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Aggregation Equation with Degenerate Diffusion

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Yao Yao

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Abstract of the Dissertation

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Professor Inwon C. Kim, Chair

Recently, there has been a growing interest in the use of nonlocal partial differential equation (PDE) to model biological and physical phenomena. In this dissertation, we study the behavior of solutions to several nonlocal PDEs, which have both an aggregation term and a degenerate diffusion term.

Chapter 1 and Chapter 2 of this dissertation are devoted to the study of the Patlak-Keller-Segel (PKS) equation and its variations. The PKS equation is a degenerate diffusion equation with a nonlocal aggregation term, which models the collective motion of cells attracted by a self-emitted chemical substance. While the global well-posedness and finite-time blow-up criteria are well known, the asymptotic behaviors of solutions are not completely clear.

In Chapter 1, we investigate qualitative and asymptotic behavior of solutions for the PKS equation when the solution exists globally in time. The challenge in the analysis consists of the nonlocal aggregation term as well as the degeneracy of the diffusion term which generates compactly supported solutions. Using maximum-principle type arguments as well as energy argument, we prove the finite propagation property of general solutions, and several results regarding asymptotic behaviors of solutions.

In Chapter 2, we consider the PKS equation with general power-law interaction kernel, and focus on the cases where the solution blows up in finite time. We study radially symmetric finite time blow-up dynamics from both the numerical and asymptotic aspect, and show that the solution exhibits three kinds of blow-up behavior: self-similar with no

mass concentrated at the core, imploding shock solution and near-self-similar blow-up with a fixed amount of mass concentrated at the core. Computation are performed for a variety of parameters using an arbitrary Lagrangian Eulerian method with adaptive mesh refinement.

Chapter 3 discusses the study on an aggregation-diffusion equation with smooth interaction kernel in the periodic domain. This equation represents the generalization to $m > 1$ of the McKean–Vlasov equation where here the "diffusive" portion of the dynamics are governed by *Porous medium* self–interactions. We focus primarily on $m \in (1, 2]$ with particular emphasis on $m = 2$. In general, we establish regularity properties and, for small interaction, exponential decay to the uniform stationary solution. For $m = 2$, we obtain essentially sharp results on the rate of decay for the entire regime up to the (sharp) transitional value of the interaction parameter.

This dissertation has been resulted in the publications [CKY, KY, Y, YB].

The dissertation of Yao Yao is approved.

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For my family and friends

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Acknowledgments

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CHAPTER 1

Asymptotic Behavior for Patlak-Keller-Segel Equation with Degenerate Diffusion

1.1 Introduction

In this chapter we study solutions of a nonlocal aggregation equation with degenerate diffusion, given by

$$\rho_t = \Delta\rho^m + \nabla \cdot (\rho\nabla(\rho * V)) \text{ in } \mathbb{R}^d \times [0, \infty), \tag{1.1.1}$$

with initial data $\rho_0 \in L^1(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$. Here $m > 1$, $d \geq 3$ and $*$ denotes the convolution operator. In the absence of the aggregation term (when $V = 0$), our equation becomes the well-known *Porous medium equation* (PME):

$$\rho_t - \Delta(\rho^m) = 0. \tag{1.1.2}$$

Note that, formally, the mass of solutions is preserved over time:

$$\int_{\mathbb{R}^d} \rho(\cdot, 0)dx = \int_{\mathbb{R}^d} \rho(\cdot, t)dx \text{ for all } t > 0.$$

Nonlocal aggregation phenomena have been studied in various biological applications such as population dynamics [BCM, BCM, GM, TBL] and Patlak-Keller-Segel (PKS) models of chemotaxis [KS2, LL, P, FLP]. In the context of biological aggregation, ρ represents the population density which is locally dispersed by the diffusion term, while V is the interaction kernel that models the long-range attraction. Recently, there has been a growing interest in models with degenerate diffusion to include anti-overcrowding effects (see for example [TBL, BCM]). Mathematically, the equation models competition between diffusion and nonlocal aggregation.

Throughout this chapter, we will focus on the following two types of potentials:

(A) (PKS-model) $V(x)$ is a *Newtonian potential*:

$$V(x) = \mathcal{N} := -\frac{c_d}{|x|^{d-2}}, \tag{1.1.3}$$

where $c_d := \dfrac{1}{(d-2)\sigma_d}$, with σ_d: the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

(B) (regularized Newtonian potential)

$$V(x) = (\mathcal{N} * h)(x), \tag{1.1.4}$$

where $*$ denotes convolution and $h(x)$ is a radial function in $L^1(\mathbb{R}^d : (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ which is continuous and radially decreasing.

Note that (A)-(B) cover all attractive potentials V whose Laplacian is nonnegative and radially decreasing. These restrictions on ΔV turn out to be necessary for obtaining the preservation of radial monotonicity (see Proposition 1.4.3) as well as the mass comparison principle in Section 1.5.

The global wellposedness and finite time blow-up results for (1.1.1) has been well studied (see [B, H] for review articles), however the asymptotic and qualitative behaviors of solutions are not completely known. Below we briefly summarize the global existence/finite time blow-up criteria for (1.1.1).

When V is the regularized Newtonian potential (B), we have an *a priori* L^∞ bound of $\Delta(\rho * V)$ via the inequality $\|\Delta(\rho * V)\|_{L^\infty(\mathbb{R}^d)} \leq \|\rho\|_{L^1(\mathbb{R}^d)} \|\Delta V\|_{L^\infty(\mathbb{R}^d)}$. This suggests that $\|\rho(\cdot, t)\|_{L^\infty(\mathbb{R}^d)}$ can at most grow exponentially in time, hence we should expect the weak solution to exist globally in time. A rigorous proof of the global existence can be found in [BRB]. Moreover, for any mass size A, the existence of a stationary solution with mass A is proven in [L] and [B2], however it is unknown whether the stationary solution with mass A is unique.

When $V = \mathcal{N}$, the existence/blow-up criteria is more delicate due to the singularity of $\mathcal{N}$ at the origin. To study the well-posedness of (1.1.1), the following *free energy* functional

(1.1.5) plays an important role, where the first term is usually referred to as the *entropy* and the latter term is referred to as the *interaction energy*. When $m > 1$, the free energy is given by

$$\mathcal{F}(\rho) = \int_{\mathbb{R}^d} \left(\frac{1}{m-1}\rho^m + \frac{1}{2}\rho(\rho * \mathcal{N}) \right) dx, \tag{1.1.5}$$

while for $m = 1$ the first term in the integrand is replaced by $\rho \log \rho$. The free energy of a weak solution to (1.1.1) is non-increasing in time; indeed, it is shown that (1.1.1) is the gradient flow for $\mathcal{F}$ with respect to the 2-Wasserstein metric (see for example [AGS] and [CMV]).

To link the entropy term and the interaction term together, the key observation in [BCL2] is the sharp Hardy-Littlewood-Sobolev inequality, which bounds the interaction energy by the L^{m_c}-norm of ρ:

$$\left| \int_{\mathbb{R}^d} \rho(\rho * \mathcal{N}) dx \right| \leq C^* \|\rho\|_{L^1(\mathbb{R}^d)}^{2/d} \|\rho\|_{L_c^m(\mathbb{R}^d)}^{m_c}, \tag{1.1.6}$$

where C^* is a constant only depending on the dimension d, and $m_c := 2 - 2/d$. Therefore one should expect the solution exhibits different behavior for $m > m_c$ and $1 < m < m_c$. This is indeed the case, and the global existence and finite time blow-up results for different m are summarized below.

Supercritical regime: For $1 \leq m < 2 - 2/d$, the problem is *supercritical*: the diffusion is dominant at low concentrations and the aggregation is dominant at high concentration. As a result supercritical and critical problems with singular kernels may exhibit finite time blow-up phenomena [DP, HV, S1, BlCM]. On the other hand solutions globally exist with small mass and relatively regular initial data, and here the diffusion dominates at large length scale (see [C] and [S2]). Indeed using the entropy dissipation method similar to [CJMTU], it is shown that the solutions with small L^1 and $L^{(2-m)d/2}$- norms converge to the self-similar Barenblatt profile [LS1, LS2, B1].

Subcritical regime: On the other hand, in the *subcritical* regime ($m > 2 - 2/d$), the diffusion is dominant at high concentration. For this reason the weak solution exists globally in time regardless of its mass size [S1, BCL2, BRB]. Since aggregation dominates in low concentration, one can show that there are compactly supported stationary solutions for any mass size (see Proposition 1.2.1)). When $V = \mathcal{N}$, for any given mass size, uniqueness of radial stationary solution is proved in [LY] for the PKS model. However it was unknown whether this stationary solution is an attractor.

Critical regime: When $m = m_c$, the right hand side of the inequality (1.1.6) becomes $C^* \|\rho\|_{L^1(\mathbb{R}^d)}^{2/d} \|\rho\|_{L^m(\mathbb{R})^d}^m$, which is a multiple of the entropy, where the multiplication constant depends on m, d and the mass of ρ. This suggests that the behavior of the solution depends on its mass. Making use of the inequality (1.1.6), it is proved in [BCL2] (and generalized by [BRB] for more general kernels) that there exists a critical mass M_c only depending on d, which sharply divides the possibility of finite time blow up and global existence.

∘ *Critical mass:*

When the mass is equal to the critical mass M_c, it is proved in [BCL2] that weak solutions with bounded initial data exist globally in time. Moreover, they show the global minimizers of the free energy functional $\mathcal{F}$ have zero free energy, and are given by the one-parameter family

$$\eta_R(x) = \frac{1}{R^d}\eta_1(\frac{x}{R}) \tag{1.1.7}$$

subject to translations. Here η_1 is the unique radial classical solution to

$$\frac{m}{m-1}\Delta\eta_1^{m-1} + \eta_1 = 0 \text{ in } B(0,1), \quad \text{with } \eta_1 = 0 \text{ on } \partial B(0,1). \tag{1.1.8}$$

It was unknown that whether this family of stationary solutions attract some solutions.

∘ *Subcritical mass:*

When $0 < M < M_c$, the weak solution exists globally in time, as long as its initial L^m-norm is finite [BCL2, BRB]. Moreover it has been proved in [BCL2] that there exists a dissipating self-similar solution, with the same scaling as the porous medium equation. However it was unknown whether this self-similar solution would attract all solutions in the intermediate asymptotics.

∘ *Supercritical mass:* For every $M > M_c$, it is proved in [BCL2] that there exist some initial data of mass M such that the L^m-norm of the corresponding solution blows up in finite time. Their proof is based on the Virial identity, which only applies to the initial data whose free energy is negative. Recently, using mass comparison techniques, it is proved in [BK] that all radial solution with mass $M > M_c$ must blow up in finite time, regardless of its initial free energy.

Although the well-posedness and blow-up criteria of (1.1.1) are well-known, a lot of questions concerning the qualitative behavior of solutions remain to be answered, such as finite speed of propagation and asymptotic behavior of solutions. Here the difficulty lies in the lack of sufficient control on pointwise behavior of solution as well as the presence of the free boundary, which calls for new methods.

In this chapter we investigate qualitative and asymptotic behavior of solutions for (1.1.1) when the solution exists globally in time; for the cases that leads to a finite time blow-up, the asymptotic behavior for radial solutions will be studied in Chapter 2. The main tools in our analysis in this chapter are various types of comparison principles, together with energy argument. While maximum-principle type arguments are natural to parabolic PDEs, the classical maximum principle does not hold with (1.1.1) due to the nonlocal aggregation term, and therefore the standard comparison principle and the corresponding viscosity solutions theory do not apply. Instead we establish order-preserving properties of several associated quantities: the radial monotonicity (Section 1.4), the mass concentration (Section 1.5), and the rearranged mass concentration for non-radial solutions (Section 1.6)). Most results in this chapter come from a joint work with Inwon Kim [KY], while Section 1.7.4–Section 1.7.6 are taken from [Y]. Our main results are summarized below.

1.1.1 Summary of results

Let us begin with stating properties of radial stationary solutions of (1.1.1):

Theorem 1.1.1 (Properties of radial stationary solutions)**.** *Let V be given by (A) or (B) and let $m > 2 - \frac{2}{d}$. Let ρ_A be a non-negative radial stationary solution of* (1.1.1) *with $\int \rho_A(x)dx = A > 0$. Then* *(a) ρ_A is radially decreasing, compactly supported and smooth in its support (Proposition 1.2.1);*

(b) ρ_A is uniquely determined for any given A (Theorem 1.2.2 and Theorem 1.2.4).

When V is given by (A), the uniqueness of radial stationary solution comes from the well-known results of Lieb and Yau [LY]. Their proof is based on the fact that the mass function satisfies an ODE with uniqueness properties; this property fails when V is given by (B). Instead, we look at the dynamic equation (1.1.1), and prove uniqueness out of asymptotic convergence towards a stationary solution. A more direct proof of uniqueness and the uniqueness of general (possibly non-radial) stationary solutions are interesting open questions. We also mention a recent preprint [BDF], which studies another type of diffusion-aggregation equation: here authors use eigenvalue methods to prove the uniqueness of one-dimensional stationary solutions.

Next we show several results concerning the qualitative behavior of solutions, which will be used in the rest of the chapter:

Theorem 1.1.2 (Qualitative properties of solutions)**.** *Let V be given by (A) or (B), and assume $m > 1$. Let $\rho(x, t)$ be a weak solution to* (1.1.1)*, which is uniformly bounded in $\mathbb{R}^d \times [0, T)$. Then the following holds:*

(a) For any $\delta > 0$, ρ is uniformly continuous in $\mathbb{R}^d \times [\delta, T)$; (Theorem 1.3.1)

(b) [Finite propagation property] $\{\rho > 0\}$ expands over time period τ with maximal rate of $C\tau^{-1/2}$ (Theorem 1.3.1);

(c) If $\rho(\cdot, 0)$ is radial and radially decreasing, then so is $\rho(\cdot, t)$ for any $t \in [0, T)$ (Theorem 1.4.2).

Both properties (b) (the finite propagation property of the general solutions) and (c) (the preservation of radial monotonicity) are new, to the best of the authors' knowledge, for any type of diffusion-aggregation equation. For the first-order aggregation equation ((1.1.1) without the diffusion term), property (c) has been recently shown in [BGL] for the same class of potentials, via the method of characteristics.

We now turn to the discussion of asymptotic behavior of solutions.

Theorem 1.1.3 (Asymptotic behavior: subcritical regime)**.** *Let V be given by (A) or (B), $m > 2 - \frac{2}{d}$, and let $\rho(x,t)$ be the solution to* (1.1.1) *with initial data $\rho_0(x) \in L^1(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ which has mass $A > 0$. Then*

(a) If ρ_0 is radially symmetric and compactly supported, then the support of ρ, $\{\rho(\cdot,t) > 0\}$ stays inside of a large ball $\{|x| \leq R\}$ for all $t \geq 0$, where R depends on m, d, V and the initial data ρ_0 (Corollary 1.7.1);

(b) Let ρ_A be a radial stationary solution with mass A. Then If ρ_0 is radially symmetric and compactly supported, then ρ converges to ρ_A exponentially fast in p-Wasserstein distance for all $p > 1$ (Corollary 1.7.4), and $\|\rho(\cdot,t) - \rho_A\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $t \to \infty$ (Corollary 1.7.5).

(c) For every $0 < t < 1$ we have

$$\|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq c(m,d,A,V)t^{-\alpha},$$

where $\alpha := \frac{d}{d(m-1)+2}$. Note that c does not depend on $\|\rho(\cdot,0)\|_\infty$. (Proposition 1.7.7).

The proof of above theorem is based on the *mass comparison*, i.e. maximum-principle type arguments on the mass concentration of solutions (see Proposition 5.3). The mass comparison property have been previously observed for PKS models ([BKLN]; also see a recent preprint of [CLW]). However the property has not been fully taken advantage of, perhaps because of the success of entropy method for the Keller-Segel model.

Our method also provides interesting results for asymptotic behavior of radial and non-radial solutions in the supercritical regime, when the solution starts from sufficiently less concentrated initial data in comparison to a re-scaled stationary profile. (For the definition of "less concentrated than", see Definition 1.5.2) We point out that in our result the mass does not need to be small as required in previous literature (e.g. see [B1]), and provides an explicit description of solutions which are "sufficiently scattered" so that it does not blow up in finite time.

Theorem 1.1.4 (Asymptotic behavior: supercritical regime)**.** *Let $V(x)$ be given by (A) or (B), and suppose $1 < m < 2 - \frac{2}{d}$. Assume ρ_0 is radially symmetric, compactly supported and has mass A. Then there exists a sufficiently small constant $\delta > 0$ depending on d, m, A and V, such that if*

$$\rho_0(\lambda) \prec \delta^d \mu_A(\delta\lambda),$$

where "$\prec$" is defined in Definition 1.5.2 and $\mu_A(\lambda)$ is given in (1.7.22)*, then the weak solution ρ with initial data ρ_0 exists globally and converges to the Barenblatt profile in rescaled variables algebraically fast(Corollary 1.7.11).*

When V is the Newtonian potential, in the critical regime, by constructing explicit barriers in the mass comparison sense and using energy method, we obtain the following results for radial solutions of (1.1.1) with critical mass M_c:

Theorem 1.1.5. (Asymptotic behavior: critical regime with critical mass) *Suppose $V = \mathcal{N}$ and $m = 2-2/d$. Let $\rho(x,t)$ be the weak solution to* (1.1.1) *with initial data ρ_0, where ρ_0 has critical mass M_c. Assume that ρ_0 is continuous, radially symmetric, compactly supported, and satisfies $\nabla \rho_0^m \in L^2(\mathbb{R}^d)$. Then $\rho(\cdot,t) \to \rho_{R_0}$ in $L^\infty(\mathbb{R}^d)$ as $t \to \infty$ for some $R_0 > 0$, where ρ_{R_0} is a stationary solution defined in* (1.1.7)*. (Theorem 1.7.14)*

For radial solutions with subcritical mass, mass comparison again gives convergence towards the dissipating self-similar solution. For general (possibly non-radial) initial data, we use a maximum-principle type method to prove that when the mass is sufficiently small, every compactly supported stationary solution must be radially symmetric. This leads to the following asymptotic convergence result:

Theorem 1.1.6. (Asymptotic behavior: critical regime with subcritical mass) *Suppose $V = \mathcal{N}$ and $m = 2 - 2/d$. Let $\rho(x,t)$ be the weak solution to* (1.1.1) *with mass $A < M_c$, where the initial data ρ_0 is continuous and compactly supported. Then we have*

(a) *If $\rho(\cdot,0)$ is radially symmetric, then as $t \to \infty$, $\rho(\cdot,t)$ converges to the dissipating self-similar solution $\mathcal{U}_A$ as defined in* (1.7.41)*, where the Wasserstein distance between $\rho(\cdot,t)$ and $\mathcal{U}_A$ decays algebraically fast as $t \to \infty$. (Corollary 1.7.19)*

(b) *For general (non-radial) solutions, if the mass $A < M_c/2$ is sufficiently small, then*

$$\lim_{t\to\infty} \|\rho(\cdot,t) - \mathcal{U}_A\|_p = 0 \textit{ for all } 1 \le p \le \infty.$$

(Corollary 1.7.24)

1.2 Properties of the radially symmetric stationary solutions

In this section we consider non-negative radially symmetric stationary solutions of (1.1.1) in the subcritical regime (i.e. $m > 2 - 2/d$). Here the Euler-Lagrange equation is given by

$$\frac{m}{m-1}\rho^{m-1} + \rho * V = C \quad \text{ in } \{\rho > 0\}, \tag{1.2.1}$$

where the constant C may be different in different positive components of ρ. When V is given by (A) or (B), for any mass $A > 0$, the existence of a stationary solution ρ with mass A is proven in [B2], which is an application of [L]. We include the proof in Proposition 1.2.1 for the sake of completeness.

To investigate the qualitative property of the radial stationary solution, it is helpful to introduce the following *mass function*:

$$M(r) := \int_{B(0,r)} \rho(x)dx.$$

Since both ρ and V are radially symmetric, we may slightly abuse the notation and write $\rho * V$ as a function of r. When $V = \mathcal{N}$, due to the divergence theorem and radial symmetry of ρ and V, we readily obtain

$$\frac{\partial}{\partial r}(\rho * V)(r) = \frac{M(r)}{\sigma_d r^{d-1}}. \tag{1.2.2}$$

where σ_d is the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. Similarly, when V is given by (B), for all radially symmetric function ρ, we have that $\rho * V$ is radially symmetric, and

$$\frac{\partial}{\partial r}(\rho * V)(r) = \frac{\tilde{M}(r)}{\sigma_d r^{d-1}}, \tag{1.2.3}$$

where $\tilde{M}(r) := \int_{B(0,r)} \rho * \Delta V dx$. Note that in both cases, we have $\partial_r(\rho * V) \geq 0$.

Proposition 1.2.1. *Let V given by (A) or (B) and suppose $m > 2 - \frac{2}{d}$. Then there exists a radially symmetric, nonnegative solution $\rho \in L^1(\mathbb{R}^d)$ of (1.2.1). Moreover, (a) ρ is smooth in its positive set; (b) ρ is radially decreasing; and (c) ρ is compactly supported.*

Proof. 1. Existence of the stationary solution ρ follows from [L]: the proof is given in Section A.1.1 of the appendix.

2. To show (a) for $V = \mathcal{N}$, note that the right hand side of (1.2.2) is continuous since $f(r) := \dfrac{M(r)}{\sigma_d r^{d-1}}$ is continuous for all $r > 0$, and $f(r) \to 0$ as $r \to 0$. By (1.2.2), $\rho * V$ is differentiable in the positive set of ρ, which implies that ρ^{m-1} (hence ρ) is also differentiable in the positive set of ρ. Therefore $\dfrac{M(r)}{r^{d-1}}$ is now twice differentiable, hence we can repeat this argument and conclude. When V is given by (B), we can apply the same argument on (1.2.3) and conclude.

3. By differentiating (1.2.1) we have

$$\frac{m}{m-1}\frac{\partial}{\partial r}\rho^{m-1} = -\frac{\partial}{\partial r}(\rho * V) \quad \text{ in } \{\rho > 0\}. \tag{1.2.4}$$

Due to (1.2.2)-(1.2.3) the right hand side of (1.2.4) is negative, and thus we conclude (b).

4. It remains to check (c). Note that (b) yields that ρ has simply connected support. Hence (1.2.1) yields

$$\rho(r) = (C - \rho * V(r))^{\frac{1}{m-1}}.$$

When $V = \mathcal{N}$ the proof is similar to that of Theorem 5 in [LY]: since $\rho * V$ vanishes at infinity, we have

$$\rho * V(r) = -\int_r^\infty \frac{M(s)}{s^{d-1}} ds = -\frac{M(r)}{(d-2)r^{d-2}} - \int_r^\infty \frac{c_d}{d-2}\rho(s) s ds, \tag{1.2.5}$$

where c_d is the volume of a ball with radius 1 in $\mathbb{R}^d$. Note that

$$\rho * V(r) \leq 0 \quad \text{ and } \quad -\rho * V(r) \sim \frac{1}{r^{d-2}} \text{ as } r \to \infty. \tag{1.2.6}$$

If $C = 0$, (1.2.6) implies that

$$\rho(r) = (-\rho * V(r))^{\frac{1}{m-1}} \sim r^{-\frac{d-2}{m-1}},$$

where the exponent is greater than $-d$ when $m > 2 - \frac{2}{d}$, which contradicts the finite mass property of ρ. Therefore C must be negative and thus $\rho(r)$ needs to touch zero for some r.

When $V = \mathcal{N} * h$, we have $\rho * V = (\rho * \mathcal{N}) * h$. Since $h \in L^1(\mathbb{R}^d)$ and is radially decreasing, using (1.2.5) we have $\rho * V(x) \sim \dfrac{1}{|x|^{d-2}}$ as $|x| \to \infty$ as well, hence (c) follows from the same argument as above. $\square$

Next we state the uniqueness of the radial stationary solution when $V = \mathcal{N}$.

Theorem 1.2.2 ([LY]). *Let $V = \mathcal{N}$, and suppose $m > 2 - \frac{2}{d}$. Then for all choices of mass $A > 0$, the radial stationary solution for* (1.1.1) *with mass A is unique. Moreover, the stationary solution is the global minimizer for the free energy functional* (1.1.5)*.*

This theorem follows from a minor modification from the proof of Theorem 5 in [LY], which proves uniqueness of the stationary solution of a slightly different problem. Their proof consists of two steps: for a given mass, they first show the global minimizer of (1.1.5) is unique, and secondly they prove every radial stationary solution is a global minimizer for some mass.

We slightly digress here and state the following corollary, which is an immediate consequence of Theorem 1.2.2 and the homogeneity of $\mathcal{N}$, hence the proof will be omitted.

Corollary 1.2.3. *Let V and m be as in Theorem 1.2.2, and let ρ_M be the radial solution of* (1.2.1) *with mass M. Then*

$$\rho_M(x) = a\rho_1(a^{-\frac{m-2}{2}}x) \text{ with } a := M^{\frac{2}{d(m-2+2/d))}}. \tag{1.2.7}$$

In particular if $A < B$ then $\max \rho_B \geq \max \rho_A$ and the following dichotomy of behavior is observed: (see Figure 1.1)

(a) When $m \geq 2$, $\{\rho_A > 0\} \subseteq \{\rho_B > 0\}$.

(b) When $2 - \frac{2}{d} < m \leq 2$, $\{\rho_B > 0\} \subseteq \{\rho_A > 0\}$.

Coming back to the uniqueness of stationary solution, we point out that when V is given by (B), the uniqueness proof in [LY] cannot be generalized, and here the difficulty lies in

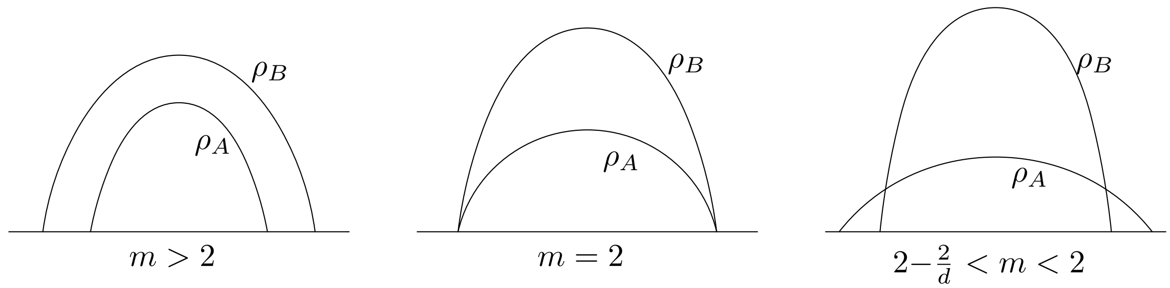


Figure 1.1: Stationary solutions with different mass for different m, where $\int \rho_A dx < \int \rho_B dx$.

the second step: In the case where $V = \mathcal{N}$, for any radial stationary solution ρ, its mass function $M(r) := \int_{|x|\le r} \rho(x)dx$ solves a second order ODE

$$\Big(\frac{m}{m-1}\big(\frac{M'(r)}{\sigma_d r^{d-1}}\big)^{m-1}\Big)' = \frac{M(r)}{\sigma_d r^{d-1}},$$

where $M(0) = 0$ is prescribed. It follows that $M(r)$ is unique for a given $\rho(0) = \lim_{r\to 0} \frac{M'(r)}{\sigma_d r^{d-1}}$, which implies that ρ can be uniquely determined by $\rho(0)$. This property then allows both the radial stationary solutions and the global minimizers to be parametrized by their values at the center of mass (see Lemma 12, [LY]).

However, when V is given by (B), $M(r)$ solves a nonlocal ODE, where different stationary solutions may have the same center density. Thus the above argument in [LY] cannot be applied to prove the second step, necessitating an alternative approach. Instead of dealing with the stationary equation (1.2.1) directly, we will consider the dynamic equation (1.1.1) and prove the uniqueness of the radial stationary solution by their asymptotic convergence. Indeed the following theorem is one of the main results in this chapter.

Theorem 1.2.4. *[Corollary 1.7.6] Let V be given by (B), and suppose $m > 2 - \frac{2}{d}$. Then for any $A > 0$, the radial stationary solution of* (1.1.1) *with mass A is unique.*

1.3 Regularity of solutions and finite speed of propagation

In this section, several regularity properties will be derived for general weak solutions of (1.1.1), including the finite propagation property. We point out that the results in this section hold for general (non-radial) solutions.

Theorem 1.3.1. *Suppose $m > 1$. Let V be given by (A) or (B), and let ρ be a weak solution of (1.1) with its initial data ρ_0 bounded with compact support. Further suppose ρ is uniformly bounded in $\mathbb{R}^d \times [0, T]$. Then*

(a) For any $\delta > 0$, ρ is uniformly continuous in $\mathbb{R}^d \times (\delta, T]$.

(b) [Finite propagation property] Suppose $\{x : \rho(\cdot, t) > 0\} \subset B_R(0)$. Then

$$\{x : \rho(\cdot, t+h) > 0\} \subset B_{R+Ch^{1/2}}(0) \text{ for } 0 < h < 1,$$

where the constant $C > 0$ depends on m, d, ρ_0 and $\|\Delta V\|_1$.

Proof. 1. Let us consider the case $V = \mathcal{N}$. This is the most singular case and parallel (and easier) arguments hold for V given by (B). Let

$$C_0 = \sup\{\rho(x, t) : (x, t) \in \mathbb{R}^n \times [0, T]\}.$$

Observe that by treating the convolution term $\Phi := V * \rho$ as *a priori* given, ρ solves

$$\rho_t = \Delta(\rho^m) + \nabla \cdot (\rho \nabla \Phi). \tag{1.3.1}$$

Also, for all $t \in [0, T)$, Φ satisfies

$$|\nabla\Phi|(\cdot, t) \leq C_0 \int_{|y|\leq 1} |\nabla\mathcal{N}|(y)dy + \|\rho(\cdot, t)\|_1 \sup_{|y|\geq 1} |\nabla\mathcal{N}(y)| \leq C_1, \tag{1.3.2}$$

where C_1 depends on C_0, the L^1 and sup-norm of ρ, and the dimension d. Also

$$|\Delta\Phi|(\cdot, t) \leq \|\rho\|_{L^\infty} \leq C_0 \quad \text{for all } t \in [0, T). \tag{1.3.3}$$

The bounds (1.3.2)-(1.3.3) and Theorem 6.1 of [D] yields the uniform continuity of ρ in $\mathbb{R}^d \times [\delta, T)$.

2. Next we prove (b). First of all let us point out that the standard comparison principle holds between weak sub- and supersolutions of (1.3.1). For the case of time-independent potential $\Phi(x, t) \equiv \Phi(x)$, Proposition 3.4 of [BH] asserts that the comparison principle between weak sub- and supersolution of (1.3.1) holds if the potential function Φ is independent of the time variable and $|\nabla\Phi|, |\Delta\Phi| \leq C$. The proof in [BH] is based on an approximation

of the original problem (1.3.1) by a sequence of regularized problems which satisfy the comparison principle (see sections 4, 7, 8 of [BH]). This argument straightforwardly extends to our (time-dependent potential) case, and one can verify that comparison principle between weak sub- and supersolution of (1.3.1) holds.

We will now construct a supersolution of (1.3.1) to compare with ρ over a small time period to prove the finite propagation property. First observe that the pressure function defined by $u := \frac{m}{m-1}\rho^{m-1}$ formally satisfies the PDE

$$u_t = (m-1)u\Delta u + |\nabla u|^2 + \nabla u \cdot \nabla \Phi + (m-1)u\nabla \Phi. \tag{1.3.4}$$

Based on this observation, we will first construct a supersolution of (1.3.4), and use the pressure-density transformation to construct the corresponding supersolution of (1.3.1). Let us define

$$\tilde{U}(x,t) := A \inf_{|x-y|\leq C-Ct} e^{-Ct}(|y| + \omega t - B)_+,$$

where $\omega = (1 + (m-1)(d-1))A$, and the constants B and C will be chosen later.

Let $\Sigma := \{|x| \leq 2B\} \times [0, \omega^{-1}B]$. Due to Proposition 2.13 in [KL], $\tilde{U}$ is a viscosity (or weak) supersolution of (1.3.4) if C is chosen to be larger than $\max(C_0, C_1)$ given in (1.3.2)-(1.3.3). In other words, $\tilde{U}$ satisfies

$$\tilde{U}_t \geq (m-1)\tilde{U}\Delta\tilde{U} + |\nabla\tilde{U}|^2 + C|\nabla\tilde{U}| + C\tilde{U} \quad \text{ in } \{\tilde{U} > 0\} \cap \Sigma,$$

and the outward normal velocity $V_{x,t}$ of the set $\{\tilde{U} > 0\}$ at $(x,t) \in \partial\{\tilde{U} > 0\}$ satisfies

$$V_{x,t} = \omega + C \geq A + C \geq |\nabla\tilde{U}| + C.$$

Hence $\tilde{\rho} := (\frac{m-1}{m}\tilde{U})^{1/(m-1)}$ satisfies

$$\tilde{\rho}_t \geq \Delta(\tilde{\rho}^m) + C|\nabla\tilde{\rho}| + \frac{C}{m-1}\tilde{\rho}$$

in the domain Σ, in the viscosity sense (see [KL] for the definition of viscosity solutions of (1.3.1)).

Moreover, observe that $\tilde{\rho}^{m-1} \sim \tilde{U}$ is Lipschitz continuous in space, and continuous in space and time. Using this regularity of $\tilde{\rho}$ as well as the above estimates on the derivatives

of Φ, it follows that $\tilde{\rho}$ is a weak supersolution of (1.1.1) in Σ, if we choose C greater than $(m-1)C_1$. More precisely the following is true: for all times $0 < t \leq \omega^{-1}B$ and for any smooth, nonnegative function
$\psi(x,t) : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ with $\{\psi(\cdot,t) > 0\} \subset \{|x| \leq 2B\}$ for $0 \leq t \leq \omega^{-1}B$, we have

$$\int \tilde{\rho}(\cdot,t)\psi(\cdot,t)dx \geq \int \tilde{\rho}(\cdot,0)\psi(\cdot,0)dx + \int\int (\tilde{\rho}^m \Delta\psi + \tilde{\rho}\psi_t - \tilde{\rho}\nabla\Phi \cdot \nabla\psi)dxdt.$$

Now suppose $\{\rho(\cdot,t_0) > 0\} \subset B_R(0)$ for some $t_0 \in [0,T]$. Let us compare ρ with $\tilde{\rho}$ in $\Sigma := \{|x| \leq 2B\} \times [t_0, t_0+h]$, with $B = R + h^{1/2}$ and $A = 2C_0 h^{-1/2}$.

Since $\{\rho(\cdot,t_0) > 0\} \subset B_R(0)$ and $\rho \leq C_0$, we have $\rho \leq \tilde{\rho}$ at $t = t_0$. Therefore comparison principle asserts that $\rho \leq \tilde{\rho}$ at $t = t_0 + h$. In particular

$$\{\rho(\cdot,t_0+h) > 0\} \subset \{\tilde{\rho}(\cdot,t_0+h) > 0\} = B_{R+Mh^{1/2}}(0),$$

where $M = h^{1/2} + h\omega$ and $\omega = (1+(m-1)(d-1))A$. This proves (b). □

Remark 1.3.2. Due to [BRB], when $m > 2 - \frac{2}{d}$, there exists a global weak solution ρ of (1.1) with initial data ρ_0. Moreover, ρ is uniformly bounded in $\mathbb{R}^d \times (0,\infty)$ due to Theorem 10 in [BRB], so in that case we may let $T = \infty$.

We finish this section with an approximation lemma which links case (A) and (B). Let

$$h^\epsilon := \epsilon^{-d} h(\frac{x}{\epsilon})$$

with h being the standard mollifier in $\mathbb{R}^d$ with unit mass, and let ρ^ϵ be the corresponding solution of (1.1.1) with $V = \mathcal{N} * h^\epsilon$ and with initial data ρ_0. Then Lemma 8 in [BRB] yields that $\{\rho^\epsilon\}_{\epsilon>0}$ are uniformly bounded for $t \in [0,T]$ for some T. This bound as well as Theorem 6.1 of [D] yields that the family of solutions $\{\rho^\epsilon\}$ are equi-continuous in space and time. This immediately yields the following result:

Proposition 1.3.3. *Let ρ_0 be as given in Theorem 1.3.1. Let $V = \mathcal{N} * h^\epsilon$ and let ρ^ϵ be the corresponding weak solution of* (1.1.1) *with initial data ρ_0. Let ρ be the unique solution to* (1.1.1) *with $V = \mathcal{N}$ and initial data ρ_0, and assume ρ exists for $t \in [0,T)$, where $T > 0$ may be infinite. Then the solutions ρ^ϵ locally uniformly converge to ρ in $\mathbb{R}^d \times [0,T)$.*

1.4 Monotonicity-preserving property for radial solutions

In this section, we show that when V is given by (A) or (B), solutions with radially decreasing initial data remains radially decreasing for all future times. (Here we say a function $u : \mathbb{R}^d \to \mathbb{R}$ is *radially decreasing* if $u(x)$ is radially symmetric and is a decreasing function of $|x|$.) The main step in the proof is the maximum-principle type argument applied to the double-variable function

$$\Psi(x, y; t) := \rho(x, t) - \rho(y, t) \text{ in } \{|x| \geq |y|\} \times [0, \infty)$$

to ensure that Ψ cannot achieve a positive maximum at a positive time.

We begin with an observation on the convolution term; the proof is in Section A.1.2 of the appendix.

Lemma 1.4.1. *Let $V(x)$ be given by (B). Let $u(x)$ be a bounded non-negative radially symmetric function in $\mathbb{R}^d$ with compact support. Further suppose $u(x)$ is* <u>not</u> *radially decreasing, i.e. there exists $a_1 = (\alpha, 0, ..., 0)$ and $b_1 = (\beta, 0, ..., 0)$ with $\alpha, \beta > 0$ such that*

$$u(b_1) - u(a_1) = \sup_{|a|<|b|} u(b) - u(a) > 0. \tag{1.4.1}$$

Then we have

$$(u * \Delta V)(b_1) - (u * \Delta V)(a_1) \leq \|\Delta V\|_{L^1(\mathbb{R}^d)}(u(b_1) - u(a_1)).$$

Theorem 1.4.2. *Let $V(x)$ be given by (A) or (B). Suppose that the initial data $\rho(x, 0) \in L^1(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ is* radially decreasing. *We assume a weak solution ρ exists for $t \in [0, T)$, where T may be infinite. Then $\rho(x, t)$ is radially decreasing for all $t \in [0, T)$.*

Proof. 1. Without loss of generality we assume that V is given by (B), and $\rho(x, 0)$ is positive and smooth. Then a classical solution $\rho(\cdot, t)$ exists for all $t \geq 0$, and we want to show $\rho(\cdot, t)$ is radially decreasing for all $t \geq 0$. When $V = \mathcal{N}$, we can use mollified Newtonian kernel to approximate $\mathcal{N}$; and for general radially decreasing initial data, we can use positive and smooth functions to approximate $\rho(x, 0)$. Then the result follows via Proposition 1.3.3.

2. Radial symmetry of ρ for all $t > 0$ directly follows from the uniqueness of weak solution. To prove that ρ is radially decreasing for all time, let us define

$$w(t) := \sup_{|a|\leq|b|} \rho(b,t) - \rho(a,t).$$

Note that $\rho(x,0)$ being radially decreasing is equivalent with $w(0) = 0$. We will use a maximum-principle type argument to show that $w(t) = 0$ for all $t \geq 0$, which proves the theorem. We point out that $w(t)$ is continuous in t, and uniformly bounded for $t \in [0,\infty)$, since ρ is uniformly bounded and uniformly continuous in $\mathbb{R}^d \times [0,\infty)$.

Suppose $w \not\equiv 0$ in $\mathbb{R}^d \times [0,\infty)$. Then for any $\lambda > 0$ the function $w(t)e^{-\lambda t}$ has a positive maximum at $t = t_1$ for some $t_1 > 0$. We will show that this cannot happen when we choose $\lambda > 2\|\rho\|_{L^\infty}\|\Delta V\|_{L^1}$.

