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**Affine structure on the Teichmüller spaces and period
maps for Calabi-Yau manifolds**

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Feng Guan

2014

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ABSTRACT OF THE DISSERTATION

Affine structure on the Teichmüller spaces and period maps for Calabi-Yau manifolds

by

Feng Guan

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2014

Professor Kefeng Liu, Chair

In this thesis, we prove that the Hodge metric completion of the Teichmüller space of polarized and marked Calabi–Yau manifolds is a complex affine manifold. As applications, we show that the extended period map from the completion space is injective into the period domain, that the completion space is a bounded domain of holomorphy and admits a complete Kähler–Einstein metric.

The dissertation of Feng Guan is approved.

Per Kraus

Robert Greene

Kefeng Liu, Committee Chair

University of California, Los Angeles

2014

To my family

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CHAPTER 1

Introduction

In this thesis, if a compact projective manifold M of complex dimension $n \geq 3$ has a trivial canonical bundle and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$, then we call it a Calabi–Yau manifold. A polarized and marked Calabi–Yau manifold is a triple consisting of a Calabi–Yau manifold M , an ample line bundle L over M , and a basis of the integral middle homology group modulo torsion, $H_n(M, \mathbb{Z})/\text{Tor}$.

We use the moduli space of equivalence classes of marked and polarized Calabi–Yau manifolds as the Teichmüller space for the deformation of the complex structure on the polarized and marked Calabi–Yau manifold M , and we denote the Teichmüller by \mathcal{T} . Let D be the period domain of polarized Hodge structures of the n -th primitive cohomology of M . The period map $\Phi : \mathcal{T} \rightarrow D$ is given by assigning to each point in \mathcal{T} the Hodge structure of the corresponding marked and polarized Calabi–Yau manifold.

1.1 A brief review of the Torelli Problem

The Torelli problem, which has a history of more than one hundred years, asks whether the period map is an injection. The very first idea to study the periods of abelian varieties on Riemann surfaces goes back to Riemann. In the year 1914, Torelli asked whether two complex curves are isomorphic if they have the same periods in his work [30]. Then In [35], Weil reformulated the Torelli problem as follows: Suppose for two Riemann surfaces, there exists an isomorphism of their Jacobians which preserves the canonical polarization of the Jacobians, is it true that the two Riemann surfaces are isomorphic. Andreotti proved Weil’s

version of the Torelli problem in [1].

Another important achievement about the Torelli problem, conjectured by Weil in [36], was the proof of the global Torelli Theorem for K3 surfaces, essentially due to Shafarevich and Piatetski-Shapiro in [21]. Andreotti's proof is based on specific geometric properties of Riemann surfaces. The approach of Shafarevich and Piatetski-Shapiro is based on the arithmeticity of the mapping class group of a K3 surface. It implies that the special K3 surfaces, the Kummer surfaces, are everywhere dense subset in the moduli of K3 surfaces. Shafarevich and Piatetski-Shapiro observed that the period map has degree one on the set of Kummer surfaces, which implies the global Torelli theorem.

In this thesis, as an application of the holomorphic affine structure on the Teichmüller space, we give a proof of global Torelli theorem of marked and polarized Calabi-Yau manifolds. In [31] Verbitsky used an approach similar to ours in his proof of the global Torelli theorem for hyperKähler manifolds.

The literature about the Torelli problem is enormous. Many authors made very substantial contributions to the general Torelli problem. We believe that it is impossible to give a complete list of all the achievements in this area and its applications.

1.2 Holomorphic affine structure on the Teichmüller space

One of our essential constructions is the holomorphic affine structure on the Teichmüller space, which can be outlined as follows: fix a base point $p \in \mathcal{T}$ and its Hodge structure $\Phi(p) = \{H_p^{k,n-k}\}_{k=0}^n$ as the reference Hodge structure in D . We identify the unipotent subgroup N_+ with its orbit in \check{D} and define $\check{\mathcal{T}} = \Phi^{-1}(N_+) \subseteq \mathcal{T}$. We first show that $\Phi : \check{\mathcal{T}} \rightarrow N_+ \cap D$ is a bounded map with respect to the Euclidean metric on N_+ , and that $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety. Then by applying Riemann extension theorem, we conclude that $\Phi(\mathcal{T}) \subseteq N_+ \cap D$. Using this property, we then show Φ induces a global holomorphic map $\tau : \mathcal{T} \rightarrow \mathbb{C}^N$, which actually gives a local coordinate map around each point in \mathcal{T} by using local Torelli theorem for Calabi-Yau manifolds. Thus $\tau : \mathcal{T} \rightarrow \mathbb{C}^N$ induces a global

holomorphic affine structure on \mathcal{T} . It is not hard to see that $\tau = P \circ \Phi : \mathcal{T} \rightarrow \mathbb{C}^N$ is a composition map with $P : N_+ \rightarrow \mathbb{C}^N \simeq H_p^{n-1,1}$ a natural projection map into a subspace, where $N_+ \simeq \mathbb{C}^d$ with the fixed base point $p \in \mathcal{T}$.

Let \mathcal{Z}_m^H be the Hodge metric completion of the smooth moduli space \mathcal{Z}_m and let \mathcal{T}_m^H be the universal cover of \mathcal{Z}_m^H with the universal covering map $\pi_m^H : \mathcal{T}_m^H \rightarrow \mathcal{Z}_m^H$. Lemma 5.1 shows that \mathcal{Z}_m^H is a connected and complete smooth complex manifold, and thus \mathcal{T}_m^H is a connected and simply connected complete smooth complex manifold. We also obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}_m^H & \xrightarrow{\Phi_m^H} & D \\ \downarrow \pi_m & & \downarrow \pi_m^H & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m}^H} & D/\Gamma, \end{array}$$

where $\Phi_{\mathcal{Z}_m}^H$ is the continuous extension map of the period map $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \rightarrow D/\Gamma$, i is the inclusion map, i_m is a lifting of $i \circ \pi_m$, and Φ_m^H is a lifting of $\Phi_{\mathcal{Z}_m}^H \circ \pi_m^H$. We prove that there is a suitable choice of i_m and Φ_m^H such that $\Phi = \Phi_m^H \circ i_m$. It is not hard to see that Φ_m^H is actually a bounded holomorphic map from \mathcal{T}_m^H to $N_+ \cap D$.

Proposition 1.1. *For any $m \geq 3$, the complete complex manifold \mathcal{T}_m^H is a complex affine manifold, which is bounded domain in \mathbb{C}^N . Moreover, the holomorphic map $\Phi_m^H : \mathcal{T}_m^H \rightarrow N_+ \cap D$ is an injection. As a consequence, the complex manifolds \mathcal{T}_m^H and $\mathcal{T}_{m'}^H$ are biholomorphic to each other for any $m, m' \geq 3$.*

This proposition allows us to define the complete complex manifold \mathcal{T}^H with respect to the Hodge metric by $\mathcal{T}^H = \mathcal{T}_m^H$, the holomorphic map $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}^H$ by $i_{\mathcal{T}} = i_m$, and the extended period map $\Phi^H : \mathcal{T}^H \rightarrow D$ by $\Phi^H = \Phi_m^H$ for any $m \geq 3$. By these definitions, Proposition 1.1 implies that \mathcal{T}^H is a complex affine manifold and that $\Phi^H : \mathcal{T}^H \rightarrow N_+ \cap D$ is a holomorphic injection. The main result of this thesis is the following.

Theorem 1. *The complete complex affine manifold \mathcal{T}^H is the completion space of \mathcal{T} with respect to the Hodge metric, and it is a bounded domain \mathbb{C}^N . Moreover, the extended period map $\Phi^H : \mathcal{T}^H \rightarrow N_+ \cap D$ is a holomorphic injection.*

1.3 Applications of the holomorphic affine structure on the Teichmüller space

The first important application of 1 is the global Torelli theorem on polarized and marked Calabi-Yau manifolds. We easily deduce that the period map $\Phi = \Phi^H \circ i_{\mathcal{T}} : \mathcal{T} \rightarrow D$ is also injective since it is a composition of two injective maps. This is the global Torelli theorem for the period map from the Teichmüller space to the period domain. In the case that the period domain D is Hermitian symmetric and that it has the same dimension as \mathcal{T} , the above theorem implies that the extended period map Φ^H is biholomorphic, in particular it is surjective.

The second application is the following geometric property of the completion space of Teichmüller space.

Theorem 2. *The completion space \mathcal{T}^H is a bounded domain of holomorphy in \mathbb{C}^N ; thus there exists a complete Kähler-Einstein metric on \mathcal{T}^H .*

To prove this theorem, we construct a plurisubharmonic exhaustion function on \mathcal{T}^H by using Proposition 6.8, the completeness of \mathcal{T}^H , and the injectivity of Φ^H . This shows that \mathcal{T}^H is a bounded domain of holomorphy in \mathbb{C}^N . The existence of the Kähler-Einstein metric follows directly from a theorem in Mok-Yau in [15].

The third application is that the Hodge bundles over the completion space of Teichmüller space are trivial bundles. We prove this property by a direct construction of the global holomorphic frames of Hodge bundles over \mathcal{T}^H . As a corollary of the triviality of the Hodge bundles, we prove in Theorem 15 that for any $1 \leq k \leq n$ there is an anti-holomorphic vector bundle \tilde{F}^k over \mathcal{T}^H such that $H^n(M, \mathbb{C}) = F^k \oplus \tilde{F}^k$ is a splitting of vector bundles.

1.4 Organization of this thesis

This thesis is organized as follows.

In Chapter 2, we review the local deformation theory of polarized Calabi-Yau manifolds, which will be needed for the construction of the global holomorphic affine structure on the Teichmüller space

In Chapter 3, we review the definition of the period domain of polarized Hodge structures and briefly describe the construction of the Teichmüller space of polarized and marked Calabi-Yau manifolds, the definition of the period map and the Hodge metrics on the moduli space and the Teichmüller space respectively.

In Chapter 4, we show that the image of the period map is in $N_+ \cap D$ and we construct a holomorphic affine structure on the Teichmüller space.

In Chapter 5, we prove that there exists a global holomorphic affine structure on \mathcal{T}_m^H and that the map $\Phi_m^H : \mathcal{T}_m^H \rightarrow D$ is an injective map.

In Chapter 6, we define the completion space \mathcal{T}^H and the extended period map Φ^H . We then show our main result that \mathcal{T}^H is the Hodge metric completion space of \mathcal{T} and that Φ^H is a holomorphic injection.

As applications, we prove the global Torelli theorem for Calabi-Yau manifolds on the Teichmüller space and that \mathcal{T}^H is a bounded domain of holomorphy in \mathbb{C}^N . Also we prove that the Hodge bundles over the completed Teichmüller space are trivial bundles.

CHAPTER 2

Local deformation theory of Calabi-Yau manifolds

In this chapter, we review the local deformation theory of polarized Calabi-Yau manifolds, which will be needed for the construction of the global holomorphic affine structure on the Teichmüller space in Section 4. In Section 2.1, we briefly review the basic local deformation theory of complex structures. In Section 2.2, we recall the local Kuranishi deformation theory of Calabi-Yau manifolds, which depends on the Calabi-Yau metric in a substantial way. In Section 2.3, we describe a local family of the canonical holomorphic $(n, 0)$ -forms as a section of the Hodge bundle F^n over the local deformation space of Calabi-Yau manifolds, from which we obtain an expansion of the family of holomorphic $(n, 0)$ -classes as given in Theorem 6.17. This simple expansion is what we need for the construction of holomorphic affine flat structure on the Teichmüller space.

Most of the results in this section are standard now in the literatures, and can be found in [17], [29], and [28]. For reader's convenience, we also briefly review some arguments. We remark that one may use a more algebraic approach to Theorem 6.17 by using the local Torelli theorem and the Griffiths transversality.

2.1 Local deformation of complex structure

Let X be a smooth manifold of dimension $\dim_{\mathbb{R}} X = 2n$ and let J be an integrable complex structure on X . We denote by $M = (X, J)$ the corresponding complex manifold, and $\partial, \bar{\partial}$ the corresponding differential operators on M .

Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be a $T^{1,0}M$ -valued smooth $(0, 1)$ -form. For any point $x \in M$,

and any local holomorphic coordinate chart (U, z_1, \dots, z_n) around x . Let us express $\varphi = \varphi_j^i d\bar{z}_j \otimes \frac{\partial}{\partial z_i} = \varphi^i \partial_i$, where $\varphi^i = \varphi_j^i d\bar{z}_j$ and $\partial_i = \frac{\partial}{\partial z_i}$ for simplicity. Here we use the standard convention to sum over the repeated indices. We can view φ as a map

$$\varphi : \Omega^{1,0}(M) \rightarrow \Omega^{0,1}(M)$$

such that locally we have

$$\varphi(dz_i) = \varphi^i \quad \text{for } 1 \leq i \leq n.$$

We use φ to describe deformation of complex structures. Let

$$\Omega_\varphi^{1,0}(x) = \text{span}_{\mathbb{C}}\{dz_1 + \varphi(dz_1), \dots, dz_n + \varphi(dz_n)\}, \quad \text{and}$$

$$\Omega_\varphi^{0,1}(x) = \text{span}_{\mathbb{C}}\{d\bar{z}_1 + \bar{\varphi}(d\bar{z}_1), \dots, d\bar{z}_n + \bar{\varphi}(d\bar{z}_n)\},$$

if $\Omega_\varphi^{1,0}(x) \cap \Omega_\varphi^{0,1}(x) = 0$ for any x , then we can define a new almost complex structure J_φ by letting $\Omega_\varphi^{1,0}(x)$ and $\Omega_\varphi^{0,1}(x)$ be the eigenspaces of $J_\varphi(x)$ with respect to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively, and we call such φ a Beltrami differential.

It was proved in [19] that the almost complex structure J_φ is integrable if and only if

$$\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]. \quad (2.1)$$

If (2.1) holds, we will call φ an integrable Beltrami differential and denote by M_φ the corresponding complex manifold. Please see Chapter 4 in [17] for more details about the deformation of complex structures.

Let us recall the notation for contractions and Lie bracket of Beltrami differentials. Let (U, z_1, \dots, z_n) be the local coordinate chart defined above, and $\Omega = f dz_1 \wedge \dots \wedge dz_n$ be a smooth $(n, 0)$ -form on M , and $\varphi \in A^{0,1}(M, T^{1,0}M)$ be a Beltrami differential. We define

$$\varphi \lrcorner \Omega = \sum_i (-1)^{i-1} f \varphi^i \wedge dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_n.$$

For Beltrami differentials $\varphi, \psi \in A^{0,1}(M, T^{1,0}M)$, with $\varphi = \varphi^i \partial_i$ and $\psi = \psi^k \partial_k$, recall that the Lie bracket is defined as

$$[\varphi, \psi] = \sum_{i,k} (\varphi^i \wedge \partial_i \psi^k + \psi^i \wedge \partial_i \varphi^k) \otimes \partial_k,$$

where $\partial_i \varphi^k = \frac{\partial \varphi^k}{\partial z_i} d\bar{z}_l$ and $\partial_i \psi^k = \frac{\partial \psi^k}{\partial z_i} d\bar{z}_l$.

For k Beltrami differentials $\varphi_1, \dots, \varphi_k \in A^{0,1}(M, T^{1,0}M)$, with $\varphi_\alpha = \varphi_\alpha^i \partial_i$ and $1 \leq \alpha \leq k$, we define

$$\varphi_1 \wedge \dots \wedge \varphi_k = \sum_{i_1 < \dots < i_k} \left(\sum_{\sigma \in S_k} \varphi_{\sigma(1)}^{i_1} \wedge \dots \wedge \varphi_{\sigma(k)}^{i_k} \right) \otimes (\partial_{i_1} \wedge \dots \wedge \partial_{i_k}),$$

where S_k is the symmetric group of k elements. Especially we have

$$\wedge^k \varphi = k! \sum_{i_1 < \dots < i_k} (\varphi^{i_1} \wedge \dots \wedge \varphi^{i_k}) \otimes (\partial_{i_1} \wedge \dots \wedge \partial_{i_k}).$$

Then we define the contraction,

$$\begin{aligned} (\varphi_1 \wedge \dots \wedge \varphi_k) \lrcorner \Omega &= \varphi_1 \lrcorner (\varphi_2 \lrcorner (\dots \lrcorner (\varphi_k \lrcorner \Omega))) \\ &= \sum_{I=(i_1, \dots, i_k) \in A_k} (-1)^{|I| + \frac{(k-1)(k-2)}{2}} f \left(\sum_{\sigma \in S_k} \varphi_{\sigma(1)}^{i_1} \wedge \dots \wedge \varphi_{\sigma(k)}^{i_k} \right) \wedge dz_{I^c}, \end{aligned}$$

where A_k is the index set

$$A_k = \{(i_1, \dots, i_k) \mid 1 \leq i_1 < \dots < i_k \leq n\}.$$

Here for each $I = (i_1, \dots, i_k) \in A_k$, we let $|I| = i_1 + \dots + i_k$ and $dz_{I^c} = dz_{j_1} \wedge \dots \wedge dz_{j_{n-k}}$ where $j_1 < \dots < j_{n-k}$ and $j_\alpha \neq i_\beta$ for any α, β . With the above notations, for any Beltrami differentials $\varphi, \psi \in A^{0,1}(M, T^{1,0}M)$ one has the following identity which was proved in [28], [29],

$$\partial((\varphi \wedge \psi) \lrcorner \Omega) = -[\varphi, \psi] \lrcorner \Omega + \varphi \lrcorner \partial(\psi \lrcorner \Omega) + \psi \lrcorner \partial(\varphi \lrcorner \Omega). \quad (2.2)$$

The following notation will be needed in the construction of the local canonical family of holomorphic $(n, 0)$ -classes.

$$e^\varphi \lrcorner \Omega = \sum_{k \geq 0} \frac{1}{k!} \wedge^k \varphi \lrcorner \Omega. \quad (2.3)$$

By direct computation, we see that $e^\varphi \lrcorner \Omega = f(dz_1 + \varphi(dz_1)) \wedge \cdots \wedge (dz_n + \varphi(dz_n))$ is a smooth $(n, 0)$ -form on M_φ .

2.2 Local deformation of Calabi-Yau manifold

For a point $p \in \mathcal{T}$, we denote by (M_p, L) the corresponding polarized and marked Calabi-Yau manifold as the fiber over p . Yau's solution of the Calabi conjecture implies that there exists a unique Calabi-Yau metric h_p on M_p , and the imaginary part $\omega_p = \text{Im } h_p \in L$ is the corresponding Kähler form. First by using the Calabi-Yau metric we have the following lemma,

Lemma 2.4. *Let Ω_p be a nowhere vanishing holomorphic $(n, 0)$ -form on M_p such that*

$$\left(\frac{\sqrt{-1}}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} \Omega_p \wedge \overline{\Omega_p} = \omega_p^n. \quad (2.5)$$

Then the map $\iota : A^{0,1}(M, T^{1,0}M) \rightarrow A^{n-1,1}(M)$ given by $\iota(\varphi) = \varphi \lrcorner \Omega_p$ is an isometry with respect to the natural Hermitian inner product on both spaces induced by ω_p . Furthermore, ι preserves the Hodge decomposition.

Let us briefly recall the proof. We can pick local coordinates z_1, \dots, z_n on M such that $\Omega_p = dz_1 \wedge \cdots \wedge dz_n$ locally and $\omega_p = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$, then the condition (2.5) implies that $\det[g_{i\bar{j}}] = 1$. The lemma follows from direct computations.

Let $\partial_{M_p}, \bar{\partial}_{M_p}, \bar{\partial}_{M_p}^*, \square_{M_p}, G_{M_p}$, and \mathbb{H}_{M_p} be the corresponding operators in the Hodge theory on M_p , where $\bar{\partial}_{M_p}^*$ is the adjoint operator of $\bar{\partial}_{M_p}$, \square_{M_p} the Laplace operator, and G_{M_p} the corresponding Green operator. We let \mathbb{H}_{M_p} denote the harmonic projection onto the kernel of \square_{M_p} . We also denote by $\mathbb{H}^{p,q}(M_p, E)$ the harmonic (p, q) -forms with value in a holomorphic vector bundle E on M_p .

By using the Calabi-Yau metric we have a more precise description of the local deformation of a polarized Calabi-Yau manifold. First from Hodge theory, we have the following

identification

$$T_p^{1,0}\mathcal{T} \cong \mathbb{H}^{0,1}(M_p, T_{M_p}^{1,0}).$$

From Kuranishi theory we have the following local convergent power series expansion of the Beltrami differentials, which is now well-known as the Bogomolov-Tian-Todorov theorem.

Theorem 3. *Let $\varphi_1, \dots, \varphi_N \in \mathbb{H}^{0,1}(M_p, T_{M_p}^{1,0})$ be a basis. Then there is a unique power series*

$$\varphi(\tau) = \sum_{i=1}^N \tau_i \varphi_i + \sum_{|I| \geq 2} \tau^I \varphi_I \quad (2.6)$$

which converges for $|\tau| < \varepsilon$ small. Here $I = (i_1, \dots, i_N)$ is a multi-index, $\tau^I = \tau_1^{i_1} \dots \tau_N^{i_N}$ and $\varphi_I \in A^{0,1}(M_p, T_{M_p}^{1,0})$. Furthermore, the family of Beltrami differentials $\varphi(\tau)$ satisfy the following conditions:

$$\begin{aligned} \bar{\partial}_{M_p} \varphi(\tau) &= \frac{1}{2} [\varphi(\tau), \varphi(\tau)], \\ \bar{\partial}_{M_p}^* \varphi(\tau) &= 0, \\ \varphi_I \lrcorner \Omega_p &= \partial_{M_p} \psi_I, \end{aligned} \quad (2.7)$$

for each $|I| \geq 2$ where $\psi_I \in A^{n-2,1}(M_p)$ are smooth $(n-2, 1)$ -forms. By shrinking ε we can pick each ψ_I appropriately such that $\sum_{|I| \geq 2} \tau^I \psi_I$ converges for $|\tau| < \varepsilon$.

Remark 2.8. The coordinate $\{\tau_1, \dots, \tau_N\}$ depends on the choice of basis $\varphi_1, \dots, \varphi_N \in \mathbb{H}^{0,1}(M_p, T_{M_p}^{1,0})$. But one can also determine the coordinate by fixing a basis $\{\eta_0\}$ and $\{\eta_1, \dots, \eta_N\}$ for $H^{n,0}(M_p)$ and $H^{n-1,1}(M_p)$ respectively. In fact, Lemma 2.4 implies that there is a unique choice of $\varphi_1, \dots, \varphi_N$ such that $\eta_k = [\varphi_k \lrcorner \eta_0]$, for each $1 \leq k \leq N$.

Theorem 3 was proved in [29], and in [28] in a form without specifying the Kuranishi gauge, the second and the third condition in (2.7). This theorem implies that the local deformation of a Calabi-Yau manifold is unobstructed. Here we only mention two important points of its proof. For the convergence of $\sum_{|I| \geq 2} \tau^I \psi_I$, noting that $\partial_{M_p} \psi_I = \varphi_I \lrcorner \Omega_p$ and $\bar{\partial} \varphi_I \lrcorner \Omega_p = 0$, we can pick $\psi_I = \partial_{M_p}^* G(\varphi_I \lrcorner \Omega_p)$. It follows that

$$\|\psi_I\|_{k,\alpha} \leq C(k, \alpha) \|\varphi_I \lrcorner \Omega_p\|_{k-1,\alpha} \leq C'(k, \alpha) \|\varphi_I\|_{k-1,\alpha}.$$

The desired convergence follows from the estimate on φ_I . We note that the convergence of (2.6) follows from standard elliptic estimate. See [29], or Chapter 4 of [17] for details.

