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Identification, Estimation and Testing of Auction Models

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ABSTRACT OF THE DISSERTATION

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by

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The first chapter establishes a way of inferring risk aversion in a first-price auction (FPA) model when an entry decision is endogenous. Bidders' risk aversion is captured by a parameter in constant relative risk aversion utility functions and the parameter is then partially identified in a set under a "monotonicity" condition. The recovery of the partially identified risk aversion parameter is concluded in a confidence set (CS). The CS is constructed by inverting a test dealing with many inequality restrictions of quantiles. In the spirit of Andrews and Shi (2013), we implement quantile selection to address possible slackness. Asymptotic results show desired properties of size and power against fixed (and some local) alternatives. Confidence sets perform fairly well in finite samples and the comparison of results highlights the necessity of quantile selection. The inference is illustrated by using US Forest Service timber auction data and detects considerable risk aversion.

Exogenous entry is a convenient assumption to make for identification and inference in auction models. The second chapter examines this assumption and develops a test against endogenous entry in first-price auctions with risk aversion. The approach also takes auction observed heterogeneity into account. The desirable property of asymptotic size and consistency of the test is proven, and Monte Carlo simulations approve the test at finite samples. The application to US Forest Service timber auctions brings in interesting implication: entry is exogenous with lower entry level but is endogenous with higher entry level. The result also suggests a relatively stronger risk

aversion attitude than what is obtained in the literature.

The third chapter shows nonparametric identification and estimation of private value distribution and density functions in first-price auctions with endogenous entry. In the model, symmetric bidders face a nontrivial entry cost and a binding reserve price. We identify latent structures by solving a two stage game, and estimate density functions (point-wisely) by using and comparing two different methods. Monte Carlo experiments show good performance of our estimators.

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Chapter 1

Introduction

Auctions have been applied in trading a wide range of objects, involving immense amount of value. As researchers, we would like to learn the fundamental primitive to understand bidders' behaviour at our best.

Two sources can mainly influence bidders' decision making. One is bidders' risk attitude. A higher level of risk aversion induces a bidder to bid closer to her true value, and therefore increases a seller's benefit. The other is bidders' private values to objects on sale. Knowledge of the value distribution or density is decisive in designing an optimal auction mechanism.

One significant feature of this dissertation is that it places considerable attention on endogeneity. This consideration makes the research more appealing as endogeneity is a popular issue wherever we employ data in economics. On the other hand, introduction of endogeneity fails some main results in the literature. So we need to seek different solutions under new approaches.

In particular, under exogenous entry, observed different numbers of bidders offer an ideal source of variation while keeping the underlying value distributions intact. Similar to the way of estimating a panel model with fixed effect, the uncertainty of value distribution can be well removed by some sort of "difference" techniques, so that one can reach identification of other unknowns such as utility functions. Then one may come back to recover the value distribution at the final stage. With endogenous entry, however, there is no certain way to separate the uncertainty of unknown value distributions with other unknowns, and therefore, we need to think hard about how much we can recover and what we can still learn, perhaps under some other weak conditions.

The first two chapters deal with the problem of learning risk aversion with endogenous entry. In the first chapter, “Inference of Risk Aversion in First-Price Auctions with Endogenous Entry”, the alternative main assumption we impose is a monotonicity condition, under which the number of entrant bidders indicate the ranking of value distributions in the sense of first-order stochastic dominance (FOSD). The idea of recovering risk aversion is to see which (range of) values of the risk aversion parameter can support the main assumption. This idea naturally leads us to developing a test for FOSD of unobserved value distributions, by employing recovered value quantiles at all levels. The technique of Chapter one establishes recovery of an identified set with multiple values of parameters as solutions.

A natural question to ask now is whether it is necessary to relax the assumption of exogenous entry, or in other words, when it is justified to use the assumption of exogenous entry to estimate risk aversion. The second chapter “Testing Exogenous Entry in First-Price Auctions with Risk Averse Bidders” addresses this question. The main object of Chapter two is to test the assumption of exogenous entry, and the by-product is that we obtain the estimate for the risk aversion parameter over a subset of data where exogenous entry is likely.

In the first two chapters, we do not discuss the recovery of private value distributions or density functions, or they are treated as some nuisance parameters. In the third chapter, “Identification and Estimation of First-Price Auctions with Entry”, we turn to the private value part. There we assume risk neutrality of bidders, yet we are facing a non-trival endogeneity problem: the sample selection comes from both an entry cost and a binding reserve price. We need to solve the game in two stages to take care of the two sources of endogeneity, and estimate value distribution and density accordingly. Chapter three basically shows a nonparametric approach to first-price auctions with the first stage of bidders playing a mixed entry strategy.

The solutions that this dissertation tries to provide apply to situations where entry is non-negligible in auctions. One can realize the application is indeed broad by seeing that it usually takes bidders a good amount of time and money to prepare for bids and decide to enter or not in reality. A large proportion of existing literature hinges on one restriction that entry is exogenous. The approach provided in this dissertation is different and more general. Hence, it is a good reference for researchers to carry out robust inference and institutions to make effective policies.

Chapter 2

Inference of Risk Aversion in First-Price Auctions with Endogenous Entry

2.1 Introduction

An auction provides a transaction format for the trading and allocating of various objects, such as the rights of exploiting resources and the procurement of constructing highways. The extensive application calls for a deep understanding of primitives in an auction, and meanwhile provides rich examples and data for research. To adapt to various sophisticated auction models, new and effective econometrics methods have been invented. Meanwhile, as is often the case, one single problem in an auction can be modeled several different ways in economics. To compare them and choose which model to use, a practitioner may consult econometricians with data, to examine, for instance, major assumptions of different candidates.

Estimating agents' risk aversion is an important issue in empirical microeconomics study. The role risk aversion plays in auctions is relevant to mechanism design. Generally speaking, a first-price auction (FPA) is ranked by revenue over an ascending auction with risk averse bidders, whereas the two formats have the same rank with risk neutral ones. Also, since bidders generally bid more aggressively as risk aversion increases, risk aversion may undercut the screening effect of a reserve price and thus reduce the optimal reserve price (possibly down to 0). Our primary interest is the risk aversion attitude of bidders represented by their von Neumann-Morgenstern (vNM) utility function. By virtue of identification results from Guerre, Perrigne and Vuong (2009), this function is observationally equivalent to a member from a parametrized family of utility functions under mild assumptions.

As for the other underlying primitive of bidders' (conditional) value distribution, we keep it nonparametrically specified. It has been shown (e.g., by Campo et al. (2011)) that a utility function is not identifiable in this scenario. To solve this problem, Guerre, Perrigne and Vuong (2009) propose an exogenous participation condition that assumes that bidders' private values are (conditionally) independent on the number of active bidders. On the other hand, empirical study shows significantly selective entry to auctions: 25% in the US Minerals Management Service (Hendricks, Pinkse and Porter, 2003) and 28% in the Texas Department of Transportation (DOT) (Li and Zheng, 2009). By ignoring endogenous entry, inference of underlying primitives would suffer selection bias. Alternatively, Guerre, Perrigne and Vuong (2009) and Haile, Hong and Shum (2003) propose instrument variable approaches to control for endogeneity. The validity of instruments thus raises to be an issue yet to be addressed formally. It is worth noticing that Fang and Tang (2014) consider testing risk aversion with endogenous entry in ascending auctions. The use of their inference is rather limited, as it is not very informative at different degrees of risk aversion.

Motivated by a robust inference against endogeneity, one general approach which has become popular recently, especially in treatment effect study, is to start with only a few "minimal" assumptions agreed on by almost everyone, and see how much of our interest in the unknown can be recovered from observables with those assumptions. More often than not, it comes with non-unique solutions as in an identified set characterized by a certain type of bounds containing objects of the unknown, and sharpness of the bounds depends on modeling strategies and data sizes. Inference based on sharp bounds may be fairly informative.

Inspired by this approach, this paper relaxes the assumption of exogenous entry by assuming first-order stochastic dominance (FOSD) of (conditional) private value distributions, which are not observable. The bounds obtained by this restriction turn out to be nontrivial but fairly informative in some data generation processes (DGPs). To preserve consistency, the test for FOSD employs many quantiles of values recovered from quantiles of obervables as instruments. For inference faced with many (in)equality conditions in general, the main challenge is that one must deal with nuisance parameters as slackness of infinite dimensions. Meanwhile, one may want to employ (infinitely) many conditions for sharp inference. The question then is how to select the most informative restrictions. Some papers that address these technical challenges include: Andrews and Shi (2013), Chernozhukov, Lee and Rosen (2013) and Lee, Song and Whang (2013), among others. We implement selection over many inequalities on quantiles in the spirit of Andrews and Shi (2013). The rate at which the test can detect local alternatives mainly depends on uniform rates from nonparametric estimation of quantiles.

The rest of this paper is organized as follows. Section 2.2 describes the environment of an FPA model with specification of entry and shows an identification set of risk aversion. Section 2.3 details the nonparametric (kernel) estimation of quantiles and density from bids and shows the asymptotic properties of estimators, which are used in section 2.4, for testing. Section 2.4 constructs a confidence set for the true risk aversion parameter by inverting a test of FOSD, and shows the asymptotic consistency of the test. Section 2.5 illustrates the main idea of inference by Monte Carlo simulation and an empirical application to US Forest Service (USFS) timber auction sales. Section 2.6 concludes and discusses possible extensions. Omitted proofs and details are reserved for the appendix.

2.2 Partial Identification of Risk Aversion

2.2.1 An FPA Model with Endogenous Entry

There is one unit of object for sale in one single auction $A_i, i = 1, 2, 3, \ldots, L$. Auction $A_i's$ are independent from each other. By convention, we characterize an auction game in two stages. At stage one of entry, there is an imperfect signal $S_i \in \mathcal{S}$ observed by potential bidders in total of N_i^* within A_i. Yet $S_i's$ are unobserved to econometricians. By the end of stage one, potential bidders make decisions of entry to stage two of bidding. Each potential bidder possesses an identical vNM utility function $U(\cdot)$ with $U(0) = 0$, $U'(\cdot) > 0$, and $U''(\cdot) \leq 0$.

Entry incurs a cost associated with bid preparation and information acquisition. We assume that the entry cost is the same level of K for all bidders in all auctions. To avoid negative wealth after paying the entry cost, without loss of generality we assume that bidders are endowed with initial wealth K. This is rather a normalization. The inference with bidders having initial wealth larger than K would also work, though by some proper normalization. Again, K is assumed observed by bidders but not econometricians. A bidder who actually enters the second stage becomes active.

Bidding behavior with entry has been studied recently by, for example, Gentry and Li (2014), Li and Zheng (2009) and Fang and Tang (2014). Bidders adopt different participation strategies at the entry stage according to their knowledge. Basically, mixed entry (entry with a positive probability) occurs if there is no private information observed at the entry stage, whereas a pure entry decision is made by a cut-off value if such information is available at the entry stage. Differently from common approaches described in the literature, we assume that entry of bidders is sequential and observed by all (other) bidders including potential ones. Heuristically, each (identical a priori) potential bidder prior to entry in auction A_i is assigned randomly a different number from 1 to N_i^*. At the entry stage, bidder j decides to enter or not and then bidder $j+1$ does the same after observing bidder j's entry decision, for $j = 1, 2, \ldots, N_i^* - 1$. The reason to consider observed sequential entry here is that in USFS timber auctions for example, potential bidders usually conduct "cruises" of a tract after entry but before bidding, as a way of investigating timber values. The "cruises" usually last for days and thus are likely to be observed in a given tract.

For active bidder j in auction i, she will draw her private value V_{ij} independently from a distribution $F_{V_i|s_i}(\cdot \mid s_i)$ ($F_{s_i}(\cdot)$ hereafter), where $V_{ij} \in [\underline{v}(s_i), \overline{v}(s_i)]$. Then she formulates and submits her bid b_{ij}. For simplicity , we assume there are no binding reserve prices.

Suppose there are eventually n_i (out of N_i^*) active bidders at the second stage of bidding within auction A_i, where $n_i \equiv n(s_i) \in \mathcal{N} = \left\{2, 3, ..., n^N\right\}$, a finite set. With our normalization of initial wealth, the entry cost, the utility function and observed sequential entry, one can easily see that our bidding model falls into an FPA framework without initial wealth or entry cost but with known number of active bidders. In particular, an active bidder's expected payoff function coincides with the one under the model specification proposed by Campo et al. (2011). Hence the bid b_{ij} solving (active) bidder $j's$ optimization problem satisfies, as in Campo et al. (2011), that

$$v_{ij} = b_{ij} + \Lambda^{-1}\left(\frac{1}{n_i - 1}\frac{G_{s_i}(b_{ij})}{g_{s_i}(b_{ij})}\right), \tag{2.1}$$

where $G_{s_i}(b_{ij})$ ($g_{s_i}(b_{ij})$) is the conditional distribution (density) of b_{ij}, and $\Lambda^{-1}(\cdot)$ is the inverse function of $\Lambda(\cdot)$ where $\Lambda(\cdot) = \frac{U(\cdot)}{U'(\cdot)}$.

2.2.2 Identification Assumptions

Inheriting the basic identification structure from Guerre, Perrigne and Vuong (2009) or Campo et al. (2011), we impose Assumptions 1 and 2 for primitives in our model with unobserved heterogeneity.

Assumption 1. $U(\cdot) \in \mathcal{U}_R$, $F_s(\cdot) \in \mathcal{F}_R$ and $G_n(\cdot) \in \mathcal{G}_R$ where $\mathcal{U}_R$, $\mathcal{F}_R$ and $\mathcal{G}_R$ are defined in Appendix A.1.

Assumption 2. Given any $s \in \mathcal{S}$, for $n(s) \in \mathcal{N}$, $F_s \in \mathcal{F}_R$ and $G_n \in \mathcal{G}_R$, there exists $\theta \in \Theta \equiv (0, 1]$ such that $U(x) = x^\theta$ solves the structural equation (2.1).

Assumption 1 ensures that an observed bid distribution $G_n \in \mathcal{G}_R$ can be rationalized by $[U, F_s] \in \mathcal{U}_R \times \mathcal{F}_R$. Assumption 2 specifies CRRA for $\mathcal{U}_R$.

One way to think about the role that an unobserved signal s plays here is to consider a possible form of correlation between S_i and V_{ij}.

Assumption 3. The set $\mathcal{S}$ is a totally ordered set. Its order is induced by (FOSD)

$$V_{s'} \succ_{FOSD} V_s \Rightarrow s' > s. \tag{2.2}$$

Note that the assumption of conditional independence can be treated as a very particular version of Assumption 3 and thus is much stronger. Moreover, the assumption of FOSD is weaker than that of affiliation, as pointed out by Gentry and Li (2014).

For an active bidder observing signal s in an n-(active)bidder auction, it is easy to verify that her bidding function can be written as

$$b_{s,n,\theta}(v) = \frac{1}{H_{s,n,\theta}(v)} \int_0^v x h_{s,n,\theta}(x) dx = v - \int_0^v \frac{H_{s,n,\theta}(x)}{H_{s,n,\theta}(v)} dx, \tag{2.3}$$

where $H_{s,n,\theta}(v) = [F_s(v)]^{(n-1)/\theta}$, and $h_{s,n,\theta}(v)$ is its density. Hence, the bidder's expected payoff at the entry stage can be shown as

$$\int_0^\infty [\int_0^v \frac{H_{s,n,\theta}(x)}{H_{s,n,\theta}(v)} dx]^\theta dF_s(v) \equiv \pi_{s,n}(\theta). \tag{2.4}$$

With the payoff of non-participation being $U(K) = K^\theta$, we get $n_i \equiv n(s_i) = \max\{n : \pi_{s_i,n}(\theta) - K^\theta \geq 0\}$.

Next comes the additional assumption for the "value" of information s.

Assumption 4. $\forall \theta \in \Theta$ and for $\forall s, s' \in \mathcal{S}$, such that $s < s'$, one of the two specifications below holds and is known to econometricians.

1. ("Good" news for larger signals) $\pi_{s,n}(\theta)$ is increasing with s, and $\pi_{s',n(s)+1}(\theta) > K^\theta$.
2. ("Bad" news for larger signals) $\pi_{s,n}(\theta)$ is decreasing with s, and $\pi_{s,n(s')+1}(\theta) > K^\theta$.

Assumption 4 can be understood as restrictions to the domain of $\mathcal{F}_R$ such that a larger signal can only be either "good" news or "bad" news for bidders. For example, under specification 1 a larger signal increases bidders' expected payoff at the entry stage; furthermore, the increase is supposed to be profitable enough so that more potential bidders are induced to enter. The opposite occurs under specification 2.

It is not surprising to see that the result of expected payoff purely increasing with signals, as in Marmer, Shneyerovb and Xu (2013), does not necessarily hold here, since we do not assume independence of bidders' signals. Here we provide some intuition of why the monotonicity in Assumption 4 may work in either direction. A larger signal

s' could discourage entry, as it may shift mass of value density to the higher end so much so that competition becomes extremely fierce among high value bidders. As such, $\pi_{s,n}(\theta)$ may decrease with s, and it is possible to have specification 2 hold. On the other hand, the first order stochastic dominance due to a wider support is likely to cause specification 1. We will provide one case for each specification in Assumption 4 as a data generation process (DGP) in section 2.5 to fit the intuition.

Consequently, we can obtain a "monotonic entry" result. To simplify, we impose an additional Assumption 5 of a sufficiently large number of potential bidders to ensure that the monotonicity is strong, and we summarize the result in Proposition 1. The proof is omitted as the statement is quite straightforward.

Assumption 5. The maximal element n^N of the finite set $\mathcal{N}$ satisfies that $n^N < N_i^*, i = 1, 2, ..., L$.

Proposition 1. *Under Assumptions 1-3 and 5,* $n_i(\equiv n(s_i))$

(a) is strictly increasing with s_i *under specification 1 of Assumption 4,*

(b) and is strictly decreasing with s_i *under specification 2 of Assumption 4.*

2.2.3 An Identification Set of Risk Aversion

With $U^0(x) = x^{\theta^0}$, $\theta^0 \in \Theta$, the previous necessary condition (2.1) now is

$$v_{ij} = b_{ij} + \theta^0 \frac{1}{n_i - 1} \frac{G_{s_i}(b_{ij})}{g_{s_i}(b_{ij})} \equiv b_{ij} + \theta^0 X_{ij}, \tag{2.5}$$

where $X_{ij} \equiv \frac{1}{n_i-1} \frac{G_{s_i}(b_{ij})}{g_{s_i}(b_{ij})}$. Or equivalently, if $s_i = s$, it is necessary that

$$V_s(\tau) = b_s(\tau) + \theta^0 X_s(\tau), \forall \tau \in [0, 1], \tag{2.6}$$

where $X_s(\tau) = \frac{1}{n(s)-1} \frac{\tau}{g_s(b_s(\tau))}$, and $Y(\tau)$ stands for the τ quantile of a generic random variable Y.

$\forall s, s' \in \mathcal{S}$, such that $s < s'$, we know $V_{s'} \succ_{FOSD} V_s$ by Assumption 3. So we have

$$b_s(\tau) + \theta^0 X_s(\tau) = V_s(\tau) \leq V_{s'}(\tau) = b_{s'}(\tau) + \theta^0 X_{s'}(\tau), \forall \tau \in [0, 1]. \tag{2.7}$$

By proposition 1, there exist $n \neq n'$ such that $n(s) = n$, and $n(s') = n'$. This indicates that (2.7) can be expressed as

$$[b_{n'}(\tau) - b_n(\tau)] + \theta^0[X_{n'}(\tau) - X_n(\tau)] \geq 0, \ \forall \tau \in [0,1]. \tag{2.8}$$

Without loss of generality, consider that specification 1 in Assumption 4 holds and is known to econometricians. Thus (2.8) holds $\forall n, n' \in \mathcal{N}$, such that $n' > n$. This leads to an identification set of θ, denoted by Θ_I.

$$\Theta_I \equiv \{\theta : [b_{n'}(\tau) - b_n(\tau)] + \theta[X_{n'}(\tau) - X_n(\tau)] \geq 0, \ \forall \tau \in [0,1], \tag{2.9}$$
$$\forall n', n \in \mathcal{N}, \quad n' > n, \quad \forall G_n, G_{n'} \in \mathcal{G}_R, \quad \theta \in \Theta\}.$$

Remark 1. As for the two specifications in Assumption 4, an econometrician may not always know which one holds. The uncertainty may not be a big problem when $\Theta_I = \emptyset$ under a misspecification from Assumption 4. Inference without knowledge of the correct specification is illustrated by one application with real data as in section 2.5.3.

