Title
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Journal
Journal of Mathematics Teacher Education, 21(1)

ISSN
1386-4416

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Publication Date
2018-02-01

DOI
10.1007/s10857-016-9348-9

Peer reviewed

This is the accepted manuscript version of the article: Wilkerson, M. H., Bautista, A., Tobin, R., Cao, Y., & Brizuela, B. (In Press). More than meets the eye: Patterns and shifts in Middle School Mathematics Teachers' Descriptions of Models. To appear in *Journal of Mathematics Teacher Education*.

More than Meets the Eye: Patterns and Shifts in Middle School Mathematics Teachers' Descriptions of Models

Abstract. Though modeling is a popular topic in mathematics education, the field's definition of *model* is diverse. Less is known about what teachers identify as mathematical models, even though it is teachers who ultimately enact modeling activities in the classroom. We asked nine middle school teachers with a variety of academic backgrounds and teaching experience to collect data related to one familiar physical phenomenon, cooling liquid. We then asked each to construct a model of that phenomenon, describe why it was a model, and identify whether a variety of artifacts representing the phenomenon also counted as models during a semi-structured interview. We sought to identify: What do mathematics teachers attend to when describing what constitutes a model? And, how do their attentions *shift* as they engage in different activities related to models? Using content analysis, we documented what *features* and *purposes* teachers attended to over the course of the interview. When constructing their own model, they focused on the visual form of the model and what quantitative information it should include. When deciding whether particular representational artifacts constituted models, they focused on how those representations reflected the system under study, and whether those representations could help to further understand the system. These findings suggest the teachers had multiple understandings of models, which were active at different times and reflected different perspectives toward modeling. This has implications for research, teacher education, and professional development.

More than Meets the Eye: Patterns and Shifts in Middle School Mathematics Teachers' Descriptions of Models

Modeling is an important practice across science, technology, engineering, and mathematics (Imbrie, et al. 2004; NRC, 2012), and there have been international calls to integrate mathematical modeling into pre-collegiate instruction (e.g., OECD, 2013). But, *modeling* is an ill-defined construct. Colloquial use of the term "model" is broad. Even within the mathematics education literature, definitions of models and modeling differ dramatically (Blum & Niss, 1991; Borromeo-Ferri & Lesh, 2013; Hoyles, Noss, Kent, & Bakker, 2010; Lesh & Doerr, 2003a; Gravemeijer, 2002; for a review, see Sriraman & Kaiser, 2006).

Investigating what teachers *themselves* understand mathematical models to be can shed light on how modeling-related standards and curricula might be enacted in the classroom, and can inform the design of teacher professional development (PD). In this study, we explore: (1) what features and purposes of models teachers attend to when constructing a mathematical model or deciding whether something is a mathematical model, and, (2) how their attention to different features and purposes shifts as they engage in different tasks. We interviewed nine middle school mathematics teachers as they constructed a model of hot coffee cooling in a cup, and chose which of a variety of artifacts (graphs, verbal descriptions, diagrams, equations) represented models of that same coffee cooling situation. We find that teachers' ideas of models may be both more diverse and more flexible than some research has suggested, and highlight ways in which teacher educators may build on teachers' existing resources to elicit more robust knowledge about models.

Background

The term *mathematical model* is used in a variety of ways within the educational research community. A model can be a representation (such as the use of the number line to represent the real numbers), an ideal example of some mathematical method (a model solution), a learner's conceptualization or simplified understanding of a situation of interest (a situation model), or a description of a real world or theoretical system using the language and tools of mathematics.

There is reasonable agreement among mathematics educators (Blum, 2002; Kaiser, Blomhøj, & Sriraman, 2006) and current policy documents (e.g., OECD, 2013) that *mathematical modeling* is an iterative process that involves: identifying and mathematically representing the key components and relationships of a mathematical problem situation; employing mathematical or extra-mathematical methods and tools to obtain a solution, generate a prediction, or otherwise better understand that situation; validating results against situational data or knowledge; and sharing, generalizing, revising, or refining the model as needed. Detailed descriptions of this cycle vary (Blum, 2015).

Our Definition of Models and Modeling

In this paper, we adopt a broad conceptualization of *mathematical modeling* as the iterative development, testing, and refinement of mathematical descriptions of open-ended situations. Operationally, we explore teachers' understanding of *mathematical models* through the artifacts that emerge during or are employed as part of the process of modeling. This is, of course, a simplification: we view modeling as an ongoing process of sensemaking, and specific representations are just one of the tools learners may use to advance model-based inquiry. However, as Lesh and Caylor (2007) note: "it may be possible for a mathematical model to function purely within the mind of an individual learner… [but] in practice… mathematical models seldom have much power unless they are expressed using some (and usually several) type(s) of representational media" (p. 176). Examining modeling through the lens of representations reflects the problems, materials, tools, and knowledge a learner has available and chooses to mobilize during the process of modeling, as well as the modeler's purposes, decisions and priorities (Blum & Niss, 1991; Lehrer & Schauble, 2000).

Teachers' Understandings and Professional Development

In implementing modeling activities in the K-12 classroom, teachers need to be aware of what mathematical content, modeling approaches, and mathematical tools and methods are likely to yield productive pathways for learners faced with a given problem. The choices will differ depending on what students are interested in emphasizing in their model and on their prior knowledge. Teachers must also consider what learning objectives (with regard to mathematical content, mathematical practices, and the situation modeled) a given approach might highlight (Doerr, 2007).

One line of research into the knowledge of modeling that teachers bring to this challenging task has investigated what teachers themselves experience when working on specific modeling tasks. Verschaffel, De Corte, and Borghart (1997) found that teachers often provided solutions to word problems that did not connect to practical problem constraints (such as suggesting fractional numbers of indivisible objects). Erdogan (2010) found that primary teacher education students had trouble perceiving mathematical functions as tools that can be used to solve modeling problems. A series of studies has compared teachers' approaches against Blum and Leiß's (2007) modeling cycle (which specifies modeling as involving phases such as simplifying/structuring and mathematizing a real-world situation, or validating a mathematical model against real results) and found that teachers encountered blockages or focused only on certain steps (e.g., Blomhøj & Kieldsen, 2006; Soon & Cheng, 2013). Other studies have documented teachers' frustration with the open-ended and integrated nature of modeling tasks (e.g., Biembengut & Hein, 2010; Lingefjard, 2002, Ng 2013) and a preference for simpler modeling tasks over more complex ones (Kuntze, 2010).

A second line of research has investigated teachers' understandings of mathematical modeling in the context of long-term professional development programs (such as LEMA, Maaß & Gurlitt, 2011; PRIMAS, Maaß, Artigue, Doorman, Krainer, & Ruthven, 2013; COM^2 and DISUM, Blum & Leiß, 2007). These projects focus on pedagogical content knowledge and explore interconnections among teachers' mathematical modeling competence, classroom experiences, and pedagogical orientations. Borromeo-Ferri and Blum (2010) described how teachers' *mathematical thinking styles* — their preference for analytic, visual, or integrated approaches — were related to which phases of the modeling cycle those teachers emphasized when working with students. Kaiser and Maaß (2007) found that teachers' willingness to enact modeling activities in their classrooms is related to their beliefs about the nature of mathematics as a sensemaking process. Similarly, teachers' beliefs about teaching as *transmission* or as supporting students in a *learning process* can influence their enactment of modeling activities (Maaß, 2009).

A third line of research has more directly investigated teachers' definitions of mathematical modeling. Some, using questionnaire items (such as "I could explain what is happening in the different phases of mathematical modeling"; Kuntze, 2011, p. 286) or self-reports during PD (Borromeo-Ferri & Blum, 2010), suggest teachers have impoverished understandings of models and modeling. Others suggest these understandings may be more robust. Ärlebäck (2010) found in an interview study that teachers' beliefs about mathematical models and modeling changed as they spoke about particular problems, such that they were inconsistent and even contradictory over the course of the interview. Our own work (Bautista, Wilkerson-Jerde, Tobin, & Brizuela, 2014) revealed that teachers held diverse notions of mathematical models that were not necessarily consistent, even across questions related to the same modeling task.

Research that explores how teachers manage mathematical modeling within the context of their own classrooms using ethnographic and mixed methods (Lesh & Doerr, 2003b) supports this more flexible characterization of teacher understanding. In contrast to a view of teachers' ideas as static and classifiable, these studies reveal that teachers successfully navigate tensions between modeling, mathematical, and pedagogical issues and flexibly adapt their instruction to student needs (de Oliveira & Barbosa, 2013; Doerr, 2007). Chapman (2007) found that some teachers engage their students deeply in specific aspects of modeling using word problems, even if they do not engage in full-fledged modeling activities. Doerr and English (2006) found that teachers who led in-classroom modeling activities learned new mathematical content and shifted toward a more student-centered pedagogical approach over the course of a lesson. Similar shifts in teacher knowledge resulting from practice were documented in Barbosa (2001) and Holmquist and Lingefjärd (2003). Maaß (2011) found that teachers who recognized the real-world relevance and pedagogical utility of mathematical modeling intended to enact those approaches in their classrooms, even if they found the activities difficult and time consuming.

