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Theoretical Structure of Plasma Equations*

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In high-temperature plasmas, collisions are very infrequent. Thus the charged particles travel on independent orbits determined by the electromagnetic field. At first sight this would seem completely different from a conventional fluid where particles are closely hemmed in by their neighbors. However, there can exist collective modes of motion in which the particles interact with each other by altering the fields.

In this paper a new method is developed for the solution of the linearized transport equation. By facilitating direct use of the properties of particle orbits, a considerable simplification is achieved. In particular, a variational expression is derived for determining stability which is rigorous in the limit of small Larmor radius.

1. INTRODUCTION

INVESTIGATIONS of the stability of a fully ionized plasma are usually based on the magnetohydrodynamic equations[1] (M-H approximation). The essential features of these equations are the assumptions of scalar pressure P and an adiabatic invariant

$$(d/dt)(P\rho_m^{-\gamma}) = 0. \tag{1}$$

Chew, Goldberger, and Low[2] have proposed a different approximation (C-G-L approximation) in which the pressure is not scalar and there are two adiabatic invariants

$$\frac{d}{dt}\left(\frac{P_\perp}{\rho_m B}\right) = 0, \tag{2}$$

$$\frac{d}{dt}\left(\frac{P_\parallel B^2}{\rho_m^3}\right) = 0. \tag{3}$$

$P_\parallel$ and $P_\perp$ are pressures parallel and perpendicular to the direction of the magnetic field **B**. Neither the accuracy nor the conditions for the validity of these approximations are completely understood.

The transport equation provides a complete description of the plasma and a deductive basis for approximate descriptions. The M-H and C-G-L approximations consist of taking moments of the transport equation. Each moment equation is a differential equation that involves only macroscopic variables. The adiabatic invariants are introduced arbitrarily to terminate the set of moment equations. To investigate the stability of a particular plasma configuration, the moment plus adiabatic invariant equations are solved subject to suitable boundary conditions.

The essential feature of the present approach is that the transport equation is solved at the outset to give the perturbed distribution function in terms of the perturbed field variables. The macroscopic variables calculated from the solution must then satisfy Maxwell's equations. In the limit of small Larmor radius, these become integro-differential equations. This approach leads to more complex macroscopic equations, but it has the virtue of being correct when the M-H and C-G-L approximations are not.

A convenient way of treating stability problems is to calculate the change in energy which results from a small perturbation. In the M-H approximation this method leads to a variational principle.[3] The energy change calculated by the present procedure of solving the transport equation also will be shown to give a variational principle. It is bounded below by the M-H energy change and above by the C-G-L energy change.

2. SOLUTION OF THE TRANSPORT EQUATION

At sufficiently high temperatures the plasma can be described by the collision-free transport equation[4]

$$\frac{\partial f_i}{\partial t} + \mathbf{v}\cdot\nabla f_i + \frac{\mathbf{F}_i}{m_i}\cdot\nabla_v f_i = 0, \tag{4}$$

where

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[1] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) **A223**, 348 (1954); M. N. Rosenbluth, "Stability of the pinch," Los Alamos Scientific Laboratory Rept. LA-2030 (April, 1956).

[2] Chew, Goldberger, and Low, Proc. Roy. Soc. (London) **236**, 112 (1956).

[3] Frieman, Kruskal, Kulsrud, and Bernstein, NYO-7315, Project Matterhorn Rept. PM-S-25, (Princeton University, Princeton, New Jersey, March, 1957); Proc. Roy. Soc. (London) (to be published).

[4] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

$$\mathbf{F}_i = q_i\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right).$$

E and **B** are macroscopic fields that satisfy Maxwell's equations. They are produced by the charge density

$$\rho_e = \sum_i q_i \int f_i \, d^3v,$$

and the current density

$$\mathbf{j} = \sum_i q_i \int f_i \mathbf{v} \, d^3v.$$

$f_i(\mathbf{x}, \mathbf{v}; t)$ is the distribution function for the ith particle species. (For present purposes, $i = 1$ for electrons and $i = 2$ for ions; the subscript i will be deleted except where it is essential.)

The stationary state whose stability is to be considered satisfies

$$\mathbf{v}\cdot\nabla f_0 + \frac{q}{mc}(\mathbf{v} \times \mathbf{B})\cdot\nabla_v f_0 = 0. \tag{5}$$

The linearized time-dependent equation that determines the stability of this state is

$$\frac{\partial}{\partial t}\delta f + \mathbf{v}\cdot\nabla\delta f + \frac{q}{mc}(\mathbf{v} \times \mathbf{B})\cdot\nabla_v\delta f = -\frac{\delta\mathbf{F}\cdot\nabla_v f_0}{m}. \tag{6}$$

This equation describes both the dynamical and statistical aspects of the problem.