2.3 Nonparametric Estimation of Quantiles and Asymptotic Properties

We aim to construct a CS to cover the true parameter θ^0 with probability greater or equal to $1 - \alpha$ for $\alpha \in (0,1)$. The CS is derived by inverting a test in section 2.4, which adopts quantiles appearing in (2.8) as instruments. The asymptotic consistency of our test relies on properties of those nonparametrically estimated quantiles which are shown in this section.

Suppose $\mathcal{N} = \{n, n'\}$, where $n < n'$. Let $\pi(l) = Pr(n_i = l)$, $l \in \mathcal{N}$, $i = 1, 2, ..., L$. Define $\triangle X(\tau) \equiv X_{n'}(\tau) - X_n(\tau)$, and $\triangle b(\tau) \equiv b_{n'}(\tau) - b_n(\tau)$. It follows that the identification condition in (2.9) now becomes

$$\triangle b(\tau) + \theta \triangle X(\tau) \geq 0, \quad \forall \tau \in [0,1]. \tag{2.10}$$

To save notations, let $Q(\tau, \theta) \equiv \triangle b(\tau) + \theta \triangle X(\tau)$. Let $K(u) : \mathbb{R} \to \mathbb{R}$ be a kernel-like function. Define $K_h(u) = h^{-1} K(\frac{u}{h})$, where $h = o(1)$ is a bandwidth. To have consistent estimators, we require the following (standard) assumption on $K(\cdot)$.

Assumption 6. (a) $|K(u)| \leq \overline{K} < \infty$ and $\int |K(u)| du \leq \mu < \infty$

(b) For some $\Lambda_1 < \infty$ and $L < \infty$, $K(u) = 0$ for $|u| > L$ and $\forall u, u' \in R$, $|K(u) - K(u')| \leq \Lambda_1 |u - u'|$.

(c) $Lh \to \infty$, $(\frac{Lh}{\ln L})^{\frac{1}{2}} h^R \to 0$.

Consider the following estimators:

$$\widehat{\pi}(n) = \frac{1}{L}\sum_{i=1}^{L} 1(n_i = n)$$

$$\widehat{G_n}(b) = \frac{1}{\widehat{\pi}(n)nL}\sum_{i=1}^{L}\sum_{j=1}^{n} 1(n_i = n)1(b_{ij} \leq b)$$

$$\widehat{b_n}(\tau) = \widehat{G_n}^{-1}(\tau) \equiv \inf_b \left\{b : \widehat{G_n}(b) \geq \tau\right\}$$

$$\widehat{g_n}(b) = \frac{1}{\widehat{\pi}(n)nL}\sum_{i=1}^{L}\sum_{j=1}^{n} 1(n_i = n)K_h(b - b_{ij})$$

$$\widehat{X_n}(\tau) = \frac{1}{n-1}\frac{\tau}{\widehat{g_n}(\widehat{b_n}(\tau))}$$

$$\widehat{Q}(\tau,\theta) = \triangle\widehat{b}(\tau) + \theta\triangle\widehat{X}(\tau) = [\widehat{b_{n'}}(\tau) - \widehat{b_n}(\tau)] + \theta[\widehat{X_{n'}}(\tau) - \widehat{X_n}(\tau)].$$

$\forall \tau_1, \tau_2$ such that $0 < \tau_1 < \tau_2 < 1$, by Guerre, Perrigne and Vuong (2000, Proposition 1), $\forall n \in \mathcal{N}$, there is a fixed inner compact interval, say $[b_{n,1}, b_{n,2}]$, such that $[b_n(\tau_1), b_n(\tau_2)] \subset (b_{n,1}, b_{n,2})$, and $[b_{n,1}, b_{n,2}] \subset [\underline{b}_n, \overline{b}_n]$. Hereafter, we use τ_1 and τ_2 for a generic arbitrary fixed inner compact interval $[\tau_1, \tau_2] \in [0,1]$.

Lemma 1 in Appendix A.2 shows uniform convergence results of estimators defined above with convergence rates. Based on Lemma 1, we are able to show asymptotic normality of relevant statistics of quantiles.

Proposition 2. *Under Assumptions 1 and 6, $\forall n \in \mathcal{N}$, and $\forall \tau \in [\tau_1, \tau_2]$, $\forall b \in [b_{n,1}, b_{n,2}]$, let $V_n(\tau) = \frac{\tau^2}{(n-1)^2}\frac{1}{n\pi(n)g_n^3(b_n(\tau))}\int K^2(u)du$, and then*

(a) $(Lh)^{1/2}(\widehat{g_n}(b) - g_n(b)) \to_d N(0, \frac{g_n(b)}{n\pi(n)}\int K^2(u)du)$,

(b) $(Lh)^{1/2}(\widehat{X_n}(\tau) - X_n(\tau)) \to_d N(0, V_n(\tau))$.

Corollary 1. *Under Assumptions 1 and 6,* $\forall\tau \in [\tau_1, \tau_2]$ *and* $\forall\theta \in \Theta$, $(Lh)^{1/2}(\widehat{Q}(\tau,\theta) - Q(\tau,\theta)) \to_d N(0, \theta^2 V(\tau))$*, where* $V(\tau) = V_n(\tau) + V_{n'}(\tau)$.

As for choosing a bandwidth, one typical way is to set $h = cL^{-\chi}$, where c is a constant. It is easy to verify that $\chi = \frac{1}{2R+1}$ satisfies Assumption 6. Moreover, it can be easily seen that such a value of χ leads to the optimal convergence rate $(Lh)^{1/2} = L^{\frac{R}{2R+1}} \equiv a_L$.

2.4 Confidence Sets and Their Asymptotic Consistency

2.4.1 Construction of CS's

By the identification result of risk aversion in section 2.2, $\forall\theta \in \Theta$, the null and alternative hypotheses of our test can be written as

$$H_0 : Q(\tau,\theta) \geq 0, \forall\tau \in (0,1),$$
$$H_1 : Q(\tau,\theta) < 0, \text{for some } \tau \in (0,1).$$

Consider a Cramer-Von Mises-type(CvM) statistic,

$$T_L(\theta) = \int_0^1 [\frac{\widetilde{Q}(\tau,\theta)}{(\widetilde{V}(\tau,\theta))^{\frac{1}{2}}}]_-^2 d\tau, \tag{2.11}$$

where $[x]_- = 0$ if $x \geq 0$, and $[x]_- = -x$ if $x < 0$; $\widetilde{Q}_L(\tau,\theta) = a_L\widehat{Q}_L(\tau,\theta)$; $\widetilde{V}_L(\tau,\theta) \equiv \theta^2\widehat{V}_L(\tau) = \theta^2(\widehat{V_n}(\tau) + \widehat{V_{n'}}(\tau))$, where $\widehat{V_n}(\tau) = \frac{\tau^2}{(n-1)^2}\frac{1}{n\widehat{\pi}(n)\widehat{g_n}^3(b_n(\tau))}\int K^2(u)du$.

Let $c_{L,1-\alpha}(\theta)$ be the critical value for a test with nominal significance level α. Then the nominal $1-\alpha$ CS for the true parameter θ^0 is

$$CS_L = \{\theta \in \Theta : T_L(\theta) \leq c_{L,1-\alpha}(\theta)\}. \tag{2.12}$$

2.4.1.1 A Strategy of Obtaining Critical Values

Define $Q^*(\tau,\theta) = a_L Q(\tau,\theta)$. Rewrite equation (2.23) as

$$T_L(\theta) = \int_0^1 [\frac{\widetilde{Q}_L(\tau,\theta) - Q_L^*(\tau,\theta)}{\widetilde{V}_L(\tau,\theta)^{\frac{1}{2}}} + \frac{Q_L^*(\tau,\theta)}{\widetilde{V}_L(\tau,\theta)^{\frac{1}{2}}}]_-^2 d\tau. \tag{2.13}$$

Note that there is no consistent estimator for $Q_L^*(\tau,\theta)$. We replace $Q_L^*(\tau,\theta)$ in (2.13) by a data-dependent quantile selection(QS) function $\varphi_L(\cdot,\theta)$.

Define

$$\xi_L(\theta,\tau) \equiv \kappa_L^{-1}\widetilde{Q}(\tau,\theta)\widetilde{V}(\tau,\theta)^{-\frac{1}{2}}, \tag{2.14}$$

where κ_L is a sequence diverging as $L \to \infty$, and our QS function is defined as

$$\varphi_L(\cdot,\theta) = B_L\mathbf{1}(\xi_L(\tau,\theta) > 1), \tag{2.15}$$

where B_L is a non-decreasing positive sequence.

Assumption 7. $\kappa_L \to \infty$ and $B_L/\kappa_L \to 0$ as $L \to \infty$.

It is necessary for $\kappa_L \to \infty$ to deliver consistency of our test. The size of the test is correct as long as B_L is non-decreasing.

With the QS function $\varphi_L(\cdot,\theta)$, we define

$$T_L(\varphi_L,\theta) = \int_0^1 [\frac{\widetilde{Q}_L(\tau,\theta) - Q_L^*(\tau,\theta)}{\widetilde{V}_L(\tau;\theta)^{\frac{1}{2}}} + \frac{\varphi_L(\tau,\theta)}{\widetilde{V}_L(\tau;\theta)^{\frac{1}{2}}}]_-^2 d\tau. \tag{2.16}$$

To obtain critical values, we will simulate the randomness of $\widetilde{Q}_L(\tau,\theta) - Q_L^*(\tau,\theta)$ by using (re-centered) bootstrap $\widetilde{Q}_L^\dagger(\tau,\theta) - \widetilde{Q}_L(\tau,\theta)$, so that we can define

$$T_L^\dagger(\varphi_L,\theta) = \int_0^1 [\frac{\widetilde{Q}_L^\dagger(\tau,\theta) - \widetilde{Q}_L(\tau,\theta)}{\widetilde{V}_L(\tau;\theta)^{\frac{1}{2}}} + \frac{\varphi_L(\tau,\theta)}{\widetilde{V}_L(\tau;\theta)^{\frac{1}{2}}}]_-^2 d\tau. \tag{2.17}$$

Then the critical value is obtained by

$$c_{L,1-\alpha}^{QS}(\varphi_L,\theta) \equiv c_{L,1-\alpha}^0(\varphi_L,\theta), \tag{2.18}$$

where $c_{L,1-\alpha}^0(\varphi_L,\theta)$ is the $1-\alpha$ quantile of $T_L^\dagger(\varphi_L,\theta)$.

To appreciate the virtue of QS at finite samples in power performance, we compare QS CS's with plug-in(PA) CS's, which ignores slackness and quantile selection and generates the critical value defined by

$$c_{L,1-\alpha}^{PA}(\theta) \equiv c_{L,1-\alpha}^0(0,\theta). \tag{2.19}$$

2.4.2 Asymptotic Size

2.4.2.1 A Useful Bound Result

To derive asymptotic coverage results, we first consider an infeasible estimator

$$\breve{T}_L(\theta) = \int_0^1 [\frac{\widetilde{Q}_L^{\dagger}(\tau,\theta) - \widetilde{Q}_L(\tau,\theta)}{\widetilde{V}_L(\tau,\theta)^{\frac{1}{2}}} + \frac{Q_L^*(\tau,\theta)}{\widetilde{V}_L(\tau,\theta)^{\frac{1}{2}}}]_-^2 d\tau \tag{2.20}$$

The infeasible estimator $\breve{T}_L(\theta)$ simulates the randomness of $T_L(\theta)$, as if we knew $Q_L^*(\tau,\theta)$. The next result shows that this approximation performs well asymptotically in probability.

Theorem 1. *Under Assumptions 1 and 6,* $\forall \theta \in \Theta_I$, $\forall x \in R$ *and* $\forall \delta > 0$, *we have*

(a) $\limsup_{L\to\infty} \quad [P(T_L(\theta) > x) - P(\breve{T}_L(\theta) + \delta > x)] \leq 0,$

(b) $\liminf_{L\to\infty} \quad [P(T_L(\theta) > x) - P(\breve{T}_L(\theta) - \delta > x)] \geq 0.$

2.4.2.2 Asymptotic Size

Now let us show asymptotic coverage probability results. Denote a CS constructed by QS and PA with sample size L by CS_L^{QS} and CS_L^{PA} respectively. The idea is to show that critical values obtained by using $\varphi_L(\cdot,\theta)$ are no smaller than by using $Q_L^*(\cdot,\theta)$ asymptotically in probability.

Theorem 2. *Under Assumptions 1, 6 and 7,* $\forall \theta \in \Theta_I$,

(a) $\liminf_{L\to\infty} \quad P(\theta \in CS_L^{QS}) \geq 1 - \alpha,$

(b) $\liminf_{L\to\infty} \quad P(\theta \in CS_L^{PA}) \geq 1 - \alpha.$

Remark 2. It may be interesting and useful to consider whether the QS CS is conservative by designing particular configurations in the spirit of Andrews and Shi (2013, Assumption GMS2).

2.4.3 Power Against Fixed Alternatives

Denote the underlying probability space by $(\Omega, \mathcal{F}, P)$. $\forall \theta \in \Theta$, define

$$\mathfrak{X}(\theta) \equiv \{\omega \in \Omega : \quad Q(\tau,\theta)(\omega) < 0, \tau \in (0,1)\}. \tag{2.21}$$

The fixed alternative $\theta_* \in \Theta$, which the test is against, is specified as below, to ensure that the "violation" of the null hypothesis under θ_* cannot be negligible in probability.

Assumption 8. $P(\mathfrak{X}(\theta_*)) > 0$.

Theorem 3. *Under Assumption 1, 6 and 8,*

(a) $\lim_{L\to\infty} P(\theta_* \in CS_L^{QS}) = 0,$

(b) $\lim_{L\to\infty} P(\theta_* \in CS_L^{PA}) = 0.$

Remark 3. The intuition for the consistent rejection is that the value of test statistic $T_L(\theta_*)$ goes to infinity (at rate a_L) whereas critical values obtained through $T_L(\varphi_L, \theta_*)$ (or $T_L(0, \theta_*)$) are $O_p(1)$.

The proof of our results in this section is given in Appendix A.3. Some, but not all, local alternatives drifting to Θ_I at the rate a_L can also be detected by our test. We relegate the part of our test that shows power against a_L^{-1}-local alternatives to Appendix A.4.

2.5 Implementation and Numerical Study

2.5.1 Computation Implementation

Here we provide details for computation implementation. First, to deal with a general (finite) set $\mathcal{N}$, suppose $\mathcal{N} = \left\{n^1, n^2, ..., n^N; n^1 < n^2 <, ..., < n^N\right\}$. Conceptually, Θ_I should be the intersection of all sets each of which is identified by using a subset from $\mathcal{N}$ with cardinality greater than or equal to 2. One equivalent way of implementation, by transitivity of FOSD, is to apply the procedures previously (for $\{n, n'\}$) for pairs $\left\{n^1, n^2\right\}$, $\left\{n^2, n^3\right\}$,..., and $\left\{n^{N-1}, n^N\right\}$ respectively, and then the corresponding test statistic becomes

$$T_L(\theta) = \sum_{k=1}^{N-1} \int_0^1 [\frac{\widetilde{Q}_L(\tau^k, \theta)}{(\widetilde{V}_L(\tau^k, \theta))^{\frac{1}{2}}}]_-^2 d\tau^k, \tag{2.22}$$

where k indexes the kth pair. And the analogous extension applies to $T_L(\varphi_L, \theta)$.

Second, the test statistic we will use (by uniform continuity of $\mathcal{F}_R$ in Assumption 1) is

$$T_L(\theta) = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} [\frac{\widetilde{Q}_L(\tau_m, \theta)}{(\widetilde{V}_L(\tau_m, \theta))^{\frac{1}{2}}}]_-^2, \tag{2.23}$$

where $\mathcal{T}_M \equiv \{\tau_m\}_{m=1}^M$ is a collection of grids "uniformly" distributed between $\tau_1 = 0.05$ and $\tau_2 = 0.95$ for all implementation.

In computation, we truncate the infinite sum at M_L, where $M_L \to \infty$, as $L \to \infty$, so we get an approximate test statistic

$$\overline{T}_{M_L}(\theta) = \frac{1}{M_L} \sum_{m=1}^{M_L} [\frac{\widetilde{Q}_L(\tau_m, \theta)}{(\widetilde{V}_L(\tau_m, \theta))^{\frac{1}{2}}}]_-^2. \tag{2.24}$$

Third, we use bootstrap to simulate the randomness, and obtain

$$\overline{T}^{\dagger}_{M_L}(\varphi_L, \theta) = \frac{1}{M_L} \sum_{m=1}^{M_L} [\frac{\widetilde{Q}^{\dagger}_L(\tau_m, \theta) - \widetilde{Q}_L(\tau_m, \theta)}{\widetilde{V}_L(\tau_m, \theta)^{\frac{1}{2}}} + \frac{\varphi_L(\tau_m, \theta)}{\widetilde{V}_L(\tau_m, \theta)^{\frac{1}{2}}}]_-^2, \tag{2.25}$$

where $\widetilde{Q}^{\dagger}_L(\tau_m, \theta)$ is the analogue of $\widetilde{Q}_L(\tau_m, \theta)$ by using bootstrap samples.

Also, for some higher order refinement, the asymptotic variance of $\widetilde{V}(\tau, \theta)$ is expressed up to the "second" order in implementation as $\widehat{V}(\tau, \theta) = h\widehat{V}_b + \theta^2 \widehat{V}(\tau)$, where $\widehat{V}_b(\tau) = \widehat{V}_{b,n}(\tau) + \widehat{V}_{b,n'}(\tau)$, and $\widehat{V}_{b,n}(\tau) = \frac{\tau(1-\tau)}{(\widehat{g}_n(\widehat{b}_n(\tau)))^2}$. Alternatively, a bootstrap counterpart $\widetilde{V}_L(\tau, \theta)$, denoted by $\widetilde{V}^{\dagger}_L(\tau, \theta)$, is also employed for comparison.

We use the tri-weight kernel function

$$K(u) = \frac{35}{32}(1-u^2)^3 \mathbf{1}(| u | \leq 1) \tag{2.26}$$

for kernel estimators. The parameters for QS procedure are chosen based on experimentations, which are also robust against various choices, as

$$\kappa_L = (0.1 \ln(Tb))^{\frac{1}{2}}, \quad B_L = (0.4 \ln(Tb)/\ln(\ln(Tb)))^{\frac{1}{2}}, \tag{2.27}$$

where Tb denotes the number of bids over all L auctions.

2.5.2 Monte Carlo Simulation

2.5.2.1 Basic Setup

For programming convenience, we assign the number of auctions such that the total number of bids of an n-bidder auction is the same across $n \in \mathcal{N}$, and this configuration is also close to the type of real data. We do experiments with $L = 174$, 870, and 1740 and with M_L=100, 200 and 300, respectively.

By rule-of-thumb,

$$h = 1.06c\widehat{\sigma_b}nb^{-\frac{1}{5}}, \tag{2.28}$$

where nb and $\widehat{\sigma_b}$ denote number and estimated standard deviation of total bids (in a given context). Results are reported for $c = 0.5, 1, 1.5$ and 2.

We will first establish finite sample coverage properties of our CS for risk aversion. The number of simulation repetitions used to compute size and power is 300. For each original simulation repetition, the critical value is simulated by using 299 repetitions by bootstrap. All results are reported for nominal 95% confidence sets. Also we compare finite sample performance of QS and PA in confidence sets.

2.5.2.2 Type 1 DGP

A random variable V_s indexed by s is from uniform $[0, s]$ distribution. We consider five different values for the signal s: $s \in \mathcal{S} = \{102,103,104,105,106\}$. Assumption 3 can be verified easily. $n \in \mathcal{N} = \{2, 3, 4, 5, 6\}$.

