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Motion due to Dynamic Density Constraints

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by

Brent Alan Woodhouse

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ABSTRACT OF THE DISSERTATION

Motion due to Dynamic Density Constraints

by

Brent Alan Woodhouse

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2018

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We consider the continuity equation for a population density subject to (i) a density upper-bound that depends on space and time and (ii) a velocity that minimizes the kinetic energy. A solution is constructed via the Wasserstein minimizing movement scheme for a corresponding time-dependent energy. Motion of the solution is driven by a decreasing density constraint.

With a few assumptions, we prove this solution moves according to a free boundary problem of modified Hele-Shaw type that depends on the density constraint. In order to do this, we utilize a modified porous medium equation as an approximation to the original problem. Viscosity solution arguments are used to prove that given a decreasing density constraint, the porous medium equation solutions converge to the Hele-Shaw free boundary problem solution. By analyzing the Wasserstein gradient flow structure of the time-dependent energies involved, we next show that the porous medium equation solutions also converge to a solution of the original problem, thus identifying it with the Hele-Shaw description.

In addition, we consider complications of the analysis without each assumption, perform numerical simulations supporting the results, and explore some limiting situations of the dynamics.

The dissertation of Brent Alan Woodhouse is approved.

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PUBLICATIONS

Inwon Kim, Norbert Pozar, Brent Woodhouse. Singular Limit of the Porous Medium Equation with a Drift. Accepted by *Advances in Mathematics*, January 2018.

F. Blanchet-Sadri and Brent Woodhouse, “Strict Bounds for Pattern Avoidance.” Theoretical Computer Science, Vol. 506, 2013, Pages 17 - 28.

Chapter 1

Introduction

1.1 The Density Constraint Model

Consider a large number of stationary individuals spread out in space, sampled from some initial probability density ρ_0. Suppose the system of individuals moves so as to minimize the sum of their kinetic energies. With nothing acting on the system, it will remain stationary in time. However, due to environmental factors, we impose a hard density constraint $m(x,t)$ on the population, which can be dependent on both space and time. When m is small enough, this constraint forces the individuals to spread out further in space in a manner that minimizes their kinetic energy.

It is more convenient from a partial differential equations point of view to take a low resolution approach to this problem (ignoring individuals) and consider a population density $\rho(x,t) : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$, starting from a given initial density ρ_0 with total mass 1, and satisfying both the density constraint

$$\rho(x,t) \leq m(x,t) \tag{1.1}$$

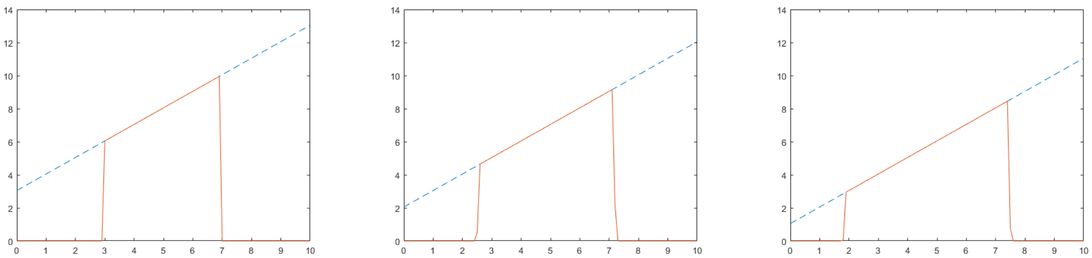
and (weakly) the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{1.2}$$

where $v(x,t)$ is chosen so that for almost every time t, it minimizes the kinetic energy at time t:

$$v(\cdot,t) = \arg\min_u \int_{\mathbb{R}^d} [\rho u^2](x,t)\, dx$$

over all velocities u that do not cause immediate violation of the density constraint (1.1). (See Definition 2.1 for a rigorous description of the solution.) We refer to this system as the



(a) $t = 0$ (b) $t = 1$ (c) $t = 2$

Figure 1: Density constraint (dotted line) and solution of the DCM (solid graph) changing over time.

density constraint model (DCM).

See Figure 1 for a simple one-dimensional example, in which $\rho_0(x) = m(x, 0)\chi_{[3,7]}$ and the density constraint $m = 3 + x - t$ decreases over time and causes expansion of the support. For some two-dimensional examples, see Appendix C.

A similar problem has been previously studied by Bertrand Maury, Aude Roudneff-Chupin, and Filippo Santambrogio ([MRS]) with a uniform density constraint $m \equiv 1$, in which case a change of density is not driven by a decreasing density constraint, but by a prescribed drift $\Psi(x)$ for individuals. In that case the continuity equation (2.4) is replaced by

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$

where v is chosen to minimize

$$\int_{\mathbb{R}^d} [\rho(v - \Psi)^2](x, t).$$

Allowing a density constraint that varies in both space and time could be combined with a drift and source term, but we prefer to isolate it and study its effects separately. The goal is to obtain a better understanding of how the shape of the density constraint affects the dynamics of our solution. In describing a solution of the DCM theoretically and numerically, we utilize both viscosity solution theory and optimal transport theory (extending the approach and results of [AKY]). The strategy is to utilize an appropriate limit of solutions of carefully chosen porous medium equations. In both viscosity solution theory and optimal transport theory, the varying density constraint causes additional challenges that push the limits of the approach, requiring additional assumptions and motivating new techniques.

1.2 Main results

Here we organize the main results with their major assumptions. Note the corresponding statements later in the text often involve a few more technical details than what is given here.

We will reference the following assumptions on the density constraint m:

(M1) m is continuously differentiable in space and time, and there exist constants M_-, M_+, and α such that

$$0 < M_- \leq m \leq M_+, \quad |\partial_t m| \leq \alpha.$$

(M2) m is decreasing over time ($\partial_t m < 0$)

(M3) There exists k_0 such that for all $k \geq k_0$, the function $(m(\cdot, t))^{-k}$ is convex in $\mathbb{R}^d$ for all $t \geq 0$.

Remark 1.1. (i) Assumption (M2) ensures that the solution of (FB-M) in Theorem 1.2 is well-defined. See Section 5.2 for further discussion of a free boundary problem description without this assumption.

(ii) Assumption (M3) is the condition necessary for λ-convexity along some generalized geodesic of the porous medium equation energies, which ensures uniqueness and convergence of the corresponding minimizing movement scheme, as well as a desirable rate of convergence. We expect Theorem 1.2 holds even without this assumption (and numerical simulations support this), but approximation via the porous medium equation appears to require it. Note (M3) holds when for all large enough k,

$$\Delta(m^{-k}) = km^{-k-2}((k+1)|\nabla m|^2 - m\Delta m) \geq 0.$$

In particular (M3) fails to hold when, at some point, $|\nabla m| = 0$ but $\Delta m > 0$, in which case m has a strict local minimum. A local maximum is certainly acceptable.

We work in $\mathcal{P}_2(\mathbb{R}^d)$, the set of probability measures on $\mathbb{R}^d$ endowed with the 2-Wasserstein distance. In order to come up with a solution of the DCM, we define the following energy functional on measures.

$$E_\infty(\rho,t) = I_{K_t}(\rho) := \begin{cases} 0 & \text{if } \rho \in K_t \\ +\infty, & \text{if } \rho \notin K_t, \end{cases}$$

with

$$K_t = \{\rho \in \mathcal{P}_2(\mathbb{R}^d) \text{ with } \rho(x) \le m(x,t) \text{ for a.e. } x\}.$$

Given a time-step τ, let $\rho_\tau^0 = \rho_0$ and for $n \ge 1$, iteratively define

$$\rho_\tau^n \in \operatorname*{arg\,min}_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} \left[E_\infty(\rho, \tau n) + \frac{1}{2\tau} W_2^2(\rho, \rho_\tau^{n-1}) \right] = \operatorname*{arg\,min}_{\rho \in \mathcal{P}_2(\mathbb{R}^d),\, \rho \in K_{\tau n}} W_2^2(\rho, \rho_\tau^{n-1}). \tag{1.3}$$

Note we must have $\rho_\tau^n \le m(\cdot, \tau n)$. This is the analogue of the (traditionally time-independent) minimizing movement scheme for the time-dependent energy E_∞.

The first main result is the following.

Theorem 1.1. *Let ρ_0 be a probability measure with compact support. Assuming (M1), the ρ_τ^n are well-defined and narrowly converge to a solution ρ_∞ of the DCM, such that for each point (x,t), either $\rho_\infty(x,t) = m(x,t)$ or $\rho_\infty(x,t) = \rho_0(x,t)$.*

Note at this point we have a theoretical solution, but it would be ideal to describe its behavior more directly. To this end, we show the following free boundary problem for a pressure $p(x,t)$ provides a dynamic description of this solution to the DCM when $\rho_0 = m(\cdot, 0)$:

$$\begin{cases} -\nabla \cdot (m \nabla p) = -\partial_t m & \text{in } \{p(\cdot,t) > 0\} \\ V = |\nabla p| & \text{on } \partial\{p(\cdot,t) > 0\}. \end{cases} \tag{FB-M}$$

Here V is the outward normal velocity of $\{p(\cdot,t) > 0\}$. Due to possible issues of regularity of $\partial\{p(\cdot,t) > 0\}$, (FB-M) must be interpreted in the viscosity solution sense.

Note

$$-\nabla \cdot (m \nabla p) = -m \Delta p - \nabla m \cdot \nabla p$$

is uniformly elliptic by (M1). Also, assumption (M2) implies the solution p to

$$-\nabla \cdot (m\nabla p) = -\partial_t m$$

with zero boundary conditions is indeed positive on any domain.

Here is the second major result.

Theorem 1.2. *Assume (M1), (M2), and (M3). Let Ω_0 be a compact set with locally Lipschitz boundary such that $\rho_0 = m(\cdot, 0)\chi_{\Omega_0}$ is a probability measure. There exists a family of sets Ω_t such that any viscosity solution p of (FB-M) starting with $\{p(\cdot, 0) > 0\} = \{\rho_0 = m(\cdot, 0)\}$ shares the same set $\Omega_t = \overline{\{p(\cdot, t) > 0\}}$. The following is then a solution to the DCM:*

$$\rho_V(x, t) = m(x, t)\chi_{\Omega_t}.$$

In particular, (FB-M) describes the boundary velocity of the support of the solution of the DCM.

As in [AKY], to prove this theorem, we cannot directly link this free boundary problem with the DCM. Instead, the following modified porous medium equations provide an approximation to the free boundary problem and link the viscosity solution theory and the optimal transport theory.

Consider the porous medium equation for density $\rho_k : \mathbb{R}^d \times [0, T] \to \mathbb{R}$,

$$\partial_t \rho + \nabla \cdot (\rho(-\nabla p)) = 0, \tag{PME-M}$$

where $k > 1$ is given, the pressure is related by

$$p = P_k(\rho) := \frac{k}{k-1}\left(\frac{\rho}{m}\right)^{k-1},$$

and $\rho(\cdot, 0) = \rho_{0,k}$.

Using viscosity solution arguments as in [K], we can show convergence of the ρ_k to ρ_V from Theorem (1.2). Here (M2) is used because the replacement for (FB-M) when $m_t < 0$

in some places but $m_t > 0$ in others involves an obstacle problem for which current types of viscosity solution arguments are insufficient.

Theorem 1.3. *Suppose (M1), (M2), let Ω_0 be a compact set with locally Lipschitz boundary such that $\rho_{0,k} = \rho_0 = m(\cdot, 0)\chi_{\Omega_0}$ is a probability measure. With Ω_t and ρ_V as in Thm (1.2), as $k \to \infty$, the ρ_k converge to ρ_V locally uniformly in $\mathbb{R}^d \setminus \partial\Omega_t$ at each time $t > 0$.*

Instead on the optimal transport side, convergence of the minimizing movement scheme also holds in the 2-Wasserstein metric. In particular we need (M3) below to ensure convexity of the corresponding energies for ρ_k along generalized geodesics, so that convergence for the minimizing movement schemes for (PME-M) to a unique limit holds, and a useful rate of convergence can be attained, allowing the proof of the following.

Theorem 1.4. *Suppose (M1), (M3), let Ω_0 be a compact set such that $\rho_{0,k} = \rho_0 = m(\cdot, 0)\chi_{\Omega_0}$ is a probability measure. With ρ_∞ from Theorem 1.1, the ρ_k converge to ρ_∞ uniformly in time in 2-Wasserstein distance with convergence rate*

$$\sup_{t\in[0,T]} W_2(\rho_k(t), \rho_\infty(t)) \leq \frac{C(T, M_-, M_+, \alpha)}{k^{1/24}}.$$

Note Theorem 1.2 is an immediate corollary of Theorems 1.3 and 1.4.

Remark 1.2. In all of the results, the assumption that ρ_0 is a probability measure can be easily relaxed to allow ρ_0 to have any positive total mass.

We summarize the functions and limits involved with Figure 2; note the blue text and dotted lines also indicate the content of Chapters 2 - 4.

Overall, the primary contributions of this work are (i) identifying a new type of system with a hard density constraint in which solutions can be described by free boundary problems of Hele-Shaw type, (ii) pushing the boundaries of gradient flow ideas regarding time-dependent energies, especially those in which the domain changes over time, and in the process, (iii) exploring limitations of viscosity solution and optimal transport theory as applied to the new system.

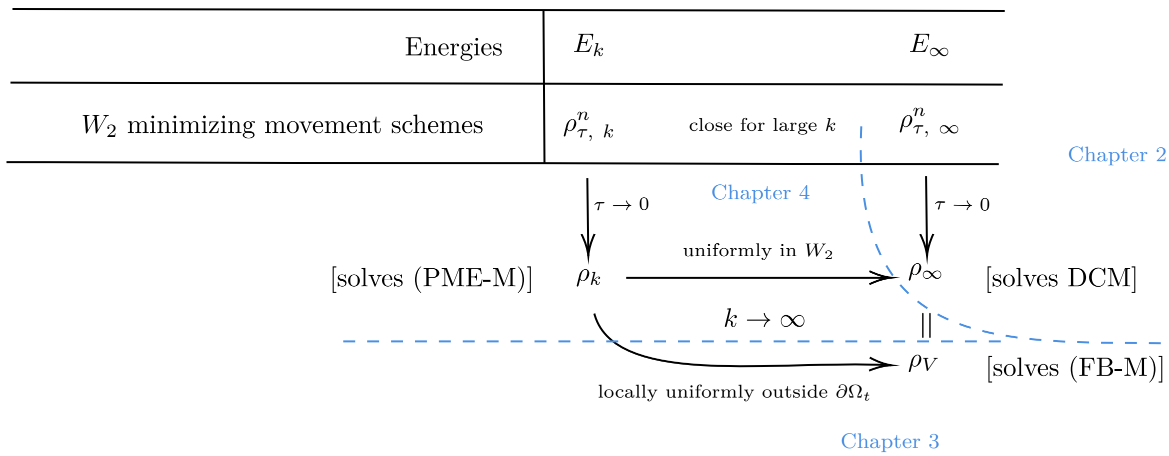


Figure 2: Primary functions and limits

1.3 Related literature

Most models of congestion phenomena use a form of soft congestion, modeling how to directly adjust velocity based on the density of nearby areas. Instead the density constraint model is an example of a hard congestion model as discussed in [MRSV], which starts from the core microscopic idea that two individuals may not occupy the same place at the same time and considers the macroscopic version thereof. The resulting velocity is a compromise between the density constraint and the desire to remain stationary.

Similarly [MMS] considers extensions of Moreau sweeping processes (in which particles are constrained to stay inside moving convex sets) to probability measures constrained by some moving boundary in space (as well as allowing other effects such as a hard density constraint $\rho \leq 1$). Instead of restricting the range of the density with some combination of $m = 0$ and $m = 1$, the DCM restricts the range of the density with the moving boundary m. (Note we do not consider $m = 0$ in this work, but a version of Theorem 1.1 allowing $m = 0$ on convex sets is feasible.) Solutions in [MMS] are achieved as limits of a prediction-correction or "catching-up" method, projecting back into the feasible set at each step. The scheme (1.3) used for the DCM can be viewed in the same manner.

The convergence as $k \to \infty$ of the porous medium equation to a Hele-Shaw type problem (sometimes called the "stiff pressure limit") was first considered in [CF], [EHKO], and in

[BC] on $\mathbb{R}^n$ with $m \equiv 1$. For the original porous medium equation ($\partial_t \rho = \Delta \rho^k$) with fixed boundary, when ρ_0 is a characteristic function of a compact set, see [GQ] and [K]. Similar convergence has been shown when the equation involves a source term ([PQV], [KP], [MPQ]) or a drift ([AKY], [KPW]) or even for aggregation-diffusion equations ([CKY]).

Much of the recent work in this area has been focused on extending these results to allow some sort of external density while describing the stiff pressure limit. When $m \equiv 1$ and external density is ρ^E, the outward normal velocity is adjusted from $V = f(p,x)$ to $V = \frac{f(p,x)}{1-\min(\rho^E,1)}$ to allow for external density. See [KP], [MPQ], [KPW]. For the DCM, this would allow more general initial data $\rho_0 \leq m$ and adjust the boundary velocity in (FB-M) from $V = |\nabla p|$ to $V = \frac{m|\nabla p|}{m-\rho_0}$, with

$$\rho_V = m\chi_{\Omega_t} + \rho_0 \chi_{\mathbb{R}^d \setminus \Omega_t}.$$

When ρ_0 touches m, "$V = \infty$" and instantaneous expansion or nucleation of a pressure region can take place. We expect the extension of Theorem 1.2 to allow external density also holds, but construction of the necessary barriers for large k has been ultimately unyielding due to the inhomogeneity in space of the term $\nabla \cdot (m(x,t)\nabla \rho))$ and insufficiency of approximation with radial solutions as was possible in [KP] and [KPW].

For some Hele-Shaw problems similar to (FB-M), regularity is fairly well-understood. See for instance [CJK] and its references. If the initial boundary is Lipschitz continuous with small Lipschitz constant, then for small uniform positive time the solution is smooth. We expect the same holds for (FB-M), though we do not pursue such regularity concerns here.

A few other authors have considered scenarios capturing some effects seen in (FB-M). The thesis of John DeIonno ([I]) considers a location-dependent source in a Hele-Shaw free boundary problem with $m \equiv 1$. If the source is f, then the pressure equation is $-\nabla p = f$. See [GS] for some discussion of generating Hele-Shaw motion by considering liquid between two plates that are pushed together over time (m decreasing over time but constant in space). Choosing $m = e^{-ct}$ for instance is roughly equivalent to the case of $f = -m_t/m = c$. In both of these scenarios one can to some degree describe the resulting support of the pressure using the Baiocchi transform $w = \int_0^t p(\cdot, s)\,ds$ and associated variational inequalities. Both [GS] and [GV] describe the same set as the variational inequalities using a concept from potential theory called "Balayage" (sweeping of measures without changing external potentials). The precise correlation between Balayage and Wasserstein projection is still unclear; a partial description can be found in the thesis of Aude Roudneff-Chupin ([R]), Section 5.2.

Finally, there are numerous studies of limits of JKO-type minimizing movement schemes for probability measures producing Wasserstein gradient flows and solutions to various PDEs. The primary reference here is [AGS]; however, its assumptions do not permit time-dependence of the energies. Thus when analyzing the minimizing movement schemes for (PME-M) and E_k with $k < \infty$, we turn to the more recent work [FV], which reproduces some of the same theory from [AGS] Chapters 2 - 4 but for time-dependent energies, discussing convergence and rate of convergence of the minimizing movement schemes given λ-convexity of the energy along generalized geodesics. Note the energy E_∞ does not fit the framework of [FV] due to time-dependence of the domain, so we provide arguments that reflect the specific energy under consideration in Chapter 2.

Chapter 2

A Solution of the Density Constraint Model

In this chapter, we analyze the minimizing movement scheme for E_∞ and prove Theorem 1.1, which says this minimizing movement scheme converges to a solution of the Density Constraint Model. To do so we build off of similar analysis from [MRS] with some new arguments.

First we restate the Density Constraint Model with a rigorous description of the velocity constraints. We look for a solution to the continuity equation

$$\rho_t + \nabla \cdot (\rho u) = 0$$

in some large convex set $\Omega \subset \mathbb{R}^d$ starting from ρ_0. Specifically, this should be interpreted in the weak sense, so for all $\varphi \in C_c^\infty(\Omega \times [0,T])$,

$$\int_0^T \int_\Omega (\partial_t \varphi + \nabla \varphi \cdot u)\rho \, dx + \int_\Omega \varphi(0,x)\rho_0 \, dx = 0. \tag{2.1}$$

However, we want to enforce $\rho \leq m$, thus

$$-\nabla \cdot (mu) = -\nabla \cdot (\rho u) = \rho_t \leq m_t \qquad \text{on } \{\rho = m\}$$

When u is smooth, multiplying by a test function in the pressure space

$$H^1_{\rho(\cdot,t)} = \{q \in H^1(\Omega), q \geq 0 \text{ a.e.}, \, q(x) = 0 \text{ a.e. in } \{\rho(\cdot,t) < m(\cdot,t)\}\},$$

integrating over Ω, and integrating by parts, we obtain

$$\int_\Omega mu \cdot \nabla q \leq \int_{\Omega_t} m_t q.$$

This calculation motivates the following feasible set (allowing an appropriate class of velocities which are not necessarily smooth):

$$C_{\rho,t} = \left\{ v \in (L^2(\Omega))^d, \int_\Omega mv \cdot \nabla q \le \int_\Omega m_t\, q \text{ for all } q \in H^1_{\rho(\cdot,t)} \right\}. \tag{2.2}$$

Definition 2.1. Consider a population density $\rho(x,t) : \Omega \times [0, \infty) \to \mathbb{R}$, starting from a given initial density $\rho_0 : \Omega \to \mathbb{R}$ with total mass 1. We say ρ is a solution of the Density Constraint Model if the density constraint

$$\rho(x,t) \le m(x,t) \tag{2.3}$$

is satisfied and the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{2.4}$$

holds in the weak sense of Eq. (2.1), where $v(x,t)$ is chosen so that for almost every time t, it minimizes the kinetic energy at time t:

$$v(\cdot, t) = \underset{u \in C_{\rho,t}}{\arg\min} \int_{\mathbb{R}^d} [\rho u^2](x,t)\, dx. \tag{2.5}$$

Here the minimum is over all velocities u that do not cause immediate violation of the density constraint $\rho \le m$, in the sense that $C_{\rho,t}$ contains all admissible velocities under consideration. (Note for each t the feasible set $C_{\rho,t}$ is a closed convex set, and as $m > 0$, the inner product $(u, v) = \int m(\cdot, t) u \cdot v$ gives rise to a Hilbert space, so the minimizer exists.)

This definition of a solution of the DCM is in the same spirit as the definition of the Wasserstein metric. To explain this, recall for probability measures ρ_0, ρ_1, and $\Pi(x, y)$ the collection of all transport plans between ρ_0 and ρ_1,

$$W_2(\rho_0, \rho_1) = \min \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\gamma(x,y) : \gamma \in \Pi(\rho_0, \rho_1) \right\}.$$

By the Benamou-Brenier formula, an equivalent formulation is

$$W_2(\rho_0, \rho_1) = \min_v \int_0^1 \int_{\mathbb{R}^d} |v_t(x)|^2 \, d\mu_t(x) \, dt,$$

where the minimum is over all velocities v_t such that (i) the integral above is well-defined and (ii) v_t interpolates between μ_0 and μ_1 in the sense that

$$\partial_t \mu_t + \nabla(v_t \mu_t) = 0 \qquad \text{on } \mathbb{R}^d \times [0, 1],$$

with $\mu_i = \rho_i$ for $i = 0, 1$. Note Eq. (2.5) has a similar interpretation for the quantity to be minimized, though working in $C_{\rho,t}$ is more restrictive.

We construct a solution to the DCM using a Wasserstein minimizing movement scheme for a specific energy. The feasible measures are

$$K_t := \{\rho \text{ a probability measure on } \Omega \text{ with } \rho(x) \leq m(x, t) \text{ for a.e. } x\}.$$

Define the energy

$$E_\infty(\rho, t) = I_{K_t}(\rho) := \begin{cases} 0 & \text{if } \rho \in K_t \\ +\infty, & \text{if } \rho \notin K_t, \end{cases}$$

We use the following minimizing movement scheme for E_∞. Given a time-step τ, let $\rho_\tau^0 = \rho_0$ and for $n \geq 1$, iteratively define

$$\rho_\tau^n \in \operatorname*{arg\,min}_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} \left[I_{K_{\tau n}}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_\tau^{n-1}) \right] = \operatorname*{arg\,min}_{\rho \in K_{\tau n}} W_2^2(\rho, \rho_\tau^{n-1}).$$

This enforces $\rho_\tau^n \leq m(\cdot, \tau n)$ with minimal movement in the W_2 sense.

For existence, consider a minimizing sequence $\rho^\ell \in K_{\tau n}$. The sequence has a uniform bound on $W_2(\rho^\ell, \rho_\tau^{n-1})$, so it is a tight sequence of probability measures. Thus there exists a subsequence of the ρ^ℓ that converges weakly to some probability measure ρ. Because $W_2(\cdot, \rho_\tau^{n-1})$ is lower semicontinuous, ρ achieves the desired minimum. Finally, taking limits of $\rho^\ell \leq \rho_\tau^{n-1}$ implies $\rho \leq \rho_\tau^{n-1}$ so that $\rho \in K_{\tau n}$.

See [DMSV] Section 5 for various properties of this type of Wasserstein projection. We state two of these properties in our context. The first provides a sense of efficiency for the Wasserstein projection.

Lemma 2.1 ([DMSV], Lemma 5.1)**.** *Let γ be the optimal plan from ρ_τ^n to ρ_τ^{n-1}. If $(x_0, y_0) \in spt(\gamma)$, then $\rho_\tau^n = m(\cdot, \tau n)$ a.e. in $B_R(y_0) \cap \Omega$, where $R = |y_0 - x_0|$.*

Based on the above lemma, the ρ_τ^n are saturated wherever ρ_0 is adjusted.

Lemma 2.2 (Immediate from [DMSV], Proposition 5.2)**.** *Let $\Omega_0 = \{\rho_0 = m(\cdot, 0)\}$, so that*

$$\rho_0 = m(\cdot, 0)\chi_{\Omega_0} + \rho_0 \chi_{\Omega \setminus \Omega_0}.$$

If we define

$$S_\tau^n = \{\rho_\tau^n = m(\cdot, \tau n)\},$$

then the S_τ^n are measurable, increasing in n, and we have uniqueness, specifically,

$$\rho_\tau^n = m(\cdot, \tau n)\chi_{S_\tau^n} + \rho_0 \chi_{\Omega \setminus S_\tau^n}.$$

Remark 2.1. A standard approach for gradient flows along the lines of [AGS] is insufficient here, since E_∞ is time-dependent and issues of convexity along generalized geodesics may arise. The recent article [FV] provides a similar framework for time-dependent energies, but it cannot handle a domain (K_t in this case) which is potentially changing in time. Thus we provide some alternative approaches that better fit the specific energy E_∞ under consideration.

Here is the main result of this section, corresponding to Theorem 1.1.

