the above definition, the distribution of equity holdings, Λ_t, is a relevant state at an individual level but does not determine aggregate variables. In other words, the distribution of equity is not an object in the aggregate state space. Intuitively, this is because of the assumption that investment opportunities are i.i.d across investors.

Worker's policy functions. The maximization of (3.10) is standard and yields

$$L_t^s = \left[\frac{w_t}{\kappa}\right]^{\frac{1}{\nu}} ,$$

where L^s stands for labor supply. Next, we characterize firms' decisions and equilibrium prices (w_t, r_t).

Firms' equilibrium and equilibrium prices. From firms' maximization problem (3.1), labor demand is given by

$$L_t^d = \left[\frac{(1-\alpha) A_t}{w_t}\right]^{\frac{1}{\alpha}} k_t ,$$

thus, using equilibrium $L_t^d = L_t^s$, the equilibrium real wage is

$$\kappa^{\frac{\alpha}{\alpha+v}} \left[(1-\alpha) A_t\right]^{\frac{v}{\alpha+v}} k_t^{\frac{\alpha v}{\alpha+v}} = w_t^*.$$

Lastly, define the equilibrium return on capital, using firms' first-order conditions:

$$r_t^* = \alpha A_t \left[\frac{(1-\alpha) A_t}{\kappa}\right]^{\frac{(1-\alpha)}{\alpha+v}} k_t^{\frac{(\alpha-1)\nu}{v+\alpha}}. \tag{3.13}$$

Next, after solving firms' and workers' optimal policies, we characterize the solution for entrepreneurs.

Entrepreneurs. We define the right-hand side of savers' and investors' budget constraints as $\omega_t^s \equiv (r_t + q_t\lambda)\, n_t^s$ and $\omega_t^i = (r_t + q_t^i\lambda)\, n_t^i$. From (3.2), we obtain the intertemporal marginal rate of substitution given by

$$M_{t,t+1}^j = \beta \left(\frac{c_{j,t+1}}{c_{j,t}}\right)^{\rho-1} \left(\frac{V_{j,t+1}}{\mu_{j,t}}\right)^{1-\gamma-\rho},$$

and therefore we need to solve for the value function. We proceed with a guess and verify strategy. The next proposition characterizes optimal policies for investors and savers:

Proposition 14. *[Policy functions] (i) Savers' optimal decision rules are*

$$\frac{c_{s,t}}{\omega_{s,t}} = 1 - \xi_{s,t}\ , \tag{3.14}$$

$$q_t n_{s,t+1} = \xi_{s,t}\omega_{s,t}\ . \tag{3.15}$$

(ii) Investors' optimal decision rules are, when $q_t > 1$

$$\frac{c_{i,t}}{\omega_{i,t}} = 1 - \xi_{i,t} \ , \tag{3.16}$$

$$q_t^R n_{i,t+1} = \xi_{i,t}\omega_{i,t} \ , \tag{3.17}$$

$$i_{t+1} = \frac{n_{i,t+1} + (\phi_t - 1)\,\lambda n_t}{(1-\theta)} \ , \tag{3.18}$$

where $\xi_{s,t}$ and $\xi_{i,t}$ are unknown functions of the aggregate state of the economy that must be solved using equilibrium conditions. When $q_t = 1$, savers' and investors' policies are identical.

Proof. See appendix. □

Returns. Following KM, we define the returns on wealth according to the alternative entrepreneurs' investment opportunity set.

$$R_{t+1}^{ss} = \frac{r_{t+1} + q_{t+1}\lambda}{q_t}, \ R_{t+1}^{is} = \frac{r_{t+1} + q_{t+1}\lambda}{q_t^R}, \ R_{t+1}^{ii} = \frac{r_{t+1} + q_{t+1}^i\lambda}{q_t^R}; \ R_{t+1}^{si} = \frac{r_{t+1} + q_{t+1}^i\lambda}{q_t}.$$

Here, R_{t+1}^{ss} is the return on wealth obtained by a saver who continues as a saver in the next period. In this case, the saver obtains—after holding $1/q_t$ units of equity—the net return on capital r_{t+1} plus the undepreciated portion of equity —valued at the spot price q_{t+1}. Similarly, R^{si} represents the return for a saver who receives an investment opportunity in the next period; recall this occurs with probability π. In this case, however, the undepreciated equity is valued at q_{t+1}^i, since he will face an investment opportunity and thus a weighted price: A fraction ϕ is non-resellable —valued at q_t^R— and a fraction $1-\phi$ is resaleable at the spot price q_t. Both R^{is} and R^{ii} follow the same logic.

From here, we can decompose the conditional return on wealth for investors and savers. In particular, we compute the conditional return on wealth faced by entrepreneurs and decompose it between a "market value" that corresponds to fluctuations in market prices and a "liquidity component" that arises from the frictions in the model. The idea is to express all of the returns as a spread relative to a benchmark market return, R^{ss}_{t+1}. The return R^{ss}_{t+1} is the appropriate benchmark, because it is the market valuation of the future flow of capital dividends.

Let R^{i}_{t+1} be the investor's return on wealth conditional on his individual state. In the next proposition, we characterize a decomposition of this return into a "market component" and a "liquidity component"

Proposition 15. *[Decomposition] The investor's return on wealth R^{i}_{t+1} can be decomposed as*

$$R^{i}_{t+1} = \frac{l_{t+1}}{q^R_t} + \eta_t R^{ss}_{t+1} \ , \tag{3.19}$$

where $l_{t+1} = -\pi \frac{\lambda\left(q_{t+1} - q^R_{t+1}\right)(1-\phi_{t+1})}{q^R_t}$ *and* $\eta_t = \left(\frac{(1-\theta)q_t}{1-\theta q_t}\right)\left(\equiv \frac{q_t}{q^R_t}\right)$. *Also , the return on wealth faced by a saver, R^{s}_{t+1}, can be decomposed as*

$$R^{s}_{t+1} = \frac{l_{t+1}}{q_t} + R^{ss}_{s} \tag{3.20}$$

Proof. See appendix. □

We define the intercept term $l_{t+1} \leq 0$ as the "liquidity component", which shows up both in (3.19) and (3.20), and captures the following intuition: There is a probability π that the entrepreneur—regardless of his initial type—will become an investor in the next period. In such a case, a fraction $(1-\phi)$ of the λ remaining equity becomes illiquid. Thus, if he becomes an investor, his internal valuation of this illiquid equity is q_{t+1}^R instead of the spot price q_{t+1}; the entrepreneur could have replaced that unit at price q_{t+1}^R. Since he values that illiquid portion at price q_{t+1}^R instead of q_{t+1}, a fraction $(1-\phi)$ loses $\left(q_{t+1}-q_{t+1}^R\right)$ with probability π relative to the benchmark return. In either formula, the term l_{t+1} is scaled by either $1/q_t^R$ or $1/q_t$ depending on which was the initial cost of capital—for investors and savers, respectively. The term $\eta_t \geq 1$ in (3.19) is a multiplier on the market return over the investor's portfolio. This term simply captures the idea that for this entrepreneur, it costs him q_t^R to build the capital that costs q_t in the market. The greater the discrepancy, the higher $q_t > 1$, the greater this multiplier. As we explained earlier, the internal cost for the investor $q_t^R \leq 1 \leq q_t$ captures the idea that the investor is effectively exploiting an arbitrage.

This concludes our specification of the model environment, definition of the equilibrium concept, and characterization of the optimal policies. We now briefly discuss about the role of liquidity in the model, then we we investigate the quantitative properties of the model.

3.2.5 Discussion

The optimal financing problem subject to the enforcement constraints is essentially static; hence, we can use a static analysis to understand the effects of liquidity shocks. We illustrate the model's main mechanism using this approach.

Whenever $q_t > 1$, external financing allows investing entrepreneurs to arbitrage. In such instances, the entrepreneur wants to finance investment projects as much as possible. We have already shown that when constraints are binding, the entrepreneur owns the fraction $(1 - q_t)$ of the investment, but finances only $(1 - \theta q_t)$. If he uses less external financing, he misses the opportunity to obtain more equity. To gain intuition, it is convenient to abstract from the consumption decision and assume that the investor entrepreneur uses $x_t \equiv q_t \phi_t \lambda n_{i,t}$ to finance the down payment. Thus, $i_t^d = x_t$.

The constraints impose a restriction on the amount of external equity that may be issued, $i_t^s \leq \theta i_t$. External financing is obtained by selling i_t^s equity at a price q_t, so the amount of external funds for the project is $i_t^f = q_t i_t^s$. Since $i_t = i_t^f + i_t^d$, external financing satisfies

$$i_t^f \leq q_t \theta \left(i_t^f + x_t \right). \tag{3.21}$$

Figure 3.1 illustrates the simple intuition behind the liquidity channel—i.e., how changes in the amount x_t of sales of equity corresponding to older projects affect investment by restricting the amount of external financing for new projects. Panel (a) in Figure 3.1 plots the right- and left-hand sides of restriction on outside funding given by inequality (8). Outside funding, i_t^f, is restricted to lie in the area where the affine function is above the 45-degree line. Since $q_t > 1$, the left panel shows that the liquidity constraint imposes a cap on the capacity to raise funds to finance the project. Panel (b) shows the effects of a decreases in x_t without considering any price effects. The fall in the down payment reduces the intercept of the function defined by the right-hand side of Figure 3.1. External funding, therefore, falls together with the whole scale of the project. Since investment falls, the price of q must rise such that the demand for saving instruments falls to match the fall in supply. The increase in the price of equity implies that the amount financed externally is higher. The effect on the price increases the slope and intercept, which partially counterbalances the original effect. This effect is captured in panel (c) of Figure 3.1.

3.3 Quantitative analysis

In this section, we explore the models quantitative predictions by solving it numerically using a global solution method. In the following seven subsections, we first describe the calibration, then perform several exercises to shed light on all of the models properties.

Calibration. We divide parameters into three categories and fix the time period to a quarter. Table 3.1 reports the list of parameters and values. We set the capital share to α =0.36 and the depreciation rate 1-λ = 0.025; these two numbers are well established in the RBC literature. The arrival of investment opportunities π is set to 0.012, a slightly greater number than that of Del Negro et al. (2016). Using π = 0.009 as in their paper, implies that the economy spends considerable time in the constrained region—i.e., the enforcement constraint almost always binds. Since they solve their model by log-linearizing around a steady state in which financial constraints bind, it is reasonable for them to use a small value for π. In contrast, we choose a π that is close to theirs, but with this value the economy also spends some time in the unconstrained equilibrium. In other words, we exploit the global solution method to study an economy in which constraints occasionally bind. The calibrated π allows us to match the observed fact the vast majority of firms do not exhibit investment rates greater than 20%.[2] Next, we calibrate θ following the extensive literature that analyzes financing constraints à-la KM. We set a value of 0.77—very similar to that of Del Negro et al. (2016), who use 0.792. This implies that the upper bound for the spot price of equity q_t is $1/\theta \approx 1.3$. Notice that this parametrization satisfies $0 \leq \theta \leq 1 - \pi = 0.988$.

In terms of preferences, we calibrate β to 0.99 to match an annual real interest rate of around 1.7% at an annual frequency. The EIS and the risk-aversion parameters are from the asset-pricing literature, primarily following Bansal and Yaron (2004a). In particular, we calibrate $\gamma = 10 > \psi = 1.5 > 1$, which implies that agents feature a preference for early resolution for uncertainty. We explore the implications for the EIS below. Lastly, the Frisch elasticity $(1/v)$ is set to 1, which is commonly used in the RBC literature and is in range of the macro estimates of Chetty et al. (2011), and the parameter κ is calibrated to obtain an average 30% of working hours.

Finally, in terms of aggregate shocks, we first calibrate $\rho_A = 0.95$ and $\sigma_A = 0.01$, again following the RBC literature and the behavior of US productivity quarterly time series. The support of liquidity shocks (i.e., the maximum and minimum values for ϕ) is set to $[\phi_L, \phi_H] = [0.05, 0.5]$, which yields an unconditional mean liquidity of 0.275. This is a smaller value than that of Del Negro et al. (2016), who calibrate a steady-state level for liquidity at 0.4. However, it is worth emphasizing that the concept of liquidity in their paper differs from KM: They introduce a positive supply of government bonds to their analysis, and they calibrate the steady-state level of liquidity to match the convenience yield in the data. In this

[2] Cooper et al. (1999) show that, using the US longitudinal survey (which covers US manufacturing industries, but accounts for only about 45% of aggregate investment), 87% of firms display investment rates lower than 23%. The calibrated $\pi = 0.012$ implies that 6% of firms have investment rates between 17% and 28%.

paper, instead, we choose a lower unconditional mean of liquidity, since we do not introduce government bonds. The last parameter for the aggregate shocks is the correlation between aggregate productivity A and liquidity ϕ, which we set to 0.75. Bigio (2015) and Kurlat (2013) study the relationship between liquidity and aggregate productivity when the former is determined by the solution to a problem of asymmetric information. Both papers provide theoretical results in which the relation between productivity and liquidity is positive. It is worth emphasizing that quantitative results are not sensibly affected by this correlation.

[Table 3.1 about here: Calibration]

We finish the calibration section with the following observation. We purposefully avoid calibrating the three key parameters of KM's framework (i.e. $\theta, \pi, E[\phi]$) to match asset-pricing or business-cycle dynamics. Instead, we adopt the practice of setting parameters to reasonable micro evidence and evaluate the macroeconomic performance. This allows us to do a comprehensive assessment of KM without departing from previous results in the literature.

Equilibrium through the state space. We start with a description of the behavior of policy functions and endogenous variables across the state space. All figures display the stock of capital in the horizontal axis. Upper panels display the variable of interest in the 20% percentile of the productivity level ("low productivity states"), while lower panels display the 80% percentile ("high productivity states"). Similarly, left and right panels display the 20% and 80% percentiles of the liquidity level, respectively.

Figure 3.3 illustrates the spot price of equity q, the internal cost of capital for investors q^R, and their liquidity-weighted price q^i. The figure shows the inverse relationship between the spot price of capital and the internal cost. The logic, as explained above, is that investors own a fraction $1-\theta$ of final investment whereas they only put $1-q\theta$ of the funds in that project. This means that as q increases, they fund a smaller share of their projects, because they can sell claims to investment at a higher price. The figure then explains how q varies with TFP, the capital stock, and liquidity. The lower the capital stock, the higher the returns on capital. Therefore, when capital is low, savers have a high demand for investment, so for a fixed level of liquidity, the price of capital must be high to clear the market. Similarly, Figure 3.3 shows that the higher the productivity, the higher the q. In contrast, when liquidity is low, holding the return to capital fixed, the supply of investment projects is scarce. This is why the panels on the left-hand side of the figure show how lower liquidity increases q. This is an important feature of the model, and we explain below how it determines risk-premium dynamics.

We next describe the policy functions for the consumption-wealth ratio. As shown in (3.16) and (3.14), these policy functions are associated with each agent's marginal valuation of capital. When the EIS is greater than one, agents save more when returns are higher—that is, the intertemporal substitution effect dominates the wealth effect. This is the case in our calibration, and this force dominates the behavior of the marginal propensity to consume. Figure 3.4 shows that the consumption-wealth ratio is an increasing function of capital since the return to capital is decreasing in the capital stock[3]. The same is true when productivity is relatively low: Returns are low, so the propensity to consume is high. However, the effect

[3] I.e., decreasing returns to scale in capital.

of liquidity is different. When liquidity is abundant, the price of equity is low. In that case, savers and investors feature similar consumption-wealth ratios. In contrast, when liquidity is scarce, investors and savers face different effective prices of capital. For the saver, capital is expensive and he faces low returns; as a result, he consumes a higher portion of his wealth. For the investor, on the other hand, this is time to increase their investment rate and exploit their arbitrage opportunities.[4]

We now study investment dynamics. Figure 3.5 displays the investment rate (I/Y) throughout the state space. As we did with the previous policy functions, we first observe that the investment rate is decreasing in the capital stock dimension. This feature depends on the EIS: Under the calibration in which EIS>1, agents invest at lower rates when returns are lower. (We extend this logic below, when we analyze the role of the EIS in the model.) On the other hand, both productivity and liquidity operate in the intuitive direction: The lower the productivity/liquidity the lower the investment rate. Thus, negative liquidity shocks depress investment rates. However, we can readily observe from this figure that the effect of a liquidity shock is not quantitatively important. We explore this feature further when we study the impulse-response dynamics of a liquidity shock.

Equity returns and liquidity. Next, we focus on equity return dynamics. In particular, we study the return on equity and its decomposition into a market return, a liquidity factor, and an amplification term. We start by analyzing the market return on equity—namely $E_t\left[\left(r_{t+1}+\lambda q_{t+1}\right)/q_t\right]$. By constructing this return, we can study the equity premium in the

[4]There is a point at which productivity is sufficiently low, returns on equity are sufficiently low and the economy operates under slack constraints. At such a point—and at lower levels of productivity—consumption policies are equivalent for savers and investors.

KM model and contrast it with previous asset-pricing literature.

Expected returns are the sum of expected dividend yield, $E_t\left[r_{t+1}/q_t\right]$, and expected capital gains, $\lambda E_t\left[q_{t+1}/q_t\right]$. Figure 3.6 displays these three objects. With respect to capital, there is a nonmonotonic relationship in total expected returns on equity. In principle, both the dividend yield and capital gains could explain this nonmonotonic relationship, because both $E_t\left[r_{t+1}\right]$ and $E_t\left[q_t\right]$ are decreasing in the capital stock—yet, the numerator term $E_t\left[r_{t+1}\right]$ dominates.[5] Hence, the overall nonmonotonic relationship between total expected returns and capital is explained by capital gains. This is because as the economy accumulates capital, the liquidity constraint is less likely to bind, and hence q_t approaches 1. Therefore, the term $E_t\left[q_{t+1}/q_t\right]$ increases, because prices can only increase as $q_t \downarrow 1$. In other words, the direction of future change in prices is only upwards; q is below its mean and expected to increase. In contrast, when capital is low, the price of equity is high. However, the economy is expected to accumulate capital, and thus, its price is expected to fall. Figure 3.6 also shows that expected returns are lower when productivity is higher. This effect is unambiguous: It appears in both expected capital gains and dividend yields. But this nonmonotonic relationship of expected capital gains with respect to capital depends on the level of productivity. This is simply because different levels of productivity affect the probability that constraints bind, which ultimately dictates future changes in q. Notice that although expected returns can exhibit negative values throughout the state space, the unconditional mean is 3.7% in annual terms.[6]

Importantly, liquidity affects expected capital gains—and therefore the expected return

[5] Dividend yields are decreasing in capital, simply due to decreasing returns to scale in the stock of capital.

[6] We elaborate on the quantitative properties below.

on equity. In particular, the higher the liquidity, the higher the expected capital gains. This feature is a key counterfactual element in the model: We usually think of periods of abundant liquidity as tranquil times, during which expected returns on risky assets are low. This is not the working logic in the KM model, however, in which tranquil times are associated with lower levels of q—and thus expected changes in q are higher. This is a crucial point in this model, which also dominates equity premium dynamics, as we discuss below.

We next discuss the liquidity component, $E_t\left[l_{t+1}/q_t^R\right]$, of returns and the amplification (η_t) that arise in an investor's return on wealth, as demonstrated in proposition 4 (equations 3.19 and 3.20). Figure 3.7 shows that the liquidity component is decreasing in both the stock of capital and in the level of productivity. As mentioned above, this last feature is in line with the logic proposed by Bigio (2015) and Kurlat (2013). We find that, conditional on a level of aggregate liquidity, the liquidity component of returns is smaller (in absolute value) when productivity is low. However, notice that this component is relatively stable across the state space, as opposed to the "market" component described above. Also, the wedge is decreasing in capital due to decreasing returns to scale: The higher the level of capital, the lower the returns on capital; this implies that endogenous responses to the price of equity are less pronounced. Thus, the endogenous wedge between q_t and q_t^R is smaller, which decreases $E_t\left[l_{t+1}\right]$.

Lastly, Figure 3.8 shows the amplification term $\eta_t = q_t/q_t^R$. This term is useful, because it illustrates the segments of the state space where the economy is constrained. The intuition for the behavior of η_t is very similar than to the intuition for q_t since q_t ultimately characterizes the behavior of q_t^R—and therefore the ratio q_t/q_t^R. Thus, the idea of η_t being decreasing in capital is also similar; again, it follows due to decreasing returns to scale—i.e., returns on capital are lower when the level of capital is higher. Also, by definition, the economy is unconstrained in states in which $\eta_t = 1$—i.e., when $q_t = 1$—, and the level of $\eta_t > 1$ gives a sense of proportion between q_t and q_t^R. In fact, this proportion depends on the value of θ, the parameter that governs the enforcement constraint. The kink observed in the right panels of Figure 3.8 is the precise point at which the constraint binds. Observing this feature is one of the advantages of solving the model using global solution methods and of understanding the state space of an economy with occasionally binding constraints—as in recent macro models such as Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013).

The model does not explicitly introduce a one-period risk-free bond. However, we can price such an asset by using the savers' stochastic discount factor, which would be the marginal agent pricing it[7]

$$\frac{1}{R_{f,t}} = E_t\left[M^s_{t,t+1}\right]. \tag{3.22}$$

Figure 3.9 shows that liquidity has overall negligible effects on the implied real interest. However, it does fluctuate with capital and productivity. Those fluctuations are similar to that of the neoclassical framework. That is, when the level of capital is high, the economy is expected to have a lower growth rate. Lower expected growth in consumption implies that real rates are lower in order to clear the goods market. Productivity level is simply a scaling effect for this intuition: The higher the productivity level, the higher the growth rate—and therefore also the risk-free interest rate. As shown in the figure, interest rates

[7]Investors are not the marginal agent who prices a riskless asset, because they have the opportunity to invest in capital. Thus, $E\left[M^s\right] > E\left[M^i\right]$ always.

could be negative when productivity is low. However, the unconditional mean is 1.7% in annual terms, as we report in Table 3.2 and 3.3 (described below).