For the third condition in (2.7), by using the first two conditions in (2.7), for example we have in the case of $|I| = 2$,

$$\bar{\partial}_{M_p}\varphi_{ij} = [\varphi_i, \varphi_j] \text{ and } \bar{\partial}_{M_p}^*\varphi_{ij} = 0. \quad (2.9)$$

Then by using formula (2.2) and Lemma 2.4, we get that

$$[\varphi_i, \varphi_j] \lrcorner \Omega_p = \partial_{M_p}(\varphi_i \wedge \varphi_j \lrcorner \Omega_p)$$

is ∂_{M_p} exact. It follows that $\bar{\partial}_{M_p}(\varphi_{ij} \lrcorner \Omega_p) = (\bar{\partial}_{M_p}\varphi_{ij}) \lrcorner \Omega_p$ is also ∂_{M_p} exact. Then by the $\partial\bar{\partial}$ -lemma we have

$$\bar{\partial}_{M_p}(\varphi_{ij} \lrcorner \Omega_p) = \bar{\partial}_{M_p}\partial_{M_p}\psi_{ij}$$

for some $\psi_{ij} \in A^{n-2,1}$. It follows that

$$\varphi_{ij} \lrcorner \Omega_p = \partial_{M_p}\psi_{ij} + \bar{\partial}_{M_p}\alpha + \beta$$

for some $\alpha \in A^{n-1,0}(M_p)$ and $\beta \in \mathbb{H}^{n-1,1}(M_p)$. By using the condition $\bar{\partial}_{M_p}^*\varphi_{ij} = 0$ and Lemma 2.4, we have

$$\varphi_{ij} \lrcorner \Omega_p = \partial_{M_p}\psi_{ij} + \beta.$$

Because φ_{ij} is not uniquely determined by condition (2.9), we can choose φ_{ij} such that the harmonic projection $\mathbb{H}(\varphi_{ij}) = 0$. Then by using Lemma 2.4 again, we have

$$\varphi_{ij} \lrcorner \Omega_p = \partial_{M_p}\psi_{ij}.$$

Thus there exists a unique φ_{ij} which satisfies all three conditions in (2.7). We can then proceed by induction and the same argument as above to show that the third condition in (2.7) holds for all $|I| \geq 2$. See [29] and [28] for details.

Theorem 3 will be used to define the local holomorphic affine flat coordinates $\{\tau_1, \dots, \tau_N\}$ around p , for a given orthonormal basis $\varphi_1, \dots, \varphi_N \in \mathbb{H}^{0,1}(M_p, T_{M_p}^{1,0})$. Sometimes we also denote by M_τ the deformation given by the Beltrami differential $\varphi(\tau)$.

2.3 Local canonical section of holomorphic $(n, 0)$ -classes

By using the local deformation theory, in [29] Todorov constructed a canonical local holomorphic section of the line bundle $H^{n,0} = F^n$ over the local deformation space of a Calabi-Yau manifold at the differential form level. We first recall the construction of the holomorphic $(n, 0)$ -forms in [29].

Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential and let M_φ be the Calabi-Yau manifold defined by φ . We refer the reader to Section 2.1 for the definition of the contraction $e^\varphi \lrcorner \Omega_p$.

Lemma 2.10. *Let Ω_p be a nowhere vanishing holomorphic $(n, 0)$ -form on M_p and $\{z_1, \dots, z_n\}$ is a local holomorphic coordinate system with respect to J such that*

$$\Omega_p = dz_1 \wedge \dots \wedge dz_n$$

locally. Then the smooth $(n, 0)$ -form

$$\Omega_\varphi = e^\varphi \lrcorner \Omega_p$$

is holomorphic with respect to the complex structure on M_φ if and only if $\partial_{M_p}(\varphi \lrcorner \Omega_p) = 0$.

Proof. The proof in [29] is by direct computations, here we give a simple proof.

Being an $(n, 0)$ -form on M_φ , $e^\varphi \lrcorner \Omega_p$ is holomorphic on M_φ if and only if $d(e^\varphi \lrcorner \Omega_p) = 0$. For any smooth $(n, 0)$ -form Ω_p and Beltrami differential $\varphi \in A^{0,1}(M, T^{1,0}M)$, we have the following formula,

$$d(e^\varphi \lrcorner \Omega_p) = e^\varphi \lrcorner (\bar{\partial}_{M_p} \Omega_p + \partial_{M_p}(\varphi \lrcorner \Omega_p)) + (\bar{\partial}_{M_p} \varphi - \frac{1}{2}[\varphi, \varphi]) \lrcorner (e^\varphi \lrcorner \Omega_p),$$

which can be verified by direct computations. In our case the Beltrami differential φ is integrable, i.e. $\bar{\partial}_{M_p} \varphi - \frac{1}{2}[\varphi, \varphi] = 0$ and Ω_p is holomorphic on M_p . Therefore we have

$$d(e^\varphi \lrcorner \Omega_p) = e^\varphi \lrcorner (\partial_{M_p}(\varphi \lrcorner \Omega_p)),$$

which implies that $e^\varphi \lrcorner \Omega_p$ is holomorphic on M_φ if and only if $\partial_{M_p}(\varphi \lrcorner \Omega_p) = 0$. \square

Now we can construct the canonical family of holomorphic $(n, 0)$ -forms on the local deformation space of Calabi-Yau manifolds.

Proposition 2.11. *We fix on M_p a nowhere vanishing holomorphic $(n, 0)$ -form Ω_p and an orthonormal basis $\{\varphi_i\}_{i=1}^N$ of $\mathbb{H}^1(M_p, T^{1,0}M_p)$. Let $\varphi(\tau)$ be the family of Beltrami differentials given by (2.7) that defines a local deformation of M_p which we denote by M_τ . Let*

$$\Omega_p^c(\tau) = e^{\varphi(\tau)} \lrcorner \Omega_p. \quad (2.12)$$

Then $\Omega_p^c(\tau)$ is a well-defined nowhere vanishing holomorphic $(n, 0)$ -form on M_τ and depends on τ holomorphically.

Proof. We call such family the canonical family of holomorphic $(n, 0)$ -forms on the local deformation space of M_p . The fact that $\Omega(\tau)^c$ is a nowhere vanishing holomorphic $(n, 0)$ -form on the fiber M_τ follows from its definition and Lemma 2.10 directly. In fact we only need to check that $\partial_{M_p}(\varphi(\tau) \lrcorner \Omega_p) = 0$. By formulae (2.6) and (2.7) we know that

$$\varphi(\tau) \lrcorner \Omega_p = \sum_{i=1}^N \tau_i (\varphi_i \lrcorner \Omega_p) + \sum_{|I| \geq 2} \tau^I (\varphi_I \lrcorner \Omega_p) = \sum_{i=1}^N \tau_i (\varphi_i \lrcorner \Omega_p) + \partial_{M_p} \left(\sum_{|I| \geq 2} \tau^I \psi_I \right).$$

Because each φ_i is harmonic, by Lemma 2.4 we know that $\varphi_i \lrcorner \Omega_p$ is also harmonic and thus $\partial_{M_p}(\varphi_i \lrcorner \Omega_p) = 0$. Furthermore, since $\sum_{|I| \geq 2} \tau^I \psi_I$ converges when $|\tau|$ is small, we see that $\partial_{M_p}(\varphi(\tau) \lrcorner \Omega_p) = 0$ from formula (2.7). The holomorphic dependence of $\Omega_p^c(\tau)$ on τ follows from formula (6.16) and the fact that $\varphi(\tau)$ depends on τ holomorphically. \square

From Theorem 3 and Proposition 2.11 we get the expansion of the deRham cohomology classes of the canonical family of holomorphic $(n, 0)$ -forms. This expansion will be important in the construction of the holomorphic affine structure on the Teichmüller space. We remark that one may also directly deduce this expansion from the local Torelli theorem for Calabi-Yau manifold and the Griffiths transversality.

Theorem 4. Let $\Omega_p^c(\tau)$ be a canonical family defined by (6.16). Then we have the following expansion for $|\tau| < \epsilon$ small,

$$[\Omega_p^c(\tau)] = [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + A(\tau), \quad (2.13)$$

where $\{[\varphi_1 \lrcorner \Omega_p], \dots, [\varphi_N \lrcorner \Omega_p]\}$ give a basis of $H^{n-1,1}(M_p)$ and $A(\tau) = O(|\tau|^2) \in \bigoplus_{k=2}^n H^{n-k,k}(M_p)$ denotes terms of order at least 2 in τ .

Proof. By Theorem 3 and Proposition 2.11 we have

$$\begin{aligned} \Omega_p^c(\tau) &= \Omega_p + \sum_{i=1}^N \tau_i (\varphi_i \lrcorner \Omega_p) + \sum_{|I| \geq 2} \tau^I (\varphi_I \lrcorner \Omega_p) + \sum_{k \geq 2} \frac{1}{k!} (\wedge^k \varphi(\tau) \lrcorner \Omega_p) \\ &= \Omega_p + \sum_{i=1}^N \tau_i (\varphi_i \lrcorner \Omega_p) + \partial_{M_p} \left(\sum_{|I| \geq 2} \tau^I \psi_I \right) + a(\tau), \end{aligned} \quad (2.14)$$

where

$$a(\tau) = \sum_{k \geq 2} \frac{1}{k!} (\wedge^k \varphi(\tau) \lrcorner \Omega_p) \in \bigoplus_{k \geq 2} A^{n-k,k}(M). \quad (2.15)$$

By Hodge theory, we have

$$\begin{aligned} [\Omega_p^c(\tau)] &= [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + \left[\mathbb{H}(\partial_{M_p}(\sum_{|I| \geq 2} \tau^I \psi_I)) \right] + [\mathbb{H}(a(\tau))] \\ &= [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + [\mathbb{H}(a(\tau))]. \end{aligned} \quad (2.16)$$

Let $A(\tau) = [\mathbb{H}(a(\tau))]$, then (2.15) shows that $A(\tau) \in \bigoplus_{k=2}^n H^{n-k,k}(M)$ and $A(\tau) = O(|\tau|^2)$ which denotes the terms of order at least 2 in τ . \square

In fact we have the following expansion of the canonical family of $(n, 0)$ -classes up to order 2 in τ ,

$$[\Omega_p^c(\tau)] = [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + \frac{1}{2} \sum_{i,j} \tau_i \tau_j [\mathbb{H}(\varphi_i \wedge \varphi_j \lrcorner \Omega_p)] + \Xi(\tau),$$

where $\Xi(\tau) = O(|\tau|^3)$ denotes terms of order at least 3 in τ , and $\Xi(\tau) \in \bigoplus_{k=2}^n H^{n-k,k}(M)$.

This will not be needed in this thesis.

CHAPTER 3

Teichmüller space and period map of polarized and marked Calabi–Yau manifolds

In Section 3.1, we recall the definition and some basic properties of the period domain. In Section 3.2, we discuss the construction of the Teichmüller space of Calabi–Yau manifolds based on the works of Popp [20], Viehweg [33] and Szendrői [27] on the moduli spaces of Calabi–Yau manifolds. In Section 3.3, we define the period map from the Teichmüller space to the period domain. We remark that most of the results in this section are standard and can be found from the literature in the subjects.

3.1 Period domain of polarized Hodge structures

We first review the construction of the period domain of polarized Hodge structures. We refer the reader to §3 in [22] for more details.

A pair (M, L) consisting of a Calabi–Yau manifold M of complex dimension n with $n \geq 3$ and an ample line bundle L over M is called a polarized Calabi–Yau manifold. By abuse of notation, the Chern class of L will also be denoted by L and thus $L \in H^2(M, \mathbb{Z})$. Let $\{\gamma_1, \dots, \gamma_{h^n}\}$ be a basis of the integral homology group modulo torsion, $H_n(M, \mathbb{Z})/\text{Tor}$ with $\dim H_n(M, \mathbb{Z})/\text{Tor} = h^n$.

Definition 3.1. *Let the pair (M, L) be a polarized Calabi–Yau manifold, we call the triple $(M, L, \{\gamma_1, \dots, \gamma_{h^n}\})$ a polarized and marked Calabi–Yau manifold.*

For a polarized and marked Calabi–Yau manifold M with background smooth manifold

X , we identify the basis of $H_n(M, \mathbb{Z})/\text{Tor}$ to a lattice Λ as in [27]. This gives us a canonical identification of the middle dimensional de Rham cohomology of M to that of the background manifold X , that is,

$$H^n(M) \cong H^n(X),$$

where the coefficient ring can be \mathbb{Q} , \mathbb{R} or \mathbb{C} . Since the polarization L is an integer class, it defines a map

$$L : H^n(X, \mathbb{Q}) \rightarrow H^{n+2}(X, \mathbb{Q}), \quad A \mapsto L \wedge A.$$

We denote by $H_{pr}^n(X) = \ker(L)$ the primitive cohomology groups, where the coefficient ring can also be \mathbb{Q} , \mathbb{R} or \mathbb{C} . We let $H_{pr}^{k, n-k}(M) = H^{k, n-k}(M) \cap H_{pr}^n(M, \mathbb{C})$ and denote its dimension by $h^{k, n-k}$. We have the Hodge decomposition

$$H_{pr}^n(M, \mathbb{C}) = H_{pr}^{n,0}(M) \oplus \cdots \oplus H_{pr}^{0,n}(M). \quad (3.2)$$

It is easy to see that for a polarized Calabi–Yau manifold, since $H^2(M, \mathcal{O}_M) = 0$, we have

$$H_{pr}^{n,0}(M) = H^{n,0}(M), \quad H_{pr}^{n-1,1}(M) = H^{n-1,1}(M).$$

The Poincaré bilinear form Q on $H_{pr}^n(X, \mathbb{Q})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d -closed n -forms u, v on X . Furthermore, Q is nondegenerate and can be extended to $H_{pr}^n(X, \mathbb{C})$ bilinearly. Moreover, it also satisfies the Hodge-Riemann relations

$$Q(H_{pr}^{k, n-k}(M), H_{pr}^{l, n-l}(M)) = 0 \quad \text{unless } k+l = n, \quad \text{and} \quad (3.3)$$

$$(\sqrt{-1})^{2k-n} Q(v, \bar{v}) > 0 \quad \text{for } v \in H_{pr}^{k, n-k}(M) \setminus \{0\}. \quad (3.4)$$

Let $f^k = \sum_{i=k}^n h^{i, n-i}$, denote $f^0 = m$, and $F^k = F^k(M) = H_{pr}^{n,0}(M) \oplus \cdots \oplus H_{pr}^{k, n-k}(M)$, from which we have the decreasing filtration $H_{pr}^n(M, \mathbb{C}) = F^0 \supset \cdots \supset F^n$. We know that

$$\dim_{\mathbb{C}} F^k = f^k, \quad (3.5)$$

$$H_{pr}^n(X, \mathbb{C}) = F^k \oplus \overline{F^{n-k+1}}, \quad \text{and} \quad H_{pr}^{k, n-k}(M) = F^k \cap \overline{F^{n-k}}. \quad (3.6)$$

In terms of the Hodge filtration, the Hodge-Riemann relations (3.3) and (3.4) are

$$Q(F^k, F^{n-k+1}) = 0, \quad \text{and} \quad (3.7)$$

$$Q(Cv, \bar{v}) > 0 \quad \text{if} \quad v \neq 0, \quad (3.8)$$

where C is the Weil operator given by $Cv = (\sqrt{-1})^{2k-n}v$ for $v \in H_{pr}^{k,n-k}(M)$. The period domain D for polarized Hodge structures with data (3.5) is the space of all such Hodge filtrations

$$D = \{F^n \subset \cdots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (3.5), (3.7) \text{ and } (3.8) \text{ hold}\}.$$

The compact dual \check{D} of D is

$$\check{D} = \{F^n \subset \cdots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (3.5) \text{ and } (3.7) \text{ hold}\}.$$

The period domain $D \subseteq \check{D}$ is an open subset. From the definition of period domain we naturally get the Hodge bundles on \check{D} by associating to each point in \check{D} the vector spaces $\{F^k\}_{k=0}^n$ in the Hodge filtration of that point. Without confusion we will also denote by F^k the bundle with F^k as the fiber for each $0 \leq k \leq n$.

Remark 3.9. We remark the notation change for the primitive cohomology groups. As mentioned above that for a polarized Calabi–Yau manifold,

$$H_{pr}^{n,0}(M) = H^{n,0}(M), \quad H_{pr}^{n-1,1}(M) = H^{n-1,1}(M).$$

For the reason that we mainly consider these two types of primitive cohomology group of a Calabi–Yau manifold, by abuse of notation, we will simply use $H^n(M, \mathbb{C})$ and $H^{k,n-k}(M)$ to denote the primitive cohomology groups $H_{pr}^n(M, \mathbb{C})$ and $H_{pr}^{k,n-k}(M)$ respectively. Moreover, we will use cohomology to mean primitive cohomology in the rest of the paper.

3.2 Construction of the Teichmuller space

We first recall the concept of Kuranishi family of compact complex manifolds. We refer to page 8-10 in [24], page 94 in [20] and page 19 in [33] for equivalent definitions and more details.

A family of compact complex manifolds $\pi : \mathcal{W} \rightarrow \mathcal{B}$ is *versal* at a point $p \in \mathcal{B}$ if it satisfies the following conditions:

1. If given a complex analytic family $\iota : \mathcal{V} \rightarrow \mathcal{S}$ of compact complex manifolds with a point $s \in \mathcal{S}$ and a biholomorphic map $f_0 : V = \iota^{-1}(s) \rightarrow U = \pi^{-1}(p)$, then there exists a holomorphic map g from a neighbourhood $\mathcal{N} \subseteq \mathcal{S}$ of the point s to \mathcal{T} and a holomorphic map $f : \iota^{-1}(\mathcal{N}) \rightarrow \mathcal{W}$ with $\iota^{-1}(\mathcal{N}) \subseteq \mathcal{V}$ such that they satisfy that $g(s) = p$ and $f|_{\iota^{-1}(s)} = f_0$ with the following commutative diagram

$$\begin{array}{ccc} \iota^{-1}(\mathcal{N}) & \xrightarrow{f} & \mathcal{W} \\ \downarrow \iota & & \downarrow \pi \\ \mathcal{N} & \xrightarrow{g} & \mathcal{B}. \end{array}$$

2. For all g satisfying the above condition, the tangent map $(dg)_s$ is uniquely determined.

If a family $\pi : \mathcal{W} \rightarrow \mathcal{B}$ is versal at every point $p \in \mathcal{B}$, then it is a *versal family* on \mathcal{B} . If a complex analytic family satisfies the above condition (1), then the family is called *complete* at p . If a complex analytic family $\pi : \mathcal{W} \rightarrow \mathcal{B}$ of compact complex manifolds is complete at each point of \mathcal{B} and versal at the point $p \in \mathcal{B}$, then the family $\pi : \mathcal{W} \rightarrow \mathcal{B}$ is called the *Kuranishi family* of the complex manifold $V = \pi^{-1}(p)$. The base space \mathcal{B} is called the *Kuranishi space*. If the family is complete at each point in a neighbourhood of $p \in \mathcal{B}$ and versal at p , then the family is called a *local Kuranishi family* at $p \in \mathcal{B}$.

Let (M, L) be a polarized Calabi–Yau manifold. For any integer $m \geq 3$, we call a basis of the quotient space $(H_n(M, \mathbb{Z})/\text{Tor})/m(H_n(M, \mathbb{Z})/\text{Tor})$ a level m structure on the polarized Calabi–Yau manifold. For deformation of polarized Calabi–Yau manifold with level m structure, we reformulate Theorem 2.2 in [27] to the following theorem, in which we only put the statements we need in this paper. One can also look at [20] and [33] for more details about the construction of moduli spaces of Calabi–Yau manifolds.

Theorem 5. *Let M be a polarized Calabi–Yau manifold with level m structure with $m \geq 3$. Then there exists a connected quasi-projective complex manifold \mathcal{Z}_m with a versal family of*

Calabi–Yau manifolds,

$$\mathcal{X}_{\mathcal{Z}_m} \rightarrow \mathcal{Z}_m, \quad (3.10)$$

which contains M as a fiber and is polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$.

The Teichmüller space is the moduli space of equivalence classes of the marked and polarized Calabi–Yau manifolds. More precisely, a polarized and marked Calabi–Yau manifold is a triple (M, L, γ) , where M is a Calabi–Yau manifold, L is a polarization on M , and γ is a basis of $H_n(M, \mathbb{Z})/\text{Tor}$. Two triples (M, L, γ) and (M', L', γ') are equivalent if there exists a biholomorphic map $f : M \rightarrow M'$ with

$$\begin{aligned} f^*L' &= L, \\ f^*\gamma &= \gamma, \end{aligned}$$

then $[M, L, \gamma] = [M', L', \gamma'] \in \mathcal{T}$. Because a basis γ of $H_n(M, \mathbb{Z})/\text{Tor}$ naturally induces a basis of $(H_n(M, \mathbb{Z})/\text{Tor})/m(H_n(M, \mathbb{Z})/\text{Tor})$, we have a natural map $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$. In Theorem 2.5 and Corollary 2.8 of [2], the authors proved, that $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$ is a universal covering map, and consequencely, that \mathcal{T} is the universal cover space of \mathcal{Z}_m for some m .

In fact as the same construction in Section 2 of [27], we can also define the Teichmüller space \mathcal{T} to be a quotient space of the universal cover of the Hilbert scheme of Calabi–Yau manifolds by special linear group $SL(N, \mathbb{C})$. Under this construction, Teichmüller space \mathcal{T} is automatically simply connected, and there is a natural covering map $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$. Then by the uniqueness of universal covering space, these two constructions of \mathcal{T} are equivalent to each other.

We denote by $\varphi : \mathcal{U} \rightarrow \mathcal{T}$ the pull-back family of the family (3.10) via the covering π_m .

Proposition 3.11. *The Teichmüller space \mathcal{T} is a connected and simply connected smooth complex manifold and the family*

$$\varphi : \mathcal{U} \rightarrow \mathcal{T}, \quad (3.12)$$

which contains M as a fiber, is local Kuranishi at each point of \mathcal{T} .

Proof. For the first half, because \mathcal{Z}_m is a connected and smooth complex manifold, its universal cover \mathcal{T} is thus a connected and simply connected smooth complex manifold. For the second half, we know that the family (3.10) is a versal family at each point of \mathcal{Z}_m and that π_m is locally biholomorphic, therefore the pull-back family via π_m is also versal at each point of \mathcal{T} . Then by the definition of local Kuranishi family, we get that $\mathcal{U} \rightarrow \mathcal{T}$ is local Kuranishi at each point of \mathcal{T} . \square

Remark 3.13. We remark that the family $\varphi : \mathcal{U} \rightarrow \mathcal{T}$ being local Kuranishi at each point is essentially due to the local Torelli theorem for Calabi–Yau manifolds. In fact, the Kodaira–Spencer map of the family $\mathcal{U} \rightarrow \mathcal{T}$

$$\kappa : T_p^{1,0}\mathcal{T} \rightarrow H^{0,1}(M_p, T^{1,0}M_p),$$

is an isomorphism for each $p \in \mathcal{T}$. Then by theorems in page 9 of [24], we conclude that $\mathcal{U} \rightarrow \mathcal{T}$ is versal at each $p \in \mathcal{T}$. Moreover, the well-known Bogomolov–Tian–Todorov result ([28] and [29]) implies that $\dim_{\mathbb{C}}(\mathcal{T}) = N = h^{n-1,1}$. We refer the reader to Chapter 4 in [17] for more details about deformation of complex structures and the Kodaira–Spencer map.

Note that the Teichmüller space \mathcal{T} does not depend on the choice of m . In fact, let m_1 and m_2 be two different integers, and $\mathcal{U}_1 \rightarrow \mathcal{T}_1$, $\mathcal{U}_2 \rightarrow \mathcal{T}_2$ be two versal families constructed via level m_1 and level m_2 structures respectively as above, and both of which contain M as a fiber. By using the fact that \mathcal{T}_1 and \mathcal{T}_2 are simply connected and the definition of versal family, we have a biholomorphic map $f : \mathcal{T}_1 \rightarrow \mathcal{T}_2$, such that the versal family $\mathcal{U}_1 \rightarrow \mathcal{T}_1$ is the pull back of the versal family $\mathcal{U}_2 \rightarrow \mathcal{T}_2$ by f . Thus these two families are biholomorphic to each other.