For the following, let t_τ^n be the unique optimal transport function from τ_τ^n to τ_τ^{n-1}, and define the discrete velocities $v_\tau^n = \dfrac{i - t_\tau^n}{\tau}$ and energies $E_\tau^n = \rho_\tau^n v_\tau^n$. Interpolate the discrete values $(\rho_\tau^n, v_\tau^n, E_\tau^n)_{n\geq 0}$ by the piecewise constant functions defined by

$$\begin{cases} \rho_\tau(t,\cdot) = \rho_\tau^n \\ v_\tau(t,\cdot) = v_\tau^n \quad \text{ if } t \in ((n-1)\tau, n\tau] \\ E_\tau(t,\cdot) = E_\tau^n \end{cases} \tag{2.6}$$

Theorem 2.3. *Assume (M1), which uses parameters α and M_-, and fix a final time $T > 0$. Let ρ_0 be a probability measure supported in $B_R(0)$ and take $\Omega = B_{3KR}(0)$ with K a constant dependent on T, α, M_-, d as explained in Lemma 2.6. Suppose (ρ_τ^n) is constructed following the above miniminizing movement scheme in Ω with energy E_∞ and the W_2 metric. Then there exists a family of probability densities $(\rho(\cdot,t))_t$ and a family of velocities $(u(\cdot,t))_t$ such that $(\rho_\tau(\cdot,t), E_\tau(\cdot,t))$ narrowly converges to $(\rho(\cdot,t), \rho(\cdot,t)u(\cdot,t))$ for a.e. $t \in [0,T]$. Moreover, (ρ, u) satisfies the continuity equation (in the weak sense of (2.1)):*

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) &= 0 \quad \text{on } \Omega\times[0,T], \\ \rho(\cdot,0) &= \rho_0 \quad \text{on } \Omega \end{cases}$$

such that for a.e. $t \in [0,T]$,

$$u(\cdot,t) \text{ minimizes } \int_\Omega m(\cdot,t)|v|^2 \text{ over all } v \in C_{\rho(\cdot,t)}.$$

That is, ρ solves the DCM with initial data ρ_0.

Proof. We follow the proof of ([MRS], Theorem 2.4), replacing the density bound by m instead of 1, and modifying the analysis accordingly. Note Lemma 2.7 is substantially more involved in this general case.

Given two probabilities μ and ν on $\overline{\Omega}$, we always have

$$\frac{1}{2}W_2^2(\mu,\nu) = \max\left\{\int_\Omega \varphi\, d\mu + \int_\Omega \psi\, d\nu, \varphi,\psi \in C^0(\overline{\Omega}) : \varphi(x)+\psi(y) \leq \frac{1}{2}|x-y|^2\right\},$$

the maximum being always realized by a pair of c-concave conjugate functions (φ,ψ) with

$\varphi = \psi^c$ and $\psi = \varphi^c$, where the c-transform of a function χ is defined through

$$\chi^c(y) = \inf_{x\in\Omega} \frac{1}{2}|x-y|^2 - \chi(x)$$

(with generalizations to other costs c rather than the square of the distance). We will call Kantorovitch potential from μ to ν (resp. from ν to μ) any c-concave function φ (resp. ψ) such that (φ, φ^c) (resp. (ψ^c, ψ)) realizes such a maximum. We have uniqueness of the optimal pair as soon as the support of one of the two measures is the whole domain $\overline{\Omega}$.

Lemma 2.4. *Fix t_0, t_1 and let $\overline{\rho} \in K_{t_0}$.*

(i) The functional $\phi(\rho) = W_2(\rho, \overline{\rho})$ admits a unique minimizer ρ_{min} over $\rho \in K_{t_1}$,

(ii) There exists a Kantorovitch potential $\overline{\varphi}$ to $\overline{\rho}$, such that:

$$\int_\Omega \overline{\varphi}\rho \geq \int_\Omega \overline{\varphi}\rho_{min} \qquad \textit{for all } \rho \leq m(\cdot, t_1) \textit{ a.e.}$$

Proof. (i) Existence holds as argued before Lemma 2.1; uniqueness holds as in Lemma 2.2.

(ii) We first assume that $\overline{\rho} > 0$ a.e., which implies that the Kantorovich potential $\overline{\varphi}$ from $\rho_{\min}$ to $\overline{\rho}$, satisfying $\overline{\varphi}(x_0) = 0$ (with x_0 any fixed point in Ω), is unique. Let us define a small perturbation of $\rho_{\min}$: let $\rho \leq m(\cdot, t_1)$ be a probability density, $\varepsilon > 0$, and $\rho_\varepsilon := \rho_{\min} + \varepsilon(\rho - \rho_{\min})$. As $\rho_{\min}$ minimizes $\phi(\rho)$, we have:

$$W_2(\rho_\varepsilon, \overline{\rho}) \geq W_2(\rho_{\min}, \overline{\rho}). \tag{2.7}$$

Let $(\varphi_\varepsilon, \psi_\varepsilon)$ be Kantorovich potentials associated to $\overline{\rho}$ and ρ_ε. We have

$$\frac{1}{2}W_2^2(\rho_\varepsilon, \overline{\rho}) = \int_\Omega \varphi_\varepsilon(x)\rho_\varepsilon(x)\,dx + \int_\Omega \psi_\varepsilon(y)\overline{\rho}(y)\,dy$$

$$\frac{1}{2}W_2^2(\rho_{\min}, \overline{\rho}) \geq \int_\Omega \varphi_\varepsilon(x)\rho_m(x)\,dx + \int_\Omega \psi_\varepsilon(y)\overline{\rho}(y)\,dy,$$

where ψ_ε is a Kantorovitch potential from ρ_ε to $\overline{\rho}$. Thus

$$\frac{1}{2}\left(W_2^2(\rho_\varepsilon,\overline{\rho}) - W_2^2(\rho_{\min},\overline{\rho})\right) \leq \int_\Omega \varphi_\varepsilon(x)(\rho_\varepsilon - \rho_m)(x)\,dx = \varepsilon \int_\Omega \varphi_\varepsilon(x)(\rho - \rho_{\min})(x)\,dx,$$

so based on (2.7),

$$\int_\Omega \varphi_\varepsilon(x)(\rho - \rho_{\min})(x)\,dx \geq 0 \qquad \text{for all admissible } \rho.$$

Sending ε to zero, φ_ε converges to the unique Kantorovich potential $\overline{\varphi}$ from $\rho_{\min}$ to $\overline{\rho}$. This gives

$$\int_\Omega \psi^c(x)(\rho - \rho_{\min})(x)\,dx \geq 0 \text{ for all admissible } \rho.$$

We now prove the general case. Let $\overline{\rho}_\delta > 0$ a.e., $\overline{\rho}_\delta \leq m(\cdot, t_1)$ a.e., such that $\overline{\rho}_\delta$ converges to $\overline{\rho}$ when δ tends to 0. Using (i), there exists a unique minimizer $\rho_{\min,\delta}$ of $\phi_\delta(\rho) := I_{K_{t_1}}(\rho) + \frac{1}{2\tau}W_2^2(\rho,\overline{\rho}_\delta)$, and it converges to $\rho_{\min}$ as δ tends to 0. Moreover, we have proved that:

$$\int_\Omega \overline{\varphi}_\delta(x)(\rho - \rho_{\min,\delta})(x)\,dx \geq 0 \text{ for all admissible } \rho,$$

with $\overline{\varphi}_\delta$ that converges to a Kantorovich potential $\overline{\varphi}$. Taking the limit as $\delta \to 0$, we obtain the desired inequality.

□

Lemma 2.5. *The optimal transport functions take the form of a pressure gradient:*

$$v_\tau^n = -\nabla p_\tau^n \qquad \textit{with } p_\tau^n \in H^1_{\rho_\tau^n}.$$

Proof. By the previous lemma, there exists a Kantorovich potential $\overline{\varphi}$ from ρ_τ^n to ρ_τ^{n-1} such that ρ_τ^n is a solution of the minimizing problem:

$$\rho_\tau^n \in \operatorname{argmin}_{\rho \in K_{\tau n}} \left\{ \int_\Omega \overline{\varphi}(x)\rho(x)\,dx \right\},$$

which imposes:

$$\begin{cases} \rho_\tau^n = m(\cdot, \tau n) & \text{on } [\overline{\varphi} < \ell] \\ \rho_\tau^n \le m(\cdot, \tau n) & \text{on } [\overline{\varphi} = \ell] \\ \rho_\tau^n = 0 & \text{on } [\overline{\varphi} > \ell], \end{cases}$$

where $\ell \in \mathbb{R}$ is chosen such that ρ_τ^n satisfies $\int_\Omega \rho_\tau^n \, dx = 1$.

We can then define a pressure like function

$$p_\tau^n(x) := \left(\ell - \frac{\overline{\varphi}}{\tau}\right)_+ = \left(\ell - \frac{\overline{\varphi}(x)}{\tau}\right)_+,$$

which satisfies $p_\tau^n \ge 0$, and $p_\tau^n = 0$ on $[\rho_\tau^n < m(\cdot, \tau n)]$, therefore $p_\tau^n \in H^1_{\rho_\tau^n}$. Since we have

$$v_\tau^n = \frac{i - t_\tau^n}{\tau} = \frac{\nabla \overline{\varphi}}{\tau},$$

we get the desired decomposition for optimal transport functions.

□

Similar to (2.6), we interpolate the discrete values p_τ^n to form the piecewise constant pressure function given by

$$p_\tau(t, \cdot) = p_\tau^n \qquad \text{if } t \in ((n-1)\tau, n\tau].$$

Then we can write $p_\tau \in H^1_{\rho_\tau}$, and this works for all times.

Let us now define the densities $\tilde{\rho}_\tau(t)$ that interpolate the discrete values (ρ_τ^n) along geodesics:

$$\tilde{\rho}_\tau(t) = \left(\frac{t - (n-1)\tau}{\tau}(\mathrm{id} - t_\tau^n) + t_\tau^n\right)_{\#} \rho_\tau^n. \tag{2.8}$$

We also define $\tilde{v}_\tau(t, \cdot)$ as the unique velocity field such that $\tilde{v}_\tau(t, \cdot) \in \mathrm{Tan}_{\tilde{\rho}_t} \mathcal{P}_2(\mathbb{R}^d)$ and $(\tilde{\rho}_\tau, \tilde{v}_\tau)$ satisfy the continuity equation. As before, we define $\tilde{E}_\tau = \tilde{\rho}_\tau \tilde{v}_\tau$.

In order to give some a priori bounds on these curves, pressures, and velocities, we show that the sequence $(\rho_\tau^n)_n$ satisfies a discrete H^1 estimate on its variation. First we need a uniform bound on the supports of the densities in the minimizing movement scheme.

Lemma 2.6. *For all small enough τ, the ρ_τ^n are uniformly compactly supported inside $B_{KR}(0)$ for some constant K dependent on T, α, M_-, d.*

Proof. Recall ρ_0 is assumed compactly supported in $B_R(0)$.

The primary restriction on ρ_τ^n is due to Lemma 2.2, which specifies that wherever ρ_τ^n has adjusted the measure from ρ_0, it must be saturated, i.e., equal to $m(\cdot, \tau n)$. Because we assume $M_- \leq m(\cdot, \tau n)$ in (M1), the maximum area of the support of ρ_τ^n is the support of ρ_0, $B_R(0)$, plus at most area $1/M_-$ where ρ_τ^n is saturated.

What we must rule out, therefore, is an extension from $B_R(0)$ with finite area that reaches for infinity. The primary tool to do so is Lemma 2.1. This extension must proceed in stages, $\rho_0 \to \rho_\tau^1$, $\rho_\tau^1 \to \rho_\tau^2$, etc., reaching further at each stage.

To analyze such an extension, suppose r is such that ρ_τ^{n-1} is supported in $B_r(0)$ and suppose the maximum radius in the support of ρ_τ^n is $r + \varepsilon$, achieved at $x_0 \in \mathbb{R}^d$, so $|x_0| = r + \varepsilon$. Let γ be the optimal plan from ρ_τ^n to ρ_τ^{n-1}. There must exist $y_0 \in B_r(0)$ such that $(x_0, y_0) \in \mathrm{spt}(\gamma)$. By Lemma 2.1, $\rho_\tau^n = m(\cdot, \tau n)$ a.e. in $B_{|y_0 - x_0|}(y_0)$.

In fact we need a stronger result than this from Lemma 2.1. Define

$$S = \{y \in B_r(0) : \text{ for some } x \in B_{\varepsilon/2}(x_0), (x, y) \in \mathrm{spt}(\gamma)\}.$$

Let $|S|$ be the measure of S in $\mathbb{R}^d$. From here on we ignore constants related to the measure of balls in $\mathbb{R}^d$ for clarity. Note on average mass from S moving into $B_{\varepsilon/2}(x_0)$ must move at least distance $|S|^{1/d} + \frac{\varepsilon}{4}$. Thus Lemma 2.1 indicates ρ_τ^n is actually supported (and equal to $m(\cdot, \tau n)$) on a set $G = \mathrm{spt}(\rho_\tau^n) \setminus B_r(0)$ of measure at least

$$|G| \geq C\left[(|S|^{1/d} + \varepsilon/4)^d - (|S|^{1/d})^d\right] \geq C\varepsilon |S|^{\frac{d-1}{d}}. \tag{2.9}$$

See Figure 3 for one possible illustration.

Now consider how much mass is available from S to fill G so that

$$\rho_\tau^n = m(\cdot, \tau n) \geq M_-$$

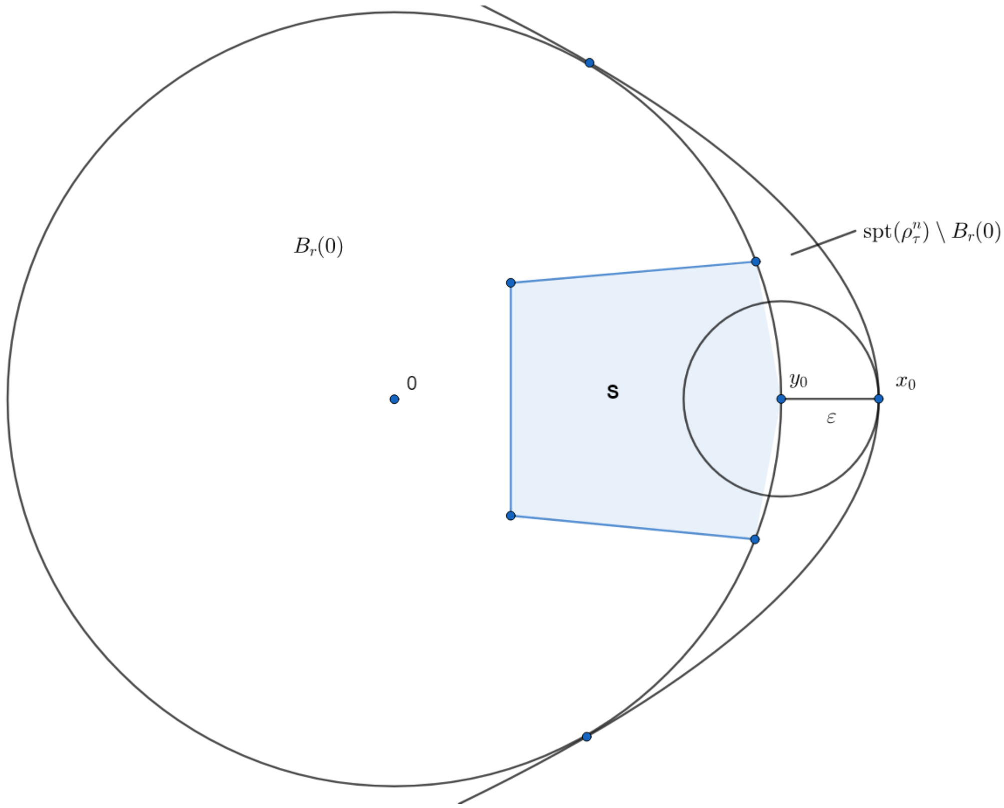


Figure 3: One possible illustration of the objects in Lemma 2.6.

on this outer set. From (M1), we have $|\partial_t m| \le \alpha$, thus the mass available is at most $\alpha\tau|S|$. Comparing with 2.9, it must be that

$$\alpha\tau|S| \ge M_-(C\varepsilon|S|^{\frac{d-1}{d}}).$$

Thus

$$\varepsilon \le C|S|^{1/d}\tau \le Cr\tau,$$

with C dependent on α, M_-, d.

Iterating a support extension of size $C\tau r$ starting from ρ_0 supported in $B_R(0)$, we conclude that ρ_τ^n is supported inside a ball of radius

$$R(1 + C\tau)^n \le R(1 + C\tau)^{T/\tau},$$

which has limit as $\tau \to 0$ of Re^{CT}, so the statement of the lemma holds with $K := 2e^{CT}$, with C dependent on R, M_-, and d.

□

Lemma 2.7. *There exists a constant C independent of τ such that for all small enough τ,*

$$\sum_n \frac{W_2^2(\rho_\tau^n, \rho_\tau^{n-1})}{\tau} \leq C.$$

Proof. Due to the previous lemma, we can assume ρ_τ^{n-1} is supported on $B_{KR}(0)$.

We devise a transport map η_n from ρ_τ^{n-1} to a candidate for ρ_τ^n that is directed radially outward from 0. Let z be a unit vector in an arbitrary direction in $\mathbb{R}^d$. We must move mass $m(hz, \tau(n-1))$ for $h \in [0, KR]$ to some candidate density ρ with $\rho(hz) \leq m(hz, \tau n)$ for all $h \geq 0$. The natural candidate is such that the mass at radius $r_1 > 0$ moves to radius r_2 (with $r_2 > r_1$), which is chosen to satisfy mass conservation:

$$\int_0^{r_1} m(hz, \tau(n-1))(h^{n-1}\, dh) = \int_0^{r_2} m(hz, \tau n)(h^{n-1}\, dh).$$

Estimating r_2, using (M1) in that $M_- \leq m$ and $|\partial_t m| \leq \alpha$,

$$\frac{1}{d}(r_2^d - r_1^d)M_- \leq \int_{r_1}^{r_2} m(hz, \tau k)(h^{d-1}\, dh) = \int_0^{r_1} [m(hz, \tau(n-1)) - m(hz, \tau n)](h^{d-1}\, dh) \leq \alpha {r_1}^d \tau,$$

thus

$$r_2 \leq r_1 \left(1 + \frac{d\alpha}{M_-}\tau\right)^{1/d} \leq \left(1 + \frac{\alpha}{M_1}\tau\right) r_1,$$

where the last inequality holds for small enough τ (depending on d). In particular, as $r_1 \leq KR$, defining the new constant $\tilde{C} = \frac{\alpha KR}{M_1}$, the transport map η_n shifts mass at most a distance $\tilde{C}\tau$. Also note by picking τ small enough we ensure $r_2 \leq 2KR$, so the image of η_n remains inside Ω, which was chosen in the statement of Theorem 2.3 as $B_{3KR}(0)$.

Now we must estimate the total W_2^2 cost for η_n. Using $m \leq M_+$,

$$W_2^2((\eta_n)_\# \rho_\tau^{n-1}, \rho_\tau^{n-1}) \leq (\text{surface area of unit ball in } R^n)(2R)^{n-1} \int_0^{2R} (M_+)(\tilde{C}\tau)^2\, dh \leq C\tau^2.$$

Thus

$$W_2^2(\rho_\tau^n, \rho_\tau^{n-1}) \leq C\tau^2,$$

which gives the desired discrete H^1 estimate.

□

Remark 2.2. If Ω had a more complicated geometry than a large enough ball, then Lemma 2.7 is much more challenging to prove, as directly constructing a candidate transport map or plan appears to be necessarily dependent on the geometry of Ω. If the transport map or plan forces a non-negligible amount of mass (specifically more than $\mathcal{O}(\tau)$) to move distance more than $\mathcal{O}(\tau)$, then we cannot recover Lemma 2.7 and the minimizing movement scheme will likely not converge, at least in the sense considered here. Similar problems can arise if the lower bound $0 < M_- \leq m$ in (M1) is not assumed.

Recall the metric derivative of a time-dependent function f with respect to a metric d is defined as

$$|f'(t)|_d := \lim_{s \to t} \frac{d(\rho(s), \rho(t))}{|s - t|}.$$

Returning to the interpolated densities along geodesics from (2.8), $\tilde{\rho}_\tau(t)$ is an absolutely continuous curve in the Wasserstein space and its velocity on the time interval $[(n-1)\tau, n\tau]$ is given by the ratio $W_2(\rho_\tau^{n-1}, \rho_\tau^n)/\tau$. Hence, the L^2 norm of its velocity on $[0, T]$ is given by

$$\int_0^T |(\tilde{\rho}_\tau)'|_{W_2}^2(t)\, dt = \sum_n \frac{W_2^2(\rho_\tau^n, \rho_\tau^{n-1})}{\tau}. \tag{2.10}$$

Comparing Eq. (2.10) and Lemma 2.7, for small enough τ, the L^2 velocity norm is uniformly bounded independent of τ. This gives compactness of the curves $\tilde{\rho}_\tau$, as well as a Hölder estimate on their variations (since $H^1 \subset C^{0,1/2}$).

Lemma 2.8. *We have the following a priori estimates:*

(i) v_τ *is* τ*-uniformly bounded in* $L^2((0,T), L^2_{\rho_\tau}(\Omega))$*,*

(ii) p_τ *is* τ*-uniformly bounded in* $L^2((0,T), H^1(\Omega))$*,*

(iii) E_τ and $\tilde{E}_\tau$ are τ-uniformly bounded measures.

Proof. (i) We have the following equalities:

$$\begin{aligned}\int_0^T\int_\Omega \rho_\tau|v_\tau|^2 &= \sum_n \int_{(n-1)\tau}^{n\tau}\int_\Omega \rho_\tau^n|v_\tau^n|^2\\ &= \sum_n\left(\int_{(n-1)\tau}^{n\tau} dt\right)\left(\int_\Omega \rho_\tau^n(x)\frac{|x-t_\tau^n(x)|^2}{\tau^2}\,dx\right)\\ &= \sum_n \tau\frac{W_2^2(\rho_\tau^{n-1},\rho_\tau^n)}{\tau^2}=\frac{1}{\tau}\sum_n W_2^2(\rho_\tau^{n-1},\rho_\tau^n).\end{aligned}$$

Thus (i) holds by Lemma 2.7.

(ii) Since we have shown $\nabla p_\tau = -v_\tau$, we have by (i),

$$\int_0^T\int_\Omega \rho_\tau|\nabla p_\tau|^2 = \int_0^T\int_\Omega \rho_\tau|v_\tau|^2 \le C.$$

If we set m_τ to be the piecewise function with $m_\tau(x,t)=m(x,\tau n)$ for $t\in[\tau(n-1),\tau n]$, then $p_\tau=0$ on $[\rho_\tau<m_\tau]$, and as $c_1\le m$,

$$\int_0^T\int_\Omega |\nabla p_\tau|^2 \le \frac{1}{c_1}\int_0^T m_\tau|\nabla p_\tau|^2 = \frac{1}{c_1}\int_0^T \rho_\tau|\nabla p_\tau|^2 \le C.$$

(iii) Using previous estimates and Cauchy-Schwartz,

$$\begin{aligned}\int_0^T\int_\Omega|\tilde{E}_\tau| &= \int_0^T\int_\Omega \tilde{\rho}_v|\tilde{v}_\tau| \le \int_0^T\left(\int_\Omega \tilde{\rho}_\tau|\tilde{v}_\tau|^2\right)^{1/2}\left(\int_\Omega \rho_\tau\right)^{1/2} \le \int_0^T\left(\int_\Omega \tilde{\rho}_v|\tilde{v}_\tau|^2\right)^{1/2}\\ &\le \sqrt{T}\left(\int_0^T\int_\Omega \rho_\tau|v_\tau|^2\right)^{1/2} \le C.\end{aligned}$$

The proof for E_τ is similar.

□

Lemma 2.9 ([MRS], Lemma 3.5)**.** *Assume that μ and ν are absolutely continuous measures, whose densities are bounded by the same constant C. Then, for all functions $f\in H^1(\Omega)$, we* *have the following inequality:*

$$\int_\Omega f\, d(\mu - \nu) \leq \sqrt{C}||\nabla f||_{L^2(\Omega)} W_2(\mu, \nu).$$

Now we proceed with the proof of Theorem 2.3.

Step 1: convergence of $(\tilde{\rho}_\tau, \tilde{E}_\tau)$ *and* (ρ_τ, E_τ). We have proved that $\tilde{\rho}_\tau$ and $\tilde{E}_\tau$ are τ-uniformly bounded measures, thus there exists (ρ, E) such that $(\tilde{\rho}_\tau, \tilde{E}_\tau)$ converges narrowly to (ρ, E). We will show that (ρ_τ, E_τ) converges to the same limit as $(\tilde{\rho}_\tau, \tilde{E}_\tau)$.

First we show the convergence of ρ_τ. The curves $\tilde{\rho}_\tau$ converge uniformly in $[0, T]$ with respect to the W_2-distance. The curves ρ_τ and $\tilde{\rho}_\tau$ coincide on every time of the form kn. Note ρ_τ is constant on every interval $(n\tau, n\tau]$, while $\tilde{\rho}_\tau$ is uniformly Hölder continuous of exponent $1/2$, which implies $W_2(\tilde{\rho}_\tau(t), \rho_\tau(t)) \leq C\tau^{1/2}$. This proves that ρ_τ converges uniformly to the same limit as $\tilde{\rho}_\tau$.

We now consider a function $f \in C_c^\infty([0, T] \times \Omega)$, and prove that $\int_0^T \int_\Omega f(\tilde{E}_\tau - E_\tau)$ converges to 0 as τ converges to 0. We have $\tilde{\rho}_\tau(t, \cdot) = (T_t)_\# \rho_\tau^n$, where

$$T_t = (t - (n-1)\tau)v_\tau^n + t_\tau^n.$$

Therefore

$$\tilde{\rho}_\tau(\cdot, t+h) = (T_t + hv_\tau^n)_\# \rho_\tau^n = ((id + hv_\tau^n \circ T_t^{-1}) \circ T_t)_\# \rho_\tau^n = (\mathrm{id} + hv_\tau^n \circ T_t^{-1})_\# \rho_\tau(t, \cdot),$$

which implies that $t_{\tilde{\rho}_\tau(\cdot,t)}^{\tilde{\rho}_\tau(\cdot,t+h)} = \mathrm{id} + hv_\tau^n \circ T_t^{-1}$. We can then express $\tilde{v}_\tau$ explicitly:

$$\tilde{v}_\tau(\cdot, t) = \lim_{h \to 0} \frac{t_{\tilde{\rho}_\tau(\cdot,t)}^{\tilde{\rho}_\tau(\cdot,t+h)} - \mathrm{id}}{h} = \lim_{h \to 0} \frac{hv_\tau^n \circ T_t^{-1}}{h} = v_\tau^n \circ T_t^{-1},$$

and obtain

$$\begin{aligned} \int_\Omega f(x,t)\tilde{\rho}_\tau(x,t)\tilde{v}_\tau(x,t)\, dx &= \int_\Omega f(t, T_t(x))\rho_\tau^n(x)\tilde{v}_\tau(T_t(x), t)\, dx \\ &= \int_\Omega f(T_t(x), t)\rho_\tau^n(x) v_\tau^n(x)\, dx. \end{aligned}$$

Hence

$$\begin{aligned}
\int_0^T \int_\Omega f(\tilde{E}_\tau - E_\tau) &\leq \sum_n \int_{\tau n}^{\tau(n+1)} \int_\Omega |f(x,t) - f(T_t(x),t)|\, |v_\tau^n(x)| \rho_\tau^n(x)\, dx\, dt \\
&\leq \sum_n \int_{\tau n}^{\tau(n+1)} \int_\Omega (\text{Lip } f)|x - T_t(x)|\, |v_\tau^n(x)| \rho_\tau^n(x)\, dx\, dt \\
&\leq \sum_n \int_{\tau n}^{\tau(n+1)} \int_\Omega (\text{Lip } f)|v_\tau^n(x)|^2 \rho_\tau^n(x)\, dx\, dt \\
&\leq C(\text{Lip } f)\tau.
\end{aligned}$$

This proves the desired convergence.

