

# Lawrence Berkeley National Laboratory

## LBL Publications

### Title

APPROXIMATIONS FOR HAND CALCULATORS USING SMALL INTEGER COEFFICIENTS

### Permalink

<https://escholarship.org/uc/item/0qr0p86x>

### Author

Derenzo, Stephen E.

### Publication Date

1975-03-01

0 0 0 0 4 3 0 3 3 4 2

Submitted to Computer Physics  
Communications

RECEIVED LBL-3804  
Preprint c.1  
LAWRENCE LABORATORY

JUL 18 1975

LIBRARY AND  
DOCUMENTS SECTION

APPROXIMATIONS FOR HAND CALCULATORS USING  
SMALL INTEGER COEFFICIENTS

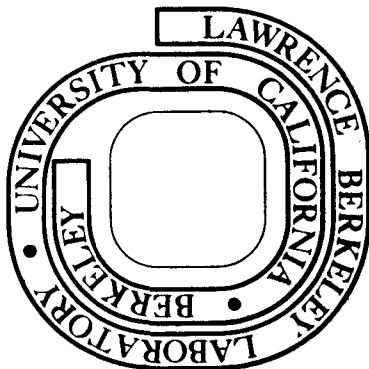
Stephen E. Derenzo

March 1975

Prepared for the U. S. Energy Research and  
Development Administration under Contract W-7405-ENG-48

**For Reference**

Not to be taken from this room



LBL-3804  
c.1

## **DISCLAIMER**

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

APPROXIMATIONS FOR HAND CALCULATORS USING  
SMALL INTEGER COEFFICIENTS

Stephen E. Derenzo  
Lawrence Berkeley Laboratory  
University of California  
Berkeley CA 94720

ABSTRACT

Methods are presented for deriving approximations containing small integer coefficients. This approach is useful for electronic hand calculators and programmable calculators, where it is important to minimize the number of keystrokes necessary to evaluate the function. For example, the probability  $P(x)$  of exceeding  $x$  standard deviations of either sign (Gaussian probability integral) is approximated by

$$P(x) \approx \text{EXP} \left[ - \frac{(83x + 351)x + 562}{703/x + 165} \right]$$

with a relative error less than 0.042% over the range  $0 < x < 5.5$

( $1 > P(x) > 4 \times 10^{-8}$ ). Other examples presented are the functional inverse of  $P(x)$ ; the Klein-Nishina cross section for Compton scattering; photoelectric cross sections in  $\text{H}_2\text{O}$ , Bone, Fe, NaI, and Pb; and the pair production cross section in Pb.

## 1. Introduction

By the use of suitable approximations most functions can be conveniently evaluated on automatic digital computers.<sup>1,2</sup> However, these approximations (usually polynomials or rational functions) are often inconvenient for hand calculators because many keystrokes are required to enter the coefficients. In this paper we describe methods for deriving approximations containing small integer coefficients, which substantially reduce the number of keystrokes required. This approach is also important for programmable calculators, where the stored programs are usually limited to a certain number of keystrokes.

In many cases such approximations can be evaluated as rapidly and will generally be as accurate as interpolation from tables, eliminating the need for tables in those cases. Although graphical representation permits ready interpolation it has limited accuracy, especially when the function spans many decades.

The method consists of four parts: (1) selecting a suitable form for the approximation, (2) fitting the approximation to the function, (3) eliminating unnecessary terms in the approximation and (4) determining small integer coefficients that give a fit not substantially worse than the best fit of (2).

## 2. Method

### 2.1 Selecting a suitable form

This part of the method rests heavily on the existing body of approximation theory<sup>1,2</sup>, but a few comments seem appropriate. (1) Many

electronic calculators are equipped with  $\sqrt{x}$ ,  $\sin x$ ,  $\cos x$ ,  $\tan x$ ,  $e^x$ ,  $\ln x$ ,  $x^y$ , etc., keys and these should be considered if the function to be approximated resembles one of them. (2) There is much merit however, in the polynomial, which (when arranged according to Horner's rule) has a repetitive pattern that lends itself to a rapid keystroke pace. Moreover, electronic calculators can evaluate polynomials as rapidly as the keys are depressed but this is not so for the transcendental functions. (3) Asymptotic limits are important in the selection of a form. For example, the Klein-Nishina formula (discussed in Section 3.3) approaches a constant value at low photon energy  $E$  and decreases as  $E^{-1}$  at large values of  $E$ . This suggests a form such as

$$g(E) = \frac{a_1 + a_2 E + a_3 E^2}{a_4 + a_5 E + a_6 E^2 + a_7 E^3}$$

Note that no simple power series can satisfy these limits.

## 2.2 Fitting the approximation to the function

After a form  $g(x)$  has been chosen, its (unknown) coefficients  $a_1$  to  $a_N$  must be selected so that  $g(x)$  fits the function  $f(x)$  to be approximated. The usual criterion is the minimax (or least maximum) error criterion, requiring that the largest deviation of  $|d(x)|$  be minimized, where

$$d(x) = w(x)[g(x) - f(x)] \quad (1)$$

and  $w(x)$  is a weighting function.<sup>3</sup> Under this criterion, the function  $d(x)$  oscillates about zero with equal positive and negative excursions. For rational approximating forms Chebyshev's Theorem gives the minimum

number of excursions that are necessary and sufficient for a best approximation.<sup>2</sup>

Unfortunately, the minimax criterion does not lend itself to the minimization code used in this work,<sup>4</sup> as the code assumes that the function to be minimized is locally quadratic in the coefficients  $a_i$ . It was found, however that the code could minimize  $D$  given by:

$$D = \sqrt{\sum_{j=1}^M d(x_j)^4} \quad (2)$$

where the base points  $x_j$  were chosen with sufficient density that  $d(x_j)$  was a reasonable representation of  $d(x)$ . Moreover, the resulting deviations  $d(x_j)$  oscillated about zero with very nearly equal positive and negative excursions and had the necessary minimum number of excursions for a best fit (see figures). Thus, while the best fit coefficients given in Section 3 may not be unique, no other values can yield a significantly better fit.<sup>5</sup>

In the event that the deviations  $d(x_j)$  are larger than the required accuracy, it is necessary to go back and improve the form of the approximation. This usually means increasing the number of terms and consequently increasing the number of coefficients.

### 2.3 Eliminating unnecessary terms

As a rule, we started with a form that contained a sufficient number of terms to give a good fit. Then the computer code set each coefficient in turn to zero while all others were varied to minimize  $D$ .

If the best of these fits was acceptable, the related coefficient was set permanently to zero and the process was automatically repeated to try to eliminate other terms.

#### 2.4 Determining small integer coefficients

The methods described in this section assume that the approximation remains numerically unchanged when all coefficients are multiplied by a common factor. For example, this assumption is valid for power series (provided they are divided by a single coefficient) and more generally for the rational approximations, but not for expansions in transcendental functions.