Contributions of the Current Study

We seek to better understand these dynamic aspects of teachers' knowledge about modeling in a way that is not tied to specific classroom enactments. We do so through the grounded and systematic documentation of what features and purposes of models teachers attend to while engaged themselves in different modeling-related tasks over the course of one think-aloud interview.

Theoretical Framework

Such an approach warrants a theoretical framework that is focused on the detailed description and analysis of individual knowledge. Here, we leverage a "knowledge in pieces" (diSessa, 1993; Wagner, 2010) or "conceptual ecology" perspective (diSessa, 2002; Posner, Strike, Hewson, & Gertzog, 1982). This perspective examines knowledge as an assemblage of "resources" (Hammer, Elby, Scherr, & Redish 2005) that are activated and loosely connected to one another in different ways at particular moments in time. As situations that require particular collections of resources become more common, those become more tightly connected. However, resources within a given collection can be re-arranged as new situations arise. This forms an ecology over time, so that different contextual cues—settings, representations, social circumstances—yield different in-the-moment understandings of a situation.

Focusing on resources as the *building blocks* of knowledge can provide insight into how understandings are structured and coordinated over time in ways that stage-like, categorical, or static descriptions of knowledge cannot (Smith, diSessa, & Roschelle, 1992). This perspective has been leveraged in the mathematics education literature to explore phenomena including transfer (Wagner, 2010), abstraction (Hoyles, Noss, & Pozzi, 2001), and the process of modeling a physical situation (Iszak, 2004). Similarly, it can lend insight into why some research finds teachers to be underprepared to teach modeling in laboratory studies while others find them to be adept their own classrooms, and why some find teachers' knowledge to be relatively stable while others find changes and contradictions even within the same interview or activity session.

We conjecture that certain situations are likely to activate particular clusters of resources that correspond to different characteristic understandings of models and modeling (*patterns*): for example, as a way to emphasize particular mathematical concepts or procedures (what Kaiser & Maaß refer to as having a "schematic-oriented understanding of mathematics"; 2007, p. 9); a way to offer students multiple ways to work on a problem ("process-oriented"; 2007, p. 9); or as involving only particular phases of a larger cycle (Blomhøj & Kieldsen, 2006; Soon & Cheng, 2013). Furthermore, we conjecture that while these characteristic understandings may appear stable as teachers engage in and talk about a particular task, the same teacher can also *shift* dramatically in his or her understanding across contexts. In other words, an individual teacher may have multiple, apparently distinct interpretations of modeling that draw from the same set of resources, but manifest differently across tasks.

Exploring the nature of these interpretations and the conditions under which they are activated can serve as a first step toward understanding how teachers' interpretations can become *integrated* into a more robust conceptualization of modeling. Pratt and Noss (2002), for example, used a resources perspective to track how a learner brought together multiple understandings of 'randomness' (as *unfair, irregular, unpredictable,* or *unsteerable*) to develop a coherent, robust understanding of the law of large numbers. Here, we attempt to elicit and document the patterns and shifts in teacher's understandings of mathematical modeling:

1. *Patterns*: What features and purposes of models do teachers attend to when constructing a mathematical model, or deciding whether something is a mathematical model?
2. *Shifts*: How does teachers' attention to different features and purposes shift as they engage in different tasks?

Method

We conducted semi-structured clinical interviews (Ginsburg, 1997) with nine middle school teachers. Interviews allow researchers to analyze the nature of participants' thought processes, including the understandings they bring to bear on a problem and the ways in which those understandings are structured, coordinated, and shift over time and in response to probing questions.

Participants

Participants were selected from a cohort of 56 grades 5-9 public school mathematics teachers enrolled in a PD program in the Northeast of the United States (US)[1]. We began by recruiting small groups of 4 to 5 teachers who had backgrounds in mathematics, mathematics education, science, or humanities/social sciences respectively. As teachers responded, we contacted additional participants from less represented backgrounds as needed. We intended to recruit 2-3 participants from each background, but more teachers in the PD program had backgrounds in the social sciences or humanities than other fields, and these teachers were more willing to participate in the study. We recruited nine participants whose experience teaching mathematics ranged from 1 to 28 years, and with a diversity of academic backgrounds, though mostly from the humanities and social sciences (see Table 1).

[INSERT TABLE 1 HERE]

Setting

Interviews were conducted after the conclusion of the PD program. The program consisted of three graduate-level, semester-long "hybrid" online and face-to-face courses focused on mathematical content and students' mathematical thinking and learning (Teixidor-i-Bigas, Schliemann, & Carraher, 2013; https://sites.tufts.edu/poincare/). The main goal of these courses was to foster teacher learning, rather than specifically to research teachers' ideas about mathematical models. One cross-cutting theme of the program, however, was the importance of exploring connections between mathematics and science. The PD program often asked teachers to build or analyze mathematical models and solve applied problems involving quantities (e.g., weight, cost, area, length, time).

It is likely that this PD program influenced the participant teachers' ideas about modeling and representations. Still, there was wide diversity in what they identified as mathematical models. The interview task was not typical of the course activities participants completed. For example, teachers had not been asked to predict trends or collect and model empirical data. We reassured all participants that the interview was not an assessment. None of the interviewers had served as primary instructors for the participants, and participants were informed that their responses would not be shared with their primary instructors.

Interview: Topic Selection

Our interview focused on liquid cooling—specifically, what happens to the temperature of hot coffee left on a table to cool (Rees & Viney, 1988)—because it features "everyday" science and clear mathematical patterns. We hoped an interdisciplinary task might elicit a more complex and robust set of understandings about modeling from teachers than a purely mathematical prompt. We expected the prompt's scientific themes might inspire teachers to also consider how *scientific* models might inform their treatment of *mathematical* models.

We also expected the familiar nature of this task to elicit multiple ways of approaching the problem. Coffee cooling involves many factors such as what container the coffee is in, whether the coffee is stirred, or how ambient temperatures may affect the process of cooling. It exhibits a non-linear mathematical relationship that we expected would be mathematically challenging, yet at least qualitatively accessible. Also, this problem would allow us to explore the participants' relative attention to the empirical or data-driven versus theoretical nature of modeling. Liquid cooling can be explored easily at home by collecting and analyzing data to develop an empirical model, but is also likely to connect to rich experiences and prior knowledge about liquid cooling which could inform a theoretical model.

Interview Protocol

We conducted interviews with each teacher at a time and location that was convenient for them. Each interview lasted around 40-60 minutes, and was videotaped with two cameras positioned to capture participants' gestures and written work, and the participants' face and gestures, respectively. Four of the authors of this paper conducted the interviews, and had been involved as designers and/or facilitators in the PD program, though we avoided pairing of participants and their course facilitators. Audio from each interview was transcribed to aid analysis.

Our interview protocol was designed through an iterative process that lasted several months and included piloting preliminary versions of the interview with three pre-service student teachers. Our final interview sequence was as follows. First, we emailed each participant and asked him or her to predict what would happen to the temperature of hot coffee that is left on a table. After teachers responded, we sent them a thermometer by mail and asked them to test their prediction by carrying out an exploration. We asked them to take notes to bring to the face-to-face interview. We were purposefully vague regarding what annotations or inscriptions they could produce during the exploration.

During the face-to-face interview, we asked teachers first to describe what they noticed during the exploration, to share and explain any annotations and inscriptions they had made, and to describe how they would communicate their findings to others. Next, we asked them to construct a model of what they noticed, and asked follow up questions about the model they produced. This was the first introduction of the word "model" during the interview. Finally, we presented the teachers with a collection of representational artifacts (graphs, diagrams, verbal descriptions, mathematical equations, etc.; see Table 2). We asked them to sort these into what they believed did or did not represent models of coffee cooling, and to explain their reasoning.

This sequence of tasks was designed to provoke the shifts in teachers' ideas about models that are the focus of our second research question.

We center our analysis on the interview tasks focused on constructing and sorting models (Parts 2 and 3; the full protocol and the initial email text are included in Appendix A). We found that these tasks produced teachers' most intentional descriptions of what models are and what they are for, and displayed the clearest evidence of shifts in the features and purposes of models to which teachers attended.