At $t = t_1$, due to our assumption on $w(t)$, there exists $a_1 = (\alpha, 0, ..., 0)$ and $b_1 = (\beta, 0, ..., 0)$ such that $\alpha < \beta$ and

$$\rho(b_1,t_1) - \rho(a_1,t_1) = w(t_1) > 0. \quad \text{(See Figure 1.2)} \tag{1.4.2}$$

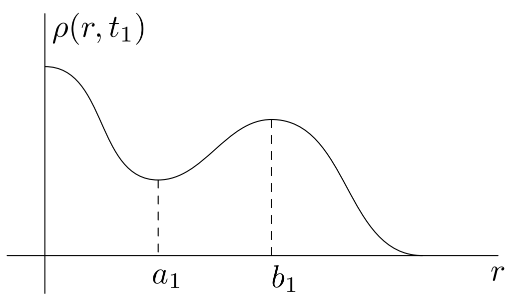


Figure 1.2: Graph of ρ at time t_1

Moreover by definition $\rho(b_1,t) - \rho(a_1,t) \leq w(t)$, and thus

$$\frac{d}{dt}((\rho(b_1,t) - \rho(a_1,t))e^{-\lambda t}) = 0 \quad \text{at } t = t_1,$$

which means

$$\rho_t(b_1,t_1) - \rho_t(a_1,t_1) = \lambda(\rho(b_1,t_1) - \rho(a_1,t_1)). \tag{1.4.3}$$

Further observe that $\rho(\cdot,t_1)$ has a local minimum (in space only) at a_1 and a local maximum at b_1. This yields

$$\nabla\rho(a_1,t_1) = 0 \ \text{ and } \ \nabla\rho(b_1,t_1) = 0,$$

as well as

$$\Delta\rho^m(a_1,t_1) \geq 0 \quad \text{and} \quad \Delta\rho^m(b_1,t_1) \leq 0.$$

To get a contradiction, recall that ρ is a classical solution of (1.1.1), which yields

$$\begin{aligned}
\rho_t(b_1,t_1) - \rho_t(a_1,t_1) &= \Delta\rho^m(b_1,t_1) + \nabla\cdot(\rho\nabla(\rho * V))(b_1,t_1) \\
&\quad -\Delta\rho^m(a_1,t_1) - \nabla\cdot(\rho\nabla(\rho * V))(a_1,t_1) \\
&\leq \rho(b_1,t_1)(\rho * \Delta V)(b_1,t_1) - \rho(a_1,t_1)(\rho * \Delta V)(a_1,t_1) \\
&= \rho(b_1,t_1)[(\rho * \Delta V)(b_1,t_1) - (\rho * \Delta V)(a_1,t_1)] \qquad (1.4.4)\\
&\quad +(\rho(b_1,t_1) - \rho(a_1,t_1))(\rho * \Delta V)(a_1,t_1).
\end{aligned}$$

In order to bound the first term in (1.4.4), we apply Lemma 1.4.1,which gives

$$(\rho * \Delta V)(b_1,t_1) - (\rho * \Delta V)(a_1,t_1) \leq \|\Delta V\|_{L^1(\mathbb{R}^d)}(\rho(b_1,t_1) - \rho(a_1,t_1)), \qquad (1.4.5)$$

and for the second term we use

$$(\rho * \Delta V)(a_1,t_1) \leq \|\rho\|_{L^\infty(\mathbb{R}^d)}\|\Delta V\|_{L^1(\mathbb{R}^d)}. \qquad (1.4.6)$$

Due to the estimates (1.4.5)-(1.4.6), (1.4.4) yields that

$$\rho_t(b_1,t_1) - \rho_t(a_1,t_1) \leq 2\|\rho\|_{L^\infty(\mathbb{R}^d)}\|\Delta V\|_{L^1(\mathbb{R}^d)}\big(\rho(b_1,t_1) - \rho(a_1,t_1)\big),$$

which contradicts (1.4.3) since we chose λ to be strictly greater than $2\|\rho\|_{L^\infty}\|\Delta V\|_{L^1}$. □

The following proposition states that in the previous theorem, the condition that ΔV is radially decreasing is indeed necessary.

Proposition 1.4.3. *Let $V(x)$ be radially symmetric, $\Delta V \geq 0$, with ΔV continuous, but not radially decreasing. Then there exists a radially decreasing initial data ρ_0 such that the solution $\rho(x,t)$ of (1.1) starting with initial data ρ_0 does not preserve the radial monotonicity over time.*

Proof. Since ΔV is not radially decreasing, we can find $x_1, x_2 \in \mathbb{R}^d$, such that $0 < |x_1| < |x_2|$, and $\Delta V(x_1) < \Delta V(x_2)$.

For a small $\epsilon > 0$, let $\rho_0(x)$ be given as below:

$$\rho_0(x) = \epsilon \chi_{B(0,x_2+1)} * \phi(x) + \frac{1}{\epsilon^d}\phi(\frac{x}{\epsilon}),$$

where χ_E is the characteristic function of E and ϕ is a radially symmetric mollifier with unit mass and supported in $B(0, r_0)$, where $r_0 < \min\{1, |x_1|/2\}$. Note that in a small space-time neighborhood of x_1 and x_2, ρ solves a uniformly parabolic equation, and thus is smooth.

Since $\Delta\rho^m(x_i, 0) = \nabla\rho(x_i, 0) = 0$ for $i = 1, 2$, we have

$$\rho_t(x_i, 0) = \rho_0(x_i)(\rho_0 * \Delta V)(x_i), \quad i = 1, 2.$$

Since $\rho_0(x_1) = \rho_0(x_2)$, to show $\rho_t(x_1, 0) < \rho_t(x_2, 0)$, it suffices to prove

$$(\rho_0 * \Delta V)(x_1) < (\rho_0 * \Delta V)(x_2). \tag{1.4.7}$$

Note that $\rho_0 * \Delta V$ locally uniformly converges to $\Delta V(x)$ as $\epsilon \to 0$. Since $\Delta V(x_1) < \Delta V(x_2)$, if we let ϵ be sufficiently small, we would have (1.4.7). In particular $\rho(x_1, t) < \rho(x_2, t)$ for small $t > 0$, which means $\rho(x, t)$ stops being radially monotone as soon as $t > 0$. □

1.5 Mass Comparison for radial solutions

In this section our goal is to prove that *mass comparison* property holds for radial solutions of (1.1.1), which is a order preserving property for the mass concentration, and will be defined momentarily. Indeed we will prove this property for (1.5.1), which is slightly more general than (1.1.1) since it could have an extra drift term. This extra term will be useful later in Section 1.7.6 since performing continuous rescaling for (1.1.1) usually introduces an extra drift term.

Let us consider the following PDE

$$\rho_t = \Delta\rho^m + \nabla \cdot (\rho\nabla(\rho * V + \Phi)), \tag{1.5.1}$$

Throughout this section we assume $m > 1$, V is given by (A) or (B), and $\Phi \in C^2(\mathbb{R}^d)$ is radially symmetric.

For a radially symmetric function $\rho(x,t)$, we define its mass function $M(r,t;\rho)$ by

$$M(r,t;\rho) := \int_{B(0,r)} \rho(x,t)dt, \tag{1.5.2}$$

and we may write it as $M(r,t)$ if the dependence on the function ρ is clear. The following lemma describes the PDE formally satisfied by the mass function.

Lemma 1.5.1 (Evolution of Mass Function)**.** *Let $\rho(x,t)$ be a non-negative smooth radially symmetric solution to* (1.5.1)*. Let $M(r,t) = M(r,t;\rho)$ be as defined in* (1.5.2)*. Then $M(r,t)$ satisfies*

$$\frac{\partial M}{\partial t} = \sigma_d r^{d-1} \partial_r \big(\frac{\partial_r M}{\sigma_d r^{d-1}}\big)^m + \partial_r M \frac{\tilde{M}}{\sigma_d r^{d-1}} + \partial_r M \partial_r \Phi, \tag{1.5.3}$$

where

$$\tilde{M}(r,t;\rho) := \int_{B(0,r)} \rho * \Delta V dx \tag{1.5.4}$$

if V is given by (B). When $V = \mathcal{N}$, $\tilde{M}(r,t;\rho)$ is set to coincide with M.

Proof. Due to divergence theorem and radial symmetry of ρ, direct computation yields

$$\frac{\partial M}{\partial t} = \sigma_d r^{d-1} \big[\partial_r \rho^m + \rho\big(\partial_r(\rho * V) + \partial_r \Phi\big)\big]. \tag{1.5.5}$$

Note that radial symmetry of ρ also gives

$$\rho(r) = \frac{\partial_r M}{\sigma_d r^{d-1}}, \tag{1.5.6}$$

from which we obtain the first term of (1.5.3).

It remains to write $\partial_r(\rho * V)$ in terms of M. When V is given by (A), i.e. $V = \mathcal{N}$, divergence theorem yields

$$\partial_r(\rho * \mathcal{N}) = \frac{\int_{B(0,r)} (\Delta\rho * \mathcal{N}) dx}{\sigma_d r^{d-1}} = \frac{M(r,t)}{\sigma_d r^{d-1}}. \tag{1.5.7}$$

When V is given by (B), we can similarly obtain

$$\partial_r(u * V) = \frac{\int_{B(0,r)} (u * \Delta V) dx}{\sigma_d r^{d-1}} = \frac{\tilde{M}(r,t)}{\sigma_d r^{d-1}}, \tag{1.5.8}$$

where $\tilde{M}(r,t;u)$ is as defined in (1.5.4). Plug (1.5.6), (1.5.7) and (1.5.8) into equation (1.5.5), and then we can obtain (1.5.3). □

Definition 1.5.2. *Let ρ_1 and ρ_2 be two non-negative radially symmetric functions in $L^1(\mathbb{R}^d)$. We say ρ_1 is* less concentrated than *ρ_2, or $\rho_1 \prec \rho_2$, if*

$$\int_{B(0,r)} \rho_1(x)dx \leq \int_{B(0,r)} \rho_2(x)dx \quad \text{for all } r \geq 0.$$

Definition 1.5.3. *Let $\rho_1(x,t)$ be a non-negative, radially symmetric function in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, which is C^1 in its positive set. We say ρ_1 is a* supersolution of (1.5.1) in the mass comparison sense *if $M_1(r,t) := M(r,t;\rho_1)$ is a supersolution of (1.5.3), i.e. $M_1(r,t)$ and $\tilde{M}_1(r,t) := \tilde{M}(r,t;\rho_1)$ satisfy*

$$\frac{\partial M_1}{\partial t} \geq \sigma_d r^{d-1}\partial_r\Big(\frac{\partial_r M_1}{\sigma_d r^{d-1}}\Big)^m + \partial_r M_1 \frac{\tilde{M}_1}{\sigma_d r^{d-1}} + \partial_r M_1 \partial_r \Phi, \tag{1.5.9}$$

in the positive set of ρ_1.

Similarly we can define a subsolution *of* (1.5.1) *in the mass comparison sense.*

Proposition 1.5.4 (mass comparison). *Suppose $m > 1$, V is given by (A) or (B), and $\Phi \in C^2(\mathbb{R}^d)$ is radially symmetric. Let $\rho_1(x,t)$ be a supersolution and $\rho_2(x,t)$ be a subsolution of* (1.5.1) *in the mass comparison sense for $t \in [0,T]$. Further assume that both ρ_i are bounded, and ρ_i's preserve their mass over time, i.e., $\int \rho_1(\cdot,t)dx$ and $\int \rho_2(\cdot,t)dx$ stay constant for all $0 \leq t \leq T$. Then their mass functions are ordered for all times: i.e., if $\rho_1(x,0) \succ \rho_2(x,0)$, then we have $\rho_1(x,t) \succ \rho_2(x,t)$ for all $t \in [0,T]$.*

Proof. Let $M_i(r,t)$ be the mass function for ρ_i, where $i = 1,2$. We claim that $M_1(r,t) \geq M_2(r,t)$ for all $r \geq 0$ and $t \in [0,T]$, which proves the proposition.

For the boundary conditions of M_i, note that

$$\begin{cases} M_1(0,t) = M_2(0,t) = 0 \quad \text{for all } t \in [0,T], \\ \lim_{r\to\infty} \big(M_1(r,t) - M_2(r,t)\big) = \int_{\mathbb{R}^d} \big(\rho_1(x,0) - \rho_2(x,0)\big)dx \geq 0 \quad \text{for all } t \in [0,T]. \end{cases}$$

As for initial data, we have $M_1(r,0) \geq M_2(r,0)$ for all $r \geq 0$.

For given $\lambda > 0$, we define

$$w(r,t) := \big(M_2(r,t) - M_1(r,t)\big)e^{-\lambda t},$$

where λ is a large constant to be determined later. Suppose the claim is false, then w attains a positive maximum at some point (r_1, t_1) in the domain $(0,\infty) \times (0,T]$. Moreover, since the mass of both ρ_1 and ρ_2 are preserved over time and thus are ordered, we know that (r_1, t_1) must lie inside the positive set for both ρ_1 and ρ_2, where M_i's are $C^{2,1}_{x,t}$.

Since w attains a maximum at (r_1, t_1), the following inequalities hold at (r_1, t_1):

$$w_t \geq 0 \implies \partial_t(M_2 - M_1) \geq \lambda(M_2 - M_1) \tag{1.5.10}$$

$$w_r = 0 \implies \partial_r M_1 = \partial_r M_2 > 0 \tag{1.5.11}$$

$$w_{rr} \leq 0 \implies \partial_{rr} M_1 \geq \partial_{rr} M_2 \tag{1.5.12}$$

Now we will analyze the terms on the right hand side of (1.5.9) one by one. For the first term, (1.5.11) and (1.5.12) imply that

$$\partial_r\Big(\frac{\partial_r M_2}{\sigma_d r^{d-1}}\Big)^m - \partial_r\Big(\frac{\partial_r M_1}{\sigma_d r^{d-1}}\Big)^m \leq 0 \quad \text{ at } (r_1, t_1). \tag{1.5.13}$$

If V is given by the Newtonian potential, we have

$$\partial_r M_1 \frac{(\tilde{M}_2 - \tilde{M}_1)}{\sigma_d r^{d-1}} = \rho_1(r_1,t_1)(M_2 - M_1) \leq \rho_{\max}(M_2 - M_1), \tag{1.5.14}$$

where $\rho_{\max} := \max\{\sup_{\mathbb{R}^d\times[0,T]} \rho_1, \sup_{\mathbb{R}^d\times[0,T]} \rho_2\}$ is finite by assumption on ρ_1 and ρ_2.

Alternatively, if V is given by (B), we claim the following inequality holds:

$$\partial_r M_1 \frac{(\tilde{M}_2 - \tilde{M}_1)(r_1,t_1)}{\sigma_d r^{d-1}} \leq \rho_{\max}\|\Delta V\|_{L^1(\mathbb{R}^d)}(M_2 - M_1)(r_1,t_1). \tag{1.5.15}$$

To prove the claim, note that $\tilde{M}_2 - \tilde{M}_1$ can be rewritten as

$$\begin{aligned} \tilde{M}_2(r_1,t_1) - \tilde{M}_1(r_1,t_1) &= \int_{\mathbb{R}^d} \big((\rho_2 - \rho_1) * \Delta V\big)\, \chi_{B(0,r_1)} dx \\ &= \int_{\mathbb{R}^d} (\rho_2 - \rho_1)\, (\chi_{B(0,r_1)} * \Delta V) dx. \end{aligned} \tag{1.5.16}$$

Note that $\Delta V \geq 0$ is radially decreasing due to assumption (B), thus $\chi_{B(0,r_1)} * \Delta V$ is non-negative, radially decreasing and has maximum less than or equal to $\|\Delta V\|_{L^1}$. Therefore we can use a sum of bump functions to approximate $\chi_{B(0,r_1)} * \Delta V$, where the sum of the heights is less than $\|\Delta V\|_{L^1}$. Hence

$$\tilde{M}_2(r_1,t_1) - \tilde{M}_1(r_1,t_1) \leq \|\Delta V\|_{L^1(\mathbb{R}^d)} \sup_{x \in \mathbb{R}^d} (M_2 - M_1)(x,t_1) = \|\Delta V\|_{L^1(\mathbb{R}^d)} (M_2 - M_1)(r_1,t_1),$$

which proves the claim (1.5.15). Finally, for the last term in (1.5.9), equation (1.5.11) immediately yields that

$$\partial_r M_2 \, \partial\Phi - \partial_r M_1 \, \partial\Phi = 0 \text{ at } (r_1,t_1). \tag{1.5.17}$$

Now we subtract (1.5.9) with the corresponding equation for the subsolution. Due to the inequalities (1.5.13), (1.5.15) and (1.5.17), we obtain the following inequality when V is given by (B):

$$\partial_t(M_2 - M_1) \leq \rho_{\max}\|\Delta V\|_1 (M_2 - M_1),$$

where $\|\Delta V\|_1$ is replaced by 1 when V is Newtonian.

Hence if we choose $\lambda > \rho_{\max}\|\Delta V\|_1$ in the beginning of the proof, the inequality above will contradict (1.5.10). □

Next we prove a simple corollary of Proposition 1.5.4, which we will use later in proving the asymptotic behavior of radial solutions in the subcritical regime. In the absence of the extra drift term, observe that (1.1.1) can be written as a transport equation

$$\rho_t + \nabla \cdot (\rho \vec{v}) = 0,$$

where the *velocity field* $\vec{v}$ is defined by

$$\vec{v}(x,t;\rho) := -\frac{m}{m-1}\nabla(\rho^{m-1}) - \nabla(\rho * V). \tag{1.5.18}$$

Then the mass function for a radial solution of (1.1.1) satisfies

$$\frac{\partial}{\partial t} M(r,t) = -\rho(r,t) \int_{\partial B(0,r)} (\vec{v} \cdot \vec{n}) ds. \tag{1.5.19}$$

The above observation along with Proposition 1.5.4 immediately yields the following corollary:

Corollary 1.5.5. *Suppose $m > 1$. Let V be given by (A) or (B). Let $\rho_0(x)$ be a continuous radially symmetric function, which is differentiable in its positive set. We assume that the velocity field of ρ_0 is pointing inside everywhere, i.e., for $\vec{v}$ as defined in* (1.5.18),

$$\vec{v}(x;\rho_0) \cdot \Big(-\frac{x}{|x|}\Big) \geq 0 \quad \textit{in } \{\rho_0 > 0\}. \tag{1.5.20}$$

Let ρ be the weak solution of (1.1.1) *with initial data $\rho_0 \prec \rho(\cdot, 0)$. Then $\rho_0 \prec \rho(\cdot, t)$ for all $t \geq 0$.*

Proof. Let us define

$$\rho_1(x,t) := \rho_0(x) \text{ for } (x,t) \in \mathbb{R}^d \times [0,\infty).$$

Then (1.5.19) and (1.5.20) yield that ρ_1 is a subsolution of (1.5.3). Therefore, Proposition 1.5.4 applies to ρ and ρ_1 and so we are done. □

1.6 A comparison principle for general solutions

In Section 1.5, we showed that the mass comparison holds between radial solutions of (1.1.1). Although mass comparison does not hold directly for non-radial solutions, in this subsection we will use symmetrization techniques to show that it is possible to control the L^p-norms of non-radial solutions in terms of the L^p-norms of radial ones.

Let us recall that, for any nonnegative measurable function f that vanishes at infinity, the *symmetric decreasing rearrangement* f^* is given by

$$f^*(x) := \int_0^\infty \chi_{\{f>t\}^*}(x)dt, \tag{1.6.1}$$

where Ω^* denotes the symmetric rearrangement of a measurable set Ω of finite volume in $\mathbb{R}^d$.

In this subsection we consider general (non-radial) solutions of (1.1.1). Our goal is to prove the following result:

Theorem 1.6.1. *Suppose $m > 1$. Let V be given by (A) or (B), and let ρ be the weak solution to* (1.1.1) *with initial data $\rho(x,0) = \rho_0(x)$. Let $\bar{\rho}$ be the weak solution to* (1.1.1) *with initial data $\bar{\rho}(x,0) = \rho_0^*(x)$. Assume $\bar{\rho}$ exists for $t \in [0,T)$, where T may be infinite. Then $\rho^*(\cdot,t) \prec \bar{\rho}(\cdot,t)$ for all $t \in [0,T)$.*

As an application of Theorem 1.6.1, we will show that solutions of (1.1.1) with initial data in L^1 immediately regularize in L^∞ (see Proposition 1.7.7.)

The proof of Theorem 1.6.1, which we divide into several subsections follows that of the corresponding theorem for solutions of (1.1.2) (see Chapter 10 of [V]). The theorem in [V] is proved by taking the semi-group approach and applying the Crandall-Liggett Theorem. The challenge lies in the fact that our operator in (1.1.1) is not a contraction, in either L^1 or L^∞. For this reason the proof requires an additional approximation of our equation with one with fixed drift: see (1.6.5).

1.6.1 Implicit time discretization for PME with a drift

Consider the following equation

$$\rho_t = \Delta\rho^m + \nabla\cdot(\rho\nabla\Phi), \tag{1.6.2}$$

where Φ is a function given *a priori* such that $\Phi(x,t) \in C(\mathbb{R}^d \times [0,\infty))$, and $\Phi(\cdot,t) \in C^2(\mathbb{R}^d)$ for all t.

Following the proof in the case of (1.1.2) in [V], we approximate (1.6.2) via an implicit discrete-time scheme. For a small constant $h > 0$, U_i is recursively defined as the solution of the following equation:

$$\frac{U_i - U_{i-1}}{h} = \Delta U_i^m + \nabla\cdot(U_i\nabla\Phi_i), \quad i = 1, 2, \ldots \tag{1.6.3}$$

where $U_0 = \rho(\cdot,0), \Phi_i = \Phi(\cdot, ih)$. Now define

$$\rho_h(\cdot,t) := U_i(\cdot) \quad \text{for } (i-1)h < t \le ih, \quad i = 1, 2, \ldots \tag{1.6.4}$$

The following result states that our approximation scheme is valid: the proof is in Section A.1.3 of the appendix.

Proposition 1.6.2. *Let $\rho_0 \in L^1(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$, and let ρ_h be defined by* (1.6.4)*. Then there exists a function $\rho \in L^\infty([0,\infty); L^1(\mathbb{R}^d))$ such that*

$$\sup_{0\le t\le T} \|\rho(\cdot,t) - \rho_h(\cdot,t)\|_{L^1(\mathbb{R}^d)} \to 0$$

for any $T > 0$. Moreover, ρ coincides with the unique weak solution for (1.6.2)*.*

1.6.2 Rearrangement comparison

For a given function $\rho(x) : \mathbb{R}^d \to \mathbb{R}$, let us define ρ^* as given in (1.6.1).

Consider the following equation, where $f(x,t) \in C([0,\infty); L^1(\mathbb{R}^d))$ is a given function:

$$\rho_t = \Delta\rho^m + \nabla \cdot (\rho\nabla(f * V)). \tag{1.6.5}$$

Theorem 1.6.3. *Suppose $m > 1$. Let V be given by (A) or (B), and let ρ be the weak solution to* (1.6.5) *with initial data $\rho(x,0) = \rho_0(x)$. Let $\bar\rho$ be the weak solution to the symmetrized problem*

$$\rho_t = \Delta\rho^m + \nabla \cdot (\rho\nabla(f^* * V)), \tag{1.6.6}$$

with initial data $\bar\rho(x,0) = \rho_0^*(x)$. Then $\bar\rho$ is radially decreasing, and

$$\rho^*(\cdot,t) \prec \bar\rho(\cdot,t) \text{ for all } t > 0.$$

Due to Proposition 1.6.2, to prove Theorem 1.6.3 it suffices to show the following Proposition; see Section A.1.4 in the appendix for the proof.

Proposition 1.6.4. *Suppose V is given by (B) and $m > 1$. Let $u \in D$ (the domain D is defined in* (A.1.7)*) be the weak solution of*

$$-h\Delta u^m - h\nabla \cdot (u\nabla(f * V)) + u = g, \tag{1.6.7}$$

where $f, g \in L^1(\mathbb{R}^d)$ are nonnegative. Also, let $\bar u \in D$ be the solution to the symmetrized problem, i.e. $\bar u$ solves (1.6.7) *with f, g replaced by $\bar f$ and $\bar g$ respectively, where $\bar f$ and $\bar g$ are radially decreasing, have the same mass as f and g respectively, and satisfy $f^* \prec \bar f$ and $g^* \prec \bar g$. Then $u^* \prec \bar u$.*

Proof of Theorem 1.6.3:

The radial monotonicity of $\bar\rho$ can be shown via a similar argument as in Theorem 1.4.2: in fact the argument is easier here since $f^* * \Delta V$ is a radially decreasing function.

Next we prove $\rho^* \prec \bar\rho$ for all $t \geq 0$. Let U_i be the discrete solution for the original problem, and let V_i be the discrete solution for the symmetrized problem. Due to Proposition 1.6.2 it suffices to prove that $U_i^* \prec V_i$ for all $i \in \mathbb{N}$. Here U_i solves

$$\frac{U_i - U_{i-1}}{h} = \Delta U_i^m + \nabla \cdot (U_i \nabla (f_i * V)), \tag{1.6.8}$$

where $U_0 = u(\cdot, 0), f_i = f(\cdot, ih)$, and V_i solves

$$\frac{V_i - V_{i-1}}{h} = \Delta V_i^m + \nabla \cdot (V_i \nabla (f_i^* * V)), \tag{1.6.9}$$

where $V_0 = u^*(\cdot, 0)$. Since $U_0^* \prec V_0$, by applying Proposition 1.6.4 inductively we can conclude. Lastly when $V = \mathcal{N}$, we can use a mollified Newtonian kernel to approximate $\mathcal{N}$, and the result follows via Proposition 1.3.3. □

Now we are ready to prove our main result:

Proof for Theorem 1.6.1: Let us first prove the theorem when V is given by (B), where we have global existence of solutions. Let $\rho_1(\cdot, t) := \rho^*(\cdot, t)$ for all $t \geq 0$, where $\rho(x, t)$ is the weak solution of (1.1.1) with initial data $\rho(x, 0) = \rho_0(x)$. For $i > 1$, we let ρ_i be the weak solution to the following equation:

$$(\rho_i)_t = \Delta(\rho_i)^m + \nabla \cdot (\rho_i \cdot \nabla(\rho_{i-1} * V)), \tag{1.6.10}$$

with initial data $\rho_i(x, 0) = \rho^*(x, 0)$. Observe that $\rho_i(\cdot, t)$ is radially decreasing for all $i \in \mathbb{N}^+, t \geq 0$.

By Theorem 1.6.3, we have $\rho_i \prec \rho_{i+1}$ for all $i \in \mathbb{N}$. Hence we have

$$\rho^*(\cdot, t) = \rho_1(\cdot, t) \prec \rho_2(\cdot, t) \prec \rho_3(\cdot, t) \prec \ldots, \quad \text{for all } t. \tag{1.6.11}$$

Due to Theorem 1.3.1, $\{\rho_i\}$ is locally uniformly continuous in space and time. Hence by the Arzela-Ascoli Theorem any subsequence of $\{\rho_i\}$ locally uniformly converges to a function $\bar{\rho}$ along a subsequence. On the other hand $\bar{\rho}$ is the unique weak solution for (1.1.1) with initial data $\bar{\rho}(x, 0) = \rho_0^*(x)$. This means that the whole sequence $\{\rho_i\}$ locally uniformly converges to $\bar{\rho}$. Now we can conclude due to (1.6.11).

When $V = \mathcal{N}$, we can use a mollified Newtonian kernel to approximate $\mathcal{N}$, and the result follows via Proposition 1.3.3. □

Corollary 1.6.5. *Suppose $m > 1$. Let V be given by (A) or (B), and let ρ be the weak solution of* (1.1.1) *with initial data $\rho_0(x)$. Let $\bar{\rho}$ be the solution to the symmetrized problem, i.e. $\bar{\rho}$ is the weak solution to* (1.1.1) *with initial data $\rho_0^*(x)$. Assume $\bar{\rho}$ exists for $t \in [0, T)$, where T may be infinite. Then for any $p \in (1, \infty]$ we have*

$$\|\rho(\cdot, t)\|_{L^p(\mathbb{R}^d)} \le \|\bar{\rho}(\cdot, t)\|_{L^p(\mathbb{R}^d)}, \quad \textit{for all } t \in [0, T).$$

1.7 Asymptotic behavior for solutions existing global-in-time

In this section, we investigate the asymptotic behaviors for solutions which exist globally in time. These results are applications of the mass comparison in Section 1.5 and the comparison principles for general solutions in Section 1.6. Most of our results in this section are concerned with radial solutions, except Section 1.7.2 and Section 1.7.6.

1.7.1 Subcritical regime: exponential convergence towards stationary solution for radial solutions

In the subcritical regime ($m \ge 2 - 2/d$), the weak solution is known to exist globally in time, however the asymptotic behavior of the solutions are unknown. When V is given by (A) or (B), our goal is to use the mass comparison property established in Section 1.5 to prove the asymptotic convergence of radial solutions towards the stationary solution. First let us prove a confinement result, which is an application of Corollary 1.5.5. It says that when the initial data is radially symmetric and compactly supported, the support of the solution will stay in some fixed large ball for all times.

Proposition 1.7.1 (Compact radial solutions stay compact)**.** *Let V be given by (A) or (B), and let $m > 2 - \frac{2}{d}$. Let ρ solve* (1.1.1) *with a continuous, radially symmetric and compactly supported initial data $\rho(x, 0)$. Then there exists $R > 0$ depending on $m, d, \|\Delta V\|_1$ and $\rho(\cdot, 0)$, such that*

$$\{\rho(\cdot, t) > 0\} \subset \{|x| \le R\} \textit{ for all } t > 0.$$

Proof. 1. We will first assume that $\rho(0, 0) > 0$. Let $A := \int_{\mathbb{R}^d} \rho(x, 0)dx$, and let $\rho_A(x)$

be a radial stationary solution with mass A. For any continuous radial initial data with $\rho(0,0)>0$, we can choose $a>0$ sufficiently small, such that

$$\rho_1(x,t) := a^d \rho_A(ax) \prec \rho(x,0).$$

Our aim is to show that the velocity field of $\rho_1(x,t)$ is pointing towards the inside all the time, i.e.,

$$v(r,t;\rho_1) := \vec{v}(r,t;\rho_1)\cdot\frac{-x}{|x|} = \frac{\partial}{\partial r}\rho_1^{m-1}(r) + \frac{\partial}{\partial r}(\rho_1 * V) \geq 0. \tag{1.7.1}$$

Let us assume that V is given by (B); the argument for V given by (A) is parallel and easier. Recall that the stationary solution $\rho_A(x,t)$ satisfies the following equation in its positive set:

$$\frac{m}{m-1}\frac{\partial \rho_A^{m-1}}{\partial r} + \frac{\tilde{M}(r;\rho_A)}{\sigma_d r^{d-1}} = 0. \tag{1.7.2}$$

Therefore it follows that

$$\frac{m}{m-1}\frac{\partial}{\partial r}\rho_1^{m-1}(r) = a^{(m-1)d+1}\frac{m}{m-1}\frac{\partial \rho_A^{m-1}}{\partial r}(ar) = -a^{(m-1)d+1}\frac{\tilde{M}(ar;\rho_A)}{\sigma_d (ar)^{d-1}}.$$

Secondly observe that $\tilde{M}(r,t;\rho_1)$ satisfies

$$\begin{aligned}
\tilde{M}(r,t;\rho_1) &= \int_{B(0,r)}\int_{\mathbb{R}^d} a^d \rho_A(ay)\Delta V(y-x)\,dy\,dx \\
&= \int_{B(0,ar)}\int_{\mathbb{R}^d} \rho_A(y) a^{-d}\Delta V(a^{-1}(y-x))\,dy\,dx \\
&\geq \int_{B(0,ar)} \rho_A * \Delta V\,dx \quad (\text{since } a^{-d}\Delta V(a^{-1}x) \succ \Delta V \text{ when } 0<a<1) \\
&= \tilde{M}(ar;\rho_A).
\end{aligned}$$

(Note that when V is given by (A), direct computation yields $M(r,t;\rho_1) = M(ar;\rho_A)$.)

Due to (1.2.3) and the above inequalities, it follows that

$$\begin{aligned}
v(r,t;\rho_1) &= \frac{\partial}{\partial r}\rho_1^{m-1}(r) + \frac{\partial}{\partial r}(\rho_1 * V) \\
&\geq \frac{\partial}{\partial r}\rho_1^{m-1}(r) + \frac{\tilde{M}(r;\rho_1)}{\sigma_d r^{d-1}} \qquad (1.7.3)\\
&\geq (1-a^{d(m-2+2/d)})a^{d-1}\frac{\tilde{M}(ar;\rho_A)}{\sigma_d (ar)^{d-1}}. \qquad (1.7.4)
\end{aligned}$$

Since $m > 2 - 2/d$, the above inequality yields that the inward velocity field $v(r, \rho_1) \geq 0$ when $a < 1$. Therefore Corollary 1.5.5 implies that $\rho(\cdot, t) \succ \rho_1$ for all $t \geq 0$. Since ρ and ρ_1 have the same mass A, it follows that

$$\{\rho(\cdot, t) > 0\} \subset \{\rho_1(\cdot, t) > 0\} \text{ for all } t > 0,$$

and we can conclude.

2. The assumption $\rho(0, 0) > 0$ can indeed be removed, since $\rho(0, t)$ would still become positive in finite time even if $\rho(0, 0) = 0$. This is because, for the porous medium equation (1.1.2), it is a well-known fact that the solution will have a positive center density after finite time: this can be verified, for example, by maximum-principle type arguments using translations of Barenblatt solutions [V].

Note that a radial solution of (1.1.2) is a subsolution for (1.1.1) in the mass comparison sense. Hence one can compare ρ with a solution ψ of (1.1.2) with initial data ρ_0 and apply Proposition 1.5.4 to conclude that $\psi \prec \rho$. Now our assertion follows due to the continuity of ψ and ρ at the origin. □

Theorem 1.7.2 (Exponential convergence of radial solutions with the Newtonian potential)**.** *Let $m > 2 - \frac{2}{d}$ and let V be given by (A) or (B). For given $\rho_0 \geq 0$: a continuous, radially symmetric function with compact support, let $\rho(x, t)$ be the solution to* (1.1.1) *with initial data ρ_0. Next let $\rho_A(x)$ be a radial stationary solution with mass $A := \int \rho_0(x)dx$. Then $M(r, t) := M(r, t; \rho)$ satisfies*

$$|M(r, t) - M(r; \rho_A)| \leq C_1 e^{-\lambda t},$$

where C_1 depends on ρ_0, A, m, d, V, and the rate λ only depends on A, m, d, V.

Proof. 1. We will only prove the case when V satisfies (B); the case for (A) can be proven with a parallel (and easier) argument. Note that we may assume $\rho_0(0) > 0$ since otherwise $\rho(0, t)$ will become positive in finite time as explained in step 3 of the proof of Corollary 1.7.1.

2. Let ρ_A be a stationary solution with the same mass as ρ_0, given as in the proof of Corollary 1.7.1. Since ρ_0 is compactly supported, continuous and with $\rho_0(0) > 0$, we can

find a sufficiently small constant $a > 0$ such that

$$a^d \rho_A(ax) \prec \rho_0 \quad \text{ and } a^{-d}\rho_A(a^{-1}x) \succ \rho_0.$$

3. With the above choice of a, we next construct a self-similar subsolution $\phi(x,t)$ of (1.5.3) with initial data $\phi(x,0) = a^d \rho_A(ax)$ such that $M_\phi(\cdot,t) := M(\cdot,t;\phi)$ converges exponentially to $M(\cdot;\rho_A)$ as $t \to \infty$.

Here is the strategy of construction of $\phi(x,t)$. Due to (1.7.4), for all $0 < a < 1$, the inward velocity field $v(r) := v(r; a^d\rho_A(ax))$ given by (1.7.1) satisfies

$$v(r) \geq (1 - a^{d(m-2+2/d)})a^d r \frac{\tilde{M}(ar;\rho_A)}{\sigma_d (ar)^d} \geq 0.$$

Observe that $\dfrac{d\tilde{M}(ar;\rho_A)}{\sigma_d(ar)^d}$ equals the average of $\rho_A * \Delta V$ in the ball $\{|x| \leq ar\}$. By Proposition 1.2.1, ρ_A (hence $\rho_A * \Delta V$) is radially decreasing, and thus we have

$$C_1 \leq \frac{\tilde{M}(ar;\rho_A)}{\sigma_d(ar)^d} \leq C_2 \quad \text{in } \{\rho_A > 0\}, \tag{1.7.5}$$

where C_1, C_2 only depend on A, d, m, V. This gives a lower bound for the inward velocity field v

$$v(r) \geq C_1 a^d (1 - a^{d(m-2+2/d)}) r. \tag{1.7.6}$$

We will use the above estimate to construct a subsolution $\phi(r,t)$ of (1.5.3). Let us define

$$\phi(r,t) = k^d(t)\ \rho_A\big(k(t)r\big), \tag{1.7.7}$$

where the scaling factor $k(t)$ solves the following ODE with initial data $k(0) = a$:

$$k'(t) = C_1 (k(t))^{d+1}(1 - (k(t))^{d(m-2+2/d)}). \tag{1.7.8}$$

Since $m > 2 - 2/d$, $k'(t) > 0$ when $0 < k < 1$, and since $k = 1$ is the only non-zero stationary point for the ODE (1.7.8), for $0 < k(0) < 1$ we have $\lim_{t\to\infty} k(t) = 1$. Since

$$C_1 k^d (1 - k^{d(m-2+2/d)}) = -C_1 d(m - 2 + 2/d)(1 - k) + o(1 - k),$$

it follows that

$$0 \leq 1 - k(t) \lesssim e^{-C_1 d(m-2+2/d)t}, \tag{1.7.9}$$

which implies

$$0 \le M(r;\rho_A) - M_\phi(r,t) \lesssim e^{-C_1 d(m-2+2/d)t}. \tag{1.7.10}$$

Next we claim that ϕ is a subsolution of (1.5.3), i.e.,

$$\frac{\partial M_\phi}{\partial t} \le \sigma_d r^{d-1}\frac{\partial}{\partial r}\Big((\frac{\partial M_\phi}{\partial r}\frac{1}{\sigma_d r^{d-1}})^m\Big) + \Big(\frac{\partial M_\phi}{\partial r}\frac{1}{\sigma_d r^{d-1}}\Big)\tilde{M}_\phi \quad \text{in } \{\phi > 0\}. \tag{1.7.11}$$

To prove the claim, first note that by definition of $\phi(r,t)$ we have $M_\phi(r,t) = M(k(t)r;\rho_A)$. Hence, due to (1.7.8) and definition of ϕ, the left hand side of (1.7.11) can be written as

$$\begin{aligned} \frac{\partial M_\phi}{\partial t}(r,t) &= \partial_r M(k(t)r;\rho_A)\ k'(t)r \\ &= \sigma_d r^d \rho_A(k(t)r)k^{d-1}(t)k'(t) \qquad (1.7.12)\\ &= \sigma_d r^d \phi(r,t)C_1 k^d(1-k^{d(m-2+2/d)}). \qquad (1.7.13) \end{aligned}$$

On the other hand, we can proceed in the same way as (1.7.6), replacing a by k, to obtain

$$\frac{m}{m-1}\frac{\partial}{\partial r}\phi^{m-1} + \frac{\tilde{M}_\phi}{\sigma_d r^{d-1}} \ge C_1 k^d(1-k^{d(m-2+2/d)})r.$$

Therefore

$$\begin{aligned} \text{RHS of (1.7.11)} &= \sigma_d r^{d-1}\frac{\partial}{\partial r}\phi^m + \phi\tilde{M}_\phi \\ &= \sigma_d r^{d-1}\phi\big(\frac{m}{m-1}\frac{\partial}{\partial r}\phi^{m-1} + \frac{\tilde{M}_\phi}{\sigma_d r^{d-1}}\big) \\ &\ge \sigma_d r^d \phi C_1 k^d(1-k^{d(m-2+2/d)}), \end{aligned}$$

thus M_ϕ indeed satisfies (1.7.11), and the claim is proved.

4. Similarly one can construct a supersolution of (1.5.3). Let us define

$$\eta(r,t) := k^d(t)\ \rho_A\big(k(t)r\big),$$

where $k(t)$ solves the following ODE with initial data $k(0) = \frac{1}{a}$:

$$k'(t) = C_2 k^{d+1}(1-k^{d(m-2+2/d)}),$$

where C_2 is defined in (1.7.5). Arguing parallel to those in as in step 3 yields that η is a supersolution of (1.5.3) and

$$0 \le M_\eta(r,t) - M(r;\rho_A) \lesssim e^{-C_2 d(m-2+2/d)t}, \text{ for all } r > 0. \tag{1.7.14}$$

5. Lastly we compare ϕ, η with the weak solution ρ of (1.1.1). Since

$$\phi(\cdot, 0) \prec \rho(\cdot, 0) \prec \eta(\cdot, 0) \quad \text{(see Figure 1.3)},$$

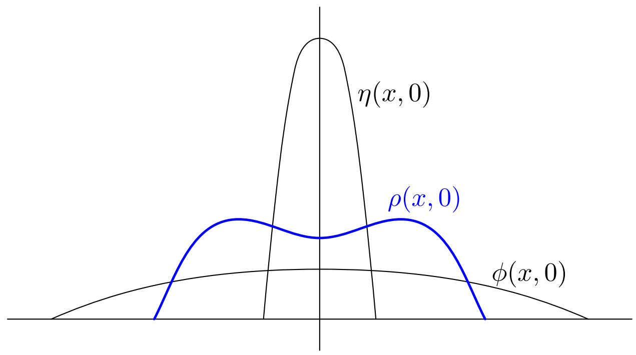


Figure 1.3: Initial data for ϕ, ρ and η

Proposition 1.5.4 yields that

$$M_\phi(\cdot, t) \le M(\cdot, t) \le M_\eta(\cdot, t). \tag{1.7.15}$$

By (1.7.10) and (1.7.14), we obtain

$$|M(r, t) - M(r; \rho_A)| \lesssim e^{-C_1 d(m-2+2/d)t} \text{ for } r \ge 0.$$

□

Using the explicit subsolution and supersolution constructed in the proof of Theorem 1.7.2, we get exponential convergence of ρ/A towards ρ_A/A in the p-Wasserstein metric, which is defined below. Note that the Wasserstein metric is natural for this problem, since as pointed out in [AGS] and [CMV], the equation (1.1.1) is a gradient flow of the energy (1.1.5) with respect to the 2-Wasserstein metric.

Definition 1.7.3. *Let μ_1 and μ_2 be two (Borel) probability measure on $\mathbb{R}^d$ with finite p-th moment. Then the* p-Wasserstein distance *between μ_1 and μ_2 is defined as*

$$W_p(\mu_1, \mu_2) := \Big(\inf_{\pi \in \mathcal{P}(\mu_1, \mu_2)} \Big\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dxdy) \Big\} \Big)^{\frac{1}{p}},$$

where $\mathcal{P}(\mu_1, \mu_2)$ is the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ_1 and second marginal μ_2.