We now define a scale factor  $b_1$  that is allowed to take on the integer values 1, 2, 3, .... Renaming the best fit coefficients  $a_1$  to  $a_N$  from Section 2.2 so that the coefficient closest to zero is  $a_1$ , the scaled best fit coefficient values are given by:

$$b_i = a_i b_1 / a_1,$$

and

$$|b_i| \geq b_1.$$

Clearly, in the limit of large integer values of  $b_1$  it is possible to round all the other coefficients to their nearest integer values and still remain very close to the best fit approximation. This suggests a straightforward integer search algorithm that consists of tabulating  $D$  (and the deviations  $d(x_j)$ ) for  $b_1 = 1, 2, 3, \dots$  where in each case the best fit values of  $b_2$  to  $b_N$  are rounded to the nearest integer. Usually, the resulting values of  $D$  are far from monotonic and it is possible to stop the search at a downward fluctuation in  $D$  that corresponds to



an acceptable fit.

The above search method was not used in this paper because, for a given  $b_1$ , the integer values of  $b_2$  to  $b_N$  closest to the best fit are usually not the best integer values (i.e., those that result in the lowest value of  $D$ ).

By searching the space of  $b_2$  to  $b_N$  it is often possible to find a set of integer values that result in a lower value of  $D$  because the variation in each coefficient from its best fit value has been nearly compensated by the variations in the other coefficients.

As an example of the need for such a search, consider eq. 11 of Section 3.2. The best fit coefficients are ( $b_1 \equiv 1$ ),  $b_2 = 260.40\dots$ ,  $b_3 = 503.60\dots$ ,  $b_4 = 134.16\dots$ ,  $b_5 = 543.36\dots$  ( $D = 0.412$ ) and varying these coefficients by less than 0.15% to their nearest integer values  $b_2 = 260$ ,  $b_3 = 504$ ,  $b_4 = 134$ ,  $b_5 = 543$  yields a rather poor fit ( $D = 6.8$ ). A complete search of the integer values of  $b_2$  to  $b_5$  for the lowest value of  $D$  results in coefficients that are far from their best fit values:  $b_2 = 280$ ,  $b_3 = 572$ ,  $b_4 = 144$ ,  $b_5 = 603$  but  $D = 0.520$ , not much larger than the best fit value.

Unfortunately, a straightforward search over a wide range of integer coefficient values requires calculating  $D$  an unacceptably large number of times (typically  $10^7$ ). Moreover, it is not obvious from the values of  $D$  how far each coefficient should be stepped.

A more efficient algorithm was therefore devised that restricted the search (as much as possible) to the volume  $V'$  within which  $D < D'$  where  $D'$  is the lowest value of  $D$  yet achieved during the integer coefficient search. The appendix is an example of this algorithm as used in this

work to search  $V'$  and determine the best integer values of  $b_2$  to  $b_N$  for each successive integer value  $b_1$ . Although it only covers the case  $N = 4$ , it is clear from its structure how it may be modified to handle any other value of  $N$ . It is hoped that the way in which it was written is self-explanatory.

This procedure permits a complete search for the smallest integer coefficients that result in an acceptable fit, subject to the condition that all subspaces of  $b_1$  to  $b_N$  have a single minimum value of  $D$ . It can search a deep, narrow valley while avoiding regions too far from the valley to be fruitful. The procedure involves typically  $10^4$  to  $10^6$  evaluations of  $D$ , depending on the number of coefficients and the efficiency of the minimizing code.

### 3. Examples

The examples that follow were chosen largely on the basis of their usefulness to physicists and engineers. No claim is made that the approximating forms are the best that could be chosen, only that no smaller integer coefficients can be used in those forms to give a significantly better fit. In each example the approximation with integer coefficients has deviations that are within a factor of two of those that result from using the best fit coefficients (this point is illustrated in the figures).

In our experience an integer coefficient fit that approaches the best fit involves fewer keystrokes than a fit using smaller integers but one or two additional terms.

The approximations make considerable use of polynomial forms and these have been arranged according to Horner's rule to minimize the number of keystrokes and the need for intermediate storage.

### 3.1 Gaussian Probability Integral

The probability  $P(x)$  of exceeding  $x$  standard deviations of either sign is given by:

$$P(x) = \sqrt{2/\pi} \int_x^{\infty} e^{-x^2/2} dx \quad x \geq 0 \quad (3)$$

#### Approximation 1:

$$\tilde{P}_1(x) = \text{EXP} \left[ - \frac{(83x + 351)x + 562}{703/x + 165} \right] \quad (4)$$

$$\text{Error: } \left| \frac{\tilde{P}_1(x) - P(x)}{P(x)} \right| < 0.042\%$$

Range:  $0 < x < 5.5$  (Fig. 1) ( $1 > P(x) > 3.8 \times 10^{-8}$ )

Number of keystrokes<sup>6</sup> = 26.

#### Approximation 2:

$$\tilde{P}_2(x) = \sqrt{2/\pi} \left( \frac{1}{x} \right) \text{EXP} (-x^2/2 - 0.94/x^2) \quad (5)$$

$$\text{Error: } \left| \frac{\tilde{P}_2(x) - P(x)}{P(x)} \right| < 0.040\%$$

Range:  $x > 5.5$  (Fig. 2) ( $P(x) < 3.8 \times 10^{-8}$ )

Number of keystrokes<sup>6</sup> = 20.

In this example, the asymptotic limits

$$P(x) \approx \text{EXP}(-\sqrt{2/\pi} x), \quad x \text{ small} \quad (6a)$$

$$P(x) \approx \sqrt{2/\pi} \left(\frac{1}{x}\right) \text{EXP}(-x^2/2), \quad x \text{ large} \quad (6b)$$

suggested the form  $\tilde{P}(x) = \text{EXP}(p_n/p_{n-2})$ , where  $p_n$  is a polynomial (in  $x$ ) of degree  $n$  and  $p_{n-2}$  is a polynomial of degree  $n-2$ .

In the interval  $0 < x < 5.5$  the form  $\ln P(x) = p_5/p_3$  gave an excellent fit and it was possible to eliminate four terms to yield the form:

$$\ln[\tilde{P}_1(x)] = - \frac{((b_1x + b_2)x + b_3)x}{b_4x + b_5} \quad (7)$$

with best fit values<sup>7</sup> ( $b_1 \equiv 1$ ),  $b_2 = 4.20075 \pm 0.00020$ ,

$b_3 = 6.72175 \pm 0.00083$ ,  $b_4 = 1.988778 \pm 0.000075$ ,  $b_5 = 8.39964 \pm 0.00036$ .

This approximation  $\ln \tilde{P}_1(x)$  was fit to  $\ln P(x)$  on a set of 60 base points  $x_i$  from 0 to 5.5 with increments of 0.005 at the lower end increasing to increments of 0.25 at the upper end.

For  $b_1 = 1$  a search of the  $b_2, b_3, b_4, b_5$  integer space yielded no acceptable  $D$  values and the coefficients were scaled by setting  $b_1$  to

increasingly larger integer values. The best integer coefficient fit between  $b_1 = 1$  and  $b_1 = 99$  occurred at  $b_1 = 83$  (Eqn. 4).