Step 2: existence and candidacy of the limit velocity.

Let us prove that E is absolutely continuous with respect to ρ. Let θ be a scalar measure, and F a vectorial measure: the function

$$\Theta : (\theta, F) \mapsto \begin{cases} \int_0^T \int_\Omega \frac{|F|^2}{\theta} & \text{if } F << \theta \text{ for a.e. } t \in [0,T], \\ \\ +\infty & \text{otherwise,} \end{cases}$$

is lower semicontinuous for the weak-$\star$ convergence of measures. Since we have shown the τ-uniform bound

$$\int_0^T \int_\Omega \frac{|E_\tau|^2}{\rho_\tau} = \int_0^T \int_\Omega \rho_\tau |v_\tau|^2 \leq C,$$

we have $\Theta(\rho, E) < +\infty$. Therefore E is absolutely continuous with respect to ρ, and there exists $u(\cdot,t) \in L^2(\rho(\cdot,t))$ such that $E = \rho u$. Moreover, $(\rho, \rho u)$ (weakly) satisfies the continuity equation as the limit of $(\tilde{\rho}_\tau, \tilde{E}_\tau)$.

Next we show $u \in C_{\rho(\cdot,t)}$. Let $t_0 \in (0,T)$, $h > 0$, and $q \in H^1_{\rho(\cdot,t_0)}$. This implies

$$\rho(\cdot, t_0) = m(\cdot, t_0)$$

wherever $q > 0$. Using the continuity equation and integration by parts,

$$\begin{aligned}\int_{t_0}^{t_0+h}\int_\Omega \nabla q(x)\cdot u(x,t)\rho(x,t)\,dx\,dt &= \int_\Omega\int_{t_0}^{t_0+h}\nabla q(x)\cdot(-\partial_t\rho(x,t))\,dt\,dx\\ &= \int_\Omega(\rho(x,t_0+h)-\rho(x,t_0))q(x)\,dx.\end{aligned}$$

As $\rho(\cdot,t_0+h)\in K_{t_0+h}$, $\rho(\cdot,t_0+h)\le m(\cdot,t_0+h)$. Thus where $q>0$,

$$\lim_{h\to 0^+}\frac{\rho(x,t_0+h)-\rho(x,t_0)}{h}\le \lim_{h\to 0^+}\frac{m(x,t_0+h)-m(x,t_0)}{h} = m_t(x,t_0)$$

and

$$\frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega \nabla q(x)\cdot u_t(x,t)\rho(x,t)\,dx\,dt = \int_\Omega \frac{1}{h}(\rho(x,t_0+h)-\rho(x,t_0))q(x)\,dx \le \int_\Omega m_t q(x)\,dx.$$

Taking a limit of the left hand side as $h\to 0$, for a.e. t_0,

$$\int_\Omega \nabla q(x)\cdot u(x,t_0)\rho(x,t_0)\,dx \le \int_\Omega m_t q(x)\,dx,$$

or

$$\int_\Omega m(x,t_0)\nabla q(x)\cdot u(x,t_0)\,dx \le \int_\Omega m_t q(x)\,dx.$$

In particular, $u\in C_{\rho(\cdot,t)}$.

Similarly, integrating from t_0-h to t_0, and repeating the above, we find the opposite inequality, so for any $q\in H^1_{\rho(\cdot,t_0)}$,

$$\int_\Omega m(x,t_0)\nabla q(x)\cdot u(x,t_0)\,dx = \int_\Omega m_t q(x)\,dx. \tag{2.11}$$

Step 3: the limit velocity satisfies $u = P_{C_\rho}(0)$. As $p_\tau\in L^2([0,T],H^1(\Omega))$, there exists p such that p_τ weakly converges to p in $L^2([0,T],H^1(\Omega))$. Clearly $p\ge 0$ a.e., so to show $p(\cdot,t)\in H^1_{\rho(\cdot,t)}$ it suffices to show that $p(\cdot,t)=0$ on $[\rho(\cdot,t)<m(\cdot,t)]$.

We consider the average functions

$$p_\tau^{a,b} = \frac{1}{b-a}\int_a^b p_\tau(\cdot,t)\,dt \qquad p^{a,b} = \frac{1}{b-a}\int_a^b p(\cdot,t)\,dt$$

Since $p_\tau = 0$ on $[\rho_\tau < m_\tau]$, we have

$$\begin{aligned}
0 = \int_a^b \int_\Omega p_\tau(x,t)(m_\tau(x,t) - \rho_\tau(x,t))\,dx\,dt \quad &= \quad \frac{1}{b-a}\int_a^b \int_\Omega p_\tau(x,t)(m_\tau(x,t) - m_\tau(x,a))\,dx\,dt \\
&\quad + \frac{1}{b-a}\int_a^b \int_\Omega p_\tau(x,t)(m_\tau(x,a) - \rho_\tau(x,a))\,dx\,dt \\
&\quad + \frac{1}{b-a}\int_a^b \int_\Omega p_\tau(x,t)(\rho_\tau(x,a) - \rho_\tau(x,t))\,dx\,dt.
\end{aligned}$$

We will take limits of this expression both as $b \to a$ and as $\tau \to 0$. As $b \to a$, since m is continuous in time, the first integral vanishes. Taking a limit of the second integral as $\tau \to 0$,

$$\int_\Omega p_\tau^{a,b}(x)(m_\tau(x,a) - \rho_\tau(x,a))\,dx \to \int_\Omega p^{a,b}(x)(m(x,a) - \rho(x,a))\,dx,$$

as $p_\tau^{a,b}$ weakly converges in $H^1(\Omega)$, and thus strongly in $L^2(\Omega)$, to $p^{a,b}$, and $\rho_\tau(\cdot,a)$ weak-$\star$ converges in $L^\infty(\Omega)$ to $\rho(\cdot,a)$. Then taking a limit as $b \to a$, for every Lebesgue point a of $p(x,\cdot)$, $p^{a,b} \to p(\cdot,a)$, thus for these a,

$$\int_\Omega p^{a,b}(x)(m(x,a) - \rho(x,a))\,dx\,dt \to \int_\Omega p(x,a)(m(x,a) - \rho(x,a))\,dx.$$

For the third integral, we use Lemma 2.9:

$$\begin{aligned}
\int_a^b \int_\Omega p_\tau(x,t)\left(\rho_\tau(x,t) - \rho_\tau(x,a)\right)\,dx\,dt \quad &\leq \quad \int_a^b ||\nabla p_\tau(\cdot,t)||_{L^2(\Omega)} W_2(\rho_\tau(\cdot,a), \rho_\tau(\cdot,t))\,dt \\
&\leq \quad C\sqrt{b-a}\left(\int_a^b ||\nabla p_\tau(\cdot,t)||^2_{L^2(\Omega)}\,dt\right)^{\frac{1}{2}} \left(\int_a^b dt\right)^{\frac{1}{2}} \\
&\leq \quad C(b-a)\left(\int_a^b ||\nabla p_\tau(\cdot,t)||^2_{L^2(\Omega)}\,dt\right)^{\frac{1}{2}}.
\end{aligned}$$

As $\int_0^T ||\nabla p_\tau(\cdot,t)||^2_{L^2(\Omega)}\,dt$ is τ-uniformly bounded, $||\nabla p_\tau(\cdot,t)||^2_{L^2(\Omega)}$ weakly converges to a measure μ. Therefore, except for a zero measure set of points $a \in [0,T]$, we have as $b \to a$,

$$\lim_{\tau \to 0} \frac{1}{b-a} \int_a^b \int_\Omega p_\tau(x,t)(\rho_\tau(x,a) - \rho_\tau(x,t))\, dx\, dt \le C\sqrt{\mu([a,b])} \to 0.$$

Thus taking both limits of the sum of these three integrals, we recover

$$\int_\Omega p(x,a)(m(x,a) - \rho(x,a))\, dx = 0$$

for almost every $a \in [0,T]$.

So we have shown $p(\cdot,t) \in H^1_{\rho(\cdot,t)}$. Note $E_\tau = \rho_\tau v_\tau = \rho_\tau(-\nabla p_\tau)$ converges to $E = \rho(-\nabla p)$. As $E = \rho u$, we have shown $u = -\nabla p$ for $p(\cdot,t) \in H^1_{\rho(\cdot,t)}$.

In particular, using $q = p$ in (2.11), and the norm and inner product in the weighted L^2 Hilbert space,

$$||u||^2_{m(\cdot,t_0)} = -\langle u, \nabla p\rangle_{m(\cdot,t_0)} = -\int_\Omega m(x,t_0)\nabla p(x) \cdot u(x,t_0)\, dx = \int_\Omega (-m_t(\cdot,t_0))p(x)\, dx.$$

Given any $v \in C_{\rho(\cdot,t_0)}$, we use the fact that $p \in H^1_{\rho(\cdot,t_0)}$ for the upper bound

$$\begin{aligned} \langle u, u-v\rangle_{m(\cdot,t_0)} &= ||u||^2_{m(\cdot,t_0)} + \int_\Omega m(\cdot,t_0)\nabla p(\cdot,t_0)\cdot v \\ &\le \int_\Omega (-m_t(\cdot,t_0))p(\cdot,t_0)\, dx + \int_\Omega m_t(\cdot,t_0)p(\cdot,t_0)\, dx \\ &= 0. \end{aligned}$$

This implies the gradient flow velocity u minimizes $\int m|v|^2$ over $v \in C_{\rho(\cdot,t_0)}$, which completes the proof of Theorem 2.3.

□

Corollary 2.10. *Let ρ_∞ denote the limit density in Theorem 2.3. For almost every t,*

$$W_2(\rho_\tau(\cdot,t), \rho_\infty(\cdot,t)) \to 0.$$

Proof. 2-Wasserstein convergence is equivalent to narrow convergence plus convergence of the second moments. As both ρ_τ and ρ_∞ were shown to be compactly supported in some large ball, convergence of the second moments follows. $\square$

Chapter 3

The Viscosity Solution Approach

In this chapter, we define the viscosity solution for the free boundary problem under consideration, show it satisfies a comparison principle, and prove the modified porous medium equation solutions converge to this viscosity solution. Following the results of Chapter 4, we will later conclude that this free boundary problem describes the solution of the DCM from Chapter 2.

3.1 Viscosity solutions of (FB-M)

Recall the free boundary problem (FB-M),

$$\begin{cases} -\nabla \cdot (m\nabla p) &= -m_t \quad \text{in } \{p(\cdot,t) > 0\}, \\ V &= |\nabla p| \quad \text{on } \partial\{p(\cdot,t) > 0\}. \end{cases}$$

Note due to (M1) the equation $-\nabla \cdot (m\nabla p) = -m_t$ can also be written as

$$-\Delta p - \nabla(\log m) \cdot \nabla p + \partial_t(\log m) = 0.$$

As the set $\{p(\cdot,t) > 0\}$ evolves according to the free boundary problem, its boundary may become less regular and go through topological singularities, in which case a well-defined classical solution p may not exist. However, the maximum principle for $-\nabla \cdot (m\nabla p) = -m_t$ still allows enough control over the solution by comparing against smooth test functions on the interior and on the boundary.

Definition 3.1. For a function $f : D \to \mathbb{R}$, define its *upper and lower semi-continuous envelopes*

$$f^*(x,t) := \lim_{r\to 0} \sup_{\substack{(y,s)\in D \\ |(y,s)-(x,t)|\le r}} f(y,s), \qquad f_*(x,t) := \lim_{r\to 0} \inf_{\substack{(y,s)\in D \\ |(y,s)-(x,t)|\le r}} f(y,s).$$

- A nonnegative uppersemicontinuous function $u : \overline{Q} \to \mathbb{R}$ is a *viscosity subsolution of (FB-M)* with initial data u_0 if

  (a) $u = u_0$ at $t = 0$,

  (b) for each $\tau > 0$,

$$\{u > 0\} \cap \{t \leq \tau\} \subset \overline{\{u > 0\} \cap \{t < \tau\}}$$

  (c) for every $\phi \in C^{2,1}(Q)$ that has a local maximum zero of $u - \phi$ in $\overline{\{u > 0\}} \cap \{t \leq t_0\}$ at (x_0, t_0),

    (i) if $(x_0, t_0) \in \{u > 0\}$,

$$\left[-\Delta\phi - \nabla(\log m) \cdot \nabla\phi + \partial_t(\log m)\right](x_0, t_0) \leq 0,$$

    (ii) if $(x_0, t_0) \in \partial\{u > 0\}$, $u(x_0, t_0) = 0$, and $|\nabla\phi(x_0, t_0)| \neq 0$, then

$$\min\left(-\Delta\phi - \nabla(\log m) \cdot \nabla\phi + \partial_t(\log m), \phi_t - |\nabla\phi|^2\right)(x_0, t_0) \leq 0.$$

- A nonnegative lowersemicontinuous function $v : \overline{Q} \to \mathbb{R}$ is a *viscosity supersolution of (FB-M)* with initial data u_0 if

  (a) $v = v_0$ at $t = 0$

  (b) for every $\phi \in C^{2,1}(Q)$ that has a local minimum zero of $v - \phi$ in $\mathbb{R}^d \times (0, t_0]$ at (x_0, t_0),

    (i) if $(x_0, t_0) \in \{v > 0\}$,

$$\left[-\Delta\phi - \nabla(\log m) \cdot \nabla\phi + \partial_t(\log m)\right](x_0, t_0) \geq 0,$$

    (ii) if $(x_0, t_0) \in \partial\{v > 0\}$, $v(x_0, t_0) = 0$, as well as

$$|\nabla\phi(x_0, t_0)| \neq 0, \text{ and for some ball } B \text{ centered at } (x_0, t_0), \tag{3.1}$$

$$\{\phi > 0\} \cap \{v > 0\} \cap B \neq \emptyset,$$

then

$$\max\left(-\Delta\phi - \nabla(\log m)\cdot\nabla\phi + \partial_t(\log m), \phi_t - |\nabla\phi|^2\right)(x_0, t_0) \geq 0.$$

- p is a *viscosity solution of (FB-M)* with initial data u_0 if u_* is a viscosity supersolution of (FB-M) with initial data u_0 and u^* is a viscosity subsolution of (FB-M) with initial data u_0.

Remark 3.1. When no initial data is referenced, we use these definitions without condition (a) for subsolutions or supersolutions.

Similar definitions apply to viscosity solutions on a bounded set E, where we replace all instances of $\{v > 0\}$, for example, with $\{v > 0\} \cap E$.

3.2 Comparison principle for (FB-M)

Definition 3.2. We say two non-negative functions $u, v : \mathbb{R}^d \to \mathbb{R}$ are *strictly separated*, denoted by $u \prec v$, if $\overline{\{u > 0\}} \subset \{v > 0\}$ and $u < v$ in $\overline{\{u > 0\}}$.

Theorem 3.1. *Suppose (M1) and (M2). Let u be a viscosity subsolution and v be a viscosity supersolution of (FB-M) with initial data u_0, v_0 respectively. If $u_0 \prec v_0$, then the solutions remain strictly separated,*

$$u(\cdot, t) \prec v(\cdot, t) \qquad \textit{for all } t \geq 0.$$

The proof is similar enough to ([K], Section 2) that we focus on major barriers that have been modified to reflect the dependence on the density constraint m and outline the rest.

Proof. (i) Regularization

To regularize solutions, we use the open sets

$$D_r := \{(y, s) : (|y - x| - r)_+^2 + |s - t|^2 < r^2\}.$$

Note that when $2r < R$, $\overline{D}_r(x, t) \subset B_R(x, t)$. For given $\bar{r}, \delta > 0$, denote the sup-convolution and inf-convolution of a subsolution u and a supersolution v with respect to $D_{\overline{r}}$ by

$$\overline{u}(x,t) := \sup_{(y,\tau)\in D_{\overline{r}}(x,t)} u(y,s), \qquad \underline{v}(x,t) := \inf_{(y,s)\in D_{\overline{r}-\delta t}(x,t)} v(y,s). \tag{3.2}$$

Suppose for the sake of contradiction that u crosses v from below. Then there exists a finite crossing time

$$T := \sup\{t : u(\cdot,\tau) \prec v(\cdot,\tau), 0 < \tau < t\}.$$

Fix r, δ small enough and consider the contact time

$$t_0 := \sup\{t : \overline{u}(\cdot,t) \prec \underline{v}(\cdot,t)\},$$

with $0 < t_0 \leq T$.

(ii) Geometry at time of contact

Consider any point $P_0 = (x_0, t_0) \in \partial\{\underline{v} > 0\}$. Then there is a point $P_2 = (x_2, t_2) \in \partial\{v > 0\}$ such that at P_0 the set $\{\underline{v} > 0\}$ has an exterior space-time ellipsoid E_0 defined by

$$(x - x_2)^2 + (t - t_2)^2 \leq (r - \delta t)^2$$

and at P_2 the set $\{v > 0\}$ has an interior space-time ball B_2 or radius $r - \delta t_0$ centered at P_0. Let H_0 denote the tangent hyperplane of E_0 at P_0. The inward normal vector to H_0 with respect to E_0 at P_0 takes the form (ν, σ) for some unit vector ν. We call this σ the advancing speed of the free boundary of $\underline{v}$ at P_0.

Lemma 3.2. *The advancing speed σ of the free boundary of $\underline{v}$ satisfies $\sigma \geq \delta$.*

Proof. If $\sigma < \delta$, then at P_2, the interior space-time ball B_2 of $\{v > 0\}$ has negative advancing velocity. By the lower semicontinuity of v, $v(P_2) = 0$.

For small $\tau > 0$, we construct a barrier $h(x,t)$ on $B_2 \cap [t_2 - \tau, t_2]$ such that

$$\begin{cases} -\Delta h - \nabla(\log m) \cdot \nabla p + \partial_t(\log m) < 0 & \text{outside } \frac{1}{4}B_2, \\ \\ 0 < h < v & \text{inside } \frac{1}{4}\overline{B}_2, \\ \\ \{h > 0\} = B_2,\ |\nabla h| \neq 0 & \text{on } \partial B_2. \end{cases}$$

Say B_2 has radius R. Since m is smooth and $m_t < 0$, there exist constants $\kappa_1, \kappa_2 > 0$ such that $|\nabla(\log m)| \leq \kappa_1, -\partial_t(\log m) \geq \kappa_2$ in a large compact set. Let $c_R = \kappa_1 + \frac{4(n-1)}{R}$ and $C > 0$ to be fixed later. We select the following base function for $0 \leq r \leq R$:

$$h_1(r) = C(e^{-c_R r} - e^{-c_R R}),$$

and define $h : \mathbb{R}^n \times [t_2 - \tau, t_2] \to \mathbb{R}$ by

$$h(x,t) = h_1\left(\frac{|x - x_2|}{\sqrt{R^2 - (t - t_2)^2}}\right).$$

Outside $\frac{1}{4}B_2$, this satisfies

$$\begin{aligned} -\Delta h - \nabla(\log m) \cdot \nabla p + \partial_t(\log m) &= -\partial_{rr} h_1 - \left(\nabla(\log m) + \frac{(n-1)}{r}\right)\partial_r h_1 + \partial_t(\log m) \\ &\leq Ce^{-c_R r}\left(-c_R^2 + c_R\left(\kappa_1 + \frac{4(n-1)}{R}\right)\right) - \kappa_2 \\ &\leq Ce^{-c_R r}(-c_R^2 + c_R^2) - \kappa_2 < 0. \end{aligned}$$

As $v > 0$ in B_2 and v is lower semicontinuous, v has a positive lower bound in $\frac{1}{4}\overline{B}_2$. Thus we can pick $C > 0$ so that $0 < h < v$ inside $\frac{1}{4}\overline{B}_2$. Moreover, h is positive inside B_2, 0 on ∂B_2, and negative outside B_2, with $|\nabla h| \neq 0$ on ∂B_2.

Since v has negative advancing velocity, there exists $\tau > 0$ such that $B_2 \cap [t_2 - \tau, t_2] \subset \{v > 0\}$. Because $h < 0$ outside B_2, $v - h > 0$ on $\partial\{v > 0\} \cap \{t \leq t_0\}$ except at P_2. Applying the maximum principle for the uniformly elliptic operator $-\frac{1}{m}\nabla \cdot (m\nabla)$, we conclude $v - h > 0$ inside B_2. Thus $v - h$ has its local minimum zero at P_2 in $\overline{\{v > 0\}} \cap \{t \leq t_0\}$. But this contradicts the definition of v as viscosity supersolution, since at P_2,

$$-\Delta h + \nabla(\log m) \cdot \nabla h - \partial_t(\log m) < 0 \quad \text{ and } \quad h_t - |\nabla h|^2 \leq -|\nabla h|^2 < 0.$$

□

By property (b) of the subsolution definition, $\overline{\{\overline{u}(\cdot, t) > 0\}}$ does not jump outward discontinuously in time. By Lemma 3.2, $\overline{\{\underline{v}(\cdot, t) > 0\}}$ does not jump inward discontinuously in time. Thus there exists a point $P_0 = (x_0, t_0)$ where the non-negative maximum of $\overline{u} - \underline{v}$ is attained in $\overline{\{\overline{u} > 0\}} \cap \{t \leq t_0\}$. Using the maximum principle for the elliptic problem and a barrier argument, it follows that

$$\overline{u}(P_0) = \underline{v}(P_0) = 0.$$

The point P_0 is the contact point of the free boundaries of $\overline{u}$ and $\underline{v}$ at $t = t_0$.

From the geometry of the regularization, at P_0, $\{\overline{u} > 0\}$ has an interior space-time ball of radius r_1 centered at $P_1 \in \partial\{u > 0\}$. Also at P_1, $\{u > 0\}$ has an exterior space-time ball B_1 of radius r centered at P_0.

For simplicity of notation, shift and rotate so that $P_0 = (0, t_0)$ and $\overline{P_0 P_1} = d_1 e_1$, where $e_1 = (1, 0, \ldots, 0)$. Let H be the tangent hyperplane to the interior ball of $\overline{u}$ at P_0. The internal normal vector to H with respect to $\{\overline{u} > 0\}$ at P_0 takes the form (e_1, σ) for some σ. Let $\alpha \in (0, \pi/2)$ be such that $\sigma = \tan\alpha$. Then s is the advancing speed of $\{\overline{u} > 0\}$ and

$$P_1 = (x_1, t_1) = (r\cos\alpha, \vec{0}, t_0 + r\sin\alpha), \quad \vec{0} \in \mathbb{R}^{n-1}.$$

Similarly, the set $\{\underline{v} > 0\}$ has an exterior space-time ball B centered at $P_2 \in \partial\{v > 0\}$, and at P_2 the set $\{v > 0\}$ has an interior space-time ball B_2 centered at P_0.

Lemma 3.3. *The hyperplane H is neither horizontal nor vertical.*

Proof. By Lemma 3.2, H is not vertical.

Suppose for the sake of contradiction that H is horizontal. Let $\delta > 0$. Shifting the time variable for convenience, there is a subsolution w of (FB-M) in the cylinder $C_1 = B_1 \times [0, \delta]$ with $P_1 = (0, \delta) \in \partial\{w(\cdot, \delta) > 0\}, 0 \in \mathbb{R}^n$, which takes value 0 on the bottom, and less than δ on the lateral boundary.

We build smooth $\phi(x, t)$ in C_1 such that

(i) $-\nabla \cdot (m\nabla\phi) > -m_t$ in C_1,

(ii) $w \leq \phi$ on the lateral boundary and bottom of C_1,

(iii) $\phi_t - |\nabla\phi|^2 > 0$ on $\partial\{\phi > 0\}$.

Since the radius r of B_2 and δ can be made arbitrarily small, we can treat m, m_t, and ∇m as effectively constant in C_1. Thus for some $c_1, c_2 > 0$ and given vector $\vec{b}$, (i) becomes

$$-c_1\Delta\phi - \vec{b} \cdot \nabla\phi > c_2,$$

Consider

$$\phi(x, t) := \left[g\left(|x| + \frac{c_1}{2c_2}t\right)\right]_+$$

where g is chosen below. Then letting $c_3 = |\vec{b}|$, (i) is satisfied if

$$-c_1\left(g'' + \frac{n-1}{r}g'\right) - c_3|g'| > c_2. \tag{3.3}$$

Consider

$$g(r) = -\frac{c_2}{c_1}\frac{1}{2n}(r^2 - 1) + \delta\left(2 - r^{-2n}\right).$$

Note $|g'|$ has an extra factor of r compared to the other terms in Inequality (3.3); the radius of B_2 can be taken small enough that the $|g'|$ term is then negligible compared to the other terms. As g was chosen to satisfy

$$-\left(g'' + \frac{n-1}{r}g'\right) > \frac{c_2}{c_1},$$

we have satisfied (i).

Note (ii) holds because $g(1) > \delta$ and $g(r) < 0$ for r near 0. Checking (iii), note that for small δ, $0 < g' < 2\frac{c_2}{c_1}$ on $\partial\{\phi > 0\}$ so that

$$\phi_t - |\nabla\phi|^2 = \frac{c_1}{2c_2}g' - (g')^2 = (g')\left(\frac{c_1}{2c_2} - g'\right) > 0 \qquad \text{on } \partial\{\phi > 0\},$$

which contradicts the definition of w as a subsolution. □

(iii) Control of gradients with barriers

Lemma 3.4. *In any nontangential cone K,*

$$\liminf_{x\to 0, x\in K} \frac{\overline{u}(x,t_0)}{\sigma(x_1)_+} \geq 1.$$

Proof. As in [[K], Lemma 2.6], with the replacement of the following test function:

With B_1 the space-time ball (centered at 0 for convenience), $P_1 = (\cos\alpha e_1, \sin\alpha)$ and boundary speed $\sigma = \tan\alpha > 0$, given $\varepsilon > 0$ we build smooth radial ϕ such that

(i) $-\nabla\cdot(m\nabla\phi) > -m_t$ outside $\frac{1}{4}B_1$,

(ii) $\phi(x,t) > 0$ in $2B_1 \setminus B_1$, $\phi(x,t) = 0$ on ∂B_1,

(iii) $\frac{\phi_t}{|\nabla\phi|}(P_1) = \sigma$, but $\phi_r(P_1) = \sigma(1-\varepsilon)$ so that

$$(\phi_t - |\nabla\phi|^2)(P_1) > 0.$$

Note it suffices for the proof in [K] that $\phi > 0$ on $2B_1 \setminus B_1$ instead of $\phi > 0$ outside of B_1.

As the radius of B_1 can be taken arbitrarily small, it suffices to treat the case when m, m_t, and ∇m are constant, as well as so that the $\nabla m \cdot \nabla\phi$ term has negligible contribution compared to other terms in (i). Thus given $c > 0$, to satisfy (i), we look for a radial function to satisfy

$$-\phi_1'' - \frac{n-1}{r}\phi_1' > c.$$

The following is a radial solution outside $\frac{1}{4}B_1$ with value zero when $|x| = 1$:

$$\phi_1(x) = c(-3 + 4|x| - |x|^2) + \begin{cases} 1 + \log|x| - |x|^{-1} & \text{if } n = 2 \\ 2 - |x|^{2-n} - |x|^{1-n} & \text{if } n > 2 \end{cases}$$

The choice of coefficients in the first part ensure $\phi_{1,r} > 0$ at $|x| = 1$, so that $\phi_1 < 0$ just inside B_1 and $\phi_1 > 0$ in $2B_1 \setminus B_1$. Extend ϕ_1 to $\frac{1}{4}B_1$ so that $\phi > 0$ in $2B_1 \setminus B_1$, $\phi(x,t) = 0$ on ∂B_1, and $\phi < 0$ inside B_1.

