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A Cosserat Theory for Solid Crystals – with Application to Fiber-Reinforced Plates

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Professor David Steigmann, Chair
Professor Lawrence Craig Evans
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Abstract

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The focus of this thesis is to understand the behavior of composite plates reinforced with rigid bars that are free to twist and bend with respect to the medium. Such composites are ubiquitous in nature and in industry, especially with increased interest in modeling biological elements as well as nano-technology. Structured fabrics abound in nature and industry – from cytoskeletons to kevlar sheets.

In the first part of this thesis existing theory on such materials is reviewed. A particular microstructure - that of a fiber-reinforced medium - is the subject of further study. Such a medium is treated as a special case of a nematic elastomer with constrained directors. The salient feature of such a material is the presence of only along-fiber derivatives in the problem, precluding certain boundary data.

The second part of the thesis focuses on the development of a two-dimensional plate theory from the three-dimensional fiber reinforced medium studied previously. The resulting two-director model shows behavior similar to a nematic elastomer with directors that are unable to shear. This is then specialized to the case of a laminate with a single family of fibers. To obtain an understanding of the theory it is applied to a simple controllable deformation; that of a fiber-reinforced plate bent into a cylindrical shell. The orientation of the fibers is selected to result in both bending and twist. This controllable deformation helps us understand the possible boundary data and its effect on the solution. The effect of boundary data on existence and uniqueness of solution is highlighted through the example problem.

Cosserat; Dimension Reduction.

To Nitin and Tanvi Chaubey

With gratitude.

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Chapter 1

Introduction

1.1 Introduction

The focus of this thesis is two-fold – to understand the mechanics of a continuum with a simple microstructure and to develop, from the three-dimensional theory, a theory for a fiber-reinforced plate. Continuum theories of physics are developed to model the macroscopic behavior of continuous media. Continuum physics focuses on the behavior of a collection of specified particles (a 'body') as it moves through space and time (a 'motion'). The geometry of the connected body as it undergoes such a motion is the focus of kinematics. The kinematics of the body is independent of the material of which it is composed and is thus a branch of differential geometry. In order to complete the analysis, the particles are assumed to interact in some specified way (the constitutive theory that differentiates various material behaviors). It is assumed that the continuum resists any deformation that tends to change the metric ('strain') of the body. Thus measures of strain need to be developed. By assuming that the stress derives from a stored energy function we are able to differentiate the material behavior be it rubber elasticity or metal plasticity (although the latter problem is complicated by dissipation).

In the classical theory discussed above, the only unknown is the deformation. Using the idea that the behavior of the body should be independent of the observer, the strain energy function depends only on the (dimensionless) deformation gradient and is thus independent of any inherent length-scales in the problem. Such a length-scale may be present in a given medium either due to the nature of the medium (microstructure) or the presence of a small relative dimension of the body under consideration (theories of rods, plates and shells for instance, as will be explained in the subsequent section).

After a quick review of a generalized nonlinear theory, a specialization – the so-called Cosserat-continuum – will be the focus of this thesis. In this approach, the continuum has additional structure which is introduced into the theory by defining additional (director) fields on the body. The strain energy of an elastic body is assumed to depend not only on the gradient of deformation, but also on the gradients of these director fields.

Such an extension allows the systematic modeling of several interesting phenomena in physics including media with microstructure, porous media (including bones), plasticity theory [36], plates and shells [72], liquid jets, geomaterials, fabrics and fiber composites [91] and is a active current area of continuum mechanical research [4],[20],[80],[100].

1.2 Generalized continua

In the mathematical literature, homogenization techniques are often employed to systematically obtain the partial differential equations governing the effective macroscopic behavior. The homogenization problem for nonlinear systems of equations is a very difficult problem and will not be attempted here. The approach followed here is to model the phenomenology of the continuum directly at the macroscopic level. The 1960s saw a revived interest in the study of generalized continua; largely due to the rationalization of continuum mechanics by Truesdell and co-workers [106].

In recent years research into such theories has gained impetus due to applications to various new materials, including bio-materials. With the development of experimental and micro-fabrication techniques, it is possible to better characterize and manufacture such media. While a body in classical elasticity theory is completely defined by the position (and hence deformation) under a motion, the theory of 'generalized media' admits continua which, either by their material nature (liquid crystals and elastomers, for instance) or as an aspect of their modeling (thin bodies, for instance) possess additional structure.

Two approaches to generalizing continuum mechanics by incorporating the length scale into these more complicated media were developed. In one approach, the generalized continuum is assumed to possess additional degrees of freedom (Cosserat type theories with one or more director fields). In the second, higher gradients of the deformation field were incorporated into the strain energy of the elastic medium (Toupin's couple-stress theory [103],[104]). Toupin [103], Mindlin and Tiersten [70] and Koiter [60] developed theories of 'couple-stress' involved higher gradients of the deformation field as opposed to the independent director fields of the Cosserats [18], Eringen [31], [32], Truesdell and Erickssen [27].

To this end, each material point in the body is now assumed, in the most general case, to be composed of a micro-continuum. Many general theories have been developed but most have proved too intractable to be of practical use; a few successful theories are a balance of generality and predictive ability. Such non-local higher gradient theories also incorporate length-scale into the problem. A comparison between the theories will be made subsequently, in Chapter 2.

Voigt [108] was the first, in 1887, to consider continua with additional degrees of freedom at each material point. His view was of a microstructure able to locally resist a moment i.e., each point in the body was itself assumed to behave as a rigid body. Thus, analogous to the stresses that result in a strained body, such bodies are able to resist local moments, which he referred to as couple-stresses. The other approach was to view the microstructure as involving some sort of deformation at each material point. In other words, the macroscopic continuum is composed of microcontinua at each material point.

The earliest formalization of this idea was with the publication of *Thèorie des Corps deformables* in 1909 [18]. The Cosserat brothers, Eugène and François, included microstructure by defining (in addition to position) a triad of 'directors' at each material point in the continuum. The director-gradients were constrained to be rotations; thus allowing for a rigid 'microrotation' at each point of the medium, independent of the deformation of the body. This differs from the higher gradient theories in that the director and position fields may be independent.

Depending on the physics to be modeled, various constraints on the deformation and director fields (as well as between the fields) are usually introduced. For instance, while modeling nematic elastomers, inextensibility constraints are often imposed on the director fields. Details of such models will be developed in Chapter 2. In this work, the main focus is on fibre-reinforced media in which the fibers are rigid with respect to the surrounding media – the director field will be completely specified via a rotation field.

Couple-stresses are the focus of Toupin's couple-stress theory [103] which emerged as as generalization of Noll's mechanics of the 'simple material' [76]. In a 'simple body', as defined by Noll , if the gradient of a deformation is the identity then such a deformation does not alter the physical response at a material point. In other words, only the first gradient of deformation appears in the constitutive description.

In Toupin's view of the generalized continuum, constitutive dependence was extended to include higher gradients of the deformation field. Thus, when the gradients up to Nth order are retained the material was considered to be 'of grade N'. When the rotation at each point is not independent but constrained to match the rotation of the deformation gradient then the resulting constrained Cosserat theory can be seen as a specialization of Toupin's grade 2 material. The next important contribution to the development of 'Cosserat elasticity' was due to Reissner [84] who developed the earliest finite deformation counterpart of Toupin's theory.

In addition to modeling microstructure, Cosserat theories are often employed to study another problem with a characteristic length scale – theories pertaining to plates and shells.

In such theories, the length-scale (associated with the thickness of the small dimension) appears in the equations through the directors [27],[40],[41],[43],[44] or through notions of micropolar continua to model plates and shells [30], [60],[70],[83]. The body is then modeled by a representative surface (often the mid-surface when symmetric) along with some number of directors. The next section considers the literature on theories of plates and shells, especially pertaining to a view of the thin structure through dimension reduction in the ambient space. In Chapter 3 the general theory of the 3D Cosserat continuum will be considered.

1.3 Theories of elasticity for thin bodies

Thin structures, including rods, membranes and shells, are often encountered in nature as well as engineering. Two dimensional theories for membranes (with resistance to extension), shells (resistant to bending and extension at differing order) and plates (flat shells) will be our focus in the discussion to follow. The literature on this subject is huge and the review provided here is far from comprehensive. For much more detailed insight into the subject the reader is referred to the work of Ciarlet [13], Antmann [5] and Naghdi [72].

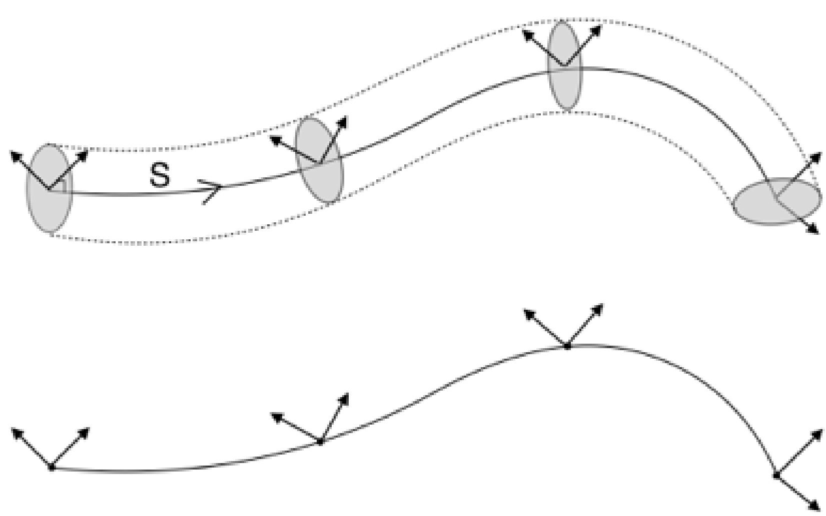

Figure 1.1: Modeling a rod as a Cosserat continuum

Broadly speaking, there are two standard approaches to the analyses of thin structures. The first approach – often called 'direct' or 'intrinsic' theories – is to consider the body as inherently of lower dimension. For instance, a shell would be considered a two-dimensional continuum embedded in $\mathbb{R}^3$. Balance laws and constitutive assumptions are prescribed for the body in the reduced dimension with the vestigial dimension modeled using director fields. The most general way in which to incorporate microstructure into continuum theory is, as proposed by Mindlin, to consider at each material point in the body a deformable micro-medium.

Let us try to understand the problem further by focusing on a couple of examples. A rod is a 1-D manifold of 2D deformable points (i.e., 2 directors at each point) $\mathcal{M}^1 + (\mathbf{d}_1(X,t), \mathbf{d}_2(X,t))$. In figure 1.1, the directors are constrained to remain orthogonal – i.e., a Kirchoff rod. Similarly, a shell may be considered a 2D manifold of 1D deformable points (single director). Additionally, an electrically polarizable 3D continuum can be assumed to have, in addition to position, a director that accounts for polarization.

Irrespective of the physics that results in the director, the mathematics proceeds similarly. Often, kinematic constraints are placed upon the director fields (for instance in a Kirchoff shell the assumption is that plane sections remain plane and this is mathematically incorporated into the theory by assuming the director field remains normal to the shell as in figure 4.1). The Kirchoff plate theory is derivable from three dimensional elasticity [96].

In the most general case at each point in the body N gradients and hence N independent directors could be defined. In the most general case (Eringen) inertial mass is assigned to the director fields. In the discussion that follows, such a situation will not be considered.

In Naghdi's approach to shell, the length-scale associated with the thinness of a shell is incorporated into the theory through a single director field. Such lower-dimensional theories are posed for mathematical ease of analysis and computation. The selection of the 'best' such theory relates to how closely the theory relates to the three dimensional theory it purports to approximate. Mathematical justification of the theory is achieved either by comparison to the solutions from the three dimensional theory (by extension of the the 2D theory or restriction of the 3D theory). This is the method used by Koiter for linear shells.

Another approach (used by Naghdi [72]) is a 'hierarchical' method in which a restriction on the 3D deformation field is applied to obtain a 2D theory. This is related to what Podio-Guidugli [82] calls 'the method of internal constraint', where the allowable class of deformation is restricted to those satisfying a constraint (the normal remains normal post-deformation under the Kirchoff-Love hypothesis for instance thus disallowing shear in the plate). In reality, since experimental work on a truly two-dimensional body is impossible – since however thin, a shell is essentially three-dimensional – experimental determination of material coefficients is challenging. One solution is homogenization [8] and numerical simulation.

The other is to consider a body with one dimension (thickness) much smaller than the others and consider some sort of limit theory. Here the equations of classical nonlinear elasticity are used derive the corresponding theory. Koiter used a consistent order-expansion to achieve this. Extensions to the Koiter theory to finite deformations is due to Steigmann [98],[96]. Rigorous theories for thin structures have also been the subject of much investigation in the mathematical community.

Another approach is formal asymptotic analysis in which energy is expanded with respect to the thickness parameter. Such a Taylor expansion cannot in any way be expected to converge to the correct three-dimensional model. Finally, methods of Gamma convergence attempt to show that models converge to a particular (assumed) limit model in some sense. This limit model must be known in advance and cannot be obtained through Gamma convergence techniques.

The approaches followed are asymptotic expansion [13] or methods of Gamma convergence [37]. While these methods are able to rigorously justify available limit theories for either flexural or extensional deformation in shells, so far all such methods have failed for coupled theories, successfully modeled by Koiter in the linear case. Progress into systematically developing a predictive coupled theory, by extending Koiter's methods, to the finite deformation case is due to Steigmann [96].

Two dimensional theories for membranes (able to resist extensional deformations) in which the strain energy depends only on the deformation of a representative surface were studied by Pipkin and co-workers. Such theories are usually ill-posed in that they are not elliptic; in other words, they do not satisfy the relevant ellipticity or Legendre-Hadamard equations.