In the interval  $x > 5.5$  the asymptotic form (Eqn. 6b) was modified slightly (Eqn. 5) to reduce the maximum deviation (Fig. 2). This single parameter fit was done by hand.

### 3.2 Inverse of the Gaussian Probability Integral

Defining  $P(x)$  by Eqn. 3:

#### Approximation 1:

$$\tilde{x}_1 = \sqrt{\frac{((4y + 100)y + 205)y^2}{(2y + 56)y + 192} + 131} \quad (8)$$

where  $y = -\ln(P)$

Error:  $|x_1 - x| < 1.3 \times 10^{-4}$

Range:  $1 > x > 2 \times 10^{-7}$  ( $0 < x < 5.2$ ) (Fig. 3)

Number of keystrokes<sup>6</sup> = 38.

The approximation was fit to 59 function values over the range

$1 > P > 1.5 \times 10^{-7}$

#### Approximation 2:

$$\tilde{x}_2 = \sqrt{\frac{((2y + 280)y + 572)y}{(y + 144)y + 603}} \quad (9)$$

where  $y = -\ln(P)$

Error:  $|x_2 - x| < 4 \times 10^{-4}$

Range:  $2 \times 10^{-7} > P > 10^{-112}$  ( $5.2 < x < 22.6$ ) (Fig. 4)

Number of keystrokes<sup>6</sup> = 30.

The approximation was fit to 26 function values over the range  
 $6 \times 10^{-7} > P > 1.5 \times 10^{-102}$ .

The forms used were

$$\tilde{x}_1 = \sqrt{\frac{((2b_1y + b_2)y + b_3)y^2}{(b_1y + b_4)y + b_5} + b_6} \quad (10)$$

with best fit coefficients<sup>7</sup> ( $b_1 \equiv 1$ ),  $b_2 = 48.8740 \pm 0.0024$ ,  
 $b_3 = 95.976 \pm 0.015$ ,  $b_4 = 27.4283 \pm 0.0016$ ,  $b_5 = 91.446 \pm 0.012$ ,  
 $b_6 = 61.231 \pm 0.025$ , and

$$\tilde{x}_2 = \sqrt{\frac{((2b_1y + b_2)y + b_3)y}{(b_1y + b_4)y + b_5}} \quad (11)$$

with best fit coefficients<sup>7</sup> ( $b_1 \equiv 1$ ),  $b_2 = 260.403 \pm 0.016$ ,  
 $b_3 = 503.60 \pm 0.62$ ,  $b_4 = 134.1596 \pm 0.0087$ ,  $b_5 = 543.36 \pm 0.37$ .

### 3.3 Klein-Nishina Formula

The Klein Nishina formula describes the narrow beam attenuation of photons by Compton scattering on free electrons.<sup>8</sup> Its exact expression is given by:

$$\sigma_{KN} = 2\pi r_0^2 \left[ \frac{((\alpha + 9)\alpha + 8)\alpha + 2}{\alpha^2(1 + 2\alpha)^2} + \frac{(\alpha - 2)\alpha - 2}{2\alpha^3} \ln(1 + 2\alpha) \right] \quad (12)$$

where  $\alpha = E/m_e c^2$  and  $r_0$  is the classical electron radius ( $2.8179 \times 10^{-13}$  cm). Although this expression may be evaluated directly, approximately 60 keystrokes are required.

Approximation 1:

$$\tilde{\sigma}_1 = K \frac{(E + 28) E + 16}{((E + 54) E + 134) E + 24} \quad (13)$$

E is the photon energy in MeV

$$K = 0.6000 Z/A \text{ (cm}^2/\text{gm)}$$

$$K = 0.9964 Z \text{ (barns/atom)}$$

$$K = 0.9964 \text{ (barns/electron)}$$

$$\text{Error: } \left| \frac{\tilde{\sigma}_1 - \sigma_{KN}}{\sigma_{KN}} \right| < 0.56\%$$

Range:  $0 < E < 100$  MeV. (Fig. 5)

Number of keystrokes<sup>6</sup> = 31, using the K values given in Table I.

The approximation was fit to 52 function values from  $E = 0$  to 100 MeV. It is sufficiently accurate for most practical purposes. Note that electron binding effects (not included in the Klein-Nishina formula nor even in most tables of Compton cross sections) can be of the order of several percent.<sup>9</sup>

The form used was

$$K \frac{(E + b_1)E + b_2}{((E + b_3)E + b_4)E + b_5} \quad (14)$$

with best fit coefficients<sup>7</sup>  $K = 0.9667 \pm 0.0020$  barns/electron

$$b_1 = 23.718 \pm 0.098, \quad b_2 = 13.186 \pm 0.050, \quad b_3 = 45.62 \pm 0.22,$$

$$b_4 = 108.08 \pm 0.64, \quad b_5 = 19.206 \pm 0.097.$$

Approximation 2:

$$\tilde{\sigma}_2 = \frac{1 + \ln(y^2)}{2y} \quad (15)$$

where  $y = E/0.2555$  MeV.

$$\text{Error: } \left| \frac{\tilde{\sigma}_2 - \sigma_{KN}}{\sigma_{KN}} \right| < 0.60\%$$

Range:  $E > 100$  MeV.

Number of keystrokes<sup>6</sup> = 21, using the K values given in Table I. This approximation is the well-known high energy limit of Eqn. (12).

### 3.4 Photoelectric cross sections in H<sub>2</sub>O, Bone, Fe, NaI, and Pb

It is well known that the photoelectric cross sections may be approximated by expansions in inverse powers of the photon energy,<sup>9</sup> and we have used the same form for our approximations (Table II). Each of these approximations was fit to typically 25 data points from Ref. 9. The deviations  $d(x_j)$  (not shown) are not smooth functions of  $x_j$  because the data are partially based on experimental measurements. Moreover, as stated in Ref. 9 these cross sections have not been established with accuracies much better than 5%.



### 3.5 Pair production cross section in Pb

Approximation:

$$\tilde{\sigma}_p \text{ (cm}^2\text{/gm)} \approx \frac{(x^2 + 9)x - 1}{((4x + 55)x - 168)x + 358} \quad (16)$$

where  $x = \log_{10}(E)$  and  $E$  is the photon energy in MeV.

Error:  $|\tilde{\sigma}_p - \sigma_p| < 8 \times 10^{-4} \text{ cm}^2\text{/gm}$

Range:  $1.5 \text{ MeV} < E < 10^5 \text{ MeV}$  (Fig. 7).

Number of keystrokes = 28.

The approximation  $\tilde{\sigma}_p$  was fit to 26 data points  $\sigma_p$  from 1.5 MeV to  $10^5$  MeV (Ref. 9). The deviations  $d(x_j)$  are not smooth functions of  $x$  because the data were only given to three significant figures. Also, the lower energy data points are quite sparse and the error bound given above is only an estimate.

#### Acknowledgements

I am grateful to P. Concus and B. Pardoe for helpful discussions.  
Work supported by the U.S. National Institutes of Health.