Finally we extend it to the space-time ball B_1 (without the top and bottom) by

$$\phi(x,t) := \phi_1\left(\frac{x}{\sqrt{1-t^2}}\right), -1 < t < 1.$$

In particular this choice gives $\frac{\phi_t}{|\nabla\phi|}(P_1) = \sigma$ since $P_1 = (\cos\alpha e_1, \sin\alpha)$ and $\sigma = \tan\alpha$. Multiplying by an appropriate positive constant depending on σ allows $\phi_r(P_1) = \sigma(1 - \varepsilon)$.

□

Finally, it is possible to derive a contradiction as in [K] page 20 by construction of an appropriate test function: Given $\varepsilon > 0$, we use smooth radial ϕ such that

(i) $-\nabla \cdot (m\nabla\phi) < -m_t$ outside $\frac{1}{4}B_2$,

(ii) $\phi(x,t) > 0$ in $2B_2 \setminus B_2$, $\phi(x,t) = 0$ on ∂B_2,

(iii) $-\frac{\phi_t}{\phi_r} < \sigma(1-\varepsilon) < -\phi_r$ on $\partial B_2 \cap \{t_2 - \tau \le t \le t_2\}$ so that

$$\phi_t - |\nabla\phi|^2 < 0.$$

Here it suffices to take a positive constant dependent on σ times the negative of the construction in Lemma 3.4.

□

3.3 (PME-M) weak solutions and pressure properties

Recall the porous medium equation for density $\rho_k : \mathbb{R}^d \times [0,T] \to \mathbb{R}$,

$$\partial_t \rho + \nabla \cdot (\rho(-\nabla p)) = 0, \qquad \text{(PME-M)}$$

where $k > 1$ is given, the pressure is related by

$$p_k = P_k(\rho) := \frac{k}{k-1}\left(\frac{\rho}{m}\right)^{k-1},$$

and $\rho(\cdot, 0) = \rho_0$.

The equation (PME-M) has a corresponding equation for the pressure p_k where $p_k > 0$:

$$p_t = (k-1)p(\nabla \cdot (m\nabla p) - m_t) + |\nabla p|^2. \qquad \text{(PME-M p)}$$

Definition 3.3. A classical solution of (PME-M p) is a nonnegative function $p \in C^{2,1}(\overline{\{p > 0\}})$ such that

(i) p solves (PME-M p) in $\{p > 0\}$,

(ii) p has a free boundary $\Gamma = \partial\{p > 0\}$ which is a $C^{2,1}$ hypersurface, and

(iii) Γ evolves with outer normal velocity $V = |\nabla p|$.

In Appendix A, we show (PME-M) has a weak, continuous solution ρ_k. As the relevant equations have a maximum principle, when comparing with any $\phi \in C^{2,1}(\mathbb{R}^d \times (0,\infty))$, we have the following properties:

(i) If $p_k - \phi$ has a local maximum zero in $\{t \le t_0\}$ at (x_0, t_0), then

$$(\phi_t - (k-1)\phi(\nabla \cdot (m\nabla\phi) - m_t) - |\nabla\phi|^2)(x_0, t_0) \le 0. \qquad (3.4)$$

(ii) If $p_k - \phi$ has a local minimum zero in $\{p_k > 0\} \cap \{t \le t_0\}$ at (x_0, t_0), then

$$(\phi_t - (k-1)\phi(\nabla \cdot (m\nabla\phi) - m_t) - |\nabla\phi|^2)(x_0, t_0) \ge 0. \qquad (3.5)$$

(iii) Any classical solution of (PME-M p) that lies below p_k at some time cannot cross p_k at a later time.

See [KL] for further properties when $m \equiv 1$ and drift is present. Here property (ii) only clearly holds in $\{p_k > 0\}$ to match up with the notion of weak solutions. The next lemma shows the resulting behavior at the free boundary is still as we expect.

Lemma 3.5. *Fix $k \geq 2$ and assume (M1) and (M2). Suppose that ϕ is a smooth function and $p_k - \phi$ has a local minimum zero in $\overline{\{p_k > 0\}}$ at $(x_0, t_0) \in \partial\{p_k > 0\}$. If ϕ satisfies (3.1), then*

$$\left[\phi_t - |\nabla \phi|^2\right](x_0, t_0) \geq 0.$$

Similarly, if $p_k - \phi$ has a local maximum zero in $\overline{\{p_k > 0\}}$ at $(x_0, t_0) \in \partial\{u > 0\}$, then

$$\left[\phi_t - |\nabla \phi|^2\right](x_0, t_0) \leq 0.$$

First we need to construct a specific barrier used in the proof.

Lemma 3.6. *Assume (M1) and (M2). Fix values $\varepsilon > 0$, $k > 0$, $\gamma > 0$ and a point $x_0 \in \mathbb{R}^d$. For small enough ε, there exists $\eta > 0$ depending on ε such that we can construct a classical subsolution u of (PME-M)$_k$ in $E_\eta := B_\eta(x_0) \times [-\eta, \eta]$ with $P_0 := (x_0, 0)$ on its free boundary, which moves with normal velocity γ, and satisfies*

$$\gamma \geq |\nabla u(P_0)| - \varepsilon.$$

Proof. We start from the source-type Barenblatt solutions for the standard porous medium equation, which in pressure form are given by

$$S(x, t; \tau, C) = (t + \tau)^{-\lambda dk} \left(C - \gamma \frac{x^2}{(t + \tau)^{2\lambda}}\right)_+, \tag{3.6}$$

where $\lambda = (kd + 2)^{-1}$, $\gamma = \lambda/2$, and C and τ are arbitrary. These are classical moving free boundary solutions of the porous medium equation, which in pressure form is

$$S_t = (k-1)S\Delta S + |\nabla S|^2. \tag{3.7}$$

Here S is centered at 0, the parameter C controls the size of the support of S, and τ controls the advancing speed of the free boundary of S. Select C such that S is supported in $B_R(0)$ and select τ such that the boundary velocity of S at P_0 is ξ. Here R and ξ will be determined later.

Let $r(t) = \mu - \nu t$, with μ, ν parameters to be chosen, and define

$$u(x,t) = \sup_{y \in B_{r(t)}(x)} S(y,t) = S\left(\left(1 - \frac{r(t)}{|x|}\right)x, t\right) \qquad \text{in } E_\eta,$$

with η chosen less than $R/2$. Then

$$u_t = S_t - r'(t)\nabla S \cdot \frac{x}{|x|} = S_t + r'(t)|\nabla S| = S_t - \nu|\nabla S|.$$

Since S is a solution of the porous medium equation Eq. (3.7),

$$u_t = S_t - \nu|\nabla S| = (k-1)S\Delta S + |\nabla S|^2 - \nu|\nabla S|. \tag{3.8}$$

Note $\eta < R/2$ implies E_η is bounded away from the origin with $1/|x| \leq 2/R$. Therefore,

$$\frac{\partial u}{\partial x_j} = \frac{\partial S}{\partial u} + \mu|\nabla S|\mathcal{O}(1/R)$$

and $\nabla u = \nabla S + \mathcal{O}(\mu)$. Taking another derivative and estimating likewise, $\Delta u = \Delta S + \mathcal{O}(\mu)$. Moreover, in E_η,

$$S \leq 2\eta \sup_{E_\eta} |\nabla u| = \mathcal{O}(\eta)$$

and $|\nabla u|$ is maximal at the boundary, where $|\nabla S| \leq \xi = \mathcal{O}(1)$. Thus Eq. 3.8 can be rewritten

$$u_t = (k-1)u(\Delta u + \nabla(\log m)\cdot\nabla u - \partial_t(\log m)) + |\nabla u|^2 - \nu|\nabla u| + \mathcal{O}(\mu) + \mathcal{O}(\eta). \tag{3.9}$$

Now we select the parameters appropriately. Assume $\varepsilon < \inf_{E_\eta \cap \{u>0\}} |\nabla u|/6$ for some small value of η. Let

$$\nu = \varepsilon/3, \;\; \xi = \gamma + \nu > 0.$$

Select η small enough so that in E_η, with $\mathcal{O}(\mu)$ and $\mathcal{O}(\eta)$ terms from (3.9),

$$|\mathcal{O}(\mu) + \mathcal{O}(\eta)| < \varepsilon \inf_{E_\eta \cap \{u>0\}} |\nabla u|/3,$$

and $R = |x'| - \mu = 1 - \mu$. Finally, refine η so that

$$\sup_{E_\eta \cap \{u>0\}} |\nabla u| - \inf_{E_\eta \cap \{u>0\}} |\nabla u| < \varepsilon \inf_{E_\eta \cap \{u>0\}} |\nabla u|$$

By choice of ν,

$$\nu |\nabla u| \geq \frac{\varepsilon \inf_{E_\eta \cap \{u>0\}} |\nabla u|}{3}$$

and by the upper bound on ε, we also have

$$\nu |\nabla u| \leq \frac{\varepsilon \sup_{E_\eta \cap \{u>0\}} |\nabla u|}{3} \leq \frac{\varepsilon \inf_{E_\eta \cap \{u>0\}} |\nabla u|}{2}.$$

Combining estimates,

$$-\varepsilon \inf_{E_\eta \cap \{u>0\}} |\nabla u| \leq -\nu |\nabla u| + \mathcal{O}(\eta) + \mathcal{O}(\mu) \leq 0.$$

Thus Eqn (3.9) implies that in E_η,

$$(k-1)u(\Delta u + \nabla(\log m) \cdot \nabla u - \partial_t(\log m)) + |\nabla u|^2 - \varepsilon \inf_{E_\eta \cap \{u>0\}} |\nabla u| \leq u_t$$

$$\leq (k-1)u(\Delta u + \nabla(\log m) \cdot \nabla u - \partial_t(\log m)) + |\nabla u|^2,$$

so u is a subsolution of $(\text{PME-M})_k$ and the free boundary of u has boundary velocity at P_0 satisfying

$$\gamma \geq |\nabla u(P_0)| - \varepsilon.$$

$\square$

Proof. **(of Lemma 3.5)**

First note the second result is trivial, as substituting $\phi(x_0, t_0) = 0$ in the inequality (i) yields the desired conclusion.

Now we focus on the first inequality. Let $v = p_k$ for convenience, as the same idea applies for any supersolution of (PME-M p). In order to regularize the boundary, we obtain this result first for the inf-convolution $\underline{v}$ defined in Eq. (3.2), then send first $\delta \to 0$ and then $\overline{r} \to 0$ in its definition to obtain the result for v. Thus we suppose $\underline{v} - \phi$ has a local minimum in $\overline{\{\underline{v} > 0\}}$ at $P_0 = (x_0, t_0) \in \partial\{\underline{v} > 0\}$ with ϕ a smooth function satisfying (3.1).

By adding $\varepsilon(t - t_0) - \varepsilon(x - x_0)^2$ to ϕ if needed, we can assume that $\underline{v} - \phi$ has a strict local minimum of zero at P_0. By (3.1), ϕ_+ is nontrivial in $\{\underline{v} > 0\}$ with a smooth free boundary near P_0. Let H be the hyperplane tangent to $\{\phi > 0\}$ at $(x_0, 0)$, with (ν, γ) the inward normal to H with $|\nu| = 1$.

Let $\beta = |\nabla\phi|(x_0, 0) > 0$. Arguing by contradiction, suppose instead of the desired inequality that

$$\left[\phi_t - |\nabla\phi|^2\right](x_0, t_0) < 0.$$

Then for some $\sigma > 0$,

$$\gamma = \frac{\phi_t}{|\nabla\phi|} < \beta - \sigma. \tag{3.10}$$

For small $\eta < 1$,

$$\underline{v} \geq \phi \qquad \text{in } B_\eta(x_0) \times [-\eta, 0]. \tag{3.11}$$

By the regularity of the free boundary of ϕ at P_0, the set $\{x : \phi(x, t_0) > 0\}$ has a space interior ball B_0 with $x_0 \in \partial B_0$. Define

$$\gamma_1 := \begin{cases} \gamma + \sigma/4 & \text{if } \gamma \geq 0 \\ \beta/2 & \text{otherwise} \end{cases}$$

Applying Lemma 3.6 with parameter $\gamma = \gamma_1$, there exists $\eta > 0$ and a classical subsolution u of (PME-M) in $B_\eta(x_0) \times [-\eta, \eta]$ with initial support inside B_0, outward normal velocity γ_1 at $(x_0, 0)$ and for some $0 < \varepsilon < \min(\sigma/4, \beta/4)$,

$$\gamma_1 \geq |\nabla u(x_0, 0)| - \varepsilon. \tag{3.12}$$

We claim that u lies under $\underline{v}$ in $B_\eta(x_0) \times [-\eta, 0]$ for sufficiently small η. Then it is sufficient to note that u crosses $\underline{v}$ at $(x_0, 0)$, which contradicts the fact that u is a subsolution and $\underline{v}$ is a supersolution.

To verify the claim, note by Eq. (3.10) and Inequality (3.12),

$$\begin{aligned} |\nabla u|(x_0, 0) \quad &< \quad \begin{cases} \beta - \frac{\sigma}{2} & \text{if } \gamma \geq 0, \\ 3\beta/4 & \text{otherwise.} \end{cases} \\ &< \quad \beta = |\nabla \phi|(x_0, 0). \end{aligned}$$

By Inequality (3.12), the support of u moves with normal speed larger than the speed of ϕ at $(x_0, 0)$. So by the regularity of ϕ and u as well as their ordering at $t = 0$, for small enough η,

$$\overline{\{u > 0\}} \subset \{\phi > 0\} \text{ in } B_\eta(x_0) \times [-\eta, 0],$$

Therefore, $u \leq \phi \leq \underline{v}$ in $B_\eta(x_0) \times [-\eta, 0]$ for small enough η, as claimed.

□

3.4 Convergence of (PME-M) solutions as $k \to \infty$

Definition 3.4. The *upper and lower half-relaxed limits* $\liminf_*$ and $\limsup^*$ of a sequence of locally bounded functions $p_k(x, t)$ are defined as

$$\limsup_{k\to\infty}{}^* p_k(x,t) := \lim_{r\to 0} \sup_{\substack{k \geq r^{-1} \\ |(y,s)-(x,t)| \leq r}} p_k(y,s), \qquad \liminf_{k\to\infty}{}_* p_k(x,t) := \lim_{r\to 0} \inf_{\substack{k \geq r^{-1} \\ |(y,s)-(x,t)| \leq r}} p_k(y,s).$$

Let Ω_0 be a compact set in $\mathbb{R}^n$ with smooth boundary, and take weak solutions p_k corresponding to solutions ρ_k of (PME-M)$_k$ with initial data $\rho_0 = m_0\chi_{\Omega_0}$. This means the p_k are weak solutions of

$$\frac{1}{m}(\nabla \cdot (m\nabla p_0)) = -\Delta p_0 - \nabla(\log m) \cdot \nabla p_0 + \partial_t(\log m) = 0 \qquad \text{in } \Omega_0,$$

with $p(\cdot, 0) = P_k(\rho_0)$. In order to show p_k converges to a viscosity solution of (FB-M), we utilize the half-relaxed limits

$$q_1(x,t) := \limsup{}^* p_k \qquad q_2(x,t) := \liminf{}_* p_k.$$

Since $p_k(\cdot, 0) = P_k(\rho_0)$, it follows that $q_1 = q_2 = p_0$ at $t = 0$, where p_0 solves

$$\nabla(m(\cdot, 0)\nabla p) = m_t$$

on Ω_0.

Lemma 3.7. *The upper half-relaxed limit q_1 is a viscosity subsolution of (FB-M), and the lower half-relaxed limit q_2 is a viscosity supersolution of (FB-M).*

Proof. First we argue that q_2 is a supersolution. To prove property (b) in the supersolution definition, let ϕ be a smooth function satisfying (3.1) such that $q_2 - \phi$ has a local minimum zero at $(x_0, t_0) \in \overline{\{q_1 > 0\}}$. Adding $\varepsilon(t-t_0) - \varepsilon(x-x_0)^2$ to ϕ if necessary, we may assume that $q_2 - \phi$ has a strict local minimum zero at $(x_0, t_0) \cap B_r(x_0, t_0)$ for small $r > 0$. Then for large enough k, along a subsequence, $p_k - \phi$ has its minimum at (x_k, t_k) in $\overline{\{p_k > 0\}} \cap B_r(x_0, t_0)$ with $(x_k, t_k) \to (x_0, t_0)$.

If $(x_0, t_0) \in \{q_2 > 0\}$, then $(x_k, t_k) \in \{p_k > 0\}$ for large k. By property (3.5) of p_k,

$$\left[\frac{1}{k-1}(\phi_t - |\nabla\phi|^2) - \phi(\Delta\phi + \nabla(\log m) \cdot \nabla\phi - \partial_t(\log m))\right](x_k, t_k) \geq 0,$$

and as $\phi(x_0, t_0) > 0$, we can take a limit as $k \to \infty$ to obtain

$$\Delta\phi + \nabla(\log m) \cdot \nabla\phi - \partial_t(\log m) \leq 0.$$

If instead $(x_0, t_0) \in \partial\{q_2 > 0\}$, suppose for the sake of contradiction that

$$\max\left(-\Delta\phi - \nabla(\log m) \cdot \nabla\phi + \partial_t(\log m), \phi_t - |\nabla\phi|^2\right)(x_0, t_0) < 0.$$

For large enough k,

$$[\phi_t - (k-1)\phi(\Delta\phi + \nabla(\log m) \cdot \nabla\phi + \partial_t(\log m))]\,(x_k, t_k) < 0,$$

so $(x_k, t_k) \in \partial\{p_k > 0\}$. But this contradicts Lemma 3.5, hence

$$\max\left(-\Delta\phi - \nabla(\log m) \cdot \nabla\phi + \partial_t(\log m), \phi_t - |\nabla\phi|^2\right)(x_0, t_0) \geq 0.$$

Now we argue that q_1 is a subsolution. The argument that q_1 satisfies property (c) of the viscosity subsolution definition is parallel to the corresponding argument above for the supersolution property (b).

Property (b) of the subsolution definition (continuous expansion) follows from the construction of a smooth supersolution which prevents jumps in pressure support at the boundary. Suppose for some constants $f, r_0 > 0$, and (x_0, t_0) given we know that $p_k(\cdot, t_0) = 0$ in $B_{r_0}(x_0)$ and $p_k \leq f$ on the parabolic boundary of $B_{2r_0}(x_0) \times [t_0, t_0 + T]$. For all small enough r_0, we construct a supersolution v such that $v = 0$ in $B_{r_0/4}(x_0) \times [t_0, t_0 + T]$. For each point $(y, t) \in \partial\{q_1(\cdot, t) > 0\}$, we apply the above to a sequence of x_0 approaching y_0 from outside $\{q_1 > 0\}$ so that this result gives the desired continuous expansion property.

This construction is a modification of [AKY], Theorem B.1. For convenience, assume that $(x_0, t_0) = (0, 0)$. Let $\kappa_1, \kappa_2 > 0$ such that in a large ball, $|\nabla(\log m)| \leq \kappa_1$ and $-\partial_t(\log m) \leq \kappa_2$. The core spatial part of the barrier is a radial function $w(x) = w_1(|x|)$ with $w_1(r_0/2) = 0$,

$w_1(r_0) = f$, and in $B_{2r_0} \setminus B_{r_0/2}$,

$$\Delta w + \nabla(\log m) \cdot \nabla w - \partial_t(\log m) < 0. \tag{3.13}$$

Let $c(r_0) = \kappa_1 + \frac{2(n-1)}{r_0}$. One such choice is

$$w_1(r) = -C_1 e^{-c(r_0)r} - \frac{\kappa_2}{n} r^2 + C_2.$$

For the following C_1, C_2 we have $w_1(r_0/2) = 0$ and $w_1(r_0) = f$:

$$C_1 = \frac{f + \frac{3}{4}\frac{\kappa_2}{n} r_0^2}{e^{-c(r_0)r_0/2} - e^{-c(r_0)r_0}} > 0, \qquad C_2 = \frac{\kappa_2}{n} r_0^2 + \frac{\frac{3}{4}\frac{\kappa_2}{n} r_0^2 + f}{e^{c(r_0)r_0/2} - 1} > 0.$$

We extend w smoothly from $[r_0/2, 2r_0]$ to $[0, 2r_0]$ so that $w = 0$ on $[0, r_0/4]$.

To verify (3.13), we compute

$$\Delta\left(-\frac{\kappa_2}{n}|x|^2\right) + \kappa_1 \left|\nabla\left(-\frac{\kappa_2}{n}|x|^2\right)\right| \leq -2\kappa_2 + \kappa_1 \frac{\kappa_2}{n}(2r)$$

and

$$\begin{aligned}
\Delta\left(-e^{-c(r_0)|x|}\right) + \kappa_1 \left|\nabla\left(-e^{-c(r_0)|x|}\right)\right| &= e^{-c(r_0)r}\left(-(c(r_0))^2 + c(r_0)\left(\kappa_1 - \frac{(n-1)}{r}\right)\right) \\
&\leq e^{-c(r_0)r}\left(-(c(r_0))^2 + c(r_0)\left(\kappa_1 + \frac{2(n-1)}{r_0}\right)\right) \\
&\leq e^{-c(r_0)r}(-(c(r_0))^2 + (c(r_0))^2) \\
&= 0.
\end{aligned}$$

Thus for small enough r_0 (depending on κ_1, κ_2), w satisfies (3.13).

Also,

$$w_1'(r) = C_1 c(r_0) e^{-c(r_0)r} - \frac{2\kappa_2}{n} r.$$

This is decreasing as r increases, and

$$w_1'(4r_0) \geq \frac{\frac{3}{4}\frac{\kappa_2}{n} r_0^2}{e^{c(r_0)r_0/2} - 1}\left(\frac{2(n-1)}{r_0}\right) - \frac{\kappa_2}{n} 8r_0 > 0,$$

where the second inequality holds for small enough r_0 (depending on κ_1, κ_2). In particular for $r \in [r_0/2, 4r_0]$, $w_1'(r) \geq 0$ and $w_1(r) \geq w_1(r_0) = f$.

Now introducing the time-dependence, define

$$v(x,t) = w(R(t)|x|),$$

where $R(t)$ is to be determined with $R(0) = 1$, $1 \leq R(t) \leq \frac{3}{2}$. Then the second radial derivatives of w receive the factor $(R(t))^2$, while the first radial derivatives receive the factor $R(t)$. Since $R(t) \leq (R(t))^2$, by possibly decreasing r_0, we can ensure as in the construction of w that

$$\Delta v + \nabla(\log m) \cdot \nabla v - \partial_t(\log m) \leq 0.$$

The choice of $R(t)$ will ensure $v_t \geq 2|\nabla v|^2$, which combined with the above, ensures that v is a supersolution of $(\text{PME-M p})_k$. This holds if

$$\frac{R'(t)}{(R(t))^2} \geq 2\frac{w'(R(t)r)}{r},$$

so it suffices to take $R(t) = \frac{1}{1-Lt}$, where

$$\frac{R'(t)}{(R(t))^2} = L := \frac{8}{r_0} \sup_{[r_0/2,4r_0]} w'(r) \geq 2 \sup_{[r_0/2,2r_0]} \frac{w'(R(t)r)}{r},$$

with the last inequality based on $1 \leq R(t) \leq \frac{3}{2}$. Last of all, pick some final time T to ensure that for $t \in [0,T]$,

$$R(t) = \frac{1}{1-Lt} \leq \frac{3}{2}.$$

By construction of v, $p_k \leq v$ on the parabolic boundary of $B_{2r_0}(0) \times [0,T]$. Thus by the comparison principle for $(\text{PME-M})_k$,

$$p_k \leq v \text{ in } B_{2r_0}(0) \times [0,T].$$

Also by construction of v, we have $v = 0$ inside $B_{r_0/4}(0) \times [0,T]$. Thus we conclude that

$p_k = 0$ in $B_{r_0/4}(0) \times [0, T]$, as desired. This completes the proof of property (b) from the subsolution definition.

□

3.5 Uniform convergence

Here we follow ([AKY], Section 3).

Theorem 3.8. *(a) There exists a unique evolution of compact sets $\{\Omega_t\}_{t>0}$ such that any viscosity solution p of (FB-M) satisfies $\Omega_t = \overline{\{p(\cdot, t) > 0\}}$ for each $t > 0$.*

(b) For each $t > 0$, the Hausdorff distance $d_H(\Omega_t, \overline{\{p_k(\cdot, t) > 0\}})$ goes to zero as $k \to \infty$, and $\limsup_{k\to\infty} p_k(\cdot, t)$ is uniformly bounded.

Proof. Starting from p_0, because $\partial\Omega_0$ is assumed to be locally Lipschitz, we can construct a sequence of initial data $q_0^{-,\ell}, q_0^{+,\ell}$ for pressure such that

(i) $q_0^{-,\ell} \prec p_0 \prec q_0^{+,\ell}$ for each ℓ,

(ii) $q_0^{\pm,\ell}$ converges uniformly to p_0 as $\ell \to \infty$,

(iii) $\{q_0^{\pm,\ell} > 0\}$ converges uniformly to $\{p_0 > 0\}$ in Hausdorff distance as $\ell \to \infty$.

Let $q_k^{\pm,\ell}$ be the corresponding solutions of (PME-M p)$_k$ with initial data $q_0^{\pm,\ell}$. As before, we consider the half-relaxed limits

$$q_1^{pm} := \limsup^* q_k^{\pm,\ell}, \qquad q_2^{pm} := \liminf_* q_k^{\pm,\ell}.$$

By Lemma 3.7, q_1 is a viscosity subsolution of (FB-M) and q_2 is a viscosity supersolution of (FB-M), with their appropriate initial data.

Define

$$V(x, t) := (\inf\{v : v \text{ is a viscosity supersolution of (FB-M) with } p_0 \prec v(\cdot, 0)\})_*$$

and

$$U(x,t) := \sup\{u : u \text{ is a viscosity subsolution of (FB-M) with } u(\cdot,0) \prec p_0\}.$$

By the definitions of U and V, we conclude

$$q_1^{pm,\ell} \leq U, \quad V \leq q_2^{pm,\ell}.$$

Also, by property (i) and the comparison principle for (FB-M) and (PME-M) (given in Lemma A.2),

$$q_k^{-,\ell} \leq q_k^{+,\ell} \quad \text{and} \quad q_1^{-,\ell} \leq q_2^{+,\ell}. \tag{3.14}$$

Writing the L^1 contraction property for weak solutions ρ_1, ρ_2 of (PME-M)$_k$ in terms of pressure $p_i = \frac{k}{k-1}\left(\frac{\rho_i}{m}\right)^{k-1}$,

$$||m(\cdot,t)(p_1^{1/(k-1)} - p_2^{1/(k-1)})(\cdot,t)||_{L^1(\mathbb{R}^d)} \leq ||m(\cdot,0)(p_1^{1/(k-1)} - p_2^{1/(k-1)})(\cdot,0)||_{L^1(\mathbb{R}^d)}.$$

In particular,

$$||m(\cdot,t)((q_k^{-,\ell})^{1/(k-1)} - (q_k^{+,\ell})^{1/(k-1)})(\cdot,t)||_{L^1(\mathbb{R}^n)} \leq ||m(\cdot,0)((q_0^{-,\ell})^{1/(k-1)} - (q_0^{+,\ell})^{1/(k-1)})||_{L^1(\mathbb{R}^n)}.$$

Now we can consider half-relaxed limits as $k \to \infty$, and use the comparisons (3.14) to conclude

$$\begin{aligned} ||m(\cdot,t)\chi_{S^\ell(t)}||_{L^1(\mathbb{R}^n)} &\leq \limsup_{k\to\infty} ||m(\cdot,t)((q_k^{-,\ell})^{1/(k-1)} - (q_k^{+,\ell})^{1/(k-1)})(\cdot,t)||_{L^1(\mathbb{R}^n)} \\ &\leq ||m(\cdot,0)\chi_{S^\ell(0)}||_{L^1(\mathbb{R}^n)}, \end{aligned}$$

where for $t \geq 0$,

$$S^\ell(t) := \{q_2^{+,\ell}(t) > 0\} \setminus \{q_1^{-,\ell}(t) > 0\}.$$

Finally, taking limits as $\ell \to \infty$, using (ii) and (iii), as well as the lower bound on m, and noting $\{V(\cdot,t)>0\}$ is open, we can define

$$\Omega_t := \overline{\{V(\cdot,t)>0\}} = \overline{\{U(\cdot,t)>0\}}.$$

Also by this reasoning, for any viscosity solution p of (FB-M) with initial data p_0, the comparison principle forces $U \le p \le V$, and the above inequalities ensure

$$\Omega_t = \overline{\{p(\cdot,t)>0\}}.$$

For part (b), note we have shown q_1 is a subsolution of (FB-M) with initial data ρ_0, $U \le q_1 \le V$, and

$$\Omega_t = \overline{\{q_1(\cdot,t)>0\}}.$$

Therefore, $d_H(\Omega_t, \overline{\{p_k(\cdot,t)>0\}}) \to 0$ as $k \to \infty$.