Corollary 1.7.4. *Let $\rho, \rho_A, A, C_1, \lambda$ be as given in Theorem 1.7.2. Then for all $p > 1$, we have*

$$W_p\Big(\frac{\rho(\cdot,t)}{A}, \frac{\rho_A}{A}\Big) \le C_1 e^{-\lambda t}.$$

Proof. Before proving the corollary, we state some properties for Wasserstein distance, which can be found in [Vi]. For two probability densities f_0, f_1 on $\mathbb{R}^d$, the p-Wasserstein distance between them coincides with the solution of Monge's optimal mass transportation problem. Namely,

$$W_p(f_1, f_0) = \Big(\inf_{T\#f_0=f_1} \int_{\mathbb{R}^d} f_0(x)|x - T(x)|^p dx\Big)^{\frac{1}{p}}, \tag{1.7.16}$$

where T is a map from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $T\#f_0 = f_1$ stands for "the map T transports f_0 onto f_1", in the sense that for all bounded continuous function h on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} h(x) f_1(x) dx = \int_{\mathbb{R}^d} h(T(x)) f_0(x) dx.$$

Let ϕ be the subsolution constructed in the proof of Theorem 1.7.2, where we proved the radius of supp ϕ converges to the radius of supp ρ_A exponentially in time. Note that $\phi(\cdot,t)$ is a rescaling of ρ_A, hence the convergence of support implies that there is a map $T_\phi(\cdot,t)$ transporting $\phi(\cdot,t)$ onto ρ_A with $\sup_{x\in\{\rho(\cdot,t)>0\}} |x - T_\phi(x)|$ decaying exponentially in time. Once we find such T_ϕ, we can use (1.7.16) to show that $W_p(\phi(\cdot,t), \rho_A)$ decays exponentially.

Without loss of generality, we assume the mass $A = 1$ for the rest of the proof to avoid dividing ρ and ρ_A by A every time. The transport map T_ϕ can be explicitly constructed as

$$T_\phi(x,t) = \frac{x}{|x|} M_\phi^{-1}\big(M(|x|; \rho_A), t\big).$$

Recall that ϕ is a rescaling of ρ_A, defined as $\phi(x,t) = (k(t))^d \rho_A(k(t)x)$, where $k(t) < 1$ for all t, and $k(t)$ converges exponentially to 1. In this case the T_ϕ defined above can be greatly simplified as $T_\phi(x,t) = x/k(t)$. That gives us the following upper bound bound for $W_p(\phi(\cdot,t), \rho_A)$:

$$W_p(\phi(\cdot,t), \rho_A) \le \Big(\int_{\mathbb{R}^d} \rho_A |x - \frac{x}{k(t)}|^p dx\Big)^{\frac{1}{p}} \le R(1 - \frac{1}{k(t)}),$$

where R is the radius of support of ρ_A. Due to the estimate of $k(t)$ in (1.7.9), we obtain the exponential decay

$$W_p(\phi(\cdot,t),\rho_A) \le C_1 e^{-\lambda t}.$$

We can apply the same argument to the supersolution $\eta(\cdot,t)$ as well.

To show that $W_p(\rho(\cdot,t),\rho_A)$ decays with the same rate, it is natural to consider the following map $T(\cdot,t)$ which transports ρ_A onto $\rho(\cdot,t)$:

$$T(x,t) = \frac{x}{|x|} M^{-1}(M(|x|;\rho_A),t).$$

Then we have $|T(x,t)-x| = |M^{-1}(M(r;\rho_A),t)-r|$, where $r=|x|$. Due to (1.7.15), we have

$$M_\phi^{-1} \ge M^{-1} \ge M_\eta^{-1},$$

which gives

$$|T(x,t)-x| \le \max\{|T_\phi(x,t)-x|, |T_\eta(x,t)-x|\}.$$

Hence we conclude that

$$W_p(\rho(\cdot,t),\rho_A) \le W_p(\phi(\cdot,t),\rho_A) + W_p(\eta(\cdot,t),\rho_A) \le C_1 e^{-\lambda t}.$$

□

In fact one can also obtain the uniform convergence of $\rho(\cdot,t)$ to ρ_A in sup-norm, however the convergence rate would depend on the modulus of continuity of ρ. Theorem 1.7.2 and the uniform continuity of ρ and ρ_A, as well as the fact that ρ_A is compactly supported, yield the following:

Corollary 1.7.5. *Let ρ,ρ_A, C_1 and λ be as given in Theorem 1.7.2. Then we have*

$$\lim_{t\to\infty} \|\rho(x,t)-\rho_A(x)\|_{L^\infty(\mathbb{R}^d)} = 0.$$

Note that uniqueness of ρ_A is not required in the proof of Theorem 1.7.2. Indeed, uniqueness of ρ_A follows from the asymptotic convergence of ρ: if there are two radial stationary solutions ρ_A^1 and ρ_A^2 with the same mass, Corollary 1.7.5 implies $\rho(\cdot,t)\to\rho_A^i$ in L^∞ norm for $i=1,2$ when ρ is given as in Theorem 1.7.2. This immediately establishes the uniqueness of the radial stationary solution.

Corollary 1.7.6. *Let V be given by (A) or (B), and let $m > 2 - \frac{2}{d}$. Then for all $A > 0$, the radial stationary solution ρ_A for* (1.1.1) *with $\int \rho_A(x)dx = A$ is unique.*

1.7.2 Subcritical regime: instant regularization in L^∞ for general solutions

We present the following regularization result as a corollary of the mass comparison result (Theorem 1.5.4) and the rearrangement comparison for general solutions (Theorem 1.6.1). It says that for initial data $\rho_0 \in L^1(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$, no matter how large the L^∞ norm of ρ_0 is, $\|\rho(\cdot,t)\|_\infty$ will always be bounded by $t^{-\alpha}$ for some short time.

Proposition 1.7.7. *Let V be given by (A) or (B), and let $m > 2 - 2/d$. Let $\rho(x,t)$ be the weak solution for* (1.1.1)*, with initial data $\rho_0 \in L^1(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$. Let us denote $A = \|\rho_0\|_1$ and $\alpha := \frac{d}{d(m-1)+2}$. Then there exists $c = c(m,d,A,V)$ and $t_0 = (2c)^{1/\alpha} > 0$ such that we have $\rho(\cdot,t) \in L^\infty(\mathbb{R}^d)$ with*

$$\|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le c(m,d,A,V)t^{-\alpha} \text{ for all } 0 < t < t_0.$$

Proof. By Corollary 1.6.5, it suffices to prove the inequality when ρ_0 is radially symmetric. Also, in this proof we denote $c(m,d,A,V)$ to be all constants which only depend on m, d, A, V.

Let ρ_A be the radial stationary solution of (1.1.1) with mass A. Note that ρ_A is radially decreasing, and thus $\rho_A(0) > 0$. Since u_0 is a radial function in L^∞, we can scale ρ_A to make it more concentrated than u_0, i.e. we choose $0 < a < 1$ to be sufficiently small such that

$$u_0 \prec a^{-d}\rho_A(a^{-1}x).$$

As in the proof of Theorem 1.7.2, let us define

$$\eta(r,t) := k^d(t)\ \rho_A\big(k(t)r\big),$$

where $k(t)$ solves the following ODE with initial data $k(0) = a^{-1}$:

$$k'(t) = c(m,d,A,V)k^{d+1}(1 - k^{d(m-2+2/d)}).$$

Here $c(m, d, A, V)$ corresponds to C_2 in the proof for Theorem 1.7.2. It was shown in the proof that

$$\rho(\cdot, t) \prec \eta(\cdot, t) \quad \text{for all } t \geq 0,$$

which in particular yields that

$$\|\eta(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \geq \|\rho(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \quad \text{for all } t \geq 0.$$

Observe that, by definition,

$$h(t) := \|\eta(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} = k^d(t)\rho_A(0) = c(m, d, A, V)k^d(t).$$

Therefore to prove our proposition it is enough to show

$$h(t) \leq f(t) := c(m, d, A, V)t^{-\alpha} \quad \text{for all } h(0) > 0 \text{ and } t \in [0, t_0],$$

where t_0 is chosen such that $f(t) \geq 2$. Note that $h(t)$ solves

$$\begin{aligned} h'(t) &= c(m, d, A, V)k^{d-1}k' \\ &= c(m, d, A, V)h^2\big(1 - h^{m-2+2/d}\big). \end{aligned}$$

In particular when $h(t) \geq 2$, h satisfies the following inequality

$$h'(t) \leq -c(m, d, A, V)h^{m+2/d}.$$

Since $f(t)$ solves the above ODE with equality, we obtain $h(t) \leq f(t)$ for $0 \leq t \leq t_0$. Now we are done. □

1.7.3 Supercritical regime: algebraic convergence towards Barenblatt profile for radial solutions

In this subsection, we consider the asymptotic behavior of radial solutions in the supercritical regime, i.e. for $1 < m < 2 - \frac{2}{d}$. In this case the diffusion overrides the aggregation and thus the solution is expected to behave similar to that of Porous Medium Equation (PME) in the long run. In fact recently it was shown in [B1] (and also in [S1]), by making use of entropy method as well as functional inequalities, that the solution of (1.1) with a general class of V and with small mass and small $L^{(2-m)d/2}$ norm converges to the self-similar Barenblatt solution $\mathcal{U}(x,t)$ with algebraic rate,

$$\mathcal{U}(x,t) = t^{-\beta d}(C - \frac{(m-1)\beta}{2m}|x|^2 t^{-2\beta})_+^{\frac{1}{m-1}}, \tag{1.7.17}$$

where C is some constant such that $\| \mathcal{U}(\cdot,0)\|_1 = \|\rho(\cdot,0)\|_1$.

Here we will give a complementary result to [B1] and [S1] in the case of radial solutions, by using mass comparison (Proposition 1.5.4). We point out that in our result the mass does not need to be small, and we provide an explicit description of solutions which are "sufficiently scattered" so that they do not blow up in finite time. Of course the method presented in [B1] is much more delicate and yields optimal convergence results for general solutions with small mass in the supercritical regime.

Let ρ be the weak solution to (1.1.1). Following [V], we re-scale ρ as follows:

$$\mu(\lambda,\tau) = (t+1)^{\alpha}\rho(x,t+1); \quad \lambda = x(t+1)^{-\beta}; \quad \tau = \ln(t+1), \tag{1.7.18}$$

where

$$\alpha = \frac{d}{d(m-1)+2}, \quad \beta = \alpha/d.$$

Then $\mu(\lambda,0) = \rho(x,0)$, and $\mu(\lambda,\tau)$ is a weak solution of

$$\mu_\tau = \Delta\mu^m + \beta\nabla\cdot(\mu\nabla\frac{|\lambda|^2}{2}) + e^{(1-\alpha)\tau}\nabla\cdot(\mu\nabla(\mu * (\mathcal{N} * \tilde{h}(\lambda,\tau))), \tag{1.7.19}$$

where

$$\tilde{h}(\lambda,\tau) := e^{d\beta\tau}\Delta V(\lambda e^{\beta\tau}). \tag{1.7.20}$$

(When $V = \mathcal{N}$ one should replace the last term by $e^{(1-\alpha)\tau}\nabla\cdot(\mu\nabla(\mu * \mathcal{N}).)$

In the absense of the last term, equation (1.7.19) is a Fokker-Plank equation

$$\mu_\tau = \Delta\mu^m + \beta\nabla\cdot(\mu\nabla\frac{|\lambda|^2}{2}), \tag{1.7.21}$$

which is known to converge to the stationary solution μ_A exponentially, where μ_A has the mass $A := \|\mu(\cdot,0)\|_1$ and satisfies

$$\frac{m}{m-1}\mu_A^{m-1} = (C - \beta\frac{|\lambda|^2}{2})_+ \text{ for some } C > 0. \tag{1.7.22}$$

In Therorem 1.7.10, we will prove for $m < 2 - 2/d$, if the initial data is sufficiently less concentrated than μ_A, then $\mu(\cdot,\tau)$ also converges to the same limit μ_A exponentially as $\tau \to \infty$. We begin by defining the following mass functions:

$$M^\mu(r,\tau) := M(r,\tau;\mu) \text{ and } \tilde{\mathcal{M}}(r,\tau;f) := \int_{B(0,r)} f * \tilde{h}(\cdot,\tau)d\lambda,$$

where M is as given in (1.5.2), $\tilde{h}$ is as given in (1.7.20), and f is an arbitrary function. Note that for $V = \mathcal{N}$, $\tilde{h}(\cdot,\tau)$ is the delta function for all τ, hence $\tilde{\mathcal{M}} \equiv M$.

Then M^μ satisfies the following PDE in the positive set of μ:

$$M^\mu_\tau = \sigma_d r^{d-1}(\frac{\partial M^\mu}{\partial r}\frac{1}{\sigma_d r^{d-1}})\Big[\frac{m}{m-1}\frac{\partial}{\partial r}((\frac{\partial M^\mu}{\partial r}\frac{1}{\sigma_d r^{d-1}})^{m-1}) + \beta r + e^{(1-\alpha)\tau}\frac{\tilde{\mathcal{M}}(r,\tau;\mu)}{\sigma_d r^{d-1}}\Big] \quad (1.7.23)$$

We first check that the mass comparison holds for re-scaled equations:

Proposition 1.7.8. *Let $V(x)$ be given by (A) or (B), and let $m < 2 - \frac{2}{d}$. Assume $\mu_1(\lambda,\tau)$ is a subsolution and $\mu_2(\lambda,\tau)$ is a supersolution of* (1.7.23). *Further assume that $\int \mu_1(\cdot,\tau)d\lambda$ and $\int \mu_2(\cdot,\tau)d\lambda$ stay constant for all $t \geq 0$. Then the mass is ordered for all times, i.e.,*

$$\text{if } \mu_1(\lambda,0) \prec \mu_2(\lambda,0), \text{ then we have } \mu_1(\lambda,\tau) \prec \mu_2(\lambda,\tau) \text{ for all } \tau > 0.$$

Proof. Let $\rho_i(x,t)$ be the corresponding re-scaled versions of μ_i. Then ρ_1 and ρ_2 are respectively a subsolution and a supersolution of (1.5.3). The proof then follows from Proposition 1.5.4 and from the fact that

$$M(r,\tau;\mu_i) = e^{(\alpha-\beta)\tau}M(re^{\beta\tau}, e^\tau;\rho_i).$$

□

Next we state a technical lemma which is used later in the proof of the convergence theorem. The proof is in Section A.1.5 of the appendix.

Lemma 1.7.9. *Let $k(t)$ solve the ODE*

$$k'(t) = C_1 k(1-k^\alpha) + C_2 k^{d+1} e^{-\beta t}, \quad (1.7.24)$$

where C_1, C_2, α, β are positive constants. Then there exists a constant $\delta > 0$ such that if $0 < k(0) < \delta$, then $k(t) \to 1$ exponentially as $t \to \infty$.

Now we are ready to prove the main theorem. We will first prove it for radially decreasing solutions.

Theorem 1.7.10. *Let $V(x)$ be given by (A) or (B), let $1 < m < 2 - \frac{2}{d}$ and let μ_A be as given in* (1.7.22)*. Suppose $\mu_0(\lambda)$ is radially decreasing, compactly supported and has mass A. Then there exists a constant $\delta > 0$ depending on d, m, μ_0 and V, such that if*

$$\mu_0(\lambda) \prec \delta^d \mu_A(\delta\lambda),$$

then the weak solution $\mu(\lambda, \tau)$ of (1.7.19) *with initial data μ_0 exists for all $\tau > 0$. Furthermore, $M(r, \tau; \mu)$ defined in* (1.5.2) *converges to $M(r, \tau; \mu_A)$ exponentially as $t \to \infty$ and uniformly in r.*

Proof. The proof of theorem is analogous to that of Theorem 1.7.2: we will construct a self-similar subsolution $\phi(\lambda, \tau)$ and supersolution $\eta(\lambda, \tau)$ to (1.7.19), both of which converge to μ_A exponentially.

Observe that (1.7.19) can be written as a transport equation

$$\mu_t + \nabla \cdot (\mu \vec{v}) = 0,$$

where the *velocity field* $\vec{v}$ is given by

$$\vec{v} := \frac{m}{m-1} \nabla(\mu^{m-1}) + \beta\lambda + e^{(1-\alpha)\tau} \nabla(\mu * (\mathcal{N} * \tilde{h}(y, \tau)).$$

Hence the inward velocity field $v(r, \tau; \mu) := -\vec{v} \cdot \frac{x}{|x|}$ for the rescaled PDE (1.7.19) is

$$v(r, \tau; \mu) = \frac{m}{m-1} \frac{\partial}{\partial r}(\mu^{m-1}) + \beta r + e^{(1-\alpha)\tau} \frac{\tilde{\mathcal{M}}(r, \tau; \mu)}{\sigma_d r^{d-1}}.$$

We first construct a subsolution $\phi(\lambda, \tau)$ with the scaling factor $k(\tau)$ to be determined later:

$$\phi(\lambda, \tau) := k^d(\tau) \mu_A\big(k(\tau)\lambda\big).$$

Since μ_A satisfies (1.7.22), the inward velocity field of ϕ is then given by

$$v(r,\tau;\phi) = (1-k^{d(m-1)+2})\beta r + e^{(1-\alpha)\tau}\frac{\tilde{\mathcal{M}}(r,\tau;\phi)}{\sigma_d r^{d-1}}.$$

Note that the last term of $v(r,\tau;\phi)$ is always non-negative, and thus $v(r,\tau;\phi) \geq (1-k^{d(m-1)+2})\beta r$. That motivates us to choose $k(\tau)$ to be the solution of the following equation

$$k'(\tau) = \beta k(1-k^{d(m-1)+2}), \tag{1.7.25}$$

with initial data $k(0)$ sufficiently small such that $\phi(\cdot,0) \prec \mu_A$ and $\phi(\cdot,0) \prec \mu(\cdot,0)$. One can proceed as in the proof of Theorem 1.7.2 to verify ϕ is indeed a subsolution. Moreover, it can be easily checked that $k(\tau) \to 1$ exponentially as $\tau \to \infty$, hence $M(r,\tau;\phi)$ converges to $M(r;\mu_A)$ exponentially as $\tau \to \infty$ and uniformly in r.

Next we turn to the construction of a supersolution of the form

$$\eta(\lambda,\tau) := k^d(\tau)\mu_A\big(k(\tau)\lambda\big).$$

Here the main difficulty comes from the aggregation term, which might cause finite time blow-up of the solution. To find an upper bound of the inward velocity field, we first need to control $\tilde{\mathcal{M}}(r,\tau,k^d\mu_A(k\lambda))$:

$$\begin{aligned}
\tilde{\mathcal{M}}(r,\tau;k^d\mu_A(k\lambda)) &= \int_{B(0,r)} k^d\mu_A(k\cdot) * e^{d\beta\tau}\Delta V(e^{\beta\tau}\cdot)(\lambda)d\lambda \\
&\leq \|\Delta V\|_1 \int_{B(0,r)} k^d\mu_A(k\lambda)d\lambda \\
&= \|\Delta V\|_1 \int_{B(0,kr)} \mu_A(\lambda)d\lambda \ \leq C(kr)^d/\sigma_d,
\end{aligned}$$

where the first inequality is due to Riesz's rearrangement inequality and the fact that μ_A is radially decreasing, and C is some constant that does not depend on k, r, τ.

The above inequality gives the following upper bound for the inward velocity field of η:

$$v(r,\tau;\eta) \leq (1-k^{d(m-1)+2})\beta r + Ck^d e^{(1-\alpha)\tau} r.$$

Therefore if we let $k(t)$ solve the following ODE

$$k'(\tau) = \beta k(1-k^{d(m-1)+2}) + Ck^{d+1}e^{(1-\alpha)\tau}, \tag{1.7.26}$$

and choose the initial data $k(0)$ such that $\mu(\cdot,0) \prec \eta(\cdot,0) = k^d(0)\mu_A\big(k(0)\lambda\big)$, then η would be a supersolution to (1.7.19).

Let us choose $k(0) = \delta$, where δ is given in the assumption of this theorem. Due to Lemma 1.7.9, $k(\tau) \to 1$ exponentially as $\tau \to \infty$ when δ is sufficiently small, hence it follows that $M(r,\tau;\eta)$ converges to $M(r;\mu_A)$ exponentially.

Since the supersolution η exists globally, we claim that the weak solution μ exists globally as well. Suppose not: then due to Theorem 4 of [BRB], μ has a maximal time interval of existence T^*, and $\lim_{\tau \nearrow T^*} \|\mu(\cdot,\tau)\|_\infty = \infty$. On the other hand, Proposition 1.7.8 yields that

$$\mu(\cdot,\tau) \prec \eta(\cdot,\tau) \text{ for all } \tau < T^*. \tag{1.7.27}$$

Note that Proposition 1.4.2 implies that μ is radially decreasing for all $\tau < T^*$, which gives

$$\|\mu(\cdot,\tau)\|_\infty \le \|\eta(\cdot,\tau)\|_\infty \text{ for all } \tau < T^*. \tag{1.7.28}$$

The above inequality implies that $\lim_{\tau \nearrow T^*} \|\eta(\cdot,\tau)\|_\infty = \infty$, which contradicts the fact that $\|\eta(\cdot,\tau)\|_\infty$ is uniformly bounded for all τ.

Once we have the global existence of μ, Proposition 1.7.8 yields that

$$\phi(\cdot,\tau) \prec \mu(\cdot,\tau) \prec \eta(\cdot,\tau) \text{ for all } \tau \ge 0.$$

Since both ϕ and η converge exponentially towards μ_A as $\tau \to \infty$, we can conclude. □

Making use of the rearrangement comparison introduced in Section 1.6, we obtain the following generalization of Theorem 1.7.10, where the first part applies to nonradial solutions, and the second part applies to radial solutions which do not need to be radially decreasing.

Corollary 1.7.11. *Let $V(x)$ be given by (A) or (B), and $1 < m < 2 - \frac{2}{d}$. For a nonnegative function μ_0 in $L^1(\mathbb{R}^d)$, define $A := \int \mu_0(\lambda)d\lambda$, and let $\mu_A(\lambda)$ be as given in* (1.7.22). *Then the following holds:*

(a) there exists a small constant $\delta > 0$ depending on d, m, μ_0 and V, such that if

$$\mu_0^*(\lambda) \prec \delta^d \mu_A(\delta\lambda),$$

then the weak solution $\mu(\lambda,\tau)$ of (1.7.19) *with initial data μ_0 exists for all $\tau > 0$.*

(b) Let μ_0 is as given in (a) and also is radially symmetric and compactly supported, then $M(r,\tau;\mu)$ defined in (1.5.2) *converges to $M(r,\tau;\mu_A)$ exponentially as $\tau \to \infty$ and uniformly in r.*

Proof. Let $\bar{\mu}(\lambda,\tau)$ be the weak solution to (1.7.19) with initial data $\mu_0^*(\lambda)$. Then $\bar{\mu}(\cdot,0)$ meets the assumptions for Theorem 1.7.10, which implies the global existence of $\bar{\mu}$. Due to Corollary 1.6.5, $\|\mu(\cdot,\tau)\|_\infty \le \|\bar{\mu}(\cdot,\tau)\|_\infty$ for all τ during the existence of μ; hence the uniform boundedness of $\bar{\mu}$ yields that μ cannot blow up and thus must exist globally in time. This proves (a).

Now suppose μ_0 is radially symmetric and compactly supported, and μ_0 satisfies the assumption in (a) such that the corresponding solution μ exists globally in time. In this case we can construct subsolution and supersolution as in the proof for Theorem 1.7.10 to prove (b). □

If we rescale back to the original space and time variables, Corollary 1.7.11(b) immediately yields the algebraic convergence of mass function for the solution to (1.1.1).

Corollary 1.7.12. *Let V, m, μ and μ_0 be as given in Corollary 1.7.11, and let ρ be given by* (1.7.18). *Let $\mathcal{U}(x,t)$ be the self-similar Barenblatt solution as given in* (1.7.17). *Then ρ is a weak solution to* (1.1.1), *and ρ vanishes to zero as $t \to \infty$ with algebraic decay. In particular if ρ_0 is radially symmetric then*

(a) $|M(r,t) - M(r,t;\mathcal{U})| \le Ct^{-\gamma}$, for all $r \ge 0$, for some C, γ depending on ρ_0, m, d and V.

(b) for all $p > 1$ we have

$$W_p\left(\frac{\rho(\cdot,t)}{A}, \frac{\mathcal{U}(\cdot,t)}{A}\right) \le Ct^{-\gamma},$$

where C, γ depend on $\rho(x,0), m, d$ and V.

Proof. From the proof of Theorem 1.7.10, we have $W_p(\frac{\mu(\cdot,\tau)}{A}, \frac{\mu_A}{A}) \lesssim e^{-\gamma\tau}$ for some γ depending on $\rho(x,0)$, m, d and V, and the proof is analogous with the proof of Corollary 1.7.4. Now we scale back, and the above inequality becomes

$$W_p(\frac{\rho(,\tau)}{A}, \frac{\rho_A}{A}) \lesssim (t+1)^{-\gamma}.$$

□

1.7.4 Critical regime with critical mass: convergence towards stationary solution for radial solutions

In this subsection, we consider the equation (1.1.1) with $V = \mathcal{N}$ and $m = m_c := 2-2/d$, and prove that every radial solution with critical mass M_c and continuous, compactly supported initial data will be eventually attracted to some stationary solution within the family (1.1.7).

If the initial data is bounded above and has the critical mass M_c, then it is proved in [BCL2] that the weak solution to (1.1.1) exists globally in time. In the next lemma we prove the solution indeed has a global (in time) L^∞ bound. In addition, if the initial data is radially symmetric and compactly supported, then the support of the solution would stay uniformly bounded in time.

Lemma 1.7.13. *Suppose $V = \mathcal{N}, d \geq 3$ and $m = 2 - 2/d$. Consider the problem* (1.1.1) *with initial data $\rho_0 \in L^1_+(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$, where ρ_0 is continuous and has critical mass M_c. Then the L^∞ norm of the weak solution $\rho(x,t)$ is globally bounded, i.e. there exists $K > 0$ depending on $\|\rho_0\|_{L^\infty(\mathbb{R}^d)}$ and d, such that $\|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq K_1$ for all $t \geq 0$.*

If ρ_0 is radially symmetric and compactly supported in addition to the assumptions above, then there exists some $R_2 > 0$, such that $\{\rho(\cdot,t) > 0\} \subseteq B(0,R_2)$ for all $t \geq 0$, where R_2 depend on d and ρ_0.

Proof. In order to bound the L^∞ norm of $\rho(\cdot,t)$, we first consider equation (1.1.1) with symmtetrized initial data, which is described below. Let $\bar{\rho}(\cdot,t)$ be the solution to (1.1.1) with initial data ρ_0^*, where $\rho_0^*(\cdot)$ is the radial decreasing rearrangement of ρ_0, as defined in (1.6.1).

Since $\bar{\rho}$ has a radially symmetric initial data and has mass M_c, due to [BCL2], we readily obtain that $\bar{\rho}$ exists globally in time, and $\bar{\rho}$ is radially symmetric for all $t \geq 0$. We first prove that there is a global L^∞ bound for $\bar{\rho}$.

Since $\|\bar{\rho}(\cdot,0)\|_\infty = \|\rho_0\|_\infty < \infty$, we can choose $R_1 > 0$ depending on $\|\rho_0\|_\infty$, where R_1 is sufficiently small such that $\rho_0^* \prec \eta_{R_1}$, where η_{R_1} is as defined in (1.1.7). Then the mass comparison result in Proposition 1.5.4 yields that

$$\bar{\rho}(\cdot,t) \prec \eta_{R_1} \text{ for all } t \geq 0. \tag{1.7.29}$$

Now we go back to the original solution ρ, and compare ρ with $\bar{\rho}$. Theorem 1.6.3 yields that

$$\rho^*(\cdot,t) \prec \bar{\rho}(\cdot,t) \text{ for all } t \geq 0.$$

Combining the above two inequalities together, we readily obtain that

$$\rho^*(\cdot,t) \prec \eta_{R_1} \text{ for all } t \geq 0,$$

which implies that $\rho^*(0,t) \leq \eta_{R_1}(0)$ for $t \geq 0$. Note that $\rho^*(\cdot,t)$ is radially decreasing for all $t \geq 0$ by definition, and η_{R_1} is radially decreasing due to [BCL2]. Hence the above inequality implies that

$$\|\rho(\cdot,t)\|_\infty = \|\rho^*(\cdot,t)\|_\infty \leq \|\eta_{R_1}\|_\infty = R_1^{-d}\eta_1(0) \quad \text{for } t \geq 0, \tag{1.7.30}$$

thus ρ has a global L^∞ bound $R_1^{-d}\eta_1(0)$, where η_1 is as defined in (1.1.7).

Next we hope to show that if ρ_0 is radially symmetric and compactly supported in addition to the conditions above, the support of $\rho(\cdot,t)$ will stay in some compact set for all time. We first prove it for the case where $\rho_0(0) > 0$. Due to the continuity of ρ_0, we have ρ_0 is uniformly positive in a neighborhood of 0. This enables us to choose $R_2 > 0$ sufficiently large such that $\rho_0 \succ \eta_{R_2}$, where η_{R_2} is as defined in (1.1.7). Proposition 1.5.4 again gives us $\rho(\cdot,t) \succ \eta_{R_2}$ for all $t \geq 0$, which implies that

$$\text{supp } \rho(\cdot,t) \subseteq \text{supp } \eta_{R_2} = B(0,R_2) \text{ for all } t \geq 0.$$

If $\rho_0(0) = 0$, we claim that after some finite time t_1, $\rho(0,t_1)$ becomes positive, and $\rho(\cdot,t_1)$ has a compact support, where t_1 depends on d and ρ_0. Then we can take t_1 as the starting time and argue as in the case $\rho_0(0) > 0$. The proof of the claim is the same as in step 2 in the proof of Proposition 1.7.1 and will thus be omitted. $\square$

Next we prove that under the conditions in Lemma 1.7.13, every radial solution converges to some stationary solution in the family (1.1.7) as $t \to \infty$. To do this we investigate the free energy functional (1.1.5), and make use of the following result proved in [BCL2] and [BRB]: Let u be a weak solution to (1.1.1), then it satisfies the following energy dissipation inequality for almost every t during its existence:

$$\mathcal{F}(\rho(t)) + \int_0^t \int_{\mathbb{R}^d} \rho \left| \frac{m}{m-1} \nabla \rho^{m-1} + \nabla \mathcal{N} * \rho \right|^2 dxdt \leq \mathcal{F}(\rho_0). \tag{1.7.31}$$

Theorem 1.7.14. *Suppose $V = \mathcal{N}, d \geq 3$ and $m = 2 - 2/d$. Let $\rho(x,t)$ be the weak solution to* (1.1.1) *with critical mass M_c and nonnegative initial data ρ_0, where ρ_0 is continuous, radially symmetric and compactly supported, and satisfies $\nabla \rho_0^m \in L^2(\mathbb{R}^d)$. Then there exists $R_0 > 0$ depending on ρ_0 and d, such that $\rho(\cdot, t) \to \eta_{R_0}$ in $L^\infty(\mathbb{R}^d)$ as $t \to \infty$, where η_{R_0} is as defined in* (1.1.7).

Proof. Due to Lemma 1.7.13, we obtain the existence of a weak solution globally in time, which has a global L^∞ bound. In addition, by treating $\rho * \mathcal{N}$ as an *a priori* potential in (1.1.1) and applying the continuity result in [D], we obtain that $\rho(x,t)$ is uniformly continuous in space and time in $[\tau, \infty)$ for all $\tau > 0$.

Our preliminary goal is to find a time sequence $\{t_n\}$ which increases to infinity, such that $\rho(t_n)$ uniformly converges to some stationary solution as $n \to \infty$. Note that $\mathcal{F}(\rho(\cdot, t))$ is non-increasing for almost every t due to (1.7.31), and is bounded below as $t \to \infty$. This enables us to find a time sequence $\{t_n\}$ which increases to infinity, such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \rho(t_n) \left| \frac{m}{m-1} \nabla \rho(t_n)^{m-1} + \nabla \mathcal{N} * \rho(t_n) \right|^2 dx = 0. \tag{1.7.32}$$

We will slightly abuse the notation and denote $\rho(t_n)$ by ρ_n. Note that $\{\rho_n\}$ is uniformly bounded and equicontinuous, hence Arzelà-Ascoli theorem enables us to find a subsequence of $\{\rho_n\}$, such that

$$\rho_n \to \rho_\infty \quad \text{uniformly in } n, \tag{1.7.33}$$

where ρ_∞ is some radially symmetric and continuous function. Moreover, Lemma 1.7.13 implies that the support of $\{\rho_n\}$ all stays in some fixed compact set, hence we have ρ_∞

is compactly supported as well, and it has mass M_c. We will prove that ρ_∞ is indeed a stationary solution later.

We next claim that $\{\nabla\rho_n^m\}$ are uniformly bounded in $L^2(\mathbb{R}^d)$. To prove the claim, note that

$$\int_{\mathbb{R}^d} |\nabla\rho_n^m + \rho_n \nabla\mathcal{N} * \rho_n|^2 dx \leq \|\rho_n\|_\infty \int_{\mathbb{R}^d} \rho_n \left|\frac{m}{m-1}\nabla\rho_n^{m-1} + \nabla\mathcal{N} * \rho_n\right|^2 dx,$$

where the right hand side is uniformly bounded for all n. In addition, since $\{\rho_n\}$ are uniformly bounded and are all supported in some $B(0,R)$, we know $\int_{\mathbb{R}^d} \rho_n |\nabla\mathcal{N} * \rho_n|^2 dx$ is also uniformly bounded for all n. Therefore triangle inequality yields the uniform boundedness of $\int_{\mathbb{R}^d} |\nabla\rho_n^m|^2 dx$, which proves the claim.

The uniform boundedness of $\{\nabla\rho_n^m\}$ in $L^2(\mathbb{R}^d)$ implies that $\{\nabla\rho_\infty^m\}$ is in $L^2(\mathbb{R}^d)$ as well. Moreover, we can find a subsequence of $\{\rho_n\}$ (which we again denote by $\{\rho_n\}$ for the simplicity of notation), such that

$$\nabla\rho_n^m \rightharpoonup \nabla\rho_\infty^m \text{ as } n \to \infty \text{ weakly in } L^1(\mathbb{R}_L^d : \mathbb{R}^d). \tag{1.7.34}$$

Using (1.7.32) and the two convergence properties (1.7.33) and (1.7.34), we can proceed in the same way as Lemma 10 in [CJMTU] and prove that ρ_∞ satisfies

$$\int_{\mathbb{R}^d} \rho_\infty \left|\frac{m}{m-1}\nabla u_\infty^{m-1} + \nabla\mathcal{N} * \rho_\infty\right|^2 dx = 0, \tag{1.7.35}$$

which implies that ρ_∞ is a radial stationary solution to (1.1.1), hence is indeed in the family (1.1.7).

Next we will prove that $\rho(\cdot,t) \to \rho_\infty$ uniformly in $L^\infty(\mathbb{R}^d)$ as $t \to \infty$. In order to prove this, we make use of the monotonicity of the second moment of $\rho(\cdot,t)$ in time. By combining the following Virial identity

$$\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2 \rho(x,t) dx = 2(d-2)\mathcal{F}[\rho(t)] \text{ for all } t \tag{1.7.36}$$

with the fact that the minimizer of $\mathcal{F}$ has free energy 0, it is shown in [BCL2] that

$$M_2[\rho(\cdot,t)] := \int_{\mathbb{R}^d} |x|^2 \rho(x,t) dt \text{ is non-decrasing in } t. \tag{1.7.37}$$

This implies that any subsequence of $\rho(\cdot, t)$ can converge to only one limit: if not, then we can find some another sequence $\{t'_n\}$ increasing to infinity, such that $\rho(t'_n)$ converges to another stationary solution ρ'_∞ uniformly as $n \to \infty$, where ρ'_∞ is also in the family (1.1.7). Since $\rho(t'_n)$ are uniformly bounded and uniformly compactly supported, we have $M_2[\rho(t'_n)] \to M_2[\rho'_\infty]$. On the other hand for the time sequence $\{t_n\}$ we have $M_2[\rho(t_n)] \to M_2[\rho_\infty]$, hence (1.7.37) implies that ρ_∞ and ρ'_∞ must have the same second moment. Since both ρ_∞ and ρ'_∞ are within the family (1.1.7), they can have the same second moment only if they are the same stationary solution. □

Remark 1.7.15. Since the proof is done by extracting a subsequence of time, we are unable to obtain the rate of the convergence. We also point out that the above proof is for radial solution only; for general initial data the difficulty lies in the fact that we are unable to bound the solution in some compact set uniform in time.

1.7.5 Critical regime with subcritical mass: convergence towards self-similar solution for radial solutions

In this subsection, we assume $V = \mathcal{N}$ and $m = 2 - 2/d$. We prove that every radial solution with subcritical mass and compactly supported initial data would converge to some self-similar solution which is dissipating with the same scaling as the solution of the porous medium equation.

Let ρ be the weak solution to (1.1.1), with mass $A \in (0, M_c)$. Following [V] and [BCL2], let μ be as defined in (1.7.18), which is a a continuous rescaling of ρ according to the scaling for the porous medium equation. Then $\mu(\lambda, 0) = \rho(x, 0)$, and $\mu(\lambda, \tau)$ solves the following rescaled equation in the weak sense:

$$\mu_\tau = \Delta\mu^m + \nabla\cdot(\mu\nabla\frac{|\lambda|^2}{2d}) + \nabla\cdot(\mu\nabla(\mu * \mathcal{N})). \tag{1.7.38}$$

It is pointed out in Theorem 5.2 of [BCL2] that the free energy associated to the rescaled problem (1.7.19) is

$$\mathcal{G}(\mu(\cdot, t)) := \int_{\mathbb{R}^d} \left(\frac{m}{m-1}\mu^m + \frac{1}{2}\mu(\mathcal{N} * \mu) + \frac{|\lambda|^2\mu}{2d}\right) d\lambda, \tag{1.7.39}$$

and for any mass $A \in (0, M_c)$, there is a unique minimizer μ_A of $\mathcal{G}$ in $\mathcal{Z}_A$ subject to translation, where $\mathcal{Z}_A := \{h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d) : \|h\|_1 = M \text{ and } \int_{\mathbb{R}^d} |x|^2 h(x)dx \leq \infty\}$. In addition, μ_A is continuous, radially decreasing and has a compact support, and μ_A satisfies

$$\frac{m}{m-1}\frac{\partial}{\partial r}\mu_A^{m-1} + \frac{r}{d} + \frac{M(r;\mu_A)}{\sigma_d r^{d-1}} = 0 \tag{1.7.40}$$

in its positive set, where the mass function M is as defined in (1.5.2).

Since μ_A is a stationary solution of (1.7.19), if we go back to the original scaling, μ_A gives a self-similar solution of (1.1.1):

$$\mathcal{U}_A(x,t) = (t+1)^{-1}\mu_A(\frac{x}{(t+1)^{1/d}}), \tag{1.7.41}$$

It is then asked in [BCL2] and [B2] that whether this self-similar solution attracts all global solutions.

We will first prove that all radial solutions to the rescaled equation (1.7.19) converge to μ_A. The following lemma construct a family of explicit solutions to (1.7.19), which all converge to μ_A exponentially fast as $\tau \to \infty$.

Lemma 1.7.16 (**A family of explicit solutions**). *Suppose $V = \mathcal{N}$, $d \geq 3$ and $m = 2-2/d$. For $0 < A < M_c$, we denote by μ_A the stationary solution of* (1.7.19)*. Let $\bar{\mu}$ be defined as*

$$\bar{\mu}(\lambda,\tau) := \frac{1}{R^d(\tau)}\mu_A(\frac{\lambda}{R(\tau)}), \tag{1.7.42}$$

where $R(\tau)$ solves the ODE

$$\begin{cases} \dot{R}(\tau) = \frac{1}{d}(\frac{1}{R^d} - 1)R \\ R(0) = R_0, \end{cases} \tag{1.7.43}$$

where $R_0 > 0$ is a constant. Then for any $R_0 > 0$, $\bar{\mu}(\lambda,\tau)$ is a weak solution to (1.7.19)*.*

Proof. Since $\bar{\mu}$ is a self-similar function, it can be easily verified that $\bar{\mu}$ solves the following transport equation

$$\bar{\mu}_\tau + \nabla \cdot (\bar{\mu}\frac{\dot{R}(\tau)}{R(\tau)}\lambda) = 0.$$

On the other hand, note that (1.7.19) can also be written as a transport equation

$$\mu_\tau = \nabla \cdot (\mu\vec{v}),$$

where

$$\vec{v} = \frac{m}{m-1}\nabla \mu^{m-1} + \frac{\lambda}{d} + \frac{M(|\lambda|,\tau;\mu)}{\sigma_d |\lambda|^{d-1}}\frac{\lambda}{|\lambda|}.$$

Therefore, to prove that $\bar{\mu}$ solves (1.7.19), it suffices to verify that

$$-\frac{\dot{R}(\tau)}{R(\tau)}r = \frac{m}{m-1}\underbrace{\frac{\partial}{\partial r}\bar{\mu}^{m-1}}_{T_1} + \frac{r}{d} + \underbrace{\frac{M(r,\tau;\bar{\mu})}{\sigma_d r^{d-1}}}_{T_2} \quad \text{for } 0 \le r \le R(\tau). \tag{1.7.44}$$

Since $\bar{\mu}$ is a rescaling of μ_A,

$$T_1 = \frac{1}{R^{(m-1)d+1}}\left(\frac{\partial}{\partial r}\mu_A^{m-1}\right)\left(\frac{r}{R(\tau)}\right) = \frac{1}{R^{d-1}}\left(\frac{\partial}{\partial r}\mu_A^{m-1}\right)\left(\frac{r}{R(\tau)}\right),$$

where in the last inequality we used the fact that m is the critical power, i.e. $m = 2 - 2/d$. For T_2 in (1.7.44), the definition of $\bar{\mu}$ gives

$$T_2 = \frac{1}{R(\tau)^{d-1}}\frac{M(\frac{r}{R(\tau)};\mu_A)}{\sigma_d(\frac{r}{R(\tau)})^{d-1}} \quad \text{for } 0 \le r \le R(\tau).$$

Now recall that μ_A satisfies (1.7.40) in its positive set, which implies

$$\begin{aligned}
\text{RHS of (1.7.44)} &= \frac{1}{R(\tau)^{d-1}}\left(\frac{m}{m-1}\frac{\partial}{\partial r}\mu_A^{m-1}\left(\frac{r}{R(\tau)}\right) + \frac{r}{dR(\tau)} + \frac{M(\frac{r}{R(\tau)};\mu_A)}{\sigma_d(\frac{r}{R(\tau)})^{d-1}}\right) + \frac{1}{d}\left(1 - \frac{1}{R^d(\tau)}\right)r \\
&= \frac{1}{d}\left(1 - \frac{1}{R^d(\tau)}\right)r \\
&= -\frac{\dot{R}(\tau)}{R(\tau)}r,
\end{aligned}$$

where the last equality comes from the definition of R in (1.7.43). This verifies that (1.7.44) is indeed true, which completes the proof. □

Next we use the family of explicit solution constructed above as barriers, and perform mass comparison between the real solution and the barriers.