Lastly, we compute the (log) equity premium to evaluate whether the model can reproduce sizable premiums, in line with the empirical evidence. In particular, we compute

$$rp_t = E_t \left[\log \left(\frac{r_{t+1} + \lambda q_{t+1}}{q_t} \right) - \log \left(R_{f,t} \right) \right]. \tag{3.23}$$

We express the premium in logs to make it comparable with the literature (see below), and present the results in figure 3.10. The equity premium inherits the behavior of expected returns—which, as explained above, are dictated by expected capital gains in the price of equity. The evidence suggests that premiums are usually countercyclical: high in bad times and low in good times. In Figure 3.10, we observe that premiums obey this logic with respect to capital and productivity. In other words, in good time—i.e., high levels of capital and productivity—expected returns are low. This is not the case, however, for liquidity. As mentioned above, higher levels of aggregate liquidity imply lower prices and higher expected capital gains and, therefore, premiums. Interestingly, the figure shows the nonmonotonic relationship around the point at which constraints binds. This is a particular feature of this model relative to the frictionless economy. A similar body of literature that studies how balance-sheet dynamics affect risk premia also predicts a non-monotonic relationship between excess returns and the distribution of wealth.[8] This feature allows the framework to capture the substantial volatility of excess returns—a feature well documented in the data

[8] Brunnermeier and Sannikov (2016) provide a comprehensive review.

and one that the standard frictionless RBC economy cannot account for.

At this point, it is useful to further study the role of ψ and its implications over excess returns. This is useful, because in leading asset-pricing models the signs of risk prices are quite sensible to the parametrization of ψ (all other things equal[9]). When the EIS is less than one, the consumption-wealth ratio is a decreasing function of capital. This translates into higher investment rates in periods when capital is high. Thus, on average, prices for equity are higher, and constraints are more likely to bind. For example, with ψ=0.5—and keeping other parameters equal—constraints are very likely to bind throughout the state space. That is, $q > 1$ almost always. Also, as is well known, the lower the EIS ψ, the higher the mean risk-free rate.

Furthermore, when the EIS is lower than one, the equity premium is higher the greater the level of capital. This is simply because the sensitivity of the interest rate with respect to the capital stock is higher, and therefore an increase in the capital stock reduces the risk-free interest rate relatively more than the expected return on equity. Thus, expected excess returns are higher. It is worth emphasizing that the relationship with respect to liquidity and productivity is the same as in the case of ψ greater than one.

In conclusion, the model offers a counter-factual result in terms of the impact of liquidity on expected returns: Liquidity carries the wrong sign in the price of liquidity risks. This result is independent of the EIS, and it fundamentally depends on expected fluctuations in the price of equity. The model can reproduce a higher mean excess return and a considerably higher volatility relative to the frictionless benchmark. We next assess the quantitative properties

[9]For example, Bansal and Yaron (2004a) show that when ψ is less than one, the price of volatility shocks changes the sign.

of business cycles.

Business-cycle properties. We now study the quantitative properties of the business cycle predicted by the model. To do so, we follow both the RBC literature and the production-based asset pricing literature—Jermann (1998), Campanale et al. (2010), Croce (2014), among others—and compute key asset pricing and business cycle moments.

[Table 3.2 about here: Asset prices and business cycles]

Table 3.2 displays the results and the comparison to two other well-known models in the production-based asset pricing literature. The model predicts fluctuations of investment almost 2.5 times more volatile than output, which is much more than the 1.46 predicted by the frictionless RBC. This is not that far from what we observe in the data, especially taking into account that our model does not exhibit adjustment costs to investment, as in Jermann (1998) or Croce (2014). It is well known that this statistic could be easily matched by adding such costs to investment, but we purposefully abstract from those to focus strictly on the q-theory mechanism embedded in the KM framework.

We also see that the unconditional mean for the risk-free interest rate is above the one observed in the data, but definitely smaller than the one predicted by the frictionless benchmark.[10] In the other models we compare with, the low interest rates is driven by the introduction of a small persistent component into the expected growth rates (Croce (2014)) or the calibrated process for the habits in preferences (Jermann (1998)).

The unconditional expected excess return on equity is 1.05% per annum. This magnitude is

[10] Although we could reduce the mean risk free rate by increasing β, we keep it at 0.99 to improve the speed of convergence of the algorithm.

considerably smaller than the one observed in the data. However, the model predicts a sizable amount of variability in equity returns—and thus in expected excess returns. Although the standard deviation we obtain is not as high as in the data, it is a considerable improvement relative to the frictionless case in which variability of both returns and expected returns is very small. Although the model predicts substantial variation, it is important to bear in mind that the directions of such fluctuations are counterfactual to the evidence, as explained in our description of expected excess returns over the state space.

Lastly, in Figure 3.2, we show the empirical distribution of excess returns, both in the data and in the model. This figure complements Table 3.2. It is evident that excess returns in the data display larger fluctuations than in the model; that is, excess returns are more volatile than in the model. One potential avenue to address this is to introduce exogenous second-order shocks to productivity (see Schneider (2017)). In this way, fluctuations in premiums would be attributed in part to the endogenous component driven by occasionally binding constraints, and also to exogenous second order shocks. It is useful to stress that the case of the frictionless case RBC predicts a degenerate distribution very close to zero.

Decomposition. We next study the decomposition of returns. As established by (3.19) and (3.20), returns on wealth can be decomposed into three terms: an intercept (or liquidity component), a slope (or amplification component) and the market return on equity. In Table 3.3, we report the moments for each of these variables. We use the expression $E[E_t(x_{t+1})]$ for a given variable x because we first compute the conditional expectation across the state space, using the Markov transition probabilities, and then compute the unconditional expectation using the corresponding distribution—i.e. unconditional or conditional distribution.[11] The appendix contains a detailed description of how we compute the unconditional and conditional distributions.

[Table 3.3 about here: Decomposition]

As shown in Table 3.3, the compensation for liquidity is negative, and relatively smaller than the pure market return on equity (in absolute terms). Intuitively, this is because there is a strictly positive probability (π) that a fraction $(1-\phi)$ of the equity will be valued at a price (q^R) that is inferior to that of the spot price (q). This is always the case, since there is always a probability of switching between constrained and unconstrained parts of the state space. In fact, the system spends 85% of the time in the constrained part of the state space.

Table 3.3 illustrates the key counter-factual element of the model: The conditionally constrained expected return on equity is considerably lower than the conditionally unconstrained one. This is is counterfactual, because we usually think that constraints are slack in tranquil times —i.e., when premiums are low. However, the volatility of equity returns is higher in the constrained part of the state space. Thus, when financial constraints bind, returns on equity are on average more volatile. This last aspect of the model is promising, since it can potentially contribute to explaining the observed large fluctuations in excess returns.

The average effect of liquidity is notably similar across the state space—i.e., it does not change significantly between constrained and unconstrained state space. Qualitatively, both $E\left[E_t\left(l_{t+1}\right)\right]/q_t^R$ and $E\left[E_t\left(l_{t+1}\right)\right]/q_t$ are on average higher, in absolute terms, when the econ-

[11] We would have used the law of iterated expectations if we had only included the unconditional column.

omy is constrained. However, this difference is quantitatively mild. In other words, the contribution of liquidity to total returns on wealth is relatively small and similar across the state space.

The last element to analyze is the amplification factor η. From Table 3.3, we see that the unconditional η is 1.94, while the conditionally constrained mean is somewhat higher at 2.07. These values for η imply a somewhat unrealistic expected return on wealth for investors, since η multiplies the gross market return and does not affect the intercept—i.e., the liquidity component. Thus, we see that the expected return on investor's wealth is

$$E\left[\frac{E_t\left(l_{t+1}\right)}{q_t^R}+\eta_t E_t\left(\frac{r_{t+1}+q_{t+1}}{q_t}\right)\right]=1.95$$

Although we don't have benchmark data to compare this number with, it seems to be relatively large. This is because η_t works as an amplification of gross returns.

Impulse response functions. We next examine the economy's to a liquidity shock. We compute impulse-response functions (IRFs) using a simulation algorithm described in the appendix. In Figure 3.12, we show the evolution of the economy after a relatively big liquidity shock. The shock reduces liquidity from $\phi_t = 0.275$—the steady-state level—, to $\phi_t = 0.05$; these results are in line with the intuition we developed above. We find that the response in output is a drop of 0.12% (approximately 0.5% annualized). This is mainly explained by a drop of 9.5% in the level of investment or, equivalently a 9.8% drop in the *investment rate*: After the shock, the economy moves from a 23% investment rate to a 20.5% rate.

Both hours and wages drop, although consumption increases after the shock. The logic is that the shock affects the supply of capital, which reduces investment and, therefore, capital accumulation. Output also drops, but since we have a closed economy with flexible prices, consumption must increase to clear the goods market.

After the liquidity shock, we observe that the probability that constraints will bind increases up to one. The price of equity, q_t, increases about 15% above it's mean. The expected excess return, however, drops around 10% from it's mean. This is the main counterfactual element we highlighted above: Periods of scarce liquidity are associated with lower expected excess returns. As noted above, this incorrect sign in the price for liquidity risk is not associated with the entrepreneur's EIS. Instead, it is embedded in the logic that liquidity shocks affect the supply of equity without affecting its demand. As a result, the price for risky equity increases, and investors expect that it will decrease in the near future.

Theoretical moments and invariant distribution. We conclude our quantitative analysis by analyzing the theoretical moments of endogenous variables. Table 3.4 shows the theoretical moments. We complement this table with Figure 3.11. We provide a detailed discussion in the appendix of how we compute the invariant distribution, but the idea is to solve an eigenvalue problem associated with the Markov chain. In the table, we report moments computed with three measures: unconditional, conditionally constrained, and conditionally unconstrained. We observe that capital, investment and output have lower means under the conditionally constrained distribution. Also, the conditionally constrained second moments of capital, output, and investment are relatively higher than the unconditionally constrained moments. This is consistent with the intuition we elaborate on throughout the text.

[Table 3.4 about here: Theoretical moments]

Interestingly, the skewness and kurtosis of the endogenous variables inform us that the conditionally unconstrained distribution is negatively skewed and displays higher levels of kurtosis. The intuition is that the unconstrained state is transitory; indeed, in our calibration the economy spends 87% of the time in the constrained state. Figure 3.11 illustrates a similar intuition and is complementary to Table 3.4. At first blush, we can see that the conditionally constrained distribution assigns smaller measures to relatively higher levels of capital.

3.4 Conclusion

In this paper, we propose a stripped-down version of KM. Our objective is to evaluate the quantitative predictions of liquidity shocks through the lens of asset prices and business cycles. Our main conclusion is that liquidity shocks, coupled with financing constraints, are a promising avenue of research that can potentially capture highly nonlinear risk premium dynamics.

However, at the core of the KM mechanism, we find that negative liquidity shocks increase the price of equity, and this translates into risk premium dynamics. Our results complement and reinforce other findings in the literature (e.g., Shi (2015)). We extend our analysis to study risk premium dynamics and highlight that although fluctuations in this variable are considerably more realistic than in the frictionless benchmark, the directions of those fluctuations are at odds with the evidence.

Finally, we observe that the fact that liquidity shocks operate solely on the "supply side" is a major restriction of this setup. A potential avenue of future research could shed light on how liquidity affects the demand side of the credit market—that is, contractions in the demand for credit would be associated with higher expected excess returns and lower asset prices.

3.5 Tables and Figures

Table 3.1: Calibration

Parameters (quarterly)			
1. *Technology*		**Value**	**Description**
	α	0.36	capital share
	λ	0.975	1-depreciation
	π	0.012	investment opportunities
	θ	0.77	borrowing constraint
2. *Preferences*			
	β	0.99	time preference
	γ	7.5	risk aversion
	ψ	1.5	EIS
	ν	1	Frisch elasticity
	κ	8.5	scaling hours
3. *Aggregate shocks*			
	ρ_A	0.95	AR(1) productivity
	σ_A	0.01	standard deviation $\log(A_t)$
	φ	0.75	$\text{corr}(A, \phi)$
	ϕ_L	0.05	minimum level of liquidity
	ϕ_H	0.5	maximum level of liquidity

NOTES: The table shows the model calibration at a quarterly frequency. We describe the procedure in the main text.

Table 3.2: Asset prices and business cycles

Model/moments	$\sigma_{\Delta c}/\sigma_{\Delta y}$	$\sigma_{\Delta i}/\sigma_{\Delta y}$	E[log $R_{f,t}$]	E[log R^e − log $R_{f,t}$]	std(log $R_{f,t}$)	std(log R^e)
KM	0.91	2.40	1.72	1.05	0.89	9.32
Frictionless	0.84	1.46	4.01	0.01	0.26	0.29
Croce2014	0.81	3.61	0.94	5.25	0.94	12.47
Jerman1998	0.49	2.64	0.82	6.18	11.46	19.86
Data	0.71	4.49	0.65	4.71	1.86	20.89

NOTES: This table shows the main business cycle statistics. We compute the moments by simulating the model for 500,000 periods (we burn 10,000 periods). $\sigma_{\Delta j}/\sigma_{\Delta y}$ is the standard deviation of j=consumption, investment growth over output growth. KM is the model presented in the main text. We also report theoretical moments for $E[R_f]$ and $E[R^e]$ in table 3.3. Frictionless is the solution when there is neither incentive compatibility nor resellability constraints. Jermann1998 is (Jermann, 1998) Table 1, Benchmark Model (habits+adjustment costs to investment). Croce14 is Croce (2014) Table 3, column 3 (EIS>1). Data is from Croce (2014), which is annual data from 1929-2008.

Table 3.3: Decomposition

Variable	Unconditional		Constrained		Unconstrained	
	Mean	St.dev.	Mean	St. Dev	Mean	St. dev.
$E\left[E_t\left(l_{t+1}\right)\right]/q_t^R$	-0.0079	0.0084	-0.0088	0.0086	-0.0021	0.0020
$E\left[E_t\left(l_{t+1}\right)\right]/q_t$	-0.0035	0.0020	-0.0037	0.0019	-0.0021	0.0019
$E\left[E_t\left(\left(\frac{r_{t+1}+q_{t+1}}{q_t}\right)\right)\right]$	1.0093	0.0507	1.0026	0.0483	1.0561	0.0416
$E\left[R_{f,t}\right]$	1.0044	0.0062	1.0040	0.0062	1.0073	0.0054
η_t	1.9406	1.1414	2.0767	1.1596	1	0

NOTES: This table shows the moments under unconditional, conditionally constrained, and conditionally unconstrained distributions of the state variables. Numbers are expressed in decimal units and in quarterly terms. We discuss in the appendix how we compute unconditional, conditionally constrained, and unconditionally constrained distributions.

Table 3.4: Theoretical moments

Variable	*Moments*			
	Mean	St.dev.	Skewness	Kurtosis
Panel A. Unconditional Distribution				
K_{t+1}	10.941	1.606	-0.154	1.404
A_t	1.000	0.030	0.071	2.524
ϕ_t	0.275	0.139	0	1.788
q_t	1.091	0.071	0.358	1.944
Y_t	1.051	0.098	-0.009	2.071
I_t	0.243	0.027	-0.151	2.512
Panel B. Conditionally constrained distribution				
K_{t+1}	10.735	1.591	0.053	1.405
A_t	0.997	0.031	0.131	2.658
ϕ_t	0.248	0.128	0.157	1.969
q_t	1.110	0.065	0.289	1.943
Y_t	1.035	0.090	0.035	2.100
I_t	0.239	0.025	-0.177	2.557
Panel C. Conditionally unconstrained distribution				
K_{t+1}	12.391	0.714	-2.132	7.645
A_t	1.021	0.0315	-0.641	2.997
ϕ_t	0.457	0.038	-0.547	2.423
q_t	1	0	0	0
Y_t	1.159	0.069	-1.146	4.822
I_t	0.272	0.019	-0.595	2.9157

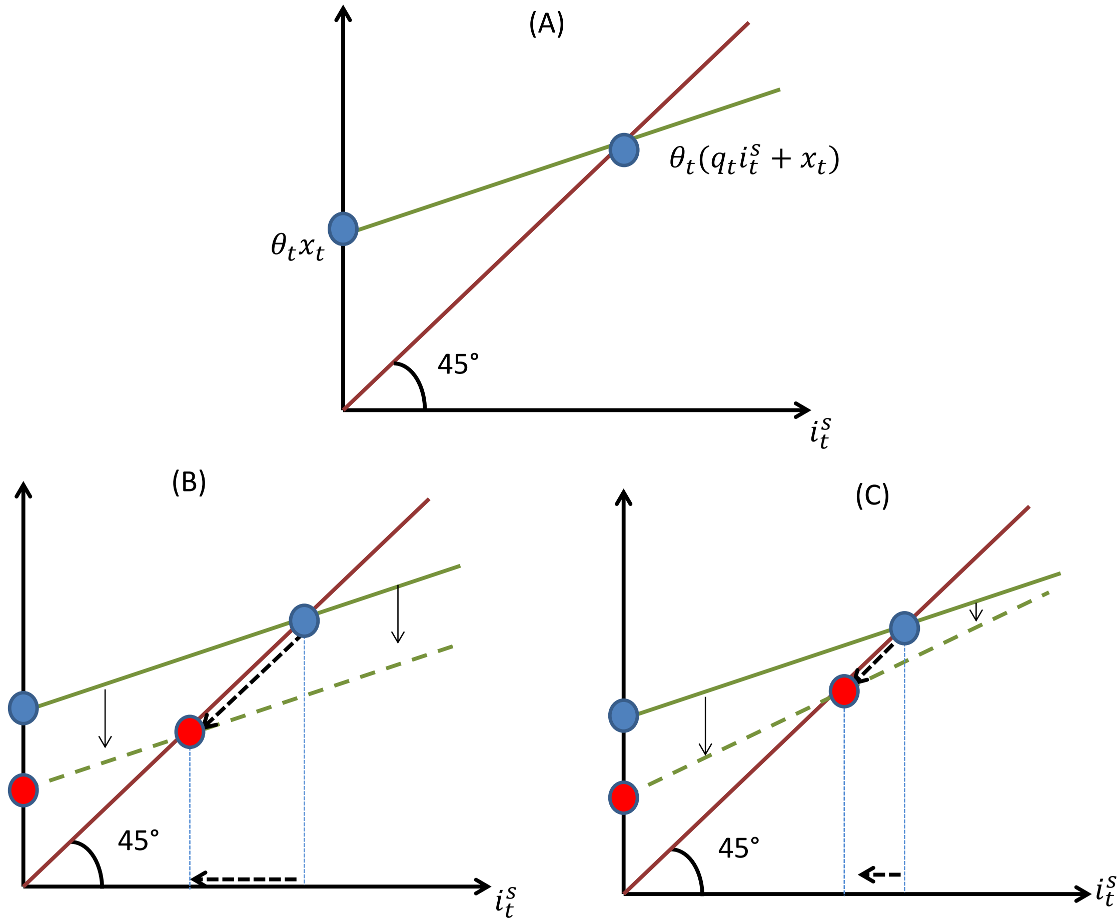
NOTES: This table reports the theoretical moments of relevant endogenous variables in the model. Panel A reports the moments computed with the unconditional distribution of the state variables. Panel B reports the moments computed with the conditionally constrained distribution (i.e., conditional con $q > 1$). Panel C Reports the moments under the conditionally unconstrained distribution (i.e., conditional o $q = 1$). We report in the appendix how we compute these distributions. Kurtosis and skewness are

$$kurt_{\pi}\left[z\left(s\right)\right] = \frac{\mathbb{E}_{\pi}\left[\left(z\left(s\right)-\mathbb{E}_{\pi}\left(z\left(s\right)\right)\right)^{4}\right]}{\left(\mathbb{E}_{\pi}\left[\left(z\left(s\right)-\mathbb{E}_{\pi}\left(z\left(s\right)\right)\right)^{2}\right]\right)^{2}}$$

$$skew_{\pi}\left[z\left(s\right)\right] = \frac{\mathbb{E}_{\pi}\left[\left(z\left(s\right)-\mathbb{E}_{\pi}\left(z\left(s\right)\right)\right)^{3}\right]}{\left(\mathbb{E}_{\pi}\left[\left(z\left(s\right)-\mathbb{E}_{\pi}\left(z\left(s\right)\right)\right)^{2}\right]\right)^{3/2}}$$

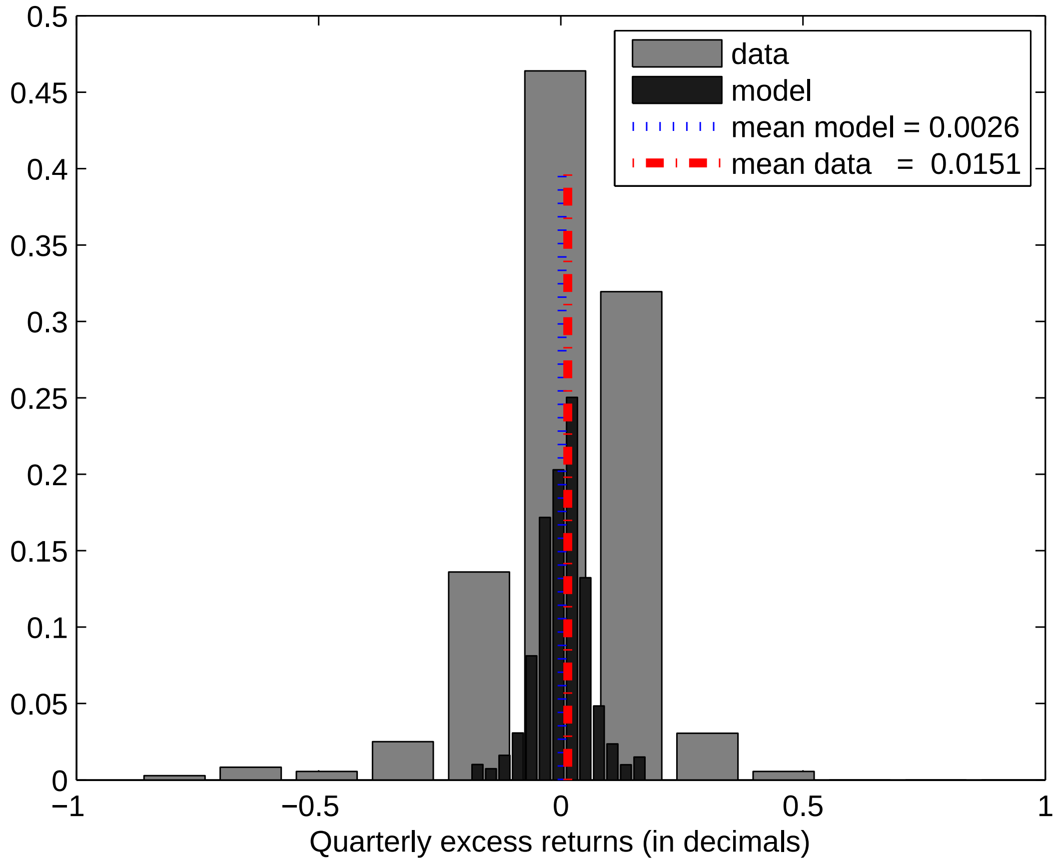
where π is the corresponding measure (unconditional, conditionally constrained, or conditionally unconstrained).