[INSERT TABLE 2 HERE]

Analysis

We analyzed interviews through a collaborative, iterative, bottom-up process (Chi, 1997). The first step focused on describing in detail what teachers attended to when constructing or sorting mathematical models. We tagged and developed descriptors for each criterion participants cited when constructing a particular type of model, and when sorting the representations we provided.

We then consolidated the descriptors and looked for themes across them. We found that each descriptor identified particular *features* or *purposes* of models. *Features* were particular ways teachers identified what a model should be (such as visually appealing) or show (such as data points or mathematical trends). *Purposes* were ways teachers identified they might use a model (for example, to make predictions or calculate expected values). As an example of the distinction, one participating teacher, Jacob, when evaluating a graph to determine whether it represented a mathematical model, stated, "It's not super accurate. I don't know at four minutes, I don't know if that's really that temperature. Sort of a rough model. But I still call it a model." This statement was tagged as evidence that Jacob was attending to a particular *feature,* specifically *accuracy*, in deciding whether to call the graph a model. When evaluating another graph, Jacob said, “This looks like a graph of the person's original data. They haven't made predictions… I don’t see anything indicating that there's, that we could extrapolate additional points or uh look for values in between. So uh that looks like somebody has just recorded their data, it doesn't look like they've uh, they've represented their data but it doesn't look like they've modeled it.” This statement was tagged as evidence that Jacob was attending to a specific *purpose*, that models are used to *make predictions or comparisons*, when deciding whether the graph represented a model. We provide explanations and examples of all descriptors in the next section.

Informed by the theoretical perspective described above, we recognized this emergent grouping of teachers’ attentions toward *features* and *purposes* as representing patterns in teachers’ activation of conceptual resources that inform their perspectives toward mathematical modeling. The division between features and purposes is not sharp, and these two types of attention often co-occurred. Nevertheless the distinction offered us traction to explore our two research questions, and shed light on both the complexity and diversity of teachers’ understandings as well as the regularity with which we identified teachers’ shifts towards a generally richer, and often more contextual and process-oriented conception of models during the sorting task.

For each participating teacher, we coded portions of the interview corresponding to the construction and sorting tasks for the presence or absence of each feature and purpose descriptor. One author conducted an analysis of all nine participants. A second author conducted an independent analysis of three randomly selected constructing and three randomly selected sorting excerpts (one third of our total coded data). Agreement on the presence or absence of each available code for each excerpt was 83%, and rose to 92% after discussion between coders.

Agreement on presence only was 76%, and rose to 89% after discussion. We describe coding in more detail in Appendix B.

Findings

We report our findings in three parts. Part I describes the specific *features* and *purposes* that we identified most frequently, and provides examples of each. Part II presents three case studies representative of the *patterns* that emerged in our coding analysis, and that illustrate different degrees to which those patterns *shifted* across different parts of the interview. Finally, Part III reports more briefly on our analyses of the remaining six participants, and reviews cross-participant trends during the interviews.

Part I: Identifying Features and Purposes of Models

We identified seven features and four purposes that were explicitly attending to by at least four participants when exploring or justifying whether a given representation was a model.

Features: What models are and what they look like. We identified as a *feature* any quality or property of a representation that teachers attended to when describing what constitutes a model at any point during the interview. A heuristic for identifying a feature is whether it was described as something a model (representation) should or should not "have," or a way a model should or should not "be." Table 3 lists all the features we identified that at least four participants attended to during the interview:

1. Is a *Visual* representation that uses iconography or spatial arrangement to communicate data, trends, or relationships;
2. *Includes Trend* or underlying quantitative relationships exhibited by the system;
3. *Includes Data* collected from the system under investigation;
4. Is *Easy to Interpret* by its intended audience because it is conventional or clear;
5. Includes a Description of the Situation modeled;

6. Is *Accurate* with respect to the data; and
7. *Includes Factors* that influence or may influence quantitative outcomes.

[INSERT TABLE 3 HERE]

Purposes: What models are for and how we use them. We identified as a *purpose* any intended or potential role that teachers suggested a model could or should serve. A heuristic for identifying a purpose is whether it was described as something a person should or could "do" with a model, or what function it could serve for someone who constructs or works with it. We identified four different purposes for models that at least four participants attended to, featured in Table 4:

1. *Explore Causal Relationships* that underlie the quantitative patterns observed;
2. *Explore Trends* that can be extrapolated from the data or theorized about the system;
3. *Make Predictions* or *Comparisons* about what would happen in future or different situations; and
4. *Generate Values* that approximate or predict the quantitative data exhibited by the system.

[INSERT TABLE 4 HERE]

Part II: Case Studies of Shifts in Attention

We interpret the specific features and purposes identified in the last section as resources for reasoning about modeling. In this section, we illustrate and describe *patterns* in how those resources were leveraged by three of our participants, and *shifts* in those patterns across the constructing and sorting tasks. Sophia shifted her attention more than any other participant, attending to more contextual features and purposes when sorting versus constructing. In contrast, Olivia's attention remained relatively stable during both tasks. Jill shifted only a moderate amount, expanding her attention to include a number of potential purposes for models during the sorting task. Together, these cases illustrate both the diversity we found in participant responses during the interview, as well as some patterns in how participants' attention to different features and purposes expanded as they moved between the tasks.

Each excerpt is annotated to identify features and purposes that were tagged within each segment of text. Text in the transcript that corresponds to a given descriptor code is underlined, and the corresponding code is listed directly to the right. For example, in the first excerpt, the text "And it's meant to represent what's going on" is identified as participant attention to the *feature* (F) *Includes a Description of the Situation*. Similarly, *purpose* codes are identified with a (P) followed by the specific code.

Sophia: Dramatic Expansion of Attention. Sophia had been teaching sixth grade for six years at the time of our interview, and had been teaching mathematics for only 1 year. When we asked her to create a model of the coffee cooling, she sketched a graph representing exponential decay (Figure 1).

[INSERT FIGURE 1 HERE]

When we asked Sophia to describe why what she constructed was a good model of coffee cooling, she emphasized its visual and representational nature:

S: So usually when you think of a model, like a model house, or a model car, it's smaller than in real life. And it's meant to represent what's going on. Um… so this [constructed graph] represents, um, high temperature and low temperature by using a visual model if higher things [data points] being up on the page [higher on the graph] and lower things being down on the page. So the... yeah, the numbers are represented with the height, kind of model, I guess.	*F–Incl. Description of Situation* *F-Visual*

Sophia's description of the model as fundamentally a visual representation persisted even when the interviewer asked Sophia if she would make any changes to the model in order to make it predictive. She suggested that the graph could be manipulated, for example by adjusting the initial height of the graph to different temperatures. When prompted about whether there were any other ways to develop a predictive model, Sophia responded, “I don’t think of anything else I would add to the graph.” These responses suggest a view of models based on (predominantly graphical) features they should possess, rather than their purposes.

However, during the sorting task, Sophia attended to many new features and purposes when deciding whether each representation provided should be considered a model. Instead, she focused on whether those representations could be used to understand or reconstruct the “big idea” – what she later described as the underlying quantitative relationship between time and coffee temperature. In the excerpt below, Sophia had identified two algebraic formulae as constituting models, but had excluded the verbal description. The interviewer asked her what the difference was between these representations.

S: Um… the difference is that [algebraic formula] doesn't summarize sort of the conclusions that you would reach based on the formula. And this one [verbal description] does. This [verbal] one is like a summary of what you should be getting the big main idea from the experiment.	*F-Includes Trend*
That's missing on these [algebraic formulae]. These are just series of steps, and they are telling you what each variable represents.	*F-Includes Factors*
They are saying use these steps, and it will work. But it doesn't say what will happen if you use these steps.	*F-Includes Trend*
I: So, and still these two [algebraic formulae] are models, and that one [verbal] is not. Right?	
S: Yes. Because if you use this [algebraic] model to generate your data table and your graph, you would have a sense of how they interact. And you should be able to get this big idea. So this [algebraic formula] is a model because I could just start with this and this is the only piece of paper I have, and I could get this big idea. But I would have something to get there.	*P-Generate Values* *P-Explore Trends*

In contrast to her previous emphasis on graphing and visual salience, here Sophia identified the ability to generate and explore quantities and the interactions between them as a goal of models and modeling. This new treatment of modeling existed alongside her previous emphasis on visual salience; when Sophia finished sorting the representations into “models” and “not models”, she further divided those identified as “models” into two subgroups:

S: Alright. Um… uh… er… this group is all visual. It all has, either has pictures or has some way of organizing the data for a visual. So even with the table, even though it doesn't have pictures, it's showing visual order. Um... these [algebraic formulae] don't show as much visual order. Um... but they are still models. Because you can get, because it does eventually communicate to the big idea. Each of these [formulae] by themselves could help you get to this big idea. Except I don't like that coffee cup [diagram], coz it doesn't have quite all of it. It just tells you that it's cooling. That's all. It doesn't get to everything.	*F-Visual* *P-Explore Trends*

This alternative way of describing models seemed directly prompted by Sophia's exposure to algebraic formulae during the sorting task, given the specificity of her justification for sorting those particular representations as models. While this short case analysis emphasizes Sophia's new attention to the ability to generate quantities and explore trends, we found a number of additional conceptualizations of models in Sophia's overall interview. This shift is reflected in Table 5, which summarizes the variety of new features and purposes she explicitly mentioned during the sorting task that she had not mentioned previously—indeed, more new attentions than any other participant.