□

Rewriting the convergence results in terms of density in the patch case, we have the following.

Corollary 3.9. *Let Ω_t be as given in Theorem 3.8, and ρ_k be solutions of (PME-M) starting from $\rho_0 = m(\cdot,0)\left(\frac{k-1}{k}p_0\right)^{1/(k-1)}$. Then for each $t>0$,*

(i) $\limsup \rho_k(\cdot,t) \le m(\cdot,t)$,

(ii) $\overline{\{\rho_k(\cdot,t)>0\}}$ *converges uniformly to Ω_t in Hausdorff distance,*

(iii) $\rho_k(\cdot,t)$ *converges locally uniformly to $m(\cdot,t)$ in the interior of Ω_t, and to 0 in $(\Omega_t)^c$.*

The same results also hold for ρ_k with initial data $m(\cdot,0)\chi_{\Omega_0}$.

Chapter 4

Wasserstein convergence for modified porous medium equation

4.1 (PME-M) solutions as gradient flows

Recall in Chapter 2 we identified the limit of the W_2 minimizing movement scheme for E_∞ as a solution of the DCM. In order to identify this solution with the viscosity solution discussed in Chapter 3, we utilize solutions ρ_k to the modified porous medium equation

$$\rho_t - \nabla \cdot \left(\rho \, \nabla \left[\left(\frac{\rho}{m} \right)^k \right] \right) = 0, \qquad \text{(PME-M)}$$

then consider a limit as $k \to \infty$. This is a reasonable approach since the solutions ρ_k can be obtained by using a W_2 minimizing movement scheme with an energy that also approaches E_∞.

The relevant energy is $E_k : \mathcal{P}_2(\mathbb{R}^d) \times [0, \infty) \to \mathbb{R}$ defined by

$$E_k(\rho, t) = \int_{\mathbb{R}^d} \frac{1}{k+1} \left(\frac{\rho}{m(\cdot, t)} \right)^{k+1}.$$

In particular, notice the time-dependence of this energy. The primary reference for gradient flows given corresponding λ-convex energies is [AGS], but its framework does not directly allow for this time-dependence. We naturally worked for a while to develop such theory, allowing time dependent energies, in order to produce results similar to those below, and later discovered that this was already completed by Lucas Ferreira and Julio Valencia-Guevara in [FV]. The assumptions and results from [FV] that are relevant for further analysis below are listed in Appendix B.

First we must check that the energy $\mathcal{E} = E_k$ satisfies the assumptions (E1) through (E5) in Appendix B. The relevant space here is $X = \mathcal{P}_2(\mathbb{R}^d)$ and the metric d is the 2-Wasserstein metric. Throughout we reference parameters M_-, M_+, α from assumption (M1).

For (E1), consider $\rho = M_- \chi_{B_r(0)}$ where r is such that $\int \rho = 1$. Then for any $t > 0$, $\rho \leq M_- \leq m(\cdot, t)$ by (M1), so

$$\mathcal{E}(\rho, t) \leq \int \frac{1}{k+1} \chi_{B_r(0)} \, dx < \infty.$$

Thus $\mathcal{E}$ is proper. Since $m \leq M_+$, the argument of $\mathcal{E}$ has superlinear growth at infinity, thus general results on integral functionals show that $\mathcal{E}$ is lower semicontinuous. ([AGS], Remark 9.3.8).

For (E2), let $t_1, t_2 \in [0, \infty)$. If $\mathcal{E}(\rho, t_1) < \infty$, then using $M_- \leq m \leq M_+$,

$$\mathcal{E}(\rho, t_2) \leq \left(\frac{M_+}{M_-} \right)^k \mathcal{E}(\rho, t_1) < \infty.$$

Thus the domain of $\mathcal{E}(\cdot, t)$ is independent of t.

For (E3), fix ρ and estimate

$$\begin{aligned} |\partial_t \mathcal{E}(\rho, t)| &= \left| \partial_t \left(\int_{\mathbb{R}^d} \frac{1}{k+1} (m(\cdot, t))^{-k} \rho^{k+1} \right) \right| \\ &= \frac{k}{k+1} \int_{\mathbb{R}^d} |\partial_t m(\cdot, t)| (m(\cdot, t))^{-(k-1)} \rho^{k+1} \\ &\leq \frac{\alpha k}{M_-} \mathcal{E}(\rho, t). \end{aligned}$$

Here we use $|m_t| \leq \alpha$ from (M1). Thus with $C := \frac{\alpha k}{M_-}$, it follows that $\mathcal{E}(\rho, t) \leq e^{CT} \mathcal{E}(\rho, 0)$ and

$$|\partial_t \mathcal{E}(\rho, t)| \leq C \mathcal{E}(\rho, t) \leq C e^{CT} \mathcal{E}(\rho, 0),$$

so we can take β proportional to e^{CT}.

(E4) is clear as $\rho \geq 0$ implies $\mathcal{E} \geq 0$.

Finally, recall (M3): there exists k_0 such that the function $(m(\cdot, t))^{-k}$ is convex in $\mathbb{R}^d$ for all $t \geq 0$ and all $k \geq k_0$. This is assumed in order to guarantee (E5), which uses the shorthand

$$\mathbf{E}(t, \tau, \rho; q) := \mathcal{E}(q, t) + \frac{d^2(\rho, q)}{2\tau}$$

and says the following:

(E5): There exists $\lambda : [0,\infty) \to \mathbb{R}$ in $L^\infty_{\text{loc}}([0,\infty))$ such that: given points $\rho, q_0, q_1 \in X$, there exists a curve $\gamma : [0,1] \to X$ satisfying $\gamma(0) = q_0, \gamma(1) = q_1$, and

$$\mathbf{E}(t,\tau,\rho;\gamma(s)) \le (1-s)\mathbf{E}(t,\tau,\rho;q_0) + s\mathbf{E}(t,\tau,\rho;q_1) - \frac{1+\tau\lambda(t)}{2\tau} s(1-s)d^2(q_0,q_1),$$

for $0 < \tau < \frac{1}{\lambda_T}$ and $s \in [0,1]$, where $\lambda_T = \max\{0, -\inf_{t\in[0,T]} \lambda(t)\}$.

We use results from sources which studied λ-convexity along geodesics of functionals of the form

$$\Phi(\eta) = \int F(x,\eta(x))\,dx.$$

for smooth F. This λ-convexity is strongly connected to a Wasserstein evolution variational inequality. For the next lemma the adjoint of F is $H(x,\xi) = \xi F(x, 1/\xi)$.

Lemma 4.1 ([FM] Section 3, [DS]). *Let $F \in C^2$ be given and assume there exists $C > 0$ such that*

$$\eta F_{\eta\eta}(x,\eta) \le C \qquad \textit{and} \qquad |\nabla_x F_\eta(x,\eta)| \le C \qquad \textit{for all } (x,\eta).$$

Assume further that there is some $\kappa \in \mathbb{R}$ such that $H_\kappa(x,\xi) = H(x,\xi) - \frac{\kappa}{2}|x|^2$ is (jointly) convex. Let S_ϕ denote the solution operator of the evolution equation

$$\eta_t = \nabla(\eta \nabla F_\eta).$$

For arbitrary $\rho \in \mathcal{P}_2(\mathbb{R}^d)$, the curve $s \mapsto S^s_\Phi$ is absolutely continuous on $[0,\infty]$ and satisfies the evolution variational inequality

$$\frac{1}{2}\frac{d^+}{d\sigma}\bigg|_{\sigma=s} W_2(S^\sigma_\Phi\rho,\tilde\rho)^2 + \frac{\kappa}{2} W_2(S^s_\Phi\rho,\tilde\rho)^2 \le \Phi(\tilde\rho) - \Phi(S^s_\Phi\rho)$$

for all $s > 0$, with respect to any comparison measure $\tilde\rho \in \mathcal{P}_2(\mathbb{R}^d)$ for which $\Phi(\tilde\rho) < \infty$. This in turn is used to prove Φ is λ-convex along geodesics, for any $\lambda \ge \kappa$.

To apply this lemma to (PME-M), let $F = a(x)\eta^m$; then $H = a(x)\xi^{1-m}$ and the most relevant entries of the Hessian are

$$\partial_{x_i x_i} H_\kappa(x,\xi) = a_{x_i x_i}(x)\xi^{1-m} - \kappa, \quad \partial_{x_i x_j} H_\kappa(x,\xi) = a_{x_i x_j}(x)\xi^{1-m} \quad \text{if } i \neq j.$$

Taking $\kappa = 0$, the adjoint H is (jointly) convex specifically when a is convex. For our purposes, $a(x) = (m(x,t))^{1-k}$. Thus (M3) is sufficient to guarantee the convexity of H in Lemma 4.1. Without assumption (M3), [FM] gives a good indication that (E5) does not hold. Without (E5), the uniqueness and convergence of the minimizing movement scheme for E_k is not guaranteed, and a good rate of convergence is not attainable.

Finally to check the other conditions of Lemma 4.1,

$$\eta F_{\eta\eta} = k\left(\frac{\rho}{m}\right)^k,$$

For each fixed k, this is bounded based on (M1) (as $M_- \leq m$) and Theorem A.8. Similarly, $|\nabla_x F_\eta|$ is bounded based on (M1) (as $|\nabla m|$ is bounded) and Theorem A.8. Thus using (M3) and Lemma (4.1) we conclude (E5) with $\lambda = 0$ for all large enough k.

Having verified (E1) - (E5), we are ready to define the minimizing movement scheme and provide results related to its convergence. Given $\tau > 0$, take ρ_τ^0 with $E_k(\rho_\tau^0, 0) < \infty$ and define for $n \geq 1$,

$$\rho_\tau^n \in \operatorname*{arg\,min}_{q\in X} \mathbf{E}(\tau n, \tau, \rho_\tau^{n-1}; q) = \operatorname*{arg\,min}_{\rho\in\mathcal{P}_2(\mathbb{R}^d)} \left[\frac{1}{2\tau}W_2^2(\rho_\tau^{n-1}, \rho) + E_k(\rho, \tau n)\right]. \tag{4.1}$$

In particular, note how the energy is evaluated at time τn.

By Lemma B.1, the minimizers ρ_τ^n exist and are unique. Note (E5) is used to obtain uniqueness. Given the ρ_τ^n, define the approximate solutions

$$\underline{\rho}_\tau(t) := \rho_\tau^{n-1}, \quad \overline{\rho}_\tau(t) := \rho_\tau^n, \qquad \text{for } t \in [\tau(n-1), \tau n).$$

Here we collect results from Appendix B when applied to E_k.

Theorem 4.2. *(i) Let $E_k(\rho_0, 0) < \infty$. Suppose ρ_τ^0 is chosen to satisfy the conditions*

$$\lim_{\tau \to 0} W_2(\rho_\tau^0, \rho_0) = 0, \qquad \sup_\tau \mathcal{E}(\rho_\tau^0, 0) < \infty,$$

the approximate solutions $\overline{\rho}_\tau$ and $\underline{\rho}_\tau$ converge locally uniformly as $\tau \to 0$ to a function $\rho : [0, \infty) \to \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\rho(0) = \rho_0$. Moreover, ρ is independent of the choice of family ρ_τ^0.

(ii) Given two initial data $u_0, v_0 \in D$, if we let $u(t), v(t)$ be limits of the minimizing movement scheme (4.1) with initial data u_0 and v_0 respectively as given in part (i), then

$$W_2(u(t), v(t)) \leq W_2(u_0, v_0).$$

(iii) Define the piecewise constant functions in time

$$\rho_\tau(x, t) := \rho_\tau^n(x) \qquad \text{for } t \in [\tau n, \tau(n+1)).$$

If $\rho_\tau^0 = \rho_0$, there exists a constant $C > 0$ dependent on m and T such that

$$W_2(\rho_\tau(\cdot, t), \rho(\cdot, t)) \leq C\sqrt{\tau}.$$

Corollary 4.3. *Taking limits in k and using Corollary 2.10, properties (ii) and (iii) also hold for the case $k = \infty$ (the minimizing movement scheme from Chapter 2).*

Lemma 4.4. *Suppose $E_k(\rho_0, 0) < \infty$. The limit of the minimizing movement scheme (4.1) is a weak solution of (PME-M) with initial data ρ_0.*

Proof. Consider a time-independent energy of the form

$$\mathcal{F}(\rho) = \int_{\mathbb{R}^d} F(x, \rho(x))\, dx$$

with smooth $F(x, z) : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ satisfying (E1) through (E5) with $\lambda = 0$. It is well known (e.g., [AGS] Chapter 11) that the limit ρ of the minimizing movement scheme for $\mathcal{F}$

is a weak solution of

$$\partial_t \rho - \nabla \cdot \left(\rho \nabla \left(\frac{\delta \mathcal{F}}{\delta \rho} \right) \right) = 0, \tag{4.2}$$

where

$$\frac{\delta \mathcal{F}(\rho)}{\delta \rho} = F_z(x, \rho)$$

is the first variation of $\mathcal{F}$.

Compare the derivative

$$\frac{d}{dt}\mathcal{F}(\rho(t)) = \int_{\mathbb{R}^d} \frac{\delta \mathcal{F}(\rho)}{\delta \rho} \partial_t \rho$$

with the gradient flow identity

$$\mathcal{F}(\rho(t)) - \mathcal{F}(\rho_0) = -\frac{1}{2}\int_0^t |\rho'|^2(s)\,ds - \frac{1}{2}\int_0^t |\partial \mathcal{F}|^2(\rho(s))\,ds.$$

For a time dependent energy $\mathcal{E}(\rho, t) = \int_{\mathbb{R}^d} E(x, \rho, t)$, these are adjusted as follows (see Theorem B.3 for the gradient flow identity):

$$\frac{d}{dt}\mathcal{E}(\rho(t), t) = \int_{\mathbb{R}^d} \left(\partial_t \mathcal{E}(\rho(t), t) + \frac{\delta \mathcal{E}(\rho)}{\delta \rho} \partial_t \rho \right)$$

and

$$\mathcal{E}(\rho(t), t) - \mathcal{E}(\rho_0, 0) = \int_0^t \partial_t \mathcal{E}(\rho(s), s)\,ds - \frac{1}{2}\int_0^t |\rho'|^2(s)\,ds - \frac{1}{2}\int_0^t |\partial \mathcal{E}(s)|^2(\rho(s))\,ds.$$

Both see the addition of precisely the $\partial_t \mathcal{E}$ term. Thus the equation for $\partial_t \rho$ maintains the same structure, and ρ is a weak solution of

$$\partial_t \rho - \nabla \cdot \left(\rho \nabla \left(\frac{\delta \mathcal{E}(x, \rho, t)}{\delta \rho} \right) \right) = 0.$$

For the choice

$$\mathcal{E}(\rho, t) = \int_{\mathbb{R}^d} \frac{m(x,t)}{k+1} \left(\frac{\rho}{m(x,t)} \right)^{k+1} dx,$$

the relevant term in the equation is

$$\frac{\delta \mathcal{E}(x,\rho,t)}{\delta \rho} = \left(\frac{\rho}{m(x,t)} \right)^k,$$

so ρ is a weak solution of (PME-M).

As assumed in Theorem 4.2 part (i), the initial data for the minimizing movement scheme tends to ρ_0, so ρ has initial data ρ_0. □

4.2 One-step estimate for Wasserstein convergence

Here we develop some one-step estimates similar to ([AKY], Section 4).

Recall

$$E_k(\rho,t) := \int_{\mathbb{R}^d} \frac{m(\cdot,t)}{k+1} \left(\frac{\rho}{m(\cdot,t)} \right)^{k+1} dx$$

and given the sets

$$K_t := \left\{ \rho \in \mathcal{P}_2(\mathbb{R}^d) \text{ with } \rho(x) \le m(x,t) \text{ for a.e. } x \right\},$$

let

$$E_\infty(\rho,t) = \begin{cases} 0 & \text{if } \rho \in K_t \\ +\infty, & \text{if } \rho \notin K_t, \end{cases}$$

Fix $\rho_0 \in \mathcal{P}_2(\mathbb{R}^n)$ with $\rho_0 \le m(\cdot,0)$. For each $\tau > 0$ and each k (including $k = \infty$), consider the first step of the minimizing movement scheme:

$$\mu_{\tau,k} := \underset{\rho \in \mathcal{P}_2(\mathbb{R}^d)}{\arg\min} \left[\frac{1}{2\tau} W_2(\rho,\rho_0) + E_k(\rho,\tau) \right]. \tag{4.3}$$

The strategy is first to control the amount of mass in $\mu_{\tau,k}$ above the density constraint $m(\cdot,\tau)$ (which is a hard constraint for $k = \infty$, but only applies in a rough sense for $k < \infty$).

Lemma 4.5. *Let $1 < k < \infty$. Given $\rho_0 \in \mathcal{P}_2(\mathbb{R}^n)$ with $\rho_0 \le m(\cdot,0)$, define $\mu_{\tau,k}$ as above. There exists C depending on m but independent of τ and k such that the following estimate* *holds:*

$$\int_{\mathbb{R}^d} (\mu_{\tau,k} - m(\cdot,\tau))_+ \, dx \le \frac{C}{\sqrt{k+1}}.$$

(Here f_+ denotes the positive part of f.)

Proof. We adjust C freely as necessary in the following.

Note $M_- \chi_{B_r(0)}$ for appropriate r with $|B_r(0)| = 1/M_-$ is a candidate for the minimizer $\mu_{\tau,k}$, with

$$E_k(M_- \chi_{B_r}(0), \tau) \le \frac{C}{k+1}$$

so there exists some $C > 0$ dependent only on m such that $E_k(\mu_{\tau,k}, \tau) < C$. Due to bounds on m in (M1), this means

$$\int_{\mathbb{R}^d} \left(\frac{\mu_{\tau,k}(x)}{m(x,\tau)} \right)^{k+1} \, dx \ \le C. \tag{4.4}$$

As $k+1 > 2$,

$$\begin{aligned} \int_{\{\mu_{\tau,k} \ge m(\cdot,\tau)\}} \left(\frac{\mu_{\tau,k}}{m(\cdot,\tau)} \right)^{k+1} \quad &\ge \quad \int_{\{\mu_{\tau,k} \ge m(\cdot,\tau)\}} \left(\frac{1}{m(\cdot,\tau)^{k+1}} \right) (m(\cdot,\tau) + (\mu_{\tau,k} - m(\cdot,\tau)))^{k+1} \\ &\ge \int_{\{\mu_{\tau,k} \ge m(\cdot,\tau)\}} \left(\frac{1}{m(\cdot,\tau)^{k+1}} \right) \left(\frac{(k+1)(k)}{2} (m(\cdot,\tau)^{k-1})(\mu_{\tau,k} - m(\cdot,\tau))^2 \right). \end{aligned}$$

Combining this with inequality (4.4) and bounding m^2 in the denominator using $m \le M_+$, it follows that

$$\int_{\mathbb{R}^d} (\mu_{\tau,k} - m(\cdot,\tau))_+^2 \le \frac{C}{k+1}.$$

As $\int_{\mathbb{R}^d} \mu_{\tau,k} = 1$ and $m \ge M_-$, we must have $|\{\mu_{\tau,k} - m(\cdot,\tau) \ge 0\}| \le \frac{1}{M_-}$, in which case the Cauchy-Schwarz inequality gives

$$\begin{aligned} \int_{\mathbb{R}^d} (\mu_{\tau,k} - m(\cdot,\tau))_+ \quad &= \quad \int_{\mathbb{R}^d} (\mu_{\tau,k} - m(\cdot,\tau)) \chi_{\{\mu_{\tau,k} \ge m(\cdot,\tau)\}} \\ &\le \quad \left(|\{\mu_{\tau,k} - m(\cdot,\tau) \ge 0\}| \int_{\mathbb{R}^d} (\mu_{\tau,k} - m(\cdot,\tau))_+^2 \right)^{1/2} \\ &\le \quad \frac{C}{\sqrt{k+1}}. \end{aligned}$$

□

Now that we know the mass of $\mu_{\tau,k}$ above $m(\cdot,\tau)$ is small, we put together a probability measure $\tilde{\mu}_{\tau,k}$ that is close to $\mu_{\tau,k}$ in Wasserstein distance and does not exceed $m(\cdot,\tau)$.

Lemma 4.6. *For all large enough k, there exists a probability density $\tilde{\mu}_{\tau,k}$ such that*

$$\tilde{\mu}_{\tau,k} \leq m(\cdot,\tau) \qquad \text{and} \qquad W_2^2(\mu_{\tau,k},\tilde{\mu}_{\tau,k}) \leq \frac{C}{\sqrt{k+1}}.$$

Proof. With C from Lemma 4.5, take $a := \frac{C}{\sqrt{k+1}}$. Let k be large enough that $a \leq \frac{1}{2}M_-$. We break $\mu_{\tau,k}$ into

$$\mu^1_{\tau,k}(x) = \min\{\mu_{\tau,k}(x), m(\cdot,\tau) - a\} \qquad \text{and} \qquad \mu^2_{\tau,k}(x) := (\mu_{\tau,k}(x) - (m(\cdot,\tau) - a))_+,$$

then construct $\tilde{\mu}_{k,\tau}$ by leaving $\mu^1_{k,\tau}$ alone and redistributing $\mu^2_{k,\tau}$.

First we need to estimate the mass of $\mu^2_{\tau,k}$. We claim

$$|\{\mu_{\tau,k} > m(\cdot,\tau) - a\}| \leq \frac{1}{M_- - a} \leq \frac{2}{M_-}. \tag{4.5}$$

If the first inequality of (4.5) fails,

$$1 = \int \mu_{\tau,k} > \int \min\{\mu_{\tau,k}, m(\cdot,\tau) - a\} \geq \frac{1}{M_- - a} \cdot (m - a) \geq 1,$$

which is impossible. The second inequality in (4.5) is due to the choice of a. Thus

$$\begin{aligned} \int \mu^2_{\tau,k} &\leq (\text{mass of } \mu_{\tau,k} \text{ above } m) + |\{\mu_{\tau,k} > m(\cdot,\tau) - a\}| \cdot a \\ &\leq \left(1 + \frac{2}{M_-}\right) a. \end{aligned}$$

Based on this estimate, we pick

$$g(x) = \frac{1}{2\left(1 + \frac{2}{M_-}\right)} \chi_{B(0,R(d))},$$

where $R(d) \leq 1$ is the dimensional constant such that $\int g = 1$, and define

$$\tilde{\mu}_{\tau,k} = \mu^1_{\tau,k}(x) + (g * \mu^2_{\tau,k})(x).$$

Note $\tilde{\mu}_{\tau,k}$ is a non-negative probability measure, since convolution with g merely redistributed some mass of $\mu_{\tau,k}$. By Young's inequality,

$$||g * \mu^2_{\tau,k}|| \leq ||\mu^2_{\tau,k}||_{L^1}||g||_{L^\infty} \leq \left(1 + \frac{2}{M_-}\right) a \cdot \frac{1}{2\left(1 + \frac{2}{M_-}\right)} \leq a,$$

thus

$$\tilde{\mu}_{\tau,k} \leq \mu^1_{\tau,k} + a \leq m(\cdot,\tau) - a + a = m(\cdot,\tau).$$

Finally, we need to estimate the Wasserstein distance. We heuristically describe a transport plan from $\mu_{\tau,k}$ to $\tilde{\mu}_{\tau,k}$. This plan keeps the mass of $\mu^1_{\tau,k}$ at its original location, which incurs no W_2^2 cost. Also, we redistribute mass in $\mu^2_{\tau,k}$ at each point x evenly in the disk $B(x, R(d))$. The W_2^2 cost of this redistribution is bounded by

$$\left(\int \mu^2_{\tau,k}\right)(R(d))^2 \leq Ca = \frac{C}{\sqrt{k+1}}.$$

□

Using these ideas, it is possible to bound the Wasserstein distance between $\mu_{\tau,k}$ and $\mu_{\tau,\infty}$ for large k.

Proposition 4.7. *Let $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ with $\rho_0 \leq m(\cdot,0)$. Let $\mu_{\tau,k}$ and $\mu_{\tau,\infty}$ be defined by (4.3). Then*

$$W_2^2(\mu_{\tau,k}, \mu_{\tau,\infty}) \leq \frac{C}{k^{1/4}},$$

where C depends on m but not τ or k.

Proof. We argue by contradiction, in which case, for arbitrarily large $A_0 > 0$ there exist

$k > 1$ and $\tau > 0$ such that

$$W_2^2(\mu_{\tau,k}, \mu_{\tau,\infty}) = Am^{-1/4}, \qquad \text{where } A > A_0. \tag{4.6}$$

We will construct a new probability measure $\eta \in \mathcal{P}_2(\mathbb{R}^d)$ such that the following inequality holds if A_0 is chosen to be sufficiently large:

$$\left[E_k(\eta, \tau) + \frac{1}{2\tau} W_2^2(\rho_0, \eta)\right] + \left[E_\infty(\eta, \tau) + \frac{1}{2\tau} W_2^2(\rho_0, \eta)\right] < \tag{4.7}$$

$$\left[E_k(\mu_{\tau,k}, \tau) + \frac{1}{2\tau} W_2^2(\rho_0, \mu_{\tau,k})\right] + \left[E_\infty(\mu_{\tau,\infty}, \tau) + \frac{1}{2\tau} W_2^2(\rho_0, \mu_{\tau,\infty})\right].$$

Thus η must be a better candidate for at least one of $\mu_{\tau,k}$ or $\mu_{\tau,\infty}$, contradicting one of their definitions.