Table I. K values for the Klein-Nishina approximation (Eqn. 9)

Material	Z	K = 0.6Z/A (cm <sup>2</sup> /gm)	K = 0.9964Z (barns/atom)	density (gm/cm <sup>3</sup> )
H <sub>2</sub>	1	0.5953	0.996	0.0763 (0.070) <sup>a</sup>
C	6	0.2997	5.98	2.25
H <sub>2</sub> O	—	0.3330	—	1
N	7	0.2999	6.97	—
O	8	0.3000	7.97	—
Al	13	0.2891	12.95	2.692
Si	14	0.2991	13.95	2.4
Ar	18	0.2704	17.93	1.65 (1.40) <sup>a</sup>
Ca	20	0.2994	19.92	1.54
Fe	26	0.2793	25.90	7.86
Ge	32	0.2645	31.88	5.4
NaI	—	0.2563	—	3.67
Xe	54	0.2468	53.8	3.5 (3.06) <sup>a</sup>
W	74	0.2415	73.7	19.3
Pb	82	0.2375	81.7	11.35
U	92	0.2319	91.7	18.7

<sup>a</sup>Solid (liquid)

Table II. Photoelectric cross sections.

Material	Density (g/cm <sup>3</sup> )	Energy range (MeV)	Photoelectric cross section (cm <sup>2</sup> /gm) <sup>c</sup>	Error <sup>d</sup>		Keystrokes <sup>e</sup>
				e <sub>1</sub> (%)	e <sub>2</sub> (cm <sup>2</sup> /gm)	
H <sub>2</sub> O	1	> 0.01	$((x/7+35)x-97)x+196)x/10^7$	1.1	$1 \times 10^{-5}$	23
Compact bone	~ 2	> 0.01	$((20x-31)x+52)x/10^6$	2.7	$1.5 \times 10^{-5}$	18
Fe	7.87	> 0.01	$((-x/23+22)x-11)x+25)x/10^5$	2.9	$2 \times 10^{-5}$	24
NaI	3.67	> 0.0332 <sup>a</sup>	$((-x+107)x+62)x+82)x/7 \times 10^4$	2.8	$2 \times 10^{-5}$	23
Pb	11.35	0.0159-0.088 <sup>b</sup>	$((-x/26+8)x+32)x^2/10^4$	1.4	0	19
Pb	11.35	> 0.088 <sup>a</sup>	$((-x+52)x+93)x+42)x/10^4$	1.4	$1 \times 10^{-5}$	21

<sup>a</sup>above K edge.

<sup>b</sup>between L1 and K edges.

<sup>c</sup> $x = 1/E$  in units MeV<sup>-1</sup>.

<sup>d</sup>Error is the greater of e<sub>1</sub> and e<sub>2</sub>.

<sup>e</sup>Ref. 6.

APPENDIXEXAMPLE OF INTEGER COEFFICIENT SEARCH PROCEDURE FOR 4 COEFFICIENTS

```

A1 = 1
DETERMINE BEST FIT VALUES OF A2,A3,A4 BY MINIMIZING D

RENAME A1,A2,A3,A4 SO THAT A1 IS CLOSEST TO ZERO

LOOP B1=1,2,3 . . . . .
B2=A2·B1/A1
B3=A3·B1/A1
B4=A4·B1/A1
D'=D EVALUATED AT B1, [B2], [B3], [B4]

LOOP J2=0,1
LOOP I2=1,2,3,. . . . .
E1=B1
*E2 = [B2] + J2 + (-1)J2 · I2
**MINIMIZE D BY VARYING PARAMETERS E3 AND E4 (HOLDING E1 AND E2 FIXED)
***IF D > 1.2 D', EXIT I2 LOOP AND TAKE NEXT J2

LOOP J3=0,1
LOOP I3=1,2,3,. . . . .
F1=E1
F2=E2
F3 = [E3] + J3 + (-1)J3 · I3
MINIMIZE D BY VARYING PARAMETER F4 (HOLDING F1, F2, AND F3 FIXED)
IF D > 1.2 D', EXIT I3 LOOP AND TAKE NEXT J3

```

(CONTINUED)

\*For J2=0, the I2 loop sets E2 to successive values [B2] + 1, [B2] + 2, . . . . where [B2] is the integer part of B2. For J2 = 1 the I2 loop sets E2 to successive values [B2], [B2]-1, [B2]-2, . . . .

\*\*For efficiency, the starting values E3 and E4 are determined whenever possible by a linear extrapolation of previous best fit values of E3 and E4 (obtained at this point in the code) as a function of E2. Moreover, this minimization can be skipped if the D value associated with E1, E2 and the extrapolated values of E3 and E4 is less than D'. Similar efficiencies are employed for the minimizations in all the other loops.

\*\*\*This test assumes that the preceding step has found the true minimum rather than a local minimum. If the minimum D is greater than 1.2 D' then this value of E2 (and all subsequent values) define a subspace within which D > 1.2 D', and the I2 loop is ended.

```
LOOP J4=0,1
LOOP I4=1,2,3,.....
G1=F1
G2=F2
G3=F3
G4 = [F4] + J4 + (-1)J4 · I4
CALCULATE D USING PARAMETERS G1, G2, G3, AND G4
IF D > 1.2 D', EXIT I4 LOOP AND TAKE NEXT J4
If D ≤ 1.2 D', SET D'=D AND PRINT OUT ALL RELEVANT QUANTITIES
ASSOCIATED WITH THIS FIT

NEXT I4
NEXT J4
NEXT I3
NEXT J3
NEXT I2
NEXT J2
STOP PROCEDURE IF D' IS SUFFICIENTLY SMALL
NEXT B1
```

References

1. C. Hastings, Approximations for Digital Computers, Princeton University Press, Princeton, NJ (1955).
2. J. F. Hart et al, Computer Approximations, Wiley Press, New York, N.Y. (1968).
3. For cases where the absolute error  $g(x)-f(x)$  is important,  $w(x)$  is equal to a constant. For cases where the relative error  $\frac{g(x)-f(x)}{f(x)}$  is important,  $w(x)$  is proportional to  $1/f(x)$ . Note that the overall normalization of  $d(x)$  is arbitrary.
4. Stephen Derenzo, Lawrence Radiation Laboratory Group A Programming Note P-190, Berkeley (1969). Most other minimizing codes can also be used.
5. Henceforth in this paper the term best fit coefficients is also taken to mean those that result from the minimization of  $D$ .
6. The number of keystrokes varies somewhat from calculator to calculator. In estimating the number of keystrokes we have assumed that (i) the independent variable has already been entered, (ii) storage, retrieval and the functions  $1/x$ ,  $x^2$ ,  $\log_{10}x$ ,  $e^x$ , and  $\sqrt{x}$  may be performed in a single keystroke, and (iii) expressions may be evaluated using a register stack and reverse Polish notation (on calculators with parenthesis, one or two additional keystrokes may be required).
7. The number given after the  $\pm$  symbol is the amount that the corresponding coefficient must be varied from its best fit value to double the value of  $D$ , holding all other coefficients constant.