Olivia: Relative Stability of Attention. Olivia had been teaching middle school mathematics for seven years. In preparation for the interview, she brought in a table of data she recorded. After discussing the activity with a colleague, she decided to also construct a graph to bring to the interview (Figure 2). When we asked Olivia to construct a model during the interview, she stated that the graph she brought should be considered a model. When asked why, she emphasized that it clearly and accurately communicated data and quantitative trends.

[INSERT FIGURE 2 HERE]

O: I guess in some way you could it’s, it explains what the table is and I think it's easier to understand and see that, what you predicted about the coffee doing and what actually happened. I think it's easier to see it this way in the graph than it is to see it in the table. Or it's easier because you can take a quick glance and say oh yea, okay that's what it's doing. Whereas here [table], well wait a minute what are you doing, it's a little bit harder to keep track of. I: Okay. So you have to do more work reading a table than just seeing a graph. O: Yep. I: Okay. Great. Um, next question is how could you decide whether this is a good model? O: Um, I don't know. Um. I guess if you're, I guess a good model would accurately display your data.	*F-Easy to Interpret* *F–Accurate* *F–Includes Data*

During the sorting task, Olivia continued to focus on the features she described as important when constructing and describing her own model of coffee cooling: whether they displayed data, were expressed as pictures and graphs, or made evident a quantitative trend.

O: Um, let's see I'm going to say that this [indicating Newton's Law, verbal description, instantiated formula, data table] is more, I guess I think of models as numbers and pictures and this [verbal] is more words I guess, but I don't know I guess the table could be a model [puts table to side]. And then, I don't know if I'd put that [mug picture] there. This [physical diagram] is a model but I guess it doesn't really represent what's happening at certain times, it just makes sense yea, of course heat is going to come off the cup and come off both sides and everything else. A model [system dynamics diagram], but definitely more of a scientific, you'd definitely have to look at that to understand what it's saying a little bit. And I guess I'll put the table over here with the [places in “not models” group] um, I don't know what to call it. Words.	*F–Includes Data* *F–Visual* *F–Includes Trend* *F–Easy to Interpret*

The only time Olivia attended explicitly to the *purposes* of a given representation was toward the end of the sorting task. She identified two representations, the physical diagram and the system dynamics diagram, as “scientific” and therefore a special type of model. When probed further, Olivia noted that these two representations could be used to understand or explain the scientific phenomenon, even if they were not easily accessible to her own students.

O: I think if you were trying to explain the cup of coffee and the rate of loss of heat and temperature, I think these things [graphs] would speak to them a little bit more. This [mug] they might not really understand what they're doing with it. They might probably think these [indicating dots on mug image] are like ants or something. But they're in sixth grade, so you know that would make sense. But I think the rest of this [graphs], I think this they would comprehend easily.	*P–Explore Causal Relationships* *F–Easy to Interpret*

These patterns are reflective of a more general stability we found in Olivia's approach toward models throughout the interview. During the sorting activity, Olivia did explicitly mention a purpose that she hadn't before – that models can be used to explain the causal relationships that underlie the cooling trend exhibited (Table 5). However, this emerged as a post-hoc justification for her sorting decisions. Other representations that could also be used to understand the scientific system, such as the verbal description, were dismissed because they lacked certain features, such as numbers or a visual emphasis.

Jill: Moderate Expansion of Attention. Jill had been teaching for four years, and teaching mathematics specifically for three years. She arrived for our interview with a data table and graph (Figure 3). While describing the patterns she identified, Jill also explored whether she could develop an algebraic representation of the trend. She quickly abandoned this approach, indicating dissatisfaction with a linear approximation, so that the interview could move forward.

[INSERT FIGURE 3 HERE]

When asked to create a model of what she noticed during the coffee cooling exploration, Jill asked, "a model different from a graph?", indicating (like Olivia) that she viewed the graph she had already produced as a model. She noted that she had her students in mind, something that she continued to emphasize.

J: [Points at graph] This is the way that I'd do it and that I would expect my sixth graders to do it as well. I: Okay. And why is that? J: Um, because <u>I think it's pretty conventional, I think most people understand what it means</u>, and it <u>shows all the data</u>. I: Okay. J: <u>Graphically</u>.	*F–Easy to Interpret* *F–Includes Data* *F-Visual*

Later, when asked whether she could think of any other possible models she would like to generate to represent what she noticed, Jill still emphasized that models should describe data. However, more than Sophia or Olivia, Jill acknowledged early on that different audiences, uses, and data types could affect what counts as a model in a given situation. This is consistent with her early attempts to generate multiple representations of her data, and later identification of a single representation – the graph – as appropriate for her own students.

J: Um, I think it depends on who your audience is and what the, <u>what we're trying to communicate through the data</u>. Um, I don't know if in other circumstances if it was a percentage then maybe a circle graph would be more appropriate, or <u>depending on the data you're trying to communicate</u>. I think it all depends on the model that you want to use.	*F–Includes Data*

Although there are no specific purpose codes identified in the last interview excerpt, it is clear that Jill was already attending to the purposes that models could serve. When she started sorting the representations we provided, Jill attended to these purposes closely.

J: [Newton's Law] Um, I don't quite understand Newton's Law of Cooling, <u>but I think it's a model to explain</u>. I think that's why I had a hard time trying to come up, is that what I was supposed to come up with? I: No. You weren't. Don't worry. It took Newton to come up with that! But you would call it a model? J: Yes. [instantiated formula] Um, yes, I would call this a model. I guess, um, I'm defining model for myself as <u>something that we could either display data</u> or what's the word I'm looking for, <u>something like get data from so if I plugged in a certain temperature or minute, that I'd be able to figure out a certain output</u>.	*P–Explore Causal Relationships* *F–Includes Data* *P–Generate Values*

Like Sophia, Jill began attending to several new purposes for models during the sorting task. However, whereas Sophia also began attending to new features of those representations, Jill attended to fewer features in general, and only those that she had previously attended to, such as whether the representations displayed data, were easy to interpret, or were visual. Instead, Jill's attention shifted to the many potential uses for each representation, even if she ultimately decided those representations should not be included as models.

Part III: Themes Across Interviews

The similarities and differences evident in Sophia, Olivia, and Jill's interviews were echoed in the other six interviews. Table 6 presents the descriptors we identified during the construction and sorting tasks for each of the remaining participants, with characteristic excerpts. There is an evident trend toward an *expansion of attention* during the sorting task. The teachers leveraged more, and more diverse, resources when deciding whether artifacts they were presented with could constitute models than when they constructed their own. This expansion of attention systematically reflected an increase in participants' *focus on purpose*, and *focus on context-related features* (describing the situation, including relevant factors) during the sorting task. This expansion of attention is evident in the different and more purposeful descriptions of mathematical models and modeling offered by teachers during the sorting task of the interview.

[INSERT TABLE 6 HERE]

It is not too surprising that we saw a shift in what teachers considered to be a model during the sorting versus the construction task – it makes sense that having diverse examples can provide teachers an opportunity to reflect and expand upon what they considered to count as a model. More interesting were the specific patterns in the expansion of what it is teachers attended to. We observed a shift in participant attention toward *purposes* versus (or in addition to) *features* of models during the sorting phase. During the construction task, only four participants mentioned specific purposes when justifying why their constructions reflected a mathematical model. During the sorting task, all nine participants mentioned specific purposes.

Even those participants who had attended to specific purposes during the construction task attended to at least one new purpose during the sorting task.

In contrast, many participants attended to fewer or the same set of *features* when sorting versus constructing models. When teachers did attend to new features during the sorting task, those features involved context setting rather than generic features of models. In other words, when constructing, many teachers attended to whether particular representations were visual, included data, or were easy to interpret throughout the interview. However, when sorting, many only attended to specific aspects of the modeled context, such as whether the representation described details about the situation to be modeled, or described particular parameters related to that situation. Table 6 reveals some consistency in these expansions across participants.