The dynamical aspects of the stability problem have been clarified by a treatment[5] which proceeds from the solution of the orbit equations

$$\frac{d\mathbf{v}'}{dt'} = \frac{q}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}' \times \mathbf{B}\right), \qquad \frac{d\mathbf{x}'}{dt'} = \mathbf{v}'. \tag{7}$$

To calculate macroscopic variables, such as charge density, current density, etc., it is essential to know the weighting function for the particles. A more convenient treatment of the statistical aspects of the problem based on a solution of the transport equation has been given by Chandrasekhar, Kaufman, and Watson.[6]

In the present approach, Eq. (6) is solved by making time integrations along the unperturbed orbits. This procedure does not emphasize one of the dynamical or statistical aspects at the expense of the other. Equation (6) can be written as

$$\frac{d}{dt}\delta f[\mathbf{x}(t)\mathbf{v}(t); t] = -\frac{\delta\mathbf{F}\cdot\nabla_v f_0}{m},$$

so that

$$\delta f(\mathbf{x}, \mathbf{v}; t) = -\frac{1}{m}\int_{t'=t_0}^{t'=t}(\delta\mathbf{F}\cdot\nabla_v f_0)\, dt' + \delta f(t_0). \tag{8}$$

The integrand must be evaluated along the unperturbed orbit at $\mathbf{x}'(\mathbf{x}, \mathbf{v}; t')$, $\mathbf{v}'(\mathbf{x}, \mathbf{v}; t')$, where these functions satisfy

$$\frac{d\mathbf{v}'}{dt'} = \frac{q}{mc}(\mathbf{v}' \times \mathbf{B}), \qquad \frac{d\mathbf{x}'}{dt'} = \mathbf{v}', \tag{9}$$

and the initial conditions $\mathbf{x}'(\mathbf{x}, \mathbf{v}; t) = \mathbf{x}$; $\mathbf{v}'(\mathbf{x}, \mathbf{v}; t) = \mathbf{v}$.

The relevant physical problem defined by Eq. (8) is an initial-value problem. An arbitrary initial displacement is made from the state described by f_0, and one wishes to know the subsequent time evolution. This type of problem has been solved by Laplace transforms.[7] The equations for the Laplace transforms of the initial-value problem are the same as for an eigenvalue problem where $\delta f = f_1(\mathbf{x}, \mathbf{v})\cdot e^{pt}$, $\delta\mathbf{B} = \mathbf{B}_1(\mathbf{x})\, e^{pt}$, etc., except for some inhomogeneous terms from the initial conditions. The Laplace transforms will have poles at the eigenvalues $p = p_n$. There are no additional singularities in which p has a positive real part. The significant time dependence of the solution of the initial-value problem will be of the form $t^l\, e^{p_n t}$, where l is the order of the pole at $p = p_n$. For stability considerations it is sufficient to know whether p_n has a positive real part so that we shall only consider the eigenvalue problem. As long as p has a positive real part, there are no convergence difficulties in Eq. (8). Furthermore, in this case, the arbitrary function $\delta f(t_0)$ in Eq. (8) drops out if $t_0 \to -\infty$.

For an unperturbed magnetic field $\mathbf{B}(\mathbf{x})$ that is not constant, Eqs. (5) and (9) cannot be solved exactly. The appropriate expansion parameters are $\lambda_i = a_i/L$ where $a_i = m_i v_\perp c/q_i B$ are the radii of gyration and $L \gg a_i$ is a characteristic length of the problem. (For example, $L = B/|\nabla B|$.) In the calculation of δf there are the additional parameters p/ω, where $\omega = qB/mc$ is the cyclotron frequency, p/ω_p, where $\omega_p = (4\pi Nq^2/m)^{\frac{1}{2}}$ is the plasma frequency, and L_D/L, where $L_D = (\Theta/4\pi Nq^2)^{\frac{1}{2}}$ is the Debye length. These parameters are assumed

[5] M. N. Rosenbluth and C. Longmire, Ann. Phys. (N. Y.) 1, 120 (1957).

[6] Chandrasekhar, Kaufman, and Watson, Ann. Phys. (N. Y.) 2, 435 (1957).

[7] L. Landau, J. Phys. U. S. S. R. 10, 25 (1946).

to be of the same order as λ. Each parameter is proportional to q^{-1}.

For Maxwell's equations we need to compute $\delta\mathbf{j}$, the perturbed current. Because $\delta\mathbf{j}$ is proportional to $q\delta f$, it would seem that δf must be calculated to order λ^{n+1} to determine $\delta\mathbf{j}$ to order λ^n. However, to calculate the pressure tensor

$$\delta\mathsf{P} = \sum_i m_i \int \mathbf{v}\mathbf{v}\delta f_i \, d^3v \tag{10}$$

to order λ^n requires δf only to order λ^n. As any solution for δf must satisfy the moment equation

$$p\rho_m \delta\mathbf{V} = -\nabla\cdot\delta\mathsf{P} + \frac{1}{c}[\delta\mathbf{j}\times\mathbf{B} + \mathbf{j}\times\delta\mathbf{B}], \tag{11}$$

$\delta\mathbf{j}\times\mathbf{B}$ can be inferred from $\delta\mathsf{P}$. This is clearly simpler than the direct calculation of $\delta\mathbf{j}$.[6]

Our procedure is then as follows: $\delta\mathbf{E}$ is to be regarded as the fundamental variable. $\delta\mathbf{B}$ can be expressed in terms of it by Maxwell's equations. The distribution function is then calculated by Eq. (8) and the charge density (or alternately the component $\delta j_\parallel$) is computed directly from it in terms of $\delta\mathbf{E}$. The two components of current perpendicular to $\mathbf{B}$ are given by Eq. (11). The Maxwell equations $\nabla\times\delta\mathbf{B} = (4\pi/c)\,\delta j$ and $\nabla\cdot\delta\mathbf{E} = 4\pi\delta\rho_e$ now provide the system to be solved for $\delta\mathbf{E}$. (It is often convenient to use the latter equation to eliminate $\delta\mathbf{E}_\parallel$.) Equation (8) will be solved to order λ^0, so that the resulting equations should be analogous to magnetohydrodynamics in which the charge q does not appear.

We now return to a brief discussion of the equilibrium state.