Figure 3.1: Intuition behind the effects



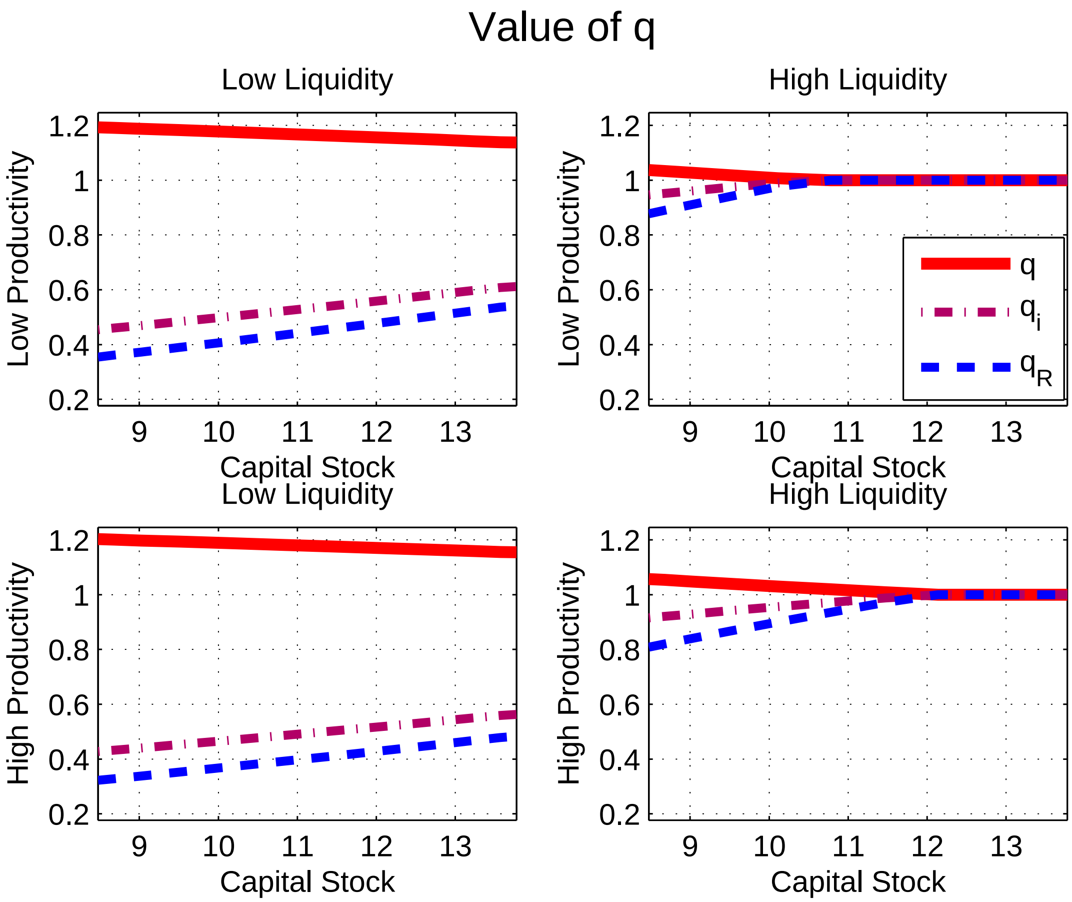
NOTES: Panel (A) shows how borrowing constraints impose a cap on the amount of equity that can be sold to finance a downpayment. Panel (B) shows how liquidity shocks affect the amount of resources available as a down payment. Panel (C) shows the price effect of the shock, which may or may not reinforce the liquidity shock.

Figure 3.2: Empirical distribution of excess returns



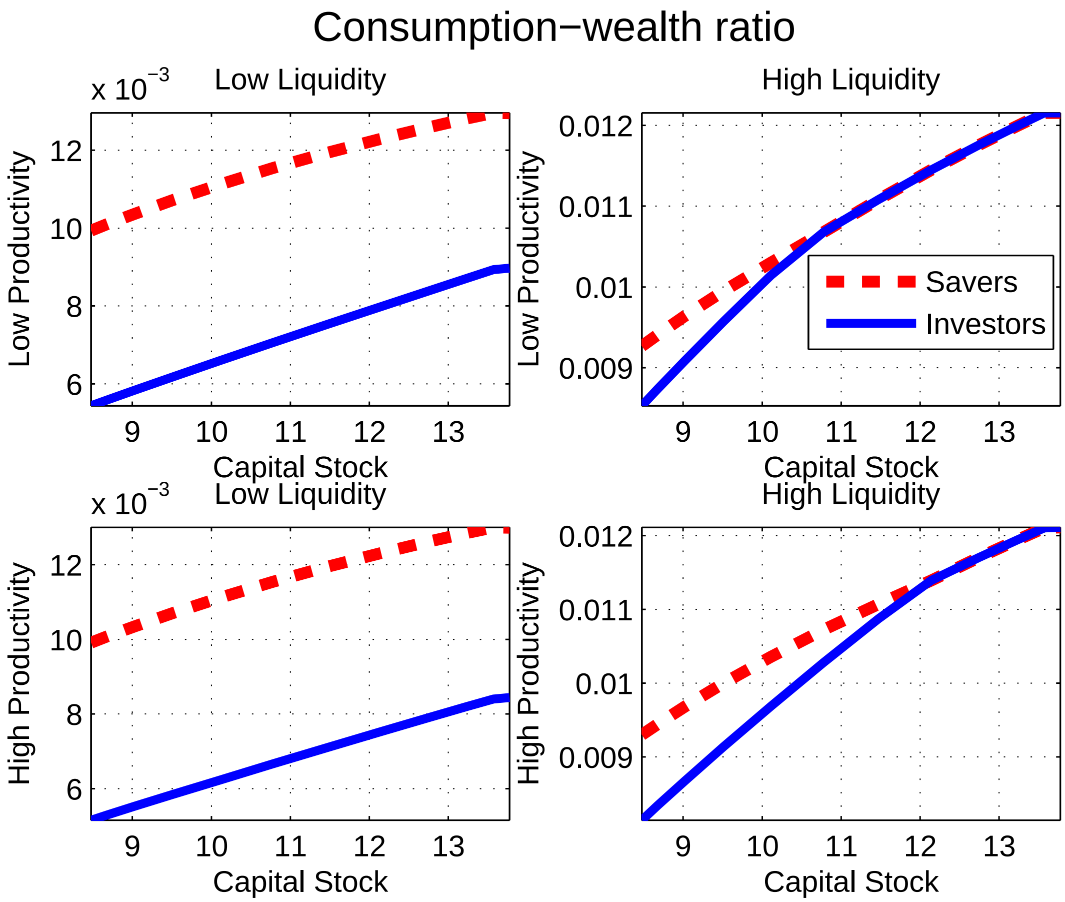
NOTES: This figure shows the empirical distribution for the risk premium(i.e., equation 3.10) in the data and in the model. The model distribution is obtained by simulating the model 500,000 periods (we burn the first 10,000 periods). Data are constructed exactly as in Croce (2014), but at a quarterly frequency and with an extended time window (1927q1-2016q4 instead of 1929-2004). For the time interval considered in Croce (2014) and reported in table 3.2, the data used in this figure yield very similar numbers: $E[\log R_{f,t}] = 0.62\%$, $E[\log R^e - \log R_{f,t}] = 5.5\%$, $std(\log R_{f,t}) = 1.93\%$, $std(\log R^e) = 19.55\%$, all annualized.

Figure 3.3: Price of equity, q_t



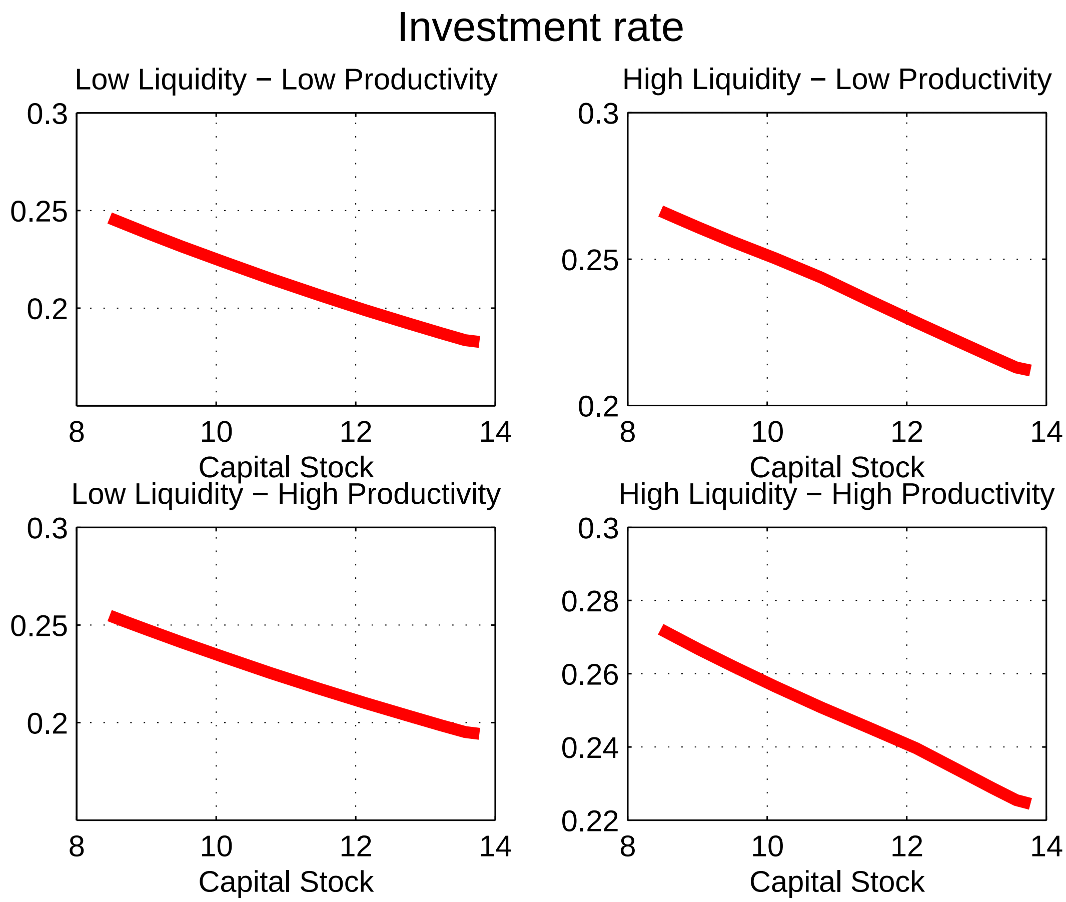
NOTES: This figure displays q across the state space. The upper-left panel represents the variable when liquidity and productivity are at their 20% percentiles. The upper-right panel represents the variable when liquidity is at its 80% percentile and productivity is at its 20% percentile. The lower-right panel represents the variable when liquidity is at its 20% percentile and productivity is at its 80% percentile. The upper-left panel represents the variable when liquidity and productivity are both at their 80% percentiles. Percentile values are $A^{(20)} = 0.97$, $A^{(80)} = 1.03$, $\phi^{(20)} = 0.1192$, $\phi^{(80)} = 0.4308$. In all subplots, the x-axis represents the level of the endogenous state variable, K. We provide details about the statistical properties of K in Table 3.3.

Figure 3.4: Consumption-wealth ratio



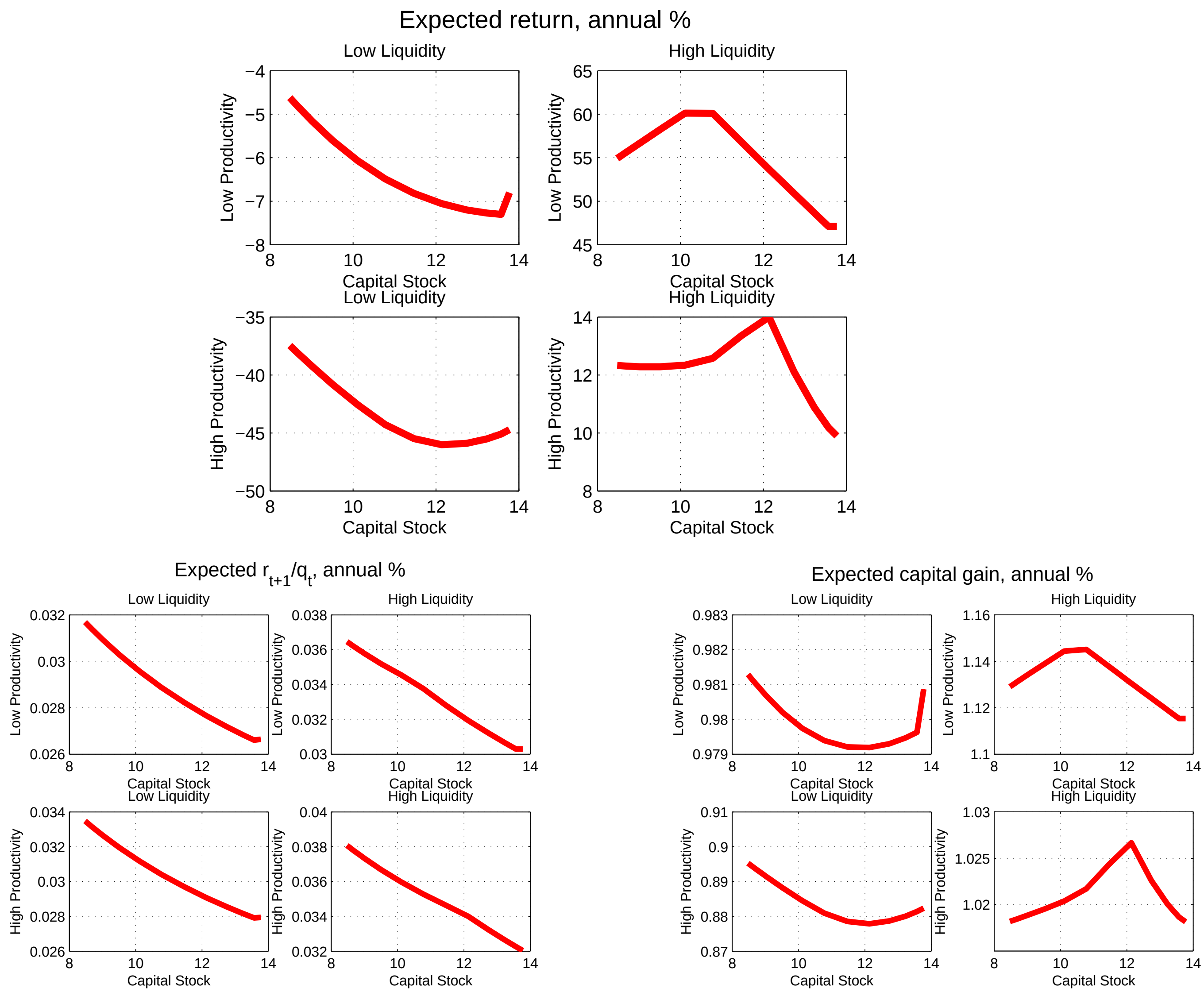
NOTES: This figure displays the consumption-wealth ratio across the state space. The upper-left panel represents the variable when liquidity and productivity are at their 20% percentiles. The upper-right panel represents the variable when liquidity is at its 80% percentile and productivity is at its 20% percentile. The lower-right panel represents the variable when liquidity is at its 20% percentile and productivity is at its 80% percentile. The upper-left panel represents the variable when liquidity and productivity are both at their 80% percentiles. Percentile values are $A^{(20)} = 0.97$, $A^{(80)} = 1.03$, $\phi^{(20)} = 0.1192$, $\phi^{(80)} = 0.4308$. In all subplots, the x-axis represents the level of the endogenous state variable, K. We provide details about the statistical properties of K in Table 3.3.

Figure 3.5: Investment rate I/Y



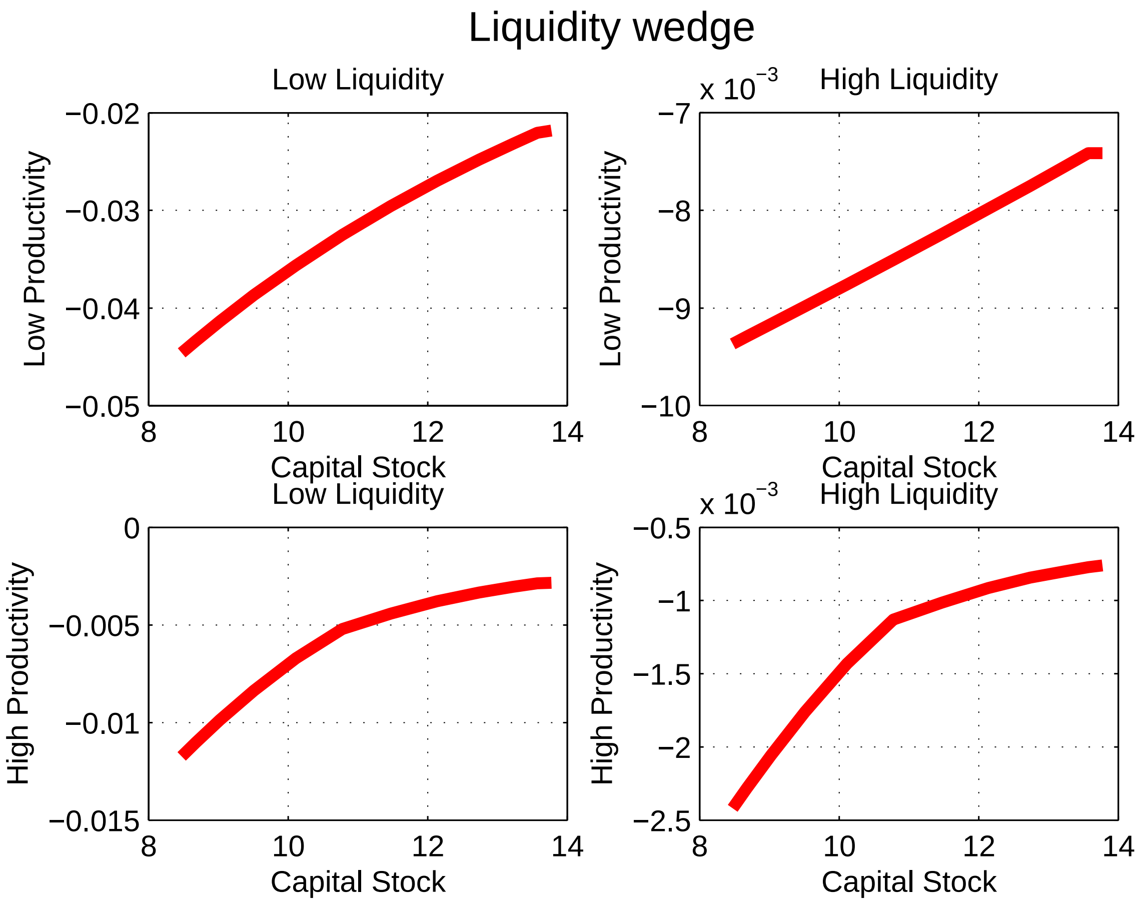
NOTES: This figure displays the investment rate I/Y across the state space. The upper-left panel represents the variable when liquidity and productivity are at their 20% percentiles. The upper-right panel represents the variable when liquidity is at its 80% percentile and productivity is at its 20% percentile. The lower-right panel represents the variable when liquidity is at its 20% percentile and productivity is at its 80% percentile. The upper-left panel represents the variable when liquidity and productivity are both at their 80% percentiles. Percentile values are $A^{(20)} = 0.97$, $A^{(80)} = 1.03$, $\phi^{(20)} = 0.1192$, $\phi^{(80)} = 0.4308$. In all subplots, the x-axis represents the level of the endogenous state variable, K. We provide details about the statistical properties of K in Table 3.3.