Let η be the midpoint on the generalized geodesic between $\mu_{\tau,\infty}$ and $\tilde{\mu}_{\tau,k}$ (from Lemma 4.6) with reference measure ρ_0. To control Wasserstein distances, we use the 1-convexity of W_2^2 along generalized geodesics. At the midpoint, this says

$$W_2^2(\rho_0, \eta) \le \frac{1}{2} W_2^2(\rho_0, \tilde{\mu}_{\tau,k}) + \frac{1}{2} W_2^2(\rho_0, \mu_{\tau,\infty}) - \frac{1}{4} W_2^2(\tilde{\mu}_{\tau,k}, \mu_{\tau,\infty}).$$

Combining this with $W_2^2(\mu_{k,\tau}, \tilde{\mu}_{k,\tau}) \le Ck^{-1/2}$ from Lemma (4.6), $W(\rho_0, \mu_{k,\tau}) \le C$ (because $\mu_{k,\tau}$ is a minimizer), assumption (4.6), and the triangle inequality,

$$\begin{aligned}
W_2^2(\rho_0, \eta) \;&\le\; \frac{1}{2}\left(W_2(\rho_0, \mu_{\tau,k}) + W_2(\mu_{\tau,k}, \tilde{\mu}_{\tau,k})\right)^2 + \frac{1}{2} W_2^2(\rho_0, \mu_{\tau,k}) \\
&\qquad\qquad\qquad\qquad - \frac{1}{4}\left(W_2(\mu_{\tau,k}, \mu_{\tau,\infty}) - W_2(\mu_{\tau,k}, \tilde{\mu}_{\tau,k})\right)^2 \\
&\le\; \frac{1}{2} W_2^2(\rho_0, \mu_{\tau,k}) + \frac{1}{2} W_2^2(\rho_0, \mu_{\tau,\infty}) - \frac{1}{4} W_2^2(\mu_{\tau,k}, \mu_{\tau,\infty}) + \frac{C}{k^{1/2}}(1 + W_2(\mu_{\tau,k}, \mu_{\tau,\infty})) \\
&\le\; \frac{1}{2} W_2^2(\rho_0, \mu_{\tau,k}) + \frac{1}{2} W_2^2(\rho_0, \mu_{\tau,\infty}) + \frac{1}{k^{1/2}}(C - C'A^2).
\end{aligned}$$

Since $\tilde{\mu}_{\tau,k}$ and $\mu_{\tau,\infty}$ are less than $m(\cdot, \tau)$, $\eta \le m(\cdot, \tau)$. Thus

$$E_k(\eta, \tau) + E_\infty(\eta, \tau) \le E_k(\mu_{\tau,k}, \tau) + E_\infty(\mu_{\tau,\infty}, \tau) + \frac{C}{k+1}.$$

(Here we estimate $E_k(\eta, \tau) \leq \frac{C}{k+1}$ just as is done at the start of Lemma 4.5.)

Adding ($\frac{1}{\tau}$ times the) W_2^2 inequality and the energy inequality, for large enough k and A_0 (since $A > A_0$), we recover (4.7).

□

4.3 Wasserstein convergence

Now we use the notation $\rho^n_{\tau,k}$ for the n-th step of the minimizing movement scheme for E_k, for $k \geq 2$ (including $k = \infty$). The one-step estimate can be strengthened to hold after n steps for any $n \leq \frac{T}{\tau}$:

Lemma 4.8. *Fix $\tau > 0$. There exists $C > 0$ dependent on m and T but not τ or k such that for all $n \leq \frac{T}{\tau}$*

$$W_2(\rho^n_{\tau,k}, \rho^n_{\tau,\infty}) \leq C\sqrt{\tau} \qquad \textit{when } k \geq C\tau^{-12}.$$

Proof. Define the additional measures

$$\eta^n_\tau := \underset{\rho \in \mathcal{P}_2(\mathbb{R}^d)}{\arg\min} \left\{ E_k(\rho) + \frac{1}{2\tau} W_2^2(\rho^{n-1}_{\tau,\infty}, \rho) \right\} \qquad \text{for } n \geq 2.$$

By the one-step estimate in Proposition 4.7 starting from $\rho^{n-1}_{\tau,\infty}$, for all $n \geq 2$,

$$W_2(\rho^n_{\tau,\infty}, \eta^n_\tau) \leq \delta := Ck^{-1/8}.$$

For $n \geq 0$, define $d_n = W_2(\rho^n_{\tau,k}, \rho^n_{\tau,\infty})$. In [FV] Theorem 4.4 (or see Lemma B.5), we see that the 2-Wasserstein distance between two discrete solutions is controlled:

$$W_2^2(\rho^n_{\tau,k}, \eta^n_\tau) \leq W_2^2(\rho^{n-1}_{\tau,k}, \rho^{n-1}_\infty) + C\tau \leq d^2_{n-1} + C\tau.$$

Using the triangle inequality and the above estimate

$$d_n \leq W_2(\rho^n_{\tau,k}, \eta^n_\tau) + W_2(\eta^n_\tau, \rho^k_{\tau,\infty}) \leq W_2(\rho^n_{\tau,k}, \eta^n_\tau) + \delta \leq \sqrt{d^2_{n-1} + C\tau} + \delta.$$

Initially the one-step estimate gives $d_1 \leq \delta$, then this inequality controls the growth of the family $(d_n)_{n\geq 1}$.

Proceeding as in the proof of [AKY] Theorem 4.2 part 4 with $a_n = C\tau$ shows

$$d_{T/h} \leq C\sqrt{h}$$

for $\delta \leq k^{3/2}$ (so by the definition of δ, $k > m^{-12}$), with C dependent on m and T.

□

Theorem 4.9. *Assuming (M1) and (M3), for any $T > 0$,*

$$\lim_{k\to\infty} \sup_{t\in[0,T]} W_2(\rho_k(t), \rho_\infty(t)) = 0,$$

with convergence rate

$$\sup_{t\in[0,T]} W_2(\rho_k(t), \rho_\infty(t)) \leq \frac{C(T, M_-, M_+, \alpha)}{k^{1/24}}.$$

Proof. This is immediate from Lemma 4.8, Theorem 4.2 part (iii) and Corollary 4.3, with the $k^{-1/24}$ power due to the $k \geq \tau^{-12}$ requirement and the $\sqrt{\tau}$ bound. □

Together with Corollary 3.9, this completes the proof of Theorem 1.2.

Chapter 5

Related Topics

This chapter contains some discussion, simulations, and applications of the results from the previous chapters. In parts the discussion here is not quite as rigorous or fully fleshed out as previous chapters in order to express the new ideas and major challenges involved.

5.1 Order of compression and long term behavior

Suppose (M1) and (M2) and consider our solution of the DCM starting from some fixed $\rho_0 = m(\cdot, 0)\chi_{\Omega_0}$, where the minimizing movement scheme takes

$$\rho_\tau^n = \operatorname*{arg\,min}_{\rho \in \mathcal{P}_2(\mathbb{R}^d) : \rho \leq m(\cdot, \tau n)} W_2(\rho, \rho_\tau^{n-1}).$$

The value of Ω_T and $\rho(x, T) = m(x, T)\chi_{\Omega_T}$ often depends on the behavior of $m(x, t)$ at intermediate times $0 \leq t \leq T$, not merely the value of $m(x, T)$. To illustrate how different choices of intermediate m yield different solutions at time T, consider in $\mathbb{R}^2$,

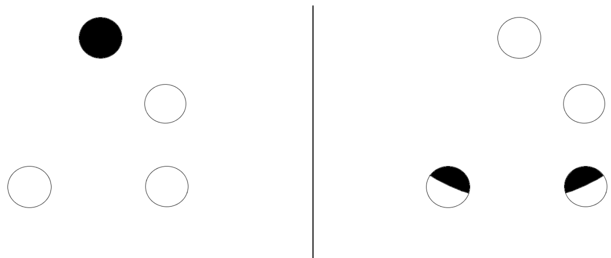
$$m_1(x, t) = \varepsilon + (1 - t)_+ \chi_{B_1((0,6))} + \frac{1}{\varepsilon}(\varepsilon - t)_+ \chi_{B_1((2,3))} + \chi_{B_1((-2,0))} + \chi_{B_1((2,0))},$$

and

$$m_2(x, t) = \varepsilon + (1 - t)_+ \chi_{B_1((0,6))} + \frac{1}{2}(2 - t)_+ \chi_{B_1((2,3))} + \chi_{B_1((-2,0))} + \chi_{B_1((2,0))}.$$

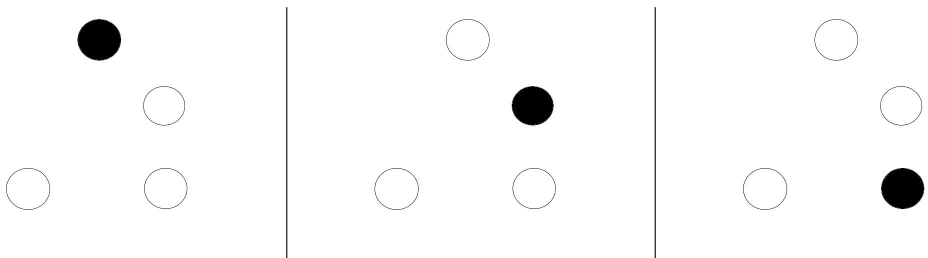
These are not smooth, but can be smoothed out to satisfy (M1) while maintaining the following behavior.

Take $\rho_0 = (\varepsilon + 1)\chi_{B_1((0,6))}$ and consider m_1 first. By time $T := 2$, all of the mass exceeding density ε of ρ_0 inside $B_1((0, 6))$ will have spread out to the nearest locations where $m(\cdot, 1)$ permits. Thus some portion depending on ε of $B_1((-2, 0))$ and $B_1((2, 0))$ will be filled, in



(a) $t = 0$ (b) $t \geq 1$

Figure 4: Solution for m_1.



(a) $t = 0$ (b) $t = 1$ (c) $t \geq 2$

Figure 5: Solution for m_2.

a symmetrical fashion with regards to reflection across the y axis, producing $\rho(x, T)$. See Figure 4 (in the pictures we do not represent the ε density outside the balls).

Now consider instead using m_2. By $t = 1$, the mass exceeding density ε of ρ_0 from $B_1((0, 6))$ will saturate most of $B_1((2, 3))$, depending on ε, as this ball is significantly closer than those at $(\pm 1, 0)$. Then by time $T = 2$, mass from $B_1((2, 3))$ will move on to the next available location, which is primarily in $B_1((\pm 2, 0))$. But $B_1((2, 0))$ is closer than $B_1((-2, 0))$ to $B_1((2, 3))$, so $\rho(x, T)$ will consist of a significant amount of mass in $B_1((2, 0))$, none in $B_1((-2, 0))$, and $\rho = \varepsilon$ elsewhere where appropriate. See Figure 5. Clearly this differs from the symmetry of $\rho(x, 2)$ in the case of $m = m_1$ above. However, $m_1(x, 0) = m_2(x, 0)$ and $m_1(x, t) = m_2(x, t)$ for all $t \geq 2$.

For this reason, it is not convenient in general to discuss exact long time behavior of solutions to the DCM. However, consider taking a uniformly convex potential Φ, so there exists some $\lambda > 0$ such that $D^2\Phi(x) \geq \lambda I$ for all $x \in \mathbb{R}^d$, and adding a drift $\Psi = -\nabla\Phi$ to the system, as was originally used in [MRS]. Then the new energy takes the form

$$E_\infty(\rho, t) = \int \rho\Phi + \begin{cases} 0 & \text{if } \rho \leq m(\cdot, t), \\ \infty & \text{otherwise,} \end{cases}$$

and as argued in Section 2, assuming (M1), we claim the minimizing movement scheme

$$\rho_\tau^n = \operatorname*{arg\,min}_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} \left\{ E_\infty(\rho, \tau n) + W_2(\rho, \rho_\tau^{n-1}) \right\}$$

narrowly converges to a solution of the corresponding DCM with drift Φ.

Suppose $m(x,t) \downarrow m_\infty(x)$ as $t \to \infty$. Then this solution tends to the global minimizer of E_∞, which is $\rho_S(x) := (m_\infty \chi_{\mathcal{O}})(x)$, with the set

$$\mathcal{O} := \{x \in \mathbb{R}^d : (m_\infty \Phi)(x) \leq C\},$$

where C is chosen so that $|\mathcal{O}| = 1$.

In fact, this convergence is exponentially fast over time in 2-Wasserstein distance. The analogue of Corollary 4.3, assuming (M2) and (M3), has

$$W_2(u(\cdot, t), v(\cdot, t)) \leq e^{-\lambda t} W_2(u(\cdot, 0), v(\cdot, 0)).$$

So in this case,

$$W_2(\rho_\infty(\cdot, t), \rho_S(\cdot)) \leq W_2(\rho_0, \rho_S) e^{-\lambda t}.$$

5.2 Assumption (M2) and obstacle problems

Assume only (M1) and consider looking for a viscosity solution description of the solution from Chapter 2 to the DCM. Given $m_t < 0$, (M2) is satisfied and the desired viscosity solution is (FB-M) or a variant where the outward normal velocity reflects an external density. If $m_t(x,t) \geq 0$ for all x, then no compression is occuring, no pressure is necessary and ρ remains constant in order to minimize $\int \rho v^2$.

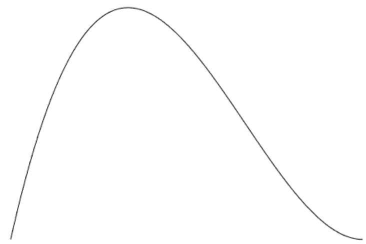


Figure 6: Sample pressure profile given $m_t < 0$ on the left side and $m_t > 0$ on the right.

The interesting behavior occurs when $m_t(x,t) \geq 0$ in some places and $m_t(x,t) \leq 0$ in others. In this case, it appears that some of the additional mass that must be moved due to $m_t(x,t) \leq 0$ (where $\rho = m$) is moved to dynamically fill in excess space made available where $m_t(x,t) > 0$. The Darcy's law type boundary velocity $V = |\nabla p|$ as in (FB-M) only applies where the boundary of the saturated set $\{\rho(\cdot,t) = m(\cdot,t)\}$ is moving outward. Elsewhere, this boundary may be moving inward, and the rate at which it moves is limited not by $|\nabla p|$ at the boundary, since pressure will become degenerate in the sense that $\nabla p = 0$, but by the amount of additional mass available and the condition that $p \geq 0$. This creates an obstacle problem for the pressure p and the region $\Omega_t := \overline{\{\rho(\cdot,t) = m(\cdot,t)\}}$.

Thus at least when the dimension $d = 1$, one can propose the following free boundary problem description of the solution:

$$\begin{cases} \nabla \cdot (m\nabla p) = m_t & \text{in } \{p(\cdot,t) > 0\}, \\ \min\left(\frac{m}{m - \min(\rho^E, m)}|\nabla p| - V, |\nabla p|\right) = 0 & \text{on } \partial\{p(\cdot,t) > 0\}, \end{cases} \tag{5.1}$$

Here ρ^E is the external density just outside $\{p(\cdot,t) > 0\}$.

See Figure 6 for an example solution profile in one dimension. Note the tail on the right hand side, ending with $|\nabla p| = 0$ that is supported by compression on the left hand side.

Roughly speaking the obstacle problem condition $\min\left(\frac{m}{m-\min(\rho^E,m)}|\nabla p| - V, |\nabla p|\right) = 0$ indicates that either $|\nabla p| > 0$, in which case $V = \frac{m}{m-\min(\rho^E)}|\nabla p|$, or $|\nabla p| = 0$, in which case $V \leq \frac{m}{m-\min(\rho^E,m)}|\nabla p|$ is not directly specified. For obstacle problems like this, the solution p smoothly joins with the obstacle at the boundary of the non-coincidence set. Since the obstacle is $p = 0$ outside Ω_t, this forces $|\nabla p| = 0$ at such points.

All of this works well in $d = 1$ when $m_t < 0$ on at least one side of all intervals in Ω_t. In this case, one can construct sufficient barriers for (5.1), prove a comparison principle, and show existence and uniqueness of viscosity solutions (in the sense that all solutions share $\overline{\{p(\cdot, t) > 0\}}$), even allowing for "$V = \infty$" and nucleation of pressure regions.

However, if $m_t > 0$ on both sides of an interval in Ω_t, then the outward normal velocity at *both* ends is not directly specified by (5.1). Now suppose ρ_0 and m are symmetric such that pressure takes the form in Figure 7. Suppose also that m_t increases slightly over time in a symmetric fashion in the middle, so that less mass can be supported on the ends as time passes. The solution of the DCM will shrink the support of the pressure in a symmetric fashion, maintaining $|\nabla p| = 0$ on the boundary.

But nothing in the free boundary problem (5.1) prohibits more of the mass from the region where $m_t < 0$ being shifted toward the left than the right or vice versa. Thus viscosity solutions of (5.1) exist for the same choice of m in which the left hand boundary moves inward at a slower rate than the right hand boundary, or vice versa, and $|\nabla p| = 0$ is still maintained on the boundaries. Of all such solutions, the solution of the DCM is the one which minimizes movement, i.e., minimizes $\int \rho v^2$. In this example, it is the symmetric solution. Thus the viscosity solution description (5.1) can describe the outward normal velocities of the support of the solution ρ_∞ of the DCM from Chapter 2 on boundaries where $|\nabla p| > 0$, but does not pinpoint these outward normal velocities in all cases where $|\nabla p| = 0$.

In higher dimensions, the issue is more pronounced, as entire connected pieces of $\partial\Omega_t$ may lie inside $m_t < 0$ such that $|\nabla p| = 0$ all along the piece of the boundary. Which parts of the boundary where $|\nabla p| = 0$ should shift inward at which velocities? As demonstrated above, solutions of (5.1) appear to exist in which different regions of this boundary move inward while maintaining $|\nabla p| = 0$, indicating non-uniqueness. Either the description (5.1) requires some additional condition that more precisely identifies the obstacle problem in question, or it cannot fully describe the solution of the DCM in such cases.

It is also worth noting the following critical technical challenge for $d > 1$: given a point

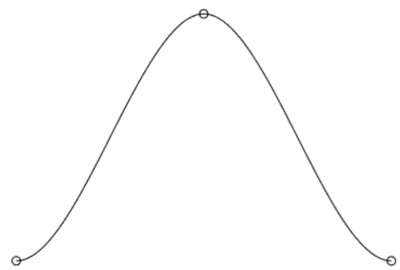


Figure 7: Sample symmetric pressure profile given $m_t > 0$ on both ends and $m_t < 0$ in the middle.

(x_0, t_0) in the interior of Ω_{t_0}, construct a smooth subsolution u of (5.1) with $u(x_0, t) > 0$ for some interval $t_0 \leq t \leq t_0 + \varepsilon$. Given (M2), one can create local subsolutions that suffice without much difficulty. However, without (M2), if $m_t(x_0) > 0$, pressure near x_0 is entirely supported by a region where $m_t < 0$, which is no longer a local problem, and can be highly depend on the geometry of Ω_{t_0} as well as precise estimates for the obstacle problem on Ω_{t_0}. Note the algorithm described in Section 5.3.2 may provide some assistance here; it converts this problem into questions related to random walks on subsets of Ω_t with smooth boundary.

5.3 Numerical approximations and investigations

5.3.1 Discrete Wasserstein projections

We simulated the minimizing movement scheme for E_∞ numerically in one and two dimensions.

In one dimension, this took the following form: fix $\delta > 0$ and break up $[a, b]$ into cells of length δ with centers x_i, $1 \leq i \leq n$. Each cell has an associated mass $\rho(x_i)$ initially assigned according to ρ_0, with plenty of space at the boundary of the grid to avoid mass hitting the boundary.

To perform the Wasserstein projection step

$$\rho_\tau^n = \operatorname*{arg\,min}_{\rho\in\mathcal{P}_2(\mathbb{R}^d):\rho\leq m(\cdot,\tau n)} W_2^2(\rho,\rho_\tau^{n-1}),$$

we considered all transfers of mass y_{ij} from cell i to cell j, with $y_{ii}=0$, and represented the minimization problem as a linear programming problem. The quantity to minimize is then

$$\sum_{i,j=1}^{n} y_{ij}|x_i-x_j|^2,$$

subject to the inequalities in each cell $\rho_\tau^n(x_i)\leq m(x_i,\tau n)$, implemented as

$$\rho_\tau^{n-1}(x_i)-\sum_{i=1}^{n} y_{ij}+\sum_{j=1}^{n} y_{ji}\leq m(x_i,\tau n).$$

Similarly, in two dimensions, we used a square grid and implemented the same linear programming problem. The linear programming problems were then solved by Matlab's linprog function, which performs a simplex algorithm on the dual problem. As $\delta\to 0$ and $\tau\to 0$, the simulation should approach the solution to the DCM from Chapter 2. We observed that all solutions became saturated in any modified cells, and only spread mass directly to adjacent cells, as they should. We then adjusted the algorithm so that only mass transfers between adjacent cells were considered, in order to improve execution speed.

The main disadvantage here is that the simulation can become rather slow for small enough δ, especially in 2D. However, it is certainly fast enough to obtain a good idea of the behavior in 2D, as well as excellent estimates on the outward normal velocities in 1D given various choices of m and ρ_0.

The normal velocity was calculated as follows: starting from time $t=0$, focus on the cell immediately to the left of the left-most saturated cell. Repeat the projection step with small τ until this cell becomes saturated ($\rho=m$) and record the time T. Then

$$V\approx\frac{\delta}{T}.$$

As $\delta \to 0$ and $\tau \to 0$, this value should approach the theoretical normal velocity $V = |\nabla p|$ (or $V = \frac{m}{m - \min(\rho_0, m)}|\nabla p|$ with external density ρ_0). If velocity is correct for the case without external density, it will certainly be correct with external density as well, so we only tested the former.

The results for a few different choices of m and δ, with $\tau = \delta/100$, starting from

$$\rho_0 = m(\cdot, 0)\chi_{[2.5,7.5]}$$

with $a = 0$, $b = 10$, are recorded in Table 1. The first choice of m was used to calibrate the system and check roughly how much error is expected. Error is due primarily to slight overflow onto the next cell (tending to zero as $\tau \to 0$), the difference between the discrete Wasserstein projection and actual continuous Wasserstein projection, and any error introduced by the linprog function. We noticed in particular, if there was a cell directly in the center of an interval, the linprog function would choose the minimizer that sent all its excess mass to the left. This is a perfectly accurate minimizer for the discrete scheme, but slightly affects the normal velocity in a way that tends to zero as $\delta \to 0$.

The theoretical V is found by solving $\nabla \cdot (m\nabla p) = m_t$ and calculating $V = |\nabla p|$ at the cell just to the left of $x = 2.5$. Note these results numerically confirm Theorem 1.2 in that the calculated boundary velocities do come very close to those predicted by (FB-M).

The last choice of m has a local minimum and thus does not satisfy (M3), but the data still supports (FB-M). Similar results hold for many other choices of m that do not satisfy (M3). Thus we expect (FB-M) describes the solution for all m satisfying (M1) and (M2), and the necessity of (M3) for λ-convexity to guarantee the uniqueness and convergence of the minimizing movement scheme for (PME-M) is primarily a limitation of using the modified porous medium equation as an approximation which allows identification of the minimizing movement scheme and the viscosity solution.

V at left boundary	$\delta = 0.25$	$\delta = 0.05$	$\delta = 0.01$	Theoretical Value
$m(x,t) = 1 - t$	2.630	2.439	2.484	2.500
$m(x,t) = 1 + x - t$	0.672	0.647	0.611	0.610
$m(x,t) = 1 + x - 2t$	1.344	1.293	1.224	1.220
$m(x,t) = 1 - \frac{1}{10}(x-4)^2 - t$	2.857	2.623	2.690	2.718
$m(x,t) = 1 + \frac{1}{10}(x-4)^2 - t$	2.500	2.326	2.320	2.321

Table 1: Outward normal velocities for various m at $t = 0$.

5.3.2 Quick stochastic algorithm

Building off the IDLA algorithm suggested for the drift problem in [MRSV], we exhibit a simple numerical algorithm that can be used to estimate the viscosity solution given (M1). This density will satisfy

$\rho(x,t) = m(x,t)\Omega_t$, where $\Omega_t = \{p(\cdot,t) > 0\}$ with p solving (FB-M),

$$\begin{cases} -\nabla \cdot (m\nabla p) &= -m_t \quad \text{in } \{p(\cdot,t) > 0\} \\ V &= |\nabla p| \quad \text{on } \partial\{p(\cdot,t) > 0\}. \end{cases}$$

We expand the primary equation

$$-m\Delta p - \nabla m \cdot p = -m_t. \tag{5.2}$$

Given Ω_t for some time t, the probabilistic form of the solution p can be written in terms of the corresponding stochastic process X_s in $\mathbb{R}^n$ ([OK], Section 9.1) defined by the stochastic differential equation

$$dX_s = \sqrt{2m(X_s)}\, dB_s + (\nabla m)\, ds, \tag{5.3}$$

(B_s is a standard Brownian motion) as

$$p(x,t) = E^x\left[\int_0^{\tau_{\Omega_t}} -m_t\, ds\right].$$

Here the stochastic process X_s starts at $X_0 = x$, and τ_{Ω_t} is the exit time of X_s from Ω_t.

Suppose $\rho(x,t) = m(x,t)\chi_{\Omega_t}$ and t increases by ε. Without further correction, $\rho = m(x,t)$ exceeds the prescribed limit $m(x, t+\varepsilon)$ on Ω_t. The correction we choose should move the additional mass (roughly $\varepsilon m_t(x,t)$ at x) outside Ω_t, and in doing so expand Ω_t by roughly the effect of the boundary velocity $V = |\nabla u|$ over time $[t, t+\varepsilon]$.

If we start X_0 with a density given by the excess mass $-m_t$, and let X_s solve (5.3), from the form of (5.2), it is well known that the the first-time hitting distribution of X_s on $\partial\Omega_t$ matches $m|\nabla p|$ (see for instance, [D]). Combining this with the fact that $V = \frac{m|\nabla p|}{m}$ in the relevant free boundary problem above, we are led to the following algorithm:

At time $t+\varepsilon$ when ρ exceeds the prescribed limit, divide $\mathbb{R}^n$ into small lattice cells, and determine the amount of mass in each. Randomly select a cell, suppose in position x, and start a new stochastic process X_s solving (5.3), starting from x. When this process reaches a cell that is not full, move as much of the excess mass from x to this new cell as possible. Repeat this process until all the excess mass is moved outside Ω_t, move time forward ε again, and repeat the procedure as desired.

In actual execution, one step of solving the SDE (5.3) numerically in $\mathbb{R}^d$ takes the form

$$X_s(s+ds) = X_s(s) + (\nabla m(X_s,t))(ds) + \left(\sqrt{2m(X_s,s)}\right)\left(\sqrt{ds}\right)(r_1,\ldots,r_n),$$

where $r_1,\ldots,r_n$ are independent and randomly generated according to the standard normal distribution centered at 0 with standard deviation 1.

Note if m is space-dependent, in the drift portion of the SDE (5.3), (∇m) points in the direction in which m is most rapidly increasing. Thus X_s will move somewhat randomly, while drifting in the direction m is most quickly increasing with a speed proportional to $|\nabla m|$. Thus overall more excess mass will likely be transferred to regions near $\partial\Omega_t$ that have larger value of m, but note the larger value of m will also decrease the outward normal velocity V at such points because more space must be filled with excess mass.

Due to possible irregularity of $\partial\Omega_t$, it is unclear if this algorithm always converges precisely to the viscosity solution's support Ω_t corresponding to (FB-M). However, it is simple to execute, appears to be accurate, and is faster than directly calculating the minimizing movement scheme as in the previous section.

Moreover, if we do not assume (M2), then this stochastic algorithm still functions well, with some mass from regions where $m_t < 0$ filling some regions where $m_t > 0$ in a manner that appears to correspond to the obstacle problem solution, at least when $\partial\Omega_t$ is smooth. Perhaps this could be helpful in defining a new form of viscosity solution for the problem without (M2).