To solve Eq. (5) it is convenient to use a local cylindrical coordinate system with the z axis in the direction of $\mathbf{B} = B\mathbf{n}$. A solution of the form

$$f_0 = f(\mathbf{x}, \mathbf{v}) + \lambda g(\mathbf{x}, \mathbf{v}) + O(\lambda^2)$$

is substituted into

$$\mathbf{v}\cdot\nabla f_0 + \omega\frac{\partial f_0}{\partial\phi} = 0.$$

Coefficients of λ^n are set equal to zero. From the coefficient of λ^0, $\partial f/\partial\phi = 0$. Therefore, $f = f(\mathbf{x}, v_\perp, v_\parallel)$. For present purposes we assume $f = f(\mathbf{x}, v^2)$, i.e., an initially isotropic pressure tensor. From the coefficient of λ it follows that

$$\mathbf{n}\cdot\nabla f = 0 \tag{12}$$

and

$$f_0 = f(\mathbf{x}, v^2) + \frac{1}{\omega}(\mathbf{v}\cdot\mathbf{n}\times\nabla f) + \lambda h(\mathbf{x}; {v_\perp}^2, v_\parallel). \tag{13}$$

The solution of Eq. (9) is only required to order λ^0. To this order, the particle undergoes circular motion about a guiding center that follows a field line, i.e.,

$$\mathbf{v}' = V_\parallel\mathbf{n} + V_\perp\left[\mathbf{e}_1\cos\int^{t'}\omega\,dt'' \pm \mathbf{e}_2\sin\int^{t'}\omega\,dt''\right], \tag{14}$$

where

$${V_\parallel}^2 = v^2 - \frac{2}{m}\mu B,$$

$${V_\perp}^2 = \frac{2}{m}\mu B,$$

$$\mathbf{e}_1 = R_1(\mathbf{n}\cdot\nabla)\mathbf{n} \quad\text{and}\quad \mathbf{e}_2 = \mathbf{n}\times\mathbf{e}_1.$$

The magnetic moment $\mu = m{V_\perp}^2/2B$ and v^2 are constants of the motion.[5] If B is constant along the orbit, $V_\perp = v_\perp$ and $V_\parallel = v_\parallel$ are constants. Otherwise, the particle orbit may have turning points because $V_\parallel = 0$ when

$$B(\mathbf{x}') = (v^2/{v_\perp}^2)B(\mathbf{x}). \tag{15}$$

δf is calculated by substituting Eq. (13) into Eq. (8); with the assumed exponential time dependence,

$$\begin{aligned}-\delta f = {}& \frac{c}{p}\int_{t'=-\infty}^{t'=t}\frac{(\mathbf{v}'\times\nabla\times\delta\mathbf{E})\cdot(\mathbf{B}\times\nabla f)}{B^2}\,dt' \\ & + c\int_{t'=-\infty}^{t'=t}\frac{\delta\mathbf{E}\cdot\mathbf{B}\times\nabla f}{B^2}\,dt' \\ & + \frac{2q}{m}\int_{t'=-\infty}^{t'=t}(\mathbf{v}'\cdot\delta\mathbf{E})\frac{\partial f}{\partial v^2}\Big)_c\,dt' \\ & + \lambda\frac{q}{m}\int_{t'=-\infty}^{t'=t}(\mathbf{n}\cdot\delta\mathbf{E})\frac{\partial h}{\partial v_\parallel}\,dt' + O(\lambda),\end{aligned} \tag{16}$$

where

$$\frac{\partial f}{\partial v^2}\Big)_c = \frac{\partial f}{\partial v^2} + \frac{1}{\omega}\left(\mathbf{v}'\cdot\mathbf{n}\times\nabla\frac{\partial f}{\partial v^2}\right)$$

is the value of $\partial f/\partial v^2$ at the guiding center and remains constant during a Larmor period. As $\delta f \sim [(\delta F_\parallel/\lambda) + \delta E_\perp][1 + O(\lambda) + \cdots]$, the charge density will be $\delta\rho_e \sim [1/L][(\delta E_\parallel/\lambda^2) + (\delta E_\perp/\lambda)][1 + O(\lambda)\cdots] \gg \nabla\cdot\delta\mathbf{E}$. Therefore, $\delta E_\parallel/\delta E_\perp \sim \lambda$. The physical reason for this is that $\delta E_\parallel$ can cause a large charge density. If it is to be slowly varying it must be small. $\delta E_\perp$ cannot produce a correspondingly large charge density because the $\delta\mathbf{E}_\perp\times\mathbf{B}$ drifts are independent of the sign of q. It is convenient to introduce a two-component displacement vector $\delta\boldsymbol{\xi} = \boldsymbol{\xi}e^{pt}$ such that

$$\delta\mathbf{E} = \frac{p}{c}\mathbf{B} \times \delta\xi + \delta\mathbf{E}_{\parallel}. \tag{17}$$

Bearing in mind the discussion in the foregoing, we treat $\delta E_{\parallel}$ as of order $\lambda\delta\xi$. It is clear that the component of ξ parallel to the field $\mathbf{B}$ is irrelevant and we set it equal to zero in the subsequent algebra.