We show in Appendix A.5 that $\pi_{s,n}$ is an increasing function of s for any given n. Thus we adopt specification 1 as in Assumption 4 to obtain a monotonic entry result as $n(s = 102) = 2$, $n(s = 103) = 3$,..., and $n(s = 106) = 6$.

We set $\theta_0 = 0.2$. As shown in Appendix A.5, $\Theta_I = (0, 0.2262]$. In particular in Figure A.1, the bounds for Θ_I are flat. We show the size and power properties of our confidence sets at $\theta = 0.2262$ and $\theta = 0.3262$, respectively.

Table 2.1 and 2.2 summarize the findings of the experiments we implement. Generally speaking, testing by using bootstrapped $\widetilde{V}^{\dagger}(\cdot,\cdot)$ behaves better than using $\widetilde{V}(\cdot,\cdot)$, although the former is over-sized at small samples. As expected, QS is more powerful than PA by the existence of slackness of restrictions shown in Figure A.1. The power of QS or PA increases reasonably fast as sample size increases. Both tables suggest power increases as bandwidth becomes larger.

2.5.2.3 Type 2 DGP

$V_s \sim F_s(v)$, where $F_s(v) = v^s, v \in [0, 1]$, $s \in \mathcal{S} = \{96, 97, 98, 99, 100\}$. Assumption 3 can be easily verified. $n \in \mathcal{N} = \{2, 3, 4, 5, 6\}$.

We show in Appendix A.5 that $\pi_{s,n}$ is a decreasing function of s when $s > 1$, for any given n. Thus we adopt specification 2 as in Assumption 4 to obtain a monotonic entry result as $n(s = 100) = 2$, $n(s = 99) = 3$,..., and $n(s = 96) = 6$.

We set $\theta_0 = 0.5$. As shown in Appendix A.5, $\Theta_I = [0.5, 1]$. In particular in Figure A.2, the bounds for Θ_I are peak in the sense of Andrews and Shi (2013). We show the size and power properties of our confidence sets at $\theta = 0.5$ and $\theta = 0.4$, respectively.

The finding is summarized in Tables 2.3 and 2.4. The advantage of QS over PA remains. This again ought to hold by the peak nature of bound. Testing by using bootstrapped $\widetilde{V}^{\dagger}(\cdot,\cdot)$ possesses power about twice as high as $\widetilde{V}(\cdot,\cdot)$. Interestingly, power shows the best with bandwidth chosen at an intermediate level of $c = 1$ in both QS and PA CS's.

2.5.3 Empirical Application to USFS Timber FPA

We apply our inference to first-price timber auction sales held by the USFS. To make the main assumptions of private values appealing, we select certain timber auction sales to use by a few criteria described here. We only focus on sales between 1982 and 1990, as the policy change after 1981 limited opportunities for resale or subcontracting thereby reducing the common value element (Haile, Hong and Shum (2003)). Another way we eliminate the effect of resale is to control the length of the contract during which bidders are allowed to act upon their rights to harvest timber. We pick sales with contract length below 24 months. It has also been suggested by, e.g., Baldwin, Marshall and Richard (1997) that lump-sum sales may introduce some common value components in auctions whereas those components are less likely to exist in scaled sales. For this reason, we disregard lump-sum sales.

It has been popular to consider observed heterogeneity of tract characteristics, for instance appraised value made by USFS, in empirical applications. However, since there is no such observed heterogeneity in our model, we decide to apply to auction sales with appraised value less than \$ 1,000,000, to limit the variation of value distribution due to appraised value. Also, we exclude auction sales which are set aside for small business and are of salvage titles. We choose $\mathcal{N} = \{2, 3, 4, 5, 6, 7\}$, as auctions with number of bidders larger than seven are relatively fewer.

We end up with 421 auctions. The numbers of auctions for two to seven bidder auctions are 160, 88, 64, 44, 19 and 46, respectively. The test implementation is done under specifications 1 and 2 as in Assumption 4, respectively. In each specification, we apply our method testing 10 values of $\theta \in \{0.1, 0.2, ..., 1\}$, respectively. The lower θ is, the more risk averse bidders are. In particular, bidders are risk neutral when $\theta = 1$. As the QS method outperforms the PA method in both theory and simulation, we only implement testing by QS. To acquire inference precision, we increase the number of bootstrap repetitions up to $M = 4999$. The pair of QS parameters are adopted the same as in Monte Carlo simulation, therefore is the same as for any other setting as before, if not particularly specified. The results by specification 1 and 2 are listed in Tables 2.5 and 2.6, respectively. Note that QS 1 and QS 2 refer to testing procedures by using $\widetilde{V}(\cdot,\cdot)$ and $\widetilde{V}^{\dagger}(\cdot,\cdot)$, respectively.

At a significance level of 5%, in Table 2.5 there is no evidence against $\theta \leq 0.5$ from either QS1 or QS2 under specification 1. P-value drops as θ increases. In particular QS2 rejects the range of $\theta > 0.5$ except for using small bandwidth 0.5. Considering the performance at finite samples, especially the better power property by using $\widetilde{V}^{\dagger}(\cdot,\cdot)$, we suggest to make decisions based on QS2. In Table 2.6 under specification 2, each value of θ we considered is rejected. This uniform rejection result suggests that specification 2 is wrong so that Θ admits no θ fitting into our model under other assumptions.

Our test turns to reject mild risk aversion with $\theta > 0.5$ and especially risk neutrality. It may be worth comparing our findings with others' work done with relatively stronger restriction. For example, Campo et al. (2011) estimate θ between 0.5560 and 0.7783 in FPA, by imposing different polynomial quantile specifications for $\mathcal{F}_R$. Using both FPA and ascending auction data, Lu and Perrigne (2008) nonparametrically estimate a utility function which can be approximated by a CRRA function of $\theta = 0.65$, which is based on the assumption that bidders in the two auction formats share the same value distribution function.

2.6 Conclusion and Discussion

This paper shows a way of making inference for risk aversion in FPA with endogenous entry. The assumptions for identification here are flexible in the sense that most of the relevant assumptions made previously in the literature for (uniquely) identifying risk aversion can be treated as special versions of our own. The identification result of risk aversion is characterized by a set featured with inequality constraints. We propose a way of constructing confidence sets for the true parameter in the identified set, by employing specified quantiles as "instruments". Inspired by the moment selection idea, we select informative "instruments" in a similar way to deliver binding constraints, which make a test more powerful in general.

The asymptotic size and power properties show the consistency of our test. The test performs well at finite samples via Monte Carlo simulations. The application to USFS auction data turns out in favor of strong risk aversion.

We have not considered auction-specific observed heterogeneity other than the number of entrants. Possible sources for observed heterogeneity include appraisal value, estimated sales value, estimated manufacturing cost and bidding firms' existing inventory. Inclusion of such additional factors may provide more informative conditions for building our test. The downside is that it may suffer from the "curse of dimensionality" by adding those factors and hence may undermine the power of our test.

Conceivably, one may be interested in knowing whether the true θ is contained in a given interval. To solve this problem, one may generalize our testing procedure by following the research of Santos (2008), which is motivated by a conditional moment equality problem in nonparametric instrumental variables (IV).

Table 2.1: Finite Sample Performance for Type 1 DGP by using $\widetilde{V}(\cdot,\cdot)$

L	c	θ=0.2263(Size) QS	PA	θ=0.3263(Power) QS	PA
174	0.5	0.003	0.000	0.003	0.003
	1.0	0.067	0.037	0.047	0.017
	1.5	0.050	0.013	0.097	0.033
	2.0	0.057	0.017	0.103	0.057
870	0.5	0.023	0.010	0.037	0.027
	1.0	0.043	0.017	0.177	0.103
	1.5	0.060	0.030	0.203	0.123
	2.0	0.067	0.023	0.240	0.147
1740	0.5	0.043	0.013	0.123	0.053
	1.0	0.047	0.020	0.303	0.190
	1.5	0.033	0.017	0.317	0.203
	2.0	0.047	0.010	0.400	0.263

Table 2.2: Finite Sample Performance for Type 1 DGP by using $\widetilde{V}^{\dagger}(\cdot,\cdot)$

L	c	θ=0.2263(Size) QS	PA	θ=0.3263(Power) QS	PA
174	0.5	0.030	0.017	0.020	0.007
	1.0	0.110	0.063	0.097	0.043
	1.5	0.080	0.040	0.150	0.067
	2.0	0.037	0.017	0.167	0.097
870	0.5	0.037	0.017	0.070	0.037
	1.0	0.040	0.023	0.227	0.130
	1.5	0.077	0.033	0.280	0.177
	2.0	0.087	0.023	0.373	0.253
1740	0.5	0.043	0.017	0.120	0.070
	1.0	0.043	0.020	0.340	0.237
	1.5	0.030	0.027	0.470	0.350
	2.0	0.053	0.023	0.610	0.493

Table 2.3: Finite Sample Performance for Type 2 DGP by using $\widetilde{V}(\cdot,\cdot)$

L	c	θ=0.5(Size) QS	θ=0.5(Size) PA	θ=0.4(Power) QS	θ=0.4(Power) PA
174	0.5	0.003	0.003	0.010	0.003
	1.0	0.013	0.007	0.023	0.017
	1.5	0.020	0.007	0.023	0.013
	2.0	0.033	0.017	0.050	0.013
870	0.5	0.017	0.013	0.143	0.077
	1.0	0.037	0.020	0.190	0.113
	1.5	0.047	0.020	0.173	0.120
	2.0	0.017	0.000	0.147	0.063
1740	0.5	0.017	0.003	0.377	0.243
	1.0	0.027	0.010	0.397	0.267
	1.5	0.020	0.003	0.277	0.177
	2.0	0.033	0.010	0.247	0.163

Table 2.4: Finite Sample Performance for Type 2 DGP by using $\widetilde{V}^{\dagger}(\cdot,\cdot)$

L	c	θ=0.5(Size) QS	θ=0.5(Size) PA	θ=0.4(Power) QS	θ=0.4(Power) PA
174	0.5	0.050	0.020	0.070	0.043
	1.0	0.067	0.037	0.123	0.060
	1.5	0.070	0.023	0.097	0.033
	2.0	0.053	0.020	0.093	0.043
870	0.5	0.053	0.010	0.367	0.230
	1.0	0.083	0.017	0.403	0.280
	1.5	0.057	0.020	0.363	0.250
	2.0	0.030	0.003	0.290	0.167
1740	0.5	0.040	0.013	0.687	0.517
	1.0	0.043	0.013	0.733	0.577
	1.5	0.023	0.003	0.577	0.413
	2.0	0.040	0.003	0.473	0.350

Table 2.5: P-values for Testing Bidders' Risk Aversion in USFS Auction Sales under Specification 1

θ	c	QS1	QS2	θ	c	QS1	QS2
0.1	0.5	0.555	0.200	0.6	0.5	0.308	0.080
	1.0	0.518	0.138		1.0	0.196	0.043
	1.5	0.552	0.160		1.5	0.211	0.043
	2.0	0.651	0.203		2.0	0.249	0.030
0.2	0.5	0.545	0.216	0.7	0.5	0.237	0.075
	1.0	0.516	0.185		1.0	0.142	0.040
	1.5	0.563	0.236		1.5	0.154	0.028
	2.0	0.655	0.175		2.0	0.174	0.028
0.3	0.5	0.570	0.241	0.8	0.5	0.187	0.060
	1.0	0.481	0.153		1.0	0.104	0.023
	1.5	0.540	0.108		1.5	0.107	0.025
	2.0	0.600	0.125		2.0	0.116	0.008
0.4	0.5	0.486	0.208	0.9	0.5	0.156	0.052
	1.0	0.397	0.113		1.0	0.076	0.020
	1.5	0.429	0.115		1.5	0.072	0.000
	2.0	0.489	0.068		2.0	0.085	0.002
0.5	0.5	0.385	0.125	1.0	0.5	0.121	0.033
	1.0	0.311	0.095		1.0	0.063	0.010
	1.5	0.325	0.080		1.5	0.055	0.027
	2.0	0.359	0.058		2.0	0.070	0.000

Table 2.6: P-values for Testing Bidders' Risk Aversion in USFS Auction Sales under Specification 2

θ	c	QS1	QS2	θ	c	QS1	QS2
0.1	0.5	0.000	0.000	0.6	0.5	0.000	0.000
	1.0	0.000	0.000		1.0	0.000	0.000
	1.5	0.000	0.000		1.5	0.002	0.000
	2.0	0.000	0.000		2.0	0.001	0.000
0.2	0.5	0.000	0.000	0.7	0.5	0.000	0.000
	1.0	0.000	0.000		1.0	0.000	0.000
	1.5	0.000	0.000		1.5	0.002	0.000
	2.0	0.000	0.000		2.0	0.003	0.000
0.3	0.5	0.000	0.000	0.8	0.5	0.000	0.000
	1.0	0.000	0.000		1.0	0.001	0.000
	1.5	0.001	0.000		1.5	0.002	0.000
	2.0	0.001	0.000		2.0	0.001	0.000
0.4	0.5	0.000	0.000	0.9	0.5	0.000	0.000
	1.0	0.001	0.000		1.0	0.001	0.000
	1.5	0.002	0.000		1.5	0.002	0.000
	2.0	0.000	0.000		2.0	0.001	0.000
0.5	0.5	0.000	0.000	1.0	0.5	0.000	0.000
	1.0	0.001	0.000		1.0	0.001	0.000
	1.5	0.002	0.000		1.5	0.002	0.000
	2.0	0.001	0.000		2.0	0.002	0.000

Chapter 3

Testing Exogenous Entry in First-Price Auctions with Risk Averse Bidders

3.1 Introduction

It is known that a first-price auction model with risk averse bidders is not identified. Specifically, the confounding effect due to bidders' risk aversion and private value distributions, which are both unknown to researchers, complicates recovery of the two primary model primitives. This problem of non-identification remains even with parametrization for risk aversion such as constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA), as shown in Campo et al. (2011). To disentangle the problem, a common strategy that researchers play with is to assume exogenous entry, under which participation serves an instrument for identification. For example, Guerre et al. (2009) are able to recover bidders' utility function by exploring variation of numbers of entrant bidders; In a similar approach, Campo et al. (2011) identify and consistently estimate the utility function under either CRRA or CARA, and afterward recover the value distributions with the obtained risk aversion parameter(s).

Ignoring endogeneity of entry may induce severe bias to estimation and inference results as decisions of entry made by bidders play a role of self-selection: different levels of entry indicate different information to bidders with respect to their private values. As such, auctions with different numbers of entrants may not be compatible with each other in values to bidders. One can refer to Gentry and Li (2014) for analysis on selective entry in auction models. Moreover, empirical finding suggests that entry in auctions is not negligible.

It is thus important to investigate the validity of the fundamental assumption of exogenous entry. Conceptually, one can think of testing independence of private value distributions on number of entrants (or conditional independence with some other observed heterogeneity). This paper investigates the assumption in a weaker form: it considers testing independence on a prescribed quantile level. The reason for testing with this weaker condition is twofold. First, rejection of the test would be significant evidence against the independence about the entire distribution. Second, as identification and estimation of Campo et al. (2011) in fact hinges on the conditional independence of a single quantile of values on the number of entrants, our test can immediately address the justification of their approaches. To illustrate the idea of testing, risk aversion is characterised by one parameter in a particular utility function form.

As for contributions to econometrics, this paper establishes a way of testing (conditional) independence of unobserved values at a certain quantile to the number of entrants. The test of (conditional) independence of two sets of variables both observed have been well studied, such as in Su and White (2012). Also, it would not be a big problem if bidders' utility function were known, since we could just recover the unknown value quantile at equilibrium and treat it as observable. The challenge we are facing here is that the value is not observed or even directly estimable, yet we wish to develop a consistent test on independence which we can use, for instance, to guide our making inference on risk aversion. One by-product the testing procedure generates is the consistent estimation of risk aversion under the null hypothesis.

The rest of this paper is organized as follows. Section 3.2 describes the model environment and introduces the model condition for exogenous entry with risk aversion. Section 3.3 discusses testing idea and procedure and shows asymptotic consistency of the test. Section 3.4 evaluates performance of the proposed test by Monte Carlo simulation. Section 3.5 applies the test to US Forest Service (USFS) timber auction sales. Section 3.6 concludes.

3.2 The Model

The researcher observes a random sample of L different auctions. In each auction, the observables are auction-specific covariates $Z \in \mathcal{Z}$, number of (entrant) bidders I and the equilibrium bid for each bidder $b_j, j = 1, ..., I$.

Each Bidder draws her private value from a distribution $F(\cdot \mid Z, I)$. One example of covariates Z that can vary private value distributions is the appraised values from government. For simplicity, we assume the dimension of Z is one. Variation of I can be also associated with variation of value distributions as the level of entry may reflect additional information of unobserved heterogeneity which is observed only to bidders. Each bidder possesses the same CRRA utility function $U(x) = x^{\theta}, \theta \in (0,1)$ so that risk aversion is identical among bidders.

Under mild conditions (see, e.g., Campo et al. (2011)), we have the celebrated first-order condition as

$$v_i(Z,I) = b_i(Z,I) + \theta(\frac{1}{I-1}\frac{G(b_i(Z,I) \mid Z,I)}{g(b_i(Z,I) \mid Z,I)}). \tag{3.1}$$

We can further obtain an equivalent expression by evaluating (4.2) at a specific level τ of a quantile, $\tau \in [0,1]$, such that

$$V_\tau(Z,I) = b_\tau(Z,I) + \theta \frac{1}{I-1}\frac{\tau}{g(b_\tau(Z,I) \mid Z,I)} \equiv b_\tau(Z,I) + \theta X_\tau(Z,I). \tag{3.2}$$

Now by differentiating (3.2) with two different level of entrants I_1 and I_2, we can have

$$\Delta V_\tau(Z) = \Delta b_\tau(Z) + \theta \Delta X_\tau(Z), \tag{3.3}$$

where $\Delta V_\tau(Z) = V_\tau(Z, I_2) - V_\tau(Z, I_1)$, and $\Delta b_\tau(Z)$ and $\Delta X_\tau(Z)$ are defined similarly.

3.3 The Consistent Test of Exogenous Entry

With exogenous entry, $V \perp I \mid Z$ so that $\Delta V_\tau(z) = 0$, $\forall z \in \mathcal{Z}$. This suggests us to consider the null and alternative hypotheses of the test as

$$H_0 : \Delta V_\tau(z) = 0, \forall z \in \mathcal{Z},$$
$$H_1 : \Delta V_\tau(z) \neq 0, \text{for some } z \in \mathcal{Z}.$$

3.3.1 Nonparametric Estimation of Quantiles

To implement the test, we need to construct a test statistic based on $\Delta V_\tau(z)$ and evaluate it under both null and alternative hypotheses. We first estimate $b_\tau(Z,I)$ and $X_\tau(Z,I)$ (and hence $\Delta b_\tau(Z)$ and $\Delta X_\tau(Z)$) by nonparametric kernel estimation.

Assumption 9 (Kernel). The kernel function K is compactly supported on [-1,1], has at least R derivatives on its support, the derivatives are Lipschitz, and $\int K(u)du = 1, \int u^k K(u)du = 0$ for $k = 1, ..., R-1$.

The estimators for conditional quantile $\widehat{b}_\tau(Z,I)$ and conditional density $\widehat{g}(\cdot \mid Z,I)$ are standard kernel estimators as in, e.g., Pagan and Ullah (1999) and Marmer and Shneyerov (2012). Then for $X_\tau(Z,I)$, we have $\widehat{X}_\tau(Z,I) = \frac{1}{I-1}\frac{\tau}{\widehat{g}(\widehat{b}_\tau(Z,I)|Z,I)}$.

By plugging the estimators in (3.2) and taking into account of estimation error, our model constraint becomes

$$\widehat{\Delta b}_\tau(Z) + \theta\widehat{\Delta X}_\tau(Z) - \Delta V_\tau(Z) + \xi_\tau(Z) = 0, \tag{3.4}$$

where $\xi_\tau(Z) \equiv [\Delta b_\tau(Z) - \widehat{\Delta b}_\tau(Z)] + \theta[\Delta X_\tau(Z) - \widehat{\Delta X}_\tau(Z)]$.