Shifts in use of the specific descriptors we identified correspond to qualitative shifts in teachers' more general descriptions of models. Many participants expressed an understanding of models and modeling as a way to communicate data and trends during the construction task. Jacob, Liam, Audrey, and Heidi illustrate this trend. However, during the sorting task many participants offered several different descriptions of models and modeling. For example, Heidi began to distinguish between models of data and situations. Violet described how different representations (all of which she characterized as models) describe processes, data, or specific details about the situation, and Audrey expressed her understanding of models as representations that allow her to make predictions about the behavior of coffee temperature.

Discussion

Our broad motivation for this study was to better understand what some middle school mathematics teachers understand to be models, given the varied colloquial and technical uses of the term both within and beyond mathematics education. We analyzed what teachers *attended* to when deciding whether a particular mathematical artifact counts as a model or not, as a window into their understandings of models and modeling more generally. Rather than describing teachers' understandings in terms of existing frameworks or definitions of modeling, we drew from a knowledge-in-pieces/conceptual ecology perspective to document the complex, and often changing, notions of modeling that teachers brought to bear as they engaged in different aspects of modeling practice.

We found that teachers leveraged a number of resources to think about models and modeling at different points in time; these resources fell into one of two general types. Model *features* concerned how models should look, what type of information they should include, and whether they are interpretable by a particular audience. When teachers attended to features, they spoke about models as artifacts that demonstrate or communicate mathematical data or trends. Model *purposes* concerned how models can be used to explore new mathematical relationships, understand scientific processes, or make predictions. When teachers attended to purposes, they decided whether or not a given artifact represented a model based on whether it might aid someone in the process of comparing, making decisions, or better understand a situation.

That we found teachers attending to these two particular aspects of models supports existing research that has shown teachers often approach or think about modeling tasks through a particular orientation or lens. However, in our study, *all* participants demonstrated elements of understanding that included aspects of *both* of these orientations at different points in time. In other words, they granted at least some attention to models both as static, demonstrative objects and as tools for sensemaking. Indeed, we found that teachers attended to different combinations of features and purposes throughout the interview. This echoes Arleback's (2009) finding that teachers' understandings of modeling cannot be neatly categorized into categories of beliefs.

Furthermore, despite considerable diversity among participants, certain types of shifts in attention happened consistently as teachers moved from the constructing to sorting tasks. When constructing models of data they had collected, many teachers focused primarily on features of models: such as that they should include data, include important trends, be visual, and be easily interpretable. These resources led teachers to express understandings of models and modeling as a way to demonstrate or communicate data and mathematical ideas. However, when deciding whether a wide collection of artifacts represented mathematical models, teachers attended to what purposes a model serves, and they argued that models should be contextualized and reflect important aspects of the situation under study. Some participants like Heidi, Jacob, and Walter explicitly described multiple model types or functions. Some, like Liam, explicitly noted that being confronted with a large collection of artifacts caused them to reflect upon and expand their own definitions of models.

The diverse and dynamic character of teachers' understandings unfolded both across and within participants. No two participants cited the same collection of features and purposes; and all teachers shifted in what they attended to during different tasks within the interview. This suggests that studies of teachers' understandings of models that seek to describe a singular orientation or preference, or document teachers' performance on a particular task, only illuminate one dimension of a complex territory of knowledge. It may also help reconcile research that focuses on teachers' performance on and preferences for modeling tasks, often concluding that teachers are unprepared and unwilling to enact activities in the classroom, with other research that suggests that teachers can in fact enact these activities successfully or in piecemeal, yet productive, ways. Just as work has been done to explore differences between existing frameworks and definitions of modeling (e.g., Kaiser & Sririman, 2006; Maaß, 2010), more work should be done to investigate the ways in which *teachers'* modeling knowledge may or may not align with those frameworks and definitions.

The changes we observed may have been prompted by the nature of the tasks themselves, the amount of time participants spent talking about models in the PD program, or by participants' expectations about what the tasks were designed to do. During the initial tasks, teachers may have focused on *demonstrating* mathematical knowledge, either to us as affiliates of their PD program or to their own imagined students. However, when they saw the diversity of models we shared with them during the second phase of the interview, they seemed to recognize that our interests were not, in fact, about specific topics from the course or their knowledge of relevant educational standards. Regardless, our findings suggest a complexity to teachers' understandings of models that deserves more attention and grounded analyses.

Limitations and Future Work

Our study was a small, exploratory examination of teachers' understandings of models. Our study also involved a very specific type of modeling task, and our interview procedure relied on the use of mathematical artifacts and representations which were intended to serve as a lens into teachers' ideas about models and modeling. This close coupling of representation and modeling is difficult to disentangle in our results. However, models are inextricably linked to and enacted through representations, especially when considering mathematical modeling in classroom practice.

Because we were interested in teachers' existing definitions of models, we focused on descriptions for models throughout the interview. However, an obvious next step would be to explore teacher thinking during the sorts of extended, iterative modeling activities that we hope they would implement in their own classes. It is likely that teachers' understandings would shift yet again in such contexts, and would be influenced by classroom context, student needs, and resources available (Doerr, 2007; Lesh & Doerr, 2003b;) as well as by administrative and policy-level pressures (e.g., Maaß, et al. 2013). Additional work is needed to understand how to access teachers' "untapped" knowledge about the nature and purposes of models in ways that translate into improved teaching practice.

Implications for Teacher Education and Professional Development

There are many calls to strengthen teachers' understandings of modeling, in particular to emphasize the role of modeling as a purposeful, exploratory meaning-making endeavor rather than a pursuit of a particular algorithm or result (Doerr, 2007; Kaiser & Maaß, 2007). Our findings suggest that teachers' understandings of models as sensemaking tools can be easily accessed even if teachers initially describe models in a static way. Specifically, we found that exposing teachers to a variety of models from mathematics and from related fields prompted them to more directly consider how models are connected to the problems for which they are developed, and how they are used to make progress in understanding not only the quantitative dynamics of a system, but also the causal relationships that drive them.

The patterns we found in what teachers prioritize about models – specifically, their visual nature and their close connection to data and mathematical trends – also have implications for the design of teacher education and PD programs. Our data suggest that teachers' understandings of mathematical models are more sophisticated than can be represented as a sequence or trajectory. Even as teachers' views of models as visual objects to describe data may be relatively stable in ways identified by the literature, these views may be complemented with others that may not be as stable or well-articulated, but are productive for modeling practice and teaching mathematical modeling. It may make sense to grant careful attention to how teachers describe models, with the expectation that new and powerful understandings can be revealed as teachers work across different modeling tasks, model types, and pedagogical considerations.

Conclusions

Although mathematical modeling is becoming an increasingly important component of international mathematics curricula and standards, little is known about what teachers themselves consider to be models. Building on theories of knowledge that emphasize the contextual and dynamic nature of knowledge, we hypothesized and confirmed that teachers were likely to exhibit *different* understandings of what a mathematical model is and what it is for when engaged in different tasks. These different understandings reveal teachers' knowledge of mathematical models as a system of interconnected, at times implicit, knowledge that can and should be accessed and built upon by educators.

Acknowledgements

This research was supported in part by the National Science Foundation, Grant #DRL-0962863. We would like to thank Ken Wright, the anonymous reviewers, and the Editor of JMTE for their feedback on prior versions of this manuscript. Findings presented in this paper represent the work of the authors and not necessarily the funding agency, colleagues, or reviewers.