In the first term of Eq. (16)

$$\frac{c}{p}\frac{(\mathbf{v}' \times \nabla \times \delta\mathbf{E})\cdot(\mathbf{B} \times \nabla f)}{B^2} = \frac{V_{\parallel}}{B}[\nabla \times (\delta\xi \times \mathbf{B})]\cdot\nabla f + O(\lambda).$$

From the vector identities

$$\begin{aligned}\nabla\cdot[(\delta\xi \times \mathbf{B}) \times \nabla f] &= \nabla f\cdot[\nabla \times (\delta\xi \times \mathbf{B})] - (\delta\xi \times \mathbf{B})\cdot\nabla \times \nabla f \\ &= \nabla\cdot[\mathbf{B}(\delta\xi\cdot\nabla f) - \delta\xi(\mathbf{B}\cdot\nabla f)] \\ &= \mathbf{B}\cdot\nabla(\delta\xi\cdot\nabla f) + \delta\xi\cdot\nabla f\nabla\cdot\mathbf{B}\end{aligned}$$

and the relations

$$\nabla\cdot\mathbf{B} = \mathbf{B}\cdot\nabla f = \nabla \times \nabla f = 0,$$

the first term is reduced to

$$\int_{t'=-\infty}^{t'=t} V_{\parallel}\mathbf{n}\cdot\nabla(\delta\xi\cdot\nabla f)\,dt'. \tag{16.1}$$

In the second term

$$c\frac{(\delta\mathbf{E}\cdot\mathbf{B} \times \nabla f)}{B^2} = p(\mathbf{n} \times \delta\xi)\cdot(\mathbf{n} \times \nabla f) = p(\delta\xi\cdot\nabla f);$$

after an integration by parts, the second term is

$$\delta\xi\cdot\nabla f - \int_{t'=-\infty}^{t'=t} V_{\parallel}\mathbf{n}\cdot\nabla(\delta\xi\cdot\nabla f)\,dt'. \tag{16.2}$$

In the third term

$$\mathbf{v}'\cdot\delta\mathbf{E} = \frac{p}{c}(\mathbf{v}'\cdot\mathbf{B} \times \delta\xi) + \mathbf{v}'\cdot\delta\mathbf{E}_{\parallel} = \frac{mp}{q}\left(\delta\xi\cdot\frac{d\mathbf{v}'}{dt'}\right) + \mathbf{v}'\cdot\delta\mathbf{E}_{\parallel};$$

after an integration by parts, the third term is

$$2p(\delta\xi\cdot\mathbf{v})\frac{\partial f}{\partial v^2} - 2p\int_{t'=-\infty}^{t'=t}[\mathbf{v}'\cdot(\mathbf{v}'\cdot\nabla)\delta\xi + p(\delta\xi\cdot\mathbf{v}')]\frac{\partial f}{\partial v^2}\Big)_c dt' + \frac{2q}{m}\int_{t'=-\infty}^{t'=t}(\mathbf{v}'\cdot\delta\mathbf{E}_{\parallel})\frac{\partial f}{\partial v^2}\Big)_c dt'.$$

The integrands next must be averaged over a Larmor period. $\partial f/(\partial v^2)_c$ is constant during this time to $O(\lambda)$, and the oscillatory parts of $\mathbf{v}'$ in Eq. (14) give zero except where a product $\cos^2$ or $\sin^2$ obtains. Thus, averaged over a period, we see that

$$\overline{\mathbf{v}'\mathbf{v}'} = \frac{V_{\perp}^2}{2}I + \left(V_{\parallel}^2 - \frac{V_{\perp}^2}{2}\right)\mathbf{nn};$$

I is the unit dyadic, and $\mathbf{n}$ the unit vector in the direction of $\mathbf{B}$. The result for the third term is

$$-p\int_{t'=-\infty}^{t'=t}[(2V_{\parallel}^2 - V_{\perp}^2)\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\delta\xi + V_{\perp}^2\nabla\cdot\delta\xi]\frac{\partial f}{\partial v^2}\,dt' + 2p(\delta\xi\cdot\mathbf{v})\frac{\partial f}{\partial v^2} + \frac{2q}{m}\int_{t'=-\infty}^{t'=t} V_{\parallel}\delta E_{\parallel}\frac{\partial f}{\partial v^2}\,dt'.$$

The integrations now are to be carried out along the zero-order particle paths $d\mathbf{x}'/dt' = V_{\parallel}\mathbf{n}$. As $df/dt' = V_{\parallel}\mathbf{n}\cdot\nabla f = 0$, $\partial f/\partial v^2$ can be taken outside the integrals.

The result for the distribution function is

$$-\delta f = \delta\xi\cdot\nabla f + 2p(\delta\xi\cdot\mathbf{v})\frac{\partial f}{\partial v^2} + \frac{2q}{m}\frac{\partial f}{\partial v^2}\int_{t'=-\infty}^{t'=t} V_{\parallel}\delta E_{\parallel}\,dt' - p\frac{\partial f}{\partial v^2}\int_{t'=-\infty}^{t'=t}[(2V_{\parallel}^2 - V_{\perp}^2)\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\delta\xi + V_{\perp}^2\nabla\cdot\delta\xi]\,dt' + O(\lambda)\cdots. \tag{18}$$

Since $\delta\mathbf{B} = \nabla \times \delta\xi \times \mathbf{B}$, the lines of force move such that each point on a line is displaced by the local value of $\delta\xi$ which is perpendicular to the line.[8] Equation (18) is of the form $-\delta f = \delta\xi\cdot\nabla f + \delta\mathbf{v}\cdot\nabla_v f$. This means that particles are all displaced by the local value of $\delta\xi$ so that they remain "frozen" on the line of force.