5.4 Mesa problem solution as a result of uniform decreasing density

Let $f \geq 0$ be a bounded function with $f \in L^1(\mathbb{R}^d)$ and consider the porous medium equation with initial data f:

$$\begin{cases} \rho_t = \Delta(\rho^k) & \text{in } \mathbb{R}^d \times (0, \infty), \\ \rho(x,0) = f(x) & x \in \mathbb{R}^d. \end{cases}$$

In [CF], it is shown that under the assumption that f has a star-shaped profile, with solutions ρ_k to the above, $\rho_\infty := \lim_{k\to\infty} \rho_k$ exists and is given by

$$\rho_\infty(x) = \begin{cases} 1 & \text{if } x \in A, \\ f(x) & \text{if } x \notin A, \end{cases} \tag{5.4}$$

where A is the noncoincidence set of the solution of the variational inequality

$$\left.\begin{array}{rcl} -\Delta w & \geq & f - 1 \\ w & \geq & 0 \\ (\Delta w + f - 1)w & = & 0 \end{array}\right\} \text{almost everywhere in } \mathbb{R}^d. \tag{5.5}$$

We refer to this limiting problem, which is a singular perturbation problem about $k = \infty$, as the mesa problem.

If $f \leq 1$ everywhere, note the solution $\rho_\infty(x) = f(x)$. The interesting behavior occurs in regions where $f > 1$, in which case the excess mass exceeding 1 is moved to nearby locations according to this variational inequality, creating a mesa.

Using the framework of the density constraint model, it is possible to reproduce the solution of the mesa problem, providing a nice characterization of the mesa problem solution. To this end, consider a uniform density in space that decreases over time, starting from $m(x,0) = \max_{\mathbb{R}^n} f$ and for any fixed final time T, decreasing to $m(x,T) = 1$. For instance, take

$$m(x,t) = \left(\max_{\mathbb{R}^n} f\right) - t \tag{5.6}$$

with T chosen so that $m(x,T) \equiv 1$. Now we consider starting from $\rho_0 = f$ and solving the DCM up until time T. We will show the result is

$$m(x,T) = \rho_\infty$$

as given in Eq. 5.4. One can interpret the mesa problem in the framework of the DCM as the result of a jump discontinuity in time of the density constraint $m(x,t)$. So it is possible to uniformly interpolate for such a jump discontinuity (at least using m which is homogeneous in space) and obtain the same result.

We utilize the viscosity solution description in (FB-M) of the DCM solution, generalized to allow external density $\rho_0 \leq m(\cdot, 0)$. This is

$$\begin{cases} -\nabla(m\nabla p) &= & -m_t & \text{in } \{p(\cdot,t) > 0\}, \\ V &= & \frac{m|\nabla p|}{m-\rho_0} & \text{on } \partial\{p(\cdot,t) > 0\}. \end{cases}$$

Note the general version with external density was not proven in previous chapters, but it is much easier in this case since m is constant in space, in which case the analysis is similar to [KP], for instance. As in Chapter 3, we define the family of sets $\Omega_t := \overline{\{p(\cdot,t) > 0\}}$.

Consider the Baiocchi transform of the viscosity solution pressure, modified to account for the density constraint:

$$w_V(x,t) := \int_0^t (mp)(x,s)\, ds.$$

Now we ask, does $w_V(\cdot, T)$ solve a variational inequality similar to (5.5)? As $m > 0$, $p \geq 0$ and the pressure support is expanding (assuming (M2)), we have $\{w_V > 0\} = \{p > 0\}$.

Moreover, based on the free boundary problem that p solves,

$$-\Delta w_V(x,t) = \int_0^t -\Delta(mp)(x,s)\,ds = \int_0^t -\nabla(m\nabla p)(x,s)\,ds = \int_0^t -m_t(x,s)\chi_{\Omega_s}(x)\,ds.$$

For convenience we define the accumulated source

$$f_V := \int_0^t -m_t(x,s)\chi_{\Omega_s}(x)\,ds$$

Continuing with an estimate using our choice of m in (5.6),

$$-\Delta w_V(x,T) = f_V \geq \int_0^T -m_t(x,s)\,ds = m(x,T) - m(x,0) = f(x) - 1,$$

with equality on $\left\{x : f(x) = \max_{\mathbb{R}^d} f\right\}$. Thus we can summarize with $w_V(\cdot,T)$ solving the variational problem

$$\left.\begin{array}{rcl} -\Delta u & \geq & f_V \\ u & \geq & 0 \\ (\Delta u + f_V)u & = & 0 \end{array}\right\} \text{almost everywhere in } \mathbb{R}^d. \tag{5.7}$$

These variational problems (5.5) and (5.7) have a comparison principle when the source functions are ordered (see [GV]), so

$$w_V(\cdot,T) \geq w.$$

By conservation of mass in each case, $|\{w_V(\cdot,T) > 0\}| = |\{w > 0\}|$. Combining these observations, we conclude that

$$\{w_V(\cdot,T) > 0\} = \{w > 0\},$$

as claimed. Therefore it is possible to recover the mesa problem solution by considering the DCM using a decreasing uniform-in-space density constraint over time.

Finally, we observe one can define a more general mesa problem using (PME-M) instead of the regular porous medium equation. The solution of this problem is not yet understood; the analysis of [CF] fails to generalize directly as the standard semiconvexity estimate for the porous medium equation is unclear for (PME-M). Perhaps the same sort of variational inequality description applies with Δw replaced by $(-\nabla(m\nabla w))$. The analysis above breaks down when m is not uniform in space and it is not clear how exactly m should be chosen.

5.5 Open problems

We conclude by discussing some related open problems.

- Uniqueness for the DCM.

  Note uniqueness among solutions with pressures in $L^2([0,T];H^1(\Omega))$ was shown for the corresponding problem with $m \equiv 1$ and drift was shown in ([MM], Theorem 2.4). However, for the DCM there are complications due to m that prevent a characterization of all admissible velocities as gradients of H^1 functions, which is the cornerstone of the approach. Also the corresponding version of ([MM], Lemma 2.1) for the DCM is not quite sufficient to proceed with the argument. See also [S], which mentions similar uniqueness questions which are unresolved.

- Without assuming (M2) and/or (M3), can one prove a descriptive characterization of the solution of the DCM from Chapter 2 similar to (FB-M)?

  Without (M2), see the discussion in Section 5.2 and 5.3.2. Without (M3), we note that numerically, the minimizing movement scheme appears to correspond to (FB-M) in 1D, though the method of proof using the modified porous medium equation has issues with convexity.

- Discontinuities in m.

  Temporal discontinuities with $m_+ \leq m_-$ as well as some choices of ρ_0 produce a sort of general mesa problem (see [CF] or [EHKO]) when considering the limit of the porous medium equation. Perhaps the corresponding variational inequality with operator $\nabla \cdot (m\nabla)$ instead of Δ offers a description (analysis of [CF] fails to generalize directly as the standard PME semiconvexity estimate is unclear for (PME-M)).

In particular, taking limits of smooth choices of m that approximate these discontinuities is rather interesting. For example, one could smoothly decrease m_+ to m_- over time ε in various ways. Based on the observation in Section 5.1, it seems not all convergent choices of m will cause convergence of density to the mesa problem limit. Do any such choices preserve the limit?

- Extensions to allow $m = 0$ on subsets.

  Convex sets with $m = 0$ are handled in [MMS]. This could potentially be combined with the analysis in Chapter 2 to obtain a solution to the DCM, with some complications in Lemma 2.7. A viscosity solution description may be possible along the lines of [K2] Section 3, but it is questionable whether this is equal to a limit of solutions of (FB-M) as $m \to 0$, as in the previous discussion.

- Including $\rho \leq m$, drift, and source.

  Let $f, \Psi : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ be smooth. The function f is a source/sink effect and Ψ is a prescribed velocity at each point. The new porous medium-type equation for density is

$$\rho_t + \nabla \cdot (\rho\Psi + \rho(-\nabla p)) = f, \tag{5.8}$$

where $k > 1$ is given, the pressure is related by

$$p = P_k(\rho) := \frac{k}{k-1}\left(\frac{\rho}{m}\right)^{k-1},$$

and $\rho(\cdot, 0) = \rho_{0,k}$. The analogue of (M2) is then

$$-m_t + f - \nabla \cdot (m\Psi) \geq 0. \tag{5.9}$$

We expect the limit of the ρ_k will be a viscosity solution ρ_V of

$$\begin{cases} -\nabla \cdot (m\nabla p) = -m_t + mf - \nabla \cdot (m\Psi) & \text{in } \{p(\cdot, t) > 0\}, \\ V = \nu \cdot \Psi + \frac{m}{m - \max(\rho_E, m)}|\nabla p| & \text{on } \partial\{p(\cdot, t) > 0\}, \end{cases} \tag{FB-C}$$

with appropriately chosen p_0, and ρ_E solving

$$\rho_t + \nabla \cdot (\rho \Psi) = f$$

from initial data ρ_0 outside Ω_0.

Assuming (M2), we expect the region $\Omega_t := \{\rho(\cdot, t) = m(\cdot, t)\}$ travels around following Ψ and being compressed by m while expanding a bit further along the way due to compression as a result of m_t, f, and $\nabla \cdot (m\Psi)$. Characteristics that enter $\{\rho = m\}$ never leave. If $x \in \Omega_{t_1}$, but soon after Ω moves on and $(x, t_2) \notin \Omega_{t_2}$, then $\rho(x, t_2) = 0$. In this case we set $\rho_E = 0$. Thus Ω_t absorbs any mass along its path, and no mass leaves Ω_t. So one can write

$$\rho_\infty = m\chi_{\{p>0\}} + \rho_E \chi_{\{p=0\}}.$$

When $f \equiv 0$ and there exists a potential Φ with $\Psi = -\nabla\Phi$, one can construct a minimizing movement scheme for corresponding energies as in E_k, E_∞ but with the additional drift term $\int \rho\Phi$. Convergence to the DCM with drift Ψ should follow by adding arguments for the drift as in [MRS] to Chapter 2. For Wasserstein convergence of the ρ_k in this case as in Section 4, assumption (M3) is likely still necessary.

- An Inverse Problem / Control Theory Problem : Given initial data ρ_0 supported on Ω_0 and a target density $\rho(\cdot, T)\chi_{\mathcal{O}}$ with $\Omega_0 \subset \mathcal{O}$, how can one choose $m(x, t)$ to most effectively compress ρ_0 to match the target density? One metric for effectiveness is minimal total kinetic energy over $[0, T]$. This could also be rephrased to allow a choice of drift, in which case the long term behavior discussed in Section 5.1 may offer an initial approach.

Appendix A

Classical theory for the modified porous medium equation

Here we follow [BH] and establish existence, uniqueness, and continuity of solutions of the Dirichlet problem for (PME-M),

$$\rho_t = \nabla \cdot \left(m \nabla \left[\left(\frac{\rho}{m} \right)^k \right] \right).$$

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the nonlinear evolution problem

$$\begin{aligned} \rho_t &= \nabla \cdot \left(m \nabla \left[\left(\tfrac{\rho}{m} \right)^k \right] \right) && \text{in } \Omega \times [0, \infty), \\ \rho &= 0 && \text{on } \partial\Omega \times [0, \infty), \\ \rho(x, 0) &= \rho_0(x) && \text{in } \Omega. \end{aligned} \tag{A.1}$$

where $m : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is smooth, satisfies (M1), and $\rho_0 \in L^\infty(\Omega)$ is non-negative with $\rho_0 = 0$ on $\partial\Omega$.

Remark A.1. We can effectively consider (PME-M) as a problem on $\mathbb{R}^n$ instead of Ω without the Dirichlet boundary condition when Ω is chosen much larger than $\{\rho_0 > 0\}$.

For convenience, let $\phi(\rho) = \rho^k$. In particular $\phi(0) = 0$ and $\phi'(0) = 0$. Similar existence, uniqueness, and continuity results apply to equations with other increasing nonlinearities with these properties.

Weak solutions and uniqueness

We first define weak solutions for (A.1). Given $T > 0$, let $Q_T = \Omega \times [0, T]$. Here the relevant function spaces are $L^2(0, T; H^1(\Omega))$, the Hilbert space with inner product

$$(u, v)_{L^2(0,T;H^1(\Omega))} = \int_{Q_T} uv + \int_{Q_T} \nabla u \cdot \nabla v$$

and $V_2(Q_T)$, the Banach space with norm

$$|u|^2_{V_2(Q_T)} = \text{ess sup}_{0\leq t\leq T}\int_\Omega\int_\Omega u^2(t) + \int_{Q_T}|\nabla u|^2.$$

Definition A.1. We say that a function $\rho : [0,\infty) \to L^1(\Omega)$ is a *weak solution* of (A.1) if, setting $w = (\rho/m)^k$ for convenience, it satisfies:

(i) $\rho \in C([0,t]; L^1(\Omega)) \cap L^\infty(Q_t)$ for all $t > 0$

(ii) $\rho(t;\rho_0) \in C(\overline{\Omega})$ and $\rho(t;u_0) = 0$ on $\partial\Omega$ for $t > 0$

(iii) $w \in V_2(Q_t)$ for all $t \in (0,\infty)$;

(iv) for all $\varphi \in C_{2,1}(\overline{Q})$ with $\varphi = 0$ on $\partial\Omega \times [0,\infty)$ and all $t \in (0,\infty)$,

$$\int_\Omega \rho(t)\varphi(t) = \int_\Omega \rho_0\varphi(0) + \int_0^t\int_\Omega(\rho\varphi_t - m\nabla w\cdot\nabla\varphi + (\nabla m\cdot\nabla w)\varphi).$$

Similarly we say ρ is a weak solution of (PME-M) (on $\mathbb{R}^n$) if it satisfies the above without the pieces due to the Dirichlet boundary condition.

Lemma A.1. *(L^1 contraction) Given two non-negative weak solutions ρ_1, ρ_2 of (PME-M), for all $t > 0$,*

$$||\rho_2(\cdot,t) - \rho_1(\cdot,t)||_{L^1(\mathbb{R}^n)} \leq ||\rho_2(\cdot,0) - \rho_1(\cdot,0)||_{L^1(\mathbb{R}^n)}.$$

Proof. Arguing as in ([V], Section 3.2.3), it is enough to approximate and prove the inequality for smooth positive solutions on a bounded domain with zero boundary data. In this case, by the divergence theorem and the zero boundary conditions,

$$\frac{d}{dt}\int(\rho_2(\cdot,t) - \rho_1(\cdot,t))\,dx = \int\nabla\cdot\left(m\nabla\left[\left(\frac{\rho_2(\cdot,t)}{m}\right)^k\right]\right) - \nabla\cdot\left(m\nabla\left[\left(\frac{\rho_1(\cdot,t)}{m}\right)^k\right]\right) = 0.$$

To deduce the L^1 bound, let U be the solution of (PME-M) with initial data

$$\max(\rho_1(\cdot,0), \rho_2(\cdot,0)).$$

Then $U - \rho_1 \geq (\rho_2 - \rho_1)_+$, with equality at $t = 0$. Thus applying the above to U and ρ_1,

$$\int (\rho_2(\cdot,t) - \rho_1(\cdot,t))_+ \leq \int (U(\cdot,t) - \rho_1(\cdot,t)) = \int (U(\cdot,0) - \rho_1(\cdot,0)) = \int (\rho_2(\cdot,0) - \rho_1(\cdot,0))_+.$$

Swapping ρ_2 and ρ_1 bounds the negative part; combining both parts yields the desired inequality.

□

The proof above also gives the comparison principle.

Lemma A.2. *(Comparison principle) Let ρ_1 and ρ_2 be two non-negative weak solutions of (PME-M) with given initial data. If $\rho_1(\cdot,0) \leq \rho_2(\cdot,0)$ a.e., then $\rho_1 \leq \rho_2$ a.e.*

In particular, there is at most one weak solution for given initial data.

Existence and continuity

The classical strategy for existence for (A.1) is take a sequence of uniformly parabolic problems that approach (A.1) in the limit. Let $\phi(\rho) = \rho^k$. We consider

$$\begin{aligned} \rho_t &= \nabla \cdot \left(m \nabla \phi_\varepsilon \left(\tfrac{\rho}{m} \right) \right) && \text{in } Q_T, \\ \rho &= 0 && \text{on } \partial\Omega \times [0,T), \\ \rho(x,0) &= \rho_0(x) && \text{in } \Omega, \end{aligned} \tag{A.2}$$

where $\phi_\varepsilon \in C^\infty(\mathbb{R}^+)$, $\phi_\varepsilon(0) = 0$, $\phi_\varepsilon'(s) \geq C(\varepsilon) > 0$ for $s \in [0,K]$, and

$$(\phi_\varepsilon^{-1}(s))' \leq (\phi^{-1}(s))'$$

for $s \in [0, \phi(2K)]$, where K is the uniform L^∞-bound of ρ_ε given in Lemma A.4 below and ϕ_ε and ϕ_ε' converge to ϕ and ϕ' on all compact subsets of $\mathbb{R}^+$ as $\varepsilon \to 0$, and where

$$\rho_{0\varepsilon} \in C^\infty(\overline{\Omega}) \text{ with } 0 \leq \rho_{0\varepsilon} \leq ||\rho_0||_{L^\infty(\Omega)},$$

with $\rho_{0\varepsilon} = 0$ on $\partial\Omega$ with $||\rho_{0\varepsilon} - \rho_0||_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$. (For instance, ϕ_ε can be chosen as a convolution of ϕ with a partition of unity.)

Lemma A.3. *(Comparison principle for approximation) Let ρ_1 and $\rho_2 \in C^{2,1}(\overline{Q}_T)$ be two solutions of (A.2) with initial functions $\rho_1(\cdot,0) \leq \rho_2(\cdot,0)$. Then $\rho_1(\cdot,t) \leq \rho_2(\cdot,t)$.*

Proof. Let $z = \rho_1 - \rho_2$. Then z satisfies the linear problem

$$\begin{aligned} z_t &= \nabla \cdot (m\nabla(a_\varepsilon z)) && \text{in } Q_T, \\ z &= 0 && \text{on } \partial\Omega \times [0,T), \\ z(x,0) &= \rho_{01\varepsilon}(x) - \rho_{02\varepsilon}(x) && \text{in } \Omega, \end{aligned} \tag{A.3}$$

where

$$a_\varepsilon(x,t) := \int_0^1 \phi'_\varepsilon \left(\theta \frac{\rho_{1\varepsilon}}{m}(x,t) + (1-\theta)\frac{\rho_{2\varepsilon}}{m}(x,t)\right) d\theta.$$

The rest of the proof is identical to ([BH], pages 866 - 867).

□

Lemma A.4. *Let $u_\varepsilon \in C^{2,1}(\overline{Q}_T)$ solve (A.2). Then*

$$0 \leq \rho_\varepsilon \leq K \qquad \text{in } \overline{Q}_T,$$

with K independent of T and ε.

Proof. The lower bound is obvious. For the upper bound, we construct time-dependent supersolutions of (A.2) which are uniformly bounded with respect to ε and T. Then the result follows from Lemma A.3.

To this end, consider an approximate pressure $p = \phi_\varepsilon(\rho/m)$. Rewriting the density equation in terms of pressure, it takes the form

$$m(\phi_\varepsilon^{-1})'(p)p_t + m\phi_\varepsilon^{-1}(p)(\log m) = m\Delta p + \nabla m \cdot \nabla p.$$

Due to global assumptions on m and ϕ_ε, there exist pressure supersolutions q_ε, uniformly bounded independent of ε and T, which take the form

$$(R^2(t) - C|x|^2)_+$$

for quickly growing $R(t)$ and large enough C. Then translating back to density, the desired supersolutions are

$$\rho = m(\phi_\varepsilon^{-1}(q_\varepsilon)).$$

As $(\phi_\varepsilon^{-1}(s))' \leq (\phi^{-1}(s))'$, ρ is also uniformly bounded with respect to ε and T.

□

Lemma A.5. *([LSU], Theorem 7.4) The uniformly parabolic problem (A.2) has a unique classical solution ρ_ε which is in $C^{2+\alpha}(\overline{Q}_T)$ for all $\alpha \in (0, 1)$.*

Before the existence proof, we need a few a priori estimates.

Lemma A.6. *Let $0 \leq t - \tau < t < T$. Then there exists $C(\tau) > 0$ such that*

$$\int_{t-\tau}^{t} \int_\Omega \left| \nabla \phi_\varepsilon \left(\frac{\rho_\varepsilon}{m} \right) \right|^2 \leq C(\tau).$$

Proof. Simply multiply the equation for ρ_ε by $\phi_\varepsilon(\rho_\varepsilon)$, integrate by parts over $\Omega \times (t, t + \tau)$, and use the assumption that m is bounded below away from zero. □

With $\phi(\rho) = \rho^k$, the assumption that $\phi_\varepsilon \to \phi$, and finally noting Lemma A.4, we clearly have

$$||\phi_\varepsilon(\rho_\varepsilon)||_{L^\infty}(T - \tau, T; H^1(\Omega)) \leq C(\tau), \quad 0 < \tau \leq T, \tag{A.4}$$

with $C(\tau)$ independent of T.

Finally, we use a strong equicontinuity property for solutions of uniformly parabolic equations:

Lemma A.7. *As in ([DB], Thm 6.2)*

(i) For every $\tau > 0$, there exists a continuous nondecreasing function $\omega_\tau(\cdot)$, independent of

T and ε, with $\omega_\tau(0) = 0$ such that

$$|\rho_\varepsilon(x_1, t_1) - \rho_\varepsilon(x_2, t_2)| \leq \omega_\tau\left(|x_1 - x_2| + |t_1 - t_2|^{1/2}\right)$$

for all $(x_i, t_i) \in \overline{\Omega}_T \times [\tau, T]$.

(ii) If $\rho_0 \in C(\overline{\Omega})$, then $\{\rho_\varepsilon\}$ is equicontinuous on $\overline{\Omega} \times [0, T]$.

Now we can show the main existence result.

Theorem A.8. *Suppose $\rho_0 \in L^\infty(\Omega)$ is non-negative and $m \in C^2(\Omega)$. Then there exists a weak solution of (A.1) which is continuous in any set $\overline{\Omega} \times [\tau, T]$ with $\tau > 0$, and satisfies*

$$0 \leq \rho \leq C \qquad \text{on } Q_T.$$

The constant C and the modulus of continuity do not depend on T.

Proof. Based on the a priori estimates, there exists a limit function $u \in L^\infty(Q_T) \cap C(\overline{\Omega} \times (0, T])$ and a subsequence of $\{\rho_\varepsilon\}$ (denoted by $\{\rho_\varepsilon\}$ for simplicity) such that

(i) $\rho_\varepsilon \to \rho$ uniformly on all sets of the form $\overline{\Omega} \times [\tau, T]$ with $\tau > 0$,

(ii) $\rho_\varepsilon \to \rho$ strongly in $L^2(Q_T)$ and a.e.,

(iii) $\phi_\varepsilon\left(\frac{\rho_\varepsilon}{m}\right) \to \phi\left(\frac{\rho}{m}\right)$ weakly in $L^2(0, T; H^1(\Omega))$.

Here (i) follows from Lemma A.7 while (ii) follows from (i) and the bound (A.4). Part (iii) is due to Lemma A.6, part (ii), and Lebesgue's dominated convergence theorem.

Now we need to check that ρ satisfies Definition A.1. Note solutions ρ_ε will satisfy the integral equation

$$\int_\Omega \rho_\varepsilon(t)\varphi(t) = \int_\Omega \rho_{0\varepsilon}\varphi(0) + \int_0^t \int_\Omega (\rho_\varepsilon \varphi_t - m\nabla w_\varepsilon \cdot \nabla\varphi + (\nabla m \cdot \nabla w_\varepsilon)\varphi),$$

for test functions φ and $w_\varepsilon := \phi(\rho_\varepsilon/m)$. Using properties (i) - (iii) above, we can take a limit and conclude that ρ satisfies the desired integral equation.

From (i), it follows that $u \in C((0,T]; L^1(\Omega))$. Let $\eta > 0$. As $\rho_{0\varepsilon} \to \rho_0$ in L^1, there exists $\varepsilon > 0$ such that $||\rho_{0\varepsilon} - \rho_0||_{L^1(\Omega)} \leq \eta$. By the L^1 contraction result, if we define $\tilde{\rho}_\varepsilon$ as the solution of (A.1) with initial data $\rho_{0\varepsilon}$, then

$$||\tilde{\rho}_\varepsilon(t) - \rho(t)||_{L^1(\Omega)} \leq ||\rho_{0\varepsilon} - \rho_0||_{L^1(\Omega)} \leq \eta.$$

Finally, by Lemma A.7, there exists $t_0 > 0$ such that $||\tilde{\rho}_\varepsilon(t) - \rho_{0\varepsilon}||_{L^1(\Omega)} \leq \eta$ for all $0 \leq t \leq t_0$. Together we estimate

$$||\rho(t) - \rho_0||_{L^1(\Omega)} \leq ||\rho(t) - \tilde{\rho}_\varepsilon(t)||_{L^1(\Omega)} + ||\tilde{\rho}_\varepsilon(t) - \rho_{0\varepsilon}||_{L^1(\Omega)} + ||\rho_{0\varepsilon} - \rho_0||_{L^1(\Omega)} \leq 3\eta.$$

Thus $\rho \in C([0,t]; L^1(\Omega))$.

The other properties in Definition (A.1) follow directly from the corresponding results in (i) - (iii).

$\square$

Appendix B

Assumptions & results for time-dependent gradient flows from [FV]

Here we state the definitions and results from [FV] that are relevant to various arguments in Section 4.

Let $\mathcal{E} : X \times [0,\infty) \to (-\infty,\infty]$ be a time-dependent functional. The gradient flow corresponding to $\mathcal{E}$ with initial data ρ_0 is a family $\rho(t) \in X$ of functions for $t \in [0,\infty)$ that corresponds to

$$\begin{cases} \rho'(t) &= -\nabla\mathcal{E}(\rho(t),t), \quad t > 0 \\ \rho(0) &= \rho_0. \end{cases} \tag{B.1}$$

Let (X,d) be a complete separable metric space. We say $\mathcal{E}$ is proper if there exists $\rho_0 \in X$ such that $\mathcal{E}(\rho) < \infty$. For a proper functional $\mathcal{E} : X \to (-\infty,\infty]$, define the local slope $|\partial\mathcal{E}|$ of $\mathcal{E}$ at ρ by

$$|\partial\mathcal{E}|(\rho) = \limsup_{q\to\rho} \frac{(\mathcal{E}(\rho)-\mathcal{E}(q))_+}{d(\rho,q)}.$$

(f_+ denotes the positive part of f.) The metric derivative of ρ is

$$|\rho'(t)| := \lim_{s\to t} \frac{d(\rho(s),\rho(t))}{|s-t|}.$$

Using these concepts, the metric formulation of a time-dependent gradient flow is given by the variational inequality

$$\frac{d}{dt}(\mathcal{E}(\rho(t),t)) \leq \partial_t\mathcal{E}(\rho(t),t) - \frac{1}{2}|\partial\mathcal{E}(t)|^2(\rho(t)) - \frac{1}{2}|\rho'|^2(t). \tag{B.2}$$

In particular, $\mathcal{E}(\rho(t),t)$ is no longer necessarily decreasing over time, as it is in the case of gradient flows of energies that do not depend on time, the difference being due to the $\partial_t\mathcal{E}$ term.

We will also make use of the shorthand

$$\mathbf{E}(t,\tau,\rho;q) := \mathcal{E}(q,t) + \frac{d^2(\rho,q)}{2\tau}.$$

The Moreau-Yosida approximation of $\mathcal{E}$ is

$$\mathscr{E}_{t,\tau}(\rho) := \inf_{q\in X} \mathbf{E}(t,\tau,\rho;q).$$

We reference a number of assumptions made on the energy $\mathcal{E}$ in [FV]:

(E1) For each $t \geq 0$, $\mathcal{E}(\cdot,t)$ is proper and lower semicontinuous with respect to the metric $d(\cdot,\cdot)$.