Figure 3.6: Expected return $E\left[\left(r_{t+1}+\lambda q_{t+1}\right)/q_t\right]$, capital gains and dividend yield



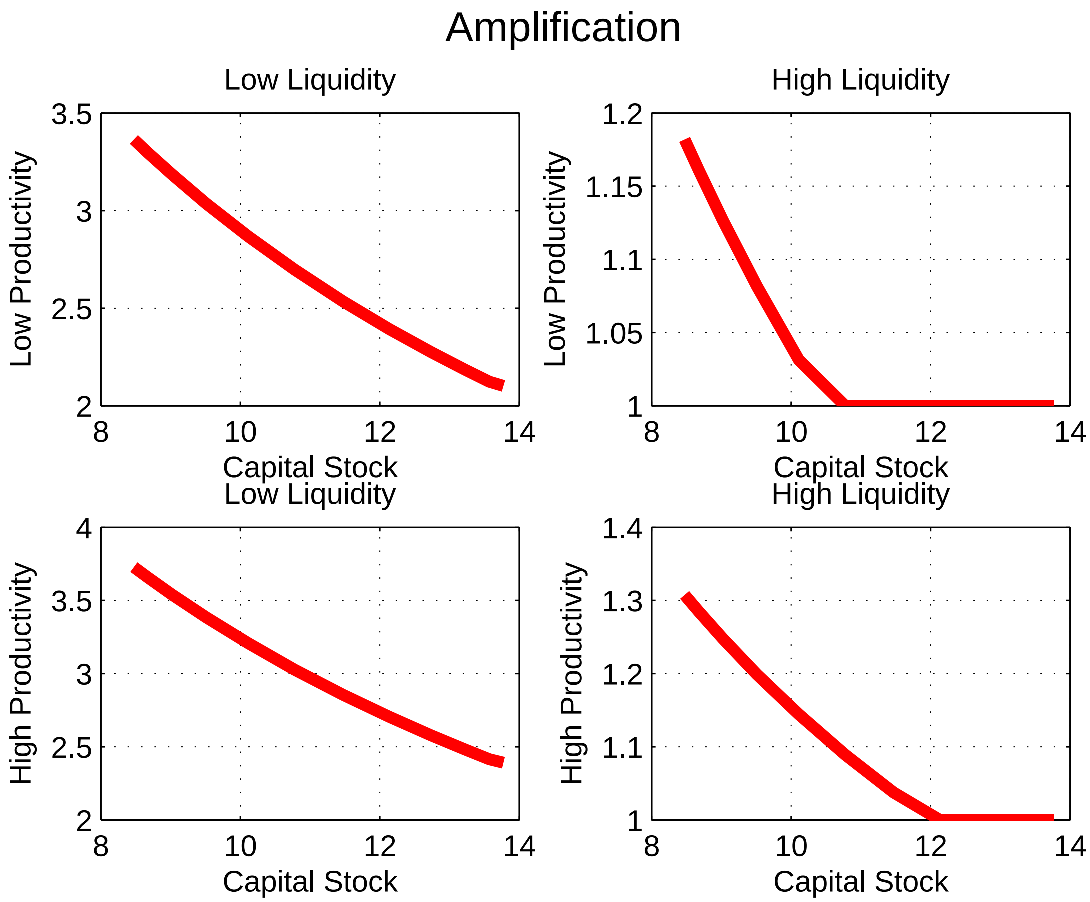
NOTES: This figure displays the market return on equity, the dividend yield and the capital gains across the state space. The upper-left panel represents the variable when liquidity and productivity are at their 20% percentiles. The upper-right panel represents the variable when liquidity is at its 80% percentile and productivity is at its 20% percentile. The lower-right panel represents the variable when liquidity is at its 20% percentile and productivity is at its 80% percentile. The upper-left panel represents the variable when liquidity and productivity are both at their 80% percentiles. Percentile values are $A^{(20)} = 0.97$, $A^{(80)} = 1.03$, $\phi^{(20)} = 0.1192$, $\phi^{(80)} = 0.4308$. In all subplots, the x-axis represents the level of the endogenous state variable, K. We provide details about the statistical properties of K in Table 3.3.

Figure 3.7: Liquidity component $E_t \left[l_{t+1} \right] / q_t^R$



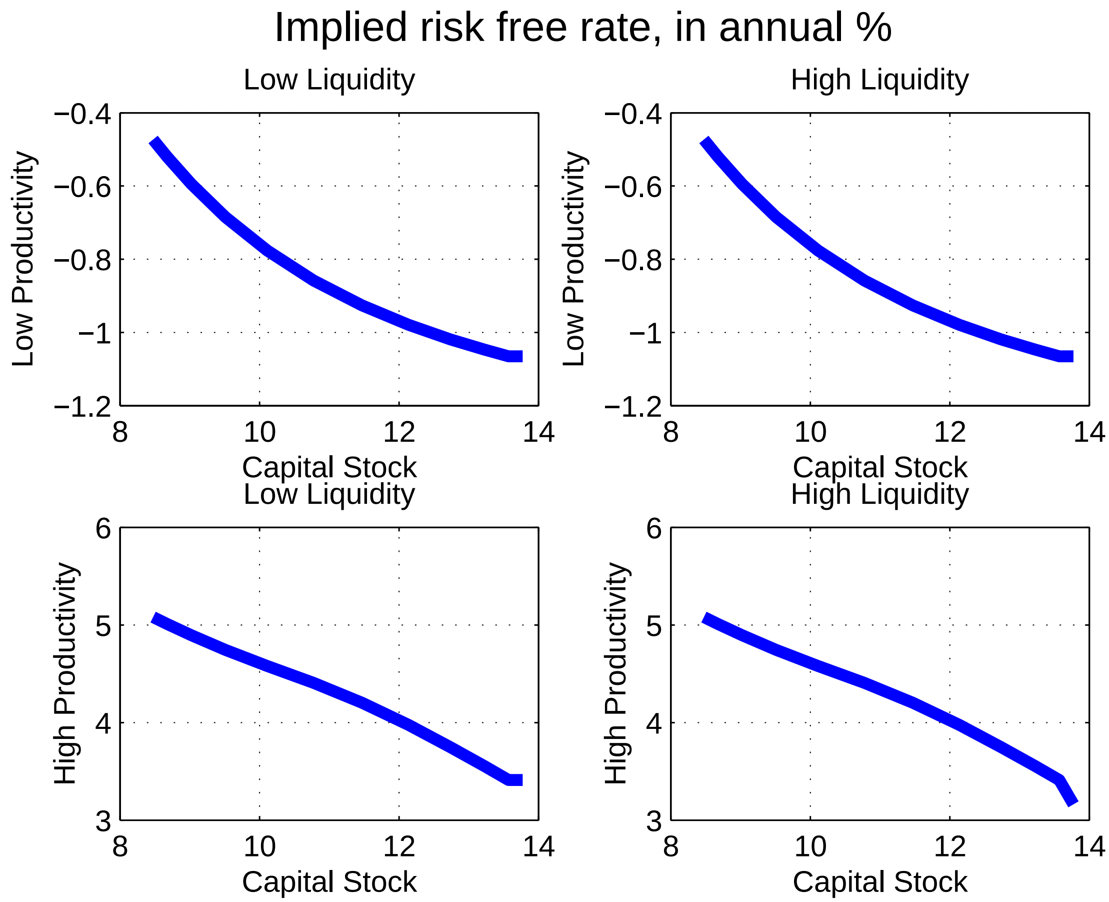
NOTES: This figure displays the liquidity component across the state space. The upper-left panel represents the variable when liquidity and productivity are at their 20% percentiles. The upper-right panel represents the variable when liquidity is at its 80% percentile and productivity is at its 20% percentile. The lower-right panel represents the variable when liquidity is at its 20% percentile and productivity is at its 80% percentile. The upper-left panel represents the variable when liquidity and productivity are both at their 80% percentiles. Percentile values are $A^{(20)} = 0.97$, $A^{(80)} = 1.03$, $\phi^{(20)} = 0.1192$, $\phi^{(80)} = 0.4308$. In all subplots, the x-axis represents the level of the endogenous state variable, K. We provide details about the statistical properties of K in Table 3.3.

Figure 3.8: Amplification,η_t



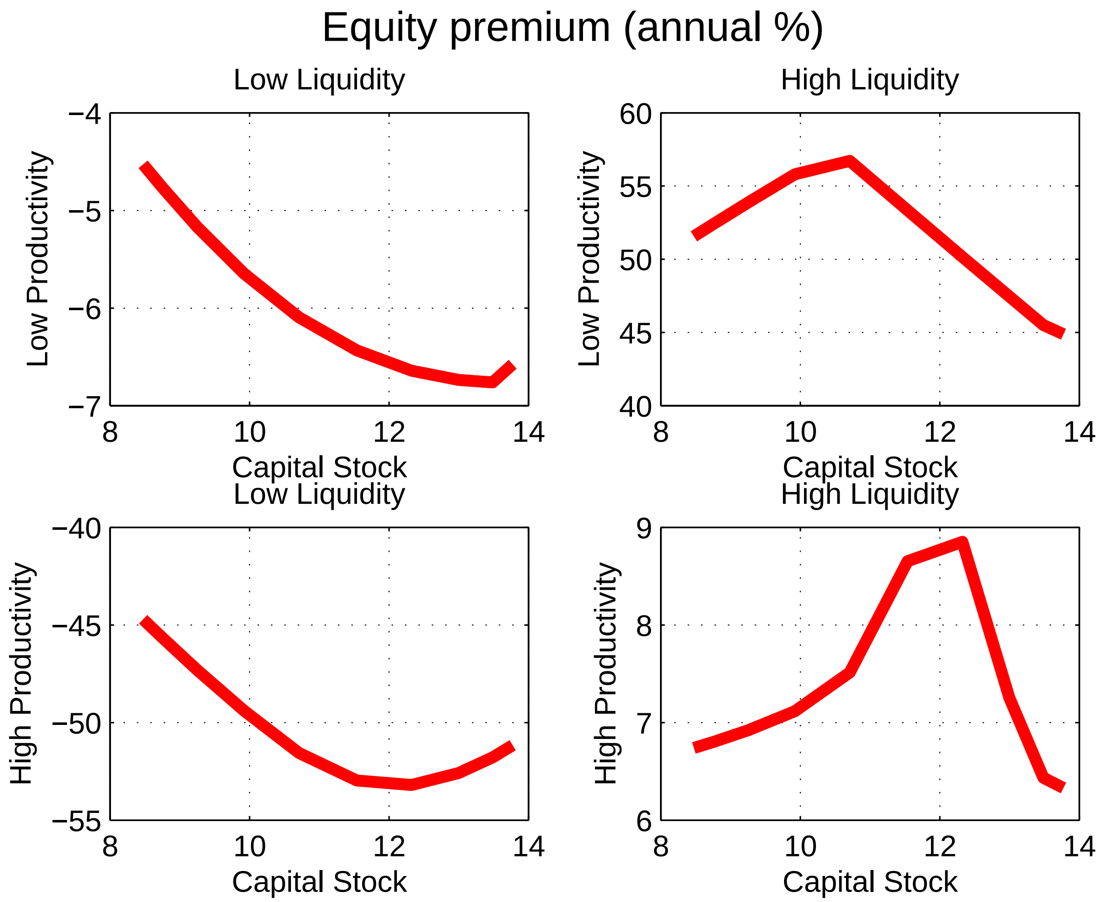
NOTES: This figure displays amplification term across the state space. The upper-left panel represents the variable when liquidity and productivity are at their 20% percentiles. The upper-right panel represents the variable when liquidity is at its 80% percentile and productivity is at its 20% percentile. The lower-right panel represents the variable when liquidity is at its 20% percentile and productivity is at its 80% percentile. The upper-left panel represents the variable when liquidity and productivity are both at their 80% percentiles. Percentile values are $A^{(20)} = 0.97$, $A^{(80)} = 1.03$, $\phi^{(20)} = 0.1192$, $\phi^{(80)} = 0.4308$. In all subplots, the x-axis represents the level of the endogenous state variable, K. We provide details about the statistical properties of K in Table 3.3.

Figure 3.9: Real interest rate



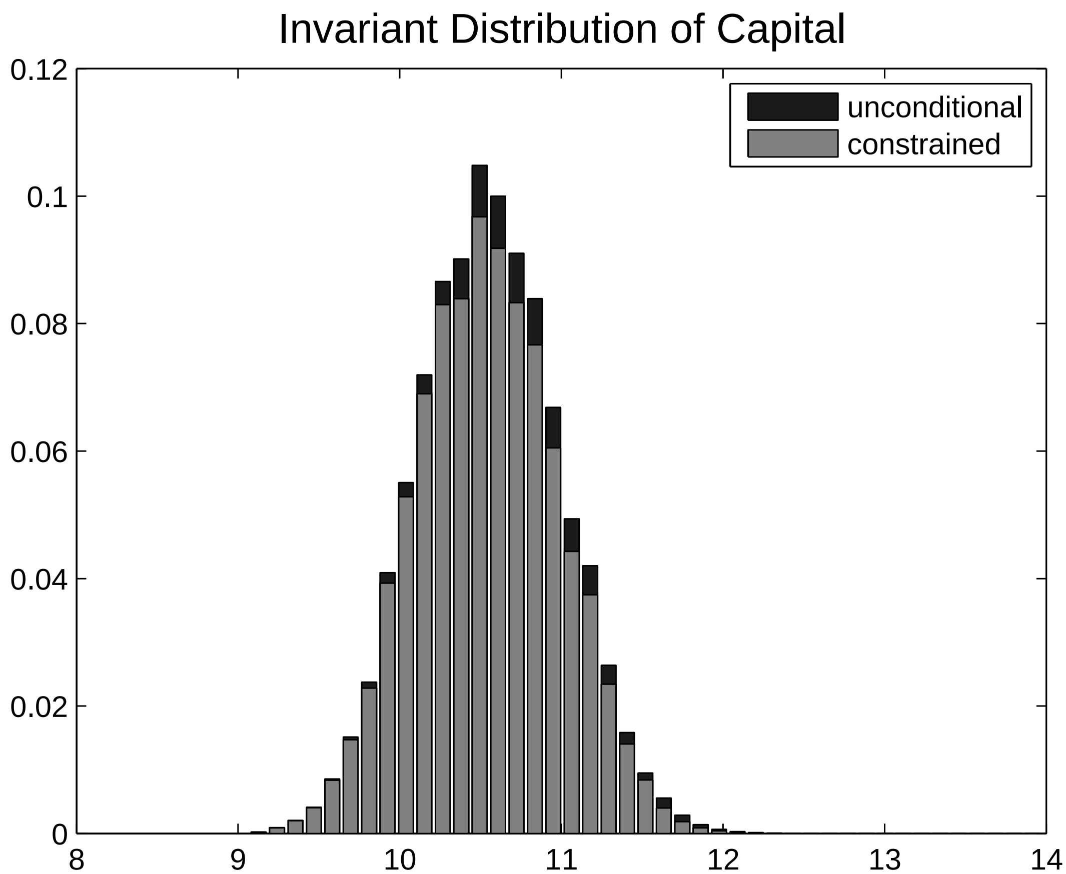
NOTES: This figure displays the real risk-free interest rate across the state space. The upper-left panel represents the variable when liquidity and productivity are at their 20% percentiles. The upper-right panel represents the variable when liquidity is at its 80% percentile and productivity is at its 20% percentile. The lower-right panel represents the variable when liquidity is at its 20% percentile and productivity is at its 80% percentile. The upper-left panel represents the variable when liquidity and productivity are both at their 80% percentiles. Percentile values are $A^{(20)} = 0.97$, $A^{(80)} = 1.03$, $\phi^{(20)} = 0.1192$, $\phi^{(80)} = 0.4308$. In all subplots, the x-axis represents the level of the endogenous state variable, K. We provide details about the statistical properties of K in Table 3.3.

Figure 3.10: Equity premium



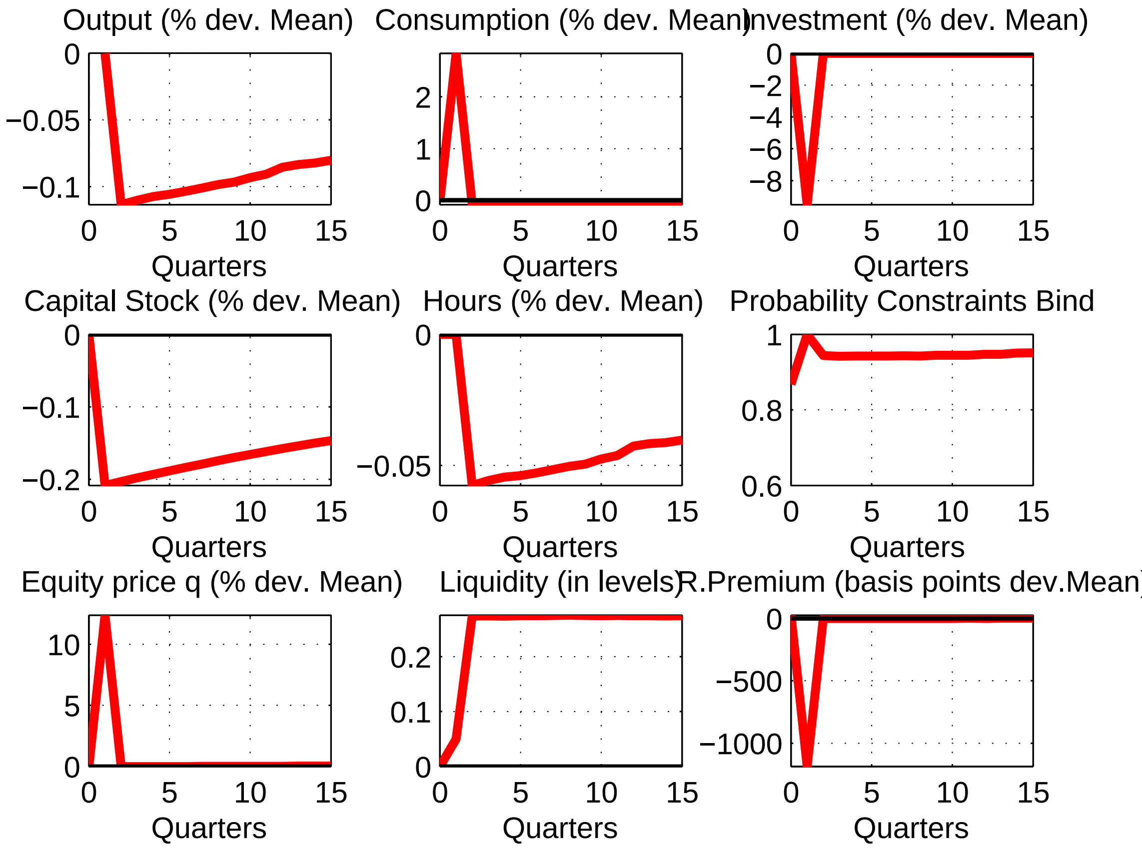
NOTES: This figure displays the equity premium across the state space. The upper-left panel represents the variable when liquidity and productivity are at their 20% percentiles. The upper-right panel represents the variable when liquidity is at its 80% percentile and productivity is at its 20% percentile. The lower-right panel represents the variable when liquidity is at its 20% percentile and productivity is at its 80% percentile. The upper-left panel represents the variable when liquidity and productivity are both at their 80% percentiles. Percentile values are $A^{(20)} = 0.97$, $A^{(80)} = 1.03$, $\phi^{(20)} = 0.1192$, $\phi^{(80)} = 0.4308$. In all subplots, the x-axis represents the level of the endogenous state variable, K. We provide details about the statistical properties of K in Table 3.3.

Figure 3.11: Invariant distribution of capital



NOTES: This figure shows the distribution of the endogenous state variable. The x-axis represents the level of capital and the y-axis the measure. The gray area is the conditionally constrained distribution (i.e., when constraints bind), while the black area represents the unconditional distribution. We provide the details on the construction of these distributions in the appendix.

Figure 3.12: Impulse-response functions



NOTES: This figure shows the impulse response functions after a liquidity shock. As shown in the figure, liquidity moves from its mean value of 0.275 to 0.05. We compute the response of the economy with 60,000 Monte carlo simulations, starting from the unconditional expectation of the state vector.