A particular particle is identified by the fact that it has velocity $\mathbf{v}$ and position $\mathbf{x}$ at time t. The zero-order particle orbit $\mathbf{x}'(\mathbf{x}, \mathbf{v}; t')$ simply follows a field line $\mathbf{B}$ which forms a closed loop. The orbits will always be at least quasi-periodic in a finite system. That is, they will return arbitrarily close to the starting point. Thus,

$$\mathbf{x}'(t' + \tau) = \mathbf{x}'(t'), \tag{19}$$

where

$$\tau = 2\int_{l_1}^{l_2}\frac{dl}{|V_{\parallel}|}.$$

[8] W. A. Newcomb, Princeton University Observatory, Tech. Rept. No. 1 (1955).

l is a coordinate that measures the length along the particle orbit. For particle orbits that have no turning points, l_1 and l_2 are determined by $B(\mathbf{x})$. They depend on $\mathbf{x}$, but not on $\mathbf{v}$. $(l_2 - l_1)$ is the distance around the closed loop that brings one back to the same point, e.g., once around a torus. If the particle orbit has turning points, l_1 and l_2 are their positions. These quantities may be determined by solving Eq. (15). They depend on $\mathbf{x}$ and the velocity direction $\cos\theta = \mathbf{v}\cdot\mathbf{n}/v$, but not on the magnitude of $\mathbf{v}$.

The integrals in Eq. (18) can be reduced to integals over a single period. That is, if $\Psi(t'+\tau) = \Psi(t')$, then

$$\int_{t'=-\infty}^{t'=t} \Psi(t')e^{pt'}\,dt' \doteq \frac{1}{1-e^{-p\tau}}\int_{t'=t-\tau}^{t'=t} \Psi(t')e^{pt'}\,dt'.$$

The limiting forms of Eq. (18) are easily obtained for the cases $p\tau \ll 1$ and $p\tau \gg 1$. They are as follows:

for $p\tau \ll 1$,

$$-\delta f = \xi\cdot\nabla f - \frac{2}{\tau}\frac{\partial f}{\partial v^2}\int_{l_1}^{l_2}[(2V_\parallel^2 - V_\perp^2)\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\xi + V_\perp^2\nabla\cdot\xi]\frac{dl'}{|V_\parallel|} + O(p\tau)\cdots; \qquad (20)$$

for $p\tau \gg 1$,

$$-\delta f = \delta\xi\cdot\nabla f + 2p(\delta\xi\cdot\mathbf{v})\frac{\partial f}{\partial v^2} - [(2v_\parallel^2 - v_\perp^2)\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\delta\xi + v_\perp^2\nabla\cdot\delta\xi]\frac{\partial f}{\partial v^2} + O\left(\frac{1}{p\tau}\right) + O(\lambda). \qquad (21)$$

In both cases the equation $\nabla\cdot\delta\mathbf{E} = 4\pi\delta\rho_e$ shows that the contribution of the first integral in Eq. (18), while of order λ^0, is of higher order in $p\tau$ and $1/p\tau$, respectively, and may be omitted.

3. MARGINAL STABILITY EIGENVALUE PROBLEM

$\delta\mathbf{E}$ satisfies Maxwell's equation

$$\mathbf{B}\times\nabla\times\nabla\times\delta\mathbf{E} + \left(\frac{p}{c}\right)^2\mathbf{B}\times\delta\mathbf{E} = \frac{4\pi p}{c^2}(\delta\mathbf{j}\times\mathbf{B}), \qquad (22)$$

where $\delta\mathbf{j}\times\mathbf{B}$ is related to the pressure tensor by Eq. (11). The pressure tensor can now be calculated to order $p\tau$ by substituting Eq. (20) for δf into Eq. (10). It is convenient to introduce as velocity variables the constants of the motion, v^2 and

$$\alpha = \frac{2\mu}{mv^2} = \frac{1-\cos^2\theta}{B(\mathbf{x})}. \qquad (23)$$

In these variables

$$-\delta f = \xi\cdot\nabla f - 2v^2\frac{\partial f}{\partial v^2}\frac{J(\mathbf{x},\alpha)}{K(\mathbf{x},\alpha)}, \qquad (20.1)$$

where

$$J(\mathbf{x},\alpha) = \int_{l_1(\mathbf{x},\alpha)}^{l_2(\mathbf{x},\alpha)}\left\{\left[1-\frac{3}{2}\alpha B(l')\right]\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\xi + \frac{\alpha B(l')}{2}\nabla\cdot\xi\right\}\frac{dl'}{[1-\alpha B(l')]^{\frac{1}{2}}}$$

and

$$K(\mathbf{x},\alpha) = \int_{l_1(\mathbf{x},\alpha)}^{l_2(\mathbf{x},\alpha)}\frac{dl'}{[1-\alpha B(l')]^{\frac{1}{2}}}.$$

The result for the pressure tensor is

$$\delta\mathsf{P} = -(\xi\cdot\nabla P)I + \delta P_\perp I + (\delta P_\parallel - \delta P_\perp)\mathbf{nn}, \qquad (24)$$

where

$$P = \frac{4\pi}{3}\sum_i m_i\int f_i(\mathbf{x},v^2)v^4\,dv$$

is the unperturbed pressure,

$$\delta P_\perp = -\frac{15}{4}PB^2(\mathbf{x})\int_{\alpha=0}^{\alpha=1/B(\mathbf{x})}\frac{J(\mathbf{x},\alpha)\alpha\,d\alpha}{K(\mathbf{x},\alpha)[1-\alpha B(\mathbf{x})]^{\frac{1}{2}}}, \qquad (24.1)$$

and

$$\delta P_\parallel = -\frac{15}{2}PB(\mathbf{x})\int_{\alpha=0}^{\alpha=1/B(\mathbf{x})}\frac{J(x,\alpha)[1-\alpha B(\mathbf{x})]^{\frac{1}{2}}}{K(\times,\alpha)}\,d\alpha. \qquad (24.2)$$