To circumvent this problem, many approaches involve the regularization of the energy by adding in *ad hoc* bending terms [51],[52],[53] to regularize the problem. Pipkin and Steigmann [94] used 'tension field theory' to account for membrane behavior. Steigmann shows a consistent way to achieve this by extending the Koiter theory to obtain a model of order h^3 that yields a well-posed minimization problem in its own right. This model yields Koiter's model in the small strain limit and is consistent with three dimensional elasticity.

1.4 Fibre-reinforced continua

Fibre-reinforced materials have vast application – from the modeling of radial tires in vehicles to biological elements like the cytoskeleton. Now, with the development of techniques for experimenting on and manufacturing nano-materials, another avenue for research into such composites has opened up. A large literature exists on fibre-reinforced materials and the following is a brief summary.

One of earliest considerations of fibre-reinforced media was due to Adkins and co-workers [1], [46] who treated the case of inextensible fibers spread on surfaces within the medium but later extended the theory to the case of fibers uniformly distributed with the medium.

A careful generalization of the theory of fibre reinforced materials was due to Spencer [91]. In Spencer's treatment, the effect of the fibers is to introduce anisotropy. Thus he used the theory of nonlinear anisotropic elasticity to model such media; perhaps with the inclusion of constraints such as inextensibility and incompressibilty. The fiber direction introduces a preferred direction and hence certain symmetry into the continuum (usually transverse isotropy in his treatment). Spencer's treatment treats fibers as as material lines that convect with the continuum and thus the fibers do not have bending or shear resistance. The theory of Adkins *et al.* can be achieved as a special case of Spencer's theory.

Later, Spencer and Soltados [92] extended this theory to fibers that have stiffness and thus twist and shear with respect to the medium. This approach was based on the nonlinear strain gradient theory [60], [70], [103] discussed in the previous section. Thus the fibers are still material curves convected with the body. del'Isolla and Steigmann [99], [100] also considered fabrics where the gradient was completely known from the deformation of the surface. In other words, the fibers did not twist or admit shear deformations.

Steigmann [98], [97] removed some of the restrictions of fibers as material curves by modeling the fibers as Kirchoff rods embedded in the medium. By considering stiff fibers that cannot deform but only rotate he developed the theory of fibers resistant to bending and shear using Cosserat theory. While the fibers were allowed to bend and stretch, they were assumed to extend with the continuum. This theory is a constrained Cosserat theory rather than a a strain gradient theory allowing for the independent deformation of fibre and material surface.

1.5 Objectives and Overview of thesis

The main objective of the thesis is to develop the mechanics of a thin plate composed of a nematic elastomer with fibers that have a directionality. From this theory the case for a plate reinforced by a single family of fibers is then developed.

A generalized continuum theory including directed fibers is the main focus of this thesis. Chapter 2 describes the current state of the art on the theory of a continuum with additional structure. In such a theory, a director field is described over the three-dimensional body. In Chapter 3 the special case of a three dimensional continuum with fibers aligned in the reference configuration is the subject of study, The comparison to the field is not assumed to be aligned in the reference configuration.

In Chapter 4 a plate theory in which the plate is composed of a set of interacting Cosserat rods is described. This also leads to a theory of fibre-reinforced laminate with one family fibers will also be considered in Chapter 4. This chapter considers the development of a theory of Cosserat plates that is well-posed with respect to continuum mechanics in three dimensional space. In this thesis a two dimensional reduction of that theory will be developed and certain non-classical behavior examined for the case of one and two families of fibers and for bodies that can resist bending and in which the fibers are oriented out of plane. Such a theory has much in common with the theory of nematic elastomers. Application of the theory to laminates where there are layers with oriented families of fibers will also be considered in Chapter 4.

An example of the resulting mechanics can be found in Chapter 5.

Chapter 2

Nonlinear Elasticity of a Medium with Microstructure

2.1 Introduction

Continuum mechanics is a nonlinear field theory developed to model the physics associated with the macroscopic behavior of deformable media. The two aspects of study are the kinematics (the study of the displacement of the body within a posed boundary value problem) and the kinetics (the study of the stresses that result in the body due to these displacements) while the former is a branch of differential geometry independent from the type of the material under consideration. In continuum mechanics a 'body' is a collection of particles that interacts to form a cohesive abstract object. The interaction between the particles reflects the physics of the problem. When modeling the physics of a continuum the level of detail included should be sufficient to represent the level of physics in the problem but no so much as to make the the problem intractable. Biological processes are quite often modeled well by simple models – a tendon by a spring, a cell by a fluid membrane, etc. In modeling the tendon as a spring with only mass and elastic resistance, any phenomena that is distributed along the length cannot be studied. Thus, we are attempting to develop models that are as simple as possible, but no simpler. In reality, behavior of many continua is more complicated and requires the inclusion of additional fields that model the various length scales present in the problem. This could be due to the presence of microstructure or due to the development of simpler models by dimension reduction. Modeling and analysis of such problems has been an active area of study within the continuum mechanics, material science and mathematical communities. In continuum mechanics these effects are included through director and gradient theories, while in the mathematical PDE literature it is through homogenization (the development of a macroscopic theory that is a limit of a microscopic theory). However, such a technique for highly nonlinear systems is an open and extremely difficult problem.

In this chapter we will discuss the extension of nonlinear elasticity to include media with microstructure. Later the specialization to Cosserat elasticity is examined. The effect of the microstructure is to introduce a length scale into the problem. A consistent development is presented below. In this thesis, it is assumed that the constitutive behavior of the material incorporates the imposed length scales due to microstructure. While the methods of continuum mechanics can be used to arrive at the form of the energy, experimental work is required to obtain the material constants. With more complicated theories – both higher gradient and director – much more extensive experimental work is necessary. The study of microstructure in continuum mechanics has been the source of much interest both experimentally and theoretically. The notion of microstructure involves activity at a length scale different from that of the body. While many approaches have been adopted to understand such phenomena, the problem of microstructure is a difficult one. In this chapter one approach and a simplified method of attack will be studied. The ideas here can be extended to a broad range of problems (early work in plasticity was one motivation to pursue this line of research). This chapter attempts to systematically present one method of handling microstructures through the development of a continuum that has more 'structure' than just position.

It is hoped that a good understanding of the methods discussed here will be of use in tackling the far more complicated problems involving dissipation and evolving microstructures of, say, plasticity [62]. The nonlinear analysis of finite plasticity theory poses extreme challenges to accurate experimental work and therefore cannot really progress without a better experimental program based on modern theoretical developments. In order to better understand the methods and analysis involved, this thesis focusses on a far simpler with a fixed microstructure and no dissipation. Thus, the appearance of a microstructure (such as in problems studied by Ball and James [7]) are not the focus here.

To motivate the idea of such a continuum, consider a rod. A one dimensional model would be able only to represent kinematics associated with stretching and bending, but not twist. To study more complicated kinematics we need more information or 'degrees of freedom' at each material point in the body. In a similar fashion, a problem involving more than a single scale (microstructure) requires the introduction of length scale into the model. This can be achieved by higher gradient theories (in which higher derivatives of displacement field appear) or, as will be demonstrated in the sections below, by introducing 'director fields'. The theory here is a single director theory and is useful in the study of elastomers. Extension to multi-director theories is straightforward but is an easy enough conceptual problem and adds nothing to the discussion here. The other way to consider microstructure is gradient theories in which higher gradients of position field appear.

The mechanics of oriented media (Toupin [103]) can be used to understand the behavior of elastic rods and shells (theories in which one dimension is an order of magnitude smaller than the other dimensions), of materials with microstructure (liquid crystals, nematic elastomers, fabrics), plasticity – essentially the study of dislocated media – can be seen in this framework but the dissipation associated with the problem makes it far more difficult. The first to study continuously dislocated media was Gunther in 1958.

At every instant in time, there is thus defined, at every point a second set of vectors $\mathbf{d}_\alpha$ $\alpha = 1, 2, ..., m$ – the directors – that help incorporate the more complicated physics into the problem.

The most general way in which to incorporate microstructure into continuum theory is, as proposed by Mindlin, to consider at each material point in the body a deformable micro-medium. Let us try to understand the problem further by focusing on a couple of examples. The rod is a 1-D manifold of 2D deformable points (i.e., 2 directors at each point) $\mathcal{M}^1 + (\mathbf{d}_1(X,t), \mathbf{d}_2(X,t))$. Similarly, a shell may be considered a 2D manifold of 1D deformable points (single director). Additionally, an electrically polarizable 3D continuum can be assumed to have, in addition to position, a director that accounts for polarization. Irrespective of the physics that results in the director, the mathematics proceeds similarly. Often, kinematic constraints are placed upon the director fields (for instance in a Kirchhoff shell the assumption is that plane sections remain plane and this is mathematically incorporated into the theory by assuming the director field remains normal to the shell).

2.2 Kinematics

Continuum mechanics can be seen as the intersection of many branches of mathematics. The first of these is differential geometry to which kinematics has strong links. When studying the kinematics of a body we are interested only in it's possible embeddings in space and not any notion of the material behavior of the material the body is composed of. In the discussion here (and in the rest of the thesis) the assumption is made that the natural space is a three dimensional Euclidean point space. In conjunction with this we define the time on the real line. However this thesis deals with equilibrium phenomena and thus time as a variable is generally suppressed in our discussion.

The ideas developed in this chapter are general and relate to problems more general than the ones considered subsequently. Below some of these ideas are made more precise.

Body, configuration, particle

A body $\mathcal{B}$ is a set of points that can be put into one-to-one correspondence with a region in Euclidean point-space. Each point $X \in \mathcal{B}$ is called a 'material point' or 'particle'. The region occupied by the collection of these points (the body) is known as a 'configuration' of the body. It is assumed here to form a three-dimensional differentiable manifold, with sufficiently smooth boundary. In such a configuration the location of each particle X is defined through the map $\boldsymbol{\chi}$. $\mathcal{B}$ is assumed here to be compact and connected.

$$\mathbf{x} = \boldsymbol{\chi}(X, t) = \boldsymbol{\chi}_t(X) \tag{2.1}$$

An event

At each material point $X \in \mathcal{B}$ let the current position, with respect to a selected coordinate system, be $\mathbf{x}$. A reference configuration, which may be occupiable, can be conveniently selected as the position occupied by the body at some selected time t_0.

Motion and deformation

The kinematics of the body are determined by the change in the configuration of the body over time. It is usual, but not necessary to denote the configuration at some time, t_0, to be the reference configuration, $\mathcal{R}_0$. If this is done, the selected configuration is said to be 'occupiable'.

$$\mathbf{X} \equiv \mathbf{x}(t_0) = \boldsymbol{\chi}(X, t_0) = \boldsymbol{\chi}_{t_0}(X) \tag{2.2}$$

At the time, t, the location of the body if given by some $\mathbf{x} = \chi(X, t)$. Thus a 'motion' of $\mathcal{B}$ is a smooth, one-parameter family of configurations, where the parameter, t, represents time. We are particularly interested in the deformation of the body between two such configurations. Since the mapping is one-to-one, it is possible to write the deformation in the current configuration in terms of the reference configuration:

$$\mathbf{x} = \mathbf{x}(X, t) = \boldsymbol{\chi}_t(\boldsymbol{\chi}_{t_0}^{-1}(\mathbf{X})) = \boldsymbol{\chi}_t(\mathbf{X}) \tag{2.3}$$

We assume that the deformations are twice differentiable since we are not looking at shock wave propagation here.

Sometimes it is convenient to formulate the equations of mechanics with respect to a preferred reference configuration. Such a formulation is referred to as *Lagrangian* as opposed to a formulation based on the current configuration of the body, the *Eulerian* formulation.

The configurations of the body can be expressed in terms of selected co-ordinate systems. There is no reason that the origin or the co-ordinate system be fixed between the configurations. Let us assume the basis $\{\mathbf{E}_A\}$ (where A can be $1, 2, 3$) in the reference configuration and $\{\mathbf{e}_i\}$ (where i can be $1, 2, 3$) in the current configuration.

$$\mathbf{X} = X_A \mathbf{E}_A, \text{ summation on } A \quad \text{and} \quad \mathbf{x} = x_i \mathbf{e}_i, \text{ summation on } i \tag{2.4}$$

It is customary and convenient but not necessary to discuss the deformation of the body with respect to a 'reference configuration', R_0, which may be occupiable (i.e., under the motion, for some time t_0, $R_0 = \mathcal{B}$) or not, 'stress-free' or not. For the purpose of ease, in the discussions that follow, we will assume that the reference configuration is occupiable at the time $t = 0$.

Microstructure

A general director theory

In the most general case at each point in the body N gradients and hence N independent directors could be defined. In the most general case (Eringen) inertial mass is assigned to the director fields. In the discussion that follows, such a situation will not be considered. The ideas of Toupin will be closely followed.

Hence, we now assume the existence of an independent field – the director field – over the body. This field accounts for micro-deformation at X. By increasing the degrees of freedom at each point, a richer class of materials can be considered. A summary of using this for both thin bodies and dislocated crystals follows. $\mathbf{d}_i : \mathcal{B} \to E$:

$$\mathbf{d} = \boldsymbol{\eta}(\mathbf{X}, t) = \boldsymbol{\eta}_t(\mathbf{X}) \tag{2.5}$$

The directors are constrained to be orthogonal

The generalized continuum introduced in the previous chapter is now specialized to the case where only rigid rotations of each point in the continuum is allowed. In this model, each point of the body is assumed to behave like a rigid body and is capable of independent rotation. Thus the directors remain orthogonal to each other. The unknown fields are now deformation and director field. But since the directors are always orthogonal, the unknown is actually a rotation tensor. The theory of Kirchhoff rods provides the background for the theory presented below [5].