Proposition 1.7.17. *Suppose $V = \mathcal{N}, d \ge 3$ and $m = 2 - 2/d$. Let $\mu(\lambda,\tau)$ be a radially symmetric weak solution to (1.7.19) with mass $0 < A < M_c$, where the initial data $\mu(\cdot,0)$ is nonnegative, continuous and compactly supported. Then as $\tau \to \infty$, the mass function of μ converges to the mass function of μ_A exponentially, i.e.*

$$\sup_r |M(r,\tau;\mu) - M(r,\tau;\mu_A)| \le Ce^{-\tau},$$

where μ_A is as defined in (1.7.40), and C depends on d, A and $\mu(\cdot,0)$.

Proof. Without loss of generality we assume that $\mu(0,0) > 0$. (When $\mu(0,0) = 0$, from the same discussion in 1.7.13, $\mu(0,\tau)$ will become positive after some finite time.) Then we can find R_{01} sufficiently small and R_{02} sufficiently large, such that

$$\frac{1}{R_{02}^d}\mu_A(\frac{\cdot}{R_{02}}) \prec \mu(\cdot,0) \prec \frac{1}{R_{01}^d}\mu_A(\frac{\cdot}{R_{01}}),$$

where in the first inequality we used that $\mu(0,0) > 0$, and in the second inequality we used $\|\mu(\cdot,0)\|_\infty < \infty$.

Let $\mu_1(\lambda,\tau)$ and $\mu_2(\lambda,\tau)$ be defined as in (1.7.42), with $R(0)$ equal to R_{01} and R_{02} respectively. Then Lemma 1.7.16 says that both μ_1 and μ_2 are solutions to (1.7.19). Note that (1.7.19) is a special case of (1.5.1), hence the mass comparison result in Proposition 1.5.4 holds here as well, which gives

$$\mu_2(\cdot,\tau) \prec \mu(\cdot,\tau) \prec \mu_1(\cdot,\tau) \text{ for all } \tau \geq 0,$$

or in other words,

$$M(\cdot,\tau;\mu_2) \leq M(\cdot,\tau;\mu) \leq M(\cdot,\tau;\mu_1) \text{ for all } \tau \geq 0.$$

It remains to show that

$$\sup_r |M(r,\tau;\mu_i) - M(r;\mu_A)| \leq Ce^{-\tau} \text{ for } i = 1,2.$$

Recall that both μ_i's are scalings of μ_A with scaling coefficient $R_i(\tau)$, hence

$$|M(r,\tau;\mu_i) - M(r;\mu_A)| = |M(\frac{r}{R_i(\tau)};\mu_A) - M(r;\mu_A)|. \tag{1.7.45}$$

Since μ_A is bounded and compactly supported, it suffices to show that $R_i(\tau) \to 1$ exponentially as $r \to \infty$. Recall that $\dot{R}_i = \frac{1}{d}(\frac{1}{R_i^d} - 1)R_i$ with initial data R_{0i} for $i = 1,2$, a simple calculation reveals that $|R_i(\tau) - 1| \leq C_i e^{-\tau}$, where C_i depends on R_{0i}. This implies that the right hand side of (1.7.45) decays like $e^{-\tau}$, which completes the proof. □

Making use of the explicit barriers μ_1 and μ_2 constructed in the proof of Proposition 1.7.17, we get exponential convergence of μ/A towards the μ_A/A in the p-Wasserstein metric as defined in Definition 1.7.3. The proof is parallel to the proof of Corollary 1.7.4 and will be omitted here.

Corollary 1.7.18. *Let $V = \mathcal{N}, d \geq 3$, and $m = 2 - \frac{2}{d}$. Let $\mu(\lambda, \tau)$ and μ_A be as given in Proposition 1.7.17. Then for all $p > 1$, we have*

$$W_p(\frac{\mu(\cdot,\tau)}{A}, \frac{\mu_A}{A}) \leq Ce^{-\tau},$$

where C depends on d and $\mu(\cdot, 0)$.

Rescaling back to the original space and time variables, we have

$$\rho(x,t) = \frac{1}{t+1}\mu\left(\frac{x}{(t+1)^{1/d}}, \ln(t+1)\right).$$

Thus Corollary 1.7.4 immediately yields the algebraic convergence towards the dissipating self-similar solution (1.7.41):

Corollary 1.7.19. *Let $\rho(x,t)$ be a radial solution to* (1.1.1) *with mass $0 < A < M_c$, where the initial data $\rho(\cdot, 0) \in L^1_+(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ is continuous and compactly supported. Let $\mathcal{U}_A$ be the dissipating self-similar solution with mass A defined in* (1.7.41)*. Then ρ/A converges to $\mathcal{U}_A/A$ in Wasserstein distance algebraically fast as $t \to \infty$. More precisely,*

$$W_p(\frac{\rho(\cdot,t)}{A}, \frac{\mathcal{U}_A}{A}) \leq Ct^{-(d-1)/d},$$

where C depends on d, A and $\rho(\cdot, 0)$.

1.7.6 Critical regime with subcritical mass: convergence towards self-similar solution for non-radial solutions with small mass

In this subsection, we consider the rescaled equation (1.7.19) with general (possibly non-radial) initial data. The key result here is that when the mass $A < M_c$ is sufficiently small, the radially symmetric stationary solution μ_A as defined in (1.7.40) is the unique compactly supported stationary solution (in rescaled variables). Then a similar argument as in Theorem 1.7.14 shows that every solution to (1.7.19) with small mass and compactly supported initial data converges to μ_A. After scaling back to the original variables, we immediately obtain the convergence towards the self-similar solution if the mass is small.

We first prove a L^∞-regularization result, saying that if the initial mass is small, then the L^∞ norm of solution to (1.7.19) will become small after unit time, regardless of the L^∞

norm of the initial data. We point out that a similar L^∞-regularization result is proved in [SS2] for the 2D case with linear diffusion, using a De Giorgi type method.

Lemma 1.7.20. *Suppose $V = \mathcal{N}, d \geq 3$ and $m = 2 - 2/d$. Let $\mu(\lambda, \tau)$ be a weak solution to (1.7.19) with mass $0 < A < M_c/2$, where the initial data $\mu_0 \in L^1_+(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ is continuous. Then we have*

$$\|\mu(\cdot, \tau)\|_\infty \leq K_A := CA^{2/d} \text{ for all } \tau \geq 1, \tag{1.7.46}$$

where C is some constant depending only on d.

Proof. Similar argument as the proof of Lemma 1.7.13 yields that

$$\mu^*(\cdot, \tau) \prec \bar{\mu}(\cdot, \tau) \text{ for all } \tau \geq 0, \tag{1.7.47}$$

where $\bar{\mu}(\cdot, \tau)$ is the solution to (1.7.19) with initial data μ_0^*. Since μ_0^* is radially symmetric and bounded above, we can find R_0 sufficiently small, such that $\mu_0^* \prec \frac{1}{R_0^d}\mu_A(\frac{\cdot}{R_0})$, where μ_A is as defined in (1.7.40). It then follows from Proposition 1.5.4 and Lemma 1.7.16 that

$$\bar{\mu}(\cdot, \tau) \prec \frac{1}{R(\tau)^d}\mu_A(\frac{\cdot}{R(\tau)}) \text{ for all } \tau \geq 0, \tag{1.7.48}$$

where $R(\tau)$ satisfies the ODE (1.7.43) with initial data $R(0) = R_0$. Combining (1.7.47) and (1.7.48), we obtain that

$$\|\mu(\cdot, \tau)\|_\infty = \|\mu^*(\cdot, \tau)\|_\infty \leq \frac{1}{R(\tau)^d}\|\mu_A\|_\infty \text{ for all } \tau \geq 0.$$

In order to bound the right hand side of the above inequality, we first find an upper bound for $1/R(\tau)^d$. It can be readily verified that $\tilde{R}(\tau) = \min\{\frac{1}{2}\tau^{\frac{1}{d+1}}, \frac{1}{2}\}$ is a subsolution to (1.7.43) for any $R_0 > 0$, which implies that $R(\tau) \geq \tilde{R}(\tau) \geq \frac{1}{2}$ for all $\tau \geq 1$, thus $\frac{1}{R(\tau)^d} \leq 2^d$ for all $\tau \geq 1$.

Next we will estimate $\|\mu_A\|_\infty$. Note that μ_A is radially decreasing for any $0 < A < M_c$, moreover $\|\mu_A\|_\infty = \mu_A(0)$ is increasing with respect to A. Therefore we readily obtain a rough bound $\|\mu_A\|_\infty \leq C_1$ for all $0 < A < M_c/2$, where $C_1 = \mu_{M_c/2}(0)$ only depends on d.

Note that this rough bound of $\|\mu_A\|_\infty$ gives us an upper bound for the velocity field given by the interaction term, namely

$$\partial_r(\mu_A * \mathcal{N}) = \frac{M(r;\mu_A)}{\sigma_d r^{d-1}} \leq \frac{C_1 r}{d}. \tag{1.7.49}$$

To refine the bound for $\|\mu_A\|_\infty$, we compare μ_A with $\tilde{\mu}_A$, where $\tilde{\mu}_A$ is the radial stationary solution to the following equation

$$\mu_\tau = \Delta\mu^m + \nabla\cdot\big(\mu\nabla\frac{(1+C_1)|\lambda|^2}{2d}\big). \tag{1.7.50}$$

Making use of (1.7.49), mass comparison yields that $\mu_A \prec \tilde{\mu}_A$, which implies $\mu_A(0) \leq \tilde{\mu}_A(0)$. On the other hand note that (1.7.50) is a Fokker-Planck equation, whose stationary solution is given by

$$\tilde{\mu}_A = \Big(C_A - \frac{(1+C_1)(m-1)}{2dm}|\lambda|^2\Big)_+^{1/(m-1)},$$

where $C_A > 0$ is the unique constant such that $\|\tilde{\mu}_A\|_1 = A$. A simple algebraic manipulation shows that $\tilde{\mu}_A(0) \leq CA^{2/d}$, where $C > 0$ depends only on d, therefore we can conclude. □

The next lemma shows that if the mass is small, any solution with compactly supported initial data will eventually be confined in some small disk.

Lemma 1.7.21. *Suppose $V = \mathcal{N}, d \geq 3$ and $m = 2 - 2/d$. Then for any $R_0 > 0$, there exists some sufficiently small $A_0 > 0$, such that all weak solutions to* (1.7.19) *with continuous and compactly supported initial data and mass $0 < A < A_0$ will be eventually confined in $B(0, R_0)$.*

Proof. Let $\mu(\lambda,\tau)$ be a weak solution to (1.7.19) with continuous and compactly supported initial data and mass $0 < A < A_0$, where A_0 is a small constant depending on R_0 and d to be determined later.

In the proof of this lemma we take $\tau = 1$ to be the starting time, in order to take advantage of the estimate (1.7.46). Our goal is to show that if the support of $\mu(\cdot,1)$ is contained in some disk $B(0,R)$ where $R > R_0 - K_A$ and K_A is as defined in (1.7.46), then there exists some time $T > 1$ to be determined later, such that

$$\text{supp } \mu(\cdot,\tau) \subset B(0, R+K_A) \text{ for all } \tau \in [1,T], \tag{1.7.51}$$

moreover at time T the support can be fit into some disk smaller than $B(0,R)$, namely

$$\text{supp }\mu(\cdot,T)\subset B(0,R-K_A/2). \tag{1.7.52}$$

By taking T as the starting time and repeating this procedure, we know that eventually the support will be confined in $B(0,R_0)$.

In order to deal with non-radial solution, we shall construct barriers in the density sense instead of in mass sense. Although comparison principle in density sense does not directly hold for (1.7.19) due to the nonlocal term, if we treat $V(\lambda,\tau):=\mu*\mathcal{N}$ as a fixed *a priori* potential, then (1.7.19) becomes

$$\mu_\tau=\Delta\mu^m+\nabla\cdot\left(\mu\nabla\big(\frac{|\lambda|^2}{2d}+V(\lambda,\tau)\big)\right), \tag{1.7.53}$$

which is a porous medium equation with a drift, and the weak solutions to it enjoy the comparison principle due to [BH]. It follows from (1.7.46) that the following estimates of V holds:

$$\Delta V(\lambda,\tau)\le\sup_{\lambda,\tau}\mu\le K_A \text{ for } \lambda\in\mathbb{R}^d,\tau\ge1,$$

and

$$|\nabla V(\lambda,\tau)|\le\sup_{\mu,\lambda}(\mu*\frac{1}{\sigma_d|\lambda|^{d-1}})\le C(d)A^{\frac{3}{d}-\frac{2}{d^2}}\quad\text{for } \lambda\in\mathbb{R}^d,\tau\ge1.$$

Note that in both estimates above, the right hand side will go to zero as $A\to0$. We also point out that if $R\gg A^{\frac{3}{d}-\frac{2}{d^2}}$, then ∇V will be dominated by $\nabla\frac{|\lambda|^2}{2d}$ around $r=R$.

Next we will construct some explicit supersolution $\tilde\mu$ to (1.7.53). More precisely, we hope to find a continuous radially decreasing function $\tilde\mu$ defined in $\{r>R-K_A\}\times[1,T]$ for some T, such that $\tilde\mu$ satisfies the following inequality

$$\tilde\mu_\tau\ge\partial_{rr}\tilde\mu^m+(\partial_r+\frac{d-1}{r})(\frac{\tilde\mu r}{d})+\tilde\mu K_A+|\partial_r\tilde\mu|C(d)A^{\frac{3}{d}-\frac{2}{d^2}}\quad\text{for all } r>R-K_A,\tau\in[1,T], \tag{1.7.54}$$

while $\tilde\mu$ also satisfies the initial condition

$$\tilde\mu(r,0)\ge K_A\quad\text{for all } R-K_A\le r\le R, \tag{1.7.55}$$

and the boundary condition

$$\tilde\mu(r,\tau)\ge K_A\quad\text{at } r=R-K_A \text{ for all } \tau\in[1,T]. \tag{1.7.56}$$

The inequalities (1.7.54)–(1.7.57) guarantees that $\tilde{\mu}$ is a supersolution to (1.7.53). If A is small enough such that $R > CA^{\frac{3}{d}-\frac{2}{d^2}}$ for some large constant C depending on d, one can check that

$$\tilde{\mu}(\lambda, \tau) = \left[2K_A - \tau(r - (R - K_A))\right]_+^{1/m}$$

satisfies the inequalities (1.7.54)–(1.7.56) for $1 \leq \tau \leq 4$, hence comparison principle yields that $\mu \leq \tilde{\mu}$ in $\{r > R - K_A\}$ for all $\tau \in [1, 4]$.

The reason we choose $\tilde{\mu}$ as above is that its support will shrink after some time: note that its support stays in $B(0, R + K_A)$ for $\tau \in [1, 4]$, and most importantly, at $\tau = 4$, the support of $\tilde{\mu}$ can be fit into a disk smaller than $B(0, R)$, namely

$$\text{supp } \tilde{\mu}(\cdot, 4) \subset B(0, R - K_A/2). \tag{1.7.57}$$

Since comparison property gives that supp $\mu(\cdot, \tau) \subset$ supp $\tilde{\mu}(\cdot, \tau)$ for all $\tau \in [1, 4]$, we immediately obtain (1.7.51) and (1.7.52), which complete the proof. □

Making use of the above two lemmas, in the next theorem we show that when the mass is sufficiently small, there cannot be any non-radial stationary stationary solutions.

Theorem 1.7.22. *Suppose $V = \mathcal{N}, d \geq 3$ and $m = 2 - 2/d$. Then when $0 < A < M_c/2$ is sufficiently small, the compactly supported stationary solution to* (1.7.19) *is unique.*

Proof. Due to Corollary 1.7.4, we know that for any $0 < A < M_c$, there does not exist any compactly supported radial stationary solution other than μ_A. Hence it suffices to prove that when A is sufficiently small, every compactly supported stationary solution is radially symmetric.

Suppose $\nu_A(\lambda)$ is a compactly supported stationary solution to (1.7.19), which is not radially symmetric. Since ν_A is stationary, it satisfies

$$\frac{m}{m-1}\nu_A^{m-1} + \nu_A * \mathcal{N} + \frac{|\lambda|^2}{d} = C \text{ in } \overline{\{\nu_A > 0\}}, \tag{1.7.58}$$

where different positive components of ν_A may have different C's. Heuristically, the idea is to argue that the term $\nu_A * \mathcal{N}$ must be more "roundish" than $\frac{m}{m-1}\nu_A^{m-1}$ if ν_A is non-radial, thus get a contradiction.

We point out that (1.7.58) implies that ν_A is continuous in $\mathbb{R}^d$ and smooth inside its positive set. This enables us to find two points $a, b \in \mathbb{R}^d$ in the same connected component of $\overline{\{\nu_A > 0\}}$, satisfying $|a| = |b|$ and

$$\nu_A(a) - \nu_A(b) = \sup_{|x|=|y|} (\nu_A(x) - \nu_A(y)) > 0. \tag{1.7.59}$$

We claim that when A is sufficiently small, the following inequality holds

$$\frac{m}{m-1} \left| \nu_A^{m-1}(a) - \nu_A^{m-1}(b) \right| > |(\nu_A * \mathcal{N})(a) - (\nu_A * \mathcal{N})(b)|, \tag{1.7.60}$$

then (1.7.60) would contradict (1.7.58).

We start with the left hand side of (1.7.60): Lemma 1.7.20 implies that both $\nu_A(a)$ and $\nu_A(b)$ are much smaller than 1 when A is small. Since $0 < m - 1 < 1$, it follows that

$$\frac{m}{m-1} \left| \nu_A^{m-1}(a) - \nu_A^{m-1}(b) \right| > |\nu_A(a) - \nu_A(b)|.$$

if A is sufficiently small. In order to prove (1.7.60), it suffices to show that

$$|(\nu_A * \mathcal{N})(a) - (\nu_A * \mathcal{N})(b)| < |\nu_A(a) - \nu_A(b)|. \tag{1.7.61}$$

We introduce a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ which is a rotation that maps a to b. Then radial symmetry of $\mathcal{N}$ yields that $(\nu_A * \mathcal{N})(b) = ((\nu_A \circ T) * \mathcal{N})(a)$.

In addition, T being a rotation implies that $|T(x)| = |x|$ for any $x = \mathbb{R}^d$, hence from the way we choose a and b, we have $|\nu_A(T(x)) - \nu_A(x)| \leq \nu_A(a) - \nu_A(b)$ for any $x \in \mathbb{R}^d$. Thus

$$\begin{aligned} |(\nu_A * \mathcal{N})(a) - (\nu_A * \mathcal{N})(b)| &= \left| (\nu_A * \mathcal{N})(a) - ((\nu_A \circ T) * \mathcal{N})(a) \right| \\ &\leq \int_{\mathbb{R}^d} \left| \nu_A(y) - \nu_A(T(y)) \right| |\mathcal{N}(a-y)| dy \\ &\leq (\nu_A(a) - \nu_A(b)) \int_{B(0,R)} |\mathcal{N}(y)| dy, \end{aligned}$$

where $B(0, R)$ is the smallest disk that contains the support of ν_A. Now we make use of Lemma 1.7.21, which shows that we can fit the support of ν_A into an arbitrarily small disk by letting A be sufficiently small. Therefore we can choose R such that $\int_{B(0,R)} |\mathcal{N}(y)| dy < 1/2$, then let A be sufficiently small such that supp $\nu_A \subset B(0, R)$. This gives us (1.7.61), which leads to a contradiction and hence completes the proof. □

Remark 1.7.23. For general $0 < A < M_c$, we are unable to prove the uniqueness of the compactly supported stationary solution. The difficulty lies in the fact that for larger mass we are only able to show the support lies in a disk with radius $O(1)$. Hence instead of (1.7.61), we can only obtain $|(\nu_A * \mathcal{N})(a) - (\nu_A * \mathcal{N})(b)| < C|\nu_A(a) - \nu_A(b)|$, where C might be a large constant, which stops us from getting a contradiction.

Once we obtain the uniqueness of compactly supported stationary solution for small mass, the following corollary shows that all solution with compactly supported initial data must converge to this unique stationary solution as $\tau \to \infty$.

Corollary 1.7.24. *Suppose* $V = \mathcal{N}, d \geq 3$ *and* $m = 2 - 2/d$. *Let* $\mu(\lambda, \tau)$ *be a weak solution to* (1.7.19) *with mass* $0 < A < M_c/2$ *being sufficiently small, where the initial data* $\mu(\cdot, 0)$ *is nonnegative, continuous and compactly supported. Then as* $\tau \to \infty$*, we have*

$$\|\mu(\cdot, \tau) - \mu_A(\cdot)\|_\infty \to 0, \tag{1.7.62}$$

where μ_A *is as defined in* (1.7.40).

Proof. The proof is similar as the proof of Theorem 1.7.14, and actually it is simpler here since there is a unique stationary solution, instead of a family of stationary solution in the case of Theorem 1.7.14.

When the initial data $\mu(\cdot, 0)$ is bounded and compactly supported, Lemma 1.7.20 and Lemma 1.7.21 shows that $\mu(\cdot, \tau)$ would be uniformly bounded and stay in some fixed compact set for all $\tau \geq 1$. In addition, the continuity result in [D] indicates that $\mu(\lambda, \tau)$ is uniformly continuous in space and time in $\mathbb{R}^d \times [1, \infty)$.

As a result, for any time sequence τ_n that increases to infinity, using the same argument as in the proof of Theorem 1.7.14, we can extract a subsequence τ_{n_k} such that $\mu(\cdot, \tau_{n_k})$ uniformly converges to some continuous function μ_∞, where μ_∞ is a compactly supported stationary solution. Theorem 1.7.22 ensures that μ_∞ must coincide with μ_A when A is sufficiently small, yielding that $\mu(\cdot, \tau)$ indeed converges to μ_A uniformly as $\tau \to \infty$. □

Remark 1.7.25. Since $\mu(\cdot, 0)$ is confined in some compact set for all time, (1.7.62) implies that $\|\mu(\cdot, \tau) - \mu_A(\cdot)\|_p \to 0$ as $\tau \to \infty$ for all $p \geq 1$. Now if we scale back to the original variables, it immediately follows that $\|\rho(\cdot,t)-\mathcal{U}_A(\cdot)\|_p \to 0$ as $t \to \infty$ for all $p \geq 1$, where $\mathcal{U}_A$ is the dissipating self-similar solution as defined in (1.7.41). However the rate of convergence here is unknown, since the proof is done by extracting a subsequence of time.

CHAPTER 2

Blow-up Dynamics for Patlak-Keller-Segel Equation with Degenerate Diffusion

2.1 Introduction

2.1.1 Background

In this chapter, which is a joint work with Andrea Bertozzi [YB], we continue studying the aggregation-diffusion equation as in Chapter 1, however from a different perspective. Note that in Chapter 1, we discussed the qualitative behavior and asymptotic behavior of solutions to (1.1.1) when the solutions exist globally in time. In this chapter we focus on the finite-time blow-up case, and our goal is to numerically and asymptotically study the blow-up dynamics of (2.1.1) when the solution blows up in finite time. Moreover, in this chapter we consider more general kernels with power-law form, other than the Newtonian potential; and our spatial domain is bounded instead of the whole $\mathbb{R}^d$. The equation we consider in this chapter is

$$u_t = \Delta u^m - \nabla \cdot (u \nabla (K * u)) \text{ in } [0, T) \times \Omega, \tag{2.1.1}$$

with Neumann boundary condition, where $m \geq 1$, $\Omega = B(0, R) \subset \mathbb{R}^d$, and K is a radially symmetric potential with power-law form, i.e.

$$K(x) = \frac{1}{|x|^\gamma},$$

where K is either equal to or less singular than the Newtonian kernel at the origin, i.e. $\gamma \leq d - 2$.

As we have summarized in Chapter 1, the solution of (2.1.1) exhibits different behavior for different powers of m [DP, HV, S1, BlCM]: the problem is *supercritical* for $1 \leq m < 2-2/d$, where the solution may exhibit finite time blow-up phenomena; while for $m > 2-2/d$ the problem is *subcritical* and the solution is globally bounded for all time. Recently the notion of criticality is generalized to general power-law kernels $K = \dfrac{1}{|x|^\gamma}$ in [BRB]. For $d \geq 3$ and $\gamma \leq d-2$, they prove that the critical power m is given by $\dfrac{d+\gamma}{d}$. When $m > \dfrac{d+\gamma}{d}$ the solution stays uniformly bounded for all time, while when $m < \dfrac{d+\gamma}{d}$ there may be a finite-time blow-up. Moreover, at the critical power, they prove that there exists a critical mass M_c which sharply divides the possibility of finite time blow up and global existence.

In this chapter we only discuss the case $d \geq 3$, and the reason to omit dimension $d = 2$ is as follows. When $d = 2$, if K is equal to the Newtonian potential $\frac{1}{2\pi}\ln|x|$, the critical exponent is given by $m = 1$, and (2.1.1) becomes the original Patlak-Keller-Segel equation, which has been well studied both asymptotically and numerically [BCKSV, HV, CS, L]; if K is less singular than the Newtonian potential, then for any $m \geq 1$, the problem is in the subcritical regime, where all solutions have a global L^∞ bound and do not blow-up.

Once the existence/blow-up results are proven for (2.1.1), it is natural and interesting to examine the asymptotic behavior of the blow-up profile. Existing results only cover the following two special cases: one is the case without the diffusion term, and the other case is where K is the Newtonian potential. We review these cases in detail in the following discussion. The main goal of this chapter is to study the behavior of the blow-up solution for general power-law kernel and power-law degenerate diffusion.

In the absence of the diffusion term Δu^m, (2.1.1) becomes the aggregation equation, which arises in biological swarming models and aggregation in material science. It is rigorously proved in [BCL1] that the local vs. global well-posedness is distinguished by an Osgood condition on the kernel K. In particular, when the kernel K is given by $|x|^\gamma$, the solution has a finite time blow-up for $0 < \gamma < 2$, and has an infinite time blow-up for $\gamma \geq 2$. For this power-law kernel, some asymptotic results for radial blow-up solutions are obtained in [HB1, HB2]: when $\gamma < 2$, they show the radial solution blows up in finite time and exhibits a second type self-similarity, while for $\gamma > 2$ the aggregation happens in infinite time and exhibits a concentration of mass along a collapsing δ-ring. We point out a difference of the self-similar blow-up between (2.1.1) and the aggregation equation: although (2.1.1) and the aggregation equation both exhibit self-similar blow-up behavior, the self-similarity for aggregation equation is of second type, while the self-similarity for (2.1.1) is indeed of first type. This is because in (2.1.1) the three terms u_t, Δu^m and $\nabla \cdot (u \nabla (K * u))$ should be of the same order, which gives one more equation than the aggregation equation and thus uniquely fixed the scaling.

With the presence of a linear diffusion term Δu, when K is the Newtonian potential, the problem is supercritical when $d > 2$, and critical when $d = 2$. For $2 < d < 10$, the asymptotic blow-up behaviors are carefully studied in [BCKSV]. They showed that there are two stable blow-up modality, one is self-similar and the other one is non-self-similar and Burger-like. When $d = 2$ the blow-up behavior is more subtle. For critical mass $M = M_c$, it is shown in [KS] that the L^∞ norm of solution grows to infinity as $t \to \infty$, where $u_{\max} \sim e^{2\sqrt{2t}}$. For supercritical mass $M > M_c$, according to asymptotic expansions computed in [CS, L] (and [HV] for a similar model), as $t \to T$, the solution is "near-self-similar" and blows up in the form

$$u(r,t) \sim R(t)^2 \bar{u}(rR(t)) + 1_{\{r > R(t)\}} f(r), \tag{2.1.2}$$

where $R(t) \sim (T-t)^{-1/2} g(T-t)$, where $g(T-t)$ is some logarithmic correction term; and $f(|x|)$ is some locally integrable function in $\mathbb{R}^2$ which has a singularity at the origin.

When the equation (2.1.1) has a nonlinear diffusion term Δu^m and Newtonian potential K, the problem is *critical* when $m = 2 - 2/d$, where $d \geq 3$. For solutions with supercritical mass $M > M_c$, some results regarding the asymptotic behavior of blow-up solution are obtained in [BL]: they prove that there exists a self-similar blow-up solution when the mass ranges in some bounded interval $(M_c, M_2]$ for some threshold M_2, however the stability of those self-similar blow-up solutions remains unclear. When the mass is above M_2, they prove that there is no exact self-similar blow-up solution, and the blow-up scaling is still open.

In this chapter we use refined numerics compared with asymptotic analysis to understand blowup behavior in a radially symmetric setting. Thus for completeness we review related numerical results, many of which do not discuss the blow up problem. There are a number of approaches to solving (2.1.1) for the special case $m = 1$ and K being a Newtonian potential. These methods include the finite-element or discontinuous Galerkin methods presented in [E, EI, EK, F2, M, Sa, SS], the moving mesh method described in [BCR], a mass-transport steepest descent scheme in [BCC], a stochastic particle approximation method in [HS], and a composite particle-grid numerical method in [F]. However, among those approaches, few of them compute the blow-up profile: since (2.1.1) can blow-up in finite time in a diminishing length scale, it is a challenging numerical problem to capture the solution behavior precisely. There are two papers [BCKSV] and [BCR] directly addressing the blow-up profile, and both of their numerical methods rely on K being Newtonian: [BCKSV] directly solves for the mass function $M(r,t) := \int_0^r u(r) r^{d-1} dr$, which satisfies a local PDE when K is Newtonian; [BCR] deals with the parabolic-parabolic Keller-Segel problem, where the drift potential can be directly solved from a parabolic PDE. Their methods no longer work when K is a general power-law kernel $|x|^{-\gamma}$, and some efficient way to compute the convolution $K * u$ is needed.

2.1.2 Summary of results

As we mention above, there are few results addressing the dynamic behavior of the blow-up solutions to (2.1.1), and that motivates our study. Sections 2-4 investigate the possible asymptotic behavior of blow-up solutions to (2.1.1), and formally show that there are three different ways of blow-up. These asymptotic results are accompanied by numerical simulations, and in Section 5 we outline our numerical method, which is an arbitrary Lagrangian Eulerian method with adaptive mesh refinement. Our results are summarized below.

Asymptotics for blow-up solutions with supercritical power m (i.e. $1 \le m < \frac{d+\gamma}{d}$)

For supercritical m, we show that there are two kinds of possible blow-up behaviors. One of them is self-similar, where the scaling of the blow-up is

$$u(x,t) \sim (T-t)^{-\beta} w\left(\frac{x}{(T-t)^{\alpha}}\right) \text{ as } t \to T, \tag{2.1.3}$$

where the power $\alpha, \beta > 0$ are computed in Section 2.2 (see (2.2.2)). We point out that as t approaches the blow-up time T, the mass of the peak area goes to zero, indicating that no mass is concentrating at the origin.

Another possible blow-up behavior is an imploding smoothed out shock wave which collapses into a Dirac mass at the origin at the blow-up time. More specifically, u forms a delta concentration on an imploding spherical surface, thus the mass function $M(r,t) := \int_0^r u(y,t)y^{d-1}dy$ is forming an imploding shock. In Section 2.3 we show that the scaling associated to this kind of blow-up is

$$u(r,t) \sim Q(t)\ \varphi\left(\frac{r - R(t)}{\delta(t)}\right) \tag{2.1.4}$$

where $\lim_{t\to T} Q(t) = \infty$, $\lim_{t\to T} R(t) = 0$, and $\delta(t) \ll R(t)$ as $t \to T$. We compute the scaling for $Q(t), R(t)$ and $\delta(t)$ in Section 2.3.1 (see (2.3.6) and (2.3.9)), and derive the equation satisfied by φ in Section 2.3.3. We point out that while the same blow-up behavior is discovered in [BCKSV] for $m = 1$ and Newtonian K, this is the first work to find out the blow-up profile φ. Note that in this case a finite amount of mass is driven to the origin at the blow-up time, indicating that this type of blow-up is intrinsically different from the self-similar blow-up.

Another difference between the two type of blow-up is as follows. As long as $1 < m < \frac{\gamma+d}{\gamma}$ is in the supercritical regime, the self-similar blow-up can happen for any m, d, γ with a suitable initial data. However the non-self-similar blow-up requires an extra condition: in Section 2.3.2, we formally derive that the non-self-similar blow-up can only happen when the extra condition $\gamma > d - 3$ is satisfied, in addition to m being supercritical. Numerical evidence suggests that this extra condition is indeed required. Note that when both conditions are satisfied, the blow-up behavior depends on the initial data: the solutions with radially decreasing initial data may tend to blow-up self-similarly, while solutions with ring-shaped initial data may tend to blow-up like a Burger shock.

Asymptotics for blow-up solutions with critical power m (i.e. $m = \frac{d+\gamma}{d}$)

When m is critical, numerical evidence suggests that u is self-similar in the peak region. However the maximum density here does not grow like $(T-t)^{-\beta}$ as in (2.1.3), and is $(T-t)^{-\beta}g(T-t)$ instead, where $g(T-t)$ is some logarithm correction. We assume that as

$t \to T$, u is of the form

$$u(r,t) = \frac{1}{R(t)^d}\bar{u}(\frac{r}{R(t)}) + 1_{\{r>R(t)\}}f(r), \tag{2.1.5}$$

where $R(t) \ll (T-t)^\alpha$ as $t \to T$, here α is as in (2.2.2). We then obtain some preliminary results, indicating that $\bar{u}$ is the stationary solution to (2.1.1) with mass M_c. This implies that the mass in the peak area converges to M_c as $t \to T$, which is verified by numerical simulations. It is an interesting open problem to find the exact scaling for $R(t)$. We point out that this type of near-self-similar blow-up behavior is not new, and it is previously observed in the semilinear heat equation [GK] and Patlak-Keller-Segel problem in 2D [HV, CS, L].

Numerical method

In order to numerically compute the blow-up profile of (2.1.1), we use an arbitrary Lagrangian Eulerian method with adaptive mesh refinement. The main idea is to split equation (2.1.1) into two steps, one contains the aggregation part and the other contains the diffusion part. For the aggregation part, we adopt the method described in [HB1] and let the mesh move with the particles. The Lagrangian method is important for advection to avoid numerical diffusion [HB1]. Then we perform an adaptive mesh refinement and use an implicit finite volume scheme to solve the degenerate diffusion equation on a fixed mesh, due to its stability and easiness to implement. The advantage of our method is that we can compute the solution until the time is very close to the blow-up time, where the maximum density can be as large as 10^{80} and the characteristic spatial scale of the solution is 10^{-27}, while maintaining sufficient resolution to capture the detailed asymptotics of the blow-up profile.

2.2 Self-similar blow-up for supercritical power m

In this section we focus on the self-similar blow-up that happens for supercritical m, i.e. $m > \frac{d+\gamma}{d}$. Figure 2.1 gives a typical example of a self-similar blow-up, where no mass is concentrating at the origin as the time approaches the blow-up time. We determine the scaling for the blow-up, and formally derive the equation satisfied by the blow-up profile.

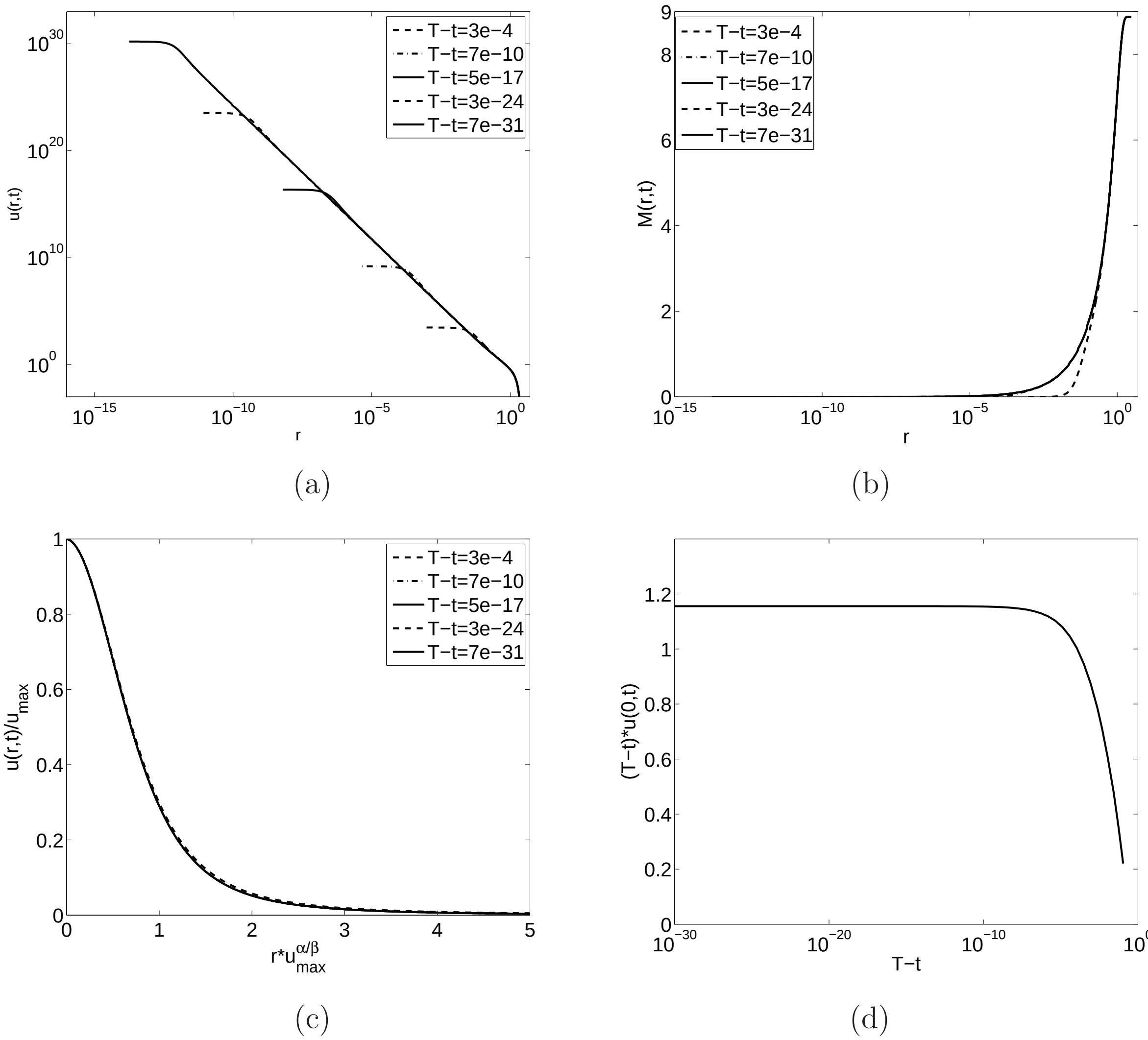


Figure 2.1: Time evolution of a self-similar blow-up solution with radially symmetric initial data, with Newtonian potential, $d = 3$ and $m = 1.2$ (supercritical). Figure (a) shows a log-log plot of the solution, which blows up at the origin at a finite-time T. Figure (b) shows the mass function $M(r,t)$ at different time, suggesting that no mass is concentrating at the origin as $t \to T$. Here $M(r,t)$ denotes the mass inside the ball $B(0,r)$ at time t. Figure (c) shows the rescaled solution, (here the scaling is determined in (2.2.2)), which converges to some blow-up profile. Figure (d) shows the evolution of $(T-t)u(0,t)$ as a function of $T-t$, which suggests that $u(0,t) \sim (T-t)^{-1}$ as $t \to T$.

2.2.1 Computing the exponents

We assume that as $t \to T$, u blows up at the origin self-similarly with the following form

$$u(x,t) \sim (T-t)^{-\beta} w(\frac{x}{(T-t)^{\alpha}}) \text{ as } t \to T. \tag{2.2.1}$$

In this subsection, our goal is to compute the exponents α and β.