To change from the independent variable $\delta\mathbf{E}$ to $\delta\xi$, we can substitute $\delta\mathbf{E} = p(\mathbf{B}\times\delta\xi)/c + O(\lambda)$ and $\delta\mathbf{B} = \nabla\times(\delta\xi\times\mathbf{B}) + O(\lambda)$ into Eq. (22). For the marginal stability case $p\to 0$, the eigenvalue equations are

$$\mathbf{B}\times\nabla\times\nabla\times(\mathbf{B}\times\xi) - (\nabla\times\mathbf{B})\times\nabla\times(\mathbf{B}\times\xi) = 4\pi\nabla\cdot\delta\mathsf{P}. \qquad (25)$$

There are two equations for the two components of ξ. In the case of a cylindrical pinch, $\mathbf{n}\cdot\nabla\mathbf{B} = 0$ everywhere, and none of the particle orbits has turning points. For displacements, $\xi = \xi_k e^{ikl}$, $J(\mathbf{x},\alpha) = 0$ unless $k = 0$. For $k \neq 0$, $\delta\mathsf{P}$ reduces to a scalar so that magnetohydrodynamics is valid. In the case $k = 0$, singularities develop in the

equations.[9] A complete treatment of this would require carrying the expansions to higher order in λ.

4. ENERGY PRINCIPLE FOR STABILITY PROBLEMS

The eigenvalue equations for the marginal stability case can be derived from a varational principle. The appropriate functional is the energy change that results from the perturbation

$$W = \int \left\{ \frac{\delta \mathbf{E} \cdot \delta \mathbf{j}}{2p} + \frac{|\delta \mathbf{B}|^2}{8\pi} \right\} d^3\mathbf{x}. \tag{26}$$

This can be expressed as a functional of ξ using Eqs. (11) and (24). In the marginal stability case

$$W = W_0 + W_1, \tag{27}$$

where

$$W_0 = \frac{1}{2} \int \left\{ -\xi \cdot \nabla (\xi \cdot \nabla P) + \frac{1}{c} \mathbf{j} \cdot \xi \times \nabla \times (\xi \times B) + \frac{1}{4\pi} |\nabla \times \xi \times \mathbf{B}|^2 \right\} d^3\mathbf{x} \tag{27.1}$$

and

$$W_1 = -\frac{1}{2} \int \{ (\delta P_\parallel - \delta P_\perp) \mathbf{n} \cdot (\mathbf{n} \cdot \nabla) \xi + \delta P_\perp \nabla \cdot \xi \} \, d^3\mathbf{x}. \tag{27.2}$$

A more explicit form for W_1 is obtained by substituting $\delta P_\parallel$, $\delta P_\perp$ from Eqs. (24.1) and (24.2), and carrying out the volume integration over flux tubes; i.e., $d^3\mathbf{x} = dl \, d\Phi / B(l)$. Φ is a coordinate which labels a given flux tube. $d\Phi = B \, dS$ indicates the amount of flux contained therein. It is a convenient variable, as a particle orbit lies in a definite flux tube. The order of integration of the variables l,α can be interchanged, i.e.,

$$\int dl \int_{\alpha=0}^{\alpha=1/B(\mathbf{x})} d\alpha \rightarrow \int_{\alpha=0}^{\alpha=1/B_m} d\alpha \int_{l_1(\Phi,\alpha)}^{l_2(\Phi,\alpha)} dl, \tag{28}$$

where $B_m(\Phi)$ is the minimum value of B for a particular flux tube. The resultant expression for W_1 is

$$W_1 = \frac{15}{4} \int P \, d\Phi \int_{\alpha=0}^{\alpha=1/B_m} d\alpha \, [J(\Phi, \alpha)]^2 / K(\Phi, \alpha). \tag{27.3}$$

W_1 is clearly a positive definite quantity. The term W_0 obtains in the M-H approximation as well as the C-G-L approximation. The difference between these approximations and the present result, which is correct to order λ, lies entirely in W_1. The condition that W be a minimum with respect to arbitrary variations in ξ turns out to be just that Eq. (25), the zero frequency equation, is satisfied. Thus, if $W_{min} = 0$, the stability is marginal, and we can conclude that $W_{min} > 0$ is a necessary condition for stability.

A brief outline of the proof for sufficiency will now be given. Again we consider the change in energy which would result from a given perturbation $\delta\mathbf{E}$ with an assumed frequency p. In general, p may be complex, but has a positive real part. The energy change is then found to have oscillatory parts as well as a monotonically changing part. In order for a solution to exist, both parts of the energy change must vanish because there is assumed to be no external energy source. We restrict our attention to the monotonically changing part

$$W' = \int \left\{ \left[\frac{1}{4} \left[\frac{\delta \mathbf{E} \cdot \delta \mathbf{j}^* + \delta \mathbf{E}^* \cdot \delta \mathbf{j}}{p + p^*} \right] + \frac{1}{2} \left(\frac{\delta \mathbf{B} \cdot \delta \mathbf{B}^*}{8\pi} \right) \right] \right\} d^3\mathbf{x}. \tag{29}$$