Assumption 10 (Bandwidth). Let h be the bandwidth in kernel estimation. $h \to 0, Lh^2 \to \infty$, and $(Lh^2)^{1/2}h^R \to 0$.

By standard arguments of nonparametric kernel estimation, some immediate results are summarized here. As we will use a bootstrap version of the test, we do not pursue the explicit forms of asymptotic variance in this paper.

Proposition 3. *Under Assumptions 9 and 10,*

(a) $(Lh)^{1/2}(\widehat{\Delta b}_\tau(Z) - \Delta b_\tau(Z)) \to_d N(0, V_{b;Z})$,

(b) $(Lh^2)^{1/2}(\widehat{\Delta X}_\tau(Z) - \Delta X_\tau(Z)) \to_d N(0, V_{X;Z})$*, and thus*

(c) $(Lh^2)^{1/2}\xi_\tau(Z) \to_d N(0, \theta^2 V_{X;Z,I})$*,*

where $V_{b;Z}$ *and* $V_{X;Z}$ *are finite and can be consistently estimated.*

3.3.2 An Infeasible Estimator of the CRRA Coefficient

In the case where $\Delta V_\tau(Z)$ were known to researchers, we could apply the method of weighted least squares or simply ordinary least squares to estimate θ. Specifically, regression based on (3.4) and observables returns the infeasible estimator $\widehat{\theta}^0$ of θ as

$$\widehat{\theta}^0 = -(\sum_{i=1}^{L} \widehat{\Delta X}_\tau^2(Z_i))^{-1} \sum_{i=1}^{L} \widehat{\Delta X}_\tau(Z_i)(\widehat{\Delta b}_\tau(Z_i) - \Delta V_\tau(Z_i)). \tag{3.5}$$

We can obtain the consistency and asymptotic normality of $\widehat{\theta}^0$ by following the approach provided by Campo et al. (2011), as $\Delta V_\tau(Z) \equiv 0$, Z-a.s., by their assumption of exogenous entry.

Proposition 4. *Under Assumption 9 and 10,*

$$(Lh)^{1/2}(\widehat{\theta}^0 - \theta) \to_d N(0, V_\theta),$$

where V_θ *is finite and can be consistently estimated.*

Note that the convergence rate of the estimator is faster that that of $\widehat{\Delta X}_\tau(Z)$. This is because (3.4) would still hold with integration of it over (Z, I), and integration increases convergence rate.

3.3.3 The Test Statistic and Consistency

The test statistic is constructed by using a naive estimator $\widehat{\theta}$ for θ, i.e., estimating θ by assuming $\Delta V_\tau(Z) = 0$, Z-a.s.. Therefore,

$$\widehat{\theta} = -(\sum_{i=1}^{L} \widehat{\Delta X}_\tau^2(Z_i))^{-1} \sum_{i=1}^{L} \widehat{\Delta X}_\tau(Z_i)(\widehat{\Delta b}_\tau(Z_i)). \tag{3.6}$$

Given that Z is exogenous, we can assume that $\bar{z} \equiv E(Z)$ is known without loss of generality. Conceptually, one may consider forming a consistent test statistic by using $\widehat{\Delta V}(Z)$ at (possibly infinitely) many different values of Z. One nice thing here is that we do not have to do so; instead, it suffices in general to consider $\widehat{\Delta V}(Z)$ at a particular value of Z. Indeed, this idea works because of the fact that $\widehat{\theta}$ is not consistent to θ under H_1. Thus, with the estimator $\widehat{\theta}$, we consider the following test statistic.

$$\widehat{\Delta V}_\tau(\bar{z}) = \widehat{\Delta b}_\tau(\bar{z}) + \widehat{\theta}\widehat{\Delta X}_\tau(\bar{z}). \tag{3.7}$$

Assumption 11. Under the alternative hypothesis, $\Delta V_\tau(\bar{z}) - \Delta X_\tau(\bar{z})[\mu_{\Delta X_\tau^2}]^{-1}\mu_{\Delta X_\tau \Delta V_\tau} = C \neq 0$, where $\mu_{\Delta X_\tau^2} = E[\Delta X_\tau^2(Z)]$ and $\mu_{\Delta X_\tau \Delta V_\tau} = E[\Delta X_\tau(Z)\Delta V_\tau(Z)]$.

Note that Assumption 11 does not depend on the risk aversion parameter θ. Also, one can realize that this assumption is mild by looking at a slightly stronger condition: $\Delta V_\tau(Z)$ is not a (deterministically) linear function of $\Delta X_\tau(Z)$.

The following theorem shows the desirable properties and thus the consistency of the test.

Theorem 4. *Under Assumptions 9-11, we have* $(Lh^2)^{1/2}\widehat{\Delta V}_\tau(\bar{z}) \to_d N(0, \theta^2 V_{X;\bar{z}})$ *under* H_0, *and* $(Lh^2)^{1/2}\widehat{\Delta V}_\tau(\bar{z})$ *diverges under* H_1. *Therefore, for a given* $\alpha \in (0,1)$, *there exists a constant* $c_{1-\alpha}$, *such that* $P[(Lh^2)^{1/2}\widehat{\Delta V}_\tau(\bar{z}) \leq c_{1-\alpha} \mid H_0]) = 1-\alpha$ *and* $P[(Lh^2)^{1/2}\widehat{\Delta V}_\tau(\bar{z}) \leq c_{1-\alpha} \mid H_1] = 0$, *as* $L \to \infty$.

Proof. With $\widehat{\theta}$ obtained previously, we write

$$\begin{aligned}
\widehat{\Delta V}_\tau(\bar{z}) &= \widehat{\Delta b}_\tau(\bar{z}) + \widehat{\theta}\widehat{\Delta X}_\tau(\bar{z}) \\
&= \Delta b_\tau(\bar{z}) + \theta \Delta X_\tau(\bar{z}) + [\widehat{\Delta b}_\tau(\bar{z}) - \Delta b_\tau(\bar{z}) + \theta(\widehat{\Delta X}_\tau(\bar{z}) - \Delta X_\tau(\bar{z}))] \\
&\quad + (\widehat{\theta} - \theta)\widehat{\Delta X}_\tau(\bar{z}) \\
&= \Delta V_\tau(\bar{z}) - \xi_\tau(\bar{z}) + (\widehat{\theta} - \widehat{\theta}^0)\widehat{\Delta X}_\tau(\bar{z}) + (\widehat{\theta}^0 - \theta)\widehat{\Delta X}_\tau(\bar{z}) \\
&= \underbrace{(\widehat{\theta}^0 - \theta)\widehat{\Delta X}_\tau(\bar{z})}_{A_1} + \underbrace{\Delta V_\tau(\bar{z}) - \widehat{\Delta X}_\tau(\bar{z})[\sum_{i=1}^{L} \widehat{\Delta X}_\tau^2(Z_i)]^{-1}[\sum_{i=1}^{L} \widehat{\Delta X}_\tau(Z_i)\Delta V_\tau(Z_i)]}_{A_2} \\
&\quad - \xi_\tau(\bar{z}),
\end{aligned}$$

where the last equality holds due to (3.5) and (3.6).

Now by Propositions 3 and 4, $A_1 = O_p((Lh)^{-1/2})$, and $(Lh^2)^{1/2}\xi_\tau(Z) \to_d \mathcal{T} \sim N(0, \theta^2 V_{X;Z})$. Hence, $(Lh^2)^{1/2}\widehat{\Delta V}_\tau(\bar{z}) \to_d \mathcal{T} + (Lh^2)^{1/2}A_2$. It can be easily checked that $A_2 = C + o_p(1)$. Also note that $A_2 = 0$ under H_0. The proof is completed. □

3.3.4 Bootstrap the Critical Value

To circumvent estimation of the asymptotic variance under the null and to get refined finite sample results, we obtain critical values for testing by bootstrap. For a sample of size L containing $\{Z_i, I_i\}_{i=1}^{L}$, we first randomly draw with replacement L samples. Then for each $\bar{l}$ selected in the first step, $\bar{l} = 1, ..., L$, we randomly draw with replacement $I_{\bar{l}}$ bids from $\{b_{\bar{l}j}\}_{j=1}^{I_{\bar{l}}}$. With such an bootstrap sample, we calculate a bootstrap analogue of test statistic $\widehat{\Delta V}^{\dagger}_{\tau}(\bar{z})$.

Let M be the number of bootstrap samples, and let $c^{\dagger}_{\alpha}$ be the α empirical quantile of $\{\widehat{\Delta V}^{\dagger}_{m,\tau}(\bar{z})\}_{m=1}^{M}$. The bootstrap percentile confidence interval is constructed as $CI^{BP}_{1-\alpha} = [c^{\dagger}_{\alpha/2}, c^{\dagger}_{1-\alpha}/2]$. Our decision rule is to reject H_0 under significance level of α, if $\widehat{\Delta V}_{\tau}(\bar{z}) \notin CI^{BP}_{1-\alpha}$.

3.4 Monte Carlo Simulation

We use a tri-weight kernel for estimation. The bandwidth we employ is basically by the rule-of-thumb, i.e., $h = 1.06c\sigma n^{-1/5}$, where c is a constant for robustness check, and σ is an adaptive measure of spread of Z defined as min(standard deviation, interquartile range/1.349), and n is the number of observations. We have examined experiments under $c = 0.5$, 1 and 2. The results of tests are robust against choices of c, so we only report for $c = 1$. The number of simulation repetitions is 400. For each original simulation repetition, the critical values are simulated by using 199 repetitions by bootstrap. Results are reported for nominal significance level of 1%, 5% and 10%.

There are two types of data generating process (DGP) from which we simulate data. In the first type, the exogenous variable Z has a normal distribution $N(10,1)$, and the variable I takes values of 4 and 5 in the same probability 1/2. For model primitives, we let $\theta = 0.5$ and $F_V(v \mid Z, I) \sim U[0, s_{Z,I}]$, where $s_{Z,I} = (Z + aI)^2$. The DGP satisfies H_0 under $a = 0$. The bigger a is, the more the DGP favors H_1. We take $\tau = 0.5$. The sample size is indicated by the number of auctions $L \in \{100, 200, 400\}$.

The results are reported in Table 3.1. The test clearly recognizes endogenous entry being different from exogenous entry. The power of the test is reasonably good at finite samples, and increases significantly as sample size grows.

Table 3.1: Empirical Size and Power of the Bootstrap Test ($V \sim U[0, s_{Z,I}]$)

L	Significance	a=0 (size)	a=1 (power)	a=2 (power)
100	0.01	0.010	0.170	0.348
100	0.05	0.058	0.383	0.668
100	0.10	0.125	0.550	0.790
200	0.01	0.003	0.275	0.660
200	0.05	0.035	0.548	0.853
200	0.10	0.093	0.680	0.930
400	0.01	0.018	0.435	0.918
400	0.05	0.045	0.710	0.970
400	0.10	0.098	0.815	0.985

The consistency of the test justifies the inference for risk aversion. Therefore, we also report the performance of $\widehat{\theta}$ under H_0, in Table 3.2.

Table 3.2: Simulated Bias and MSE of $\widehat{\theta}$ under H_0 ($V \sim U[0, s_{Z,I}]$)

L	Bias	MSE
100	-0.0519	0.1844
200	-0.0374	0.1657
400	-0.0193	0.1381

For the second type of DGP, the differences from the first one are that $Z \sim U[2,4]$, and that $F_V(v \mid Z, I) = v^{s_{Z,I}}, v \in [0,1]$, where $s_{Z,I} = Z(1 + a(7 - I))$.[1 2] The rest of configuration of the second type follow that of the first one. The results are reported in Table 3.3. Similarly, given the consistency, the performance of $\widehat{\theta}$ is shown in Table 3.4 as a by-product.

3.5 Empirical Application

In application, we choose data of USFS timber auctions collected by Professor Philip Haile.[3] The observed Z stands for appraisal value announced by USFS. We select ap-

[1]The reason that $s_{Z,I}$ is negatively related with I is that one can show that the expected payoff for a bidder is decreasing with $s_{Z,I}$ when $s_{Z,I} > 1$. Also one can show the relationship is increasing in the first type of DGP.

[2]We choose the number “7” as this is the biggest number of bidders in our empirical data.

[3]Available at http://www.econ.yale.edu/ pah29/timber/timber.htm

Table 3.3: Empirical Size and Power of the Bootstrap Test ($F_V(v \mid Z, I) = v^{s_{Z,I}}$)

L	Level	a=0 (size)	a=1 (power)	a=2 (power)
100	0.01	0.005	0.338	0.583
100	0.05	0.053	0.570	0.803
100	0.10	0.095	0.718	0.870
200	0.01	0.010	0.588	0.748
200	0.05	0.068	0.798	0.908
200	0.10	0.145	0.858	0.945
400	0.01	0.003	0.845	0.943
400	0.05	0.035	0.955	0.990
400	0.10	0.115	0.975	1.000

Table 3.4: Simulated Bias and MSE of $\widehat{\theta}$ under H_0 ($F_V(v \mid Z, I) = v^{s_{Z,I}}$)

L	Bias	MSE
100	0.0599	0.1630
200	0.0377	0.1420
400	0.0079	0.1109

propriate observations to reduce the chance of violating our model assumption. For example, sales that we pick are between 1982 and 1990, as the policy change after 1981 limited opportunities for resale and subcontracting thereby reducing the common value element Haile et al. (2003). Besides, we use sales with contract length only below 24 months, and we disregard lump-sum sales. The summary statistics are listed in Table 3.5 for the selected observations.

Table 3.5: Summary Statistics

		Bids($)		Appraisal value ($)	
Auctions of	Number	Mean	Std	Mean	Std
2_bidder	159	754748.6	568438.1	441349.0	269424.1
3_bidder	88	915551.9	856711.7	455688.6	269234.7
4_bidder	64	1170819.0	1415073.0	491831.2	288176.9
5_bidder	44	1008253.0	833504.5	453104.0	291272.5
6_bidder	19	1372551.0	1237427.0	514426.5	290936.5
7_bidder	46	2163731.0	2005032.0	493294.6	256505.7

As the auctions with bidder number between five to seven are even fewer than 50 for each category, we decide to further drop those observations. We increase bootstrap replication number to 1000. The results are summarized in Table 3.6.

Table 3.6: The Bootstrap P-Values and Estimate of $\widehat{\theta}$

Auctions of	P_value	$\widehat{\theta}$
2 v.s. 3 bidders	0.375	0.354 (0.230)
3 v.s. 4 bidders	0.002	—

Based on the test evidence, the high p-value obtained by using 2- and 3-bidder auctions leads to no rejection of the null hypothesis of exogenous entry. Therefore, it is sensible to estimate θ under the null: the estimate for θ is 0.354, with standard error of 0.230 obtained by 1000 bootstrap resamples. Clearly, we can significantly reject risk neutrality where $\theta = 1$, and the evidence against risk neutrality is stronger than that in Campo et al. (2011) where the estimate for θ is generally between 0.5 and 0.8.[4] On the other hand, the null hypothesis can be rejected significantly with 3- and 4-bidder auctions by the extremely small p-value, so we do not list estimation for θ.

The test suggests entry is exogenous with a lower entry level but is endogenous with a higher entry level. The reason that the endogeneity may change with entry levels is that the common unobserved signals that bidders observe upon their entry may be associated either with private value distributions or with entry costs, or both. When signals are associated with entry costs but not with value distributions, numbers of participants may vary with entry costs, yet the underling value distributions can be invariant as they are independent of entry costs; when signals are associated with value distributions, variation of numbers of participants are likely to reflect the variation of value distributions. Therefore, the variation bidders face may be mostly from entry costs when the number of bidders is relatively small, and the variation may be mostly from value distributions when the number of bidders is relatively big.

[4]Our estimate for θ is also lower than that obtained in Lu and Perrigne (2008). However, Lu and Perrigne (2008) do not report the standard error for their $\widehat{\theta}$, so we can not do a statistical comparison.

3.6 Conclusion

This paper examines the main assumption of exogenous entry which is extensively adopted in the existing literature. Checking this assumption is critical as the estimation of risk aversion is invalid and misleading if exogenous entry is violated. The testing procedure is based on a semiparametric estimation approach. The estimation is consistent under the null hypothesis but is inconsistent under the alternative, and this provides the base to construct and evaluate our test.

We show that the test is consistent and performs well in finite samples. Also, the test justifies the estimation of the risk aversion parameter when exogenous entry assumption is employed. The estimation result significantly rejects risk neutrality, agreeing with existing findings. Empirical testing results reveal various possible factors affecting bidders' entry.

Chapter 4

Identification and Estimation of First-Price Auctions with Entry

4.1 Introduction

Identification and estimation of latent value distributions of bidders are useful in auctions. The knowledge of bidders' private values will help make a better auction design such as an optimal reserve price. The basic idea of identification hinges on equilibrium conditions that are derived from auction theory, as those conditions provide a foundation to recover the unknown from observables. Identification procedures also inspire and guide estimation, parametrically or nonparametrically. One has witnessed fruitful progress in the recent auction literature, such as Guerre et al. (2000), Athey and Haile (2002), Krasnokutskaya (2011) and Hubbarda et al. (2012), among others.

A large part of existing work relies on the assumption that entry to an auction is exogenous for bidders, i.e., the number of entrant bidders is taken as given. This assumption simplifies a bidder's strategy as the number of entrant opponents for her is fixed. Yet this simplification restricts a bidder's choice of participation to an auction. In contrast, participation in the paper is determined endogenously by auction primitives. As such, the number of entrant bidders is stochastic to bidders. Hence, our approach is more general. Also, we will show in this paper that ignoring the fact of endogenous entry can lead to severe bias in recovering value distribution and density functions.

To relax the assumption of exogenous entry, we assume that the entry cost is known to researchers so that we are able to model bidders' (mixed) entry strategy into an auction game. The assumption of a known entry cost has been adopted by Li and Zheng (2009) and Fang and Tang (2014). In particular, it is worth comparing our study with Li and Zheng (2009). Li and Zheng (2009) do not consider a binding reserve price, so the number of entrant bidders is the same as the observed number of actual bidders. With a binding reserve price, however, the two numbers are not necessarily the same, as a bidder with a value lower than the reserve price will not actually bid. So in our approach, the selection of actual bidders is via both entry and reserve price.

Several other papers considering endogenous entry in auctions include Gentry and Li (2014) and Fang and Tang (2014). Yet their methods cannot apply to our model, as Gentry and Li (2014) assume that bidders have observed i.i.d. signals prior to entry and Fang and Tang (2014) target at ascending auctions which are strategically equivalent to second price auctions.

The goal of our identification and estimation is at the value cumulative distribution function (CDF) and probability density function (PDF). Our identification can divide into two stages. At stage one, we identify mixed entry strategy by an indifference condition. After we discover entry strategy, the identification of latent value distribution and density functions proceeds in the spirit of Guerre et al. (2000). As for estimation, we recover value density function by two methods, i.e., the pseudo value (PV) method and the quantile based (QB) method, and compare the result from each method with Guerre et al. (2000) and Marmer and Shneyerov (2012) respectively, as the latter two papers initiate PV and QB methods respectively but both take entry as exogenous.

The paper is organized as follows. Section 4.2 introduces the independent private value auction model with mixed entry. Section 4.3 shows the procedures of identification. section 4.4 constructs estimators and briefly discusses nonparametric asymptotic properties of the main estimators. Section 4.5 examines finite sample properties of estimators for CDF and PDF of private values. Section 4.6 contains possible extension and concluding remarks. Some technical details for estimation are contained in the

Appendix B.

4.2 The Model

For each auction, one single and indivisible object is auctioned, and there are N potential bidders, $N \geq 2$. Each bidder needs to pay a fixed cost C to participate in an auction. Prior to entry, bidders do not know their private values yet. Entrant bidders draw their private values independently from a common distribution $F(\cdot)$, which is absolutely continuous with density $f(\cdot)$ and support $[\underline{v}, \overline{v}] \subset \mathbb{R}_+$. Bids submitted by entrant bidders are collected simultaneously. Bidders do not know the number of entrants when submitting bids. A reservation price p_0 is announced by the seller, where $p_0 \in [\underline{v}, \overline{v}]$. $N, C, F(\cdot)$ and p_0 are common knowledge. All bidders are identical ex ante and the game is symmetric. Each bidder is assumed to be risk neutral.