There is another easier way to show that \mathcal{T} does not depend on the choice of m . Let m_1 and m_2 be two different integers, and $\mathcal{X}_{\mathcal{Z}_{m_1}} \rightarrow \mathcal{Z}_{m_1}$, $\mathcal{X}_{\mathcal{Z}_{m_2}} \rightarrow \mathcal{Z}_{m_2}$ be two versal families with level m_1 and level m_2 structures respectively, and $\mathcal{U}_1 \rightarrow \mathcal{T}_1$, $\mathcal{U}_2 \rightarrow \mathcal{T}_2$ be two versal families constructed via level m_1 and level m_2 structures respectively as above, and both of which contain M as a fiber as above. Then \mathcal{T}_1 is the universal cover of \mathcal{Z}_{m_1} and \mathcal{T}_2 is the universal cover of \mathcal{Z}_{m_2} . Let us consider the product of two integers $m_{1,2} = m_1 m_2$, and the versal family

$\mathcal{X}_{\mathcal{Z}_{m_{1,2}}} \rightarrow \mathcal{Z}_{m_{1,2}}$. Then the moduli space $\mathcal{Z}_{m_{1,2}}$ with level $m_{1,2}$ structure is a covering space of both \mathcal{Z}_{m_1} and \mathcal{Z}_{m_2} . Let \mathcal{T} be the universal cover space of $\mathcal{Z}_{m_{1,2}}$, and $\mathcal{U} \rightarrow \mathcal{T}$ the pull back family from $\mathcal{X}_{\mathcal{Z}_{m_{1,2}}}$ as above. Since $\mathcal{Z}_{m_{1,2}}$ is a covering space of both \mathcal{Z}_{m_1} and \mathcal{Z}_{m_2} , we conclude that \mathcal{T} is universal cover of \mathcal{Z}_{m_1} , and \mathcal{T} is also the universal cover of \mathcal{Z}_{m_2} . Thus we proved that the universal covers of \mathcal{Z}_{m_1} and \mathcal{Z}_{m_2} are the same, that is, $\mathcal{T} = \mathcal{T}_1 = \mathcal{T}_2$ with \mathcal{U} the pull back versal family over \mathcal{T} .

In the rest of the paper, we will simply use “level m structure” to mean “level m structure with $m \geq 3$ ”.

3.3 The period map and the Hodge metric on the Teichmüller space

For any point $p \in \mathcal{T}$, let M_p be the fiber of family $\varphi : \mathcal{U} \rightarrow \mathcal{T}$, which is a polarized and marked Calabi–Yau manifold. Since the Teichmüller space is simply connected and we have fixed the basis of the middle homology group modulo torsions, we identify the basis of $H_n(M, \mathbb{Z})/\text{Tor}$ to a lattice Λ as in [27]. This gives us a canonical identification of the middle dimensional cohomology of M to that of the background manifold X , that is, $H^n(M, \mathbb{C}) \simeq H^n(X, \mathbb{C})$. Therefore, we can use this to identify $H^n(M_p, \mathbb{C})$ for all fibers on \mathcal{T} . Thus we get a canonical trivial bundle $H^n(M_p, \mathbb{C}) \times \mathcal{T}$.

The period map from \mathcal{T} to D is defined by assigning to each point $p \in \mathcal{T}$ the Hodge structure on M_p , that is

$$\Phi : \mathcal{T} \rightarrow D, \quad p \mapsto \Phi(p) = \{F^n(M_p) \subset \cdots \subset F^0(M_p)\}$$

We denote $F^k(M_p)$ by F_p^k for simplicity.

The period map has several good properties, and one may refer to Chapter 10 in [34] for details. Among them, one of the most important is the following Griffiths transversality:

the period map Φ is a holomorphic map and its tangent map satisfies that

$$\Phi_*(v) \in \bigoplus_{k=1}^n \text{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k) \quad \text{for any } p \in \mathcal{T} \text{ and } v \in T_p^{1,0}\mathcal{T}$$

with $F^{n+1} = 0$, or equivalently, $\Phi_*(v) \in \bigoplus_{k=0}^n \text{Hom}(F_p^k, F_p^{k-1})$.

In [6], Griffiths and Schmid studied the so-called Hodge metric on the period domain D . We denote it by h . In particular, this Hodge metric is a complete homogeneous metric. Let us denote the period map on the moduli space by $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \rightarrow D/\Gamma$, where Γ denotes the global monodromy group which acts properly and discontinuously on the period domain D . By local Torelli theorem for Calabi–Yau manifolds, we know that $\Phi_{\mathcal{Z}_m}, \Phi$ are both locally injective. Thus it follows from [6] that the pull-backs of h by $\Phi_{\mathcal{Z}_m}$ and Φ on \mathcal{Z}_m and \mathcal{T} respectively are both well-defined Kähler metrics. By abuse of notation, we still call these pull-back metrics the *Hodge metrics*.

CHAPTER 4

Holomorphic affine structure on the Teichmüller space

In Section 4.1, we review some properties of the period domain from Lie group and Lie algebra point of view. In Section 4.2, we fix a base point $p \in \mathcal{T}$ and introduce the unipotent space $N_+ \subseteq \check{D}$, which is biholomorphic to \mathbb{C}^d . Then we show that the image $\Phi(\mathcal{T})$ is bounded in $N_+ \cap D$ with respect to the Euclidean metric on N_+ . In Section 4.3, using the property that $\Phi(\mathcal{T}) \subseteq N_+$, we define a holomorphic map $\tau : \mathcal{T} \rightarrow \mathbb{C}^N$. Then we use local Torelli theorem to show that τ defines a local coordinate chart around each point in \mathcal{T} , and this shows that $\tau : \mathcal{T} \rightarrow \mathbb{C}^N$ defines a holomorphic affine structure on \mathcal{T} .

4.1 Preliminary

Let us briefly recall some properties of the period domain from Lie group and Lie algebra point of view. All of the results in this section is well-known to the experts in the subject. The purpose to give details is to fix notations. One may either skip this section or refer to [6] and [22] for most of the details.

The orthogonal group of the bilinear form Q in the definition of Hodge structure is a linear algebraic group, defined over \mathbb{Q} . Let us simply denote $H_{\mathbb{C}} = H^n(M, \mathbb{C})$ and $H_{\mathbb{R}} = H^n(M, \mathbb{R})$. The group of the \mathbb{C} -rational points is

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

which acts on \check{D} transitively. The group of real points in $G_{\mathbb{C}}$ is

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

which acts transitively on D as well.

Consider the period map $\Phi : \mathcal{T} \rightarrow D$. Fix a point $p \in \mathcal{T}$ with the image $o := \Phi(p) = \{F_p^n \subset \cdots \subset F_p^0\} \in D$. The points $p \in \mathcal{T}$ and $o \in D$ may be referred as the base points or the reference points. A linear transformation $g \in G_{\mathbb{C}}$ preserves the base point if and only if $gF_p^k = F_p^k$ for each k . Thus it gives the identification

$$\check{D} \simeq G_{\mathbb{C}}/B \quad \text{with} \quad B = \{g \in G_{\mathbb{C}} \mid gF_p^k = F_p^k, \text{ for any } k\}.$$

Similarly, one obtains an analogous identification

$$D \simeq G_{\mathbb{R}}/V \hookrightarrow \check{D} \quad \text{with} \quad V = G_{\mathbb{R}} \cap B,$$

where the embedding corresponds to the inclusion $G_{\mathbb{R}}/V = G_{\mathbb{R}}/G_{\mathbb{R}} \cap B \subseteq G_{\mathbb{C}}/B$. The Lie algebra \mathfrak{g} of the complex Lie group $G_{\mathbb{C}}$ can be described as

$$\mathfrak{g} = \{X \in \text{End}(H_{\mathbb{C}}) \mid Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\}.$$

It is a simple complex Lie algebra, which contains $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. With the inclusion $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$, \mathfrak{g}_0 becomes Lie algebra of $G_{\mathbb{R}}$. One observes that the reference Hodge structure $\{H_p^{k, n-k}\}_{k=0}^n$ of $H^n(M, \mathbb{C})$ induces a Hodge structure of weight zero on $\text{End}(H^n(M, \mathbb{C}))$, namely,

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k, -k} \quad \text{with} \quad \mathfrak{g}^{k, -k} = \{X \in \mathfrak{g} \mid XH_p^{r, n-r} \subseteq H_p^{r+k, n-r-k}\}.$$

Since the Lie algebra \mathfrak{b} of B consists of those $X \in \mathfrak{g}$ that preserves the reference Hodge filtration $\{F_p^n \subset \cdots \subset F_p^0\}$, one thus has

$$\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k, -k}.$$

The Lie algebra \mathfrak{v}_0 of V is $\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. With the above isomorphisms, the holomorphic tangent space of \check{D} at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \bigoplus_{k \geq 1} \mathfrak{g}^{-k, k}$. Then one gets the holomorphic isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. We take the unipotent group $N_+ = \exp(\mathfrak{n}_+)$.

As $\text{Ad}(g)(\mathfrak{g}^{k,-k})$ is in $\bigoplus_{i \geq k} \mathfrak{g}^{i,-i}$ for each $g \in B$, the sub-Lie algebra $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \subseteq \mathfrak{g}/\mathfrak{b}$ defines an $\text{Ad}(B)$ -invariant subspace. By left translation via $G_{\mathbb{C}}$, $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$ gives rise to a $G_{\mathbb{C}}$ -invariant holomorphic subbundle of the holomorphic tangent bundle at the base point. It will be denoted by $T_{o,h}^{1,0}\check{D}$, and will be referred to as the holomorphic horizontal tangent bundle at the base point. One can check that this construction does not depend on the choice of the base point. The horizontal tangent subbundle at the base point o , restricted to D , determines a subbundle $T_{o,h}^{1,0}D$ of the holomorphic tangent bundle $T_o^{1,0}D$ of D at the base point. The $G_{\mathbb{C}}$ -invariance of $T_{o,h}^{1,0}\check{D}$ implies the $G_{\mathbb{R}}$ -invariance of $T_{o,h}^{1,0}D$. As another interpretation of this holomorphic horizontal bundle at the base point, one has

$$T_{o,h}^{1,0}\check{D} \simeq T_o^{1,0}\check{D} \cap \bigoplus_{k=1}^n \text{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k). \quad (4.1)$$

In [22], Schmid call a holomorphic mapping $\Psi : M \rightarrow \check{D}$ of a complex manifold M into \check{D} *horizontal* if at each point of M , the induced map between the holomorphic tangent spaces takes values in the appropriate fibre $T^{1,0}\check{D}$. It is easy to see that the period map $\Phi : \mathcal{T} \rightarrow D$ is horizontal since $\Phi_*(T_p^{1,0}\mathcal{T}) \subseteq T_{o,h}^{1,0}D$ for any $p \in \mathcal{T}$. Since D is an open set in \check{D} , we have the following relation:

$$T_{o,h}^{1,0}D = T_{o,h}^{1,0}\check{D} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \hookrightarrow \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+. \quad (4.2)$$

Remark 4.3. With a fixed base point, we can identify N_+ with its unipotent orbit in \check{D} by identifying an element $c \in N_+$ with $[c] = cB$ in \check{D} ; that is, $N_+ = N_+(\text{base point}) \cong N_+B/B \subseteq \check{D}$. In particular, when the base point o is in D , we have $N_+ \cap D \subseteq D$.

Let us introduce the notion of an adapted basis for the given Hodge decomposition or the Hodge filtration. For any $p \in \mathcal{T}$ and $f^k = \dim F_p^k$ for any $0 \leq k \leq n$, we call a basis

$$\xi = \{\xi_0, \xi_1, \dots, \xi_N, \dots, \xi_{f^{k+1}}, \dots, \xi_{f^{k-1}}, \dots, \xi_{f^2}, \dots, \xi_{f^1-1}, \xi_{f^0-1}\}$$

of $H^n(M_p, \mathbb{C})$ an *adapted basis for the given Hodge decomposition*

$$H^n(M_p, \mathbb{C}) = H_p^{n,0} \oplus H_p^{n-1,1} \oplus \dots \oplus H_p^{1,n-1} \oplus H_p^{0,n},$$

if it satisfies $H_p^{k,n-k} = \text{Span}_{\mathbb{C}} \{\xi_{f^{k+1}}, \dots, \xi_{f^{k-1}}\}$ with $\dim H_p^{k,n-k} = f^k - f^{k+1}$. We call a basis

$$\zeta = \{\zeta_0, \zeta_1, \dots, \zeta_N, \dots, \zeta_{f^{k+1}}, \dots, \zeta_{f^{k-1}}, \dots, \zeta_{f^2}, \dots, \zeta_{f^1-0}, \zeta_{f^0-1}\}$$

of $H^n(M_p, \mathbb{C})$ an *adapted basis for the given filtration*

$$F^n \subseteq F^{n-1} \subseteq \dots \subseteq F^0$$

if it satisfies $F^k = \text{Span}_{\mathbb{C}} \{\zeta_0, \dots, \zeta_{f^k-1}\}$ with $\dim_{\mathbb{C}} F^k = f^k$. Moreover, unless otherwise pointed out, the matrices in this paper are $m \times m$ matrices, where $m = f^0$. The blocks of the $m \times m$ matrix T is set as follows: for each $0 \leq \alpha, \beta \leq n$, the (α, β) -th block $T^{\alpha, \beta}$ is

$$T^{\alpha, \beta} = [T_{ij}(\tau)]_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n-1}, f^{-\beta+n+1} \leq j \leq f^{-\beta+n-1}}, \quad (4.4)$$

where T_{ij} is the entries of the matrix T , and f^{n+1} is defined to be zero. In particular, $T = [T^{\alpha, \beta}]$ is called a *block lower triangular matrix* if $T^{\alpha, \beta} = 0$ whenever $\alpha < \beta$.

Remark 4.5. We remark that by fixing a base point, we can identify the above quotient Lie groups or Lie algebras with their orbits in the corresponding quotient Lie algebras or Lie groups. For example, $\mathfrak{n}_+ \cong \mathfrak{g}/\mathfrak{b}$, $\mathfrak{g}^{-1,1} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$, and $N_+ \cong N_+B/B \subseteq \check{D}$. We can also identify a point $\Phi(p) = \{F_p^n \subseteq F_p^{n-1} \subseteq \dots \subseteq F_p^0\} \in D$ with its Hodge decomposition $\bigoplus_{k=0}^n H_p^{k,n-k}$, and thus with any fixed adapted basis of the corresponding Hodge decomposition for the base point, we have matrix representations of elements in the above Lie groups and Lie algebras. For example, elements in N_+ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in B can be realized as nonsingular block upper triangular matrices.

We shall review and collect some facts about the structure of simple Lie algebra \mathfrak{g} in our case. Again one may refer to [6] and [22] for more details. Let $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be the Weil operator, which is defined by

$$\theta(X) = (-1)^p X \quad \text{for } X \in \mathfrak{g}^{p,-p}.$$

Then θ is an involutive automorphism of \mathfrak{g} , and is defined over \mathbb{R} . The $(+1)$ and (-1) eigenspaces of θ will be denoted by \mathfrak{k} and \mathfrak{p} respectively. Moreover, set

$$\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.$$

The fact that θ is an involutive automorphism implies

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}.$$

Let us consider $\mathfrak{g}_c = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$. Then \mathfrak{g}_c is a real form for \mathfrak{g} . Recall that the killing form $B(\cdot, \cdot)$ on \mathfrak{g} is defined by

$$B(X, Y) = \text{Trace}(\text{ad}(X) \circ \text{ad}(Y)) \quad \text{for } X, Y \in \mathfrak{g}.$$

A semisimple Lie algebra is compact if and only if the Killing form is negative definite. Thus it is not hard to check that \mathfrak{g}_c is actually a compact real form of \mathfrak{g} , while \mathfrak{g}_0 is a non-compact real form. Recall that $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$ is the subgroup which corresponds to the subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$. Let us denote the connected subgroup $G_c \subseteq G_{\mathbb{C}}$ which corresponds to the subalgebra $\mathfrak{g}_c \subseteq \mathfrak{g}$. Let us denote the complex conjugation of \mathfrak{g} with respect to the compact real form by τ_c , and the complex conjugation of \mathfrak{g} with respect to the compact real form by τ_0 .

The intersection $K = G_c \cap G_{\mathbb{R}}$ is then a compact subgroup of $G_{\mathbb{R}}$, whose Lie algebra is $\mathfrak{k}_0 = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}_c$. With the above notations, Schmid showed in [22] that K is a maximal compact subgroup of $G_{\mathbb{R}}$, and it meets every connected component of $G_{\mathbb{R}}$. Moreover, $V = G_{\mathbb{R}} \cap B \subseteq K$.

We know that in our cases, $G_{\mathbb{C}}$ is a connected simple Lie group, B a parabolic subgroup in $G_{\mathbb{C}}$ with \mathfrak{b} as its Lie algebra. The Lie algebra \mathfrak{b} has a unique maximal nilpotent ideal \mathfrak{n}_- . It is not hard to see that

$$\mathfrak{g}_c \cap \mathfrak{n}_- = \mathfrak{n}_- \cap \tau_c(\mathfrak{n}_-) = 0.$$

By using Bruhat's lemma, one concludes \mathfrak{g} is spanned by the parabolic subalgebras \mathfrak{b} and $\tau_c(\mathfrak{b})$. Moreover $\mathfrak{v} = \mathfrak{b} \cap \tau_c(\mathfrak{b})$, $\mathfrak{b} = \mathfrak{v} \oplus \mathfrak{n}_-$. In particular, we also have

$$\mathfrak{n}_+ = \tau_c(\mathfrak{n}_-).$$

As remarked in §1 in [6] of Griffiths and Schmid, one gets that \mathfrak{v} must have the same rank of \mathfrak{g} as \mathfrak{v} is the intersection of the two parabolic subalgebras \mathfrak{b} and $\tau_c(\mathfrak{b})$. Moreover, \mathfrak{g}_0 and \mathfrak{v}_0 are also of equal rank, since they are real forms of \mathfrak{g} and \mathfrak{v} respectively. Therefore, we can choose a Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0$ is also a Cartan subalgebra of \mathfrak{v}_0 . Since $\mathfrak{v}_0 \subseteq \mathfrak{k}_0$, we also have $\mathfrak{h}_0 \subseteq \mathfrak{k}_0$. A Cartan subalgebra of a real Lie algebra is a maximal abelian subalgebra. Therefore \mathfrak{h}_0 is also a maximal abelian subalgebra of \mathfrak{k}_0 , hence \mathfrak{h}_0 is a Cartan subalgebra of \mathfrak{k}_0 . Summarizing the above, we get

Proposition 4.6. *There exists a Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and \mathfrak{h}_0 is also a Cartan subalgebra of \mathfrak{k}_0 .*

Remark 4.7. As an alternate proof of Proposition 4.6, to show that \mathfrak{k}_0 and \mathfrak{g}_0 have equal rank, one realizes that that \mathfrak{g}_0 in our case is one of the following real simple Lie algebras: $\mathfrak{sp}(2l, \mathbb{R})$, $\mathfrak{so}(p, q)$ with $p + q$ odd, or $\mathfrak{so}(p, q)$ with p and q both even. One may refer to [5] and [24] for more details.

Proposition 4.6 implies that the simple Lie algebra \mathfrak{g}_0 in our case is a simple Lie algebra of first category as defined in §4 in [25]. In the upcoming part, we will briefly derive the result of a simple Lie algebra of first category in Lemma 3 in [26]. One may also refer to [37] Lemma 2.2.12 at pp. 141-142 for the same result.

Let us still use the above notations of the Lie algebras we consider. By Proposition 4, we can take \mathfrak{h}_0 to be a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and \mathfrak{h}_0 is also a Cartan subalgebra of \mathfrak{k}_0 . Let us denote \mathfrak{h} to be the complexification of \mathfrak{h}_0 . Then \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{k}$.

Write $\mathfrak{h}_0^* = \text{Hom}(\mathfrak{h}_0, \mathbb{R})$ and $\mathfrak{h}_{\mathbb{R}}^* = \sqrt{-1}\mathfrak{h}_0^*$. Then $\mathfrak{h}_{\mathbb{R}}^*$ can be identified with $\mathfrak{h}_{\mathbb{R}} := \sqrt{-1}\mathfrak{h}_0$ by duality using the restriction of the Killing form B of \mathfrak{g} to $\mathfrak{h}_{\mathbb{R}}$. Let $\rho \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$, one can define the following subspace of \mathfrak{g}

$$\mathfrak{g}^\rho = \{x \in \mathfrak{g} \mid [h, x] = \rho(h)x \text{ for all } h \in \mathfrak{h}\}.$$

An element $\varphi \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ is called a root of \mathfrak{g} with respect to \mathfrak{h} if $\mathfrak{g}^\varphi \neq \{0\}$.

Let $\Delta \subseteq \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ denote the space of nonzero \mathfrak{h} -roots. Then each root space

$$\mathfrak{g}^\varphi = \{x \in \mathfrak{g} \mid [h, x] = \varphi(h)x \text{ for all } h \in \mathfrak{h}\}$$

belongs to some $\varphi \in \Delta$ is one-dimensional over \mathbb{C} , generated by a root vector e_φ .