As in the previous case, we use Eq. (11) to find $\delta \mathbf{j}_\perp \cdot \delta \mathbf{j}_\parallel$, $\delta \mathbf{V}$, and δP are calculated by taking the appropriate moments of Eq. (18). The result for W' is of the form

$$2W' = W_0(\xi, \xi^*) + W_1(\xi, \xi^*) + p^* p \int \frac{\rho_m}{2} \xi^* \cdot \xi \, d^3\mathbf{x} + W_1'(\xi, \xi^*; E_\parallel, E_\parallel^*; p, p^*). \tag{29.1}$$

The first two terms are just the zero frequency limit, i.e., Eq. (27). The third term is a positive definite kinetic energy. The fourth term is a complicated correction to the potential energy. The major part of the proof consists in showing that $W_1' > 0$ for any ξ, unless $p = 0$ and $\delta E_\parallel = 0$, in which case $W_1' = 0$, by definition. After introducing the variables Φ and α as before, W_1' is expressed in the form

$$W_1' = \int d\Phi \int dv \int d\alpha \int_{l_1}^{l_2} dl \cdot \left\{ X^*(\alpha, l, v) \int_{t=-\infty}^{0} Y(\alpha, t, v) e^{pt} \, dt + (\text{complex conjugate}) \right\}. \tag{29.2}$$

If we fix our attention on a particular v and α, there is a well-defined relation between t and l. We may then write X and Y as Fourier series in t because the orbits are periodic. The first two integrations can

[9] M. N. Rosenbluth, "Theory of pinch effect—stability and heating," *Second Geneva Conference on the Peaceful Uses of Atomic Energy* (1958), Paper No. 1074.

be performed explicitly, and the result displayed in terms of the expansion coefficients for ξ and $E_\parallel$. In this form it is straightforward to show that the contribution from all elements $dv\ d\alpha$ is positive.

We may therefore conclude that if $W_0 + W_1 > 0$ for all ξ, then so is W', and no instability is possible. Thus the condition $W_{\min} > 0$, where W is given by Eq. (27), is a necessary and sufficient condition for stability.

5. RELATION TO APPROXIMATE METHODS

In this section we show that Eq. (27) for W, which is correct to order λ, is bounded below by the M-H approximation and above by the C-G-L approximation. Physical reasons for this are given.

Lower Bound of W

According to the Schwarz inequality

$$\int \left(\frac{J}{K}\right)^2 K\, d\alpha \geq \left[\int J\, d\alpha\right]^2 \Big/ \int K\, d\alpha.$$

Therefore, a lower bound for W_1 is

$$W_L = \frac{15}{4}\int P\, d\Phi \left\{\frac{\left[\int_{\alpha=0}^{\alpha=1/B_m} J(\Phi,\alpha)\, d\alpha\right]^2}{\int_{\alpha=0}^{\alpha=1/B_m} K(\Phi,\alpha)\, d\alpha}\right\}. \tag{30}$$

It is now possible to perform the α integrations. The result is discussed below. For the magneto-hydrodynamic approximation[3]

$$W_{MH} = W_0 + W_2,$$

where W_0 is formally the same as Eq. (27.1) and

$$W_2 = \frac{\gamma}{2}\int P(\nabla\cdot\xi)^2\, d^3\mathbf{x}. \tag{31}$$

In this case ξ is defined by $\delta\mathbf{V} = p\xi$, where $\delta\mathbf{V}$ is the macroscopic velocity. Therefore, ξ has three components, whereas the ξ in W has only two components. However, W_0 is stationary with respect to variations in $\xi_\parallel$, and we may choose $\xi_\parallel$ so that W_2 is also minimized. The condition for this is

$$\nabla\cdot\xi = \int_{l_1}^{l_2} \nabla\cdot\xi_\perp \frac{dl}{B(l)} \Big/ \int_{l_1}^{l_2} \frac{dl}{B(l)}, \tag{32}$$

which is constant along a flux tube. By substituting this value for $\nabla\cdot\xi$ into Eq. (31), it can be shown that $W_2 = (3\gamma/5)W_L$. For a monatomic gas, $\gamma = 5/3$, so that

$$W \geq W_{MH}. \tag{33}$$

Upper Bound of W

This time the Schwarz inequality is applied to Eq. (27.3) in the form

$$\frac{[J(\Phi,\alpha)]^2}{K(\Phi,\alpha)} \leq \int_{l_1(\alpha)}^{l_2(\alpha)} \frac{\left\{\left[1 - \frac{3}{2}\alpha B(l')\right]\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\xi + \frac{\alpha B(l')}{2}\cdot\xi\right\}^2}{[1-\alpha B(l')]^{\frac{1}{2}}}\, dl'. \tag{34}$$

If the right-hand side of this inequality is substituted for J^2/K in Eq. (27.3), the α integrations can be carried out and we obtain an upper bound for W_1:

$$W_U = \frac{1}{2}\int P[3(\mathbf{n}\cdot\mathbf{n}\cdot\nabla\xi)^2 - 2(\mathbf{n}\cdot\mathbf{n}\cdot\nabla\xi)\nabla\cdot\xi + 2(\nabla\cdot\xi)^2]\, d^3\mathbf{x}. \tag{35}$$

When $\omega,\ \omega_p \gg p \gg 1/\tau$, Eq. (21) is applicable for δf. The pressure tensor has the same form as Eq. (24), with

$$\begin{aligned}\delta P_\perp &= P[\mathbf{n}\cdot\mathbf{n}\cdot\nabla\delta\xi - 2\nabla\cdot\delta\xi],\\ \delta P_\parallel &= -P[2\mathbf{n}\cdot\mathbf{n}\cdot\nabla\delta\xi + \nabla\cdot\delta\xi].\end{aligned} \tag{36}$$