Bidders who enter and bid higher than p_0 are called actual bidders. We denote the number of actual bidders by n^*, and their bids by $\mathbf{b}^*$. Hence, $\mathbf{b}^* = \{b_i\}_{i=1}^{n^*}$. What are observed by researchers are n^*, $\mathbf{b}^*$ and C.

In a symmetric Bayesian Nash equilibrium, each potential bidder adopts a mixed strategy to entry: entry with probability q and no entry with probability $1-q$. Upon entry, a bidder who draws a value lower than p_0 submits any value less than p_0. For an entrant bidder who draws $v \geq p_0$, her expected payoff by bidding $s(v)$ is

$$(v - s(v))(qF(v) + 1 - q)^{N-1}. \tag{4.1}$$

By the first order condition, we obtain a first order differential equation

$$v - s(v) = \frac{s'(v)}{f(v)} \frac{qF(v) + 1 - q}{q(N-1)}, \tag{4.2}$$

with the initial condition $s(p_0) = p_0$.

4.3 Identification

4.3.1 Identification of the Mixed Entry Strategy

We first show that the probability of entry q is identified through an indifference condition. In the indifference condition, the ex ante payoff can be expressed as a function of the identifiable. By (4.2), the ex ante expected payoff for making a decision of entry, denoted by Π, is

$$\begin{aligned}\Pi &= \int_{p_0}^{\overline{v}} (v - s(v))(qF(v) + 1 - q)^{N-1} f(v) dv \\ &= \int_{p_0}^{\overline{v}} \frac{s'(v)(qF(v) + 1 - q)^N}{q(N-1)} dv. \end{aligned} \tag{4.3}$$

Let $G^*(\cdot)$ be the distribution of observed b_i^*, and it follows that,

$$G^*(b^*) = \frac{F(v) - F(p_0)}{1 - F(p_0)}, \tag{4.4}$$

where $v = s^{-1}(b^*)$. Next, define $r = q(1 - F(p_0))$, and it is easy to verify that

$$qF(v) + 1 - q = rG^*(b^*) + 1 - r. \tag{4.5}$$

Plugging (4.5) into (4.3) and by using the fact that $db^* = s'(v)dv$, we get

$$\Pi = \int_{p_0}^{\overline{b}} \frac{(rG^*(b^*) + 1 - r)^N}{q(N-1)} db^*, \tag{4.6}$$

where $\overline{b} = s(\overline{v})$.

The identification of N and r follows from identification of parameters of binomial distributions as $n^* \sim Binomial(N, r)$. In addition, $\overline{b}$ and $p_0 = s(\underline{v})$ can be identified from observed bids. As a result, q is identified being the solution to the indifference condition $\int_{p_0}^{\overline{b}} \frac{(rG^*(b^*)+1-r)^N}{(N-1)q} db^* = C$, as C is known.

4.3.2 Identification of $F(v)$ and $f(v)$

Next we will discuss identification of $F(v)$, for $v \in [p_0, \overline{v}]$. By differentiating with respect to b^* at (4.4), we first obtain $g^*(b^*) = \frac{f(v)}{(1-F(p_0)s'(v))}$, i.e.,

$$\frac{s'(v)}{f(v)} = \frac{1}{(1-F(p_0))g^*(b^*)}. \tag{4.7}$$

Then we plug (4.7) into (4.2) and meanwhile use (4.5) to get

$$v = b^* + \frac{rG^*(b^*)+1-r}{r(N-1)g^*(b^*)} \equiv \xi(b^*). \tag{4.8}$$

For a given $v \geq p_0$, we can solve b^* by $b^* = \xi^{-1}(v)$. Then we define naturally $F^*(v) \equiv G^*(b^*)$, as the value distribution function for actual bidders. Hence, $F(v) = F(p_0) + (1-F(p_0))F^*(v)$ by (4.4), where $F(p_0) = 1 - r/q$ by the definition of r.

As for identification of $f(v)$, for $v \in [p_0, \overline{v}]$, there are two methods we implement. The first method, called PV, is to first recover pseudo values for actual bidders by $v^* = \xi(b^*)$, and then to identify value density for actual bidders $f^*(v)$ based on v^*. Lastly, by definition, $F^*(v) = \frac{F(v)-F(p_0)}{1-F(p_0)}$, so we can identify $f(v)$ by $f(v) = (1-F(p_0))f^*(v)$.

The second method, called QB (for quantile based), is to recover $f(v)$ at a particular value $v \in [p_0, \overline{v}]$, by the unique quantile level τ with which v is associated. Specifically, consider the τ-quantile $V^*(\tau)$ of V^* (for actual bidders) and the τ-quantile $b^*(\tau)$ of b^*. Since the reverse bidding function $\xi(b^*)$ is monotone, an equivalent expression to (4.8) is

$$V^*(\tau) = b^*(\tau) + \frac{r\tau+1-r}{r(N-1)g^*(b^*(\tau))}. \tag{4.9}$$

By differentiation with respect to τ, and the fact that $b^{*\prime}(\tau) = 1/g^*(b^*(\tau))$, we get

$$\begin{aligned} V^{*\prime}(\tau) &= \frac{1}{g^*(b^*(\tau))} + \frac{1}{r(N-1)} \frac{rg^*(b^*(\tau)) - (r\tau+1-r)g^{*\prime}(b^*(\tau))g^{*^{-1}}(b^*(\tau))}{g^{*2}(b(^*\tau))} \\ &= \frac{1}{g^*(b^*(\tau))}\frac{N}{N-1} - \frac{1}{r(N-1)}\frac{(r\tau+1-r)g^{*\prime}(b^*(\tau))}{g^{*3}(b^*(\tau))}. \end{aligned} \tag{4.10}$$

Now by substituting $\tau = F^*(v)$, and the fact that $V^{*\prime}(\tau) = 1/f^*(V^*(\tau))$, we derive the expression for $f^*(v)$, as reciprocal of (4.10). Finally, $f(v) = (1-F(p_0))f^*(v)$ as before.

4.3.3 Comparison with Identification under Exogenous Entry

Under exogenous entry, the selection of actual bidders out of potential bidders is only due to reserve price. In this paper, both entry and reserve prices act upon the selection. In other words, when a bidder can make a choice of participation, the exogenous entry approach is correct when each potential bidder enters for sure, and one immediately realizes that this is subject to a negligible entry cost condition: $C < \underline{C}$, where $\underline{C}$ will be specified later.

Specifically, under exogenous entry one would get the distribution function, denoted by $F^{EX}(v)$, and the density function, denoted by $f^{EX}(v)$, as

$$F^{EX}(v) = 1 - r + rF^{*}(v), \tag{4.11}$$

and

$$f^{EX}(v) = rf^{*}(v), \tag{4.12}$$

for $v \geq p_0$. Hence, $F(v) \leq F^{EX}(v)$ and $f(v) \geq f^{EX}(v)$, for $v \geq p_0$.

4.4 Nonparametric Estimation

4.4.1 Construction of Estimators

Suppose we have L auctions in total. We use subscript l for auction l, and il for the ith bidder in auction l. First come some straightforward estimators: $\widehat{N} = \max_{l} n_l^{*}$, $\widehat{\overline{b}} = \max_{i,l} b_{il}^{*}$, $\widehat{p_0} = \min_{i,l} b_{il}^{*}$, $\widehat{r} = \sum_{l=1}^{L} n_l^{*}/\widehat{N}L$. $\widehat{q}$ follows by solving the previous indifference condition with all other estimators plugged in, and $\widehat{F(p_0)} = 1 - \widehat{r}/\widehat{q}$.

The estimator for $G^{*}(b)$ is the typical empirical distribution function, i.e.,

$$\widehat{G^{*}}(b) = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{n_l}\sum_{i=1}^{n_l} 1(b_{il}^{*} \leq b), \tag{4.13}$$

and this enables us to formulate estimator $\widehat{b^{*}}(\tau) = \widehat{G^{*}}^{-1}(\tau) \equiv \inf\left\{b : \widehat{G^{*}}(b) \geq \tau\right\}$. We employ a kernel estimator for $g^{*}(b)$. Let $K(u) : \mathbb{R} \to \mathbb{R}$ be a kernel-like function. Define $K_h(u) = h^{-1}K(\frac{u}{h})$, where $h = o(1)$ is a bandwidth. Then we get, (see, e.g., Pagan and Ullah (1999))

$$\widehat{g^*}(b) = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{n_l}\sum_{i=1}^{n_l}K_h(b - b_{il}^*). \tag{4.14}$$

Now by (4.8), we can obtain the conditional CDF estimator $\widehat{F^*}(v)$ following the recipe given in identification. Then, $\widehat{F}(v) = \widehat{F(p_0)} + (1 - \widehat{F(p_0)})\widehat{F^*}(v)$.

Moreover, we can obtain the pseudo value estimator $\widehat{V}_{il}$, by plugging b_{il}^* into (4.8). To be clear, we denote the estimators of (conditional) PDF using PV and QB by $f^{PV}(v)$ ($f^{*PV}(v)$) and $f^{QB}(v)$ ($f^{*QB}(v)$) respectively. The estimator $f^{*PV}(v)$ is also of a kernel form,

$$f^{*PV}(v) = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{n_l}\sum_{i=1}^{n_l}K_h(v - \widehat{V}_{il}). \tag{4.15}$$

As for $f^{*QB}(v)$, we need an additional derivative estimator for $g^{*\prime}(b)$ as

$$\widehat{g^{*\prime}}(b) = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{n_l}\sum_{i=1}^{n_l}K_h'(b - b_{il}^*), \tag{4.16}$$

where $K_h'(u) = \frac{1}{h^2}K'(\frac{u}{h})$ and $K'(u)$ denotes the derivative of $K(u)$. Now we obtain $\widehat{V^{*\prime}}(\tau)$ as a plug-in estimator by (4.10), and $f^{*QB}(v) = 1/\widehat{V^{*\prime}}(\tau)$, where $\tau = \widehat{F^*}(v)$. Finally, $f^{PV}(v) = (1 - \widehat{F(p_0)})f^{*PV}(v)$ and $f^{QB}(v) = (1 - \widehat{F(p_0)})f^{*QB}(v)$.

4.4.2 A Brief Discussion of Asymptotic Results

The assumptions for estimation are mild and standard. So we just list them as below, where $R \geq 2$.

Assumption 12 (Smoothness). $f(v)$ is bounded away from zero and admits up to R continuous derivatives over $[\underline{v}, \overline{v}]$.

Assumption 13 (Kernel). K is compactly supported on $[-1,1]$, has at least R derivatives which are Lipschitz, and $\int K(u)du = 1$, $\int u^k K(u)du = 0$ for $k = 1, ..., R-1$.

Assumption 14 (Bandwidth). $Lh \to \infty$, and $(Lh^3)^{1/2}h^R \to 0$.

Besides, to make the entry game non-degenerate, we require $C \in [\underline{C}, \overline{C}]$, where $\underline{C}$ and $\overline{C}$ will be specified later.

Under the assumptions, asymptotic results for our kernel estimators can be obtained by standard techniques as in, e.g., Pagan and Ullah (1999). Also see Guerre et al.

(2000) and Marmer and Shneyerov (2012) for relevant discussion. Here, we just briefly discuss and summarize the main (point-wise) results.

First, note that the estimators for $\widehat{N}$, $\widehat{\overline{b}}$, $\widehat{p_0}$ and $\widehat{r}$ converge at rates greater or equal to $L^{1/2}$, so do the empirical distribution $\widehat{G^*}(b)$ and the derived $\widehat{q}$ and $\widehat{F(p_0)}$. Hence, all of them will not affect the nonparametric asymptotic results.

One technical issue is that $\lim_{b \downarrow p_0} g^*(b) \to \infty$. We deal with it by a change of variable in the way of Guerre et al. (2000). We define $b^\dagger = (b^* - p_0)^{0.5}$, and replace b^* by $b^\dagger$. Some more details will be presented in the Appendix B.

As the recovery of $\widehat{F^*}(v)$ and thus $\widehat{F}(v)$ rely on the link function defined as in (4.8), the asymptotic property of $\widehat{F}(v)$ depends on that of the estimator for $g^*(\cdot)$ appearing in (4.8). Specifically, $(Lh)^{1/2}(\widehat{F}(v) - F(v)) \to_d N(0, W_1)$, where $N(0, W)$ represents the normal distribution with mean zero and variance W.

For $f^{PV}(v)$, one can show that, similar to Guerre et al. (2000), $(Lh^3)^{1/2}(f^{PV}(v) - f(v)) \to_d N(0, W_2)$; for $f^{QB}(v)$, one can show that, similar to Marmer and Shneyerov (2012), $(Lh^3)^{1/2}(f^{QB}(v) - f(v)) \to_d N(0, W_3)$. The two asymptotic normality results hold only when $v \in \mathcal{V}$, where $\mathcal{V}$ is a closed inner subsect of $[\underline{v}, \overline{v}]$.

Here, we do not pursue the explicit formulae for W_1, W_2 or W_3 as above. One can do this by following Guerre et al. (2000) and Marmer and Shneyerov (2012). It is actually easier for one to get consistent estimators for those W's by bootstrap in inference. Also, note that the convergence rate for PDF is lower than that of CDF, as expected.

4.5 Monte Carlo Experiments

For simulation, we need to calculate bidding function $s(v)$, given $F(\cdot), p_0, N$ and C. By solving the first order differential equation (4.2) with the initial condition, we get for $v \geq p_0$,

$$s(v) = v - \frac{1}{(qF(v) + 1 - q)^{N-1}} \int_{p_0}^{v} (qF(x) + 1 - q)^{N-1} dx. \tag{4.17}$$

Next, we plug (4.17) into (4.3), and get

$$\Pi = \int_{p_0}^{\overline{v}} \int_{p_0}^{v} (1 - q + qF(x))^{N-1} dx dv$$
$$= \int_{p_0}^{\overline{v}} (1 - q + qF(v))^{N-1} (1 - F(v)) dv, \quad (4.18)$$

so that we can solve the unknown q by using the indifference condition $\Pi = C$. Then we plug the solved q into (4.17) to solve the bidding function and thus to simulate bids. We drop observations with $s(v) < p_0$. Now we can respond to the range of $[\underline{C}, \overline{C}]$ we have previously assumed for entry cost: as (4.18) is an decreasing function of q, $\underline{C} = \int_{p_0}^{\overline{v}} (F(v))^{N-1}(1 - F(v))dv$, and $\overline{C} = \int_{p_0}^{\overline{v}} (1 - F(v))dv$.

For model primitives, we set $F(v) = v^{\beta}$, where $v \in [0,1]$, so that $f(v) = \beta v^{\beta-1}$. We set the reserve price at 0.2 and the entry cost at 0.08. We do experiments with three values of β: 0.5, 1 and 2. For each value of β, we report results for v at 0.3, 0.4,...,0.9, and N at 4, 5 and 6. As for kernel estimation, we take the tri-weight kernel $K(u) = 35/32(1 - u^2)^3 \mathbf{1}(| u | \leq 1)$, and we choose the bandwidth by the rule of thumb as in Marmer and Shneyerov (2012).

We will evaluate finite sample performance of $\widehat{F}(v)$, $f^{QB}(v)$ and $f^{PV}(v)$ by the coverage of bootstrap percentile confidence intervals (CIs) and simulated bias and MSE. The number of auctions in one experiment is 1000. The number of experiment replications is 300, and for each replication we draw 199 bootstrap samples when needed.

To have a preliminary view of the bootstrap percentile CIs of the estimators, we show 95% CIs at one of 300 experiments in Figures 4.1-4.3.[1] More precisely, the CIs' performance is summarized by simulated coverage probabilities as in Tables 4.1-4.3. This is useful for inference. In general, the coverage probabilities are close to the corresponding nominal levels, except at values close to the boundary of 1. The PV method causes a bit over-coverage whereas the QB method causes a bit under-coverage.

Tables 4.4-4.6 show the simulated bias and MSE of estimators. The findings are interestingly consistent: compared with the PV method, the QB method is inferior when $\beta = 2$, but is superior when $\beta = 0.5$ especially in MSE; when $\beta = 1$, QB works

[1] In particular, we choose the first of 300 simulations.

better with smaller values whereas PV works better with larger values.

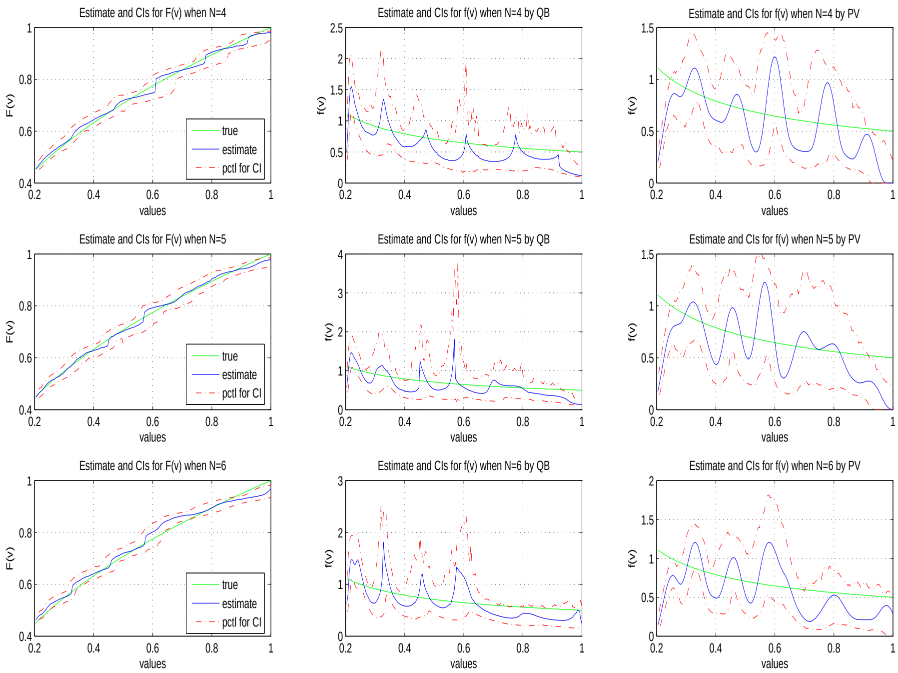
We also compare the average of our estimators with the average of the estimators under exogenous entry. As Figures 4.4-4.6 show, estimation under exogenous entry overestimates CDF but underestimates PDF. Moreover, the bias is exaggerated as N increases. The average of estimate under the two alternative methods considering endogenous entry seem to perform equally well at values away from boundaries.

4.6 Discussion and Conclusion

This paper shows how we can recover bidders' private value CDF and PDF under endogenous entry, in the presence of additional information of entry costs. As one may have noticed, the auction objects are treated homogenous across all sales. To allow for possible observed heterogeneity, we can introduce relevant characteristics, say X, and rewrite everything conditional on X as needed. Given a specific characteristic value of x, we may apply a nonparametric kernel smoothing method for data with x. As such, the recovered objects will be conditional CDF and PDF.