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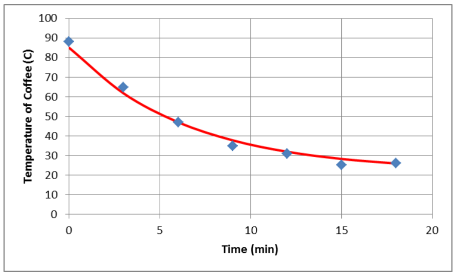
Table 1. Participant details

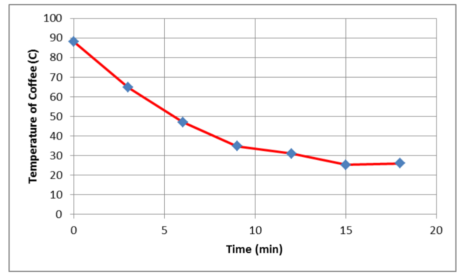
Name	Grade(s) Taught	Years of Teaching	Years Teaching Math	Academic Background
Audrey	5	15	12	BS Education
Heidi	6 & 7	9	9	BS Math Education ME Elementary Education
Jill	6	4	3	MA Special Education
Jacob	9	5	5	BA Psychology
Liam	7	11	11	BA Political Science MA Secondary Social Studies
Olivia	6	7	7	BS Business Administration
Sophia	6	6	1	BA Religious Studies MA Theological Studies
Violet	6	28	28	MA Middle School Science
Walter	8	6	6	BS Mathematics

Table 2. Representations of coffee cooling problem used for sorting task

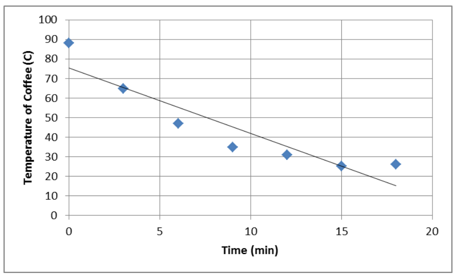
Scatterplot – Exponential



Scatterplot – Connected



Scatterplot – Linear



Scatterplot

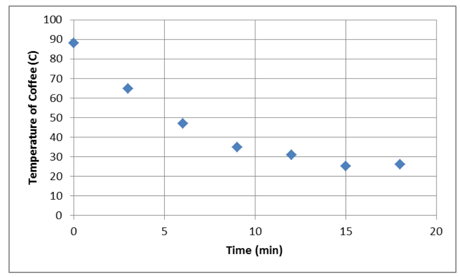


Diagram – Physical

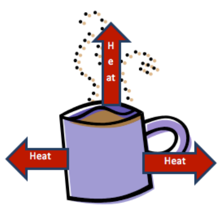
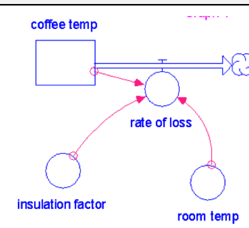


Diagram – System Dynamics



Formula – General*

Newton's Law of Cooling
U(t) = T + (u0 – T)ekt, k<0
In this formula, T represents the temperature of the surrounding medium, u0 is the initial temperature of the heated object, t is the length of time in minutes, k is a negative constant, and u represents the temperature at time t.

Formula – Instantiated*

The formula that represents the temperature of the coffee over time is given by , where T is temperature in degrees Celsius, and x is time in minutes.

Data Table

Time (min)	Temperature (C)
0	88
3	65
6	47
9	35
12	31
15	25
18	26

Verbal Description

The temperature of the coffee will fall exponentially until it reaches room temperature. The rate at which it falls will be affected by a number of factors:
Coffee inside a cup with more insulation will cool more slowly. Coffee with more surface area exposed to the air will cool more quickly. If you stir the coffee, it will cool more quickly.

* There were no differences in how participants sorted Formula-General and Formula-Instantiated. These are consolidated in our analysis.

Table 3. Features commonly cited by participants over the course of the interview

Feature	# Participants	Examples
Visual	9	"For me a model is being able to not so much read, but see it visually." (Audrey) "And so by using Geogebra, I could represent the information visually." (Walter)
Includes Trend	9	"Because some kids are able to see a trend looking at the table and just numbers, and some kids are easier for them to see the visual." (Sophia) "Because you, or my students, or whatever might not be able to see the trend here [table]. So that doesn't really say much for them, or give much for them." (Violet)
Includes Data	9	"I think the data is probably accurately represented but you can cert-, I can at least see a curve here. And this person put a straight line. So I just don't think that's the best way of describing what's going on here. Um, but it's certainly a visual model of the data." (Jacob) "I call it a model because it shows um I guess what do I want to say, graphically it shows the data. So thats what I'd call it, shows the data." (Olivia)
Easy to Interpret / Conventional	6	"…it's a good model if the person you are sharing it with is able to see and understand the data that's being presented." (Heidi) "I think it [the graph] is pretty conventional, I think most people understand what it means." (Jill)
Includes Description of Situation	6	"…if they knew that it was in a ceramic mug, there was not cream and sugar, from a drip coffee machine, then hopefully the model would work for them." (Liam) "I think this [verbal representation] is interesting because often times when I talk about different real world situations, that we have to take into account different things." (Olivia)
Accurate	5	"That [linear approximation] doesn't seem to be a very accurate way of describing what's going on." (Walter) "…to be accurate is not easy because I did such small intervals of time, a lot of times I got the same temperature over and over again. Uh, which I don't think is completely accurate, I do think it was going down even though, but you know when you read it, it's really hard to tell if there's a .5 or something in there." (Jacob)
Includes Key Factors	5	"This is … better than that coffee cup one, because it's getting at more factors." (Sophia) "I guess you could say this is a model of the factors of the problem." (Heidi)

Table 4. Purposes commonly cited by participants over the course of the interview

Purpose	# Participants	Examples
Explore Causal Relationships	7	“This definitely represents the situation and describes what's going on. Um, which also gets to another factor that I didn't mention earlier, which was my data was collected with the cover on, so I wonder if the cover is off, my guess is this is what's happening, heat would escape from the top and perhaps um, the temperature would drop more quickly. So here, I think this is a model that describes the problem.” (Heidi) “I think if you were trying to explain the cup of coffee and the rate of, of loss of heat and temperature, I think these things would speak to them a little bit more.” (Olivia)
Explore Trends	6	“You can see the trend here and the specifics there so I don't think one is better. And this one [scatter] is just not connected and that's ok too. Because then you can choose to best fit or exact fit.” (Violet) “This is again not a very good model. I think the data is probably accurately represented but you can, I can at least see a curve here. And this person put a straight line. So I just don't think that's the best way of describing what's going on here.” (Jacob)
Make Predictions or Comparisons	4	“I feel like the other thing I'd want to do is I would want to um, test this in a colder temperature. And see, because what might end up happening is it might just it, I don't know. If it's accurate, then but like I said, I didn't see now that might, if a room was cold, colder, it might actually change this.” (Audrey) “So it perhaps these aren't models in the same way that these are, but they um it's not just one set of observations. It's uh, you couldn't use these directly in order to make as detailed predictions as the others would allow, but they do describe in slightly different ways the system that's uh, that contributes to the loss of heat.” (Walter)
Generate Values	4	“I guess, um, I'm defining model for myself as something that we could either display data or what's the work I'm looking for, something like get data from so if I plugged in a certain temperature or minute, that I'd be able to figure out a certain output.” (Jill) “…if you use this model to generate your data table and your graph, you would have a sense of how they interact. And you should be able to get this big idea. So this is a model because I could just start with this and this is the only piece of paper I have, and I could get this big idea.” (Sophia)

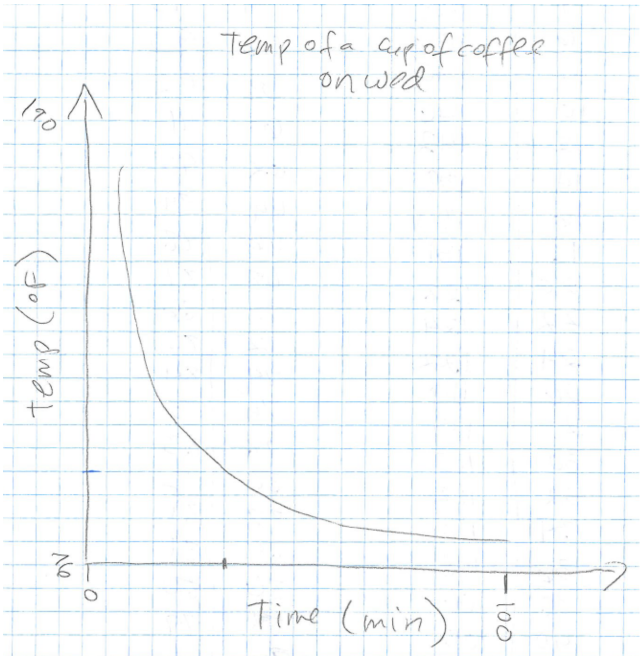


Figure 1. Sophia's constructed model, generated during the interview.

Table 5. Summaries of shifts found in interviews with Sophia, Olivia, and Jill.