Since the involution θ is a Lie-algebra automorphism fixing \mathfrak{k} , we have $[h, \theta(e_\varphi)] = \varphi(h)\theta(e_\varphi)$ for any $h \in \mathfrak{h}$ and $\varphi \in \Delta$. Thus $\theta(e_\varphi)$ is also a root vector belonging to the root φ , so e_φ must be an eigenvector of θ . It follows that there is a decomposition of the roots Δ into $\Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}$ of compact roots and non-compact roots with root spaces $\mathbb{C}e_\varphi \subseteq \mathfrak{k}$ and \mathfrak{p} respectively. The adjoint representation of \mathfrak{h} on \mathfrak{g} determines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^\varphi.$$

There also exists a Weyl base $\{h_i, 1 \leq i \leq l; e_\varphi, \text{ for any } \varphi \in \Delta\}$ with $l = \text{rank}(\mathfrak{g})$ such that $\text{Span}_{\mathbb{C}}\{h_1, \dots, h_l\} = \mathfrak{h}$, $\text{Span}_{\mathbb{C}}\{e_\varphi\} = \mathfrak{g}^\varphi$ for each $\varphi \in \Delta$, and

$$\tau_c(h_i) = \tau_0(h_i) = -h_i, \quad \text{for any } 1 \leq i \leq l;$$

$$\tau_c(e_\varphi) = \tau_0(e_\varphi) = -e_{-\varphi} \quad \text{for any } \varphi \in \Delta_{\mathfrak{k}}; \quad \tau_0(e_\varphi) = -\tau_c(e_\varphi) = e_\varphi \quad \text{for any } \varphi \in \Delta_{\mathfrak{p}}.$$

With respect to this Weyl base, we have

$$\begin{aligned} \mathfrak{k}_0 &= \mathfrak{h}_0 + \sum_{\varphi \in \Delta_{\mathfrak{k}}} \mathbb{R}(e_\varphi - e_{-\varphi}) + \sum_{\varphi \in \Delta_{\mathfrak{k}}} \mathbb{R}\sqrt{-1}(e_\varphi + e_{-\varphi}); \\ \mathfrak{p}_0 &= \sum_{\varphi \in \Delta_{\mathfrak{p}}} \mathbb{R}(e_\varphi + e_{-\varphi}) + \sum_{\varphi \in \Delta_{\mathfrak{p}}} \mathbb{R}\sqrt{-1}(e_\varphi - e_{-\varphi}). \end{aligned}$$

Lemma 4.8. *Let Δ be the set of \mathfrak{h} -roots as above. Then for each root $\varphi \in \Delta$, there is an integer $-n \leq k \leq n$ such that $e_\varphi \in \mathfrak{g}^{k, -k}$. In particular, if $e_\varphi \in \mathfrak{g}^{k, -k}$, then $\tau_0(e_\varphi) \in \mathfrak{g}^{-k, k}$ for any $-n \leq k \leq n$.*

Proof. Let φ be a root, and e_φ be the generator of the root space \mathfrak{g}^φ , then $e_\varphi = \sum_{k=-n}^n e^{-k, k}$, where $e^{-k, k} \in \mathfrak{g}^{-k, k}$. Because $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{g}^{0, 0}$, the Lie bracket $[e^{-k, k}, h] \in \mathfrak{g}^{-k, k}$ for each k . Then the condition $[e_\varphi, h] = \varphi(h)e_\varphi$ implies that

$$\sum_{k=-n}^n [e^{-k, k}, h] = \sum_{k=-n}^n \varphi(h)e^{-k, k} \quad \text{for each } h \in \mathfrak{h}.$$

By comparing the type, we get

$$[e^{-k,k}, h] = \varphi(h)e^{-k,k} \quad \text{for each } h \in \mathfrak{h}.$$

Therefore $e^{-k,k} \in \mathfrak{g}^\varphi$ for each k . As $\{e^{-k,k}\}_{k=-n}^n$ forms a linear independent set, but \mathfrak{g}^φ is one dimensional, thus there is only one $-n \leq k \leq n$ with $e^{-k,k} \neq 0$. \square

Let us now introduce a lexicographic order (cf. pp.41 in [37] or pp.416 in [25]) in the real vector space $\mathfrak{h}_\mathbb{R}$ as follows: we fix an ordered basis e_1, \dots, e_l for $\mathfrak{h}_\mathbb{R}$. Then for any $h = \sum_{i=1}^l \lambda_i e_i \in \mathfrak{h}_\mathbb{R}$, we call $h > 0$ if the first nonzero coefficient is positive, that is, if $\lambda_1 = \dots = \lambda_k = 0, \lambda_{k+1} > 0$ for some $1 \leq k < l$. For any $h, h' \in \mathfrak{h}_\mathbb{R}$, we say $h > h'$ if $h - h' > 0$, $h < h'$ if $h - h' < 0$ and $h = h'$ if $h - h' = 0$. In particular, let us identify the dual spaces $\mathfrak{h}_\mathbb{R}^*$ and $\mathfrak{h}_\mathbb{R}$, thus $\Delta \subseteq \mathfrak{h}_\mathbb{R}$. Let us choose a maximal linearly independent subset $\{e_1, \dots, e_s\}$ of $\Delta_\mathfrak{p}$, then a maximal linearly independent subset $\{e_{s+1}, \dots, e_l\}$ of $\Delta_\mathfrak{t}$. Then $\{e_1, \dots, e_s, e_{s+1}, \dots, e_l\}$ forms a basis for $\mathfrak{h}_\mathbb{R}^*$ since $\text{Span}_\mathbb{R} \Delta = \mathfrak{h}_\mathbb{R}^*$. Then define the above lexicographic order in $\mathfrak{h}_\mathbb{R}^* \simeq \mathfrak{h}_\mathbb{R}$ using the ordered basis $\{e_1, \dots, e_l\}$. In this way, we can also define

$$\Delta^+ = \{\varphi > 0 : \varphi \in \Delta\}; \quad \Delta_\mathfrak{p}^+ = \Delta^+ \cap \Delta_\mathfrak{p}.$$

Similarly we can define $\Delta^-, \Delta_\mathfrak{p}^-, \Delta_\mathfrak{t}^+$, and $\Delta_\mathfrak{t}^-$. Then one can conclude the following lemma from Lemma 2.2.10 and Lemma 2.2.11 at pp.141 in [37],

Lemma 4.9. *Using the above notation, we have*

$$(\Delta_\mathfrak{t} + \Delta_\mathfrak{p}^\pm) \cap \Delta \subseteq \Delta_\mathfrak{p}^\pm; \quad (\Delta_\mathfrak{p}^\pm + \Delta_\mathfrak{p}^\pm) \cap \Delta = \emptyset.$$

If one defines

$$\mathfrak{p}^\pm = \sum_{\varphi \in \Delta_\mathfrak{p}^\pm} \mathfrak{g}^\varphi \subseteq \mathfrak{p},$$

then $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ and $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] = 0, [\mathfrak{p}^+, \mathfrak{p}^-] \subseteq \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}^\pm] \subseteq \mathfrak{p}^\pm$.

Definition 4.10. *Two different roots $\varphi, \psi \in \Delta$ are said to be strongly orthogonal if and only if $\varphi \pm \psi \notin \Delta \cup \{0\}$, which is denoted by $\varphi \perp \psi$.*

For the real simple Lie algebra $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ which has a Cartan subalgebra \mathfrak{h}_0 in \mathfrak{k}_0 , the maximal abelian subspace of \mathfrak{p}_0 can be described as in the following lemma, which is a slight extension of a lemma of Harish-Chandra in [7]. One may refer to Lemma 3 in [26] or Lemma 2.2.12 at pp.141–142 in [37] for more details. For reader's convenience we give the detailed proof.

Lemma 4.11. *There exists a set of strongly orthogonal noncompact positive roots $\Lambda = \{\varphi_1, \dots, \varphi_r\} \subseteq \Delta_{\mathfrak{p}}^+$ such that*

$$\mathfrak{a}_0 = \sum_{i=1}^r \mathbb{R} (e_{\varphi_i} + e_{-\varphi_i})$$

is a maximal abelian subspace in \mathfrak{p}_0 .

Proof. Let φ_1 be the minimum in $\Delta_{\mathfrak{p}}^+$, and φ_2 be the minimal element in $\{\varphi \in \Delta_{\mathfrak{p}}^+ : \varphi \perp \varphi_1\}$, then we obtain inductively an maximal ordered set of roots $\Lambda = \{\varphi_1, \dots, \varphi_r\} \subseteq \Delta_{\mathfrak{p}}^+$, such that for each $1 \leq k \leq r$

$$\varphi_k = \min\{\varphi \in \Delta_{\mathfrak{p}}^+ : \varphi \perp \varphi_j \text{ for } 1 \leq j \leq k-1\}.$$

Because $\varphi_i \perp \varphi_j$ for any $1 \leq i < j \leq r$, we have $[e_{\pm\varphi_i}, e_{\pm\varphi_j}] = 0$. Therefore $\mathfrak{a}_0 = \sum_{i=1}^r \mathbb{R} (e_{\varphi_i} + e_{-\varphi_i})$ is an abelian subspace of \mathfrak{p}_0 . Also because a root can not be strongly orthogonal to itself, the ordered set Λ contains distinct roots. Thus $\dim_{\mathbb{R}} \mathfrak{a}_0 = r$.

Now we prove that \mathfrak{a}_0 is a maximal abelian subspace of \mathfrak{p}_0 . Suppose towards a contradiction that there was a nonzero vector $X \in \mathfrak{p}_0$ as follows

$$X = \sum_{\alpha \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda} \lambda_{\alpha} (e_{\alpha} + e_{-\alpha}) + \sum_{\alpha \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda} \mu_{\alpha} \sqrt{-1} (e_{\alpha} - e_{-\alpha}), \quad \text{where } \lambda_{\alpha}, \mu_{\alpha} \in \mathbb{R},$$

such that $[X, e_{\varphi_i} + e_{-\varphi_i}] = 0$ for each $1 \leq i \leq r$. We denote $c_{\alpha} = \lambda_{\alpha} + \sqrt{-1}\mu_{\alpha}$. Because $X \neq 0$, there exists $\psi \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda$ with $c_{\psi} \neq 0$. Also ψ is not strongly orthogonal to φ_i for some $1 \leq i \leq r$. Thus we may first define k_{ψ} for each ψ with $c_{\psi} \neq 0$ as the following:

$$k_{\psi} = \min_{1 \leq i \leq r} \{i : \psi \text{ is not strongly orthogonal to } \varphi_i\}.$$

Then we know that $1 \leq k_\psi \leq r$ for each ψ with $c_\psi \neq 0$. Then we define k to be the following,

$$k = \min_{\psi \in \Delta_{\mathbf{p}}^+ \setminus \Lambda \text{ with } c_\psi \neq 0} \{k_\psi\}. \quad (4.12)$$

Here, we are taking the minimum over a finite set in the (4.12) and $1 \leq k \leq r$. Moreover, we get the following non-empty set,

$$S_k = \{\psi \in \Delta_{\mathbf{p}}^+ \setminus \Lambda : c_\psi \neq 0 \text{ and } k_\psi = k\} \neq \emptyset. \quad (4.13)$$

Recall the notation $N_{\beta,\gamma}$ for any $\beta, \gamma \in \Delta$ is defined as follows: if $\beta + \gamma \in \Delta \cup \{0\}$, $N_{\beta,\gamma}$ is defined such that $[e_\beta, e_\gamma] = N_{\beta,\gamma} e_{\beta+\gamma}$; if $\beta + \gamma \notin \Delta \cup \{0\}$ then one defines $N_{\beta,\gamma} = 0$. Now let us take k as defined in (4.12) and consider the Lie bracket

$$\begin{aligned} 0 &= [X, e_{\varphi_k} + e_{-\varphi_k}] \\ &= \sum_{\psi \in \Delta_{\mathbf{p}}^+ \setminus \Lambda} \left(c_\psi (N_{\psi, \varphi_k} e_{\psi+\varphi_k} + N_{\psi, -\varphi_k} e_{\psi-\varphi_k}) + \bar{c}_\psi (N_{-\psi, \varphi_k} e_{-\psi+\varphi_k} + N_{-\psi, -\varphi_k} e_{-\psi-\varphi_k}) \right). \end{aligned}$$

As $[\mathbf{p}^\pm, \mathbf{p}^\pm] = 0$, we have $\psi + \varphi_k \notin \Delta$ and $-\psi - \varphi_k \notin \Delta$ for each $\psi \in \Delta_{\mathbf{p}}^+$. Hence, $N_{\psi, \varphi_k} = N_{-\psi, -\varphi_k} = 0$ for each $\psi \in \Delta_{\mathbf{p}}^+$. Then we have the simplified expression

$$0 = [X, e_{\varphi_k} + e_{-\varphi_k}] = \sum_{\psi \in \Delta_{\mathbf{p}}^+ \setminus \Lambda} \left(c_\psi N_{\psi, -\varphi_k} e_{\psi-\varphi_k} + \bar{c}_\psi N_{-\psi, \varphi_k} e_{-\psi+\varphi_k} \right). \quad (4.14)$$

Now let us take $\psi_0 \in S_k \neq \emptyset$. Then $c_{\psi_0} \neq 0$. By the definition of k , we have ψ_0 is not strongly orthogonal to φ_k while $\psi_0 + \varphi_k \notin \Delta \cup \{0\}$. Thus we have $\psi_0 - \varphi_k \in \Delta \cup \{0\}$. Therefore $c_{\psi_0} N_{\psi_0, -\varphi_k} e_{\psi_0-\varphi_k} \neq 0$. Since $0 = [X, e_{\varphi_k} + e_{-\varphi_k}]$, there must exist one element $\psi'_0 \neq \psi_0 \in \Delta_{\mathbf{p}}^+ \setminus \Lambda$ such that $\varphi_k - \psi_0 = \psi'_0 - \varphi_k$ and $c_{\psi'_0} \neq 0$. This implies $2\varphi_k = \psi_0 + \psi'_0$, and consequently one of ψ_0 and ψ'_0 is smaller than φ_k . Then we have the following two cases:

(i). if $\psi_0 < \varphi_k$, then we find $\psi_0 < \varphi_k$ with $\psi_0 \perp \varphi_i$ for all $1 \leq i \leq k-1$, and this contradicts to the definition of φ_k as the following

$$\varphi_k = \min\{\varphi \in \Delta_{\mathbf{p}}^+ : \varphi \perp \varphi_j \text{ for } 1 \leq j \leq k-1\}.$$

(ii). if $\psi'_0 < \varphi_k$, since we have $c_{\psi'_0} \neq 0$, we have

$$k_{\psi'_0} = \min_{1 \leq i \leq r} \{i : \psi'_0 \text{ is not strongly orthogonal to } \varphi_i\}.$$

Then by the definition of k in (4.12), we have $k_{\psi'_0} \geq k$. Therefore we found $\psi'_0 < \varphi_k$ such that $\psi'_0 < \varphi_i$ for any $1 \leq i \leq k-1 < k_{\psi'_0}$, and this contradicts with the definition of φ_k .

Therefore in both cases, we found contradictions. Thus we conclude that \mathfrak{a}_0 is a maximal abelian subspace of \mathfrak{p}_0 . \square

For further use, we also state a proposition about the maximal abelian subspaces of \mathfrak{p}_0 according to Ch V in [8],

Proposition 4.15. *Let \mathfrak{a}'_0 be an arbitrary maximal abelian subspaces of \mathfrak{p}_0 , then there exists an element $k \in K$ such that $k \cdot \mathfrak{a}_0 = \mathfrak{a}'_0$. Moreover, we have*

$$\mathfrak{p}_0 = \bigcup_{k \in K} \text{Ad}(k) \cdot \mathfrak{a}_0,$$

where Ad denotes the adjoint action of K on \mathfrak{a}_0 .

4.2 Boundedness of the period map

Now let us fix the base point $p \in \mathcal{T}$ with $\Phi(p) \in D$. Then according to the above remark, N_+ can be viewed as a subset in \check{D} by identifying it with its orbit in \check{D} with base point $\Phi(p)$. Let us also fix an adapted basis $(\eta_0, \dots, \eta_{m-1})$ for the Hodge decomposition of the base point $\Phi(p) \in D$. Then we can identify elements in N_+ with nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. We define

$$\check{\mathcal{T}} = \Phi^{-1}(N_+).$$

At the base point $\Phi(p) = o \in N_+ \cap D$, the tangent space $T_o^{1,0}N_+ = T_o^{1,0}D \simeq \mathfrak{n}_+ \simeq N_+$, then the Hodge metric on $T_o^{1,0}D$ induces an Euclidean metric on N_+ . In the proof of the following lemma, we require all the root vectors to be unit vectors with respect to this Euclidean metric.

Because the period map is a horizontal map, and the geometry of horizontal slices of the period domain D is similar to Hermitian symmetric space as discussed in detail in [6],

the proof the following theorem is basically an analogue of the proof of the Harish-Chandra embedding theorem for Hermitian symmetric spaces, see for example [14].

Theorem 6. *The restriction of the period map $\Phi : \check{\mathcal{T}} \rightarrow N_+$ is bounded in N_+ with respect to the Euclidean metric on N_+ .*

Proof. We need to show that there exists $0 \leq C < \infty$ such that for any $q \in \check{\mathcal{T}}$, $d_E(\Phi(p), \Phi(q)) \leq C$, where d_E is the Euclidean distance on N_+ .

For any $q \in \check{\mathcal{T}}$, there exists a vector $X^+ \in \mathfrak{g}^{-1,1} \subseteq \mathfrak{n}_+$ and a real number T_0 such that $\beta(t) = \exp(tX^+)$ defines a geodesic $\beta : [0, T_0] \rightarrow N_+ \subseteq G_{\mathbb{C}}$ from $\beta(0) = I$ to $\beta(T_0)$ with $\pi_1(\beta(T_0)) = \Phi(q)$, where $\pi_1 : N_+ \rightarrow N_+B/B \subseteq \check{D}$ is the projection map with the fixed base point $\Phi(p) = o \in D$. In this proof, we will not distinguish $N_+ \subseteq G_{\mathbb{C}}$ from its orbit $N_+B/B \subseteq \check{D}$ with fixed base point $\Phi(p) = o$.

Consider $X^- = \tau_0(X^+) \in \mathfrak{g}^{1,-1}$, then $X = X^+ + X^- \in (\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}) \cap \mathfrak{g}_0$. For any $q \in \check{\mathcal{T}}$, there exists T_1 such that $\gamma = \exp(tX) : [0, T_1] \rightarrow G_{\mathbb{R}}$ defines a geodesic from $\gamma(0) = I$ to $\gamma(T_1) \in G_{\mathbb{R}}$ such that $\pi_2(\gamma(T_1)) = \Phi(q) \in D$, where $\pi_2 : G_{\mathbb{R}} \rightarrow G_{\mathbb{R}}/V \simeq D$ denotes the projection map with the fixed base point $\Phi(p) = o$.

Let $\Lambda = \{\varphi_1, \dots, \varphi_r\} \subseteq \Delta_{\mathfrak{p}}^+$ be a set of strongly orthogonal roots given in Proposition 4.15. We denote $x_{\varphi_i} = e_{\varphi_i} + e_{-\varphi_i}$ and $y_{\varphi_i} = \sqrt{-1}(e_{\varphi_i} - e_{-\varphi_i})$ for any $\varphi_i \in \Lambda$. Then

$$\mathfrak{a}_0 = \mathbb{R}x_{\varphi_1} \oplus \dots \oplus \mathbb{R}x_{\varphi_r}, \quad \text{and} \quad \mathfrak{a}_c = \mathbb{R}y_{\varphi_1} \oplus \dots \oplus \mathbb{R}y_{\varphi_r},$$

are maximal abelian spaces in \mathfrak{p}_0 and $\sqrt{-1}\mathfrak{p}_0$ respectively.

Since $X \in \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1} \subseteq \mathfrak{p}_0$, by Proposition 4.15, there exists $k \in K$ such that $X \in Ad(k) \cdot \mathfrak{a}_0$. As the adjoint action of K on \mathfrak{p}_0 is unitary action and we are considering the length in this proof, we may simply assume that $X \in \mathfrak{a}_0$ up to a unitary transformation. With this assumption, there exists $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq r$ such that

$$X = \lambda_1 x_{\varphi_1} + \lambda_2 x_{\varphi_2} + \dots + \lambda_r x_{\varphi_r}$$

Since \mathfrak{a}_0 is commutative, we have

$$\exp(tX) = \prod_{i=1}^r \exp(t\lambda_i x_{\varphi_i}).$$

Now for each $\varphi_i \in \Lambda$, we have $\text{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{C})$ with

$$h_{\varphi_i} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{-\varphi_i} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

and $\text{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{R})$ with

$$\sqrt{-1}h_{\varphi_i} \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad x_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y_{-\varphi_i} \mapsto \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Since $\Lambda = \{\varphi_1, \dots, \varphi_r\}$ is a set of strongly orthogonal roots, we have that

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}}(\Lambda) &= \text{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}_{i=1}^r \simeq (\mathfrak{sl}_2(\mathbb{C}))^r, \\ \text{and } \mathfrak{g}_{\mathbb{R}}(\Lambda) &= \text{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{-\varphi_i}, \sqrt{-1}h_{\varphi_i}\}_{i=1}^r \simeq (\mathfrak{sl}_2(\mathbb{R}))^r. \end{aligned}$$

In fact, we know that for any $\varphi, \psi \in \Lambda$ with $\varphi \neq \psi$, $[e_{\pm\varphi}, e_{\pm\psi}] = 0$ since φ is strongly orthogonal to ψ ; $[h_{\varphi}, h_{\psi}] = 0$, since \mathfrak{h} is abelian; $[h_{\varphi}, e_{\pm\psi}] = \sqrt{-1}[[e_{\varphi}, e_{-\varphi}], e_{\pm\psi}] = \sqrt{-1}[e_{-\varphi}, [e_{\varphi}, e_{\pm\psi}]] = 0$.

Then we denote $G_{\mathbb{C}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda)) \simeq (SL_2(\mathbb{C}))^r$ and $G_{\mathbb{R}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda)) = (SL_2(\mathbb{R}))^r$, which are subgroups of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$ respectively. With the fixed reference point $o = \Phi(p)$, we denote $D(\Lambda) = G_{\mathbb{R}}(\Lambda)(o)$ and $S(\Lambda) = G_{\mathbb{C}}(\Lambda)(o)$ to be the corresponding orbits of these two subgroups, respectively. Then we have the following isomorphisms,

$$D(\Lambda) = G_{\mathbb{R}}(\Lambda) \cdot B/B \simeq G_{\mathbb{R}}(\Lambda)/G_{\mathbb{R}}(\Lambda) \cap V, \quad (4.16)$$

$$S(\Lambda) \cap (N_+B/B) = (G_{\mathbb{C}}(\Lambda) \cap N_+) \cdot B/B \simeq (G_{\mathbb{C}}(\Lambda) \cap N_+)/((G_{\mathbb{C}}(\Lambda) \cap N_+ \cap B). \quad (4.17)$$

With the above notations, we will show that (i). $D(\Lambda) \subseteq S(\Lambda) \cap (N_+B/B) \subseteq \check{D}$; (ii). $D(\Lambda)$ is bounded inside $S(\Lambda) \cap (N_+B/B)$.

Notice that since $X \in \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$. By Proposition 4.8, we know that for each pair of roots $\{e_{\varphi_i}, e_{-\varphi_i}\}$, either $e_{\varphi_i} \in \mathfrak{g}^{-1,1} \subseteq \mathfrak{n}_+$ and $e_{-\varphi_i} \in \mathfrak{g}^{1,-1}$, or $e_{\varphi_i} \in \mathfrak{g}^{1,-1}$ and $e_{-\varphi_i} \in \mathfrak{g}^{-1,1} \subseteq \mathfrak{n}_+$.

For notation simplicity, for each pair of root vectors $\{e_{\varphi_i}, e_{-\varphi_i}\}$, we may assume the one in $\mathfrak{g}^{-1,1} \subseteq \mathfrak{n}_+$ to be e_{φ_i} and denote the one in $\mathfrak{g}^{1,-1}$ by $e_{-\varphi_i}$. In this way, one can check that $\{\varphi_1, \dots, \varphi_r\}$ may not be a set in $\Delta_{\mathfrak{p}}^+$, but it is a set of strongly orthogonal roots in $\Delta_{\mathfrak{p}}$. Therefore, we have the following description of the above groups,

$$\begin{aligned} G_{\mathbb{R}}(\Lambda) &= \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda)) = \exp(\text{Span}_{\mathbb{R}}\{x_{\varphi_1}, y_{\varphi_1}, \sqrt{-1}h_{\varphi_1}, \dots, x_{\varphi_r}, y_{-\varphi_r}, \sqrt{-1}h_{\varphi_r}\}) \\ G_{\mathbb{R}}(\Lambda) \cap V &= \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda) \cap \mathfrak{v}_0) = \exp(\text{Span}_{\mathbb{R}}\{\sqrt{-1}h_{\varphi_1}, \dots, \sqrt{-1}h_{\varphi_r}\}) \\ G_{\mathbb{C}}(\Lambda) \cap N_+ &= \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda) \cap \mathfrak{n}_+) = \exp(\text{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \dots, e_{\varphi_r}\}); \\ G_{\mathbb{C}}(\Lambda) \cap B &= \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda) \cap \mathfrak{b}) = \exp(\text{Span}_{\mathbb{C}}\{h_{\varphi_1}, e_{-\varphi_1}, \dots, h_{\varphi_r}, e_{-\varphi_r}\}). \end{aligned}$$

Thus by the isomorphisms in (4.16) and (4.17), we have

$$\begin{aligned} D(\Lambda) &\simeq \prod_{i=1}^r \exp(\text{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}) / \exp(\text{Span}_{\mathbb{R}}\{\sqrt{-1}h_{\varphi_i}\}), \\ S(\Lambda) \cap (N_+B/B) &\simeq \prod_{i=1}^r \exp(\text{Span}_{\mathbb{C}}\{e_{\varphi_i}\}). \end{aligned}$$

Let us denote $G_{\mathbb{C}}(\varphi_i) = \exp(\text{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}) \simeq SL_2(\mathbb{C})$, $S(\varphi_i) = G_{\mathbb{C}}(\varphi_i)(o)$, and $G_{\mathbb{R}}(\varphi_i) = \exp(\text{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}) \simeq SL_2(\mathbb{R})$, $D(\varphi_i) = G_{\mathbb{R}}(\varphi_i)(o)$. On one hand, each point in $S(\varphi_i) \cap (N_+B/B)$ can be represented by

$$\exp(ze_{\varphi_i}) = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \quad \text{for some } z \in \mathbb{C}.$$

Thus $S(\varphi_i) \cap (N_+B/B) \simeq \mathbb{C}$. On the other hand, denote $z = a + bi$ for some $a, b \in \mathbb{R}$, then

$$\begin{aligned} \exp(ax_{\varphi_i} + by_{\varphi_i}) &= \begin{bmatrix} \cosh |z| & \frac{\bar{z}}{|z|} \sinh |z| \\ \frac{z}{|z|} \sinh |z| & \cosh |z| \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \frac{z}{|z|} \tanh |z| & 1 \end{bmatrix} \begin{bmatrix} \cosh |z| & 0 \\ 0 & (\cosh |z|)^{-1} \end{bmatrix} \begin{bmatrix} 1 & \frac{\bar{z}}{|z|} \tanh |z| \\ 0 & 1 \end{bmatrix} \\ &= \exp \left[\left(\frac{z}{|z|} \tanh |z| \right) e_{\varphi_i} \right] \exp \left[(-\log \cosh |z|) h_{\varphi_i} \right] \exp \left[\left(\frac{\bar{z}}{|z|} \tanh |z| \right) e_{-\varphi_i} \right]. \end{aligned}$$

So elements in $D(\varphi_i)$ can be represented by $\exp[(z/|z|)(\tanh |z|)e_{\varphi_i}]$, i.e. the lower triangular matrix

$$\begin{bmatrix} 1 & 0 \\ \frac{z}{|z|} \tanh |z| & 1 \end{bmatrix},$$

in which $\frac{z}{|z|} \tanh |z|$ is a point in the unit disc \mathfrak{D} of the complex plane. Therefore the $D(\varphi_i)$ is a unit disc \mathfrak{D} in the complex plane $S(\varphi_i) \cap (N_+B/B)$. Therefore

$$D(\Lambda) \simeq \mathfrak{D}^r \quad \text{and} \quad S(\Lambda) \cap N_+ \simeq \mathbb{C}^r.$$

So we have obtained both (i) and (ii). As a consequence, we get that for any $q \in \check{\mathcal{T}}$, $\Phi(q) \in D(\Lambda)$. This implies

$$d_E(\Phi(p), \Phi(q)) \leq \sqrt{r}$$

where d_E is the Euclidean distance on $S(\Lambda) \cap (N_+B/B)$.