We first compute the order for the two term u_t and Δu^m as $t \to T$:

$$u_t \sim (T-t)^{-(\beta+1)},$$

$$\Delta u^m \sim (T-t)^{-2\alpha-\beta m}.$$

For the term $\nabla \cdot (u\nabla(u * K))$, we begin with estimating the order for $u * K$:

$$\begin{aligned}(u*K)(x,t) &= \int_{\mathbb{R}^d} (T-t)^{-\beta} w(\frac{y}{(T-t)^{\alpha}}) \frac{1}{|x-y|^{\gamma}} dy \\ &= (T-t)^{-\beta+\alpha d} \int_{\mathbb{R}^d} w(z) \frac{1}{|x - z(T-t)^{\alpha}|^{\gamma}} dz \quad (\text{let } z = \frac{y}{(T-t)^{\alpha}}) \\ &= (T-t)^{-\beta+\alpha d - \gamma\alpha} (w * \frac{1}{|x|^{\gamma}})(x(T-t)^{-\alpha}),\end{aligned}$$

which implies that

$$\nabla \cdot (u\nabla(u*K)) \sim (T-t)^{-2\alpha-2\beta+\alpha d-\gamma\alpha}.$$

Since we are looking for a self-similar blow-up profile, we want u_t, Δu^m and $\nabla\cdot(u\nabla(u*K))$ to have the same order, which gives the following equations

$$-(\beta+1) = -2\alpha - \beta m = -2\alpha - 2\beta + \alpha d - \gamma\alpha.$$

Since there are two equations and two unknowns, we can explicitly solve for α, β in terms of γ, m and d:

$$\begin{cases} \alpha = \dfrac{2-m}{(m-1)(d-\gamma) - 2(m-2)}, \\ \beta = \dfrac{d-\gamma}{(m-1)(d-\gamma) - 2(m-2)}. \end{cases} \tag{2.2.2}$$

If the solution blows up, then α and β are both positive, which implies that the blow-up can only happen when $m < 2$. Also note that the mass in the ball $B(0, (T-t)^{\alpha})$ is of the order $(T-t)^{-\beta+d\alpha}$, which cannot go to infinity as $t \to T$ since we start with a finite mass.

This gives a necessary condition for the solution to blow-up, which is $-\beta + d\alpha \geq 0$, or in other words,

$$m \leq \frac{d+\gamma}{d}. \tag{2.2.3}$$

Note that $\frac{d+\gamma}{d}$ is exactly the critical exponent for (2.1.1) given by [BRB].

2.2.2 Self-similar blow-up profile

Assuming (2.2.1), we want to find the equation w satisfies. Plug w into (2.1.1), and let $y = x(T-t)^{-\alpha}$, we have that as $t \to T$,

$$\begin{aligned} u_t &\approx (T-t)^{-\beta-1}[\beta w(y) + \alpha y \cdot \nabla w(y)], \\ \Delta u^m &\approx (T-t)^{-2\alpha-\beta m}\Delta w(y), \\ \nabla \cdot (u\nabla(u * \frac{1}{|x|^\gamma})) &\approx (T-t)^{-2\alpha-2\beta+\alpha d-\gamma\alpha}\nabla \cdot (w\nabla(w * \frac{1}{|x|^\gamma}))(y). \end{aligned} \tag{2.2.4}$$

Therefore formally speaking, w should satisfies the following equation:

$$\beta w + \alpha y \cdot \nabla w = \Delta w^m - \nabla \cdot (w\nabla(w * \frac{1}{|x|^\gamma})). \tag{2.2.5}$$

Figure 2.2 shows the blow-up profile and scaling for different powers m, and Figure 2.3 illustrates that the blow-up profile indeed satisfies (2.2.5) for both Newtonian and non-Newtonian kernel K. It would be interesting to study the full behavior of (2.2.5), although it is outside the scope of our study. The case for $m = 1$ and Newtonian potential V is covered in [BCKSV].

2.2.3 Limit function outside the blow-up region

From the numerical simulation in Figure 2.1(a), we can see that for every $r > 0$, $u(r,t)$ converges to some limit $\psi(r)$ as $t \to T$. Now we will formally compute the outer solution $\psi(r)$.

Since $\psi(r)$ is stationary as $t \to T$, we have that ψ satisfies the following equation, where

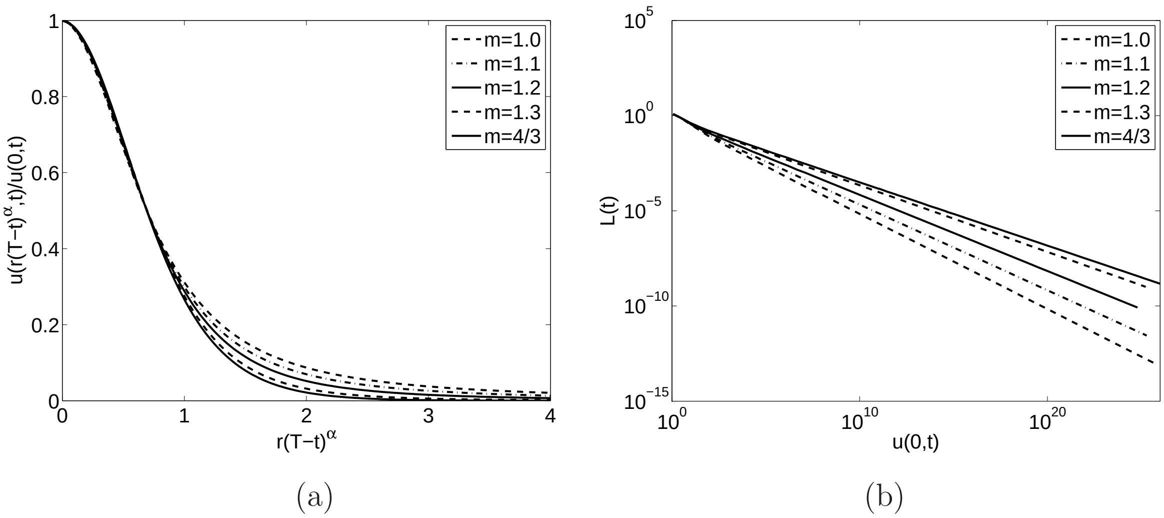
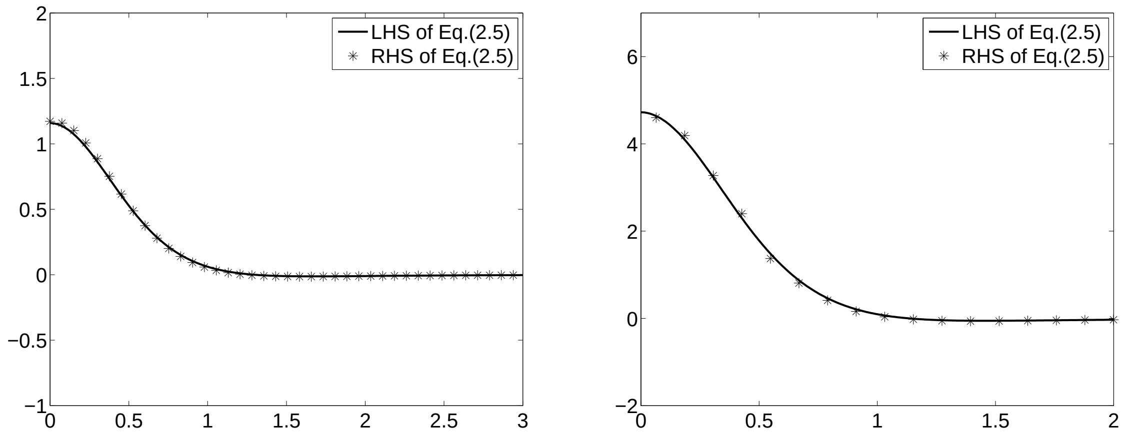


Figure 2.2: Behavior of solutions with different power m, with Newtonian potential in $d = 3$. (a) Blow-up profile for different m. (b) log-log plot of the height $u(0,t)$ and the width $L(t)$ for different m, where $L(t)$ is defined as the radius at half the height of $u(0,t)$. The slopes of the lines are in good agreement with the theoretically predicted values of -0.5, -0.45, -0.4, -0.35 and $-1/3$.



(a) (b)

Figure 2.3: Verification of (2.2.5) for the blow-up profile. The solid line is the plot of the left hand side of (2.2.5), and the stars represent the right hand side of (2.2.5). (a) Here the parameters are $m = 1.2$, $d = 3$ and $\gamma = 1$ (Newtonian). (b) Here the parameters are $m = 1.1$, $d = 3$ and $\gamma = 0.5$, where the kernel is less singular than the Newtonian kernel.

we slightly abuse the notation and write ψ as a radially symmetric function on $\mathbb{R}^d$:

$$\nabla(\psi^{m-1} + \psi * \frac{1}{|x|^\gamma}) = 0. \tag{2.2.6}$$

Since $\psi(r)$ appears to be a straight line on the log-log graph in Figure 2.1(a), we assume it takes the form $\psi(r) \sim r^{-a}$, where $a > 0$. Plug it into (2.2.6) and solve for a, we obtain $a = \frac{d-\gamma}{2-m}$, which implies the tail should satisfy

$$\psi(r) \sim r^{-\frac{d-\gamma}{2-m}}.$$

2.3 Non-self-similar blow-up for supercritical power m

When the initial data is not radially decreasing, it is possible for the solution to blow-up in finite time in a non-self-similar way. More precisely, the solution consists of an imploding smoothed out shock wave in the mass variable, which collapses into a delta function at the origin in finite time. Figure 2.4 gives an typical example of a non-self-similar blow-up.

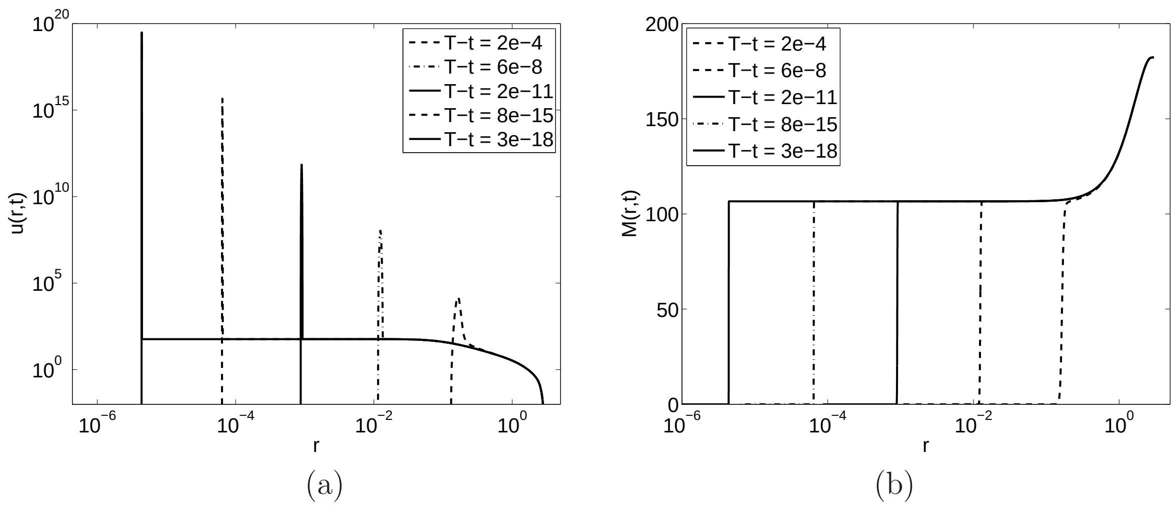


Figure 2.4: Time evolution of a non-self-similar blow-up solution with radially symmetric initial data, with Newtonian potential, $d = 3$ and $m = 1.2$(supercritical). Figure (a) shows a log-log plot of the solution, which blows up at the origin at a finite-time T. Figure (b) shows the mass function $M(r,t)$ at different time, indicating that there is a fixed amount of mass concentrating at the origin as $t \to T$. Here $M(r,t)$ denotes the mass inside the ball $B(0,r)$ at time t.

We assume $u(r,t)$ have the following blow-up profile

$$u(r,t) \sim Q(t)\, \varphi\left(\frac{r-R(t)}{\delta(t)}\right), \tag{2.3.1}$$

where $\lim_{t\to T} Q(t) = \infty$, $\lim_{t\to T} R(t) = 0$, and $\delta(t) \ll R(t)$ as $t \to T$. Figure 2.5 shows the re-centered and rescaled solution indeed converge to some function φ.

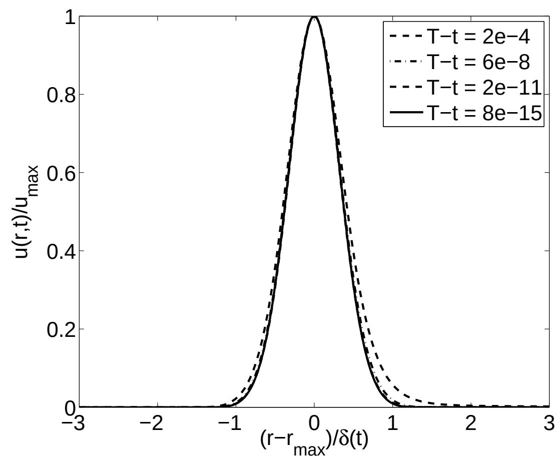


Figure 2.5: Simulation of a non-self-similar blow-up with radially symmetric initial data in three dimensions, with Newtonian potential and $m = 1.2$. The figure shows the re–centered and rescaled solution indeed converge to some function φ. The x-axis represents $(r - r_{max})/\delta(t)$, and the y-axis is the normalized density. Here $\delta(t)$ is computed according to (2.3.6).

2.3.1 Scaling for non-self-similar blow-up

We first figure out the relation between Q, δ and R. Since we only consider a radially symmetric solution, (2.1.1) can be written as

$$u_t = \underbrace{\partial_r^2 u^m}_{T_1} + \underbrace{\frac{d-1}{r}\partial_r u^m}_{T_2} - \underbrace{\partial_r u\ \partial_r(\frac{1}{|x|^\gamma} * u)}_{T_3} - \underbrace{u(\Delta\frac{1}{|x|^\gamma} * u)}_{T_4}. \tag{2.3.2}$$

Note that in the neighborhood of the peak, we have $\frac{1}{r}\partial_r u^m \sim Q^m/(R\delta)$ and $\partial_r^2 u^m \sim Q^m/\delta^2$, which gives $T_2 \ll T_1$, due to our assumption that $\delta(t) \ll R(t)$ as t goes to the blow-up time T. Thus T_2 becomes asymptotically irrelevant and can be ignored. We will require T_1, T_3,

T_4 to be of the same size in the neighborhood of the peak. For T_1, the previous discussion gives

$$T_1 \sim \frac{Q^m}{\delta^2}. \tag{2.3.3}$$

We next estimate the order of T_4. When $\gamma < d-2$, (i.e. the kernel is less singular than the Newtonian kernel), a direct computation gives

$$\Delta \frac{1}{|x|^\gamma} = \gamma(\gamma+2-d)\frac{1}{|x|^{\gamma+2}},$$

hence to obtain the order of T_4, it suffices to look at $(\frac{1}{|x|^{\gamma+2}} * u)(x)$ when $|x| - R(t) = O(\delta(t))$. For C large, we have

$$(\frac{1}{|x|^{\gamma+2}} * u)(x) \sim \int_{B(0,C\delta)} \frac{1}{|y|^{\gamma+2}} Q dy \sim Q\delta^{d-2-\gamma}. \tag{2.3.4}$$

hence when $|x| - R(t) = O(\delta(t))$, the computation above implies

$$T_4 \sim u\ (\frac{1}{|x|^{\gamma+2}} * u) \sim Q^2 \delta^{d-2-\gamma}. \tag{2.3.5}$$

Note that (2.3.5) holds for Newtonian kernel as well, since when $\gamma = d-2$, $\Delta|x|^{-\gamma}$ becomes a multiple of the delta function.

Finally, due to divergence theorem, we can evaluate the order of T_3 as follows

$$T_3 \sim \frac{Q}{\delta}\frac{1}{R^{d-1}} \int_{B(0,|x|)} \Delta \frac{1}{|x|^\gamma} * u dx.$$

Note that the integrand quickly vanishes to 0 as $|x| - R \gg \delta$. And when $|x| - R = O(\delta)$, the computation for T_4 yields that $\Delta|x|^{-\gamma} * u \sim \delta^{d-2-\gamma} u$, hence $\int_{B(0,|x|)} \Delta|x|^{-\gamma} * u dx \sim \delta^{d-2-\gamma} M$, where M is the mass of u around the peak. Recall that we assume that M is of order unity, which implies

$$T_3 \sim Q R^{1-d} \delta^{d-3-\gamma}.$$

Since we assume T_1, T_3, T_4 are of the same order, we finally obtain that $Q(t), R(t)$ and $\delta(t)$ should satisfy the following relation

$$R(t) \sim Q(t)^{-\frac{d-\gamma+m-2}{(d-\gamma)(d-1)}}, \quad \delta(t) \sim Q(t)^{\frac{m-2}{d-\gamma}}. \tag{2.3.6}$$

Now we compute the order of $Q(t)$ in terms of $T-t$, where T is the blow-up time. From the previous computation, when $|x|-R=O(\delta)$,

$$\text{RHS of (2.2)} \sim Q^{m-\frac{2(m-2)}{d-\gamma}}, \tag{2.3.7}$$

and the order of the left hand side is

$$\text{LHS of (2.2)} \sim \dot{Q} + Q\frac{d}{dt}(\frac{r-R(t)}{\delta(t)}) \sim \dot{Q}Q^{\frac{d+\gamma-md}{(d-\gamma)(d-1)}}. \tag{2.3.8}$$

Combining (2.3.7) and (2.3.8), we obtain

$$\dot{Q}Q^{\frac{d+\gamma-md}{(d-\gamma)(d-1)}} \sim Q^{m-\frac{2(m-2)}{d-\gamma}},$$

which implies

$$Q(t) \sim (T-t)^{-\frac{(d-\gamma)(d-1)}{(d-\gamma)(md-m-d+2)-(d-2)(m-2)}}. \tag{2.3.9}$$

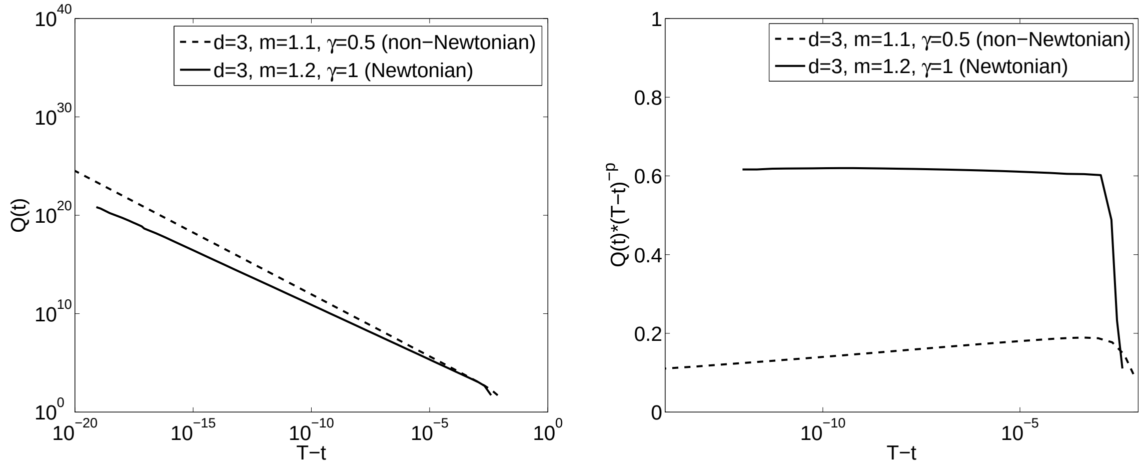
In the special case of the Newtonian potential, $d-\gamma = 2$, and $Q(t)$ simplifies to $Q(t) \sim (T-t)^{-\frac{2(d-1)}{md}}$, which is in agreement with the result in [BCKSV] for the special case $m=1$, and our work generalize their result to general m, d amd γ.

Figure 2.6(a) shows that when the solution blows up self-similarly, the maximum density $Q(t)$ indeed has the same scaling as (2.3.9). However, Figure 2.6(b) suggests that there is a difference between the Newtonian kernel and kernels less singular than Newtonian: for the Newtonian kernel (i.e. $\gamma = d-2$), numerical simulation suggests that $Q(t)(T-t)^{2(d-1)/md}$ converges to a constant as t goes to the blow-up time T, while for less singular kernel (i.e. $\gamma < d-2$) we have $Q(t)(T-t)^{-p}$ goes to 0 slowly as $t \to T$, where p is the power in (2.3.9).

2.3.2 Requirements for the parameters

In this subsection, we derive some requirements for the parameters d, m and γ in order for the non-self-similar blow-up to happen. Recall that in (2.3.1), we assumed that $Q \to \infty, \delta \ll R \to 0$ as $t \to T$. Here $\delta \ll R \to 0$ as $t \to T$ implies that the following conditions on m, d and γ are required

$$m \leq \frac{d+\gamma}{d} \quad \text{and} \quad \gamma < d-2+m. \tag{2.3.10}$$



(a) (b)

Figure 2.6: (a) Log-log plot of the maximum density $Q(t)$ versus $(T-t)$, where T is the blow-up time. The slopes of the lines are in good agreement with the theoretically predicted values in (2.3.9). (b) Plot of $Q(t)(T-t)^{-p}$ versus $(T-t)$, where p is the exponent as given in (2.3.9).

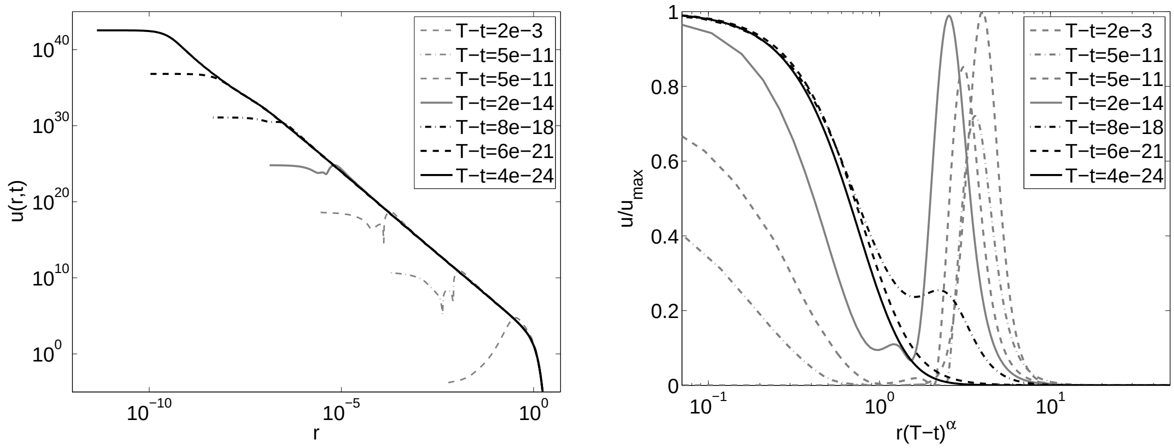
Note that the first requirement coincides with the criteria for supercritical m, hence is automatically satisfied in the supercritical regime. We point out that the second requirement can indeed be removed since we assume K is no more singular than the Newtonian potential at the origin, i.e. $\gamma \leq d-2$. Once the above conditions are met, $Q \to \infty$ as $t \to T$ will be automatically satisfied since $m \geq 1$.

Next we argue that an extra requirement is needed besides (2.3.10). Recall that in equation (2.3.4), we assumed that for $|x| = R + O(\delta)$, $(|x|^{-(\gamma+2)} * u)(x)$ is approximately equal to $\int_{B(0,C\delta)} |y|^{-(\gamma+2)} u(x-y)dy$ when C is of order unity and sufficiently large, which is comparable to $\delta^{-(\gamma+2)}Q\delta^d$. This requires that the tail of the kernel K be small such that the contribution from the term $\int_{\mathbb{R}^d \setminus B(0,C\delta)} |y|^{-(\gamma+2)} u(x-y)dy$ can be negligible, which implies that

$$\delta^{-(\gamma+2)}Q\delta^d \gg R^{-(\gamma+2)}QR^{d-1}\delta,$$

which simplifies to the following extra requirement

$$\gamma > d-3. \tag{2.3.11}$$



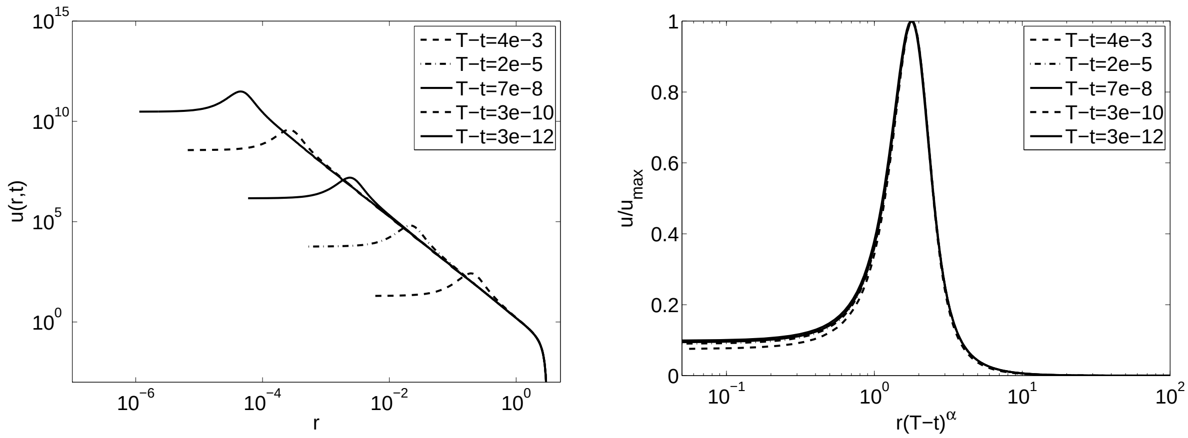
(a) (b)

Figure 2.7: Behavior of solutions with $d = 5, \gamma = 1$ and $m = 1.1$, where the condition (2.3.10) is satisfied but (2.3.11) is not. Even with a very singular ring-shaped initial data, it does not blow-up as an imploding shock. Figure (a) shows a log-log plot of the solution, which blows up at the origin at a finite-time T in a self-similar way with the scaling as in section 2. Figure (b) shows the rescaled solution, which eventually converges to some radially decreasing blow-up profile.

Figure 2.7 provides numerical evidence that this extra requirement (2.3.11) is indeed valid. When all the conditions in (2.3.10) are met but not (2.3.11), even we start with a very singular ring-shaped initial data, the solution still behaves according to the scale for self-similar solution in Section 2, and eventually converges to a self-similar profile that is radially decreasing.

When both conditions (2.3.10) and (2.3.11) are met, the solution may blow-up in either a self-similar way or non-self-similar way, depending on its initial data. The examples shown in Figure 2.1 and 2.4 indeed have the same m, d and γ, and the only difference is that we start with a radially decreasing initial data in Figure 2.1 and a ring-shaped initial data in Figure 2.4. In Figure 2.8, we carefully choose the initial data that is close to the separatrix between the self-similar one and non-self-similar one. The blow-up turns out to be self-similar with the scaling as in Section 2, where the blow-up profile is not radially decreasing.

We point out that this blow-up profile is unstable, and eventually the solution will be either attracted to a radially decreasing blow-up profile, or an imploding shock wave. This result is in agreement with [BCKSV] where $m = 1$ and K is Newtonian, where they conjectured that this separatrix has connection with the unstable blow-up modality that has exactly one linearly unstable mode.



(a) (b)

Figure 2.8: Blow-up profile in $d = 3$, $\gamma = 1$ and $m = 1.2$ (supercritical), with the initial condition chosen to be very close to the separatrix between the self-similar blow-up and non-self-similar blow-up. This blow-up profile is unstable and will be eventually attracted to either the non-self-similar blow-up or the self-similar blow-up with a radially decreasing profile. Figure (a) shows a log-log plot of the solution, and Figure (b) is the rescaled solution.

2.3.3 Similarity profile for Newtonian kernel

When K is the Newtonian potential, when the solution blows up non-self-similarly according to (2.3.1), numerical evidence in Figure 2.5 suggests that the rescaled and re-centered solution converges to some blow-up profile φ. In this subsection, our goal is to find the equation that φ satisfies. We point out that this result is new even for Newtonian kernel: although the scaling of Q, R and δ has been studied in [BCKSV] for $m = 1$ and the Newtonian potential K, this is the first work to investigate the blow-up profile φ and find the equation it satisfies.

Let e_1 be the unit vector $(1, 0, \ldots, 0)$, and we investigate the behavior of solution to (2.1.1) near the point $((R(t) + y\delta(t))e_1, t)$, which corresponds to $\varphi(y)$. As $t \to T$, we have

$$u_t((R + y\delta)e_1, t) \approx \dot{Q}Q^{-\frac{2+md-2d}{2(d-1)}}\varphi'(y), \tag{2.3.12}$$

and

$$\Delta u^m((R + y\delta)e_1, t) \approx \frac{Q^m}{\delta^2}(\varphi^m(y))''. \tag{2.3.13}$$

The estimation of the last term $\nabla \cdot (u\nabla(u * K))$ is as follows. Note that when K is the Newtonian potential $\frac{1}{|x|^{d-2}}$, ΔK is $(2-d)\omega_{d-1}\delta(x)$ in the distribution sense, where $\delta(x)$ is the delta function and ω_{d-1} is the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. Hence

$$\begin{aligned}
\frac{\partial}{\partial r}(u * \frac{1}{|x|^\gamma})(R + y\delta, t) &= \frac{\displaystyle\int_{B(0,R+y\delta)} \Delta u * \frac{1}{|x|^\gamma}dx}{|\partial B(0, R + y\delta)|} \\
&= \frac{(2-d)\displaystyle\int_{B(0,R+y\delta)} udx}{R^{d-1}} \\
&\approx (2-d)\omega_{d-1}Q\delta \int_{-\infty}^{y} \varphi(z)dz,
\end{aligned}$$

where in the last line we used the fact that $u(r,t)$ is very small when $|r - R| \gg \delta$. and as a result,

$$\nabla \cdot (u\nabla(u * K))((R + y\delta)e_1, t) \approx (2-d)\omega_{d-1}\big(Q^2\varphi'(y)\int_{-\infty}^{y} \varphi(z)dz + Q^2\varphi^2(y)\big). \tag{2.3.14}$$

Combining (2.3.12), (2.3.13) and (2.3.14) together, we obtain that φ satisfies the following equation

$$-\frac{1}{d}\varphi'(y) = (\varphi^m(y))'' - (2-d)w_{d-1}\big(\varphi'(y)\int_{-\infty}^{y} \varphi(z)dz + \varphi^2(y)\big). \tag{2.3.15}$$

Figure 2.9 provides numerical evidence that the rescaled and re-centered blow-up profile indeed satisfies (2.3.15).

2.4 Near-self-similar blow-up for critical power m

When $m = \frac{d+\gamma}{d}$, it is proved in [BRB] that there exists a critical mass M_c depending on γ and d, such that the solution to (2.1.1) exists globally in time for $M < M_c$, while for any

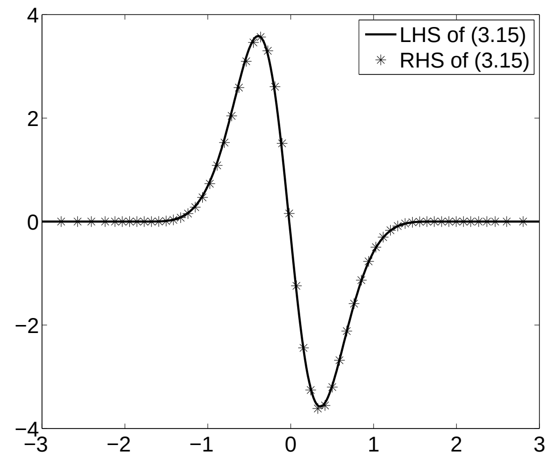


Figure 2.9: Verification of Eq.(2.3.15) for the blow-up profile of a non-self-similar blow-up, where $m = 1.2$, $d = 3$ and $\gamma = 1$. The solid line is the left hand side of Eq.(2.3.15), and the stars represent the right hand side of Eq.(2.3.15).

$M > M_c$ there exists a solution with mass M that blows up in finite time. In addition, for Newtonian potential, by using a comparison principle on the mass concentration, it is proved in [BK] that every radial solution with mass $M > M_c$ must blow up in finite time. We point out that their method can be generalized to the general power-law kernel $K = |x|^{-\gamma}$ as well.

In this section we let $m = \frac{d+\gamma}{d}$ be the critical power, and we study the blow-up behavior for solution with supercritical mass $M > M_c$. Let u be the weak solution to (2.1.1) with supercritical mass, which blows up at some finite time T. Figure 2.10 is a typical result of the simulation. While Figure 2.10(c) suggests that the blow-up is self-similar in its peak region, it is no longer of the form (2.2.1): suppose that u blows up with the form (2.2.1) as $t \to T$, then the same argument as in Section 2 would imply that the α and β given in (2.2.2) are the only possible exponents, and hence $(T-t)^\beta u(0,t)$ should converge to some finite number $w(0)$ as $t \to T$. However, numerical simulation of $(T-t)^\beta u(0,t)$ in Figure 2.10(d) suggests that this is not true, since $(T-t)^\beta u(0,t)$ is slowly increasing to infinity as $t \to T$, instead of converging to a constant.

Because of the self-similarity of u in the peak region, we assume that as $t \to T$, u is of the form

$$u(r,t) = \frac{1}{R(t)^d}\bar{u}(\frac{r}{R(t)}) + 1_{\{r>R(t)\}}f(r), \tag{2.4.1}$$

where $R(t) \ll (T-t)^\alpha$ as $t \to T$, here $\alpha = \frac{2-m}{(m-1)(d-\gamma)-2(m-2)}$ is as in (2.2.2).

It remains to determine $R(t)$, $\bar{u}$ and $f(r)$. If T is the blow-up time, we first introduce the following similarity variables

$$y = x(T-t)^{-\alpha}, \tau = -\ln(T-t),$$

and $U(y,\tau) := (T-t)^\beta u(x,t)$, where α, β are given by (2.2.2). Then a quick computation reveals that $U(y,\tau)$ satisfies the following equation

$$U_\tau = \Delta U^m - \nabla \cdot (U\nabla(U * \frac{1}{|y|^\gamma})) - \alpha\nabla U \cdot y - \beta U. \tag{2.4.2}$$

Since we assume that $R(t) \ll (T-t)^\alpha$ as $t \to T$, we would expect that there is an inner layer of size $\epsilon(\gamma)$ in (2.4.2), where $\epsilon(\tau) \to 0$ as $\tau \to \infty$, however $\epsilon(\tau)$ should be bigger than any decaying power-law function as $\tau \to \infty$. Moreover, we expect that U is self-similar in this inner-layer and contains a fixed amount of mass in the inner-layer. Hence we introduce another scaling

$$\xi = \frac{y}{\epsilon(\tau)},$$

and

$$\tilde{U}(\xi,\tau) := \epsilon(\tau)^d U(y,\tau).$$

Then $\tilde{U}(\xi,\tau)$ satisfies

$$\tilde{U}_\tau = \epsilon^{-d(m-1)-2}\big(\Delta\tilde{U}^m - \nabla\cdot(\tilde{U}\nabla(\tilde{U} * \frac{1}{|\xi|^\gamma})\big) + \big(\alpha\nabla\tilde{U}\cdot\xi + \beta\tilde{U}\big) + \dot{\epsilon}\epsilon^{-1}\big(\nabla\tilde{U}\cdot\xi + d\tilde{U}\big). \tag{2.4.3}$$

After performing this rescaling, we expect $\tilde{U}$ to converge to some stationary blow-up profile $\bar{U}(\xi)$, hence we assume that $\tilde{U}_\tau \to 0$ as $t \to \infty$. As τ goes to infinity, note that the terms on the right hand side of (2.4.3) are not of the same order, due to the assumption that $\epsilon(\tau)$ slowly decays to 0 as $\tau \to \infty$. Recall that we only consider the case $m \geq 1$, which implies that $\epsilon^{-d(m-1)-2} \gg \dot{\epsilon}\epsilon^{-1} \gg 1$. Hence $\bar{U}$ satisfies the equation

$$\Delta\bar{U}^m - \nabla\cdot(\bar{U}\nabla(\bar{U} * \frac{1}{|\xi|^\gamma})) = 0. \tag{2.4.4}$$

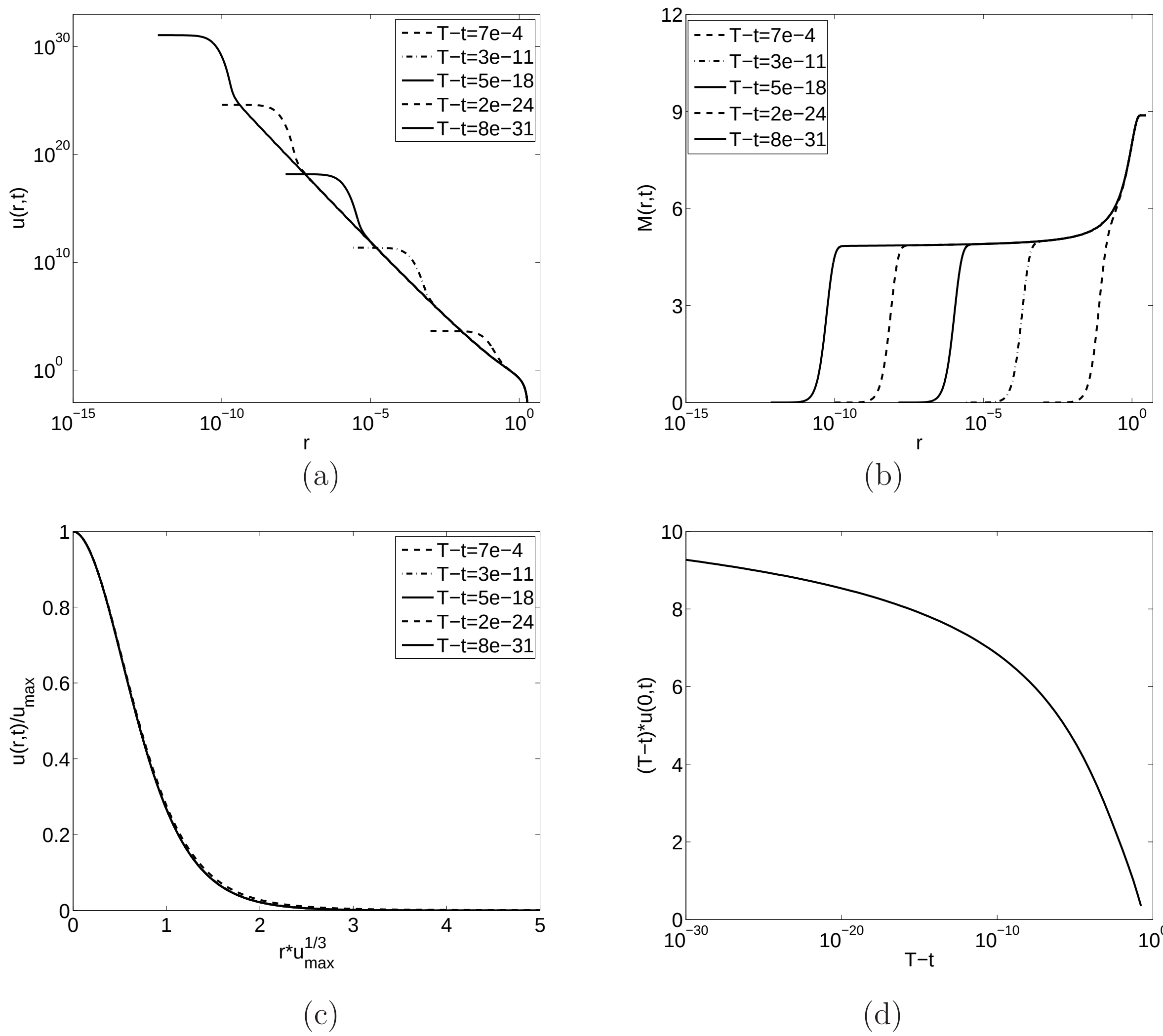


Figure 2.10: Time evolution of a near-self-similar blow-up solution with radially symmetric initial data, with Newtonian potential, $d = 3$ and $m = 4/3$ (critical). Figure (a) shows a log-log plot of the solution, which blows up at the origin at a finite-time T. Figure (b) shows the mass function $M(r,t)$ at different time, indicating that there is a fixed amount of mass contained in the peak area as $t \to T$. Here $M(r,t)$ denotes the mass inside the ball $B(0,r)$ at time t. Figure (c) is a rescaling of the peak area, which indicates that the peak area is self-similar and it converges to some profile. Figure (d) shows the evolution of $(T-t)u(0,t)$ as a function of $T-t$, which suggests that $u(0,t) \sim (T-t)^{-1} f(T-t)$ as $t \to T$, where f is some logarithmic correction term.

For Newtonian potential, it is proved in [BCL2] that the only radially symmetric solution to (2.4.4) has mass M_c, where M_c is the critical mass. That suggests that $\bar{U}$ is the unique stationary solution for (2.1.1) with critical mass M_c. Hence as $t \to T$, the peak should contain exactly the critical mass M_c, which fits our observation in Figure 2.10 (b). Moreover, Figure 2.11 suggests that the rescaled blow-up profile indeed coincides with the stationary solution.

It is an interesting question to solve for the logarithmic corrector $\epsilon(\tau)$; this has been done for $m = 1$ and the Newtonian kernel K in [HV, L, CS], however it is an open problem for general m and K.

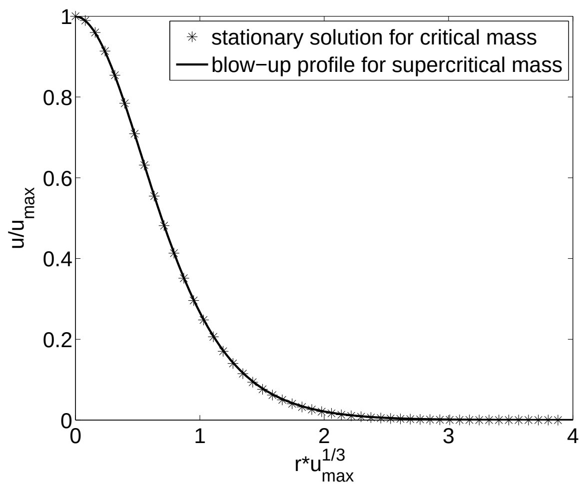


Figure 2.11: Comparison between the rescaled blow-up profile and the stationary solution with critical mass. The solid line is the rescaled blow-up profile of a a near-self-similar blow-up solution with radially symmetric initial data, with Newtonian potential, $d = 3$ and $m = 4/3$ (critical). The star symbol is the (rescaled) stationary solution with critical mass M_c.