(E2) The domain $D := \{\rho \in X : \mathcal{E}(\rho,t) < \infty\}$, is time-independent.

(E3) There exist $\rho^* \in X$ and a function $\beta : [0,\infty) \to [0,\infty)$ with $\beta \in L^1_{\text{loc}}([0,\infty))$ such that, for each $\rho \in D$, the function $t \to \mathcal{E}(\rho,t)$ satisfies

$$|\mathcal{E}(\rho,t) - \mathcal{E}(\rho,s)| \leq (1 + d^2(\rho,\rho^*)) \left(\int_s^t \beta(r)\, dr\right).$$

(E4) For each $T > 0$, there exists a $\rho^* \in X$ and $\tau^*(T) = \tau^* > 0$ such that the function $t \mapsto \mathscr{E}_{t,\tau^*}(\rho^*)$ is bounded from below in $[0,T]$.

(E5) There is a function $\lambda : [0,\infty) \to \mathbb{R}$ in $L^\infty_{\text{loc}}([0,\infty))$ such that: given points $\rho, q_0, q_1 \in X$, there exists a curve $\gamma : [0,1] \to X$ satisfying $\gamma(0) = q_0, \gamma(1) = q_1$, and

$$\mathbf{E}(t,\tau,\rho;\gamma(s)) \leq (1-s)\mathbf{E}(t,\tau,\rho;q_0) + s\mathbf{E}(t,\tau,\rho;q_1) - \frac{1+\tau\lambda(t)}{2\tau} s(1-s) d^2(q_0,q_1),$$

for $0 < \tau < \frac{1}{\lambda_T}$ and $s \in [0,1]$, where $\lambda_T = \max\{0, -\inf_{t\in[0,T]} \lambda(t)\}$.

Definition B.1. Let $\rho_0 \in X$ and $\mathcal{E} : X \times [0,\infty) \to (-\infty,\infty]$ be a functional satisfying (E1), (E2), and (E3). We say that an absolutely continuous curve $\rho : [0,\infty) \to X$ is a solution of (B.1) (i.e. a gradient flow of $\mathcal{E}$) if $\rho(0) = \rho_0$, the function $t \to \mathcal{E}(\rho(t),t)$ is absolutely continuous,

$$|\rho'|, |\partial\mathcal{E}(\cdot)|(\rho(\cdot)) \in L^2_{\mathrm{loc}}([0,\infty)),$$

and the variational inequality (B.2) holds.

The relevant minimizing movement scheme for a partition with fixed step size $\tau > 0$ is defined by an initial choice of ρ_τ^0 and for $n \geq 1$,

$$\rho_\tau^n \in \arg\min_{q\in X} \mathbf{E}(\tau n, \tau, \rho_\tau^{n-1}; q).$$

Lemma B.1 ([FV], Lemma 3.1)**.** *Suppose (E1), (E4), and (E5) and $\rho_\tau^0 \in D$. Then the sequence ρ_τ^n exists and is uniquely defined.*

In particular (E5) is used to obtain uniqueness.

Given the ρ_τ^n, define the approximate solutions

$$\underline{\rho}_\tau(t) := \rho_\tau^{n-1}, \quad \overline{\rho}_\tau(t) := \rho_\tau^n, \qquad \text{for } t \in (\tau(n-1), \tau n).$$

Theorem B.2 ([FV], Theorem 4.4)**.** *Assume (E1) through (E5) and let $\rho_0 \in \overline{D}$. Given the conditions*

$$\lim_{\tau\to 0} d(\rho_\tau^0, \rho_0) = 0, \qquad \sup_\tau \mathcal{E}(\rho_\tau^0, 0) < \infty,$$

the approximate solutions $\overline{\rho}_\tau$ and $\underline{\rho}_\tau$ converge locally uniformly to a function $\rho : [0,\infty) \to X$ satisfying $\rho(0) = \rho_0$. Moreover, ρ is independent of the choice of family ρ_τ^0.

Theorem B.3 ([FV], Theorem 5.4)**.** *Assume (E1) through (E5). The limit ρ in the above theorem is locally absolutely continuous and its metric derivative $|\rho'|$ belongs to $L^2_{loc}([0,\infty))$. Moreover, if the function $t \mapsto \mathcal{E}(\rho, t)$ is differentiable for $\rho \in D$, its time-derivative is upper semicontinuous in the ρ-variable, and the property*

$$t_n \downarrow t, d(\rho_n, \rho) \to 0 \text{ as } n \to \infty \Longrightarrow \liminf_{n\to\infty} \frac{\mathcal{E}(\rho_n, t_n) - \mathcal{E}(\rho_n, t)}{t_n - t} \geq \partial_t \mathcal{E}(\rho, t)$$

holds, then the function $t \mapsto \mathcal{E}(\rho(t), t)$ is absolutely continuous and satisfies the identity

$$\mathcal{E}(\rho(t), t) - \mathcal{E}(\rho_0, 0) = \int_0^t \partial_t \mathcal{E}(\rho(s), s)\, ds - \frac{1}{2}\int_0^t |\rho'|^2(s)\, ds - \frac{1}{2}\int_0^t |\partial \mathcal{E}(s)|^2(\rho(s))\, ds.$$

In particular, ρ is a solution of (B.1) in the sense of Definition B.1.

Lemma B.4 ([FV], Section 5.1)**.** *Assume (E1) through (E5) and also that $\lambda(t)$ is continuous in time. Then given two initial data $u_0, v_0 \in D$, if we let $u(t), v(t)$ be solutions to (B.1) as discussed above with initial data u_0 and v_0 respectively,*

$$d(u(t), v(t)) \leq e^{-\int_0^t \lambda(s)\, ds} d(u_0, v_0).$$

We need to continue estimates from [FV] to obtain a specific rate of convergence of the minimizing movement scheme to the solution.

Lemma B.5. *Assume (E1) through (E5) and $\lambda(t)$ is continuous in time. Define the piecewise constant functions in time*

$$\rho_\tau(x, t) := \rho_\tau^n \qquad \textit{for } t \in [\tau n, \tau(n+1)].$$

If $\rho_\tau^0 = \rho_0$ and $\mathcal{E}(\rho_0, 0) < \infty$, there exists a constant $C > 0$ dependent on $\mathcal{E}$, λ, and T such that for all $t \in [0, T]$,

$$d(\rho_\tau(\cdot, t), \rho(\cdot, t)) \leq C\sqrt{\tau}.$$

Proof. From the proof of [FV] Theorem 4.4, given two partitions $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$,

$$d^2(\overline{\rho}_{\boldsymbol{\tau}}(t), \overline{\rho}_{\boldsymbol{\eta}}(t)) \leq 3 d_{\boldsymbol{\tau},\boldsymbol{\eta}}(t, t) + 3C(|\boldsymbol{\tau}| + |\boldsymbol{\eta}|) \tag{B.3}$$

with

$$d_{\boldsymbol{\tau},\boldsymbol{\eta}}(t, t) \leq \left(d^2(\rho_{\boldsymbol{\tau}}^0, \rho_{\boldsymbol{\eta}}^0) + C(|\boldsymbol{\tau}| + |\boldsymbol{\eta}|) + \int_0^t e^{2\alpha_{\boldsymbol{\tau},\boldsymbol{\eta}}(t)} (G^+_{\boldsymbol{\tau},\boldsymbol{\eta}}(t) + G^+_{\boldsymbol{\eta},\boldsymbol{\tau}}(t))\, dt \right)^{1/2} + C(|\boldsymbol{\tau}| + |\boldsymbol{\eta}|).$$

Here

$$\alpha_{\boldsymbol{\tau},\boldsymbol{\eta}}(t) := \int_0^t \tilde{\lambda}_{\boldsymbol{\tau}}(s) + \tilde{\lambda}_{\boldsymbol{\eta}}(s)\, ds,$$

where $\tilde{\lambda}_{\boldsymbol{\tau}}(t) := \lambda(t_{\boldsymbol{\tau}}^n)$. Moreover, by [FV] Proposition 4.3,

$$\int_0^T G^+_{\boldsymbol{\tau},\boldsymbol{\eta}}(t)\, dt \leq C(|\boldsymbol{\tau}| + |\boldsymbol{\eta}|).$$

Tracing all these estimates back to (B.3), it follows that

$$d^2(\overline{\rho}_{\boldsymbol{\tau}}(t), \overline{\rho}_{\boldsymbol{\eta}}(t)) \leq C(|\boldsymbol{\tau}| + |\boldsymbol{\eta}|).$$

Sending $|\boldsymbol{\eta}| \to 0$ and taking square roots yields the claimed rate of convergence. $\square$

Appendix C

Numerical simulations

Figure 8 shows a few snapshots of the 2D Wasserstein projection calculation with $\rho_0 = 2$ in $[3,7] \times [3,7]$ and the pyramid

$$m(x,t) = 2 + (0.5)\max(|x-5|, |y-5|) - t.$$

The right column is an overhead view.

Figure 9 shows a few snapshots of the 2D Wasserstein projection calculation with $\rho_0 = 2 + \frac{x}{2}$ in $[3,7] \times [3,7]$ and the plane

$$m(x,t) = 2 + \frac{x}{2} - 2t.$$

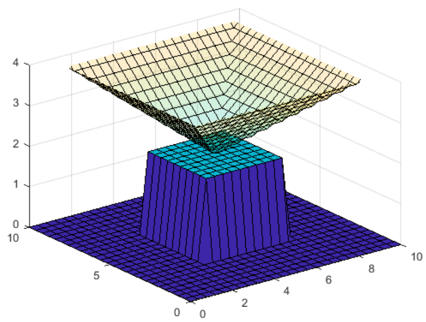
The right column is an overhead view.

Figure 10 shows a few snapshots of the 2D Wasserstein projection calculation with $\rho_0 = 42$ in $[3,7] \times [3,7]$ and the paraboloid

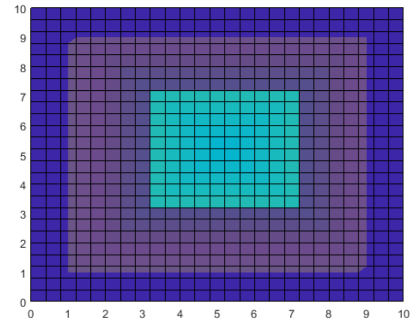
$$m(x,t) = 50 - (x-5)^2 - (y-5)^2 - 25t.$$

The right column is an side view. In the second row, the maximum in the center exceeds the original density slightly as some mass shifts inward to fill the gap.

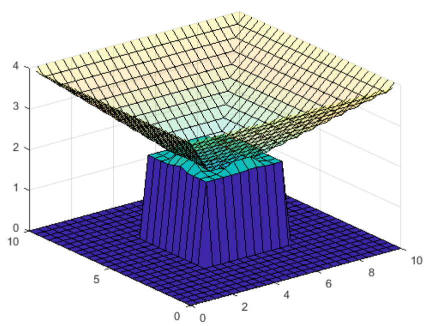
In general, the discrete Wasserstein projection is not unique, as excess mass could be moved to any of the closest available cells. This explains the minor asymmetry and irregularites near the boundary in the simulations. All of these simulations use a spatial step dx = 0.4 and time step dt = 0.01. These can certainly be made smaller at the cost of additional time to execute the simulations.



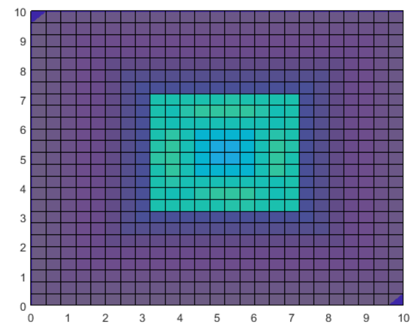
(a) $t = 0$



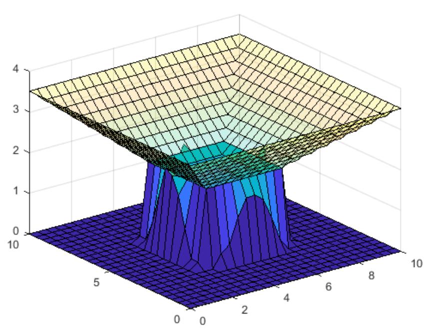
(b) $t = 0$



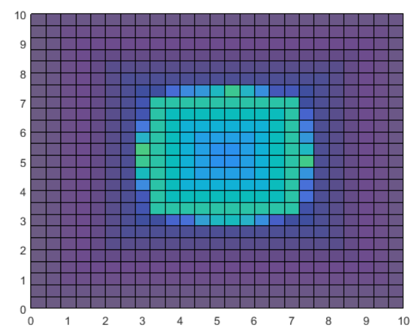
(c) $t = 0.5$



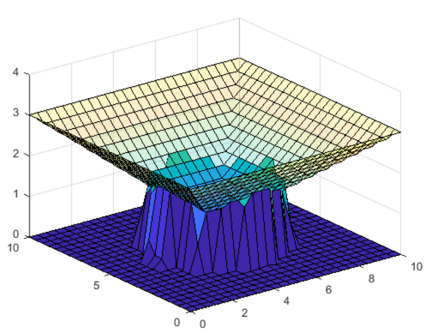
(d) $t = 0.5$



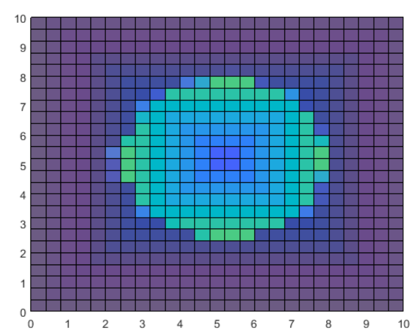
(e) $t = 1$



(f) $t = 1$

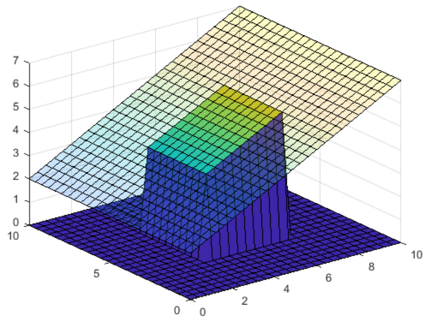


(g) $t = 1.5$

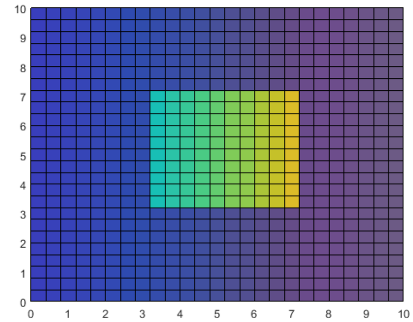


(h) $t = 1.5$

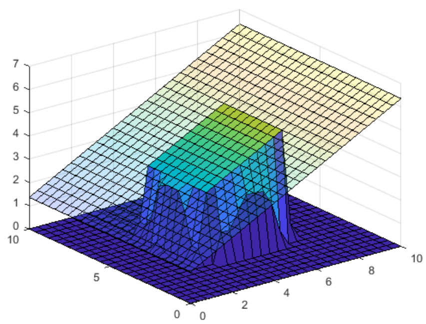
Figure 8: Simulation 1



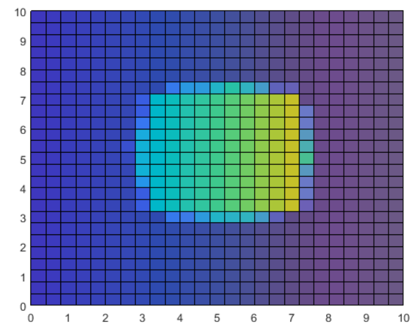
(a) $t = 0$



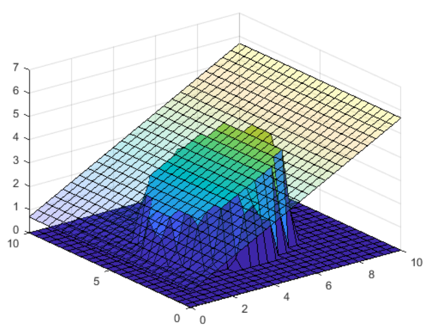
(b) $t = 0$



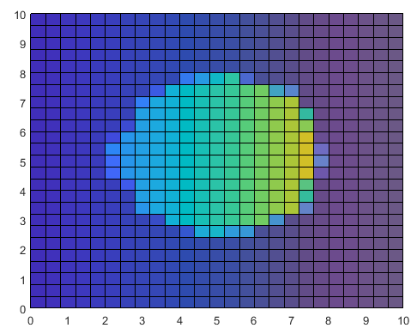
(c) $t = 0.33$



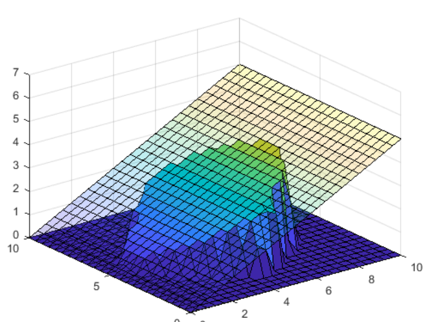
(d) $t = 0.33$



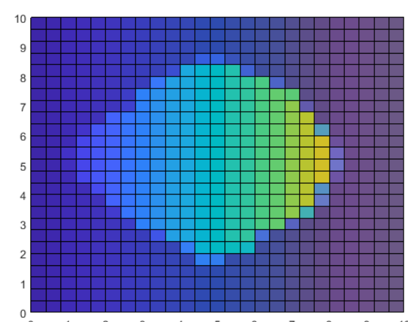
(e) $t = 0.67$



(f) $t = 0.67$

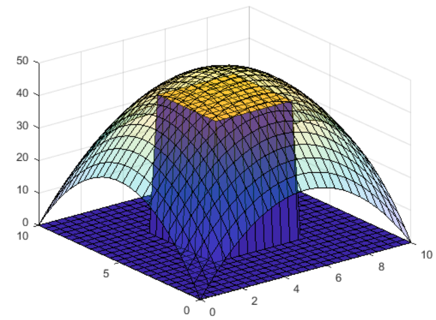


(g) $t = 1$

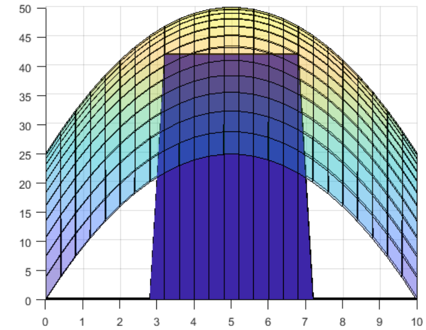


(h) $t = 1$

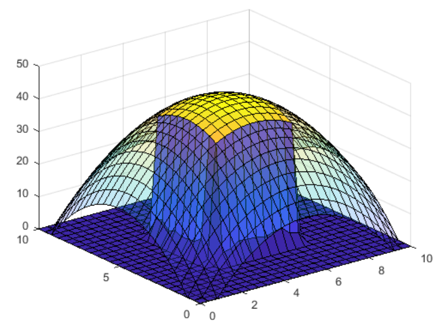
Figure 9: Simulation 2



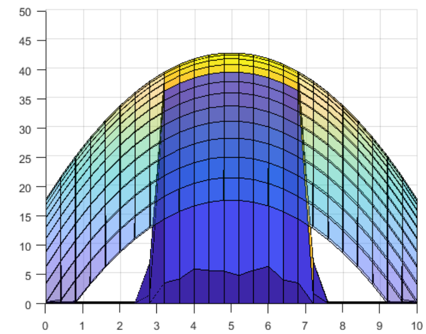
(a) $t = 0$



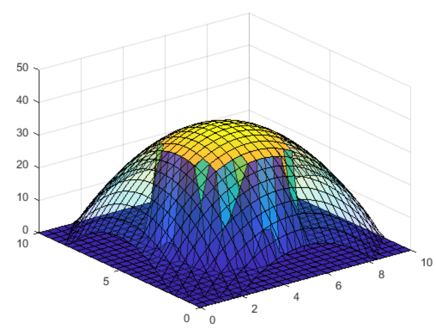
(b) $t = 0$



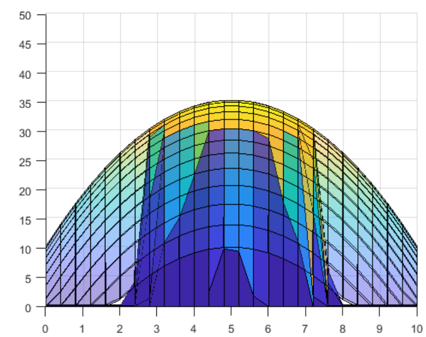
(c) $t = 0.3$



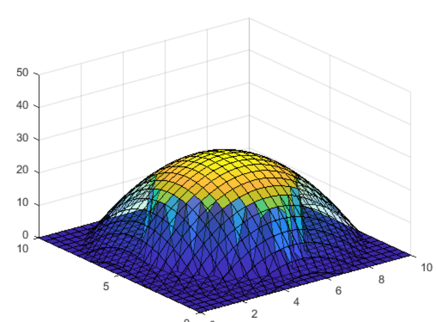
(d) $t = 0.3$



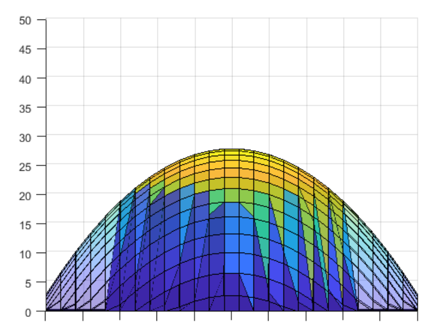
(e) $t = 0.6$



(f) $t = 0.6$



(g) $t = 0.9$



(h) $t = 0.9$

Figure 10: Simulation 3

References

[AGS] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, 2008.

[AKY] Damon Alexander, Inwon Kim, and Yao Yao. Quasi-static evolution and congested crowd transport. *Nonlinearity*, 27(4):823, 2014.

[BC] Philippe Bénilan and Michael G Crandall. The continuous dependence on phi of solutions of $u_t - \Delta\varphi(u) = 0$. *Indiana University Mathematics Journal*, 30(2):161–177, 1981.

[BH] M. Bertsch and D. Hilhorst. A density dependent diffusion equation in population dynamics: Stabilization to equilibrium. *SIAM Journal of Mathematical Analysis*, 1986.

[CF] Luis Caffarelli and Avner Friedman. Asymptotic behavior of solutions of $u_t = \Delta u^m$ as $m \to \infty$. *Indiana University mathematics journal*, 36(4):711–728, 1987.

[CJK] Sunhi Choi, David Jerison, and Inwon Kim. Local regularization of the one-phase hele-shaw flow. *Indiana University Mathematics Journal*, 58(6):2765–2804, 2009.

[CKY] Lincoln Chayes, Inwon Kim, and Yao Yao. An aggregation equation with degenerate diffusion: qualitative property of solutions. *SIAM Journal of Mathematical Analysis*, 45(5):2995–3018, 2013.

[D] J. Doob. *Classical Potential Theory and its Probabilistic Counterpart*. Springer-Verlag, 1984.

[DB] E. DiBenedetto. Continuity of weak solutions to a general porous media equation. *Indiana Univ. Math. J.*, 1983.

[DMSV] Guido De Philippis, Alpar Richard Meszaros, Filippo Santambrogio, and Bozhidar Velichkov. Bv estimates in optimal transportation and applications. *Arch. Ration. Mech. Anal.*, 2016.

[DS] S. Daneri and G. Savare. Eulerian calculus for the displacement convexity in the wasserstein distance. *SIAM Journal on Mathematical Analysis*, 40(3):1104–1122, 2008.

[EHKO] C. M. Elliott, M. A. Herrero, J. R. King, and J. R. Ockendon. The mesa problem: Diffusion patterns for $u_t = \nabla \cdot (u^m \nabla u)$ as $m \to \infty$. *IMA Journal of Applied Mathematics*, 37(2):147–154, 1986.

[FM] Marco Di Francesco and Daniel Matthes. Curves of steepest descent are entropy solutions for a class of degenerate convection–diffusion equations. *Calculus of Variations and Partial Differential Equations*, 50(1):199–230, May 2014.

[FV] Lucas Ferreira and Julio Valencia Guevara. Gradient flows of time-dependent functionals in metric spaces and applications to pdes. 185, 09 2015.

[GQ] O. Gil and Fernando Quirós. Convergence of the porous media equation to hele-shaw. *Nonlinear Analysis: Theory, Methods, & Applications*, 44(8):1111–1131, 2001.

[GS] Bjorn Gustafsson and Makoto Sakai. Properties of some balayage operators, with applications to quadrature domains and moving boundary problems. *Nonlinear Analysis: Theory, Methods & Applications*, 22(10):1221 – 1245, 1994.

[GV] Bjorn Gustafsson and A. Vasil'ev. *Conformal and Potential Analysis in Hele-Shaw Cells*. Advances in Mathematical Fluid Mechanics. Birkhäuser Basel, 2006.

[I] John DeIonno. Quasi-variational inequalities for source-expanding hele-shaw problems. 2013.

[K] Inwon C. Kim. Uniqueness and existence results on the hele-shaw and the stefan problems. *Archive for rational mechanics and analysis*, 168(4):299–328, 2003.

[K2] Inwon C. Kim. A free boundary problem arising in flame propagation. *Journal of Differential Equations*, 191(2):470 – 489, 2003.

[KL] H. K. Lei. Degenerate diffusion with a drift potential: a viscosity solutions approach, joint work with inwon kim, truncated version. *arXiv preprint arXiv:1011.4502*, 2010.

[KP] Inwon Kim and Norbert Pozar. Porous medium equation to hele-shaw flow with general initial density. *arXiv preprint arXiv:1509.06287*, 2015.

[KPW] Inwon Kim, Norbert Pozar, and Brent Woodhouse. Singular limit of the porous medium equation with a drift. 08 2017.

[LSU] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type*. American Mathematical Society, 1968.

[MM] Simone Di Marino and Alpár Richárd Mészáros. Uniqueness issues for evolution equations with density constraints. *Mathematical Models and Methods in Applied Sciences*, July 2016.

[MMS] Simone Di Marino, Bertrand Maury, and Filippo Santambrogio. Measure sweeping processes, 2015.

[MPQ] Antoine Mellet, Benoît Perthame, and Fernando Quiros. A hele-shaw problem for tumor growth. *arXiv preprint arXiv:1512.06995*, 2015.

[MRS] Bertrand Maury, Aude Roudneff-Chupin, and Filippo Santambrogio. A macroscopic crowd motion model of gradient flow type. *Mathematical Models and Methods in Applied Sciences*, 20(10):1787–1821, 2010.

[MRSV] Bertrand Maury, Aude Roudneff-Chupin, Filippo Santambrogio, and Juliette Venel. Handling congestion in crowd motion modeling. *arXiv preprint arXiv:1101.4102*, 2011.

[OK] Bernt Oksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer, 1979.

[PQV] Benoît Perthame, Fernando Quirós, and Juan Luis Vázquez. The hele–shaw asymptotics for mechanical models of tumor growth. *Archive for Rational Mechanics and Analysis*, 212(1):93–127, 2014.

[R] Aude Roudneff-Chupin. Modelisation macroscopique de mouvements de foule. 2011.

[S] Filippo Santambrogio. Crowd motion and evolution pdes under density constraints, 2018.

[V] Juan Luis Vázquez. *The porous medium equation: mathematical theory*. Oxford University Press, 2007.