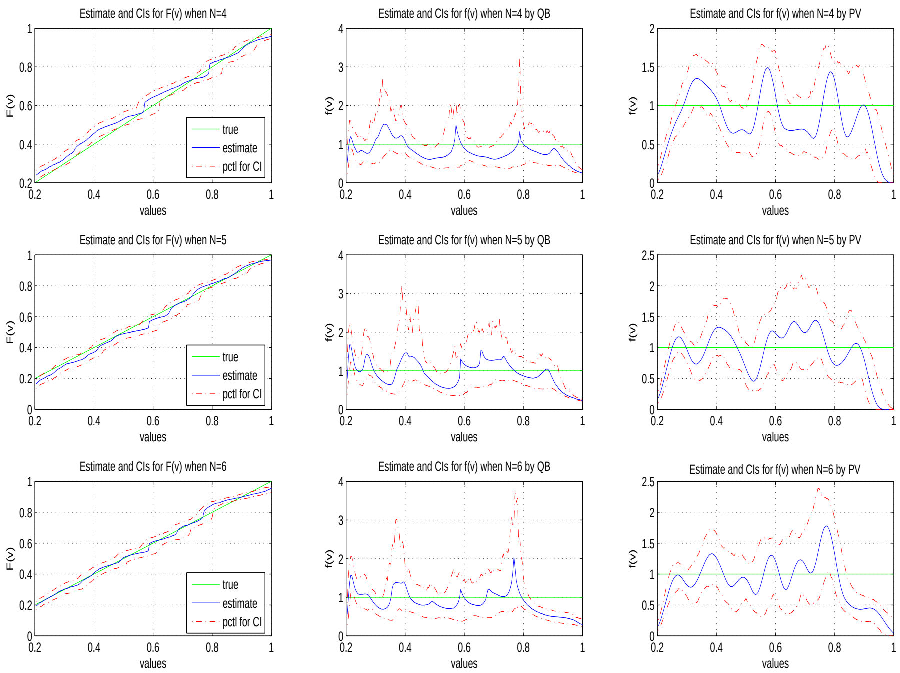
In all, this paper establishes a method to recover the distribution and density of primitive values, from observed bids selected by entry and a reserve price. The basic idea of identification is shown as the solution to a two-stage entry and bidding game, and the estimation immediately follows by nonparametric approaches. We show the estimated functions perform reasonably well at finite samples, but could be severely misleading if endogenous entry is ignored.

Figure 4.1: Estimate and 95% CIs for $F(v)$ and $f(v)$ at $\beta = 0.5$.



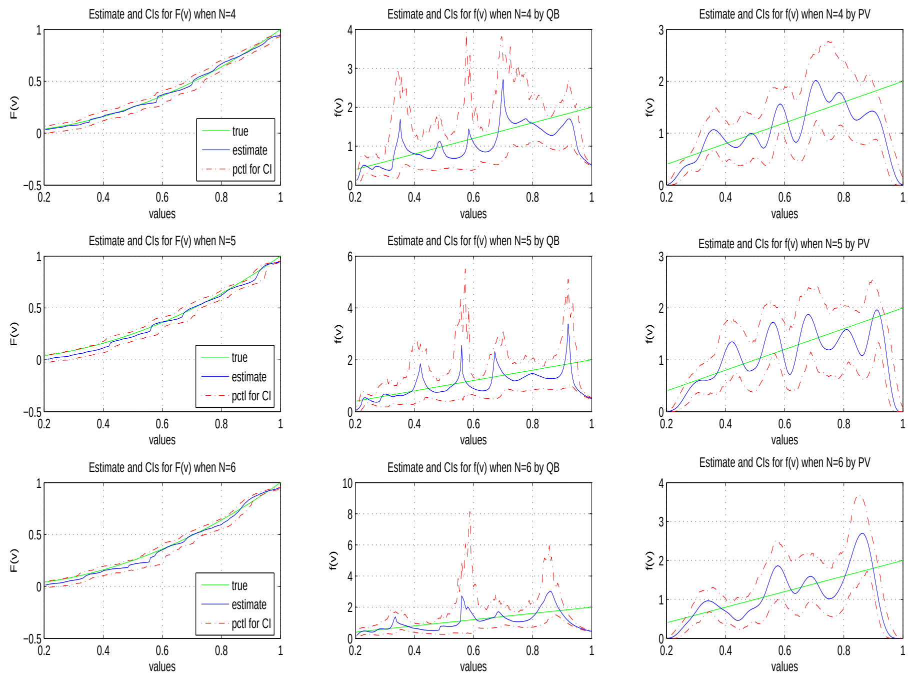
Note: The legend for $f(v)$ are the same as that for $F(v)$.

Figure 4.2: Estimate and 95% CIs for $F(v)$ and $f(v)$ at $\beta = 1$.



Note: The legend for $f(v)$ are the same as that for $F(v)$.

Figure 4.3: Estimate and 95% CIs for $F(v)$ and $f(v)$ at $\beta = 2$.



Note: The legend for $f(v)$ are the same as that for $F(v)$.

Table 4.1: Simulated Coverage Probabilities for $\beta = 0.5$.

Estimator	v 0.3	0.4	0.5	0.6	0.7	0.8	0.9
N=4, CL=0.99							
$\widehat{F}(v)$	0.953	0.970	0.993	0.997	1.000	0.997	1.000
$f^{QB}(v)$	0.980	0.987	0.973	0.973	0.980	0.990	0.943
$f^{PV}(v)$	0.990	0.990	0.990	0.990	1.000	0.997	0.953
N=5, CL=0.99							
$\widehat{F}(v)$	0.960	0.983	0.993	0.993	1.000	0.993	0.997
$f^{QB}(v)$	0.973	0.983	0.990	0.997	0.990	0.987	0.957
$f^{PV}(v)$	0.983	0.993	1.000	0.997	0.997	0.997	0.960
N=6, CL=0.99							
$\widehat{F}(v)$	0.983	0.993	1.000	0.997	0.997	0.997	0.960
$f^{QB}(v)$	0.990	0.997	0.983	0.987	0.977	0.993	0.973
$f^{PV}(v)$	0.990	0.997	0.997	0.997	1.000	0.997	0.970
N=4, CL=0.95							
$\widehat{F}(v)$	0.870	0.893	0.940	0.963	0.967	0.973	0.983
$f^{QB}(v)$	0.950	0.907	0.933	0.900	0.920	0.947	0.763
$f^{PV}(v)$	0.930	0.957	0.970	0.973	0.987	0.987	0.817
N=5, CL=0.95							
$\widehat{F}(v)$	0.830	0.910	0.930	0.980	0.960	0.980	0.960
$f^{QB}(v)$	0.950	0.930	0.910	0.970	0.950	0.930	0.750
$f^{PV}(v)$	0.947	0.950	0.987	0.987	0.983	0.993	0.850
N=6, CL=0.95							
$\widehat{F}(v)$	0.947	0.950	0.987	0.987	0.983	0.993	0.850
$f^{QB}(v)$	0.970	0.970	0.933	0.967	0.933	0.930	0.843
$f^{PV}(v)$	0.947	0.950	0.967	0.980	0.960	0.987	0.853
N=4, CL=0.90							
$\widehat{F}(v)$	0.760	0.843	0.890	0.940	0.923	0.933	0.947
$f^{QB}(v)$	0.887	0.837	0.870	0.817	0.857	0.883	0.590
$f^{PV}(v)$	0.873	0.890	0.937	0.937	0.967	0.973	0.673
N=5, CL=0.90							
$\widehat{F}(v)$	0.760	0.860	0.870	0.970	0.930	0.930	0.940
$f^{QB}(v)$	0.930	0.840	0.860	0.930	0.850	0.820	0.640
$f^{PV}(v)$	0.887	0.913	0.937	0.983	0.957	0.980	0.687
N=6, CL=0.90							
$\widehat{F}(v)$	0.887	0.913	0.937	0.983	0.957	0.980	0.687
$f^{QB}(v)$	0.937	0.890	0.890	0.903	0.863	0.850	0.710
$f^{PV}(v)$	0.900	0.923	0.927	0.953	0.930	0.957	0.743

Note: CL is for confidence level.

Table 4.2: Simulated Coverage Probabilities for $\beta = 1$.

	v						
Estimator	0.3	0.4	0.5	0.6	0.7	0.8	0.9
N=4, CL=0.99							
$\widehat{F}(v)$	0.960	0.987	0.993	0.997	0.997	1.000	0.997
$f^{QB}(v)$	1.000	0.990	0.980	0.980	0.983	0.990	0.967
$f^{PV}(v)$	0.993	1.000	0.987	1.000	0.993	0.993	0.920
N=5, CL=0.99							
$\widehat{F}(v)$	0.950	0.973	0.997	0.993	0.993	0.993	0.983
$f^{QB}(v)$	0.983	0.977	0.993	0.980	0.983	0.987	0.940
$f^{PV}(v)$	0.973	0.980	0.993	0.993	0.990	0.993	0.917
N=6, CL=0.99							
$\widehat{F}(v)$	0.958	0.967	0.982	0.996	1.000	1.000	0.999
$f^{QB}(v)$	0.990	0.990	0.989	0.994	0.985	0.994	0.978
$f^{PV}(v)$	0.996	0.996	0.994	0.990	0.986	1.000	0.968
N=4, CL=0.95							
$\widehat{F}(v)$	0.897	0.950	0.950	0.960	0.973	0.963	0.963
$f^{QB}(v)$	0.967	0.967	0.913	0.933	0.913	0.937	0.763
$f^{PV}(v)$	0.947	0.977	0.947	0.960	0.983	0.963	0.763
N=5, CL=0.95							
$\widehat{F}(v)$	0.850	0.913	0.967	0.960	0.967	0.963	0.960
$f^{QB}(v)$	0.917	0.927	0.953	0.927	0.927	0.930	0.830
$f^{PV}(v)$	0.930	0.960	0.970	0.967	0.977	0.970	0.810
N=6, CL=0.95							
$\widehat{F}(v)$	0.879	0.913	0.922	0.967	0.985	0.981	0.982
$f^{QB}(v)$	0.940	0.918	0.960	0.935	0.933	0.961	0.863
$f^{PV}(v)$	0.957	0.979	0.967	0.965	0.958	0.976	0.825
N=4, CL=0.90							
$\widehat{F}(v)$	0.817	0.867	0.877	0.917	0.933	0.930	0.900
$f^{QB}(v)$	0.930	0.920	0.877	0.877	0.837	0.873	0.650
$f^{PV}(v)$	0.913	0.937	0.903	0.930	0.933	0.920	0.657
N=5, CL=0.90							
$\widehat{F}(v)$	0.767	0.850	0.930	0.903	0.907	0.910	0.923
$f^{QB}(v)$	0.863	0.860	0.887	0.893	0.883	0.863	0.697
$f^{PV}(v)$	0.857	0.900	0.927	0.933	0.913	0.910	0.677
N=6, CL=0.90							
$\widehat{F}(v)$	0.806	0.850	0.864	0.913	0.936	0.933	0.958
$f^{QB}(v)$	0.904	0.888	0.867	0.893	0.860	0.907	0.726
$f^{PV}(v)$	0.899	0.926	0.936	0.932	0.918	0.942	0.708

Note: CL is for confidence level.

Table 4.3: Simulated Coverage Probabilities for $\beta = 2$.

Estimator	v 0.3	0.4	0.5	0.6	0.7	0.8	0.9
N=4, CL=0.99							
$\widehat{F}(v)$	0.957	0.977	0.983	1.000	0.990	0.993	0.993
$f^{QB}(v)$	0.987	0.993	0.980	0.997	0.987	0.987	0.957
$f^{PV}(v)$	0.997	0.993	1.000	1.000	0.983	0.990	0.953
N=5, CL=0.99							
$\widehat{F}(v)$	0.937	0.973	0.990	0.987	1.000	0.983	1.000
$f^{QB}(v)$	0.977	0.990	0.983	0.970	0.993	0.993	0.987
$f^{PV}(v)$	0.980	0.997	0.990	0.987	0.993	0.990	0.960
N=6, CL=0.99							
$\widehat{F}(v)$	0.910	0.967	0.990	0.983	0.997	0.993	0.997
$f^{QB}(v)$	0.993	0.990	0.977	0.993	0.983	0.997	0.987
$f^{PV}(v)$	0.987	0.983	0.993	1.000	1.000	1.000	0.983
N=4, CL=0.95							
$\widehat{F}(v)$	0.857	0.893	0.943	0.943	0.943	0.957	0.950
$f^{QB}(v)$	0.950	0.930	0.907	0.930	0.897	0.950	0.893
$f^{PV}(v)$	0.957	0.960	0.963	0.970	0.953	0.960	0.850
N=5, CL=0.95							
$\widehat{F}(v)$	0.787	0.877	0.913	0.950	0.970	0.973	0.980
$f^{QB}(v)$	0.913	0.947	0.917	0.907	0.960	0.937	0.920
$f^{PV}(v)$	0.913	0.977	0.970	0.960	0.980	0.957	0.880
N=6, CL=0.95							
$\widehat{F}(v)$	0.767	0.847	0.943	0.950	0.967	0.953	0.953
$f^{QB}(v)$	0.940	0.943	0.913	0.940	0.933	0.963	0.927
$f^{PV}(v)$	0.963	0.963	0.963	0.970	0.970	0.967	0.927
N=4, CL=0.90							
$\widehat{F}(v)$	0.773	0.830	0.907	0.903	0.897	0.920	0.913
$f^{QB}(v)$	0.877	0.870	0.863	0.863	0.813	0.910	0.820
$f^{PV}(v)$	0.920	0.923	0.933	0.920	0.907	0.913	0.777
N=5, CL=0.90							
$\widehat{F}(v)$	0.697	0.817	0.880	0.903	0.927	0.920	0.917
$f^{QB}(v)$	0.867	0.863	0.853	0.837	0.890	0.883	0.883
$f^{PV}(v)$	0.863	0.927	0.907	0.923	0.937	0.900	0.837
N=6, CL=0.90							
$\widehat{F}(v)$	0.690	0.747	0.877	0.897	0.917	0.910	0.903
$f^{QB}(v)$	0.887	0.867	0.823	0.863	0.820	0.913	0.880
$f^{PV}(v)$	0.917	0.940	0.927	0.923	0.950	0.927	0.850

Note: CL is for confidence level.

Table 4.4: Simulated Bias and MSE of Estimators for $\beta = 0.5$

v		Bias			MSE		
		$\widehat{F}(v)$	$f^{QB}(v)$	$f^{PV}(v)$	$\widehat{F}(v)$	$f^{QB}(v)$	$f^{PV}(v)$
	N=4						
0.3		-0.0009	-0.0125	0.0123	0.0003	0.0535	0.0257
0.4		-0.0010	-0.0495	-0.0010	0.0003	0.0607	0.0401
0.5		0.0000	-0.0108	0.0412	0.0003	0.0520	0.0641
0.6		0.0005	-0.0564	-0.0112	0.0002	0.0395	0.0602
0.7		-0.0001	-0.0589	0.0062	0.0002	0.0505	0.0632
0.8		-0.0012	-0.0425	-0.0005	0.0002	0.0596	0.0787
0.9		-0.0009	-0.1682	-0.2054	0.0002	0.0461	0.0744
	N=5						
0.3		0.0000	-0.0051	0.0070	0.0003	0.0478	0.0246
0.4		-0.0002	-0.0418	0.0031	0.0003	0.0515	0.0409
0.5		0.0005	-0.0562	-0.0093	0.0002	0.0472	0.0533
0.6		0.0006	-0.0659	-0.0019	0.0003	0.0455	0.0540
0.7		0.0000	-0.0591	0.0072	0.0002	0.0390	0.0686
0.8		0.0002	-0.0611	-0.0339	0.0002	0.0547	0.0660
0.9		-0.0015	-0.1566	-0.1810	0.0002	0.0464	0.0655
	N=6						
0.3		0.0098	0.0742	-0.0031	0.0005	0.0594	0.0259
0.4		0.0069	0.0068	-0.0138	0.0003	0.0527	0.0433
0.5		0.0044	-0.0281	-0.0037	0.0003	0.0468	0.0589
0.6		0.0031	-0.0361	-0.0220	0.0003	0.0425	0.0608
0.7		0.0007	-0.0665	-0.0436	0.0002	0.0276	0.0559
0.8		-0.0025	-0.0429	-0.0177	0.0002	0.0359	0.0645
0.9		-0.0046	-0.1186	-0.1515	0.0003	0.0330	0.0596

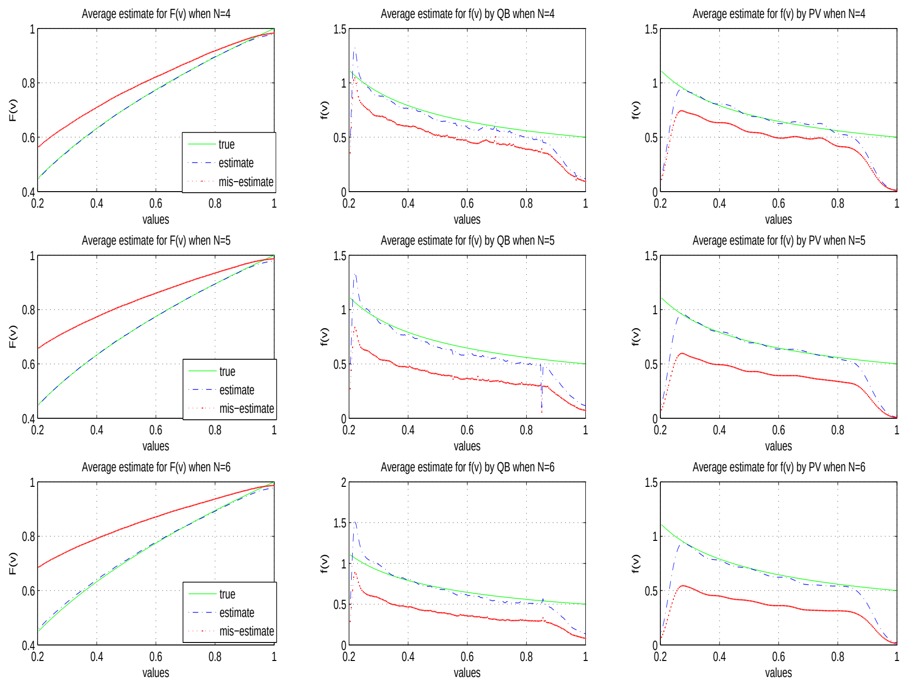
Table 4.5: Simulated Bias and MSE of Estimators for $\beta = 1$

v		Bias			MSE		
		$\widehat{F}(v)$	$f^{QB}(v)$	$f^{PV}(v)$	$\widehat{F}(v)$	$f^{QB}(v)$	$f^{PV}(v)$
	N=4						
0.3		0.0002	-0.0211	0.0042	0.0006	0.0504	0.0206
0.4		0.0007	-0.0214	0.0162	0.0006	0.0801	0.0413
0.5		0.0000	-0.0344	-0.0102	0.0005	0.1015	0.0601
0.6		-0.0003	-0.0523	-0.0229	0.0004	0.0628	0.0560
0.7		0.0000	-0.0362	0.0121	0.0003	0.0544	0.0825
0.8		-0.0018	-0.0365	-0.0127	0.0002	0.0583	0.0784
0.9		0.0011	-0.1677	-0.2464	0.0002	0.0658	0.1008
	N=5						
0.3		0.0012	-0.0184	0.0039	0.0005	0.0520	0.0191
0.4		0.0011	-0.0266	0.0222	0.0005	0.0798	0.0446
0.5		0.0005	-0.0795	-0.0236	0.0004	0.0774	0.0629
0.6		0.0002	-0.0579	0.0034	0.0004	0.0634	0.0682
0.7		0.0005	-0.0505	-0.0049	0.0003	0.0757	0.0845
0.8		0.0000	-0.0673	-0.0179	0.0003	0.0516	0.0782
0.9		0.0007	-0.1797	-0.2328	0.0002	0.0613	0.0904
	N=6						
0.3		-0.0015	-0.0017	0.0049	0.0006	0.0640	0.0221
0.4		-0.0007	-0.0315	0.0088	0.0006	0.0819	0.0414
0.5		-0.0005	-0.0534	0.0154	0.0005	0.0654	0.0665
0.6		-0.0014	-0.0773	-0.0155	0.0004	0.0723	0.0791
0.7		-0.0006	-0.0195	0.0211	0.0004	0.0965	0.1079
0.8		0.0017	-0.0531	-0.0099	0.0003	0.0645	0.0889
0.9		-0.0002	-0.1510	-0.1992	0.0002	0.0590	0.0803