2.5 Numerical method

The numerical method we use is a combination of a Lagrangian method and an Eulerian method. The main idea is to split the equation (2.1.1) into two steps, one with the aggregation part only and the next with the diffusion part only, and we run those two steps alternatively. More precisely, in every time step $[t, t+dt]$, we first use the method of characteristic to solve the aggregation equation

$$v_t = -\nabla \cdot (v\nabla(K * v)) \quad \text{for } t \in [t, t+dt], \tag{2.5.1}$$

where the time step is chosen to be small enough such that the characteristics do not intersect. After reconstructing the density from the particle locations and performing an adaptive mesh refinement, we then use an implicit finite-difference scheme to solve the following diffusion equation for another time step,

$$w_t = \Delta w^m \quad \text{for } t \in [t, t+dt], \tag{2.5.2}$$

where the initial data of w is taken from the result of the aggregation step, namely $w(x,t) = v(x, t+dt)$. Then we set $v(x, t+dt) = w(x, t+dt)$ and start the next time step.

2.5.1 Advection Step

For the advection step, due to the underlying transport structure of (2.5.1), it can be solved by method of characteristics. To do this we follow the method in [HB1], which we present here for the sake of completeness.

Assume the radial domain is partitioned into the intervals $0 = r_0 < r_1 < \ldots < r_N = R$, where the mass of u is m_i in the ring $B(0, r_{i+1})\backslash B(0, r_i)$ for $i = 0, \ldots, N-1$. We then approximate u by a system of N delta rings located at radius $r_0, \ldots, r_N$ with mass $m_1, \ldots, m_N$ respectively. Note that (2.5.1) is a transport equation, where the outward velocity field at radius r is given by

$$v(r) = -\frac{\partial}{\partial r}(u * K).$$

Hence the ith ring is moving inwards with velocity

$$\frac{d}{dt} r_i(t) = \sum_{j=1}^{N} m_j v_{r_j}(r_i), \tag{2.5.3}$$

where $v_{r_j}(r_i)$ is the outward velocity at r_i caused by a delta ring with unit mass located at radius r_j. $v_R(r)$ is given by the following integral

$$
\begin{aligned}
v_R(r) &= -\int_0^\pi K'(\sqrt{R^2+r^2-2rR\cos\theta})\frac{R\cos\theta - r}{\sqrt{R^2+r^2-2rR\cos\theta}}(R\sin\theta)^{d-2}\omega_{d-1}d\theta/(\omega_d R^{d-1}) \\
&= -\gamma\frac{w_{d-1}}{w_d}\int_0^\pi \frac{R\cos\theta - r}{R(R^2+r^2-2rR\cos\theta)^{\gamma/2+1}}(\sin\theta)^{d-2}d\theta,
\end{aligned}
$$

Due to the homogeneity of the kernel K, we can define $\rho = \dfrac{\min\{r,R\}}{\max\{r,R\}}$, then $v(r)$ becomes

$$
v_R(r) = \begin{cases} -\gamma\dfrac{w_{d-1}}{w_d}R^{-\frac{\gamma}{2}-1}I_1(\rho), & \text{if } r \le R \\[2ex] -\gamma\dfrac{w_{d-1}}{w_d}r^{-\frac{\gamma}{2}-1}I_2(\rho), & \text{if } r > R, \end{cases}
$$

where the two auxiliary functions $I_1(\rho), I_2(\rho)$ are defined by

$$
\begin{aligned}
I_1(\rho) &= \int_0^\pi \frac{\cos\theta - \rho}{(1+\rho^2-2\rho\cos\theta)}(\sin\theta)^{d-2}d\theta, \\
I_2(\rho) &= \int_0^\pi \frac{\rho\cos\theta - 1}{(\rho^2+1-2\rho\cos\theta)}(\sin\theta)^{d-2}d\theta,
\end{aligned}
$$

and we only need to perform numerical integration for $I_1(\rho)$ and $I_2(\rho)$ once, for $0 \le \rho \le 1$, which reduce the complexity to $O(N^2)$ to evaluate the right hand side of (2.5.3). Once we have the velocity of each delta ring in (2.5.3), we use the classical forth order Runge-Kutta method to evaluate the position at the next time step.

2.5.2 Regridding and interpolation

2.5.2.1 Reconstruct density from particle locations

After the aggregation step, we have new locations of the $\delta-$rings. Assume the i-th $\delta-$ring is now located at radius r_i. We can reconstruct the density u from the particle location as following. We denote by $\bar{u}_i$ the the average density in the ring $[r_i, r_{i+1}]$, then $\bar{u}_i$ is given by $\frac{m_i}{|B(0,r_{i+1})\backslash B(0,r_i)|}$, where m_i is the mass of the i-th δ-ring.

2.5.2.2 Adaptive mesh refinement

Since we are interested in the blow-up profile, we perform an adaptive mesh refinement that efficiently captures the scaling of the blow-up without losing resolution.

Figure 2.12 shows that for both self-similar blow-up and non-self-similar blow-up, the density changes slowly outside of the singularity area, due to the fact that the blow-up is localized. Thus we dedicate a fixed portion of the grid to the singularity area and a fixed portion outside of that area.

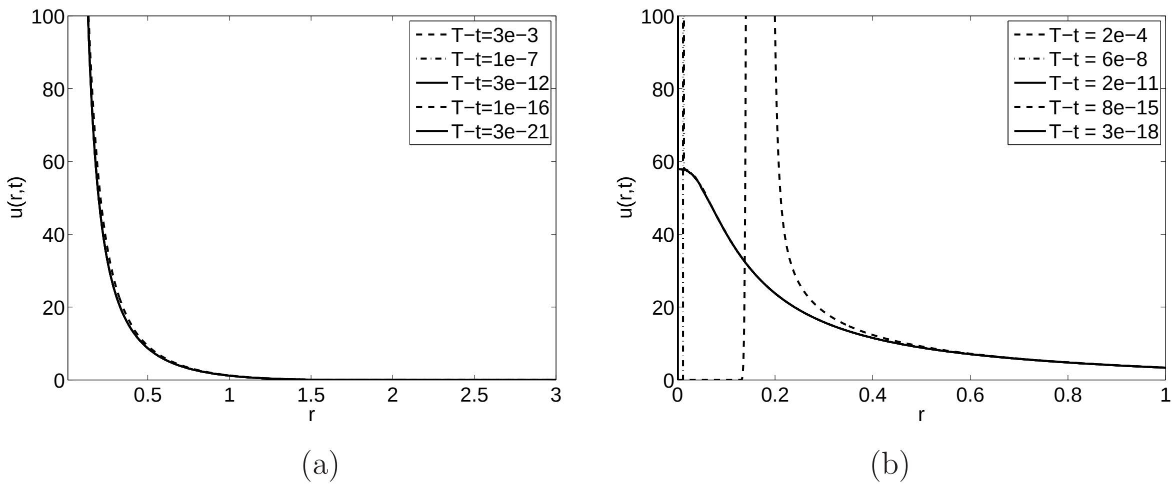


Figure 2.12: Behavior of solutions away from the blow-up point, where $d = 3, \gamma = 1$ and $m = 1.2$. Figure (a) shows the tail of solution in the case of self-similar blow-up, and Figure (b) is for non-self-similar blow-up.

More precisely, we first find the peak location r_{max}, then we locate the peak area $[r_1, r_2]$ to be the part where $u(r) \geq u(r_{max})/1000$. This interval would contain the singularity area in the case for the self-similar blow-up, but to make it also work for the non-self-similar blow-up, we enlarge the interval to $[r_L, r_R]$, where $r_L = \max\{0, r_{max} - 2(r_{max} - r_1)\}$, $r_R = r_{max} + 2(r_2 - r_{max}))$.

We start with an initial grid of $N = N_0$ points, in the example here $N_0 = 500$. After locating the interval $[r_L, r_R]$ using the method above, we partition $[r_L, r_R]$ into $N/2$ equal-length grids, and partition the remaining set $[0, a]\backslash[r_L, r_R]$ into another $N/2$ equal-length grids. Note that the length of $[r_L, r_R]$ will go to zero as the time approaches the blow-up time, so the grid size inside the interval $[r_L, r_R]$ will be much smaller than outside, which might introduce some numerical error. To ensure that the size of two neighboring grids are comparable, we refine the grid using the strategy similar to [B]: when the size of an outer grid is two times more than the size of its neighboring inner grid, we divide the k inner-most outer grids in half. We perform this procedure iteratively until all neighboring intervals have ratio between 1/2 and 2. We point out that the grid size depends logarithmly on ρ_{max}: In the computation we choose $k = 8$, and the size of the grid grows from 500 to around 900 as the maximum density reaches 10^{50}.

2.5.2.3 Interpolation

Regridding is followed by interpolation. Given the old cell average $\bar{u}_i$, we will interpolate $u(r)$ for $0 \le r \le a$, then we could use $u(r)$ to compute the cell average u'_i on the new grid.

One way to perform the interpolation is to simply let $u(\frac{r_i+r_{i+1}}{2}) = \bar{u}_i$ and apply a cubic spline interpolation. However, this interpolation does not preserve the mass, nor does it preserve positivity. On the other hand, the simplest mass and positivity preserving interpolation is to make $u(r)$ a piecewise function with value $\bar{u}_i$ in the i-th ring. However this method is only first-order accurate, and we hope to find some more accurate interpolation method that is volume-preserving. More precisely, given the old cell average $\bar{u}_i$, our goal is to find $u(r)$, such that

$$\begin{aligned} \text{minimize} \quad & \int_{B(0,r)} |\nabla u|^2 dx \\ \text{subject to} \quad & u(r) \ge 0 \text{ for } 0 \le r \le a \\ \text{and} \quad & \int_{B(r_{i+1})\backslash B(r_i)} u(x)dx = \bar{u}_i |B(r_{i+1})\backslash B(r_i)| \end{aligned} \tag{2.5.4}$$

We realize that this kind of interpolation is a 1D and simpler version of the pycnophylactic interpolation performed in [T], which we will briefly describe here.

To find the solution to the minimization problem (2.5.4), we use a finite volume scheme to solve the heat equation $u_t = \Delta u$ on the refined mesh, where the initial data are taken to be the piecewise constant function with value $\bar{u}_i$ on each old grid. After each time step, we adjust u such that both the positivity and the volume-preserving restrictions are met. More precisely, we add a different constant to the value in each old cell, to ensure the density is non-negative and the mass in every old cell remain unchanged. Then we repeat the above steps again until the $u(r)$ does not change.

2.5.3 Degenerate diffusion step

To solve the porous medium equation (2.5.2) on a fixed grid, we apply a fully implicit finite-volume scheme, and use Newton's method for solving the nonlinear equation. While these are standard procedures for dealing with degenerate diffusion equation (see [KA] for example), we briefly sketch the details here for the sake of completeness.

Assume the radial domain is partitioned into the intervals $0 = r_0 < r_1 < \ldots < r_N = R$. We denote by $U_i(t)$ the average of $u(x,t)$ in the ring $B(0,r_{i+1})\backslash B(0,r_i)$ at time t, and denote by $\vec{U}(t)$ the vector $(U_0(t), \cdots, U_{n-1}(t))$. Our goal is to find $\vec{U}(t+\Delta t)$, such that it solves the following nonlinear equation

$$\frac{\vec{U}(t+\Delta t) - \vec{U}(t)}{\Delta t} = A[\vec{U}^m(t+\Delta t)], \tag{2.5.5}$$

here A is a finite-volume discretization of the Laplace operator in radial coordinates, and the m-th power in $\vec{U}^m$ is understood in a component-wise sense.

We first explicitly write down the linear operator A. For any radially symmetric function $v(x)$, note that the radial derivative at r_i can be approximated by

$$\partial_r v(r_i) \approx \frac{V_i - V_{i-1}}{(r_{i+1} - r_{i-1})/2} \text{ for } 1 \le i \le N-1,$$

where V_i is the average of $v(x)$ in the ring $B(0,r_{i+1})\backslash B(0,r_i)$. For the radial derivative at the boundaries, we have $\partial_r v(x_0) = \partial_r v(x_N) = 0$. Note that divergence theorem gives

$$\int_{B(0,r_{i+1})\backslash B(0,r_i)} \Delta v dx = \partial_r v(r_{i+1})|\partial B(0,r_{i+1})| - \partial_r v(r_i)|\partial B(0,r_i)|.$$

Hence

$$(AV)_i = \frac{\left(\frac{v_{i+1} - v_i}{(r_{i+2} - r_i)/2}\right) r_{i+1}^{d-1} - \left(\frac{v_i - v_{i-1}}{(r_{i+1} - r_{i-1})/2}\right) r_i^{d-1}}{(r_{i+1}^d - r_i^d)/d} \text{ for } 1 \le i \le N-2,$$

and for the two boundaries we have

$$(AV)_0 = \frac{(v_1 - v_0)2d}{r_1 r_2},$$

$$(AV)_{N-1} = -\left(\frac{v_{N-1} - v_{N-2}}{(r_N - r_{N-2})/2}\right) \frac{r_{N-1}^{d-1}}{(r_N^d - r_{N-1}^d)/d}.$$

Next we use Newton's method to solve for $\vec{U}(t + \Delta t)$, where the iteration is performed as follows. For the initial step, we take $\vec{U}^{(0)}$ as $\vec{U}(t)$. Assuming $\vec{U}^{(k)}$ is known, by linearizing $\vec{U}^m(t + \Delta t)$ around $\vec{U}^{(k)}$, we obtain

$$\vec{U}^m(t + \Delta t) \approx m(\vec{U}^{(k)})^{m-1}\vec{U}(t + \Delta t) - (m-1)(\vec{U}^{(k)})^m,$$

hence $\vec{U}^{(k+1)}$ is given by

$$\vec{U}^{(k+1)} = \left(1 - m\Delta t A(\vec{U}^{(k)})^{m-1}\right)^{-1} \left(\vec{U}(t) - (m-1)\Delta t A(\vec{U}^{(k)})^m\right). \tag{2.5.6}$$

We point out that it only takes $O(N)$ steps to invert the $N \times N$ matrix $1 - m\Delta t A(\vec{U}^{(k)})^{m-1}$, due to its tridiagonal nature. Making use of (2.5.6), we solve for $\vec{U}^{(k+1)}$ iteratively for $k = 0, 1, \ldots$, until some $\vec{U}^{(k+1)}$ approximately solves (2.5.5) with error below some predetermined threshold. Then we stop the iteration and simply let $\vec{U}(t + \Delta t) = \vec{U}^{(k+1)}$. Typically the iteration would stop in less than 10 iterations, since the Newton's method has quadratic convergence.

2.5.4 Adaptive time step

Finally, when the computation in one time step is finished, we update Δt to control the growth rate of the maximum density. In the computation we would multiply Δt by 1.1 if the maximum density increases less than 0.02% in one time step, and divide Δt by 1.1 if the maximum density increases more than 0.1%. We point out that this simple criteria is indeed sufficient for our problem, due to the self-similarity of solutions in the peak area. Figure 2.13 shows an example of a log-log plot of the time step versus the maximum density.

2.6 Conclusions and remarks

We have studied the blowup behavior of radial solutions to the aggregation-diffusion equation $u_t = \Delta u^m - \nabla \cdot (u \nabla K * u)$ in dimension $d \geq 3$ for the kernel $K(x) = |x|^{-\gamma}$, where K is

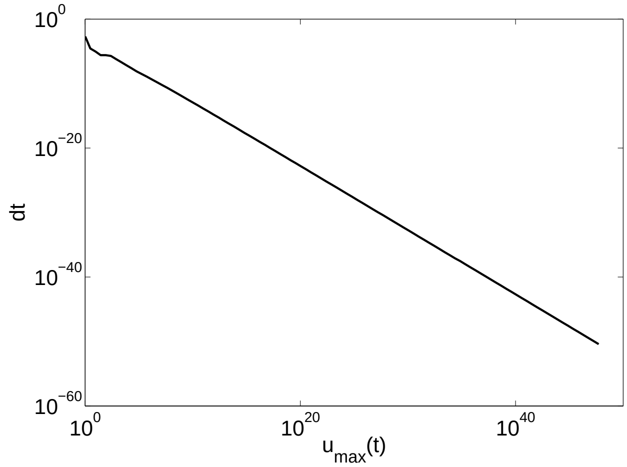


Figure 2.13: Log-log plot of the time step versus the maximum density, where $d = 3, \gamma = 1$ and $m = 4/3$. In the asymptotic regime, the best fit has slopes of value -1.003, which is in good agreement with the theoretically predicted value -1.

either equal to or less singular than the Newtonian kernel, i.e. $\gamma \leq d - 2$. Note that the dimension $d = 2$ is omitted in this chapter, since when $d = 2$ and K satisfies the above condition, (2.1.1) is either the well-studied Patlak-Keller-Segel model, or in the subcritical regime where solutions do not blow-up. For $d \geq 3$, formal asymptotic results and numerical observations both show that for supercritical m (i.e. $1 < m < \frac{d+\gamma}{d}$), the solution may blow-up either self-similarly or like a Burger shock; while for critical m (i.e. $m = \frac{d+\gamma}{d}$) and supercritical mass, the solution exhibits a near-self-similar blow-up behavior.

A number of problems regarding the blow-up behavior of solutions remain unsolved. First, for supercritical m, numerical observation suggests that when m, d and γ are fixed, there is a stable self-similar blow-up profile, and at least one unstable self-similar blow-up profile (see Figure 2.8). It would be interesting to know whether there exists a stable blow-up profile that attracts all self-similar blow-up solutions. For the case $m = 1$ with Newtonian potential, the stability of blow-up profile is studied in [BCKSV]. They proved that there exist a countable family of self-similar blow-up modalities $\{H_n\}$ for $n = 0, 1, 2, ...$, where H_0 gives a stable blow-up profile, and all the other H_n are unstable. However their eigenvalue method does not directly generalize to our equation, due to the nonlinear diffusion term.

For critical m with supercritical mass, numerical observation suggests that the solution may blow-up in a near-self-similar way, however the exact scaling for the blow-up remains open. Although the scaling is derived for $m = 1, d = 2$ and Newtonian potential K in [HV, L, CS], their arguments does not generalize to (2.1.1) for $m > 1$. Here the difficulty lies in the nonlinear diffusion term, and also in the fact that unlike the $m = 1$ case, the stationary solution with $m > 1$ has a compact support.

The numerical method is an arbitrary Lagrangian Eulerian method with adaptive mesh refinement. The advantage of our method is that we can compute to very high spatial resolution. Using around 1000 spatial points, we can compute the solution until the maximum density reaches 10^{80} and the characteristic spatial scale of the solution reaches 10^{-27}. We point out that our method preserves the L^1 norm of the solution, which is an important property especially for critical m, since the behavior of the solution depends on its mass.

Finally we note that it would be interesting to try to apply the numerical method to other problems that also have a non-local term with power-law interaction. Although the local well-posedness result in [BRB] is only established for kernels K that are less singular than (or equal to) the Newtonian kernel, preliminary numerical results suggests that our algorithm also works when K is more singular than Newtonian kernel. Thus we might be able to apply our numerical method to the fractional porous medium equation introduced in [CV], which is an aggregation equation with a repulsive kernel $K = -|x|^{-\gamma}$, where $2 - d < \gamma < d$.

CHAPTER 3

An Aggregation Equation with Diffusion in the Periodic Domain

3.1 Introduction

In this chapter we study weak solutions of the following equation in the periodic domain:

$$\rho_t = \Delta(\rho^m) + \theta L^{d(2-m)} \nabla \cdot (\rho \nabla (V * \rho)) \quad \text{in } \mathbb{T}_L^d \times [0, \infty), \tag{3.1.1}$$

where $*$ stands for convolution, and the space domain is the d-dimension torus with scale L, defined as $\mathbb{T}_L^d := \Big[-\dfrac{L}{2}, \dfrac{L}{2}\Big]^d$ with periodic boundary condition. We assume that V smooth and integrable (for precise conditions, see **(V1)-(V2)** in Section 3.3), and that θ is a positive constant. The primary focus of this work concerns the cases $m \in (1, 2]$ – especially $m = 2$. In addition, we remark that a goal of interest (not always achieved) is to acquire results uniform in L for $L \gg 1$.

This chapter is a joint work with Lincoln Chayes and Inwon Kim [CKY]. Before stating the results we obtained, we first briefly mention the similarity and differences between (3.1.1) and the equations (1.1.1) and (2.1.1) studied in Chapter 1 and 2. It is evident that all of these three equations are of aggregation-diffusion type, and indeed they all share the same degenerate diffusion term $\Delta\rho^m$ with $m > 1$. Their differences lies in the kernel and spatial domain: instead of a singular interaction kernel in Chapter 1 and 2, we focus on smooth interaction kernels V in this chapter, and therefore one should expect all solutions to exist globally in time, which is indeed true and proved in [BS]. Although finite time blow-up is never an issue for (3.1.1), many questions remain to be answered regarding the qualitative and asymptotic behavior of its solutions. Given the global existence of solutions, it is natural to ask whether there is some regularity result that holds uniform in time, which motivates our study in Section 3.2. Moreover, due to the periodic nature of this problem, one can easily see that any constant solution is a stationary solution. This fact leads to the following questions: (a) Are the constant solutions the only stationary solutions? (b) If so, will all the solutions converge to some constant solution? If so, at what rate? We will try to answer these questions in Section 3.4 and Section 3.5. Below we summarize the known results on (3.1.1) and discuss the results we obtained.

Firstly let us point out that formally (and in actuality) the mass of the solution to (3.1.1) is preserved over time. Without loss of generality, we can thus assume $\int \rho(x,0)dx = 1$ throughout this chapter, and results for other normalizations can be obtained by scaling.

When V satisfies $V(x) = V(-x)$, (3.1.1) is a gradient flow of the following energy with respect to the Wasserstein metric:

$$\mathcal{F}_\theta(\rho) := \int_{\mathbb{T}^d_L} \frac{1}{m-1}(\rho^m - \rho) + \frac{1}{2}\theta L^{d(2-m)}\rho(V * \rho)dx. \tag{3.1.2}$$

Note that as $m \to 1$, the first term in the integrand of $\mathcal{F}_\theta$ converges to $\rho \log \rho$ which we refer to as the $m = 1$ case. Using above energy structure, the existence and uniqueness properties of (3.1.1), in some appropriate Sobolev space, has been obtained in [BS] (also see [S1] and [BRB] for relevant results).

Compared to the well-posedness theory based on energy methods, few results has been known for pointwise behaviors of solutions, due to the lack of regularity estimates: the difficulty for regularity analysis lie mainly in the fact that the solutions are not necessarily positive (i.e., *strictly* positive) due to the degenerate diffusion. This is what we address in the first part of this chapter. In addition, in the non-compact setting, the plausible limiting solutions tend to be trivial; here, since mass is conserved, even in the "worst" of cases, there is always the uniform stationary state. Most of the rest of this work is concerned with the approach to the asymptotic state.

• *Regularity properties* Due to the degenerate diffusion, one cannot expect smooth solutions of (3.1.1): even for (PME), Hölder regularity is optimal, as verified by the self-similar (Barenblatt) solutions (see [V]). On the other hand, the solution of (PME) is indeed Hölder continuous (again, see [V]), which motivates the question of Hölder regularity of the solution of our problem (3.1.1).

Note that, if we choose V as a mollifier approximating the Dirac delta function, formally the nonlocal term approximates

$$\nabla \cdot [\theta L^{d(m-2)} \nabla V * \rho] = \theta L^{d(m-2)} \nabla \cdot [\rho \nabla \rho] = \theta L^{d(m-2)} \Delta(\rho^2).$$

Therefore it is plausible that, at least when ρ is bounded from above, diffusion dominates when $m < 2$ and the aggregation dominates when $m > 2$. Indeed we will show that, when $m < 2$, the effect of the aggregation term is weak enough that it is possible to locally approximate solutions of (3.1.1) with those of (PME). As a result, Hölder regularity of solutions of (3.1.1) for $m < 2$ follows. Let us mention that the Hölder regularity result obtained here is, to the best of the authors' knowledge, one of the first such result addressing Hölder regularity of solutions for degenerate parabolic equation with (either local or nonlocal) drift. Note that the classical result of DiBenedetto [D2] does not apply to our equation, since the assumptions which are crucial for [D2] fail here. As for $m \geq 2$, we show that solutions are continuous "uniformly in time", based on the result of Dibenedetto ([D]). For all $m > 1$, we also show that the L^∞ norm of solution is uniformly bounded from above depending on the L^1 and L^∞ norm of the initial data (see Theorem 3.2.1) which is of independent interest.

• *Asymptotic behavior* Our next result, partly an application of the first result, is on the asymptotic behavior of solutions of (3.1.1) in the periodic domain $\mathbb{T}^d_L$. We work in a periodic domain because, primarily, we are interested in finite volume problems and $\mathbb{T}^d_L$ provides the most convenient boundary conditions. Even though asymptotic behavior for $m < 2$ has been studied before in various references (e.g., [S1], and [HV] for a more singular interaction kernel) this is one of the first such result for these type of domains to the best of the authors' knowledge. One difficulty specific to the periodic setting is that the radial symmetry is not preserved over time, and thus exact (non-constant) solutions – always useful in these contexts – are not readily available. We also point out that in the case $m \geq 2$, there exist solutions which assumes zero value, possibly with compact support. Asymptotic behavior of such solutions are, in general, an interesting and difficult question, even for radial solutions in $\mathbb{R}^d$ (see [KY]).

Intuitively, one expects that when the diffusion term is "dominant" in (3.1.1), the solutions would converge to the constant solution as time goes to infinity. We show that this is indeed the case when $1 < m < 2$ and θ is sufficiently small. However, with most interactions (specifically, V being *not* of positive type) there is a linear instability that sets in at some $\theta^\sharp = \theta^\sharp(m) < \infty$ which is determined by the minimal coefficient in the Fourier series of V (see Section 4). It is not hard to show that for all m, when $\theta > \theta^\sharp$, the functional in (3.1.2) has non–constant minimizers (and the constant solution is not a minimizer – in fact, not even a *local* minimizer). However as has been shown explicitly for $m = 1$ under reasonable conditions – pertinently $d \geq 2$ – this "transition" occurs at some $\theta_{\mathrm{T}} < \theta^\sharp$ [CP]. Presumably, this argument holds in great generality. It is therefore somewhat surprising that for $m = 2$ the transition occurs exactly at $\theta = \theta^\sharp$.

More precisely, for $m = 2$, we show that for $\theta < \theta^\sharp$ (the *subcritical* case), the constant solution is the only minimizer and is stable. Indeed we can actually show that for all bounded initial data $\rho(x, 0)$, the dynamical solution $\rho(\cdot, t)$ will converge to the constant solution ρ_0 exponentially fast in L^2-norm. See Section 4 for detailed discussion on critical and supercritical case. When $1 < m < 2$, the energy is no longer in the form of an L^2–norm, and our Fourier-transform based approach does not generate a transitional value for θ. However, when θ is sufficiently small, similar approach used by one of the authors in [CP] yields that the constant solution is the global minimizer. Moreover, we show when θ is sufficiently small, the solution uniformly and exponentially converges to the constant solution.

3.2 Hölder continuity of the solution of PME with a drift

In this section, we study the regularity of the porous medium equation with a drift, where the drift potential may depend on time:

$$\rho_t = \Delta(\rho^m) + \nabla \cdot (\rho \nabla \Phi) \quad \text{in } \Omega, \tag{3.2.1}$$

with Neumann boundary condition on $\partial\Omega$. Here we may assume Ω is a bounded open set in $\mathbb{R}^d$, where $d \geq 1$, but all the results in this section certainly hold for periodic domain $\mathbb{T}^d_L$ as well. We assume $1 < m < 2$, the initial data $\rho(x,0) \in L^\infty(\Omega) \cap L^1(\Omega)$, the potential $\Phi(x,t) \in C(\Omega \times \mathbb{R}^+)$, and that $\Phi(\cdot,t) \in C^2(\Omega)$ for all $t \geq 0$.

Before even stating the main result, we will first prove that $\rho \in L^\infty(\Omega \times \mathbb{R}^+)$. When Φ does not depend on t, Bertch and Hilhorst in [BH] proved a uniform L^∞ bound of ρ by comparing ρ with an explicit supersolution which does not depend on t. When Φ is a function of both x and t, using arguments similar to those in [KL], we aquire an L^∞ bound for ρ which doesn't depend on t:

Theorem 3.2.1. *Suppose $m > 1$. Let ρ be the unique weak solution of* (3.2.1) *with Neumann boundary condition, with initial data $\rho(x,0) \in L^\infty(\Omega) \cap L^1(\Omega)$. We assume that the potential $\Phi(x,t)$ satisfies $\Phi(x,t) \in C(\Omega \times \mathbb{R}^+)$, and $\Phi(\cdot,t) \in C^2(\Omega)$ for all t with uniformly bounded norm. Then there exists $M > 0$, such that $\|\rho(\cdot,t)\|_{L^\infty(\Omega)} \leq M$ for all t, where M depends on $\|\rho(x,0)\|_{L^\infty(\Omega)}$, $\|\rho(x,0)\|_{L^1(\Omega)}$, $\sup_{t\in[0,\infty)} \|\Phi(\cdot,t)\|_{C^2(\Omega)}$, and m.*

Proof. We begin with implementing the following scaling: Let

$$\tilde{\rho}(x,t) = a^{\frac{1}{m-1}} \rho(x, at),$$

where $0 < a < 1$. Let us choose a sufficiently small such that

$$a < \min\Big\{ \big(\frac{1}{\|\rho(x,0)\|_{L^\infty(\Omega)}}\big)^{m-1}, \big(\frac{c_0}{\|\rho(x,0)\|_{L^1(\Omega)}}\big)^{m-1}, \frac{1}{\|\Phi\|_{C^2(\Omega)}} \Big\}, \tag{3.2.2}$$

where c_0 is a sufficiently small constant – certainly less than 1 – depending only on m and d and whose precise value will be determined later. By choosing a in this way, we have both $\|\tilde{\rho}(x,0)\|_{L^1(\Omega)} \leq c_0 < 1$ and $\|\tilde{\rho}(x,0)\|_{L^\infty(\Omega)} \leq 1$ and, moreover, that $\tilde{\rho}$ is a viscosity solution to the following PDE:

$$\tilde{\rho}_t = \Delta \tilde{\rho}^m + \nabla \cdot (\tilde{\rho} \nabla \tilde{\Phi}), \tag{3.2.3}$$

where $\tilde{\Phi} := a\Phi$. From the definition of a we know $\|\tilde{\Phi}(\cdot,t)\|_{C^2(\Omega)} \leq 1$ for all t.

Our preliminary goal is to show $\|\tilde{\rho}(x,1)\|_{L^\infty(\Omega)} \leq 1$; then we can take $\tilde{\rho}(x,1)$ as the new initial data and iterate the argument to get a uniform bound for all time.

We will introduce another variable v, which is bigger than $\tilde{\rho}$ and is of order unity in $\Omega \times [0,1]$. Let v be the viscosity solution to the following equation

$$v_t = \nabla \cdot (mv^{m-1}\nabla v + v\nabla\tilde{\Phi}), \tag{3.2.4}$$

with initial data $v(x,0) = \tilde{\rho}(x,0) + \frac{1}{2}e^{-1}$. Since v solves the same equation as $\tilde{\rho}$ with bigger initial data, we can apply the comparison principle for the porous medium equation with drift, which was established in Theorem 2.21 of [KL]. This comparison principle immediately implies $v(x,t) \geq \tilde{\rho}(x,t)$ for all (x,t), hence it suffices to show $\|v(\cdot,1)\|_{L^\infty(\Omega)} \leq 1$.

One can check easily that $\tilde{v}(x,t) := [\|v(\cdot,0)\|_{L^\infty(\Omega)}]e^{Kt}$ – where $K := \sup_{t\in[0,\infty)} \|\tilde{\Phi}(\cdot,t)\|_{C^2(\Omega)}$ – is a classical supersolution to (3.2.4) and hence also a viscosity supersolution. Noting that the initial data of v satisfies, for all x, $\frac{1}{2}e^{-1} \leq v(x,0) \leq 1 + \frac{1}{2}e^{-1}$, the comparison principle gives the following upper bound for v:

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq [\|v(\cdot,0)\|_{L^\infty(\Omega)}]e^{Kt} \leq (1 + \frac{1}{2}e^{-1})e^t.$$

Similarly we can find a classical subsolution which gives the lower bound

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \geq [\|v(\cdot,0)\|_{L^\infty(\Omega)}]e^{-Kt} \geq \frac{1}{2}e^{-1}e^{-t}.$$

Combining the two inequalities above, we have

$$v(x,t) \in [\frac{1}{2}e^{-2}, e + \frac{1}{2}] \quad \text{for all } x \in \Omega, t \in [0,1].$$

We would like to refine the estimate above and get a better estimate at $t = 1$. By treating the diffusion coefficients mv^{m-1} in (3.2.4) as an *a priori* function, – which we denote by $b(x,t)$ – then we may say that v solves a linear equation of divergence form, where the diffusion coefficient is of (the order of) size unity:

$$v_t = \nabla \cdot (b(x,t)\nabla v + v\nabla\tilde{\Phi}), \tag{3.2.5}$$

where $b(x,t) := mv^{m-1}(x,t) \in [m(\frac{1}{2}e^{-2})^{m-1}, m(e + \frac{1}{2})^{m-1}]$ for all $x \in \Omega$, $t \in [0,1]$.

In particular, since (3.2.5) is linear, we can decompose v as $v_1 + v_2$, such that v_1 solves (3.2.5) with initial data $v_1(x,0) = \tilde{\rho}(x,0)$, and v_2 solves (3.2.5) with initial data $v_2(x,0) = \frac{1}{2}e^{-1}$. We claim that $v_1(x,1)$ and $v_2(x,1)$ are both bounded by $\frac{1}{2}$, for all $x \in \Omega$.

For v_1, first note that due to the divergence form of (3.2.5), the L^1 norm of v_1 is conserved, i.e. $\|v_1(\cdot,1)\|_{L^1(\Omega)} = c_0$. Since b is bounded above and below away from zero, then by [LSU] (see Theorem 10.1, pp. 204), $v_1(\cdot,1)$ is Hölder continuous, where the Hölder exponent and coefficient do not depend on c_0, as long as $c_0 < 1$. So if we choose c_0 to be sufficiently small, we have $v_1(x,1) < \frac{1}{2}$ for all $x \in \Omega$.

For v_2, we can directly evaluate the necessary L^∞ bounds:

$$\sup_x v_2(x,1) \le e^{\|\Delta\tilde{\Phi}\|_\infty} \sup_x v_2(x,0) \le e\ \frac{1}{2}e^{-1} = \frac{1}{2}$$

(where again, on the basis of continuity, we may now talk about the supremum).

Combining the two estimates together, we have $\sup_x v(x,1) \le 1$, which implies $\sup_x \tilde{\rho}(x,1) \le 1$ from our discussion above. Also, for $0 < t < 1$ we have $\rho(x,t) \le v(x,t) \le e + 1/2$. Then by treating $\tilde{\rho}(x,1)$ as initial data and iterating the same argument, we get $\sup_x \tilde{\rho}(x,t) \le e + 1/2$ for all t, i.e.,

$$\rho(x,t) \le (e + \frac{1}{2})a^{-\frac{1}{m-1}} \quad \text{for all } x \in \Omega, t \ge 0.$$

Now plugging in the definition of a in the above and the bound becomes

$$\rho(x,t) \le (e+\frac{1}{2}) \max\left\{ \|\rho(x,0)\|_{L^\infty(\Omega)},\ \frac{\|\rho(x,0)\|_{L^1(\Omega)}}{c_0},\ \|\Phi\|_{C^2(\Omega)}^{\frac{1}{m-1}} \right\} \text{ in } \Omega \times [0,\infty).$$

□

Remark 3.2.2. In the statement of Theorem 3.2.1, we assumed that Ω is a bounded open set, with Neumann boundary conditions. The same proof also applies to Dirichlet boundary condition. Indeed, the L^∞ bound we obtained is independent with the size of Ω, and the same proof works as well when $\Omega = \mathbb{R}^d$. However, ostensibly, the L^∞ norm of ρ should be of the order L^{-d} and, even if true in the initial data, we cannot establish that this order is preserved at later times.

Since $\rho(x,t)$ is uniformly bounded for all (x,t), DiBenedetto has shown in [D] that $\rho(\cdot,t)$ is continuous uniformly in t:

Theorem 3.2.3 ([D]). *For any $m > 1$, let ρ be the weak solution to* (3.2.1) *with initial data $\rho(x, 0) \in L^\infty(\Omega) \cap L^1(\Omega)$. Let the potential $\Phi(x, t)$ satisfy $\Phi(x, t) \in C(\Omega \times \mathbb{R})$, $\Phi(\cdot, t) \in C^2(\Omega)$ for all t, moreover $\sup_t \|\Phi(\cdot, t)\|_{C^2(\Omega)]} < \infty$. Then for all $\tau > 0$, $\rho(x, t)$ is uniformly continuous in $\Omega \times [\tau, \infty)$, and the continuity is uniform in x and t.*

Now we want to show when $1 < m < 2$, for all $\tau > 0$, $\rho(x, t)$ is uniformly Hölder continuous in space and time in $\Omega \times [\tau, \infty)$. Our main theorem of this section is stated as following:

Theorem 3.2.4. *Let $1 < m < 2$. Let ρ be a viscosity solution of* (3.2.1)*, with initial data $\rho(x, 0)$. We make the following assumptions on $\rho(\cdot, 0)$ and Φ:*

1. *$\|\rho(\cdot, 0)\|_\infty \le M_1$ and $\int_\Omega \rho(x, 0)dx \le M_1$.*
2. *$\Phi(x, t) \in C(\Omega \times \mathbb{R})$, and $\|\Phi(\cdot, t)\|_{C^2(\Omega)} \le M_2$ for some $M_2 > 0$ for all $t \ge 0$.*

Then for any $0 < \tau < \infty$, u is Hölder continuous in $\Omega \times [\tau, \infty)$, where the Hölder exponent and coefficient depends on τ, m, d, M_1 and M_2.

Proof. To prove the Hölder continuity of ρ, our goal is to show that for any $(x_0, t_0) \in \Omega \times [\tau, \infty)$,

$$\text{osc}_{B(x_0, a^2) \times [t_0, t_0 + a^4]} \rho \le C a^\gamma \tag{3.2.6}$$

for some $C, \gamma > 0$ not depending on a, for a satisfying $0 < a < \min\{\frac{2-m}{2c}, \sqrt{\tau}\}$ (where c is a constant to be determined soon).

Bearing in mind that we want to zoom in on the profile and look at the oscillation in a small neighborhood, it makes sense to start with a parabolic scaling with scaling factor a. Let

$$\tilde{\rho}(x, t) := \rho(ax, a^2 t + (t_0 - a^2))), \tag{3.2.7}$$

and our goal (3.2.6) would transform into

$$\text{osc}_{B(\frac{x_0}{a}, a) \times [1, 1+a^2]} \tilde{\rho} \le C a^\gamma. \tag{3.2.8}$$

Here $\tilde{\rho}(x,t)$ is defined in the domain $\tilde{\Omega}\times[0,\infty)$, where $\tilde{\Omega} := \{x\in\mathbb{R}^d : ax\in\Omega\}$. and, it is noted, the early portion of the time domain had been omitted. We readily see that $\tilde{\rho}$ is the viscosity solution to

$$\tilde{\rho}_t = \Delta\tilde{\rho}^m + \nabla\cdot(\tilde{\rho}\nabla\tilde{\Phi}) \quad \text{in } \tilde{\Omega}\times[0,\infty). \tag{3.2.9}$$

Here, the initial data reads $\tilde{\rho}(x,0) = \rho(ax, t_0 - a^2)$, which has an *a priori* L^∞ bound depending on m, d, M_1, M_2 due to Theorem 3.2.1. Moreover, in the above $\tilde{\Phi}(x,t) := \Phi(ax, a^2t + (t_0 - a^2)))$ and hence $|\nabla\tilde{\Phi}|$ is bounded by aM_2. We wish to compare $\tilde{\rho}$ with w, where w is the viscosity solution to the porous medium equation

$$w_t = \Delta w^m \quad \text{in } \tilde{\Omega}\times[0,\infty), \tag{3.2.10}$$

with initial data $w(\cdot,0)\equiv\tilde{\rho}(\cdot,0)$. Since (3.2.9) and (3.2.10) only differ by the term $\nabla\cdot(\tilde{\rho}\nabla\tilde{\Phi})$, we would expect

$$|\tilde{\rho} - w| \le Ca^\beta \text{ in } \tilde{\Omega}\times[1,2], \tag{3.2.11}$$

for some $C > 0, 0 < \beta < 1$ depending on m, d, M_1, M_2.

The main part of this proof will be devoted to proving (3.2.11) is indeed true. Without loss of generality, we can assume that $\tilde{\rho}(x,t)$ is a classical solution. First, if the initial data $\tilde{\rho}(x,0)$ is uniformly positive, then $\tilde{\rho}(x,t)$ will be a classical solution for all time. This is because $\tilde{\rho}$ will stay positive for any time period $[0,T]$ (since $\inf_{\tilde{\Omega}\times[0,T]}\tilde{\rho}(x,t) \ge \exp(-t\sup_{t\in[0,T]}\|\Delta\tilde{\Phi}\|_\infty)\inf_{x\in\tilde{\Omega}}\tilde{\rho}(x,0))$, which implies that (3.2.9) is uniformly parabolic for $t\in[0,T]$ and hence the weak solution $\tilde{\rho}$ is classical.