To complete the proof, we only need to show that $S(\Lambda) \cap (N_+B/B)$ is totally geodesic in N_+B/B . In fact, the tangent space of N_+ at the base point is \mathfrak{n}_+ and the tangent space of $S(\Lambda) \cap N_+B/B$ at the base point is $\text{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \dots, e_{\varphi_r}\}$. Since $\text{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \dots, e_{\varphi_r}\}$ is a sub-Lie algebra of \mathfrak{n}_+ , the corresponding orbit $S(\Lambda) \cap N_+B/B$ is totally geodesic in N_+B/B . Here the basis $\{e_{\varphi_1}, e_{\varphi_2}, \dots, e_{\varphi_r}\}$ is an orthonormal basis with respect to the pull-back Euclidean metric. \square

Although not needed in the proof of the above theorem, we can also show that the above inclusion of $D(\varphi_i)$ in D is totally geodesic in D with respect to the Hodge metric. In fact, the tangent space of $D(\varphi_i)$ at the base point is $\text{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}\}$ which satisfies

$$\begin{aligned} [x_{\varphi_i}, [x_{\varphi_i}, y_{\varphi_i}]] &= 4y_{\varphi_i}, \\ [y_{\varphi_i}, [y_{\varphi_i}, x_{\varphi_i}]] &= 4x_{\varphi_i}. \end{aligned}$$

So the tangent space of $D(\varphi_i)$ forms a Lie triple system, and consequently $D(\varphi_i)$ gives a totally geodesic in D . The fact that the exponential map of a Lie triple system gives a

totally geodesic in D is from (cf. [8] Ch 4 §7), and we note that this result still holds true for locally homogeneous spaces instead of only for symmetric spaces. And the pull-back of the Hodge metric on $D(\varphi_i)$ is $G(\varphi_i)$ invariant metric, therefore must be the Poincare metric on the unit disc. In fact, more generally, we have

Lemma 4.18. *If \tilde{G} is a subgroup of $G_{\mathbb{R}}$, then the orbit $\tilde{D} = \tilde{G}(o)$ is totally geodesic in D , and the induced metric on \tilde{D} is \tilde{G} invariant.*

Proof. Firstly, $\tilde{D} \simeq \tilde{G}/(\tilde{G} \cap V)$ is a quotient space. The induced metric of the Hodge metric from D is $G_{\mathbb{R}}$ -invariant, and therefore \tilde{G} -invariant. Now let $\gamma : [0, 1] \rightarrow \tilde{D}$ be any geodesic, then there is a local one parameter subgroup $S : [0, 1] \rightarrow \tilde{G}$ such that, $\gamma(t) = S(t) \cdot \gamma(0)$. On the other hand, because \tilde{G} is a subgroup of $G_{\mathbb{R}}$, we have that $S(t)$ is also a one parameter subgroup of $G_{\mathbb{R}}$, therefore the curve $\gamma(t) = S(t) \cdot \gamma(0)$ also gives a geodesic in D . Since geodesics on \tilde{D} are also geodesics on D , we have proved \tilde{D} is totally geodesic in D . \square

In order to prove Corollary 4.22, we first show that $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of \mathcal{T} with $\text{codim}_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) \geq 1$.

Lemma 4.19. *Let $p \in \mathcal{T}$ be the base point with $\Phi(p) = \{F_p^n \subseteq F_p^{n-1} \subseteq \dots \subseteq F_p^0\}$. Let $q \in \mathcal{T}$ be any point with $\Phi(q) = \{F_q^n \subseteq F_q^{n-1} \subseteq \dots \subseteq F_q^0\}$, then $\Phi(q) \in N_+$ if and only if F_q^k is isomorphic to F_p^k for all $0 \leq k \leq n$.*

Proof. For any $q \in \mathcal{T}$, we choose an arbitrary adapted basis $\{\zeta_0, \dots, \zeta_{m-1}\}$ for the given Hodge filtration $\{F_q^n \subseteq F_q^{n-1} \subseteq \dots \subseteq F_q^0\}$. Recall that $\{\eta_0, \dots, \eta_{m-1}\}$ is the adapted basis for the Hodge filtration $\{F_p^n \subseteq F_p^{n-1} \subseteq \dots \subseteq F_p^0\}$ for the base point p . Let $[A^{i,j}(q)]_{0 \leq i,j \leq n}$ be the transition matrix between the basis $\{\eta_0, \dots, \eta_{m-1}\}$ and $\{\zeta_0, \dots, \zeta_{m-1}\}$ for the same vector space $H^n(M, \mathbb{C})$, where $A^{i,j}(q)$ are the corresponding blocks. Recall that elements in N_+ and B have matrix representations with the fixed adapted basis at the base point: elements in N_+ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in B can be realized as nonsingular block upper triangular matrices. Therefore $\Phi(q) \in N_+ = N_+B/B \subseteq \check{D}$ if and only if its matrix representation $[A^{i,j}(q)]_{0 \leq i,j \leq n}$

can be decomposed as $L(q) \cdot U(q)$, where $L(q)$ is a nonsingular block lower triangular matrix with identities in the diagonal blocks, and $U(q)$ is a nonsingular block upper triangular matrix. By basic linear algebra, we know that $[A^{i,j}(q)]$ has such decomposition if and only if $\det[A^{i,j}(q)]_{0 \leq i,j \leq k} \neq 0$ for any $0 \leq k \leq n$. In particular, we know that $[A(q)^{i,j}]_{0 \leq i,j \leq k}$ is the transition map between the bases of F_p^k and F_q^k . Therefore, $\det([A(q)^{i,j}]_{0 \leq i,j \leq k}) \neq 0$ if and only if F_q^k is isomorphic to F_p^k . \square

Lemma 4.20. *The subset $\check{\mathcal{T}}$ is an open dense submanifold in \mathcal{T} , and $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of \mathcal{T} with $\text{codim}_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) \geq 1$.*

Proof. From Lemma 4.19, one can see that $\check{D} \setminus N_+ \subseteq \check{D}$ is defined as a analytic subvariety by equations

$$\det[A^{i,j}(q)]_{0 \leq i,j \leq k} = 0 \quad \text{for each } 0 \leq k \leq n.$$

Therefore N_+ is dense in \check{D} , and that $\check{D} \setminus N_+$ is an analytic subvariety, which is close in \check{D} and $\text{codim}_{\mathbb{C}}(\check{D} \setminus N_+) \geq 1$. We consider the period map $\Phi : \mathcal{T} \rightarrow \check{D}$ as a holomorphic map to \check{D} , then $\mathcal{T} \setminus \check{\mathcal{T}} = \Phi^{-1}(\check{D} \setminus N_+)$ is the pre-image of the holomorphic map Φ . So $\mathcal{T} \setminus \check{\mathcal{T}}$ is also an analytic subvariety and a close set in \mathcal{T} . Because \mathcal{T} is smooth and connected, if $\dim(\mathcal{T} \setminus \check{\mathcal{T}}) = \dim \mathcal{T}$, then $\mathcal{T} \setminus \check{\mathcal{T}} = \mathcal{T}$ and $\check{\mathcal{T}} = \emptyset$. But this contradicts to the fact that the reference point $p \in \check{\mathcal{T}}$. So we have $\dim(\mathcal{T} \setminus \check{\mathcal{T}}) < \dim \mathcal{T}$, and consequently $\text{codim}_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) \geq 1$. \square

Remark 4.21. We can also prove this lemma in a more direct manner. Let $p \in \mathcal{T}$ be the base point. For any $q \in \mathcal{T}$, let us still use $[A(q)^{i,j}]_{0 \leq i,j \leq n}$ as the transition matrix between the adapted bases $\{\eta_0, \dots, \eta_{m-1}\}$ and $\{\zeta_0, \dots, \zeta_{m-1}\}$ to the Hodge filtration at p and q respectively. In particular, we know that $[A(q)^{i,j}]_{0 \leq i,j \leq k}$ is the transition map between the bases of F_p^k and F_q^k . Therefore, $\det([A(q)^{i,j}]_{0 \leq i,j \leq k}) \neq 0$ if and only if F_q^k is isomorphic to F_p^k . We recall the inclusion

$$D \subseteq \check{D} \subseteq \check{\mathcal{F}} \subseteq Gr(f^n, H^n(X, \mathbb{C})) \times \dots \times Gr(f^1, H^1(X, \mathbb{C}))$$

where $\check{\mathcal{F}} = \{F^n \subseteq \dots \subseteq F^1 \subseteq H^n(X, \mathbb{C}) \mid \dim_{\mathbb{C}} F^k = f^k\}$ and $Gr(f^k, H^n(X, \mathbb{C}))$ is the complex vector subspaces of dimension f^k of $H^n(X, \mathbb{C})$. For each $1 \leq k \leq n$ the points in $Gr(f^k, H^n(X, \mathbb{C}))$ whose corresponding vector spaces are not isomorphic to the reference vector space F_p^k form a divisor $Y_k \subseteq Gr(f^k, H^n(X, \mathbb{C}))$. Now we consider the divisor $Y \subseteq \prod_{k=1}^n Gr(f^k, H^n(X, \mathbb{C}))$ given by

$$Y = \sum_{k=1}^n \left(\prod_{j < k} Gr(f^j, H^n(X, \mathbb{C})) \times Y_k \times \prod_{j > k} Gr(f^j, H^n(X, \mathbb{C})) \right).$$

Then $Y \cap D$ is also a divisor in D . Since by Lemma 4.19, we know that $\Phi(q) \in N_+$ if and only if F_q^k is isomorphic to F_p^k for all $0 \leq k \leq n$, so we have $\mathcal{T} \setminus \check{\mathcal{T}} = \Phi^{-1}(Y \cap D)$. Finally, by local Torelli theorem for Calabi–Yau manifolds, we know that $\Phi : \mathcal{T} \rightarrow D$ is a local embedding. Therefore, the complex codimension of $(\mathcal{T} \setminus \check{\mathcal{T}})$ in \mathcal{T} is at least the complex codimension of the divisor $Y \cap D$ in D .

Corollary 4.22. *The image of $\Phi : \mathcal{T} \rightarrow D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+ .*

Proof. According to Lemma 4.20, $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of \mathcal{T} and the complex codimension of $\mathcal{T} \setminus \check{\mathcal{T}}$ is at least one; by Theorem 6, the holomorphic map $\Phi : \check{\mathcal{T}} \rightarrow N_+ \cap D$ is bounded in N_+ with respect to the Euclidean metric. Thus by the Riemann extension theorem, there exists a holomorphic map $\Phi' : \mathcal{T} \rightarrow N_+ \cap D$ such that $\Phi'|_{\check{\mathcal{T}}} = \Phi|_{\check{\mathcal{T}}}$. Since as holomorphic maps, Φ' and Φ agree on the open subset $\check{\mathcal{T}}$, they must be the same on the entire \mathcal{T} . Therefore, the image of Φ is in $N_+ \cap D$, and the image is bounded with respect to the Euclidean metric on N_+ . As a consequence, we also get $\mathcal{T} = \check{\mathcal{T}} = \Phi^{-1}(N_+)$. \square

4.3 Holomorphic affine structure on the Teichmüller space

We first review the definition of complex affine manifolds. One may refer to page 215 of [18] for more details.

Definition 4.23. *Let M be a complex manifold of complex dimension n . If there is a coordinate cover $\{(U_i, \varphi_i); i \in I\}$ of M such that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine*

transformation on \mathbb{C}^n whenever $U_i \cap U_k$ is not empty, then $\{(U_i, \varphi_i); i \in I\}$ is called a complex affine coordinate cover on M and it defines a holomorphic affine structure on M .

Let us still fix an adapted basis $(\eta_0, \dots, \eta_{m-1})$ for the Hodge decomposition of the base point $\Phi(p) \in D$. Recall that we can identify elements in N_+ with nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix, and element in B with nonsingular block upper triangular matrices. Therefore $N_+ \cap B = Id$. By Corollary 4.22, we know that $\mathcal{T} = \Phi^{-1}(N_+)$. Thus we get that each $\Phi(q)$ can be uniquely determined by a matrix, which we will still denote by $\Phi(q) = [\Phi_{ij}(q)]_{0 \leq i, j \leq m-1}$ of the form of nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. Thus we can define a holomorphic map

$$\tau : \mathcal{T} \rightarrow \mathbb{C}^N \cong H_p^{n-1,1}, \quad q \mapsto (1, 0)\text{-block of the matrix } \Phi(q) \in N_+,$$

$$\text{that is } \tau(q) = (\tau_1(q), \tau_2(q), \dots, \tau_N(q)) = (\Phi_{10}(q), \Phi_{20}(q), \dots, \Phi_{N0}(q))$$

Remark 4.24. If we define the following projection map with respect to the base point and the its pre-fixed adapted basis to the Hodge decomposition,

$$P : N_+ \cap D \rightarrow H_p^{n-1,1} \cong \mathbb{C}^N, \quad F \mapsto (\eta_1, \dots, \eta_N)F^{(1,0)} = F_{10}\eta_1 + \dots + F_{N0}\eta_N, \quad (4.25)$$

where $F^{(1,0)}$ is the $(1,0)$ -block of the unipotent matrix F , according to our convention in (4.4), then $\tau = P \circ \Phi : \mathcal{T} \rightarrow \mathbb{C}^N$.

Proposition 4.26. *The holomorphic map $\tau = (\tau_1, \dots, \tau_N) : \mathcal{T} \rightarrow \mathbb{C}^N$ defines a coordinate chart around each point $q \in \mathcal{T}$.*

Proof. Recall that we have the generator $\{\eta_0\} \subseteq H^0(M_p, \Omega_{M_p}^n)$, the generators $\{\eta_1, \dots, \eta_N\} \subseteq H^1(M_p, \Omega^{n-1}(M_p))$, and the generators $\{\eta_{N+1}, \dots, \eta_{m-1}\} \in \bigoplus_{k \geq 2} H^k(M_p, \Omega^{n-k}(M_p))$.

On one hand, the 0-th column of the matrix $\Phi(q) \in N_+$ for each $q \in \mathcal{T}$ gives us the following data:

$$\begin{aligned} \Omega : \mathcal{T} \rightarrow F^n; \quad \Omega(q) &= (\eta_0, \dots, \eta_{m-1})(\Phi_{00}(q), \Phi_{10}(q), \dots, \Phi_{N0}(q), \dots)^T \\ &= \eta_0 + \tau_1(q)\eta_1 + \tau_2(q)\eta_2 + \dots + \tau_N(q)\eta_N + g_0(q) \in F_q^n \simeq H^0(M_q, \Omega^n(M_q)), \end{aligned}$$

where $g_0(q) \in \bigoplus_{k \geq 2} H^k(M_p, \Omega^{n-k}(M_p))$.

The 1-st to N -th columns of $\Phi(q) \in N_+$ give us the following data:

$$\theta_1(q) = \eta_1 + g_1(q), \quad \dots, \quad \theta_N(q) = \eta_N + g_N(q) \in F_q^{n-1},$$

where $g_k(q) \in \bigoplus_{k \geq 2} H^k(M_p, \Omega^{n-k}(M_p))$, such that $\{\Omega(q), \theta_1(q), \dots, \theta_N(q)\}$ forms a basis for F_q^{n-1} for each $q \in \mathcal{T}$.

On the other hand, by local Torelli theorem, we know that for any holomorphic coordinate $\{\sigma_1, \dots, \sigma_N\}$ around q , $\{\Omega(q), \frac{\partial \Omega(q)}{\partial \sigma_1}, \dots, \frac{\partial \Omega(q)}{\partial \sigma_N}\}$ forms a basis of F_q^{n-1} .

As both $\{\Omega(q), \theta_1(q), \dots, \theta_N(q)\}$ and $\{\Omega(q), \frac{\partial \Omega(q)}{\partial \sigma_1}, \dots, \frac{\partial \Omega(q)}{\partial \sigma_N}\}$ are bases for F_q^{n-1} , there exists $\{X_1, \dots, X_N\}$ such that $X_k = \sum_{i=1}^N a_{ik} \frac{\partial}{\partial \sigma_i}$ for each $1 \leq k \leq N$ such that

$$\theta_k = X_k(\Omega(q)) + \lambda_k \Omega(q) \quad \text{for } 1 \leq k \leq N. \quad (4.27)$$

Note that we have $X_k(\Omega(q)) = X_k(\tau_1(q))\eta_1 + \dots + X_k(\tau_N(q))\eta_N + X_k(g_0(q))$ and $\theta_k = \eta_k + g_k(q)$, where $X_k(g_0(q)), g_k(q) \in \bigoplus_{k \geq 2} H^k(M_q, \Omega^{n-k}(M_q))$. By comparing the types of classes in (4.27), we get

$$\lambda_k = 0, \quad \text{and} \quad X_k(\Omega(q)) = \theta_k(q) = \eta_k + g_k(q) \quad \text{for each } 1 \leq k \leq N. \quad (4.28)$$

Since $\{\theta_1(q), \dots, \theta_N(q)\}$ are linearly independent set for $F_q^{n-1}(M_q)$, we know that $\{X_1, \dots, X_N\}$ are also linearly independent in $T_q^{1,0}(\mathcal{T})$. Therefore $\{X_1, \dots, X_N\}$ forms a basis for $T_q^{1,0}(\mathcal{T})$. Without loss of generality, we may assume $X_k = \frac{\partial}{\partial \sigma_k}$ for each $1 \leq k \leq N$. Thus by (4.28), we have

$$\frac{\partial \Omega(q)}{\partial \sigma_k} = \theta_k = \eta_k + g_k(q) \quad \text{for any } 1 \leq k \leq N.$$

Since we also have

$$\frac{\partial \Omega(q)}{\partial \sigma_k} = \frac{\partial}{\partial \sigma_k}(\eta_0 + \tau_1(q)\eta_1 + \tau_2(q)\eta_2 + \dots + \tau_N(q)\eta_N + g_0(q)),$$

we get $\left[\frac{\partial \tau_i(q)}{\partial \sigma_j} \right]_{1 \leq i, j \leq N} = I_N$. This shows that $\tau_* : T_q^{1,0}(\mathcal{T}) \rightarrow T_{\tau(q)}(\mathbb{C}^N)$ is an isomorphism for each $q \in \mathcal{T}$, as $\{\frac{\partial}{\partial \sigma_1}, \dots, \frac{\partial}{\partial \sigma_N}\}$ is a basis for $T_q^{1,0}(\mathcal{T})$. \square

Thus the holomorphic map $\tau : \mathcal{T} \rightarrow \mathbb{C}^N$ defines local coordinate map around each point $q \in \mathcal{T}$. In particular, the map τ itself gives a global holomorphic coordinate for \mathcal{T} . Thus the transition maps are all identity maps. Therefore,

Theorem 7. *The global holomorphic coordinate map $\tau : \mathcal{T} \rightarrow \mathbb{C}^N$ defines a holomorphic affine structure on \mathcal{T} .*

Remark 4.29. This affine structure on \mathcal{T} depends on the choice of the base point p . Affine structures on \mathcal{T} defined in this ways by fixing different base point may not be compatible with each other.

CHAPTER 5

Hodge metric completion of the Teichmüller space and global Torelli theorem

In Section 5.1, given $m \geq 3$, we introduce the Hodge metric completion \mathcal{T}_m^H of the Teichmüller space with level m structure, which is the universal cover of \mathcal{Z}_m^H , where \mathcal{Z}_m^H is the completion space of the smooth moduli space \mathcal{Z}_m with respect to the Hodge metric. We denote the lifting maps $i_m : \mathcal{T} \rightarrow \mathcal{T}_m^H$, $\Phi_m^H : \mathcal{T}_m^H \rightarrow D$ and $\mathcal{T}_m := i_m(\mathcal{T})$. We prove that Φ_m^H is a bounded holomorphic map from \mathcal{T}_m^H to $N_+ \cap D$. In Section 5.2, we first define the map τ_m^H from \mathcal{T}_m^H to \mathbb{C}^N and its restriction τ_m on the submanifold \mathcal{T}_m . We then show that τ_m is also a local embedding and conclude that τ_m defined a global holomorphic affine structure on \mathcal{T}_m . Then the affineness of τ_m shows that the extension map τ_m^H is also defines a holomorphic affine structure on \mathcal{T}_m^H . In Section 5.3, we prove that τ_m^H is an injection by using the Hodge metric completeness and the global holomorphic affine structure on \mathcal{T}_m^H . As a corollary, we show that the holomorphic map Φ_m^H is an injection.

5.1 Definitions and basic properties

Recall in Section 3.2, \mathcal{Z}_m is the smooth moduli space of polarized Calabi–Yau manifolds with level m structure, which is introduced in [27]. The Teichmüller space \mathcal{T} is the universal cover of \mathcal{Z}_m .

By the work of Viehweg in [33], we know that \mathcal{Z}_m is quasi-projective and consequently we can find a smooth projective compactification $\overline{\mathcal{Z}}_m$ such that \mathcal{Z}_m is open in $\overline{\mathcal{Z}}_m$ and the complement $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ is a divisor of normal crossing. Therefore, \mathcal{Z}_m is dense and open in $\overline{\mathcal{Z}}_m$.

with the complex codimension of the complement $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ at least one. Moreover as $\overline{\mathcal{Z}}_m$ is a compact space, it is a complete space.

Recall at the end of Section 3.3, we pointed out that there is an induced Hodge metric on \mathcal{Z}_m . Let us now take \mathcal{Z}_m^H to be the completion of \mathcal{Z}_m with respect to the Hodge metric. Then \mathcal{Z}_m^H is the smallest complete space with respect to the Hodge metric that contains \mathcal{Z}_m . Although the compact space $\overline{\mathcal{Z}}_m$ may not be unique, the Hodge metric completion space \mathcal{Z}_m^H is unique up to isometry. In particular, $\mathcal{Z}_m^H \subseteq \overline{\mathcal{Z}}_m$ and thus the complex codimension of the complement $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is at least one. Moreover, we can prove the following lemma.

Lemma 5.1. *The Hodge metric completion \mathcal{Z}_m^H is a dense and open smooth submanifold in $\overline{\mathcal{Z}}_m$ and the complex codimension of $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is at least one.*

Proof. We will give two different proofs. The first one is more conceptual which uses extension of the period map, while the second one is more elementary which only uses basic definition of metric completion.