Table 4.6: Simulated Bias and MSE of Estimators for $\beta = 2$

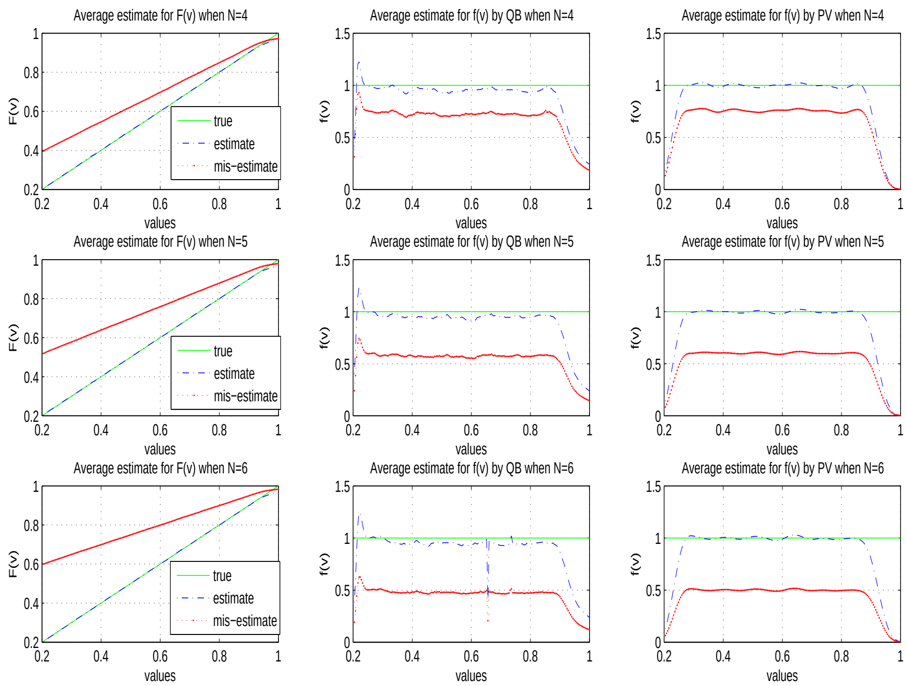
v		Bias			MSE		
		$\widehat{F}(v)$	$f^{QB}(v)$	$f^{PV}(v)$	$\widehat{F}(v)$	$f^{QB}(v)$	$f^{PV}(v)$
	N=4						
0.3		0.0055	-0.0332	-0.0067	0.0009	0.0353	0.0151
0.4		0.0045	-0.0779	0.0019	0.0009	0.0853	0.0399
0.5		0.0039	-0.1379	-0.0261	0.0010	0.1514	0.0764
0.6		0.0039	-0.0905	-0.0034	0.0007	0.1524	0.0820
0.7		0.0026	-0.0852	-0.0269	0.0007	0.1513	0.1242
0.8		0.0041	-0.0646	-0.0196	0.0005	0.1281	0.1426
0.9		0.0006	-0.1081	-0.2384	0.0003	0.1268	0.1363
	N=5						
0.3		0.0050	-0.0328	-0.0120	0.0009	0.0291	0.0143
0.4		0.0054	-0.0684	-0.0092	0.0009	0.0770	0.0431
0.5		0.0033	-0.0966	0.0097	0.0009	0.1770	0.0853
0.6		0.0020	-0.1288	-0.0103	0.0009	0.1632	0.0899
0.7		0.0029	-0.0820	-0.0311	0.0007	0.2296	0.1282
0.8		0.0007	-0.0664	-0.0194	0.0005	0.1748	0.1454
0.9		0.0004	-0.0589	-0.1689	0.0004	0.1465	0.1143
	N=6						
0.3		0.0027	-0.0203	-0.0014	0.0010	0.0301	0.0149
0.4		0.0029	-0.0561	-0.0106	0.0010	0.1557	0.0456
0.5		-0.0007	-0.0744	0.0083	0.0010	0.5198	0.0861
0.6		0.0014	-0.0437	-0.0070	0.0009	1.8395	0.0986
0.7		0.0025	-0.1005	-0.0226	0.0007	0.1908	0.1504
0.8		-0.0005	-0.0717	0.0280	0.0007	0.1522	0.1488
0.9		0.0010	-0.0737	-0.1676	0.0004	0.1518	0.1333

Figure 4.4: Comparison of Simulated Average Estimate for $F(v)$ and $f(v)$ when $\beta = 0.5$.



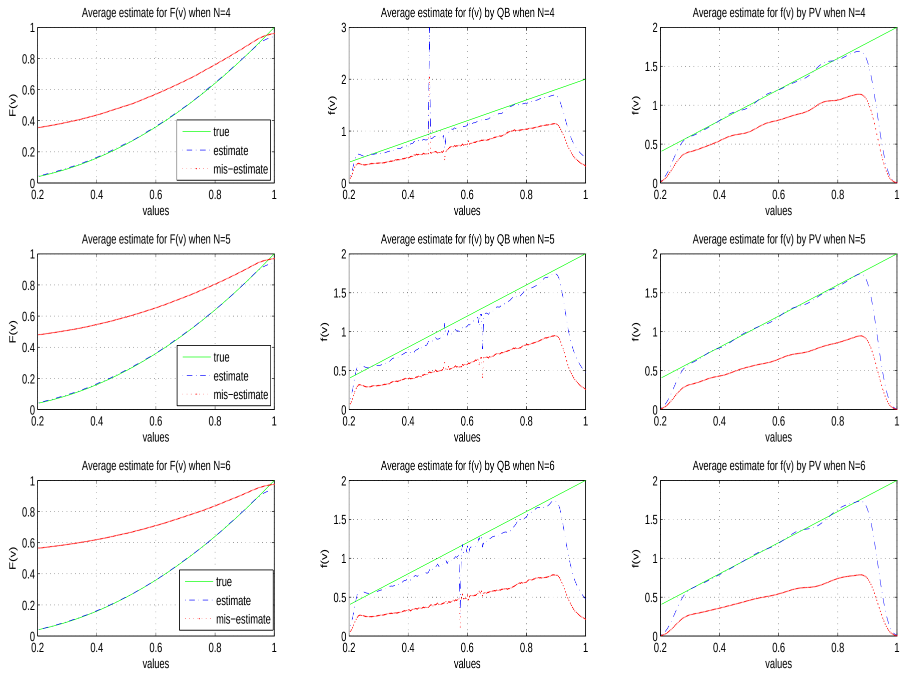
Note: We use "mis-estimate" to denote estimation under exogenous entry. The legend for $f(v)$ are the same as that for $F(v)$.

Figure 4.5: Comparison of Simulated Average Estimate for $F(v)$ and $f(v)$ when $\beta = 1.0$.



Note: We use "mis-estimate" to denote estimation under exogenous entry. The legend for $f(v)$ are the same as that for $F(v)$.

Figure 4.6: Comparison of Simulated Average Estimate for $F(v)$ and $f(v)$ when $\beta = 2.0$.



Note: We use "mis-estimate" to denote estimation under exogenous entry. The legend for $f(v)$ are the same as that for $F(v)$.

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Appendix A

Appendix to Chapter 1

A.1 Definition of $\mathcal{U}_R$, $\mathcal{F}_R$ and $\mathcal{G}_R$ as in Guerre, Perrigne and Vuong (2009)

Definition 1. For $R \geq 1$, let $\mathcal{U}_R$ be the set of utility functions $U(\cdot)$ that satisfy the following conditions:

(a) $U : [0, +\infty) \to [0, +\infty), U(0) = 0$ and $U(1) = 1$.

(b) $U(\cdot)$ is continuous on $[0, +\infty)$ and admits $R+2$ continuous derivatives on $(0, +\infty)$ with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ on $(0, +\infty)$.

(c) $\lim_{x \downarrow 0} \Lambda^r(x)$ is finite for $1 \leq r \leq R+1$, where $\Lambda^r(\cdot)$ is the rth derivative of $\Lambda(\cdot)$.

Definition 2. For $R \geq 1$, let $\mathcal{F}_R$ be the set of distributions $F_s(\cdot), s \in \mathcal{S}$, that satisfy the following conditions:

(a) $F_s(\cdot)$ is a cumulative distribution function (c.d.f.) with support of the form $[\underline{v}(s), \overline{v}(s)]$, where $0 \leq \underline{v}(s) < \overline{v}(s) < +\infty$.

(b) $F_s(\cdot)$ admits $R+1$ continuous derivatives on $[\underline{v}(s), \overline{v}(s)]$.

(c) $f_s(\cdot) > 0$ on $[\underline{v}(s), \overline{v}(s)]$.

Definition 3. For $R \geq 1$, let $\mathcal{G}_{\mathcal{R}}$ be the set of distributions $G_n(\cdot), n \in \mathcal{N}$, that satisfy the following conditions:

(a) G_n is a c.d.f. with support of the form $[\underline{b}_n, \overline{b}_n]$, where $0 \leq \underline{b}_n < \overline{b}_n < +\infty$.

(b) G_n admits R continuous derivatives on $[\underline{b}_n, \overline{b}_n]$

(c) $g_n > 0$ on $[\underline{b}_n, \overline{b}_n]$

(d) g_n admits $R+1$ continuous derivatives on $(\underline{b}_n, \overline{b}_n]$

(e) $\lim_{b \downarrow \underline{b}_n} d^r[G_n/g_n]/db^r$ exists and is finite for $r = 1, ..., R+1$.

A.2 Proof of Results in Section 2.3

Lemma 1. *Under Assumptions 1 and 6, $\forall n \in \mathcal{N}$, and $\forall \theta \in \Theta$*

(a) $\sup_{b \in [\underline{b}_n, \overline{b}_n]} | \widehat{G_n}(b) - G_n(b) |= O_p(L^{-\frac{1}{2}})$.

(b) $\sup_{\tau \in (0,1)} | \widehat{b_n}(\tau) - b_n(\tau) |= O_p(L^{-\frac{1}{2}})$.

(c) $\sup_{b \in [b_{n,1}, b_{n,2}]} | \widehat{g_n}(b) - g_n(b) |= O_p((\frac{Lh}{\ln L})^{-\frac{1}{2}} + h^R)$.

(d) $\sup_{[\tau_1, \tau_2]} | \widehat{X_n}(\tau) - X_n(\tau) |= O_p((\frac{Lh}{\ln L})^{-\frac{1}{2}} + h^R)$.

(e) $\sup_{[\tau_1, \tau_2]} | \widehat{Q}(\tau, \theta) - Q(\tau, \theta) |= O_p((\frac{Lh}{\ln L})^{-\frac{1}{2}} + h^R)$.

Proof. We refer readers to Marmer and Shneyerov (2012, Lemma 1), where similar results are derived and proved in the context of conditional quantiles. □

A.2.1 Proof of Proposition 2

Proof. Again, a similar proof can be found as in Marmer and Shneyerov (2012, Lemma 2) for result 1 in Proposition 2.

For result 2, the mean-value theorem leads us to

$$
\begin{aligned}
\widehat{X_n}(\tau) - X_n(\tau) &= \frac{\tau}{n-1}[\frac{1}{\widehat{g_n}(\widehat{b_n}(\tau))} - \frac{1}{g_n(b_n(\tau))}] \\
&= \frac{\tau}{n-1}[\frac{1}{\widehat{g_n}(\widehat{b_n}(\tau))} - \frac{1}{g_n(\widehat{b_n}(\tau))} + \frac{1}{g_n(\widehat{b_n}(\tau))} - \frac{1}{g_n(b_n(\tau))}] \\
&= \frac{\tau}{n-1}[\frac{1}{\tilde{g_n}^2(\widehat{b_n}(\tau))}(\widehat{g_n}(\widehat{b_n}(\tau)) - g_n(\widehat{b_n}(\tau))) \\
&+ \frac{1}{g_n^2(\tilde{b_n}(\tau))} g_n^{(1)}(\tilde{b_n}(\tau))(\widehat{b_n}(\tau) - b_n(\alpha))].
\end{aligned}
$$

By Assumption 1(boundedness of g_n and its derivative), Lemma 1 and the first result in Proposition 2, we have

$$(Lh)^{1/2}(\widehat{X_n}(\tau) - X_n(\tau)) = \frac{\tau}{n-1}\frac{1}{g_n^2(\widehat{b_n}(\tau))}(Lh)^{1/2}(\widehat{g_n}(\widehat{b_n}(\tau)) - g_n(\widehat{b_n}(\tau))) + O_p(h^{1/2})$$
$$\to_d N(0, \widetilde{V_n}(\tau)),$$

where $\widetilde{V_n}(\tau) = \frac{\tau^2}{(n-1)^2}\frac{g_n(\widehat{b_n}(\tau))}{n\pi(n)\tilde{g_n}^4(\widehat{b_n}(\tau))}\int K^2(u)du$. It is easy to verify that $V_n(\tau) = \widetilde{V_n}(\tau) + o_p(1)$, by consistency results from Lemma 1. So the claim is proved. □

A.2.2 Proof of Corollary 1

Proof. Following Marmer and Shneyerov (2012, Lemma 2), one can show that $\widehat{g_n}(b)$ and $\widehat{g_{n'}}(b)$ are asymptotically independent which implies $\widehat{X_n}(\tau)$ and $\widehat{X_{n'}}(\tau)$ are asymptotically independent.

Then note that

$$\widehat{Q}(\tau) - Q(\tau) = [(\widehat{b_{n'}}(\tau) - b_{n'}(\tau)) - (\widehat{b_n}(\tau) - b_n(\tau))]$$
$$+ \theta[(\widehat{X_{n'}}(\tau) - X_{n'}(\tau)) - (\widehat{X_n}(\tau) - X_n(\tau))].$$

Therefore, Lemma 1 and Proposition 2 apply to the expression above gives

$$(Lh)^{1/2}(\widehat{Q}(\tau) - Q(\tau)) = \theta[(Lh)^{1/2}(\widehat{X_{n'}}(\tau) - X_{n'}(\tau))$$
$$- (Lh)^{1/2}(\widehat{X_n}(\tau) - X_n(\tau))] + o_p(1)$$
$$\to_d N(0, \theta^2 V(\tau)).$$

□

A.3 Proof of Results in Section 2.4

Lemma 2. *Under Assumptions 1 and 6, $\forall \theta \in \Theta$, $\forall \tau \in [\tau_1, \tau_2]$, and $\forall u \in \mathbb{R}$,*
$\limsup_{L\to\infty} | P(\widetilde{Q}_L^{\dagger}(\tau,\theta) - \widetilde{Q}_L(\tau,\theta)) - P(\widetilde{Q}_L(\tau,\theta) - Q_L^{*}(\tau,\theta)) | \to_p 0$

Proof. Given the asymptotic results derived in section 2.3, we refer readers to Marmer and Shneyerov (2012, Theorem 3). □

A.3.1 Proof of Theorem 1

Proof. By Lemma 2, $\forall \tau \in (0,1)$,

$$\widetilde{Q}_L^{\dagger}(\tau,\theta) - \widetilde{Q}_L(\tau,\theta) \to_d \widetilde{Q}_L(\tau,\theta) - Q_L^*(\tau,\theta), \tag{A.1}$$

and it is easy to verify that $V_n(\tau) = \widetilde{V_n}(\tau) + o_p(1)$, by consistency results from Lemma 1. Hence, $V(\tau,\theta) = \widetilde{V}_L(\tau,\theta) + o_p(1)$, where $V(\tau,\theta) \equiv \theta^2 V(\tau)$.

By the almost sure representation theorem, e.g., see **?**, Theorem 9.4, there exists a probability space and random quantities $\nu_L(\cdot)$, $\nu_0(\cdot)$ and $W_L(\cdot)$ defined on it such that (i)$\widetilde{Q}_L^{\dagger}(\cdot,\theta) - \widetilde{Q}_L(\cdot,\theta)$ and $\nu_L(\cdot)$ have the same distribution, (ii)$\widetilde{Q}_L(\cdot,\theta) - Q_L^*(\cdot,\theta)$ and $\nu_0(\cdot)$ have the same distribution, (iii)$\widetilde{V}_L(\cdot,\theta)$ and $W_L(\cdot)$ have the same distribution, and (iv) $\sup_{\tau\in(0,1)} \mid \nu_L(\tau) - \nu_L(\tau) \mid \to 0$ a.s. and $\sup_{\tau\in(0,1)} \mid W_L(\tau) - V(\tau,\theta) \mid \to 0$ a.s.

Now define

$$S_L = \int_0^1 [\frac{\nu_L(\tau)}{W_L(\tau)^{\frac{1}{2}}} + \frac{Q_L^*(\tau,\theta)}{W_L(\tau)^{\frac{1}{2}}}]_-^2 d\tau, \tag{A.2}$$

and

$$S_{L,0} = \int_0^1 [\frac{\nu_0(\tau)}{V(\tau)^{\frac{1}{2}}} + \frac{Q_L^*(\tau,\theta)}{V(\tau)^{\frac{1}{2}}}]_-^2 d\tau. \tag{A.3}$$

By construction, S_L and $\check{T}_L(\theta)$ have the same distribution, and $S_{L,0}$ and $T_L(\theta)$ have the same distribution. And to prove part (a), it suffices to show that $S_L - S_{L,0} \to 0$ a.s., by Andrews and Shi (2013, Theorem 1). For a proof for the sufficient condition, we refer readers to Andrews and Shi (2013, Theorem 1).

□

A.3.2 Proof of Theorem 2

Proof. As stated before, based on the bound from Theorem 1, the key to proving Theorem 2 is to show that critical values obtained by using $\varphi_L(\cdot,\theta)$ are no smaller asymptotically in probability. In other words, we wish to show that, $\forall \theta \in \Theta_I$,

$$\lim_{L\to\infty} P[c_{1-\alpha}(\varphi_L,\theta) < c_{1-\alpha}(Q_L^*,\theta)] = 0. \tag{A.4}$$

We can prove this result by showing that

$$\begin{aligned}
&P[c_{1-\alpha}(\varphi_L(\theta_L),\theta_L) < c_{1-\alpha}(Q_L^*(\theta_L),\theta_L)]\\
\leq &P[\varphi_L(\theta_L,\tau) > Q_L^*(\theta_L,\tau)]\\
\leq &P[\xi_L(\theta,\tau) > 1, \quad Q_L^*(\theta_L,\tau) < B_L]\\
\leq &P[\kappa_L^{-1}\widetilde{Q}_L(\tau,\theta_L)\widetilde{V}_L(\tau,\theta_L)^{-\frac{1}{2}} > 1, \quad Q_L^*(\theta_L,\tau) < B_L]\\
\leq &P[(\widetilde{Q}_L(\tau,\theta) - Q_L^*(\tau,\theta))\widetilde{V}_L(\tau,\theta_L)^{-\frac{1}{2}} + Q_L^*(\tau,\theta_L)\widetilde{V}_L(\tau,\theta_L)^{-\frac{1}{2}} > \kappa_L, \quad Q_L^*(\theta_L,\tau) < B_L]\\
\leq &P[Q_L^*(\tau,\theta_L)\widetilde{V}_L(\tau,\theta_L)^{-\frac{1}{2}} > \kappa_L - O_p(1), \quad Q_L^*(\theta_L,\tau) < B_L]\\
= &o(1).
\end{aligned}$$

The last equality follows by Assumption 7. □

A.3.3 Proof of Theorem 3

Proof. Note that part (a) follows immediately part (b) as $\varphi_L(\theta_*,\tau) \geq 0$ and as $T_L(\varphi_L,\theta_*)$ is decreasing with φ_L.

For part (b), We first prove $T_L(\theta_*) \to \infty$.

$$\begin{aligned}
T_L(\theta_*) &= \int_0^1 [\frac{\widetilde{Q}_L(\tau,\theta_*) - Q_L^*(\tau,\theta_*)}{\widetilde{V}_L(\tau,\theta_*)^{\frac{1}{2}}} + \frac{Q_L^*(\tau,\theta_*)}{\widetilde{V}_L(\tau,\theta_*)^{\frac{1}{2}}}]_-^2 d\tau\\
&= \int_0^1 [O_p(1) + \frac{Q_L^*(\tau,\theta_*)}{\widetilde{V}_L(\tau,\theta_*)^{\frac{1}{2}}}]_-^2 d\tau\\
&= \int_0^1 [O_p(1) + a_L\frac{Q(\tau,\theta_*)}{\widetilde{V}_L(\tau,\theta_*)^{\frac{1}{2}}}]_-^2 d\tau\\
&\geq a_L^2 \int_{\mathfrak{X}(\theta_*)} [o_p(1) + \frac{Q(\tau,\theta_*)}{\widetilde{V}_L(\tau,\theta_*)^{\frac{1}{2}}}]_-^2 d\tau\\
&\to a_L^2 \int_{\mathfrak{X}(\theta_*)} [\frac{Q(\tau,\theta_*)}{V(\tau,\theta_*)^{\frac{1}{2}}}]_-^2 d\tau
\end{aligned}$$

The integration is positive bounded from 0 by Assumption 8 and $V(\tau,\theta_*) = O(1)$. This together with the fact that $a_L^2 \to \infty$ shows what we want.