	Constructing	**Sorting**
Sophia	F Visual F Includes Data F Easily Interpreted	F Visual F Includes Trend F Easily Interpreted F Describes Situation F Accurate F Includes Features P Explore Causal Rltns P Explore Trends P Generate Values
Olivia	F Visual F Includes Trend F Includes Data F Easily Interpreted F Accurate	F Visual F Includes Trend F Includes Data F Easily Interpreted F Includes Features P Explore Causal Rltns
Jill	F Visual F Includes Trend F Includes Data F Easily Interpreted	F Visual F Includes Trend F Includes Data F Easily Interpreted F Includes Features P Explore Causal Rltns P Explore Trends P Generate Values

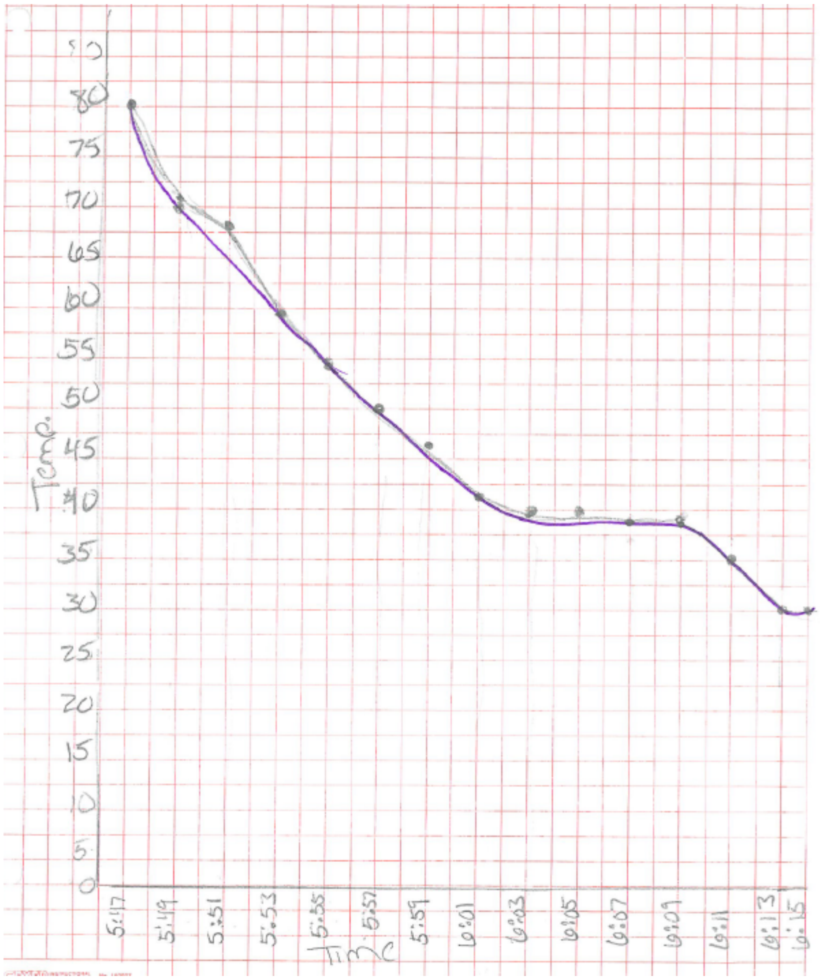


Figure 2. Olivia’s constructed model.

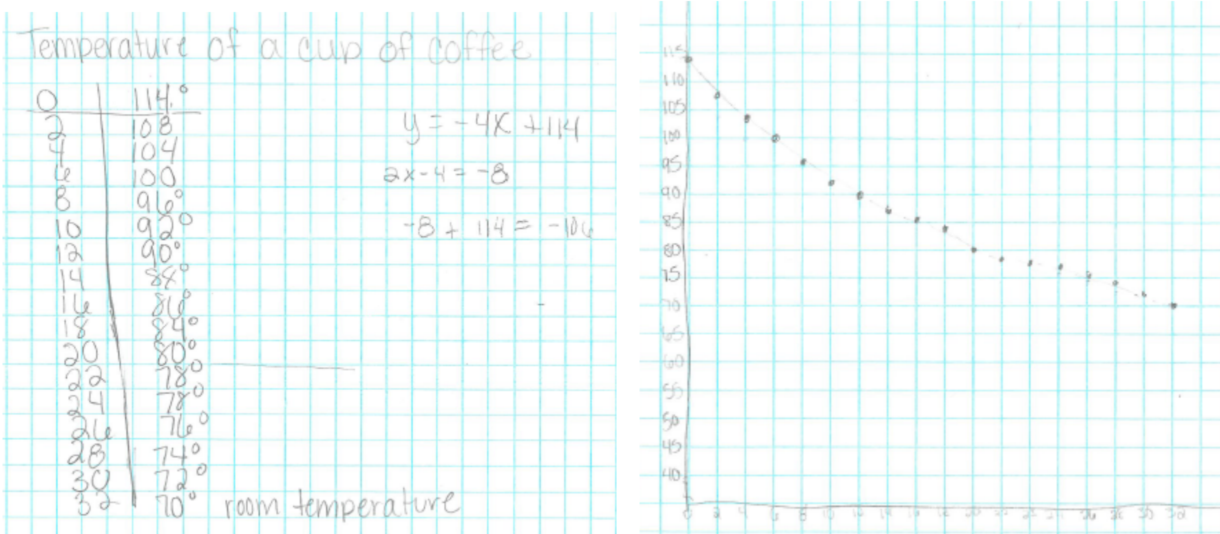


Figure 3. Jill’s constructed models.

Table 6. Summaries of non-focal participant interviews

	Construction	Sorting	Representative Interview Excerpts
Audrey	F Visual F Includes Trend P Predict/Compare	F Visual F Includes Trend F Includes Data F Describes Situation P Expl. Causal Rltns P Predict/Compare	*Construction Task* A: I mean I think it does, it shows that um that almost like, it's almost like the new car rule. You drive it off the lot and it drops. Like it dramatically, it dropped a whole lot faster at the beginning than I thought it was going to. […] I know that eventually it's going to be, you know, but then looking too at what the room temperature was, which would have been about 25, it's getting close to that so these aren't gonna drop that much faster to get here. So that huge drop is right in that first, like, 15 minutes. You've lost um, what, 40 degrees Celsius, which would be, what if that's 95, so you're going to go from 200 degrees down to 130. So you're losing 70, I mean, you're losing, now it's lukewarm as opposed to really piping hot. Which also says if you don't want to burn your tongue, all you have to do is wait 5 minutes. *Sorting Task* A: Um, I mean I think if I'm looking at it me personally these [graphs] would be more models. [I: Okay.] This one here is more of a picture. Um, and this is a data chart that shows the information, but it is part of a mo- they are both models, but this would like I said I think. This doesn't show, this is showing how heat loss works but it doesn't really talk about data. I: Okay. And this [table] is just data? A: That's just data. And this is showing kind of everything. Like giving you a visual as to, like I said, you should be able to, you look at this and say alright, most of my heat loss is coming out of the top, I'm going to lose a little bit here, here's some data but I don't really see how it looks yet, looking at these I know that I'm going to have a big loss at the beginning and then it's going to even out, so those would be like I said models.
Heidi	F Visual F Includes Data F Easily Interpreted P Predict/Compare	F Includes Trend F Includes Data F Describes Situation P Expl. Causal Rltns P Predict/Compare P Generate Values	*Construction Task* I: How do you decide what's a good model? So like if this, whether that's a good model or it could be better or worse? H: Um, well I guess it depends on the purpose of the model, I think it's a good model if the person you are sharing it with is able to see and understand the data that's being presented. Um, and I'd rate models on how effectively they communicate that data. Um. So, so I think because this model clearly shows that the temperature is dropping more quickly in the beginning than the end that it's probably a good model. *Sorting Task* H: [referencing verbal description] I would say that this is a description of a situation, but I wouldn't call it a model.[referencing sysdyn] I guess you could say this is a model of the factors of the problem. Um, it's a visual like almost like a flow chart of, um, impacting factors to the problem. Um, I guess different models would give you different information, so you could get different information from this model than you can for this model [references coffee cup] and it seems like it's drastically different information that you could get from these picture models than you can from the actual data. I: And you called this one, I think it was not necessarily a model of the data, but a model of the problem I think you said, so would you say - H: Right, and I think this is very similar in that it models the problem. So I guess maybe the formulas model what would happen to the data. Or maybe it describes what the data would look like.