By using Eq. (27.2), we find that in this case the potential energy change is $W = W_0 + W_U$. [Equation (26) includes a term $\rho_m p^2\xi^2$, which is clearly the kinetic energy; it vanishes in the marginal stability case and is omitted here.] Thus an upper bound for the marginal stability criterion is just the potential energy change for large p. The upper bound also has a direct interpretation in terms of the C-G-L approximation. Equations (2) and (3) are differentiated and linearized,

$$\frac{B^2}{\rho_m^3}[p\delta P_\parallel + (\delta\mathbf{V}\cdot\nabla)P] + \frac{2P}{\rho_m^3}B\cdot[p\delta\mathbf{B} + \delta\mathbf{V}\cdot\nabla\mathbf{B}] - \frac{3B^2P}{\rho_m^4}[p\delta\rho_m + \delta\mathbf{V}\cdot\nabla\rho_m] = 0. \tag{37}$$

$$\frac{1}{\rho_m B}[p\delta P_\perp + \delta\mathbf{V}\cdot\nabla P) - \frac{P}{\rho_m B^3}\mathbf{B}\cdot[p\delta\mathbf{B} + \delta\mathbf{V}\cdot\nabla\mathbf{B}] - \frac{P}{\rho_m^2 B}[p\delta\rho_m + \delta\mathbf{V}\cdot\nabla\rho_m] = 0.$$

After substituting $\delta\mathbf{V} = p\delta\xi$, $\delta\mathbf{B} = \nabla\times\delta\xi\times\mathbf{B}$, and $p\delta\rho_m + \delta\mathbf{V}\cdot\nabla\rho_m = -\rho_m\nabla\cdot\delta\mathbf{V}$, Eq. (36) is obtained for $\delta P_\perp$ and $\delta P_\parallel$. However, in the C-G-L theory these expressions contain a $\xi_\parallel$. But $\delta\mathbf{E}$ and $\delta\mathbf{B}$ are unaffected by any component of $\delta\xi$ along the field lines, and the motion of particles is due entirely

to these fields. Therefore, a more reasonable version of the C-G-L theory would appear to be obtained by considering $\xi_\parallel$ to be zero in Eq. (36). With this interpretation the upper bound Eq. (35) agrees with C-G-L.

6. PHYSICAL INTERPRETATION

The potential energy of Eq. (27.3) can be given a more transparent form. To this end, we calculate the change in kinetic energy T, which a particular particle will experience from the perturbation, by making use of the invariance of the action integral[5]

$$\delta A = \delta \int_{l_1}^{l_2} V_\parallel \, dl = \delta \int_{l_1}^{l_2} \left[\frac{2}{m}(T - \mu B)\right]^{\frac{1}{2}} dl = 0. \tag{38}$$

The integration is carried out along the perturbed orbit rather than the unperturbed one as heretofore. μ is still a constant of the motion,[5] but T is not.

After expressing the change in B and dl on the moving line in terms of ξ, one finds for the change in kinetic energy of a particle

$$\delta T = 2T \frac{J(\Phi, \alpha)}{K(\Phi, \alpha)}. \tag{39}$$

After some algebra, Eq. (27.3) becomes

$$W_1 = \frac{5}{4} \sum_{\substack{\text{all} \\ \text{particles}}} \frac{(\delta T)^2}{T}. \tag{40}$$

From Eqs. (30) and (31) we see that the M-H approximation corresponds to

$$W_2 = \frac{5}{4} \sum_{\substack{\text{all} \\ \text{particles}}} \frac{\langle \delta T \rangle^2}{T}, \tag{41}$$

where the brackets indicate an average with respect to α over all particles of energy T on a line of force. Such an average is of course inherent in a fluid-type equation of state where all the particles on a line are supposed to remain in thermal equilibrium.

Similarly, the C-G-L upper limit can be written as

$$W_U = \frac{5}{4} \sum_{\substack{\text{all} \\ \text{particles}}} \frac{(\delta T_U)^2}{T}, \tag{42}$$

where δT_U is the energy change predicted by using, instead of Eq. (38), the equation

$$\delta(V_\parallel \, dl) = 0. \tag{43}$$

In other words, the energy change for a particle at a particular point on a field line is calculated as if the fields were behaving all along its orbit just as they are locally. This provides a physical understanding of the meaning of the neglect of the third-moment pressure transport terms, which is the essence of the C-G-L theory. This approximation should agree with the high-frequency limit, where the past history is forgotten.

Thus, in the correct theory, one must average along each particle's orbit and then sum over all particles. These operations are replaced by a double average in the M-H approximation and by a double sum in the C-G-L approximation. In view of this interpretation, the inequalities which we find are not surprising.[10]

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[10] *Note added in proof:* An alternate method of deriving the energy change W_1 due to a slow perturbation ($p \to 0$) is to calculate $\delta P_\perp$ and $\delta P_\parallel$ by summing the contribution from all particles as given by Eq. (39). Substituting $\delta P_\perp$ and $\delta P_\parallel$ into Eq. (27.2) gives our final result of Eq. (27.3) for W_1. By circumventing the equations of motion this gives a much simpler derivation of W_1. However, it does not permit as strong a formulation of the necessary and sufficient conditions for stability.

Basically the same results as obtained in this paper have been derived independently by Kruskal and Oberman [Phys. Fluids 1, 275 (1958)] from a completely different point of view, i.e., a thermodynamic argument.