3.6 Appendix

Proof of proposition 1 (Investor's constraint)

We need to show that the budget constraint for investors, when constraints binds, is given by

$$c_{i,t} + q_t^R n_{t+1} = \left(r_t + \lambda q_t^i\right) n_t, \tag{3.24}$$

where $q_t^i = \phi_t q_t + (1-\phi_t)\, q_t^R$. As shown in 3.3, the entrepreneur's budget constraint is

$$c_t + i_t^d = r_t n_t + q_t \left(\Delta e_{t+1}^{-} - \Delta e_{t+1}^{+}\right).$$

Therefore, when constraints bind, we use $\left(\Delta e_{t+1}^{-} - \Delta e_{t+1}^{+}\right) = \lambda \phi_t n_t$

$$c_t + i_t^d = (r_t + q_t \lambda \phi_t)\, n_t.$$

Also

$$\begin{aligned} i_t^d &= (1 - q_t \theta)\, i_t, \\ &= (i_t - i_t^s)\, q_t^R \ , \end{aligned}$$

So, using the law of motion for capital

$$\begin{aligned} c_t + q_t^R i_t - q_t^R i_t^s &= (r_t + q_t \lambda \phi_t)\, n_t, \\ c_t + (k_{t+1} - \lambda k_t)\, q_t^R - q_t^R i_t^s &= (r_t + q_t \lambda \phi_t)\, n_t. \end{aligned}$$

Now, using the identity, $k_{t+1} = n_{t+1} - e^+_{t+1} + e^-_{t+1}$, we have

$$c_t + n_{t+1}q^R_t + \left(-e^+_{t+1} + e^-_{t+1} - i^s_t\right) q^R_t - q^R_t \lambda k_t = \left(r_t + q_t \lambda \phi_t\right).$$

Lastly, we use $e^-_{t+1} = \lambda e^-_t + \Delta e^-_{t+1} + i^s_t$ and obtain the desired result (3.24). The computation for saver follows the exact same steps. □

Proof of proposition 3 (Policy functions). We first establish the solution for the value functions. Recall that the value function for agent j has the form

$$V_{j,t} = \left[c^{\rho}_{j,t} + \beta E\left[V^{1-\gamma}_{j,t+1}\right]^{\frac{\rho}{1-\gamma}}\right]^{\frac{1}{\rho}}.$$

We do a monotone transformation to ease the algebra

$$V_{j,t} = \left(\rho v_{j,t}\right)^{\frac{1}{\rho}},$$

so

$$v_{j,t} = \frac{c^{\rho}_{j,t}}{\rho} + \beta E\left[v^{\frac{(1-\gamma)}{\rho}}_{j,t+1}\right].$$

The intertemporal marginal rate of substitution is

$$\begin{aligned} M_{j,t+1} &= \frac{\frac{\partial v_{j,t}}{\partial c_{j,t+1}}}{\frac{\partial v_{j,t}}{\partial c_{j,t}}}, \\ &= \beta\left(\frac{c_{j,t+1}}{c_{j,t}}\right)^{\rho-1}\left(\frac{v^{\frac{1}{\rho}}_{j,t}}{E_t\left[v^{\frac{(1-\gamma)}{\rho}}_{j,t+1}\right]^{\frac{1}{1-\gamma}}}\right)^{1-\gamma-\rho}. \end{aligned} \tag{3.25}$$

We guess

$$v_{j,t} = \frac{1}{\rho}\left(a_{j,t}\omega_{j,t}\right)^{\rho}.$$

So the certainty equivalent can be written as

$$E_t\left[v_{j,t+1}^{\frac{(1-\gamma)}{\rho}}\right]^{\frac{1}{1-\gamma}} = \left(\frac{1}{\rho}\right)^{\frac{1}{\rho}} E_t\left[\pi\left(a_{i,t}\omega_{i,t}\right)^{1-\gamma} + (1-\pi)\left(a_{s,t}\omega_{s,t}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

We consider the case of savers; the investors case is exactly the same. Substitute the guess in (3.25)

$$M_{s,t+1} = \beta\left(\frac{c_{s,t+1}}{c_{s,t}}\right)^{\rho-1}\left(\frac{a_{s,t}\omega_{s,t}}{E_t\left[\pi\left(a_{i,t+1}\omega_{i,t+1}\right)^{1-\gamma} + (1-\pi)\left(a_{s,t+1}\omega_{s,t+1}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{1-\gamma-\rho}. \tag{3.26}$$

Now we use $\frac{c_{s,t+1}}{c_{s,t}} = \frac{c_{s,t+1}}{\omega_{s,t+1}}\frac{\omega_{s,t+1}}{\omega_{s,t}}\frac{\omega_{s,t}}{c_{s,t}} = \frac{1-\xi_{s,t+1}}{1-\xi_{s,t}}R^{ss}_{t+1}\xi_{s,t}$.(Where we have used the identity $\frac{\omega_{s,t+1}}{\omega_{s,t}} = R^{ss}_{t+1}\xi_{s,t}$.) Then we can plug (3.26) in the first-order conditions and solve. The last step is to substitute

$$
\begin{aligned}
a_{s,t} &= (1-\xi_{s,t})^{\frac{1-\rho}{\rho}} \\
a_{i,t} &= (1-\xi_{i,t})^{\frac{1-\rho}{\rho}} \\
\frac{\omega_{i,t+1}}{\omega_{s,t}} &= \xi_{s,t} R^{si}_{t+1} \\
\frac{\omega_{s,t+1}}{\omega_{s,t}} &= \xi_{s,t} R^{ss}_{t+1} \\
\frac{\omega_{i,t+1}}{\omega_{i,t}} &= \xi_{i,t} R^{ii}_{t+1} \\
\frac{\omega_{s,t+1}}{\omega_{i,t}} &= \xi_{i,t} R^{is}_{t+1} \\
R^{s}_{t+1} &= \pi R^{si}_{t+1} + (1-\pi) R^{ss}_{t+1} \\
R^{i}_{t+1} &= \pi R^{ii}_{t+1} + (1-\pi) R^{is}_{t+1}
\end{aligned}
$$

Together with the FOCs for investor, they define a system of equations in (ξ_s, ξ_i). □

Verification. We now need to verify that our guess actually verifies the value function. In the way we wrote the recursion, it is clear to see that any pair $\xi_{i,t}$, $\xi_{s,t}$ that satisfies solving the system of difference equations guarantee the functional form guessed for v_i and v_s. Notice that the envelope condition is already embedded in the recursion.

Uniqueness. The proof of uniqueness of the policy functions consists of checking that the recursion is in fact a contraction. That is, that is it satisfies the Blackwell's sufficient conditions. A general treatment of this technical aspect can be found in Epstein and Zin (1989a).

Proof of proposition 4 (Decomposition). The proof consists in substituting $R^{ss}_{t+1}, R^{si}_{t+1}, R^{is}_{t+1}$, and R^{ii}_{t+1} in R^{i}_{t+1} and R^{s}_{t+1}.We start with R^{i}_{t+1}

$$
\begin{aligned}
R^{i}_{t+1} &= \pi R^{ii}_{t+1} + (1-\pi) R^{is}_{t+1} \ , \\
&= \pi \left(\frac{r_{t+1} + q^{i}_{t+1}\lambda}{q^{R}_{t}} \right) + (1-\pi) \left(\frac{r_{t+1} + q_{t+1}\lambda}{q^{R}_{t}} \right), \\
&= \pi \left[\frac{r_{t+1} + q^{i}_{t+1}\lambda}{q^{R}_{t}} - \frac{r_{t+1} + q_{t+1}\lambda}{q^{R}_{t}} \right] + \frac{r_{t+1} + q_{t+1}\lambda}{q^{R}_{t}}, \\
&= \pi \left[\frac{\left(q^{i}_{t+1} - q_{t+1}\right)\lambda}{q^{R}_{t}} \right] + \frac{r_{t+1} + q_{t+1}\lambda}{q^{R}_{t}}, \\
&= \pi \left[\frac{\left(\phi_{t+1} q_{t+1} + (1-\phi_{t+1}) q^{R}_{t+1} - q_{t+1}\right)\lambda}{q^{R}_{t}} \right] + \frac{r_{t+1} + q_{t+1}\lambda}{q^{R}_{t}}, \\
&= \pi\lambda \left[\frac{(\phi_{t+1} - 1) q_{t+1} + (1-\phi_{t+1}) q^{R}_{t+1}}{q^{R}_{t}} \right] + \frac{r_{t+1} + q_{t+1}\lambda}{q^{R}_{t}} \ , \\
&= -\pi\lambda \frac{\left(q_{t+1} - q^{R}_{t+1}\right)(1-\phi_{t+1})}{q^{R}_{t}} + \frac{q_t}{q_t} \frac{r_{t+1} + q_{t+1}\lambda}{q^{R}_{t}} \ , \\
&= -\pi \frac{\lambda \left(q_{t+1} - q^{R}_{t+1}\right)(1-\phi_{t+1})}{q^{R}_{t}} + \left(\frac{(1-\theta) q_t}{1-\theta q_t} \right) \frac{r_{t+1} + \lambda q_{t+1}}{q_t} \ , \\
&= l_{t+1} + \eta_t R^{ss}_{t},
\end{aligned}
$$

which is equation (3.19). The exact same steps for $R^{s}_{t+1} = \pi R^{ss}_{t+1} + (1-\pi) R^{si}_{t+1}$ deliver equation (3.20). □

Numerical solution. We start by constructing a tensor grid for the state space (A, K, Φ), and the objective is to project all endogenous variables onto it. Our reported results are for $D_A = 14, D_K = 20, D_\Phi = 14$. So the grid has $D_A \times D_K \times D_\Phi = 3920$ points. Results are robust to a lower degree of coarseness in the grid.

Next, we guess values for the endogenous values. We pick a random number generator for the ξ_s and ξ_i at every point of the state space (an educated guess would be β). We guess the first value of the aggregate endogenous state variable $(K')^{(0)} = \beta (r + \lambda) K$. We then start an iterative loop to solve for the policy functions, until convergence. The loop is as follows:

1. We start from the steady-state level of capital without constraints, and compute the transition probabilities.

2. Compute expectations and obtain an updated vector for ξ_i, ξ_s.

3. Compute the excess demand in the equity market, to check whether the constraint is binding.

4. If there is excess demand, we compute the q that clears the equity market in each state. Otherwise, $q = 1$.

5. Compute the policy functions c_i, c_s, n_i', n_s', L and prices with the values we obtained for q, q^R, q^i and w.

6. Verify these policy functions clear the goods market.

7. Compute $(K')^{(1)} = [\pi n_i' + (1-\pi) n_s'] K$, and iterate until converge, i.e., $\left\| (K')^{(I)} - (K')^{(I-1)} \right\| \leq tol$, where tol is a small number.

Invariant distribution. We construct the exogenous Markov chain in the following way. First, we discretize the state space of exogenous productivity, which follows an AR(1) (in logs). For this, we proceed in line with Tauchen (1985, 1991). This give us a matrix P^A, in which each element is $P^A_{i,j} = \Pr\{A_{t+1} = a_j | A_t = a_i\}$.

We then compute the joint Markov chain for liquidity and productivity. We denote this joint transition probability matrix as $P^{A,\phi}$, where an element of the matrix represents

$P_{i,j}^{A,\phi} = \Pr\left\{\phi_{t+1} = \phi_j, A_{t+1} = a_j | \phi_t = \phi_i, A_t = a_i\right\}$ (abusing notation, i is a pair) and is given by

$$P_{i,j}^{A,\phi} = P_{i,j}^{A}\left(\varphi P_{i,j}^{A(\phi')} + (1-\varphi)\,U_{i,j}\right),$$

where $P_{i,j}^{A(\phi')}$ is an auxiliary matrix we construct to capture the correlation between productivity and liquidity, given by φ. This mapping is simply the order of elements in the liquidity vector Φ, and $U_{i,j}$ is a uniform probability matrix. $P^{A,\phi}$ has all the desired properties of a transition probability matrix.

The solution of the model gives us a transition matrix $P^{K',A,\phi} = \mathbb{P}$, where an element is

$$P_{i,j}^{K',A,\phi} = \Pr\left\{K_{t+1} = K_j; \phi_{t+1} = \phi_j, A_{t+1} = a_j | K_t = K_i, \phi_t = \phi_i, A_t = a_i\right\}$$

Then, the invariant distribution, μ, simply solves the following eigenvalue problem

$$\mu'\left(I - \mathbb{P}\right) = 0$$

where I is the identity matrix and the vector[12] $\mu = \lim_{t\to\infty}\mu_t = \mathbb{P}\left(K_\infty, A_\infty, \phi_\infty\right)$, with $\mu'_{t+1} = \mu'_t\mathbb{P}$. This vector has dimension $\left(1 \times \left(N_K \times N_A \times N_\phi\right)\right)$. The problem is guaranteed to have a solution because $\mathbb{P}$ is a Markov chain (positive semi-definite and rows sum to 1), so we select the eigenvector associated with the (unique) unitary eigenvalue. Then, we can compute unconditional theoretical moments of endogenous variables using this unconditional measure, following theorem 2.2.3 in Ljungqvist and Sargent (2012). That is, for any random

[12] We normalize the vector to ensure that the sum of its elements is equal to one.

variable $z(s)$, we compute the unconditional mean as $\mathbb{E}_{\mu}[z(s)] = \sum_i^N \mu_i z_i$ (where μ is normalized to be a probability measure.)

We compute the conditionally constrained distribution as follows. Let $\mathcal{C} = \{s \in S : q(s) > 1\}$ be the set in which constraints bind (i.e., the spot price of capital is greater than one). Then, we define the following random variable (an indicator function):

$$1_{\mathcal{C}}(s) = \begin{cases} 1 & \text{if } s \in \mathcal{C} \\ 0 & \text{if } s \notin \mathcal{C} \end{cases}$$

Then

$$\widetilde{\mu} = \mathbb{P}(K_{\infty}, A_{\infty}, \phi_{\infty} | \mathcal{C}) = \frac{\mathbb{E}_{\mu}[1_{\mathcal{C}}(s)]}{\sum_{\{i:s \in \mathcal{C}\}} \mu_i}$$

We can compute the conditional expectation numerically, by constructing this indicator function across the state space. Then we can compute, for example, the conditionally constrained mean for a given random variable z as $\mathbb{E}_{\widetilde{\mu}}[z(s)] = \sum_i^N \widetilde{\mu}_i z_i$. Figure (3.11) displays the results for μ and $\widetilde{\mu}$ (each bar represents the size of the measure for each point in the K state). The conditionally unconstrained distribution can be computed in a similar way, by simply defining the complement of $\mathcal{C}$. In Table 3.4 we report statistics using these three measures.

Impulse-response functions. We construct the impulse response functions (IRF) using a standard simulation method. In Figure (3.12), we report the IRFs to a liquidity shock. We start the economy in the unconditional means. We then simulate multiple paths of the exogenous Markov chain to obtain a sequences of $\left\{A_t^{(j)}, \phi_t^{(j)}\right\}_{t=0}^{T}$, in each simulation $j = 1, ..., J$. We simulate these paths with $\phi_0 = 0.05$ (i.e., in period 0 there is a liquidity shock that moves liquidity from 0.275 to 0.5, as mentioned in the text). In each simulation, we use the model solution to obtain the responses (i.e., the paths) of endogenous variables in each simulation. We then average paths across simulations. That is, for a given variable z, we have J simulated paths $\left\{z_0^{(j)}, z_1^{(j)}, ..., z_T^{(j)}\right\}_{j=1}^{J}$, so we take the mean $\frac{1}{J}\sum_{j=1}^{J} z^j = z^{irf}$., and we obtain the sequence z^{irf}.

Chapter 4

A Macrofinance View of U.S. Sovereign CDS Premiums

4.1 Introduction

The credit crisis brought about a visible change in the sovereign credit default swaps (CDS) of economically developed countries. Near zero trading volumes at near zero premiums in late 2007 expanded to active trading at substantial premiums of hundreds of basis points. Although the crisis subsided, the sovereign CDS premiums remain elevated, and are nowhere close to pre-crisis levels. The question that we address in this paper is what risks are so richly compensated in these markets.

At a first blush, the answer seems to be obvious. After all, CDS are designed to insure against default. But let us consider the United States as the most stark example. At the height of the crisis the cost of the five-year protection was 100 bps, and it has traded around 20 bps since 2014. Is the U.S. default so likely, or is the expected loss so severe to justify such premiums? According to basic reasoning, the answer would be no. For instance, some observers believe that the U.S. is not going to default at all as it can either "inflate away" its debt obligations, or increase taxes, or both. Furthermore, by the standard replication argument, the CDS premium cannot be too different from the credit spread, which is the difference between a par yield on a bond of the credit name and that of a U.S. Treasury, and in the case of the U.S., is mechanically zero at any maturity.

There are a lot of reasons to think that frictions may arise from various institutional features of the CDS markets, such as margin requirements, counterparty risk, capital constraints, and credit event determination. Such frictions could be responsible for a part of the observed premium. We do not disagree with such a view. Rather than giving a full explanation of the observed premium, our objective in this paper is to establish a quantitative benchmark for the compensation based on default risk only.

These initial arguments prompt us to make a first step towards developing a formal macro-based framework that allows us to evaluate the likelihood of and risk premium associated with a sovereign default. The advantage of such an approach is that it allows to study the impact of monetary and fiscal policies, and does not require an ability to replicate an asset in order to value it.

Because it is a first step, we keep our setting as simple as possible. We directly specify the dynamics of many key variables, such as aggregate output, consumption growth, and government expenditures. What holds it all together and allows us to investigate the questions of interest is the government budget constraint (GBC). The government can tax aggregate output and issue new nominal debt to finance its expenditures and repay its outstanding debt. Thus, the GBC determines endogenously the relation between the issued debt and taxes.

We specify monetary policy via a Taylor rule that determines behavior of inflation. In an endowment economy monetary policy usually does not have real effects. In contrast, in our setting with the GBC featuring nominal debt, inflation affects real quantities. Fiscal policy responds to the amount of outstanding debt and expected growth in the economy.

Our model endogenously allows for states of the economy in which budget balance can no longer be restored by raising taxes or by eroding the real value of debt by creating inflation. In such situations, the government will have no other choice other than to default on its debt. We refer to such a scenario as a *fiscal default*. Episodes of fiscal stress arise in our model because we assume that an increase in the tax rate has a small, negative effect on future long-term output growth. Attempts to achieve a balanced budget by raising taxes thus may come with a slowdown in taxable income, which can further exacerbate fiscal conditions. Fiscal default then arises when taxes cannot be raised further without reducing future tax revenues, in the spirit of a Laffer curve. This trade-off prompts our specification of a maximum amount of debt outstanding, which is related to the expenditure and tax rates, and ultimately determines the timing of default.

We complement our model with a representative agent who has Epstein and Zin (1989b) preferences and uses her marginal rate of substitution to value assets. Consumption features time-varying conditional mean similar to Bansal and Yaron (2004b). These assumptions allow us to value nominal defaultable securities using inflation and timing of default implied by the GBC and policy rules.

Qualitatively, we find that the model provides significant insights into the macroeconomic determinants of CDS premiums on U.S. Treasury debt. In the model, episodes of high government debt endogenously correspond to investors' high marginal utility states. When the government's expenditures rise, the likelihood that it finds itself close to a fiscal limit, a state in which further tax increases will reduce tax income, becomes more realistic. Default probabilities, and the likelihood of incurring losses on government debt thus increase in high marginal utility states. Writers of insurance against government debt thus face required payments in high marginal utility states. In order to be compensated for exposure to that risk, they earn high risk premia. Despite potentially small average losses on government debt, and thus small average payments for insurers, they occur in the worst of all states.

Within the context of our model, risk premiums thus make up a substantial part of CDS premiums beyond expected losses.

We use our model to explore an endogenous rare and severe event that may affect the U.S. economy. Despite the severity of default, exogenous consumption and, therefore, the marginal rate of substitution are not affected. In this sense, we are contemplating a mechanism that is similar to that of Barro (2006), although without the rare disasters in consumption. One implication of this setting is that the derived CDS premiums are likely to be conservative.

Quantitatively, we find that our model can generate episodes of persistently elevated CDS premiums similar to the recent U.S. experience. In simulations, our model produces CDS premiums of up to a 100 bps on an annual basis. This is similar to peak values of U.S. CDS premiums around the financial crisis in 2008. Perhaps more importantly, however, our model predicts episodes of persistently elevated CDS premiums even during calmer times. This is because in our setup with recursive preferences, investors will anticipate and dislike occasional shocks to default probabilities, which will result in an elevation of CDS premiums. The model is thus consistent with the notion that CDS premiums reflect investors' rational forecasts of the likelihood of U.S. fiscal stress.

We use the model to revisit the idea of avoiding default by increasing taxes or inflating the government debt away. We represent these notions by changing the fiscal and monetary policy stance, respectively. Raising less debt or responding to inflation less aggressively leads to a decline in the average probability of default and to an increase in CDS premiums. This happens because changes in the government's policy stance also increases the volatility of taxes and inflation, respectively, implying higher risk premiums. We also evaluate changes in the debt duration that serves as a metaphor for the combination of the Federal reserve Board's quantitative easing and U.S. Treasury's debt maturity extension programs. Our model implies that shortening duration leads to an increase in CDS premiums due to rollover risk.

Notation. We use capital letters to denote the levels of the variables. Lowercase letters are used for their logs. The changes in the variables are denoted by Δ.

Literature

Our work adds a macrofinance perspective to the growing literature on sovereign default and the pricing of sovereign default risk. While there is considerable interest in sovereign default both in macro and in finance, these literatures have evolved somewhat separately. Our paper is a first step towards synthesizing insights from macro and finance and distilling them into a quantitative framework that relies on standard building blocks.

There are a number of papers in the finance literature that are based on the contingent claims approach (CCA), which was originally developed to analyze defaultable corporate debt. In this approach a bond is treated as a (short put) option on the value of a firm's unlevered assets. Default is triggered by a combination of a firm's difficulty in servicing debt and the provisions of bankruptcy laws. When applying the CCA to sovereign debt, unlevered assets are replaced with the present value of future output. The key difficulty is that there is no bankruptcy law at the sovereign level, so the cause and timing of default is not clear.

Strategic default takes place when penalties such as limited access to international debt markets, trade sanctions, etc. are outweighed by the debt burden. In the CCA framework, these considerations lead to default when the present value of output under default exceeds the present value under continuation of debt service (Kulatilaka and Marcus, 1987). Gibson and Sundaresan (2005) endogenize the strategic default trigger and the resulting risk premiums (credit spreads) by embedding a bargaining game between the sovereign and the creditors. The issue with this approach is that there is inconclusive empirical evidence regarding the impact of penalties on sovereign defaults. In our model, the government defaults when it runs out of available debt-servicing tools (issue new debt, inflate debt, tax more), and, thus, can no longer meet its long-term financial obligations.

Affine models of sovereign default are focused on estimating a realistic model of a default probability and default risk premium in emerging economies using an intensity-based approach. Duffie, Pedersen and Singleton (2003) estimate a model of Russian credit spreads. Pan and Singleton (2008) estimate risk-adjusted default arrival rate and loss given default using sovereign CDS. Ang and Longstaff (2013) estimate a joint affine model of U.S. CDS, U.S. states, and Eurozone sovereigns. We use our model to provide the economic underpinning of defaults and to distinguish between risk-adjusted and actual probabilities of default (recovery is fixed at a constant for simplicity, but this can be easily extended).

Augustin and Tedongap (2016) value eurozone CDS from the perspective of an Epstein-Zin agent as well. The key difference from our approach is that they also follow an intensity-based approach, that is, they assume a function connecting a sovereign's default probability to expected consumption growth and macro volatility. In our model, default probability is determined endogenously via interaction between fiscal policy and the GBC combined with monetary policy.

Bhamra, Kuehn and Strebulaev (2010), Chen (2010) and Chen, Collin-Dufresne and Goldstein (2009) have linked models of endogenous corporate default with habit-based and recursive preferences to value corporate bonds. The valuation mechanism in our paper shares much with theirs as a high default risk premium generates substantial credit spreads while keeping default probabilities realistically low. In contrast to this line of work, we focus on sovereign default, which entails a different default trigger. Borri and Verdelhan (2012) use a risk-sensitive consumption-based model based on habit preferences to study sovereign default premiums in emerging markets.