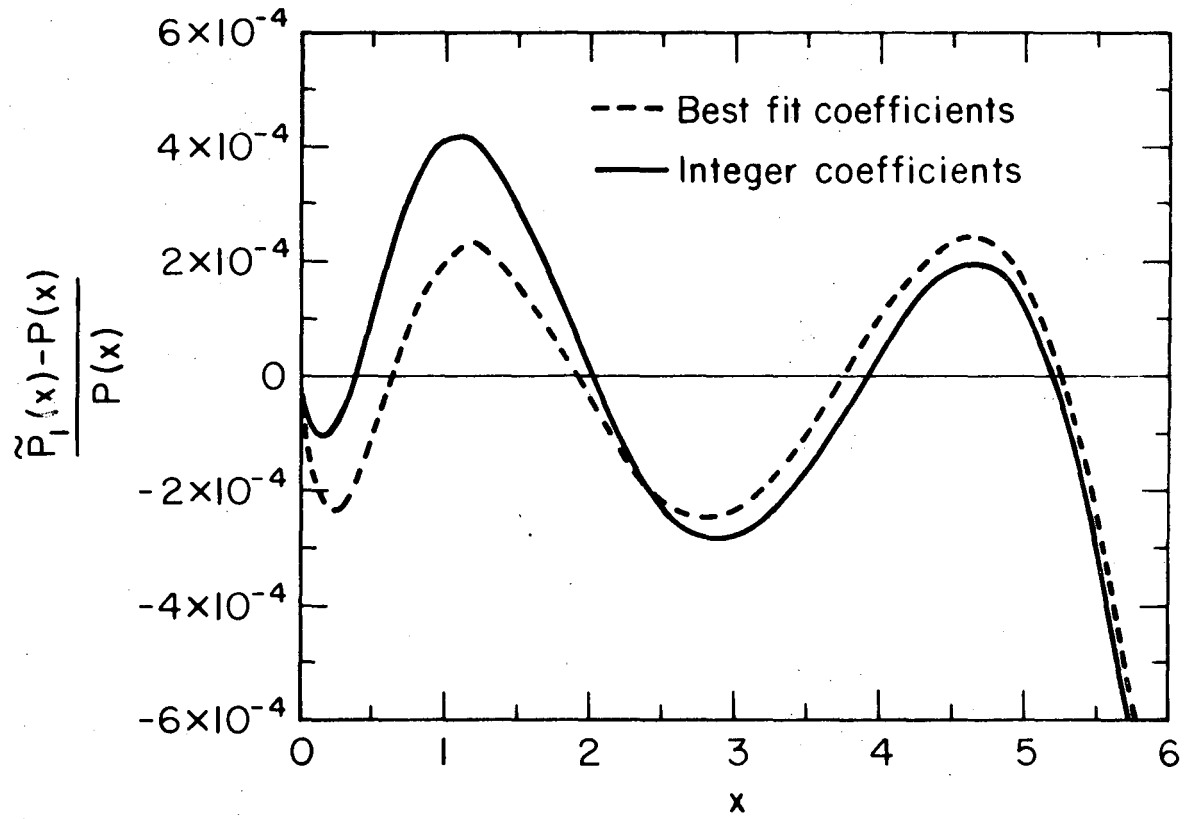
8. O. Klein and Y. Nishina, *Nature* 122, 398 (1928). Formulas are also available in the American Institute of Physics Handbook, 3rd edit., D. E. Gray edit., McGraw-Hill, New York (1972), p. 8-197.
9. J. H. Hubbell, Photon Cross Sections, Attenuation Coefficients, and Energy Absorption Coefficients from 10 keV to 100 GeV, U.S. National Bureau of Standards report number NSRDS-NBS 29 (1969).

Figure Captions

1. Solid curve- relative deviation between approximation  $\tilde{P}_1(x)$  with integer coefficients (Eqn. 4) and the Gaussian probability  $P(x)$  of exceeding  $x$  standard deviations of either sign (Eqn. 3). Dashed curve- same but with best fit coefficients (Eqn. 7).
2. Relative deviation between approximation  $\tilde{P}_2(x)$  (Eqn. 5) and the Gaussian probability  $P(x)$  of exceeding  $x$  standard deviations of either sign (Eqn. 3).
3. Solid curve- deviation between approximation  $\tilde{x}_1$  with integer coefficients (Eqn. 8) and the number of standard deviations associated with the Gaussian probability  $P(x)$  of exceeding  $|x|$ . Dashed curve- same but with best fit coefficients (Eqn. 10).
4. Solid curve- deviation between approximation  $\tilde{x}_2$  with integer coefficients (Eqn. 9) and the number of standard deviations associated with the Gaussian probability  $P(x)$  of exceeding  $|x|$ . Dashed curve- same but with best fit coefficients (Eqn. 11).
5. Solid curve- relative deviation between approximation  $\tilde{\sigma}_1$  with integer coefficients (Eqn. 13) and the Klein-Nishina formula (Eqn. 12). Dashed curve- same but with best fit coefficients. For descending values of photon energy below 0.01 MeV both curves descend monotonically. At  $E = 0$  the dashed curve reaches  $-2.3 \times 10^{-3}$  and the solid curve reaches  $-1.5 \times 10^{-3}$ .
6. Relative deviation between approximation  $\tilde{\sigma}_2$  (Eqn. 15) and the Klein-Nishina formula (Eqn. 12).