Assume a coordinate system attached to the fibers – the vectors $\{\mathbf{D}_i\}$ and $\{\mathbf{d}_i\}$ are associated with the fibers in the reference and current configurations respectively. The notation $\mathbf{D}$ refers to the direction along fiber and the transverse direction is represented by greek subscripts i.e., $\{\mathbf{d}_\alpha\}$. So $\{\mathbf{D}_i\} = \{\mathbf{D}, \mathbf{D}_\alpha\}$.

The rod is considered to be a deformable directed curve the kinematics of which is fully described by a rotation field (modulo a rigid translation) at each material point.

$$\mathbf{d}_i = \mathbf{R}\mathbf{D}_i \tag{2.6}$$

Gradients of the two fields

The position vector $\mathbf{v} \in \mathcal{T}$ and are material vectors are with respect to position (equilibrium deformations are under consideration and hence time dependence is suppressed for now).

A vector-valued filed $\mathbf{v} : \mathcal{B} \to \nu$ is differentiable if there exists unique $D\mathbf{v}$ such that:

$$\mathbf{v}(X,t) = \mathbf{v}(X_0,t) + D\mathbf{v}(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0) + o(|x-x_0|^2) \tag{2.7}$$
$$d\mathbf{v} = D\mathbf{v}d\mathbf{x} \tag{2.8}$$
$$Div(\mathbf{v}) = tr(D\mathbf{v}) \tag{2.9}$$
$$curl(\mathbf{v}) = Div(\mathbf{v}\times\mathbf{c})\forall\mathbf{c} \tag{2.10}$$

By the fact that $J > 0$,

$$\mathbf{x} = \chi(\mathbf{X},t) = \chi_t(\mathbf{X}) = \chi_t(\mathbf{x}_0) + \mathbf{F}(\mathbf{X}-\mathbf{X}_0) + o(|x-x_0|^2) \tag{2.11}$$

where $\mathbf{F} \equiv \frac{\partial\chi_t}{\partial\mathbf{X}} = D\chi_t$ is the deformation gradient, x_0 the initial and x the current position. Note that $\chi, \mathbf{F}$ etc depend on the choice of reference configuration U_0.

$$\frac{\partial\chi_t}{\partial\mathbf{X}}$$

By definition on differentiability,

$$\boldsymbol{\eta}(\mathbf{X},t) = \boldsymbol{\eta}(\mathbf{X}_0,t) + \mathbf{G}(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0) + o(|x-x_0|^2 \tag{2.12}$$

where $\mathbf{G} = \frac{\partial\boldsymbol{\eta}_t}{\partial\mathbf{X}}$.

$$\frac{\partial\eta_i}{\partial X_A} = G_{iA} = \frac{\partial\eta_i}{\partial x_j}\frac{\partial x_j}{\partial X_A} = \frac{\partial\eta_i}{\partial x_j}F_{jA} \tag{2.13}$$

where $\mathbf{H} \equiv \frac{\partial\boldsymbol{\eta}_t}{\partial\mathbf{X}}$ is the gradient of the director field with respect to the current configuration.

$$\mathbf{G} = \mathbf{HF} \tag{2.14}$$

Deformation gradient, metrics and Strain

Taking the differential of equation 2.3,

$$dx_i = \frac{\partial x_i}{\partial X_A}dX_A \tag{2.15}$$

The tensor $F_iA \equiv \frac{\partial x_i}{\partial X_A}$ represents the deformation gradient, the tangent map between the reference and current manifold, and can be expressed as

$$\mathbf{F} = \frac{\partial x_i}{\partial X_A}\mathbf{e}_i \otimes \mathbf{E}_A \tag{2.16}$$

Lines

The length of a differential element on each manifold is obtained from the metric induced by the norm:

$$d\mathbf{x} \cdot d\mathbf{x} = ||d\mathbf{x}||^2 = x_i x_i \tag{2.17}$$

Thus the length of a differential element on the configurations is related as

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{2.18}$$

$$d\mathbf{x} \cdot d\mathbf{x} = ||d\mathbf{x}||^2 = \mathbf{F}d\mathbf{X} \cdot \mathbf{F}d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^T\mathbf{F}d\mathbf{X} = d\mathbf{X} \cdot \mathbf{C}d\mathbf{X} \tag{2.19}$$

where $\mathbf{C} = C_{AB}\mathbf{E}_A \otimes \mathbf{E}_B \equiv \mathbf{F}^T\mathbf{F}$ is known as the 'Left Cauchy-Green tensor'. This yields the measure on the current configuration and is a measure of strain.

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{2.20}$$

$$d\mathbf{x} \cdot d\mathbf{x} = ||d\mathbf{x}||^2 = \mathbf{F}d\mathbf{X} \cdot \mathbf{F}d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^T\mathbf{F}d\mathbf{X} = d\mathbf{X} \cdot \mathbf{C}d\mathbf{X} \tag{2.21}$$

Thus the metric from this norm is

Area

$$da\mathbf{n} = d\mathbf{x} \times d\mathbf{y} = \mathbf{F}d\mathbf{X} \times \mathbf{F}d\mathbf{Y} = \mathbf{F}^*(d\mathbf{X} \times \mathbf{Y}) = \mathbf{F}^*\mathbf{N}dA \tag{2.22}$$

Volume

$$dv = JdV \tag{2.23}$$

$$J = \det(\mathbf{F}) = \frac{1}{6}e_{ijk}e_{ABC}F_{iA}F_{jB} = F_{kC} \tag{2.24}$$

This is also assumed to assign each material point X to distinct points $\mathbf{x}$ (matter does not interpenetrate) and thus:

$$J(\mathbf{X}, t) \equiv \det(D\chi_t(\mathbf{X})) > 0 \tag{2.25}$$

Mass

m: Here assumed a non-negative sigma measure on $\mathcal{B}$.

Contact forces and stresses

Continuum theories are marked by the interaction of the body with the surrounding medium and the region external to the body. Ignoring body forces (for example, gravitational), this manifests in the form of contact stresses and couples. The traction, $\mathbf{t}$, and the couple stresses $\mathbf{c}$ act on the boundary as we will see in the virtual work statement below. They also mark

The strain due to imposed kinematic restrictions manifests itself as stress. In the generalized continuum theory stresses result due to both tractions and moments.

2.3 Variational principle and Principle of least action

Variational problem

It is assumed here that the fibers are rigid and thus can be modeled by rods within an elastic medium. The rod is a one dimensional element defined by position as well as a rotation. It is assumed that the fibers deform with the medium. For such a fiber-reinforced medium, the strain energy function, is assumed to depend on the two fields ($\boldsymbol{\chi}$ and $\mathbf{P}$) and their gradients ($D\boldsymbol{\chi}$ and $\mathbf{S}$): $W = W(\boldsymbol{\chi}(x), D\boldsymbol{\chi}(x), \mathbf{P}(x), \mathbf{S}(x); x)$.

Where $\mathbf{S} = S_{iAB}\mathbf{e}_i \otimes \mathbf{E}_A \otimes \mathbf{E}_B$ and $S_{iAB} = R_{iA,B}$

Using the notation $\mathbf{w} \equiv (\boldsymbol{\chi}, \mathbf{P})$ and $\mathbf{w}' \equiv (D\boldsymbol{\chi}, \mathbf{P})$, the problem is to find the pair $\hat{\mathbf{u}} = (\mathbf{u}, \mathbf{R})$ in the admissible class $\mathcal{A}$ that minimize the net stored energy in the body.

Thus the problem here is to study the following variational problem:

$$\underset{w \in A}{\text{Min}}\ I[w] = \int_U W\left(\mathbf{R}^T D\mathbf{u}, \mathbf{R}^T \mathbf{S}\right) dv$$

We are interested here in the conditions under which a minimizing sequences $(\mathbf{u}_n, \mathbf{R}_n) \in \mathbb{R}^3 \times SO(3)$ exist (or fail to exist). In the next sections we consider a few simplifying cases.

$$W : \mathbb{R}^n \times \mathbb{M}^{n\times n} \times SO(3) \times \mathbb{M}^{n\times n} \times U \to \mathbb{R}$$

The strain energy is assumed to satisfy the following conditions, thus restricting it's form: Strain measures as conjugates to internal stress and couple stress fields.

Invariance under superposed rigid motions

The discussion that follows pertains to any point in the domain and hence all points. For simplicity could also assume a homogeneous strain energy function. The strain energy function is the same under experiments conducted by two separate observers who are assumed to agree on distances between points and on lengths of time intervals. Imposing this condition leads to a restriction in the form the strain energy function can take. For simplicity the effect of the translation and the rotation will be considered separately.

Translational invariance

The strain energy function is the same when considered by two observers who are at an arbitrary distance $\mathbf{a}$ apart. To the first observer, the point $x \in U$ is at $\boldsymbol{\chi}$ whereas to the second observer the same point is at $\boldsymbol{\chi} + \mathbf{a}$. The restriction on the strain energy function is thus:

$$W(\boldsymbol{\chi} + \mathbf{a}, D\boldsymbol{\chi}, \mathbf{P}, \mathbf{S}) = W(\boldsymbol{\chi}, D\boldsymbol{\chi}, \mathbf{P}, \mathbf{S})$$

To obtain a necessary condition, assume that $\mathbf{a} = -\boldsymbol{\chi}(x)$. Thus the strain energy cannot depend directly on the deformation and be frame invariant.
We have the reduced form of strain energy function: $W(D\boldsymbol{\chi}(x), \mathbf{P}(x), \mathbf{S}; x)$

Rotational invariance

The strain energy function is the same when considered by two observers whose positions are linked by the arbitrary rotation $\mathbf{Q}$. To the first observer, the location of point $x \in U$ and its Cosserat rotation are as above whereas to the second observer it is rotated through $\mathbf{Q}$. The restriction on the strain energy function is thus:

$$W(\mathbf{Q}D\boldsymbol{\chi}, \mathbf{P}, \mathbf{S}) = W(D\boldsymbol{\chi}, \mathbf{QP}, \mathbf{QS})$$

To obtain a necessary condition, select $\mathbf{Q} = \mathbf{P}^T(x)$. Then:

$$W(D\boldsymbol{\chi}, \mathbf{P}, \mathbf{S}) = W\left(D\boldsymbol{\chi}, I, \mathbf{P}^T\mathbf{S}\right)$$

We have the reduced form of strain energy function: $W\left(\mathbf{P}^T D\boldsymbol{\chi}(x), \mathbf{P}^T\mathbf{S}(x); x\right)$. The second entry here is the analog of the 'Q-tensor' in liquid crystal theory. Let us elaborate on it.

$$\boldsymbol{\Gamma} = \mathbf{R}^T\mathbf{S} = \Gamma_{DC}\mathbf{E}_D \otimes \mathbf{E}_C, \quad \Gamma_{DC} = \frac{1}{2}e_{BAD}R_{iA}R_{iB,C} \tag{2.26}$$

$$\gamma_{D(C)} = \frac{1}{2}e_{BAD}R_{iA}R_{iB,C} \tag{2.27}$$

$$\boldsymbol{\Gamma} = \gamma_c \otimes \mathbf{E}_C \tag{2.28}$$

Reference configuration is stress-free

The response to the strain fields ($\mathbf{E} = \mathbf{R}^T D\mathbf{u}$ and $\boldsymbol{\kappa} = \mathbf{R}^T \boldsymbol{R}'$) are the stress ($W_{\boldsymbol{E}}$) and the couple stress ($W_{\boldsymbol{\kappa}}$). If $D\boldsymbol{\chi} = I, \mathbf{P} = I$ then, $W(\mathbf{E}, \boldsymbol{\kappa})|_{\mathbf{E}=I, \kappa=\mathbf{0}} = 0$

Admits a Taylor expansion

The energy function is smooth and can be expanded about the zero-stress configuration.

$$I = \int_U W(\mathbf{E}, \boldsymbol{\kappa}) dv \tag{2.29}$$

Would like suitable conditions on W that would ensure the existence of minimizers in eqn. (3.3).

2.4 Some special cases and applications

It is useful to see how a broad range of physics can be modeled by director theory. Liquid crystals, for instance, are now ubiquitous – making their appearance in flat screen television to bio-membranes. They are often described as being an intermediate state of matter in that the material is a fluid with orientation like a crystal. Mathematically, this can be modeled by position and director field and thus can be modeled by the methods described in this chapter. The dependence of $D\mathbf{u} = \mathbf{F}$ in the energy is restricted to the volume invariant, $det(\mathbf{F}) = J$ due to the in-plane fluidity assumed. Thus the energy dependence is through J alone. Additionally, the director field and all its gradients appear in the energy. It is usual to constrain the directors to be unit vectors i.e., $\mathbf{dd} = 1$ and make the dependence invariant to the change $\mathbf{d} \to -\mathbf{d}$.
The simplest such energy is the Oseen-Frank energy in which all the gradients of $\mathbf{d}$ appear. Nematic elastomers, are a generalization of liquid crystal theory where the material has solid-like properties as well as a liquid-crystal-like orientation

The methodology adopted is broadly to apply the constraints that model the physics. Identify the tensors upon which energy must depend. Then impose invariant structure (material symmetry considerations) to obtain a general form of energy. Usually simpler models that fit in the framework are then selected based on the problem being analyzed.

In the next chapter we will look more closely at a special case in which the fibers are assumed to be Kirchhoff rods and hence only an along fiber derivative appears in the energy function.

Gradient theories in which higher gradients of position field appear are special cases of the a Cosserat theory where the directors are assumed to be inertia-less. In this case the constraint imposed is that the rotation in the Cosserat theory correspond to the rotation in the polar decomposition of deformation gradient.