Jacob	F Visual F Includes Trend F Includes Data F Easily Interpreted F Accurate	F Visual F Includes Trend F Includes Data F Easily Interpreted F Describes Situation F Accurate F Includes Features P Explore Trends P Generate Values	*Excerpt from Construction Task* I: And what is it about that that would make you feel, that you would call it a model? J: Uh, well if modeling was happening to the temperature, in comparison to the time and that's what the original question was, what's your hypothesis. As to what will happen. Um, temperature versus time. I: And, so you had this, you'd have this representation, you'd have this graph, how would you decide whether you thought that was a good model for the, for the situation? J: Um, I'd have to tease this data a little bit. I know that one of the issues I ran into is that the model is really hard to read. Um, and to be accurate is not easy because I did such small intervals of time, a lot of times I got the same temperature over and over again. Uh, which I don't think is completely accurate, I do think it was going down even though, but you know when you read it, it's really hard to tell if there's a .5 or something in there. So I tried not to do that because it would be more speculative than accurate. *Excerpt from Sorting Task* I: Okay. And let's look at the ones that you classified as not models. J: Alright well, see I don't want to classify it as not models, they're just a different type of model. [I: Okay, well, so-] Because that word model is loaded. You know I: Well tell me about that how, how is it loaded? What is it loaded with in your view? J: Um, because you have something going on. Well, the first thing you can do is describe it. You're modeling it with words. So that's a, a um linguistic model. you can then you know, collect the data. That's also a model of the phenomenon. Before I had data, all I had was a cup cooling down. But I could have described what was happening, you know, and then I put in data so that's a new kind of model. Then we have the visual model. So the word model in my mind makes me want to go to a visual, but I don't think that's what it necessarily means. I think that model can be any representation, um, of such a phenomenon.
Liam	F Visual F Includes Trend F Includes Data	F Visual F Includes Data F Accurate F Includes Features P Expl. Causal Rltns P Explore Trends P Predict/Compare	*Excerpt from Construction Task* L: I can't think of a better model than a graph. So I can take, I can draw the curve. Um, a little sloppy but um, and then I would probably you know use different colors maybe highlighter and show this different sort of sections that I am noticing. so I would maybe as far as making a model I would want to use different color or some sort of pattern or some way to show you know the dashed line or dotted line or something to show that these are sort of different things are happening about those points. *Excerpt from Sorting Task* L: So here is a model again I am calling it a model because it allows me to predict some values that weren't measured. Um, it's certainly less accurate than other models. They create a linear line between two of the points, some of the points above it, and some below it. And it also would, if I extend this out, it would tell me at some point around twenty two minutes, the coffee will be frozen. I: Ok. Right. So this one goes in... L: it's a model. But it's not a very good model. So fourth pile. I: Ok. Fourth pile, bad models. L: Um. this is text. um... it explains the factors, and it explains um, how it's not linear. Um, if it's some other factors. So this is a model in the same way this is a model. It helps people understand the factors, but it's not a picture. Now I am really testing my own notions what makes a model.

Violet	F Visual F Includes Trend F Includes Data F Describes Situation P Expl. Causal Rltns	F Visual F Includes Trend F Includes Data F Easily Interpreted F Describes Situation P Expl. Causal Rltns P Explore Trends	*Excerpt from Construction Task* I: But the question is can you create a model for the things that you notice? V: A model. I did. Isn't that [graph] a model? [I: Alright.] A table is a model. Say it again? I: Can you create one model for the things you notice? V: I think I did. [I: Right, ok.] But the only other thing, this is bad drawing but we could draw convection currents [draws on paper]. And show the that this is 76 degrees, and that this is 160 degrees. And that there's a lot of fast molecules. And here they're sort of slower. And I don't know, these rise up and get heavier and come back down. I don't know. [I: Okay.] Something like that. I: So ok. Would you call this representation a model? V: Maybe. A model I think of a 3d thing. So that would maybe be the actual cup. *Excerpt from Sorting Task* I: So why do you think that they all represent a model? V: Because they do. They all represent the same thing. Temperature versus time in a coffee cup cooling. I: Right. Alright. Okay. V: One with pictures, one with words, one with graphs. I: … Is there anything that you think well this model is fine, but it's not showing this aspect of the information? V: Well the pictures show the process. They don't show information and rates and they don't, you'd still be left with so how does this cup of coffee cool. What does it look like. So that doesn't really show it in the best way. It shows the process but not the specifics. I: Can you please clarify what you mean by the process? V: The loss of temperature is going to come out of the coffee cup and it's gonna depend on how it's insulated and what the room temperature is. But it doesn't tell me how that's gonna look. Same as this it says it's gonna be lost this way, but how fast, how slow, what's the rate. It doesn't tell you specifics.
Walter	F Visual F Includes Trend F Includes Data F Accurate P Predict/Compare	F Includes Trend F Includes Data F Describes Situation F Accurate F Includes Features P Expl. Causal Rltns P Explore Trends P Predict/Compare	*Excerpt from Construction Task* W: Um, there's mathematical ways that you could calculate the goodness of fit of the data to the curve. By looking at the standard, by looking at the deviations the absolute value, you could look at your model of the curve, look at each piece of data that you know a value for, calculate that difference, square it and then uh to um, and then take the average and use that infromation. If you end up with a really large number your model might not be so good, but if you get something that fits it almost exactly then every point that you predict is very close to uh what you observed in the real world then your model is probably a good one. *Excerpt from Sorting Task* W: Uh, so things that are not models I just called these two [data and table] not models because they're just the collection of specific uh pieces of information specific to one um, to one person's set of observations. Everything over in this pile is [models pile] describes the situation in a little bit more detail, but in different ways. Um, these three that I started with are more mathematical models of varying [graphs connected or fit] complexity. These [formulae, then puts Newton down] this one here is specific to coffee, you've got an equation that describes how coffee cools, so this is uh similar to uh to the top one. Because that's how I believe coffee would cool. Uh, Newton's law of cooling, sort of governs that this is more extensible and can be applied to uh, to any liquid cooling in any setting I believe, who knows you could even describe gas as it heats up, I don't know if it works in reverse or not but that goes with that [sets over, but offset from, graphs and instantiated formula] And then these three [mug, sysdyn, verbal] are describing that system in different ways. So it perhaps these aren't models in the same way that these are, but they um it's not just one set of observations. It's uh, you couldn't use these directly in order to make as detailed predictions as the others would allow, but they do describe in slightly different ways the system that's uh, that contributes to the loss of heat.

Appendix A. Interview protocol.

Part 0: Email sent to participants

We are going to investigate the temperature of a hot cup of coffee that is left on the table. Please answer the following question and email back your response: What is your prediction of how the temperature of the coffee will change as time goes by? Please explain why you think that. For now, please rely only on your own understandings and ideas. Refrain from consulting resources such as books, the Internet, or other people.

Once we received their response, we mailed each participant a thermometer and asked him or her to conduct an experiment:

Now, actually try it out. Investigate the temperature of a hot cup of coffee that is left on the table and send me the data you collect along with any details you think I should know about the conditions in which the experiment took place. Please keep any notes you make and bring to our interview, or take pictures of them and email them.

Part I: Interview Description Task

Here is the data you collected.
What do you notice?
How would you communicate what you notice to somebody else?

Part II: Interview Construction Task

Can you create one model for the things you noticed?
What makes you call that a model?
How would you decide whether that's a good model?
If the goal of the model were to predict coffee temperature, how would you change your model?
If there were more information available, would that affect your model? (And: what other kind of information would you use?)
If someone else made a similar investigation, could your model be used to describe the other data? [Possible follow up: could you use your same process to describe the other data?]

Part III: Interview Sorting Task

Present teachers with different representations of the coffee cooling situation. (See Table 2).
Put them all on the table randomly.
Please create two groups: those that represent a model and those that do not.
Is there any other way that you would group the models?
Explain why you grouped them in this way. [Related or as follow up: What criteria did you use to make your grouping?]

For each group of representations they created, ask:

Is this a model of the data?
Why do you think this represents a model?
Is there any aspect of the representation that you think does not show the information provided in the best possible way?
If yes, what would you change about the representation?
If it's not a representation of the model, is there something you would add?
Would you use this in your classroom? Why?

Appendix B. Nature of coding disagreements.

In this paper, some of the findings we report are based on aggregate patterns revealed by thematic coding analysis. We agree with Hammer and Berland (2013) that it is important to not only describe the degree of coding agreement in such analyses, but also to describe the nature of coding disagreement. In this analysis, there were two types of disagreement of note.

In the first type of disagreement, the second coder assigned more codes to transcript excerpts than the first coder did, in general. Specifically, the second coder identified more utterances she deemed worthy of being assigned a code than the first coder. This type of disagreement was rare (4 out of 66 possible), but constituted all but one of the disagreements after discussion. Two of the disagreements were feature codes, and two were purpose codes. Since the first coder was responsible for analyzing the rest of the data, this suggests that our findings may slightly underestimate the total number of features and purposes attended to by participants during the interview. However, this would not affect our findings about shifts in attention from features to purposes.

The second type of disagreement emerged around whether one particular utterance referred to Including a Description of the Situation or Including Trend. The participant wanted to "model what is happening to the temperature over time;" one coder argued the participant was referring to the decrease in temperature as a trend, while the other argued the participant was referring to the coffee cooling situation more generally. This disagreement happened only once during interrater procedures.