Similarly to the CCA framework, strategic default is also at the core of the international macroeconomics literature on sovereign default, in the spirit of Eaton and Gersovitz (1981). Recent work along these lines includes Arellano (2008); Arellano and Ramanarayanan (2012); Yue (2010). This important line of work solves general equilibrium endowment models of small open economies in which governments default strategically in the best interest of households and analyzes the implications for sovereign credit spreads. Our paper differs from that work along several dimensions. From a quantitative viewpoint, we operate in a risk-sensitive framework in which risk premia make up a sizeable component of spreads. Further, we emphasize the limitations of fiscal instruments for restoring budget balance in default.

In the latter respect, our work is closer to work by Leeper (2013) on fiscal uncertainty and debt limits. Bi and Leeper (2013) and Bi and Traum (2012) analyze business cycle models that explicitly allow for fiscal limits and apply them to the recent episode of heightened sovereign risk in Greece. In contrast to our work, they do not focus on CDS premiums or spreads, and do not operate in a risk-sensitive framework. Moreover, we emphasizes a growth channel of fiscal policy via elevated tax rates depressing future growth prospects, which is absent in their work. This channel emerges endogenously from recent work linking long-run risks with fiscal policy in models of endogenous growth (Croce, Kung, Nguyen and Schmid, 2012; Croce, Nguyen and Schmid, 2013), and is consistent with the empirical evidence, as documented in Easterly and Rebelo (1993) and Mendoza and Tesar (1998). In this respect, our work is closest to Chen and Verdelhan (2015), who examine the links between taxation and sovereign risk, but do not focus on U.S. CDS premiums as we do.

4.2 A Primer on U.S. Sovereign CDS

We start by providing a basic background on corporate CDS. This information motivates our interest in sovereign CDS and we use it to explain important differences between the two types of contracts.

4.2.1 Corporate CDS

Prior to the introduction of the Big and Small Bang protocols in 2009, a long position in a corporate CDS contract required no payments upfront, quarterly premiums, and, in case of a credit event, delivery of allowed bonds of the corporate entity, or a cash payment with the amount determined in a CDS auction in exchange for the full par (notional) paid in cash.

The Big and Small Bang protocols have codified the use of bond auctions to determine the payments by the long party. They take place within 30 days after a credit event. The auctions allow delivery of any bond of a defaulted company from a pre-specified list leading to the cheapest-to-deliver option. The value of this option should be small for corporate names because their bonds tend to trade at approximately the same price after a credit event (Chernov, Gorbenko and Makarov, 2013).

The protocols also established standardized CDS premiums (100 bps for investment grade and 500 bps for speculative grade entities). The standardized CDS premiums simplified the netting and offsetting of positions but introduced the need to pay an upfront fee to ensure that the present values of all the cash flows line up. The CDS contracts continue to be quoted on a par basis (zero payment upfront). For this reason, we ignore all these institutional details in the paper.

It is easy to obtain a back-of-the-envelope estimate of the quarterly premiums using the replication argument applied to par bonds. Par bonds have coupon payments such that the bond value is equal to par immediately after a coupon payment. Assuming that par bonds of matching maturity are available for both the entity and U.S. Treasury, consider shorting the corporate bond, and buying the Treasury bond. Because these are par bonds, there are no upfront payments. The running payment is the difference between higher corporate and lower Treasury coupons, known as the credit spread. In the case of a credit event, the Treasury bond can be sold at the par value, while the short position in the corporate bond requires the purchase of the bond in the marketplace and delivering it to the original owner.

In practice, par bonds may not be available, so it could be difficult to find bonds with matching maturity, or corporate bonds could be much more expensive to short due to their scarcity. All these complications introduce the non-zero difference between the CDS premium and a bond's credit spread, known as the CDS-bond basis (Blanco, Brennan and Marsh, 2005; Longstaff, Mithal and Neis, 2005). Typically, the basis is positive, reflecting the cost of shorting a corporate bond. Because these costs vary with a trading party, there is always "basis arbing" activity in the marketplace. As a result, with the exception of short-lived periods of stress, the basis is very close to zero.

To summarize, if one were to take a macro-fundamental view of the determinants of CDS premiums, there would be no new information relative to credit spreads obtained from bonds. All the differences between the CDS premiums and credit spreads come from differences in the institutional features of CDS and bond markets, liquidity, and the lack of a perfect match between the terms of the two types of instruments.

4.2.2 Sovereign CDS

Figure 4.1 displays history of the U.S. CDS premiums for the most liquid contracts, which are the five-year ones. The premiums rapidly increased from 0.2 bps in October 2007 to 20 bps during the Lehman Brothers crisis in September 2008. They continued escalating until they peaked at 100 bps in March 2009. As the first round of quantitative easing went into effect, the premium came down and reached the levels seen during the Lehman Brothers default by October 2009. Thereafter, the premiums varied between 20 and 65 bps. The premiums started declining in the middle of 2012 and most recently settled at about 20 bps, which is 100 times larger than the pre-crisis level. In Figure 4.1, we also highlight some of the events associated with the variations in the cost of protection.

The replication argument applied naively to a sovereign CDS contract would imply zero premiums for the U.S. sovereign CDS (U.S. CDS for short) regardless of a contract's maturity. This stark implication clashes with the evidence and prompts us to focus specifically on U.S. CDS as opposed to similar contracts for other developed economies.

In fact, the replication argument is not wrong, it simply is not applicable in this case. Corporate CDS could by valued via replication because cashflows on a risk-free combination of a bond and its CDS are the same as those on a US Treasury bond of matching maturity if a Treasury bond is risk-free. If a Treasury bond is risky then a risk-free combination of a Treasury bond and its CDS cannot be replicated.

This lack of replication implies that one needs to use an equilibrium setting to determine the CDS premium. An equilibrium setup, discussed in the next section, will naturally bring out potential economic causes of a sovereign credit event. Our primary interest lies in how such avenues as monetary and fiscal policies could trigger a credit event and how risks of these contingencies are priced.

Failure to pay could be another trigger of payments on the CDS contracts and received a lot of attention during the congressional debt ceiling debacles of 2011, 2013, and 2015. Many observers believe that one reason for the high U.S. CDS premiums is the chance of default due to the debt ceiling. Indeed, Figure 4.1 shows that the premiums increased from 40 bps to 60 bps during the first debt ceiling debacle of 2011. However, they declined from 45 bps to 25 bps during the second debt ceiling crisis in 2013, and moved briefly between 15 bps and 25 bps during 2015's debacle. We find the debt ceiling avenue to be the least interesting economically because it is a hardwired outcome of a political decision-making process (although the state of the economy may have an impact on a specific stance of politicians). Furthermore, recovery is likely to be close to 100% in the case of such a technical credit event, so it is unlikely to have a material impact on the magnitude of the premiums.

There could be non-credit-related risks that we do not account for in our model, but are potentially responsible for the U.S. CDS premium. First, U.S. CDS are denominated in euros (EUR). The rationale for such a feature is to separate the sovereign risk that the contract ensures from the payments made on this contract. Because U.S. Treasuries are denominated in U.S. dollars (USD), the currency of all deliverable bonds is mismatched with the currency of a contract. This feature complicates the ability to replicate the U.S. CDS using traded securities. Because the date of a credit event is uncertain, one cannot use a currency forward or swap contracts to perfectly offset EUR payments with the ones in USD. While less liquid, the USD-denominated contracts started trading in August 2010 to mitigate this issue. Figure 4.1 contrasts the difference between the EUR and USD contracts, which offers a sense of how large the foreign exchange premium can be. It averages 8 bps for the five-year contract with a standard deviation of 4 bps.

Second, the contracts may command a liquidity premium because they are not the most actively traded ones. We review a number of measures to gauge liquidity of the U.S. CDS market. According to Augustin (2014), with a gross notional amount of $3 trillion, sovereign CDS constitute about 11% of the overall credit derivatives market. Dealers have the largest market share of 70%. In particular, the average gross (net) notional amount of outstanding

U.S. CDS is \$17 (\$3.2) billion. To gain further insight into the trading activity of the U.S. CDS, we report our crude measure of liquidity in Figure 4.2. Because CDS contracts on the Italian government are the most actively traded sovereign CDS, we report the ratio of the weekly net notional amount of U.S. CDS to that of Italian CDS.[1] The average ratio is 18% and it ranges between 6.5% in the beginning of the sample in 2008 to 33% in late 2011 at the peak of the anxieties regarding the European credit crisis and the U.S. fiscal uncertainty. So, clearly the contract is not the most liquid one, but nonetheless has a considerable trading activity.

Third, the Basel III capital charge rule may impact the magnitude of the CDS premium even if there is absolutely no credit risk. Dealers are allowed to buy protection against sovereign default to reduce a capital charge associated with their counterparty risk exposure. As pointed out by Klingler and Lando (2015), a sovereign protection seller would require a positive CDS premium even if the sovereign is riskless because of capital constraints. Anecdotally, some dealers began to implement the rule voluntarily in 2013. Klingler and Lando (2015) empirically attribute a fraction of CDS premiums to this effect in their sample from 2010 to 2014.

Fourth, there is legal risk associated with the credit event determination by a committee comprised of 15 voting members: 10 from the sell side and five from the buy side. At present, there is poor understanding of the incentives of committee participants and how this may affect the decision of whether a credit event took place or not. Last, but not least, there is a risk of uncertain recovery that is determined by the bond auction with a cheapest-to-deliver

[1] We are indebted to Patrick Augustin for sharing his data that was hand-collected from the Depository Trust and Clearing Corporation (DTCC).

option.

4.3 The Model

In our model, we use a standard framework to link nominal debt, taxes, inflation, and aggregate growth to fiscal and monetary policy through the government's budget constraint. The government can maintain the budget balance either by issuing new debt, or raising inflation or taxes. Fiscal default arises when the government can no longer service its debt, rendering it insolvent. As a result, investors may want to buy protection against default events through sovereign CDS contracts.

As we pointed out earlier, we cannot use the standard replication argument to value CDS when Treasuries are themselves subject to credit risk. We therefore complement our setup with a global investor with Epstein and Zin (1989b) preferences who uses her marginal rate of substitution to value assets. This allows us to value any financial security.

In this section, we describe the details of our model. We start with the pricing kernel, which we derive from the global investor's preferences and her aggregate consumption process. Next, we describe he dynamics of the aggregate economy and government. Then we specify the interaction of the government's fiscal and monetary policy stance with the real economy. We conclude with the valuation of defaultable securities such as CDS.

4.3.1 Valuation of Financial Assets

We assume the representative agent with recursive preferences:

$$
\begin{aligned}
U_t &= \left[(1-\beta)C_t^{\rho} + \beta\mu_t(U_{t+1})^{\rho}\right]^{1/\rho}, \\
\mu_t(U_{t+1}) &= E_t(U_{t+1}^{\alpha})^{1/\alpha},
\end{aligned}
$$

where $\rho < 1$ captures time preferences (intertemporal elasticity of substitution is $1/(1-\rho)$), and $\alpha < 1$ captures risk aversion (relative risk aversion is $1-\alpha$). Aggregate consumption is denoted by C_t.

With this utility function, the real pricing kernel is:

$$
M_{t+1} = \beta(C_{t+1}/C_t)^{\rho-1}(U_{t+1}/\mu_t(U_{t+1}))^{\alpha-\rho}.
$$

In our model, we assume the economy is cashless and we use money as a unit of account only. Correspondingly, P_t denotes the price level. The agent is using the nominal pricing kernel $M_{t+1}^{\$} = M_{t+1}\Pi_{t+1}^{-1}$, where $\Pi_t = P_t/P_{t-1}$ is the inflation rate, to value nominal assets. We provide the determinants of endogenous inflation below.

Consumption is assumed to have the following dynamics:

$$
\begin{aligned}
\Delta c_{t+1} &= \nu + x_t + \sigma_c\varepsilon_{t+1} \\
x_{t+1} &= \varphi_x x_t + \sigma_x\varepsilon_{t+1},
\end{aligned}
$$

where the shock ε_{t+1} is $\mathcal{N}(0,1)$. This assumption is similar to Bansal and Yaron (2004b), Model I, by allowing for a time-varying conditional mean in consumption growth. The shock to consumption growth and its expectation are perfectly correlated for simplicity. ν captures the deterministic trend growth rate.

4.3.2 The Government and the Economy

We assume that output Y_t evolves as follows:

$$\Delta y_{t+1} = \nu + \varphi_y(\tau_t - \tau) + \sigma_y \varepsilon_{t+1}, \tag{4.1}$$

where $\tau_t = \log \mathcal{T}_t$ is the (log) tax rate at time t and τ is its unconditional mean. The trend growth rate of output growth is set to that of consumption growth, ν, to ensure a balanced growth path. We assume the existence of one single tax rate and remain agnostic about its precise nature. This tax rate is time-varying and its dynamics arise endogenously through the fiscal authority's response to debt, as specified below.[2] An identical shock to output and consumption serves as a modelling shortcut to the resource constraint that arises in general equilibrium models.

Importantly, we assume that deviations of the prevailing tax rate from the mean affect future growth prospects, through the parameter φ_y. Consistent with the evidence (Croce, Kung, Nguyen and Schmid, 2012; Jaimovich and Rebelo, 2012), φ_y will be negative and small in our calibration, so that raising taxes will depress future growth prospects. While

[2] One might worry that the economy can attain infinite output as the tax rate approaches zero. In practice, such a scenario is not feasible due to the endogenous nature of taxes in our model.

we assume this link directly, our specification is in the spirit of the literature on endogenous growth and taxation in which an elevated tax burden endogenously decelerates growth through its effect on innovation (Rebelo, 1991; Croce, Nguyen and Schmid, 2013).

Let G_t be the government expenditures as a fraction of output. Its log dynamics are given as follows:

$$g_{t+1} = (1 - \varphi_g)g + \varphi_g g_t - \sigma_g \varepsilon_{t+1}.$$

The minus sign in front of the volatility coefficient σ_g highlights the perfect negative correlation between shocks to output and expenditures, so that a bad shock to the economy corresponds to an increase in expenditures.

In order to finance expenditures, the government raises taxes and issues nominal debt. For simplicity, we assume that the government directly taxes output, so that the tax revenue in levels at time t is given by $\mathcal{T}_t Y_t$. We view this specification as a tractable way to capture the link between taxation and the aggregate economy. We assume that the government issues nominal debt with a face value N_t. The real face value of debt as a fraction of output is:

$$B_t = (N_t/P_t)/Y_t.$$

The government finances its expenditures with two types of bonds: short-term with a price of Q_t^s and long-term with a price of Q_t^ℓ per \$1 of face value. Short-term bonds mature in one period. We think of the short-term bond as a monetary policy instrument. We model long-term debt so as to allow for more realistic modeling of default and to be able to give an account of the quantitative easing episode within the context of our setup. For tractability, we assume that short- and long-term bonds are issued in constant proportion: the nominal amounts are $N_t^s = \omega N_t$ and $N_t^\ell = (1-\omega)N_t$, respectively. Variation in ω can represent shifts in the overall maturity structure of government debt held by the public, such as those induced by the quantitative easing program of the Federal Reserve. We explore these variations later in the paper.

To retain a stationary environment with long-term debt, we model it via a sinking fund provision in the spirit of Leland (1994). A long-term bond specifies a coupon payment γ every period and requires a fraction λ of the debt to be repaid every period. This amounts to a constant amortization rate of the bond. Although this is perpetual debt, it has an implicit maturity that is determined by the repayment rate λ. If $\lambda = 1$, this simplifies the bond to the one-period one; if $\lambda < 1$, then the implicit bond maturity is longer and proportional to $1/\lambda$.

In the absence of default, the properties of debt and taxes are connected via the GBC:

$$\begin{aligned} \mathcal{T}_t Y_t + Q_t^\ell (N_t^\ell - (1-\lambda)N_{t-1}^\ell)/P_t + Q_t^s N_t^s/P_t \\ = (\gamma+\lambda)N_{t-1}^\ell/P_t + N_{t-1}^s/P_t + G_t Y_t. \end{aligned} \tag{4.2}$$

The GBC requires that government expenditures $G_t Y_t$ and due payments on short- and long-term debt (coupon payments and amorization) $(\gamma+\lambda)N_{t-1}^\ell/P_t + N_{t-1}^s/P_t$ have to be covered either by tax income $\mathcal{T}_t Y_t$ or by issuing new short- or long-term debt $Q_t^\ell(N_t^\ell - (1 -$

$\lambda)N^{\ell}_{t-1})/P_t + Q^s_t N^s_t/P_t$. The GBC implies the following tax rate:

$$\mathcal{T}_t = G_t - Q_t B_t + CF_t B_{t-1}\Pi_t^{-1}(Y_t/Y_{t-1})^{-1},$$

where $Q_t \equiv \omega Q^s_t + (1-\omega)Q^{\ell}_t$ is the market value of one unit of the debt-to-GDP ratio, and $CF_t \equiv \omega + (1-\omega)(\gamma + \lambda + (1-\lambda)Q^{\ell}_t)$ is the promised cash flow per one unit of debt.

We capture the monetary and fiscal policy stance by means of policy rules. In case of the monetary policy, this is achieved by a standard Taylor rule linking the nominal short-term interest rate to macroeconomic variables. In line with the literature, we assume that the central bank responds to inflation and output growth, which we view as corresponding to the output gap in the New-Keynesian literature.

In the case of fiscal policy, we assume that the government sets the amount of new debt issued in response to the amount of debt outstanding and expected economic conditions x_t. Mechanically, then, the prevailing tax rate has to be such as to establish budget balance in the GBC.

Our specification is related to policy rules examined in the recent literature on monetary-fiscal interactions (Bianchi and Ilut, 2014; Leeper, 1991, 2013; Schmitt-Grohe and Uribe, 2007). In particular, it is shown by Schmitt-Grohe and Uribe (2007) that, in a rich New Keynesian dynamic stochastic general equilibrium model, policy rules of this sort lead to welfare levels that are quantitatively indistinguishable from those stemming from optimal Ramsey policies, in which fiscal and monetary policies are designed to maximize welfare. Relatedly, Cuadra, Sanchez and Sapriza (2010) show how debt and tax dynamics that are consistent with our specification arise endogenously in Ramsey optimal fiscal policies when the government has the option to default.

Summarizing, the government controls the real debt and nominal interest rate through the fiscal and monetary policies, respectively, as follows:

$$
\begin{aligned}
b_t &= \rho_0 + \rho_b b_{t-1} + \rho_x x_t + \xi_t^b, && \text{(Fiscal policy)} \\
-q_t^s &= \delta_0 + \delta_\pi \pi_t + \delta_y \Delta y_t + \xi_t^q, && \text{(Monetary policy)}
\end{aligned}
$$

where $\pi_t = \log \Pi_t$ is the (log) inflation rate. Intuitively, the parameter ρ_b determines how fast the government intends to pay back outstanding debt. Similarly, we allow for the possibility that the government increases public debt in bad times by responding to x_t. The parameter $\rho_x < 0$ determines the intensity of this interaction. Innovations $\xi_t^b \sim \mathcal{N}(0, \sigma_b^2)$ and $\xi_t^q \sim \mathcal{N}(0, \sigma_q^2)$ capture the uncertainty about the future indebtedness of the government and monetary policy, respectively. As is well-known, obtaining determinacy imposes restrictions on the parameters of both the fiscal and the monetary policy rules (see Leeper, 1991, 2013), which we discuss in the calibration section.

Given the real pricing kernel, the Taylor rule implies the dynamics of inflation as in Gallmeyer, Hollifield, Palomino and Zin (2007a). This reflects the fact that the nominal short rate implied by the nominal pricing kernel and that implied by the Taylor rule must be consistent. In this line of work, which evolves around endowment economies with fully flexible pricing mechanisms, monetary policy has no scope to affect real variables. In our setting, the GBC is the channel through which monetary policy influences real quantities because it affects the real value of outstanding debt, which in turn impacts the tax rate and output growth.

4.3.3 Fiscal Default

We think of government default in the model in the sense of fiscal default, namely scenarios in which budget balance can no longer be restored by further raising taxes, as opposed to mere technical defaults resulting from the political decision-making process. Our model captures the negative effect of taxes on the tax base by means of the output growth equation (4.1). This effect limits the future stream of surplus the government can generate in any state, and thus the maximal amount of debt it can repay.

Limits to raising taxes arise frequently in macroeconomic models with distortionary taxes in the context of Laffer curves. Laffer curves relate the government's tax revenue to the prevailing tax rate. While they typically start out increasing for low tax rates, they often reach the "slippery slope" (Trabandt and Uhlig, 2011), where raising tax rates actually lowers tax revenue, so that tax policy becomes an ineffective budget-balancing tool. This is because distortionary taxation tends to negatively affect the tax base, such as in the case of labor taxes, where excessive taxation reduces work incentives.

To capture this Laffer curve intuition, we introduce two notions of expected surplus. One is the present value of tax receipts minus the expenditures:

$$S_t = E_t \sum_{j=1}^{\infty} M_{t,t+j} (\mathcal{T}_{t+j} - G_{t+j}) Y_{t+j} / Y_t. \tag{4.3}$$

Note that S_t coincides with the market value of debt, $Q_t B_t$, only if there is no default. The second one is expected sustainable surplus. It corresponds to the maximal tax rate, $\mathcal{T}_t^*$, that is feasible without lowering tax revenues:

$$S_t^* = E_t \sum_{j=1}^{\infty} M_{t,t+j}(\mathcal{T}_{t+j} \wedge \mathcal{T}_t^* - G_{t+j}) Y_{t+j}^* / Y_t^*,$$

and $\mathcal{T}_t^*$ solves $S_t = S_t^*$. The notation Y_t^* highlights the different dynamics of output if the tax rate changes from the one prescribed by the GBC. If $\mathcal{T}_t > \mathcal{T}_t^*$, then the shrinking tax base would decrease the surplus.