Chapter 3

The case of a fibre-reinforced medium

3.1 A fiber reinforced continuum with twist

Laminated sheets and fabrics have been studied extensively by the continuum mechanics and partial differential equations communities [**tartar1990h** , 92, 95]. The simplest approach to modeling such continua is to assume that the fibers establish a directionality in the medium.

In this chapter a specific restriction to the theory developed in the preceding chapter is considered. It is assumed that the medium is reinforced by thick, initially straight rods aligned along a direction $\boldsymbol{D}$. Under the motion, the material point is both deformed to $\boldsymbol{\chi}_t(\boldsymbol{X})$ as well as the three directors mapped to $\boldsymbol{d}_i = \boldsymbol{R}\boldsymbol{D}_i$. It is assumed that the fibers are all initially aligned along a single direction. The theory as developed could apply to a nematic elastomer with a director field that does not tilt. A particular simplification would be the case in which the director is aligned along the normal to the body, say. Extensions to this theory to a nematic elastomer where the directors are free to tilt will not be considered here and will be addressed in future work.

The problem being considered here is of an elastic body reinforced by uniformly distributed unidirectional fibers, aligned along, say, $\mathbf{D}$. This body is assumed to occupy a reference domain $R_0 \subset \mathbb{R}^3$. Each material point, $X \in R_0$, is mapped to its location in the current configuration, R, by the deformation $\mathbf{u}$. The gradient of this deformation $\mathbf{F} = D\mathbf{u}$ then maps material vectors between the two configurations.

In this model, the embedded fibers are allowed to shear and twist with respect to the matrix and hence their position is not completely determined by $\mathbf{F}$. Another field is defined on the body that accounts for the fiber deformations .To simplify the problem, it is assumed here that the fibers are initially straight, untwisted and are extremely rigid with respect to the surrounding medium. These assumptions imply the following kinematics: the fibers can be treated as 1D Kirchhoff rods that stretch with the matrix but are rotated, rigidly,

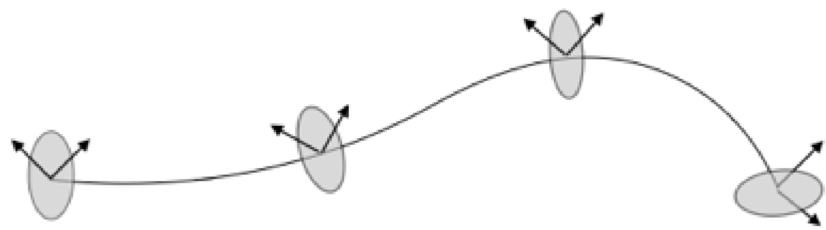

Figure 3.1: A Kirchhoff rod

through $\mathbf{R}$ during the deformation (i.e., the micro deformation tensor in this case consists of a rotation, $\mathbf{R}$ and an axial stretch, λ – in other words, as the body deforms under the map $\mathbf{u}$, the fibers in the medium rotate with respect to the matrix but elongate with it).

The problem being considered here is of an elastic body reinforced by uniformly distributed unidirectional fibers, aligned along, say, $\mathbf{D}$. This body is assumed to occupy a reference domain $R_0 \subset \mathbb{R}^3$. Each material point, $X \in R_0$, is mapped to its location in the current configuration, R, by the deformation $\mathbf{u}$. The gradient of this deformation $\mathbf{F} = D\mathbf{u}$ then maps material vectors between the two configurations.

In this model, the embedded fibers are allowed to shear and twist with respect to the matrix and hence their position is not completely determined by $\mathbf{F}$. Another field is defined on the body that accounts for the fiber deformations .To simplify the problem, it is assumed here that the fibers are initially straight, untwisted and are extremely rigid with respect to the surrounding medium. These assumptions imply the following kinematics: the fibers can be treated as 1D Kirchhoff rods that stretch with the matrix but are rotated, rigidly, through $\mathbf{R}$ during the deformation (i.e., the micro deformation tensor in this case consists of a rotation, $\mathbf{R}$ and an axial stretch, λ – in other words, as the body deforms under the map $\mathbf{u}$, the fibers in the medium rotate with respect to the matrix but elongate with it).

Assume a coordinate system attached to the fibers – the vectors $\{\mathbf{D}_i\}$ and $\{\mathbf{d}_i\}$ are associated with the fibers in the reference and current configurations respectively. The notation $\mathbf{D}$ refers to the direction along fiber and the transverse direction is represented by Greek subscripts i.e., $\{\mathbf{d}_\alpha\}$. So $\{\mathbf{D}_i\} = \{\mathbf{D}, \mathbf{D}_\alpha\}$.

The reinforcing bars are assumed to be Kirchhoff rods. The rod is parametrized along its center line by s. So $[s_1, s_2] \times \mathbb{R} \ni (s,t) \mapsto \mathbf{r}(s,t), \mathbf{d}_1(s,t), \mathbf{d}_2(s,t) \in \mathbb{E}^3$. $\mathbf{d}_i = \{\mathbf{d}, \mathbf{d}_2, \mathbf{d}_3\}$

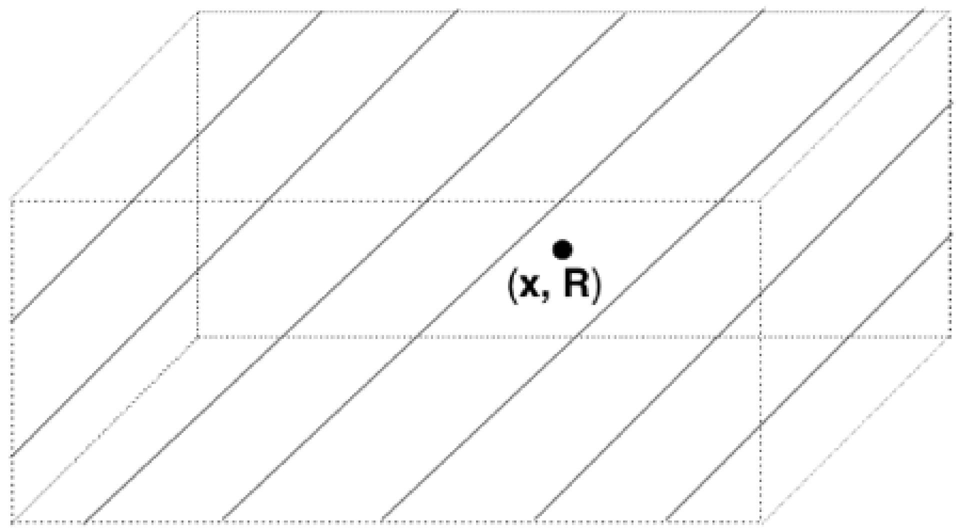

Figure 3.2: A single family of fibres in a 3D continuum

where $\mathbf{d}$ is tangent to the centerline. The along-fiber derivative

$$()' = \frac{\partial}{\partial S}() \tag{3.1}$$

Then, by definition of an axial vector, we have:

$$\mathbf{d}' = ax(\mathbf{R}'\mathbf{R}^T) \times \mathbf{d} \tag{3.2}$$

The strain energy function and equilibrium solutions

In this case, in place of $\mathbf{S} = D\mathbf{R}$, only the along fiber derivative, $\mathbf{R}'$ appears in the energy. The strain energy for the body is:

$$I = \int_{R_0} W(\mathbf{Du}, \mathbf{R}, \mathbf{R}')dv$$

Imposing invariance under observer, the energy must have the form $W(\mathbf{R}^T\mathbf{Du}, \mathbf{R}^T\mathbf{R}')$ where $()' = \frac{d()}{ds}$,

$$\boldsymbol{R}' = \frac{d\mathbf{R}}{ds} = \boldsymbol{SD}$$

$$W(\mathbf{E}, \boldsymbol{\kappa}), \quad \text{where} \quad \boldsymbol{\kappa} = \kappa_i \mathbf{D}_i \text{ is the axial vector of } \mathbf{R}^T\mathbf{R}' = \mathbf{R}^T\mathbf{SD}$$

Problem: Looking for the pair $(\mathbf{u}, \mathbf{R})$, $\mathbf{u} : R_0 \to \mathbb{R}^3, \mathbf{R} : R_0 \to SO(3)$, and hence $(\mathbf{E}, \boldsymbol{\kappa})$, where $\mathbf{E} = \mathbf{R}^T D\mathbf{u}, \kappa_i = \frac{1}{2} e_{ijk} \mathbf{D}_k \cdot \mathbf{D}_k \mathbf{R}^T \mathbf{R}' \mathbf{D}_j$ that minimize the functional, I,

$$I = \int_{R_0} W(\mathbf{E}, \boldsymbol{\kappa}) dv \;\; \text{subject to the constraint in equation(\textbf{??})} \tag{3.3}$$

Would like suitable conditions on W that would ensure the existence of minimizers in eqn. (3.3).

The problem being considered here is of an elastic body reinforced by uniformly distributed unidirectional fibers, aligned along, say, $\mathbf{D}$. This body is assumed to occupy a reference domain $R_0 \subset \mathbb{R}^3$. Each material point, $X \in R_0$, is mapped to its location in the current configuration, R, by the deformation $\mathbf{u}$. The gradient of this deformation $\mathbf{F} = D\mathbf{u}$ then maps material vectors between the two configurations.

In this model, the embedded fibers are allowed to shear and twist with respect to the matrix and hence their position is not completely determined by $\mathbf{F}$. Another field is defined on the body that accounts for the fiber deformations .To simplify the problem, it is assumed here that the fibers are initially straight, untwisted and are extremely rigid with respect to the surrounding medium. These assumptions imply the following kinematics: the fibers can be treated as 1D Kirchhoff rods that stretch with the matrix but are rotated, rigidly, through $\mathbf{R}$ during the deformation (i.e., the micro deformation tensor in this case consists of a rotation, $\mathbf{R}$ and an axial stretch, λ – in other words, as the body deforms under the map $\mathbf{u}$, the fibers in the medium rotate with respect to the matrix but elongate with it).

3.2 Balance Laws

The potential energy, $\mathrm{E} = \int_\xi W \,\mathrm{d}v$ and the load potential, L from tractions, $\mathbf{t}$ and moments, $\mathbf{m}_i$ on the fibers:

$$L = \int_{\partial\xi_t} \mathbf{t} \cdot \boldsymbol{\chi} \,\mathrm{d}a + \int_{\partial\xi_c} \mathbf{m}_i \cdot \mathbf{d}_i \,\mathrm{d}a \tag{3.4}$$

These fiber moments are related to couples through:

$$\mathbf{c} = ax[(\mathbf{D}_i \otimes \mathbf{m}_i)\mathbf{R} - \mathbf{R}^t(\mathbf{m}_i \otimes \mathbf{D}_i)] \tag{3.5}$$

The virtual work statement equals the power of the loads so:

$$\dot{L} = \int_{\partial\xi_t} \mathbf{t} \cdot \dot{\boldsymbol{\chi}} \,\mathrm{d}a + \int_{\partial\xi_c} \mathbf{c} \cdot \boldsymbol{\omega} \,\mathrm{d}a \tag{3.6}$$

Variational problem

It is assumed here that the fibers are rigid and thus can be modeled by rods within an elastic medium. The rod is a one dimensional element defined by position as well as a rotation. It is assumed that the fibers deform with the medium. For such a fiber-reinforced medium, the strain energy function, is assumed to depend on the two fields ($\boldsymbol{\chi}$ and $\mathbf{P}$) and their gradients ($D\boldsymbol{\chi}$ and $\frac{\partial \mathbf{P}}{\partial x_1}$): $W = W(\boldsymbol{\chi}(x), D\boldsymbol{\chi}(x), \mathbf{P}(x), \frac{\partial \mathbf{P}}{\partial x_1}(x); x)$.

$$W : \mathbb{R}^n \times \mathbb{M}^{n\times n} \times SO(3) \times \mathbb{M}^{n\times n} \times U \to \mathbb{R}$$

Using the notation $\mathbf{w} \equiv (\boldsymbol{\chi}, \mathbf{P})$ and $\mathbf{w}' \equiv (D\boldsymbol{\chi}, \mathbf{P})$, the problem is to find the pair $\hat{\mathbf{u}} = (\mathbf{u}, \mathbf{R})$ in the admissible class $\mathcal{A}$ that minimize the net stored energy in the body.

Thus the problem here is to study the following variational problem:

$$\underset{w\in A}{\text{Min}}\, I[w] = \int_U W\left(\mathbf{R}^T D\mathbf{u}, \mathbf{R}^T \frac{\partial \mathbf{R}}{\partial x}\right)$$

We are interested here in the conditions under which a minimizing sequences $(\mathbf{u}_n, \mathbf{R}_n) \in \mathbb{R}^3 \times SO(3)$ exist (or fail to exist). In the next sections we consider a few simplifying cases.

The strain energy is assumed to satisfy the following conditions, thus restricting it's form: Strain measures as conjugates to internal stress and couple stress fields.