Since $\overline{\mathcal{Z}}_m$ is a smooth manifold and \mathcal{Z}_m is dense in $\overline{\mathcal{Z}}_m$, we only need to show that \mathcal{Z}_m^H is open in $\overline{\mathcal{Z}}_m$.

For the first proof, we can use a compactification $\overline{D/\Gamma}$ of D/Γ and a continuous extension of the period map $\Phi_{\mathcal{Z}_m} \rightarrow D/\Gamma$ to

$$\overline{\Phi}_{\mathcal{Z}_m} : \overline{\mathcal{Z}}_m \rightarrow \overline{D/\Gamma}.$$

Then since D/Γ is complete with respect to the Hodge metric, the Hodge metric completion space $\mathcal{Z}_m^H = \overline{\Phi}_{\mathcal{Z}_m}^{-1}(D/\Gamma)$. Notice that D/Γ is open and dense in the compactification in $\overline{D/\Gamma}$, \mathcal{Z}_m^H is therefore an open subset of $\overline{\mathcal{Z}}_m$.

To define the compactification space $\overline{D/\Gamma}$, there are different approaches from literature. One natural compactification is given by Kao-Usuai following [23] as given in [32] and discussed in [4, Page 2], where one attaches to D a set $B(\Gamma)$ of equivalence classes of limiting mixed Hodge structures, then define $\overline{D/\Gamma} = (D \cup B(\Gamma))/\Gamma$ and extend the period map $\Phi_{\mathcal{Z}_m}$ continuously to $\overline{\Phi}_{\mathcal{Z}_m}$. An alternative and more recent method is to use the reduced limit

period mapping as defined in [4, Page 29, 30] to enlarge the period domain D . Different approaches can give us different compactification spaces \overline{D}/Γ . Both compactifications are applicable to our above proof of the openness of $\mathcal{Z}_m^H = \overline{\Phi}_{\mathcal{Z}_m}^{-1}(D/\Gamma)$.

For the second proof, we will directly show the open smoothness of \mathcal{Z}_m^H . Since $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ is a normal crossing divisor in $\overline{\mathcal{Z}}_m$, any point $q \in \overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ has a neighborhood $U_q \subset \overline{\mathcal{Z}}_m$ such that $U_q \cap \mathcal{Z}_m$ is biholomorphic to $(\Delta^*)^k \times \Delta^{N-k}$. Given a fixed reference point $p \in \mathcal{Z}_m$, and any point $p' \in U_q \cap \mathcal{Z}_m$, there is a piecewise smooth curve $\gamma_{p,p'} : [0, 1] \rightarrow \mathcal{Z}_m$ connecting p and p' , since \mathcal{Z}_m is a path connected manifold. There is also a piecewise smooth curve $\gamma_{p',q}$ in $U_q \cap \mathcal{Z}_m$ connecting p' and q . Therefore, there is a piecewise smooth curve $\gamma_{p,q} : [0, 1] \rightarrow \overline{\mathcal{Z}}_m$ connecting p and q with $\gamma_{p,q}((0, 1)) \subset \mathcal{Z}_m$. Thus for any point $q \in \overline{\mathcal{Z}}_m$, we can define the Hodge distance between q and p by

$$d_H(p, q) = \inf_{\gamma_{p,q}} L_H(\gamma_{p,q}),$$

where $L_H(\gamma_{p,q})$ denotes the length of the curve $\gamma_{p,q}$ with respect to the Hodge metric on \mathcal{Z}_m , and the infimum is taken over all piecewise smooth curves $\gamma_{p,q} : [0, 1] \rightarrow \overline{\mathcal{Z}}_m$ connecting p and q with $\gamma_{p,q}((0, 1)) \subset \mathcal{Z}_m$.

Then \mathcal{Z}_m^H is the set of all the points q in $\overline{\mathcal{Z}}_m$ with $d_H(p, q) < \infty$, and the complement set $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m^H$ contains all the points q in $\overline{\mathcal{Z}}_m$ with $d_H(p, q) = \infty$. Now let $\{q_k\}_{k=1}^\infty \subset \overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m^H$ be a convergent sequence and assume that the limit of this sequence is q_∞ . Then the Hodge distance between the reference point p and the limiting point q_∞ is clearly $d(p, q_\infty) = \infty$ as well, since $d(p, q_k) = \infty$ for all k . Thus $q_\infty \in \overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m^H$. Therefore, the set of the points of Hodge infinite distance from p is a close set in $\overline{\mathcal{Z}}_m$. We conclude that \mathcal{Z}_m^H is an open submanifold of $\overline{\mathcal{Z}}_m$. \square

Remark 5.2. We recall some basic properties about metric completion space we are using in this paper. We know that the metric completion space of a connected space is still connected. Therefore, \mathcal{Z}_m^H is connected.

Suppose (X, d) is a metric space with the metric d . Then the metric completion space of (X, d) is unique in the following sense: if \overline{X}_1 and \overline{X}_2 are complete metric spaces that both

contain X as a dense set, then there exists an isometry $f : \overline{X}_1 \rightarrow \overline{X}_2$ such that $f|_X$ is the identity map on X . Moreover, as mentioned above that the metric completion space \overline{X} of X is the smallest complete metric space containing X in the sense that any other complete space that contains X as a subspace must also contains \overline{X} as a subspace.

Moreover, suppose \overline{X} is the metric completion space of the metric space (X, d) . If there is a continuous map $f : X \rightarrow Y$ which is a local isometry with Y a complete space, then there exists a continuous extension $\overline{f} : \overline{X} \rightarrow Y$ such that $\overline{f}|_X = f$.

In the rest of the paper, unless otherwise pointed out, when we mention a complete space, the completeness is always with respect to the Hodge metric.

Let \mathcal{T}_m^H be the universal cover of \mathcal{Z}_m^H with the universal covering map $\pi_m^H : \mathcal{T}_m^H \rightarrow \mathcal{Z}_m^H$. Thus \mathcal{T}_m^H is a connected and simply connected complete smooth complex manifold with respect to the Hodge metric. We will call \mathcal{T}_m^H the *Hodge metric completion space with level m structure* of \mathcal{T} . Since \mathcal{Z}_m^H is the Hodge metric completion of \mathcal{Z}_m , there exists the continuous extension map $\Phi_{\mathcal{Z}_m}^H : \mathcal{Z}_m^H \rightarrow D/\Gamma$. Moreover, recall that the Teichmüller space \mathcal{T} is the universal cover of the moduli space \mathcal{Z}_m with the universal covering denoted by $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$. Thus we have the following commutative diagram

$$\begin{array}{ccccc}
 \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}_m^H & \xrightarrow{\Phi_m^H} & D \\
 \downarrow \pi_m & & \downarrow \pi_m^H & & \downarrow \pi_D \\
 \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m}^H} & D/\Gamma,
 \end{array} \tag{5.3}$$

where i is the inclusion map, i_m is a lifting map of $i \circ \pi_m$, π_D is the covering map and Φ_m^H is a lifting map of $\Phi_{\mathcal{Z}_m}^H \circ \pi_m^H$. In particular, Φ_m^H is a continuous map from \mathcal{T}_m^H to D . We notice that the lifting maps $i_{\mathcal{T}}$ and Φ_m^H are not unique, but there exists a suitable choice of i_m and Φ_m^H such that $\Phi = \Phi_m^H \circ i_m$. We will fix the choice of i_m and Φ_m^H such that $\Phi = \Phi_m^H \circ i_m$ in the rest of the paper. Let us denote $\mathcal{T}_m := i_m(\mathcal{T})$ and the restriction map $\Phi_m = \Phi_m^H|_{\mathcal{T}_m}$. Then we also have $\Phi = \Phi_m \circ i_m$.

Proposition 5.4. *The image \mathcal{T}_m equals to the preimage $(\pi_m^H)^{-1}(\mathcal{Z}_m)$.*

Proof. Because of the commutativity of diagram (5.3), we have that $\pi_m^H(i_m(\mathcal{T})) = i(\pi_m(\mathcal{T})) = \mathcal{Z}_m$. Therefore, $\mathcal{T}_m = i_m(\mathcal{T}) \subseteq (\pi_m^H)^{-1}(\mathcal{Z}_m)$. For the other direction, we need to show that for any point $q \in (\pi_m^H)^{-1}(\mathcal{Z}_m) \subseteq \mathcal{T}_m^H$, then $q \in i_m(\mathcal{T}) = \mathcal{T}_m$.

Let $p = \pi_m^H(q) \in i(\mathcal{Z}_m)$, Let $x_1 \in \pi_m^{-1}(i^{-1}(p)) \subseteq \mathcal{T}$ be an arbitrary point, then $\pi_m^H(i_m(x_1)) = i(\pi_m(x_1)) = p$ and $i_m(x_1) \in (\pi_m^H)^{-1}(p) \subseteq \mathcal{T}_m^H$.

As \mathcal{T}_m^H is a connected smooth complex manifold, \mathcal{T}_m^H is path connected. Therefore, for $i_m(x_1), q \in \mathcal{T}_m^H$, there exists a curve $\gamma : [0, 1] \rightarrow \mathcal{T}_m^H$ with $\gamma(0) = i_m(x_1)$ and $\gamma(1) = q$. Then the composition $\pi_m^H \circ \gamma$ gives a loop on \mathcal{Z}_m^H with $\pi_m^H \circ \gamma(0) = \pi_m^H \circ \gamma(1) = p$. One can show that here is a loop Γ on \mathcal{Z}_m with $\Gamma(0) = \Gamma(1) = i^{-1}(p)$ such that

$$[i \circ \Gamma] = [\pi_m^H \circ \gamma] \in \pi_1(\mathcal{Z}_m^H),$$

where $\pi_1(\mathcal{Z}_m^H)$ denotes the fundamental group of \mathcal{Z}_m^H . Because \mathcal{T} is universal cover of \mathcal{Z}_m , there is a unique lifting map $\tilde{\Gamma} : [0, 1] \rightarrow \mathcal{T}$ with $\tilde{\Gamma}(0) = x_1$ and $\pi_m \circ \tilde{\Gamma} = \Gamma$. Again since $\pi_m^H \circ i_m = i \circ \pi_m$, we have

$$\pi_m^H \circ i_m \circ \tilde{\Gamma} = i \circ \pi_m \circ \tilde{\Gamma} = i \circ \Gamma : [0, 1] \rightarrow \mathcal{Z}_m.$$

Therefore $[\pi_m^H \circ i_m \circ \tilde{\Gamma}] = [i \circ \Gamma] \in \pi_1(\mathcal{Z}_m)$, and the two curves $i_m \circ \tilde{\Gamma}$ and γ have the same starting points $i_m \circ \tilde{\Gamma}(0) = \gamma(0) = i_m(x_1)$. Then the homotopy lifting property of the covering map π_m^H implies that $i_m \circ \tilde{\Gamma}(1) = \gamma(1) = q$. Therefore, $q \in i_m(\mathcal{T})$, as needed. \square

Since \mathcal{Z}_m is an open submanifold of \mathcal{Z}_m^H and π_m^H is a holomorphic covering map, the preimage $\mathcal{T}_m = (\pi_m^H)^{-1}(\mathcal{Z}_m)$ is a connected open submanifold of \mathcal{T}_m^H . Furthermore, because the complex codimension of $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is as least one in \mathcal{Z}_m^H , the complex codimension of $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is also as least one in \mathcal{T}_m^H .

First, it is not hard to see that the restriction map Φ_m is holomorphic. Indeed, we know that $i_m : \mathcal{T} \rightarrow \mathcal{T}_m$ is the lifting of $i \circ \pi_m$ and $\pi_m^H|_{\mathcal{T}_m} : \mathcal{T}_m \rightarrow \mathcal{Z}_m$ is a holomorphic covering map, thus i_m is also holomorphic. Since $\Phi = \Phi_m \circ i_m$ with both Φ , i_m holomorphic and i_m locally invertible, we can conclude that $\Phi_m : \mathcal{T}_m \rightarrow D$ is a holomorphic map. Moreover, we have $\Phi_m(\mathcal{T}_m) = \Phi_m(i_m(\mathcal{T})) = \Phi(\mathcal{T})$ as $\Phi = i_m \circ \Phi_m$. In particular, as $\Phi : \mathcal{T} \rightarrow N_+ \cap D$ is

bounded, we get that $\Phi_m : \mathcal{T} \rightarrow N_+ \cap D$ is also bounded in N_+ with the Euclidean metric. Thus Φ_m^H is also bounded. Therefore applying Riemann extension theorem, we get

Proposition 5.5. *The map Φ_m^H is a bounded holomorphic map from \mathcal{T}_m^H to $N_+ \cap D$.*

Proof. According to the above discussion, we know that the complement $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is the pre-image of $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ of the covering map π_m^H . So $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is a analytic subvariety of \mathcal{T}_m^H , with complex codimension at least one and $\Phi_m : \mathcal{T}_m \rightarrow N_+ \cap D$ is a bounded holomorphic map. Therefore, simply applying the Riemann extension theorem to the holomorphic map $\Phi_m : \mathcal{T}_m \rightarrow N_+ \cap D$, we conclude that there exists a holomorphic map $\Phi'_m : \mathcal{T}_m^H \rightarrow N_+ \cap D$ such that $\Phi'_m|_{\mathcal{T}_m} = \Phi_m$. We know that both Φ_m^H and Φ'_m are continuous maps defined on \mathcal{T}_m^H that agree on the dense subset \mathcal{T}_m . Therefore, they must agree on the whole \mathcal{T}_m^H , that is, $\Phi_m^H = \Phi'_m$ on \mathcal{T}_m^H . Therefore, $\Phi_m^H : \mathcal{T}_m^H \rightarrow N_+ \cap D$ is a bounded holomorphic map. \square

5.2 Holomorphic affine structure on the Hodge metric completion space

In this section, we still fix the base point $\Phi(p) \in D$ with $p \in \mathcal{T}$ and an adapted basis $(\eta_0, \dots, \eta_{m-1})$ for the Hodge decomposition of $\Phi(p)$. We defined the global coordinate map $\tau : \mathcal{T} \rightarrow \mathbb{C}^N$, which is a holomorphic affine local embedding. Let us now define

$$\tau_m := P \circ \Phi_m : \mathcal{T}_m \rightarrow \mathbb{C}^N,$$

where $P : N_+ \cap D \rightarrow \mathbb{C}^N$ is the projection map defined in (4.25). Then as $\Phi = \Phi_m \circ i_m$, we get $\tau = P \circ \Phi = P \circ \Phi_m \circ i_m = \tau_m \circ i_m$.

In the following lemma, we will crucially use the fact that the holomorphic map $\tau : \mathcal{T} \rightarrow \mathbb{C}^N$ is a local embedding.

Lemma 5.6. *The holomorphic map $\tau_m : \mathcal{T} \rightarrow \mathbb{C}^N$ is a local embedding. In particular, $\tau_m : \mathcal{T}_m \rightarrow \mathbb{C}^N$ defines a global holomorphic affine structure on \mathcal{T}_m .*

Proof. Since $i \circ \pi_m = \pi_m^H \circ i_m$ with $i : \mathcal{Z}_m \rightarrow \mathcal{Z}_m^H$ the natural inclusion map and π_m, π_m^H both universal covering maps, i_m is a lifting of the inclusion map. Thus i_m is locally biholomorphic. On the other hand, we showed that τ is a local embedding. We may choose an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of \mathcal{T}_m such that for each $U_\alpha \subseteq \mathcal{T}_m$, i_m is biholomorphic on U_α and thus the inverse $(i_m)^{-1}$ is also an embedding on U_α . Obviously we may also assume that τ is an embedding on $(i_m)^{-1}(U_\alpha)$. In particular, the relation $\tau = \tau_m \circ i_m$ implies that $\tau_m|_{U_\alpha} = \tau \circ (i_m)^{-1}|_{U_\alpha}$ is also an embedding on U_α . In this way, we showed τ_m is a local embedding on \mathcal{T}_m . Therefore, since $\dim_{\mathbb{C}} \mathcal{T}_m = N$, $\tau_m : \mathcal{T}_m \rightarrow \mathbb{C}^N$ defines a local coordinate map around each point in \mathcal{T}_m . In particular, the map τ_m itself gives a global holomorphic coordinate for \mathcal{T}_m . Thus the transition maps are all identity maps. Therefore, $\tau_m : \mathcal{T}_m \rightarrow \mathbb{C}^N$ defines a global holomorphic affine structure on \mathcal{T}_m . \square

Let us define $\tau_m^H := P \circ \Phi_m^H : \mathcal{T}_m^H \rightarrow \mathbb{C}^N$, where $P : N_+ \cap D \rightarrow \mathbb{C}^N$ is still the projection map defined in (4.25). Then we easily see that $\tau_m^H|_{\mathcal{T}_m} = \tau_m$. We also have the following,

Lemma 5.7. *The holomorphic map $\tau_m^H : \mathcal{T}_m^H \rightarrow \mathbb{C}^N \cong H_p^{n-1,1}$ is a local embedding.*

Proof. The proof uses mainly the affineness of $\tau_m : \mathcal{T}_m \rightarrow \mathbb{C}^N \cong H_p^{n-1,1}$. By Proposition 5.4, we know that \mathcal{T}_m is dense and open in \mathcal{T}_m^H . Thus for any point $q \in \mathcal{T}_m^H$, there exists $\{q_k\}_{k=1}^\infty \subseteq \mathcal{T}_m$ such that $\lim_{k \rightarrow \infty} q_k = q$. Because $\tau_m^H(q) \in H_p^{n-1,1}$, we can take a neighborhood $W \subseteq H_p^{n-1,1}$ of $\tau_m^H(q)$ with $W \subseteq \tau_m^H(\mathcal{T}_m^H)$.

Consider the projection map $P : N_+ \rightarrow \mathbb{C}^N$ with $P(F) = F^{(1,0)}$ the $(1,0)$ block of the matrix F , and the decomposition of the holomorphic tangent bundle

$$T^{1,0}N_+ = \bigoplus_{0 \leq l < k \leq n} \text{Hom}(F^k/F^{k+1}, F^l/F^{l+1}).$$

In particular, the subtangent bundle $\text{Hom}(F^n, F^{n-1}/F^n)$ over N_+ is isomorphic to the pull-back bundle $P^*(T^{1,0}\mathbb{C}^N)$ of the holomorphic tangent bundle of \mathbb{C}^N through the projection P . On the other hand, the holomorphic tangent bundle of \mathcal{T}_m is also isomorphic to the holomorphic bundle $\text{Hom}(F^n, F^{n-1}/F^n)$, where F^n and F^{n-1} are pull-back bundles on \mathcal{T}_m via Φ_m from $N_+ \cap D$. Since the holomorphic map $\tau_m = P \circ \Phi_m$ is a composition of P and

Φ_m , the pull-back bundle of $T^{1,0}W$ through τ_m is also isomorphic to the tangent bundle of \mathcal{T}_m .

Now with the fixed adapted basis $\{\eta_1, \dots, \eta_N\}$, one has a standard coordinate (z_1, \dots, z_N) on $\mathbb{C}^N \cong H_p^{n-1,1}$ such that each point in $\mathbb{C}^N \cong H_p^{n-1,1}$ is of the form $z_1\eta_1 + \dots + z_N\eta_N$. Let us choose one special trivialization of

$$T^{1,0}W \cong \text{Hom}(F_p^n, F_p^{n-1}/F_p^n) \times W$$

by the standard global holomorphic frame $(\Lambda_1, \dots, \Lambda_N) = (\partial/\partial z_1, \dots, \partial/\partial z_N)$ on $T^{1,0}W$. Under this trivialization, we can identify $T_o^{1,0}W$ with $\text{Hom}(F_p^n, F_p^{n-1}/F_p^n)$ for any $o \in W$. Then $(\Lambda_1, \dots, \Lambda_N)$ are parallel sections with respect to the trivial affine connection on $T^{1,0}W$. Let $U_q \subseteq (\tau_m^H)^{-1}(W)$ be a neighborhood of q and let $U = U_q \cap \mathcal{T}_m$. Then the pull back sections $(\tau_m^H)^*(\Lambda_1, \dots, \Lambda_N) : U_q \rightarrow T^{1,0}U_q$ are tangent vectors of U_q , we denote them by $(\mu_1^H, \dots, \mu_N^H)$ for convenience.

According to the proof of Lemma 5.6, we know that the restriction map τ_m is a local embedding. Therefore the tangent map $(\tau_m)_* : T_{q'}^{1,0}U \rightarrow T_o^{1,0}W$ is an isomorphism, for any $q' \in U$ and $o = \tau_m(q')$. Moreover, since τ_m is a holomorphic affine map, the holomorphic sections $(\mu_1, \dots, \mu_N) := (\mu_1^H, \dots, \mu_N^H)|_U$ form a holomorphic parallel frame for $T^{1,0}U$. Under the parallel frames (μ_1, \dots, μ_N) and $(\Lambda_1, \dots, \Lambda_N)$, there exists a nonsingular matrix function $A(q') = (a_{ij}(q'))_{1 \leq i \leq N, 1 \leq j \leq N}$, such that the tangent map $(\tau_m)_*$ is given by

$$(\tau_m)_*(\mu_1, \dots, \mu_N)(q') = (\Lambda_1(o), \dots, \Lambda_N(o))A(q'), \quad \text{with } q' \in U \text{ and } o = \tau_m(q') \in D.$$

Moreover, since $(\Lambda_1, \dots, \Lambda_N)$ and (μ_1, \dots, μ_N) are parallel frames for $T^{1,0}W$ and $T^{1,0}U$ respectively and τ_m is a holomorphic affine map, the matrix $A(q') = A$ is actually a constant nonsingular matrix for all $q' \in U$. In particular, for each $q_k \in U$, we have $((\tau_m)_*\mu_1, \dots, (\tau_m)_*\mu_N)(q_k) = (\Lambda_1(o_k), \dots, \Lambda_N(o_k))A$, where $o_k = \tau_m(q_k)$. Because the tangent map $(\tau_m^H)_* : T^{1,0}U_q \rightarrow T^{1,0}W$ is a continuous map, we have that

$$\begin{aligned} (\tau_m^H)_*(\mu_1^H(q), \dots, \mu_N^H(q)) &= \lim_{k \rightarrow \infty} (\tau_m)_*(\mu_1(q_k), \dots, \mu_N(q_k)) = \lim_{k \rightarrow \infty} (\Lambda_1(o_k), \dots, \Lambda_N(o_k))A \\ &= (\Lambda_1(\bar{o}), \dots, \Lambda_N(\bar{o}))A, \quad \text{where } o_k = \tau_m(q_k) \text{ and } \bar{o} = \tau_m^H(q). \end{aligned}$$

As $(\Lambda_1(\bar{\omega}), \dots, \Lambda_N(\bar{\omega}))$ forms a basis for $T_{\bar{\omega}}^{1,0}W = \text{Hom}(F_p^n, F_p^{n-1}/F_p)$ and A is nonsingular, we can conclude that $(\tau_m^H)_*$ is an isomorphism from $T_q^{1,0}U_q$ to $T_{\bar{\omega}}^{1,0}W$. This shows that $\tau_m^H : \mathcal{T}_m^H \rightarrow \mathbb{C}^N \cong H_p^{n-1,1}$ is a local embedding. \square

Theorem 8. *The holomorphic map $\tau_m^H : \mathcal{T}_m^H \rightarrow \mathbb{C}^N$ defines a global holomorphic affine structure on \mathcal{T}_m^H .*

Proof. Since $\tau_m^H : \mathcal{T}_m^H \rightarrow \mathbb{C}^N$ is a local embedding and $\dim \mathcal{T}_m^H = N$, thus the same arguments as the proof of Lemma 5.6 can be applied to conclude τ_m^H defines a global holomorphic affine structure on \mathcal{T}_m^H . \square

It is important to note that the flat connections which correspond to the global holomorphic affine structures on \mathcal{T} , on \mathcal{T}_m or on \mathcal{T}_m^H are in general not compatible to the corresponding Hodge metrics on them.