For general initial data $\tilde{\rho}(x,0)$, we can use approximation as follows. Let $\tilde{\rho}_n$ and w_n solve (3.2.9) and (3.2.11) respectively with initial data $\tilde{\rho}(x,0)+2^{-n}$; n sufficiently large. As discussed above, $\tilde{\rho}_n$ would be a sequence of classical solutions. If we can obtain $|\tilde{\rho}_n - w_n| < Ca^\beta$ for all n, (where C, β doesn't depend on n), then (3.2.11) would hold for $\tilde{\rho}$ and w as well, since as $n\to\infty$, comparison principle yields $\tilde{\rho}_n(x,t)\searrow\tilde{\rho}$ and $w_n(x,t)\searrow w$ uniformly in x, t.

Note that one cannot directly compare ρ with w, due to the fact that the term $\nabla\cdot(\tilde{\rho}\nabla\tilde{\Phi})$ contains $\nabla\tilde{\rho}\cdot\nabla\tilde{\Phi}$ and hence does not have any *a priori* bound. In order to bound this term, it will help to change from the density variable $\tilde{\rho}$ to the pressure variable $\tilde{u}$. Let

$$\tilde{u} = \frac{m}{m-1}\tilde{\rho}^{m-1},$$

then (3.2.9) becomes

$$\tilde{u}_t = (m-1)\tilde{u}\Delta\tilde{u} + |\nabla\tilde{u}|^2 + \nabla\tilde{u}\cdot\nabla\tilde{\Phi} + (m-1)\tilde{u}\Delta\tilde{\Phi}, \tag{3.2.12}$$

which will enable us to use $|\nabla\tilde{u}|^2$ plus a constant to control the term $\nabla\tilde{u}\cdot\nabla\tilde{\Phi}$: Recall that $|\nabla\tilde{\Phi}| < aM_2$, which gives us the following bound

$$|\nabla\tilde{u}\cdot\nabla\tilde{\Phi}| \le aM_2|\nabla\tilde{u}| \le a\big[|\nabla\tilde{u}|^2 + \frac{1}{4}(M_2)^2\big].$$

Also, due to the fact that $(m-1)\tilde{u}(x,t) \le C_1$ in $\tilde{\Omega}\times[0,2]$, (where C_1, which depends on m, d, M_1 and M_2, is related to the L^∞ bounds on ρ) we obtain

$$|(m-1)\tilde{u}\Delta\tilde{\Phi}| \le a^2C_1M_2 \le aC_1M_2.$$

Putting the above two bounds together, and by choosing c such that $c > C_1M_2 + (M_2/2)^2$, $\tilde{u}$ will satisfy the following inequality

$$\tilde{u}_t \ge (m-1)\tilde{u}\Delta\tilde{u} + (1-ca)|\nabla\tilde{u}|^2 - ca \quad \text{for all } x\in\tilde{\Omega}, t\in[0,2]. \tag{3.2.13}$$

Note that we assumed $a < (2-m)/(2c)$ in the beginning of the proof, we have $ca < (2-m)/2$.

In order to make (3.2.13) look similar to the porous medium equation in the pressure form, we apply the rescaling $u_1 = (1-ca)\tilde{u}$. Then u_1 satisfies

$$(u_1)_t \ge (m^- - 1)u_1\Delta u_1 + |\nabla u_1|^2 - ca(1-ca) \quad \text{for all } x\in\tilde{\Omega}, t\in[0,2], \tag{3.2.14}$$

where

$$m^- := \frac{m-1}{1-ca} + 1. \text{ (hence } ca < (2-m)/2 \text{ implies that } 1 < m^- < 2) \tag{3.2.15}$$

Now (3.2.14) has the same form as the porous medium equation in the pressure form, minus an extra constant term $ca(1-ca)$. To take advantage of the existence and regularity results for equations with divergence form, we change the pressure variable back to the density variable (however here the power is m^- instead of m), i.e., we define ρ_1 such that

$$(1-ca)\tilde{u} = u_1 = \frac{m^-}{m^- - 1}\rho_1^{m^- -1}, \tag{3.2.16}$$

or in other words,

$$\rho_1 = \Big(\frac{m}{m^-}\Big)^{m^-}\tilde{\rho}^{1-ca} = \Big(\frac{1+ca}{1+ca/m}\Big)^{m^-}\tilde{\rho}^{1-ca}. \tag{3.2.17}$$

Due to the positivity of $\tilde{u}$, we know ρ_1 is positive as well. Hence when we plug (3.2.16) into (3.2.14), after canceling a positive power of ρ_1 on both sides, we obtain

$$(\rho_1)_t > \Delta\rho_1^{m^-} - ca(1-ca)\rho_1^{2-m^-} \quad \text{in } \tilde{\Omega}\times[0,2], \tag{3.2.18}$$

Note that the term $ca(1-ca)\rho_1^{2-m^-}$ has an *a priori* upper bound: since $2-m^- > 0$ and ρ_1 is given by (3.2.17), we have $c(1-ca)\rho_1^{2-m^-} < M$, for some constant M depending on m, d, M_1, M_2.

Let us denote by ρ^- the weak solution of

$$(\rho^-)_t = \Delta(\rho^-|\rho^-|^{m^- -1}) - Ma, \tag{3.2.19}$$

with initial data the same as $\rho_1(x,0)$, which is

$$\rho^-(x,0) = \Big(\frac{m}{m^-}\Big)^{m^-}\tilde{\rho}(x,0)^{1-ca} \tag{3.2.20}$$

Since $\tilde{\Omega}$ is a bounded domain, we have $Ma \in L^p(\tilde{\Omega})$ for all $p \geq 1$, and the existence of weak solution of (3.2.19) is guaranteed by Theorem 5.7 in [V]. That theorem also gives us a comparison result that, a.e., $\rho_1 \geq \rho^-$.

Moreover, note that the "a.e." above can in fact be removed, since both $\tilde{\rho}$ and ρ^- are continuous in $\tilde{\Omega}\times[0,2]$: the continuity of $\tilde{\rho}$ is given by Theorem 3.2.3, and the continuity of ρ^- is given by Theorem 11.2 of [DGV]. Therefore we have the following comparison between ρ^- and $\tilde{\rho}$:

$$\rho^- \leq \Big(\frac{m}{m^-}\Big)^{m^-}\tilde{\rho}^{1-ca} \quad \text{in } \tilde{\Omega}\times[0,2] \tag{3.2.21}$$

Since $m/m^- = 1 + O(a)$, and $\tilde{\rho}$ is bounded in $\tilde{\Omega}\times[0,2]$, (3.2.21) implies that $\tilde{\rho} - \rho^- \geq -C_1 a$, where C_1 depend on m, d, M_1, M_2.

Analogous to the definition to ρ^-, we define ρ^+ to be the weak solution of

$$(\rho^+)_t = \Delta((\rho^+)^{m^+}) + Ma, \tag{3.2.22}$$

with initial data

$$\rho^+(x,0) = \big(\frac{m}{m^+}\big)^{m^+} \tilde{\rho}(x,0)^{1+ca}, \tag{3.2.23}$$

where

$$m^+ := \frac{m-1}{1+ca} + 1. \text{ (hence } 1 < m^+ < 2) \tag{3.2.24}$$

Then analogous argument would lead to $\tilde{\rho} - \rho^+ \leq C_1 a$. Summarizing, we have obtained

$$\rho^- - C_1 a \leq \tilde{\rho} \leq \rho^+ + C_1 a \quad \text{in } \tilde{\Omega} \in [0,2], \tag{3.2.25}$$

where C_1 depends on m, d, M_1, M_2.

To prove (3.2.11), it suffices to show $|\rho^\pm - w| \leq O(a^\beta)$ for some $\beta > 0$, which is proved in the following lemma.

Lemma 3.2.5. *Let $1 < m < 2$. Let w be the viscosity solution of the porous medium equation*

$$w_t = \Delta w^m \quad \textit{in } \tilde{\Omega} \times [0, \infty) \tag{3.2.26}$$

where the initial data $w(x,0)$ satisfies $w(x,0) = \tilde{\rho}(x,0)$.

Let ρ^- and ρ^+ be the weak solutions to (3.2.19) and (3.2.22) respectively, where $0 < a < (2-m)/(2c)$ is a small constant, and the initial data is given by (3.2.20) and (3.2.23). Then

$$|\rho^\pm - w| \leq Ca^\beta \quad \textit{in } \tilde{\Omega} \times [1,2], \tag{3.2.27}$$

where C and β depends on d, m, M_1, M_2.

The proof of Lemma 3.2.5 is the content of the appendix in Section A.2. Putting Lemma 3.2.5 and (3.2.25) together, we obtain (3.2.11), and we will use this to (immediately) prove (3.2.8).

Since w solves the porous medium equation, Theorem 7.17 in [V] gives us the Hölder continuity of w:

$$\text{osc}_{B(x,a)\times[1,1+a^2]} w \leq Ca^\alpha, \text{ for all } x \in \tilde{\Omega}, \tag{3.2.28}$$

where C and α depends on $\|w(\cdot,0)\|_\infty$ (and hence depends on m, d, M_1, M_2).

By putting (3.2.28) and (3.2.11) together, we obtain

$$\mathrm{osc}_{B(x,a)\times[1,1+a^2]}\tilde{\rho} \leq Ca^\gamma, \text{ for all } x \in \tilde{\Omega}, \tag{3.2.29}$$

where C depends on m, d, M_1, M_2, and $\gamma = \min\{\alpha, \beta\}$ (hence also depends on m, d, M_1, M_2). Hence (3.2.8) is proved. □

Remark 3.2.6. For $m \geq 2$, Hölder continuity of the solution to (3.2.1) is still open. Indeed, concerning the present approach – which closely parallels that of [KL], [K] – when $m > 2$ we have that $m^- = 1 + (m-1)/(1-ca) > 2$. Hence the "inhomogeneous" term in (3.2.18), which is proportional to $\rho^{(2-m^-)}$, would actually be divergent in places where $\rho \to 0$. This indicates that another approach will be required.

3.3 Application to aggregation equation with degenerate diffusion

In the following two sections, we study (3.1.1) in the domain $\mathbb{T}_L^d$, the d-dimension torus of scale L. Here θ is a non-negative constant, and, of course, $*$ denotes convolution in $\mathbb{T}_L^d$. We make the following assumptions on $V(x)$:

(V1) $V(x) = V(-x)$ for all $x \in \mathbb{T}_L^d$.

(V2) $V(x) \in C^2(\mathbb{T}_L^d)$, with $\|V(x)\|_{C^2(\mathbb{T}_L^d)} = C$ for some constant $C < \infty$.

Moreover, we have in mind $V : \mathbb{R}^d \to \mathbb{R}$ compactly supported with the diameter of the support smaller than L. In particular we do not envision "wrapping" effects and $\int_{\mathbb{T}_L^d} |V|dx$ may be regarded as independent of L.

Our goal in this section is to show the Hölder continuity of the weak solution to (3.1.1) for $1 < m < 2$, and uniform continuity of the weak solution when $m = 2$. First, we state the definition of weak solution to (3.1.1) and a existence theorem from [BS].

Definition 3.3.1 (Weak Solution)**.** *Let $m > 1$, and let us assume that $\rho(x,0)$ is non-negative, with $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ and consider a potential V that satisfies the assumptions **(V1)** and **(V2)**. A function $\rho : \mathbb{T}_L^d \times [0,T] \to [0,\infty)$ is a weak solution to* (3.1.1) *if $\rho \in L^\infty(\mathbb{T}_L^d \times [0,T])$, $\rho^m \in L^2(0,T,H^1(\mathbb{T}_L^d))$ (i.e., $\|\rho(\cdot,t)\|_{H^1(\mathbb{T}_L^d} \in L^2(0,T)$) and $\rho_t \in L^2(0,T,H^{-1}(\mathbb{T}_L^d))$ and for all test function $\phi \in H^1(\mathbb{T}_L^d)$, for almost all $t \in [0,T]$,*

$$< \rho_t(t), \phi > + \int_{\mathbb{T}_L^d} \nabla(\rho^m(t)) \cdot \nabla\phi + \theta L^{d(2-m)} \rho(t)(\nabla V * \rho(t)) \cdot \nabla\phi dx = 0. \tag{3.3.1}$$

In [BS], existence and uniqueness of weak solution are proved:

Theorem 3.3.2 (Bertozzi-Slepčev)**.** *Let $m > 1$ and consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x,0)$ be a nonnegative function in $L^\infty(\mathbb{T}_L^d)$. Then the problem* (3.1.1) *has a unique weak solution on $\mathbb{T}_L^d \times [0,T]$ for all $T > 0$, and furthermore $\rho \in C(0,T,L^p(\mathbb{T}_L^d))$ for all $p \in [1,\infty)$.*

By treating $\theta L^{d(2-m)} \rho * V$ as an *a priori* potential, we can apply our results in Section 2, and obtain L^∞ bound of ρ which does not depend on T, together with uniform continuity of ρ, and Hölder continuity of ρ for $1 < m < 2$.

Theorem 3.3.3. *Let $m > 1$ and consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x,t)$ be the unique weak solution to* (3.1.1) *given by Theorem 3.3.2, with nonnegative initial data $\rho(x,0) \in C(\mathbb{T}_L^d)$, which satisfies $\int_{\mathbb{T}_L^d} \rho(x,0)dx = 1$. Then $\|\rho(x,t)\|_{L^\infty(\mathbb{T}_L^d \times [0,\infty))}$ is bounded, where the bound only depend on $\sup_x \rho(x,0)$, θ, $\|V\|_{C^2}$ and L.*

Proof. To begin with, note that Theorem 3.3.2 guarantees the existence and uniqueness of the weak solution to (3.1.1), which we denote by by ρ. Now we treat $\Phi := \theta L^{d(2-m)} \rho * V$ as an *a priori* potential, and we obtain the following estimate of Φ assumption **(V2)**:

$$\begin{aligned} \|\Phi(\cdot,t)\|_{C^2(\mathbb{T}_L^d)} &\leq \theta L^{d(2-m)} \|\rho(\cdot,t)\|_{L^1(\mathbb{T}_L^d)} \|V\|_{C^2(\mathbb{T}_L^d)} \\ &\leq \theta L^{d(2-m)} \|V\|_{C^2(\mathbb{T}_L^d)} \\ &= \theta L^{d(2-m)} C \quad \text{for all } t \geq 0. \end{aligned}$$

We denote by ρ_1 the unique weak solution to the equation

$$(\rho_1)_t = \Delta\rho_1^m + \nabla \cdot (\rho_1 \nabla\Phi) \tag{3.3.2}$$

with initial data $\rho_1(\cdot, 0) \equiv \rho(\cdot, 0)$, where the existence and uniqueness is proved in [BH]. Theorem 3.2.1 implies $\sup_x \|\rho_1(\cdot, t)\|$ is bounded uniformly in t. Moreover, note that ρ also satisfies the weak equation for (3.3.2), hence ρ must coincide with ρ_1, which yields a uniform bound of ρ which doesn't depend on time. □

Applying Theorem 3.2.3 to (3.3.2), we have the continuity of ρ uniformly in t for $m > 1$ – in particular (in light of Theorem 3.3.5 below) for the case $m = 2$.

Theorem 3.3.4. *Let $m > 1$ and consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x, t)$ be the unique weak solution to* (3.1.1) *given by Theorem 3.3.2, with nonnegative initial data $\rho(\cdot, 0)$ satisfying $\|\rho(\cdot, 0)\|_{L^\infty(\mathbb{T}_L^d)} < \infty$, and $\|\rho(\cdot, 0)\|_{L^1(\mathbb{T}_L^d)} = 1$. Then for any $\tau > 0$, ρ is continuous in $\mathbb{T}_L^d \times [\tau, \infty)$, where the continuity is uniform in both x and t.*

Proof. Follows immediately from the above reasoning, Theorem 3.3.3 and Theorem 3.2.3 □

Applying Theorem 3.2.4 to (3.3.2), with $\Phi = \theta L^{d(2-m)} \rho * V$ we have the Hölder continuity of ρ for $1 < m < 2$.

Theorem 3.3.5. *Let $1 < m < 2$ and consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x, t)$ be the unique weak solution to* (3.1.1) *given by Theorem 3.3.2, with nonnegative initial data $\rho(x, 0)$ satisfying $\|\rho(\cdot, 0)\|_{L^\infty(\mathbb{T}_L^d)} < \infty$, and $\|\rho(\cdot, 0)\|_{L^1(\mathbb{T}_L^d)} = 1$. Then for any $\tau > 0$, u is Hölder continuous in $\mathbb{T}_L^d \times (\tau, \infty)$, where the Hölder exponent and coefficient depend on τ, m, d, θ, L and C and the L^∞ norm of the initial condition.*

Proof. Follows immediately from the preceding reasoning, Theorem 3.3.3 and Theorem 3.2.4.

□

3.4 The case $m = 2$: analysis via normal modes

In this section, we will use Fourier Transform to study the PDE in (3.1.1), and this method works best when $m = 2$. We continue to assume, without loss of generality that

$\|\rho(x,0)\|_{L^1(\mathbb{T}_L^d)} = 1$, however from the perspective of *functional analysis*, the homogeneity of the special case $m = 2$ makes even this stipulation redundant.

The dynamics in (3.1.1) is governed by gradient flow for the "free energy" functional

$$\mathcal{F}_\theta(\rho) = \int_{\mathbb{T}_L^d} \rho^2 + \frac{1}{2}\theta\rho(\rho * V)dx. \quad (3.4.1)$$

For the analysis of the functional $\mathcal{F}_\theta$, since we are assuming $\rho(x,0)$ integrates to 1, we shall denote by $\mathscr{P}$ the class of probability densities on $\mathbb{T}_L^d$ which also belong to $L^2(\mathbb{T}_L^d)$, i.e.

$$\mathscr{P} := \{f \in L^1(\mathbb{T}_L^d) \cap L^2(\mathbb{T}_L^d) : \|f\|_{L^1(\mathbb{T}_L^d)} = 1\}. \quad (3.4.2)$$

Special to the case $m = 2$ is that the functional $\mathcal{F}_\theta(\cdot)$ can be expressed in a simpler form if we express ρ in terms of its Fourier modes. We write

$$\hat{\rho}(k) = \int_{\mathbb{T}_L^d} \rho(x)\mathrm{e}^{-ik\cdot x}dx$$

where k is of the form $k = \frac{2\pi}{L}\vec{n}$ with $\vec{n} \in \mathbb{Z}^d$. With these conventions we have

$$\rho(x) = \frac{1}{L^d}\sum_k \hat{\rho}(k)\mathrm{e}^{ik\cdot x}$$

and, in terms of these variables, (3.4.1) becomes

$$\mathcal{F}_\theta(\rho) = \frac{1}{L^d}\sum_k |\hat{\rho}(k)|^2(1 + \frac{1}{2}\theta\hat{V}(k)). \quad (3.4.3)$$

On the basis of (3.4.3), a salient value of θ emerges: We denote this value by $\theta^\sharp$, which is defined via

$$[\theta^\sharp]^{-1} := \frac{1}{2}\max_{k\neq 0}\{|\hat{V}(k)|; \hat{V}(k) < 0\}. \quad (3.4.4)$$

Formally $\theta^\sharp$ may be designated as $+\infty$ in case $\hat{V}(k) \geq 0$ for all $k \neq 0$ – i.e. if V is (essentially) of positive type. For the purposes of the present discussion, we shall assume otherwise. Different values of θ separate our problem into 3 cases:

1. (subcritical) When $\theta < \theta^\sharp$, we have $1 + \frac{1}{2}\theta\hat{V}(k) > 0$ for all $k \in \mathbb{Z}^d$, then under the restriction $\hat{\rho}(0) = 1$, it is manifest that global minimizer for $\mathcal{F}_\theta(\rho)$ in $\mathscr{P}$ is the constant solution

$$\rho_0(x) := \frac{1}{L^d}\int_{\mathbb{T}_L^d} \rho(x,0)dx \equiv \frac{1}{L^d}. \quad (3.4.5)$$

2. (critical) When $\theta = \theta^\sharp$, we have still have $1 + \frac{1}{2}\theta\hat{V}(k) \geq 0$ for all $k \in \mathbb{Z}^d$ however now there is a set $\mathbb{K}^\sharp$ (containing at least two elements) defined by the condition that for $k \in \mathbb{K}^\sharp$, $1 + \frac{1}{2}\theta^\sharp\hat{V}(k) = 0$. In this case the global minimizers for $\mathcal{F}_\theta(\rho)$ in $\mathscr{P}$ take the form

$$\rho(x) = \rho_0 + \sum_{k \in \mathbb{K}^\sharp} c_k \mathrm{e}^{ik\cdot x}, \tag{3.4.6}$$

where $c_{-k} = \overline{c}_k$ and, of course, subject to the restriction that the resultant quantity is non–negative.

3. (supercritical) When $\theta > \theta^\sharp$, we have $1 + \frac{1}{2}\theta\hat{V}(k) < 0$ for some $k \in \mathbb{Z}^d$. In this case the constant solution ρ_0 is not even a local minimizer of $\mathcal{F}_\theta$ in $\mathscr{P}$, let alone global minimizer.

Remark 3.4.1. The above – which is *manifest* for $m = 2$ – is in sharp contrast to the cases $m \neq 2$. In particular, for general m there is an analogous quantity $\theta^\sharp$ given by

$$[\theta^\sharp]^{-1} := \frac{1}{m} \max_{k \neq 0} \{|\hat{V}(k)|; \hat{V}(k) < 0\}$$

where items (1) – (3) are suggested. However, the following was shown for $m = 1$ and, presumably holds for all $m \neq 2$: While for $\theta < \theta^\sharp$, the constant solution has "some stability" (c.f. [CP] Theorem 2.11 for the case $m = 1$) there is a $\theta_{\mathrm{T}} < \theta^\sharp$ where global considerations come into play. In particular, at $\theta = \theta_{\mathrm{T}}$, there is a non–uniform minimizer for $\mathcal{F}_{\theta_{\mathrm{T}}}(\cdot)$ which is degenerate with the uniform solution. Moreover, for $\theta > \theta_{\mathrm{T}}$ (which implies, in particular, at $\theta = \theta^\sharp$) the uniform solution is no longer a minimizer.

3.4.1 The subcritical case, when m=2

In the subcritical case, the constant solution ρ_0 is the only global minimizer of $\mathcal{F}_\theta$ in $\mathscr{P}$. Our goal in this section is to show for every non–negative initial data $\rho(x, 0) \in L^\infty(\mathbb{T}_L^d)$ which integrates to 1, the weak solution $\rho(x, t)$ converges to ρ_0 exponentially in $L^2(T_L^d)$ as $t \to \infty$, where ρ_0 is as given in (3.4.5).

By formally taking the time derivative of the free energy functional, a simple calculation indicates that e.g., at least for classical solutions to (3.1.1), the free energy is always non–increasing:

$$\frac{d}{dt}\mathcal{F}_\theta(\rho) = -\int_{\mathbb{T}_L^d} \rho\big|\nabla(\frac{m}{m-1}\rho^{m-1} + \theta L^{d(2-m)}\rho * V)\big|^2 dx. \tag{3.4.7}$$

In [BS], it is proved that (3.4.7) is indeed true in the integral sense:

Lemma 3.4.2 (Bertozzi–Slepčev)**.** *Consider* V *that satisfies the assumptions* ***(V1)*** *and* ***(V2)****. Let* $\rho(x,t)$ *be a weak solution of* (3.1.1) *in* $\mathbb{T}_L^d \times [0,T]$*. Then for almost all* $\tau \in [0,T]$*,*

$$\mathcal{F}_\theta(\rho(\cdot,0)) - \mathcal{F}_\theta(\rho(\cdot,\tau)) \geq \int_0^\tau \int_{\mathbb{T}_L^d} \rho|\nabla(\frac{m}{m-1}\rho^{m-1} + \theta L^{d(2-m)}\rho * V)|^2 dxdt \tag{3.4.8}$$

Remark 3.4.3. Theorem 3.3.4 implies that $\rho(\cdot,t)$ is a continuous function of t, hence $\mathcal{F}_\theta(\rho(\cdot,t))$ is continuous in t as well. Therefore (3.4.8) indeed holds for all $\tau \in [0,T]$ and, moreover, (3.4.7) may be regarded as a *differential inequality*.

In the following lemma, we show when $\theta < \theta^\sharp$, the free energy will decay to the free energy of the global minimizer as $t \to \infty$.

Lemma 3.4.4. *Suppose* $m = 2$ *and consider* V *that satisfies the assumptions* ***(V1)*** *and* ***(V2)****. Further suppose that* $\theta < \theta^\sharp$*, where* $\theta^\sharp$ *is as given in* (3.4.4) *– including* $\theta^\sharp = \infty$ *if* V *is of positive type. Let* $\rho(x,t)$ *be the weak solution to* (3.1.1) *on* $[0,\infty)\times\mathbb{T}_L^d$*, with non-negative initial data* $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ *which integrates to 1. Then* $\mathcal{F}_\theta(\rho) \to \mathcal{F}_\theta(\rho_0)$ *as* $t \to \infty$*, where* ρ_0 *is the uniform solution* (*as given in* (3.4.5))*.*

Proof. By Lemma 3.4.2, we know $\mathcal{F}_\theta(\rho(t))$ is a continuous and decreasing function of t, whose limit is bounded below by $\mathcal{F}_\theta(\rho_0)$, since ρ_0 is the global minimizer of $\mathcal{F}_\theta$ in $\mathscr{P}$ when $\theta < \theta^\sharp$. Hence we can send τ to infinity in (3.4.8), which gives

$$\int_0^\infty \int_{\mathbb{T}_L^d} \rho|\nabla(2\rho + \theta\rho * V)|^2 dxdt < \infty. \tag{3.4.9}$$

Then there exists an increasing sequence of time $(t_n)_{n=1}^\infty$, where $\lim_{n\to\infty} t_n = \infty$, such that

$$\lim_{n\to\infty} \int_{\mathbb{T}_L^d} \rho(x,t_n)|\nabla(2\rho(x,t_n) + \theta\rho(x,t_n) * V)|^2 dx = 0. \tag{3.4.10}$$

To avoid clutter, in what follows, we shall abbreviate $\rho(\cdot, t_n)$ by ρ_n. Recall that Theorem 3.2.1 gives us a uniform bound of $\|\rho_n\|_{L^\infty(\mathbb{R}^d)}$. In addition, by [D], (ρ_n) is uniformly equicontinuous, hence Arzelà-Ascoli Theorem enables us to find a subsequence of ρ_n (which we again denote by ρ_n for notational simplicity), and a continuous function ρ_∞, such that

$$\lim_{n\to\infty} \|\rho_n - \rho_\infty\|_{L^\infty(\mathbb{T}^d_L)} = 0, \tag{3.4.11}$$

We next claim that $\|\nabla\rho_n^{3/2}\|_{L^2(\mathbb{T}^d_L)}$ is bounded uniformly in n. To prove the claim, we first note that

$$\int_{\mathbb{T}^d_L} \left|\frac{4}{3}\nabla\rho_n^{3/2} + \rho_n^{1/2}\nabla(\theta\rho_n * V)\right|^2 dx = \int_{\mathbb{T}^d_L} \rho_n\left|2\nabla\rho_n + \nabla(\theta\rho_n * V)\right|^2 dx \to 0. \tag{3.4.12}$$

To obtain the uniform L^2 bound for $\nabla\rho_n^{3/2}$, due to the triangle inequality, it suffices to prove a uniform L^2 bound for $\rho_n^{1/2}\nabla(\theta\rho_n * V)$, which is true since ρ_n is uniformly bounded in n and $\|V\|_{C^2(\mathbb{T}^d_L)} < \infty$ due to **(V2)**, hence the claim is proved.

As a consequence of the claim, we obtain weak convergence of $\nabla\rho_n^{3/2}$ in L^2 (along another subsequence) And, it is clear, the limit is just $\nabla\rho_\infty^{3/2}$ due to the uniform convergence of the (ρ_n). (Moreover, this places $\nabla\rho_\infty^{3/2} \in L^2(\mathbb{T}^d_L : \mathbb{R}^d)$). Thus:

$$\nabla\rho_n^{3/2} \rightharpoonup \nabla\rho_\infty^{3/2} \text{ as } n \to \infty \text{ weakly in } L^2(\mathbb{T}^d_L : \mathbb{R}^d). \tag{3.4.13}$$

Let

$$B_n := \frac{4}{3}\nabla\rho_n^{3/2} + \rho_n^{1/2}\nabla(\theta\rho_n * V).$$

Then (3.4.11) and (3.4.13) and an additional uniform convergence argument identifying the weak limit of $\rho_n^{1/2}\nabla(\theta\rho_n * V)$, implies that B_n weakly converges to B_∞ in L^2, where

$$B_\infty := \frac{4}{3}\nabla\rho_\infty^{3/2} + \rho_\infty^{1/2}\nabla(\theta\rho_\infty * V).$$

On the other hand, recall that (3.4.12) gives us that $B_n \to 0$ strongly in L^2, thus we have B_∞ is indeed 0 i.e.,

$$\int_{\mathbb{T}^d_L} \left|\frac{4}{3}\nabla\rho_\infty^{3/2} + \rho_\infty^{1/2}\nabla(\theta\rho_\infty * V)\right|^2 dx = \int_{\mathbb{T}^d_L} \rho_\infty\left|2\nabla\rho_\infty + \nabla(\theta\rho_\infty * V)\right|^2 dx = 0. \tag{3.4.14}$$

In particular, then, $\nabla(\rho_\infty + \frac{1}{2}\theta\rho_\infty * V)$ is zero a.e. on the support of ρ_∞. Now ρ_∞ certainly admits a weak derivative which, clearly, is non–zero only on the support of ρ_∞. Thus, from the preceding, we can write

$$\int_{\mathbb{T}_L^d} \nabla\rho_\infty \cdot \nabla(\rho_\infty + \frac{1}{2}\theta\rho_\infty * V)dx = 0. \tag{3.4.15}$$

Now, we wish to express the above as a Fourier sum which requires some additional justification. To this end we claim that ρ_∞ is Lipschitz continuous – i.e., in $W^{1,\infty}(\mathbb{T}_L^d)$ – which places both entities in $L^2(\mathbb{T}_L^d)$ and vindicates the use of explicit formulas.

The equation $\nabla\rho_\infty = -\frac{1}{2}\theta\nabla(V * \rho_\infty)$ valid on the support of ρ shows that in the various *components* where ρ_∞ is positive, it is at least C^2. Indeed, in general, Hypothesis **(V2)** immediately implies $\|\rho_\infty(x) * V\|_{C^2(\mathbb{T}_L^d)} \leq \|\rho_\infty\|_{L^1}\|V\|_{C^2(\mathbb{T}_L^d)}$ so whenever ρ_∞ satisfies this ($m = 2$ version of the Kirkwood–Monroe) equation, we have Lipschitz continuity with uniform constant. We shall denote this constant by κ. Now suppose that $x, y \in \mathbb{T}_L^d$ have $\rho_\infty(x)$ and $\rho_\infty(y)$ positive. Let us assume, ostensibly, that x and y belong to different components. On the (shortest) line joining x and y, let z_x denote the first point, starting from x that is encountered on the boundary of the component of x and similarly for z_y. Then

$$\begin{aligned} |\rho_\infty(x) - \rho_\infty(y)| &= |\rho_\infty(x) - \rho_\infty(z_x) + \rho_\infty(z_y) - \rho_\infty(y)| \\ &\leq |\rho_\infty(x) - \rho_\infty(z_x)| + |\rho_\infty(z_y) - \rho_\infty(y)| \\ &\leq \kappa[|x - z_x| + |y - z_y|] \leq \kappa|x - y|; \end{aligned} \tag{3.4.16}$$

the first inequality due to $\rho_\infty(z_x) = \rho_\infty(z_y) = 0$ and the last inequality because all four points lie in order on the same line. A similar argument can be used if, e.g., $\rho_\infty(x)$ is positive and $\rho_\infty(y)$ is zero.

All of this establishes enough regularity to unabashedly express (3.4.15) in Fourier modes:

$$0 = \sum_k \frac{|k|^2}{L^d}|\hat{\rho}_\infty(k)|^2(1 + \frac{1}{2}\theta\hat{V}(k)). \tag{3.4.17}$$

By the defining property of $\theta^\sharp$ we have $1 + \frac{1}{2}\theta\hat{V}(k) > 0$ for all $k \neq 0$, thus (3.4.17) implies $\hat{\rho}_\infty(k) = 0$ for all $k \neq 0$, i.e. $\rho_\infty \equiv \rho_0$.

Now, we may use the monotonicity in time of $\mathcal{F}_\theta(\rho(t))$ and we finally have

$$\lim_{t\to\infty} \mathcal{F}_\theta(\rho(t)) = \lim_{n\to\infty} \mathcal{F}_\theta(\rho_n) = \mathcal{F}_\theta(\rho_\infty) = \mathcal{F}_\theta(\rho_0)$$

which is the stated claim. □

By combining the above result with the uniform continuity in time, we can show the solution will become uniformly positive after a sufficiently large time.

Corollary 3.4.5. *Under the assumption of Lemma 3.4.4, we have*

$$\lim_{t\to\infty} \|\rho(\cdot,t) - \rho_0\|_{L^\infty(\mathbb{T}_L^d)} = 0,$$

hence there exists $T > 0$ depending on θ, $\|V\|_{C^2(\mathbb{T}_L^d)}$ and $\rho(\cdot,0)$, such that $\rho(x,t) > \rho_0/2$ for all $x \in \mathbb{T}_L^d, t > T$.

Proof. We prove the statement in the display. Supposing that this is not the case. Then there is a sequence of times, (τ_n) and points (y_n) – $y_n \in \mathbb{T}_L^d$ – and a $\delta > 0$ such that

$$|\rho(y_n,\tau_n) - \rho_0| > \delta.$$

Now, going to a further subsequence, we have $y_n \to y_\infty$ (with $y_\infty \in \mathbb{T}_L^d$ by compactness). But, along yet a further subsequence, not relabeled, we have, according to the arguments of Lemma 3.4.4 that $\rho(\cdot,\tau_n)$ is converging uniformly and the limit *must* be ρ_0. Thus

$$\lim_{n\to\infty} \rho_n(y_n,\tau_n) = \lim_{n\to\infty} [\rho_n(y_n,\tau_n) - \rho_n(y_\infty,\tau_n)] + \lim_{n\to\infty} \rho_n(y_\infty,\tau_n) = \rho_0$$

in contradiction with the preceding display. □

Theorem 3.4.6. *Suppose $m = 2$ and $\theta < \theta^\sharp$, where $\theta^\sharp$ is as given in* (3.4.4). *Consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x,t)$ be the weak solution to* (3.1.1) *on* $[0,\infty) \times \mathbb{T}_L^d$*, with non–negative initial data $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ which integrates to 1. Then $\mathcal{F}_\theta(\rho(t))$ decays exponentially to $\mathcal{F}_\theta(\rho_0)$, where the rate depend on $\rho(x,0)$. Moreover, $\|\rho(\cdot,t) - \rho_0\|_{L^2(\mathbb{T}_L^d)} \to 0$ exponentially, i.e.*

$$0 \le \mathcal{F}_\theta(\rho(t)) - \mathcal{F}_\theta(\rho_0) \le C_1 \exp(-\frac{\rho_0 c'}{L^2} t),$$

and

$$\|\rho(t)-\rho_0\|_{L^2(\mathbb{T}_L^d)} \le C_2 \exp(-\frac{\rho_0 c'}{L^2}t),$$

where c' and C_1 and C_2 depend on θ, V and $\rho(\cdot,0)$.

Proof. By Lemma 3.4.5, there exist some $T>0$ depending on θ, V and $\rho(\cdot,0)$, such that $\rho(x,t)>\rho_0/2$ for all $x\in\mathbb{T}_L^d, t>T$. Then for all $t_2>t_1>T$, (3.4.8) becomes

$$\begin{aligned}\mathcal{F}_\theta(\rho(\cdot,t_1))-\mathcal{F}_\theta(\rho(\cdot,t_2)) &\ge \int_{t_1}^{t_2}\int_{\mathbb{T}_L^d}\frac{\rho_0}{2}|\nabla(2\rho+\theta\rho*V)|^2dxdt\\ &= 2\rho_0\int_{t_1}^{t_2}\frac{1}{L^d}\sum_k |k|^2|\hat\rho(k)|^2(1+\frac{1}{2}\theta\hat V(k))^2dt\\ &\ge \rho_0 c'\int_{t_1}^{t_2}\frac{1}{L^d}\sum_{k\neq 0}|\hat\rho(k)|^2(1+\frac{1}{2}\theta\hat V(k))dt\\ &= \rho_0 c'\int_{t_1}^{t_2}(\mathcal{F}_\theta(\rho(\cdot,t))-\mathcal{F}_\theta(\rho_0))dt, \qquad (3.4.18)\end{aligned}$$

where $c'=2\min_{k\neq 0}|k^2|(1+\frac{1}{2}\theta\hat V(k))$, which is positive when $\theta<\theta^\sharp$.

In the spirit of Remark 3.4.3 we may regard the above as a differential inequality for $g(t):=\mathcal{F}_\theta(\rho(\cdot,t))-\mathcal{F}_\theta(\rho_0)$; the inequality reads

$$-\frac{dg}{dt}\ge\rho_0 c' g(t).$$

This immediately integrates to yield $g(t)\le g(T)\exp\{-\rho_0 c'(t-T)\}$ for $t\ge T$. I.e.,

$$\mathcal{F}(\rho(\cdot,t))-\mathcal{F}(\rho_0)\le C\mathrm{e}^{-\rho_0 c' t}.$$

Since $\mathcal{F}_\theta(\rho(\cdot,t))-\mathcal{F}_\theta(\rho_0)$ is comparable to $\|\rho(t)-\rho_0\|_{L^2}$, we have $\|\rho(t)-\rho_0\|_{L^2}\to 0$ exponentially with the same rate. □

Remark 3.4.7. It is remarked that, via comparison to linearized theory, the above is essentially optimal. (The results differ by a factor of two which comes from the definition of $T =: T_{1/2}$. Using $T_\epsilon=\sup\{t>0 \mid ||\rho(\cdot,t)-\rho_0||_{L^\infty(\mathbb{T}_L^d)}>\epsilon\rho_0\}$, the *long* time asymptotic rates are actually in complete agreement.) Moreover, while for L of order unity, the result stands: c' – with or without an additional factor of two – might well be optimized at a wave number of order unity. However, as $L\to\infty$, it is clear that

$$\min_{k\neq 0}|k|^2(1+\frac{1}{2}\hat V(k))\to(\frac{2\pi}{L})^2(1+\frac{1}{2}\hat V(0)).$$

So, in particular, for large L the rate scales as $L^{-(d+2)}$ – a result which may be an artifact of our normalization.

3.4.2 Some remarks on the supercritical case, when $m = 2$

When $\theta > \theta^\sharp$, we have $1 + \frac{1}{2}\theta\hat{V}(k_0) < 0$ for some $k_0 = \frac{2\pi}{L}\vec{n}_0$, where $\vec{n}_0 \in \mathbb{Z}^d$. In other words, at least one of the coefficients of the free energy (3.4.3) is negative. In the next proposition we show that in this case the constant solution ρ_0 is not linearly stable.

Proposition 3.4.8. *Suppose $m = 2$ and $\theta < \theta^\sharp$, where $\theta^\sharp$ is as given in* (3.4.4). *Consider an interaction V that satisfies the assumptions **(V1)** and **(V2)**. Then the constant solution ρ_0 is not a local minimizer of the free energy* (3.4.3) *in $\mathscr{P}$.*

Proof. We choose $k_0 = \frac{2\pi}{L}\vec{n}_0$ such that $1 + \frac{1}{2}\theta\hat{V}(k_0) < 0$, where $\vec{n}_0 \in \mathbb{Z}^d$. We add a small pertubation $\epsilon\eta$ to the constant solution ρ_0, where

$$\eta := \cos(\frac{2\pi n_0 \cdot x}{L}).$$

Then

$$\mathcal{F}_\theta(\rho_0 + \epsilon\eta) = \mathcal{F}_\theta(\rho_0) + L^d\epsilon^2(1 + \frac{1}{2}\theta\hat{V}(k_0)),$$

which is strictly less than $\mathcal{F}_\theta(\rho_0)$ by the defining property of k_0. □

Remark 3.4.9. In fact, using the same perturbation term in the proof, we would know that when $\theta > \theta^\sharp$, any strictly positive function is not a local minimizer of the free energy (3.4.3).

In the supercritical case, while (3.4.3) immediately implies that ρ_0 is not a local minimizer of $\mathcal{F}_\theta$ in $\mathscr{P}$, it gives us little information about what is the global minimizer. The difficulty comes from the restriction $\rho(x) \geq 0$ for all x, which evidently plays an important role in the supercritical case, since any minimizer should touch zero somewhere due to Remark 3.4.9. After Fourier transform, the non-negativity of ρ actually gives us infinite numbers of restrictions, which causes the difficulty.

3.5 Exponential decay for $1 < m < 2$ and weak interaction

In this section, we continue our study of (3.1.1) with $m \in (1, 2)$ and here we will assume that θ is "small". Unfortunately, θ will *not* be uniformly small in volume. In particular, we shall require $\theta L^{d(2-m)}$ to be a small number of order unity and, under these conditions we shall acquire all the results of the previous section. We claim that without additional (physics based) assumptions – in particular H–stability of the interaction – the above condition is essentially optimal. Specifically, our cornerstone result of a unique stationary state does not hold for non–H–stable interactions when $\theta L^{d(2-m)}$ is a sufficiently *large* number of order unity. However, from an æsthetic perspective, this uniqueness result is the sole instance where $\theta L^{d(2-m)}$ must be considered small. In the aftermath of Proposition 3.5.1 and its corollary, we will only require θ itself to be a small quantity.