For rotation $\mathbf{R}$, we saw constitutive dependence must be on the skew tensor $\boldsymbol{\Gamma}$
Fiber twist

$$\boldsymbol{\Gamma} \equiv \mathbf{R}^T \frac{\partial \mathbf{R}}{\partial s} = \begin{bmatrix} 0 & \kappa_3 & \kappa_2 \\ -\kappa_3 & 0 & \kappa_1 \\ -\kappa_2 & -\kappa_1 & 0 \end{bmatrix}$$

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix}$$

First Variation, Euler-Lagrange Equations and Boundary Conditions

The Euler-Lagrange equations, $A[\mathbf{u}] = 0$ are the differential equations that correspond (in the distributional sense) to the derivative

$$I'[\mathbf{w}] = A[\mathbf{w}] = 0 \tag{3.7}$$

Let $\mathbf{w} \equiv (\boldsymbol{\chi}, \mathbf{P})$ and $\mathbf{w}' \equiv (D\boldsymbol{\chi}, \mathbf{P}')$

$$\begin{aligned} I[\mathbf{w}] &= \int_U W(\mathbf{w}')dx \\ &= \int_U W(\mathbf{E}, \boldsymbol{\gamma})dx \\ &= \int_U W\left(\mathbf{P}^T D\chi, \mathbf{P}^T \mathbf{P}'\right) dx \end{aligned}$$

$$W = \mathbf{g} \text{ on } \partial U \qquad [\text{i.e. } \boldsymbol{\chi} = \mathbf{u}_0, \mathbf{P} = \mathbf{R}_0] \tag{3.8}$$

Suppose $\hat{\mathbf{u}}$ (smooth) is a mimimizer of equation [3.7] that also satisfies $\hat{\mathbf{u}} = \mathbf{g}$ on ∂U, then $(\tau \in \mathbb{R}) : i(\tau) = I[\hat{\mathbf{u}} + \tau\hat{\mathbf{v}}]$ has min at $\tau = 0$ and so $i'(0) = 0$.

$\hat{\mathbf{v}} \equiv (\mathbf{v}, \mathbf{S})$ and so $\boldsymbol{\chi} \to \mathbf{u} + \tau\mathbf{v}$; $\mathbf{P} \to \mathbf{R} + \tau\mathbf{S}$

$$i(\tau) = \int_U W\left[(\mathbf{R} + \tau\mathbf{S})^T (D\mathbf{u} + \tau D\mathbf{v}), (\mathbf{R} + \tau\mathbf{S})^T (\mathbf{R}' + \tau\mathbf{S}')\right] dx \tag{3.9}$$

$$i'(\tau) = \int_U \left(W_{E_{AB}} \frac{\partial E_{AB}}{\partial\tau} + W_{\Gamma_{AB}} \frac{\partial \Gamma_{AB}}{\partial\tau}\right) dx \tag{3.10}$$

where:

$$\begin{aligned} \mathbf{E} &= \mathbf{R}^T D\mathbf{u} + \tau\mathbf{S}^T D\mathbf{u} + \tau\mathbf{R}^T D\mathbf{v} + \tau^2\mathbf{S}^T D\mathbf{v} \\ \frac{\partial\mathbf{E}}{\partial\tau} &= \mathbf{S}^T D\mathbf{u} + \mathbf{R}^T D\mathbf{v} + 2\tau\mathbf{S}^T D\mathbf{v} \\ \boldsymbol{\Gamma} &= \mathbf{R}^T\mathbf{R}' + \tau\mathbf{S}^T\mathbf{R}' + \tau\mathbf{R}^T\mathbf{S}' + \tau^2\mathbf{S}^T\mathbf{S}' \\ \frac{\partial\boldsymbol{\Gamma}}{\partial\tau} &= \mathbf{S}^T\mathbf{R}' + \mathbf{R}^T\mathbf{S}' + 2\tau\mathbf{S}^T\mathbf{S}' \end{aligned}$$

And thus,

$$\begin{aligned} 0 = i'(0) &= \int_U \left[W_{E_{AB}} \left(\mathbf{S}^T D\mathbf{u} + \mathbf{R}^T D\mathbf{v}\right)_{AB} + W_{\Gamma_{AB}} \left(\mathbf{S}^T\mathbf{R}' + \mathbf{R}^T\mathbf{S}'\right)_{AB}\right] dx \\ &= \int_U \left[W_{E_{AB}} \left(S_{iA}u_{i,B} + R_{iA}v_{i,B}\right) + W_{\Gamma_{AB}} \left(S_{iA}R'_{iB} + R_{iA}S'_{iB}\right)\right] dx \end{aligned}$$

$$= \int_U [(W_{E_{AB}} R_{iA} v_i)_{,B} - (W_{E_{AB}} R_{iA})_{,B} v_i + [W_{E_{AB}} u_{i,B} + W_{\Gamma_{AB}} R'_{iB}] S_{iA} + (W_{\Gamma_{AB}} R_{iA} S_{iB})' - (W_{\Gamma_{AB}} R_{iA})' S_{iB}] dx$$

Since $\Gamma_{AB} = -\Gamma_{BA}$ and the variation vanishes on the boundary,

$$-\left(W_{\Gamma_{AB}} R_{iA}\right)' S_{iB} = (W_{\Gamma_{AB}} R_{iB})' S_{iA} = (W'_{\Gamma_{AB}} R_{iB} + W_{\Gamma_{AB}} R'_{iB}) S_{iA}$$

$$\text{So } i'(0) = \int_U -\left(W_{E_{AB}} R_{iA}\right)_{,B} v_i + [W_{E_{AB}} u_{i,B} + 2W_{\Gamma_{AB}} R'_{iB} + W'_{\Gamma_{AB}} R_{iB}] S_{iA} dx$$

Which yields the system of PDEs

$$Div[\mathbf{R}^T W_{\mathbf{E}}] = \mathbf{0} \tag{3.11}$$

or in component form : $-\left(W_{E_{AB}} R_{iA}\right)_{,B} = 0$ (free index i)

$$D\mathbf{u}\left(W_{\mathbf{E}}\right)^T + 2\mathbf{R}' W_{\mathbf{\Gamma}}^T + \mathbf{R} W_{\mathbf{\Gamma}}'^T = \mathbf{0} \tag{3.12}$$

or in component form: $W_{E_{AB}} u_{i,B} + 2W_{\Gamma_{AB}} R'_{iB} + W'_{\Gamma_{AB}} R_{iB} = 0$

3.3 Constraints and Corresponding Lagrange Multipliers

In this section we will look at the effect of the constraints separately and will analyze the boundary value problems that can be posed.

Constraint: $\mathbf{P} \in SO(3)$

In the case that $\mathbf{P}$ is in $SO(3)$, i.e., a rotation: $\mathbf{P}^T\mathbf{P} = \mathbf{I}$ i.e., $P_{iA}P_{iB} = \delta_{AB}$
Introduce a penalization that imposes the constraint as $\epsilon \to 0$:

$$\tilde{I} = \int_U \beta_\epsilon \left(P_{iA}P_{iB} - \delta_{AB}\right) \qquad \mathbf{P} \to \mathbf{R} + \tau\mathbf{S} \tag{3.13}$$

Now

$$\begin{aligned}
i(\tau) &= I(\mathbf{u} + \tau\mathbf{v}) = \int_U \beta_\epsilon \left([R_{iA} + \tau S_{iA}][R_{iB} + \tau S_{iB}] - \delta_{AB}\right] \\
i'(\tau) &= \int_U \frac{\partial \beta_\epsilon}{\partial P_{jA} P_{jB}} \left[S_{iA}R_{iB} + R_{iA}S_{iB} + 2\tau S_{iA}S_{iB}\right] \\
i'(0) &= \int_U \frac{\partial \beta_\epsilon}{\partial R_{jA} R_{jB}} \left(R_{iB}S_{iA} + R_{iA}S_{iB}\right) \\
&= 2\int_U \Lambda^{AB} R_{iB} S_{iA} \\
\Lambda^{AB} &= \Lambda^{BA} \equiv \frac{\partial \beta_\epsilon}{\partial R_{iA} R_{iB}}
\end{aligned}$$

So this adds the term $2\boldsymbol{\Lambda R}$ to the right-had-side of the Euler-Lagrange equation [3.12].

$$W_{E_{AB}} u_{i,B} + 2W_{\Gamma_{AB}} R'_{iB} + W'_{\Gamma_{AB}} R_{iB} = 2\Lambda^{AB} R_{iB} \tag{3.14}$$

Constraint: $\frac{\partial u_i}{\partial x_1} R_{i\alpha} = 0 \qquad \alpha = 2, 3$

This constraint defines the interaction of the fiber with the surrounding matrix. This is achieved by assuming that the along-fiber director, $\mathbf{D}$ is material and hence is convected with the gradient map. Then

$$\int_U \beta_\epsilon \left(\frac{\partial \chi_i}{\partial x_1} P_{i\alpha} \right) dx \tag{3.15}$$

is the corrected term.

$$\begin{aligned}
i(\tau) &= \int_U \beta_\epsilon \left(\left[\frac{\partial u_i}{\partial x_1} + \tau \frac{\partial v_i}{\partial x_1} \right] [R_{i\alpha} + \tau S_{i\alpha}] \right) dx \\
i'(\tau) &= \int_U \frac{\partial \beta_\epsilon}{\partial \chi'_j P_{j\alpha}} \left[\frac{\partial u_i}{\partial x_1} S_{i\alpha} + \frac{\partial v_i}{\partial x_1} R_{i\alpha} + 2\tau \frac{\partial v_i}{\partial x_1} S_{i\alpha} \right] dx \\
\lambda_\alpha &\equiv \frac{\partial \beta_\epsilon}{\partial \chi'_i P_{i\alpha}} \\
i'(0) &= \int_U \lambda_\alpha \left[\frac{\partial u_i}{\partial x_1} S_{i\alpha} + \frac{\partial v_i}{\partial x_1} R_{i\alpha} \right] dx \\
&= \int_U (R_{i\alpha} \lambda_\alpha v_i)' - (R_{i\alpha} \lambda_\alpha)' v_i + \lambda_\alpha \frac{\partial u_i}{\partial x_1} S_{i\alpha}
\end{aligned}$$

So this adds terms $(R_{i\alpha}\lambda_\alpha)'$ and $-\lambda_\alpha \frac{\partial u_i}{\partial x_1}$ to the right-hand-side of the Euler-Lagrange equations [3.16] and [3.17] respectively:

$$-\left(W_{E_{AB}} R_{iA}\right)_{,B} = (R_{i\alpha}\lambda_\alpha)' \tag{3.16}$$

$$W_{E_{AB}} u_{i,B} + W'_{\Gamma_{AB}} R_{iB} + 2W_{\Gamma_{AB}} R'_{iB} = -\lambda_\alpha \frac{\partial u_i}{\partial x_1} \tag{3.17}$$

Equations including both constraints:

$$-\left(R_{iA} W_{E_{AB}}\right)_{,B} = (R_{i\alpha}\lambda_\alpha)' \tag{3.18}$$

$$W_{E_{AB}} u_{i,B} + W'_{\Gamma_{AB}} R_{iB} + 2W_{\Gamma_{AB}} R'_{iB} = -2\Lambda^{AB} R_{iB} - \lambda_\alpha \frac{\partial u_i}{\partial x_1} \tag{3.19}$$

Dirichlet boundary conditions

$$u_i = (u_0)_i \qquad \text{on } \partial U_u \tag{3.20}$$

$$R_i A = (R_0)_i A \qquad \text{on } \partial U_R \tag{3.21}$$

The Neumann boundary conditions

$$R_{iA} W_{E_{AB}} \nu_B = R_{i\alpha} \lambda_\alpha e_1 \qquad \text{on } \partial U_t \tag{3.22}$$

$$W_{\Gamma_{AB}} R_{iA} e_1 = 0 \qquad \text{on } \partial U_c \tag{3.23}$$

Chapter 4

Laminates

For simplicity we will deal only with flat two dimensional bodies in this chapter. The ideas here can be generalized to the case of shells in future work.

In developing the theory of plates, it would be possible to start with the assumption of a two dimensional manifold but here we will use a leading order expansion to extract a theory for the plates based on the three dimensional case discussed in the previous chapter. While it is mathematically possible to work develop the 'intrinsic' theory for the two dimensional plates, no constitutive information would be directly available since all experimental work necessarily pertains to the three dimensional body. Naghdi [72] does show how to obtain the necessary coefficients for the two dimensional object – but we will work from 3D while acknowledging that the minimizer of the 2D leading order energy need not correspond to the minimizer of the 3D problem.

As before, it is assumed that the fibers are extremely rigid with respect to the surrounding medium and are straight and untwisted in the reference configuration.

Kinematics

Let Ω represent the center-plane of the plate of thickness h.Assume a reference configuration of the plate with origin on Ω. In the discussion that follows, quantities with a $\tilde{()}$ represent quantities in 3D and without are the quantities as evaluated on Ω. The region occupied by the plate is thus $U = \Omega \oplus [-h/2, h/2]$. A normal-coordinate parameterization is used to describe the reference placement of the plate:

$$\tilde{\mathbf{x}} = \mathbf{x} + \zeta \mathbf{k} \tag{4.1}$$

where $\mathbf{u} \in \Omega$, $\zeta \in [-h/2, h/2]$ and $\mathbf{k}$ is the vector normal to the undeformed plate.

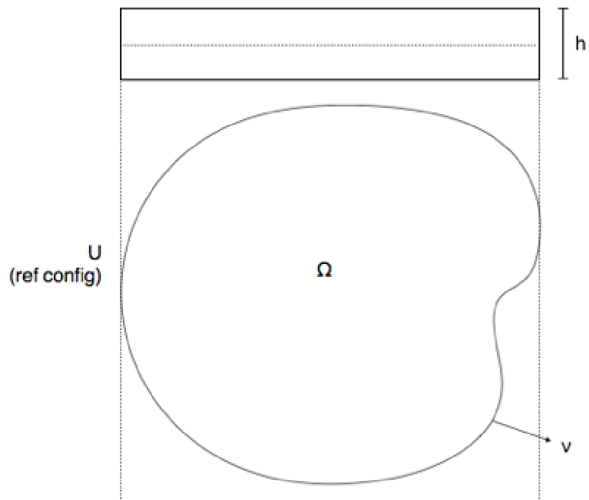

Figure 4.1: Thin plate

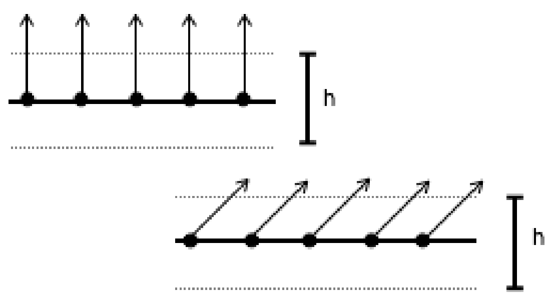

Figure 4.2: A laminate

The outward normal to the two-dimensional region Ω is assumed to be $\boldsymbol{\nu}$.