5.3 Injectivity of the period map on the Hodge metric completion space

Theorem 9. *For any $m \geq 3$, the holomorphic map $\tau_m^H : \mathcal{T}_m^H \rightarrow \mathbb{C}^N$ is an injection.*

To prove this theorem, we will first prove the following elementary lemma, where we mainly use the completeness with the Hodge metric, the holomorphic affine structure on \mathcal{T}_m^H , the affineness of τ_m^H , and the properties of Hodge metric. We remark that as \mathcal{T}_m^H is a complex affine manifold, we have the notion of straight lines in it with respect to the affine structure.

Lemma 5.8. *For any two points in \mathcal{T}_m^H , there is a straight line in \mathcal{T}_m^H connecting them.*

Proof. Let p be an arbitrary point in \mathcal{T}_m^H , and let $S \subseteq \mathcal{T}_m^H$ be the collection of points that can be connected to p by straight lines in \mathcal{T}_m^H . We need to show that $S = \mathcal{T}_m^H$.

We first show that S is a closed set. Let $\{q_i\}_{i=1}^\infty \subseteq S$ be a Cauchy sequence with respect to the Hodge metric. Then for each i we have the straight line l_i connecting p and q_i such

that $l_i(0) = p$, $l_i(T_i) = q_i$ for some $T_i \geq 0$ and $v_i := \frac{\partial}{\partial t} l_i(0)$ a unit vector with respect to the Euclidean metric on n_+ . We can view these straight lines $l_i : [0, T_i] \rightarrow \mathcal{T}_m^H$ as the solutions of the affine geodesic equations $l_i''(t) = 0$ with initial conditions $v_i := \frac{\partial}{\partial t} l_i(0)$ and $l_i(0) = p$ in particular $T_i = d_E(p, q_i)$ is the Euclidean distance between p and q_i . It is well-known that solutions of these geodesic equations analytically depend on their initial data.

Proposition 5.5 showed that $\Phi_m^H : \mathcal{T}_m^H \rightarrow N_+ \cap D$ is a bounded map, which implies that the image of Φ_m^H is bounded with respect to the Euclidean metric on N_+ . Because a linear projection will map a bounded set to a bounded set, we have that the image of $\tau_m^H = P \circ \Phi_m^H$ is also bounded in \mathbb{C}^N with respect to the Euclidean metric on \mathbb{C}^N . Passing to a subsequence, we may therefore assume that $\{T_i\}$ and $\{v_i\}$ converge, with $\lim_{i \rightarrow \infty} T_i = T_\infty$ and $\lim_{i \rightarrow \infty} v_i = v_\infty$, respectively. Let $l_\infty(t)$ be the local solution of the affine geodesic equation with initial conditions $\frac{\partial}{\partial t} l_\infty(0) = v_\infty$ and $l_\infty(0) = p$. We claim that the solution $l_\infty(t)$ exists for $t \in [0, T_\infty]$. Consider the set

$$E_\infty := \{a : l_\infty(t) \text{ exists for } t \in [0, a]\}.$$

If E_∞ is unbounded above, then the claim clearly holds. Otherwise, we let $a_\infty = \sup E_\infty$, and our goal is to show $a_\infty > T_\infty$. Suppose towards a contradiction that $a_\infty \leq T_\infty$. We then define the sequence $\{a_k\}_{k=1}^\infty$ so that $a_k/T_k = a_\infty/T_\infty$. We have $a_k \leq T_k$ and $\lim_{k \rightarrow \infty} a_k = a_\infty$. Using the continuous dependence of solutions of the geodesic equation on initial data, we conclude that the sequence $\{l_k(a_k)\}_{k=1}^\infty$ is a Cauchy sequence. As \mathcal{T}_m^H is a complete space, the sequence $\{l_k(a_k)\}_{k=1}^\infty$ converges to some $q' \in \mathcal{T}_m^H$. Let us define $l_\infty(a_\infty) := q'$. Then the solution $l_\infty(t)$ exists for $t \in [0, a_\infty]$. On the other hand, as \mathcal{T}_m^H is a smooth manifold, we have that q' is an inner point of \mathcal{T}_m^H . Thus the affine geodesic equation has a local solution at q' which extends the geodesic l_∞ . That is, there exists $\epsilon > 0$ such that the solution $l_\infty(t)$ exists for $t \in [0, a_\infty + \epsilon)$. This contradicts the fact that a_∞ is an upper bound of E_∞ . We have therefore proven that $l_\infty(t)$ exists for $t \in [0, T_\infty]$.

Using the continuous dependence of solutions of the affine geodesic equations on the

initial data again, we get

$$l_\infty(T_\infty) = \lim_{k \rightarrow \infty} l_k(T_k) = \lim_{k \rightarrow \infty} q_k = q_\infty.$$

This means the limit point $q_\infty \in S$, and hence S is a closed set.

Let us now show that S is an open set. Let $q \in S$. Then there exists a straight line l connecting p and q . For each point $x \in l$ there exists an open neighborhood $U_x \subseteq \mathcal{T}_m^H$ with diameter $2r_x$. The collection of $\{U_x\}_{x \in l}$ forms an open cover of l . But l is a compact set, so there is a finite subcover $\{U_{x_i}\}_{i=1}^K$ of l . Then the straight line l is covered by a cylinder C_r in \mathcal{T}_m^H of radius $r = \min\{r_{x_i} : 1 \leq i \leq K\}$. As C_r is a convex set, each point in C_r can be connected to p by a straight line. Therefore we have found an open neighborhood C_r of $q \in S$ such that $C_r \subseteq S$, which implies that S is an open set.

As S is a non-empty, open and closed subset in the connected space \mathcal{T}_m^H , we conclude that $S = \mathcal{T}_m^H$, as we desired. \square

Proof of Theorem 9. Let $p, q \in \mathcal{T}_m^H$ be two different points. Then Lemma 5.8 implies that there is a straight line $l \subseteq \mathcal{T}_m^H$ connecting p and q . Since $\tau_m^H : \mathcal{T}_m^H \rightarrow \mathbb{C}^N$ is affine, the restriction $\tau_m^H|_l$ is a linear map. Suppose towards a contradiction that $\tau_m^H(p) = \tau_m^H(q) \in \mathbb{C}^N$. Then the restriction of τ_m^H to the straight line l is a constant map as $\tau_m^H|_l$ is linear. By Lemma 5.7, we know that $\tau_m^H : \mathcal{T}_m^H \rightarrow \mathbb{C}^N$ is locally injective. Therefore, we may take U_p to be a neighborhood of p in \mathcal{T}_m^H such that $\tau_m^H : U_p \rightarrow \mathbb{C}^N$ is injective. However, the intersection of U_p and l contains infinitely many points, but the restriction of τ_m^H to $U_p \cap l$ is a constant map. This contradicts the fact that when we restrict τ_m^H to $U_p \cap l$, τ_m^H is an injective map. Thus $\tau_m^H(p) \neq \tau_m^H(q)$ if $p \neq q \in \mathcal{T}_m^H$. \square

Since $\tau_m^H = P \circ \Phi_m^H$, where P a the projection map and τ_m^H is injective and Φ_m^H a bounded map, we get

Corollary 5.9. *The completion space \mathcal{T}_m^H is a bounded domain in \mathbb{C}^N .*

Corollary 5.10. *The holomorphic map $\Phi_m^H : \mathcal{T}_m^H \rightarrow N_+ \cap D$ is also an injection.*

CHAPTER 6

Main result and applications

6.1 Completion space of the Teichmüller space

In this section, we define the completion space \mathcal{T}^H by $\mathcal{T}^H = \mathcal{T}_m^H$, and the extended period map Φ^H by $\Phi^H = \Phi_m^H$ for any $m \geq 3$ after proving that \mathcal{T}_m^H doesn't depend on the choice of the level m . Therefore \mathcal{T}^H is a complex affine manifold and that Φ^H is a holomorphic injection.

For any two integers $m, m' \geq 3$, let \mathcal{Z}_m and $\mathcal{Z}_{m'}$ be the smooth quasi-projective manifolds as in Theorem 5 and let \mathcal{Z}_m^H and $\mathcal{Z}_{m'}^H$ be their completions with respect to the Hodge metric. Let \mathcal{T}_m^H and $\mathcal{T}_{m'}^H$ be the universal cover spaces of \mathcal{Z}_m^H and $\mathcal{Z}_{m'}^H$ respectively, then we have the following proposition.

Proposition 6.1. *The complete complex manifolds \mathcal{T}_m^H and $\mathcal{T}_{m'}^H$ are biholomorphic to each other.*

Proof. By definition, $\mathcal{T}_m = i_m(\mathcal{T})$, $\mathcal{T}_{m'} = i_{m'}(\mathcal{T})$ and $\Phi_m = \Phi_m^H|_{\mathcal{T}_m}$, $\Phi_{m'} = \Phi_{m'}^H|_{\mathcal{T}_{m'}}$. Because Φ_m^H and $\Phi_{m'}^H$ are embeddings, $\mathcal{T}_m \cong \Phi_m^H(\mathcal{T}_m)$ and $\mathcal{T}_{m'} \cong \Phi_{m'}^H(\mathcal{T}_{m'})$. Since the composition maps $\Phi_m^H \circ i_m = \Phi$ and $\Phi_{m'}^H \circ i_{m'} = \Phi$, we get $\Phi_m^H(i_m(\mathcal{T})) = \Phi(\mathcal{T}) = \Phi_{m'}^H(i_{m'}(\mathcal{T}))$. Since Φ and \mathcal{T} are both independent of the choice of the level structures, so is the image $\Phi(\mathcal{T})$. Then $\mathcal{T}_m \cong \Phi(\mathcal{T}) \cong \mathcal{T}_{m'}$ biholomorphically, and they don't depend on the choice of level structures. Moreover, Proposition 5.4 implies that \mathcal{T}_m^H and $\mathcal{T}_{m'}^H$ are Hodge metric completion spaces of \mathcal{T}_m and $\mathcal{T}_{m'}$, respectively. Thus the uniqueness of the metric completion spaces implies that \mathcal{T}_m^H is biholomorphic to $\mathcal{T}_{m'}^H$. \square

Proposition 6.1 shows that \mathcal{T}_m^H is independent of the choice of the level m structure, and it allows us to give the following definitions.

Definition 6.2. We define the complete complex manifold $\mathcal{T}^H = \mathcal{T}_m^H$, the holomorphic map $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}^H$ by $i_{\mathcal{T}} = i_m$, and the extended period map $\Phi^H : \mathcal{T}^H \rightarrow D$ by $\Phi^H = \Phi_m^H$ for any $m \geq 3$. In particular, with these new notations, we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}^H & \xrightarrow{\Phi^H} & D \\ \downarrow \pi_m & & \downarrow \pi_m^H & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_m^H} & D/\Gamma. \end{array}$$

6.2 Main result and the global Torelli theorem

In this section, we prove the main result Theorem 10, which asserts that \mathcal{T}^H is the completion space of \mathcal{T} with respect to the Hodge metric and it is a bounded domain of holomorphy in \mathbb{C}^N . As a direct corollary, we get the global Torelli theorem of the period map from the Teichmüller space to the period domain.

Theorem 10. *The complex manifold \mathcal{T}^H is a complex affine manifold which can be embedded into \mathbb{C}^N and it is the completion space of \mathcal{T} with respect to the Hodge metric. Moreover, the extended period map $\Phi^H : \mathcal{T}^H \rightarrow N_+ \cap D$ is a holomorphic injection.*

Proof. By the definition of \mathcal{T}^H and Theorem 8, it is easy to see that \mathcal{T}_m^H is a complex affine manifold, which can be embedded into \mathbb{C}^N . It is also not hard to see that the injectivity of Φ^H follows directly from Corollary 5.10 by the definition of Φ^H . Now we only need to show that \mathcal{T}^H is the Hodge metric completion space of \mathcal{T} , which follows from the following lemma. □

Lemma 6.3. *The map $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}^H$ is an embedding.*

Proof. On one hand, define \mathcal{T}_0 to be $\mathcal{T}_0 = \mathcal{T}_m$ for any $m \geq 3$, as \mathcal{T}_m doesn't depend on the choice of level m structure according to the proof of Proposition 6.1. Since $\mathcal{T}_0 = (\pi_m^H)^{-1}(\mathcal{Z}_m)$,

$\pi_m^H : \mathcal{T}_0 \rightarrow \mathcal{Z}_m$ is a covering map. Thus the fundamental group of \mathcal{T}_0 is a subgroup of the fundamental group of \mathcal{Z}_m , that is, $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{Z}_m)$, for any $m \geq 3$. Moreover, the universal property of the universal covering map $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$ with $\pi_m = \pi_m^H|_{\mathcal{T}_0} \circ i_{\mathcal{T}}$ implies that $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}_0$ is also a covering map.

On the other hand, let $\{m_k\}_{k=1}^{\infty}$ be a sequence of positive integers such that $m_k < m_{k+1}$ and $m_k | m_{k+1}$ for each $k \geq 1$. Then there is a natural covering map from $\mathcal{Z}_{m_{k+1}}$ to \mathcal{Z}_{m_k} for each k . In fact, because each point in $\mathcal{Z}_{m_{k+1}}$ is a polarized Calabi–Yau manifold with a basis $\gamma_{m_{k+1}}$ for the space $(H_n(M, \mathbb{Z})/\text{Tor})/m_{k+1}(H_n(M, \mathbb{Z})/\text{Tor})$ and $m_k | m_{k+1}$, then the basis $\gamma_{m_{k+1}}$ induces a basis for the space $(H_n(M, \mathbb{Z})/\text{Tor})/m_k(H_n(M, \mathbb{Z})/\text{Tor})$. Therefore we get a well-defined map $\mathcal{Z}_{m_{k+1}} \rightarrow \mathcal{Z}_{m_k}$ by assigning to a polarized Calabi–Yau manifold with the basis $\gamma_{m_{k+1}}$ the same polarized Calabi–Yau manifold with the basis $(\gamma_{m_{k+1}} \pmod{m_k}) \in (H_n(M, \mathbb{Z})/\text{Tor})/m_k(H_n(M, \mathbb{Z})/\text{Tor})$. Moreover, using the versal properties of both the families $\mathcal{X}_{m_{k+1}} \rightarrow \mathcal{Z}_{m_{k+1}}$ and $\mathcal{X}_{m_k} \rightarrow \mathcal{Z}_{m_k}$, we can conclude that the map $\mathcal{Z}_{m_{k+1}} \rightarrow \mathcal{Z}_{m_k}$ is locally biholomorphic. Therefore, $\mathcal{Z}_{m_{k+1}} \rightarrow \mathcal{Z}_{m_k}$ is actually a covering map. Thus the fundamental group $\pi_1(\mathcal{Z}_{m_{k+1}})$ is a subgroup of $\pi_1(\mathcal{Z}_{m_k})$ for each k . Hence, the inverse system of fundamental groups

$$\pi_1(\mathcal{Z}_{m_1}) \longleftarrow \pi_1(\mathcal{Z}_{m_2}) \longleftarrow \cdots \longleftarrow \pi_1(\mathcal{Z}_{m_k}) \longleftarrow \cdots$$

has an inverse limit, which is the fundamental group of \mathcal{T} . Because $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{Z}_{m_k})$ for any k , we have the inclusion $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{T})$. But $\pi_1(\mathcal{T})$ is a trivial group since \mathcal{T} is simply connected, thus $\pi_1(\mathcal{T}_0)$ is also a trivial group. Therefore the covering map $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}_0$ is a one-to-one covering. This shows that $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}^H$ is an embedding. \square

Remark 6.4. There is another approach to Lemma 6.3, which is a proof by contradiction. Suppose towards a contradiction that there were two points $p \neq q \in \mathcal{T}$ such that $i_{\mathcal{T}}(p) = i_{\mathcal{T}}(q) \in \mathcal{T}^H$.

On one hand, since each point in \mathcal{T} represents a polarized and marked Calabi–Yau manifold, p and q are actually triples (M_p, L_p, γ_p) and (M_q, L_q, γ_q) respectively, where γ_p and γ_q are two bases of $H_n(M, \mathbb{Z})/\text{Tor}$. On the one hand, each point in \mathcal{Z}_m represents

a triple (M, L, γ_m) with γ_m a basis of $(H_n(M, \mathbb{Z})/\text{Tor})/m(H_n(M, \mathbb{Z})\text{Tor})$ for any $m \geq 3$. By the assumption that $i_{\mathcal{T}}(p) = i_{\mathcal{T}}(q)$ and the relation that $i \circ \pi_m = \pi_m^H \circ i_{\mathcal{T}}$, we have $i \circ \pi_m(p) = i \circ \pi_m(q) \in \mathcal{Z}_m$ for any $m \geq 3$. In particular, for any $m \geq 3$, the image of (M_p, L_p, γ_p) and (M_q, L_q, γ_q) under π_m are the same in \mathcal{Z}_m . This implies that there exists a biholomorphic map $f_{pq} : M_p \rightarrow M_q$ such that $f_{pq}^*(L_q) = L_p$ and $f_{pq}^*(\gamma_q) = \gamma_p \cdot A$, where A is an integer matrix satisfying

$$A = (A_{ij}) \equiv \text{Id} \pmod{m} \quad \text{for any } m \geq 3. \quad (6.5)$$

Let $|A_{ij}|$ be the absolute value of the ij -th entry of the matrix (A_{ij}) . Since (6.5) holds for any $m \geq 3$, we can choose an integer m_0 greater than any $|A_{ij}|$ such that

$$A = (A_{ij}) \equiv \text{Id} \pmod{m_0}.$$

Since each $A_{ij} < m_0$ and $A = (A_{ij}) \equiv \text{Id} \pmod{m_0}$, we have $A = \text{Id}$. Therefore, we found a biholomorphic map $f_{pq} : M_p \rightarrow M_q$ such that $f_{pq}^*(L_q) = L_p$ and $f_{pq}^*(\gamma_q) = \gamma_p$. This implies that the two triples (M_p, L_p, γ_p) and (M_q, L_q, γ_q) are equivalent to each other. Therefore, p and q in \mathcal{T} are actually the same point. This contradicts to our assumption that $p \neq q$.

Since $\Phi = \Phi^H \circ i_{\mathcal{T}}$ with both Φ^H and $i_{\mathcal{T}}$ embeddings, we get the global Torelli theorem for the period map from the Teichmüller space to the period domain as follows.

Corollary 6.6 (Global Torelli theorem). *The period map $\Phi : \mathcal{T} \rightarrow D$ is injective.*

6.3 Completion space of the Teichmüller space is a domain of holomorphy

In this section, as another important consequence of the main Theorem 10, we prove the following property of \mathcal{T}^H .

Theorem 11. *The completion space \mathcal{T}^H is a bounded domain of holomorphy in \mathbb{C}^N ; thus there exists a complete Kähler–Einstein metric on \mathcal{T}^H .*

We recall that a \mathcal{C}^2 function $\rho : \Omega \rightarrow \mathbb{R}$ on a domain $\Omega \subseteq \mathbb{C}^n$ is *plurisubharmornic* if and only if its Levi form is positive definite at each point in Ω . Given a domain $\Omega \subseteq \mathbb{C}^n$, a function $f : \Omega \rightarrow \mathbb{R}$ is called an *exhaustion function* if for any $c \in \mathbb{R}$, the set $\{z \in \Omega \mid f(z) < c\}$ is relatively compact in Ω . The following well-known theorem provides a definition for domains of holomorphy. For example, one may refer to [9] for details.

Proposition 6.7. *An open set $\Omega \in \mathbb{C}^n$ is a domain of holomorphy if and only if there exists a continuous plurisubharmonic function $f : \Omega \rightarrow \mathbb{R}$ such that f is also an exhaustion function.*

The following theorem in Section 3.1 of [6] gives us the basic ingredients to construct a plurisubharmonic exhaustion function on \mathcal{T}^H .

Proposition 6.8. *On every manifold D , which is dual to a Kähler C -space, there exists an exhaustion function $f : D \rightarrow \mathbb{R}$, whose Levi form, restricted to $T_h^{1,0}(D)$, is positive definite at every point of D .*

We remark that in this proposition, in order to show f is an exhaustion function on D , Griffiths and Schmid showed that the set $f^{-1}(-\infty, c]$ is compact in D for any $c \in \mathbb{R}$.

Lemma 6.9. *The extended period map $\Phi^H : \mathcal{T}^H \rightarrow D$ still satisfies the Griffiths transversality.*

Proof. Let us consider the tangent bundles $T^{1,0}\mathcal{T}^H$ and $T^{1,0}D$ as two differential manifolds, and the tangent map $(\Phi^H)_* : T^{1,0}\mathcal{T}^H \rightarrow T^{1,0}D$ as a continuous map. We only need to show that the image of $(\Phi^H)_*$ is contained in the horizontal tangent bundle $T_h^{1,0}D$.

The horizontal subbundle $T_h^{1,0}D$ is a close set in $T^{1,0}D$, so the preimage of $T_h^{1,0}D$ under $(\Phi^H)_*$ is a close set in $T^{1,0}\mathcal{T}^H$. On the other hand, because the period map Φ satisfies the Griffiths transversality, the image of Φ_* is in the horizontal subbundle $T_h^{1,0}D$. This means that the preimage of $T_h^{1,0}D$ under $(\Phi^H)_*$ contains both $T^{1,0}\mathcal{T}$ and the closure of $T^{1,0}\mathcal{T}$, which is $T^{1,0}\mathcal{T}^H$. This finishes the proof. \square

Proof of Theorem 11. By Theorem 10, we can see that \mathcal{T}^H is a bounded domain in \mathbb{C}^N . Therefore, once we show \mathcal{T}^H is domain of holomorphy, the existence of Kähler-Einstein metric on it follows directly from the well-known theorem by Mok–Yau in [15].

In order to show that \mathcal{T}^H is a domain of holomorphy in \mathbb{C}^N , it is enough to construct a plurisubharmonic exhaustion function on \mathcal{T}^H .

Let f be the exhaustion function on D constructed in Proposition 6.8, whose Levi form, when restricted to the horizontal tangent bundle $T_h^{1,0}D$ of D , is positive definite at each point of D . We claim that the composition function $f \circ \Phi^H$ is the demanded plurisubharmonic exhaustion function on \mathcal{T}^H .

By the Griffiths transversality of Φ^H , the composition function $f \circ \Phi^H : \mathcal{T}^H \rightarrow \mathbb{R}$ is a plurisubharmonic function on \mathcal{T}^H . Thus it suffices to show that the function $f \circ \Phi^H$ is an exhaustion function on \mathcal{T}^H , which is enough to show that for any constant $c \in \mathbb{R}$, $(f \circ \Phi^H)^{-1}(-\infty, c] = (\Phi^H)^{-1}(f^{-1}(-\infty, c])$ is a compact set in \mathcal{T}^H . Indeed, it has already been shown in [6] that the set $f^{-1}(-\infty, c]$ is a compact set in D . Now for any sequence $\{p_k\}_{k=1}^\infty \subseteq (f \circ \Phi^H)^{-1}(-\infty, c]$, we have $\{\Phi^H(p_k)\}_{k=1}^\infty \subseteq f^{-1}(-\infty, c]$. Since $f^{-1}(-\infty, c]$ is compact in D , the sequence $\{\Phi^H(p_k)\}_{k=1}^\infty$ has a convergent subsequence. We denote this convergent subsequence by $\{\Phi^H(p_{k_n})\}_{n=1}^\infty \subseteq \{\Phi^H(p_k)\}_{k=1}^\infty$ with $k_n < k_{n+1}$, and denote $\lim_{k \rightarrow \infty} \Phi^H(p_k) = o_\infty \in D$. On the other hand, since the map Φ^H is injective and the Hodge metric on \mathcal{T}^H is induced from the Hodge metric on D , the extended period map Φ^H is actually a global isometry onto its image. Therefore the sequence $\{p_{k_n}\}_{n=1}^\infty$ is also a Cauchy sequence that converges to $(\Phi^H)^{-1}(o_\infty)$ with respect to the Hodge metric in $(f \circ \Phi^H)^{-1}(-\infty, c] \subseteq \mathcal{T}^H$. In this way, we have proved that any sequence in $(f \circ \Phi^H)^{-1}(-\infty, c]$ has a convergent subsequence. Therefore $(f \circ \Phi^H)^{-1}(-\infty, c]$ is compact in \mathcal{T}^H , as was needed to show. \square

This section contains several applications of the results reviewed in the previous sections. In Section 6.4 we prove a general result for the extended period map to be a bi-holomorphic map from \mathcal{T}^H , the Hodge metric completion of the Teichmüller space \mathcal{T} of polarized and marked Calabi–Yau and Calabi–Yau type manifolds to the corresponding period domain;

and apply this result to the cases of K3 surfaces and cubic fourfolds. We hope to find more interesting examples in our subsequent work. In Section 6.5 we construct explicit holomorphic sections of the Hodge bundles on \mathcal{T}^H , which trivialize those Hodge bundles. In particular, for Calabi–Yau manifolds, a global holomorphic section of holomorphic $(n, 0)$ -classes on \mathcal{T}^H is constructed, which coincides with explicit local Taylor expansion in the affine coordinates at any base point p in \mathcal{T}^H . Finally in Section 6.6 we prove a global splitting property for the Hodge bundles, as well as a theorem proves that a compact polarized and marked Calabi–Yau manifold with complex structure J can not be deformation equivalent to a polarized and marked Calabi–Yau manifolds with conjugate complex structure $-J$.