These equations capture the idea that if $\mathcal{T}_t$ becomes greater than $\mathcal{T}_t^*$, then the current government policies will not be sustainable. So, the government should either adjust one of its policies, or default. We assume that the government is committed to its expenditures, as well as monetary and debt management rules. Expenditures reflect, to a large extent, various entitlement programs that are hard to renegotiate. We intentionally do not allow changes in the policy rules. By doing so we effectively assume that the Fed will never be insolvent separately from the Treasury, that is, the Fed has *fiscal support* of the Treasury (Reis, 2015). These assumptions allow us to highlight the default channel of CDS premiums. In practice, many changes may take place in an extreme fiscal situation. Studying all the possibilities is beyond the scope of this paper.

Indeed, if $\mathcal{T}_t$ becomes greater than $\mathcal{T}_t^*$, the expected surplus required to service debt exceeds the surplus that the government can sustain by committing to its policy rules. In this case, the government will no longer be able to honor its long-term financial obligations. At this stage, rational investors will not be willing to roll over the short-term debt. Being unable to access the bond market, the government has to default.

Fiscal theory of the price level (FTPL) also features a prominent role for the GBC in a similar situation of fiscal stress. FTPL requires the GBC to hold. As a result, the price level P_t is determined via the equality of the market value of debt, $Q_t B_t$, and expected surplus (4.3). Cochrane (2011a) points out that in this case reaching the top of the Laffer curve leads to fiscal inflation instead of default. As Leeper (1991); Woodford (2003) show, such a mechanism leads to determinate equilibrium only when the fiscal policy is active (locally non-Ricardian) and the monetary policy is passive.

In our model, the monetary policy rule satisfies the Taylor principle implying a unique bounded path for inflation, P_t/P_{t-1}. The fiscal policy operates on the stock of government debt and with $\rho_b < 1$ ensures unique bounded path for this stock. Thus, both policies are active, so they are uncoordinated (Cochrane, 2011a). This is feasible because we allow for default via violation of the GBC.

4.3.4 Defaultable Securities

We denote default time by:

$$t^D = \min\{t : \tau_t^* \leq \tau_t\},$$

and probability of default by P_t^D. So default will take place at time $t+1$ if $t^D = t+1$. Given the definition of the one-period ahead default probability, one can value the short-term bond as:

$$Q_t^s = E_t\left(M_{t,t+1}^{\$}\left[(1-\mathbf{1}_{\{t^D=t+1\}})+(1-L)\mathbf{1}_{\{t^D=t+1\}}\right]\right), \tag{4.4}$$

where L is the the loss given default. We can also value the long-term bond by relying on one-period ahead default probabilities via the following recursive representation:

$$Q_t^\ell = E_t\left(M_{t,t+1}^{\$}\left[(\gamma+\lambda+(1-\lambda)Q_{t+1}^\ell)(1-\mathbf{1}_{\{t^D=t+1\}})+(1-L)\mathbf{1}_{\{t^D=t+1\}}\right]\right).$$

A CDS contract has two legs: the premium leg pays the CDS premium CDS_t^T every quarter until a default takes place. It pays nothing after default. The protection leg pays a fraction of the face value of debt that is lost in default and nothing if there is no default before maturity. Accordingly, the value of the fixed payment to be made at time $t+j$ is $CDS_t^T \times E_t(M_{t,t+j}^{\$}\mathbf{1}_{\{t+j<t^D\}})$. As a result, the value of the premium leg is equal to:

$$\text{Premium}_t^T = CDS_t^T \cdot \sum_{j=1}^{T} E_t M_{t,t+j}^{\$}\mathbf{1}_{\{t+j<t^D\}}.$$

The protection leg can be represented as a portfolio of securities, each of them maturing on one of the days of the premium payment, $t+j$, and paying L if default took place between $t+j-1$ and $t+j$, and nothing otherwise. Thus,

$$\text{Protection}_t^T = L \cdot \sum_{j=1}^{T} E_t(M_{t,t+j}^{\$}\mathbf{1}_{\{t+j-1<t^D\leq t+j\}}).$$

The CDS premium CDS_t^T is determined by equalizing the values of the two legs.

Importantly, CDS premiums depend on the joint behavior of the nominal pricing kernel and default probabilities. While we specify the process for the real pricing kernel exogenously, default probabilities reflect the endogenous responses of our economy to shocks. To the extent that the endogenous dynamics of our economy are predictive of high government indebtedness in times of low consumption growth prospects, the global representative agent in our model will require compensation for potential default losses during such episodes. In other words, the prices of default-sensitive securities will reflect a risk premium beyond expected losses.

4.3.5 Discussion

In our simple model of the U.S. economy, we allow for scenarios that endogenously trigger the government's default on its debt. Before we describe the model's solution, we briefly review its ingredients.

There are four building blocks. In the first block, we describe the dynamics of the aggregate economy as given by (4.1). In the second block, we outline governement-related objects, such as the fiscal and monetary rules, as well as the GBC. in the third block, we describe the default condition that is based on the Laffer-curve argument. Finally, in the fourth block we derive a risk-sensitive pricing kernel from recursive preferences given a process for consumption growth. While blocks one and four reflect a standard structure familiar from the literature on long-run risks following Bansal and Yaron (2004b), we add a specification of the government's and the central bank's policy instruments and default event in blocks two and three. Although we do not complete the model in general equilibrium, we link all these blocks through the government budget constraint.

Inflation arises endogenously as the nominal interest rate implied by the Taylor rule has to coincide with that implied by the nominal pricing kernel. Inflation thus has real effects in our model, because it affects the real value of debt and thus the prevailing tax rate, which in turn impacts expected growth. Growing debt-financed government deficits can lead to episodes of elevated tax rates, which may trigger default.

Default probabilities are reflected in the pricing of defaultable bonds. Treasury bonds and thus the central bank's policy instrument are themselves subject to credit risk. Even the value of a hypothetical nominal bond that has no cash flow risk depends on the default probability because inflation does. This is because the combination of fiscal and monetary policies and the GBC imply that inflation depends on the risky government debt.

4.4 Quantitative Analysis

In this section, we evaluate to what extent the possibility of a U.S. fiscal default can quantitatively account for the CDS premiums observed since the onset of the recent financial crisis. We calibrate our model in a way that is quantitatively consistent with salient features of the recent U.S. monetary and fiscal experience. We check whether the calibrated model implies CDS premiums consistent with the ones in the data. Moreover, our risk-sensitive specification allows for a decomposition of CDS premiums into a default probability and a default risk premium. Finally, we can use our calibrated model as a laboratory for a set of counterfactual experiments that highlights the different channels that affect valuation of the sovereign default risk. We start by describing our calibration approach, and then illustrate the main mechanisms driving the quantitative results and counterfactuals.

4.4.1 Calibration

We report our baseline parameter choices in Table 4.1. We calibrate the model at a quarterly frequency, consistent with the availability of macroeconomic data. We need to calibrate parameters from four different groups. First, we follow the literature on long-run risks to select our preference parameters. Second, we pick parameters governing the exogenous stochastic processes in our model, such as output growth, consumption growth and, critically, government expenditures. We do so by matching time series moments of their empirical counterparts. Because our data on CDS spreads cover a relatively short and recent time period, we focus on a similar sample to construct the empirical counterparts for the macroeconomic moments. Specifically, we use the period from 2000 to 2014. Third, we choose parameters controlling the maturity and payment structure of government debt. Finally, we specify the fiscal and monetary policy rules to match the recent U.S. policy experience in a high debt environment. We remove deterministic trend by setting $\nu = 0$.

Our choice of preference parameters follows Bansal and Yaron (2004b). As is well-known, the combination of relatively high risk aversion and an intertemporal elasticity of substitution above one allows the rationalization of sizeable risk premia in many markets. In a similar vein, the calibration of the consumption growth process reflects long-run risks, and the parameter choices follow Bansal and Yaron (2004b). To calibrate G_t, we fit an autoregressive process to the GDP-government expenditures ratio, which helps us to determine its mean, autocorrelation, and volatility. Turning to the output dynamics, a critical parameter is φ_y, which is the elasticity of output growth with the respect to taxes. Intuitvely, we would expect a raise in taxation to be bad news for trend growth. By setting $\varphi_y = -0.024$, based on the empirical estimate obtained in Croce, Kung, Nguyen and Schmid (2012), our parameter choice is consistent with that notion. We choose σ_y to match the relative volatility of consumption and output growth observed in the data.

The weighted average maturity of U.S. Treasury bonds is 59 months on average, but it has been rising consistently over the past few years, reaching about 69 months by the end of 2015 (U.S., 2010). In addition, debt of maturity that is less than one year represents about 20%-30% of all outstanding debt. These numbers allow us to select the (ω, λ) combination. We pick ω to be 0.2 to match the latter fact. In order to match the long-term average maturity, we select $\lambda = 0.04$. Finally, there is little guidance about the recovery rate in a potential default of the U.S. government. Perhaps erring on the conservative side, we assume a recovery rate of 80% ($L = 0.2$) in our benchmark calibration. This is quite a bit higher than in the U.S. corporate bond market, where recovery rates around 50% are a good starting point, as reported, for example, in Chen (2010).

Our calibration of the parameters in the policy rules is quite standard. We choose the parameters of the Taylor rule following the parameterization in Gallmeyer, Hollifield, Palomino and Zin (2007a). This choice implies an average inflation rate in line with the data. In order to determine the parameters in the fiscal rule, we run a regression of the debt-to-GDP ratio on its lagged value, and a proxy for expected consumption growth. We compute an estimate of x_t from data on consumption growth using the Kalman filter and the assumed model parameters.

4.4.2 Quantitative Results

We now present quantitative results based on model simulations. The possibility of default induces strong nonlinearities in both payoffs and the discount factor. Therefore, we use a global, nonlinear solution method. Endogenous variables are approximated using Chebychev polynomials and solved for using projection methods. Appendix 4.6 outlines the procedure.

We start by discussing the macroeconomic implications of the model. Taking these as a benchmark, we proceed to examine the quantitative implications for CDS premiums.

Matching Quantities

In table 4.2, we summarize the main implications for the macroeconomic quantities. The average market value of debt to GDP ratio in the model is about 0.92, which is within one standard error of the one in the data. Identifying and dtermining one single relevant aggregate tax rate is complicated by the tax code. We use the estimates from McGrattan and Prescott (2005) as our sample statistics. Our model matches these numbers quite closely. Average inflation is matched as well.

While matching basic macroeconomic moments is important to discipline our analysis, our main interest is the potential for fiscal defaults. The results in table 4.2 provide a sense of the possibility of such events in our model. The results suggest that the unconditional mean of the debt limit is in the range of a 120% to 165% percent debt-to-GDP ratio. These numbers are well within the range of the CBO long-term debt projections (CBO, 2016). The corresponding tax limit is 70% to 97%. These are large numbers as compared to the average current tax rate. Yet the low bound is not far from that of Trabandt and Uhlig (2011). Moroever, we would expect debt and tax limits to fall during economic downturns. We confirm this intuition below.

The estimated distribution of the tax limit determines fiscal default probabilities in the model. Our benchmark calibration yields a one-year ahead default probability of 0 to 0.4%. As one external validation of this magnitudes, Moody's estimates this probability at 0.05% (Tempelman, 2011). Below we explore to what extent such a default probability can account for observed CDS premiums.

Inspecting the mechanism

To dissect the main economic mechanism underlying our quantitative results, we inspect the response of our economy to a negative one-standard deviation shock to the long-run trend, x_t. In our model, innovations to variables other than policy shocks are perfectly correlated, as is the case in a general equilibrium environment. Thus, the behavior of the variables will be driven by the properties of the long-run trend. Figure 4.3 illustrates the comovement of all our variables. The same patterns are also reflected in the unconditional correlations reported in Table 4.3.

A negative shock to the long-run consumption trend triggers a raise in government expenditures. This is consistent with the countercyclicality of government expenditures. Naturally, these expenditures give rise to financing needs. Our fiscal policy rule then requires that they are partially financed by the government issuing debt. Due to the fiscal rule, the government debt is realistically countercyclical. However, the GBC requires budget balance, so that elevated expenditures also lead to a rise in the tax rates. Our specification of the fiscal block of the model thus is consistent with countercyclical fiscal policies.

Let us now examine how fiscal and monetary policy interact in our model. First, given our specification of the process governing output growth, a higher tax rate depresses expected output growth. As a result, an accommodative central bank tries to stimulate the economy by lowering the nominal short-rate, thereby creating inflation. This is because the central bank adheres to the Taylor rule. As a result, inflation increases in response to a negative shock, generating countercyclical inflation.

While inflation displays substantial short-run volatility, it also exhibits a small, persistent component. This reflects the central bank's response to long-lasting bad news about output growth induced by a persistent rise in taxes. This small persistent component is important for generating realistic nominal term structure in the model. This is because endogenous long-run inflation risk generates negative correlation between expected inflation and consumption growth, implying a substantial inflation risk premium.

Since the tax base shrinks when taxes go up, and government expenditures increase persistently, the fiscal limit (maximal sustainable tax rate) declines, and default probabilities rise. Note that this rise in default probabilities coincides with an upward jump in the stochastic discount factor, or marginal utility. In other words, in our model episodes with high default probabilities and thus high potential losses endogenously coincide with high marginal utility times. To bear the risk of such losses, agents in our model thus require a credit risk premium to hold defaultable securities. It is this credit risk premium that allows our model to generate non-trivial CDS spreads.

Term Structures of Risk-free and Defaultable Securities

As discussed above, the standard replication approach for corporate CDS contracts does not apply in the context of U.S. sovereign CDS premiums due to the lack of a risk-free benchmark. While U.S. Treasury bonds are often conveniently interpreted as such a benchmark, the very notion of observed non-zero CDS premiums on U.S. government debt invalidates this view. When U.S. government debt is subject to credit risk itself, approaches other than replication are called for when determining CDS premiums. Our equilibrium model offers such an approach.

The pricing kernel in our model implies an equilibrium term-structure of real risk-free yields. The term structure of U.S. Treasury yields cannot serve as an empirical counterpart to these yields. There are two sources of discrepancies. First, the term structure of Treasury yields refers to nominal bonds. Second, and more importantly, these bonds are not insulated from credit risk as highlighted above. Nonetheless, one can infer a theoretical counterpart to U.S. Treasury yields from our model by using the nominal pricing kernel and accounting for a possibility of default similar to expression (4.4).

In table 4.4, we summarize three yield curves inferred from our calibrated model. We show the term structure of risk-free yields that correspond to expectations of the equilibrium *real* pricing kernel at various horizons. We report what we call the term structure of pseudo risk-free nominal yields. This curve corresponds to expectations of the equilibrium *nominal* pricing kernel. We label them pseudo risk-free, as endogenous inflation in our model reflects the risk of a government default, while the real discount factor does not. We also report the yield curve of nominal, defaultable bonds that correspond to expectations of the nominal pricing kernel accounting for government default probabilities at various horizons, that is, the term structure of *default probabilities*.

The term structure of real risk-free yields is mildly downward sloping, which is consistent with the long-run risk paradigm. In the context of our model, this is an implication of a high intertemporal elasticity of substitution. Empirically, no clear consensus about the average slope of the real term structure has yet emerged. Various researchers have interpreted the data on inflation-protected bonds (TIPS) in the U.S. as pointing to an upward sloping real yield curve, while others point to the short data sample and conflicting evidence from a longer data sample on inflation-indexed bonds in the U.K. Neither line of argument provides guidance for our purposes, as even an upward sloping term structure of real yields does not allow disentangling the effects of inflation and default risk, which is at the core of our setup.

Given the real risk-free term structure, our model generates nominal pseudo risk-free yield curves that are on average upward sloping. Thus, our model predicts a realistically upward sloping term structure of inflation expectations and, importantly, inflation risk premia. As described earlier, inflation is endogenously countercyclical in the model. Indeed, adhering to the Taylor rule requires the central bank to raise inflation in response to elevated government indebtedness, in order to restore budget balance by eroding the real debt burden. This is because high debt leads the government to raise taxes, which typically depresses long-term growth prospects, and lowers the output growth, which the central bank reacts to. Inflation thus erodes away the payoff to holding debt precisely in high marginal utility states, so that bond holders will require an inflation risk premium to hold government bonds.

We find that the term structure of nominal defaultable yields – the model counterpart of U.S. Treasury yield curves - is upward sloping as well. This curve reflects inflation expectations and an inflation risk premium adjustment, and it also accounts for the term structure of default expectations and a *default risk premium*. The default risk premium accounts for the fact that the model is naturally predictive potential government defaults that may occur during high debt episodes, which we show to endogenously coincide with high marginal utility states in the model. Notably, the defaultable term structure is steeper than the nominal pseudo risk-free curve, implying that default risk cannot be avoided by inflating away debt. Such default premia thus reflect market expectations about the limits to the ability of the central bank to restore budget balance by means of inflation. This is consistent with the empirical evidence in Hilscher, Raviv and Reis (2014) on the limited ability of inflation to balance the government budget. We explore this further in the counterfactual analysis.

Fiscal Defaults and CDS Premiums

We now examine the pricing of CDS contracts and the link between CDS premiums and the probability of a U.S. fiscal default. Table 4.5 provides the results. We report average CDS premiums from the data and from the model, in basis points. Columns (1) to (3) report various versions of the data depending on the contract denomination (EUR in (1) and (2), and USD in (3)), and sample (2007-2016 in (1), and 2010-2016 in (2) and (3)). The main differences in the averages are driven by the currency of the contract denomination rather than the sample.

The model-based averages are reported in column (4) of table 4.5. Overall, we see that our model delivers an upward sloping term structure of CDS premiums with magnitudes that are consistent with the data. They are particularly close to the USD-denominated numbers. This is natural as we ignore currency risk in our model. There are some quantitative discrepancies, but, as highlighted in section 4.2.2, we do not account for the risks associated with various institutional features of the contract in our model. On balance, the results suggests that accounting for default risk goes a long way toward explaining the magnitudes of the CDS premiums.

Traditionally, credit models better fit premiums at the longer end of the curve than at the short end. This is a standard implication of structural models of the defaultable term structure, especially consumption-based ones. We find magnitudes that are consistent with the data because our default boundary is moving over time and time is discrete, so there is no perfect anticipation of default in the next instant.

The CDS premiums we find are substantial despite modest default probabilities in the model. As in all models of defaultable securities, default spreads can be decomposed into two components: expected losses, and default risk premia. In our model, losses given default are known, so the default risk premium reflects the compensation protection sellers require in order to bear the risk of experiencing the default event in high marginal utility states. The calibrated loss is relatively small. It is conservative because the burden of fitting CDS premiums rests on the ability of our model to generate high default risk premiums.

Indeed, the results of column (5) of Table 4.5 confirm that the risk premiums are substantial. The column displays the size of CDS premiums if investors were risk-neutral. The ratio of the numbers in column (4) to the ones in column (5) reflects the magnitude of the aforementioned default premium. This ratio is approximately equal to 3 across all maturities. The default premium is so large because fiscal default endogenously is more likely to happen in high marginal utility states, so that selling default insurance earns a high covariance risk premium, akin to default risk premia in other debt markets, such as corporate bond markets. The model is thus consistent with high CDS premiums, reflecting investors' rational forecasts of the likelihood of U.S. fiscal stress.

Inflating and Taxing Away Debt

A common view is that a U.S. default is unlikely as the government can always resort to higher taxation or creating inflation to restore budget balance. We now examine a potential effect of such scenarios through the lens of our model. We represent an attempt to inflate away the debt burden with a shift towards a looser monetary policy stance. This is captured by a shift towards lower values of δ_π in the Taylor rule. Similarly, we can represent a shift towards a fiscal policy with more aggressive taxation by lowering ρ_b. This means that new debt is issued in smaller amounts, which would imply higher taxes via the GBC.

We now quantitatively evaluate variation in the policies using counterfactual analysis. Table 4.6 reports the results for the monetary policy. Loosening of the monetary policy stance has the desired effect of increasing the average inflation rate. Similarly, as expected, the average debt is reduced, which comes with a reduction in default probabilities.

Remarkably, CDS premiums rise. This happens because an increase in mean inflation is accompanied by an increase in its volatility. Larger shocks to inflation make the fiscal limit more volatile, thus it can fall more relative to its mean, as highlighted in Figure 4.3. This decrease in the fiscal limit is accompanied by an increase in default probability even though its mean declines. As a result, the risk premium amplification mechanism that we discuss in the previous section delivers larger risk premiums despite the decline in expected losses.

We obtain a similar result in the case of an attempt to "tax away" debt. As Table 4.7 shows, using taxes more aggressively to respond to economic conditions does lead to a fall in the average debt burden and default probabilities, while the average tax rate goes up. The volatility of taxes also goes up, and, again according to Figure 4.3, the fiscal limit declines relative to its mean. As a result, the same mechanism is at play.

Our counterfactual exercises illustrate some of the pitfalls associated with the notion of inflating or taxing away the government debt obligations. While these policies tend to have the desired effects for the first moments of debt, taxes, and inflation, they come with endogenous movements in second moments. These movements are priced in our risk-sensitive framework and push CDS risk premiums in the opposite direction.

Shifting Debt Duration

In our model, we represent monetary policy through a standard Taylor rule linking the short-term interest rate to inflation and output growth. In response to the recent financial crisis, the Federal Reserve increasingly relied on non-standard monetary policy instruments. Under the label of quantitative easing (QE), these measures effectively shifted the average duration of outstanding Treasuries. Arguably, the use of these instruments critically shaped the Treasury markets in our sample.

We now discuss our analysis of how shifts in debt duration affect default probabilities and CDS premiums, through the lens of our model. In our quantitative experiments, we capture shifts in debt duration by variations in λ, that is, the amortization rate of long-term debt in the government's debt portfolio. Greenwood, Hanson, Rudolph and Summers (2014) emphasize that the Fed's QE activities were offset by the Treasury's maturity extension program. As a result, the maturity of the consolidated balance sheet of the two entities experienced relatively little change. Our model is silent about the different roles of the two entities, so our exercises with varying effective maturity of debt apply to the joint implication of the Fed's QE policy and the Treasury's debt management.