Using the parametrization, the deformation gradient can be expressed:

$$\mathbf{F} = D\mathbf{x} = \nabla\mathbf{x} + D\zeta \otimes \mathbf{k} \text{ where } \nabla \text{ is the 2D gradient in the plane.} \tag{4.2}$$

In the above, $D\zeta = \mathbf{g}$ is a director on Ω that accounts for the dimension reduction. The director field is aligned along $\boldsymbol{D}$ in the reference configuration. As the fibers are assumed to behave as Kirchhoff rods, we have,

$$\boldsymbol{\kappa} = \kappa_i \mathbf{D}_i \tag{4.3}$$

with $\mathbf{D}_1 = \mathbf{D}$, and $\mathbf{D}_2, \mathbf{D}_3$ in cross-section of the rod.

4.1 The leading order approximation

The energy stored in the plate :

$$\mathcal{E} = \int_U \tilde{W}(\tilde{E}, \tilde{\kappa}) dV = \int_\Omega \int_{h/2}^{h/2} \tilde{W} d\zeta dA = \int_\Omega \int_{h/2}^{h/2} W dA \tag{4.4}$$

$$\text{where } W = \int_{-h/2}^{h/2} \tilde{W} d\zeta = h\tilde{W}(\mathbf{E}, \boldsymbol{\kappa}) + o(h) \tag{4.5}$$

The problem here is to find the two dimensional energy that emerges in the leading order:

$$E = \lim_{h \to 0} \frac{1}{h} \mathcal{E} \tag{4.6}$$

In the nonlinear elasticity theory, it is well known that the leading order expansion fails to satisfy the relevant (two-dimensional) Legendre-Hadamard inequality (semi-strict strong ellipticity) and thus fails to be quasiconvex, even when the three dimensional energy, $\mathcal{E}$ is strongly elliptic. For this reason equilibrium boundary-value problems for a membrane theory generally fail to possess energy-minimizing solutions. In such circumstances well-posedness may be restored via relaxation. However, the theory here (similar to liquid crystal theory) does not suffer from this problem and a leading order approximation is well-posed.

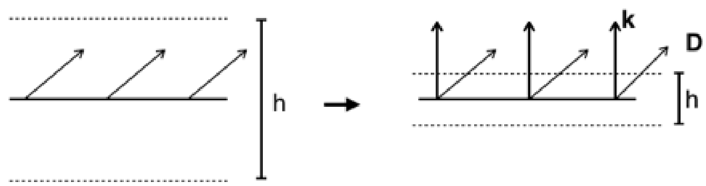

Figure 4.3: Thin plate

$$\begin{aligned} E &= \lim_{h \to 0} \frac{1}{h} \mathcal{E} = \lim_{h \to 0} \frac{1}{h} \int_\Omega [h \ \tilde{W}(\mathbf{E}, \boldsymbol{\kappa}) + O(h)] = \int_\Omega W(\mathbf{E}, \boldsymbol{\kappa}) \ dA \\ \dot{E} &= \int_\Omega W_{\mathbf{E}} \cdot \dot{\mathbf{E}} + W_{\boldsymbol{\kappa}} \cdot \dot{\boldsymbol{\kappa}} \ dA \\ &= \int_\Omega \boldsymbol{\sigma} \cdot (\dot{\boldsymbol{R}}^T \boldsymbol{F} + \boldsymbol{R}^T \dot{\mathbf{F}}) + \mathbf{M} \cdot \dot{\boldsymbol{\kappa}} \ dA \text{ where } \sigma \equiv W_{\mathbf{E}}, \ \mathbf{M} \equiv W_{\boldsymbol{\kappa}} \end{aligned} \tag{4.7}$$

The variation of $\mathbf{F}$ can be written using Eq. 4.2:

$$\dot{\mathbf{F}} = D\dot{\chi} = \nabla \dot{\boldsymbol{\chi}} + \dot{\mathbf{g}} \otimes \mathbf{k} \tag{4.8}$$

Previously, we saw that $\boldsymbol{M}$ can be expressed as the axial vector of a skew tensor $\boldsymbol{\mu}$.

We will use the following observation in the calculations below: For $\mathbf{W}, \dot{\boldsymbol{\Omega}} \in skw$. We have, axial vectors $\boldsymbol{\omega} = ax\{\boldsymbol{\Omega}\}$, $\mathbf{w} = ax\{\boldsymbol{W}\}$. In components: $\Omega_{ij} = \mathcal{E}_{jik}\omega_k$, $W_{ij} = \mathcal{E}_{jil}w_l$ and thus:

$$\begin{aligned}
\mathbf{W} \cdot \boldsymbol{\Omega} &= \Omega_{ij}W_{ij} = \mathcal{E}_{jik}\mathcal{E}_{jil}w_k\omega_l = 2\delta_{kl}w_k\omega_l \\
&= 2w_k\omega_k \\
&= 2\ \mathbf{w} \cdot \boldsymbol{\omega} \\
&= 2(ax\ \boldsymbol{\Omega}) \cdot (ax\ \mathbf{W})
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\int_\Omega \boldsymbol{\sigma} \cdot \dot{\mathbf{R}}^T\mathbf{F} + \boldsymbol{\sigma} \cdot \mathbf{R}^T\dot{\mathbf{F}} &= \int_\Omega \mathbf{F}^T\dot{\mathbf{R}} \cdot \boldsymbol{\sigma}^T + \mathbf{R}\boldsymbol{\sigma} \cdot \dot{\mathbf{F}} \\
&= \int_\Omega \dot{\mathbf{R}} \cdot \mathbf{F}\boldsymbol{\sigma}^T + \mathbf{R}\boldsymbol{\sigma} \cdot \dot{\mathbf{F}} \\
&= \int_\Omega \mathbf{R}^T\dot{\mathbf{R}} \cdot \mathbf{R}^T\mathbf{F}\boldsymbol{\sigma}^T + \mathbf{R}\boldsymbol{\sigma} \cdot \dot{\mathbf{F}} \\
&= \int_\Omega \boldsymbol{\Omega}^T \cdot \mathbf{E}\boldsymbol{\sigma}^T + \mathbf{R}\boldsymbol{\sigma} \cdot \dot{\mathbf{F}} \text{ where } \boldsymbol{\Omega} = \dot{\mathbf{R}}^T\mathbf{R} \\
&= \int_\Omega \boldsymbol{\omega} \cdot ax\{\boldsymbol{\sigma}\mathbf{E}^T - \mathbf{E}\boldsymbol{\sigma}^T\} + \mathbf{R}\boldsymbol{\sigma} \cdot \dot{\mathbf{F}} \text{ where } \boldsymbol{\Omega} = \dot{\mathbf{R}}^T\mathbf{R}
\end{aligned} \tag{4.10}$$

By analogy with Piola stress, the term

$$\mathbf{P} \equiv \mathbf{R}\boldsymbol{\sigma} \tag{4.11}$$

Consider the term

$$\begin{aligned}
\int_\Omega \mathbf{P} \cdot \dot{\mathbf{F}} &= \mathbf{P} \cdot \nabla\dot{\boldsymbol{\chi}} + \dot{\mathbf{g}} \otimes \mathbf{k}\ dA \text{ using Eq. 4.8} \\
&= \int_\Omega \mathbf{P} \cdot \nabla\dot{\boldsymbol{\chi}} + \mathbf{P} \cdot \dot{\mathbf{g}} \otimes \mathbf{k}\ dA \\
&= \int_\Omega P_{i\alpha}\frac{\partial\chi_i}{\partial X_\alpha} + P_{i3}g_i\ dA \\
&= \int_\Omega (P_{i\alpha}\chi_i)_{,\alpha} - P_{i\alpha,\alpha}\chi_i + P_{i3}g_i\ dA \\
&= \int_{\partial\Omega} \mathbf{P1}\boldsymbol{\nu} \cdot \dot{\boldsymbol{\chi}}\ dS + \int_\Omega [-div(\mathbf{P1})] \cdot \dot{\boldsymbol{\chi}} + \mathbf{Pk} \cdot \dot{\mathbf{g}}\ dA
\end{aligned} \tag{4.12}$$

$$\begin{aligned}\int_\Omega \mathbf{M}\cdot\dot{\boldsymbol{\kappa}}\;dA &= \int_\Omega M_i\cdot\dot{\kappa}_i\;dA = \frac{1}{2}\int_\Omega \mu_{ij}(\mathbf{R}^T\mathbf{R}')_{ij}\;dA\\
&= \frac{1}{2}\int_\Omega \mu_{ij}(\dot{R}_{ki}R'_{kj} + R_{ki}\dot{R}'_{kj})\;dA\\
&= \frac{1}{2}\int_\Omega \mu_{ij}[R'_{kj}\dot{R}_{ki} + (\mu_{ij}R_{ki}\dot{R}_{kj})' - \mu'_{ij}R_{ki}\dot{R}_{kj} - \mu_{ij}R'_{ki}\dot{R}_{kj}]\;dA\\
&= \frac{1}{2}\int_\Omega \mu_{ij}R'_{kj}\dot{R}_{ki} - \mu_{ij}R'_{ki}\dot{R}_{kj}\;dA + \int_\Omega (m_i\omega_i)'\;dA - \int_\Omega m'_i\omega_i\;dA\\
&= \int_{\partial\Omega} \mathbf{m}\cdot\boldsymbol{\omega}(\mathbf{D}\cdot\boldsymbol{\nu})\;dA - \int_\Omega \boldsymbol{\omega}\cdot\mathbf{m}'\;dA + \int_\Omega \mathbf{R}^T\mathbf{R}'\boldsymbol{\mu}^T\cdot R^T\dot{\mathbf{R}}\;dA
\end{aligned} \tag{4.13}$$

$$\begin{aligned}\int_\Omega \mathbf{R}^T\mathbf{R}'\boldsymbol{\mu}^T\cdot R^T\dot{\mathbf{R}} &= \int_\Omega \boldsymbol{\Omega}\cdot\boldsymbol{\mu}\boldsymbol{\Gamma}^T dA = \int_\Omega \Gamma_{ik}\;\mu_{jk}W_{ij}dA\\
&= \int_\Omega \mathcal{E}_{ikl}k_l\;\mathcal{E}_{jkn}m_n\;\mathcal{E}_{ijp}\;\omega_p\;dA\\
&= \int_\Omega \mathcal{E}_{ijp}\;k_l\;m_n\;\omega_p[\delta_{ij}\;\delta_{ln} - \delta_{in}\;\delta_{jl}]\;dA = -\int_\Omega \omega_p\mathcal{E}_{pij}\;m_i\;k_j dA\\
&= -\int_\Omega [\mathbf{m}\times\boldsymbol{\kappa}]\cdot\boldsymbol{\omega}
\end{aligned} \tag{4.14}$$

So we have:

$$\int_\Omega \mathbf{M}\cdot\dot{\boldsymbol{\kappa}}\;dA = \int_{\partial\Omega}[(\mathbf{D}\cdot\boldsymbol{\nu})\mathbf{m}]\cdot\boldsymbol{\omega}\;dA - \int_\Omega [\mathbf{m}' + \mathbf{m}\times\boldsymbol{\kappa}]\cdot\boldsymbol{\omega}\;dA \tag{4.15}$$

Constraint

It is assumed that the fibers and the matrix interact such that the fiber cross-sections remain perpendicular to the fiber axis after deformation:

$$\mathbf{0} = \mathbf{d}\cdot\mathbf{d}_\alpha = \mathbf{F}\mathbf{D}\cdot\mathbf{R}\mathbf{D}_\alpha = \mathbf{R}^T\mathbf{F}\mathbf{D}\cdot\mathbf{D}_\alpha \tag{4.16}$$