6.4 Surjectivity of the period map on the Teichmüller space

In this section we use our results on the Hodge completion space \mathcal{T}^H to give a simple proof of the surjectivity of the period maps of K3 surfaces and cubic fourfolds. First we have the following general result for polarized and marked Calabi–Yau manifolds and Calabi–Yau type manifolds,

Theorem 12. *If $\dim \mathcal{T}^H = \dim D$, then the extended period map $\Phi^H : \mathcal{T}^H \rightarrow D$ is surjective.*

Proof. Since $\dim \mathcal{T}^H = \dim D$, the property that $\Phi^H : \mathcal{T}^H \rightarrow D$ is an local isomorphism shows that the image of \mathcal{T}^H under the extended period map Φ^H is open in D . On the other hand, the completeness of \mathcal{T}^H with respect to Hodge metric implies that the image of \mathcal{T}^H under Φ^H is close in D . As \mathcal{T}^H is not empty and that D is connected, we can conclude that $\Phi^H(\mathcal{T}^H) = D$. □

It is well known that for K3 surfaces, which are two dimensional Calabi–Yau manifolds, we have $\dim \mathcal{T}^H = \dim \mathcal{T} = \dim D = 19$; for cubic fourfolds, they are Calabi–Yau type manifolds. One knows that both K3 and cubic fourfolds have smooth Teichmüller spaces, and $\dim \mathcal{T}^H = \dim \mathcal{T} = \dim D = 20$. Thus applying the above theorem, we can easily conclude that

Corollary 6.10. *Let \mathcal{T}^H be the Hodge metric completion space of the Teichmüller space for polarized and marked K3 surfaces or cubic fourfolds. Then the extended period map $\Phi^H : \mathcal{T}^H \rightarrow D$ is surjective.*

In fact, among all the Calabi–Yau or Calabi–Yau type projective hypersurfaces, K3 surfaces and cubic fourfolds are the only two satisfying the condition that the dimensions of the Teichmüller space and the period domain are the same. It would be interesting to find such examples for complete intersections in weighted projective spaces and compact homogeneous manifolds.

Remark 6.11. Let \mathcal{T} be the Teichmüller space of polarized and marked hyperkähler manifolds, $H_{pr}^2(M, \mathbb{C})$ the degree 2 primitive cohomology group, and D the period domain of weight two Hodge structures on $H_{pr}^2(M, \mathbb{C})$. Then our method can be applied without change to prove that the period map from \mathcal{T} to D is also injective. Furthermore, let \mathcal{T}^H be the Hodge completion of \mathcal{T} with respect to the Hodge metric induced from the homogeneous metric on D , then the extended period map from \mathcal{T}^H to D is also surjective. This follows from the same argument of above theorem. See [31] and [10] for different injectivity and surjectivity results for hyperkähler manifolds.

6.5 Global holomorphic sections of the Hodge bundles

In this section we prove the existence and study the property of global holomorphic sections of the Hodge bundles $\{F^k\}_{k=0}^n$ over Hodge completion space \mathcal{T}^H of Teichmüller space of polarized and marked Calabi–Yau and Calabi–Yau type manifolds. The main ingredient of proofs in this section is Theorem 4.22.

Recall that we have fixed a base point $p \in \mathcal{T}$ and an adapted basis $\{\eta_0, \dots, \eta_{m-1}\}$ for the Hodge decomposition of the base point $\Phi(p) \in D$. With the fixed base point in D , we can identify N_+ with its unipotent orbit in \check{D} by identifying an element $c \in N_+$ with $[c] = cB$ in \check{D} .

On one hand, as we have fixed an adapted basis $\{\eta_0, \dots, \eta_{m-1}\}$ for the Hodge decomposi-

tion of the base point. Then elements in $G_{\mathbb{C}}$ can be identified with a subset of the nonsingular block matrices. In particular, the set N_+ with its unipotent orbit in \check{D} , then elements in N_+ can be realized as nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. Namely, for any element $\{F_o^k\}_{k=0}^n \in N_+ \subseteq \check{D}$, there exists a unique nonsingular block lower triangular matrices $A(o) \in G_{\mathbb{C}}$ such that $(\eta_0, \dots, \eta_{m-1})A(o)$ is an adapted basis for the Hodge filtration $\{F_o^k\} \in N_+$ that represents this element in N_+ . Similarly, any elements in B can be realized as nonsingular block upper triangular matrices in $G_{\mathbb{C}}$. Moreover, as $\check{D} = G_{\mathbb{C}}/B$, we have that for any $U \in G_{\mathbb{C}}$, which is a nonsingular block upper triangular matrix, $(\eta_0, \dots, \eta_{m-1})A(o)U$ is also an adapted basis for the Hodge filtration $\{F^k(o)\}_{k=0}^n$. Conversely, if $(\zeta_0, \dots, \zeta_{m-1})$ is an adapted basis for the Hodge filtration $\{F_o^k\}_{k=0}^n$, then there exists a unique $U \in G_{\mathbb{C}}$ such that $(\zeta_0, \dots, \zeta_{m-1}) = (\eta_0, \dots, \eta_{m-1})A(o)U$.

For any $q \in \mathcal{T}$, let us denote the Hodge filtration at $q \in \mathcal{T}$ by $\{F_q^k\}_{k=0}^n$, and we have that $\{F_q^k\}_{k=0}^n \in N_+ \cap D$ by Theorem 4.22. Thus there exists a unique nonsingular block lower triangular matrices $\tilde{A}(q)$ such that $(\eta_0, \dots, \eta_{m-1})\tilde{A}(q)$ is an adapted basis for the Hodge filtration $\{F_q^k\}_{k=0}^n$.

On the other hand, for any adapted basis $\{\zeta_0(q), \dots, \zeta_{m-1}(q)\}$ for the Hodge filtration $\{F_q^k\}_{k=0}^n$ at q , we know that there exists an $m \times m$ transition matrix $A(q)$ such that

$$(\zeta_0(q), \dots, \zeta_{m-1}(q)) = (\eta_0, \dots, \eta_{m-1})A(q).$$

Moreover, we set the blocks of $A(q)$ as in (4.4) and denote the (i, j) -th block of $A(q)$ by $A^{i,j}(q)$.

According to the above discussion, as both $(\eta_0, \dots, \eta_{m-1})\tilde{A}(q)$ and $(\eta_0, \dots, \eta_{m-1})A(q)$ are adapted bases for the Hodge filtration for $\{F_q^k\}_{k=0}^n$, there exists a $U \in G_{\mathbb{C}}$ which is a block nonsingular upper triangular matrix such that

$$(\eta_0, \dots, \eta_{m-1})\tilde{A}(q)U = (\eta_0, \dots, \eta_{m-1})A(q).$$

Therefore, we conclude that

$$\tilde{A}(q)U = A(q). \tag{6.12}$$

where $\tilde{A}(q)$ is a nonsingular block lower triangular matrix in $G_{\mathbb{C}}$ with all the diagonal blocks equal to identity submatrix, while U is a block upper triangular matrix in $G_{\mathbb{C}}$. However, according to basic linear algebra, we know that a nonsingular matrix $A(q) \in G_{\mathbb{C}}$ have the decomposition of the type in (6.12) if and only if the principal submatrices $[A^{i,j}(q)]_{0 \leq i,j \leq n-k}$ are nonsingular for all $0 \leq k \leq n$.

To conclude, by Theorem 4.22, we have that $\Phi(q) \in N_+$ for any $q \in \mathcal{T}$. Therefore, for any adapted basis $(\zeta_0(q), \dots, \zeta_{m-1}(q))$, there exists a nonsingular block matrix $A(q) \in G_{\mathbb{C}}$ with $\det[A^{i,j}(q)]_{0 \leq i,j \leq n-k} \neq 0$ for any $0 \leq k \leq n$ such that

$$(\zeta_0(q), \dots, \zeta_{m-1}(q)) = (\eta_0, \dots, \eta_{m-1})A(q).$$

Let $\{F_p^k\}_{k=0}^n$ be the reference Hodge filtration at the base point $p \in \mathcal{T}$. For any point $q \in \mathcal{T}^H$ with the corresponding Hodge filtrations $\{F_q^k\}_{k=0}^n$, we define the following maps

$$P_q^k : F_q^k \rightarrow F_p^k \quad \text{for any } 0 \leq k \leq n$$

to be the projection map with respect to the Hodge decomposition at the reference point p .

Lemma 6.13. *For any point $q \in \mathcal{T}^H$ and $0 \leq k \leq n$, the map $P_q^k : F_q^k \rightarrow F_p^k$ is an isomorphism. Furthermore, P_q^k depends on q holomorphically.*

Proof. We have already fixed $\{\eta_0, \dots, \eta_{m-1}\}$ as an adapted basis for the Hodge decomposition of the Hodge structure at the base point p . Thus it is also the adapted basis for the Hodge filtration $\{F_p^k\}_{k=0}^n$ at the base point. For any point $q \in \mathcal{T}$, let $\{\zeta_0, \dots, \zeta_{m-1}\}$ be an adapted basis for the Hodge filtration $\{F_q^k\}_{k=0}^n$ at q . Let $[A^{i,j}(q)]_{0 \leq i,j \leq n} \in G_{\mathbb{C}}$ be the transition matrix between the basis $\{\eta_0, \dots, \eta_{m-1}\}$ and $\{\zeta_0, \dots, \zeta_{m-1}\}$ for the same vector space $H^n(M, \mathbb{C})$. Then we have showed that $[A^{i,j}(q)]_{0 \leq i,j \leq n-k}$ is nonsingular for all $0 \leq k \leq n$.

On the other hand, the submatrix $[A^{i,j}(q)]_{0 \leq j \leq n-k}$ is the transition matrix between the bases of F_q^k and F_p^0 for any $0 \leq k \leq n$, that is

$$(\zeta_0(q), \dots, \zeta_{f^{k-1}}(q)) = (\eta_0, \dots, \eta_{m-1})[A^{i,j}(q)]_{0 \leq j \leq n-k} \quad \text{for any } 0 \leq k \leq n,$$

where $(\zeta_0(q), \dots, \zeta_{f^k-1}(q))$ and $(\eta_0, \dots, \eta_{m-1})$ are the bases for F_q^k and F_p^0 respectively. Thus the matrix of P_q^k with respect to $\{\eta_0, \dots, \eta_{f^k-1}\}$ and $\{\zeta_0, \dots, \zeta_{f^k-1}\}$ is the first $(n-k+1) \times (n-k+1)$ principal submatrix $[A^{i,j}(q)]_{0 \leq i, j \leq n-k}$ of $[A^{i,j}(q)]_{0 \leq i, j \leq n}$. Now since $[A^{i,j}(q)]_{0 \leq i, j \leq n-k}$ for any $0 \leq k \leq n$ is nonsingular, we conclude that the map P_q^k is an isomorphism for any $0 \leq k \leq n$.

From our construction, it is clear that the projection P_q^k depends on q holomorphically. □

Now we are ready to construct the global holomorphic sections of Hodge bundles over \mathcal{T}^H . For any $0 \leq k \leq n$, we know that $\{\eta_0, \eta_1, \dots, \eta_{f^k-1}\}$ is an adapted basis of the Hodge decomposition of F_p^k for any $0 \leq k \leq n$. Then we define the sections

$$s_i : \mathcal{T}^H \rightarrow F^k, \quad q \mapsto (P_q^k)^{-1}(\eta_i) \in F_q^k \quad \text{for any } 0 \leq i \leq f^k - 1. \quad (6.14)$$

Lemma 6.13 implies that $\{(P_q^k)^{-1}(\eta_i)\}_{i=0}^{f^k-1}$ form a basis of F_q^k for any $q \in \mathcal{T}^H$. In fact, we have proved the following theorem for polarized and marked Calabi–Yau and Calabi–Yau type manifolds.

Theorem 13. *For all $0 \leq k \leq n$, the Hodge bundles F^k over \mathcal{T}^H are trivial bundles, and the trivialization can be obtained by $\{s_i\}_{0 \leq i \leq f^k-1}$ which is defined in (6.14).*

Remark 6.15. In particular, the section $s_0 : \mathcal{T}^H \rightarrow F^n$ is a global nowhere zero section of the Hodge bundle F^n for Calabi–Yau manifolds.

By using the local deformation theory for Calabi–Yau manifolds in [29], Todorov constructed a canonical local holomorphic section of the line bundle F^n over the local deformation space of a Calabi–Yau manifold. In fact, let Ω_p be a holomorphic $(n, 0)$ -form on the central fiber M_p of the family. Then there exists a coordinate chart $\{U_p, (\tau_1, \dots, \tau_N)\}$ around the base point p and a basis $\{\varphi_1, \dots, \varphi_N\}$ of harmonic Beltrami differentials $\mathbb{H}^{0,1}(M_p, T^{1,0}M_p)$, such that

$$\Omega_p^c(\tau) = e^{\varphi(\tau)} \lrcorner \Omega_p, \quad (6.16)$$

is a family of holomorphic $(n, 0)$ -forms over U_p . We can assume this local coordinate chart is the same as the affine coordinates at p we constructed, which can be achieved simply by taking p as the base point in our construction in §2 of the affine structure on \mathcal{T} . The Kuranishi family of Beltrami differentials $\varphi(\tau)$ satisfies the integrability equation which is solvable for Calabi–Yau manifolds by the Tian-Todorov lemma,

$$\bar{\partial}\varphi(\tau) = \frac{1}{2}[\varphi(\tau), \varphi(\tau)],$$

and the Taylor expansion $\varphi(\tau) = \sum_{i=1}^N \varphi_i \tau_i + O(|\tau|^2)$ converges for $|\tau|$ small by classical Kodaira-Spencer theory. We have

Lemma 6.17. *Let $\Omega_p^c(\tau)$ be a canonical family defined by (6.16). Then we have the following section of F^n over u_p ,*

$$[\Omega_p^c(\tau)] = [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + A(\tau), \quad (6.18)$$

where $\{[\varphi_i \lrcorner \Omega_p]\}_{i=1}^N$ give a basis of $H^{n-1,1}(M_p)$, and $A(\tau) = O(|\tau|^2) \in \bigoplus_{k=2}^n H^{n-k,k}(M_p)$ denotes terms of order at least 2 in τ .

Proof. Details of the proof of this lemma can be found in [2, Page 12–14], or [12, Proposition 5.1]. In fact one can directly check the following formula

$$e^{-\varphi(\tau)} \lrcorner (d(e^{\varphi(\tau)} \lrcorner \Omega_p)) = \bar{\partial}\Omega_p + \partial(\varphi(\tau) \lrcorner \Omega_p). \quad (6.19)$$

The construction of the Kuranishi family $\varphi(\tau)$ implies that $\partial(\varphi(\tau) \lrcorner \Omega_p) = 0$, and the fact that Ω_p is holomorphic on M_p implies $\bar{\partial}\Omega_p = 0$. So the right hand side of formula (6.19) is equal to 0. Then by replacing the de Rham differential operator d on the left hand side by $\partial_\tau + \bar{\partial}_\tau$ on fiber M_τ , we get $(\partial_\tau + \bar{\partial}_\tau)(e^{\varphi(\tau)} \lrcorner \Omega_p) = 0$. Note that $e^{\varphi(\tau)} \lrcorner \Omega_p$ is a $(n, 0)$ form on M_τ , and $\partial_\tau(e^{\varphi(\tau)} \lrcorner \Omega_p) = 0$, we get

$$\bar{\partial}_\tau(e^{\varphi(\tau)} \lrcorner \Omega_p) = 0.$$

Therefore $e^{\varphi(\tau)} \lrcorner \Omega_p$ is a holomorphic $(n, 0)$ -form on the Calabi–Yau manifold M_τ . The Taylor expansion (6.18) follows from the corresponding Taylor expansion of $\varphi(\tau)$. \square

Using the same notation as in Lemma 6.17, we are ready to prove the following theorem for Calabi–Yau manifolds,

Theorem 14. *Choose $[\Omega_p] = \eta_0$, then the section s_0 of F^n is a global holomorphic extension of the local section $[\Omega_p^c(\tau)]$.*

Proof. Because both s_0 and $[\Omega_p^c(\tau)]$ are holomorphic sections of F^n , we only need to show that $s_0|_{U_p} = [\Omega_p^c(\tau)]$. In fact, from the expansion formula (6.18), we have that for any $q \in U_p$

$$P_q^n([\Omega_p^c(\tau(q))]) = [\Omega_p] = \eta_0.$$

Therefore, $[\Omega_p^c(\tau(q))] = (P_q^n)^{-1}(\eta_0) = s_0(q)$ for any point $q \in U_p$. \square

As an example, if we consider the Teichmüller space of polarized and marked hyperkähler manifolds and the weight two variation of Hodge structure of hyperkähler manifolds, then the Taylor series (6.18) is a finite degree polynomial and converges globally on \mathcal{T}^H . More precisely we have the following

Example 6.20. Let \mathcal{T}^H be the Hodge completion of Teichmüller space of a polarized and marked hyperkähler manifold, and (τ_1, \dots, τ_N) be global affine coordinates with respect to the reference point p and an orthonormal basis $\{\eta_1, \dots, \eta_N\}$ of $H_{pr}^{1,1}(M_p)$, then

$$[\Omega_p^c(\tau)] = [\Omega_p] + \sum_{i=1}^{19} \tau_i \eta_i + \left(\frac{1}{2} \sum_{i=1}^{19} \tau_i^2 \right) [\bar{\Omega}_p],$$

is a global holomorphic section of F^2 over \mathcal{T}^H . In fact, in this case, \mathcal{T}^H is bi-holomorphic to D given by the period map as discussed in Section 4.1, and the affine structure on \mathcal{T}^H is induced from the affine structure on D by the Harish-Chandra embedding of D into the complex Euclidean space. The global affine coordinates on \mathcal{T}^H is induced by the Harish-Chandra embedding.

6.6 A global splitting property of the Hodge bundles

In section 6.5, we proved that the Hodge bundles $\{F^k\}_{k=0}^n$ over the Hodge metric completion space of the Teichmüller space of polarized and marked Calabi–Yau and Calabi–Yau type

manifolds are trivial bundles by directly constructing global trivializations. In particular, because the Hodge bundle F^0 is a trivial bundle over \mathcal{T}^H , we have a trivialization

$$F^0 = \mathcal{T}^H \times H^n(M, \mathbb{C}).$$

Then for any sub-bundle $V \subset F^0$, the fiber V_q at a point $q \in \mathcal{T}^H$ will be considered as a subspace of $H^n(M, \mathbb{C})$, which does not depend on the point $q \in \mathcal{T}^H$.

In this section, we directly construct global defined anti-holomorphic vector bundles \tilde{F}^k over \mathcal{T}^H , such that for any point $q \in \mathcal{T}$, the vector space $H^n(M, \mathbb{C})$ splits as

$$H^n(M, \mathbb{C}) = F_p^k \oplus \tilde{F}_q^k \quad \text{for any } q \in \mathcal{T}^H,$$

where p is the base point in \mathcal{T}^H . Then as an application of our method, in Theorem 16 we also prove that any two fibers of the versal family $\mathcal{U} \rightarrow \mathcal{T}$ can not be conjugate manifolds of each other.

The construction of vector bundles \tilde{F}^k is again based on Lemma 6.13, in fact, we have the following equivalent lemma,

Lemma 6.21. *For any $q \in \mathcal{T}^H$ and $1 \leq k \leq n$, we have that $H^n(M, \mathbb{C}) = F_p^k \oplus \overline{F_q^{n-k+1}}$.*

Proof. Firstly, the decomposition $H^n(M, \mathbb{C}) = F_p^k \oplus \overline{F_p^{n-k+1}}$ follows from the definition of the Hodge structure for any $0 \leq k \leq n$. Secondly Lemma 6.13 implies that $P_q^{n-k+1} : F_p^{n-k+1} \rightarrow F_q^{n-k+1}$ is an isomorphism for any $q \in \mathcal{T}^H$ and any $0 \leq k \leq n$. Therefore $F_p^k \cap \overline{F_q^{n-k+1}} = \{0\}$ as the projection from F_p^k to F_p^{n-k+1} is a zero map.

On the other hand, $\dim F_p^k + \dim \overline{F_q^{n-k+1}} = \dim F_p^k + \dim \overline{F_p^{n-k+1}} = \dim H^n(M, \mathbb{C})$, so we have that

$$H^n(M, \mathbb{C}) = F_p^k \oplus \overline{F_q^{n-k+1}}.$$

□

Because the reference point p is an arbitrary prefixed point on \mathcal{T} , and the Hodge filtration at each point does not depend on the choice of the reference point, Lemma 6.21 actually implies,

Corollary 6.22. *For any different points q and q' on \mathcal{T} , and $1 \leq k \leq n$, we have $H^n(M, \mathbb{C}) = F_q^k \oplus \overline{F_{q'}^{n-k+1}}$.*

Let us write $\tilde{F}^k = \overline{F^{n-k+1}}$, for each $0 \leq k \leq n$. Then we have proved the following result,

Theorem 15. *The vector bundles $\{\tilde{F}^k\}_{k=0}^n$ are globally defined anti-holomorphic vector bundles over \mathcal{T}^H such that*

$$H^n(M, \mathbb{C}) = F_p^k \oplus \tilde{F}_q^k \quad \text{for any } q \in \mathcal{T}^H.$$

Now we let M be a complex manifold with background differential manifold X and complex structure $J : T_{\mathbb{R}}X \rightarrow T_{\mathbb{R}}X$, then the complex conjugate manifold \overline{M} is a complex manifold with the same background differential manifold X and with conjugate complex structure $-J$. In fact, M and its complex conjugate manifold \overline{M} satisfy the relation $T^{1,0}M = T^{0,1}\overline{M}$ and $T^{0,1}M = T^{1,0}\overline{M}$.

Problems regarding deformation inequivalent complex conjugated complex structures have been studied, for example one may find interesting results in [11]. We will apply our results to study such problem for polarized and marked Calabi–Yau manifolds. In fact, another interesting application of Corollary 6.22 is that a polarized and marked Calabi–Yau manifold M can not be connected to its complex conjugate manifold \overline{M} by deformation of complex structure. For any point q in \mathcal{T} , let M_q denote the fiber of the versal family $\mathcal{U} \rightarrow \mathcal{T}$ at point q . Then we have the following theorem,

Theorem 16. *If $q \neq q'$ are two different points on \mathcal{T} , then $M_{q'} \neq \overline{M}_q$.*

Proof. We prove this theorem by contradiction. Suppose $M_{q'} = \overline{M}_q$, and let Ω be an holomorphic $(n, 0)$ form on M_q , then $\overline{\Omega}$ is a holomorphic $(n, 0)$ form on $M_{q'} = \overline{M}_q$. Therefore the fibers of Hodge bundles on the two points satisfy $F_q^n = \overline{F_{q'}^n} \subset \overline{F_{q'}^1}$, and

$$H^n(M, \mathbb{C}) \neq F_q^n \oplus \overline{F_{q'}^1}.$$

But this contradicts to Corollary 6.22, so $M_{q'} \neq \overline{M}_q$ as desired. \square

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