In table 4.8, we report the results. Increasing λ effectively corresponds to a shortening of debt duration. We see that in our model such shifts come with elevated default probabilities, and hence CDS spreads. The last columns give a sense of the mechanism at work, in that shortening debt duration is accompanied by increases in the volatility of taxes as well as the volatility of the market value of debt. This is consistent with the notion of elevated rollover risk. When λ goes up, the fiscal rule dictates that the government has to refinance debt, or roll over, a larger fraction of debt in episodes with depressed long run growth prospects. Intuitively, refinancing thus occurs when bond prices fall and expenditures are high, so that tax rates have to increase relatively more to restore budget balance. Clearly, more pronounced tax raises only exacerbate default probabilities, as reaching the tax limit is more likely.

4.5 Conclusion

Premiums on U.S. sovereign CDS rose to unprecedented levels during the recent financial crisis, and still remain at elevated levels today. Given the apparent size of these premiums, commentators have widely speculated whether they indeed reflect financial market expectations about an impending U.S. default. After all, casual inspection suggests that the U.S. government can always balance the budget by raising taxes or else, by inflating away the real value of debt. In this paper, we ask whether the likelihood of a fiscal default, namely a state when tax- or inflation-based finance is no longer available, justify the size of the observed premiums.

We develop an equilibrium model of the U.S. economy with a representative agent featuring recursive preferences, in which monetary and fiscal policy jointly endogenously determine the dynamics of growth, debt, taxes and inflation. Fiscal default obtains when the economy approaches the slippery slope of the Laffer curve, where a further increase of the tax rate reduces tax revenue. Our equilibrium approach allows us to value CDS contracts reflecting risk-adjusted probabilities of fiscal default, thereby overcoming the challenge that standard replication arguments for CDS pricing fail in the absence of the risk-free benchmark.

We find that our model quantitatively generates premiums on CDS contracts in line with the U.S. experience since the recent financial crisis. Annualized CDS premiums peak at around 100 bps in the model. This is because high debt and default probability episodes endogenously correspond to high marginal utility states in the model, so that selling default insurance earns high risk premia. Importantly, CDS premiums raise persistently even in response to small shocks to the likelihood of fiscal default, as investors with recursive preferences anticipate and dislike such states. Our model is thus consistent with the view that high CDS premiums reflect investors' rational forecasts of the likelihood of U.S. fiscal stress.

Our results also cast a doubt on the notion that the government can restore budget balance by simply inflating or taxing away debt. In the context of our model, elevated mean inflation and taxes do come with a reduction of average debt, and default probabilities, but similarly they bring about endogenous movements in the second moments of these variables, both in volatilities and correlations. While our partial equilibrium model merely suggests that such policies can also lead to a raise in risk premia, such movements would likely have welfare implications in a richer general equilibrium framework.

Table 4.1: Calibration

Parameter	Description	Value
1. Preferences		
β	Subjective discount factor	0.997
ρ	IES=$(1-\rho)^{-1}$	1/3
α	RRA=$(1-\alpha)$	-9
2. Exogenous processes		
σ_c	Volatility of shocks to Δc	0.014
σ_x	Volatility of LLR process	$\sigma_c \times 0.044$
φ_x	Autocorrelation of LLR	0.936
σ_y	Volatility of shocks to Δy	0.022
φ_y	Elasticity output growth - taxes	-0.024
σ_g	Volatility of G	0.075
φ_g	Autocorrelation of G	0.990
3. Different Maturities and Default		
ω	Share of short term debt	0.200
λ	Repayment rate	0.040
γ	Coupon payment	0.050
L	Losses in the default event	0.200
4. Policy parameters		
δ_π	Inflation loading coefficient	1.500
δ_y	Output growth coefficients	0.500
σ_q	Monetary policy shock	0.003
ρ_b	Debt loading coeffcient	0.960
ρ_x	Expected growth coeffcient	-0.220
σ_b	Fiscal rule shock	0.008

Notes. We describe the calibration in section 4.4.1.

Table 4.2: Macroeconomic Moments

	Data		Model	
	Mean	Std	Mean	Std
Market value of debt ($Q_t B_t$)	0.916	0.086	0.903	0.112
Taxes ($\mathcal{T}_t$)	0.326	0.031	0.354	0.124
Annual gross inflation (Π_t)	1.011	0.012	1.012	0.027
Debt limit (S_t^*)			1.414	0.119
Tax limit ($\mathcal{T}_t^*$)			0.829	0.068
Default probability (P_t^D)			0.002	0.001

Notes. This table reports basic macro moments. The empirical moments come from the BEA quarterly data and cover the sample period 2000: Q1 to 2016: Q2. All moments are annualized. To compute theoretical moments, we simulate the data at quarterly frequency 100 times for 15 years and average across simulations.

Table 4.3: Correlation

	$\log(Q_t B_t)$	τ_t	g_t	s_t^*	τ_t^*	$\log P_t^D$	$\log E_t(\Pi_{t+1})$
Δc_t	-0.305	-0.680	-0.509	0.493	0.627	-0.904	-0.478

Notes. This table reports correlations between main variables in our model. We simulate the data at quarterly frequency 100 times for 15 years and average across simulations.

Table 4.4: Term Structure

Maturity, years	Model Real yields	Pseudo risk-free yields	Yields	Data Yields
1	0.52	1.90	2.09	2.01
3	0.46	2.27	2.48	2.53
5	0.38	2.59	2.83	3.03
10	0.21	3.41	3.68	3.76

Notes. This table reports annualized mean yields across horizons of various fixed income instruments in the model and the data. The empirical moments correspond to U.S. nominal Treasury bonds in the sample between 2000: Q1 and 2016: Q2. We simulate the data at quarterly frequency 100 times for 15 years and average across simulations.

Table 4.5: CDS Spreads

Maturity, years	Data € (1)	Data € (2)	Data $ (3)	Model (4)	L·Mean(P_t^D) (5)
1	16.13	15.61	13.29	11.42	3.97
3	22.18	21.66	17.32	15.94	5.62
5	29.34	31.29	23.11	20.72	7.30
10	40.79	46.86	34.92	33.91	12.16

Notes. This table reports annualized mean CDS premiums across maturities in the data and the model. Column (1) displays EUR-denominated contracts from 2007: Q2 to 2016: Q2. Column (3) displays USD-denominated contracts from 2010: Q3 to 2106: Q2. Column (2) shows EUR-denominated contracts for the same sample as the USD-denominated ones. Column (4) shows the theoretical premiums. The table also reports theoretical expected losses on the government debt portfolio in column (5). We simulate the data at quarterly frequency 100 times for 15 years and average across simulations.

Table 4.6: Monetary Policy and CDS Premiums

	Mean				Std
δ_π	Q_tB_t	P_t^D	CDS_5	Π_t	Π_t
1.50	0.9034	0.0024	21	1.0123	0.0274
1.35	0.8829	0.0022	23	1.0181	0.0315
1.20	0.8538	0.0019	24	1.0236	0.0382

Notes. This table reports theoretical moments corresponding to various magnitudes of monetary policy response to inflation, δ_π. Means and standard deviations are annualized. We simulate the data at quarterly frequency 100 times for 15 years and average across simulations.

Table 4.7: Fiscal Policy and CDS Premiums

	Mean				Std
ρ_b	Q_tB_t	P_t^D	$CDS_t(5)$	$\mathcal{T}_t$	$\mathcal{T}_t$
0.96	0.9034	0.0024	21	0.3540	0.1237
0.94	0.8729	0.0021	23	0.3817	0.1365
0.92	0.8515	0.0019	25	0.4022	0.1443

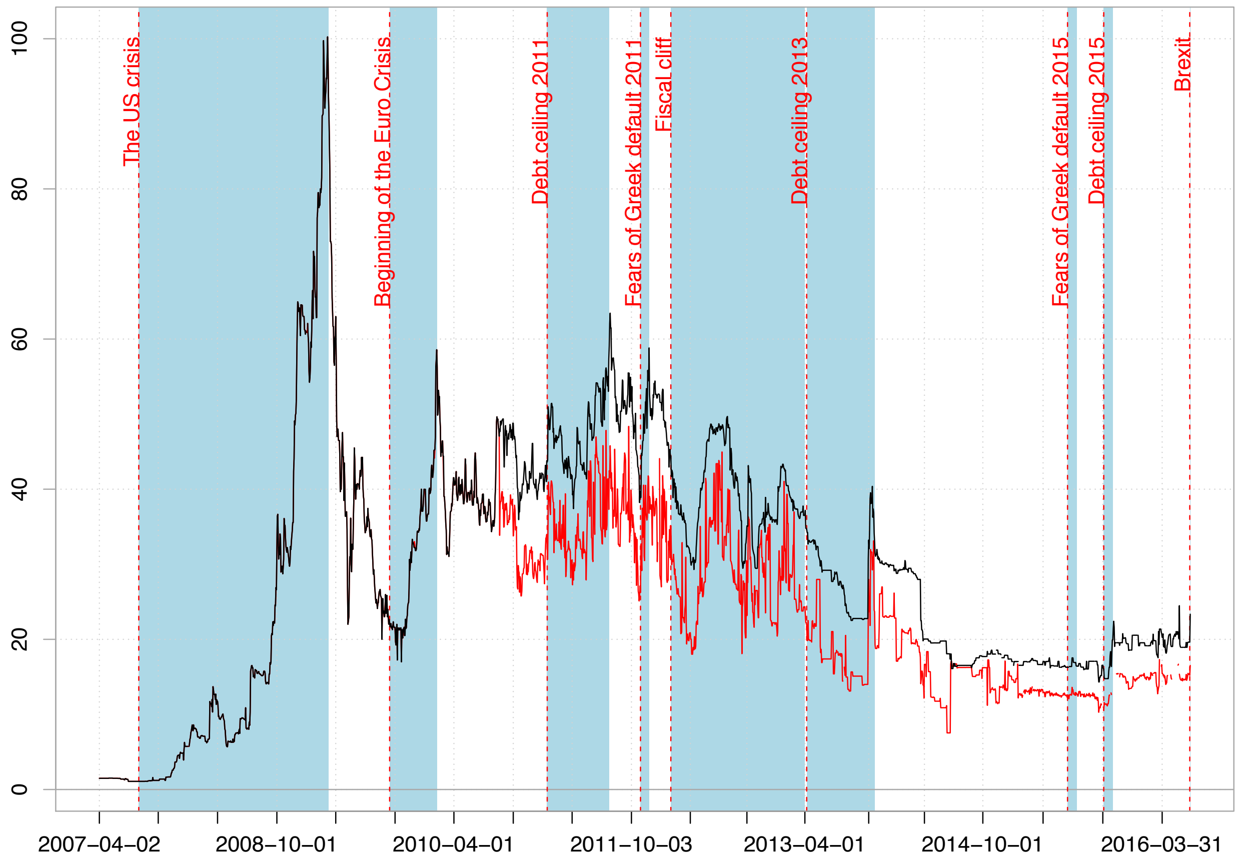
Notes. This table reports theoretical moments corresponding to various magnitudes of fiscal policy response to debt, ρ_b. Means and standard deviations are annualized. We simulate the data at quarterly frequency 100 times for 15 years and average across simulations.

Table 4.8: Debt Duration and CDS Premiums

	Mean		Std	
λ	P_t^D	$CDS_t(5)$	$\mathcal{T}_t$	Q_tB_t
0.01	0.0021	18	0.1190	0.1081
0.04	0.0024	21	0.1237	0.1124
0.16	0.0032	27	0.1436	0.1317

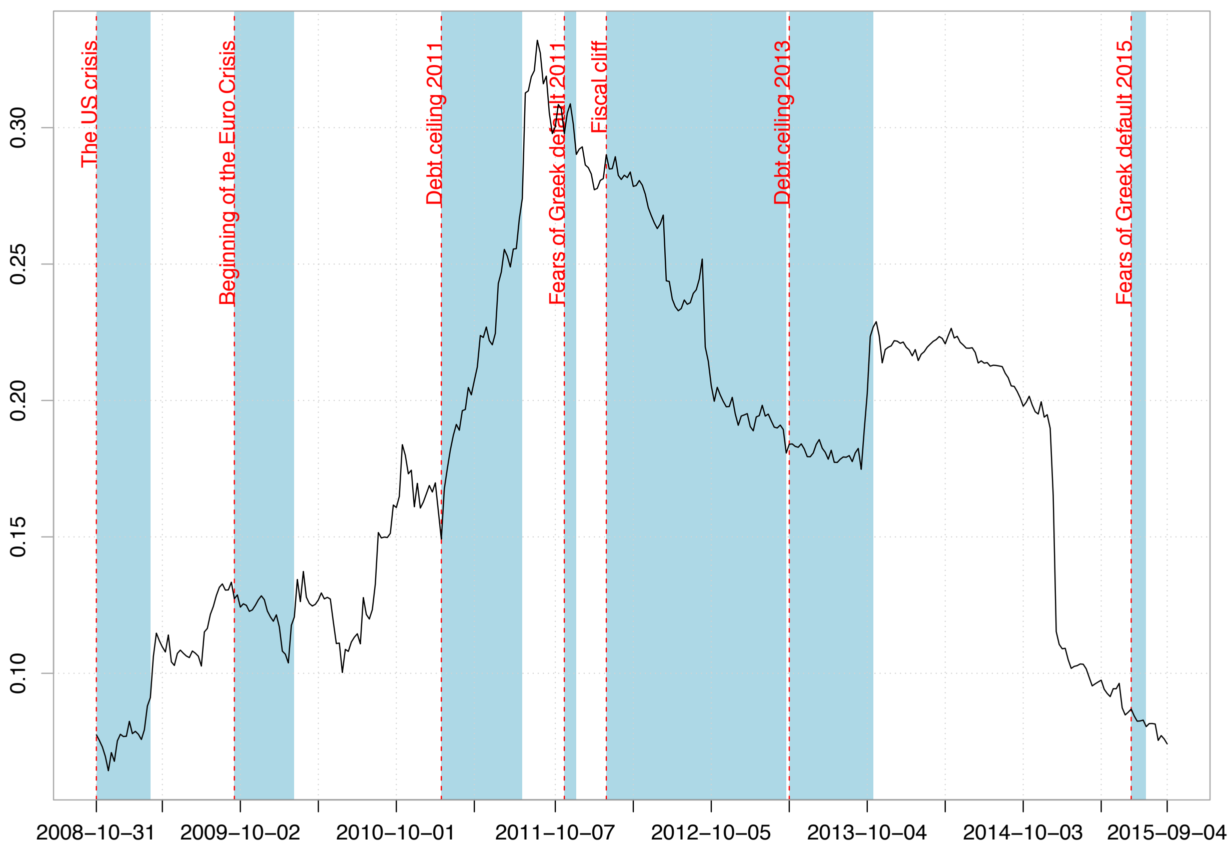
Notes. This table reports theoretical moments corresponding to various magnitudes of the amortization rate of long-term debt, λ. Means and standard deviations are annualized. We simulate the data at quarterly frequency 100 times for 15 years and average across simulations.

Figure 4.1: History of U.S. CDS premiums



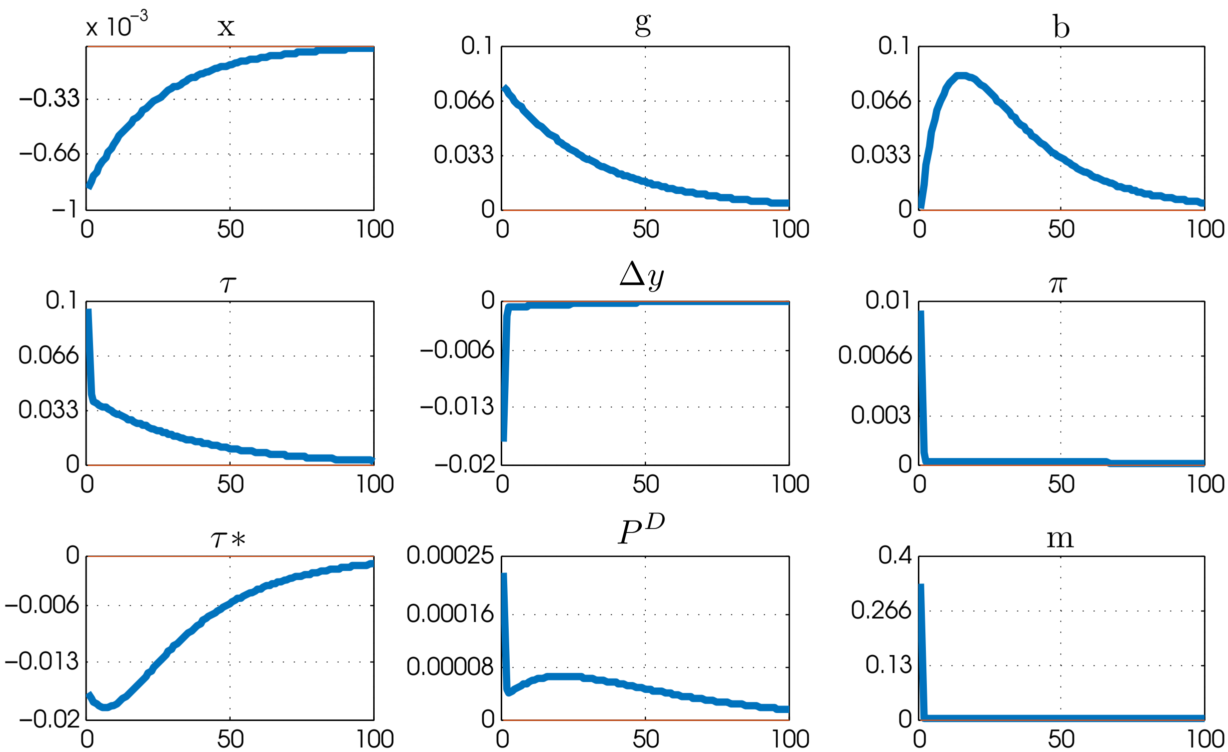
Notes. We plot the time-series of premiums on five-year contracts. The dark line represents quotes in EUR from April 2007 to June 2016, and the light one is in USD from August 2010 to June 2016. The time series are complemented by the highlights of major economic and political events during that period. The premiums are expressed in basis points per year.

Figure 4.2: Liquidity of U.S. CDS



Notes. We plot the time-series of liquidity of the U.S. CDS market. CDS contracts on Italian government are the most actively traded sovereign contracts. For this reason, our liquidity measure is equal to the ratio of the weekly net notional amount of U.S. CDS to that of Italian CDS. The time series is complemented by the highlights of major economic and political events.

Figure 4.3: Impulse Responses



Notes. We plot the responses of model variables to a one standard deviation negative innovation ε_t to the long-run consumption trend, over 100 quarters. The responses are reported in decimal deviations from the variable means.

4.6 Appendix

Computational procedure

The model is summarized by a system of expectational difference equations. Solving for the endogenous variables in the system is complicated by (i) the nonlinearities induced by the pricing kernel and the possibility of default and (ii) the endogeneity of the default boundary τ^*, which depends on present values of endogenous variables. We deal with (i) by adopting a global, nonlinear solution method based on projection techniques, and with (ii) by implementing an iterative algorithm based on Monte Carlo methods.

Our solution strategy to deal with (i) is to approximate the endogenous variables π_t, q_t^s, and q_t^l with flexible Chebyshev polynomials in the state variables $\varsigma_t = \{\tau_{t-1}, b_{t-1}, g_{t-1}, x_{t-1}, \varepsilon_t, \xi_t^b, \xi_t^q\}$. This amounts to solving for the coefficients of these polynomials that satisfy the model equations at specific points, namely the Chebyshev nodes. To find those, we choose bounds on the state variables and map those linearly into $[-1, 1]$, the domain of the Chebyshev polynomials. The bounds on the persistent stochastic variables x_t and g_t come from the Tauchen (1986) procedure to approximate an AR(1) process. Finding the coefficients of the Chebyshev polynomials at the relevant nodes thus translates into solving a nonlinear system of equations. To aid convergence, we start with lower order polynomials and successively increase the number of nodes.

To find the default boundary, we proceed as follows. For some initial default boundary $\tau_t^{*(0)} \equiv \tau^{*(0)}(\varsigma_t)$, we obtain the corresponding bond prices and inflation $q_t^{s,(0)}, q_t^{l,(0)}$, and $\pi_t^{(0)}$ using the Chebyshev collocation method described above. With these solutions at hand, we evaluate the expected sustainable (log) surplus s_t^* in any state ς_t via Monte Carlo simulations. Starting from any state ς_t, we simulate the model forward for T periods to obtain s_t^* and an updated $\tau_t^{*(1)}$ for that state. Note that this $\tau^{*(1)}(\varsigma_t)$ depends on the endogenous variables $q_t^{s,(0)}, q_t^{l,(0)}$, and $\pi_t^{(0)}$, which were obtained as functions of the initial default boundary $\tau_t^{*(0)}$. We choose T sufficiently large to accommodate the persistence of the underlying processes.

Our algorithm then iterates back and forth between the projection step and the simulation step. More precisely, starting from any $\tau_t^{*(j)}$, we obtain an updated default boundary $\tau_t^{*(j+0.5)}$ by solving the model with projection and simulating it forward. Our aim is to iterate that procedure to convergence, so that $\max_{\varsigma_t} \|\tau^{*(j)}(\varsigma_t) - \tau^{*(j+0.5)}(\varsigma_t)\| < \bar{\varepsilon}$. To facilitate convergence, we implement a relaxation scheme by introducing $\tau^{*(j+1)}(\varsigma_t) = (1-\zeta)\tau^{*(j)}(\varsigma_t) + \zeta\tau^{*(j+0.5)}(\varsigma_t)$, where ζ is a relaxation parameter. The convergence criterion becomes $\max_{\varsigma_t} \|\tau^{*(j)}(\varsigma_t) - \tau^{*(j+1)}(\varsigma_t)\| < \bar{\varepsilon}$. We also check that bond prices and inflation stabilize in the iterative process.

With the default boundary τ_t^* at hand, it is straightforward to evaluate the CDS premiums CDS_t^T on the grids. Our model statistics are computed from 100 simulations of 15 years of data, to be consistent with our empirical targets.

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