Variation of the constraint term :

$$
\begin{aligned}
\int_\Omega \lambda_\alpha \dot{\mathbf{E}}\mathbf{D}\cdot\mathbf{D}_\alpha &= \int_\Omega \dot{\mathbf{E}}\mathbf{D}\cdot\boldsymbol{\Lambda} \quad \text{where} \quad \boldsymbol{\Lambda} = \lambda_\alpha \mathbf{D}_\alpha \\
&= \int_\Omega \dot{E}_{AB} D_B \Lambda_\alpha \delta_{A\alpha} \\
&= \int_\Omega \dot{E}_{\alpha B} D_B \Lambda_\alpha = \int_\Omega [(\dot{R}^T F)_{\alpha B} + (R^T \dot{F})_{\alpha B}] D_B \Lambda_\alpha \\
&= \int_\Omega \dot{R}_{i\alpha} F_{iB} D_B \Lambda_\alpha + R_{i\alpha} \dot{F}_{iB} D_B \Lambda_\alpha \\
&= \int_\Omega \dot{\mathbf{R}}\cdot\mathbf{F}\mathbf{D}\otimes\boldsymbol{\Lambda} + \mathbf{R}\boldsymbol{\Lambda}\otimes\mathbf{D}\cdot\dot{\mathbf{F}} \\
&= \int_\Omega \mathbf{R}^T\dot{\mathbf{R}}\cdot\mathbf{R}^T\mathbf{F}\mathbf{D}\otimes\boldsymbol{\Lambda} + \mathbf{R}\boldsymbol{\Lambda}\otimes\mathbf{D}\cdot\dot{\mathbf{F}} \\
&= \int_\Omega \boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}\otimes\mathbf{E}\mathbf{D} + \mathbf{R}\boldsymbol{\Lambda}\otimes\mathbf{D}\cdot\dot{\mathbf{F}}
\end{aligned} \tag{4.17}
$$

$$
\begin{aligned}
\int_\Omega \boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}\otimes\mathbf{E}\mathbf{D}\,dA &= \int_\Omega \mathcal{E}_{\alpha BC}\omega_C \Lambda_\alpha (\mathbf{E}\mathbf{D})_B\,dA \\
&= \int_\Omega \omega_C \mathcal{E}_{C\alpha B} \Lambda_\alpha (\mathbf{E}\mathbf{D})_B\,dA \\
&= \int_\Omega \boldsymbol{\omega}\cdot(\boldsymbol{\Lambda}\times\mathbf{E}\mathbf{D})\,dA
\end{aligned} \tag{4.18}
$$

If $\boldsymbol{\Lambda} = \mathbf{R}^T\boldsymbol{\lambda}$ and for $\mathbf{E} = \mathbf{R}^T\mathbf{F}$

$$
\begin{aligned}
\boldsymbol{\Lambda}\times\mathbf{E}\mathbf{D} = \mathbf{R}^T\boldsymbol{\lambda}\times\mathbf{R}^T\mathbf{F}\mathbf{D} &= (\mathbf{R}^T)^*(\boldsymbol{\lambda}\times\mathbf{F}\mathbf{D}) \text{ where } ()^* \text{ represents cofactor} \\
&= \mathbf{R}^T(\boldsymbol{\lambda}\times\mathbf{F}\mathbf{D})
\end{aligned} \tag{4.19}
$$

So the constraint term finally reduces to:

$$
\int_\Omega \lambda_\alpha \dot{\mathbf{E}}\mathbf{D}\cdot\mathbf{D}_\alpha\,dA = \int_\Omega \mathbf{R}^T(\boldsymbol{\lambda}\times\mathbf{F}\mathbf{D}) + \mathbf{R}\boldsymbol{\Lambda}\otimes\mathbf{D}\cdot\dot{\mathbf{F}}\,dA \tag{4.20}
$$

Boundary data

In addition to tractions, $\mathbf{t}$, we assume that the plate can support boundary couples, $\mathbf{c}$, as well. Thus the boundary power term, P, is:

$$
P = \int_{\partial\Omega} (\mathbf{t}\cdot\dot{\boldsymbol{\chi}} + \mathbf{c}\cdot\boldsymbol{\omega})\,dS \tag{4.21}
$$

Extended energy

$$\bar{E} = E + \lambda_\alpha \mathbf{R}^T\mathbf{F}\mathbf{D}\cdot\mathbf{D}_\alpha = E + \mathbf{R}^T\mathbf{F}\mathbf{D}\cdot\boldsymbol{\Lambda} \tag{4.22}$$

Taking the variation: $\dot{\bar{E}} = P$

$$\begin{aligned}\int_{\partial\Omega}(\mathbf{t}\cdot\dot{\boldsymbol{\chi}} + \mathbf{c}\cdot\boldsymbol{\omega}) = &\int_{\partial\Omega}[(\mathbf{D}\cdot\boldsymbol{\nu})\mathbf{m}]\cdot\boldsymbol{\omega}\ dS \\ &+ \int_\Omega [ax\{\boldsymbol{\sigma}\mathbf{E}^T - \mathbf{E}\boldsymbol{\sigma}^T\} + \mathbf{R}^T(\boldsymbol{\lambda}\times\mathbf{F}\mathbf{D}) - \mathbf{m}' - \mathbf{m}\times\boldsymbol{\kappa}]\cdot\boldsymbol{\omega} dA \\ &+ \int_\Omega [\mathbf{R}\boldsymbol{\sigma} + \mathbf{R}\boldsymbol{\Lambda}\otimes\mathbf{D}]\cdot\dot{\mathbf{F}}\end{aligned} \tag{4.23}$$

So the stress term, $\mathbf{P}$ is modified by the constraint:

$$\mathbf{P} \equiv \mathbf{R}\boldsymbol{\sigma} + \mathbf{R}\boldsymbol{\Lambda}\otimes\mathbf{D} \tag{4.24}$$

Thus we obtain:

$$\begin{aligned}\mathbf{0} = &\int_{\partial\Omega}[\mathbf{t} - \mathbf{P}\mathbf{1}\boldsymbol{\nu}]\cdot\dot{\boldsymbol{\chi}}\ dS \\ &+ \int_{\partial\Omega}[\mathbf{c} - (\mathbf{D}\cdot\boldsymbol{\nu})\mathbf{m}]\cdot\boldsymbol{\omega})\ dS \\ &+ \int_\Omega [ax\{\boldsymbol{\sigma}\mathbf{E}^T - \mathbf{E}\boldsymbol{\sigma}^T\} + \mathbf{R}^T(\boldsymbol{\lambda}\times\mathbf{F}\mathbf{D}) - \mathbf{m}' - \mathbf{m}\times\boldsymbol{\kappa}]\cdot\boldsymbol{\omega}\ dA \\ &+ \int_\Omega [-div(\mathbf{P}\mathbf{1})\cdot\dot{\boldsymbol{\chi}}]\ dA \\ &+ \int_\Omega \mathbf{P}\mathbf{k}\cdot\dot{\mathbf{g}}\ dA\end{aligned} \tag{4.25}$$

EL equations

$$\text{For } \mathbf{P} = \mathbf{R}(\boldsymbol{\sigma} + \boldsymbol{\Lambda}\otimes\mathbf{D})\mathbf{1}, \tag{4.26}$$

$$\mathbf{P}\mathbf{k} = \mathbf{0}\ div\mathbf{P} = \mathbf{0} \tag{4.27}$$

$$\mathbf{m}' + \mathbf{m}\times\kappa = ax\{\boldsymbol{\sigma}\mathbf{E}^T - \mathbf{E}\boldsymbol{\sigma}^T\} + \mathbf{R}^T(\boldsymbol{\lambda}\times\mathbf{F}\mathbf{D}) \tag{4.28}$$

$$\tag{4.29}$$

Boundary Conditions

$$\mathbf{t} = \mathbf{R}(\boldsymbol{\sigma} + \boldsymbol{\Lambda}\otimes\mathbf{D})\mathbf{1}\boldsymbol{\nu} = \mathbf{R}\boldsymbol{\sigma}\mathbf{1}\boldsymbol{\nu} + \mathbf{R}\boldsymbol{\Lambda}(\mathbf{D}\cdot\boldsymbol{\nu}) \tag{4.30}$$

$$\mathbf{c} = (\mathbf{D}\cdot\boldsymbol{\nu})\mathbf{m} \tag{4.31}$$

Discussion

The equations developed here are a natural extension to the usual Kirchhoff rod equations where the interaction between the fibers and the surrounding medium is incorporated. $\mathbf{P}$ here is the analog of the Piola-Kirchhoff equations in the standard nonlinear elasticity theory. The first of the three balance equations above pertains to the dimension-reduction of the plate and pertains to lateral tractions. The second of the equations is the analog of the usual balance of linear momentum equation. The final equation is the Kirchhoff rod equation including the effect of the surrounding medium.

When the fibers are in the plane

In this case,

$$\mathbf{D} \cdot \mathbf{k} = 0 \tag{4.32}$$

$$\mathbf{m} \times \boldsymbol{\kappa} \text{ acts along } \mathbf{k} \text{ and so } : \tag{4.33}$$

$$\mathbf{m} \times \boldsymbol{\kappa} \cdot \boldsymbol{\omega} = \mathbf{0} \tag{4.34}$$

Euler-Lagrange equations and boundary conditions

$$\mathbf{Pk} = \mathbf{0}, \tag{4.35}$$

$$div(\mathbf{P1}) = \mathbf{0} \tag{4.36}$$

$$\mathbf{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} + \mathbf{R}\, ax(\boldsymbol{\sigma}\mathbf{E}^T - \mathbf{E}\boldsymbol{\sigma}^T) = \mathbf{0} \tag{4.37}$$

$$\text{where } \mathbf{m}' = (\nabla\mathbf{m})\mathbf{D} \text{ and } \mathbf{r}' = (\nabla\mathbf{r})\mathbf{D}$$

4.2 Summary

In this chapter the mechanics of a thin sheet composed of a material with a particular microstructure is developed. The parent medium in three-dimensions is assumed to have a family of thick aligned fibers. The medium is thus a nematic elastomer with directors that do not shear. This extension to the standard theory could have application for textured media. The equations for a fiber-reinforced laminate is then written out.

Chapter 5

Example – bending of a plate to cylindrical shell

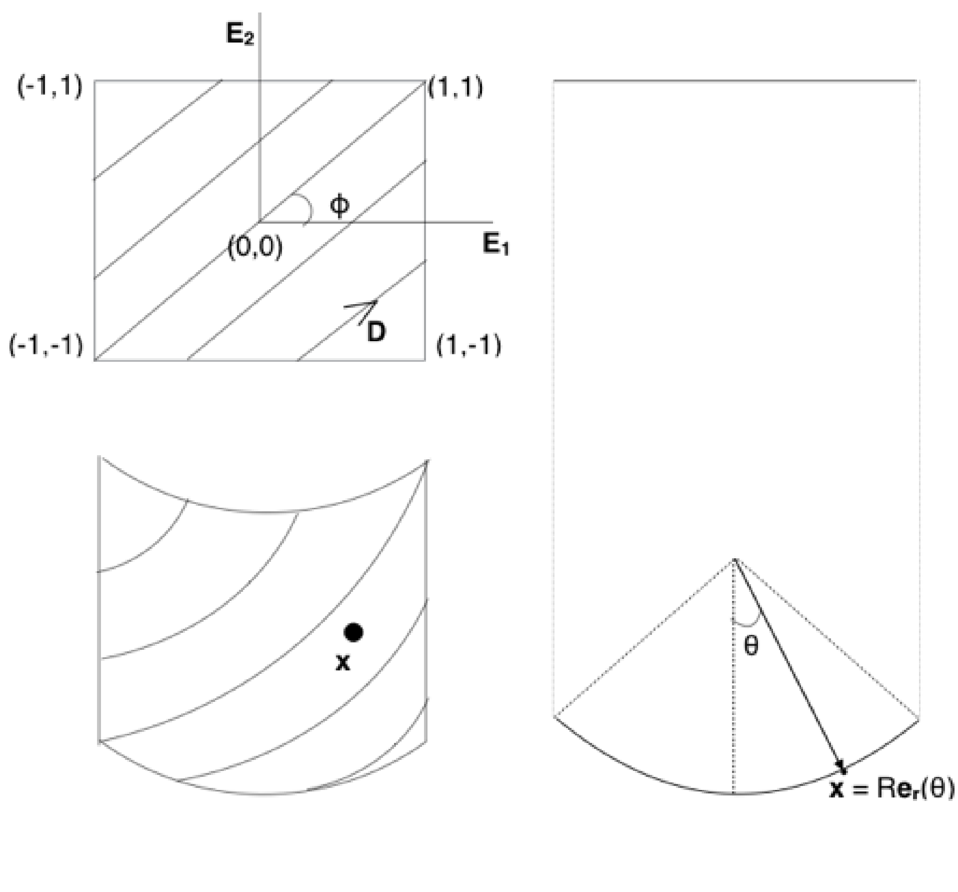

Figure 5.1: Plate to Cylinder

In order to understand the theory, we will apply it to a very simple deformation. In the example considered here a square plate, flat in the reference configuration, is bent into a cylindrical shell. First we will consider the case where the fibers, while in the plane, are at an inclination to the edges. It is assumed that the body is reinforced by a single family of fibers aligned along **D** in the reference configuration. It will be seen that the deformation cannot be maintained without both boundary tractions and couples.