Second, notice that

$$T_L(0,\theta_*) = \int_0^1 [\frac{\widetilde{Q}_L(\tau,\theta_*) - Q_L^*(\tau,\theta_*)}{\widetilde{V}_L(\tau,\theta)^{\frac{1}{2}}}]_-^2 d\tau = O_p(1),$$

and hence $c_{1-\alpha}(0,\theta_*) = O_p(1)$.

The two results above basically show that the test statistic $T_L(\theta_*)$ is bigger than the critical value $c_{1-\alpha}(0,\theta_*)$ in probability 1. Hence, part (b) holds and the proof is complete. □

A.4 Power Against a_L^{-1}-Local Alternatives

As is often the case, fixed alternatives are relatively easier to detect. This section applies the configurations of a_L^{-1}-local alternatives designed by Andrews and Shi (2013) to our setting, and shows that our test can also detect a kind of, but not all, local alternatives drifting to the identified set Θ_I at the rate of a_L. For illustration, we only consider local alternatives under a fixed DGP, whereas the setting in Andrews and Shi (2013) allows the true DGP flow with sample sizes and thus is more general.

The null and alternative hypotheses of our test for local alternative $\theta_{L,*}$ can be written as

$$H_0 : Q(\tau,\theta_{L,*}) \geq 0, \forall \tau \in (0,1),$$
$$H_1 : Q(\tau,\theta_{L,*}) < 0, \text{for some } \tau \in (0,1).$$

The local behavior of $\theta_{L,*}$ relative to Θ_I is specified in Assumption 15.

Assumption 15. For a given $\theta_0 \in \Theta_I$[1], $\theta_{L,*} = \theta_0 + \lambda a_L^{-1}(1+o(1))$ for some $\lambda \in \mathbb{R}$, where $\theta_{L,*} \in \Theta$.

A smoothness condition is needed for existence of the probability limit of $T_L(\varphi_L,\theta_{L,*})$, from which critical values are generated.

Assumption 16. The derivative $\Psi(\tau,\theta) \equiv \frac{\partial}{\partial\theta}Q(\tau,\theta)$ is continuous in a neighborhood of θ_0.

With Assumption 16, it is easy to see that the probability limit of $T_L(\varphi_L,\theta_{L,*})$ depends only on θ_0. Hence we denote the probability limit as $T_\infty(\varphi_\infty,\theta_0)$, where $\varphi_\infty(\tau,\theta_0) = \infty$ if $Q(\tau,\theta) > 0$, and $\varphi_\infty(\tau,\theta_0) = 0$ if $Q(\tau,\theta) = 0$.

[1] θ_0 is not necessarily the true parameter.

Some mild regularity conditions are imposed in Assumption 17 to ensure the non-trivial power of our test.

Assumption 17. (a) The $1-\alpha$ quantile of $T_\infty(\varphi_\infty, \theta_0)$, denoted by $c_{\infty,1-\alpha}(\varphi_\infty, \theta_0)$, is a continuity point for the distribution of $T_\infty(\varphi_\infty, \theta_0)$.

(b) The $1-\alpha$ quantile of $T_\infty(0, \theta_0)$, denoted by $c_{\infty,1-\alpha}(0, \theta_0)$, is a continuity point for the distribution of $T_\infty(0, \theta_0)$.

Proposition 5. *Define the asymptotic (limiting) distribution of* $T_L(\theta_{L,*})$ *as* J_λ, *under Assumptions 1,6, 7, and 15-17,*

(a) $\lim_{L\to\infty} P(\theta_{L,*} \notin CS_L^{QS}) = 1 - J_\lambda(c_{\infty,1-\alpha}(\varphi_\infty, \theta_0))$,

(b) $\lim_{L\to\infty} P(\theta_{L,*} \notin CS_L^{PA}) = 1 - J_\lambda(c_{\infty,1-\alpha}(0, \theta_0))$.

Proof. We refer readers to Andrews and Shi (2013, Theorem 4). □

Without loss of generality, we can specify that $\lambda = \beta\lambda_0$, for some $\beta > 0$ and $\lambda_0 \in \mathbb{R}$. We next state a sufficient condition, as in Assumption 18, for our test being asymptotically consistent against the local alternatives $\theta_{L,*}$

Assumption 18. Define $\wp(\theta_0) \equiv \{\tau \in (0,1) : Q(\tau, \theta_0) = 0, \Psi_0(\tau)\lambda_0 < 0\}$, where $\Psi_0(\tau) = \Psi(\tau, \theta_0)$. Then $\mu(\wp(\theta_0)) > 0$, where μ is the Lebsgue measure.

Assumption 18 implies that the θ_0 may lie on the "boundary" of Θ_I such that there is a set of binding constraints with its measure bounded away from 0; moreover, on the set of binding constraints, the values $\theta_{L,*}$ drift to the direction, indicated by λ_0, which detects the violation of the null hypothesis.

Theorem 5. *Under Assumptions 15-18,*

$\lim_{\beta\to\infty}(1 - J_{\beta\lambda_0}(c_{\infty,1-\alpha}(\varphi_\infty, \theta_0)) = \lim_{\beta\to\infty}(1 - J_{\beta\lambda_0}(c_{\infty,1-\alpha}(0, \theta_0)) = 1$

Proof.

$$
\begin{aligned}
T_L(\theta_{L,*}) &= \int_0^1 [\frac{\widetilde{Q}_L(\tau,\theta_{L,*}) - Q_L^*(\tau,\theta_{L,*})}{\widetilde{V}_L(\tau,\theta_{L,*})^{\frac{1}{2}}} + \frac{Q_L^*(\tau,\theta_{L,*})}{\widetilde{V}_L(\tau,\theta_{L,*})^{\frac{1}{2}}}]_-^2 d\tau \\
&= \int_0^1 [O_p(1) + \frac{Q_L^*(\tau,\theta_{L,*})}{\widetilde{V}_L(\tau,\theta_{L,*})^{\frac{1}{2}}}]_-^2 d\tau \\
&= \int_0^1 [O_p(1) + \frac{Q_L^*(\tau,\theta_0)}{\widetilde{V}_L(\tau,\theta_{L,*})^{\frac{1}{2}}} + \frac{\Psi_0(\tau)\lambda}{\widetilde{V}_L(\tau,\theta_{L,*})^{\frac{1}{2}}}]_-^2 d\tau \\
&= \beta \int_0^1 [\frac{O_p(1)}{\beta} + 0 + \frac{\Psi_0(\tau)\lambda_0}{\widetilde{V}_L(\tau,\theta_{L,*})^{\frac{1}{2}}}]_-^2 d\tau \\
&\overset{L\to\infty}{\longrightarrow} \beta \int_0^1 [\frac{O_p(1)}{\beta} + \frac{\Psi_0(\tau)\lambda_0}{V(\tau,\theta_0)^{\frac{1}{2}}}]_-^2 d\tau \\
&\geq \beta \int_{\mathfrak{V}(\theta_0)} [\frac{O_p(1)}{\beta} + \frac{\Psi_0(\tau)\lambda_0}{V(\tau,\theta_0)^{\frac{1}{2}}}]_-^2 d\tau \\
&\overset{\beta\to\infty}{\longrightarrow} \infty \times \int_{\mathfrak{V}(\theta_0)} [\frac{\Psi_0(\tau)\lambda_0}{V(\tau,\theta_0)^{\frac{1}{2}}}]_-^2 d\tau \\
&\longrightarrow \infty,
\end{aligned}
$$

where the third equality holds by Assumptions 15 and 16, and the convergence as $L \to \infty$ holds by Assumptions 1 and 16, and the final result of divergence holds by Assumption 18 and $V(\tau,\theta_0) = O(1)$.

Now we see that $J_{\beta\lambda_0}$, the asymptotic (limiting) distribution function of $T_L(\theta_{L,*})$, is divergent as $\beta \to \infty$. On the other hand, neither $c_{\infty,1-\alpha}(\varphi_\infty,\theta_0)$ nor $c_{\infty,1-\alpha}(0,\theta_0)$ depends on β. So the result is proved. □

Proposition 5 together with Theorem 5 establishes that our test has asymptotic power 1 against the kind of the a_L^{-1}-local alternatives in this section.

A.5 Details about DGP in Section 2.5

A.5.1 Calculation for Type 1 DGP

For

$$F_s(v) = \frac{v}{s}, H_{s,n}(v) = (\frac{v}{s})^{\frac{(n-1)}{\theta}}.$$

So by equation 2.3,

$$b_{s,n}(v) = v - \int_0^v (\frac{x}{v})^{\frac{(n-1)}{\theta}} dx = v - \frac{1}{\frac{(n-1)}{\theta}+1} v = \frac{\frac{(n-1)}{\theta}}{\frac{(n-1)}{\theta}+1} v.$$

By equation 2.4,

$$\pi_{s,n} = \int_0^s (\frac{1}{1+\frac{n-1}{\theta}} v)^\theta \frac{1}{s} dv = (\frac{1}{1+\frac{(n-1)}{\theta}})^\theta \frac{1}{1+\theta} s^\theta.$$

Obviously, $\pi_{s,n}$ is an increasing function of s $\forall n \in \mathcal{N}$, $\forall \theta \in \Theta$.

Next, we show $\Theta_I = (0, 0.2262]$.

Since

$$b_n(\tau) = b_n(V(\tau)) = \frac{n-1}{\theta+(n-1)} V(\tau) = \frac{s(n-1)\tau}{\theta+(n-1)},$$

and by the fact that $g_n(b_n(\tau)) b_n'(\tau) = 1$,

$$g_n(b_n(\tau)) = \frac{1}{b_n'(\tau)} = \frac{\theta+(n-1)}{s(n-1)}.$$

Now with those specifications in type 1, one can first calculate identified sets as defined in (2.9) using $\{2,3\}, \{3,4\}, \{4,5\}$ and $\{5,6\}$ for $\{n, n'\}$, respectively, and then derive Θ_I by intersecting these sets. To save space, we do not show the routine work but illustrate the basic idea by depicting the bounds for Θ_I in Figure A.1. Note that each bound depicted is the upper bound for θ for each pair of entrant numbers in Figure A.1.

A.5.2 Calculation for Type 2 DGP

For

$$F_s(v) = v^s, v \in [0,1], H_{s,n}(v) = v^{\frac{s(n-1)}{\theta}}.$$

So by equation 2.3,

$$b_{s,n}(v) = v - \int_0^v (\frac{x}{v})^{\frac{s(n-1)}{\theta}} = v - \frac{\theta}{\theta+s(n-1)} v = \frac{s(n-1)}{\theta+s(n-1)} v.$$

By equation 2.4,

$$\pi_{s,n} = \int_0^1 (\frac{\theta}{\theta+s(n-1)} v)^\theta s v^{s-1} dv = \frac{s}{s+\theta} (\frac{\theta}{\theta+s(n-1)})^\theta,$$

By studying its derivative,

$$\begin{aligned}\frac{\partial \pi_{s,n}}{\partial s} &= \frac{\theta^2}{(\theta+s)^2(\theta+s(n-1))} - \frac{\theta^2 s(n-1)}{(\theta+s)(\theta+s(n-1))^2} \\ &= C(\frac{1}{\theta+s} - \frac{s(n-1)}{\theta+s(n-1)}) \\ &= C(\frac{1}{\theta+s} - \frac{(n-1)}{\frac{\theta}{s}+(n-1)})\end{aligned}$$

where C is a generic positive constant. Note that

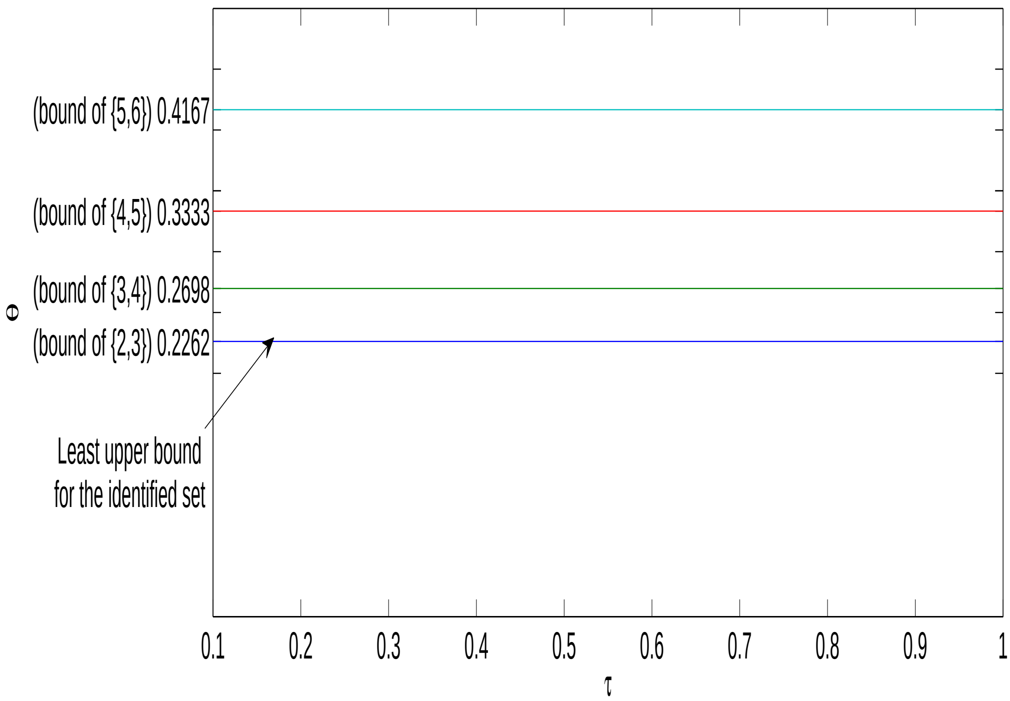
$$\left.\frac{\partial \pi_{s,n}}{\partial s}\right|_{s=1} = \frac{1}{1+\theta} - \frac{1}{1+\frac{\theta}{n-1}} \leq 0,$$

and that $\frac{\partial \pi_{s,n}}{\partial s}$ is decreasing with s. Hence, it should be obvious that $\pi_{s,n}$ is a decreasing function of s $\forall n \in \mathcal{N}$, $\forall \theta \in \Theta$.

The procedure of deriving $\Theta_I = [0.5, 1]$ simply follows that in Type 1 and thus is omitted. We depict the bounds for Θ_I in Figure A.2. Note that the greatest lower bound is also the common lower bound for θ for each pair of entrant numbers in Figure A.2.

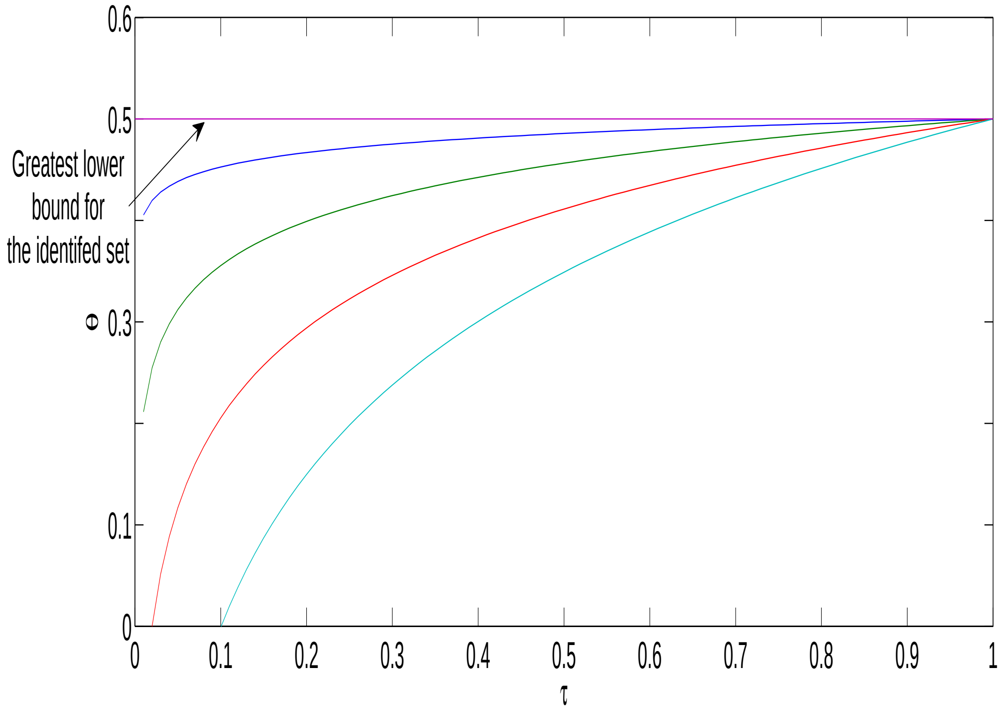
A.6 Graphing Bounds

Figure A.1: Bound for Θ_I in Type 1 DGP.



Note: Each line represents an upper bound for θ by considering a pair $\{n, n+1\}$. The intersection of sets identified by these upper bounds yields Θ_I.

Figure A.2: Bound for Θ_I in Type 2 DGP.



Note: Each curve represents a lower bound for θ by considering a pair $\{n, n+1\}$ ($n = 2$ to 5 from top to bottom), and the lower bound provided by each pair for θ is the same as 0.5, as needed uniformly across τ, which reveals the greatest lower bound for Θ_I.

Appendix B

Appendix to Chapter 3

B.1 Details of the Change of Variables

Let $b^{\dagger} = (b^{*} - p_0)^{0.5}$, and let $g^{\dagger}(\cdot)$ be the density for $b^{\dagger}$. It is easy to show that

$$V^{*}(\tau) = (b^{\dagger}(\tau))^2 + p_0 + \frac{2}{r(N-1)} \frac{(r\tau + 1 - r)b^{\dagger}(\tau)}{g^{\dagger}(b^{\dagger}(\tau))}. \tag{9'}$$

Note that

$$\begin{aligned}
&\frac{d}{d\tau}\left[\frac{(r\tau + 1 - r)b^{\dagger}(\tau)}{g^{\dagger}(b^{\dagger}(\tau))}\right] \\
=&\frac{[rb^{\dagger}(\tau) + (r\tau + 1 - r)(g^{\dagger}(b^{\dagger}(\tau)))^{-1}]g^{\dagger}(b^{\dagger}(\tau)) - (r\tau + 1 - r)b^{\dagger}(\tau)g^{\dagger\prime}(b^{\dagger}(\tau))(g^{\dagger}(b^{\dagger}(\tau)))^{-1}}{(g^{\dagger}(b^{\dagger}(\tau)))^2} \\
=&\frac{rb^{\dagger}(\tau) + (r\tau + 1 - r)(g^{\dagger}(b^{\dagger}(\tau)))^{-1}}{g^{\dagger}(b^{\dagger}(\tau))} - \frac{(r\tau + 1 - r)b^{\dagger}(\tau)g^{\dagger\prime}(b^{\dagger}(\tau))}{(g^{\dagger}(b^{\dagger}(\tau)))^3} \\
=&\frac{r\tau}{g^{\dagger}(b^{\dagger}(\tau))} + \frac{r\tau + 1 - r}{(g^{\dagger}(b(\tau)))^2} - \frac{(r\tau + 1 - r)b^{\dagger}(\tau)g^{\dagger\prime}(b^{\dagger}(\tau))}{(g^{\dagger}(b^{\dagger}(\tau)))^3}
\end{aligned}$$

So

$$V^{*\prime}(\tau) = \frac{2b^{\dagger}(\tau)N}{(N-1)g^{\dagger}(b^{\dagger}(\tau))} + \frac{2(r\tau + 1 - r)}{r(N-1)}\left[\frac{1}{(g^{\dagger}(b^{\dagger}(\tau)))^2} - \frac{b^{\dagger}(\tau)g^{\dagger\prime}(b^{\dagger}(\tau))}{(g^{\dagger}(b^{\dagger}(\tau)))^3}\right]. \tag{10'}$$

In estimation, we follow the procedures in Section 4, but replacing (4.9) and (4.10) by (9') and (10'), respectively.