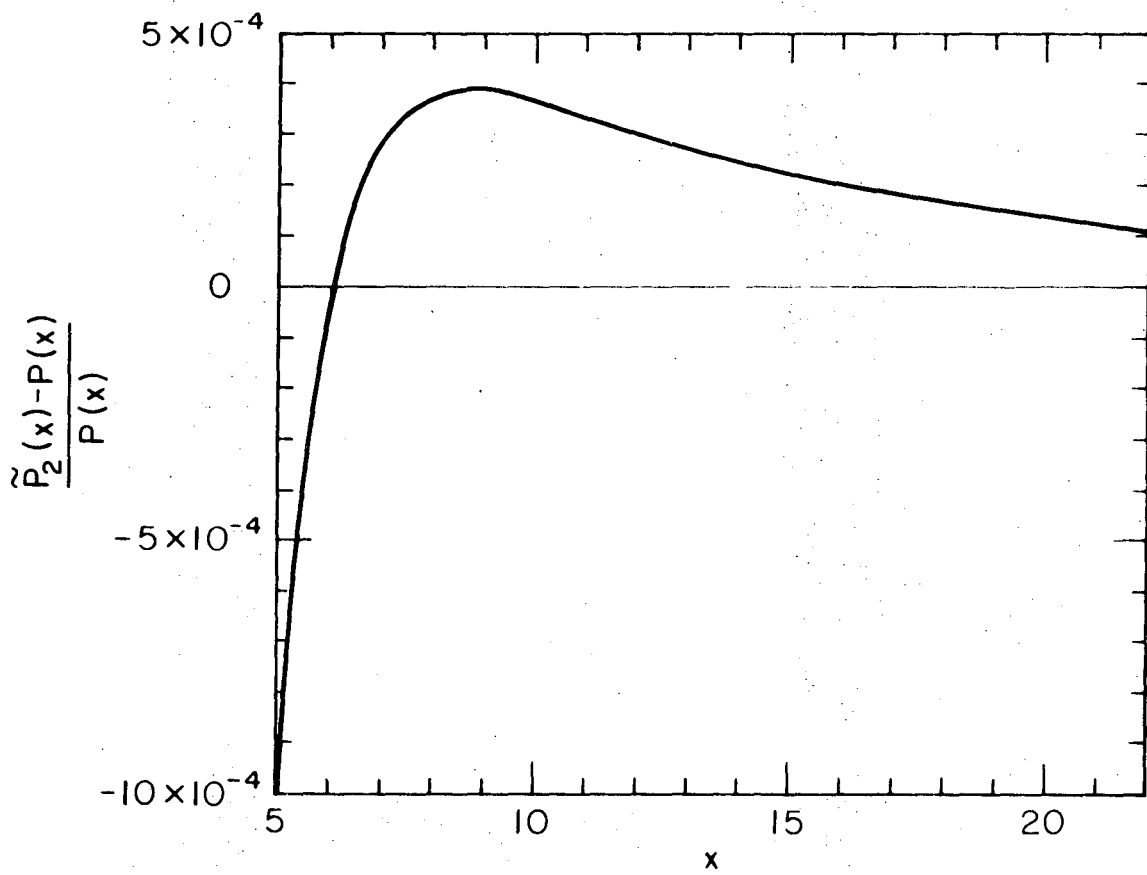


7. Deviation between approximation  $\tilde{\sigma}_p$  with integer coefficients (Eqn. 16) and the pair production cross section in Pb for 26 data points (Ref. 9). The curve is provided to guide the eye.