5.1 Fibers at an arbitrary inclination

$$\text{Plate in reference configuration: } \mathbf{X} = X_\alpha \mathbf{E}_\alpha \tag{5.1}$$

$$\text{Plate in the deformed configuration: } \mathbf{x} = R\mathbf{e}_\gamma(\theta) + x_2\mathbf{E}_2 \text{ where R is fixed.} \tag{5.2}$$

Extension of this plate prior to bending is possible but not considered, so $x_2 = X_2$

Deformation gradient:

$$\mathbf{F} = \mathbf{e}_\theta(\theta) \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_2 \tag{5.3}$$

Assume fibers are uniformly distributed at an angle, ϕ, with respect to $\mathbf{E}_1$

$$\mathbf{D} = \cos(\phi)\mathbf{E}_1 + \sin(\phi)\mathbf{E}_2 \tag{5.4}$$

$$\text{here, } \mathbf{d} = \mathbf{FD} = (\mathbf{e}_\theta \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_2)(\cos(\phi)\mathbf{E}_1 + \sin(\phi)\mathbf{E}_2) \tag{5.5}$$

$$= \cos(\phi)\mathbf{e}_\theta(\theta) + \sin(\phi)\mathbf{E}_2 \tag{5.6}$$

Choice of strain energy function

A simple form of energy when $\boldsymbol{\kappa}$ is sufficiently small, the energy can be expanded to quadratic order:

$$W(\mathbf{E}, \boldsymbol{\kappa}) = W(\mathbf{E}, \mathbf{0}) + \frac{1}{2}\boldsymbol{\kappa} \cdot W_{\boldsymbol{\kappa}\boldsymbol{\kappa}}\Big|_{\mathbf{E}=\mathbf{0}}\boldsymbol{\kappa} \tag{5.7}$$

$$= W_1(\mathbf{E}) + W_2(\mathbf{E})(\boldsymbol{\kappa} \cdot \mathbf{D})^2 + W_3(\mathbf{E})|\mathbf{1}\boldsymbol{\kappa}|^2 \tag{5.8}$$

Let us assume the medium is Neo-Hookean and the reinforcing fibers are Kirchoff rods with linear resistance to bending, F and resistance to twist, T.

$$W(\mathbf{E}, \boldsymbol{\kappa}) = \frac{\text{tr}(\mathbf{E}^T\mathbf{E}) - 3}{2}(\mathbf{E}) + \frac{T}{2}\kappa^2 + \frac{F}{2}\kappa_\alpha\kappa_\alpha \tag{5.9}$$

Response functions

$$\boldsymbol{\sigma} = W_{\mathbf{E}} \text{ i.e. } E_{AB} = R_{iA}F_{iB}$$
$$\sigma_{AB} = \frac{\partial W}{\partial E_{AB}} \tag{5.10}$$

$$\text{and } \mathbf{M} = W_{\boldsymbol{\kappa}}$$
$$M_A = \frac{\partial W}{\partial \kappa_A} \tag{5.11}$$

For the selected energy function, this yields:

$$\boldsymbol{\sigma} = \mu \mathbf{E} \tag{5.12}$$
$$\mathbf{M} = T\kappa\mathbf{D} + F\mathbf{1}\boldsymbol{\kappa} \tag{5.13}$$
$$\mathbf{m} = T\kappa\mathbf{R}\mathbf{D} + F\mathbf{R}\mathbf{1}\boldsymbol{\kappa} = T\kappa\mathbf{d} + F\underbrace{\mathbf{R}\mathbf{1}\boldsymbol{\kappa}}_{\kappa_\alpha \mathbf{D}_\alpha} = T\kappa\mathbf{d} + F\mathbf{d}\times\mathbf{d}' \tag{5.14}$$

PDEs :

Thus, $\boldsymbol{\sigma}\mathbf{E}^T$ is symmetric, $\mathbf{Ek} = \mathbf{0}$ and the equations **??** from Chapter 4 simplify to:

$$div(\mathbf{P1}) = div(\mathbf{R}\boldsymbol{\sigma}\mathbf{1} + \boldsymbol{\lambda}\otimes\mathbf{D}) = \mathbf{0} \tag{5.15}$$
$$\mathbf{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = \mathbf{0} \tag{5.16}$$

Thus, since $\mathbf{F}$ is homogeneous:

$$\mathbf{P1} = \mathbf{R}\boldsymbol{\sigma}\mathbf{1} + \boldsymbol{\lambda}\otimes\mathbf{D}\text{i.e. } P_{i\alpha} = R_{iA}\sigma_{AB}\delta_{B\alpha} + \lambda_i D_\alpha \tag{5.17}$$
$$div\mathbf{P1} = (P_{i\alpha,\alpha}) = \mathbf{0} \tag{5.18}$$
$$P_{i\alpha} = \mu F_{i\alpha} + \lambda_i D_\alpha \tag{5.19}$$
$$0 = P_{i\alpha,\alpha} = \lambda_{i,\alpha} D_\alpha \to \boldsymbol{\lambda}' = \mathbf{0} \tag{5.20}$$

Since the first of the equations is already satisfied by the chosen $\mathbf{P}$ we will use the other two equations along with boundary conditions to solve for the unknowns.

Boundary Conditions :

$$\mathbf{t} = \mathbf{P1}\boldsymbol{\nu} \tag{5.21}$$
$$\mathbf{c} = -\mathbf{R}^T\mathbf{M}(\mathbf{D}\cdot\boldsymbol{\nu}) \tag{5.22}$$

Some brief calculations yield:

$$\mathbf{d}' = -\frac{1}{R}\cos^2(\phi)\mathbf{e}_r(\theta) \tag{5.23}$$
$$\mathbf{d} = \cos\phi\ \mathbf{e}_\theta + \sin\phi\ \mathbf{E}_2 \tag{5.24}$$
$$\mathbf{d}\times\mathbf{d}' = \frac{\cos^2\phi}{R}[\cos\phi\ \mathbf{E}_2 - \sin\phi\ \mathbf{e}_\theta] = c[\cos\phi\ \mathbf{E}_2 - \sin\phi\ \mathbf{e}_\theta] \tag{5.25}$$

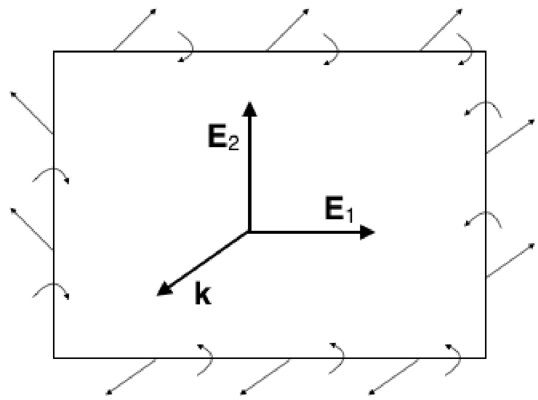

Figure 5.2: Plate with tractions and couples

Thus the moment in the fibers is:

$$\begin{aligned}\mathbf{m} &= T\kappa[\cos(\phi)\mathbf{e}_\theta(\theta) + \sin(\phi)\mathbf{E}_2] + F(\frac{\cos^3(\phi)}{R}\mathbf{E}_2 - \frac{\cos^2(\phi)\sin(\phi)}{R}\mathbf{e}_\theta(\theta)) \\ &= [T\kappa\cos(\phi) - \frac{\cos^2(\phi)\sin(\phi)}{R}F]\mathbf{e}_\theta(\theta) + [T\kappa\sin(\phi) + \frac{F}{R}\cos^3(\phi)]\mathbf{E}_2\end{aligned} \tag{5.26}$$

$$\mathbf{m}' = \frac{\cos^2\phi}{R}[\frac{\sin\phi\cos\phi}{R}F - T\kappa]e_r(\theta) = Pe_r \tag{5.27}$$

To solve for the vector of Lagrange multipliers, $\boldsymbol{\lambda} = \lambda_i\mathbf{e}_i = [\lambda_\theta\mathbf{e}_\theta + \lambda_2\mathbf{e}_2]$, we use the PDE $\mathbf{m}' = \boldsymbol{\lambda} \times \boldsymbol{\chi}'$

$$\boldsymbol{\lambda} \times \boldsymbol{\chi}' = [\lambda_r e_r + \lambda_\theta\mathbf{e}_\theta + \lambda_2\mathbf{E}_2] \times [\cos\phi\ \mathbf{e}_\theta + \sin\phi\ \mathbf{E}_2] \tag{5.28}$$

$$= \lambda_r\cos\phi\ \mathbf{k} - \lambda_r\sin\phi\ \mathbf{e}_\theta + \lambda_\theta\sin\phi\ \mathbf{e}_r - \lambda_2\cos\phi\ \mathbf{e}_r \tag{5.29}$$

$$(\lambda_\theta\sin\phi - \lambda_2\cos\phi)\mathbf{e}_r = \frac{\cos^2\phi}{R}[\frac{\sin\phi\cos\phi}{R}F - T\kappa]e_r \tag{5.30}$$

$$\lambda_\theta\sin\phi - \lambda_2\cos\phi = \frac{\cos^2\phi}{R}[\frac{\sin\phi\cos\phi}{R}F - T\kappa] = P \tag{5.31}$$

$$\text{When } \phi = 0\ :\ -\lambda_2 = -\frac{1}{R}T\kappa\ \boxed{\lambda_2 = \frac{T\kappa}{R}} \tag{5.32}$$

$$\text{When } \phi = 90^\circ\ :\ \boxed{\lambda_\theta = 0} \tag{5.33}$$

$$\begin{aligned}\boldsymbol{\lambda} &= \underbrace{(\lambda\cdot\mathbf{d})}_{0}\mathbf{d} + \underbrace{\mathbf{d}\times\lambda}_{\mathbf{m}'}\times\mathbf{d}\\ \boldsymbol{\lambda} &= \lambda_\theta\mathbf{e}_\theta + \lambda_2\mathbf{E}_2\\ \boldsymbol{\lambda}\cdot\mathbf{d} &= \underbrace{\lambda_\theta\cos\phi + \lambda_2\sin\phi}_{0}\end{aligned} \tag{5.34}$$

$$\mathbf{m}'\times\mathbf{d} = \boldsymbol{\lambda} \tag{5.35}$$

$$P\mathbf{e}_r \times (\cos\phi\ \mathbf{e}_\theta + \sin\phi\ \mathbf{E}_2) = \boldsymbol{\lambda} \tag{5.36}$$

$$P\ \cos\phi\ \mathbf{E}_2 - P\ \sin\phi\ \mathbf{e}_\theta = \boldsymbol{\lambda} \tag{5.37}$$

$$\begin{aligned}P = 0 \text{ if } 2T\kappa &= \sin 2\phi\ F\\ \kappa &= \frac{\sin 2\phi\ F}{2T}\end{aligned} \tag{5.38}$$

if not,

$$P\cos\phi = \lambda_2 \tag{5.39}$$

$$-P\sin\phi = \lambda_\theta \tag{5.40}$$

in terms of $\boldsymbol{\kappa}$

$$\lambda_2 = \frac{\cos^3\phi}{R}\underbrace{[\frac{\sin\ 2\phi}{R}F - T\kappa]}_{\neq 0} \tag{5.41}$$

$$\lambda_\theta = -\frac{\cos^2\phi\ \sin\phi}{R}[\frac{\sin\ 2\phi}{R}F - T\kappa] \tag{5.42}$$

λ_2 and λ_θ are constants!

Traction Boundary Condition :

$$\begin{aligned}\mathbf{t} &= \mathbf{R}\boldsymbol{\sigma} + \boldsymbol{\lambda}(\mathbf{D}\cdot\boldsymbol{\nu}) \text{ where } \mathbf{R}\boldsymbol{\sigma} = \mu\mathbf{F}_2\\ \mathbf{t}_1 &= \mu\cos\phi\ \mathbf{e}_\theta + \boldsymbol{\lambda}\cos\phi\end{aligned} \tag{5.43}$$

if $\mathbf{t}_1 = \mathbf{0}$ and $\cos\phi \neq 0$ then $\mu\mathbf{e}_\theta = \boldsymbol{\lambda}$ which is not possible

$$\mathbf{c} = -(\mathbf{D}\cdot\boldsymbol{\nu})\mathbf{R}^T\mathbf{M} = 0 \tag{5.44}$$

if $\mathbf{D}\cdot\boldsymbol{\nu} \neq 0$ then $\mathbf{m} = \mathbf{0}$

but if $\mathbf{m} = \mathbf{0}$ then $\mathbf{m}' = \mathbf{0}$, $\boldsymbol{\lambda} = \mathbf{0}$
$\mathbf{m} = T\kappa\mathbf{d} + F\mathbf{d} \times \mathbf{d}'$
T, F not 0
$\mathbf{m} = T\kappa[\cos\phi\ \mathbf{e}_\theta + \sin\phi\ \mathbf{E}_2] + F[$

$$\boxed{\begin{array}{c} T\kappa\cos\phi = -F\sin\phi \\ T\kappa\sin\phi = -F\cos\phi \end{array}}$$

$\kappa = -\frac{F}{T}\tan\phi = \frac{F}{T}\cot\phi$
$-\tan\phi = \cot\phi$
$\tan^2\phi = -1$ two imaginary, not possible!!

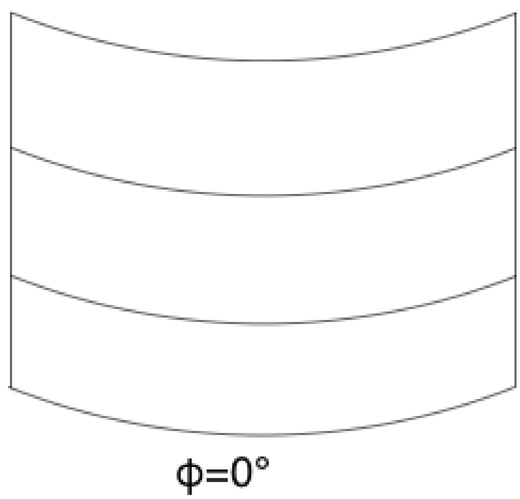

Figure 5.3: Fibers aligned along axis of cylinder

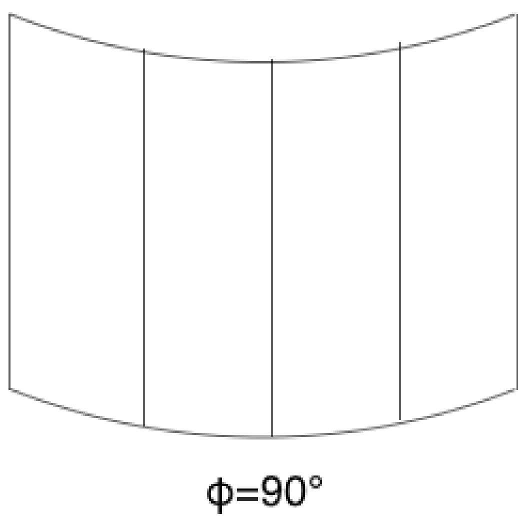

Figure 5.4: Fibers in bending

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