We start with a priliminary result (which is, actually, just a quantitative version of the argument used in Lemma 3.4.4 in the vicinity of (3.4.16)).

Proposition 3.5.1. *Consider an interaction V that satisfies the assumptions **(V1)** and **(V2)**. Let*

$$\varepsilon_0 := \theta L^{d(2-m)}$$

be a sufficiently small number of order unity. Let ρ denote any solution to the Kirkwood–Monroe equations which here read, whenever $\rho > 0$,

$$\nabla \rho^{m-1} = -\varepsilon_0 \frac{m-1}{m} \rho * \nabla V$$

and let

$$R := \|\rho\|_{L^\infty(\mathbb{T}_L^d)}.$$

Then if ε_0 is a small number of order unity then R is also a small number of order unity (if L is large). In particular,

$$R \leq \kappa_4 \max\{[\varepsilon_0]^{\frac{d}{d(m-1)+1}}, L^{-d}\}$$

with κ_4 a constant of order unity.

Proof. From the mean–field equations,

$$|\nabla\rho^{m-1}| \leq \frac{m-1}{m}\varepsilon_0 \int_{\mathbb{T}_L^d} |\nabla V(x-y)|\rho(y)dy \leq \frac{m-1}{m}\|V\|_{C_1}\varepsilon_0 =: \kappa_1\varepsilon_0.$$

Let x_0 mark the spot where ρ achieves R. Then, for all x,

$$\rho^{m-1}(x) \geq R^{m-1} - \kappa_1\varepsilon_0|x-x_0|.$$

Thus, if r is the length scale of the region about x_0 where ρ^{m-1} exceeds, a.e., $\frac{1}{2}R^{m-1}$ we have

$$r \geq \frac{R^{m-1}}{2\kappa_1\varepsilon_0}$$

provided the right hand side does not exceed L. Otherwise, obviously, $r = L$. Since ρ integrates to unity we have, assuming $r < L$,

$$1 = \int_{\mathbb{T}_L^d} \rho dx \geq \kappa_2 r^d R \geq \kappa_2 \frac{R^{d(m-1)+1}}{(2\kappa_1\varepsilon_0)^d} =: \frac{1}{\kappa_3^d}\frac{1}{\varepsilon_0^d} R^{d(m-1)+1}$$

(with κ_2 a geometric constant of order unity) and otherwise we acquire the mundane bound. After a small step, the stated bound is obtained with an appropriate definition of κ_4. □

With the above in hand, we can establish that ρ_0 is the unique stationary solution. We start with

Corollary 3.5.2. *Under the conditions stated in Proposition 3.5.1, if ε_0 is sufficiently small – but of order unity independent of L – the unique solution to the mean–field equations is $\rho = \rho_0$.*

Proof. From the mean–field equation, we may write

$$0 = \int_{\mathbb{T}_L^d} \nabla\rho \cdot \nabla(\rho^{m-1} + \varepsilon_0 \frac{m-1}{m}\rho * V)dx.$$

By recapitulating the Lipchitz continuity that was featured in the vicinity of (3.4.16) we have full justification to manipulate classically under the integral. Letting R_{ε_0} denote the upper bound on the L^∞ norm of ρ that was featured in Proposition 3.5.1. Then, pointwise a.e. on the support of ρ,

$$\nabla\rho \cdot \nabla\rho^{m-1} = \frac{m-1}{\rho^{2-m}}|\nabla\rho|^2 \geq \frac{m-1}{R_{\varepsilon_0}^{2-m}}|\nabla\rho|^2$$

since, we remind the reader, $2-m>0$. In other words,

$$0 \geq \int_{\mathbb{T}_L^d} \frac{1}{R_{\varepsilon_0}^{2-m}} |\nabla\rho|^2 + \frac{\varepsilon_0}{m} \nabla\rho \cdot \nabla(\rho * V) dx.$$

We can again go to Fourier modes and the above reads

$$0 \geq \sum_{k\neq 0} k^2 |\hat{\rho}(k)|^2 [\frac{1}{R_{\varepsilon_0}^{2-m}} + \frac{\varepsilon_0}{m}\hat{V}(k)].$$

For ε_0 sufficiently small (but of order unity independent of L) the coefficient of $|\hat{\rho}(k)|^2$ is positive for all terms so the later must vanish identically. The desired result is proved. □

Based on the fact that ρ_0 is the unique stationary solution, in the next lemma we prove that $\rho(\cdot,t)$ will converge to ρ_0 uniformly, (but not with a quantitative estimate on the *rate*.)

Lemma 3.5.3. *Suppose the conclusions in Corollary 3.5.2 are satisfied. Let $\rho(x,t)$ be the weak solution to* (3.1.1) *on $[0,\infty)\times\mathbb{T}_L^d$, with non-negative initial data $\rho(x,0)\in L^\infty(\mathbb{T}_L^d)$ which integrates to 1. Then $\sup_x|\rho(\cdot,t)-\rho_0|\to 0$ as $t\to\infty$.*

Proof. This is more or less identical to the proof of Corollary 3.4.5 based on Lemma 3.4.4 so we shall be succinct. Assuming the result false, we could find a sequence of times $t_n\to\infty$ and points $x_n\to x_\infty\in\mathbb{T}_L^d$ such that $\rho(\cdot,t_n)$ converges uniformly and yet $|\rho(x_n,\tau_n)-\rho_0|>\delta$. So, denoting by $\rho_\infty(\cdot)$ the uniform limit, we would have $|\rho_\infty(x_\infty)-\rho_0|>\delta$.

Hence, since ρ_∞ is continuous, it is definitively *not* equal to ρ_0. However, any subsequential limit must satisfy the mean–field equation and by Corollary 3.5.2 this is uniquely ρ_0 in contradiction with the preceding. This completes the proof. □

In the next lemma, we show that once ρ and ρ_0 becomes comparable, $\mathcal{F}_\theta(\rho)-\mathcal{F}_\theta(\rho_0)$ also becomes comparable with $L^{d(2-m)}\|\rho-\rho_0\|_{L^2(\mathbb{T}_L^d)}$. Indeed, as alluded to earlier, this will be proved under the weaker assumption that θ – not $\theta L^{d(m-2)}$ – is small. We start with:

Lemma 3.5.4. *Suppose that $\theta>0$ is sufficiently small (but of order unity independent of L). Let ρ be such that $\|\rho-\rho_0\|_{\mathbb{T}_L^d}<\frac{1}{2}\rho_0$. Then we have*

$$\alpha L^{d(2-m)}\|\rho-\rho_0\|^2_{L^2(\mathbb{T}_L^d)} \leq \mathcal{F}_\theta(\rho)-\mathcal{F}_\theta(\rho_0) \leq \beta L^{d(2-m)}\|\rho-\rho_0\|^2_{L^2(\mathbb{T}_L^d)} \tag{3.5.1}$$

for some $\alpha,\beta>0$ of order unity.

Proof. First, by any number of methods we have

$$\int_{\mathbb{T}_L^d\times\mathbb{T}_L^d}\rho(x)\rho(y)V(x-y)dxdy \geq -K_V\|\rho-\rho_0\|^2_{L^2(\mathbb{T}_L^d)};$$

e.g., we may take, using the Fourier decomposition, $K_V = [\theta^\sharp]^{-1}$. Similarly for a corresponding *upper* bound with a positive constant. Let us turn to the entropic–like terms.

Writing $\rho = \rho_0(1+\eta)$, our assumption implies that $|\eta| \leq \frac{1}{2}$. From this it is easy to verify that, pointwise,

$$(1+\eta)^m \geq 1 + m\eta + \frac{m(m-1)}{2}(\frac{2}{3})^{2-m}\eta^2 := 1 + m\eta + a\eta^2,$$

and for the other direction we have

$$(1+\eta)^m \leq 1 + m\eta + \frac{m(m-1)}{2}(3)^{2-m}\eta^2 := 1 + m\eta + b\eta^2.$$

Thence $\rho^m - \rho_0^m = \rho_0^m[(1+\eta)^m - 1] = \rho_0^m[(1+\eta)^m - 1 - m\eta + m\eta] \geq \rho_0^m[m\eta + a\eta^2]$. So

$$\int_{\mathbb{T}_L^d}(\rho^m - \rho_0^m)dx \geq a\rho_0^m\|\eta\|^2_{L^2(\mathbb{T}_L^d)} = aL^{d(2-m)}\|\rho-\rho_0\|^2_{L^2(\mathbb{T}_L^d)},$$

and similarly we have

$$\int_{\mathbb{T}_L^d}(\rho^m - \rho_0^m)dx \leq bL^{d(2-m)}\|\rho-\rho_0\|^2_{L^2(\mathbb{T}_L^d)}.$$

Combining this with the bounds on the energy term, the stated claim has been established.

□

Finally, in the next theorem, we prove that the free energy decays exponentially to its minimum value.

Theorem 3.5.5. *Suppose the conclusions acquired in Corollary 3.5.2 are satisfied and suppose that θ is a sufficiently small number which is of the order of unity. Let $\rho(x,t)$ be the weak solution to* (1.1.1) *on $[0,\infty)\times\mathbb{T}_L^d$, with non-negative initial data $\rho(x,0)\in L^\infty(\mathbb{T}_L^d)$ which integrates to 1. Then $\mathcal{F}_\theta(\rho(t))$ decays exponentially to $\mathcal{F}_\theta(\rho_0)$. More precisely,*

$$\mathcal{F}(\rho(\cdot,t)) - \mathcal{F}(\rho_0) \leq C_1 e^{-\rho_0^{m-1}c't} \tag{3.5.2}$$

for various constants c' and C_1. Similarly for the L^2–norm of $(\rho-\rho_0)$ with a different prefactor.

Proof. According to Lemma 3.5.3, there exist some $T > 0$ depending on θ, L, V and $\rho(\cdot, 0)$, such that $|\rho(x,t) - \rho_0| < \frac{1}{2}\rho_0$ for all $x \in \mathbb{T}_L^d, t > T$. Then for all $t_2 > t_1 > T$, we manipulate the integrand on the right hand side of (3.4.8) – the lower bound on $\mathcal{F}_\theta(\rho(\cdot,t_1)) - \mathcal{F}_\theta(\rho(\cdot,t_2))$:

$$\int_{\mathbb{T}_L^d} \rho|\nabla \frac{m}{m-1}\rho^{m-1} + \theta L^{d(2-m)}\nabla\rho * V|^2 dx \geq$$

$$\begin{aligned} &\int_{\mathbb{T}_L^d} \rho \left[\frac{1}{2}|\nabla \frac{m}{m-1}\rho^{m-1}|^2 - |\theta L^{d(2-m)}\nabla\rho * V|^2\right] dx \\ =& \int_{\mathbb{T}_L^d} \left[\frac{1}{2}m^2\rho^{2m-3}|\nabla\rho|^2 - \rho\theta^2 L^{2d(2-m)}|\nabla(\rho * V)|^2\right] dx \\ \geq& \int_{\mathbb{T}_L^d} \left[g\rho_0^{2m-3}|\nabla\rho|^2 - \frac{3}{2}\rho_0\theta^2 L^{2d(2-m)}|\nabla(\rho * V)|^2\right] dx \end{aligned} \tag{3.5.3}$$

where the value of g – which is always of order unity – depends on whether $2m-3$ is positive or not. Note that all terms are proportional to $\rho_0^{2m-3} = \rho_0^{m-1}L^{d(2-m)}$.

Going to Fourier modes, the final (spatial) integral in the above string becomes

$$\rho_0^{m-1}L^{d(2-m)} \cdot \frac{1}{L^d}\sum_k k^2|\hat{\rho}(k)|^2[g - \frac{3}{2}\theta^2|\hat{V}(k)|^2]$$

where, for sufficiently small θ, we may assert that the summand is positive.

We thus have

$$\mathcal{F}_\theta(\rho(\cdot,t_1)) - \mathcal{F}_\theta(\rho(\cdot,t_2)) \geq$$

$$\rho_0^{m-1}\beta c' \int_{t_1}^{t_2}\int_{\mathbb{T}_L^d} L^{d(2-m)}(\rho - \rho_0)^2 dx dt \;\geq\; \rho_0^{m-1}c' \int_{t_1}^{t_2} [\mathcal{F}_\theta(\rho(\cdot,t)) - \mathcal{F}_\theta(\rho_0)]dt \tag{3.5.4}$$

where in the above, β is the constant from Lemma 3.5.4 which has been conveniently absorbed into the definition of c':

$$c'\beta := \min_{k\neq 0}[k^2(g - \frac{3}{2}\theta^2|\hat{V}(k)|^2)]$$

and in the final step we have used Lemma 3.5.4.

Note that (3.5.4) has the same form as (3.4.18) therefore we can again treat it as a differential inequality as in the proof of Theorem 3.4.6. We obtain that

$$\mathcal{F}(\rho(\cdot,t)) - \mathcal{F}(\rho_0) \leq C_1 \mathrm{e}^{-\rho_0^{m-1}c't}.$$

A further application of Lemma 3.5.4 implies a similar result for the L^2–norm of $(\rho - \rho_0)$ and the proof is finished. $\square$

Remark 3.5.6. Here as in the case $m = 2$, when L is large, $c' \propto L^{-2}$ and we have the large L scaling of the rate proportional to $L^{-(2+d(m-1))}$ in agreement with a perturbative analysis. However in this case, our arguments do not provide agreement with the constant of proportionality. We also note that by Theorem 3.2.4 we have that $\rho(\cdot, t)$ is uniformly Hölder continuous in space and time for all $t \geq T$, where the Hölder coefficient and exponent depends on θ, L and V. Thus we can bound, the L^∞–norm of $\rho - \rho_0$ by some power of its L^2–norm. Hence the exponential convergence of $\|\rho - \rho_0\|_{L^2(\mathbb{T}_L^d)}$ implies the exponential convergence of $\|\rho - \rho_0\|_{L^\infty(\mathbb{T}_L^d)}$. However a bound along these lines is "even more" non–optimal since the two norms should, presumably, differ by a factor of L^d.

APPENDIX A

Additional Computations

A.1 Additional computations for Chapter 1

A.1.1 Proof of existence for ρ as given in Proposition 1.2.1

Here we will show the existence of a global minimizer for the free energy functional given in (3.1.2). First note that the kernels V given in (A) and (B) belong to $M^{\frac{d}{d-2}}$, where M^p denotes the weak L^p space.

Our proof is based on a theorem of Lions in [L]:

Theorem A.1.1 ([L])**.** *Suppose $f \in M^p(\mathbb{R}^d)$, $f \geq 0$ and consider the problem*

$$I_\lambda = \inf_{u \in K_\lambda} \Big\{ \int_{\mathbb{R}^d} \frac{1}{m-1} u^m dx - \frac{1}{2} \int_{\mathbb{R}^d} u(u * f) dx \Big\},$$

where

$$K_\lambda = \Big\{ u \in L^q(\mathbb{R}^d) \cap L^1(\mathbb{R}^d),\ u \geq 0 \text{ a.e.}, \int_{\mathbb{R}^d} u dx = \lambda \Big\} \text{ with } q = \frac{p+1}{p}.$$

Then there exists a minimizer of problem (I_λ) if and only if the following holds:

$$I_\lambda < I_\alpha + I_{\lambda-\alpha},\ \forall \alpha \in (0, \lambda). \tag{A.1.1}$$

Proposition A.1.2 ([L])**.** *Suppose there exists some $\alpha \in (0, d)$ such that $f(tx) \geq t^{-\alpha} f(x)$ for all $t \geq 1$. Then (A.1.1) holds if and only if*

$$I_\lambda < 0,\ \textit{for all } \lambda > 0. \tag{A.1.2}$$

For the rest of this subsection, we will verify that Proposition A.1.2 applies to our kernels.

Proposition A.1.3. *Let $f = V$ given by either* (1.1.3) *or* (1.1.4). *Then $f(tx) \geq t^{-\alpha} f(x)$ with $t \geq 1$ and $\alpha = d - 2$.*

Proof. When V is given by (A) the proof is straightforward, so suppose V is given by (B). Then $f = -\mathcal{N} * h$. Since h can be approximated by a sum of indicator functions, it suffices to prove the proposition for $f = -\mathcal{N} * \chi_{B(0,r)}$, where χ is the indicator function. In this case we have

$$f(x) = \begin{cases} \dfrac{1}{2(d-2)} r^2 - \dfrac{1}{2d}|x|^2 & \text{for } |x| \leq r, \\ \dfrac{1}{d(d-2)} r^d |x|^{-d+2} & \text{for } |x| > r, \end{cases} \tag{A.1.3}$$

which finishes the proof. □

Proposition A.1.4. *Let f be as in Lemma A.1.3. Suppose $m > 2 - \frac{2}{d}$ and let us define*

$$u = \frac{\lambda \chi_{B(0,R)}}{c_d R^d}$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$ and R is a constant to be chosen later. If R is sufficiently large, we have

$$E(u) := \int_{\mathbb{R}^d} \frac{1}{m-1} u^m dx - \frac{1}{2} \int_{\mathbb{R}^d} u(u * f) dx < 0,$$

and thus $I_\lambda \leq E(u) < 0$.

Proof. First note that we have

$$\int_{\mathbb{R}^d} \frac{1}{m-1} u^m dx = \int_{\mathbb{R}^d} \frac{1}{m-1} \Big(\frac{\lambda \chi_{B(0,R)}}{cR^d} \Big)^m dx \simeq \lambda^m R^{-d(m-1)}.$$

On the other hand, $\int_{B(0,R/2)} (-V) dx \simeq R^2$ if R is sufficiently large: this implies $\int_{\mathbb{R}^d} u(u * (-V)) dx \gtrsim \lambda^2 R^{-d+2}$. Since $m > 2 - \frac{2}{d}$ we have

$$E(u) = \int_{\mathbb{R}^d} \frac{1}{m-1} u^m dx - \int_{\mathbb{R}^d} u(u * (-V)) dx < 0.$$

□

A.1.2 Proof of Lemma 1.4.1

Observe that ΔV is nonnegative and radially decreasing, and thus it can be approximated in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ by the sum of bump functions of the form $c\chi_{B(0,r)}$, where $c > 0$. By linearity of convolution, it suffices to prove that for each bump function $\chi_{B(0,r)}$, where r is any positive real number, we have

$$(u * \chi_{B(0,r)})(b_1) - (u * \chi_{B(0,r)})(a_1) \le \|\chi_{B(0,r)}\|_1 (u(b_1) - u(a_1)). \tag{A.1.4}$$

Observe that

$$\begin{aligned}(u * \chi_{B(0,r)})(b_1) - (u * \chi_{B(0,r)})(a_1) &= \int_{B(b_1,r)} u(x)dx - \int_{B(a_1,r)} u(x)dx &\text{(A.1.5)}\\ &= \int_{\Omega_B} u(x)dx - \int_{\Omega_A} u(x)dx. &\text{(A.1.6)}\end{aligned}$$

Here $\Omega_A := B(a_1, r)\backslash B(b_1, r)$ and $\Omega_B := B(b_1, r)\backslash B(a_1, r)$ (see Figure A.1).

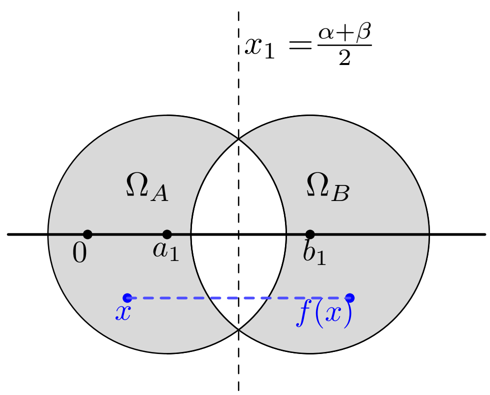


Figure A.1: The domains Ω_A and Ω_B

Note that Ω_A and Ω_B are symmetric about the hyperplane $H = \{x : x_1 = \frac{\alpha+\beta}{2}\}$. For any $x \in \Omega_A$, use $f(x)$ to denote the reflection point of x with respect to H. Then we have

$$\int_{\Omega_B} u(x)dx - \int_{\Omega_A} u(x)dx = \int_{\Omega_A} (u(f(x)) - u(x))dx.$$

Since $|x| < |f(x)|$ for $x \in \Omega_A$, we can use the assumption (1.4.1) to obtain

$$\int_{\Omega_A} (u(f(x)) - u(x))dx \le \int_{\Omega_A} (u(b) - u(a))dx \le |B(0,r)|(u(b) - u(a)),$$

which completes the proof.

A.1.3 Proof of Proposition 1.6.2

The proof of Proposition 1.6.2 is an application of the Crandall-Liggett Theorem ([CL], also see Theorem 10.16 in [V]). Let us consider the following domain:

$$D := \Big\{ \rho \in L^1(\mathbb{R}^d) : \rho^m \in W^{1,1}_{\text{loc}}(\mathbb{R}^d), \Delta\rho^m \in L^1(\mathbb{R}^d), |\nabla \rho^m| \in M^{d/(d-1)}(\mathbb{R}^d) \Big\}. \tag{A.1.7}$$

Here the Marcinkiewicz space $M^p(\mathbb{R}^d), 1 < p < \infty$, is defined as the set of $f \in L^1_{loc}(\mathbb{R}^d)$ such that

$$\int_K |f(x)| dx \leq C|K|^{\frac{p-1}{p}},$$

for all subsets K of finite measure. The minimal C in the above inequality gives a norm in this space, i.e.

$$\|f\|_{M^p(\mathbb{R}^d)} = \sup \Big\{ \text{meas}(K)^{-\frac{p-1}{p}} \int_K |f| dx : K \subset \mathbb{R}^d, \text{meas}(K) > 0 \Big\}.$$

A parallel argument as in Theorem 2.1 of [BBC] yields the existence of solutions for the discretized equation.

Lemma A.1.5 (Existence). *Let $d \geq 3$ and let $\rho_0 \in L^1(\mathbb{R}^d), \Phi \in C^2(\mathbb{R}^d)$. Then there exists a unique weak solution $\rho \in D$ of the following equation:*

$$\frac{\rho - \rho_0}{h} = \Delta \rho^m + \nabla \cdot (\rho \nabla \Phi). \tag{A.1.8}$$

The proof of the next lemma is parallel to that of Prop 3.5 in [V] for (1.1.2).

Lemma A.1.6 (L^1 contraction). *Let $\Phi \in C^2(\mathbb{R}^d)$, $\rho_{0i} \in L^1(\mathbb{R}^d)$ and let $\rho_1, \rho_2 \in D$ be the weak solutions to the degenerate elliptic equation*

$$\frac{\rho_i - \rho_{0i}}{h} = \Delta(\rho_i)^m + \nabla \cdot (\rho_i \nabla \Phi), \quad i = 1, 2. \tag{A.1.9}$$

Then ρ_1 and ρ_2 satisfy

$$\|\rho_1 - \rho_2\|_{L^1(\mathbb{R}^d)} \leq \|\rho_{01} - \rho_{02}\|_{L^1(\mathbb{R}^d)}. \tag{A.1.10}$$

Proof of Proposition 1.6.2

Proof. Let D be defined above, and define the nonlinear operator $\mathcal{A} : D \to L^1(\mathbb{R}^d)$ by

$$\mathcal{A}(\rho) = -\Delta\rho^m - \nabla(\rho\nabla\Phi),$$

Then Lemma A.1.5 and Lemma A.1.6 yield that for any $h > 0$, there is a unique solution ρ in D solving

$$h\mathcal{A}(\rho) + \rho = f.$$

Moreover the map $f \mapsto \rho$ is a contraction in $L^1(\mathbb{R}^d)$. Now arguing as in [V], the Crandall-Liggett Theorem yields the conclusion. □

A.1.4 Proof of Proposition 1.6.4

The proof of Proposition 1.6.4 is parallel to that of Theorem 11.7 in [V] for (1.1.2). First we state a lemma which deals with the extra convolution term.

Lemma A.1.7. *Let V be given by (B). Let $f \in L^1(\mathbb{R}^d)$ and $\phi \in W_0^{1,\infty}(\mathbb{R}^d)$ be non-negative functions. Then for any non-negative number a, b, we have*

$$\int_{\{a<\phi<b\}} \nabla(f * (-V)) \cdot \nabla\phi \le \int_{\{\phi^*>a\}} (f^* * \Delta V)(\max\{\phi^*, b\} - a), \tag{A.1.11}$$

where the equality is achieved if f, ϕ are both radially decreasing.

Proof. Let us define $\eta : \mathbb{R}^d \to \mathbb{R}$ by

$$\eta(x) := \begin{cases} b & \text{if } \phi(x) \ge b, \\ \phi(x) - a & \text{if } a < \phi(x) < b, \\ 0 & \text{if } \phi(x) \le a. \end{cases}$$

Then $\eta(x) \in W_0^{1,\infty}(\mathbb{R}^d)$, $\nabla\phi = \nabla\eta$ in $\{a < \phi(x) < b\}$, and $\nabla\eta = 0$ in $\mathbb{R}^d \backslash \{a < \phi(x) < b\}$. Therefore

$$\begin{aligned} \text{LHS of (A.1.11)} &= \int_{\mathbb{R}^d} \nabla(f * (-V)) \cdot \nabla\eta \\ &\le \int_{\mathbb{R}^d} (f^* * \Delta V)\eta^* = \int_{\{\phi^*>a\}} (f^* * \Delta V)(\max\{\phi^*, b\} - a), \end{aligned}$$

where the inequality comes from Riesz's rearrangement inequality. Note that we obtain an equality if $f = f^*$ and $\eta = \eta^*$. Hence the lemma is proved. □

The following lemma corresponds to Theorem 17.5 in [V].

Lemma A.1.8. *Let V be given by (B). Let $f, \bar{f}$ and g be non-negative radially decreasing functions in $L^1(\mathbb{R}^d)$, where $f \prec \bar{f}$. Let $h > 0$, and let $v_1, v_2 \in D$ be two non-negative radial decreasing functions. Assume v_1 and v_2 satisfies*

$$-h\Delta(v_1)^m - h\nabla \cdot (v_1 \nabla(f * V)) + v_1 \prec g, \tag{A.1.12}$$

$$-h\Delta(v_2)^m - h\nabla \cdot (v_2 \nabla(\bar{f} * V)) + v_2 = g. \tag{A.1.13}$$

Then we have $v_1 \prec v_2$.

Proof. Let $u_i := v_i^m$ and define $u := u_1 - u_2$, $v := v_1 - v_2$, $A(r) := \int_{B(0,r)} v(x)dx$. Our goal is to show $A(r) \leq 0$ for all $r \geq 0$.

Subtracting (A.1.12) from (A.1.13), and integrating the quantity in $B(0,r)$ yields that

$$\int_{B(0,r)} -h\Delta u dx - h\Big(v_1(r) \int_{B(0,r)} f * \Delta V dx - v_2(r) \int_{B(0,r)} \bar{f} * \Delta V dx\Big) + A(r) \leq 0, \tag{A.1.14}$$

which can be written as

$$-hc_d r^{d-1} u'(r) - hv(r) \int_{B(0,r)} f * \Delta V dx - hv_2(r) \int_{B(0,r)} (f - \bar{f}) * \Delta V dx + A(r) \leq 0. \tag{A.1.15}$$

(Here $u'(r)$ exists due to the fact that $v_i \in D$ for $i = 1, 2$, which implies that Δu is in L^1.) Since we assume $f \prec \bar{f}$, it follows that $\int_{B(0,r)}((f - \bar{f}) * \Delta V)dx \leq 0$ for all $r \geq 0$. Therefore

$$-hc_d r^{d-1} u'(r) - hv(r) \int_{B(0,r)} f * \Delta V + A(r) \leq 0 \quad \text{for all } r \geq 0. \tag{A.1.16}$$

Note that since u_i and v_i both vanish at infinity, from (A.1.16) it follows that $\lim_{r\to\infty} A(r) \leq 0$. Hence if $A(r)$ is positive somewhere, it achieves its positive maximum at some point $r_0 > 0$. At $r = r_0$ we have $v(r_0) = A'(r_0) = 0$, and (A.1.16) becomes

$$u'(r_0) \geq \frac{A(r_0)}{hc_d r^{d-1}} > 0,$$

which means $u_2 - u_1$ is strictly increasing at r_0: hence $v_2 - v_1$ will also be strictly positive in $(r_0, r_0 + \epsilon)$ for some small ϵ, which implies $A(r_0 + \epsilon) > A(r_0)$. This contradicts our assumption that $A(r)$ achieves its maximum at r_0. Therefore $A(r) \leq 0$ for all r, which means $v_2 \prec v_1$. □

Proof of Proposition 1.6.4: The proof is parallel to that of Theorem 11.7 as in [V]. For any test function $\phi \in W_0^{1,\infty}(\mathbb{R}^d)$, we have

$$h\int_{\mathbb{R}^d} \nabla u^m \cdot \nabla \phi + h\int_{\mathbb{R}^d} u\nabla(f*V)\cdot\nabla\phi + \int_{\mathbb{R}^d} u\phi = \int_{\mathbb{R}^d} g\phi, \tag{A.1.17}$$

where $\phi \in W_0^{1,\infty}(\mathbb{R}^d)$ is any test function. Now let us take $\phi(x) := (u^m(x)-t)_+$ where $t>0$, and differentiate the equation with respect to t. Then we have:

$$\underbrace{-h(\frac{d}{dt}\int_{\{u^m>t\}}|\nabla u^m|^2)}_{I_1} \underbrace{-h(\frac{d}{dt}\int_{\{u^m>t\}}\frac{m}{m+1}\nabla(f*V)\cdot\nabla(u^{m+1}))}_{I_2} + \underbrace{\int_{\{u^m>t\}} u}_{I_3} = \underbrace{\int_{\{u^m>t\}} g}_{I_4}. \tag{A.1.18}$$

Following the proof of Theorem 17.7 in [V], one can check that

$$\begin{aligned} I_1 &\le \int_{\{(u^*)^m>t\}} h\Delta((u^*)^m) \ \ \text{(with equality if } u\equiv u^*), \\ I_3 &= \int_{\{(u^*)^m>t\}} u^*, \\ I_4 &\le \sup_{|\Omega|=|\{u^m>t\}|}\int_\Omega g^* = \int_{\{(u^*)^m>t\}} g^*. \end{aligned}$$

It remains to examine I_2. Using Lemma A.1.7, it follows that

$$\begin{aligned} I_2 &= h\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\{t<u^m<t+\epsilon\}}\frac{m}{m+1}\nabla(f*(-V))\cdot\nabla(u^{m+1}) \\ &\le h\liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\{t<(u^*)^m<t+\epsilon\}}\frac{m}{m+1}(f^**\Delta V)(\max\{u^{m+1},(t+\epsilon)^{1+\frac{1}{m}}\}-t^{1+\frac{1}{m}})_+ \\ &= ht^{\frac{1}{m}}\int_{\{(u^*)^m>t\}} f^**\Delta V. \end{aligned}$$

Plugging in the four inequalities into (A.1.18), the following inequality holds for all $t\ge 0$:

$$-\int_{\{(u^*)^m>t\}} h\Delta((u^*)^m) - ht^{\frac{1}{m}}\int_{\{(u^*)^m>t\}} f^**\Delta V + \int_{\{(u^*)^m>t\}} u^* \le \int_{\{(u^*)^m>t\}} g^*. \tag{A.1.19}$$

Since $t\ge 0$ is arbitrary, the above inequality implies

$$-h\Delta((u^*)^m) - h\nabla\cdot(u^*\nabla(f^**V)) + u^* \prec g^*. \tag{A.1.20}$$

On the other hand, by assumption, $\bar{u}$ solves

$$-h\Delta(\bar{u}^m) - h\nabla\cdot(\bar{u}\nabla(\bar{f}*V)) + \bar{u} = \bar{g}, \tag{A.1.21}$$

where $\bar{f}\succ f^*$ and $\bar{g}\succ g^*$. Note that $u\in D$ implies $u^*\in D$. So we can apply Lemma A.1.8 and get $u^*\prec\bar{u}$. □

A.1.5 Proof of Lemma 1.7.9

Proof. When $0 < k < 1$, the right hand side of (1.7.24) is bounded above by $(C_1 + C_2)k$. Hence if the initial data satisfies $0 < k(0) < 1$, the inequality $k(t) \leq k(0)e^{(C_1+C_2)t}$ will hold until k reaches 1. In other words, $k(t)$ is guaranteed to be smaller than 1 until time $t_1 := -\frac{\ln k(0)}{C_1+C_2}$.

Now if we choose $k(0)$ to be sufficiently small such that $0 < k(0) < \delta$, where

$$\delta := (\alpha C_1 C_2^{-1} 2^{-d-2})^{\frac{C_1+C_2}{\beta}},$$

then t_1 would be sufficiently large such that

$$C_2 2^{d+1} e^{-\beta t_1} \leq \frac{C_1 \alpha}{2}.$$

We claim $g(t) := 1 + e^{-\epsilon(t-t_1)}$ is a supersolution of (1.7.24) for $t \geq t_1$, where $\epsilon := \min\{\beta, \frac{1}{2}C_1\alpha\}$. It is obvious that $g(t_1) > 1 \geq k(t_1)$, so it suffices to show

$$g'(t) \geq C_1 g(1 - g^\alpha) + C_2 g^{d+1} e^{-\beta t} \quad \text{for } t \geq t_1. \tag{A.1.22}$$

By definition of g, we have

$$\begin{aligned} \text{RHS of (A.1.22)} &\leq -C_1 \alpha e^{-\epsilon(t-t_1)} + C_2 2^{d+1} e^{-\beta t_1} e^{-\beta(t-t_1)} & \text{(A.1.23)} \\ &\leq -\frac{1}{2} C_1 \alpha e^{-\epsilon(t-t_1)} & \text{(A.1.24)} \\ &\leq -\epsilon e^{-\epsilon(t-t_1)} = \text{LHS of (A.1.22)}. & \text{(A.1.25)} \end{aligned}$$

Therefore $k(t) \leq 1 + e^{-\epsilon(t-t_1)}$ for all $t \geq t_1$.

To obtain the corresponding lower bound for $k(t)$, note that the last term of (1.7.24) is non-negative. Therefore if g solves $g'(t) = C_1 g(1 - g^\alpha)$ and $g(0) = k(0)$, then $|g(t) - 1| \lesssim e^{-C_1 \alpha t}$. Comparison between these two ODEs yields $k(t) \geq g(t)$ for all $t \geq 0$, which implies $k(t) \geq 1 - Ce^{-C_1 \alpha t}$. Now we can conclude that there exists C depending on C_1, C_2, α, β and $k(0)$ such that

$$|k(t) - 1| \leq Ce^{-\epsilon t}.$$

$\square$

A.2 Additional Computations for Chapter 4

Proof of Lemma 3.2.5. We will do the comparison between ρ^- and w first; the comparison between ρ^+ and w can be done in the same way.

First note that w also satisfies (3.2.9) with $\Phi \equiv 0$, therefore the inequality (3.2.25) also hold for w, namely

$$w - \rho^- \geq -C_1 a, \tag{A.2.1}$$

where C_1 depends on m, d, M_1, M_2.

We define $f := w - \rho^-$, and our goal is to obtain an upper bound for f. More precisely, we want to show there exists some constant C and β depending on m, d, M_1, M_2, such that $f(x,t) \leq Ca^\beta$ in $\tilde{\Omega} \times [1,2]$.

Our strategy is as following. First, we claim that

$$g(T) := \sup_{y \in \tilde{\Omega}} \int_{B(y,1)\cap\tilde{\Omega}} f(x,T)dx < C_0 a \text{ for all } T \in [0,2], \tag{A.2.2}$$

where C_0 depends on m, d, M_1, M_2. We will prove this claim momentarily. Once we have the claim, we know the space integral of $f(x,t)$ in any unit ball is of order a, for $0 < t < 2$. To get $f(x,t) \leq O(a^\beta)$ for $t \in [1,2]$, it suffices to show f is Hölder continuous in space with exponent and constant that are uniform in time for all $t \in [1,2]$, which is indeed true, since Theorem 11.2 of [DGV] guarantees this uniform Hölder continuity of ρ^- and w for $t \in [1,2]$.

Now it suffices to prove our claim. It is proved by writing both equations in weak form, choosing an appropriate test function and applying the Gronwall inequality. By writing both (3.2.19) and (3.2.26) in weak form and subtracting the two equations, we arrive at

$$\underbrace{\int_{\tilde{\Omega}} f(x,T)\varphi(x)dx}_{I_1} = \underbrace{\int_{\tilde{\Omega}} f(x,0)\varphi(x)dx}_{I_2} + \int_0^T \underbrace{\int_{\tilde{\Omega}} \left(w^m - \rho^-|\rho^-|^{m^- -1}\right)\Delta\varphi(x) + Ma\varphi(x)\ dx\,dt}_{I_3}, \tag{A.2.3}$$

where $\varphi \in C_0^\infty(\tilde{\Omega})$ is a test function chosen as follows. For a fixed $T > 0$, there exists $z \in \tilde{\Omega}$, such that the maximum of $\int_{B(y,1)\cap\tilde{\Omega}} f(x,T)dx$ is achieved at z. We then define

$$\varphi(x) := \mu * h^z(x),$$

where μ is a standard mollifier supported in $B(0, \frac{1}{10})$, and

$$h^z(x) := \begin{cases} 1 - |x-z|^2/2 & \text{for } |x-z| \le 1 \\ (|x-z|-2)^2/2 & \text{for } 1 < |x-z| \le 2 \\ 0 & \text{for } |x-z| > 2 \end{cases} \tag{A.2.4}$$

For such φ, we have $0 < \varphi < 1$, inside the ball $B(z,1)$ and $\int_{\tilde{\Omega}} \varphi dx < |B(z,3)| < 6^d$.

To estimate I_1, note that $\varphi(x) \ge 1/3$ in $B(z,1)$, and $f(x,T) + C_1 a \ge 0$ in $\tilde{\Omega}$, which implies

$$\begin{aligned} I_1 &= \int_{\tilde{\Omega}} (f(x,T) + C_1 a)\varphi(x) dx - \int_{\tilde{\Omega}} C_1 a \varphi(x) dx \\ &\ge \frac{1}{3} \int_{B(z,1)\cap\tilde{\Omega}} (f(x,T) + C_1 a) dx - 6^d C_1 a \\ &\ge \frac{g(T)}{3} - 6^d C_1 a. \end{aligned}$$

For I_2, since $f(x,0) = (\frac{m}{m^-})^{m^-} \tilde{\rho}(x,0)^{1-ca} - \tilde{\rho}(x,0)$, we would obtain $f(x,0) < C_2 a$, where C_2 depends on $m, \|\tilde{\rho}(\cdot,0)\|_\infty$ and c, (hence depends on m, d, M_1, M_2), which yields

$$I_2 \le C_2 a \int_{\tilde{\Omega}} \varphi(x) dx \le 6^d C_2 a.$$

Now we start to estimate I_3. Due to the definition of m^- in (3.2.15), we have $m^- - m \le 2(m-1)ca$. Also, we can derive some *a priori* bound of $\rho^-(x,t)$ and $w(x,t)$ for $t \in [1,2]$, which depend on m, d, M_1, M_2. Then we have

$$\left| w^m - \rho^- |\rho^-|^{m^- - 1} \right| \le C_3 |w - \rho^-| + C_4 a \quad \text{in } \tilde{\Omega} \times [0,2],$$

where C_3, C_4 depends on m, d, M_1, M_2. Together with the fact that $|\Delta\varphi|$ is bounded, in particular by d, in $B(z,3)$ and vanishes outside of $B(z,3)$, we obtain the following bound for I_3:

$$\begin{aligned} I_3 &\le \int_{\tilde{\Omega}} (C_3|f| + C_4 a)|\Delta\varphi| + M a \varphi dx \\ &\le dC_3 \int_{B(z,3)\cap\tilde{\Omega}} |f(x,t)| dx + 6^d (dC_4 + M) a \\ &\le dC_3 \sum_{i=1}^{c_d} \int_{B(z+x_i,1)\cap\tilde{\Omega}} |f(x,t)| dx + 6^d (dC_4 + M) a, \end{aligned} \tag{A.2.5}$$

where in the last inequality we denote by c_d the number such that $B(0,3)$ can be covered by c_d numbers of balls of radius 1, centered at $x_1, \dots x_{c_d}$ respectively. Note that c_d is a constant only depending on d.

Finally, we wish to control $\int_{B(z+x_i,1)\cap\tilde{\Omega}} |f|dx$. Note that $f \geq -C_1 a$ implies $|f| \leq f + 2C_1 a$, which yields

$$\begin{aligned}\int_{B(z+x_i,1)\cap\tilde{\Omega}} |f(x,t)|dx &\leq \int_{B(z+x_i,1)\cap\tilde{\Omega}} f dx + 2^d 2C_1 a \\ &\leq g(t) + 2^{d+1} C_1 a\end{aligned}$$

Plugging it into (A.2.5), we obtain

$$I_3 \leq dC_3 c_d g(t) + (dC_3 c_d 2^{d+1} C_1 + 6^d dC_4 + 6^d M)a$$

By putting estimates of I_1, I_2, I_3 together, we have

$$g(T) \leq C_5 \int_0^T g(t)dt + C_6 a \quad \text{for } T \in [0,2]$$

where C_5, C_6 only depend on m, d, M_1, M_2. And for initial data, we have $g(0) \leq |B(0,1)| \sup_x f(x,0) \leq 2^d C_2 a$. By Gronwall inequality, we have $g(T) \leq C_0 a$ for all $T \in [0,2]$, where C_0 only depends on m, d, M_1, M_2, hence our claim (A.2.2) is proved. $\square$

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