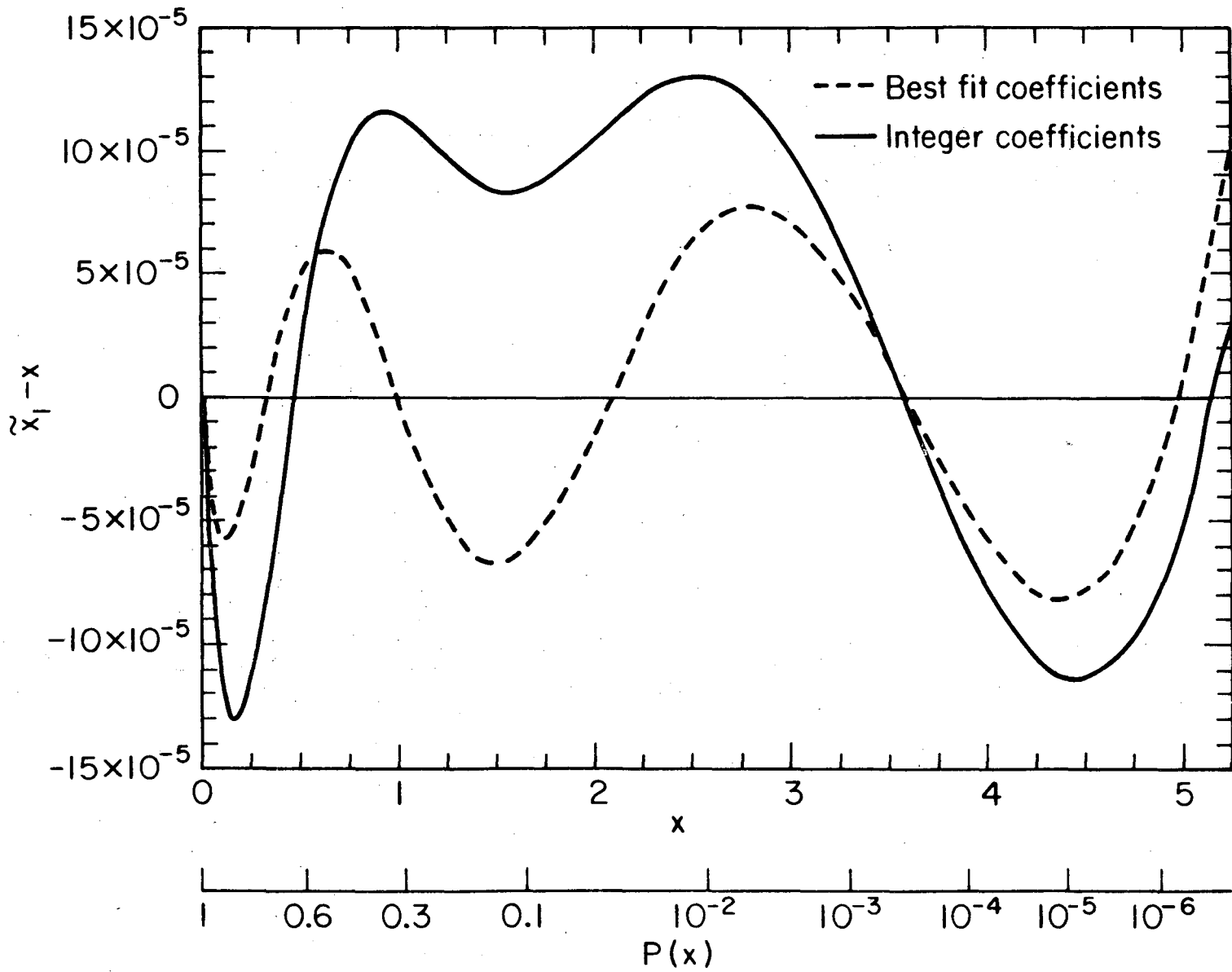
XBL754-2736

Fig. 1



XBL754-2737

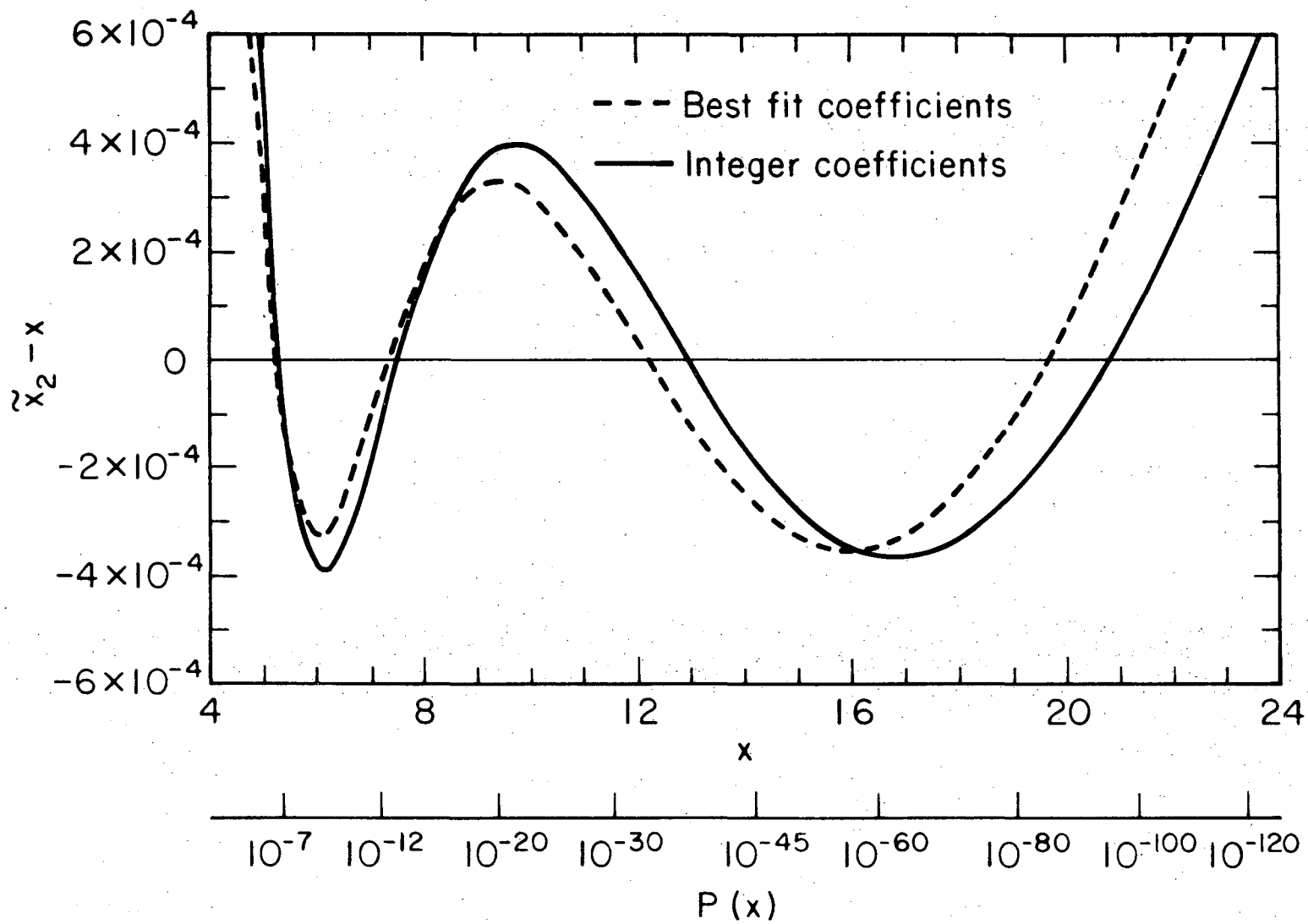
Fig. 2



XBL 754-2738

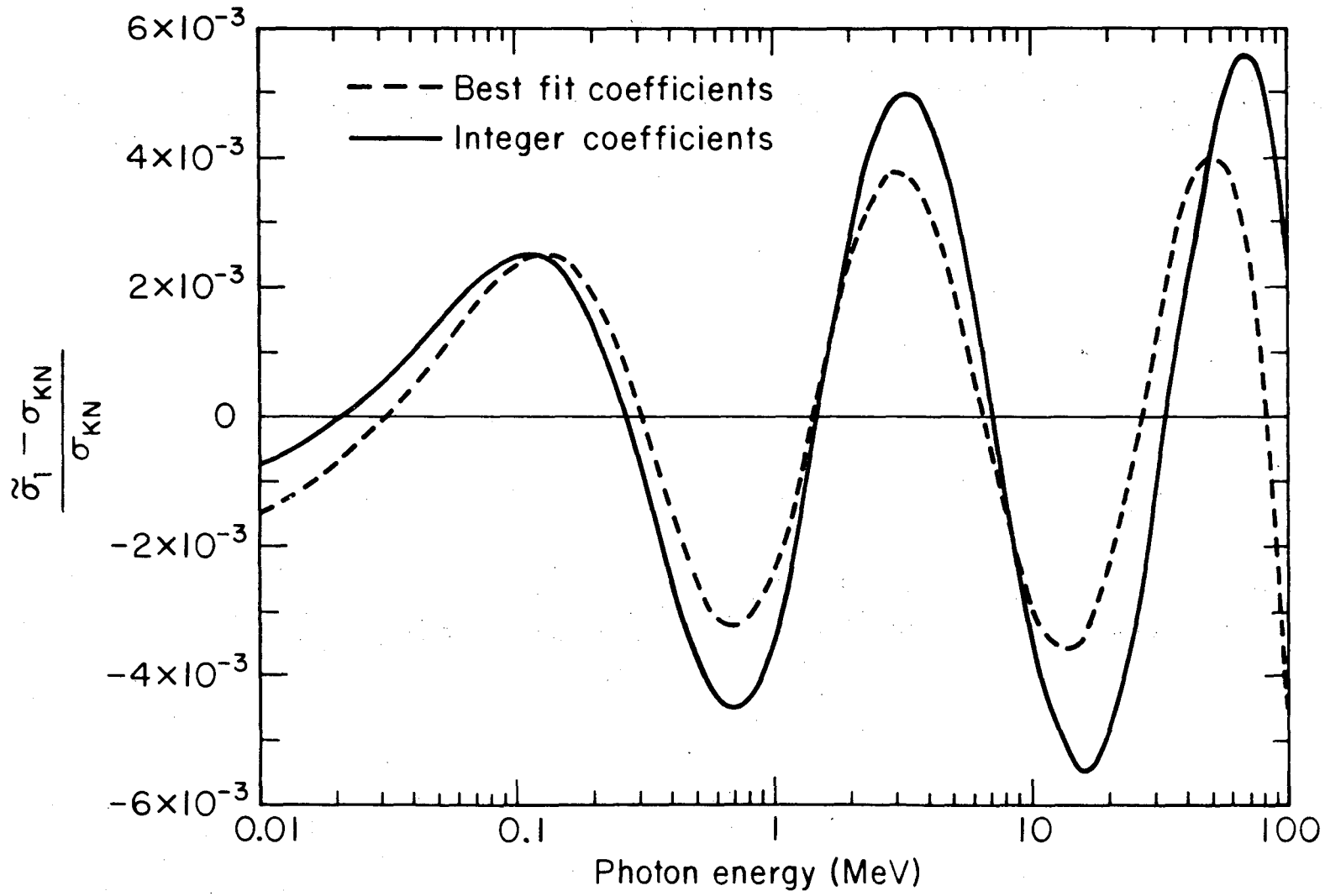
Fig. 3

00004303555  
-25-



XBL 754-2741

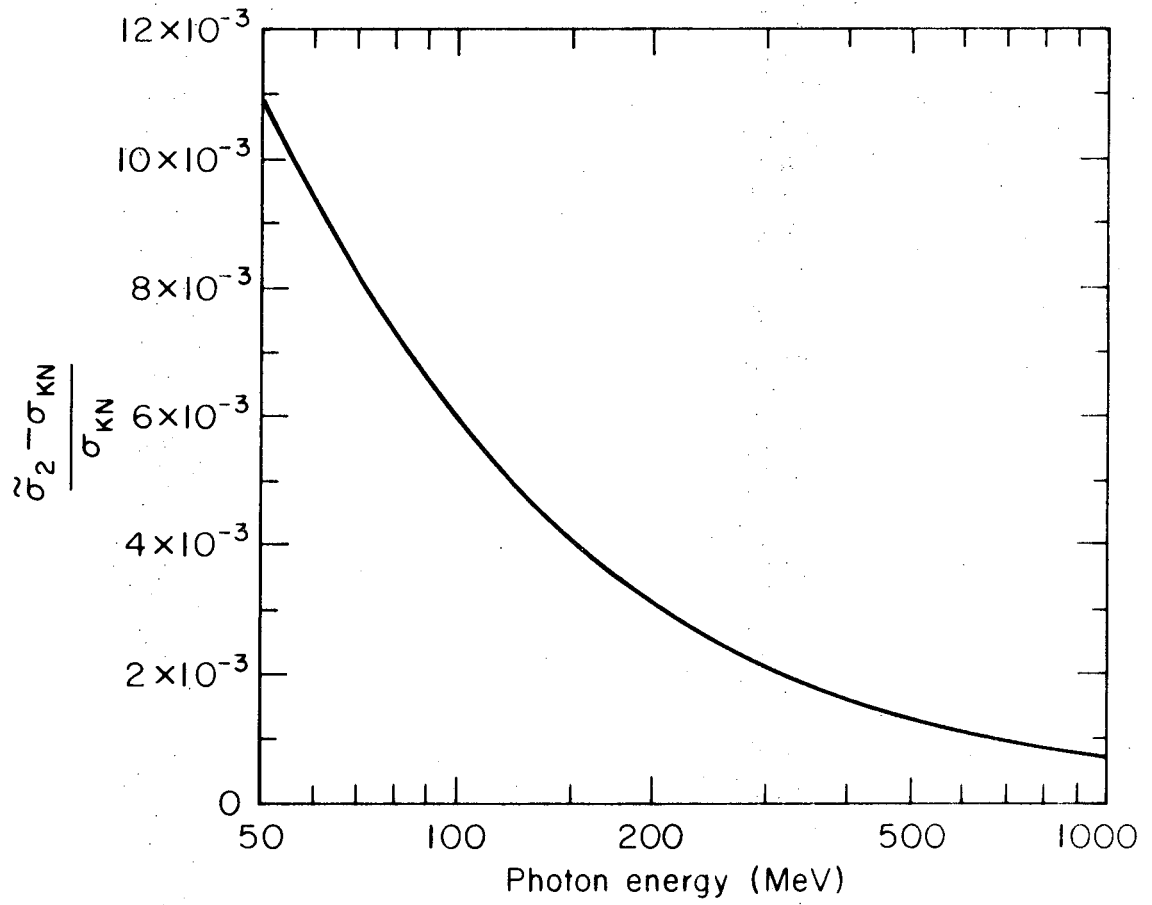
Fig. 4



XBL 754-2742

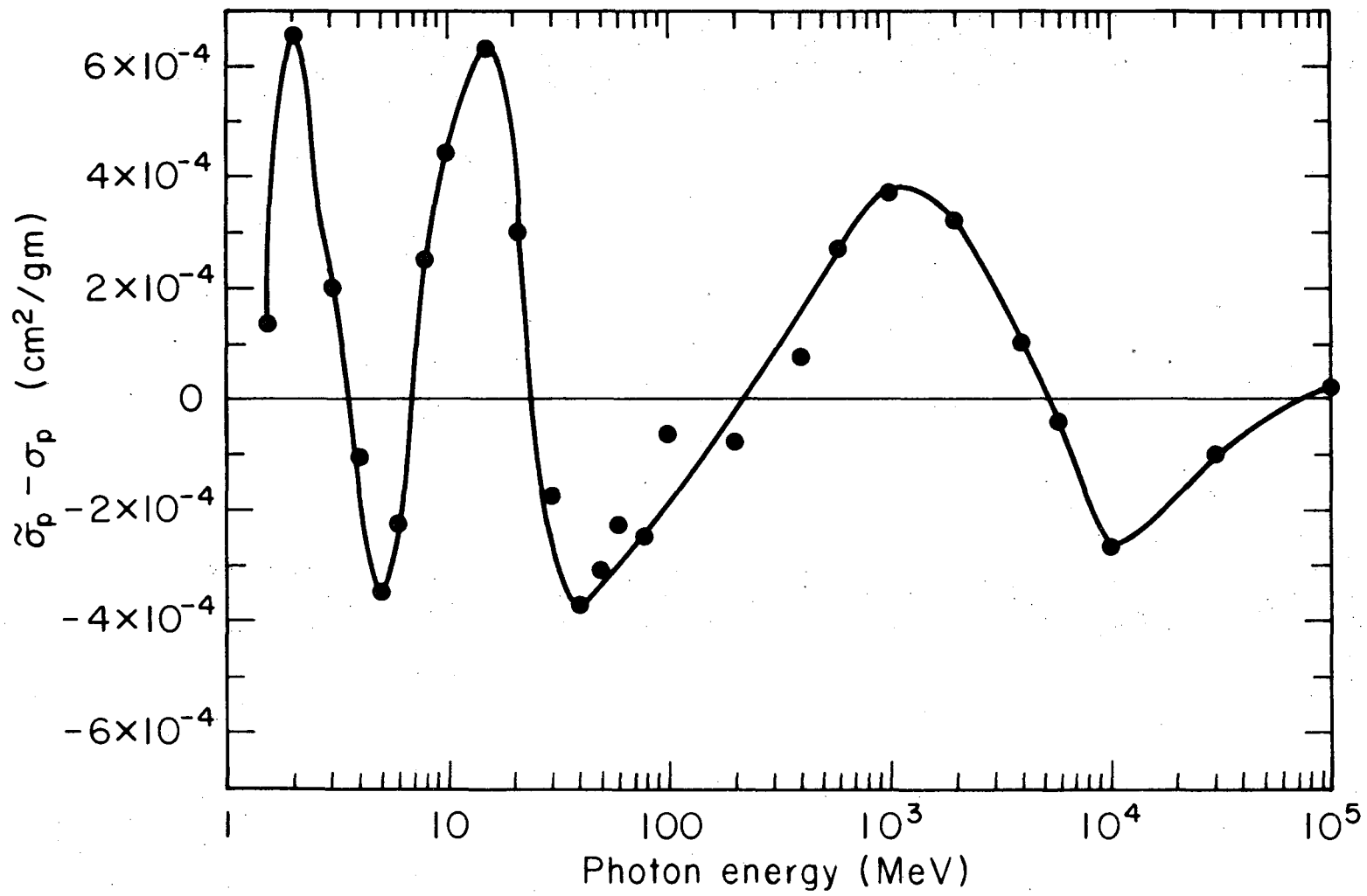
Fig. 5

00004303556  
 -27-



XBL 754 - 2740

Fig. 6



XBL754-2739

Fig. 7

00004303557  
-29-



**LEGAL NOTICE**

*This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.*

TECHNICAL INFORMATION DIVISION  
LAWRENCE BERKELEY LABORATORY  
UNIVERSITY OF CALIFORNIA  
BERKELEY, CALIFORNIA 94720